DISSERTATION

Discrete polyharmonic functions and their applications in combinatorics

Diskrete polyharmonische Funktionen und ihre Anwendung in der Kombinatorik

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Polyharmonic Functions and their Applications in Combinatorics

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Contents

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Abstracts in English, German and French

English abstract

The starting point for this thesis was the article [36] by Chapon, Fusy and Raschel, where the authors noticed that in some asymptotic expansions of lattice path models in the quarter plane the dependency on the endpoint is given by so-called discrete polyharmonic functions, and went to show that the continuous analogue holds (i.e. the heat kernel allows for an expansion using continuous polyharmonic functions).

It was already shown by Denisov and Wachtel in [51] that, under some slight technical assumptions, the asymptotics of the number of lattice paths in a cone are in a first-order approximation directly tied to discrete harmonic functions, i.e. that if $q(x;n)$ denotes the number of paths from the origin to x with n steps, we have, up to periodicity,

$$q(x;n) \sim \gamma^n \frac{V(x)}{n^c}.$$

Here, γ and c are some (real) constants, and V is a discrete harmonic function. It is now a natural question to ask whether one could extend such asymptotics, i.e. whether it would be possible to write for any $k \in \mathbb{N}$,

$$q(x;n) = \frac{\gamma^n}{n^c} \sum_{p=1}^{k-1} \frac{v_p(x)}{n^p} + \mathcal{O}\left(\frac{1}{n^k}\right), \tag{1}$$

where γ, c are constants as before, and now each $v_p(x)$ is a (discrete) polyharmonic function of degree p. This would in a sense be an analogue to the continuous case, where an asymptotic heat kernel expansion allows for a very similar representation [36, 11].

Furthermore, unlike in the continuous case, the computation of discrete polyharmonic functions in cones has, to the author's knowledge, not been studied before.

These two topics will form the main part of this thesis. After introducing some basic notions and going through some technical prerequisites in Chapters 1 and 2, it will be shown in Chapter 3 how one can construct a basis of the space of all discrete polyharmonic functions. This will be done in two ways; one of them arguably more straightforward and purely algebraic in nature, which works in any case and leads to a basis consisting of functions with algebraic generating function. The downside to this basis, however, is that the functions therein does not allow for a scaling limit, and that they do not allow an easy representation of the polyharmonic functions appearing in the asymptotics of lattice paths in the quarter plane. The second method uses decoupling functions similar as in [12], and allows us to construct discrete polyharmonic functions for all finite group models. If the correlation coefficient is an integer fraction of π, then the resulting functions will even have rational generating functions of a rather nice shape.

In Chapters 4 and 5, the question about the form of asymptotics of the number of lattice paths in the quarter plane will be addressed. For so-called orbit-summable models, which exhibit a remarkable algebraic property tied to their reflection group and were first studied in [31], we will see in Chapter 4 using a saddle point method that one can indeed find an

asymptotic expansion as in (1). In Chapter 5, a class of infinite group models is treated, using a parametrization of the kernel curve via Jacobi ϑ-functions. Perhaps surprisingly, it turns out that in this case the asymptotic expansion is somewhat more complicated, including logarithmic terms. Nonetheless, the dependency on the endpoint is still given by discrete polyharmonic functions.

Deutsche Kurzfassung

Der Ausgangspunkt für diese Dissertation war der Artikel von Chapon, Fusy und Raschel [36], in dem die Autoren bemerkten, dass die Abhängigkeit vom Endpunkt in der asymptotischen Entwicklung der Anzahl von Gitterpfaden in der Viertelebene bei manchen Modellen durch sogenannte diskrete polyharmonische Funktionen gegeben ist, und das dazugehörige Analogon im Kontinuierlichen zeigten.

Denisov und Wachtel hatten in [51] bereits gezeigt dass, unter milden technischen Voraussetzungen, die Asymptotik dieser Art von Gitterpfaden in erster Ordnung eng mit diskreten harmonischen Funktionen verknüpft ist. Bezeichnet $q(x, n)$ die Anzahl der Pfade vom Ursprung nach x mit n Schritten, so ist nämlich (abgesehen von etwaiger Periodizität)

$$q(x; n) \sim \gamma^n \frac{V(x)}{n^c}.$$

Dabei sind γ und c (reelle) Konstanten, und V ist eine diskrete harmonische Funktion. Es ist nun eine sehr natürliche Frage ob sich das erweitern lässt, also ob es möglich ist, für ein beliebiges $k \in \mathbb{N}$ eine Entwicklung der Form

$$q(x; n) = \frac{\gamma^n}{n^c} \sum_{p=1}^{k-1} \frac{v_p(x)}{n^p} + \mathcal{O}\left(\frac{1}{n^k}\right)$$

zu finden, wobei γ, c wie zuvor Konstanten sind, und jede Funktion v_p (diskret) polyharmonisch vom Grad p ist. Das wäre in einem gewissen Sinn ein Analogon zum stetigen Fall, in dem der Wärmeleitungskern im Wesentlichen dieselbe Repräsentation erlaubt [36, 11].

Abgesehen davon ist (soweit dem Autor bekannt) auch die Konstruktion von diskreten polyharmonischen Funktionen in der Viertelebene (im Gegensatz zu 'normalen', kontinuierlichen polyharmonischen Funktionen) noch nicht untersucht.

Diese zwei Fragen werden den Hauptteil dieser Dissertation ausmachen. Nachdem in den Kapiteln 1 und 2 einige Grundlagen und technische Voraussetzungen eingeführt werden, werden wir in Kapitel 3 sehen wie eine Basis aller diskreten polyharmonischen Funktionen konstruiert werden kann. Das wird auf zwei Arten passieren; die erste davon rein algebraisch und womöglich etwas direkter. Diese funktioniert in jedem Fall, führt aber zu einer Basis die von einem kombinatorischen Standpunkt aus nur wenig brauchbar ist. Die zweite Methode verwended sogenannte 'decoupling functions', ähnlich wie in [12], und erlaubt uns, diskrete polyharmonische Funktionen für alle Modelle mit endlicher Gruppe zu konstruieren. Ist π ein ganzzahliges Vielfaches des Arkustangens des sogenannten Korellationskoeffizienten, so haben die resultierenden Funktionen sogar rationale erzeugende Funktionen mit einer recht simplen Struktur.

Die Kapitel 4 und 5 sind der Frage nach der Form einer asymptotischen Entwicklung der Anzahl von Gitterpfaden in der Viertelebene gewidmet. Für sogenannte Orbit-summierbare Modelle, die aufgrund der algebraischen Eigenschaften ihrer Reflexionsgruppe bemerkenswert schöne Lösungen zulassen und zuerst in [31] untersucht wurden, werden wir in Kapitel 4 mit Hilfe der Sattelpunktmethode sehen, dass die Vermutung der Autoren von [36] tatsächlich stimmt, und erhalten eine Asymptotik wie bereits erwähnt. In Kapitel 5 betrachten wir eine Klasse von Modellen mit unendlicher Gruppe. Das Hauptwerkzeug dabei ist die Parametrisierung der resultierenden elliptischen Kurve mittels Jacobi ϑ-Funktionen. Es ist vielleicht etwas überraschend dass sich die Asymptotik in diesem Fall anders verhält; sie enthält nun auch logarithmische Terme. Was sich hingegen nicht ändert ist, dass die Abhängigkeit vom Endpunkt durch diskrete polyharmonische Funktionen gegeben ist.

Résumé en français

Le point de départ de cette thèse est l'article [36], écrit par Chapon, Fusy et Raschel. Dans cet article les auteurs remarquent que dans un développement asymptotique du nombre de chemins dans le quart de plan $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, la dépendance du point final est décrite par des fonctions dites 'polyharmoniques discrètes'. Ils montrent aussi que dans le cas continu, le noyau de la chaleur conduit à un un développement utilisant des fonctions polyharmoniques continues.

Il est déjà connu grâce à Denisov et Wachtel [51] que, sous des hypothèses plutôt faibles dans une approximation de premier ordre, ce développement asymptotique est étroitement lié aux fonctions harmoniques discrètes. Plus précisement, soit $q(x; n)$ le nombre de chemins de l'origine au point x et de longueur n, alors on peut écrire (aux questions de périodicité près)

$$q(x; n) \sim \gamma^n \frac{V(x)}{n^c},$$

où γ, c sont des constantes (réelles), et $V(x)$ est une fonction harmonique discrète. On peut alors se poser la question si on peut étendre ce résultat, c'est-à-dire écrire pour un $k \in \mathbb{N}$ quelconque

$$q(x; n) = \frac{\gamma^n}{n^c} \sum_{p=1}^{k-1} \frac{v_p(x)}{n^p} + \mathcal{O}\left(\frac{1}{n^k}\right),$$

où γ, c sont des constantes comme ci-dessus, et chaque v_p est une fonction polyharmonique (discrète) de degré p. Ce serait dans un certain sens un analogue du cas continu, où un développement du noyau de la chaleur permet un développement de cette forme [36, 11].

De plus, contrairement au cas continu, la construction des fonctions polyharmoniques discrètes dans le quart des plan n'est (à la connaissance de l'auteur) pas encore étudiée.

Ces deux questions forment la partie principale de cette thèse. Après avoir introduit certaines notions basiques et préliminaires techniques dans les chapitres 1 et 2, nous construisons dans le chapitre 3 une base de l'espace des fonctions polyharmoniques discrètes dans le quart de plan par deux méthodes différentes. La prmière est plus simple techniquement et strictement algébrique. Par contre, elle mène à une base qui est, il semble, peu applicable dans un contexte combinatoire. La deuxième approche utilise des fonctions de découplage comme dans [12], et nous permet de construire des fonctions polyharmoniques discrètes pour tous les modèles avec un groupe fini. Si π est un multiple entier de l'arc tangente du coefficient de corrélation, alors cette base est formée de fonctions dont les fonctions génératrices sont rationnelles et d'une forme assez simple.

Les chapitres 4 et 5 sont consacrés à la deuxième question, relative à la structure du développement asymptotique. Pour les modèles qui satisfont une certaine condition (nous disons dans ce cas qu'ils permettent une 'sommation orbite', ce qui signifie qu'ils ont des proprietés algébriques remarquables liées à leur groupe de réflexion [31]), nous prouvons dans le chapitre 4 la conjecture des auteurs de [36], en utilisant la méthode du points-selle. Dans le chapitre 5, nous considérons une famille de modèles avec groupes infini. L'outil principal pour traiter ces modèles est la paramétrisation de la courbe du noyau par des fonctions ϑ de Jacobi. Il se trouve, peut-être étonnamment, que pour ces modèles la structure de ce développement est vraiment plus compliquée : il contient aussi des termes logarithmiques.

Chapter 1

Introduction

> Last time, I asked: "What does mathematics mean to you?" And some people answered: "The manipulation of numbers, the manipulation of structures." And if I had asked what music means to you, would you have answered: "The manipulation of notes?"
>
> Serge Lang

The goal of this introduction is to give a brief overview over the main topics of this thesis, as well as to give a bit of context and references to recent works. We will state some principal results and sketch the main techniques used to obtain them. For precise definitions and a more technical introduction to the main concepts which will be necessary later on, see Chapter 2.

1.1 Random walks in cones

In order to give an intuition of what a random walk is, and what kind of questions one might ask themselves about them, one very commonly finds the analogy to a (very) drunk person, who happens to be somewhere and wants to find their way home. Now, due to a lack of either physical or mental capacity (or, in fact, both), this person will not just walk back home, but instead take steps in an arbitrary direction, one after another. As mathematicians, we now ask ourselves how long (i.e. how many steps) it will take for this person to get back home, or if at all[1]. For a more thorough discussion on how one or multiple drunk people, either nice or vicious, can behave/interact, their survival probability etc., see e.g. [72].

This example in fact already demonstrates very clearly the main points of a random walk[2]: we have a process, which

1. moves at (not necessarily fixed) discrete time intervals in a random way (one cannot expect the drunk person to take steps at constant time intervals, but in a finite amount of time they will surely only take a finite number of steps),

2. this movement depends on the current state, but not on anything else (if the drunk person is in front of a wall they cannot go forward, and it does not matter how they got to their current position there since they cannot remember it anyway).

Formally, this could lead to the following definition:

[1] While it might also seem natural to ask if this person will turn out to be fine, this is not a question I have found in any textbook.

[2] by which I do not mean mathematicians' lack of empathy

Definition 1. *Given a state space $\mathcal{X}$, a **random walk** is a stochastic process X_n such that*

$$\mathbb{P}\left(X_{n+1} = y \mid X_i = x_i, i = 0, \ldots, n\right) = \mathbb{P}\left(X_{n+1} = y \mid X_n = x_n\right) \quad \forall x_0, \ldots, x_n, y \in \mathcal{X}.$$

One could imagine the time between X_n and X_{n+1} to be distributed according to a suitable distribution of one's choice, but this is not important in the following. It is also interesting to note that the above definition is effectively the same as for a time-discrete Markov chain. Indeed, it turns out that the name 'random walk' is not used consistently throughout literature. In [37], for instance, a random walk is defined to be a stochastic process traversing the vertices of a graph, at each step traveling to a neighbour uniformly at random (whereas a Markov chain is defined to be the same, but the distribution does not have to be uniform). This coincides with the standard definition of a random walk on a graph (at least in the unweighted case), as used for instance in the survey paper [99], or in [127] (though for directed, weighted graphs, it is defined in the latter using the transition probabilities induced by the weights of outgoing edges; which again leads to the definition of a Markov chain in [37]).
In Donsker's famous theorem [52] about the scaling limit of random walks (although he did not use this name), the latter is just a sum of independently, identically distributed random variables. This is the same definition as is used in [51], which will be important later on. Note that this definition explicitly excludes random walks on graphs as defined above, seeing as graphs do in general not have the necessary translation symmetry. In the setting of this thesis, we will be even more specific, using the same definition as in the classical book by Spitzer [128] or in the more recent book by Lawler and Limic [100], which restricts random walks to the lattice $\mathbb{Z}^d$:

Definition 2. *Given a random variable X taking values on $\mathbb{Z}^d$ for some d, a **random walk** (with starting point $x_0 \in \mathbb{Z}^d$) is the stochastic process given by*

$$S_n := x_0 + \sum_{i=1}^{n} X_i,$$

where the X_i are independent copies of X.

Seeing as we are taking the sum of independent copies of the same random variable, it is perhaps not surprising that a kind of central limit theorem for random walks exists. This is essentially what the following famous theorem named after Donsker (based on previous work by, amongst others, Kolmogorov) states; if we let a random walk continue for long enough, and rescale appropriately, then the random walk looks like a Brownian motion, as is illustrated in Fig. 1.1.

Theorem 1.1 (Donsker)**.** *Suppose X is a d-dimensional random variable such that its second moment are finite, and with $\mathbb{E}[X] = 0$, $\mathrm{Cov}(X) = \mathrm{Id}$. Let S_n be the corresponding random walk as defined above, and let*

$$B_n(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}.$$

Then we have the convergence in distribution

$$B_n(t) \to B(t),$$

where B is the standard d-dimensional Brownian motion.

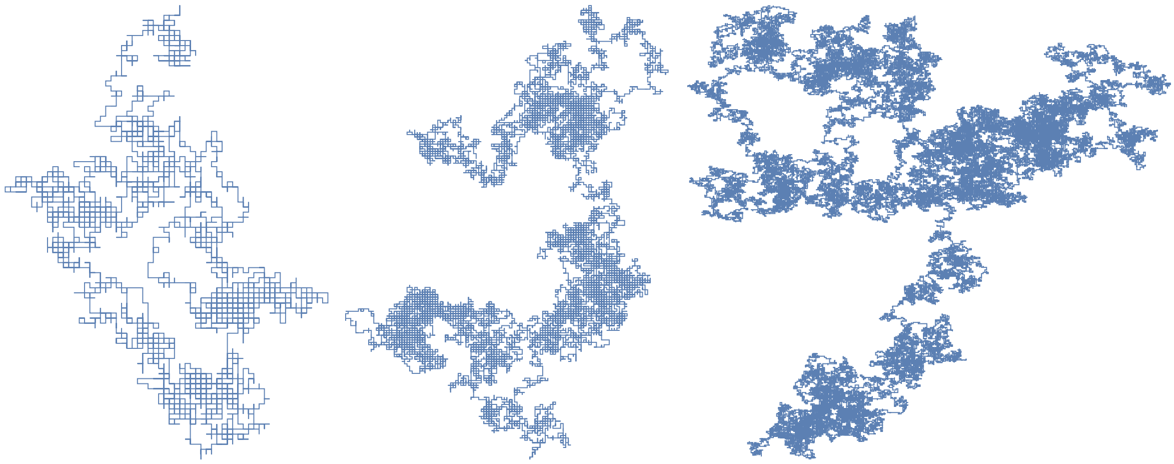


Figure 1.1: Realizations of a simple random walk (with steps $\{\rightarrow, \downarrow, \leftarrow, \uparrow\}$) with $5000, 25000$ and 100000 steps.

Here, $\lfloor nt \rfloor$ denotes the largest integer smaller than or equal to nt (a necessary requirement, as the discrete process S_n is defined only for integer indices). Note also that, if X has a covariance other than the identity matrix, via a linear transformation we can transform it to a process with identity covariance; and as linear transformations are continuous this carries over to the limit, i.e. we will obtain a Brownian motion with the same covariance. When we will be interested in the asymptotic counting of lattice paths later on, Thm. 1.1 will turn out to be very useful, as it tells us that we can approximate a long random walk by a Brownian motion. On a more intuitive level, this means that we can look at Brownian motion first, and expect many of its properties to carry over to a discrete setting.

An important special case is that of random walks in cones. A d-dimensional cone is defined by picking an open, connected subset $\Sigma \subset \mathbb{S}^{d-1}$ of the $d-1$-dimensional unit sphere, and then drawing all rays from the origin through Σ, although usually some additional conditions like convexity are imposed [51, 57]. One such cone is the quadrant, which is also a particularly nice in the sense that it satisfies all conditions in aforementioned literature. For some standard results on Brownian motion in cones see for instance [47, 11].

Random walks in cones are studied for multiple reasons. They appear in conjunction with nonintersecting paths, or paths in Weyl chambers [50, 60, 71, 94, 129], feature prominently in queueing theory [39, 38, 70, 74, 94] or finance [42], and can be used to model some other physical or biological processes [3, 15, 72]. Additionally, they are studied for their combinatorial properties; they appear for instance in conjunction with Young tableaux or the study of permutations [31, 27]. For some more examples see the references at the end of this section or in Section 1.2.2.

Given a process and a cone, one can now ask the question of how long this process will stay inside the cone. For Brownian motion, it turns out that

$$\mathbb{P}_x\left(\tau^B > t\right) \sim c_1 t^{-a} u(x), \tag{1.1}$$

where $\mathbb{P}_x$ denotes the probability depending on the starting point x of the Brownian motion, c_1 is some constant, τ^B is the first exit time of Brownian motion from the cone, a is a constant depending on the dimension and shape of the cone, and $u(x)$ is a harmonic function (see Section 1.3) on the cone [11, 47, 51, 49]. Under some moment conditions and mild technical assumptions on the shape of the cone, it is shown in [49, 51] that this translates directly to the discrete setting: given a random walk S_n, the we can define

$$V(x) := \lim_{n\to\infty} \mathbb{E}_x\left[u(S_n); \tau > n\right]. \tag{1.2}$$

This function $V(x)$ then turns out to be (discrete) harmonic (see again Section 1.3), and for the exit time τ and some constants c_2 and a (the latter again depending on the dimension and shape of the cone, as well as the random walk) we have

$$\mathbb{P}_x(\tau > n) \sim c_2 t^{-a} V(x). \tag{1.3}$$

Additionally, one has the local limit theorem

$$\mathbb{P}_x\left(\frac{S_n}{\sqrt{n}} \in D \mid \tau > n\right) \to \mu(D), \tag{1.4}$$

for compact sets D, with μ being a measure with explicit density function of $H_0 u(y) e^{-|y|^2/2}$, where H_0 is a constant and u is the same harmonic function as in (1.1). This allows us to give a first order asymptotic estimate not only of the probability that the random walk starts at a point x and survives for a certain amount of time, but also to consider the dependence on the endpoint. In particular, this leads to (ignoring possible periodicity)

$$\mathbb{P}_x\left(X_n = y, \tau > n\right) \sim \kappa n^{-a} V(x) V'(y), \tag{1.5}$$

for again κ, a some constants, and the V, V' (adjoint) discrete harmonic functions [51]. Note that the probability on the left-hand side of (1.5) is nothing but a discrete heat kernel.

In the continuous setting, a heat kernel is defined as the fundamental solution of the heat equation[3]

$$\frac{\partial u}{\partial t} = \triangle u$$

over a certain region. This heat kernel has an interpretation in the context of probability theory as well, namely it gives the transition density of Brownian motion, which is the probability density function of the transition probability kernel

$$\mathbb{P}_x(B_t \in dy, \tau > t),$$

where again τ is the first exit time [90]. This continuous heat kernel has been studied in more detail in [11, 47]; and in [36] it is shown that the continuous heat kernel admits a complete asymptotic expansion in terms of continuous polyharmonic functions for the Laplacian. One can understand Thm's 4.5 and 4.7 as discrete counterparts of this fact.

The cone which will interest us the most in the following is the quarter plane $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. The reason for this is that the combinatorics for walks in the entire or half-plane are much better understood and the respective path counting functions are a lot easier to compute explicitly via the standard kernel method, see for instance [9, 10, 133, 32]. The same applies for models on $\mathbb{Z}$ or on the half-line. Walks in the orthant, on the other hand, or generally in cones of higher dimension than 2, are understood a lot less, and many of the methods used in this thesis fail due to the existence of additional boundary terms and the resulting more complicated structure of the kernel curve (see Section 2.1.2). Walks in the quarter plane are therefore a nice middle-ground, so to say, which are not easy to understand, but still leaving us a range of tools to do interesting mathematics with. In particular, as we will be considering models with finitely many steps, it is easy to check that all regularity conditions as in [51, 49] are satisfied. It is also worth noting that for some models we have bijections from quarter plane walks to some other cone: in case of the Gouyou-Beauchamps model with steps $\{\leftarrow, \rightarrow, \nwarrow, \searrow\}$, these walks are in bijection with simple walks in the Weyl chamber $\{(x, y) : 0 \leq x \leq y\}$ [43]. Walks in Weyl chamber are of particular interest in combinatorics, seeing as they are in bijection, or at least in a close relation with a variety of different objects. See for instance [14, 13] (with a strong emphasis on the link with representation theory), [78, 60, 95, 50, 71] (in relation to reflectable random walks) and [106, 109, 108] (on highly symmetric lattice path models). For more general information about walks in Weyl chambers see e.g. [50, 60, 94].

[3] Depending on the notation there might be a factor of $\frac{1}{2}$ on the right-hand side.

1.2 Lattice path enumeration in the quadrant

While one can see a random walk as a stochastic process, it is also an interesting object from a combinatorial point of view, leading to lattice path enumeration. A lattice path is nothing but a realization of a given random walk. Alternatively, one can also give a direct definition:

Definition 3. *Given a lattice $\mathbb{Z}^d$ and a set $\mathcal{S} \subseteq \mathbb{Z}^d$ of allowed steps, a **lattice path** (of length n) is a sequence of points $x_0, x_1, \ldots, x_n \in \mathbb{Z}^d$ such that $x_k - x_{k-1} \in \mathcal{S}$ for $k = 1, \ldots, n$.*

Historically speaking, as the excellent survey article [87] describes, the reason lattice paths were first looked at were indeed due to their aptitude for modeling probabilistic processes. While it is hard to say that a single person 'invented' lattice paths, possibly the first time that something one might call a lattice path was used was in 1654, when Pascal wrote to Fermat, concerning the problem called now 'Problem of points' or 'Division of the stakes'. In this problem, two people are playing dice. Each turn, the one with a higher number gains a point, and the first one to reach a fixed number of points wins the game. However, they have to leave before the game is decided, so they have to divide the prize between them. Clearly, if at the time they are finishing one of the players has more points than the other, then dividing equally might not be fair, so the question was how to justly distribute the money.[4] While Pascal's and Fermat's proposed solution did not in fact introduce lattice paths as a geometrical object, it already contained all the important ideas: they reasoned that the division of the prize should not in fact depend on the past but only the current state of the game, and from then on took a recursive approach, in a sense counting paths for either player to victory [92].
It turns out that this problem is very closely related to the one called the 'Gambler's ruin', where at each turn not only does the winner gain a point, but he takes it from his opponent; so one of the two players will always have 0 points. Already mentioned by Pascal in a letter, it was then treated amongst others by Huygens, Bernoulli and de Moivre, as well as a bit later by Bertrand, Laplace and Lagrange.
The gambler's ruin is itself closely tied to the so-called 'Ballot problem', in which, given an election where two candidates A and B obtain fractions p and q of the total votes respectively, one wants to know the probability that the victor is ahead throughout the entire count. This problem was proposed by Bertrand and solved by André [4], although he did in fact not use the famous reflection principle named after him. It was in conjunction with this problem that Allen Whitworth made what might have been the first drawing of an actual lattice path [134].
From a modern point of view, it is clear how these three problems all relate to counting lattice paths: for the problem of points, one can consider a walk moving either upwards or to the right, and the game ending once it hits the upper or right edge of a square; for the gambler's ruin the path goes to the northeast or southeast, and the game ends if the path leaves a horizontal strip of a certain width, and for the ballot problem we are interested in what is now commonly called Dyck paths ending at a certain point. One might argue at this point that the gambler's ruin is formulated more easily as a question of probabilities rather than counting lattice paths, seeing as we do not know the length of the path, and are only interested in where it leaves the strip. Indeed, a modern solution would probably avoid the counting problem altogether and make direct use of either martingale theory or the underlying random walk.

Nowadays, lattice paths are not only studied in the context of probability, but they have instead become a main topic of interest in combinatorics. Clearly, there is a link between the two: the discrete heat kernel in a cone $\mathbb{P}_x(S_n = y, \tau > n)$ is nothing but the sum of all lattice paths of length n from x to y, staying inside this cone, weighed by their probability. We will

[4] This problem had in fact been considered earlier by Pacioli, but his approach was much more simplistic and not probabilistic in nature [117].

in the following consider more general weights ω_s on our step set $\mathcal{S}$, which we do not require to come from a probability distribution.

Furthermore, in this thesis we are not so much interested in giving the precise number of lattice paths from a start- to and endpoint, but rather in its asymptotic properties. A very quick overview of what kind of properties in particular we are interested in will be given in the following.

1.2.1 Asymptotic expansions

Given a step set $\mathcal{S} \subset \mathbb{Z}^2$ together with weights $(\omega_s)_{s\in\mathcal{S}}$, denote by $q(x, y; n)$ the number of lattice paths in the quarter plane (that is, paths in $\mathbb{Z}^2$ satisfying the additional condition that $x_i \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq n$) from x to y with n steps, where each path $x_0, \dots, x_n$ is counted by its weight

$$\prod_{i=1}^{n} \omega_{x_i - x_{i-1}}.$$

It turns out that for some models, we have (depending on the model at least for some $(x, y) \in \mathbb{Z}^2$) explicit formulas for $q(x, y; n)$ [30]. For the simple walk for example, consisting of steps $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ with equal probability, one finds that for $y = (i, j)$ we have

$$q((0,0), y) = 4^{-n} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n+i-j+2}{2}} \binom{n+2}{\frac{n-i-j}{2}} \tag{1.6}$$

for $i + j \equiv n \mod 2$, and 0 otherwise. For general models, however, we don't have an explicit formula. Instead, one can utilize (1.5) and obtain the asymptotic equivalence (see once again [51])

$$q(x, y; n) \sim \kappa \frac{\rho^n}{n^a} V(x) V'(y), \tag{1.7}$$

with ρ being an exponential factor which depends on the model. The expression $A(n) \sim B(n)$ here means asymptotic equivalence in the sense that

$$A(n) = B(n) + o\left(B(n)\right),$$

with the small Landau-o, or, equivalently, that we have

$$\lim_{n\to\infty} \frac{A(n)}{B(n)} = 1$$

Given (1.6), for instance, one can check that we have $\rho = 1$, and obtain (with the additional condition that $i + j$ be of the same parity as n, due to the periodicity of the model)

$$q((0,0), (i,j); n) \sim \frac{4}{\pi} \frac{(i+1)(j+1)}{n^3}. \tag{1.8}$$

This kind of asymptotic estimate is nothing new and has been considered by various authors in the past; an overview of relevant literature is given in Section 1.2.2. However, in particular given that for the simple walk we have an explicit formula, one could also try to find a more precise asymptotic approximation of $q((0,0), y; n)$. And indeed, by a simple series expansion of (1.6) at $n = \infty$ we obtain, as for instance in [36], that for any point (i, j) we have (again up to periodicity)

$$q((0,0), (i,j); n) = \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{v_p(i,j)}{n^p} \quad (n \to \infty), \tag{1.9}$$

where the functions $v_p(i,j)$ depend on the endpoint only. This equality is in the sense of [136, 110]: the asymptotic equality

$$A(n) = \sum_{p=1}^{\infty} B_p(n) \quad (n \to \infty)$$

means that for any $k \in \mathbb{N}$ we have

$$A(n) = \sum_{p=1}^{k} B_p(n) + o\left(B_k(n)\right).$$

In our case of the simple walk, we can compute the functions v_p in (1.9) explicitly. We have, for instance,

$$v_1(i,j) = (1+i)(1+j), \tag{1.10}$$

$$v_2(i,j) = -\frac{1}{2}(1+i)(1+j)(15+4i+2i^2+4j+2j^2), \tag{1.11}$$

$$\begin{aligned} v_3(i,j) = &\frac{1}{4}(1+i)(1+j)(317+168i+100i^2+16i^3+4i^4+168j+32ij+16i^2j \\ &+100j^2+16ij^2+8i^2j^2+16j^3+4j^4)). \end{aligned} \tag{1.12}$$

It turns out that each function v_p is what is called a discrete polyharmonic function of order p (see Sections 1.3 or 2.2.1). Additionally, each v_p is a polynomial.

Given this (and a few similar) examples of an asymptotic expansion of $q(x,y;n)$, a few questions suggest themselves:

1. For general models, can we obtain a complete asymptotic expansion of $q(x,y;n)$?
2. If so, what is the dependency of the lower order terms on the start- and endpoints x, y?
3. Will we only ever find exponential-polynomial parts (of the form $\frac{\rho^n}{n^\alpha}$) in n, or can there be more complicated (logarithmic, for instance) terms?

These questions were the starting point for [36], and also for a large portion of this thesis. In the setting of walks with small steps in the quarter plane, we will see that for a large class of models, we can answer them.

In terms of previous literature, to the author's knowledge, so far higher order estimates for asymptotics have barely been studied in connection with lattice path counting. There seem to be very recent results for walks on the half-line [48], showing that they allow for an expansion similar to (1.9). Some general information about asymptotic series can be found in [34, 65]. In the context of graph theory, for instance, some work in this direction has been done for example in [136, 110]. Before describing the approaches used and some of the results in this thesis concerning the asymptotic counting of lattice paths, we will give a short overview about previous work in this regard.

1.2.2 Earlier literature on asymptotics of lattice walk models

The kernel method and algebraic solutions

In lattice walk enumeration, a standard method to obtain asymptotic expansions is the so-called kernel method. In dimension 1, the kernel method yields algebraic expressions for the generating function of the numbers of excursions (with given length, starting and ending points), starting from which it is possible to compute arbitrarily precise asymptotic expansions of the coefficients, using standard singularity analysis, see [9, 10, 32] (although only one-term asymptotics are derived). In a few cases of dimensions 2 and 3, the kernel method (or subtle variations of it, using the idea of half-orbit sums) also yields algebraic expressions for the path generating function, for example for Kreweras' model, see [74, 30, 28, 31], Gessel's model [24], or for some three-dimensional models [23]. Again, in these cases, it is possible to deduce precise asymptotic expansions of the numbers of walks.

The kernel method and transcendental, D-finite solutions

In dimension 2 and more, the kernel method may also yield D-finite expressions for the generating functions as positive parts of rational functions, see [30, 31] for small steps in dimension 2, [23] for small steps in dimension 3 and [19] for large steps in dimension 2. Let us notice that these ideas go beyond the case of quadrant (or octant) walks and also apply, for instance, in the framework of walks in the slit plane or the three quarter plane [29, 33].

There are several ways to pursue and to deduce from these expressions the asymptotics of the number of walks. In a few cases, it is possible to extract the coefficients in an explicit way (for instance, for Gouyou-Beauchamps walks), and then to deduce asymptotic expansions starting from these closed-form expressions. Many examples are provided in [30, 31, 23, 43, 19].

Another possibility to continue is to use the modern theory of analytic combinatorics in several variables (ACSV), see for instance [43, 108, 106] for examples of applications in the framework of lattice walks. In principle, using ACSV, one can deduce from these positive part expressions full asymptotic expansions for the numbers of walks. However, the applicability of the method is still restricted to properties very similar to orbit-summability as outlined in Section 2.1.3, and the constants appearing in the prefactors of the asymptotic terms are not always easily computable.

The kernel method and non-D-finite solutions

In a small number of cases, the (iterated) kernel method also applies to more singular models, associated to non-D-finite generating functions, see for instance [107]. In those cases it is also possible to deduce some asymptotic estimates.

Weyl chambers

As mentioned in the previous section, Weyl chambers represent another popular class of cones, particularly due to its links with non-intersecting paths and other probabilistic and physical models, for which various asymptotic estimates exist. As mentioned above, we refer to [14, 13] (with a strong emphasis on the link with representation theory), [78, 60, 95, 50, 71] (in relation to reflectable random walks) and [106, 109, 108] (on highly symmetric lattice path models).

Guess and prove

In a few works, the authors were able to guess the asymptotic behavior of various lattice path sequences, see [20] and [7]. There are also various experiments by Tony Guttmann, which are mainly not published.

Probabilistic and potential theory

Another possible approach makes direct use of the link between probability theory and combinatorics, taking local limit theorems and translating them directly into combinatorial estimates.

In particular, the exponential growth of the number of excursions (resp. total number of walks) is given in [76] (resp. [77, 91]).

These rough estimates are refined in [51, 58], where the authors obtain precise asymptotics of the numbers of excursions and of the total numbers of walks (the last result under the additional hypothesis of a drift equal to zero or directed to the vertex of the cone). A summary of these methods is also given in the thesis [57].

In the particular case of quadrant walk models with infinite group, these asymptotics are worked out in [21]. In particular, the critical exponent is shown to be non-rational in all these infinite group models.

Another fruitful approach is based on harmonic functions, which are deeply related to these lattice walk asymptotics problems. Indeed, they first appear as prefactors in the asymptotic estimates [51]; more than that, their polynomial growth encodes the critical exponent of the number of walks [132, 45, 111]. This approach has been generalized in [36], where formal asymptotic expansions are derived in terms of polyharmonic functions.

Boundary value problems and applications

Following the pioneering works of Iasnogorodski and Fayolle [68], Malyshev [103], see also [70], Cohen and Boxma [39] and Cohen [38], functional equations may be written for the generating functions of various probabilities (also for counting walks), which can then be treated via boundary value problems. This method results in contour integral expressions for the generating functions, on which one may try to apply singularity analysis, see [69].

Other techniques

Finally, there are a few other techniques, different from the ones mentioned above. In dimension 3, the numbers of walks may be connected to the computation of triangle eigenvalues [16, 46]. In dimension 2, there are also hypergeometric expressions [22], which lead to precise asymptotic estimates. One may consult [18] for a nice and complete survey of lattice walk problems, containing in particular many asymptotic results.

1.2.3 Approaches in this thesis

In this thesis, the approaches used will start out from the classical kernel method. The principal idea is to utilize a functional equation for the path generating function

$$Q(x, y; t) = \sum_{t\geq 0} t^n \sum_{k,l\geq 0} x^k y^l q((0,0),(k,l);n),$$

counting paths of length n from the origin to (k, l). Note that since we are working in the quarter plane, we never have negative powers anywhere, and thus $Q(x, y; t)$ is a well-defined power series. This kind of functional equation is derived for instance in [9, 31], and reads

$$K(x, y)Q(x, y) = xy - K(x, 0)Q(x, 0) - K(0, y)Q(0, y) + tQ(0, 0), \tag{1.13}$$

where $K(x, y)$ is the so-called *kernel*, encoding the combinatorial information about the model: given a model with small steps (that is, which go at most 1 in either direction), it is defined as

$$K(x, y) = xy \left[1 - t \sum_{-1\leq i,j\leq 1} \omega_{i,j} x^i y^j\right].$$

Note also that we leave out the t in the notation to keep things shorter, so we write $Q(x, y)$ instead of $Q(x, y; t)$ and so on. For the (unweighted) simple walk, for instance, one obtains

$K(x,y) = xy\left(1 - t\left[x + y + \frac{1}{x} + \frac{1}{y}\right]\right)$. For some more general information about the kernel and its properties, see Section 2.1.2. The equation (1.13) is attractive insofar as the right-hand side has a fairly simple structure; except for the single monomial xy it has nicely separated two parts depending on only x and y respectively. From this point on, we will use two different strategies depending on the model.

Approach via orbit summation

The main strategy in [31] is that of so-called orbit summation. The authors observed that the kernel $K(x,y)$ is – in the case of the simple walk – invariant under the two transformations $(x,y) \mapsto \left(\frac{1}{x}, y\right)$ and $(x,y) \mapsto \left(x, \frac{1}{y}\right)$. These two involutions form a reflection group, which is finite; in particular its orbit reads

$$(x,y) \mapsto \left(\frac{1}{x}, y\right) \mapsto \left(\frac{1}{x}, \frac{1}{y}\right) \mapsto \left(x, \frac{1}{y}\right), \tag{1.14}$$

where the last element would once again map to (x,y). Now, adding up (1.13) when substituting the orbit of (x,y), with alternating signs (the so-called 'orbit sum') gives, seeing as most terms on the right-hand side cancel,

$$xyQ(x,y) - \frac{y}{x}Q\left(\frac{1}{x}, y\right) + \frac{1}{xy}Q\left(\frac{1}{x}, \frac{1}{y}\right) - \frac{x}{y}Q\left(x, \frac{1}{y}\right) = \frac{xy - \frac{y}{x} + \frac{1}{xy} - \frac{x}{y}}{K(x,y)}. \tag{1.15}$$

The fact that $Q(x,y)$ is a power series, i.e. it contains only positive powers of x, y, directly implies that $\frac{1}{xy}Q\left(\frac{1}{x}, \frac{1}{y}\right)$ contains only negative powers of x, y. Consequently, when extracting the positive powers of the left-hand side, it vanishes. In a similar fashion, the other two unwanted terms on the left-hand side vanish if extracting positive coefficients of either x and y, leaving us with

$$xyQ(x,y) = [x^>]\,[y^>]\,\frac{xy - \frac{y}{x} + \frac{1}{xy} - \frac{x}{y}}{K(x,y)}, \tag{1.16}$$

where $[x^>], [y^>]$ denote the linear operators extracting positive powers of x and y respectively of some series. Being able to write the generating function as in (1.16) is the defining property of orbit-summable models (though the exact shape of the right-hand side will depend on the exact model). Since $K(x,y) = 1 - tS(x,y)$, where in case of the simple walk we have $S(x,y) = \left(x + y + \frac{1}{x} + \frac{1}{y}\right)$, the denominator on the right-hand side can be seen as a geometric series in t, and we can write

$$\left[x^k y^l t^n\right] xyQ(x,y) = \left[x^k y^l\right] S(x,y)^n \left(xy - \frac{y}{x} + \frac{1}{xy} - \frac{x}{y}\right). \tag{1.17}$$

This finally allows us to compute this coefficient via a Cauchy type integral, of the form

$$\left[x^k y^l t^n\right] xyQ(x,y) = -\frac{1}{4\pi^2}\int_\gamma \frac{S(x,y)^n\left(xy - \frac{y}{x} + \frac{1}{xy} - \frac{x}{y}\right)}{x^{k+1}y^{l+1}}\mathrm{d}x\mathrm{d}y, \tag{1.18}$$

where γ is suitable contour around the origin. The main idea in Chapter 4 is to utilize a saddle point approach in order to approximate the integral in (1.18). One can select the contour γ such that the absolute value of $S(x,y)$ is maximal at a finite set of saddle points, which means that for large values of n the integral will behave approximately like its local value around these points. This will then allow us to find an asymptotic expansion of $q((0,0),(k,l);n)$, and by symmetry properties also one of the form $q((u,v),(k,l);n)$. In particular, this will allow us to show

Theorem (see Thm. 4.5)**.** *Suppose that $\mathcal{S}$ is a step set satisfying the general assumptions stated in Chapter 2 and that $\mathcal{S}$ is orbit-summable (see Section 2.1.3). Then there are a constant $c \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$, roots of unity $\alpha_i, \beta_i, \zeta_i$ as well as functions $v_p(k,l,u,v)$ such that for any $m \in \mathbb{N}$ we have*

$$q((u,v),(k,l);n) = \frac{\gamma^n}{n^c}\left[\sum_{p=1}^{m-1} \frac{v_p(k,l,u,v)\sum_{i=1}^{r} \alpha_i^{u-k}\beta_i^{v-l}\zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right)\right].$$

The $v_p(k,l,u,v)$ are polynomials precisely if the drift is zero (else they contain exponential factors). In this case they are of bidegree $c+2(p-1)$ in both (k,l) and (u,v), and of total degree $2c+4(p-1)$. Each $v_p(k,l,u,v)$ is multivariate γ-polyharmonic of degree p (see Section 2.2).

The values (and number) of the roots of unity appearing depend on the saddle points defined more closely in Section 2.1.4, as well as the periodicity properties of the model. Thm. 4.5 completely answers the three questions posed above in the case of orbit-summable models. Additionally, by finding a suitable decomposition of the functions $v_p(k,l,u,v)$, it allows us to extend (1.7) (albeit requiring far stronger assumptions). We will also see in App. B that this method extends to orbit-summable models with large steps, or in higher dimension.

In addition, we will see that we can rewrite each v_p as a sum of the form

$$v_p(u,v,k,l) = \sum_{i=1}^{k_p} h_{p,i}(u,v)g_{p,i}(k,l), \tag{1.19}$$

where the summation index k_p is bounded polynomially in p, and the $h_{p,i}$ and $g_{p,i}$ are both polyharmonic of degrees which sum up to at most p for each i, see Thm. 4.7. In particular, for $p=1$ we recover the result (1.5) of [51] (albeit requiring much stronger conditions).

Approach via elliptic functions

Another approach in dealing with the functional equation (1.13) lies in parametrizing the kernel curve $\mathcal{C}_t := \{(x,y) : K(x,y;t) = 0\}$, which is, in non-degenerate (see Chapter 2) cases and for small t, an elliptic curve [82, 56]. A parametrization of the kernel curve via elliptic functions has been used for instance in [97, 82] to show that the path generating function $Q(x,y;t)$ is not differentially finite for unweighted infinite group models. Asymptotics of lattice path enumeration using elliptic functions, however, is a fairly new concept, first being used, to the author's knowledge, in [61]. The main idea in Chapter 5 will be to find a parametrization for a class of infinite group models with steps $\{\leftarrow, \uparrow, \rightarrow, \downarrow, \nearrow\}$, the latter with an arbitrary weight a (not equal to 0, which would result in the simple walk). It turns out that in this case we can find a parametrization of the form

$$X(z) = c\frac{\vartheta(z \mid \tau)\vartheta(z+\gamma \mid \tau)}{\vartheta(z-\gamma \mid \tau)\vartheta(z+2\gamma \mid \tau)}.$$
$$Y(z) = c\frac{\vartheta(z \mid \tau)\vartheta(z-\gamma \mid \tau)}{\vartheta(z+\gamma \mid \tau)\vartheta(z-2\gamma \mid \tau)} = X(-z).$$

Here, $\vartheta(z \mid \tau)$ is the Jacobi ϑ-function. It can be written either in terms of τ or $q = e^{i\pi\tau}$, and is given by

$$\vartheta(z) = \vartheta(z|\tau) = \sum_{n=0}^{\infty}(-1)^n e^{i\pi\tau(n+\frac{1}{2})^2}\left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right)$$
$$= \vartheta(z,q) = 2i\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z).$$

Intuitively, writing the functions in terms of τ gives us more control on the geometric side, when trying to construct functions with certain periods (as the invariant $J(z)$ defined in Chapter 5), whereas the expression in terms of q is more convenient to handle on the analytic side, as we can treat $\vartheta(z,q)$ as a power series in q (which converges for $q<1$).

We can rewrite the functional equation (1.13) on the kernel curve as

$$X(z)Q\left(X(z),0\right)+Y(z)Q\left(0,y(z)\right)=X(z)Y(z). \tag{1.20}$$

As $X(z),Y(z)$ by construction satisfy $K\left(X(z),Y(z)\right)=0$, we can rewrite

$$X(z)Y(z)=\frac{1}{a}\left[\frac{1}{t}-X(z)-Y(z)-\frac{1}{X(z)}-\frac{1}{Y(z)}\right] \tag{1.21}$$

This in turn implies that we can seperate the terms depending on $X(z)$ and $Y(z)$ and find a function $J(z)$ such that we can express $J(z)$ (in some regions) as a function in $X(z)$ or as another function in $Y(z)$, formally

$$J(z)=\frac{1}{2t}-atX(z)Q\left(X(z),0\right)-X(z)-\frac{1}{X(z)} \tag{1.22}$$
$$=-\left(\frac{1}{2t}-atY(z)Q\left(0,Y(z)\right)-Y(z)-\frac{1}{Y(z)}\right). \tag{1.23}$$

Making use of periodicity properties of $X(z),Y(z)$ allows us to give an explicit expression for $J(z)$ in term of ϑ-functions, namely

$$J(z)=-\frac{\vartheta'(0,\frac{\gamma}{\pi})\vartheta(2\gamma,\tau)\vartheta(z+\frac{\pi}{2},\frac{\gamma}{\pi})}{c\vartheta(\frac{\pi}{2},\frac{\gamma}{\pi})\vartheta'(0,\tau)\vartheta(z,\frac{\gamma}{\pi})}. \tag{1.24}$$

Utilizing (1.24) together with, for example, (1.22) then lets us to obtain an expression for $X(z)$ in terms of Jacobi ϑ-functions. To turn this into an asymptotic expansion, we then make use of the Jacobi transformation. This transformation makes use of the symmetry $q\longleftrightarrow\hat{q}$ given by

$$\log(q)\log(\hat{q})=\pi^2, \tag{1.25}$$

and the associated the Jacobi identity

$$\vartheta(z,q)=\sqrt{-\frac{\log(\hat{q})}{\pi}}\exp\left(\frac{\log(\hat{q})z^2}{\pi^2}\right)\vartheta\left(\frac{i}{\pi}\log(\hat{q})z,\hat{q}\right). \tag{1.26}$$

This transformation admits a direct combinatorial interpretation, in that applying it on the theta-expression for the generating function, we will exactly obtain the expression of the function at the critical point. From here, using classical singularity analysis, we can deduce immediately a complete asymptotic expansion. In other words:

- $q=0$ $(\hat{q}=1)\longleftrightarrow$ series (Taylor) expansion at $t=0$;
- $\hat{q}=0$ $(q=1)\longleftrightarrow$ series expansion at the critical point $t=\frac{1}{\rho}$.

While the duality (1.25) is classical in physics literature, its use to obtain asymptotic expansion at criticality seems to be less studied, though there has been some work done e.g. by Kostov [96] on the six-vertex model on a random lattice.

The approach using Jacobi ϑ-functions finally allows us to show for the class of models in question the following

Theorem (see Thm. 5.9)**.** *If the angle θ, given as the arctangent of the correlation coefficient (see Section 2.1.2), has the property that π/θ is irrational, then for any $p > 0$ (not necessarily integer), there is a $\rho > 0$ such that we have*

$$q((0,0),(k,l);n) = \gamma^n \sum_{\substack{i\geq 1, j\geq m\geq 0\\ i\rho+j<p}} v_{i,j,m}(k,l)\frac{(\log n)^m}{n^{k\rho+\ell}} + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right),$$

where the $v_{i,j,m}$ are discrete γ-polyharmonic functions of order $j-m+1$. If $\pi/\theta = \frac{u}{v} \in \mathbb{Q}$ with u and v coprime integers, then the same holds with the additional condition that the summation index k be at most v.

Thm. 5.9 gives us a complete asymptotic expansion, and it is novel in the sense that it is, to the author's knowledge, the first time that logarithms appear in an asymptotic expansion of lattice path enumeration in the quadrant, nor do they occur in the continuous setting, when deriving a complete asymptotic expansion of the heat kernel [36]. Seeing as it appears that the methods used might generalize, one could conjecture that a representation as in Thm. 5.9 might hold for all models with infinite group.

1.3 Polyharmonic functions

Compared to harmonic functions, which are probably at least vaguely familiar to anyone having ever heard an introduction to (complex) analysis, polyharmonic functions have been a lot less studied. Still, there are a variety of results in particular in the continuous case; for a more thorough introduction see for instance [5]. While in particular (1-)harmonic and biharmonic functions have by now found a variety of applications in the theory of stochastic processes and physics (see for instance [102, 6, 41]), the appearance of discrete polyharmonic functions in asymptotics of path counting problems (as will be discussed in Chapters 4 and 5) was to the author's knowledge noticed only fairly recently in [36]. Correspondingly, there is only very little literature about them. On finite graphs and/or trees, they have been studied in [120, 40, 84], but it appears that describing them on infinite graphs, in particular in cones, has not been done before. Let us consider the better known continuous polyharmonic functions first, and only then proceed to the discrete case.

1.3.1 Continuous polyharmonic functions

Though most of what follows works in $\mathbb{R}^d$ for arbitrary d, we will from now on restrict ourselves to the two-dimensional setting. In this case, given a Brownian motion with (positive semidefinite) covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

then its infinitesimal generator is the Laplacian

$$\triangle = \frac{1}{2}\left(\sigma_{11}\frac{\partial^2}{\partial x^2} + 2\sigma_{12}\frac{\partial^2}{\partial x \partial y} + \sigma_{22}\frac{\partial^2}{\partial y^2}\right).$$

Definition 4. *Given an open set $A \subseteq \mathbb{R}^2$, we call a function f* **(continuous) polyharmonic of degree** *k, if*

$$\triangle^k f(x) = 0 \quad \forall x \in A, \tag{1.27}$$
$$f(x) = 0 \quad \forall x \in \partial A. \tag{1.28}$$

We call a function **(continuous) harmonic** *if it is polyharmonic of degree 1.*

Clearly, the fact whether or not a given function is polyharmonic depends on the choice of the Laplacian. In the following, we will usually have $A = \mathbb{R}^+ \times \mathbb{R}^+$. Note also that we have defined polyharmonic functions here as solutions to a Dirichlet problem; one could be more general here and allow for other boundary conditions. However, in the context of lattice path enumeration later on it will become clear why this definition suffices for our purposes.

Possibly the first person to systematically study (continuous) polyharmonic functions was E. Almansi, who in 1899 showed the following

Theorem 1.2 (Almansi, [2])**.** *Let $A \subseteq \mathbb{C}^d$ be a star-shaped domain around 0. Then we can write any function $f(x)$ which is polyharmonic of degree k[5] uniquely as*

$$f(x) = \sum_{n=0}^{k-1} |x|^{2n} h_n(x), \tag{1.29}$$

where the $h_n(x)$ are harmonic functions.

For a proof see for instance [5, Prop. 1.3]. Thm. 1.2 is interesting due to (at least) two reasons: it gives us a canonical representation of a given polyharmonic function; but it also allows us to iteratively construct polyharmonic functions of higher degree from a given sequence of harmonic functions. In this sense, one could argue the construction of polyharmonic functions in Chapter 3 works in a rather similar fashion. Similar to how discrete polyharmonic functions appear in the asymptotics of some path-counting problems, their continuous analogues occur when studying the asymptotics of exit times of Brownian motions [5, VI.], [36, Thm. 2.3].

The theory of continuous polyharmonic functions is well-developed, and computing them in a region as nice as the quarter plane is not a big challenge anymore. A common approach is to switch to polar coordinates, and then consider eigenfunctions of the resulting spherical Laplacian, as in [36, 79]. In order to find continuous (poly-)harmonic functions, the standard approach is to transform the Laplacian into polar coordinates, and then construct them via its eigenfunctions, see e.g. [36]. An alternative approach, which is closer to the one applicable in the discrete setting, is suggested in [125, App. A]: seeing as applying the Laplacian translates to multiplication with a quadratic polynomial to the Laplace transform, this makes it possible to rewrite conditions (1.27,1.28) as a functional equation in the Laplace transform. This functional equation, which is very similar to (1.49), is a rather large hint that there might be a connection between discrete and continuous polyharmonic functions; and in some cases one can use these equations to compute discrete and polyharmonic functions in very similar manners. For the harmonic and biharmonic cases this has already been done via a direct computation in [36, 2.2]; this will be explored in more detail in Section 3.4.2.

1.3.2 Discrete polyharmonic functions

In order to define discrete polyharmonic functions, one first needs a suitable discretization of a Laplacian. To give an intuition of how to do so, recall first the defining property of an infinitesimal generator of a Brownian motion. It serves in a similar fashion as a Taylor expansion in time, i.e. for a reasonably smooth function $f(x)$ we have

$$\mathbb{E}_x \left[f\left(B(t)\right) \right] \approx f(x) + \frac{1}{2} \triangle f(x). \tag{1.30}$$

If we compute the same expectation for a discrete random walk, we obtain

$$\mathbb{E}_x \left[f(X_1) \right] = \sum_s \omega_s f(x+s) = Pf(x), \tag{1.31}$$

[5] In fact, allowing for infinite summation one can locally represent any holomorphic function in this manner, cf. [5, Thm. 1.1].

with P defined as above being the standard Markov operator, see e.g. [118, 135]. Bringing (1.31) into the same form as (1.30) and forgetting about the factor $\frac{1}{2}$, we thus obtain

$$\triangle = P - \mathrm{Id}, \tag{1.32}$$

which is the standard definition for a discrete Laplacian [100, 135]. Given this operator, we can now proceed directly as in Def. 4.

Definition 5. *Given a subset A of $\mathbb{Z}^2$, we call a function f* ***(discrete) polyharmonic of degree*** *k, if*

$$\begin{aligned} \triangle^k f(x) =& 0 \quad \forall x \in A, \qquad (1.33)\\ f(x) =& 0 \quad \forall x \in A^c. \qquad (1.34) \end{aligned}$$

We call a function ***(discrete) harmonic*** *if it is polyharmonic of degree* 1.

Later on, we will sometimes have a factor t on the right-hand side of (1.32), i.e.

$$\triangle = P - t\,\mathrm{Id}\,. \tag{1.35}$$

In this case, unless the definition of the corresponding Laplacian is clear from the context, we speak of (discrete) **t-polyharmonic** functions.

As for the continuous case, in order to decide if a given function is polyharmonic or not, we first need to fix a random walk, and with it the discrete Laplacian. In the following, we will choose for A the quarter plane $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, though the above definitions (as well as the construction for a discrete Laplacian) would work for any graph.

A first intuitive reason for why polyharmonic functions might be interesting from a combinatorial point of view is that if we have a quantity which behaves similar to a lattice path and has reasonably nice asymptotics, then we will find polyharmonic functions in the coefficients. More formally, one can show

Lemma 1.3. *Let $q(x;n)$ be some combinatorial quantity depending on n and a point $x \in \mathbb{Z}^d$, and let $\mathcal{S} \subset \mathbb{Z}^d$ be some step set. Suppose that $q(x)$ satisfies a recursive relation of the form*

$$q(x;n+1) = \sum_{s\in\mathcal{S}} \omega_s q(x-s;n),$$

and that for each $m \in \mathbb{N}$ we can write

$$q(x;n) = \sum_{k=1}^{m-1} f_k(n)v_k(x) + \mathcal{O}\left(f_m(n)\right),$$

where $\lim_{n\to\infty} \frac{f_{k+1}(n)}{f_k(n)} = 0$ and $\lim_{n\to\infty} \frac{f_k(n+1)}{f_k(n)} = 1$ for all $k \geq 1$, and the Landau-$\mathcal{O}$ is to be understood with respect to n. Then, for all $k \geq 1$, $v_k(x)$ is a polyharmonic function of degree k.

The conditions on the coefficients f_k are, for example, satisfied for $f_k = \frac{(\log n)^a}{n^b}$ for any $b > 0$. We will see asymptotics of this kind in Chapters 4 and 5. A proof of Lemma 1.3 is given in App. A. In a certain sense this lemma is not yet optimal, as we will later on prove refinements of it in Lemmas 4.3, 4.6 and 5.13. However, in all of the latter, the underlying idea is the same, and we merely make use of additional knowledge of the structure of the f_k.

1.3.3 Canonical candidates/interpretation

On a bounded domain, or on a finite graph, one can find a reasonably simple probabilistic interpretation of (discrete or continuous) polyharmonic functions. To see this, first suppose we are given a Brownian motion which starts somewhere inside a bounded, open set $A \subset \mathbb{R}^d$. Suppose we are interested in the first exit time τ^B of a standard Brownian motion of A, when starting a certain point $x \in \mathbb{R}^d$. It can be shown (e.g. [53]) that, given any bounded and measurable function g defined on the boundary ∂A, then the function

$$f(x) := \mathbb{E}_x \left[g\left(\tau^{A^c}\right) \right] = \mathbb{E}\left[g\left(\tau^{A^c}\right) \mid B(0) = x \right] \tag{1.36}$$

satisfies the Dirichlet problem

$$\triangle f(x) = 0 \qquad \forall x \in A, \tag{1.37}$$
$$f(x) = g(x) \quad \forall x \in \partial A. \tag{1.38}$$

In other words, $f(x)$ is a harmonic function. This turns out to be a consequence of the fact that the infinitesimal generator of Brownian motion is given by $\frac{1}{2}\triangle$ (see e.g. [116]). In [83], Helms shows that this generalizes in a certain way: suppose instead of $f(x) =: f_1(x)$, we look at the functions

$$f_k(x) := \mathbb{E}_x \left[(\tau^B)^{k-1} g\left(B\left(\tau^B\right)\right)\right]. \tag{1.39}$$

Then it turns out that the f_k satisfy a very similar boundary value problem, namely

$$\triangle^k f_k(x) = 0 \qquad \forall x \in A, \tag{1.40}$$
$$f_k(x) = g(x) \quad \forall x \in \partial A, \tag{1.41}$$

which is to say that the f_k are polyharmonic of degree k.

If we translate this to a discrete setting, then instead of a Brownian motion we consider a finite subset A of $\mathbb{Z}^d$, and an arbitrary random walk S_n. Since the geometry of $\mathbb{Z}^d$ is a bit more involved, possibly the easiest way to formalize a boundary value problem is to define the function g on the entire complement A^c (alternatively, one would have to define the boundary depending on the step set, which is not difficult per se but would lengthen the notation). In this setting, the following discrete analogue holds:

Proposition 1.4. *Let $A \subset \mathbb{Z}^d$ be bounded, and let $(S_n)_{n\geq 0}$ be a random walk starting at some $x \in \mathbb{Z}^d$. Let τ be the exit time of S_n from A, and $g : A^c \to \mathbb{C}$ be any function. Finally, define for $k \in \mathbb{Z}^+$:*

$$f_k(x) := \mathbb{E}_x \left[\tau^{k-1} g\left(S_\tau\right)\right]. \tag{1.42}$$

Then the f_k satisfy the discrete Dirichlet problem

$$\triangle^k f_k(x) = 0 \qquad \forall x \in A, \tag{1.43}$$
$$f(x) = g(x) \quad \forall x \in A^c. \tag{1.44}$$

This means that, again, f_k is (discrete) polyharmonic of degree k. A proof of Prop. 1.4 is given in App. A. While it might at first glance be tempting to try and extend this kind of a probabilistic interpretation to the quadrant, it turns out that this is not possible: the fact that the domain A is bounded turns out to be crucial for the existence of the expected value. In fact, on a bounded domain harmonic functions with zero boundary conditions need to vanish due to the maximum value property, see e.g. [5]. This is clearly not the case for unbounded domains like the quarter plane, where we could pick, for instance, $f(x, y) = xy$. Note that the same holds true for discrete harmonic functions.

In general, it is not easy to find one, or even all positive harmonic functions on a given domain. Indeed, this leads to the theory of the so-called Martin boundary (introduced by Martin in [104]), which serves as parametrization of the set of all such functions. See for instance [135] for a introduction to this topic, and [128] for the case of homogeneous random walks. With respect to random walks on the half plane, the Martin boundary for irreducible walks was first described by Ignatiouk in [88], and for the quadrant with Loore in [89]. The latter was independently done by Kurkova and Raschel in [98] a bit later, using kernel methods similar as in this thesis. It turns out that, in our context of walks in the quarter plane, in the zero drift case there is a unique positive harmonic function, which is the one naturally appearing in the asymptotics as in (1.5). If one wants to translate this to a case with drift, then one needs to utilize a Cramér transform as in [49], as is explained in Section 2.2.4. In this case, instead of harmonic functions, in the same asymptotics we will have a t-harmonic function, and for the arising value of t this will indeed turn out to be the unique positive harmonic function [101].[6] Even if one has a canonical, unique positive harmonic function, however, it is not at all clear what a canonical candidate for a biharmonic, or generally polyharmonic function should look like. One could go ahead and talk about second order asymptotics of counting problems, but this does not seem as satisfactory as for the harmonic case. On top of this, there is no reason at all why the functions in the second order asymptotics should be positive.

1.3.4 Constructing discrete polyharmonic functions in the quarter plane

While constructing continuous polyharmonic functions (in somewhat nice regions, at least) is, as previously mentioned, not much of a challenge anymore, it had not been known previously how to systematically find discrete polyharmonic functions in the quarter plane. While there had already been work done on other graphs (see for instance [120, 40]), the first general method to compute harmonic functions in the quarter plane for models with small steps and zero drift was given in [123], via translation into a complex boundary problem. This approach has then been extended upon in [86, 85]. To give a very rough outline of the strategy, the main idea is, given a discrete harmonic function $h(k,l)$, to consider its **generating function**[7]

$$H(x,y) := \sum_{k,l\geq 0} h(k,l)x^k y^l. \tag{1.45}$$

From here on, one then finds a functional equation, which is satisfied if and only if $H(x,y)$ is the generating function of a harmonic function, which reads

$$K(x,y)H(x,y) = K(0,0)H(0,0) - K(x,0)H(x,0) - K(0,y)H(0,y). \tag{1.46}$$

Here, $K(x,y)$ is the same kernel as in Section 1.2.3[8], where the parameter t takes the value of 1[9], and again encodes the allowed steps and their weights. As previously mentioned, it will

[6] Note that the asymptotics mentioned here are arising from counting $q(x,y;n)$, i.e. one counts them by start- and endpoint. If one fixes the starting point only (i.e. counting *all* admissible paths of a certain length), then it is a priori not so clear which positive harmonic functions appear, and in fact the structure and difficulty of this problem seem to heavily depend on the direction of the drift, see e.g. [58, 59].

[7] The notations differ slightly in the literature; in e.g. [123], the quarter plane is defined as $\mathbb{Z}^+ \times \mathbb{Z}^+$, not including the axes like here, leading to a shift in the indices.

[8] In some articles the sign of the kernel is reversed, leading to different signs on the right-hand side of (1.46), see e.g. [36, 86, 114].

[9] The value the parameter t takes would be different if we were to consider t-polyharmonic functions instead (and indeed these two ts would be the same). It is, however, always a constant, unlike when computing the path counting function.

be rigorously defined in Section 2.1.2. Note that this functional equation looks very similar to the one for the path counting function (1.13). Considering how harmonic functions arise naturally from path enumeration problems, this is not entirely surprising. The main difference is, however, that unlike for the counting function we do not have a parameter t (which previously was the counting variable; and we are not counting anything here). As $K(x,y)$ is quadratic in both variables due to the assumption of having only small steps, we can find an algebraic function $X_{\pm}(y)$ such that $K(X_{\pm}(y),y)=0$, and via substitution we then obtain

$$K(X_+(y),0)H(X_+(y),0) - K(X_-(y),0)H(X_-(y),0) = 0. \tag{1.47}$$

Seeing as on some closed contour we have $X_+(y) = \overline{X_-(y)}$ (with $\bar{x}$ denoting the complex conjugate of x), this translates to

$$K(x,0)H(x,0) - K(\bar{x},0)H(\bar{x},0) = 0, \tag{1.48}$$

valid on the boundary of some domain $\mathcal{G}$ (except for a single discontinuity). This kind of boundary value problem is then solvable by standard theory, leading to a certain conformal mapping $\omega(x)$, and allowing us to give a basis of the space of harmonic functions by

$$H(x,y) = \frac{P_n(\omega(x)) - P_n(\omega(X_+(y)))}{K(x,y)}, \tag{1.49}$$

with the P_n being polynomials of degree n (see [86, Thm. 2]). This leads to a basis $(H_1^n)_{n=1}^{\infty}$ of the space of all harmonic functions. In order to compute discrete polyharmonic functions, a similar approach was suggested in [36], making use of the very similar functional equation

$$K(x,y)H(x,y) = xy\triangle H(x,y) + K(0,0)H(0,0) - K(x,0)H(x,0) - K(0,y)H(0,y). \tag{1.50}$$

Here, if $H(x,y)$ is the generating function of $h(i,j)$, then $\triangle H(x,y)$ denotes the generating function of $\triangle h(i,j)$ (i.e. it is the natural extension of the Laplacian on the space of generating functions). The idea is to iteratively solve this functional equation; solving it for $\triangle H(x,y) = H_1^k(x,y)$, with $H_1^k(x,y)$ some harmonic base function, then one will eventually find all biharmonic functions, and so on. When trying to solve (1.50) in a similar fashion as (1.46), then one obtains

$$K(X_+(y),0)H(X_+,0) - K(X_-(y),0)H(X_-(y),0) = R(X_+(y),X_-(y),y) \tag{1.51}$$

for some function R. Annoyingly, this right-hand side does not only depend on $X_{\pm}(y)$, but also on y directly, so we cannot directly bring this equation into the form (1.48). In order to compute discrete polyharmonic functions, there are now different ways to proceed. In the following, we will always consider non-degenerate models with small steps in the quarter plane (see Chapter 2).

Direct approach

Going back to (1.50), it turns out that one can argue that a solution can be found with the additional property that $K(x,0)H(x,0) = 0$. This immediately simplifies (1.50) sufficiently so we can proceed to recursively construct discrete polyharmonic functions, leading to

Theorem (see Thm. 3.7)**.** *Given a model with small steps and zero drift, let $(H_1^n)_{n\geq 1}$ be the basis of harmonic functions mentioned above. Then we can inductively construct bivariate power series $H_m^k(x,y)$ via*

$$H_{m+1}^k(x,y) := \frac{X_+ y H_m^k(X_+(y),y) - xyH_m^k(x,y)}{K(x,y)} \qquad \text{if } K(0,0)=0, \tag{1.52}$$

$$H_{m+1}^k(x,y) := \frac{xyH_m^k(x,y)}{K(x,y)} \qquad \text{if } K(0,0)\neq 0. \tag{1.53}$$

Each H^k_m is m-polyharmonic, and we have $\triangle H^k_{m+1} = H^k_m$. The set $\left(H^k_m\right)$ forms a basis of the space of all discrete polyharmonic functions.

While it might seem at first that this completely solves the question of constructing discrete polyharmonic functions, this construction turns out not to be ideal: we will see in Sec. 3.3 that these functions do not have proper behaviour in terms of their scaling limit, and are thus unlikely to appear in the asymptotics of enumeration problems.

Decoupling approach

In [36], the authors proposed utilizing so-called decoupling functions. The latter were first utilized by Tutte in [131], in order to treat functional equations arising when counting planar maps, and were discussed in the context of problems related to lattice paths (in particular, determining algebraic properties of path counting functions) by Bernardi, Bousquet-Mélou and Raschel in [12]. Roughly speaking, the idea of a decoupling function is as follows: suppose we have a rational function $R(x, y)$. If, given a kernel $K(x, y)$, we can write

$$R(x, y) = A(x) + B(y) \mod K(x, y), \tag{1.54}$$

for rational functions A, B, which is to say that we can, modulo the kernel, *decouple* the dependence on x and y, then this immediately implies that

$$R(X_+(y), y) - R(X_-(y), y) = A(X_+(y)) - A(X_-(y)). \tag{1.55}$$

This means that, if we find a suitable decoupling function for the function R appearing in (1.51), then we can once again obtain a boundary value problem of the same form as (1.48), which we already know how to solve.

In [36], decoupling functions were found for some models using an ansatz making use of a parametrization of the kernel curve, which is in fact very reasonable in that one can show that if a (sufficiently nice) decoupling function exists, then it must be of a similar shape. This can be used to show that no algebraic decoupling function exists for certain models, as will be done in Section 3.4.1. However, it is very hard to show existence of decoupling functions for a class of models using this decoupling approach. This is due to some loss of information about the group properties when using the parametrization, as will also be explained in Section 3.5.

Instead, in Section 3.4.1 we will show directly that, for models with finite group, decoupling functions for polyharmonic functions can always be found (and given explicitly). If, in particular, a certain angle θ satisfies $\pi/\theta \in \mathbb{Z}$, then the resulting polyharmonic functions even have a particular nice shape. We will arrive at

Theorem (see Thm. 3.15)**.** *Suppose our step set has finite group and $\pi/\theta \in \mathbb{Z}$. Let $H^k_1(x, y)$ be the usual basis of harmonic functions. We can then define inductively*

$$H^k_n(x, y) = \frac{xyH^k_{n-1}(x, y) - F^k_{n-1}(x) - \left[X_+ yH^k_{n-1}(X_+, y) - F^k_{n-1}(X_+, y)\right]}{K(x, y)}, \tag{1.56}$$

where $F^k_n(x)$ is a suitable decoupling function of $xyH^k_n(x, y)$, which in particular exists. Then, $H^k_n(x, y)$ a rational function, and it is discrete polyharmonic of degree n for all n, k, and satisfies $\triangle H^k_{n+1} = H^k_n$. For each n, k we can write

$$H^k_n(x, y) = \frac{p_{n,k}(x, y)}{(1 - x)^\alpha (1 - y)^\alpha}, \tag{1.57}$$

where $p_{n,k}(x, y)$ is a polynomial and $\alpha \in \mathbb{N}$ with $\alpha \leq k\pi/\theta + 2(n - 1)$.

Let us again emphasize that the decoupling functions can be computed explicitly, see (3.76). Also, note that this gives us not only a basis of polyharmonic functions, but one of a particularly nice shape; namely rational functions with nicely behaved poles. Contrary to before, these polyharmonic functions now show good limiting behaviour: we will show in Thm. 3.19 that their scaling limits are indeed continuous polyharmonic functions (which can be constructed via a functional equation approach in a completely analogous fashion). Lastly, we will see that these discrete polyharmonic functions do in fact appear in asymptotics of lattice path enumerations, i.e. they are indeed combinatorially reasonable.

Approach via a boundary value problem

Given models with infinite group, it appears that a decoupling approach as above does not work (there is to the author's knowledge no counter-example, but neither is there a formal proof of this). Therefore, going back to (1.51), one can once again make use of standard theory and try to directly solve the inhomogeneous boundary value problem. This is possible, and done explicitly for models with infinite group and $\pi/\theta = 2$ in Section 3.6. It turns out that the resulting decoupling function (and biharmonic function) is given in terms of a contour integral, and is, unlike in previously considered cases, no longer algebraic.

Direct approach using a single generating function

Another possible approach would be to, instead of a generating function $H_k(x,y)$ for each discrete polyharmonic function of order k, use a single generating function $H(x,y,z)$ defined by

$$H(x,y,z) = \sum_{k\geq 0} H_k(x,y)z^k.$$

Assuming that the functions $H_k(x,y)$ satisfy $\triangle H_{k+1}(x,y) = H_k(x,y)$ (which in particular implies that $H_0(x,y) = 0$), then one could hope that the structure of the functional equation is as simple as, or maybe even easier than that of (1.50). And indeed, iterative application of (1.50) gives us

$$\underbrace{[K(x,y) - xyz]}_{:=L(x,y,z)} H(x,y,z) = K(0,0)H(0,0,z) - K(x,0)H(x,0,z) - K(0,y)H(0,y,z). \quad (1.58)$$

Noticing that $K(x,0) = L(x,0,z)$, $K(0,y) = L(0,y,z)$ and $K(0,0) = L(0,0,z)$ and then letting $L(x,y,z)H(x,y,z) := M(x,y,z)$, this equation takes the very simple form

$$M(x,y,z) = M(0,0,z) - M(x,0,z) - M(0,y,z). \quad (1.59)$$

By looking at the right hand side and noting that all mixed products of x and y need to be 0, one sees that a power series $M(x,y,z) \in \mathbb{C}[[x,y,z]]$ solves (1.59) precisely if it can be written as a sum of power series as $M(x,y,z) = P_1(x,z) + P_2(y,z) + P_3(z)$. The problem here is, however, finding appropriate power series $P_{1,2,3}$ such that at the end we can divide by $L(x,y,z)$ and obtain a proper trivariate power series $H(x,y,z)$. While it is possible that this could be done in general and lead to a nice structure of solutions, at this point it appears that the easiest way to do so is to consider each k consecutively, which in essence boils down to iteratively computing the $H_k(x,y)$ as outlined above and detailed in Chapter 3.

To see what this looks like in a concrete example, consider the simple walk with steps $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ with weights $\frac{1}{4}$ each. We will see later on (see Thm. 3.26) that one can find a sequence of discrete polyharmonic functions for the simple walk

$$H_k(x,y) = \frac{8}{(1-x)^2(1-y)^2}\left[\frac{4}{(1-y)^2}\right]^{k-1}.$$

This enables us to write

$$\begin{aligned}
H(x, y, z) &= \sum_{k\geq 1} z^k H_k(x, y) \\
&= \frac{8}{(1-x)^2(1-y)^2} \sum_{k\geq 1} z^k \left[\frac{4}{(1-y)^2}\right]^{k-1} \\
&= \frac{8z}{(1-x)^2(1-y)^2} \frac{1}{1-\frac{4z}{(1-y)^2}}.
\end{aligned}$$

Here, the computation was largely aided by the fact that we have an explicit formula for the $H_k(x, y)$, and that they are of a very simple structure (in particular, they form a geometric sequence). In general, this is not the case, and it is not obvious how one could find an expression for $H(x, y, z)$ given only a recursive relation for the $H_k(x, y)$ as for example in Thm. 3.15.

Chapter 2

Preliminaries

> Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions
>
> Felix Klein

2.1 Walks in the Quarter Plane

2.1.1 A functional equation

In order to count lattice paths in the quarter plane, first of all we need to fix a model; which is to say a set $\mathcal{S} \subseteq \mathbb{Z}^2$ of permissible steps, together with a family of positive weights $(\omega_s)_{s\in\mathcal{S}}$. A lattice path of length n is then a sequence of points $(x_0, \ldots, x_n) \subseteq \mathcal{Q}^n$, with $\mathcal{Q} := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ being the quarter plane, such that $x_k - x_{k-1} \in \mathcal{S}$ for all $k \in \{1, \ldots, n\}$. We will count such a path by its weight, that is, the product $\prod_{k=1}^n \omega_{x_k - x_{k-1}}$. Similarly, we will denote by $|\mathcal{S}|$ the total weight $\sum_{s\in\mathcal{S}} \omega_s$ of the steps in a model.

We will in the following assume that:

1. our step set consists of small steps only, i.e. $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$,
2. our step set is non-degenerate, i.e. there is no (possibly rotated) half-plane containing all allowed steps.

In order to keep the notation short, we will in the following sometimes denote by $\mathcal{S}$ not only the set of allowed steps, but also the associated weights $(\omega_s)_{s\in\mathcal{S}}$.
Let $q((u, v), (k, l); n)$ be the (weighted) number of paths of length n from (u, v) to the endpoint (k, l). It can then be shown (see e.g. [31, Lemma 4]) that the generating function

$$Q_{u,v}(x, y; t) = \sum_{n\geq 0} t^n \sum_{k,l\in\mathbb{N}} x^k y^l q((u, v), (k, l); n) \tag{2.1}$$

satisfies the functional equation

$$\begin{aligned} &K(x, y; t)Q_{u,v}(x, y; t) \\ &\quad = x^{u+1}y^{v+1} - \underbrace{K(x, 0; t)Q_{u,v}(x, 0; t)}_{=:A(x;t)} - \underbrace{K(0, y; t)Q_{u,v}(0, y; t)}_{=:B(y;t)} + \underbrace{K(0, 0; t)Q_{u,v}(0, 0; t)}_{=:C(t)}, \end{aligned} \tag{2.2}$$

where

$$K(x, y) = xy\,[1 - tS(x, y)]\,, \tag{2.3}$$

and $S(x, y)$ is the step-counting Laurent polynomial

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} \omega_{i,j} x^i y^j. \tag{2.4}$$

For ease of notation, if $(u, v) = (0, 0)$, then we will just write $q(k, l; n)$ and $Q(x, y; t)$.
Given that we consider paths in the quarter plane only, we see that the series $Q_{u,v}(x, y; t)$ are indeed power series in x, y and t (meaning there are no negative powers appearing anywhere); in particular, since we are working with finite step sets only, they are power series in t with polynomial coefficients in x, y. In the following, seeing as we will often treat the variable t as a parameter, we will often write $K(x, y)$ and $Q(x, y)$, leaving out the last variable.

2.1.2 The kernel, its curve, and a conformal mapping

Seeing as we allow small steps only and our step set is non-degenerate, we can write the kernel $K(x, y)$ as

$$\begin{aligned} K(x, y) &= a(x)y^2 + b(x)y + c(x) && (2.5)\\ &= \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), && (2.6) \end{aligned}$$

with $a(x), b(x), c(x), \tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ all being non-zero. Note that, as previously mentioned, all of the above functions depend on the parameter t. We can use the quadratic formula to find solutions of $K(\cdot, y) = 0$ and $K(x, \cdot) = 0$, which are given by

$$X_\pm(y) = \frac{-\tilde{b}(y) \pm \sqrt{\tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y)}}{2\tilde{a}(y)}, \quad Y_\pm(x) = \frac{-b(x) \pm \sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)}. \tag{2.7}$$

Letting $D(x) := b(x)^2 - 4a(x)c(x)$, then one can show (see [125, 2.5], [70, 2.3.2]) that given a non-degenerate walk and letting $t \in (0, |\mathcal{S}|)$, then $D(x) = 0$ has pairwise different solutions x_1 to x_4 with $|x_1| \leq x_2 < 1 < x_3 \leq |x_4|$, where we let $x_4 = \infty$ if $D(x)$ is of degree 3. For $t = |\mathcal{S}|$, we have $x_2 = x_3 = 1$ precisely if the model has zero drift, i.e. if

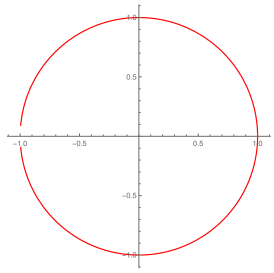
$$\sum_{s\in\mathcal{S}} i\omega_{i,j} = \sum_{s\in\mathcal{S}} j\omega_{i,j} = 0.$$

If the drift in only one direction is 0, then only either x_2 or x_3 will be equal to 1, see [70, Lemma 2.3.9].

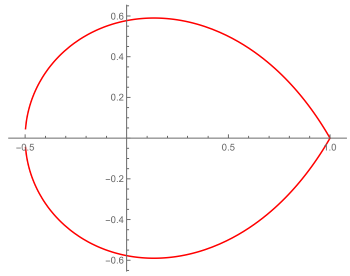
A standard approach in trying to solve the functional equation (2.2) is to restrict it to the so-called **kernel curve** $\mathcal{C}$. This curve, which also depends on t, is defined as

$$\mathcal{C} := \{(x, y) \in \overline{\mathbb{C}}^2 : K(x, y) = 0\}. \tag{2.8}$$

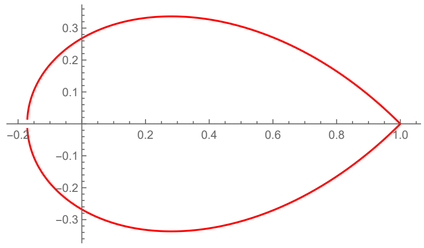
Here, $\overline{\mathbb{C}}$ is the projective closure of $\mathbb{C}$. The motivation to do so is that for $(x, y) \in \mathcal{C}$, the most complicated term $K(x, y)Q(x, y)$ vanishes. To proceed from there it is essential to know the algebraic structure of this curve, which can be either of genus 0 or 1. By (2.7), we see that we can consider this curve as a two-fold cover of $\mathbb{C}$ with branch cuts depending on the number of zeros of $D(x)$; if x_1, x_2, x_3, x_4 are all distinct, then $\mathcal{C}$ (or, more formally, its compactification, for details see [56, 2.4.1], or, using this same kernel, [82, 55]) will be of genus 1; else it will be of genus 0. For $t < |\mathcal{S}|$ this means in particular that for any non-degenerate model, we will have a curve of genus 1.



(a) The simple walk: $\mathcal{S} = \{\rightarrow, \downarrow, \leftarrow, \uparrow\}$



(b) The tandem walk: $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$



(c) The Gouyou-Beauchamps walk:
$\mathcal{S} = \{\leftarrow, \rightarrow, \nwarrow, \searrow\}$

Figure 2.1: The shape of the domain $\mathcal{G}$ for different (unweighted) models. The angle θ can be defined as the angle at which its boundary intersects the real axis at the point $(1, 1)$.

In the following, suppose that our model has zero drift, and let $|\mathcal{S}| = t = 1$, as will be the case in Chapter 3[1]. Then by the above, one can see that the discriminant is negative for $y \in [y_1, 1]$, and therefore in this range we have $X_+(y) = \overline{X_-(y)}$. Analogous results hold for $\tilde{D}(y) := \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y)$. This is in particular used in the computation of harmonic functions, as in [125, 86] and will be used in Section 3.2. The idea is to define the domain $\mathcal{G}$ as the area bounded by the curve $X_\pm([y_1, 1])$, and notice that the functional equation (2.2) will lead to a boundary value problem of the form

$$f(x) - f(\overline{x}) = 0 \tag{2.9}$$

on $\partial\mathcal{G} \setminus \{1\}$, for a function $f(x)$ which is analytic in the interior of $\mathcal{G}$ and continuous on $\overline{\mathcal{G}} \setminus \{1\}$ (cf. [36, 125]). A few examples of what $\mathcal{G}$ can look like is given in Fig. 2.1; in particular the case where $\mathcal{G}$ is the unit disk will be examined in 3.6.

In order to solve the above boundary value problem, one can construct a mapping $\omega : \mathbb{C} \to \bar{\mathbb{C}}$ which is a fundamental solution in the sense that any other solution can be written as some entire function applied to ω. An explicit formula for ω as well as some additional information are given in [125]. This ω satisfies

$$\omega(0) = 0, \quad \omega(X_+(y)) = \omega(X_-(y)) \quad \forall y \in [y_1, 1], \quad \frac{\partial \omega}{\partial x}(x) \neq 0 \quad \forall x \in \mathcal{G}^\circ. \tag{2.10}$$

In particular, ω is a conformal mapping of the domain $\mathcal{G}$. Furthermore, it has a pole-like singularity of order π/θ at $x = 1$, where θ is the inner angle at which $\partial\mathcal{G}$ intersects the x-axis. This angle is the arctangent of the so-called **correlation coefficient** of the model, see [69, 125]. It can be computed directly via

$$\theta = \arccos\left(-\frac{\sum ij\omega_{i,j}}{\sqrt{\sum i^2\omega_{i,j}}\sqrt{\sum j^2\omega_{i,j}}} \right). \tag{2.11}$$

[1]If it is not, then we can apply the Cramér transform to obtain an equivalent model with these properties, see Section 2.2.4

The angle θ happens to be closely related to the so-called group of the walk, which will be defined in Section 2.1.2 below: it turns out that its restriction to the kernel is finite if and only if $\pi/\theta \in \mathbb{Q}$) (see [70]). Additionally, in [123], [86], it is shown that π/θ determines the growth rate of the positive harmonic function. Given a model with non-zero drift, we can first apply a Cramér-transform (see 2.2.4) to get rid of the drift, and then define θ as above.

In the same manner as we have constructed the region $\mathcal{G}$ and the conformal mapping ω above, one can obtain an equivalent region $\mathcal{G}'$ by swapping the roles of x and y. While it is possible to construct a second conformal mapping $\hat{\omega}$ for $\mathcal{G}'$ in the same fashion as ω, one can also see that $\omega \circ X_+$ is a conformal mapping by [70, Cor. 5.3.5], and it has the same behaviour around 1 as ω. Finally, we note that due to (2.10), ω is an invariant in the sense of [12, Def. 4.3].

2.1.3 The group of a model and orbit summation

Consider now the two birational transformations

$$\Phi : \begin{cases} x \mapsto x, \\ y \mapsto y^{-1}\frac{c(x)}{a(x)}, \end{cases} \tag{2.12}$$

$$\Psi : \begin{cases} x \mapsto x^{-1}\frac{\tilde{c}(y)}{\tilde{a}(y)}, \\ y \mapsto y. \end{cases} \tag{2.13}$$

These two transformations, which clearly depend on our step set $\mathcal{S}$ as well as the parameter t, generate the so-called **group of a model**, denoted by $\mathcal{G}$. If $\mathcal{G}$ is finite, then, as both Φ, Ψ are involutions, any element $g \in \mathcal{G}$ can be written as either $(\Phi \circ \Psi)^k$, or as $\Psi \circ (\Phi \circ \Psi)^k$ for some k. We define $\operatorname{sgn}(g) = 1$ in the first, and $\operatorname{sgn}(g) = -1$ in the second case. This group has been central to many results in the study of lattice walks in the quarter plane, see e.g. [31, 82, 55, 69]. The main reason for this is that if we let

$$k(x, y) := \frac{K(x, y)}{xy} = 1 - tS(x, y), \tag{2.14}$$

then it follows that $k(x, y)$ is invariant under $\mathcal{G}$. In particular, given (x, y) such that $K(x, y) = 0$, then we know that $(K \circ g)(x, y) = 0$ for all $g \in \mathcal{G}$. On the kernel curve, the group therefore behaves like the so-called QRT-map in [56], consisting of horizontal and vertical switches on a biquadratic curve, see Fig. 2.2.

The invariance of $k(x, y)$ under $\mathcal{G}$ is the starting point for orbit summation methods as in [31]: we rewrite (2.2) as

$$xyk(x, y)Q(x, y) = xy - A(x) - B(y) + C. \tag{2.15}$$

Assuming now that the group is finite, and picking any $g \in \mathcal{G}$, we obtain

$$g(xy)k(x, y)Q(g(x), g(y)) = g(xy) - A(g(x)) - B(g(y)) - C. \tag{2.16}$$

Multiplying (2.16) with $\operatorname{sgn}(g)$ and taking the sum over all elements $g \in \mathcal{G}$, all terms $A(g(x))$ and $B(g(y))$ cancel (note that both Ψ, Φ change only one variable each, while sgn switches sign), we obtain (see [31, Prop. 5])

$$\sum_{g \in \mathcal{G}} \operatorname{sgn}(g)g(xy)Q(g(x), g(y)) = \frac{1}{k(x, y)} \sum_{g \in \mathcal{G}} \operatorname{sgn}(g)g(xy). \tag{2.17}$$

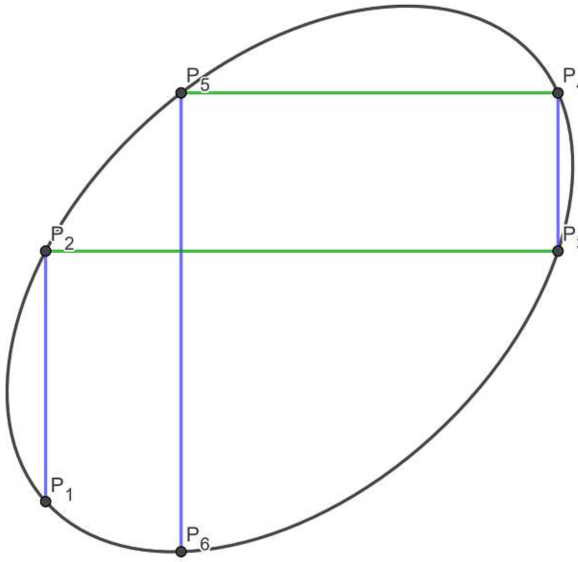


Figure 2.2: An illustration of the action of the group $\mathcal{G}$ on the kernel curve. The mappings Φ, Ψ represent switching vertically to the opposite points. Correspondingly, starting from P_1, we have $P_2 = \Phi(P_1)$, $P_3 = \Psi(P_2)$ and so on.

Depending on the exact shape of $\mathcal{G}$, often the terms of the form $Q(g(x), g(y))$ will not contribute positive powers of both x and y except for $g = \mathrm{Id}$. In this case, we say that the model is **orbit-summable**, and we can write

$$xyQ(x, y; t) = [x^>] [y^>] \frac{\sum_{g \in \mathcal{G}} \operatorname{sgn}(g) g(xy)}{k(x, y; t)}. \tag{2.18}$$

By [31, Lemma 2], we know that if a model $\mathcal{S}$ (consisting of both steps and weights ω_s) has a finite group, then the model with reversed steps $\tilde{S} := -\mathcal{S}$ has a finite group as well. Furthermore, by essentially the same argument, one can see that orbit-summability is retained when reversing the steps as well:

Lemma 2.1. *Given an orbit-summable model $\mathcal{S}$, the reversed model $\tilde{\mathcal{S}}$ is also orbit-summable.*

Proof. We note as in [31, Lemma 2] that if Φ, Ψ are the generators of the group of $\mathcal{S}$, then the generators of the group of $\tilde{\mathcal{S}}$ are given by $\iota \circ \Psi \circ \iota, \iota \circ \Phi \circ \iota$, where $\iota(x, y) = (x^{-1}, y^{-1})$. Given any element g in the group $\mathcal{G}$ of $\mathcal{S}$, denote by $\tilde{g}$ the corresponding element of the group $\tilde{\mathcal{G}}$ of $\tilde{\mathcal{S}}$. As ι is an involution, we see that $\tilde{g} = \iota \circ g \circ \iota$ for any such g, and in particular that $\tilde{g}(xy) = \iota \circ g \circ \iota(xy)$. Noticing that $g(xy) \in \mathbb{C}(x, y)$ we can deduce that $\tilde{g}(xy) = g(xy)$. From there, we obtain an equivalent equation to (2.18) by the very same combinatorial arguments. $\square$

Remark: In the literature, the group being finite is sometimes used ambiguously: in this thesis, as well as for example in [30, 31], the group being finite means it is finite as a group generated by the two birational transformations Φ, Ψ. Some other times (e.g. in [70, 82]), the group is understood as the restriction of these transformations to the kernel curve, the finiteness of which is a weaker statement (and equivalent to the fact that $\pi/\theta \in \mathbb{Q}$, where θ is the arctangent of the correlation coefficient as defined above [70, 7.1]). These two notions are indeed different, as can be seen e.g. in Example 3.6.4.

2.1.4 Saddle points

In Chapter 4 we will make use the saddle point method as described for instance in [73, VIII]. In particular, we will be interested in saddle points of $S(x, y)$, which is a Laurent polynomial with only positive coefficients. We call a point (x_0, y_0) a **dominant saddle point** if:

1. (x_0, y_0) is a local minimizer of $S(x, y)$,
2. $x_0, y_0 \in \mathbb{R}^+$.

The fact that $S(x_0, y_0) \neq 0$ follows from the positivity of coefficients. The existence is a consequence of the fact that our model is non-degenerate; which implies that $S(x, y)$ is coercive (it goes to infinity wherever we approach the boundary; see also [51, 1.5]). By definition, we know that $\frac{\partial S}{\partial x}(x_0, y_0) = \frac{\partial S}{\partial y}(x_0, y_0) = 0$.
By a short computation, one can check that

Lemma 2.2. *The dominant saddle point $s_0 = (x_0, y_0)$ is equal to $(1, 1)$ if and only if the drift of the model is 0.*

Now take any other point (x_1, y_1) on the torus $|x_1| = x_0, |y_1| = y_0$. By the positivity of the coefficients of $S(x, y)$, we know that $|S(x_1, y_1)| < S(x_0, y_0)$, unless there is a $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that each monomial term of $S(x_1, y_1)$ differs from the corresponding term in $S(x_0, y_0)$ by the same factor of ζ. Choosing α, β such that $(x_1, y_1) = (\alpha x_0, \beta y_0)$, we must therefore have $\alpha^i \beta^j = \zeta$ for all (i, j) such that $\omega_{i,j} \neq 0$. From here, it is not difficult to see that α, β and ζ must be roots of unity. It is also clear that there can only be finitely many such ζ, in a one-to-one correspondence with finitely many pairs $(x_i, y_i), 0 \leq i \leq l$ (the maximum l appearing for the 19 unweighted orbit-summable models is 3, see App. D). We will call such a point $(x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$ and $|S(x_i, y_i)| = S(x_0, y_0)$ a **saddle point associated with** (x_0, y_0). One can check directly that the (x_i, y_i) are indeed saddle points of $S(x, y)$ as well, moreover, as we will see in Lemma 4.2, the local behaviour of $S(x_0, y_0)$ and $S(x_i, y_i)$ is the same up to the factor ζ. By the same reasoning as in [73, VIII], it turns out that when computing the asymptotics of the coefficients via the Cauchy formula, the main contributions to the contour integral come from the points (x_i, y_i), as the modulus of $S(x, y)$ will be smaller elsewhere, leading to exponentially smaller terms.
Note that while this article considers mainly the 2-dimensional case, all the definitions above extend to more dimensions in a natural manner; only the structure of the group becomes more complicated as we will have more than two transformations, see e.g. [23].

2.1.5 The Jacobi ϑ-function

As discussed in Section 2.1.2, for small t we know that (seeing as we always assume our models to be non-singular) the kernel curve $\mathcal{C}$ is elliptic. In order to find a suitable parametrization in Chapter 5, we will make use of Jacobi's ϑ-function (of the first kind). This function is defined by

$$\vartheta(z, q) := 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \sin\left[(2n+1)z\right]. \tag{2.19}$$

Alternatively, one often finds it written in terms of τ, where $e^{i\pi\tau} = q$, leading to

$$\vartheta(z \mid \tau) := 2 \sum_{n=0}^{\infty} e^{i\pi\tau(n+1)^2/2} \sin\left[(2n+1)z\right]. \tag{2.20}$$

It is easy to check that $\vartheta(z, q)$ converges for $|q| < 1$ or, equivalently, for τ with positive imaginary part. It is also easy to check the elementary properties

$$\vartheta(\pi - z) = \vartheta(z), \quad \vartheta(-z) = -\vartheta(z) \quad \text{and} \quad \vartheta(z + \pi\tau) = -e^{-i\pi\tau - 2iz}\vartheta(z). \tag{2.21}$$

A parametrization of the kernel curve could also be done (and would in principle be equally useful) by Weierstraß' $\wp$-functions, as e.g. in [97]. An advantage of ϑ-functions, however, is that one has somewhat more direct control of zeros and poles, which will be useful for us [1].

Given a series representation around 0, in order to extract asymptotics we will need a series expansion around the critical point. To do so, we will make use of the so-called Jacobi-transformation. Given q and $\hat{q}$ with

$$\log(q)\log(\hat{q}) = \pi^2, \tag{2.22}$$

then we have

$$\vartheta(z,q) = \sqrt{-\frac{\log(\hat{q})}{\pi}} \exp\left(\frac{\log(\hat{q})z^2}{\pi^2}\right) \vartheta\left(\frac{i}{\pi}\log(\hat{q})z, \hat{q}\right). \tag{2.23}$$

As mentioned in the introduction, this transformation corresponds to shifting a series expansion from the origin ($q = 0$) to the critical point ($q = 1$), see e.g. [96].

2.2 Polyharmonic Functions

2.2.1 Discrete polyharmonic functions

Given a step set $\mathcal{S}$ and a discrete function f defined on the quarter plane $\mathcal{Q}$, we can define the Markov operator

$$Pf(x) := \sum_{s\in\mathcal{S}} \omega_s f(x-s). \tag{2.24}$$

If we take the discrete random walk (X_n) with the transition probabilities given by the reverse of $\mathcal{S}$, and the induced Markov chain $M_n := f(X_n)$, then we can interpret the operator P as the expectation $\mathbb{E}\left[M_{n+1} \mid M_n\right]$. One can then proceed to look at the expected change during a time step, weighted by a parameter t, which is given by

$$\triangle f(x) := (P - t\,\mathrm{Id})\, f(x). \tag{2.25}$$

As outlined in Section 1.3, the operator $\triangle$ can be viewed as the discrete equivalent of a Laplace-Beltrami operator. We call a function **(discrete) t–harmonic** if

1. $\triangle f(x) = 0$ for all $x \in \mathcal{Q}$,
2. $f(x) = 0$ for all $x \in \mathcal{Q}^c$,

where $\mathcal{Q}^c$ is the complement of $\mathcal{Q}$. Similarly, we call a function **(discrete) t–polyharmonic of degree** p if

1. $\triangle^p f(x) = 0$ for all $x \in \mathcal{Q}$,
2. $f(x) = 0$ for all $x \in \mathcal{Q}^c$.

Instead of a (poly-)harmonic function $h(i,j)$, we will in the following consider its generating function

$$H(x,y) := \sum_{i,j\geq 0} h(i,j)x^{i+1}y^{j+1}.$$

It turns out that this generating function then satisfies the functional equation

$$K(x,y)H(x,y) = xy\left[\triangle H\right](x,y) + K(0,0)H(0,0) - K(x,0)H(x,0) - K(0,y)H(0,y), \tag{2.26}$$

where $K(x, y)$ is the same kernel as defined in Section 2.1.2, but with all steps in $\mathcal{S}$ reversed, similar as in (2.24), and $\triangle H$ is the generating function of $\triangle h(i, j)$. Correspondingly, we have $\triangle H(x, y) = 0$ for harmonic functions $h(i, j)$.

Given a model $\mathcal{S}$, we can also define a discrete Laplacian $\tilde{\triangle}$ for the model with directions reversed $\tilde{\mathcal{S}}$. Seeing as $\tilde{\triangle}$ is the adjoint operator to $\triangle$ on the space $\mathcal{L}^2(\mathbb{Z}^2)$, we will call it the adjoint Laplacian. In Section 4.2.1, we will encounter functions of the form $f(x, y)$, for $x, y \in \mathbb{Z}^2$, which are polyharmonic in x and adjoint polyharmonic in y. We will call such a function $f(x, y)$ **multivariate polyharmonic** of order p if

$$\triangle^k \left(\tilde{\triangle}^{p-k} f \right) = 0$$

for all $0 \leq k \leq p$. Note that in this case, the ordering of the Laplacians does not matter as by linearity they commute, i.e. we have

$$\triangle \left(\tilde{\triangle} f(x, y) \right) = \tilde{\triangle} \Big(\triangle f(x, y) \Big). \tag{2.27}$$

2.2.2 A basis of discrete harmonic functions

In order to compute discrete harmonic functions, the approach in [125, 86, 85] makes direct use of the functional equation (2.26). Substituting $x \mapsto X_{\pm}(y)$ as in Section 2.1.2 leads to the boundary value problem

$$K(x, 0)H(x, 0) - K(\bar{x}, 0)H(\bar{x}, 0) = 0 \tag{2.28}$$

on the domain $\mathcal{G}$ (where $X_+(y) = \overline{X_-(y)}$, as discussed in Section 2.1.2. It follows (for details see aforementioned references [125, 86, 85]) that we can construct discrete (1-)harmonic functions via letting

$$K(x, y)H(x, y) = P(\omega(x)) - P\left(\omega\left(X_+(y)\right)\right), \tag{2.29}$$

where P is an entire function and ω is the conformal mapping introduced in Section 2.1.2. As will be shown in Lemma 3.3, the functions defined by

$$H_1^m(x, y) := \frac{P_m(\omega(x)) - P_m(\omega(X_+))}{K(x, y)}, \tag{2.30}$$

where

$$P_m(z) := z^m \qquad \text{if } K(0,0) = 0, \tag{2.31}$$

$$\left.\begin{array}{l} P_{2m}(z) := z^m(z - d_0)^m \\ P_{2m+1}(z) := z^{m+1}(z - d_0)^m \end{array}\right\} \qquad \text{if } K(0,0) \neq 0, \tag{2.32}$$

and $d_0 = \omega(X_+(0))$, as in the proof of Thm. 3.3, form a Schauder basis[2] of the space $\mathcal{H}_1$ of all discrete harmonic functions. This implies in particular that we have

$$\mathcal{H}_1 \cong \mathbb{C}[[z]],$$

and we will see later on that for the spaces $\mathcal{H}_n$ of discrete polyharmonic functions of degree n we have

$$\mathcal{H}_n \cong (\mathcal{H}_1)^n \cong \mathbb{C}[[z]]^n.$$

[2]That is, we can express any element uniquely not necessarily via finite, but via countable sums.

2.2.3 Continuous polyharmonic functions

The scaling limit of a random walk in the quarter plane is a Brownian motion in $\mathcal{W} := \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with (positive semidefinite) covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

whose infinitesimal generator is Laplace-Beltrami operator

$$\triangle = \frac{1}{2}\left(\sigma_{11}\frac{\partial^2}{\partial x^2} + 2\sigma_{12}\frac{\partial^2}{\partial x \partial y} + \sigma_{22}\frac{\partial^2}{\partial y^2}\right).$$

The coefficients $\sigma_{11}, \sigma_{12}, \sigma_{22}$ can be directly computed via $\mathbb{E}X^2 = \sigma_{11}, \mathbb{E}XY = \sigma_{12}, \mathbb{E}Y^2 = \sigma_{22}$ [100]. As in the discrete setting, we call a function f **(continuous) polyharmonic of degree** k**,** if

$$\begin{aligned} \triangle^k f(x) &= 0 \quad \forall x \in \mathcal{W}, \\ f(x) &= 0 \quad \forall x \in \partial\mathcal{W}, \end{aligned}$$

where $\triangle$ is the Laplace-Beltrami operator defined above. Note that this definition is exactly the same as for discrete polyharmonic functions, except for the different Laplacian. Note as well that, as previously, we always impose a Dirichlet condition, which is due to the combinatorial background of the functions considered here. Also, while in the following there might be technically an ambiguity due to the same symbol $\triangle$ used for both the continuous and discrete Laplacian, it should always be clear from the context which one is to be used.

In [125, App. A], it was shown that a continuous polyharmonic function satisfies the functional equation

$$\gamma(x,y)\mathcal{L}(f)(x,y) = \frac{1}{2}\left[\sigma_{11}\mathcal{L}_1(f)(y) + \sigma_{22}\mathcal{L}_2(f)(x)\right] + \mathcal{L}(\triangle f)(x,y), \tag{2.33}$$

where we have

$$\gamma(x,y) = \frac{1}{2}\left(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2\right), \quad \mathcal{L}(f)(x,y) = \int_0^\infty \int_0^\infty e^{-ux-vy} f(u,v)\mathrm{d}u\mathrm{d}v, \tag{2.34}$$

$$\mathcal{L}_1(f)(y) = \int_0^\infty \frac{\partial f}{\partial x}(0,v)e^{-vy}\mathrm{d}v, \qquad \mathcal{L}_2(f)(x) = \int_0^\infty \frac{\partial f}{\partial y}(u,0)e^{-ux}\mathrm{d}u, \tag{2.35}$$

see also [36, 2.2].

2.2.4 The Cramér-transform

In the following it will often be convenient to only consider models with zero drift, that is, where

$$\sum_{(i,j)\in\mathcal{S}} i\omega_{i,j} = \sum_{(i,j)\in\mathcal{S}} j\omega_{i,j} = 0.$$

If this is not the case, then we can utilize the **Cramér-transformation**[3]: we multiply each weight $\omega_{i,j}$ by a factor of $\alpha^i\beta^j$, where we choose α, β such that the drift will be 0 (for some information about this kind of transformation in a more general, not necessarily discrete context, see for instance [44, 8]). The existence of such an α, β is ensured for non-singular models, see

[3]named after H. Cramér, 1893-1985.

e.g. [51, 1.5]. The reason why this substitution is very convenient combinatorially is fairly simple; given the number $q((i,j),(k,l);n)$ of paths from (i,j) to (k,l) with n steps weighted by old weights $\omega_{i,j}$, then for the equivalent $\hat{q}$ using the new weights we have

$$\hat{q}((i,j),(k,l);n) = \alpha^{k-i}\beta^{l-j}q((i,j),(k,l);n). \tag{2.36}$$

Additionally, it turns out that the group of the model is, in a certain sense, invariant under this transformation, and in particular orbit-summability is preserved.

Lemma 2.3. *Let $\mathcal{S}$ be a model and $\hat{\mathcal{S}}$ be a Cramér-transform of $\mathcal{S}$, with weights α, β. Let $\mathcal{G}, \hat{\mathcal{G}}$ be the respective groups with generators Φ, Ψ and $\hat{\Phi}, \hat{\Psi}$. Lastly, let ι be the mapping $(x,y) \mapsto (\alpha x, \beta y)$. Then we have:*

1. *$\hat{\Phi}(x,y) = \iota^{-1} \circ \Phi \circ \iota$, $\hat{\Psi}(x,y) = \iota^{-1} \circ \Psi \circ \iota$; in particular $\mathcal{G}$ and $\hat{\mathcal{G}}$ are isomorphic,*
2. *$\hat{\mathcal{S}}$ is orbit-summable if and only if $\mathcal{S}$ is orbit-summable.*

Proof. We show the first part for Φ only, the statement for Ψ will then follow by symmetry. We can define $\hat{c}(x)$ and $\hat{a}(x)$ as in (2.5). One finds that, by definition of $\hat{\mathcal{S}}$, we will have $\hat{a}(x) = \beta a(\alpha x)$ and $\hat{c}(x) = \beta^{-1}c(\alpha x)$. From this it follows that

$$\iota^{-1} \circ \Phi \circ \iota(x,y) = \iota^{-1} \circ \Phi(\alpha x, \beta y) = \iota^{-1}\left[\alpha x, (\beta y)^{-1}\frac{c(\alpha x)}{a(\alpha x)}\right] \tag{2.37}$$

$$= \iota^{-1}\left[\alpha x, \beta y^{-1}\frac{\hat{c}(x)}{\hat{a}(x)}\right] = \left[x, y^{-1}\frac{\hat{c}(x)}{\hat{a}(x)}\right] = \hat{\Phi}(x,y). \tag{2.38}$$

The isomorphism of $\mathcal{G}$ and $\hat{\mathcal{G}}$ follows immediately and is given by $g \mapsto \iota^{-1} \circ g \circ \iota \in \hat{\mathcal{G}}$. By this isomorphism, we also see immediately that positive and negative powers of x, y are preserved, which is in turn all that matters for orbit-summability, hence we are done. □

Lastly, discrete polyharmonic functions behave well with respect to Cramér transformations: via a short computation, one can show

Lemma 2.4. *Let $\hat{\mathcal{S}}$ be a Cramér-transform of $\mathcal{S}$, with $\hat{\omega}_{i,j} = \alpha^i\beta^j\omega_{i,j}$. Let $\hat{\triangle}, \triangle$ be the associated Laplacians. We then have*

$$\hat{\triangle}\left[\alpha^{-k}\beta^{-l}f(k,l)\right] = \alpha^{-k}\beta^{-l}\triangle\left[f(k,l)\right]. \tag{2.39}$$

This directly implies that we have a bijection between the polyharmonic functions w.r.t. $\triangle$ and those w.r.t. $\hat{\triangle}$, given by adding a factor of $\alpha^{-k}\beta^{-l}$.

Chapter 3

Construction of discrete polyharmonic functions in the quarter plane

> The purpose of computing is insight, not numbers.
>
> Richard Hamming

The goal of this chapter is to give an overview of the structure of the space of discrete polyharmonic functions in the quarter plane for non-singular small-step models and an algorithm to compute them. To illustrate the functional equation approach, we will first treat the (much simpler) case of polyharmonic functions in the half-line.

There will be two main approaches discussed, one of them purely algebraic and more general, but leading to a basis of functions which might not be combinatorially relevant due to their bad behaviour with respect to a scaling limit. The other approach works for models with a finite group, where one can expand on the idea in [36] and use decoupling functions, resulting in many cases in a particularly nice basis consisting of rational functions of a fairly simple shape. In this case, one also obtains a direct link to continuous polyharmonic functions.
The structure of this chapter will be roughly as follows:

- in Section 3.1, the one-dimensional case will be treated. In particular, a way to construct of discrete polyharmonic functions on the half-line for step sets with finitely many positive steps will be derived in Prop. 3.1.
- Starting from Section 3.2, the rest of this chapter will concern models with small steps and zero drift in the quarter plane. In Section 3.2, we will give some properties of the vector spaces of discrete polyharmonic functions in the quarter plane, show that the discrete harmonic functions as constructed in Section 2.2 form a basis, and give a criterion for a set of discrete polyharmonic functions to be a basis.
- In Section 3.3, a general algorithm to construct discrete polyharmonic functions in the quarter plane is presented (Thm. 3.7), and it is shown that all possible discrete polyharmonic functions can be constructed in this manner (Thm. 3.8). An analogue of this method in the continuous case is presented in Section 3.3.2, and the relation between the discrete and continuous functional equations as well as convergence in terms of generating functions and Laplace transforms are discussed.
- In Section 3.4, an alternative construction utilizing decoupling functions is presented, which is applicable to models with finite group only (Thm. 3.15). This method leads, provided a certain parameter is integer, to a (Schauder) basis consisting of rational func-

tions. This construction is then translated to the continuous setting, and convergence properties are shown (Thm. 3.19).

- In Section 3.5, the guessing approach mentioned (but not detailed) in [36] is discussed, in which one uses an ansatz to try and find suitable decoupling functions. In particular, we deduce that the guessing approach works to decide if there is a decoupling function of a particular shape.
- In Section 3.6, a special case is examined in more detail, to show the relation between the two approaches and in particular why it does not appear promising to extend the notion of decoupling to the infinite group case.

Throughout this chapter, we will assume that we have a zero drift model where the weights sum up to 1. The construction of discrete polyharmonic functions for any other model can be reduced to this case via the Cramer transform outlined in Section 2.2.4. Also, in order to keep things more concise, the term 'polyharmonic function' will be used for such a function itself as well as for its generating function. If it is not clear from the context which is meant, it will be specified.

This chapter is largely based on the author's article [114].

3.1 Constructing discrete polyharmonic functions on the half-line

In this section, the (much easier) example of polyharmonic functions on the half-line is treated. Unlike in the quarter plane case, the only assumption we need is for the step set $\mathcal{S}$ to have finitely many steps in the positive direction.

Fundamentally, the approach we utilize in this case is the same as in the quarter plane: we deduce a functional equation which any generating function of a polyharmonic function has to satisfy, which we can then solve. So suppose that we are given a function $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, and a function $f : \mathbb{Z} \to \mathbb{C}$ which satisfies

$$\triangle f(n) = g(n) \quad \forall n \geq 0 \tag{3.1}$$
$$f(n) = 0 \qquad \forall n < 0. \tag{3.2}$$

Let $H(x), G(x)$ be their respective generating functions. We can then write

$$\sum_{i\in\mathcal{S}} \omega_i h(n+i) - h(n) = g(n) \quad \forall n \geq 0 \quad \Rightarrow \tag{3.3}$$
$$\sum_{n\geq 0}\sum_{i\in\mathcal{S}} x^{-i}\omega_i x^{i+n} h(i+n) - H(x) = G(x) \quad \Leftrightarrow \tag{3.4}$$
$$\sum_{i\in\mathcal{S}} x^{-i}\omega_i \sum_{n\geq 0} x^{i+n} h(i+n) = G(x) + H(x). \tag{3.5}$$

Looking at the left-hand side of (3.5), since $h(n) = 0$ for $n < 0$ we can see that for any $i < 0$, the second sum will simply be $H(x)$. For $i > 0$, however, in the latter sum we are missing the first i terms in comparison. In order to write down a more concise formula of (3.5), we want to define a one-dimensional kernel analogously to the 'standard' one as in Section 2.1.2. As we want our kernel to be a polynomial, let $d := \max(i \in \mathcal{S})$. As we assumed our step set to have only finitely many positive steps, we know that d is finite. We then let

$$K(x) = x^d \left(1 - \sum_{i\in\mathcal{S}} x^{-i}\omega_i\right).$$

Furthermore, let $H_k(x) := \left[x^{<k}\right] H(x)$, i.e. $H_1(x) = h(0), H_2(x) = h(0) + xh(1)$, and so on. As $\sum_{n\geq 0} x^{i+n}h(i+n) = H(x) - H_i(x)$, we can rewrite (3.5) and thus obtain:

Proposition 3.1. *Let $\mathcal{S} \subset \mathbb{Z}$ be a step set with associated weights ω_i and the additional condition that* $\max(s : s \in \mathcal{S}) = d < \infty$*. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, and let $G(x)$ be its generating function Using the definition for the kernel $K(x)$ and the $H_i(x)$ as above, we can construct all solutions of* (3.1)–(3.2) *via*

$$K(x)H(x) = -x^d G(x) - x^d \sum_{0<i, i\in\mathcal{S}} \omega_i \frac{H_i(x)}{x^i}, \tag{3.6}$$

where $H(x)$ is the generating function of h.

Remarks:

- Iterative application of Prop. 3.1 allows us to construct discrete polyharmonic functions of arbitrarily high order.
- By the definition of d, the sum on the right-hand side of (3.6) is a polynomial. As $K(0) \neq 0$ by definition, we therefore obtain proper power series solutions for $H(x)$.
- The right-hand side of (3.6) depends on the values of $h(0), \ldots, h(d)$. These are parameters which we can choose freely, therefore the space of harmonic functions will be d-dimensional (as a vector space over $\mathbb{C}$). For polyharmonic functions of degree k the solution space will be $d \times k$-dimensional (we utilize Prop. 3.1 k times, where we can choose d parameters each time).

Proof. The fact that any solution of (3.1)–(3.2) has the form as in Prop. 3.1 is a direct consequence of the above computation. The fact that the inverse holds, i.e. that each solution of (3.6) also solves (3.1)–(3.2) can be checked directly. □

While the derivation of this functional equation is essentially the same as that of 2.2, seeing as we only have a single boundary term there is no need to bother with the zero set of the kernel curve since we can just solve it directly. The appearance of multiple boundary terms is what makes the solution so much more involved in the quarter plane (or generally in the higher-dimensional) case.

An example with small steps

First, we will look at the model with step set $\mathcal{S} = \{-1, 1\}$. We have

$$K = x\left(1 - x\omega_{-1} - \frac{\omega_1}{x}\right).$$

As $d = 1$, we have, up to multiples, only one harmonic function $H_1(x)$. So let us compute it. For a harmonic function, we have $G(x) = 0$ in 3.6, so it takes the form

$$K(x)H_1(x) = -\omega_1 h(0).$$

This immediately leads to

$$H_1(x) = \frac{-\omega_1 h(0)}{K(x)}.$$

Choosing the harmonic function with $h_1(0) = 1$, we obtain

$$H_1(x) = \frac{-\omega_1}{K(x)} = \frac{1}{1 - x\frac{1}{\omega_1} + x^2\frac{\omega_{-1}}{\omega_1}},$$

with coefficients

$$1, \frac{1}{\omega_1}, \frac{\omega_1\omega_{-1} - 1}{\omega_1^2}, \frac{1 - 2\omega_1\omega_{-1}}{\omega_1^3}, \ldots$$

Wanting to compute a biharmonic function $H_2(x)$, we only need to substitute $G(x) = H_1(x)$ in (3.6), giving us

$$H_2(x) = -xH_1(x) - \frac{\omega_1 h_2(0)}{K(x)}$$

Choosing once again $h_2(1) = 1$, we now find

$$H_2(x) = \frac{\omega_1(\omega_1 + \omega_{-1}x^2)}{\omega_1 + x(\omega_{-1}x - 1))^2},$$

leading to the coefficients

$$1, \frac{2}{\omega_1}, \frac{3 - \omega_{-1}\omega_1}{\omega_1^2}, \frac{4 - \omega_1\omega_{-1}}{\omega_1^3}, \ldots$$

An example with larger steps

Consider now the model with step set $\mathcal{S} = \{-1, 1, 2\}$. In this case we have

$$K(x) = x^2 \left(1 - \omega_{-1}x - \frac{\omega_1}{x} - \frac{\omega_2}{x^2}\right),$$

and (3.6) for a harmonic function $H_1(x)$ reads

$$K(x)H_1(x) = -x^2 \left[\omega_1 \frac{h_1(0)}{x} + \omega_2 \frac{h_1(0) + xh_1(1)}{x^2}\right].$$

We see now that depending on the choice of $h_1(0)$ and $h_1(1)$, we can find different harmonic functions. Letting $h_1(0) = 1, h_1(1) = 0$, for example, we have

$$H_1(x) = \frac{\omega_2 + \omega_1 x}{\omega_2 + \omega_1 x - x^2 + \omega_{-1}x^3}.$$

This leads to the coefficients

$$1, 0, \frac{1}{\omega_2}, -\frac{\omega_1 + \omega_2\omega_{-1}}{\omega_2^2}, \frac{\omega_1^2 + \omega_2 + \omega_1\omega_2\omega_{-1}}{\omega_2^3} \ldots$$

Choosing $h_1(0) = 0, h_1(1) = 1$, on the other hand, we obtain

$$H_1(x) = \frac{\omega_2 x}{\omega_2 + (\omega_1 - 1)x + \omega_{-1}x^2},$$

with coefficients

$$0, 1, -\frac{\omega_1}{\omega_2}, \frac{\omega_1^2 + \omega_2}{\omega_2^2}, -\frac{\omega_1^3 - 2\omega_1\omega_2 - \omega_2^2\omega_{-1}}{\omega_2^3}, \ldots$$

In order to compute higher order polyharmonic functions, one can proceed just as in the previous example by iteratively utilizing (3.6).

3.2 General observations about a basis

From now on, we will always consider the setting of the quarter plane as in Chapter 2. The goal of this section is twofold: firstly in Lemma 3.3 we show that the discrete harmonic functions constructed as in [125, 85, 86] and defined in Section 2.2 are indeed a basis of the space of discrete harmonic functions. Secondly, we give a criterion for a set of discrete polyharmonic functions to be a basis in Lemma 3.4.

3.2.1 Constructing a basis of discrete harmonic functions

Denote in the following as in Section 2.2 by $\mathcal{H}_n$ the vector space of discrete n-polyharmonic functions, and by $\mathcal{H} := \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ the vector space of all discrete polyharmonic functions. Given any $\hat{H}_n \in \mathcal{H}_n$, we can identify it with the sequence $\left(\hat{H}_n, \hat{H}_{n-1}, \ldots, \hat{H}_1\right)$, where $\triangle \hat{H}_{k+1} = \hat{H}_k$, and $\triangle \hat{H}_1 = 0$. It is clear that any such sequence is uniquely defined by the corresponding $\hat{H}_n$. Now suppose that we have $\hat{H}_n, \hat{H}'_n \in \mathcal{H}_n$, such that, with their sequence representation as above, $\hat{H}_1 = \hat{H}'_1$. In this case, we have

$$\triangle^{n-1}\left[\hat{H}_n - \hat{H}'_n\right] = \hat{H}_1 - \hat{H}'_1 = 0, \tag{3.7}$$

thus $\hat{H}_n - \hat{H}'_n \in \mathcal{H}_{n-1}$. Therefore, provided that for each $\hat{H}_n \in \mathcal{H}_n$ we can find a corresponding $\hat{H}_{n+1} \in \mathcal{H}_{n+1}$, which will be shown below in Lemmas 3.3 and 3.8, one can prove the following lemma:

Lemma 3.2. *Let $\mathcal{H}_n$ be the space of real-valued, discrete n-polyharmonic functions in the quarter plane. Then we have an isomorphy of vector spaces*

$$\mathcal{H}_n \cong (\mathcal{H}_1)^n. \tag{3.8}$$

Proof. Suppose the statement holds for $k = 1, \ldots, n$ and we have $H_{n+1}, H'_{n+1} \in \mathcal{H}_{n+1}$ such that $H_n = H'_n$. Then we have

$$\triangle\left[H_{n+1} - H'_{n+1}\right] = H_n - H'_n = 0, \tag{3.9}$$

thus $H_{n+1} - H'_{n+1} \in \mathcal{H}_1$. Therefore (and utilizing in advance Thm. 3.7), we can construct an isomorphism $\mathcal{H}_{n+1}/\mathcal{H}_1 \to \mathcal{H}_n$, and the proof is complete. □

In particular, if we are given any $\hat{H}_n \in \mathcal{H}_n$, and we want to find all $\hat{H}_{n+1} \in \mathcal{H}_{n+1}$ with $\triangle \hat{H}_{n+1} = \hat{H}_n$, then this means that it suffices to find a single $\hat{H}_{n+1}$ with this property as well as all harmonic functions, because any other such $\hat{H}'_{n+1}$ can be written as $\hat{H}_{n+1} + \hat{G}_1$, for some $\hat{G}_1 \in \mathcal{H}_1$.

In Section 2.2, it was already mentioned how one can construct a set of discrete harmonic functions via a boundary value problem. In the following lemma, we will show that this set already forms a basis.

Lemma 3.3. *Let*

$$H_1^m(x,y) := \frac{P_m(\omega(x)) - P_m(\omega(X_+))}{K(x,y)}, \tag{3.10}$$

where

$$P_m(z) := z^m \qquad \text{if } K(0,0) = 0, \tag{3.11}$$

$$\left.\begin{aligned} P_{2m}(z) &:= z^m (z-d_0)^m \\ P_{2m+1}(z) &:= z^{m+1}(z-d_0)^m \end{aligned}\right\} \qquad \text{if } K(0,0) \neq 0, \tag{3.12}$$

where ω is the conformal mapping defined in Section 2.1.2, and $d_0 = \omega(X_+(0))$. Then the functions $H_1^m(x,y)$ form a Schauder basis[1] *of the space of discrete harmonic functions $\mathcal{H}_1$.*

[1]That is, we can express any function not necessarily via finite, but via countable sums.

Proof (outline). The arguments are mostly the same as in [86, Thm. 2]. From (2.26), it follows that $K(x,y)H(x,y)$ is already uniquely defined by the (univariate) boundary terms $K(x,0)H(x,0)$ and $K(0,y)H(0,y)$. The idea is to construct, using appropriate power series $P_m(x)$ in (3.10), a harmonic function for any given possible boundary condition. If $K(0,0)=0$, which is the same as saying that our model does not include a North-East step, then we cannot write $1/K(x,y)$ as a power series, but we can instead choose $X_+(0)$ such that $X_+(0)=0$, i.e. we can substitute X_+ into another power series. Therefore, we will consider two cases:

1. $K(0,0)=0$:
   In this case, substituting X_+ for x in (2.26) gives
   $$0 = K(X_+,0)H(X_+,0) + K(0,y)H(0,y). \tag{3.13}$$
   Utilizing this to substitute for $K(0,y)H(0,y)$ in (2.26), we obtain
   $$K(x,y)H(x,y) = \underbrace{K(x,0)H(x,0)}_{=:P(x)} - \underbrace{K(X_+,0)H(X_+,0)}_{=:P(X_+)}. \tag{3.14}$$
   Setting
   $$H_1^m(x,y) = \frac{\omega(x)^m - \omega(X_+)^m}{K(x,y)}, \tag{3.15}$$
   and utilizing that around 0 we have (after scaling and potentially switching x,y) $\omega(x) = \frac{x(1+p(x))}{(1-x)^{\pi/\theta}}$ (see [70, 5.3]; use that our walk is not singular), we can iteratively compute coefficients a_k such that $\sum a_j\omega(x)^k = P(x)$. To see that at the end we indeed obtain a power series, one can apply the Weierstraß preparation theorem.

2. $K(0,0)\neq 0$:
   In this case, the previous approach does not work anymore since substitution of X_+ into an arbitrary power series fails. Instead, let now $\omega(x) = \sum x^n c_n, \omega(X_+) = \sum y^n d_n$. We know that $c_1, d_1 \neq 0, c_0 = 0$ (see [70, 5.3], and notice that $p_{-1,-1}\neq 0$).
   We can now proceed by defining
   $$P_{2m}(z) = z^m(z-d_0)^m, \tag{3.16}$$
   $$P_{2m+1}(z) = z^{m+1}(z-d_0)^m. \tag{3.17}$$
   Letting
   $$H_1^m(x,y) := \frac{P_m(\omega(x)) - P_m(\omega(X_+))}{K(x,y)}, \tag{3.18}$$
   one can check that the monomial with non-zero coefficient with minimal degree in the series representation of $H_1^m(x,y)$ around 0 occurs for $k=l=m$ for m even, and $k=l+1=m$ otherwise. Note here that $\omega(x), \omega(X_0)$ have non-vanishing derivatives at 0 as $0\in\mathcal{G}^\circ$, see [36, 5.3]. From there, given arbitrary power series $Q(x), R(y)$ with $Q(0)=R(0)$, one can again iteratively build coefficients a_n such that $\sum a_nP_n(\omega(x)) = Q(x)$, $\sum b_nP_n(\omega(X_+)) = R(y)$. We have thus constructed a harmonic function with boundary terms $Q(x), R(y)$; since these were arbitrary we are done. Note that as $K(0,0)\neq 0$, the division by $K(x,y)$ is not an issue here.

□

As a corollary, we immediately obtain

Corollary 3.3.1. *We have*
$$\mathcal{H}_1 \cong \mathbb{C}[[z]].$$

3.2.2 A basis criterion for polyharmonic functions

Lemma 3.3 combined with the idea from Lemma 3.2, gives us a criterion for a family of polyharmonic functions to be a Schauder basis of $\mathcal{H}$, the space of all polyharmonic functions.

Lemma 3.4. *Let $\left(H_n^k\right)_{n,k\in\mathbb{N}}$ be a family of discrete polyharmonic functions, such that*

1. $H_1^k(x,y) = \frac{P_k(\omega(x))-P_k(\omega(X_+))}{K(x,y)}$ *as in (3.10),*
2. $\triangle H_{n+1}^k = H_n^k$,

Then, the (H_n^k), $1 \le k, 1 \le n \le m$ form a Schauder basis of $\mathcal{H}_m$, that is, given any $H_m \in \mathcal{H}_m$ there are unique $a_{n,k}$, $1 \le k$, $1 \le n \le m \in \mathbb{N}$ such that

$$H_m = \sum_{n=1}^{m}\sum_{k=1}^{\infty} a_{n,k} H_n^k. \tag{3.19}$$

Proof. We do not a priori assume that sums of the form $\sum_{n\ge 0} H_n^k$ converge as formal power series. Therefore, we will first modify our family (H_n^k), forcing this type of convergence. The main issue here is that we do not have any handle on the order of H_n^k at 0, written in the following as $\deg_0 H_n^k := \min\{u+v \mid \text{ the term } x^u y^v \text{ has non-zero coefficient }\}$. However, (2.26) implies that if H_{n+1}^k has vanishing boundary terms up to a sufficiently high order – which we can force by adding a suitable harmonic function, as in the proof of Lemma 3.3 –, then we have $\deg_0 H_{n+1}^k > \deg_0 H_n^k$. We will utilize this in order to construct a family $(\hat{H}_n^k)$ such that, for each n, we have

1. $\hat{H}_1^k = H_1^k$,
2. $\triangle \hat{H}_{n+1}^k = \hat{H}_n^k$,
3. each $\hat{H}_m^k$ can be written as a countable sum of the H_n^k for $n \le m$,
4. $\deg_0 \hat{H}_{n+1}^k$ is at least $\deg_0 \hat{H}_n^k + 1 \ge \left[\frac{k}{2}\right] + n - 1$,
5. the $\hat{H}_n^k$ with $n \le m$ form a Schauder basis of $\mathcal{H}_m$.

From this, the statement follows immediately.
We proceed by induction. For $m = 1$, we know that all the conditions are satisfied due to Lemma 3.3 (the part about the order at 0 is follows from the construction of the H_1^k). Now suppose we have found a suitable family $(\hat{H}_n^k)$ for $k \in \mathbb{Z}^+, n = 1,\dots,m$, and pick any k. We want to construct a suitable $\hat{H}_{m+1}^k$. As $H_1^k = \hat{H}_1^k$, we know that

$$\triangle^{n-1} H_n^k = \triangle^{n-1}\hat{H}_n^k, \tag{3.20}$$

that is, $G_{m-1}^k := H_m^k - \hat{H}_m^k \in \mathcal{H}_{m-1}$. By the induction hypothesis, we can therefore write G_{m-1}^k as some countable sum of the $\hat{H}_n^l$, $n \le m-1, l \in \mathbb{Z}_+$, for which we can then find a pre-image G_m^k under $\triangle$ by substituting n for $n-1$ in that sum representation. Note that, again by the induction hypothesis, in this representation we do not have any convergence issues. By definition of G_m^k, we have

$$\triangle\left[H_{m+1}^k - G_m^k\right] = H_m^k - \left(H_m^k - \hat{H}_m^k\right) = \hat{H}_m^k. \tag{3.21}$$

The idea in order to get the bound on $\deg_0 \hat{H}_{m+1}^k$ is to utilize the functional equation (2.26), which essentially tells us that, if the boundary terms are well enough behaved, then the degree at

0 of discrete polyharmonic functions will increase with their order (note that the $\deg_0 K(x,y) \leq 1$ because the model is non-singular). But due to the proof of Lemma 3.3 we already know that we can find harmonic functions where we can freely choose the boundary terms, so all we need to do is to select a suitable harmonic function $J_{m,k}$ killing the boundary terms up to sufficiently high order, and then, letting

$$\hat{H}^k_{n+1} := H^k_{n+1} - G^k_n - J_{n,k} \tag{3.22}$$

we see that the first four conditions are satisfied. All that remains to show is therefore that the $\hat{H}^k_n$ with $1 \leq n \leq m+1, k \geq 1$ form a Schauder basis of $\mathcal{H}_{m+1}$. First, we will show that this family generates $\mathcal{H}_{m+1}$. To see this, pick any element $H'_{m+1} \in \mathcal{H}_{m+1}$. By induction, we know that we can write

$$\triangle H'_{m+1} = \sum_{n=1}^{m} \sum_{k\geq 1} a_{n,k} H^k_n. \tag{3.23}$$

However, letting $H_{m+1} := \sum_{n=1}^{m} \sum_{k\geq 1} a_{n,k} H^k_{n+1}$, we have

$$\triangle \left[H'_{m+1} - H_{m+1}\right] = 0, \tag{3.24}$$

thus $H'_{m+1} - H_{m+1} \in \mathcal{H}_1$. Hence, up to addition of a harmonic function (remember that by Lemma 3.3 they can all be written as countable sum of the H^k_1), we know that our chosen H'_{m+1} can already be written as countable sum of the $\hat{H}^k_n$, and we are done.
To show uniqueness of the coefficients, suppose we have two representations

$$H_{m+1} = \sum_{n=1}^{m+1} \sum_{k=1}^{\infty} a_{n,k} \hat{H}^k_n = \sum_{n=1}^{m+1} \sum_{k=1}^{\infty} b_{n,k} \hat{H}^k_n. \tag{3.25}$$

As

$$0 = \triangle^m \left[H_{m+1} - H_{m+1}\right] = \sum_{k=1}^{\infty} (a_{m+1,k} - b_{m+1,k}) H^k_1 = 0, \tag{3.26}$$

we know by the basis property for $\mathcal{H}_1$ that $a_{m+1,k} = b_{m+1,k}$ for all k. For $n \leq m$ the equality then follows from the induction hypothesis, and we thus obtain $a_{m,k} = b_{m,k}$ for all $1 \leq m \leq n+1, k \geq 1$. □

Remarks:

- If instead of defining the H^k_1 as in Lemma 3.3 one allows for any Schauder basis of $\mathcal{H}_1$, then via Lemma 3.4 one obtains not only a sufficient, but also a necessary condition for a family of polyharmonic functions to form a Schauder basis.
- In the constructions of polyharmonic functions given below, it will often be the case that it is clear from construction that for any given n, any sum of the form $\sum_{k\geq 1} a_k H^k_n$ converges as power series. In this case, we can let $\hat{H}^k_n := H^k_n$, and the proof boils down to the very last step.
- In Sections 3.2 and 3.4, different bases will be constructed, and it is not obvious how to switch between them. Combinatorially it is odd to construct polyharmonic functions eliminating the boundary terms, which are also generally not easy to write down explicitly. One could argue that the basis constructed in Section 3.4 is therefore in some sense a canonical one, seeing as it has a particularly nice shape, but it is not clear whether something comparable exists in the infinite group case.

If we compare the functional equation (2.26) for harmonic and polyharmonic functions, then the only difference lies in the additional term of $xyH_n(x, y)$ on the right-hand side not vanishing for the latter. In terms of the boundary value problem, this means that we now want to solve

$$K\left(X_+, 0\right) H_n\left(X_+, 0\right) - K\left(X_-, 0\right) H_n\left(X_-, 0\right) = X_+ y H_{n-1}(X_+, y) - X_- y H_{n-1}(X_-, y). \tag{3.27}$$

In an ideal world, the right-hand side of the latter equation would be 0 as in the harmonic case, and this is indeed what happens for the simple walk (and, more generally, if ever $\pi/\theta = 2$ and the group is finite, as will be discussed in Section 3.6). In this case, we can proceed as before, and obtain an explicit formula for polyharmonic functions, see Thm. 3.26.

Example: the simple walk

The simple walk has the step set $\mathcal{S} = \{\uparrow, \rightarrow, \downarrow, \leftarrow\}$, each with probability $\frac{1}{4}$. We have

$$K(x, y) = xy - \frac{xy}{4}\left(x + y + x^{-1} + y^{-1}\right), \quad \omega(x) = \frac{-2x}{(1-x)^2}, \quad \omega\left(X_+\right) = -\omega(y). \tag{3.28}$$

It turns out that the right-hand side of (3.27) keeps vanishing, and thus one can iteratively construct polyharmonic functions via $H_{n+1}(x, y) := \frac{xyH_n(x,y) - X_+ y H_n(X_+, y)}{K(x,y)}$. This allows us to find an explicit expression for all resulting polyharmonic functions. This property is directly tied to the fact that $\pi/\theta = 2$, where θ is given by (2.11), which will be discussed in more detail in Section 3.6.
We therefore obtain a basis of all polyharmonic functions by letting

$$H_{n+1}^k(x, y) = \frac{xyH_n^k(x, y) - X_+ y H_n^k(X_+, y)}{K(x, y)}. \tag{3.29}$$

For an explicit formula as well as a proof, see Thm. 3.26. We can for instance compute $H_1^1 = \frac{8}{(1-x)^2(1-y)^2}$, $H_1^2 = \frac{32y}{(x-1)^2(y-1)^4}$, $H_1^3 = \frac{128y^2}{(x-1)^2(y-1)^6}$. One can show by induction that we have $H_1^k = \frac{2 \cdot 2^{2k}}{(x-1)^2(y-1)^{2k}}$.

3.3 A first way to construct polyharmonic functions

The goal of this section is to give a first method of computing a basis of discrete polyharmonic functions directly via manipulation of the functional equation (2.26), which will be done in Section 3.3.1. After looking at the examples of the tandem and the king's walk, we will then discuss in Section 3.3.2 why this method might not be ideal from a combinatorial perspective, which one can see by comparison with continuous polyharmonic functions and taking the scaling limit. This is then illustrated by the example of the tandem walk.

3.3.1 Construction of polyharmonic functions

While the computation for the simple walk turned out to be fairly simple, this was mainly due to the right-hand side of (3.27) consistently vanishing. This does not happen in general. For the tandem walk, for instance, we arrive at

$$K(X_+, 0)H_1^1(X_+, 0) - K(X_-, 0)H_1^1(X_+, 0) = \frac{y^3\sqrt{1-4y}}{(y-1)^5}. \tag{3.30}$$

The direct approach using a boundary value problem like in the harmonic case does not generally yield an explicit solution as easily as before. One could modify the structure in order to obtain a similar boundary value problem as before, using a decoupling function, which is the approach which works with finite group models and will be discussed in Section 3.4. However, we will first construct polyharmonic functions directly utilizing the functional equation (2.26), independently of whether or not the group is finite. The main idea again utilizes that in Lemma 3.3, we showed that given any power series $P(x) \in \mathbb{C}[[x]]$, we can construct a harmonic function $H(x,y)$ such that $K(x,0)H(x,0) = P(x)$. Now suppose that for one of the H_1^k constructed in the aforementioned theorem, there is a H_2^k such that $\triangle H_2^k = H_1^k$. Then, subtracting a harmonic function with the same values on the boundary, we know that there is also a $\hat{H}_2^k$, such that $\triangle \hat{H}_2^k = H_1^k$ and $K(x,0)\hat{H}_2^k(x,0) = 0$. By Lemma 3.4, if we know this $\hat{H}_2^k$ then we can reconstruct H_2^k (or indeed any other biharmonic function with image H_1^k under $\triangle$). Therefore, our strategy will be to utilize (2.26) in order to find this particular $\hat{H}_2^k$, where the assumption that $K(x,0)\hat{H}_2^k(x,0) = 0$ simplifies the equation immensely. While the construction itself is not very complicated, we will need a small technical lemma to make sure we will indeed end up with bivariate power series.

Lemma 3.5. *Suppose we have a model such that* $K(0,0) = 0$, $\frac{\partial}{\partial x}K\big|_{x=y=0} \neq 0$, *and select* $X_+(y)$ *such that* $0 = X_+(0)$. *Furthermore, let* $F(x,y)$ *be a bivariate power series, such that* $F(X_+(y), y) = 0$ *and that* $F(x,0) \neq 0$. *Then,*

$$\frac{F(x,y)}{K(x,y)} \tag{3.31}$$

is a bivariate power series in x, y.

Proof. By the Weierstraß preparation theorem, we can write

$$K(x,y) = e(x,y)(x - g(y)), \tag{3.32}$$

with e being an invertible bivariate power series, and $f_2(y) \in \mathbb{C}[[y]]$, with $f_2(0) = 0$.
We can also rewrite

$$F(x,y) = f(x,y)(x^k + x^{k-1}f_{k-1}(y) + \cdots + f_0(y)) =: f(x,y)P(x,y), \tag{3.33}$$

where again f is an invertible bivariate power series, the $f_i(y) \in \mathbb{C}[[y]]$ satisfy $f_i(0) = 0$, and $P(x,y) \in \mathbb{C}[[y]][x]$ is a polynomial in x over the ring $\mathbb{C}[[y]]$. Consequently, we have

$$\frac{F(x,y)}{K(x,y)} = \frac{f(x,y)}{e(x,y)} \cdot \frac{P(x,y)}{x - g(y)}. \tag{3.34}$$

Since $e(x,y)$ is invertible, it remains to show that the second factor is a power series in x, y. To do so, all we need to do is to show that $g(y)$ is a zero of the polynomial $P(x,y)$, i.e. that $P(g(y), y) = 0$.
By (3.33), we know that $P(X_+(y), y) = 0$ locally around 0, and by (3.32) we see that we also have $X_+(y) = g(y)$. The statement follows. □

Remark: In case $K(0,0) \neq 0$ (which is equivalent to our model having a North-East step) we do not need any additional tools, as in this case $1/K(x,y)$ is a power series anyway, so division by the kernel does not pose any problem. Also note that we must have either $K(0,0) \neq 0$, $\frac{\partial}{\partial x}K\big|_{x=y=0} \neq 0$ or $\frac{\partial}{\partial y}K\Big|_{x=y=0} \neq 0$, because otherwise our model could have no North, North-East or East steps and would therefore be singular.

Lemma 3.6. *Suppose we have an arbitrary non-singular model with small steps and zero drift such that either* $K(0,0) \neq 0$ *or* $\frac{\partial}{\partial x}K(x,y)\mid_{x=y=0} \neq 0$. *Given any bivariate power series* $G(x,y)$ *which is analytic around* $(0,0)$, *we can then construct a power series* $H(x,y)$ *such that*

1. *$H(x, y)$ is analytic around $(0, 0)$,*

2. *$\triangle H(x, y) = G(x, y)$, and*

3. *$K(x, 0)H(x, 0) = 0$*

by letting

$$H(x, y) := \frac{X_+ yG(X_+, y) - xyG(x, y)}{K(x, y)} \qquad \text{if } K(0,0) = 0, \tag{3.35}$$

$$H(x, y) := \frac{xyG(x, y)}{K(x, y)} \qquad \text{if } K(0,0) \neq 0, \tag{3.36}$$

where we select $X_+(y)$ such that $X_+(0) = 0$.

Proof. The property $\triangle H = G$ can be written in terms of the functional equation (2.26):

$$K(x, y)H(x, y) = K(x, 0)H(x, 0) + K(0, y)H(0, y) - K(0, 0)H(0, 0) - xyG(x, y). \tag{3.37}$$

The case $K(0, 0) \neq 0$ is easy, because then $1/K(x, y)$ is a power series around $(0, 0)$ and one can directly check that (2.26) is satisfied. Consider now the case $K(0, 0) = 0$, and define $K(x, y)H(x, y)$ via (3.35). The substitution $G(X_+, y)$ is valid because $X_+(0) = 0$. One can check immediately that $K(x, 0)H(x, 0) = 0$, and that (2.26) is satisfied.
All that therefore remains to do is to show that we can divide the thusly obtained power series $K(x, y)H(x, y)$ by $K(x, y)$. To do so, we can utilize Lemma 3.5. To check the conditions to apply this lemma, note that $K(X_+, y)H(X_+, y) = 0$ by construction. In order to satisfy the second condition, let $H'(x, y)$ be a harmonic function such that $K(x, 0)H'(x, 0) \neq 0$. We then have

1. $K(X_+, y)H(X_+, y) + K(X_+, y)H'(X_+, y) = 0$, as is clear for the first summand from (3.35) and for the second due to the fact that $K(X_+, y) = 0$,

2. $K(x, 0)H(x, 0) + K(x, 0)H'(x, 0) \neq 0$.

We can therefore apply Lemma 3.5 to the function $K(x, y)H(x, y) + K(x, y)H'(x, y)$. This tells us that the expression

$$\begin{aligned} \frac{K(x, y)H(x, y) + K(x, y)H'(x, y)}{K(x, y)} &= \frac{K(x, y)H(x, y)}{K(x, y)} + \frac{K(x, y)H'(x, y)}{K(x, y)} && (3.38)\\ &= H(x, y) + H'(x, y) && (3.39) \end{aligned}$$

is a power series in x, y around $(x, y) = (0, 0)$. As we already know that $H'(x, y)$ is a power series, by consequence so is $H(x, y)$. Hence, we are done. $\square$

As previously remarked, after potentially swapping x and y such that $\frac{\partial}{\partial x} K\big|_{x=y=0} \neq 0$, this covers all non-singular models with small steps and zero drift. Therefore, in the following we can assume without loss of generality that if $K(0, 0) = 0$, then $\frac{\partial}{\partial x} K(x, y) \mid_{x=y=0} \neq 0$. Utilizing Lemma 3.6, it is now easy to construct a Schauder basis of all polyharmonic functions.

Theorem 3.7. *Given a model with small steps and zero drift, and let*

$$H_1^k(x, y) := \frac{P_k(\omega(x)) - P_k(\omega(X_+))}{K(x, y)}, \tag{3.40}$$

where the polynomials P_k are defined as in the proof of Lemma 3.3. Then we can inductively construct bivariate power series H_n^k via

$$H_{n+1}^k(x,y) := \frac{X_+ y H_n^k(X_+,y) - xyH_n^k(x,y)}{K(x,y)} \qquad \text{if } K(0,0)=0, \tag{3.41}$$

$$H_{n+1}^k(x,y) := \frac{xyH_n^k}{K(x,y)} \qquad \text{if } K(0,0)\neq 0. \tag{3.42}$$

Each H_n^k is n-polyharmonic, and we have $\triangle H_{n+1}^k = H_n^k$.

Proof. By iterative application of Lemma 3.6, one sees that the resulting expressions are power series in x, y. One can then easily check that the functional equation (2.26) is satisfied. □

Remark: instead of using the two different definitions (3.41), (3.42) depending on whether or not $K(0,0)=0$, one could just use (3.41) in any case. The disadvantage of that would be, however, fairly obvious: the resulting expressions are a bit more unwieldy, and we lose a bit of niceness (i.e. if $H_1^1(x,y)$ is rational, we would normally end up with an algebraic $H_2^1(x,y)$). Also, to make things work out formally one would still have to argue why all substitutions are valid.

Theorem 3.8. *The polyharmonic functions $(H_m^k)_{m,k\in\mathbb{N}}$ constructed in Thm. 3.7 form a basis of the space $\mathcal{H}$ of all polyharmonic functions.*

Proof. By Lemma 3.4. □

Example: the tandem walk

The tandem walk is the model with step set $\mathcal{S}=\{\nwarrow,\rightarrow,\downarrow\}$, each with weight $\frac{1}{3}$. We find

$$K(x,y) = xy - \frac{x^2+y+xy^2}{3}, \quad H_1^1(x,y) = \frac{81(1-xy)}{4(x-1)^3(y-1)^3}, \tag{3.43}$$

leading to the harmonic function $h(i,j)=(i+1)(j+1)(i+j+2)$. We now want to find a biharmonic function H_2^1 such that $\triangle H_2^1 = H_1^1$. To do so, we apply the procedure from Thm. 3.8. First, we notice that $K(0,0)=0$, and that $\frac{\partial K}{\partial x}\Big|_{x=y=0}=0$, while on the other hand we have $\frac{\partial K}{\partial y}\Big|_{x=y=0} = -\frac{1}{3}\neq 0$. This is due to the fact that our model has no West, but a South step. Therefore, we need to swap the roles of x and y in (3.35). We pick our Y_+ such that $Y_+(0)=0$; which gives us

$$Y_+(x) = \frac{3x-1+(1-x)\sqrt{1-4x}}{2x}, \tag{3.44}$$

and obtain

$$H_2^1(x,y) = \frac{243x(3x^2+ax^2+2y-4xy-7x^2y-3ax^2y-2xy^2+13x^2y^2+3ax^2y^2-2x^3y^2-3x^2y^3-ax^2y^3}{8(1-x)^5(1-y)^3(x^2+y+xy^2-3xy)}, \tag{3.45}$$

where $a:=\sqrt{1-4x}$. One can check that this expression is indeed a power series which satisfies $\triangle H_2^1(x,y) = H_1^1(x,y)$, and that $H_2^1(0,y)=0$. In particular, we have

$$H_2^1(x,y) = \frac{243}{4}x + \frac{729}{4}xy + \frac{729}{2}x^2 + \frac{729}{2}xy^2 + 972x^2y + \dots \tag{3.46}$$

As we already know, $H_2^1(x,y)$ is unique with the property $\triangle H_2^1 = H_1^1$ only up to harmonic functions. And indeed, we will see in Section 3.4 that instead of this algebraic function, there is a much nicer rational biharmonic function $\hat{H}_2^1$ with $\triangle \hat{H}_2^1 = \hat{H}_1^1$.

Example: the king's walk

The king's walk is the model with step set $\mathcal{S} = \{\uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow, \leftarrow, \nwarrow\}$, each with probability $\frac{1}{8}$. We find

$$K(x,y) = xy - \frac{1+x+y+x^2+y^2+x^2y+xy^2+x^2y^2}{8}, \tag{3.47}$$

$$H_1^1(x,y) = \frac{1}{16(x-1)^2(y-1)^2}. \tag{3.48}$$

As $K(0,0) \neq 0$, we can utilize (3.36) and have

$$H_2^1(x,y) = \frac{128xy}{(x-1)^2(y-1)^2\left[8xy - (1+x+y+x^2+y^2+x^2y+xy^2+x^2y^2)\right]}. \tag{3.49}$$

3.3.2 Relations between discrete and continuous cases

For harmonic functions, since the last term of (2.33) vanishes, everything works as in the discrete case, except the calculations turn out to be a lot simpler. We can define continuous versions $x_\pm$ of $X_\pm$, which satisfy $\gamma(x_\pm(y), y) = 0$. It turns out that we have

$$x_\pm(y) = c_\pm y, \tag{3.50}$$

$$c_\pm = ce^{\pm i\theta}, \tag{3.51}$$

$$\hat{\omega}(x) = \frac{1}{x^{\pi/\theta}}, \tag{3.52}$$

where θ is the arctangent of the correlation coefficient as introduced in Section 2.1.2 and $c \in \mathbb{R}^+$. We can then construct (continuous) harmonic functions via

$$\mathcal{L}(h_1^n)(x,y) := \frac{\omega(x)^n - \omega(x_+(y))^n}{\gamma(x,y)}, \tag{3.53}$$

see also [36, Thm. 2.4]. Not very surprisingly, there is a relation between the discrete and continuous polyharmonic functions constructed in this manner. For the computations here as well as in later sections, the following lemma will be useful:

Lemma 3.9. *We have*

$$\lim_{\mu\to 0} \frac{K\left(e^{-\mu x}, e^{-\mu y}\right)}{\mu^2} = \gamma(x,y), \tag{3.54}$$

$$\lim_{\mu\to 0} X_\pm\left(e^{-\mu y}\right) = 1 + c_\pm y + \mathcal{O}(y^2). \tag{3.55}$$

Proof. Both of the results follow by a direct computation, which however in the second case is somewhat tedious. The main idea there is to write

$$X_+(e^{-z})X_-(e^{-z}) = \frac{\tilde{c}(e^{-z})}{\tilde{a}(e^{-z})}, \tag{3.56}$$

$$X_+(e^{-z})X_-(e^{-z}) = -\frac{\tilde{b}(e^{-z})}{\tilde{a}(e^{-z})}, \tag{3.57}$$

with $X_\pm(y)$ the solutions of $K(\cdot,y) = 0$ as defined in Section 2.1.2, and then use the fact that $X_+(1) = X_-(1) = 1$ in order to obtain defining equations for the first coefficients in a series expansion of $X_\pm(e^{-z})$. □

Remark: While for (3.54) it can be seen that this is a direct consequence of the drift being zero, it would be interesting to know if there is a more intuitive, or geometric way to obtain (3.55) as well.

Comparing the discrete and continuous constructions of harmonic functions (3.10) and (3.53), it is not very surprising that there is a clear relation between them.

Theorem 3.10. *We have*

$$\lim_{\mu\to 0} \mu^{k\pi/\theta-2} H_1^k\left(e^{-\mu x}, e^{-\mu y}\right) = \alpha \mathcal{L}(h_1^k)(x,y) \tag{3.58}$$

for some non-zero constant α.

Proof. Using Lemma 3.9, all we need to show is that

$$\lim_{\mu\to 0} \mu^{k\pi/\theta}\left[P_k(\omega(x)) - P_k(\omega(X_+))\right] = \hat{\omega}(x)^k - \hat{\omega}(x_+)^k. \tag{3.59}$$

But this follows immediately from the fact that $\omega(x) = \frac{\alpha+o(1)}{(1-x)^{\pi/\theta}}$ in a neighbourhood of $x = 1$ [67, 2.2] and the fact that $[z^k]P_k(z) = 1$ (see the construction of $P_k(z)$ in the proof of Lemma 3.3), as well as for the second term once again Lemma 3.9. □

Knowing that, in the sense of a scaling limit as in Thm. 3.10, we know that the discrete kernel K corresponds to the continuous kernel γ, and that the Laplace transform can be understood as the continuous analogue of a generating function, it would be reasonable to expect that the boundary term $K(x,0)H(x,0)$ corresponds in the same fashion to $\frac{\sigma_{22}}{2}\mathcal{L}_2(h)(x)$. In the following, we will see that this is indeed the case.

Lemma 3.11. *Suppose* $h(u,v)$ *and its derivatives up to order* 2 *are of exponential order, i.e. their absolute value is asymptotically bounded by* $e^{c(x+y)}$ *for some constant* c*, with* $h(u,0) = h(0,v) = 0$*. Then we have*

$$\mathcal{L}_2(h)(x) = \lim_{y\to\infty} y^2 \mathcal{L}(h)(x,y), \tag{3.60}$$

$$\mathcal{L}_1(h)(y) = \lim_{x\to\infty} x^2 \mathcal{L}(h)(x,y). \tag{3.61}$$

Proof. We will only show the first equality, the second follows by symmetry. From the computation in [125, App. A], we see that

$$\mathcal{L}(h)(x,y) = \frac{1}{y^2}\left[\mathcal{L}\left(\frac{\partial^2 h}{\partial v^2}\right)(x,y) + \mathcal{L}_2(h)(x)\right] \quad \Leftrightarrow \tag{3.62}$$

$$y^2\mathcal{L}(h)(x,y) = \mathcal{L}\left(\frac{\partial^2 h}{\partial v^2}\right)(x,y) + \mathcal{L}_2(h)(x). \tag{3.63}$$

Therefore, all that remains to show is that $\lim_{y\to\infty}\mathcal{L}\left(\frac{\partial^2 h}{\partial y^2}\right) = 0$. But this follows by monotone convergence using the growth assumption on $\frac{\partial^2 h}{\partial v^2}$. □

Remark: The condition that $h(u,v)$ and its derivatives are of exponential order will hold true for those polyharmonic functions which have their origins in asymptotics of exit times of Brownian motions, see e.g. [11, 36].

Lemma 3.12. *Let $\alpha \in \mathbb{R}$, and $H(x,y), \mathcal{L}(h)(x,y)$ be polyharmonic such that*

$$\lim_{\mu\to 0} \mu^{\alpha+2} H\left(e^{-\mu x}, e^{-\mu y}\right) = \mathcal{L}(h)(x,y). \tag{3.64}$$

Assume furthermore that $H(x,y)$ is algebraic, and that the restrictions of $H(x,y)$ at $x=0$ and $y=0$ are well-defined. Then we have

$$\lim_{\mu\to 0} \mu^{\alpha} K(e^{-\mu x},0) H(e^{-\mu x},0) = \frac{\sigma_{22}}{2} \mathcal{L}_2(h)(x), \tag{3.65}$$

$$\lim_{\mu\to 0} \mu^{\alpha} K(0,e^{-\mu y}) H(0,e^{-\mu y}) = \frac{\sigma_{11}}{2} \mathcal{L}_1(h)(y). \tag{3.66}$$

Proof. Using Lemma 3.11, we have

$$\lim_{\mu\to 0} \mu^{\alpha} KH(e^{-\mu x},0) = \lim_{\mu\to 0} \lim_{y\to\infty} \mu^{\alpha} KH(e^{-\mu x}, e^{-\mu y}), \tag{3.67}$$

$$\frac{\sigma_{22}}{2} \mathcal{L}_2(h)(x) = \lim_{y\to\infty} \gamma(x,y) \mathcal{L}(h)(x,y) \tag{3.68}$$

$$= \lim_{y\to\infty} \lim_{\mu\to 0} \mu^{\alpha} KH(e^{-\mu x}, e^{-\mu y}). \tag{3.69}$$

Thus, all we need to show is that we can exchange the order of the two limits. But this follows by the algebraicity of $H(x,y)$. □

Remark: There is a marked difference between the discrete and continuous cases in terms of the value of formal solutions. In the discrete case we work with formal power series, i.e. every formal solution of the functional equation (2.26) leads to an actual solution since we can just extract coefficients. In the continuous case, however, this is not so simple: there are formal solutions of (2.33) which turn out not to have an inverse Laplace transform. It is always possible to utilize the method given in Section 3.3 to obtain continuous (formal) solutions, by simply defining $\mathcal{L}(h_n^k)$ as the scaling limit – with an appropriate scaling factor – of the discrete polyharmonic function H_n^k, and then by some computations using Lemmas 3.11 and 3.12 one can check that (2.33) is indeed satisfied. But the resulting solutions do generally not allow for an inverse Laplace transform: due to the shape of the kernel, which we repeatedly divide by, we cannot usually find a region of the form $\{\Re(x) \geq u, \Re(y) \geq v\}$ where $\mathcal{L}(h_i^k)(x,y)$ is finite. By [121], this implies that $\mathcal{L}(h_i^k)(x,y)$ is not the Laplace transform of any function, nor of any distribution. This will be different for the method presented in Section 3.4.

Example: the scaling limit of the tandem walk

For the scaling limit of the tandem walk (see Example 3.3.1), we have

$$\gamma(x,y) = \frac{1}{3}\left(x^2 - xy + y^2\right), \quad c_{\pm} = \frac{1 \pm i\sqrt{3}}{2}, \quad \hat{\omega}(x) = \frac{1}{x^3}. \tag{3.70}$$

We obtain $\mathcal{L}(h_1^1)(x,y) = \frac{\hat{\omega}(x)-\hat{\omega}(c_+ y)}{\gamma(x,y)} = \frac{3(x+y)}{x^3 y^3}$, and one can check immediately that

$$\lim_{\mu\to 0} \mu^5 H_1^1\left(e^{-\mu x}, e^{-\mu y}\right) = \mathcal{L}(h_1^1)(x,y) \tag{3.71}$$

for H_1^1 computed in Section 3.3.1. We can then proceed to the scaling limit of H_2^1, which gives us the formal solution of (2.33)

$$\mathcal{L}(h_2^1)(x,y) = \frac{3x^3 + 2xy^2 + 2y^3}{x^3 y^5 (x^2 - xy + y^2)}, \tag{3.72}$$

of which one can check directly that there is no inverse Laplace transform. We will see in Section 3.4.2 that an advantage of the construction done in Section 3.4 using decoupling functions is that the scaling limit of the resulting discrete polyharmonic functions will always be the Laplace transform of a continuous polyharmonic function.

3.4 A second way to construct polyharmonic functions

While the method given in Section 3.3 gives us a Schauder basis of all polyharmonic functions, the resulting basis is not ideal in two senses:

1. They do not have a continuous analogue, as discussed at the end of Section 3.3.2;
2. They are often more complicated than necessary; for the king's walk we obtained a rational function which is singular on some not so easily described curve in Section 3.3.1, and for the tandem walk the functions constructed in Section 3.3.1 were not even rational. We will see that both of these models have a basis which is a lot nicer to work with.

The goal of this section is to present an alternative method to construct discrete polyharmonic functions using a decoupling approach, similar as in [12, 36, 131]. This will allow us to give a basis of polyharmonic functions for all models which finite group such that $\pi/\theta \in \mathbb{Z}$ in Thm. 3.15 in Section 3.4.1. We will see in Section 3.4.2 that an analogous method works for continuous models. Lastly, in Section 3.4.3 it will be shown that the discrete polyharmonic functions obtained via the decoupling approach behave nicely with respect to the scaling limit.

3.4.1 An approach via decoupling functions

Remember that the main issue why computing polyharmonic functions is not as easy as computing harmonic functions is that the right-hand side of (3.27) does not usually vanish, and therefore the boundary value problem approach does not immediately work. But in some cases, one can circumvent this problem by utilizing what is called a decoupling function in [12, Def. 4.7].

Definition 6. *Let $M(x, y)$ be an rational function in x, y. If we can find $F(x), G(y)$ such that*

$$F(x) + G(y) \equiv M(x, y) \mod K(x, y), \tag{3.73}$$

*then we say that F is a **decoupling function** of M.*

Here, we say that $A(x) \equiv B(x) \bmod K(x, y)$ if there are polynomials $N(x, y), D(x, y)$ such that $D(x, y)$ is not divisible by $K(x, y)$ and $A(x) - B(y) = \frac{N(x,y)}{D(x,y)} K(x, y)$.

These decoupling functions are closely related to the concept of invariants as in [12, Def. 4.3]. An example of a decoupling function will for instance be given in Section 3.4.1. The reason why finding such decoupling functions are useful can be seen directly from (3.27): if $F(x)$ is a decoupling function of $xy\triangle H(x, y)$, then we have

$$K(X_+, 0)H(X_+, 0) - F(X_+) - [K(X_-, 0)H(X_-, 0) - F(X_-)] = 0. \tag{3.74}$$

In other words, if one knows how to compute a decoupling function of $xy\triangle H(x, y)$, then one can proceed by the same arguments as for the boundary value problem outlined in Section 3.2, and it turns out that by setting $K(x, 0)H(x, 0) - F(x) = P(\omega)$ for some entire function Pone will eventually arrive at a solution for $H(x, y)$. In [36, App. C], a decoupling function is guessed using an ansatz (as illustrated in Section 3.5) in order to compute a biharmonic function for the tandem walk. It turns out, however, that such a decoupling function can be explicitly computed for any model as long as the group of the model (see Section 2.1.3) is finite.

Theorem 3.13 (see [12, Thm. 4.11]). *Suppose our step set has a finite group of order $2n$, and $M(x,y)$ is rational such that*

$$\sum_{\gamma\in\mathcal{G}} \operatorname{sgn}(\gamma)\gamma\left(M(x,y)\right) = 0. \tag{3.75}$$

Then a rational decoupling function of $M(x,y)$ is given by

$$F(x) = -\frac{1}{n}\sum_{i=1}^{n-1}\Theta^i\left[M(x,Y_+) + M(x,Y_-)\right]. \tag{3.76}$$

In the following, we will show that $xyH_n(x,y)$ will turn out to have an orbit sum of 0 for any polyharmonic H_n. This is in particular independent of whether or not the given model has a vanishing orbit sum as in [12].

Corollary 3.13.1. *Suppose the group of the step set is finite and has a series representation around* $(0,0)$*. Then any algebraic function $M(x,y)$ of the form*

$$M(x,y) = xy\frac{u(x)+v(y)}{K(x,y)} \tag{3.77}$$

allows for a decoupling function via (3.76).

Proof. For any point (x,y) such that $K(x,y)\neq 0$, (3.75) is satisfied, seeing as the denominator $\frac{1}{xy}K(x,y)$ is invariant under $\mathcal{G}$, and alternating orbit summation over the numerator leads to a telescopic sum. As the set $\{(x,y) : K(x,y)\neq 0\}$ is dense in $\mathbb{C}^2$ and $M(x,y)$ is algebraic, this implies that (3.75) is satisfied everywhere. By Thm. 3.13, we can therefore construct a decoupling function via (3.76). $\square$

If a model has a finite group, then it can be shown that $\pi/\theta\in\mathbb{Q}$ (cf [70, 7.1]). The main difference between π/θ being integer or not is that in the former case, the conformal mapping $\omega(x)$ will be rational, and thus we can construct a basis consisting of rational functions.
To make things work out nicely in this case, we need to start wich a small technical lemma.

Lemma 3.14. *Let $N(x,y)$ be a polynomial such that $N(X_+,y) = N(X_-,y) = 0$. Then, $K(x,y)|N(x,y)$.*

Proof. First we note that $N(X_+,y) = N(X_-,y) = 0$ implies that also $N(x,Y_+) = N(x,Y_-) = 0$ via the substitution $x\mapsto X_\pm(y)$. Using the notation of Section 2.2.3 of $\sigma_{11} = \mathbb{E}(X^2), \sigma_{22} = \mathbb{E}(Y^2)$, we can therefore assume that $\sigma_{11}\geq\sigma_{22}$, else we switch the roles of x and y in the following.
We write $K(x,y) = (x-X_+)(x-X_-)\tilde{a}(y)$. As $N(X_+,y) = N(X_-,y) = 0$ for any y, we know that $N(x,y)$ contains a factor $(x-X_+)(x-X_-) = x^2-(X_-+X_+)x+X_-X_+ = x^2+\frac{\tilde{b}(y)}{\tilde{a}(y)}x+\frac{\tilde{c}(y)}{\tilde{a}(y)}\in\mathbb{C}(y)[x]$. By assumption, $N(x,y)$ is a polynomial; thus it must also contain a factor $\frac{\tilde{a}(y)}{\gcd(\tilde{a}(y),\tilde{b}(y),\tilde{c}(y))}$. Therefore, it suffices to show that $\tilde{a}(y),\tilde{b}(y),\tilde{c}(y)$ have no common zero.
We have

$$\tilde{a}(y) = p_{1,1}y^2 + p_{1,0}y + p_{1,-1}, \tag{3.78}$$

$$\tilde{b}(y) = p_{0,1}y^2 - y + p_{0,-1}, \tag{3.79}$$

$$\tilde{c}(y) = p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}. \tag{3.80}$$

Now suppose there is an u such that $\tilde{a}(u) = \tilde{b}(u) = \tilde{c}(u) = 0$. Adding the three previous equations yields

$$0 = u^2 [p_{1,1} + p_{0,1} + p_{-1,1}] - u [1 - p_{1,0} - p_{-1,0}] + [p_{1,-1} + p_{0,-1} + p_{-1,-1}] . \tag{3.81}$$

As the drift is 0, we know that the coefficient of u^2 is the same as the constant, namely $\frac{1}{2}\sigma_{11}$, and the coefficient of u is σ_{22}. Since our model is non-singular, we have $\sigma_{11}, \sigma_{22} > 0$. We can therefore rewrite (3.81) as

$$\sigma_{11} u^2 - 2\sigma_{22} u + \sigma_{11} = 0. \tag{3.82}$$

Using the quadratic formula, we obtain

$$u = \frac{\sigma_{22}}{\sigma_{11}} \pm \sqrt{\left(\frac{\sigma_{22}}{\sigma_{11}}\right)^2 - 1}. \tag{3.83}$$

If $\sigma_{11} = \sigma_{22}$, then $u = 1$, but we see that $\tilde{b}(1) < 0 \leq \tilde{a}(1), \tilde{b}(1)$.
Therefore we must have $\sigma_{11} > \sigma_{22}$, so we have two complex conjugate solutions for u. However, as $\tilde{b}(0) > 0$ and $\tilde{b}(1) < 0$, we know that $\tilde{b}(y)$ can only have real solutions (note in particular that this does not change if $p_{0,1} = 0$, in which case $\tilde{b}(y)$ is linear), so $\tilde{b}(u) = 0$ cannot hold.
Hence, $\tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ cannot have a common factor, and the statement follows. □

Using the above lemma, we can now use decoupling functions to construct, in the case of a finite group with $\pi/\theta \in \mathbb{Z}$, rational discrete polyharmonic functions of a particularly nice shape.

Theorem 3.15. *Suppose our step set has finite group and $\pi/\theta \in \mathbb{Z}$. Let $H_1^k(x,y)$ be defined by (3.40). We can then define inductively*

$$H_n^k(x,y) = \frac{xyH_{n-1}^k(x,y) - F_{n-1}^k(x) - \left[X_+ y H_{n-1}^k(X_+,y) - F_{n-1}^k(X_+,y)\right]}{K(x,y)}, \tag{3.84}$$

where $F_n^k(x)$ is the decoupling function of $xyH_n^k(x,y)$ defined by (3.76), which in particular exists. Then, $H_n^k(x,y)$ is a rational function in $\mathcal{H}_n$ for all n,k, which satisfies $\triangle H_{n+1}^k = H_n^k$. For each n,k we can write

$$H_n^k(x,y) = \frac{p_{n,k}(x,y)}{(1-x)^\alpha (1-y)^\alpha}, \tag{3.85}$$

where $p_{n,k}(x,y)$ is a polynomial and $\alpha \in \mathbb{N}$.

Remarks:

- In Thm. 3.19 we will see that $\alpha \leq k\pi/\theta + 2(n-1)$.
- Defining decoupling functions and utilizing them in order to compute polyharmonic functions works, as long as the group is finite, for any π/θ (which must then automatically be rational). In particular, one can check that an analogous version of (3.76) holds. However, in the non-integer case we do not obtain polynomial functions anymore, and in particular we will not have a representation like (3.85); the main reason being that an equivalent of Lemma 3.14 does not hold. Therefore we lose information about the positioning of singularities, which will generally not only be where $x = 1$ or $y = 1$. The rest one can prove in the same manner as the corresponding points in the proof of Thm. 3.15.

Proof. We proceed by induction. In each step, we will show that:

- $H_n^k(x,y)$ is rational,
- $H_n^k(x,y)$ has its only poles at $x=1$ or $y=1$,
- $xyH_n^k(x,y)$ does not have a pole at $x=\infty$ or $y=\infty$,
- $xyH_n^k(x,y)$ has orbit sum 0 and thus admits a decoupling function $F_n^k(x)$,
- $F_n^k(x)$ has its only pole at $x=1$.

To see that $\triangle H_{n+1}^k(x,y) = H_n^k(x,y)$, one can simply plug (3.84) into the functional equation (2.26).
So consider first the case $n=1$. $H_1^k(x,y)$ being rational follows immediately from $\pi/\theta \in \mathbb{Z}$, and thus ω being rational (see [125, (3.12)]). As by construction the numerator $N_1^k(x,y)$ of $xyH_1^k(x,y)$ as defined in (3.84) satisfies $N_1^k(X_\pm)=0$, it must according to Lemma 3.14 be a multiple of $K(x,y)$, thus the only poles of $H_1^k(x,y)$ can be those coming from $\omega(x), \omega(X_+)$. Since $\omega(x)$ has its only pole at $x=1$ and $X_+(y)=1$ only if $y=1$, $H_1^k(x,y)$ can only have poles at $x=1, y=1$. Similarly, we check by a direct computation that $xyH_1^k(x,y)$ does not have a pole at $x=\infty, y=\infty$. The existence of a decoupling function $F_1^k(x)$ follows immediately from Prop. 3.13.1. Finally, we can deduce from (3.76), utilizing that $xyH_1^k(x,y)$ does not have poles at infinity and noting that $(1,1)$ is a fixed point under the group, that $F_n^k(x)$ has its only pole at $x=1$, thus the case $n=1$ is done.
Now let $n \geq 2$ and assume the theorem is already shown up to $n-1$. We then formally define as in (3.84)

$$H_n^k(x,y) := \frac{xyH_{n-1}^k(x,y) - F_{n-1}^k(x) - \left[X_+ y H_{n-1}^k(X_+,y) - F_{n-1}^k(X_+,y)\right]}{K(x,y)}. \tag{3.86}$$

First, we need to argue that $H_n^k(x,y)$ is rational. By assumption, we know that $H_{n-1}^k(x,y)$ is rational. To see that the remaining part of the numerator, that is, $X_+yH_{n-1}^k(X_+,y) - F_{n-1}^k(X_+)$, is rational, we use the defining property (3.73) of the decoupling function F_n^k, rewriting

$$G(y) = X_+ y H_{n-1}^k(X_+,y) - F_{n-1}^k(X_+) \qquad \Rightarrow \tag{3.87}$$
$$xyH_{n-1}^k(x,y) \equiv F_{n-1}^k(x) + X_+yH_{n-1}^k(X_+,y) - F_{n-1}^k(X_+) \qquad \bmod K(x,y), \tag{3.88}$$

and since $xyH_{n-1}^k(x,y)$, $F_{n-1}^k(x)$ as well as $K(x,y)$ are rational, so is $X_+yH_{n-1}^k(X_+,y) - F_{n-1}^k(X_+)$. Next, we consider the poles of $H_n^k(x,y)$. Again, by construction we have that the numerator $N_n^k(X_\pm,y)=0$, and thus by Lemma 3.14 the $K(x,y)$ in the denominator cancels. As we know by assumption that $F_{n-1}^k(x)$ has its only pole at $x=1$, and as $X_+(y)=1$ if and only if $y=1$, there will be no new poles coming from the F_{n-1}^k-parts. The same goes for $H_{n-1}^k(x,y)$ and $H_{n-1}^k(X_+,y)$, and therefore the only poles of $H_n^k(x,y)$ can be at $x=1, y=1$.
To check that $xyH_n^k(x,y)$ does not have a pole at infinity, we utilize (2.26):

$$K(x,y)H_n^k(x,y) = K(x,0)H_n^k(x,y) + K(0,y)H_n^k(0,y) - xyH_{n-1}^k(x,y).$$

If $xyH_n^k(x,y)$ had a pole at infinity, then so would $K(x,y)H_n^k(x,y)$. But as by the induction hypothesis, $xyH_{n-1}^k(x,y)$ does not have a pole at infinity; so the pole for, say, $x\to\infty$ of the left-hand side would need to cancel with $K(x,0)H_n^k(x,0)$ on the right-hand side. But the left-hand side depends on y while the $K(x,0)H_n^k(x,0)$ doesn't, so the poles cannot cancel for all values of y; a contradiction.
To see that a decoupling function of $xyH_n^k(x,y)$ exists, we split $xyH_n^k(x,y)$ in two parts. First, we notice that

$$xy\frac{xyH_{n-1}^k(x,y)}{K(x,y)} = \frac{xy}{K(x,y)}xyH_{n-1}^k(x,y), \tag{3.89}$$

and since $\frac{xy}{K(x,y)}$ is invariant under the group and we already know that $xyH^k_{n-1}(x,y)$ has a decoupling function (and thus its orbit sum is 0), we deduce that this part as well has orbit sum 0, and thus it can be decoupled by Thm. 3.13. For the rest, we notice that

$$xy\frac{-F^k_{n-1}(x) - \left[X_+yH^k_{n-1}(X_+,y) - F^k_{n-1}(X_+,y)\right]}{K(x,y)} \tag{3.90}$$

has the form $xy\frac{A(x)+B(y)}{K(x,y)}$ (note that $X_+(y)$ does in fact not depend on x), and thus its orbit sum is 0 by Cor. 3.13.1. Therefore, Thm. 3.13 gives us a decoupling function $F^k_n(x)$ of $xyH^k_n(x,y)$ via (3.76). As each summand has its poles at $x=1$ only, and Θ leaves the point $(1,1)$ invariant, we know that F^k_n has its only pole at $x=1$.
It remains to show that the order of the poles at $x,y=1$ is at most $k\cdot\pi/\theta+2(n-1)$. For $n=1$ this can again be verified directly; afterwards it follows by induction: by a short computation one can see that the order of the pole of $F(x)$ compared to the one at $x=1$ of $xyH(x,y)$ increases at most by 2, and by a similar argument for the $G(y)$ in (3.73) (see [12, Thm. 4.11] for an explicit formula) one can show the same for $X_+yH(X_+,y) - F(X_+) = G(y)$. Using (3.84) finally yields the statement. □

By Lemma 3.4, it therefore follows that the thusly constructed polyharmonic functions form a Schauder basis of the space of all polyharmonic functions.

Example: the tandem walk revisited

To illustrate the results from Section 3.4, consider once again the tandem walk, which has the step set $\mathcal{S} = \{\rightarrow, \downarrow, \nwarrow\}$, with weights $\frac{1}{3}$ each. As in Example 3.3.1, we have

$$K(x,y) = xy - \frac{xy}{3}\left(x^{-1} + y + xy^{-1}\right), \quad H^1_1(x,y) = \frac{81(1-xy)}{4(x-1)^3(y-1)^3}. \tag{3.91}$$

Coefficient extraction then led us to recover the original harmonic function from the generating series, giving us $h^1_1(i,j) = (i+1)(j+1)(i+j+2)$. In Section 3.4 we computed a biharmonic function for H^1_1, which was however not rational. Using the method presented in this section, however, we will find that there is, in fact, a rational one.
First, one can check that the group is finite and of order 6; we have

$$(x,y) \overset{\Psi}{\mapsto} \left(x, \frac{x}{y}\right) \overset{\Phi}{\mapsto} \left(\frac{1}{y}, \frac{x}{y}\right) \overset{\Psi}{\mapsto} \left(\frac{1}{y}, \frac{1}{x}\right) \overset{\Phi}{\mapsto} \left(\frac{y}{x}, \frac{1}{x}\right) \overset{\Psi}{\mapsto} \left(\frac{y}{x}, y\right) \overset{\Phi}{\mapsto} (x,y). \tag{3.92}$$

Now using (3.76), we obtain the decoupling function $F_1(x) = -\frac{81x^3}{4(1-x)^5}$. Note that this decoupling function is not the same one as is given in [36, App. C], where instead (after scaling) $F_1' = \frac{-81x^3}{4(1-x)^6}$ is given. This goes to show that the choice of a decoupling function is, due to the invariance property in (2.10), unique only up to functions of ω; in this particular case we have (up to a multiplicative constant) $F_1'(x) - F_1(x) = \omega(x)^2$. The way in which this alternative decoupling function was found is described in Section 3.5.
We can now utilize this F_1 in order to compute a biharmonic function; (3.84) directly gives us

$$H^1_2 = \frac{243(xy-1)(x+y+xy(x+y-4))}{(x-1)^5(y-1)^5}, \tag{3.93}$$

which after extracting coefficients corresponds to

$$\begin{aligned} h^1_2(i,j) = (i+1)(j+1)(&-36i - 30i^2 - 6i^3 - 36j - 44ij - 14i^2j - 2i^3j - 30j^2 \\ &- 14ij^2 + i^2j^2 + i^3j^2 - 6j^3 - 2ij^3 + i^2j^3 + i^3j^3) \end{aligned} \tag{3.94}$$

We can now use (3.76) again to obtain the next decoupling function $F_2(x) = -\frac{81x^4(1+2x)}{4(1-x)^7}$, which we can then use to compute

$$H_3^1 = \frac{p(x,y)}{(x-1)^7(y-1)^7}, \tag{3.95}$$

where $p(x,y)$ is a somewhat unwieldy polynomial of bidegree 9.

Example: the king's walk revisited

Consider now once again the king's walk with the step set $\mathcal{S} = \{\uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow, \leftarrow, \nwarrow\}$, each with probability $\frac{1}{8}$. We have, as in Section 3.3.1,

$$K(x,y) = xy - \frac{1+x+y+x^2+y^2+x^2y+xy^2+x^2y^2}{8}, \tag{3.96}$$

$$H_1^1(x,y) = \frac{1}{16(x-1)^2(y-1)^2}. \tag{3.97}$$

After coefficient extraction, we find that $h_1^1(i,j) = (i+1)(j+1)$. While the biharmonic function we obtained in Section 3.3.1 was rational, it did not have a shape which made it very easy to describe its singularities, or to extract coefficients. This will once again be very different applying the decoupling method.
The king's walk has a finite group of order 4, namely

$$(x,y) \overset{\Psi}{\mapsto} \left(x, \frac{1}{x}\right) \overset{\Phi}{\mapsto} \left(\frac{1}{x}, \frac{1}{y}\right) \overset{\Psi}{\mapsto} \left(\frac{1}{x}, y\right) \overset{\Phi}{\mapsto} (x,y). \tag{3.98}$$

It turns out that in this case, we can pick 0 as a decoupling function, as the right-hand side of (3.27) vanishes. Therefore, (3.84) gives us $\hat{H}_2^1(x,y) = \frac{128y}{3(x-1)^2(y-1)^4}$, which is essentially the same result as we obtained for the simple walk in Section 3.2.2. This is not a coincidence but rather due to the fact that we have $\pi/\theta = 2$ and a finite group. This case will be discussed in more depth in Section 3.6.2. There we will also prove Thm. 3.26, from which it follows that indeed these two models have the same discrete polyharmonic functions.

3.4.2 Continuous decoupling

The idea of decoupling in the continuous setting, as suggested in [125, 36], is very much the same as in Section 3.4. The continuous version of the boundary value problem for polyharmonic functions now reads

$$\sigma_{22}\mathcal{L}_2(h_n)(c_+y) - \sigma_{22}\mathcal{L}_2(h_n)(c_-y) = \mathcal{L}(h_{n-1}(c_+y,y)) - \mathcal{L}(h_{n-1}(c_-y,y)). \tag{3.99}$$

Our goal will now be to construct a decoupling function $f_{n-1}(x)$, such that

$$f_{n-1}(c_+y) - f_{n-1}(c_-y) = \mathcal{L}(h_{n-1})(c_+y,y) - \mathcal{L}(h_{n-1})(c_-y,y). \tag{3.100}$$

One key point to note here is that all expressions appearing in (3.99) are homogeneous[2], which follows for $\mathcal{L}(h_1^k)$ by construction, and can be checked for the others by induction. In particular, this means that the right-hand side of (3.100) is homogeneous as well; and as it depends only

[2]That is, there is some $m \in \mathbb{R}$ such that they satisfy $T(\lambda x, \lambda y) = \lambda^m T(x,y)$ for all x, y; we call this m the degree of T and (written as $\deg T$).

on y it must therefore be of the form αy^m for some α, and $m = \deg \mathcal{L}(h_{n+1})$. Consequently, we choose the ansatz $f_n = \beta y^m$, with $m = \deg \mathcal{L}(h_n)$. The equation we wish to solve thus reads

$$\beta\left[(c_+ y)^m - (c_- y)^m\right] = \alpha y^m \quad \Leftrightarrow \quad \frac{\alpha}{(c_+)^m - (c_-)^m} = \beta. \tag{3.101}$$

Remembering that $c_\pm = ce^{\pm i\pi/\theta}$, we see that this is solvable in general only if m is not an integer multiple of π/θ, as then we would have $(c_+)^m - (c_-)^m = 0$. It turns out that this constraint does in fact not matter: whenever we would run into this issue, it just so happens that α is already 0, i.e. we do not need a decoupling function (see Example 3.4.2). At this stage, no direct proof of this is known, and it would be very interesting to find a way to see this directly. But one can use the convergence properties of discrete polyharmonic functions to show that the decoupling function will be 0 in all suitable cases to circumvent this problem. Since this is essential in order to continue the procedure but we will use convergence properties which will be introduced later, this will be stated here and be proven in Section 3.4.3. An illustration of this is given in Example 3.4.2.

Lemma 3.16. *In the setting of Thm. 3.17 below, if $(c_+)^m - (c_-)^m = 0$, then the right-hand side of (3.100) vanishes. In particular, we can always find a decoupling function of the form $f_n^k(x) = \alpha x^m$, with $m = \deg \mathcal{L}h_n^k(c_+x, x)$ (where $\alpha = 0$ if $(c_+)^m = (c_-)^m$).*

Utilizing the above lemma, it is now easy to prove the continuous analogue of Thm. 3.15.

Theorem 3.17. *Suppose we have a non-singular model with zero drift, small steps and such that $\pi/\theta \in \mathbb{Z}$. Let $\mathcal{L}(h_1^k)(x, y)$ be defined by (3.53). We can then define inductively*

$$\mathcal{L}(h_n^k)(x, y) = \frac{\mathcal{L}(h_{n-1})(x, y) - f_{n-1}(x) - [\mathcal{L}(h_{n-1})(c_+y, y) - f_{n-1}(c_+y)]}{\gamma(x, y)}, \tag{3.102}$$

where $f_n(x)$ is a decoupling function as in (3.100). Then, $\mathcal{L}(h_n^k)(x, y)$ is the Laplace transform of an n-harmonic function, such that $\mathcal{L}h_n^k = h_{n-1}^k$. For each n, k we can write

$$\mathcal{L}(h_n^k)(x, y) = \frac{q_{n,k}(x, y)}{x^\alpha y^\alpha}, \tag{3.103}$$

for $\alpha \in \mathbb{N}$ and $q_{n,k}(x, y)$ a homogeneous polynomial.

Remark: We will see in Thm. 3.19 that $\alpha \le k\pi/\theta + 2(n-1)$.

Proof. For $n = 1$, the statement can be checked directly. Now suppose the statement holds for n, thus we know that $\mathcal{L}(h_n^k)(x, y) = \frac{q_{n,k}(x,y)}{x^\alpha y^\alpha}$. By Lemma 3.16 (which will be proven in Section 3.4.3), we know that we can find a decoupling function, which must either be 0 or have the same degree as $\mathcal{L}(h_n^k)(c_+y, y)$, and we can therefore formally define $\mathcal{L}(h_{n+1}^k)(x, y)$ via (3.102). One can check that each summand is homogeneous of the same degree; hence so is their sum. By construction, the numerator of (3.102) is 0 for $x = c_\pm y$; it must therefore contain a factor $\gamma(x, y) = \sigma_{11}(x - c_+y)(x - c_-y)$, so the denominator cancels. The fact that $\mathcal{L}(h_{n+1}^k)(x, y)$ is the Laplace transform of a continuous polyharmonic function such that $\triangle h_{n+1}^k(s, t) = h_n^k(s, t)$ follows from checking that the functional equation (2.33) is satisfied, and noticing that we can perform an inverse transform on monomials of the form $x^u y^v$ for $u, v \in \mathbb{R}$. □

Remark: while the construction of discrete polyharmonic functions via decoupling functions is only possible if the group is finite, there are no such restrictions in the continuous setting.

Example: the scaling limit of the tandem walk revisited

For the scaling limit of the tandem walk, we have

$$\gamma(x,y)=\frac{1}{3}\left(x^2-xy+y^2\right),\quad c_{\pm}=\frac{1\pm i\sqrt{3}}{2},\quad \hat{\omega}(x)=\frac{1}{x^3}. \tag{3.104}$$

As before, we have $\mathcal{L}(h_1^1)(x,y)=\frac{3(x+y)}{x^3y^3}$, and thus (3.100) takes the form

$$f_1^1(c_+y)-f_1^1(c_-y)=\mathcal{L}(h_1^1)(c_+y,y)-\mathcal{L}(h_1^1)(c_+y,y)=\frac{3i\sqrt{3}}{y^5}. \tag{3.105}$$

By a quick computation, one obtains $f_1(x)=\frac{-3}{x^5}$ and a biharmonic function

$$\mathcal{L}(h_2^1)(x,y)=\frac{\mathcal{L}(h_1^1)(x,y)-f_1(x)-[\mathcal{L}(h_1^1)(c_+y,y)-f_1^1(c_+y)]}{\gamma(x,y)} \tag{3.106}$$

$$=\frac{9(x+y)(x^2+y^2)}{x^5y^5}. \tag{3.107}$$

Performing the inverse Laplace transform, this gives us

$$h_2^1(s,t)=-\frac{81}{8}st(s+t)(s^2+st+t^2). \tag{3.108}$$

For computing a triharmonic function, our decoupling function must now satisfy

$$f_2^1(c_+y)-f_2^1(c_-y)=\mathcal{L}(h_2^1)(c_+y,y)-\mathcal{L}(h_2^1(c_-y,y)=\frac{243i\sqrt{3}}{4y^7}, \tag{3.109}$$

which leads to $f_2^1(x)=-\frac{243}{x^7}$ and

$$\mathcal{L}(h_3^1)(x,y)=\frac{729(x+y)(x^2-xy+y^2)(x^2+xy+y^2)}{4x^7y^7}, \tag{3.110}$$

and

$$h_3^1(s,t)=-\frac{81}{320}st(s+t)(s^2+st+t^2)^2. \tag{3.111}$$

When trying to compute a decoupling function $f_3^1(x)$ as in (3.105) and (3.109), seeing that the degree of the denominator will always increase by 2, this is where one might expect things to go wrong, as $(c_+)^9=(c_-)^9$ and thus an ansatz as above might not work. However, doing the computation one finds that

$$f_3(c_+y)-f_3(c_-y)=\mathcal{L}(h_3^1)(c_+y,y)-\mathcal{L}(h_3^1)(c_-y,y)=0, \tag{3.112}$$

thus we can pick $f_3(x)=0$ and directly obtain a 4-harmonic function

$$\mathcal{L}(h_4^1)(x,y)=\frac{2187(x^3+2x^2y+2xy^2+y^3)}{4x^7y^7}, \tag{3.113}$$

which leads to

$$h_4^1(s,t)=\frac{81}{640}s^3t^3(s+t)^3. \tag{3.114}$$

The fact that the right-hand side of (3.112) turns out to be 0 is a consequence of the convergence of discrete to continuous polyharmonic and decoupling functions, and will be shown in the next section in Thm. 3.19.

3.4.3 The scaling limit

Using Lemma 3.9, the strategy to show a general convergence of the polyharmonic functions obtained by decoupling is quite simple: we use the fact that the recursive definitions (3.84) and (3.102) have the same structure, and take the limit of each term separately. All that remains to consider are the decoupling functions. However, using once again Lemma 3.9, this turns out to be rather straightforward, too.

Lemma 3.18. *Suppose we have are given discrete and continuous polyharmonic function $H(x, y)$ and $\mathcal{L}(h)(x, y)$ respectively, and a constant α such that*

$$\lim_{\mu\to 0} \mu^\alpha H\left(e^{-\mu x}, e^{-\mu y}\right) = \mathcal{L}(h)(x, y). \tag{3.115}$$

Then, if we can construct a decoupling function $F(x)$ of $xyH(x, y)$ via (3.76), the limit

$$f(x) := \lim_{\mu\to 0} \mu^\alpha F\left(e^{-\mu x}\right) \tag{3.116}$$

exists and is a decoupling function of $\mathcal{L}(h)(x, y)$.

Remark: In the context of the construction in Thm. 3.15, we know that $F_n^k(x)$ is rational with its only pole at $x = 1$. From this, we can conclude immediately that $f(x)$ will have the form $f(x) = \frac{\beta}{x^\alpha}$, where β may or may not be 0. This, as we will see, is essentially the idea of the proof of Lemma 3.16.

Proof. To see that the limit exists, we note that α must be the order of the pole at $x = y = 1$ of $H(x, y)$, and thus also the order of the pole of $xY_\pm H(x, Y_\pm)$. Noticing that, due to (3.76), $F(x)$ consists of such summands with powers of Θ applied to them, provided that $\Theta'(x, Y_\pm) \neq 0$, we know that the maximum possible order of the pole of $F(x)$ at $x = 1$ is α. The condition about the derivative, however, is guaranteed by the parametrization of the kernel curve we will use in Section 3.5.1, which tells us that we have $\Theta(x(s)) = s/q$, and therefore the derivative wrt x can never be 0. Thus, the limit exists, and the statement follows by taking the limit of (3.74). □

We can now formulate and prove the following theorem, which shows convergence between the H_n^k and the $\mathcal{L}(h_n^k)$ defined in Sections 3.4 and 3.4.2 respectively. In doing so, we will also prove Lemma 3.16. Since we will be using Thm. 3.17 to do so, which in turn utilizes the former, it is worth taking a moment to make sure that in each induction step in the proof of Thm. 3.19 for some fixed $n + 1$, we use the statement of Thm. 3.17 for n, and then proceed to prove Lemma 3.16 for $n + 1$. We therefore do not enter any circular reasoning.

Theorem 3.19. *Let $\pi/\theta \in \mathbb{Z}$ and H_n^k, $\mathcal{L}(h_n^k)$ be defined by (3.84), (3.102) respectively. Then*

$$\lim_{\mu\to 0} \mu^{k\pi/\theta+2n} H_n^k\left(e^{-\mu x}, e^{-\mu y}\right) = \alpha_{n,k}\mathcal{L}\left(h_n^k\right)(x, y) \tag{3.117}$$

for some constants $\alpha_{n,k} \neq 0$.
Furthermore, we can write

$$H_n^k(x, y) = \frac{p_n^k(x, y)}{(1 - x)^u(1 - y)^v}, \quad \mathcal{L}(h_n^k)(x, y) = \frac{q_n^k(x, y)}{x^\alpha y^\alpha}, \tag{3.118}$$

where $u, v, \alpha \in \mathbb{Z}$ with $u, v \leq \alpha = k\pi/\theta + 2(n - 1)$, $p_n^k(x, y)$ a polynomial and $q_n^k(x, y)$ a homogeneous polynomial of degree $\pi/\theta + 2(n - 1)$.

Proof of Thm. 3.19 and Lemma 3.16. We prove the theorem and the lemma simultaneously by induction. For $n = 1$ everything can be checked by a direct computation. Now suppose everything is shown up to some n. By Lemma 3.18, we know that we can define a continuous decoupling function $f_n^k(x)$ of $\mathcal{L}(h_n^k)(x, y)$ via a the scaling limit $f_n^k(x) := \lim_{\mu\to 0} \mu^{k\pi/\theta} F_n^k(e^{-\mu x})$, and we also know that it is of the form $f(x) = \beta x^{-k\pi/\theta-2n}$, so in particular Lemma 3.16 holds for $n + 1$ as well. Having now completely proven Thm. 3.17 for $n + 1$ (where we utilized Lemma 3.16), we can take the piecewise limit of (3.84). Using in particular Lemma 3.9, and the definition (3.102) of $\mathcal{L}(h_{n+1}^k)(x, y)$, we have for $\alpha = k\pi/\theta+2n$ and (to save space) $e_x = e^{-\mu x}$, $e_y := e^{-\mu y}$,

$$\begin{aligned}
&\lim_{\mu\to 0} \mu^{\alpha+2} H_n^k\left(e_x, e_y\right) \\
&= \lim_{\mu\to 0} \frac{\mu^\alpha\left(e_x e_y K(e_x, e_y) - F_n^k(e_x) - \left[X_+(e_y)e_y H_n^k\left(X_+(e_y), e_y\right) - F_n^k\left(X_+(e_y)\right)\right]\right)}{\mu^{-2}K(e^{-\mu x}, e^{-\mu y})} \\
&= \frac{\mathcal{L}(h_n^k)(x, y) - f_n^k(x) - \left[\mathcal{L}(h_n^k)(c_+y, y) - f_n^k(c_+y)\right]}{\gamma(x, y)} \\
&= \mathcal{L}(h_{n+1}^k)(x, y).
\end{aligned}$$

The degree of $q_n^k(x, y)$ and the value of α in (3.118) can be checked by a direct computation (note that it is allowed that $q_n^k(x, y)$ be divisible by some power of x or y). From there it follows immediately that $\alpha = k\pi/\theta + 2n$ is an upper bound of u, v using (3.117). □

3.5 A guessing approach

In [36], the authors used an entirely different approach to find biharmonic functions, which may not be as easy to generalize as the method above, but is in many ways a more elementary and intuitive approach. As we will see, their guessing method using an ansatz can be shown to be effective for computation of biharmonic functions whenever decoupling is possible. Unlike the constructive approach above, this ansatz allows to rule out the existence of decoupling functions of a sufficiently nice shape. The main goal of this section is to detail this approach, which was only roughly outlined in [36], and show in Prop. 3.21 that if a sufficiently nice decoupling function exists, it can always be found using this ansatz.

3.5.1 A parametrization of the kernel curve

If we substitute $x \mapsto X_\pm(y)$ into (2.26), then we obtain (see [36, (30)])

$$K(X_+, 0)H_2(X_+, 0) - K(X_-, 0)H_2(X_-, 0) = y\left[X_+H_1(X_+, y) - X_-H_1(X_-, y)\right]. \tag{3.119}$$

Our goal is to rewrite the right-hand side of (3.119) for $H_1(x, y) = \frac{P(\omega(x))-P(\omega(X_+))}{K(x,y)}$. Seeing as we substitute $x \mapsto X_\pm$, where the denominator is 0, we need to utilize L'Hôpital's rule, which gives us (note that $K(x, y) = \tilde{a}(y)\left(x - X_+(y)\right)\left(x - X_-(y)\right)$

$$X_+yH_1(X_+, y) = \frac{yX_+(y)\omega'(X_+)}{\tilde{a}(y)\left(X_+ - X_-\right)}P'(\omega(X_+)), \tag{3.120}$$

$$X_-yH_1(X_-, y) = \frac{yX_-\omega'(X_-)}{\tilde{a}(y)\left(X_- - X_+\right)}P'(\omega(X_-)). \tag{3.121}$$

Noting that $P'(\omega(X_+)) = P'(\omega(X_-))$ due to the invariance property of ω, in order to find a decoupling function $F(x)$ it would be enough to find an F such that

$$F(X_+) - F(X_-) = \frac{yX_+(y)\omega'(X_+)}{\tilde{a}(y)\left(X_+ - X_-\right)} - \frac{yX_-\omega'(X_-)}{\tilde{a}(y)\left(X_- - X_+\right)}. \tag{3.122}$$

To do so, we utilize the fact that the kernel curve $\mathcal{C} := \{(x,y) \in \bar{\mathbb{C}}^2 : K(x,y) = 0\}$ as defined in Section 2.1.2 permits a parametrization of the form

$$x(s) = \frac{(s-s_1)(s-1/s_1)}{(s-s_0)(s-1/s_0)}, \tag{3.123}$$

$$y(s) = \frac{(\rho s-s_3)(\rho s-1/s_3)}{\rho s-s_2)(\rho s-1/s_2)}, \tag{3.124}$$

where $\rho = e^{-i\theta}$,

$$s_0 = \frac{2-(x_1+x_4)-2\sqrt{(1-x_1)(1-x_4)}}{x_4-x_1}, \tag{3.125}$$

$$s_1 = \frac{x_1+x_4-2x_1x_4-2\sqrt{x_1x_4(1-x_1)(1-x_4)}}{x_4-x_1}, \tag{3.126}$$

with similar definitions for s_2 and s_3 using y_1, y_4 instead of x_1, x_4, where the x_i and y_i are defined by the zeros of the discriminant of the kernel as in Section 2.1.2 (see [70, 31] for details). Using this parametrization, we have [67, 2.3]

$$x\left(\frac{1}{s}\right) = x(s), \quad y\left(\frac{q}{s}\right) = y(s), \tag{3.127}$$

with $q := e^{2i\theta} = 1/\rho^2$. One can deduce that the mappings $s \mapsto \frac{1}{s}, s \mapsto \frac{q}{s}$ correspond to the restriction of the group to $\mathcal{C}$, and due to the invariance properties of $x(s), y(s)$ and $\omega(x)$, we see that $\omega(x(s)) = s^{\pi/\theta} + s^{-\pi/\theta} + c$, for c some some constant. In the following, seeing as with $\omega(x)$, we know that $\omega(x) - c$ will also be a suitable conformal mapping for our purposes, we will assume that $\omega(x(s)) = s^{\pi/\theta} - s^{-\pi/\theta}$.
Using these parametrizations, one eventually finds that the right-hand side of (3.119) written in terms of s takes the form

$$c\left(s^{\pi/\theta} - s^{-\pi/\theta}\right) \frac{Q(s)}{s^2(s-1)(s+1)(s-q)(s+q)} =: B(s), \tag{3.128}$$

where $c \in \mathbb{C}$ is a constant and $Q(s)$ is a polynomial of degree 8. Noticing that the mapping $s \mapsto \frac{s}{q}$ maps X_+ to X_-, we want to find a function F such that

$$f(s) - f(s/q) = B(s). \tag{3.129}$$

Then, one would only need to find a way to write $f(s)$ – which must inherit the invariance property $f(s) = f(1/s)$ from $x(s)$ – to a function of the form $F(x(s))$, and we would have our decoupling function.

Rewriting the boundary value problem

The following computations were originally done in [122]. Writing
$H(x,y) = \frac{P(\omega(x)) - P(\omega(X_+))}{K(x,y)}$ we have, using L'Hôpital's rule,

$$\begin{aligned} &X_+ y H(X_+, y) - X_- y H(X_-, y) \\ &\qquad = \frac{yX_+(y)\omega'(y)}{\tilde{a}(y)\left[X_+(y) - X_-(y)\right]} P'(\omega(X_+(y))) - \frac{yX_-(y)\omega'(y)}{\tilde{a}(y)\left[X_-(y) - X_+(y)\right]} P'(\omega_-(y))), \end{aligned} \tag{3.130}$$

where $\tilde{a}(y)$ is defined as in Section 2.1.2. As $\omega(X_+(y)) = \omega(X_-(y))$, it therefore suffices to find a decoupling function $F(x)$ such that

$$F(X_+) - F(X_-) = \frac{yX_+(y)\omega'(X_+(y))}{\tilde{a}(y)\left[X_+(y) - X_-(y)\right]} - \frac{yX_-(y)\omega'(X_-(y))}{\tilde{a}(y)\left[X_-(y) - X_+(y)\right]}. \tag{3.131}$$

Using the parametrization, we have $X_+(y) = x(s)$, $X_-(y) = x(q/s)$. (3.131) thus becomes

$$f(x(s)) - f(x(q/s)) = \frac{y(s)x(s)\omega'(x(s))}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} - \frac{y(s)x(s)\omega'(x(q/s))}{\tilde{a}(y(s))\left[x(q/s) - x(s)\right]}. \tag{3.132}$$

In order to simplify the right-hand side of (3.132), the main idea is to utilize the fact that $\omega(x(s)) = s^{\pi/\theta} + s^{-\pi/\theta}$, thus

$$\omega'(x(s))x'(s) = \frac{\pi}{\theta}\frac{1}{s}\left(s^{\pi/\theta} - s^{-\pi/\theta}\right). \tag{3.133}$$

We therefore rewrite

$$\frac{y(s)x(s)\omega'(x(s))}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} = \underbrace{\frac{y(s)}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]}}_{:=T_1(s)} \underbrace{\frac{x(s)}{x'(s)}}_{:=T_2(s)} \underbrace{\omega'(x(s))x'(s)}_{:=T_3(s)}. \tag{3.134}$$

By utilizing the fact that

$$T_1(s) = \frac{y(s)}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} = \frac{y(s)}{\tilde{D}(y(s))}, \tag{3.135}$$

with $\tilde{D}$ the determinant from Section 2.1.2, after some computations one obtains that, for some constant c_1,

$$T_1(s) = c_1 \frac{(\rho s_3 s - 1)(\rho s - s_3)(\rho s_2 s - 1)(\rho s - s_2)}{s(\rho s - 1)(\rho s + 1)}. \tag{3.136}$$

Similarly, we have

$$T_2(s) = c_2 \frac{(s_0 s - 1)(s - s_0)(s_1 s - 1)(s - s_1)}{(s + 1)(s - 1)} \tag{3.137}$$

for some constant c_3[3]. We can therefore, after again some short computations, rewrite

$$\begin{aligned} &\frac{y(s)x(s)\omega'(x(s))}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} - \frac{y(s)x(s)\omega'(x(q/s))}{\tilde{a}(y(s))\left[x(q/s) - x(s)\right]} \\ &\qquad = \frac{y(s)}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} \frac{\pi}{\theta}\frac{1}{s}\left(s^{\pi/\theta} - s^{-\pi/\theta}\right)\left(\frac{x(s)}{x'(s)} - \frac{s^2}{q}\frac{x(q/s)}{x'(q/s)}\right). \end{aligned} \tag{3.138}$$

After simplifying the last factor, we end up with

$$\begin{aligned} &\frac{y(s)x(s)\omega'(x(s))}{\tilde{a}(y(s))\left[x(s) - x(q/s)\right]} - \frac{y(s)x(s)\omega'(x(q/s))}{\tilde{a}(y(s))\left[x(q/s) - x(s)\right]} \\ &\quad = c\left(s^{\pi/\theta} - s^{-\pi/\theta}\right) \frac{(\rho s_3 s - 1)(\rho s - s_3)(\rho s_2 s - 1)(\rho s - s_2)(s - s_4)(s - q/s_4)(s - s_5)(s - q/s_5)}{s^2(s + 1)(s - 1)(s - q)(s + q)}, \end{aligned} \tag{3.139}$$

where c, s_4, s_5 are constants.

[3]This is the only part where our assumption $y_4 \neq \infty$ comes into play; if $y_4 = \infty$ we find that the denominator of $T_1(s)$ stays the same while the numerator is quadratic.

3.5.2 The ansatz

The guessing method used by the authors of [36] is to search for a $f(s)$ of the form

$$f(s) = c\left(s^{\pi/\theta} - s^{-\pi/\theta}\right)\frac{s^p R(s)}{s - s^{-1}}, \tag{3.140}$$

where p is some constant and $R(s)$ rational. Utilizing (3.139) and the fact that (due to the invariance property of $x(s)$) we must have $f(s) = f(1/s)$, it turns out that this already implies $p = -3$, and that $R(s)$ must be a reciprocal polynomial of degree 6[4]. From here, one can simply write $R(s) = (s - z_1)(s - \frac{1}{z_1}) \dots (s - \frac{1}{z_3})$, and check to see if it is possible to find z_1, z_2, z_3 such that (3.129) holds. After some more calculations one finds that we want to solve

$$\begin{aligned}(s-q)(s+q)R(s) - s^6 q^{-2}(s-1)(s+1)R\left(q/s\right) = \\ c(\rho s_3 s - 1)(\rho s - s_3)(\rho s_2 s - 1)(\rho s - s_2)(s - s_4)(s - q/s_4)(s - s_5)(s - q/s_5)\end{aligned} \tag{3.141}$$

for some constant c. In (3.128), everything except for $s^{\pm\pi/\theta}$ is rational in s, and $s^{\pi/\theta}$ is invariant under $s \mapsto s/q$ (by definition, we have $q^{\pi/\theta} = e^{2i\theta\cdot\pi/\theta} = 1$). Therefore it is not very surprising that we can, in a sense, leave the invariant factor $\left(s^{\pi/\theta} - s^{-\pi/\theta}\right)$ alone and find a decoupling function for the remaining part only. This is formalized by the following lemma.

Lemma 3.20. *Let $t := s^{\pi/\theta}$, and let $f_1 \in \mathbb{C}(s,t)$ such that $f_1 \in \mathbb{C}(s,t)$ is a decoupling function of $b(s) = R(s)h(t)$. Then we can find $f \in \mathbb{C}(s)$ such that f is a decoupling function of $R(s)$.*

Proof. Let τ be the automorphism $s \mapsto s/q$, which by definition of q leaves t invariant. We have two cases here, depending on whether or not π/θ is rational or not, i.e. if there is some algebraic relation between s and t.

1. $\underline{\pi/\theta \notin \mathbb{Q}}$:
   In this case, as we are always working with rational functions, we can utilize the fact that s and t are algebraically independent. By invariance of t under τ, we can thus write

   $$\frac{f_1(\tau s, t)}{h(t)} - \frac{f_1(s,t)}{h(t)} = R(s). \tag{3.142}$$

   Due to the independence of s and t we can treat the right-hand side, viewed as a rational function coefficients in $\mathbb{C}(s)$, like a constant function in t.
   Now let $f_2(s,t) := \frac{f_1(s,t)}{h(t)} = \frac{u(s,t)}{v(s,t)}$, with $u, v \in \mathbb{C}(s)[t]$. We factor u, v into their irreducible components $(u_i), (v_i)$ over $\mathbb{C}(s)$. Any u_i, v_i which lies in $\mathbb{C}[t]$ must cancel, because else it would be a factor of the entire left-hand side and thus of $R(s)$, a contradiction. Thus we can assume that all u_i, v_i lie in $\mathbb{C}(s)[t] \setminus \mathbb{C}[t]$.
   Suppose there is an i, which we can assume to be 1, such that $\deg_t(v_i) > 0$. Then, as the resulting pole must cancel, we know there is a j, let us say $j = 2$, such that $v_2(s,t) = v_1(\tau(s), t)$. Proceeding inductively, we can construct a sequence (v_i) such that $v_1(\tau^{n-1}(s), t) = v_n(s,t)$. But as $\tau^n \neq \mathrm{id}$ for all $n \in \mathbb{N}$ since $q \notin \mathbb{Q}$, this procedure will never stop, i.e. we would need to have an infinite number of factors v_i, which is impossible. Therefore, we know that $v(s,t) = v(s) \in \mathbb{C}(s)$; and by looking at the degree of the left-hand side of (3.142), therefore $u(s,t) = u(s) \in \mathbb{C}(s)$ as well. Hence, $f_2 = \frac{f_1}{h} \in \mathbb{C}(s)$, and therefore it is a valid decoupling function of $R(s)$.

2. $\underline{\pi/\theta \in \mathbb{Q}}$:
   Let $\pi/\theta = \frac{m}{n}$, with $m, n \in \mathbb{N}$, $(m,n) = 1$. While before we could simply consider the

[4] In the case $y_4 = \infty$, one finds $p = -1$ and $R(s)$ of degree 2.

irreducible factors of numerator and denominator of f_2, this is not now so simple anymore, as the algebraic structure of $\mathbb{F} := \text{Quot}\left(\mathbb{C}[s,t] \Big/ \langle s^m - t^n\rangle\right)$ is not as obvious. It would, for instance, not immediately make sense to talk about the degree of an expression. Therefore, we need to work around this issue. We know that $\gcd(m,n) = 1$, and thus there are a, b such that $am + bn = 1$. Now consider the mapping[5]

$$\phi: \quad \mathbb{C}(T) \to \mathbb{F}: \quad T \mapsto s^b t^a. \tag{3.143}$$

One can check that an inverse map ϕ^{-1} is given by $(s,t) \mapsto (T^n, T^m)$, and that ϕ is in fact an isomorphism. Our automorphism τ can now be carried over to $\mathbb{C}(T)$ thus we obtain an automorphism $\sigma := \phi^{-1} \circ \tau$. As τ and ϕ both fix $\mathbb{C}$, so does σ. Therefore, we know that σ is of the form

$$\sigma: \quad T \mapsto \frac{c_1 T + c_2}{d_2 T + d_2} \tag{3.144}$$

for some c_1, c_2, d_1, d_2. From this, one obtains that $\sigma(T) = \zeta T$, where $\zeta^n = q, \zeta^m = 1$, which means $\zeta = \exp\left(2i\pi\frac{1}{m}\right)$.
Translating the decoupling property (3.142) to T, we now have, with $f_3 = \phi^{-1} \circ f_2$

$$f_3(\sigma T) - f_3(T) = R_1(T). \tag{3.145}$$

By Hilbert's Thm. 90 [70, 4.6.5] such a f_3 can exist if and only if $\operatorname{Tr}_\sigma R_1 = 0$, that is, if

$$\sum_{k=1}^{m} R_1(\zeta^k T) = 0. \tag{3.146}$$

But using that $R_1(T) := \phi^{-1}(R(s)) = R(T^n)$ and that $\zeta^n = q$, we have

$$\operatorname{Tr}_\sigma R_1 = \sum_{k=1}^{m} R\left(\zeta^{kn} T^n\right) = \sum_{k=1}^{m} R\left(\phi^{-1}(q^k s)\right) = \phi^{-1}\left[\sum_{k=1}^{m} R(q^k s)\right] = \phi^{-1}\left(\operatorname{Tr}_\tau R\right). \tag{3.147}$$

As ϕ leaves $\mathbb{C}$ fixed, we therefore know that $\operatorname{Tr}_\sigma R_1 = 0 \Leftrightarrow \operatorname{Tr}_\tau R = 0$, and thus a decoupling of $R_1(T)$ is possible if and only if a decoupling of $R(s)$ is possible.

□

Proposition 3.21. *If the function $f(x(s))$ as in* (3.132) *has a decoupling function in* $\mathbb{C}\left(s, s^{\pi/\theta}\right)$, *then we can find it using an ansatz as in* (3.140).

Proof. Using the computations from the previous section, in particular (3.133), we see that we can write $b(s) := f(x(s))$ in the form $R(s)h(t)$, with $t := s^{\pi/\theta}$ as in Lemma 3.20. Assuming that $b(s)$ has a decoupling function in $\mathbb{C}\left(s, s^{\pi/\theta}\right)$, it follows by the same lemma that we can then find a decoupling function of $R(s)$ in $\mathbb{C}(s)$. But finding such a decoupling function is, after our previous computations, equivalent to solving (3.141), i.e. to the ansatz working out. □

Remarks:

- The above allows to show for concrete models with infinite group that a decoupling function in $\mathbb{C}(x, y, \omega(x), \omega(y))$ cannot exist. However, it is not obvious how to show that an infinite group implies that there is no decoupling function. In particular, it is not at all clear how the group (as defined in Section 3.4, i.e. the birational transformations) being finite can be grasped in terms of the parametrization, where all that is left is the restriction of the group to the curve $\mathcal{C}$. Section 3.6 will contain some examples where the restriction of the group, but not the group itself is finite.

[5] This is merely a formalisation of the basic idea of adjoining a simple element T such that $T^n = s, T^m = t$, which serves the role of an n-th root of s.

- While it would be natural to assume that any decoupling function would be in $\mathbb{C}\left(s, s^{\pi/\theta}\right)$, this is not always the case. We will see an example for this in Example 3.6.4, where the ansatz will not work but we will construct a (non-algebraic) decoupling function. This might also be tied to the appearance of logarithms in the asymptotics of models with infinite group, as in Chapter 5 later. In the setting of Section 3.4, however, one checks immediately by (3.76) that if ever the group is finite, then the resulting decoupling functions will be rational in s and $s^{\pi/\theta}$.

- While this kind of ansatz seems useful to either compute decoupling functions in concrete examples or show that they don't exist, it may be difficult to use it to obtain a general criterion for models which allow for such a function: though this is not formally proven, it appears that such a function exists precisely for the models with finite group (in particular, the existence is shown in Thm. 3.15). When utilizing the parametrization, however, one loses information about the group: one knows only if the group is finite *on the kernel curve*, which is not equivalent to it being finite in general and not sufficient for a decoupling function to exist. For an example of this, see Sec. 3.6.

Example: a decoupling function for the tandem walk using the ansatz

Recall that the tandem walk was defined by the step set $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$ with weight $\frac{1}{3}$ each. We had $K(x,y) = xy - \frac{1}{3}(x^2 + y + xy^2)$, from which we obtain $x_1 = 0, x_4 = 4, y_0 = \frac{1}{4}, y_4 = \infty$, and thus $s_0 = -\frac{1}{2}(1 + i\sqrt{3}), s_1 = 1, s_2 = -1, s_3 = \frac{1}{2}(1 - i\sqrt{3})$. Furthermore, we have $\pi/\theta = 3$, and thus $q := e^{2i\theta} = -\frac{1}{2}(1 - i\sqrt{3})$ as well as $\rho := e^{-i\theta} = \frac{1}{2}(1 - \sqrt{3}) = -q$. This leads to

$$x(s) = \frac{(s-1)^2}{1+s+s^2}, \quad y(s) = \frac{-\frac{1}{2}(1-i\sqrt{3})s^2 - \frac{1}{2}(1+i\sqrt{3})s + 1}{\left(s + \frac{1}{2}(1+i\sqrt{3})\right)^2}. \tag{3.148}$$

After some computations, we see that the right-hand side of (3.141) takes the form

$$1/6i(s-1)^3(2\sqrt{3} + (\sqrt{3}+3i)s)(1+s+s^2)^2. \tag{3.149}$$

Letting $R(s) = (1-z_1)(1-z_2)(1-z_3)(1-1/z_1)(1-1/z_2)(1-1/z_3)$ and solving (3.141) for z_1, z_2, z_3 yields multiple solutions, for instance $c = -\frac{1}{3}, z_1 = 1, z_2 = i, z_3 = -\frac{1}{2}(1 + i\sqrt{3})$. This particular one leads to

$$\begin{aligned} f(s) &:= c\left(s^3 - s^{-3}\right)\frac{R(s)}{s^4 - s^2} \\ &= \frac{1 - s + 2s^2 - 3s^3 + 3s^4 - 4s^5 + 3s^6 - 3s^7 + 2s^8 - s^9 + s^{10}}{3s^5}. \end{aligned} \tag{3.150}$$

To transform this back into a function of x, we utilize the fact that

$$\frac{1}{1-x(s)} = \frac{1+s+s^2}{3s}. \tag{3.151}$$

Making an ansatz of $f(s) = a_0 + a_1\frac{1}{1-x(s)} + \cdots + a_5\frac{1}{(1-x(s))^5}$ gives $a_0 = a_1 = 0, a_2 = -18, a_3 = 99, a_4 = -162, a_5 = 81$. Putting this together, we finally obtain the decoupling function

$$f(x) = \frac{9x(2+5x+2x^2)}{(1-x)^5}. \tag{3.152}$$

Note that this decoupling function is different from the one computed in Example 3.4.1, where we obtained $F(x) = -\frac{81x^3}{4(1-x)^5}$, as already mentioned in Example 3.4.1.

Example: trying to decouple a model with infinite group

Consider the model with the step set

$$p_{1,0} = p_{0,1} = 0, \tag{3.153}$$
$$p_{1,1} = 1/4, \tag{3.154}$$
$$p_{1,-1} = p_{0,-1} = p_{-1,0} = p_{-1,1} = 1/6, \tag{3.155}$$
$$p_{-1,-1} = 1/12. \tag{3.156}$$

This model has an infinite group, as can be seen by Thm. 3.24. We have $K(x, y) = xy - \frac{1}{12}(1 + 2x + 2y + 2x^2 + 2y^2 + 3x^2y^2)$, and can compute $x_{1,4} = y_{1,4} = \frac{1}{6}(\pm 5\sqrt{3} - 9)$. As $\pi/\theta = 2$, we have $q = -1, \rho = i$. After some calculations, one finds that the right-hand side of (3.141) takes, up to a multiplicative constant, the form

$$s(\sqrt{2} - \sqrt{3} + is)(\sqrt{3} + is)(\sqrt{3} + 3is)(1 + i(\sqrt{2} - \sqrt{3})s)(-1 + s^2). \tag{3.157}$$

We can check that solving (3.141) does not give any solutions for z_1, z_2, z_3, thus we cannot find a decoupling function using the ansatz. In particular, in light of Lemma 3.20 this implies that there is no rational (in s) decoupling function. We will, however, see how one can construct a non-algebraic decoupling function for this model using a contour integral in Example 3.6.4.

3.6 The special case $\pi/\theta = 2$

In this section we will consider the special case where $\theta = \pi/2$ (θ is the angle defined in Section 2.1.2, and can be computed by (2.11)). This is done for two different reasons: firstly, we will see that in this setting we gain a number of nice properties; in particular an explicit criterion for the group to be finite (Thm. 3.24 in Section 3.6.1), in which case we the polyharmonic functions can be explicitly computed (Thm. 3.26 in Section 3.6.2). On the other hand, this setting allows us to explicitly compute decoupling functions in the case of an infinite group with comparably little effort, which are – albeit not rational – still guaranteed to exist by general theory about complex boundary value problems, see for example [75, §4],[70, 5.]. This will be done starting from Section 3.6.3. In particular, we will see that the resulting functions are not even algebraic anymore (though still D-finite).
The case $\pi/\theta = 2$ includes a number of standard models, such as the simple walk, the king's walk or the diagonal walk. It is characterized by the property

$$p_{1,1} + p_{-1,-1} = p_{1,-1} + p_{-1,1}, \tag{3.158}$$

that is, the sums of the weights of the two diagonals are the same. This is a direct consequence of (2.11). Also note that that the function $f(i, j) = (i + 1)(j + 1)$ satisfies $\triangle f(i, j) = 0$ in this case, which corresponds to the fact that, up to a multiplicative constant, $H_1^1(x, y) = \frac{1}{(1-x)^2(1-y)^2}$. This can be checked directly using the definition of the discrete Laplacian (2.2).

3.6.1 A criterion for the group to be finite

Deciding if the group of a given model is finite is in general not an easy problem, as can be seen for example by the very computationally heavy approach in [93], or by the approach in [31] where it was done for unweighted small-step models using a combination of eigenvalue properties and valuations. While for the standard models with $\pi/\theta = 2$ like the simple walk, the diagonal walk or the king's walk the group is finite, this is not always the case[6]. Take for example the model with probabilities

$$p_{1,0} = p_{0,1} = 0, \tag{3.159}$$
$$p_{1,1} = 1/4, \tag{3.160}$$
$$p_{1,-1} = p_{0,-1} = p_{0,-1} = p_{-1,1} = 1/6, \tag{3.161}$$
$$p_{-1,-1} = 1/12. \tag{3.162}$$

One can check immediately that while we have $\pi/\theta = 2$ and the restriction of the group has order 4, the group itself is infinite.
In the case of $\pi/\theta = 2$ we will show in this section that there is a very intuitive way to classify the behaviour of the group of a given model: it is finite of order 4 if the model has either a North-South or an East-West symmetry, else it is infinite. To do so, we will first show that the group is of order four precisely if one of these symmetries holds; and then afterwards show that the group being finite leads directly back to this case. We will start with a technical lemma in order to shorten later computations.

Lemma 3.22. *Suppose we have a non-degenerate model with small steps, zero drift and $\pi/\theta = 2$. If there is an $i \in \{-1, 0, 1\}$ such that $p_{1,i} = p_{-1,i}$, then we have $p_{1,j} = p_{-1,j}$ for $j \in \{-1, 0, 1\}$. Similarly, if there is an $i \in \{-1, 0, 1\}$ such that $p_{i,1} = p_{i,-1}$, then we have $p_{j,1} = p_{j,-1}$ for $j \in \{-1, 0, 1\}$.*

Proof. By a direct computation, using (3.158). □

This lemma tells us that we have a North-South symmetry in only one of the three possible ways, then our step set as a whole already has a North-South symmetry. We can now utilize this in the following

Lemma 3.23. *Suppose we have a non-degenerate model with small steps, zero drift and $\pi/\theta = 2$. Then, the group is finite of order 4 precisely if the model has either a North-South, or an East-West symmetry.*

Proof. We know from [123, Prop. 4] that the order of the group is four if and only if the determinant

$$\left| \begin{pmatrix} p_{-1,1} & p_{0,1} & p_{1,1} \\ p_{-1,0} & -\frac{1}{t} & p_{1,0} \\ p_{-1,-1} & p_{0,-1} & p_{1,-1} \end{pmatrix} \right| = 0. \tag{3.163}$$

This determinant can be explicitly computed to be

$$-p_{1,0}p_{-1,1}p_{0,-1} + p_{1,0}p_{0,1}p_{-1,-1} + p_{1,1}p_{0,-1}p_{-1,0} - p_{0,1}p_{1,-1}p_{-1,0} - \frac{p_{-1,1}p_{1,-1} + p_{1,1}p_{-1,-1}}{t}.$$

Utilizing the fact that our walk has drift 0, small steps, and (3.158), this can be simplified to

$$\frac{t-1}{t}\left[(p_{-1,1} - p_{-1,-1})(p_{-1,-1} - p_{1,-1})\right]. \tag{3.164}$$

By Lemma 3.22, the last expression is 0 precisely if our model has either a North-South, or an East-West symmetry. □

[6] The fact that $\pi/\theta = 2 \in \mathbb{Z}$ guarantees that the *restriction of the group on $\mathcal{C}$* is finite, see e.g. [67], but not that it is finite on all of $\mathbb{C}^2$.

Theorem 3.24. *Suppose we have a non-degenerate model with small steps, zero drift and $\pi/\theta = 2$. Then the group is finite of order 4 if the model has a North-South or an East-West symmetry, and it is infinite otherwise.*

Proof. Utilizing Lemma 3.23, all that remains to show is that in our setting any group that is finite must be of order 4. To see this, define $\tilde{\delta}, \tilde{\varepsilon}, \delta, \varepsilon \in \bar{\mathbb{R}}$ such that

$$\frac{c(x)}{a(x)} \overset{x\to\infty}{\to} \delta, \qquad \frac{c(x)}{a(x)} \overset{x\to 0}{\to} \varepsilon, \tag{3.165}$$

$$\frac{\tilde{c}(y)}{\tilde{a}(y)} \overset{y\to\infty}{\to} \tilde{\delta}, \qquad \frac{\tilde{c}(y)}{\tilde{a}(y)} \overset{y\to 0}{\to} \tilde{\varepsilon}, \tag{3.166}$$

where $a(x), c(x), \tilde{a}(y), \tilde{c}(y)$ are defined as in Section 2.1.2. Notice that if $\delta = \varepsilon = 0$, then this would imply $p_{1,1} = p_{-1,1} = 0$, in which case by Lemmas 3.22 and 3.23 we already know the group to be finite of order 4. In the same fashion, one sees that if $\delta = \varepsilon = \infty$, $\tilde{\delta} = \tilde{\varepsilon} \in \{0, \infty\}$ then we have a finite group of order 4. In all other cases, we find that, for sufficiently large values of (x, y), the group behaves like

$$(x, y) \mapsto (\delta\bar{x}, y) \mapsto (\delta\bar{x}, \tilde{\varepsilon}\bar{y}) \mapsto \left(\frac{\varepsilon}{\delta}x, \tilde{\varepsilon}\bar{y}\right) \mapsto \left(\frac{\varepsilon}{\delta}x, \frac{\tilde{\delta}}{\tilde{\varepsilon}}, y\right) \mapsto \dots \tag{3.167}$$

Consequently, for the group to be finite, both $\frac{\varepsilon}{\delta}, \frac{\tilde{\delta}}{\tilde{\varepsilon}}$ must be roots of unity. As they are non-negative reals, they must therefore be 1. This condition can then be checked to simplify to $p_{1,1}p_{-1,-1} = p_{1,-1}p_{-1,1}$.
We write

$$\begin{aligned} 0 = p_{1,1}p_{-1,-1} - p_{1,-1}p_{-1,1} &= p_{1,1}p_{-1,-1} - p_{1,-1}(p_{1,1} + p_{-1,-1} - p_{1,-1}) && (3.168)\\ &= (p_{1,1} - p_{1,-1})(p_{-1,-1} - p_{1,-1}), && (3.169) \end{aligned}$$

where in (3.168) we made use of (3.158). One can check by a short computation that the first condition implies a North-South and the second one an East-West symmetry; thus by Lemma 3.23 we already know that our group is of order 4. □

3.6.2 The finite group case

We use the same parametrization of the curve $\mathcal{C} := \{(x, y) \in \overline{\mathbb{C}}^2 : K(x, y) = 0\}$ as in Section 3.5.1,

$$x(s) = \frac{(s - s_1)(s - 1/s_1)}{(s - s_0)(s - 1/s_0)}, \tag{3.170}$$

$$y(s) = \frac{(\rho s - s_3)(\rho s - 1/s_3)}{(\rho s - s_2)(\rho s - 1/s_2)}, \tag{3.171}$$

with s_0, s_1, s_2, s_3 given by (3.125)–(3.126). Remember the invariance properties $x(s) = x\left(1/s\right)$ and $y(s) = (q/s)$ with $q := e^{2i\theta} = 1/\rho^2$, and that the mappings $s \mapsto \frac{1}{s}, s \mapsto \frac{q}{s}$ correspond to the restriction of the group to $\mathcal{C}$, and that we have $\omega(x(s)) = s^{\pi/\theta} + s^{-\pi/\theta}$.
In the case $\pi/\theta = 2$, the above immediately simplifies to $\rho = -i, q = -1$. By Thm. 3.24, we also know that in this case we have a North-South, or an East-West symmetry. This gives us some particularly nice properties of these models, and allows us to compute polyharmonic functions without the use of decoupling functions.

Lemma 3.25. *Suppose we have a non-degenerate model with small steps, zero drift, $\pi/\theta = 2$ and finite group. If we have an East-West symmetry, then there is a constant c such that we can write*

$$\omega(x) = c\frac{x}{(1-x)^2}, \tag{3.172}$$

and the contour $\mathcal{G}$ given by $X_\pm([y_1, y_4])$ is the unit circle.
In case of a North-South symmetry, a corresponding statement holds true for y and the corresponding conformal mapping $\hat{\omega}$ instead.

Proof. (3.172) follows from the parametrization, computing

$$\frac{x(s)}{(1-x(s))^2} = \frac{a_0 s^4 + a_1 s^3 + a_2 s^2 + a_1 s + a_0}{s^2}. \tag{3.173}$$

It turns out that we have $a_1 = 0$ if $s_0 s_1 = 0$, which we can check to be true if and only if we have a North-South symmetry. In this case we can then see that the right-hand side of (3.173) is $a_0(s^2 + s^{-2}) + a_2$. As we already know that, up to an additive constant, $\omega(x(s)) = s^2 + s^{-2}$, we thus have $\omega(x) = \frac{1}{a_0}\frac{x}{(1-x)^2} - a_2$, and seeing as we pick $\omega(x)$ such that $\omega(0) = 0$, (3.172) follows.
For the statement about the contour, [70, Thm. 5.3.3] tells us that it is a circle if $\pi/\theta = 2$ (in their notation, $r = 0$). However, considering that $\omega(x)$ must take the same values on the upper and lower half of the contour $\mathcal{G}$ by the invariance property $\omega(X_+(y)) = \omega(X_-(y))$ and given (3.172), it follows that this circle must be the unit circle. $\square$

Theorem 3.26. *For any non-degenerate model with small steps, zero drift, $\pi/\theta = 2$, finite group, and East-West symmetry, an explicit basis of polyharmonic functions is given by*

$$H_n^k(x,y) = \beta^{n-1}\omega(y)^{n-1}\frac{y^{n-1}}{(1-x)^2(1-y)^{2n}}\left[\sum_{j=0}^{k-1} s_n(j)\omega(X_+)^j\omega(x)^{k-j-1}\right], \tag{3.174}$$

for some constant β, where $s_l : \mathbb{N} \to \mathbb{N}$ is defined inductively via $s_1(j) = 1$, $s_{l+1}(j) = \sum_{i=1}^{j+1} s_l(j)$. In particular, this basis has the form

$$H_n^k(x,y) = \frac{p_{m,k}}{(1-x)^{2k}(1-y)^{2(m+k-1)}}, \tag{3.175}$$

for $p_{m,k}$ some polynomial of degree at most $2k + n - 3$.
In case of a North-South symmetry, the statement holds with x and y reversed.

Proof. To illustrate the idea, let us start computing $H_n^1(x,y)$. We know that, for some constant α, we have

$$H_1^1(x,y) = \alpha\frac{1}{(1-x)^2(1-y)^2}. \tag{3.176}$$

Since the group is finite, we can without loss of generality assume that we have an East-West symmetry; otherwise we exchange the roles of x, y in the following. In this case, by Lemma 3.25, we can rewrite

$$xyH_1^1(x,y) = xy\frac{\omega(x) - \omega(X_+)}{K(x,y)} = \beta\omega(x)\omega(y), \tag{3.177}$$

where again β is some constant. As $\omega(X_+) = \omega(X_-)$, we see immediately that we do not need a decoupling function, and instead we can continue via

$$H_2^1(x,y) = \frac{xyH_1^1(x,y) - X_+ y H_1^1(X_+,y)}{K(x,y)} \tag{3.178}$$

$$= \frac{\beta}{K(x,y)} \left[\omega(x)\omega(y) - \omega(X_+)\omega(y)\right] \tag{3.179}$$

$$= \beta\omega(y) \underbrace{\frac{\omega(x) - \omega(X_+)}{K(x,y)}}_{=H_1^1(x,y)} . \tag{3.180}$$

We can continue inductively, noticing that in each step we only gain a factor of $\beta\omega(y)$, and thus obtain

$$H_n^1(x,y) = \beta^{n-1}\omega(y)^{n-1} H_1^1(x,y) \tag{3.181}$$

$$= \frac{\alpha\tilde{\beta}^n y^{n-1}}{(1-x)^2(1-y)^{2n}}. \tag{3.182}$$

If now $k \geq 2$, then we can compute

$$H_1^k = \frac{\omega(x)^n - \omega(X_+)^n}{K(x,y)} \tag{3.183}$$

$$= \underbrace{\frac{\omega(x) - \omega(X_+)}{K(x,y)}}_{=H_1^1(x,y)} \left[\omega(x)^{n-1} + \cdots + \omega(X_+)^{n-1}\right] . \tag{3.184}$$

The computation now continues in exactly the same fashion as for $k = 1$, except one needs to carry along more terms of the form $\omega(x)^a\omega(y)^b$, with their coefficients, which is where (3.174) comes from. We have

$$xyH_n^k(x,y) - X_+ y H_n^k(X_+,y)$$
$$= \beta^m\omega(y)^m \left[\sum_{j=0}^{k-1} \omega(x)_{k-1}\omega(X_+)^j - \omega(X_+)^k \sum_{j=0}^{k-1} s_m(j)\right] . \tag{3.185}$$

Using the algebraic identity

$$a\left[\sum_{j=0}^{k-1} c_j a^{k-j-1} b^j\right] - b^n \sum_{j=0}^{k-1} c_j = (a-b) \sum_{j=0}^{k-1} \left(\sum_{i=1}^{j+1} c_i\right) a^{k-j-1} b^j \tag{3.186}$$

for $a = \omega(x)$ and $b = \omega\left(X_+\right)$ then yields the statement. □

3.6.3 The infinite group case

In the case of an infinite group, the approach as in the previous section clearly does not work, as it was dependent on the fact that we could write, up to multiplicative constants, $xyH_1^1(x,y) = \frac{xy}{(1-x)^2(1-y)^2} = \omega(x)\omega(y)$. For this, in the finite group case we utilized the special shape of ω given by Lemma 3.25, which is now unavailable. Neither can we use (3.76) to find a decoupling function and simplify our boundary value problem. However, general theory of these boundary value problems as in [75] still tells us that a decoupling function should exist, and there are methods to find them. It is therefore only natural to try and see what happens if we want to apply them here. Unfortunately, it will turn out that even in this simple case $\pi/\theta = 2$, the resulting functions are rather unwieldy, and will in general not even be algebraic anymore.

Suppose from now on that we have an arbitrary non-singular model with small steps, zero drift and $\pi/\theta = 2$. We already know that $H_1^1(x,y) = \frac{1}{(x-1)^2(y-1)^2}$ is a harmonic function for such model, and what we want to do is to compute a biharmonic function of $H_1^1(x,y)$.
Now let

$$Y_{\pm}(x) = \frac{-b(x) \mp \sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)}, \tag{3.187}$$

as in Section 2.1.2, and consider the contour Γ given by $X_{\pm}[y_1, 1]$. By [70, Lemma 6.5.1], we know that Γ is a circle, symmetric with respect to the real axis, which it intersects at 1 and at some point $-1 < p$. We let c, d be the center and radius of Γ respectively. Let, again as in Section 2.1.2, $\mathcal{G}$ be the (finite) domain bounded by Γ. Via

$$r: \quad \mathbb{C} \to \bar{\mathbb{C}}: \quad z \mapsto \frac{d^2}{z-c} + c, \tag{3.188}$$

we can define a rational mapping r such that

1. r is an involution,
2. r maps the interior $\mathcal{G}^{\circ}$ to the exterior $\mathcal{G}^{c}$ and vice versa,
3. r corresponds to complex conjugation on Γ itself.

The existence of this rational mapping is the main reason why the following computation turns out to be comparatively simple; if π/θ were not 2, then Γ would not be a circle and things would end up being more complicated.
Now define

$$L(x) := xY_+(x)H_1(x, Y_+(x)) - r(x)Y_+(x)H_1(r(x), Y_+(x)). \tag{3.189}$$

$L(x)$ describes the value of $xyH_1^1(x,y) - X_+yH_1^1(X_+,y)$ on Γ: we substitute $Y_+(x)$ for y to be on Γ in the first place, and complex conjugation corresponds to switching from one solution of $K(x, Y_+(x)) = 0$ to the other, thus we have $K(x, Y_+(x)) = K(r(x), Y_+(x)) = 0$. It is also the expression we want to find a decoupling function of; our goal is to find a Υ, which is analytic inside $\mathcal{G} \setminus \{1\}$, such that

$$\Upsilon(x) - \Upsilon(r(x)) = L(x), \quad \forall x \in \Gamma. \tag{3.190}$$

Lemma 3.27. *We have*

$$\alpha L(x) = \frac{[(-1+2d+x)]\left[p_2(x) + \sqrt{b(x)^2 - 4a(x)c(x)}\right]}{(x-1)^3} \tag{3.191}$$

for some non-zero constant α and a polynomial $p_2(x)$ of degree at most 2.

Proof. We make use of the expression for $H_1(x,y)$, and rewrite (3.189) as

$$\beta L(x) = \left[\frac{x}{(x-1)^2} - \frac{r(x)}{(r(x)-1)^2}\right] \frac{Y_+}{(Y_+-1)^2} \tag{3.192}$$

$$= \frac{(d-1)(-1+2d+x)}{d^2(x-1)} \frac{Y_+}{(Y_+-1)^2}, \tag{3.193}$$

where β is some non-zero multiplicative constant we can ignore in the following. Note here that the factor $(-1+2d+x)$ has a zero at $x = 1-2d = p$ (where p was defined to be the second intersection, other than 1, of $\mathcal{G}$ with the real axis), which will be important later on. In order to simplify $\frac{Y_+}{(Y_+-1)^2}$, seeing as $a(x)+b(x)+c(x) = K(x,1) = \rho_1(x-1)^2$ and $b(x)^2 - 4a(x)c(x) = (x-1)^2\tilde{p}(x)$ for a constant ρ_1 and a quadratic polynomial $\tilde{p}(x)$, we can check that the following identities hold, which will be useful to us:

$$a(1) = c(1) \tag{3.194}$$

$$a(1)+b(1)+c(1) = K(1,1) = 0, \tag{3.195}$$

$$a'(1)+b'(1)+c'(1) = \frac{\partial}{\partial x}K(x,1)\bigg|_{x=1}, \tag{3.196}$$

$$\frac{\partial}{\partial x}b(x)^2 - 4a(x)c(x)\bigg|_{x=1} = \frac{\partial}{\partial x}\triangle(x)\bigg|_{x=1} = 0. \tag{3.197}$$

A direct simplification yields

$$\frac{Y_+}{(Y_+-1)^2} = \frac{\left[-b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]\left[2a(x)+b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]^2}{8\left[a(x)+b(x)+c(x)\right]^2} \tag{3.198}$$

$$= \frac{\left[-b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]\left[2a(x)+b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]^2}{8\rho_1^2 a(x)(x-1)^4}. \tag{3.199}$$

In order to simplify the numerator, we write

$$\left[-b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]\left[2a(x)+b(x)+\sqrt{b(x)^2-4a(x)c(x)}\right]^2 \tag{3.200}$$

$$= -4a(x)\left[a(x)b(x)+4a(x)c(x)+b(x)c(x)\right] + a(x)(a(x)-c(x))\sqrt{b(x)^2-4a(x)c(x)}. \tag{3.201}$$

Note that the factor $a(x)-c(x)$ in front of the square root vanishes if and only if our walk has a North-South symmetry[7], which is why everything stays rational in the finite group case. We want to show that the first summand contains a factor of $(x-1)^2$. To do so, we notice that for $s_1 := a(x)b(x)+4a(x)c(x)+a(x)b(x)$, we have

$$s_1(1) = a(1)b(1)+4a(1)c(1)+b(1)c(1) \tag{3.202}$$

$$\overset{(3.194)}{=} 2a(1)\left[a(1)+b(1)+c(1)\right] \tag{3.203}$$

$$\overset{(3.195)}{=} 0, \tag{3.204}$$

as well as

$$s_1'(1) = a'(1)b(1)+b'(1)a(1)+4a'(1)c(1)+4a(1)c'(1)+b'(1)c(1)+b(1)c'(1) \tag{3.205}$$

$$= b(1)\left[a'(1)+c'(1)\right]+b'(1)\left[a(1)+c(1)\right]+4\left[a'(1)c(1)+a(1)c'(1)\right] \tag{3.206}$$

$$\overset{(3.196)}{=} -b(1)b'(1)-b'(1)b(1)+4\left[a'(1)c(1)+a(1)c'(1)\right] \tag{3.207}$$

$$\overset{(3.197)}{=} 0. \tag{3.208}$$

[7] In particular, had we picked a model with East-West, but no North-South symmetry here, then the computation here is much more complicated than necessary – we could just have swapped the roles of x and y, done the same calculations and at this point ended up with a purely rational expression.

Thus, $-4s_1(x) = (x-1)^2 p_1(x)$ for some polynomial $p_1(x)$ of degree at most 2. Next,in order to treat the second summand of (3.201), we want to show that $a(x) - c(x) = \rho(x-1)^2$ for some constant ρ. We already know due to (3.194)) that $a(1) - c(1) = 0$. To see that $a'(1) - c'(1) = 0$ as well, let $a(x) = a_0 + a_1 x + a_2 x^2, c(x) = c_0 + c_1 x + c_2 x^2$ (that is, $a_{0/1/2} = p_{-1/0/1,1}, c_{0/1/2} = p_{-1/0/1,-1}$). The expression $a'(1) - c'(1)$ thus simplifies to $2a_2 + a_1 - 2c_2 - c_1$. Utilizing that, as $\pi/\theta = 2$, we have $a_2 + c_0 = c_2 + a_0$, we can write the latter as $2(a_2 - c_2) + a_1 - c_1 = a_2 - c_2 + a_0 - c_0 + a_1 - c_1 = (a_2 + a_1 + a_0) - (c_2 + c_1 + c_0) = 0$, since we have zero drift. Therefore, we know that $a(1) - c(1) = a'(1) - c'(1) = 0$, and therefore $a(x) - c(x) = \rho(x-1)^2$ (note that $a(x) - c(x)$ is quadratic in x). Thus we obtain

$$\frac{Y_+}{(Y_+ - 1)^2} = \frac{\left[-b(x) + \sqrt{b(x)^2 - 4a(x)c(x)}\right]\left[2a(x) + b(x) + \sqrt{b(x)^2 - 4a(x)c(x)}\right]^2}{8\rho_1^2(x-1)^4} \tag{3.209}$$

$$= \frac{p_2(x)(x-1)^2 + (x-1)^2\sqrt{b(x)^2 - 4a(x)c(x)}}{\rho_3(x-1)^4} \tag{3.210}$$

$$= \frac{p_2(x) + \sqrt{b(x)^2 - 4a(x)c(x)}}{\rho_3(x-1)^2}. \tag{3.211}$$

Substituting this into (3.193) yields the statement. □

We now define

$$L_1(x) := \frac{(-1+2d+x)p_2(x)}{(x-1)^3}, \tag{3.212}$$

$$L_2(x) := \frac{(-1+2d+x)\sqrt{b(x)^2 - 4a(x)c(x)}}{(x-1)^3} = \frac{(-1+2d+x)\sqrt{-(x-x_1)(x-x_4)}}{(x-1)^2}. \tag{3.213}$$

By construction, we have $[L_1(x) + L_2(x)] = \alpha L(x)$.
We would now like to proceed by computing decoupling functions of L_1, L_2 separately. For decoupling functions of $L_{1,2}(x)$ to exist, we must have $L_{1,2}(x) + L_{1,2}(r(x)) = 0$ for $x \in \Gamma$, due to (3.190). Note that $L(x)$ satisfies this condition by construction.
The first question to ask here is in which way we define the square root. This depends on the sign of $(x-x_1)(x-x_4)$; in order to utilize our methods later we will want the expression $L_2(x)$ to be continuous on the contour Γ. Due to [70, Thm. 5.3.3], we know that $x_1 \in \mathcal{G}$, $x_4 \in \mathcal{G}^c$. We select the branch cut such that the root singularity on Γ is canceled out by the factor $(-1+2d+x) = (x-p)$, for p the left intersection of Γ with the real axis, i.e. we need to select the branch cut along the axis with the sign of $-(p-x_1)(p-x_4)$.
In both cases, there is a section of the contour Γ which lies on the side of the branch cut. Therefore, on this section Γ' of the contour we have

$$\sqrt{-(\bar{x}-x_1)(\bar{x}-x_4)} = -\overline{\sqrt{-(x-x_1)(x-x_4)}}.$$

This implies that, on Γ',

$$L(x) + L(r(x)) = L_1(x) + L_1(\overline{x}) + L_2(x) + L_2(\overline{x}) = \underbrace{L_1(x) + \overline{L_1(x)}}_{\in\mathbb{R}} + \underbrace{L_2(x) - \overline{L_2(x)}}_{\in\mathbb{C}} = 0. \tag{3.214}$$

Consequently, we know that $L_1(x) + L_1(r(x))$ must be 0 on Γ', and as it is a rational function it must thus be 0 everywhere. The same goes for $L_2(x) + L_2(r(x))$. This means that finding a decoupling function of $L(x)$ can be done in two parts:

1. We find a decoupling function of the rational function $L_1(x)$,

2. We find a decoupling function of the non-rational function $L_2(x)$.

Decoupling of the (rational) L_1

As we already know that $L_1(x) + L_1(r(x)) = 0$, this turns out to be rather straightforward: we have

$$L_1(x) = \frac{1}{2}\left[L_1(x) - L_1(r(x))\right], \tag{3.215}$$

which already gives us a rational decoupling function. One arguably gets a somewhat nicer form by utilizing an ansatz of the form

$$L_1(x) = \alpha_3\left[\frac{1}{(x-1)^3} - \frac{1}{(r(x)-1)^3}\right] + \alpha_1\left[\frac{1}{x-1} - \frac{1}{r(x)-1}\right], \tag{3.216}$$

as will be done for Example 3.6.4.

Decoupling of the (irrational) L_2

Note that the previous approach is problematic here, as the resulting function would have singularities at $x = x_1$ and $x = x_4$, which might be inside of $\mathcal{G}$. Thus we compute a decoupling function via a contour integral, which is also the standard approach given the theory of complex boundary value problems. To utilize the theory as in e.g. [75], we need a function which is continuous on Γ. This is not the case for L_2, due to its pole at $x = 1$. However, this can easily be remedied by considering instead of L_2 the function

$$L_3(x) := (x-1)(r(x)-1)L_2(x) = -d\frac{(-1+2d+x)\sqrt{-(x-x_1)(x-x_4)}}{x-(1-d)}. \tag{3.217}$$

If we find a decoupling function $\Upsilon_3(x)$ of L_3, then $\Upsilon_2(x) := \frac{\Upsilon(x)_3}{(x-1)(r(x)-1)}$ will be a decoupling function of L_2, as the denominator is invariant under $x \mapsto r(x)$. However, due to [75] we already know that such a Υ_3 exists, seeing as $L_3(x)$ is continuous and bounded on Γ (though not analytic near $x = p$, but this does not matter for us).
Since general theory guarantees us the existence of a Υ_3, analytic in $\mathcal{G}^\circ$, which decouples L_3, we can utilise the same trick as in [67]: we write the decoupling property, select a $t \in \mathcal{G}^\circ$, divide by $(x-t)$ and integrate over Γ with respect to x. The resulting equation then reads

$$\int_\Gamma \frac{\Upsilon_3(x)}{x-t}\mathrm{d}x - \int_\Gamma \frac{\Upsilon_3(r(x))}{x-t}\mathrm{d}x = \int_\Gamma \frac{L_3(x)}{x-t}\mathrm{d}x. \tag{3.218}$$

The leftmost term is, by Cauchy's integral formula, nothing but $2\pi i\Upsilon_3(t)$, and the rightmost term can be computed. The question is what to do with the middle term. We notice that $r(x)$ is an involution, sending Γ to itself (only changing the direction along which Γ is traversed), and that $r'(x) = -\frac{d^2}{(x-c)^2}$. Letting now $x = r(y)$, this integral can be written as

$$\int_\Gamma \frac{\Upsilon_3(r(x))}{x-t}\mathrm{d}x = d^2\int_\Gamma \frac{\Upsilon_3(y)}{r(y)-t}\frac{1}{(y-c)^2}\mathrm{d}y \tag{3.219}$$

$$= d^2\int_\Gamma \frac{\Upsilon_3(y)}{(r(y)-t)(y-c)}\frac{1}{y-c}\mathrm{d}y \tag{3.220}$$

$$= d^2\int_\Gamma \frac{\Upsilon_3(y)}{d^2+(c-t)(y-c)}\frac{1}{y-c}\mathrm{d}y. \tag{3.221}$$

From (3.219), we see that the only possible pole of the integrand is at $y = c$ (since $t \in \mathcal{G}^\circ$), and from (3.221) it follows that this is a simple pole. We can therefore, again by Cauchy, write

$$\int_\Gamma \frac{\Upsilon_3(r(x))}{x-t}\mathrm{d}x = 2\pi i\Upsilon(c). \tag{3.222}$$

Overall, we therefore get

$$\Upsilon_3(t) - \Upsilon_3(c) = \frac{1}{2\pi i}\int_\Gamma \frac{L_3(x)}{x-t}\mathrm{d}x. \tag{3.223}$$

Noticing that with Υ_3 any translation $\Upsilon_3 + \text{const}$ is also a solution for any constant, we can without loss of generality assume that $\Upsilon_3(c) = 0$, and obtain

$$\Upsilon_3(t) = \frac{1}{2\pi i}\int_\Gamma \frac{L_3(x)}{x-t}\mathrm{d}x. \tag{3.224}$$

Remark: We can, unfortunately, not apply calculus of residues to the integral on the right-hand side of (3.218), since it is not analytic on the branch cut. We will take a closer look at this integral in the following section.

Asymptotics of $\Upsilon_3(t)$

The goal of this section is to compute the asymptotics of $[t^n]\,\Upsilon_3(t)^8$, which will serve to show that $\Upsilon_3(t)$ cannot be an algebraic function. To do so, by standard methods about power series as presented for example in [73], we need to know the location of the singularity of $\Upsilon_3(t)$ closest to the origin.
Remember that the contour Γ is a circle in $\mathbb{C}$, going through Γ and intersecting the real axis at a second point p. For the asymptotics, we need a bit more information about the exact location of p.

Lemma 3.28. *Let p be the left intersection of the circle $X\,[y_1, 1]$ with the real axis. We have*

$$\begin{aligned} |p| &> 1 && \text{if} \quad p_{1,1} > p_{1,-1}, \\ |p| &< 1 && \text{if} \quad p_{1,1} < p_{1,-1}, \\ |p| &= 1 && \text{if} \quad p_{1,1} = p_{1,-1}. \end{aligned}$$

Proof. Let us parametrize the contour and consider the absolute value of $X_+(t)X_-(t)$, t close to 1. Note again that we have

$$f(t) := |X_0(t)|^2 = X_0(t)X_1(t) = 2\frac{c(t)}{a(t)}. \tag{3.225}$$

As we have zero drift and thus $X_0(1) = X_1(1) = 0$, we see that $f'(1) = 0$. Now all that remains to do is to check whether at this point we have a local minimum or maximum; i.e. whether we have $f''(1) > 0$ or $f''(1) < 0$ respectively. Therefore, we can use the explicit forms of $a(x), b(x)$, and in the end we obtain (using, again, that $r = 0$)

$$f''(1) = -4\frac{p_{1,-1} - p_{-1,-1}}{p_{-1,1} + p_{-1,0} + p_{-1,-1}}. \tag{3.226}$$

Since the denominator is certainly > 0, and since (again, due to the drift being zero) we have $p_{1,-1} - p_{-1,-1} = p_{1,1} - p_{-1,1}$, the statement follows.
Note lastly that if we have equality, i.e. $p_{1,1} = p_{-1,1}$, then we already know that we have an East-West symmetry, and that in this case the contour will be the unit circle. □

[8] Here, $[t^n]$ is the linear operator extracting the n-th coefficient of a power series around 0.

In the case $|p|= 1$, there is no need for all these computations, as the group is finite by Lemma 3.22 and Thm. 3.24, and we can directly compute all polyharmonic functions. If $|p|> 1$, the singularity of Υ_3 closest to 0 (and thus the one determining asymptotic behaviour) is at $x = 1$. In the case $|p|< 1$, however, Υ will have exponential growth with base $\frac{1}{p}$. We will now compute the exact shape of the resulting terms in this case.

Lemma 3.29. *Let d be the radius of the circle $X[y_1, 1]$, c its center and p its left intersection with the real axis. We then have*

$$d = \frac{(1 - x_1)(1 - x_4)}{2 - x_1 - x_4}, \tag{3.227}$$

or, equivalently,

$$p - c(x_1 + x_4) + x_1x_4 = 0. \tag{3.228}$$

Proof. One can check that, if $\pi/\theta = 2$ and we have zero drift, then

$$p = X(y_4) = X(y_1) = -\frac{\tilde{b}(y_1)}{2\tilde{a}(y_1)} = \frac{-1 + 2p_{-1,0}}{1 + 4p_{1,-1} - 4p_{-1,-1} - 2p_{-1,0}}. \tag{3.229}$$

Similarly, one sees that

$$x_1x_4 = \frac{4p_{-1,1}p_{-1,-1} - p_{-1,0}^2}{4p_{-1,1}p_{1,-1} + 4p_{1,-1}(p_{-1,-1} + p_{-1,0}) - (2p_{-1,-1} + p_{-1,0})^2}, \tag{3.230}$$

$$x_1 + x_4 = \frac{2((-1 + p_{-1,0})p_{-1,0} + p_{-1,-1}(-1 + 2p_{-1,0}) + p_{-1,1}(-1 + 4p_{-1,-1} + 2p_{-1,0}))}{4p_{-1,1}p_{1,-1} + 4p_{1,-1}(p_{1,-1} + p_{-1,0}) - (2p_{-1,-1} + p_{-1,0})^2}. \tag{3.231}$$

Putting the above together, (3.227) follows immediately. $\square$

The above Lemma 3.29 now turns to be very useful for us: the term $L_3(x)$, which we want to integrate, contains a root of the form $\sqrt{-(x - x_1)(x - x_4)}$. We integrate along the contour Γ, thus we substitute $x \mapsto c + de^{iu}$. It turns out that we now have, for any two constants a, b,

$$-x^2 - ax - b = -1 - b + a(-1 + d) + 2d - de^{iu}(2 + a - 2d + 2d\cos u). \tag{3.232}$$

However, for $a = -(x_1+x_4)$ and $b = x_1x_4$, Lemma 3.29 tells us that the constant term vanishes, i.e. we have

$$-(x - x_1)(x - x_4) = -de^{iu}(2 - x_1 - x_4 - 2d + 2d\cos u). \tag{3.233}$$

This will allow us to rewrite the square root, since the second factor is strictly real.
There are now two cases to consider: $x_4 < p$ and $x_4 > p$. As the computations are very similar, we will now only look at the former.
If $x_4 < p$, then we have $-(p - x_1)(p - x_4) > 0$, which means that we select the branch cut of the root to be along the positive real axis. Next, we will determine the sign of the real factor under the root.

Lemma 3.30. *If $x_4 < p$, then we have for any $u \in [0, 2\pi]$*

$$2 - x_1 - x_4 + 2d(-1 + \cos u) \geq 0. \tag{3.234}$$

Proof. First, notice that

$$2 - x_1 - x_4 + 2d(-1 + \cos u) \geq (1 - x_1) + (1 - x_4) - 4d. \tag{3.235}$$

Due to Lemma 3.29, we know that

$$d = \frac{(1 - x_1)(1 - x_4)}{(1 - x_1) + (1 - x_4)}. \tag{3.236}$$

The right side of (3.235) can therefore be written as

$$a + b - 4\frac{ab}{a + b}, \tag{3.237}$$

for $a = 1 - x_1, b = 1 - x_4$. Since $x_4 < -1 < x_1$, we know that $a > b > 0$, utilizing homogeneicity it therefore suffices to study the function

$$f(a) = 1 + a - 4\frac{a}{1 + a} = \frac{(1 - a)^2}{1 + a}. \tag{3.238}$$

As $f(a)$ does not have any zeros for $a > 1$, (3.234) holds. □

Due to our choice of branch cut, we have

$$\begin{aligned}
\sqrt{-(x - x_1)(x - x_4)} &= \sqrt{-de^{iu}(2 - x_1 - x_4 - 2d + 2d\cos u)} &(3.239)\\
&= \left|\sqrt{d(2 - x_1 - x_4 - 2d + 2d\cos u)}\right| \sqrt{e^{i(u+\pi)}} &(3.240)\\
&= \begin{cases} \left|\sqrt{d(2 - x_1 - x_4 - 2d + 2d\cos u)}\right| \cdot e^{i(u+\pi)/2}, & \text{if} \quad u \in [0, \pi), \\ \left|\sqrt{d(2 - x_1 - x_4 - 2d + 2d\cos u)}\right| \cdot e^{i(u-\pi)/2}, & \text{if} \quad u \in [\pi, 2\pi). \end{cases} &(3.241)
\end{aligned}$$

If we now rewrite the integral

$$\int_\Gamma \frac{L_3(x)}{x - t} \mathrm{d}x = d \int_0^{2\pi} \frac{L_3(c + de^{iu})e^{iu}}{c + de^{iu} - t} \mathrm{d}u \tag{3.242}$$

using (3.241), it turns out that the only term generating arbitrarily high powers of t is

$$\begin{aligned}
&\sqrt{2}b(-1 + 2c - t)\sqrt{(a - b)(2a(-1 + c)(c - t) + b(1 + 2(-1 + c)c - 2ct + t^2))} \\
&\qquad \times \arctan\left(\frac{\sqrt{a - b}(-1 + t)}{\sqrt{4a(-1 + c)(c - t) + 2b(1 + 2(-1 + c)c - 2ct + t^2)}}\right), \qquad (3.243)
\end{aligned}$$

where $a = d(2 - x_1 - x_4 + 2d)$, $b = 2d^2$. Up to a multiplicative constant, we can write this as

$$T(t) := (-1 + 2c - t)\sqrt{(t - x_1)(t - x_4)} \arctan\left(\frac{\sqrt{d(2 - x_1 - x_4 - 4d)}(t - 1)}{2d\sqrt{(t - x_1)(t - x_4)}}\right). \tag{3.244}$$

From here on, it remains to do a standard computation, using the theory developed e.g. in [73]. We know that $\arctan x$ has its singularities at $x = \pm i$; one can check that this is the case for $t = -p$. The singularities due to the $\sqrt{(t - x_1)(t - x_4)}$ cancel, since the series representation of $x \arctan x$ contains only even powers of x. Therefore, all we need to do is to consider the behaviour of the right-hand side of (3.244) near $t = -p$.
One can check that, after considering $\tilde{T}(t) := T(pt)$ in order to shift the pole to $t = 1$, we have an asymptotic expansion of the form

$$T(t) = c_1 + c_2 \log|1 - t| + \mathcal{O}(1 - t) \tag{3.245}$$

for some constants c_1, c_2, and thus, according to e.g. [73, Thm. VI.3], we have

$$[t^n]T(t) = \mathcal{O}\left[\frac{1}{p^n}\left(\frac{\log n}{n}\right)\right]. \tag{3.246}$$

In particular, considering the log-terms, we know that $T(t)$, and hence also $\Upsilon(x)$ and the resulting decoupling function, cannot be algebraic.

3.6.4 An example with infinite group

We will now look at how the computations above work in a concrete example with zero drift, $\pi/\theta = 2$ and infinite group. As there are no models as famous as say the simple walk or the tandem walk which belong to this group, we pick the same model as in Example 3.5.2, which was defined by

$$\begin{aligned} p_{1,0} = p_{0,1} &= 0, \\ p_{1,1} &= 1/4, \\ p_{1,-1} = p_{0,-1} = p_{-1,0} = p_{-1,1} &= 1/6, \\ p_{-1,-1} &= 1/12. \end{aligned}$$

Here, Γ is the circle defined by $|z - 1/6|^2 = \left(\frac{5}{6}\right)^2$, and therefore complex conjugation on Γ corresponds to the Möbius transform $r : z \mapsto \left(\frac{5}{6}\right)^2 \frac{1}{z-1/6} + \frac{1}{6}$. After some computation, one checks that $L_1(x)$ and $L_2(x)$ as defined in (3.212) and (3.213) now take the form

$$L_1(x) = \frac{8(2+3x)(7+6x(1+2x))}{125(1-x)^3} \tag{3.247}$$

$$L_2(x) = \frac{8(2+3x)(x-1)\sqrt{(6x-1)(3+x)}}{125(1-x)^3}. \tag{3.248}$$

For a decoupling function of $L_1(x)$, one obtains

$$\Upsilon_1(x) = \frac{4}{(x-1)^3} + \frac{96}{25(x-1)}. \tag{3.249}$$

For a decoupling function $\Upsilon_2(x)$ of $L_2(x)$, we proceed as in Section 3.6.3. We start by letting $L_3(x) = \frac{-8(2+3x)\sqrt{-1-18x-6x^2}}{25(1-6x)}$, and compute in a first step a decoupling function $\Upsilon_3(x)$ of $L_3(x)$. Using the same integration trick as above, we finally arrive at

$$2\pi i \Upsilon_3(t) + 2\pi i \Upsilon_3(1/6) = \int_\Gamma \frac{L_3(x)}{x-t} \mathrm{d}x. \tag{3.250}$$

We remember that we can always add a constant to a decoupling function without changing the decoupling property, so we can assume that $\Upsilon_3(1/6) = 0$ and thus have

$$\Upsilon_3(t) = \frac{1}{2\pi i} \int_\Gamma \frac{L_3(x)}{x-t} \mathrm{d}x. \tag{3.251}$$

Combining everything until now, we have

$$\Upsilon(x) = \frac{\frac{1}{2\pi i}\int_\Gamma \frac{L_3(t)}{t-x}\mathrm{d}t}{(x-1)(r(x)-1)} + \frac{4}{(x-1)^3} - \frac{96}{25(x-1)}. \tag{3.252}$$

Here, the interesting part is clearly the contour integral. Thus, we want to know the asymptotics of the coefficients of

$$\Upsilon_3(t) = \frac{1}{2\pi i} \int_\Gamma \frac{(x+2/3)\sqrt{-(1+18x+6x^2)}}{(x-1/6)(x-t)}. \tag{3.253}$$

By some computation, one finds that

$$[t^n]\,\Upsilon_3(t) \sim [t^n]\,(2+3t)\sqrt{1/3+6t+2t^2}\,\arctan\left(\frac{t-1}{\sqrt{1/3+6t+2t^2}}\right). \tag{3.254}$$

The singularity closest to 0 here is at $-\frac{2}{3}$, and utilizing that $\Upsilon_3\left(-\frac{2}{3}z\right) = \mathcal{O}\left(\log\frac{1}{1-z}\right)$, we can apply [73, Thm. VI.3] and have

$$[z^n]\,\Upsilon_3(z) = \left(-\frac{3}{2}\right)^n [z^n]\,\Upsilon_3\left(-\frac{2}{3}z\right) = \mathcal{O}\left(\left(-\frac{3}{2}\right)^n \frac{\log n}{n}\right). \tag{3.255}$$

As mentioned, we can therefore deduce that $\Upsilon_3(t)$, and therefore also $\Upsilon(t)$, is not algebraic.

Accordingly, in case of an infinite group, the existence of a decoupling function as postulated by general theory about boundary value problems does not appear to be very useful in terms of actual computations. If one were to drop the condition that $\pi/\theta = 2$ on top of this, then the calculations would yet again get much more complicated, as we were heavily relying on the fact that we can describe complex conjugation on the contour Γ via a simple rational transformation, which is only due to the fact that Γ is a circle.

Chapter 4

Asymptotic expansion of orbit-summable quadrant walks and polyharmonic functions

> One starts out in life trying to do mathematics, and winds up doing combinatorics.
>
> Ian Macdonald

This chapter will be structured as follows:

1. In Section 4.1, Thm. 4.1, we will compute a complete asymptotic expansion for models with finite group allowing for orbit summation (see [31]). We will see that this expansions features features inverse powers of n (the length of the paths) and a dependency on the endpoint in terms of polyharmonic functions. The example of the Gouyou-Beauchamps model will be worked out explicitly in Section 4.1.2, whereas in Section 4.1.3 the influence of periodicity of a model will be discussed.

2. In Section 4.2, we will consider the same problem where instead of $(0, 0)$, we start our paths at an arbitrary point (u, v). It turns out, maybe not surprisingly, that due to the symmetry of the problem the resulting solution is similar to that obtained in Section 4.1, leading to Thm. 4.5. Continuing from there, we can then find a decomposition of the polyharmonic functions in Thm. 4.7. This decomposition is given for the example of the simple walk in Section 4.2.2, using the basis of polyharmonic functions constructed previously in Chapter 3.

As an addition, examples of the method applied to a model with large steps and a three-dimensional model are given in App. B.1 and App. B.2. A decomposition of the polyharmonic coefficients as in Section 4.2.2 will be done for the Gouyou-Beauchamps model and the tandem walk. Lastly, the first three terms of the asymptotics for all the 19 unweighted, orbit-summable models are given in App. D.
This chapter is largely based on the author's article [113].

4.1 Quadrant walks starting at the origin

4.1.1 Full asymptotic expansion

The goal of this section is to compute the asymptotics of orbit-summable lattice walks from the origin to an arbitrary but fixed point in the quarter plane, and in particular to show the following:

Theorem 4.1. *Let $\mathcal{S}$ be a step set satisfying*

1. *our step set consists of small steps only, i.e.* $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$,
2. *our step set is non-degenerate, i.e. there is no (possibly rotated) half-plane containing all allowed steps,*
3. $\mathcal{S}$ *is orbit-summable, i.e.*

$$xyQ(x,y;t) = [x^>]\,[y^>]\,\frac{N(x,y)}{k(x,y;t)},$$

where

$$N(x,y) := \sum_{g\in\mathcal{G}} \operatorname{sgn}(g)g(xy).$$

Suppose that $s_0 = (x_0, y_0)$ is a dominant saddle point, with associated other saddle points s_i, $1 \le i \le r$ (meaning that we consider $r+1$ saddle points in total). Furthermore, let $\alpha_i, \beta_i, \zeta_i$ be the roots of unity as constructed in Section 2.1.4. Then, there are a constant $c \in \mathbb{N}$, a constant $\gamma > 0$, and γ-polyharmonic functions v_p of degree p such that for any $m \in \mathbb{N}$ we have

$$q((0,0),(k,l);n) = \frac{\gamma^n}{n^c}\left[\sum_{p=1}^{m-1} \frac{v_p(k,l)\sum_{i=1}^r \alpha_i^k \beta_i^l \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right)\right]. \tag{4.1}$$

The polyharmonic functions $v_p(k,l)$ are polynomials precisely if the drift of the model is zero, else they contain an additional factor of $x_0^{-k} y_0^{-l}$, with (x_0, y_0) the dominant saddle point. They can be computed explicitly via a Cauchy-type integral, as in (4.19).
Lastly, the constant c can be expressed using the arctangent of the correlation coefficient θ defined in Section 2.1.2 via $c = \pi/\theta$, and for the constant γ we have $\gamma = S(1,1)$.

In order to keep the proof of Thm. 4.1 reasonably concise, we will first start with two somewhat technical lemmas. In the first lemma, we will establish some periodicity properties of $S(x,y)$ and the group, in the case where we have multiple saddle points.

Lemma 4.2. *Let (x_0, y_0) be a dominant saddle point, and $(x_i, y_i) = (\alpha_i x_0, \beta_i x_0)$ be the associated ones, with $S(x_i, y_i) = \zeta_i S(x_0, y_0)$. Let furthermore $\Phi(x,y) = (x, \phi(x,y)), \Psi(x,y) = (\psi(x,y), y)$ be the generators of the group as in (2.12),(2.13). We then have, for all $x, y \in \mathbb{C}$:*

$$S(\alpha_i x, \beta_i y) = \zeta_i S(x,y), \tag{4.2}$$
$$\psi(\alpha_i x, \beta_i y) = \alpha_i \psi(x,y), \tag{4.3}$$
$$\phi(\alpha_i x, \beta_i y) = \beta_i \phi(x,y). \tag{4.4}$$

Remark: Lemma 4.2 still holds true in more than two dimensions, with a completely analogous proof.

Proof. As (x_0, y_0) is a dominant saddle point and (x_i, y_i) associated to it, we have $|\alpha_i| = |\beta_i| = |\zeta_i| = 1$, and know that for each monomial $x^k y^l$ appearing in $S(x,y)$ we have $(\alpha_i x)^k (\beta_i y)^l = \zeta_i x^k y^l$. Consequently, for all such k, l we have $\alpha_i^k \beta_i^l = \zeta_i$, and thus (4.2) holds.
We can define $\psi(x,y)$ almost everywhere by the properties

$$S(\psi(x,y), y) = S(x,y), \tag{4.5}$$
$$\psi(x,y) \neq x, \tag{4.6}$$

the exception being the points where $\frac{\partial S}{\partial x}(\cdot, y) = 0$, i.e. where x is uniquely defined by the property $S(\cdot, y) = S(x, y)$. We know that for any y, we have $\alpha_i x \neq \alpha_i \psi(x, y)$ almost everywhere. Furthermore, using (4.2), we can rewrite:

$$S(\alpha_i \psi(x, y), \beta_i y) = \zeta_i S(\psi(x, y), y) = \zeta_i S(x, y) = S(\alpha_i x, \beta_i y). \tag{4.7}$$

Note in particular that (4.2) and therefore (4.7) hold true for all $x, y \in \mathbb{C}$, not only saddle points. Since ψ is rational, we therefore have (4.3), and by symmetry also (4.4). □

Lastly, we will show that given an asymptotic representation as in Thm. 4.1 below, the functions appearing therein are indeed polyharmonic. Note that this lemma is similar to Lemma 1.3. The latter does not, however, account for the roots of unity.

Lemma 4.3. *Suppose $q(B; n)$ is a (combinatorial) quantity satisfying*

$$q(B; n+1) = \sum_{x \in \mathcal{S}} \omega_s q(x - s; n) \quad \forall B \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, n \geq 0, \tag{4.8}$$

and at the same time it is of the form

$$q(B; n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^r v_{p,i}(B) \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^{m+1}} \right) \right], \tag{4.9}$$

for all $k \geq 0$, with the ζ_i pairwise different roots of unity. Then, the $v_{p,i}$ are γ-polyharmonic of degree p.
If, additionally, for a fixed point B we know that $q(B; n) = 0$ $\forall n \in \mathbb{N}$, then $v_{p,i}(B) = 0$ for all p, i.

Proof. Substituting (4.9) into (4.8) gives us

$$q(B; n+1) = \sum_{s \in \mathcal{S}} \omega_s q(B - s; n) \quad \Leftrightarrow$$

$$\gamma^{n+1} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^r v_{p,i}(B) \zeta_i^{n+1}}{(n+1)^{p+c}} + \mathcal{O}\left(\frac{1}{n^{m+c}} \right) \right] = \gamma^n \sum_{s \in \mathcal{S}} \omega_s \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^r v_{p,i}(B - s) \zeta_i^n}{n^{p+c}} \right].$$

Extracting the terms for $p = 1$ and noticing that the others are smaller by a factor of at least $\frac{1}{n}$, we obtain

$$\gamma \frac{n^{c+1}}{(n+1)^{c+1}} \sum_{i=1}^r v_{1,i}(B) \zeta_i^{n+1} + \mathcal{O}\left(\frac{1}{n} \right) = \sum_{s \in \mathcal{S}} \omega_s \sum_{i=1}^r v_{1,i}(B - s) \zeta_i^n. \tag{4.10}$$

Letting n go to ∞, we notice that the fraction $\frac{n^{c+1}}{(n+1)^{c+1}}$ will converge to 1. This gives us

$$\gamma \sum_{i=1}^r v_{1,i}(B) \zeta_i^{n+1} + \mathcal{O}\left(\frac{1}{n} \right) = \sum_{s \in \mathcal{S}} \omega_s \sum_{i=1}^r v_{1,i}(B - s) \zeta_i^n. \tag{4.11}$$

All we need to do now is show that each of the $v_{1,i}(x)$ is γ-harmonic by itself, i.e. that (4.11) holds for each summation index separately. To do so, let us forget for a moment the part $\mathcal{O}\left(\frac{1}{n}\right)$ and solve the exact analogue to (4.11).
Since by assumption the ζ_i are all different, we know that the vectors

$$(\zeta_1^m, \dots, \zeta_l^m), \quad 0 \leq m \leq r - 1 \tag{4.12}$$

are linearly independent (written as a matrix, they give a Vandermonde matrix with determinant $\prod_{i<j}(\zeta_j - \zeta_i) \neq 0$). Therefore, the system

$$\sum_{i=1}^{r} \left(cv_{i,1}(B) - \sum_{s\in\mathcal{S}} \omega_s v_{i,1}(B-s) \right) \zeta_i^n = 0 \tag{4.13}$$

has no nontrivial solutions, and hence

$$cv_{i,1}(B) = \sum_{s\in\mathcal{S}} \omega_s v_{i,1}(B-s) \quad \forall 1 \leq i \leq l. \tag{4.14}$$

All that remains to do now is to see that the error term $\mathcal{O}\left(\frac{1}{n}\right)$ in (4.11) does not change anything. To do so, suppose now that (4.14) is not satisfied for some i. Then we know that there are arbitrarily large n such that (4.13) does not hold, i.e. its right-hand side takes a value $\varepsilon_n \neq 0$. As the ζ_i are roots of unity, there are only finitely many values which ε_n can take for different n, so we cannot have convergence of ε_n to 0. But then, choosing n large enough, (4.10) cannot hold either; a contradiction. Thus, (4.14) must hold, and we know that the $v_{1,i}(B)$ are harmonic. By induction, applying the discrete Laplacian $\triangle$ to both sides of (4.10), we argue in the same fashion that the $v_{k,i}(B)$ must be polyharmonic of degree k.
For the second part, suppose that $q(B;n) = 0$ for all n, but we have p, i such that $v_{p,i}(B) \neq 0$. Assume our p to be minimal with this property. Then, we know that

$$\sum_{i=1}^{l} v_{p,i}(B)\zeta_i^n = 0, \tag{4.15}$$

because otherwise this would be a contradiction to (4.9) for large n. But then we can utilize independence of the vectors $(\zeta_1^k, \zeta_2^k, \ldots, \zeta_l^k)$ for $k = 0, \ldots, l-1$ as before and arrive at a contradiction.
Finally, to show that the $v_{p,i}$ are polyharmonic of order p, we can apply the operator $\triangle^{p-1}$ to both sides of (4.9), notice that the first $p-1$ terms vanish by assumption and then repeat the above argument. □

Remarks:

- In the proof of Thm. 4.1 we will see that in our case, for different i the $v_{p,i}(k,l)$ differ only by a factor of $\alpha_i^{-k}\beta_i^{-l}$. Here, given a dominant saddle point (x_0, y_0), and an associated one (x_i, y_i), then α_i, β_i are the numbers such that $(x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$. This allows us to essentially talk about only a single polyharmonic function v_p for any given p; namely the one related to the dominant saddle point.

- We will see in Section 4.2.1 that we always have $\triangle v_{p+1} = v_p + r_{p-1}$, where r_{p-1} is some polyharmonic function of degree at most $p-1$. This hints at the fact that the polyharmonic functions appearing in the asymptotic expansions are not arbitrary, but do have some form of recursive structure.

We now have all ingredients ready for the proof of Thm. 4.1.

Proof (of Thm. 4.1). By the assumptions, we know that we have

$$q(0,(k,l);n) = \left[x^k y^l t^n\right] Q(x,y;t) = \left[x^{k+1} y^{l+1} t^n\right] \frac{\sum_{g\in\mathcal{G}} \operatorname{sgn}(g) g(xy)}{k(x,y;t)}. \tag{4.16}$$

As $k(x,y) = 1 - tS(x,y)$, we can rewrite

$$q(0,(k,l);n) = \left[x^{k+1}y^{l+1}t^n\right] \sum_{i=0}^{\infty} t^i S(x,y)^i N(x,y) \tag{4.17}$$
$$= \left[x^{k+1}y^{l+1}\right] S(x,y)^n N(x,y). \tag{4.18}$$

By Cauchy's formula, we have

$$q((0,0),(k,l);n) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{S(x,y)^n N(x,y)}{x^{k+2}y^{l+2}} \mathrm{d}x\mathrm{d}y, \tag{4.19}$$

with $\Gamma_{1,2}$ being closed curves around the origin. To evaluate the asymptotics of this integral, we utilize the saddle point method, as described for instance in [73, Chapter VIII].
The main idea is to conveniently choose our contours Γ_1, Γ_2 such that they make the integral as easy to compute as possible.
With this in mind, suppose that (x_0, y_0) is a dominant saddle point, and pick $\Gamma_1 = \{|x| = |x_0|\}, \Gamma_2 = \{|y| = |y_0|\}$. We know that the modulus of $S(x,y)$ on $\Gamma_1 \cap \Gamma_2$ is maximal; and the only other points where it attains the same value are the associated saddle points (x_i, y_i). At any other point, $|S(x,y)|$ will be strictly smaller – hence, when n goes to infinity, it suffices to compute the integral locally around our saddle points, since the rest will grow exponentially slower. We could hypothetically run into issues if $N(x,y)$ were to be infinite, but we will see that this is not the case at our saddle points (and for any other point, we can always just slightly shift our contour to avoid a pole). It is easy to check that, given an $\varepsilon > 0$, the set $|S(x_0,y_0) - S(x,y)| < \varepsilon$ is contained in a domain of the form $|x - x_0| < \varepsilon_1, |y - y_0| < \varepsilon_2$. As previously mentioned, the rest of the integral can be – in order to find the asymptotics – neglected, as it will be exponentially smaller. Changing our coordinates to $x = x_0 e^{is/\sqrt{n}}, y = y_0 e^{it/\sqrt{n}}$, this corresponds to a region of the form $\left|\frac{s}{\sqrt{n}}\right| < \delta_1, \left|\frac{t}{\sqrt{n}}\right| < \delta_2$, or, equivalently, $|s| < \delta_1\sqrt{n}, |t| < \delta_2\sqrt{n}$.
To find the asymptotics, it therefore suffices to compute the integrals

$$\int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} \int_{-\delta_2\sqrt{n}}^{\delta_2\sqrt{n}} F_j(s,t,k,l,n) \mathrm{d}t\mathrm{d}s, \tag{4.20}$$

where $F_j(s,t,k,l,n)$ is the expression obtained by substituting $x = x_j e^{is/\sqrt{n}}, y = y_j e^{it/\sqrt{n}}$, for (x_j, y_j) the relevant saddle points (i.e. (x_0, y_0) and the ones associated to it, as outlined in Section 2.1.4). We will see that, given any fixed $m \in \mathbb{N}$, each such integral can be written in the form

$$\frac{\gamma^n}{n^c} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} \int_{-\delta_2\sqrt{n}}^{\delta_2\sqrt{n}} e^{-Q(s,t)} \left(p_0(s,t) + \frac{p_1(s,t)}{n} + \frac{p_2(s,t)}{n^2} + \cdots + \mathcal{O}\left(\frac{1}{n^m}\right)\right) \mathrm{d}t\mathrm{d}s, \tag{4.21}$$

where $Q(s,t)$ is some (positive definite) quadratic form and the $p_j(s,t)$ are polynomials. Utilizing the error estimate in Lemma A.1 in Appendix A, we can see that instead of (4.21) we can consider the integral

$$\frac{\gamma^n}{n^c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(s,t)} \left(p_0(s,t) + \frac{p_1(s,t)}{n} + \cdots + \mathcal{O}\left(\frac{1}{n^m}\right)\right) \mathrm{d}t\mathrm{d}s, \tag{4.22}$$

since the difference between the two is exponentially small in n. Note that the coordinate transform above also allows us to compute this kind of integrals by splitting them up into Gaussian integrals and then proceeding with integration by parts, see e.g. [118].

Consequently, all we need to do is to consider the (4.22) for all saddle points with maximum absolute value (of which there are finitely many), and by computing all the expressions up to a fixed $p_j(s,t)$, we will then have obtained the asymptotics of (4.19) and at the same time shown (4.1).

In the following, we will proceed in two steps: first, we pick a dominant saddle point and show that, locally around this point, everything works out smoothly. Then, we pick any associated saddle point and show that, up to powers of roots of unity, nothing changes from the first step.

1. Suppose (x_0, y_0) is a dominant saddle point, let $\gamma := S(x_0, y_0)$ and fix a $m \in \mathbb{N}$. First, we show that $0 = |N(x_0, y_0)| < \infty$. This is due to (x_0, y_0) being a saddle point: we have $\frac{\partial S}{\partial y}(x_0, y_0) = 0$ and thus y_0 is the unique solution to $S(x_0, \cdot) = S(x_0, y_0)$. Therefore, $\phi(y_0) = y_0$, and in the same manner we can see $\psi(x_0) = x_0$. It follows immediately that the alternating orbit sum of xy evaluated at (x_0, y_0) is 0; in particular it is finite.
   Our next step is to show that we can rewrite the integrand (that is, the one in (4.19)) as in (4.22). To do so, we substitute $x \mapsto x_0 \exp \frac{is}{\sqrt{n}} =: e_s, y \mapsto y_0 \exp \frac{it}{\sqrt{n}} =: e_t$, and then separate the integral into three parts: $S(e_s, e_t)^n, N(e_s, e_t)$ and the denominator $1/e_s^{k+1} e_t^{k+1}$ (note that a power in the denominator vanishes due to the substitution rule).

   (a) Part 1: $S(e_s, e_t)^n$
   As (x_0, y_0) is a dominant saddle point of $S(x, y)$ and therefore $\frac{\partial S}{\partial x}(x_0, y_0) = \frac{\partial S}{\partial y}(x_0, y_0) = 0$, we have a Taylor expansion of $S(x, y)$ around (x_0, y_0) of the form

   $$S(x, y) = \gamma - u(x - x_0)^2 - v(x - x_0)(y - y_0) - w(y - y_0)^2 + \dots \tag{4.23}$$

   After our substitution, this gives us (note that e_s, e_t are functions of s, t and n)

   $$S(e_s, e_t) = \gamma - u\frac{s^2}{4n} - v\frac{st}{n} - w\frac{t^2}{4n} + A(s, t, n), \tag{4.24}$$

   with $A(s, t, n) = n^{-3/2} \sum_{j=0}^{2m-1} \frac{a_j(s,t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-3/2}\right)$, and the r_j homogeneous polynomials. We know that $\hat{Q}(s, t) := \left[\frac{u}{4}s^2 + vst + \frac{w}{4}t^2\right]$ is a positive definite quadratic form (seeing as our saddle point is a local minimizer of $S(x, y)$ in $\mathbb{R}^+ \times \mathbb{R}^+$). Consequently, we can write

   $$\begin{aligned} \log S(e_s, e_t) &= \log\left(\gamma - \left[\hat{Q}(s, t) - A(s, t, n)\right]\right) & (4.25)\\ &= \log \gamma - \underbrace{\frac{1}{\gamma}\hat{Q}(s, t)}_{=:Q(s,t)} + B(s, t, n), & (4.26) \end{aligned}$$

   with once again $B(s, t, n) = n^{-3/2} \sum_{j=0}^{2m-1} \frac{b_j(s,t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-3/2}\right)$, the b'_j homogeneous polynomials and $Q(s, t)$ a positive definite quadratic form. Consequently,

   $$S(e_s, e_t)^n = \exp\left[n \log S(e_s, e_t)\right] = \gamma^n \exp\left[-Q(s, t)\right] \exp\left[nB(s, t, n)\right]. \tag{4.27}$$

   Comparing this to (4.19), the first two factors are already precisely as we want them, and the last factor is of the form $\sum_{m \geq 1} \frac{q_m(s,t)}{n^{m/2}}$, with the $q_m(s, t)$ homogeneous of degree m.

   (b) Part 2: $N(e_s, e_t)$
   As we have seen that $|N(x_0, y_0)| < \infty$, and seeing as $N(x, y)$ is a rational function in x, y, it follows that we can write

   $$N(e_s, e_t) = \sum_{j=0}^{2m-1} \frac{d_j(s, t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-1}\right), \tag{4.28}$$

   with the $d_j(s, t)$ homogeneous polynomials of degree j.

(c) Part 3: $1/e_s^{k+1}e_t^{l+1}$
Lastly, we have

$$\frac{1}{e_s^{k+1}e_t^{l+1}} = x_0^{-k-1}y_0^{-l-1}\sum_{j\geq 0}\frac{1}{n^{j/2}}\frac{(-i[(k+1)s+(l+1)t])^j}{j!}. \tag{4.29}$$

Note in particular that this last factor is the only part which depends on the endpoint (k,l).

Multiplying the power series together and sorting them by powers of n, we obtain as a result the contribution of this saddle point to (4.22) of the form

$$\frac{\gamma^n}{n^c}\left[\sum_{p=0}^{m-1}\frac{1}{n^{p/2}}\frac{1}{x_0^{k+1}y_0^{l+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q(s,t)}q_p'(s,t,k,l)\mathrm{d}s\mathrm{d}t+\mathcal{O}\left(\frac{1}{n^{m-1/2}}\right)\right], \tag{4.30}$$

where the $q_{i,p}'(s,t,k,l)$ are polynomials in s,t,k,l. One can easily see, however, that $q_p'(s,t,k,l)$ is homogeneous of degree p; for odd p the double integral therefore vanishes by symmetry. Thus we can rewrite (4.30) as

$$\frac{\gamma}{n^c}\left[\sum_{p=0}^{m-1}\frac{1}{n^p}\frac{1}{x_0^{k+1}y_0^{l+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q(s,t)}q_p(s,t,k,l)\mathrm{d}s\mathrm{d}t+\mathcal{O}\left(\frac{1}{n^m}\right)\right], \tag{4.31}$$

with $q_p := q_{2p}'$. The factor $\frac{1}{n^c}$ in (4.30) stems from the fact that we obtain a factor of $1/n$ by the substitution rule, and that the integral might vanish for small values of p.

2. Suppose now that we have another saddle point $(x_i,y_i) = (\alpha_i x_0, \beta_i y_0)$ associated to (x_0,y_0), and pick ζ_i such that $S(x_i,y_i) = \zeta_i S(x_0,y_0)$ (notice that we then have $|\zeta_i|=1$, as discussed in Section 2.1.4). Our goal is now to describe the series expansion of the numerator around (x_i,y_i) using the one around (x_0,y_0). We substitute as before $x \mapsto e_s' := x_i \exp\frac{is}{\sqrt{n}}, y \mapsto e_t' := y_i \exp\frac{it}{\sqrt{n}}$. Due to Lemma 4.2, we can now conclude that the series representation (w.r.t. n at ∞) around $S(e_s',e_t')$ is the same as the one of $\zeta_i S(e_s,e_t)$, and the representation of $N(e_s',e_t')$ is the same as $\alpha_i\beta_i N(e_s,e_t)$. Lastly, the expansion of $1/e_s'^{(k+1)}e_t'^{(l+1)}$ clearly changes only by adding a factor of $\alpha_i^{k+1}\beta_i^{l+1}$ as well. Therefore, we can conclude that the contribution of the saddle point (x_i,y_i) is the same as the one of (x_0,y_0) up to a factor of $\zeta_i^n\alpha_i^{-k}\beta_i^{-k}$.

By Lemma 4.3, we deduce that the $v_p(k,l)$ are indeed γ-polyharmonic of degree p (note that for our dominant saddle point we have $\alpha=\beta=\zeta=1$).
Seeing as the only point in the construction where k,l appear is in (4.29), one finds that the $v_p(k,l)$ are, up to a factor of $x_0^{-k}y_0^{-l}$, bivariate polynomials in k,l. Due to Lemma 2.2, they are therefore polynomials if and only if the model has zero drift.
The fact that $c=\pi/\theta$ (with θ the arctangent of the correlation coefficient as defined in (2.1.2) follows from [51, Thm. 7]. □

The construction used in the proof allows us to give some further properties of the polyharmonic functions appearing in the asymptotic expansion.

Corollary 4.3.1. *The degree of the polynomial part of $v_p(k,l)$ (that is, without the factor of $x_0^{-k}y_0^{-l}$) is $c+2(p-1)$.*

Proof. By looking once again at the proof of Thm. 4.1 and in particular (4.29). From there, the statement follows immediately. □

Corollary 4.3.2. *For any orbit-summable model, we have* $\pi/\theta \in \mathbb{N}$.

Remarks:

- Thm. 4.1 holds true in higher dimensions as well, and indeed the proof translates directly. The only difference lies in the powers of n which appear: by the substitution rule $\frac{\mathrm{d}x_i}{\mathrm{d}s_i} = c_i e^{is_i/\sqrt{n}}$, one obtains an additional factor of $n^{-1/2}$. Thus, for even dimensions the constant c in (4.1) will be integer, whereas for odd dimensions it will be in $\frac{1}{2} + \mathbb{N}$. An example case for three dimensions is treated in App. B.2.

- When looking at models with large steps, the one thing that could go wrong is that the numerator might have a singularity at a saddle point. Usually, this seems not to be the case, and for a given model this condition is very easy to check. An example is treated in App. B.1.

- A table of the first three asymptotic terms for the 19 unweighted orbit-summable models is given in App. D.

4.1.2 Example: the Gouyou-Beauchamps model

In this section, we will illustrate the result of Thm. 4.1 by computing the asymptotics for the Gouyou-Beauchamps model, and in doing so find an explicit formula for the polyharmonic functions appearing therein.
The Gouyou-Beauchamps walk is defined by the step set $\{\nwarrow, \searrow, \leftarrow, \rightarrow\}$. Its step polynomial is $S(x,y) = \frac{y}{x} + \frac{x}{y} + \frac{1}{x} + x$, and solving $S_x = S_y = 0$ yields the four solutions $(x,y) = (\pm 1, \pm 1)$. We find that $S(1,1) = 4, S(-1,1) = -4$, $S(1,-1) = S(-1,-1) = 0$. Verifying that the second derivatives do not vanish, we therefore have the dominant saddle point $(1,1)$ and one other associated to it, namely $(-1,1)$, to consider. Note that the appearance of two saddle points is not at all surprising here, due to parity (or, in more general terms, periodicity) considerations: depending on the first coordinate, we can only hit a point after an even or odd number of steps. Therefore we can already expect at this point the asymptotics resulting from the saddle points to be precisely the same up to a factor of $(-1)^{k+n}$, which coincides with the statement of Thm. 4.1. Hence, we will only consider the dominant saddle point here.
The alternating orbit sum of xy can be checked to be

$$N(x,y) := -\frac{(-1+x^2)(-1+y)(x^4+y^3-x^2y-x^2y^2)}{x^3y^2}. \tag{4.32}$$

Out integrand is therefore of the form

$$\frac{S(x,y)^n N(x,y)}{x^{k+2}y^{l+2}},$$

where after letting $x \mapsto e_s := e^{is/\sqrt{n}}, y \mapsto e_t := e^{it/\sqrt{n}}$, by the substitution rule we will end up with

$$-\frac{1}{n}\frac{S(e_s,e_t)^n N(e_s,e_t)}{e_s^{k+1}e_t^{l+1}}.$$

The first factor of $-1/n$ is entirely harmless; we will therefore proceed to compute each of the factors in the second fraction separately.

1. Series representation of $S(e_s, e_t)^n$:
   First, we can compute (we let $m := \sqrt{n}$ for readability)

$$S(e_s, e_t) = 4 - \sum_{j \geq 2} \frac{a_j}{m^j} = 4 - \underbrace{\frac{s^2 + (s-t)^2}{n}}_{=:4Q(s,t)} + \underbrace{\sum_{j \geq 3} \frac{a_j}{m^j}}_{=:A(s,t,m)}, \tag{4.33}$$

   with

$$a_j := i^j \frac{s^j + (s-t)^j}{j!} (1 + (-1)^j).$$

   From here we obtain

$$\log [S(e_s, e_t)] = \log 4 - \frac{Q(s,t)}{n} + B(s,t,m), \tag{4.34}$$

   where $B(s,t,m) = \sum_{j \geq 3} \frac{b_j}{m^j}$, with

$$b_j = \sum_{i_1 + i_2 + \cdots + i_n = j} \frac{(-1/4)^n}{n} \prod_{p=1}^{n} a_{i_p},$$

   with the i_j positive integers.
   Finally, we can compute

$$\begin{aligned} S(e_s, e_t)^n &= \exp (n \log [S(e_s, e_t)]) &(4.35)\\ &= \exp (n \log [4 + B(s,t,m)] - Q(s,t)) &(4.36)\\ &= 4^n e^{-Q(s,t)} \underbrace{\exp [B(s,t,m)]}_{:=C(s,t,m)}, &(4.37) \end{aligned}$$

   where $C(s,t,m) = \sum_{k \geq 1} \frac{c_j}{j^k}$ with

$$c_j = \sum_{i_1 + \cdots + i_n = j} \frac{1}{n!} \prod_{p=1}^{n} b_{i_p},$$

   with the i_j again positive integers.

2. Series representation of $N(e_s, e_t)$:
   Using (4.32), we find that

$$N(e_s, e_t) = \sum_{j \geq 1} \frac{d_j}{m^j}, \tag{4.38}$$

   with

$$d_j := \frac{i^j}{j!} \left[-(-4)^j s^j + (2s - 3t)^j - 2^j (s-t)^j - (t - 2s)^j + (t - 4s)^j + (-2)^j (s+t)^j - (-3)^j t^j \right].$$

3. Series representation of $1/e_s^{k+1} e_t^{l+1}$:
   Lastly, we have

$$\frac{1}{e_s^{k+1} e_t^{l+1}} = \exp [-i(s(k+1) + t(l+1))/m] = \sum_{j \geq 0} \frac{1}{m^j} \underbrace{\frac{(-i((k+1)s + (l+1)t))^j}{j!}}_{=:f_j}.$$

Overall, we obtain as product of the three factors computed above

$$4^n e^{-Q(s,t)} \sum_{p\geq 0} \frac{1}{m^p} \underbrace{\left[\sum_{j_1+j_2+j_3=p} c_{j_1} d_{j_2} f_{j_3} \right]}_{=:q_p},$$

for j_1, j_2, j_3 nonnegative integers. In particular, we notice that q_p is homogeneous of degree p in s, t, and of degree p in k, l. In order to compute the contribution up to order $\mathcal{O}\left(\frac{1}{n^j}\right)$ of this saddle point to the asymptotics, all we need to do now is compute

$$4^n \frac{1}{4\pi^2 n} \sum_{p=0}^{2r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(s,t)} q_p(s,t,k,l) \mathrm{d}t\mathrm{d}s.$$

This gives us an explicit formula for the asymptotics of this model. In particular, we can check directly that all coefficients of $\frac{1}{m^k}$ for odd k vanish. By computing the integrals, we see that

$$\begin{aligned}
v_1(k,l) =& \frac{64}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l),\\
v_2(k,l) =& -\frac{32}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l)(35+2k^2+4k(2+l)+4l(3+l)),\\
v_3(k,l) =& \frac{8}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l)(25+2k^2+4k(2+l)\\
&+4l(3+l))(61+2k^2+4k(2+l)+4l(3+l)).
\end{aligned}$$

The harmonic function $v_1(k,l)$ was already computed in [13, 124].

4.1.3 Periodicity

Considering the combinatorial context, it is clear that in any asymptotic expansion as in (4.1), the discrete harmonic function $v_1(k,l)$ will always be positive. When looking at the computations in App. D, it appears as if there is an even stronger pattern; namely that $v_p(k,l)$ is positive for even p, and negative for odd p. It turns out, however, that this is not generally true; a counterexample is given for instance by computing enough terms in the expansion of the simple walk [35].
It is also a direct consequence of Thm. 4.1 that the number of saddle points is closely tied to the periodicity of the model. If we have a single saddle point, then clearly our model is aperiodic; but the number of saddle points also corresponds directly to the periodicity.

Lemma 4.4. *Suppose that our model is irreducible (that is, the step set $\mathcal{S}$ generates all of $\mathbb{Z}^2$), and that it is m-periodic. Then we have exactly $m-1$ saddle points $s_1, \ldots, s_{m-1}$ associated to our dominant saddle point $s_0 = (x_0, y_0)$. The corresponding ζ_i are – in some order – the m-th roots of unity.*

Proof. Let r be the number of saddle points. We use the representation (4.1) for the asymptotics of $q(k,l;n)$. We know that the $\alpha_i, \beta_i, \zeta_i$ are roots of unity. We can therefore pick k, l such that $\alpha_i^k = \beta_i^l = 1$ for all i. We then know, since our model is m-periodic, that there is a z, $0 \leq z \leq m-1$, such that we have

$$\chi(n) := 1 + \zeta_1^n + \cdots + \zeta_{r-1}^n = \begin{cases} 0 & n \not\equiv z \mod m, \\ m & n \equiv z \mod m. \end{cases} \tag{4.39}$$

Note that as value for $\chi(k \cdot m), k \in \mathbb{Z}$, we could pick any constant, because to compensate we can just multiply the corresponding polyharmonic functions in (4.1) with a constant factor. By definition, we know that χ is a character on $\mathbb{Z}/m\mathbb{Z}$. Therefore, it can be uniquely written as sum of irreducible characters [105], which in this case are all the m-th roots of unity. From this, and the fact that the ζ_i are pairwise different, it follows already that the ζ_i are (in some ordering) the m-th roots of unity. □

The regularity condition in Lemma 4.4 is necessary, as can be seen for the diagonal walk for instance. This walk is 2-periodic, but we still have 4 saddle points. The reason for the two extra saddle points is that there are some points the walk will never reach, which, heuristically speaking, translates to two additional conditions on k, l for which two saddle points then do not suffice to express them.

4.2 Quadrant walks with arbitrary starting point

4.2.1 Full asymptotic expansion

If we want to count walks starting from a point other than $(0,0)$, then the only thing that changes is that we have a different monomial $x^{k+1}y^{l+1}$ in the functional equation (2.2). It follows that we can proceed in exactly the same manner as before in order to obtain an expression as in (2.18), where only the sum on the right hand side changes. Hence, we can proceed in a the same manner as for Thm. 4.1, which allows us to recover the result of [51, Thm. 7], which states that the first order term in the asymptotics of the number of walks terminating at y and starting at x will be given – up to, once again, an exponential term and some power of n, and possible parity constraints – by the product of two functions; a harmonic function in x and a function in y which is adjoint harmonic. This is fairly natural, seeing as the underlying combinatorial problem is highly symmetrical: any path from x to y corresponds to a path from y to x with reversed steps. This, and a similar statement for the higher order terms, can be formalized in Thm. 4.5 below and the following Thm. 4.7.

Theorem 4.5. *Suppose that $\mathcal{S}$ is a step set satisfying the assumptions of Thm. 4.1. Then, with (x_0, y_0) being a dominant saddle point and (s_i) the associated ones such that $s_i = (x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$ and $S(x_i, y_i) = \zeta_i S(x_0, y_0)$, then there are a constant $c \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$ as well as functions $v_p(k, l, u, v)$ such that for any $m \in \mathbb{N}$ we have*

$$q((u,v),(k,l);n) = \frac{\gamma^n}{n^c}\left[\sum_{p=1}^{m-1} \frac{v_p(k,l,u,v)\sum_{i=1}^r \alpha_i^{u-k}\beta_i^{v-l}\zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right)\right]. \tag{4.40}$$

The $v_p(k, l, u, v)$ are polynomials precisely if the drift is zero. In this case they are of bidegree $c + 2(p-1)$ in both (k, l) and (u, v), and of total degree $2c + 4(p-1)$. Each $v_p(k, l, u, v)$ is multivariate polyharmonic of degree p. If the drift is not zero, then they have an additional factor of the form $x_0^{u-k} y_0^{v-l}$, where (x_0, y_0) is the dominant saddle point.

Before proving Thm. 4.5, we first show the following lemma, which is a natural extension of Lemma 4.3.

Lemma 4.6. *Suppose that $q(A, B; n)$ is a (combinatorial) quantity satisfying*

$$\sum_{s\in\mathcal{S}} \omega_s q(A-s, B; n-1) = q(A, B; n), \tag{4.41}$$

$$\sum_{s\in\mathcal{S}} \omega_s q(A, B+s; n-1) = q(A, B; n), \tag{4.42}$$

for all $A, B \in \mathbb{Z}_{\geq} \times \mathbb{Z}_{\geq 0}$, $n \geq 0$. Assume furthermore that $q(A, B; n)$ has an asymptotic representation of the form

$$q(A, B; n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^{l} v_{p,i}(A, B)\zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right) \right] \tag{4.43}$$

for all k, with the ζ_i pairwise different and of modulus 1. Then, each $v_{p,i}$ is multivariate polyharmonic of order p.

Proof. The proof works in the very same manner as the proof of Lemma 4.3; at each step we can choose whether to apply the identity (4.41) or (4.42), leading to an additional instance of $\triangle$ or $\tilde{\triangle}$ respectively. □

Proof (of Thm. 4.5). Analogous to the proof of Thm. 4.1. We will also use the same notation. The only difference lies in the term $N(x, y)$ in (4.19), which now depends on u, v and is now given by the orbit sum of $x^{u+1}y^{v+1}$ instead of xy. By the same argument as in aforementioned proof, we see that this orbit sum is also finite around a saddle point. In particular, seeing as the group consists of birational transformations, we can therefore expand $N(e_s, e_t)$ as a series with respect to n around infinity as in (4.28), where the numerators $d_j(s, t)$ are now polynomials in (u, v, s, t). By Lemma 4.2, we can also deduce that the resulting series representations of $N(e_s, e_t)$ are the same up to a factor of $\alpha_i^{u+1}\beta_i^{v+1}$ for the dominant and associated saddle points. It follows that each $v_p(k, l, u, v)$ is, up to a factor of $x_0^{u-k}y_0^{v-l}$, a polynomial in k, l, u, v. Next, we want to know the bidegree of this polynomial in (u, v) and in (k, l). To avoid doing so explicitly, we can make of the fact that in the same manner as in the proof of Thm. 4.1 we can conclude by Cor. 4.3.1 that the polynomial part $v_p(k, l, u, v)$ has bidegree $c+2(p-1)$ in k, l, where $c = \pi/\theta$. Denoting by $q(A, B; n)$ and $\tilde{q}(A, B; n)$ the number of paths from A to B in n steps with our chosen step set and its reverse respectively, we have $q((k, l), (u, v); n) = \tilde{q}((u, v), (k, l); n)$. Seeing as the parameter θ does not change when reversing the step set (see (2.11)), we can therefore conclude that $v_p(k, l, u, v)$ is of the same bidegree in (k, l) and (u, v).
Lastly, the polyharmonicity properties of the $v_p(k, l, u, v)$ are a direct consequence of Lemma 4.6. □

Remark:

- While the starting point (u, v) and the end point (k, l) of the walk end up playing a very similar role (see also Thm. 4.7), which is not at all surprising from a combinatorial point of view, this is not at all obvious from the proof: the role of (k, l) is very easily summarized as these coefficients only appear in the integrand as a factor of $x^{-k-1}y^{-l-1}$, the starting point does not appear directly as factor $x^{u+1}y^{v+1}$, but instead as its orbit sum. A priori, without the combinatorial interpretation, it does not seem to be obvious that both of these occurrences lead to a symmetrical role in the result.

In the following, we will want to describe the coefficients $v_p(k, l, u, v)$ appearing in Thm. 4.5 more precisely. We will consider the drift zero case, which is however not a real restriction as we can transform any other model via a zero drift one using the Cramer transformation discussed in Section 2.2.4. The goal will be to prove the following theorem:

Theorem 4.7. *The polynomials $v_p(k, l, u, v)$ in the zero drift case of Thm. 4.5 each have a representation of the form*

$$v_p(k, l, u, v) = \sum_{\substack{1 \leq i,j \leq p, \\ i+j \leq p+1}} a_{i,j} h_i^j(k, l) g_i^j(u, v), \tag{4.44}$$

where the $a_{i,j}$ are constants, the h_i^j are polyharmonic of degree (at most) i, and the g_i^j are adjoint polyharmonic of degree (at most) $p+1-i$.

In order to prove Thm. 4.7, it turns out to be very useful to have a polynomial basis of the space of polyharmonic functions for any given model. As the group is finite by assumption and $\pi/\theta \in \mathbb{Z}$ by Cor. 4.3.2, we can use the basis given in [114], which consists of sequences h_n^m of n-polyharmonic functions satisfying:

1. $\triangle h_{n+1}^m = h_n^m$,
2. the $h_n^m(k,l)$ are bivariate polynomials of increasing degree in both n and m: the degree will increase by 2 for each step in n, whereas it will increase by at least 2 and at most $c+1$ for each step in m.

Proof (of Thm. 4.7). Taking u, v as parameters, we can for each (u, v) write $v_p(k, l, u, v)$ as a sum of the basis functions $h_i^j(k,l)$, and obtain

$$\sum_{\substack{1\le i,j\le p,\\ i+j\le p+1}} h_i^j(k,l) g_i^j(u,v). \tag{4.45}$$

Since we know that $v_p(k, l, u, v)$ is a bivariate polynomial of bidegree $c + 2p - 1$ in both (k, l) and (u, v) (which is also where the conditions $i, j \le p$ and $i + j \le p + 1$ come from), the only thing we need to show is that g_i^j is adjoint polyharmonic of degree $p + 1 - i$ for any j. To do so, we utilize Lemma 4.6. First, consider $i = p$. We then have

$$\tilde{\triangle}\left(\triangle^{p-1} q_p(k,l,u,v)\right) = \tilde{\triangle} h_1^1(k,l) g_p^1(u,v) = h_1^1(k,l)\tilde{\triangle} g_p^1(u,v) = 0, \tag{4.46}$$

therefore $v_p^1(u, v)$ is adjoint harmonic. Seeing as the discrete Laplacians are linear, we can now proceed by induction, in each step applying the same argument as above to all terms which are polyharmonic (in (k, l)) of order j, i.e. the multiples of $h_j^1, \ldots, h_j^{p+1-j}$. The statement follows. $\square$

Remarks:

- Thm. 4.7 tells us in particular that we can write each coefficient v_p as a product of polyharmonic and adjoint polyharmonic functions, so that the degree of the former and latter adds up to at most $p+1$. This can be viewed as an extension of [51, Thm. 7], where an equivalent statement is shown for v_1 (albeit in a much more general setting).
- Using the fact that we know that bidegree of $v_p(k, l, u, v)$ as polynomials in k, l and u, v is $\pi/\theta + 2(p - 1)$ according to Thm. 4.5, we can deduce from Thm. 3.19 that the only base function h_p^j appearing in $v_p(k, l, u, v)$ can be h_p^1. From this it follows that $\triangle v_{p+1}(k, l, u, v) = v_p(k, l, u, v) + r_{p-1}(k, l, u, v)$, with r_{p-1} polyharmonic of degree at most $p - 1$.
- If the model is symmetric, then one can easily see that $a_{i,j} = a_{j,i}$ due to the symmetry of the underlying combinatorial problem ($q(A, B; n) = \tilde{q}(B, A; n)$, where $\tilde{q}$ denotes the paths with reversed steps). This holds true for an appropriate choice of basis in some other cases as well; for examples of this, see App. C.
- A priori it is not at all clear which elements of the basis h_i^j constructed in [114] actually appear in connection with some combinatorial problems, i.e. if this basis is combinatorially reasonably chosen. As we will see in Section 4.2.2 and App. D, this seems to be the case.
- We can in fact give an upper bound on the number of summands appearing in the decomposition (4.44). First of all, we know for all p that all (i, j) such that h_i^j appears in the decomposition are contained in the subset $\{(i, j) : i + j \le p + 1\}$. If one writes

the h_i^j as a table, this gives us a triangular shape of size increasing with p. Along the lines $i+j=k$, we will have functions of degree at least $2(k-1)$. We can now count the number of possible products of a given function h_i^j with a base function $\tilde{h}_m^n$ of the adjoint Laplacian. In order not to exceed the maximum degree, we can multiply h_1^1 with the entire triangle of adjoint base functions. For h_1^2 and h_2^1, we cannot multiply them with any $\tilde{h}_m^n$ with $m+n=p+1$, and so on. All in all, this gives us a maximum of $\frac{p(p+1)(p+2)(p+3)}{24}$ summands. This maximum is achieved e.g. for the simple walk for $v_{1,2,3}$. Generally, the larger the value of π/θ, the less summands we will have (since the degrees of the h_i^j increase more quickly).

In the following, we will see what this decomposition looks like in case of the simple walk. The Gouyou-Beauchamps model and the tandem walk are treated in App. C.

4.2.2 Example: the simple walk

In the following, to keep the expressions a bit shorter, we will give the expressions as after the substitution $k\mapsto k-1, l\mapsto l-1$ etc. (i.e. we have kl instead of $(k+1)(l+1)$). This corresponds to a shift of the quarter plane, where instead of $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ we now consider $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$. For the simple walk, with $\mathcal{S}=\{\rightarrow,\downarrow,\leftarrow,\uparrow\}$, we then have (after rescaling by multiplicative constants)

$$\begin{aligned}
h_1^1(k,l) =& kl,\\
h_1^2(k,l) =& kl(k-l)(k+l),\\
h_1^3(k,l) =& kl(14-5k^2+3k^4-5l^2-10k^2l^2+3l^4),\\
h_2^1(k,l) =& kl(l-1)(l+1),\\
h_2^2(k,l) =& kl(l-1)(l+1)(7+5k^2-3l^2),\\
h_3^1(k,l) =& kl(l-2)(l-1)(l+1)(l+2).
\end{aligned}$$

By symmetry, we can pick the base functions $\tilde{h}_i^j(u,v)=h_i^j(u,v)$ for the adjoint Laplacian. For the first three asymptotic terms with arbitrary starting and ending points, we obtain (again up to multiplicative constants)

$$\begin{aligned}
v_1(k,l,u,v) =& kluv,\\
v_2(k,l,u,v) =& kluv(7+2k^2+2l^2+2u^2+2v^2),\\
v_3(k,l,u,v) =& kluv(167+140k^2+12k^4+140l^2+24k^2l^2+12l^4+140u^2+40k^2u^2\\
&+24l^2u^2+12u^4+140v^2+24k^2v^2+40l^2v^2+24u^2v^2+12v^4).
\end{aligned}$$

One can check that Cor. 4.7 takes the form

$$\begin{aligned}
v_1 =& h_1^1\tilde{h}_1^1,\\
v_2 =& 4\left(h_2^1\tilde{h}_1^1+h_1^1\tilde{h}_2^1\right)+2\left(h_1^2\tilde{h}_1^1+\tilde{h}_2^1h_1^1\right)+15h_1^1\tilde{h}_1^1,\\
v_3 =& \frac{192}{5}\left(h_3^1\tilde{h}_1^1+\tilde{h}_3^1h_1^1\right)+\frac{64}{5}\left(h_2^2\tilde{h}_1^1+\tilde{h}_2^2h_1^1\right)+4\left(h_1^3\tilde{h}_1^1+\tilde{h}_1^3h_1^1\right)+64\left(h_2^1\tilde{h}_1^2+\tilde{h}_2^1h_1^2\right)\\
&+128h_2^1\tilde{h}_2^1+24h_1^2\tilde{h}_1^2+576\left(h_2^1\tilde{h}_1^1+\tilde{h}_2^1h_1^1\right)+288\left(h_1^2\tilde{h}_1^1+\tilde{h}_1^2h_1^1\right)+951h_1^1\tilde{h}_1^1.
\end{aligned}$$

As the degree of h_i^j is truly increasing by only 2 whenever we increase either i or j by one, it turns out that we have indeed $1,5$ and 15 different summands respectively. Due to the symmetry of this model, the second and third equations can be simplified a bit: for v_2, letting $g_2:=4h_2^1+2h_1^2$ (clearly, g_2 is then biharmonic) gives us

$$v_2=g_2\tilde{h}_1^1+h_1^1\tilde{g}_2+15h_1^1\tilde{h}_1^1.$$

For v_3, letting in the same manner $g_3 := \frac{192}{5}h_3^1 + \frac{64}{5}h_2^2 + 4h_1^3$, we have

$$g_3\tilde{h}_1^1 + \tilde{g}_3 h_1^1 + 144\left(g_2\tilde{h}_1^1 + \tilde{g}_2 h_1^1\right) + 64\left(h_2^1\tilde{h}_1^2 + \tilde{h}_2^1 h_1^2\right) + 951 h_1^1\tilde{h}_1^1.$$

While one can view the definition of $g_{2,3}$ as purely a crutch to make the resulting expressions shorter, they do in fact give in a sense a natural decomposition of the v_i: g_2 consists of the highest order terms in v_2, while g_3 consists of those in v_3.

Remarks:

- By the above, when taking a scaling limit, then we have

$$\lim_{\mu\to 0} m^\alpha v_i\left(\frac{x}{m}, \frac{y}{m}\right) = \lim_{\mu\to 0} m^\alpha g_i\left(\frac{x}{m}, \frac{y}{m}\right),$$

  where α is an appropriate scaling constant, which means that the g_i already give us all the terms which will not vanish in this kind of limit.

- As described in Section 2.2.3, we know that the continuous heat kernel $p_t(x, y, u, v)$ of a Brownian motion with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

  with $\sigma_{11} = \mathbb{E}[X^2], \sigma_{12} = \mathbb{E}[XY], \sigma_{22} = \mathbb{E}[Y^2]$ (the scaling limit of this model) allows for an asymptotic representation of the form

$$p_t(x, y, u, v) = \frac{1}{t^2}\sum_{k\geq 1}\frac{f_k(x, y, u, v)}{t^k}. \tag{4.47}$$

  Seeing as this representation looks almost the same as the one for the discrete case in Thm. 4.5, it is natural to compare the functions v_p to their continuous counterparts f_p. For v_1, for instance, by [51] we know that we will have (after appropriate scaling by a constant) $v_1(k, l, u, v) \to f_1(k, l, u, v)$. However, this is not at all clear for $p > 1$. For the simple walk, one can check that $v_2 \to f_2$, but this fails for $p = 3$: we have

$$\begin{aligned}&f_3(k, l, u, v)\\ &= kluv(3k^4 + 6k^2l^2 + 3l^4 + 22k^2u^2 + 6l^2u^2 + 3u^4 + 6k^2v^2 + 22l^2v^2 + 6u^2v^2 + 3v^4),\end{aligned}$$

  whereas the scaling limit of $v_3(k, l, u, v)$ turns out to be

$$kluv(3k^4 + 6k^2l^2 + 3l^4 + 10k^2u^2 + 6l^2u^2 + 3u^4 + 6k^2v^2 + 10l^2v^2 + 6u^2v^2 + 3v^4),$$

  where the coefficients of k^2u^2 and l^2v^2 do not match.

Chapter 5

Logarithmic terms in discrete heat kernel expansions in the quadrant

> The combinatory analysis in my opinion holds the ground between the theory of numbers and algebra, and is the proper passage between the realms of discontinuous and continuous quantity
>
> Percy A. MacMahon

Among the 79 quadrant walk models studied in [31], 13 have the property of admitting a so-called decoupling function, see [12]. The existence of such a function allows us to express their path counting function using Jacobi ϑ-functions as defined in (2.19) (see also [12, 61, 63, 62]). This in turn will enable us to extract properties about their asymptotic behaviour.

While this approach would work for all these decoupled models, for brevity we choose to focus in the present work on the two examples of Figure 5.1. See Propositions 5.1 and 5.6 for instances of such expressions in terms of theta functions.

The main novelty in our complete asymptotics is the presence of logarithmic terms, see for instance (5.27). Such terms did not appear yet in the lattice walk literature, nor do they appear in the continuous setting, when deriving complete asymptotic expansions of the continuous heat kernel [36].

The structure of this chapter is as follows:

- In Section 5.1, the approach will be illustrated using the example of the Kreweras model. This allows us to recover results in [74, 30, 28, 31].
- In Section 5.2, we will compute a series expansion of the generating function $Q(x, 0; t)$ around the critical point t_c. In particular, we will see that this expansion contains polynomial as well as logarithmic terms.
- In Section 5.3, we will show in Thm. 5.9 how the dependency on the endpoint translates to polyharmonic functions. In particular, we will see that due to the specific form of the expansion, we will get a stronger statement as implied by Lemma 1.3.

This chapter is largely based on [64], which is joint work with Andrew Elvey-Price and Kilian Raschel.

Figure 5.1: The models considered in this work: on the left, the algebraic (unweighted) Kreweras model will serve as an example for our approach (Section 5.1); on the right, a more generic model with associated infinite reflection group (Section 5.2). In the second model, we allow a weight a for the jump $(1,1)$, our motivation being to show how our results depend on this parameter, in particular in the limit $a \to 0$, at which the model degenerates into the simple walk (to the four nearest neighbors).

5.1 An algebraic example: the Kreweras model

5.1.1 Definition of the model

By definition, the Kreweras model corresponds to the step set $\{\leftarrow, \downarrow, \nearrow\}$, and the kernel

$$K(x,y) = xy - txy\left(xy + \frac{1}{x} + \frac{1}{y}\right).$$

The functional equation (2.2) for the path generating function (2.2) takes the form

$$K(x,y)Q(x,y) = xy - txQ(x,0) - tyQ(0,y). \tag{5.1}$$

It is known that the generating function of this model admits the following expansion:

$$Q(x,0) = \frac{1}{xt}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}\right), \tag{5.2}$$

with W being the unique power series in t solution to $W = t(2 + W^3)$, see [30, 28, 31].

5.1.2 Parametrization of the kernel curve

We want to find a parametrization of the elliptic curve $\mathcal{C}$ as defined in (2.8), meaning we want to find functions $X(z)$ and $Y(z)$ meromorphic on $\mathbb{C}$ such that

$$\mathcal{C} = \{(X(z), Y(z)) : z \in \overline{\mathbb{C}}\}. \tag{5.3}$$

Such a parametrisation has been obtained in [70, 61]. To state the result of [61], recall the Jacobi theta function $\vartheta(z)$ introduced in Section 2.1.5.

First, define $\tau \in i\mathbb{R}_+$ in terms of $t \in (0, \frac{1}{3})$ as follows:

$$t = e^{-i\gamma/3}\frac{\vartheta'(0|\tau)}{4i\vartheta(\gamma|\tau) + 6\vartheta'(\gamma|\tau)}, \tag{5.4}$$

with $\gamma = \pi\frac{\tau}{3}$. The fact that τ is defined in terms of t using an equation as (5.4) will be shown in higher generality in Lemma 5.5. Then setting

$$\begin{cases} X(z) &= e^{-4i\gamma/3}\dfrac{\vartheta(z|\tau)\vartheta(z-\gamma|\tau)}{\vartheta(z+\gamma|\tau)\vartheta(z-2\gamma|\tau)}, \\ Y(z) &= X(z+\gamma), \end{cases} \tag{5.5}$$

Lemma 3 in [61] asserts that Equation (5.3) holds. In particular, we know that $X(z)$ has its only (double) pole at $z = -\gamma$ and $Y(z)$ at $z = \gamma$, see (2.21).

We can use (5.4) to write t as a series in q, and as a consequence find an inverse, giving us

$$q = t^{9/2} + \frac{45}{2}t^{15/2} + \frac{4023}{8}t^{21/2} + \frac{184341}{16}t^{27/2} + \mathcal{O}\left(t^{27/2}\right). \tag{5.6}$$

Notice that in [61], the equivalent of (5.6) is not exactly the same due to a slightly different choice in parametrization: what is q here corresponds to $q^{3/2}$ in [61].

5.1.3 Explicit expression for the generating function

In order to obtain an expression for $Q(x, y)$, we will first find $Q(x, 0)$ explicitly, and then make use of the functional equation (5.1). To that purpose, we will use an approach utilizing an invariant for this model [12, 61, 63, 62] and recall the proof of the following:

Proposition 5.1 ([61]). *We have*

$$tX(z)Q(X(z), 0) = \frac{1}{2t} - \frac{1}{X(z)} - J(z), \tag{5.7}$$

where (with $\gamma = \pi\frac{\tau}{3}$*)*

$$J(z) = -e^{-4i\gamma/3}\frac{\vartheta(2\gamma|\tau)\vartheta'(0|\frac{\tau}{3})}{\vartheta(\pi/2|\frac{\tau}{3})\vartheta'(0|\tau)}\frac{\vartheta(z+\pi/2|\frac{\tau}{3})}{\vartheta(z|\frac{\tau}{3})}. \tag{5.8}$$

Proof. We make use of the fact that:

$$X(z)Y(z) + \frac{1}{X(z)} + \frac{1}{Y(z)} = \frac{1}{t}, \tag{5.9}$$

$$tX(z)Q(X(z), 0) + tY(z)Q(0, Y(z)) = X(z)Y(z). \tag{5.10}$$

Here, (5.9) is due to the fact that $(X(z), Y(z))$ parametrises the kernel curve $\mathcal{C}$, see (5.3), and (5.10) is a reformulation of the functional equation (5.1). Letting

$$A(x) := \frac{1}{x} + xtQ(x, 0) \quad \text{and} \quad B(y) := \frac{1}{y} + ytQ(0, y), \tag{5.11}$$

we find that

$$J(z) := \frac{1}{2t} - A(X(z)) = -\frac{1}{2t} + B(Y(z)). \tag{5.12}$$

By symmetry of our model we could in fact conclude that in our case we have $A(x) = B(x)$, and hence $J(-z) = -J(z)$, but this is not necessary for our approach.

Knowing that $J(z)$ can be expressed as a function of $X(z)$ and as a function of $Y(z)$ at the same time, it must have a range of symmetry properties:

- $J(\gamma - z) = J(z)$, because $J(z)$ is a function of $X(z)$;
- $J(-\gamma - z) = J(z)$, because $J(z)$ is a function of $Y(z)$;
- $J(2\gamma + z) = J(z)$ by the above;
- $J(z + \pi) = J(z)$ is inherited from this property of ϑ functions, see (2.21).

In particular, we know that J is doubly periodic with periods π and 2γ. We thus construct as a candidate for $J(z)$ a function which has simple poles at $\{k\pi + \ell\gamma\colon\ (k,\ell) \in \mathbb{Z}^2\}$. This way we obtain (5.8), by scaling with an appropriate constant and verifying that all poles of the difference between (5.8) and (5.12) vanish. Using this together with (5.11) and (5.12), we obtain (5.7). $\square$

From Prop. 5.1 we can deduce the representation (5.2) of $Q(x,0)$ in the following manner: Defining $W := X(\pi/2) = Y(\pi/2)$, the equation $W = t(2+W^3)$ follows from substituting $z = \frac{\pi}{2}$ into (5.9). Next, defining

$$U(z) := \frac{\vartheta(\gamma|\tau)}{\vartheta(\frac{\pi}{2}+\gamma|\tau)}\frac{\vartheta(z+\frac{\pi}{2}+\gamma|\tau)}{\vartheta(z+\gamma|\tau)},$$

we have the equations

$$\frac{J(z)}{U(z)} + \frac{1}{X(z)} = \frac{1}{W} \quad \text{and} \quad \frac{U(z)^2-1}{X(z)} = -W^2,$$

as in both cases the left-hand side is an elliptic function with no poles, where the constant value can be determined explicitly by setting z to $\frac{\pi}{2}$ or $\frac{\pi}{2}-\gamma$. Combining these three equations yields (5.2).

To keep computations short, in this section we will only give a representation for $Q(0,0)$ rather than all of $Q(x,y)$. Conveniently, $X(0) = 0$, so it suffices to take the $z \to 0$ limit of our expression for $Q(X(z),0)$. Expanding both sides of (5.7) as series in z then yields

$$Q(0,0) = \frac{q^{2/3}\vartheta(\gamma|\tau)^2}{\vartheta'(0|\tau)^2}\times \\ \left[\left(2+\frac{\vartheta''\left(\frac{\pi}{2}|\frac{\tau}{3}\right)}{2\vartheta\left(\frac{\pi}{2}|\frac{\tau}{3}\right)} - \frac{\vartheta'''(0|\frac{\tau}{3})}{6\vartheta'(0|\frac{\tau}{3})} + \frac{\vartheta'''(0|\tau)}{6\vartheta'(0|\tau)}\right) - \frac{6i\vartheta'(\gamma|\tau)-\frac{1}{2}\vartheta''(\gamma|\tau)}{\vartheta(\gamma|\tau)} - \frac{4\vartheta'(\gamma|\tau)^2}{\vartheta(\gamma|\tau)^2}\right]. \quad (5.13)$$

Rewriting this as a series in q and making use of (5.6), we obtain the generating function of Kreweras excursions (see A006335 in the OEIS)

$$Q(0,0) = 1 + 2t^3 + 16t^6 + 192t^9 + \dots$$

5.1.4 Effect of the Jacobi transformation

As previously discussed, the Jacobi transformation (2.23) involves a parameter $\hat{q}$, related to q by $\log(q)\log(\hat{q}) = \pi^2$. Using (2.23) in (5.4), we find that

$$t = \frac{\vartheta'(0,\hat{q})}{6\vartheta'(\frac{\pi}{3},\hat{q})}.$$

This in turn can be used to express $\hat{q}$ as a series in t around the critical point $\frac{1}{3}$ (the property that $\hat{q} = 0$ actually corresponds to the critical value of t turns out to be a general fact and will be proven in Lemma 5.7, see also the beginning of Section 5.2.4), which starts

$$\hat{q} = \frac{1}{\sqrt{3}}\left(\frac{1}{3}-t\right)^{1/2} - \frac{1}{\sqrt{3}}\left(\frac{1}{3}-t\right)^{3/2} + \frac{\sqrt{3}}{2}\left(\frac{1}{3}-t\right)^{5/2} - \frac{70}{27\sqrt{3}}\left(\frac{1}{3}-t\right)^{7/2} + \dots \quad (5.14)$$

Next, we use the Jacobi identity (2.23) in order to rewrite (5.13) in terms of $\hat{q}$, which, after some simplifications, yields

$$
\begin{aligned}
Q(0,0) = \frac{\vartheta\left(\frac{\pi}{3},\hat{q}\right)}{\vartheta'(0,\hat{q})^2}\Bigg[&-\frac{9\vartheta\left(\frac{2\pi}{3},\hat{q}\right)}{2} - \frac{\vartheta''\left(\frac{\pi}{3},\hat{q}\right)}{2} + \frac{\vartheta\left(\frac{\pi}{3},\hat{q}\right)\vartheta'''(0,\hat{q})}{6\vartheta'(0,\hat{q})} - \frac{4\vartheta'\left(\frac{\pi}{3},\hat{q}\right)}{\vartheta\left(\frac{\pi}{3},\hat{q}\right)} \\
&- \frac{3\vartheta\left(\frac{2}{3},\hat{q}\right)\vartheta'''(0,\hat{q}^3)}{2\vartheta'(0,\hat{q}^3)} + \frac{9\vartheta\left(\frac{2\pi}{3},\hat{q}\right)}{\vartheta\left(\frac{3i}{2}\log\hat{q},\hat{q}^3\right)}\left(\frac{1}{2}\vartheta''\left(\frac{3i}{2}\log\hat{q},\hat{q}^3\right) - i\vartheta'\left(\frac{3i}{2}\log\hat{q},\hat{q}^3\right)\right)\Bigg]. \qquad (5.15)
\end{aligned}
$$

Note in particular that the terms of the form $\vartheta^{(k)}\left(\frac{3i}{2}\log\hat{q},\hat{q}^3\right)$ are not quite as unwieldy as they appear, since the dependency of ϑ (or its derivatives) on the first component is essentially given by trigonometric functions, and we have for instance

$$
\sin\left(\frac{3i}{2}\log\hat{q}\right) = \frac{1}{2i}\left(\hat{q}^{-3/2} - \hat{q}^{3/2}\right).
$$

That a simplification of this form works is to be expected, seeing as we already know this model to be algebraic, thus all logarithms must vanish [74, 30, 28, 31]. In terms of the computation, this relies heavily on the relation $\gamma = \pi\frac{\tau}{3}$ in the parametrisation (5.5). For the model we study in Sec. 5.2, there is no similar relation between γ and τ, and thus there is no intuitive reason why the logarithms should disappear.

5.1.5 Series expansion around the critical point

Lastly, all we need to do is substitute (5.14) into (5.15) in order to find a local expansion at the critical point $t = \frac{1}{3}$, starting

$$
Q(0,0) = \frac{9}{8} - \frac{27}{8}\left(\frac{1}{3} - t\right) - 9\sqrt{3}\left(\frac{1}{3} - t\right)^{3/2} + 72\sqrt{3}\left(\frac{1}{3} - t\right)^{5/2} \pm \dots
$$

One can easily verify that this coincides with the explicit expression for $Q(0,0)$ given in [74, 30, 28, 31].

5.2 An infinite group model

We follow the exact same exposition as in Section 5.1.

5.2.1 Definition of the model

We consider the problem of quadrant walks with N, E, S, W and NE steps with a weight $a > 0$ for each NE step, see Figure 5.1. The generating function for this model was shown to be D-algebraic by Bernardi, Bousquet-Mélou and Raschel [12]. As in Section 5.1, we start with the functional equation (1.13), which takes the form

$$
\frac{K(x,y)}{xyt}Q(x,y) = R(x,y),
$$

where $K(x,y)$ is the kernel as in (2.3), and the term

$$
R(x,y) = \frac{1}{t} - \frac{1}{x}Q(0,y) - \frac{1}{y}Q(x,0)
$$

is called the remainder. The benefit of writing the equation in this form is that we know that if $K(x,y)=0$ then we also have $R(x,y)=0$, as long as the series all converge (alternatively one could write x as a formal power series of y satisfying this equation, but we choose to take the analytic approach in this work). We note that for t sufficiently small (i.e., $|t|< 1/(a+4)$), the series $Q(x,y)$ converges in the domain where $|x|, |y|< 1$.

5.2.2 Parametrisation of the zero-set of the kernel

In this subsection, we deduce a parametrisation of the kernel curve using results from [62] (which follows [68, 70, 123, 54]) for general weighted step-sets, then specialising these results to our step-set. We consider the curve $\mathcal{C} = \{(x,y) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : K(x,y) = 0\}$ as in (2.8). Under the assumption that our step-set is non-singular and that

$$0 < t < \frac{1}{S(1,1)} = \frac{1}{a+4}, \tag{5.16}$$

we have the following lemmas (see [62, Lem. 2.3]):

Lemma 5.2. *There are meromorphic functions $X, Y : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ which parametrise $\mathcal{C}$, that is*

$$\mathcal{C} = \{(X(z), Y(z)) : z \in \mathbb{C}\},$$

and numbers $\gamma, \tau \in i\mathbb{R}$ with $\Im(\pi\tau) > \Im(2\gamma) > 0$ satisfying the following conditions:

- $K(X(z), Y(z)) = 0$;
- $X(z) = X(z+\pi) = X(z+\pi\tau) = X(-\gamma - z)$;
- $Y(z) = Y(z+\pi) = Y(z+\pi\tau) = Y(\gamma - z)$;
- $|X(-\frac{\gamma}{2})|, |Y(\frac{\gamma}{2})|< 1$;
- *Counting with multiplicity, the functions $X(z)$ and $Y(z)$ each contain two poles and two roots in each fundamental domain $\{z_c + r_1\pi + r_2\pi\tau : r_1, r_2 \in [0,1)\}$.*

The following is [62, Lem. 2.5], under the assumption (5.16).

Lemma 5.3. *The complex plane can be partitioned into simply connected regions $\{\Omega_s\}_{s\in\mathbb{Z}}$ as in Figure 5.2, satisfying*

$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s} \cup \Omega_{4s+1} = \{z \in \mathbb{C} : |Y(z)|< 1\},$$
$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s-2} \cup \Omega_{4s-1} = \{z \in \mathbb{C} : |Y(z)|\geq 1\},$$
$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s-1} \cup \Omega_{4s} = \{z \in \mathbb{C} : |X(z)|< 1\},$$
$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s+1} \cup \Omega_{4s+2} = \{z \in \mathbb{C} : |X(z)|\geq 1\}.$$

Moreover, the equations

$$\pi + \Omega_s = \Omega_s,$$
$$s\pi\tau + \gamma - \Omega_{2s} \cup \Omega_{2s+1} = \Omega_{2s} \cup \Omega_{2s+1} \supset \frac{s\pi\tau + \gamma}{2} + \mathbb{R},$$
$$s\pi\tau - \gamma - \Omega_{2s} \cup \Omega_{2s-1} = \Omega_{2s} \cup \Omega_{2s-1} \supset \frac{s\pi\tau - \gamma}{2} + \mathbb{R},$$

hold for each $s \in \mathbb{Z}$.

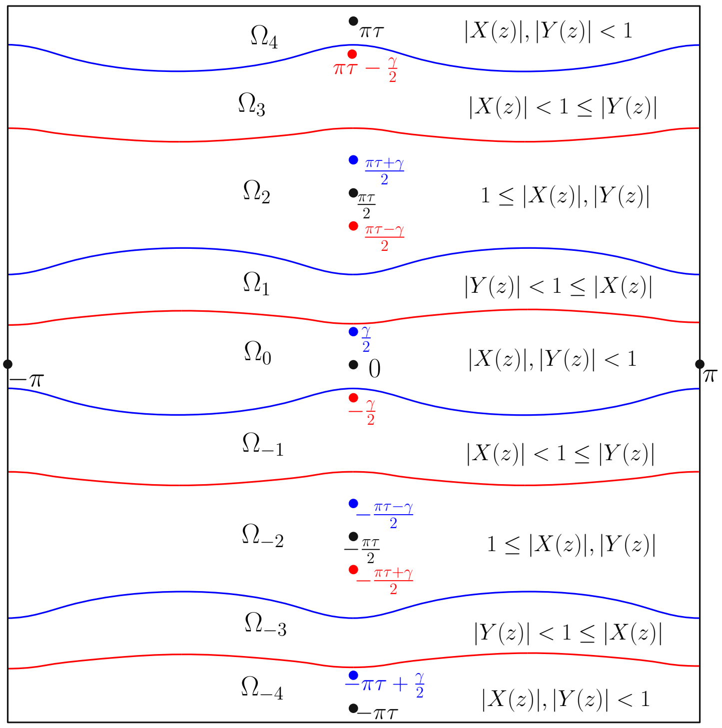


Figure 5.2: The complex plane partitioned into regions Ω_j. For z on the blue lines, $|Y(z)|= 1$, while on the red lines $|X(z)|= 1$.

The third result that we will need is [62, Prop. 2.6].

Proposition 5.4. *There are some $\alpha \in \Omega_0 \cup \Omega_{-1}$, $\beta \in \Omega_0 \cup \Omega_1$, $\delta \in \Omega_1 \cup \Omega_2$, $\epsilon \in \Omega_{-2} \cup \Omega_{-1}$ and $x_c, y_c \in \mathbb{C} \setminus \{0\}$ satisfying*

$$\begin{cases} X(z) &= x_c \dfrac{\vartheta(z-\alpha|\tau)\vartheta(z+\gamma+\alpha|\tau)}{\vartheta(z-\delta|\tau)\vartheta(z+\gamma+\delta|\tau)}, \\ Y(z) &= y_c \dfrac{\vartheta(z-\beta|\tau)\vartheta(z-\gamma+\beta|\tau)}{\vartheta(z-\epsilon|\tau)\vartheta(z-\gamma+\epsilon|\tau)}. \end{cases}$$

In the following result, we specialise the above parametrisation to our weighted step-set and thus describe the zero-set of the kernel, meaning the set $\mathcal{C}$ as introduced in (2.8):

Lemma 5.5. *Given a step-set as defined in Section 5.2.1, that is, with counting polynomial $S(x,y) = x + y + \frac{1}{x} + \frac{1}{y} + axy$, and assuming (5.16), there exists a triple $(c, \gamma, \tau) \in \mathbb{C}^3$ with $\gamma, \tau \in i\mathbb{R}$, $0 < \Im(2\gamma) < \Im(\pi\tau)$, satisfying the following equations:*

$$ac^3 = -\frac{\vartheta(4\gamma|\tau)}{\vartheta(2\gamma|\tau)}, \tag{5.17}$$

$$c^2 = \frac{\vartheta(3\gamma|\tau)}{\vartheta(\gamma|\tau)}, \tag{5.18}$$

$$\frac{c}{t} = 4\frac{\vartheta'(\gamma|\tau)\vartheta(2\gamma|\tau)}{\vartheta'(0|\tau)\vartheta(\gamma|\tau)} - 2\frac{\vartheta'(2\gamma|\tau)}{\vartheta'(0|\tau)}. \tag{5.19}$$

With the above notation, the functions $X(z)$ and $Y(z)$ given by

$$\begin{cases} X(z) &= c\dfrac{\vartheta(z|\tau)\vartheta(z+\gamma|\tau)}{\vartheta(z-\gamma|\tau)\vartheta(z+2\gamma|\tau)}, \\ Y(z) &= c\dfrac{\vartheta(z|\tau)\vartheta(z-\gamma|\tau)}{\vartheta(z+\gamma|\tau)\vartheta(z-2\gamma|\tau)} = X(-z), \end{cases}$$

satisfy the conditions of Lemmas 5.2 and 5.3.

Proof. From the definition of $X(z)$ and $Y(z)$ given in [62, App. A], the fact that the step-set is symmetric implies that $X(z) = Y(-z)$. Moreover, $\Omega_0 \cup \Omega_1$ is the connected component of $\{z \in \mathbb{C} : |Y(z)| < 1\}$ containing $\frac{\gamma}{2} + \mathbb{R}$, while $\Omega_0 \cup \Omega_{-1}$ is the connected component of $\{z \in \mathbb{C} : |X(z)| < 1\}$ containing $-\frac{\gamma}{2} + \mathbb{R}$. Combining with $X(z) = Y(-z)$, this implies that $-\Omega_0 \cup \Omega_1 = \Omega_{-1} \cup \Omega_0$. Considering the intersection $\Omega_0 = (\Omega_0 \cup \Omega_{-1}) \cap (\Omega_0 \cup \Omega_1)$, we see that $-\Omega_0 = \Omega_0$.

Now, by the definition (2.8) of $\mathcal{C}$, we have $(0,0), (\infty, 0), (0, \infty) \in \mathcal{C}$.

From Lemma 5.2, we know that $X(z)$ and $Y(z)$ each contain two roots and two poles in each fundamental domain. Consider the fundamental domain $F = \{z \in \Omega_{-1} \cup \Omega_0 \cup \Omega_1 \cup \Omega_2 : \Re(z) \in [0, \pi)\}$, and let $\alpha \in F$ and $\delta \in F$ be a root and pole of $X(z)$, respectively, which are both roots of $Y(z)$. From Lemma 5.3, we must have $\alpha \in \Omega_0$ and $\delta \in \Omega_1$. Since these are distinct, $Y(z)$ has no other roots in F, and so the complete set of roots of $Y(z)$ in $\Omega_0 \cup \Omega_1$ is $\alpha + \pi\mathbb{Z} \cup \delta + \pi\mathbb{Z}$. Note that $-\alpha \in \Omega_0$ is also a root of $Y(z)$, so we must have $-\alpha \in \alpha + \pi\mathbb{Z}$. In fact, since $\alpha \in F$, we have $\alpha = \frac{\pi}{2}$ or $\alpha = 0$. We will start by considering the case where $\alpha = 0$. Then 0 is a root of $Y(z)$, so, since $Y(\gamma - z) = Y(z)$, the value $\gamma \in \Omega_0 \cup \Omega_1$ is also a root of $Y(z)$. Since $\Re(\gamma) = 0$, we have $\gamma \in F$, so $\delta = \gamma$. Recall also that δ is a pole of $X(z)$, so $-\gamma = -\delta$ is a pole of $Y(z)$. This implies that the function $\tilde{Y}(z)$ defined by

$$\tilde{Y}(z) := Y(z)\frac{\vartheta(z+\gamma|\tau)\vartheta(z-2\gamma|\tau)}{\vartheta(z|\tau)\vartheta(z-\gamma|\tau)}$$

has at most a single pole in each fundamental domain, at the pole of $Y(z)$ other than $-\gamma$. But $\tilde{Y}(z)$ is an elliptic function with periods π and $\pi\tau$, so it cannot have only a single pole in each fundamental domain [1, p. 8]. Therefore it must have no poles, and is therefore a constant function.

Now write $\tilde{Y}(z) = c$ where $c \in \mathbb{C}$. Note $c \neq 0$ as $Y(z)$ is not the zero function. Then

$$Y(z) = c\frac{\vartheta(z|\tau)\vartheta(z-\gamma|\tau)}{\vartheta(z+\gamma|\tau)\vartheta(z-2\gamma|\tau)}$$

and

$$X(z) = Y(-z) = c\frac{\vartheta(z|\tau)\vartheta(z+\gamma|\tau)}{\vartheta(z-\gamma|\tau)\vartheta(z+2\gamma|\tau)}.$$

Equations (5.17), (5.18) and (5.19) follow from considering the equation $K(X(z), Y(z)) = 0$ at $z = 2\gamma$, γ and 0.

Finally we will discuss the case $\alpha = \frac{\pi}{2}$ instead of $\alpha = 0$. Then the functions $\hat{X}(z) := X(z + \pi/2)$ and $\hat{Y}(z) := Y(z + \pi/2)$ satisfy $\hat{X}(0) = \hat{Y}(0)$, so we can apply the rest of the proof to $\hat{X}(z)$ and $\hat{Y}(z)$, to show that they have the required properties. By carefully analysing the definition of $X(z)$ and $Y(z)$ in [62, App. A], one can prove that the $\alpha = \frac{\pi}{2}$ case actually never occurs; this is however unnecessary for the proof. □

Remark: the fact that $\frac{\vartheta(3\gamma|\tau)}{\vartheta(\gamma|\tau)} > 0 > \frac{\vartheta(4\gamma|\tau)}{\vartheta(2\gamma|\tau)}$, along with $\frac{\gamma}{\pi\tau} \in (0, \frac{1}{2})$ implies that $\frac{\gamma}{\pi\tau} \in (\frac{1}{4}, \frac{1}{3})$. Observe that the above uniformization is very similar to that (5.5) of Kreweras' model. Unlike in the latter case, however, where we had (after rescaling τ) $\alpha = 3\tau$, there is no obvious relation between γ and τ here.

5.2.3 Explicit expression for the generating function

As a consequence of this definition, we can define holomorphic functions $Q_1 : \Omega_{-1} \cup \Omega_0 \to \mathbb{C}$ and $Q_2 : \Omega_0 \cup \Omega_1 \to \mathbb{C}$ by

$$Q_1(z) = Q(X(z), 0) \qquad \text{and} \qquad Q_2(z) = Q(0, Y(z)).$$

Moreover, by symmetry, $Q_1(z) = Q_1(\gamma - z)$ and $Q_2(z) = Q_2(-\gamma - z)$.

The next step is to find a function of $X(z)$ which is equal to a function of $Y(z)$ for $z \in \Omega_0$ using the equations $K(X(z), Y(z)) = R(X(z), Y(z)) = 0$. Note that, given (1.13), this is equivalent to finding a decoupling function in the sense of [12, 4.2].

In this case, this is immediate as combining the two equations yields

$$atX(z)Q(X(z), 0) + atY(z)Q(0, Y(z)) = aX(z)Y(z) = -X(z) - Y(z) - \frac{1}{X(z)} - \frac{1}{Y(z)} + \frac{1}{t},$$

so we can now define a meromorphic function $J(z)$ on $\Omega_{-1} \cup \Omega_0 \cup \Omega_1$ by writing

$$J(z) := \begin{cases} \dfrac{1}{2t} - atX(z)Q_1(z) - X(z) - \dfrac{1}{X(z)} & \text{for } z \in \Omega_{-1} \cup \Omega_0, \\ -\left(\dfrac{1}{2t} - atY(z)Q_2(z) - Y(z) - \dfrac{1}{Y(z)}\right) & \text{for } z \in \Omega_0 \cup \Omega_1, \end{cases} \tag{5.20}$$

as these expressions are equal on Ω_0. Moreover, $J(z)$ satisfies $J(\gamma - z) = J(z) = J(z + \pi)$ for $z \in \Omega_{-1} \cup \Omega_0$ and $J(-\gamma - z) = J(z)$ for $z \in \Omega_0 \cup \Omega_1$, so $J(z - \gamma) = J(z + \gamma)$ for $z \in \Omega_0$. We can use this to extend $J(z)$ to a meromorphic function on $\mathbb{C}$ which satisfies

$$J(\gamma - z) = J(z), \quad J(-\gamma - z) = J(z) \quad \text{and} \quad J(z + \pi) = J(z).$$

This implies that J is doubly periodic, with periods π and 2γ, which will allow us to determine it exactly. We also note that by symmetry in x and y, we have $J(z) + J(-z) = 0$. Solving this exactly yields the following result, analogous to Proposition 5.1 for the Kreweras model:

Proposition 5.6. *Let c, γ, τ be defined as in Lemma 5.5. The function J in* (5.20) *is given by*

$$J(z) = -\frac{\vartheta'(0|\frac{\gamma}{\pi})\vartheta(2\gamma|\tau)\vartheta(z + \frac{\pi}{2}|\frac{\gamma}{\pi})}{c\vartheta(\frac{\pi}{2}|\frac{\gamma}{\pi})\vartheta'(0|\tau)\vartheta(z|\frac{\gamma}{\pi})}.$$

Proof. Define $I(z)$ by

$$I(z) := J(z) + \frac{\vartheta'(0|\frac{\gamma}{\pi})\vartheta(2\gamma|\tau)\vartheta(z + \frac{\pi}{2}|\frac{\gamma}{\pi})}{c\vartheta(\frac{\pi}{2}|\frac{\gamma}{\pi})\vartheta'(0|\tau)\vartheta(z|\frac{\gamma}{\pi})}.$$

Then it suffices to show that $I(z) = 0$. Using properties of ϑ (see (2.21)), we observe that $I(z)$ satisfies the same transformations as $J(z)$, namely $I(z) = -I(-z) = I(\gamma - z) = I(z + \pi)$. Moreover, by (5.20), the poles of $J(z)$ in $\Omega_{-1} \cup \Omega_0$ occur precisely at the points πn and $\gamma + \pi n$ for $n \in \mathbb{Z}$. By the definition, $I(z)$ has no other poles in $\Omega_{-1} \cup \Omega_0$, and taking $z \to 0$, we see that 0 is not a pole of $I(z)$. Hence, the transformations $I(z) = I(\gamma - z) = I(z + \pi)$ imply that $I(z)$ must be holomorphic on $\Omega_{-1} \cup \Omega_0$. Moreover, since $I(z) = -I(\gamma + z)$, this implies that $I(z)$ is holomorphic on $\mathbb{C}$, so it is a constant function. Finally we find that this constant k is 0 from $k = I(z) = -I(-z) = -k$. □

Proposition 5.6 combined with (5.20) gives an explicit expression for $Q(x, 0)$, starting from which we can extract the exact form of $Q(0, 0)$. For convenience we will write $\vartheta(z) = \vartheta(z|\tau)$ and $\tilde{\vartheta}(z) = \vartheta(z|\frac{\gamma}{\pi})$. Analysing $J(z)$ as $z \to 0$ yields the equation

$$1 + atQ(0, 0) = \frac{\vartheta(2\gamma)^2}{c^2\vartheta'(0)^2}\left(\frac{1}{2}\frac{\vartheta'(\gamma)\vartheta'(2\gamma)}{\vartheta(\gamma)\vartheta(2\gamma)} - 2\frac{\vartheta'(\gamma)^2}{\vartheta(\gamma)^2} - \frac{1}{2}\frac{\vartheta''(2\gamma)}{\vartheta(2\gamma)} + \frac{1}{6}\frac{\vartheta'''(0)}{\vartheta'(0)} + \frac{1}{2}\frac{\tilde{\vartheta}''(\frac{\pi}{2})}{\tilde{\vartheta}(\frac{\pi}{2})} - \frac{1}{6}\frac{\tilde{\vartheta}'''(0)}{\tilde{\vartheta}'(0)}\right).$$

We now describe how the coefficients of the series $Q(0,0)$ can be extracted from the solution above. Writing $s = e^{i\gamma}$, the right-hand side of the equation

$$a^2 = \frac{\vartheta(\gamma|\tau)^3\vartheta(4\gamma|\tau)^2}{\vartheta(2\gamma|\tau)^2\vartheta(3\gamma|\tau)^3} \tag{5.21}$$

is a series in q and s, while the left-hand side is constant. We can use this to write q as a series in s, which starts

$$q = a^{1/2}s^{7/2} + \left(\frac{1}{2\sqrt{a}} - \frac{3}{4}a^{3/2}\right)s^{9/2} + \mathcal{O}(s^{11/2}).$$

We can then write t (and c) as series in s using (5.17), (5.18) and (5.19). Consequently our expression for $Q(0,0)$ can be expanded as a series in s and therefore t.

5.2.4 Effect of the Jacobi transformation

Before analysing the Jacobi transformation (2.22), let us define the critical point of the model. As shown in [66, 69, 21, 51], the critical point $t_c > 0$ may be defined as the smallest positive singularity of the series $Q(0,0)$. It can be further characterized as the exponential growth of the coefficients of $Q(0,0)$, meaning that (up to a polynomial correction) $[t^n]Q(0,0) \sim 1/t_c^n$ (see (5.28) for a more precise statement). Finally, it is also the smallest value of $t > 0$ such that the Riemann surface $\mathcal{C}$ has genus 0.

What is essential for applying the Jacobi transformation is the fact that the critical point corresponds to $q = 1$ (and consequently $\hat{q} = 0$, whereas $\hat{q} = 1$, or $q = 0$, is equivalent to the regime $t = 0$). Additionally, we will also need to know that the only singularity of $Q(0,0)$ lies on the positive real axis. These two facts will be shown in the following Lemmas 5.7 and 5.8, which are true for more general models than considered here. They will be proven in Appendix A.

Lemma 5.7. *Writing q as a function of t, we have*

1. $0 < q(t) < 1$ *for* $0 < t < t_c$,
2. $q(t)$ *as a function of* t *is continuous on* $(0, t_c)$,
3. $\lim_{t\to 0} q(t) = 0$,
4. $\lim_{t\to t_c} q(t) = 1$.

Lemma 5.8. *Let $\mathcal{S}$ be a non-singular, weighted step-set, and let $Q(x,y)$ be the generating function* (2.1) *for walks in the quadrant using this step-set. Define the* period *of the model to be the maximum value k such that $Q(0,0) = Q(0,0;t) \in \mathbb{R}[t^k]$. If r is the radius of convergence of $Q(0,0)$, then the singularities of $Q(0,0)$ on the radius of convergence are precisely the points $re^{\frac{2\pi ij}{k}}$ for $j = 0, 1, \ldots, k-1$. The same result holds for the generating function $[x^a][y^b]Q(x,y)$ for any a, b for which this generating function is non-zero.*

Applying Lemma 5.8 to our case, since $a > 0$, the period k is 1, so the only singularity on the radius of convergence is on the positive real line. Moreover, Lemma 5.7 implies that q and τ have no singularity for t in the interval $[0, t_c)$. In addition, γ in Lemma 5.5 is analytic as well on $[0, t_c)$; this follows from the expression of γ in terms of two periods ω_1, ω_3 given in [62, App. A], and from the analytic behavior of these periods shown in [97, Sec. 7.4]. Together these imply that if $Q(0,0)$ has a singularity at t_c, then it is the unique singularity within the radius of convergence, so the asymptotic form of the coefficients is uniquely determined by the behaviour at this point. The same holds for all coefficients of $Q(x,0)$ in its series expansion at

$x = 0$. Again by Lemma 5.7, the point $t = t_c$ corresponds to $\hat{q} = 0$. For this reason we proceed by analysing the parameters at $\hat{q} = 0$.

It is convenient to parametrise a using the unique $k \in (0, 1)$ satisfying

$$a = \frac{1-k^2}{k^3}. \tag{5.22}$$

Writing $\beta = \frac{i}{\pi}\log(\hat{q})\gamma$, the equation (5.21) relating q and a becomes

$$a^2 = \frac{\vartheta(\beta,\hat{q})^3\vartheta(4\beta,\hat{q})^2}{\vartheta(2\beta,\hat{q})^2\vartheta(3\beta,\hat{q})^3}.$$

This equation allows $2\cos(2\beta)$ to be written as a series in $\hat{q}$, with initial terms

$$2\cos(2\beta) = k^2 - 1 - k^2(2k^2-1)(k^2+1)(k^2-3)(k^2-1)\hat{q}^2 + O(\hat{q}^4).$$

This allows us to write β itself as a series in $\hat{q}$:

$$\beta = \beta_0 + \beta_1\hat{q}^2 + \beta_2\hat{q}^4 + O(\hat{q}^6),$$

where the constant term β_0 is given by

$$\beta_0 = \frac{1}{2}\cos^{-1}\left(\frac{k^2-1}{2}\right), \tag{5.23}$$

while β_1 is given by

$$\beta_1 = \frac{1}{2}k^2(2k^2-1)(1-k^2)\sqrt{(1+k^2)(3-k^2)}. \tag{5.24}$$

It follows that t is also a series in $\hat{q}$, given by

$$\frac{1}{t} = -a - \frac{1}{a} - \frac{1}{a}\frac{\vartheta(\beta,\hat{q})^2\vartheta(5\beta,\hat{q})}{\vartheta(3\beta,\hat{q})^3}.$$

The initial terms are then

$$t = \frac{k}{k^2+3}\left(1 - \frac{(k^2+1)^2(3-k^2)^3}{k^2+3}\hat{q}^2\right) + O(\hat{q}^4).$$

In particular, the critical point t_c is given by

$$t_c = \frac{k}{k^2+3},$$

so t_c is given by an algebraic function of a. Taking the inverse of the series above yields

$$\hat{q}^2 = \frac{k^2+3}{(k^2+1)^2(3-k^2)^3}\left(1-\frac{t}{t_c}\right) + O\left(\left(1-\frac{t}{t_c}\right)^2\right).$$

Notice that the parameter β_0 is connected to another relevant parameter θ introduced in [51, 21]. Based on [51, Ex. 2], θ is computed in [21] to describe the asymptotic behavior of walks in two-dimensional cones. More precisely, let $S(x, y)$ be the step polynomial of the model as in (2.4). There exists a unique point $(x_0, y_0) \in (0, \infty)^2$ such that $\frac{\partial S}{\partial x}(x_0, y_0) = \frac{\partial S}{\partial y}(x_0, y_0) = 0$. Then the parameter θ can be defined as follows

$$\theta = \arccos\left(-\frac{\frac{\partial^2 S}{\partial x\partial y}(x_0,y_0)}{\sqrt{\frac{\partial^2 S}{\partial x^2}(x_0,y_0)\cdot\frac{\partial^2 S}{\partial y^2}(x_0,y_0)}}\right).$$

Standard computations give that $x_0 = y_0$ are both solutions to the equation $ax_0^3 = 1 - x_0^2$, hence with our notation (5.22), we have $x_0 = y_0 = k$. Accordingly, $\theta = \arccos(\frac{k^2-1}{2}) = 2\beta_0$.

5.2.5 Series expansion of $Q(x,0)$ at the critical point

In order to understand the asymptotics of the coefficients of $Q(x,0)$, we will write $Q(x,0)$ as a series in x and $\hat{q}$. Recall that an expression for $Q(x,0)$ was given parametrically by $J(z)$ and $X(z)$, see Proposition 5.6. It is important to notice that in the previously cited proposition, it is assumed that $t < \frac{1}{S(1,1)}$, see (5.16), while we now want to work with t close to $t_c > \frac{1}{S(1,1)}$. We observe that the identity in Proposition 5.6, giving an expression for the generating function in terms of theta functions, can be readily extended from $t \in (0, \frac{1}{S(1,1)})$ to $t \in (0, t_c)$ by analytic continuation, the two sides of the identities being actually analytic in that bigger domain.

Using then the Jacobi identity (2.23) on the expressions for $X(z)$ and $J(z)$ yields (see Lemma 5.5 and Proposition 5.6)

$$X(z) = d\frac{\vartheta(-z\hat{\tau},\hat{q})\vartheta(-\beta - z\hat{\tau},\hat{q})}{\vartheta(\beta - z\hat{\tau},\hat{q})\vartheta(-2\beta - z\hat{\tau},\hat{q})} \tag{5.25}$$

and

$$J(z) = -e^{\frac{i\pi\hat{\tau}z}{\beta}}\frac{\pi}{\beta d}\frac{\vartheta(2\beta,\hat{q})\vartheta(-\pi\hat{\tau}\frac{\pi+2z}{2\beta},\hat{q}^{\frac{\pi}{\beta}})\vartheta'(0,\hat{q}^{\frac{\pi}{\beta}})}{\vartheta'(0,\hat{q})\vartheta(-\pi\hat{\tau}\frac{\pi}{2\beta},\hat{q}^{\frac{\pi}{\beta}})\vartheta(-\pi\hat{\tau}\frac{z}{\beta},\hat{q}^{\frac{\pi}{\beta}})}, \tag{5.26}$$

where $d := c\exp(-\frac{4i\beta^2}{\pi\hat{\tau}})$ is determined by

$$d^2 = \frac{\vartheta(3\beta,\hat{q})}{\vartheta(\beta,\hat{q})},$$

and $d > 0$. The first few terms of d are

$$d = k - k^3(k^2-3)(k^2-1)(k^2+1)\hat{q}^2 + O(\hat{q}^4).$$

Using (5.25)–(5.26), we can expand $X(z)$ and $J(z)$ as series in $\mathbb{C}(e^{i\hat{\tau}z})[[\hat{q}]]$ and $\mathbb{C}(e^{\frac{i\pi\hat{\tau}z}{\beta}})[[\hat{q}^2,\hat{q}^{\frac{\pi}{\beta}}]]$, respectively. Writing $\hat{z} = z\hat{\tau}$, we can write $X(z)$ and $J(z)$ as series in $\hat{z}\mathbb{C}[[\hat{z},\hat{q}]]$ and $\frac{1}{\hat{z}}\mathbb{C}[[\hat{z},\hat{q}^2,\hat{q}^{\frac{\pi}{\beta}}]]$, respectively. Writing $u = \frac{\cos(\beta_0+2\hat{z})}{\cos(\beta_0)} - 1 \in \mathbb{C}[[\hat{z}]]$, these have initial terms

$$\begin{cases} X(z) &= \dfrac{ku}{u+3-k^2} + k\dfrac{3-k^2}{1+k^2}\left(\dfrac{-4\beta_1\sin(2\hat{z})}{(u+3-k^2)^2} + \dfrac{u(1+k^2)^2(1-k^2+k^4+u)}{u+3-k^2}\right)\hat{q}^2 + O(\hat{q}^4), \\ J(z) &= \dfrac{\pi}{\beta_0 k}\dfrac{\sin(2\beta_0)}{\sin(\frac{\pi\hat{z}}{\beta_0})} + J_1(\hat{z})\hat{q}^2 + \dfrac{4\pi}{\beta_0 k}\sin(2\beta_0)\sin\left(\dfrac{\pi\hat{z}}{\beta_0}\right)\hat{q}^{\frac{\pi}{\beta_0}} + O(\hat{q}^4), \end{cases}$$

where

$$J_1(\hat{z}) = \frac{\pi\sin(2\beta_0)}{\beta_0 k\sin(\frac{\pi\hat{z}}{\beta_0})}\left((1+k^2-k^4)(3-k^2)(1+k^2) + \frac{\pi\beta_1\hat{z}\cos(\frac{\pi\hat{z}}{\beta_0})}{\beta_0^2\sin(\frac{\pi\hat{z}}{\beta_0})} + \frac{2\beta_1\cos(2\beta_0)}{\sin(2\beta_0)} - \frac{\beta_1}{\beta_0}\right).$$

Taking the inverse of the first series, we can then write $\hat{z}$ as a series in $\mathbb{C}[\hat{q}^2, X(z)]$, which yields J as a series in $\frac{1}{X(z)}\mathbb{C}[[\hat{q}^2,\hat{q}^{\frac{\pi}{\beta}},X(z)]]$. Combining this with (5.20), this yields $Q(x,0)$ as a series in $\mathbb{C}[[\hat{q}^2,\hat{q}^{\frac{\pi}{\beta}},x]]$. Note, however, that β still depends on $\hat{q}$, so to complete our understanding of $Q(x,0)$ we will need to expand $\hat{q}^{\frac{\pi}{\beta}}$ as a series in $\hat{q}$. This is where logarithmic terms will appear, as $\hat{q}^{\frac{\pi}{\beta}} \in \hat{q}^{\frac{\pi}{\beta_0}}(1+\hat{q}^2\log(\hat{q})\mathbb{C}[[\hat{q},\hat{q}^2\log(\hat{q})]])$. In particular, using $\beta = \beta_0 + \beta_1\hat{q}^2 + \ldots$, we have

$$\hat{q}^{\frac{\pi}{\beta}} = \hat{q}^{\frac{\pi}{\beta_0}}\left(1 - \frac{2\pi\beta_1}{\beta_0^2}\hat{q}^2\log(\hat{q}) + \frac{2\pi^2\beta_1^2}{\beta_0^4}\hat{q}^4\log(\hat{q})^2 + \frac{2\pi(\beta_1^2-\beta_0\beta_2)}{\beta_0^3}\hat{q}^4\log(\hat{q}) + O(\hat{q}^{6-\epsilon})\right).$$

So, finally, $Q(x,0)$ is a series in $\mathbb{C}[[\hat{q}^2,x]]+\hat{q}^{\frac{\pi}{\beta_0}}\mathbb{C}[[\hat{q}^2\log(\hat{q}),\hat{q}^2,\hat{q}^{\frac{\pi}{\beta_0}},x]]$. Using the relation between $\hat{q}$ and t, we can write $\hat{q}^2$ as a series in $(t-t_c)\mathbb{C}[[(t-t_c)]]$. Hence $Q(x,0)$ is a series in

$$\mathbb{C}[[t-t_c,x]]+(t-t_c)^{\frac{\pi}{2\beta_0}}\mathbb{C}[[(t-t_c)\log(t-t_c),t-t_c,(t-t_c)^{\frac{\pi}{2\beta_0}},x]].$$

Explicitly, we can write this as:

$$Q(x,0)=A(x,1-t/t_c)+\sum_{k,\ell=0}^{\infty}\sum_{m=0}^{\ell}(1-t/t_c)^{\ell+(k+1)\frac{\pi}{2\beta_0}}\log(1-t/t_c)^m P_{k,\ell,m}(x), \tag{5.27}$$

where each $P_{k,\ell,m}(x)\in\mathbb{C}[[x]]$. The A part in (5.27) has no effect on the asymptotic expansion, this all comes from the series $P_{k,\ell,m}$. We note that $\beta_0\in(\frac{\pi}{4},\frac{\pi}{3})$, so $\frac{\pi}{2\beta_0}\in(3/2,2)$. So the leading terms in the asymptotic expansion are:

$$\begin{aligned}(1-t/t_c)^{\frac{\pi}{2\beta_0}}P_{0,0,0}(x)+(1-t/t_c)^{1+\frac{\pi}{2\beta_0}}P_{0,1,0}(x)+(1-t/t_c)^{1+\frac{\pi}{2\beta_0}}\log(1-t/t_c)P_{0,1,1}(x)\\ +(1-t/t_c)^{\frac{\pi}{\beta_0}}P_{1,0,0}(x).\end{aligned}$$

We can calculate these explicitly, for example the first term is given by

$$xP_{0,0,0}(x)=\frac{2\pi k(3+k^2)\sqrt{3+2k^2-k^4}}{\beta_0(1-k^2)}\left(\frac{3+k^2}{(3-k^2)^3(1+k^2)^2}\right)^{\frac{\pi}{2\beta_0}}\sin\left(\frac{\pi}{\beta_0}\hat{z}\right),$$

where $\hat{z}$ and x are related by

$$x=k\frac{\sin(\hat{z}-\beta_0)\sin(\hat{z})}{\sin(\hat{z}-2\beta_0)\sin(\hat{z}+\beta_0)}=\frac{ku}{u+3-k^2}.$$

It can be checked by a direct computation that the function $\sin\left(\frac{\pi}{\beta_0}\hat{z}\right)$ exactly corresponds to the generating function of the positive harmonic function for the model, as computed in [125].

The term associated to $P_{0,0,0}(x)$ determines the leading asymptotic behaviour of the coefficients:

$$[x^j][t^n]Q(x,0)\sim C\left([x^{j+1}]\sin\left(\frac{\pi}{\beta_0}\hat{z}\right)\right)n^{-1-\frac{\pi}{2\beta_0}}t_c^{-n} \tag{5.28}$$

for fixed j as $n\to\infty$, where C is a constant (only depending on a) given by

$$C=\frac{2\pi k(3+k^2)\sqrt{3+2k^2-k^4}}{\beta_0(1-k^2)\Gamma(-\frac{\pi}{2\beta_0})}\left(\frac{3+k^2}{(3-k^2)^3(1+k^2)^2}\right)^{\frac{\pi}{2\beta_0}}.$$

In order to verify that there really is a log term in this expansion, we also calculate $P_{0,1,1}(x)$ exactly. In fact this only differs from $P_{0,0,0}(x)$ by a constant multiple (dependent on a but not x):

$$xP_{0,1,1}(x)=\frac{\pi^2k^3(k^2+3)(1-2k^2)(3-k^2)(1+k^2)\left(\frac{3+k^2}{(3-k^2)^3(1+k^2)^2}\right)^{1+\frac{\pi}{2\beta_0}}}{2\beta_0^3}\sin\left(\frac{\pi}{\beta_0}\hat{z}\right).$$

The only value of k for which $P_{0,1,1}(x)=0$ is $k=\frac{1}{\sqrt{2}}$, corresponding to $a=\sqrt{2}$, $\beta_0=\frac{1}{2}\arccos(-\frac{1}{4})$ and $t_c=\frac{\sqrt{2}}{7}$, However we note that there are still log terms in the asymptotics in this case, for example $P_{0,1,2}(x)\neq 0$.

5.2.6 The limit $a \to 0$

A priori our results only apply for $a > 0$, and indeed this is necessary as some functions such as $P_{0,0,0}$ diverge for $a = 0$. Nonetheless, we see that the leading asymptotic expression (5.28) converges in a way that somewhat corresponds to the $a = 0$ case. Note that as $a \to 0$ we also have $k \to 1$, $\beta_0 \to \frac{\pi}{4}$ and $t_c \to \frac{1}{4}$. The limit of the constant C as $a \to 0$ is $\frac{4}{\pi}$. Moreover, we have

$$x = \frac{\sin(\hat{z} - \frac{\pi}{4})\sin(\hat{z})}{\sin(\hat{z} - \frac{\pi}{2})\sin(\hat{z} + \frac{\pi}{4})},$$

from which it follows that

$$\sin\left(\frac{\pi}{\beta_0}\hat{z}\right) = \sin(4\hat{z}) = \frac{4x}{(1-x)^2}.$$

So (5.28) would give

$$[x^j][t^n]Q(x,0) \sim \frac{16}{\pi}(j+1)n^{-3}4^n.$$

The only problem with this is that $[x^j][t^n]Q(x,0)$ is 0 when j and n have opposite parity, whereas for terms with the same parity the correct asymptotic formula is

$$[x^j][t^n]Q(x,0) \sim \frac{32}{\pi}(j+1)n^{-3}4^n.$$

In other words, this correctly yields the behaviour of $Q(x,0)$ around $t = t_c = \frac{1}{4}$, however there is a second critical point, $-\frac{1}{4}$ on the radius of convergence.

5.3 Polyharmonicity of coefficients

Due to (5.27) we already have a fair amount of information about the coefficients of $Q(x,0)$ at the critical point. One can now use this in order to describe the asymptotic behaviour of the (weighted) number of paths $q((0,0),(i,j);n) := q(i,j;n)$ in the quadrant from the origin to (i,j) with n steps, see (2.1). In particular, we will see in Lemma 5.11 that the dependence on the number of steps n is given in terms of a mix of powers of n and logarithms. In a similar fashion as in [113], one can then show that the dependence on the endpoint (i,j) is given in terms of so-called discrete polyharmonic functions (see [114, 113, 5, 125, 36]).

Given a step-set $\mathcal{S}$ with corresponding weights $(\omega_s)_{s\in\mathcal{S}}$, we define a discrete Laplacian operator $\triangle$ acting on functions $v : \mathbb{Z}^2 \to \mathbb{C}$ as follows:

$$\triangle v(A) = \sum_{s\in\mathcal{S}} \omega_s v(A+s) - tv(A).$$

We say that a function v is t-harmonic (resp. t-polyharmonic of order k) if for all points A in the quarter plane, $\triangle v(x) = 0$ (resp. $\triangle^k v(x) = 0$ for some positive integer k). Furthermore, in order to keep the notation compact in the following, let

$$\rho := \frac{\pi}{2\beta_0},$$

where β_0 is defined as in (5.23). As mentioned in the previous section, ρ varies (continuously) in $\left(\frac{3}{2}, 2\right)$ as a varies in $(0,\infty)$.

Our main objective in this section is to show the following result:

Theorem 5.9. *If $\rho \notin \mathbb{Q}$, then for any $p > 0$ (not necessarily integer), we have*

$$q(i,j;n) = t_c^{-n} \sum_{\substack{k\geq 1, \ell \geq m \geq 0 \\ k\rho+\ell+1<p}} v_{k,\ell,m}(i,j) \frac{(\log n)^m}{n^{k\rho+\ell+1}} + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right),$$

where the $v_{k,\ell,m}$ are discrete t_c-polyharmonic functions of order $\ell - m + 1$. If $\rho = \frac{u}{v} \in \mathbb{Q}$ with u and v coprime, then the same holds with the additional condition that the summation index k be at most v.

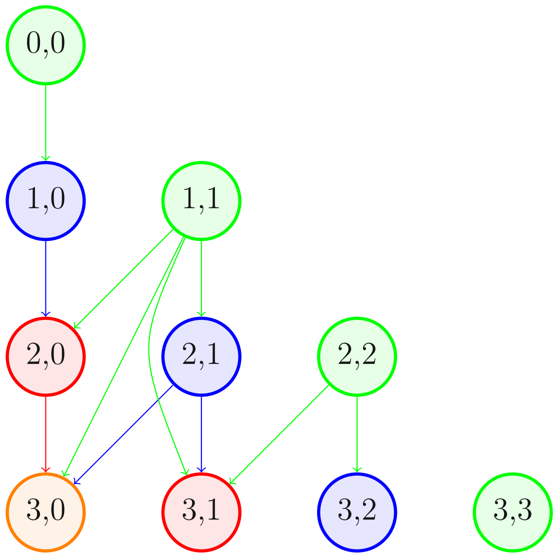


Figure 5.3: The arrows indicate a change in the coefficients of $f_{\ell,m} := \frac{(\log n)^m}{n^\ell}$ when substituting (5.30) into (5.33). We see that $f_{0,0}$, $f_{1,1}$, $f_{2,2}$ and $f_{3,3}$ (the green nodes) are not affected as we can only ever go down and to the left as indicated (see Lemma 5.10); thus their coefficients $v_{0,0}$, $v_{1,1}$, etc., are harmonic. Drawing the same diagram for $\triangle q(x;n)$, these nodes therefore disappear, and the unaffected coefficients are then those of the blue nodes. Thus, $v_{1,0}, v_{2,1}, \ldots,$ are biharmonic, and so on.

See Figure 5.3 for an illustration of Theorem 5.9 for fixed k, showing in particular the interdependency of the polyharmonic functions $v_{k,\ell,m}$. For irrational ρ we will have infinitely many such diagrams; whereas for rational ρ there will be only finitely many.

The rest of Section 5.3 is devoted to the proof of Theorem 5.9. In the following we will assume that $\rho \notin \mathbb{Q}$; otherwise we only need to bound the k in the summation indices by its denominator (as in Theorem 5.9). Using a standard transfer between the local behaviour of the generating function around the singularity and the asymptotics of the coefficients as in [73, VI.2], we know due to (5.27) that we have an asymptotic expansion of $q(i,0;n)$ and $q(0,j;n)$ (which are identical due to the symmetry of the model; this is however not necessary for the following) of the form

$$q(i,0;n) = t_c^{-n} \sum_{\substack{k\geq 1, \ell \geq m \geq 0 \\ k\rho+\ell<p}} v_{k,\ell,m}(i,0) \frac{(\log n)^m}{n^{k\rho+\ell+1}} + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right), \tag{5.29}$$

for some coefficients denoted by $v_{k,\ell,m}(i,0)$. The structure of the proof of Theorem 5.9 is as follows: first, we want to show that a similar expansion holds not only for $q(i,0;n)$, but also for $q(i,j;n)$, which will be done in Lemma 5.11. Then we will make use of this in order to show that the resulting coefficients $v_{k,\ell,m}(i,j)$ are discrete polyharmonic functions, see Lemma 5.13.

Before we start, we will state a simple lemma, which will turn out to be useful:

Lemma 5.10. *For any integer $p > \ell$, we have*

$$\frac{(\log(n+1))^m}{(n+1)^\ell} = \frac{(\log n)^m}{n^\ell} + \sum_{i=1}^{p-\ell} \sum_{j=\max(m-i,0)}^{m} c_{\ell,m,i,j} \frac{(\log n)^j}{n^{\ell+i}} + \mathcal{O}\left(\frac{\log n^m}{n^{p+1}}\right), \tag{5.30}$$

with the $c_{\ell,m,i,j}$ constants (in particular, if $m = 0$, then we will have no logarithmic parts).

Proof. The result follows immediately from writing $\log(n+1) = \log(n) + \log(1+\frac{1}{n})$ and expanding as a series in $\frac{1}{n}$. □

Note that in particular all terms inside the sum on the right-hand side have powers of the logarithm not exceeding m, and powers of n strictly smaller than $-\ell$ (this is the reason why in Figure 5.3 all arrows can only go downwards, and possibly to the left).

We can now extend the asymptotic expansion of $q(i,0;n)$ as in (5.29) to one of $q(i,j;n)$.

Lemma 5.11. *For any $(i,j) \in \mathbb{Z}^2$ and any $p > 0$, we have*

$$q(i,j;n) = t_c^{-n} \sum_{\substack{k\geq 1, \ell \geq m \geq 0 \\ k\rho+\ell+1<p}} v_{k,\ell,m}(i,j) \frac{(\log n)^m}{n^{k\rho+\ell+1}} + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right). \tag{5.31}$$

Proof. By induction on $r := \min(i,j)$. For $r < 0$, the statement is trivial because we have $q(i,j;n) = 0$ (we will use repeatedly this convention throughout the proof). For $r = 0$, the statement is precisely (5.29). So let us suppose that we already know that (5.31) holds up to a given r, and consider a point $(i, r+1)$. We can then write

$$\begin{aligned} q(i,r;n+1) = \\ &q(i,r+1;n) + q(i,r-1;n) + q(i-1,r;n) + q(i+1,r;n) + aq(i-1,r-1;n). \end{aligned}$$

By induction hypothesis and Lemma 5.10 applied to $q(i,r;n+1)$, the statement follows. □

In the following Lemmas 5.12 and 5.13, we will formalize the argumentation given in Figure 5.3. By [113] we know that, ordering the triples (k,ℓ,m) in (5.31) such that the weight functions $f_{k,\ell,m}(n) := \frac{(\log n)^m}{n^{k+\ell\rho+1}}$ are decreasing, then the corresponding coefficient functions $v_{k,\ell,m}$ are polyharmonic functions of increasing order; i.e. $v_{1,0,0}$ is harmonic, $v_{1,1,0}$ is biharmonic, $v_{1,1,1}$ is triharmonic, and so on. It turns out, however, that the fact that we know the $f_{k,\ell,m}$ explicitly allows us to greatly improve upon this statement.

Lemma 5.12. *Suppose we have $p > 0$ and a sequence of reals $a_{k,\ell,m}$ such that*

$$\sum_{\substack{k\geq 1, m\geq \ell \geq 0 \\ k\rho+\ell<p}} \frac{a_{k,\ell,m}}{n^{k\rho+\ell+1}} (\log n)^m = \mathcal{O}\left(n^{-p}\right).$$

Then, $a_{k,\ell,m} = 0$ for all triples (k,ℓ,m).

Proof. After multiplying with $n^{\rho+1}$, taking the limit $n \to \infty$ we can see immediately that $a_{1,0,0} = 0$. Proceeding by multiplying with increasing powers of n and $\log n$ and using that by assumption in each case there is only one non-zero coefficient (note that we make use of the fact that $\rho \notin \mathbb{Q}$ here), we can inductively show that $a_{k,\ell,m} = 0$ for all triples (k,ℓ,m). □

The idea behind the following Lemma 5.13 is similar as in [113, Lem. 6]. One writes $q(x;n+1)$ recursively as a sum over the step-set, and utilizes Lemma 5.10 in order to compare the coefficients. For large n, many of these coefficients will disappear, leaving us essentially with the (poly-)harmonicity of the $v_{k,\ell,m}$.

Lemma 5.13. *Suppose we have a combinatorial quantity $q(x;n)$ such that we have*

$$q(x;n+1) = \sum_{s\in\mathcal{S}} \omega_s q(x-s;n), \tag{5.32}$$

and for each $p>0$ we have

$$q(x;n) = t_c^{-n} \sum_{\substack{k\geq 1, \ell\geq m\geq 0\\ k\rho+\ell+1<p}} v_{k,\ell,m}(x)\frac{(\log n)^m}{n^{k\rho+\ell+1}} + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right). \tag{5.33}$$

Then $v_{k,\ell,m}$ is t_c-polyharmonic of order $\ell-m+1$.

Proof. To shorten notation, define the sets

$$\begin{aligned}\mathcal{U} :=&\{(k,\ell,m)\in\mathbb{Z}^3 : k\geq 1, \ell\geq m\geq 0, k\rho+\ell+1<p\},\\ \mathcal{V}_{k,\ell,m} :=&\{(i,j)\in\mathbb{Z}^2 : 1\leq i\leq m, k\rho+\ell+i+1<p, m-i\leq j\leq m\}.\end{aligned}$$

First, we notice that by (5.33) and (5.30), we have

$$q(x;n+1) = \\ t_c^{-n-1}\sum_{(k,\ell,m)\in\mathcal{U}} v_{k,\ell,m}(x)\left(\frac{(\log n)^m}{n^{k\rho+\ell+1}} + \sum_{(i,j)\in\mathcal{V}_{k,\ell,m}} c_{k,\ell,m,i,j}\frac{(\log n)^j}{n^{k\rho+\ell+i+1}}\right) + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right)$$

for some constants $c_{k,\ell,m,i,j}$. Utilizing (5.32), we now obtain

$$t_c^{-n}\sum_{s\in\mathcal{S}}\sum_{(k,\ell,m)\in\mathcal{U}}\omega_s v_{k,\ell,m}(x-s)\frac{(\log n)^m}{n^{k\rho+\ell+1}} = \\ t_c^{-n-1}\sum_{(k,\ell,m)\in\mathcal{U}} v_{k,\ell,m}(x)\left(\frac{(\log n)^m}{n^{k\rho+\ell+1}} + \sum_{(i,j)\in\mathcal{V}_{k,\ell,m}} c_{k,\ell,m,i,j}\frac{(\log n)^j}{n^{k\rho+\ell+i+1}}\right) + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right). \tag{5.34}$$

Now let us partially order the triples (k,ℓ,m) giving them the index $\ell-m$ (which means sorting them by their diagonal in Figure 5.3). We proceed inductively by the index of the triples (k,ℓ,m).

For index 0, one can check immediately (using an argument as shown in Figure 5.3; formally utilizing (5.30)) that the coefficient of $\frac{(\log n)^m}{n^{k\rho+\ell+1}}$ in the right-hand side of (5.34) is precisely $v_{k,\ell,m}(x)$. From Lemma 5.12 it follows immediately that the corresponding coefficients $v_{k,\ell,m}$ are t_c-harmonic.

Now suppose the statement is already shown for all triples of order r, and consider those of order $r+1$. We utilize the same arguments as before on $\triangle^r q(x;n)$. The equivalent of (5.34) now has the form

$$t_c^{-n}\sum_{s\in\mathcal{S}}\sum_{(k,\ell,m)\in\mathcal{U}}\omega_s \triangle^r v_{k,\ell,m}(x-s)\frac{(\log n)^m}{n^{k\rho+\ell+1}} = \\ t_c^{-n-1}\sum_{(k,\ell,m)\in\mathcal{U}} \triangle^r v_{k,\ell,m}(x)\left(\frac{(\log n)^m}{n^{k\rho+\ell+1}} + \sum_{(i,j)\in\mathcal{V}_{k,\ell,m}} c_{k,\ell,m,i,j}\frac{(\log n)^j}{n^{k\rho+\ell+i+1}}\right) + \mathcal{O}\left(t_c^{-n}\left(\frac{\log n}{n}\right)^p\right),$$

where we let the sum run over the triples with index at least $r+1$, since by induction hypothesis $\triangle^r v_{k,\ell,m}(x) = 0$ for all (k,ℓ,m) with index at most r. But from here it is again easily seen that the coefficient of $\frac{(\log n)^m}{n^{k\rho+\ell+1}}$ is nothing else than $\triangle^r v_{k,\ell,m}$ for the triples of index r; thus $\triangle^r v_{k,\ell,m}$ is harmonic and the proof is complete. □

Finally, Theorem 5.9 follows from Lemmas 5.11 and 5.13. Note that for the proof we did not make use of the fact that we are in dimension 2; the only part where we used any properties of our model in particular was in the proof of Lemma 5.11.

Chapter 6

Open problems

> We often hear that mathematics consists mainly of "proving theorems." Is a writer's job mainly that of "writing sentences?"
>
> Gian-Carlo Rota

1. In Chapter 3, discrete polyharmonic functions in the quarter plane were constructed in two fashions: in Section 3.3 via a purely algebraic approach, and in Section 3.4.1 via decoupling functions. In light of the fact that, for orbit-summable models, the only ones appearing in asymptotic expansions are those constructed in the latter fashion (see Section 4.1.3), one might ask if a similar construction to the decoupling approach (or an altogether different method leading to similar results) may be applicable for a wider class of models.

2. Utilizing Analytic Combinatorics in Several Variables (ACSV, see [119] for an introduction) would in many of the cases treated in Chapter 4 be an alternative to the saddle point method when computing an asymptotic expansion. Seeing as in many cases, first-order approximations have already been done [43, 108, 106], it might also be interesting to more formally compare the two methods and see up to which point ACSV would yield the same results, or if there are cases in which one or the other is easier to use or not applicable at all.

3. Working on a parametrization of a rational kernel curve in Section 3.5, we saw that finding a decoupling function is the same as solving a difference equation of the form

$$f(s) = f\left(\frac{s}{q}\right) = B(s),$$

as in (3.129). However, one could transition to the parametrization even sooner: utilizing (3.127), and letting $K(x(s), 0)H(x(s), 0) := f(s)$ for some harmonic function $H(x, y)$, we can then see that we must have

$$f(s) = f(qs).$$

Notice, however, that it is not a priori clear that this criterion is sufficient for $H(x, y)$ to be harmonic. It would be interesting to know if there is a deeper link of this kind of q-difference equations and the computation of polyharmonic functions. Maybe some or even a lot of the theory could be reformulated or simplified using methods as presented for example in [82, 80, 81]. Seeing as the decoupling approach only works for finite groups, and in particular not for those which only have a finite restriction to the kernel curve. Correspondingly, if one were to find a solution using such a parametrization and q-difference equations, then maybe such a solution would work independently of the finiteness of the underlying group (or at least work in different cases).

4. In the context of non-singular small-step models in the quarter plane, in this thesis we have computed asymptotics expansions of models which were either orbit-summable, or belonged to a very specific subset of those with infinite group (i.e. a simple walk together with a weighted North-East step attached). Leaving the exponential term and the polyharmonic functions aside, in the former case this expansion consisted only of powers of the path length n (see Thm. 4.1), while in the latter we also have logarithms (see 5.9). This immediately bears the question of whether the appearance of logarithms is due to the group of the model being infinite (which is the same as the generating function not being D-finite), or if there is some other way to classify the models according to their asymptotic behaviour. One might also ask if an expansion as in Thm. 5.9 generally holds. An approach using ϑ-functions seems promising to do so, though the computations might get a lot more complicated, and it is for instance not obvious how one would construct the function $J(z)$ in Section 5.2 in general.

5. When counting lattice paths with fixed start- and endpoint, there is (for an appropriate choice of t) a unique combinatorially relevant t-harmonic function for a given model, which can easily be characterized by its positivity. For higher order polyharmonic functions, however, there is no similar way to characterize the functions appearing in the asymptotics, nor a reasonable combinatorial interpretation. It would be very interesting to know if there is such a characterization, or interpretation, or at least, in some sense, a 'canonical' polyharmonic function of a given degree $p > 1$. Intuitively, one may think such a canonical function should coincide in its scaling limit with the (continuous) polyharmonic function appearing in the corresponding continuous heat kernel, but seeing as the higher order asymptotics of the discrete and continuous cases do not coincide (see Section 4.2.2), this might not be an ideal choice. Maybe the functions $g_2, g_3, \ldots$ as defined at the end of Section 4.2.2 could serve as candidates for such canonical representatives instead.

6. While so far we have always considered non-singular models with small steps in the quarter plane, there are of course a lot of ways to generalize the questions, be it that of construction of polyharmonic functions or that of a complete asymptotic expansions of lattice paths. The two obvious ways to change the setting are a change of the domain, and a change of the step set. For works on different cones (most notably the three-quarter plane or in the octant), see e.g. [26, 130, 126, 19, 25, 137, 23]. Work on domains other than cones is much rarer; for some results on walks in Lipschitz domains see for instance [112].

   Changing the step sets, one could for instance look at singular models as in [55, 86], allow larger step sets [19, 85], or even infinite ones sets (possibly requiring moment conditions and using probabilistic methods, as in e.g. [51, 57]). In one dimension, it appears there has been recent progress showing that, under some moment conditions, one always has a complete asymptotic expansion as in Thm. 4.1 [48].

   It should be noted, however, that counting lattice paths seems to get a lot more involved with the number of dimensions increasing, due to the functional equation (2.2) having more and more boundary terms. To the author's knowledge, there is no general method for treating models with an arbitrary number of dimensions at this point.

Bibliography

[1] N.I. Akhiezer and H.H. McFaden. *Elements of the Theory of Elliptic Functions*. Translations of Mathematical Monographs. American Mathematical Soc. ISBN: 9780821886779.

[2] Emilio Almansi. "Sull'integrazione dell'Equazione differenziale $\triangle 2n = 0$". In: *Annali di Matematica Pura ed Applicata* 2 (1899), pp. 1–51.

[3] Gerold Alsmeyer and Kilian Raschel. "The extinction problem for a distylous plant population with sporophytic self-incompatibility". In: *J. Math. Biol.* 78.6 (2019), pp. 1841–1874. ISSN: 0303-6812. DOI: 10.1007/s00285-019-01328-5.

[4] Désiré André. "Solution directe du problème résolu par M. Bertrand". In: *Comptes Rendus de l'Académie des Sciences* (1887), pp. 436–437.

[5] N. Aronszajn, T. M. Creese, and L. J. Lipkin. *Polyharmonic Functions*. Oxford University Press, 1983. ISBN: 0-19-853906-1.

[6] Nakhlé H. Asmar and Loukas Grafakos. "Harmonic Functions and Applications". In: *Complex Analysis with Applications*. Cham: Springer International Publishing, 2018, pp. 367–402. ISBN: 978-3-319-94063-2. DOI: 10.1007/978-3-319-94063-2_6.

[7] Axel Bacher, Manuel Kauers, and Rika Yatchak. "Continued classification of 3D lattice models in the positive octant". In: *Proceedings of the 28th international conference on formal power series and algebraic combinatorics, FPSAC 2016, Vancouver, Canada, July 4–8, 2016*. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2020, pp. 95–106. URL: dmtcs.episciences.org/6415.

[8] R. R. Bahadur and R. Ranga Rao. "On deviations of the sample mean". In: *Ann. Math. Stat.* 31 (1960), pp. 1015–1027. ISSN: 0003-4851. DOI: 10.1214/aoms/1177705674.

[9] Cyril Banderier and Philippe Flajolet. "Basic analytic combinatorics of directed lattice paths". In: *Theor. Comput. Sci.* 281.1-2 (2002), pp. 37–80. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(02)00007-5.

[10] Cyril Banderier and Michael Wallner. "Lattice paths with catastrophes". In: *Discrete Math. Theor. Comput. Sci.* 19.1 (2017). Id/No 23, p. 32. ISSN: 1365-8050. URL: dmtcs.episciences.org/3967/pdf.

[11] Rodrigo Bañuelos and Robert G. Smits. "Brownian motion in cones". In: *Probab. Theory Relat. Fields* 108.3 (1997), pp. 299–319. ISSN: 0178-8051. DOI: 10.1007/s004400050111.

[12] Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel. "Counting quadrant walks via Tutte's invariant method". In: *Combinatorial Theory* 1 (2021). Id/No 3, p. 77. ISSN: 2766-1334. DOI: 10.5070/C61055360.

[13] Philippe Biane. "Minuscule weights and random walks on lattices". In: *Quantum probability and related topics*. Singapore: World Scientific, 1992, pp. 51–65. ISBN: 981-02-1979-2.

[14] Philippe Biane. "Quantum random walk on the dual of SU(n)". In: *Probab. Theory Relat. Fields* 89.1 (1991), pp. 117–129. ISSN: 0178-8051. DOI: 10.1007/BF01225828.

[15] Sylvain Billiard and Viet Chi Tran. “A general stochastic model for sporophytic self-incompatibility”. In: *J. Math. Biol.* 64.1-2 (2012), pp. 163–210. ISSN: 0303-6812. DOI: `10.1007/s00285-011-0410-z`.

[16] B. Bogosel et al. “3D positive lattice walks and spherical triangles”. In: *J. Comb. Theory, Ser. A* 172 (2020). Id/No 105189, p. 47. ISSN: 0097-3165. DOI: `10.1016/j.jcta.2019.105189`.

[17] J.M. Borwein and P.B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*. Wiley-Interscience and Canadian Mathematics Series of Monographs and Texts. Wiley, 1987. ISBN: 9780471831389.

[18] Alin Bostan. “Computer Algebra for Lattice Path Combinatorics”. Available at `https://mathexp.eu/bostan/publications/HDR.pdf`. Habilitation à Diriger des Recherches. Université Paris 13, 2017.

[19] Alin Bostan, Mireille Bousquet-Mélou, and Stephen Melczer. “Counting walks with large steps in an orthant”. In: *J. Eur. Math. Soc. (JEMS)* 23.7 (2021), pp. 2221–2297. ISSN: 1435-9855. DOI: `10.4171/JEMS/1053`.

[20] Alin Bostan and Manuel Kauers. “Automatic classification of restricted lattice walks”. In: *Proceedings of the 21st annual international conference on formal power series and algebraic combinatorics, FPSAC 2009, Hagenberg, Austria, July 20–24, 2009*. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2009, pp. 201–215. URL: `www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAK0117`.

[21] Alin Bostan, Kilian Raschel, and Bruno Salvy. “Non-D-finite excursions in the quarter plane”. In: *J. Comb. Theory, Ser. A* 121 (2014), pp. 45–63. ISSN: 0097-3165. DOI: `10.1016/j.jcta.2013.09.005`.

[22] Alin Bostan et al. “Hypergeometric expressions for generating functions of walks with small steps in the quarter plane”. In: *Eur. J. Comb.* 61 (2017), pp. 242–275. ISSN: 0195-6698. DOI: `10.1016/j.ejc.2016.10.010`.

[23] Alin Bostan et al. “On 3-dimensional lattice walks confined to the positive octant”. In: *Ann. Comb.* 20.4 (2016), pp. 661–704. ISSN: 0218-0006. DOI: `10.1007/s00026-016-0328-7`.

[24] Mireille Bousquet-Mélou. “An elementary solution of Gessel’s walks in the quadrant”. In: *Adv. Math.* 303 (2016), pp. 1171–1189. ISSN: 0001-8708. DOI: `10.1016/j.aim.2016.08.038`.

[25] Mireille Bousquet-Mélou. “Enumeration of three-quadrant walks via invariants: some diagonally symmetric models”. In: *Canadian Journal of Mathematics* (2022), pp. 1–67. DOI: `10.4153/S0008414X22000487`.

[26] Mireille Bousquet-Mélou. “Square lattice walks avoiding a quadrant”. In: *J. Comb. Theory, Ser. A* 144 (2016), pp. 37–79. ISSN: 0097-3165. DOI: `10.1016/j.jcta.2016.06.010`.

[27] Mireille Bousquet-Mélou. “The expected number of inversions after n adjacent transpositions”. In: *Discrete Math. Theor. Comput. Sci.* 12.2 (2010), pp. 65–88. ISSN: 1365-8050. URL: `www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/article/view/1358/0.html`.

[28] Mireille Bousquet-Mélou. “Walks in the quarter plane: Kreweras’ algebraic model”. In: *Ann. Appl. Probab.* 15.2 (2005), pp. 1451–1491. ISSN: 1050-5164. DOI: `10.1214/10505160500000005`

[29] Mireille Bousquet-Mélou. “Walks on the slit plane: Other approaches”. In: *Adv. Appl. Math.* 27.2-3 (2001), pp. 243–288. ISSN: 0196-8858. DOI: `10.1006/aama.2001.0734`.

[30] Mireille Bousquet-Mèlou. "Counting walks in the quarter plane". In: *Mathematics and computer science II. Algorithms, trees, combinatorics and probabilities. Proceedings of the 2nd colloquium, Versailles-St. -Quentin, France, September 16–19, 2002.* Basel: Birkhäuser, 2002, pp. 49–67. ISBN: 3-7643-6933-7.

[31] Mireille Bousquet-Mélou and Marni Mishna. "Walks with small steps in the quarter plane". In: *Algorithmic probability and combinatorics. Papers from the AMS special sessions, Chicago, IL, USA, October 5–6, 2007 and Vancouver, BC, Canada, October 4–5, 2008.* Providence, RI: American Mathematical Society (AMS), 2010, pp. 1–39. ISBN: 978-0-8218-4783-1.

[32] Mireille Bousquet-Mélou and Marko Petkovšek. "Linear recurrences with constant coefficients: The multivariate case". In: *Discrete Math.* 225.1-3 (2000), pp. 51–75. ISSN: 0012-365X. DOI: `10.1016/S0012-365X(00)00147-3`.

[33] Mireille Bousquet-Mélou and Gilles Schaeffer. "Walks on the slit plane". In: *Probab. Theory Relat. Fields* 124.3 (2002), pp. 305–344. ISSN: 0178-8051. DOI: `10.1007/s004400200205`.

[34] N.G. de Bruijn. *Asymptotic Methods in Analysis*. Bibliotheca mathematica. Dover Publications, 1981. ISBN: 9780486642215. URL: `https://books.google.at/books?id=0qj9AgAAQBAJ`.

[35] François Chapon. Private correspondence.

[36] François Chapon, Éric Fusy, and Kilian Raschel. "Polyharmonic Functions And Random Processes in Cones". In: *International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*. 2020.

[37] Shuchi Chawla. *Lecture notes to CS787, Advanced Algorithms*. University of Wisconsin, Madison. 2015.

[38] J. W. Cohen. *Analysis of random walks*. Vol. 2. Stud. Probab. Optim. Stat. Amsterdam: IOS Press, 1992. ISBN: 90-5199-086-3.

[39] J. W. Cohen and O. J. Boxma. *Boundary value problems in queueing system analysis*. North Holland Publishing Company, 1983. ISBN: 0-444-86567-5.

[40] Joel Cohen et al. "Polyharmonic Functions on Trees". In: *American Journal of Mathematics, 124, 999-1043* 124 (2002), pp. 999–1043. DOI: `https://doi.org/10.48550/arXiv.1802.06239`.

[41] Christopher I. Connoll and Roderic A. Grupen. "The applications of harmonic functions to robotics". In: *Journal of Field Robotics* 10 (7 1993), pp. 931–946. ISSN: 0097-3165. DOI: `https://doi.org/10.1002/rob.4620100704`.

[42] Rama Cont and Adrien de Larrard. "Price dynamics in a Markovian limit order market". In: *SIAM J. Financ. Math.* 4 (2013), pp. 1–25. ISSN: 1945-497X. DOI: `10.1137/110856605`.

[43] J. Courtiel et al. "Weighted lattice walks and universality classes". In: *J. Comb. Theory, Ser. A* 152 (2017), pp. 255–302. ISSN: 0097-3165. DOI: `10.1016/j.jcta.2017.06.008`.

[44] H. Cramér. *Sur un nouveau théorème-limite de la théorie des probabilités*. French. Actual. sci. industr. 736, 5-23. (Confér. internat. Sci. math. Univ. Genève. Théorie des probabilités. III: Les sommes et les fonctions de variables aléatoires.) (1938). 1938.

[45] Philippe D'Arco, Valentina Lacivita, and Sami Mustapha. "Combinatorics meets potential theory". In: *Electron. J. Comb.* 23.2 (2016), research paper p2.28, 17. ISSN: 1077-8926. URL: `www.combinatorics.org/ojs/index.php/eljc/article/view/v23i2p28`.

[46] Joel Dahne and Bruno Salvy. "Computation of tight enclosures for Laplacian eigenvalues". In: *SIAM J. Sci. Comput.* 42.5 (2020), a3210–a3232. ISSN: 1064-8275. DOI: 10.1137/20M1326520.

[47] R. Dante DeBlassie. "Exit times from cones in $\mathbb{R}^n$ of Brownian motion". In: *Probab. Theory Relat. Fields* 74 (1986), pp. 1–29. ISSN: 0178-8051. DOI: 10.1007/BF01845637.

[48] D. Denisov, A. Tarasov, and V. Wachtel. *Asymptotic expansions for first-passage times of a random walk*. In preparation. 2023.

[49] Denis Denisov and Vitali Wachtel. "Alternative constructions of a harmonic function for a random walk in a cone". In: *Electronic Journal of Probability* 24.none (2019), pp. 1–26. DOI: 10.1214/19-EJP349.

[50] Denis Denisov and Vitali Wachtel. "Conditional limit theorems for ordered random walks". In: *Electron. J. Probab.* 15 (2010). Id/No 11, pp. 292–322. ISSN: 1083-6489. DOI: 10.1214/EJP.v15-752.

[51] Denis Denisov and Vitali Wachtel. "Random walks in cones". In: *Ann. Probab.* 43.3 (2015), pp. 992–1044. ISSN: 0091-1798. DOI: 10.1214/13-AOP867.

[52] Monroe D. Donsker. "Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems". In: *Ann. Math. Stat.* 23 (1952), pp. 277–281. ISSN: 0003-4851. DOI: 10.1214/aoms/1177729445.

[53] Joseph L. Doob. "Semimartingales and subharmonic functions". In: *Trans. Am. Math. Soc.* 77 (1954), pp. 86–121. ISSN: 0002-9947. DOI: 10.2307/1990680.

[54] Thomas Dreyfus and Kilian Raschel. "Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks". In: *Actes de la conférence «Analogies between number field and function field : algebraic and analytic approaches »*. Besançon: Presses Universitaires de Franche-Comté, 2019, pp. 41–80. ISBN: 978-2-84867-724-8.

[55] Thomas Dreyfus et al. "Walks in the quarter plane: genus zero case". In: *J. Comb. Theory, Ser. A* 174 (2020). Id/No 105251, p. 24. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2020.105251.

[56] J.J. Duistermaat. *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*. Springer Monographs in Mathematics. Springer New York, 2010. ISBN: 9781441971166.

[57] Jetlir Duraj. "Essays on Random Walk in Cones". Available at https://edoc.ub.uni-muenchen.de/27023/1/Duraj_Jetlir.pdf. PhD thesis. München: Ludwig-Maximilians-Universität, 2020.

[58] Jetlir Duraj. "Random walks in cones: the case of nonzero drift". In: *Stochastic Processes Appl.* 124.4 (2014), pp. 1503–1518. ISSN: 0304-4149. DOI: 10.1016/j.spa.2013.12.003.

[59] Jetlir Duraj et al. "Martin boundary of random walks in convex cones". In: *Ann. Henri Lebesgue* 5 (2022), pp. 559–609. ISSN: 2644-9463. DOI: 10.5802/ahl.130.

[60] Peter Eichelsbacher and Wolfgang König. "Ordered random walks". In: *Electron. J. Probab.* 13 (2008), pp. 1307–1336. ISSN: 1083-6489. DOI: 10.1214/EJP.v13-539.

[61] Andrew Elvey Price. "Counting lattice walks by winding angle". In: *Sémin. Lothar. Comb.* 84B (2020). Id/No 43, p. 12. ISSN: 1286-4889. URL: www.mat.univie.ac.at/~slc/wpapers/FPSAC2020//43.html.

[62] Andrew Elvey-Price. "Enumeration of three quadrant walks with small steps and walks on other M-quadrant cones". In: 2022. URL: https://api.semanticscholar.org/CorpusID:248177766.

[63] Andrew Elvey-Price. “Enumeration of walks with small steps avoiding a quadrant”. In: *Sémin. Lothar. Comb.* 86B (2022). Id/No 1, p. 12. ISSN: 1286-4889. URL: www.mat.univie.ac.at/~slc/wpapers/FPSAC2022//1.html.

[64] Andrew Elvey-Price, Andreas Nessmann, and Kilian Raschel. *Logarithmic terms in discrete heat kernel expansions in the quadrant.* 2023. arXiv: 2309.15209 [math.CO].

[65] Arthur Erdélyi and M. Wyman. “The asymptotic evaluation of certain integrals”. In: *Arch. Ration. Mech. Anal.* 14 (1963), pp. 217–260. ISSN: 0003-9527. DOI: 10.1007/BF00250704.

[66] G. Fayolle and K. Raschel. “On the holonomy or algebraicity of generating functions counting lattice walks in the quarter-plane”. In: *Markov Process. Relat. Fields* 16.3 (2010), pp. 485–496. ISSN: 1024-2953.

[67] G. Fayolle and K. Raschel. “Random walks in the quarter-plane with zero drift: an explicit criterion for the finiteness of the associated group”. English. In: *Markov Process. Relat. Fields* 17.4 (2011), pp. 619–636. ISSN: 1024-2953.

[68] Guy Fayolle and Roudolf Iasnogorodski. “Two coupled processors: The reduction to a Riemann-Hilbert problem”. In: *Z. Wahrscheinlichkeitstheor. Verw. Geb.* 47 (1979), pp. 325–351. ISSN: 0044-3719. DOI: 10.1007/BF00535168.

[69] Guy Fayolle and Kilian Raschel. “Some exact asymptotics in the counting of walks in the quarter plane”. In: *Proceeding of the 23rd international meeting on probabilistic, combinatorial, and asymptotic methods in the analysis of algorithms (AofA’12), Montreal, Canada, June 18–22, 2012.* Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2012, pp. 109–124. URL: www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAQ0110/0.html.

[70] L. Fayolle, R. Iasnagorodski, and V. Malyshev. *Random Walks in the Quarter Plane.* Springer International Publishing AG, 2017. ISBN: 978-3-319-50928-0.

[71] Thomas Feierl. “Asymptotics for the number of walks in a Weyl chamber of type B”. In: *Random Struct. Algorithms* 45.2 (2014), pp. 261–305. ISSN: 1042-9832. DOI: 10.1002/rsa.20467.

[72] Michael E. Fisher. “Walks, walls, wetting, and melting”. In: *J. Stat. Phys.* 34 (1984), pp. 667–730. ISSN: 0022-4715. DOI: 10.1007/BF01009436.

[73] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, 2009. ISBN: 978-0-521-89806-5.

[74] L. Flatto and S. Hahn. “Two parallel queues created by arrivals with two demands. I”. In: *SIAM J. Appl. Math.* 44 (1984), pp. 1041–1053. ISSN: 0036-1399. DOI: 10.1137/0144074.

[75] F. D. Gakhov. *Boundary Value Problems.* Pergamon Press, 1966. ISBN: 978-1483132563.

[76] Rodolphe Garbit. “Exit time of a centered random walk from a cone”. In: *C. R., Math., Acad. Sci. Paris* 345.10 (2007), pp. 587–591. ISSN: 1631-073X. DOI: 10.1016/j.crma.2007.10.016.

[77] Rodolphe Garbit and Kilian Raschel. “On the exit time from a cone for random walks with drift”. In: *Rev. Mat. Iberoam.* 32.2 (2016), pp. 511–532. ISSN: 0213-2230. DOI: 10.4171/RMI/893.

[78] Ira M. Gessel and Doron Zeilberger. “Random walk in a Weyl chamber”. In: *Proc. Am. Math. Soc.* 115.1 (1992), pp. 27–31. ISSN: 0002-9939. DOI: 10.2307/2159560.

[79] D. J. Griffiths. *Introduction to Electrodynamics 4th ed.* Cambridge University Press, 2013. ISBN: 978-1108822909.

[80] Charlotte Hardouin. “Hypertranscendance of diagonal difference systems”. In: *Compos. Math.* 144.3 (2008), pp. 565–581. ISSN: 0010-437X. DOI: 10.1112/S0010437X07003430.

[81] Charlotte Hardouin, Jacques Sauloy, and Michael F. Singer. *Galois Theories of Linear Difference Equations: An Introduction*. Mathematical Surveys and Monographs 211. American Mathematical Society, 2016. ISBN: 9781470426552.

[82] Charlotte Hardouin and Michael F. Singer. “On differentially algebraic generating series for walks in the quarter plane”. In: *Sel. Math., New Ser.* 27.5 (2021). Id/No 89, p. 49. ISSN: 1022-1824. DOI: 10.1007/s00029-021-00703-9.

[83] L. L. Helms. “Biharmonic functions and Brownian motion”. In: *J. Appl. Probab.* 4 (1967), pp. 130–136. ISSN: 0021-9002. DOI: 10.2307/3212305.

[84] Thomas Hirschler and Wolfgang Woess. “Polyharmonic functions for finite graphs and Markov chains”. In: *Frontiers in analysis and probability. In the spirit of the Strasbourg-Zürich meetings*. Cham: Springer, 2020, pp. 77–90. DOI: 10.1007/978-3-030-56409-4_4.

[85] Viet Hung Hoang. *Discrete harmonic functions for non-symmetric Laplace operators in the quarter plane*. 2022. DOI: 10.48550/ARXIV.2201.09985.

[86] Viet Hung Hoang, Kilian Raschel, and Pierre Tarrago. “Constructing discrete harmonic functions in wedges”. In: *Trans. Am. Math. Soc.* 375.7 (2022), pp. 4741–4782. ISSN: 0002-9947. DOI: 10.1090/tran/8615.

[87] Katherine Humphreys. “A history and a survey of lattice path enumeration”. In: *J. Stat. Plann. Inference* 140.8 (2010), pp. 2237–2254. ISSN: 0378-3758. DOI: 10.1016/j.jspi.2010.01.020.

[88] Irina Ignatiouk-Robert. “Martin boundary of a killed random walk on a half-space”. In: *J. Theor. Probab.* 21.1 (2008), pp. 35–68. ISSN: 0894-9840. DOI: 10.1007/s10959-007-0100-3.

[89] Irina Ignatiouk-Robert and Christophe Loree. “Martin boundary of a killed random walk on a quadrant”. In: *Ann. Probab.* 38.3 (2010), pp. 1106–1142. ISSN: 0091-1798. DOI: 10.1214/09-AOP506.

[90] K. Itô and H.P.J. McKean. *Diffusion Processes and their Sample Paths*. Classics in Mathematics. Springer Berlin Heidelberg, 2012. ISBN: 9783642620256.

[91] Samuel Johnson, Marni Mishna, and Karen Yeats. “A combinatorial understanding of lattice path asymptotics”. In: *Adv. Appl. Math.* 92 (2018), pp. 144–163. ISSN: 0196-8858. DOI: 10.1016/j.aam.2017.08.001.

[92] V.J. Katz. *A History of Mathematics: An Introduction*. Katz Series. Addison-Wesley, 1998. ISBN: 9780321016188.

[93] Manuel Kauers and Rika Yatchak. “Walks in the quarter plane with multiple steps”. In: *Proceedings of the 27th international conference on formal power series and algebraic combinatorics, FPSAC 2015, Daejeon, South Korea, July 6–10, 2015*. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2015, pp. 25–36.

[94] Wolfgang König, Neil O’Connell, and Sébastien Roch. “Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles”. In: *Electron. J. Probab.* 7 (2002). Id/No 5, p. 24. ISSN: 1083-6489.

[95] Wolfgang König and Patrick Schmid. “Random walks conditioned to stay in Weyl chambers of type C and D”. In: *Electron. Commun. Probab.* 15 (2010), pp. 286–296. ISSN: 1083-589X. DOI: 10.1214/ECP.v15-1560.

[96] Ivan K. Kostov. “Exact solution of the six-vertex model on a random lattice.” In: *Nucl. Phys., B* 575.3 (2000), pp. 513–534. ISSN: 0550-3213. DOI: 10.1016/S0550-3213(00)00060-2.

[97] Irina Kurkova and Kilian Raschel. “On the functions counting walks with small steps in the quarter plane”. In: *Publ. Math., Inst. Hautes Étud. Sci.* 116 (2012), pp. 69–114. ISSN: 0073-8301. DOI: 10.1007/s10240-012-0045-7.

[98] Irina Kurkova and Kilian Raschel. “Random walks in $(\mathbb{Z}_+)^2$ with non-zero drift absorbed at the axes”. In: *Bull. Soc. Math. Fr.* 139.3 (2011), pp. 341–387. ISSN: 0037-9484. DOI: 10.24033/bsmf.2611.

[99] Lovász László, L. Lov, and Of Erdos. “Random Walks on Graphs: A Survey”. In: Jan. 1996, pp. 1–46.

[100] Gregory F. Lawler and Vlada Limic. *Random Walk: A Modern Introduction*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010. DOI: 10.1017/CBO9780511750854.

[101] Cédric Lecouvey and Kilian Raschel. “t-Martin boundary of killed random walks in the quadrant”. In: *Séminaire de probabilités XLVIII*. Cham: Springer, 2016, pp. 305–323. DOI: 10.1007/978-3-319-44465-9_11.

[102] S. A. Lurie and V. V. Vasiliev. *The Biharmonic Problem in the Theory of Elasticity*. Taylor & Francis Ltd, 1995. ISBN: 978-2884490542.

[103] V. A. Malyshev. “An analytical method in the theory of two-dimensional positive random walks”. In: *Sib. Math. J.* 13 (1973), pp. 917–929. ISSN: 0037-4466. DOI: 10.1007/BF00971868.

[104] Robert S. Martin. “Minimal positive harmonic functions”. In: *Trans. Am. Math. Soc.* 49 (1941), pp. 137–172. ISSN: 0002-9947. DOI: 10.2307/1990054.

[105] H. Maschke. “Ueber den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen.” German. In: *Math. Ann.* 50 (1898), pp. 492–498. ISSN: 0025-5831. DOI: 10.1007/BF01444297.

[106] Stephen Melczer and Marni Mishna. “Asymptotic lattice path enumeration using diagonals”. In: *Algorithmica* 75.4 (2016), pp. 782–811. ISSN: 0178-4617. DOI: 10.1007/s00453-015-0063-1.

[107] Stephen Melczer and Marni Mishna. “Singularity analysis via the iterated kernel method”. In: *Comb. Probab. Comput.* 23.5 (2014), pp. 861–888. ISSN: 0963-5483. DOI: 10.1017/S0963548314000145.

[108] Stephen Melczer and Mark C. Wilson. “Higher dimensional lattice walks: connecting combinatorial and analytic behavior”. In: *SIAM J. Discrete Math.* 33.4 (2019), pp. 2140–2174. ISSN: 0895-4801. DOI: 10.1137/18M1220856.

[109] Marni Mishna and Samuel Simon. “The asymptotics of reflectable weighted walks in arbitrary dimension”. In: *Adv. Appl. Math.* 118 (2020). Id/No 102043, p. 18. ISSN: 0196-8858. DOI: 10.1016/j.aam.2020.102043.

[110] Thierry Monteil and Khaydar Nurligareev. “Asymptotics for Connected Graphs and Irreducible Tournaments”. In: *Extended Abstracts EuroComb 2021*. Cham: Springer International Publishing, 2021, pp. 823–828. ISBN: 978-3-030-83823-2.

[111] Sami Mustapha. “Non-D-finite walks in a three-quadrant cone”. In: *Ann. Comb.* 23.1 (2019), pp. 143–158. ISSN: 0218-0006. DOI: 10.1007/s00026-019-00413-2.

[112] Sami Mustapha and Mohamed Sifi. “Discrete harmonic functions in Lipschitz domains”. In: *Electron. Commun. Probab.* 24 (2019). Id/No 58, p. 15. ISSN: 1083-589X. DOI: 10.1214/19-ECP259.

[113] Andreas Nessmann. *Full asymptotic expansion for orbit-summable quadrant walks and discrete polyharmonic functions.* 2023. arXiv: 2307.11539 [math.CO].

[114] Andreas Nessmann. *Polyharmonic Functions in the Quarter Plane.* 2022. arXiv: 2212.07258 [math.CO].

[115] O. T. O’Meara. *Introduction to quadratic forms. 3rd corrected printing.* Vol. 117. Grundlehren Math. Wiss. Springer, Cham, 1973.

[116] Bernt Øksendal. *Stochastic Differential Equations. An Introduction with Applications.* Springer Berlin, Heidelberg, 2010. ISBN: 978-3-642-14394-6.

[117] L. Pacioli. *Summa de Arithmetica geometria proportioni et proportionalità.* Paganino de Paganini, 1523. URL: https://books.google.at/books?id=iqgPe49fhrsC.

[118] A. Papoulis and S.U. Pillai. *Probability, Random Variables, and Stochastic Processes.* McGraw-Hill series in electrical and computer engineering. McGraw-Hill, 2002. ISBN: 9780071226615. URL: https://books.google.at/books?id=cUmiDAEACAAJ.

[119] Robin Pemantle, Mark C. Wilson, and Stephen Melczer. *Analytic Combinatorics in Several Variables.* Cambridge University Press, 2023. URL: https://acsvproject.com/PemantleWilsonMelczer23.pdf.

[120] Massimo A. Picardello and Wolfgang Woess. “Boundary representations of λ-harmonic and polyharmonic functions on trees”. In: *Potential Analysis* 51 (2019), pp. 541–561.

[121] L. Poli and P. Delerue. *Le calcul symbolique à deux variables et ses applications.* French. Mém. Sci. Math. 127, 77 p. (1954). 1954.

[122] Kilian Raschel. Private correspondence.

[123] Kilian Raschel. “Counting walks in a quadrant: a unified approach via boundary value problems”. In: *J. Eur. Math. Soc. (JEMS)* 14.3 (2012), pp. 749–777. ISSN: 1435-9855. DOI: 10.4171/JEMS/317.

[124] Kilian Raschel. “Green functions for killed random walks in the Weyl chamber of Sp(4)”. In: *Ann. Inst. Henri Poincaré, Probab. Stat.* 47.4 (2011), pp. 1001–1019. ISSN: 0246-0203. DOI: 10.1214/10-AIHP405.

[125] Kilian Raschel. “Random walks in the quarter plane, discrete harmonic functions and conformal mappings”. In: *Stochastic Processes Appl.* 124.10 (2014), pp. 3147–3178. ISSN: 0304-4149. DOI: 10.1016/j.spa.2014.04.013.

[126] Kilian Raschel and Amélie Trotignon. “On walks avoiding a quadrant”. In: *Electron. J. Comb.* 26.3 (2019), research paper p3.31, 34. ISSN: 1077-8926. URL: www.combinatorics.org/ojs/index.php/eljc/article/view/v26i3p31.

[127] Daniel A. Spielman. *Lecture notes to Spectral Graph Theory and its Applications.* Yale University. 2004.

[128] F. Spitzer. *Principles of Random Walk, 2nd ed.* Springer, 1976. ISBN: 978-1-4757-4231-2.

[129] John R. Stembridge. “Nonintersecting paths, pfaffians, and plane partitions”. In: *Adv. Math.* 83.1 (1990), pp. 96–113. ISSN: 0001-8708. DOI: 10.1016/0001-8708(90)90070-4.

[130] Amélie Trotignon. “Discrete harmonic functions in the three-quarter plane”. In: *Potential Anal.* 56.2 (2022), pp. 267–296. ISSN: 0926-2601. DOI: 10.1007/s11118-020-09884-y.

[131] W. T. Tutte. “Chromatic sums revisited”. In: *Aequationes Math.* 50.1-2 (1995), pp. 95–134. ISSN: 0001-9054. DOI: 10.1007/BF01831115.

[132] N. Th. Varopoulos. “Potential theory in conical domains”. In: *Math. Proc. Camb. Philos. Soc.* 125.2 (1999), pp. 335–384. ISSN: 0305-0041. DOI: 10.1017/S0305004198002771.

[133] Michael Wallner. “Lattice Path Combinatorics”. Available at https://dmg.tuwien.ac.at/drmota/Thesis_Wallner.pdf. PhD thesis. Wien: Technische Universität Wien, 2016.

[134] A. W. Whitworth. “Arrangements of m things of one sort and n things of another sort under certain conditions of priority”. In: *Messenger of Math* 8 (1878), pp. 105–114.

[135] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*. Cambridge Tracts in Mathematics. Cambridge University Press, 2000. DOI: 10.1017/CBO9780511470967.

[136] E. M. Wright. “Asymptotic relations between enumerative functions in graph theory”. In: *Proc. Lond. Math. Soc. (3)* 20 (1970), pp. 558–572. ISSN: 0024-6115. DOI: 10.1112/plms/s3-20.3.558.

[137] Rika Yatchak. “Automated positive part extraction for lattice path generating functions in the octant”. In: *Extended abstracts of the ninth European conference on combinatorics, graph theory and applications, EuroComb 2017, Vienna, Austria, August 28 – September 1, 2017*. Amsterdam: Elsevier, 2017, pp. 1061–1067. DOI: 10.1016/j.endm.2017.07.073.

Appendices

Appendix A

Proofs

Proof of Lemma 1.3

Lemma 1.3. *Let $q(x;n)$ be some combinatorial quantity depending on n and a point $x \in \mathbb{Z}^d$, and let $\mathcal{S} \subset \mathbb{Z}^d$ be some step set. Suppose that $q(x)$ satisfies a recursive relation of the form*

$$q(x;n+1) = \sum_{s\in\mathcal{S}} \omega_s q(x-s;n),$$

and that for each $m \in \mathbb{N}$ we can write

$$q(x;n) = \sum_{k=1}^{m-1} f_k(n)v_k(x) + \mathcal{O}\left(f_m(n)\right),$$

where $\lim_{n\to\infty} \frac{f_{k+1}(n)}{f_k(n)} = 0$ and $\lim_{n\to\infty} \frac{f_k(n+1)}{f_k(n)} = 1$ for all $k \geq 1$, and the Landau-$\mathcal{O}$ is to be understood with respect to n. Then, for all $k \geq 1$, $v_k(x)$ is a polyharmonic function of degree k.

Proof. Suppose we already know that for $p = 1, \ldots, m$, v_p is p-polyharmonic. We have

$$\triangle^m q(x;n+1) = \sum_{s\in\mathcal{S}} \omega_s \triangle^m q(x-s;n) \quad \Leftrightarrow \tag{A.1}$$

$$\sum_{k=1}^{\infty} f_k(n+1)\triangle^m v_k(x) = \sum_{s\in\mathcal{S}}\sum_{k=1}^{\infty} \omega_s f_k(n)\triangle^m v_k(x) \quad \Leftrightarrow \tag{A.2}$$

$$\sum_{k=m+1}^{\infty} f_k(n+1)\triangle^m v_k(x) = \sum_{s\in\mathcal{S}}\sum_{k=m+1}^{\infty} \omega_s f_k(n)\triangle^m v_k(x) \quad \Rightarrow \tag{A.3}$$

$$\sum_{k=m+1}^{\infty} \frac{f_k(n+1)}{f_{m+1}(n)}\triangle^m v_k(x) = \sum_{s\in\mathcal{S}}\sum_{k=m+1}^{\infty} \omega_s \frac{f_k(n)}{f_{m+1}(n)}\triangle^m v_k(x-s) \quad \Rightarrow \tag{A.4}$$

$$\triangle^m v_{m+1}(x) = \sum_{s\in\mathcal{S}} \omega_s \triangle^m v_{m+1}(x-s), \tag{A.5}$$

where in the last line we take the limit $n \to \infty$. This tells us that $\triangle^m v_{m+1}(x)$ is harmonic, which the same as $v_{m+1}(x)$ being polyharmonic of degree $m+1$. $\square$

Proof of Prop. 1.4

Proposition 1.4. *Let $A \subset \mathbb{Z}^d$ be bounded, and let $(S_n)_{n\geq 0}$ be a random walk starting at some $x \in \mathbb{Z}^d$. Let τ be the exit time of S_n from A, and $g : A^c \to \mathbb{C}$ be any function. Finally, define for $k \in \mathbb{Z}^+$:*

$$f_k(x) := \mathbb{E}_x \left[\tau^{k-1} g\left(S_\tau\right)\right]. \tag{1.42}$$

Then the f_k satisfy the discrete Dirichlet problem

$$\triangle^k f_k(x) = 0 \qquad \forall x \in A, \tag{1.43}$$

$$f(x) = g(x) \quad \forall x \in A^c. \tag{1.44}$$

Proof. For $x \in A^c$ we have $\tau = 0$, therefore $f_k(x) = g(x)$ and the statement follows. So let us assume in the following that $x \in A$. We will proceed by induction.

First, suppose that $k = 1$. Let $\mathcal{S}$ be the step set of the random walk, with weights $(\omega_s)_{s\in\mathcal{S}}$. We can write, by decomposing over the first step,

$$f_1(x) = \mathbb{E}_x \left[g\left(S_\tau\right)\right] \tag{A.6}$$

$$= \sum_{s\in\mathcal{S}} \omega_s \mathbb{E}_{x+s} \left[g\left(S_\tau\right)\right] \tag{A.7}$$

$$= \sum_{s\in\mathcal{S}} \omega_s f_1(x+s), \tag{A.8}$$

thus $f_1(x)$ is discrete harmonic.

Now assume that Prop. 1.4 holds for $k = 1, \ldots, l-1$. We can then write

$$f_l(x) = \mathbb{E}_x \left[\tau^{l-1} g\left(S_\tau\right)\right] \tag{A.9}$$

$$= \sum_{s\in\mathcal{S}} \omega_s \mathbb{E}_{x+s} \left[(\tau-1)^{l-1} g\left(S_\tau\right)\right] \tag{A.10}$$

$$= \sum_{s\in\mathcal{S}} \omega_s \mathbb{E}_{x+s} \left[\sum_{i=0}^{l-1} \binom{l-1}{i} \tau^i (-1)^{l-1} g\left(S_\tau\right)\right] \tag{A.11}$$

$$= \sum_{s\in\mathcal{S}} \omega_s \sum_{i=0}^{l-1} \binom{l-1}{i} f_i(x+s). \tag{A.12}$$

Since by assumption we have $\triangle^{l-1} f_i = 0$ for $i = 1, \ldots, l-1$, it follows that

$$\triangle^{l-1} f_l(x) = \sum_{s\in\mathcal{S}} \omega_s f_l(x+s), \tag{A.13}$$

which is equivalent to $\triangle^l f_l(x) = 0$. $\square$

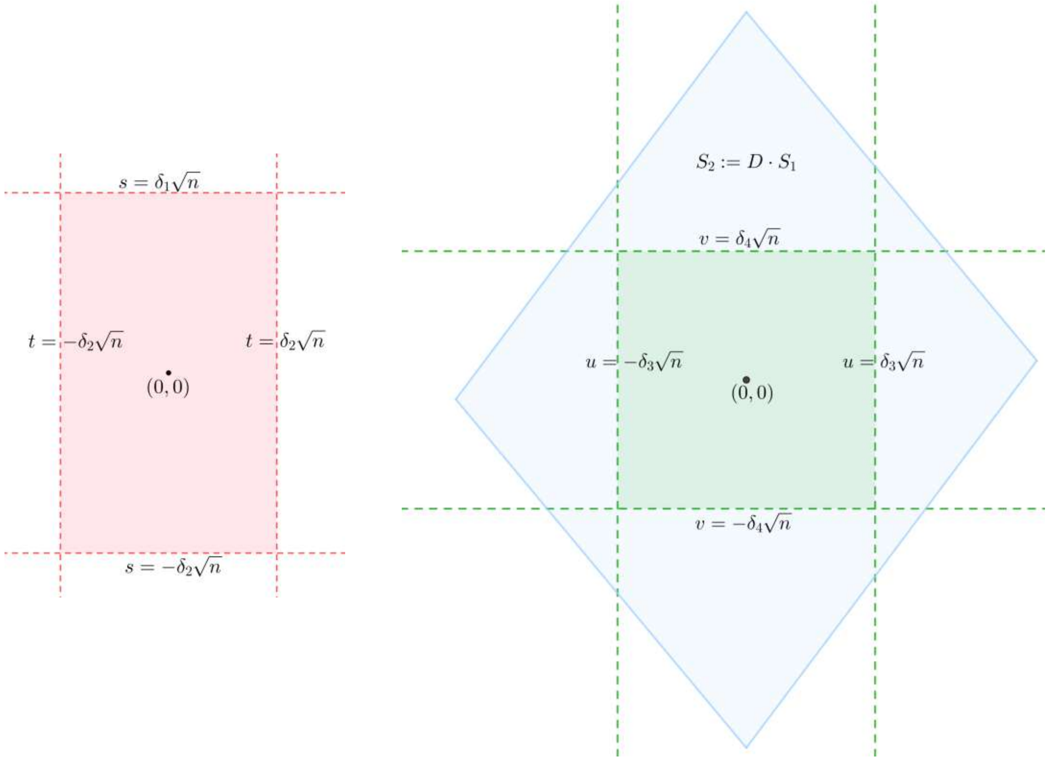


Figure A.1: The region S_1 in the (s,t)-, and its imagine in the (u,v)-plane. As we want to bound the integral of the exponential of the respective quadratic forms on the complements on these regions from above, we can consider the complement of any square inscribed into S_2 instead of that of the latter.

An error estimate for Section 4.1

Our goal in this section is to prove

Lemma A.1. *Let $Q(s,t)$ be a positive definite quadratic form. Then there are constants $c, d > 0$ such that*

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(s,t)} \mathrm{d}t\mathrm{d}s - \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} \int_{-\delta_2\sqrt{n}}^{\delta_2\sqrt{n}} e^{-Q(s,t)} \mathrm{d}t\mathrm{d}s \right| < ce^{-d\sqrt{n}}. \tag{A.14}$$

Proof. As $Q(s,t)$ is positive definite, after a suitable (linear) coordinate transformation D we can write $Q(s,t) = \lambda_1 u^2 + \lambda_2 v^2 =: Q'(u,v)$, where $(u,v) = (s,t) \cdot D$ [115]. Under this coordinate transformation, the rectangle $S_1 := \{|s| < \delta_1\sqrt{n}, |t| < \delta_2\sqrt{n}\}$ is mapped to a parallelogram $S_2 := D \cdot S_1$ containing the origin. We can then find another square S_3, with sidelength $\delta_3\sqrt{n}$, where $\delta_3 > 0$, which is completely contained in S_2 (see Fig. A.1). Consequently, we know that

$$|\det D| \int_{S_1} e^{-Q(s,t)} \mathrm{d}t\mathrm{d}s = \int_{S_2} e^{-Q'(u,v)} \mathrm{d}v\mathrm{d}u > \int_{S_3} e^{-Q'(u,v)} \mathrm{d}t\mathrm{d}s.$$

In order to show (A.14) it therefore suffices to show that the integral of $e^{-Q'(u,v)}$ over the complement of S_3 is bounded from above by a term of the form $c'e^{-d'\sqrt{n}}$, with $c', d' > 0$. As this complement can be written as $S_3^c = A_1 \cup A_2$, with $A_1 := \{|u| > \delta_3\sqrt{n}\}$ and $A_2 := \{|v| > \delta_3\sqrt{n}\}$, it is therefore enough to show that

$$\int_{A_1} e^{-Q'(u,v)} \mathrm{d}v\mathrm{d}u + \int_{A_2} e^{-Q'(u,v)} \mathrm{d}v\mathrm{d}u < c'e^{-d'\sqrt{n}}. \tag{A.15}$$

We will consider the integral over A_1, the one over A_2 can be treated in the same fashion. We have

$$\int_{A_1} e^{-Q'(u,v)}\mathrm{d}v\mathrm{d}u = \int_{|u|>\delta_3\sqrt{n}} \int_{-\infty}^{\infty} e^{-Q'(u,v)}\mathrm{d}v\mathrm{d}u \tag{A.16}$$

$$=2\int_{\delta_3\sqrt{n}}^{\infty} e^{-\lambda_1 u^2} \int_{-\infty}^{\infty} e^{-\lambda_2 v^2}\mathrm{d}v\mathrm{d}u \tag{A.17}$$

$$=2\sqrt{\frac{\pi}{\lambda_2}} \int_{\delta_3\sqrt{n}}^{\infty} e^{-\lambda_1 u^2}. \tag{A.18}$$

For n large enough, we know that $\lambda_1 u^2 > \lambda_1 u$ for $u \geq \delta_3\sqrt{n}$, thus we can utilize the (very crude, but sufficient) estimate

$$\int_{\delta_3\sqrt{n}}^{\infty} e^{-\lambda_1 u^2}\mathrm{d}u < \int_{\delta_3\sqrt{n}}^{\infty} e^{-\lambda_1 u}\mathrm{d}u = \frac{e^{-\lambda_1\delta_3\sqrt{n}}}{\lambda_1}.$$

Applying the same argument to the integral over A_2, we see that (A.15) holds for $c' = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$, and $d' = \min(\lambda_1, \lambda_2)\delta_3$. From this, (A.14) follows immediately.

□

Proof of Lemma 5.7

We want to show

Lemma 5.7. *Writing q as a function of t, we have*

1. $0 < q(t) < 1$ *for* $0 < t < t_c$,
2. $q(t)$ *as a function of t is continuous on* $(0, t_c)$,
3. $\lim_{t\to 0} q(t) = 0$,
4. $\lim_{t\to t_c} q(t) = 1$.

Proof. From [17] we know that we can write

$$q = \exp\left(-\pi K\left(\sqrt{1-k^2}\right)/K\left(k\right)\right),$$

where k is the elliptic modulus, with $0 < k < 1$, and $K(k)$ is the complete elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Using the explicit formula [97, (7.26)] for k, we see that $k \in (0,1)$ for $t \in (0, t_c)$, so the first point follows. Making use of the fact that zeros of a polynomial are generically continuous in its coefficients, we see that $q(t)$ is continuous for $t \in (0, t_c)$.

It is shown in [97, 7.4] that if $t \to 0$, then $k \to 1$, which implies that $\lim_{t\to 0} q(t) = 0$. By the same argument we can see that, using the notation as in [70, 97], if as $t \to t_c$ we have $x_2 \to x_3$, then forcibly $\lim_{t\to t_c} q(t) = 1$. But this can be seen with a straightforward adaptation of [70, Sec. 2.3] (note that for our model the discriminant vanishes for no t at either 0 or ∞). □

Remark: While not needed here, it seems plausible that $q(t)$ is in fact increasing for $t \in (0, t_c)$. A proof of this would likely come down to the study of the zeros x_1 to x_4 of the discriminant (see e.g. [97]) and get rather technical.

Proof of Lemma 5.8

In this section, our goal is to prove Lemma 5.8. To do so, we first show the preparatory Lemma A.2.

Lemma A.2. *Let S be a weighted step-set, let $Q(x,y,t)$ be the generating function for walks in the quadrant using this step-set and let r be the radius of convergence of $Q(0,0,t)$. If $\ell \in \mathbb{N}$ satisfies $[t^\ell]Q(0,0,t) \neq 0$, then each generating function $[x^a][y^b]Q(x,y,t)$ has no singularities t_c for $|t_c| \leq r$ except possibly at points $re^{\frac{2\pi ij}{\ell}}$ for $j = 0,1,\dots,\ell-1$.*

Proof. We start by writing

$$Q(x,y,t) = Q_0(x,y,t^\ell) + tQ_1(x,y,t^\ell) + \cdots + t^{\ell-1}Q_{\ell-1}(x,y,t^\ell),$$

so $t^jQ_j(x,y,t^\ell)$ counts walks whose length is j more than a multiple of ℓ. The radius of convergence of $[x^ay^b]t^jQ_j(x,y,t^\ell)$ is no less than r as its coefficients are bounded above by the coefficients of $[x^ay^b]Q(x,y,t)$. Hence each $[x^ay^b]Q_j(x,y,t)$ has no singularities in $|t| \leq r^\ell$. We will now prove that each $[x^ay^b]Q_j(x,y,t)$ has no singularities in $|t| = r^\ell$ except possibly at $t = r^\ell$. Note that $[x^ay^b]Q_j(x,y,t)$ counts weighted quadrant walks from $(0,0)$ to (a,b) which can be cut into succesive walks $w_1, w_2, w_3, \dots, w_m$ of length ℓ followed by a walk v of length j, where t counts the number of walks of length ℓ. Let Ω be a path of length ℓ from $(0,0)$ to $(0,0)$ in the quadrant and let $w_\Omega > 0$ be the weight of Ω. Now let $\tilde{Q}_j(x,y,t)$ be the generating function for walks counted by $Q_j(x,y,t)$, where none of the subpaths w_k is equal to Ω. Note then that any walk counted by $Q_j(x,y,t)$ can be uniquely constructed by taking a walk counted by $\tilde{Q}_j(x,y,t)$ and inserting any number of copies of Ω before each w_k and v. Hence

$$Q_j(x,y,t) = \frac{1}{1-w_\Omega t}\tilde{Q}_j\left(x,y,\frac{t}{1-w_\Omega t}\right).$$

Now, the series $F(t) := \tilde{Q}_j(x,y,t)$, has non-negative coefficients, so its radius of convergence $t_F > 0$ is a singularity of $F(t)$. Moreover, $F(t)$ satisfies

$$[x^a][y^b]Q_j(x,y,t) = \frac{1}{1-w_\Omega t}F\left(\frac{t}{1-w_\Omega t}\right).$$

Hence $[x^a][y^b]Q_j(x,y,t)$ has a corresponding singularity at $t_Q = \frac{t_F}{1+w_\Omega t_F} > 0$, so we must have $t_Q \geq r^\ell$. If $[x^a][y^b]Q_j(x,y,t)$ has another singularity t_0 satisfying $|t_0| = r^\ell$, then $\frac{t_0}{1-w_\Omega t_0}$ is a singularity of F, so

$$\frac{r^\ell}{|1-w_\Omega t_0|} = \frac{|t_0|}{|1-w_\Omega t_0|} \geq t_F = \frac{t_Q}{1-w_\Omega t_Q} \geq \frac{r^\ell}{1-w_\Omega t_Q},$$

hence

$$|1-w_\Omega t_0| \leq 1-w_\Omega t_Q \leq 1-w_\Omega r^\ell = 1-|w_\Omega t_0|.$$

By the triangle inequality this is only possible if $w_\Omega t_0 > 0$ i.e., $t_0 = r^\ell$. Hence $[x^a][y^b]Q_j(x,y,t)$ has no other singularities t_0 satisfying $|t_0| = r^\ell$. Therefore, the series $[x^a][y^b]t^jQ_j(x,y,t^\ell)$ has no singularities on the radius of convergence r except possibly at points $re^{\frac{2\pi ij}{\ell}}$ for $j = 0,1,\dots,\ell-1$. Since this is true for all j, it follows that the same statement holds for $[x^a][y^b]Q(x,y,t)$. $\square$

We can now proceed with the proof of Lemma 5.8.

Lemma 5.8. *Let $\mathcal{S}$ be a non-singular, weighted step-set, and let $Q(x, y)$ be the generating function* (2.1) *for walks in the quadrant using this step-set. Define the* period *of the model to be the maximum value k such that $Q(0, 0) = Q(0, 0; t) \in \mathbb{R}[t^k]$. If r is the radius of convergence of $Q(0, 0)$, then the singularities of $Q(0, 0)$ on the radius of convergence are precisely the points $re^{\frac{2\pi ij}{k}}$ for $j = 0, 1, \ldots, k-1$. The same result holds for the generating function $[x^a][y^b]Q(x, y)$ for any a, b for which this generating function is non-zero.*

Proof. First, since $Q_{a,b}(t) := [x^a][y^b]Q(x, y, t)$ is non-constant, it must have the same radius of convergence r as $Q(0, 0, t)$. Moreover, since $Q_{a,b}(t)$ has only non-negative coefficients, r must be a singularity. Since k is a period of the model, the powers of t appearing in the generating function $Q_{a,b}(t)$ must all have the same residue u modulo k. This means that for each integer j, we can write

$$Q_{a,b}(e^{\frac{2\pi ij}{k}}t) = e^{\frac{2\pi iju}{k}}Q_{a,b}(t),$$

so $re^{\frac{2\pi ij}{k}}$ is a singularity of $Q_{a,b}(t)$, as claimed. now suppose for the sake of contradiction that there is some other singularity $r\kappa$ on the radius of convergence, with $|\kappa| = 1$ but $\kappa^k \neq 1$. Let $\rho > 0$ be minimal such that $\kappa^\rho = 1$ (with $\rho := \infty$ and $\rho\mathbb{Z} := \{0\}$ if κ is not a root of unity). Then $k \notin \rho\mathbb{Z}$ since $\kappa^k \neq 1$. It follows from the maximality of k that there is some ℓ satisfying $[t^\ell]Q(0, 0, t) \neq 0$ and $\ell \notin \rho\mathbb{Z}$. From Lemma A.2, the singularity $r\kappa$ must satisfy $\kappa^\ell = 1$, but this is a contradiction as $\ell \notin \rho\mathbb{Z}$. $\square$

Appendix B

Additional examples

B.1 An example with large steps

Consider the model with steps $(1,0),(-1,0),(0,-1),(-2,1)$. In [19, Prop. 16] it is shown that

$$Q(x,y;t) = [x^> y^>] \frac{(x^2+1)(x+y)(y-x)(x^2y-2x-y)(x^3-x-2y)}{x^7y^3(1-tS(x,y))},$$

with $S(x,y)$ the step counting polynomial

$$S(x,y) = x + x^{-1} + y^{-1} + x^{-2}y.$$

We find the dominant saddle point to be at $s_0 := \left(\sqrt{3},\sqrt{3}\right)$, and one other saddle point at $s_1 = \left(-\sqrt{3},-\sqrt{3}\right)$ associated to it. We can check that we have[1], using the notation as in the proof of Thm. 4.1, $N(s_0) = 0$ (in particular it is not infinite), thus we can proceed in the same manner as in aforementioned theorem. We obtain

$$\gamma = 2\sqrt{3}, \quad c = 3,$$

and have

$$\begin{aligned}
v_1(k,l) &= \frac{16}{\pi} \cdot \sqrt{3}^{-1-k-l}(1+k)(1+l)(3+k+2l),\\
v_2(k,l) &= -\frac{2}{\pi}\sqrt{3}^{-1-k-l}(1+k)(1+l)(3+k+2l)(107+4k^2+32l+16l^2+8k(1+l)),\\
v_3(k,l) &= \frac{1}{8\pi}\sqrt{3}^{-3-k-l}(1+k)(1+l)(3+k+2l)(15205+16k^4+8672l+4976l^2+832l^3+256l^4+64k^3(1+l)\\
&\quad + 8k^2(157+48l+24l^2)+16k(149+157l+36l^2+16l^3)).
\end{aligned}$$

B.2 A three-dimensional example

Consider the model with steps $(-1,-1,-1),(-1,-1,1),(-1,1,0),(1,0,0)$. In [23, 4.3] it is shown that we have

$$Q(x,y,z;t) = \left[x^{>0}\right]\left[y^{>0}\right]\left[z^{>0}\right] \frac{\left(x-x^{-1}y-x^{-1}y^{-1}z-x^{-1}y^{-1}z^{-1}\right)\left(y-y^{-1}z-y^{-1}z^{-1}\right)\left(z-z^{-1}\right)}{xyz\left(1-tS(x,y,z)\right)},$$

with the step counting polynomial

$$S(x,y,z) = x^{-1}y^{-1}z^{-1} + x^{-1}y^{-1}z + x^{-1}y + x.$$

[1]This is in fact a consequence of $K(x,y)$ still being quadratic in y, which allows one to argue in a fashion similar as in the proof of Thm. 4.1.

We can find the dominant saddle point of $S(x,y,z)$ to be at $s_0 = \left(2^{3/4}, 2^{1/2}, 1\right)$, with 7 others associated to it. Using the notation as in (4.1), we can check that

$$\gamma = 2 \cdot 2^{3/4}, \quad c = \frac{7}{2},$$

and that we have

$$\begin{aligned}
v_1(k,l,m) =& \frac{2^6}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m),\\
v_2(k,l,m) =& -\frac{2^4}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m)(63-8k+2k^2-4l+4l^2+16m+8m^2),\\
v_3(k,l,m) =& \frac{2}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m)(5313-32k^3+4k^4-32l^3+16l^4+3040m\\
&+1776m^2+256m^3+64m^4-32k(43-3l+3l^2+12m+6m^2)\\
&+8l^2(93+16m+8m^2)+4k^2(99-4l+4l^2+16m+8m^2)-8l(103+48m+24m^2)).
\end{aligned}$$

Appendix C

Decomposition of polyharmonic functions

In this section we will, as in Sec. 4.2.2, once again give the expressions after the substitution $k \mapsto k-1$, $l \mapsto l-1$ etc.

C.1 Gouyou-Beauchamps

For the Gouyou-Beauchamps model with steps $\mathcal{S} = \{\leftarrow, \rightarrow, \nwarrow, \searrow\}$, we obtain as a basis of the polyharmonic functions (from here on, all functions will be given only up to constant multiples):

$$\begin{aligned}
h_1^1 =& kl(k+l)(k+2l), \\
h_1^2 =& kl(k+l)(k+2l)(42 - 7k^2 + k^4 - 14kl + 4k^3 l - 14l^2 - 8kl^3 - 4l^4), \\
h_1^3 =& kl(k+l)(k+2l)(52976 - 11880k^2 + 2211k^4 - 110k^6 + 3k^8 - 23760kl + 8844k^3 l - 660k^5 l + 24k^7 l \\
& - 23760l^2 + 4620k^2 l^2 - 1100k^4 l^2 + 32k^6 l^2 - 8448kl^3 - 144k^5 l^3 - 4224l^4 + 2200k^2 l^4 - 424k^4 l^4 \\
& + 2640kl^5 - 288k^3 l^5 + 880l^6 + 128k^2 l^6 + 192kl^7 + 48l^8), \\
h_2^1 =& kl(k+l)(k+2l)(-20 + k^2 + 2kl + 2l^2), \\
h_2^2 =& kl(k+l)(k+2l)(-3010 + 567k^2 - 78k^4 + k^6 + 1134kl - 312k^3 l + 6k^5 l + 1134l^2 - 84k^2 l^2 \\
& + 10k^4 l^2 + 456kl^3 + 228l^4 - 20k^2 l^4 - 24kl^5 - 8l^6), \\
h_3^1 =& kl(k+l)(k+2l)(-373 + k^2 + 3k^4 + 2kl + 12k^3 l + 2l^2 - 4k^2 l^2 - 32kl^3 - 16l^4).
\end{aligned}$$

In this case, due to the symmetry of the model we can pick $\tilde{h_n^m}(u,v) := h_n^m(u,v)$. We consider once again the first three $v_p(k,l,u,v)$ as appearing in Thm. 4.5. We obtain

$$\begin{aligned}
v_1(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v), \\
v_2(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(15 + 2k^2 + 4kl + 4l^2 + 2u^2 + 4uv + 4v^2), \\
v_3(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(1225 + 420k^2 + 20k^4 + 840kl + 80k^3 l + 840l^2 \\
& + 160k^2 l^2 + 160kl^3 + 80l^4 + 420u^2 + 48k^2 u^2 + 96klu^2 + 96l^2 u^2 + 20u^4 + 840uv \\
& + 96k^2 uv + 192kluv + 192l^2 uv + 80u^3 v + 840v^2 + 96k^2 v^2 + 192klv^2 + 192l^2 v^2 \\
& + 160u^2 v^2 + 160uv^3 + 80v^4).
\end{aligned}$$

We obtain

$$\begin{aligned}
v_1 =& h_1^1 \tilde{h_1^1}, \\
v_2 =& 2\left(h_2^1 \tilde{h_1^1} + h_1^1 \tilde{h_2^1}\right) + 95 h_1^1 \tilde{h_1^1}, \\
v_3 =& -40\left(h_3^1 \tilde{h_1^1} + h_1^1 \tilde{h_3^1}\right) + 48 h_2^1 \tilde{h_2^1} + 2400\left(h_2^1 \tilde{h_1^1} + h_1^1 \tilde{h_2^1}\right) + 140\left(h_1^2 \tilde{h_1^1} + h_1^1 \tilde{h_1^2}\right) + 36425 h_1^1 \tilde{h_1^1}.
\end{aligned}$$

Here we notice that the number of summands is noticably reduced: as the degree of h_i^j increases by 4 whenever j increases by one, there are a lot less suitable combinations of base functions.

C.2 Tandem walk

For the tandem walk, with steps $\mathcal{S} = \{\rightarrow, \downarrow, \nwarrow\}$, we have

$$\begin{aligned}
v_1(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v),\\
v_2(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(15+2k^2+4kl+4l^2+2u^2+4uv+4v^2),\\
v_3(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(1225+420k^2+20k^4+840kl+80k^3l+840l^2\\
&+160k^2l^2+160kl^3+80l^4+420u^2+48k^2u^2+96klu^2+96l^2u^2+20u^4+840uv\\
&+96k^2uv+192kluv+192l^2uv+80u^3v+840v^2+96k^2v^2+192klv^2+192l^2v^2\\
&+160u^2v^2+160uv^3+80v^4).
\end{aligned}$$

We can compute

$$\begin{aligned}
v_1(k,l,u,v) =& kluv(k+l)(u+v),\\
v_2(k,l,u,v) =& kl(k+l)uv(u+v)(20+3k^2+3kl+3l^2+3u^2+3uv+3v^2),\\
v_3(k,l,u,v) =& kl(k+l)uv(u+v)(881+375k^2+12k^3+18k^4+375kl+18k^2l+36k^3l+375l^2-18kl^2\\
&+54k^2l^2-12l^3+36kl^3+18l^4+375u^2+45k^2u^2+45klu^2+45l^2u^2-12u^3+18u^4\\
&+375uv+45k^2uv+45kluv+45l^2uv-18u^2v+36u^3v+375v^2+45k^2v^2+45klv^2\\
&+45l^2v^2+18uv^2+54u^2v^2+12v^3+36uv^3+18v^4).
\end{aligned}$$

We can check that, although the model is not symmetric, we can let $\tilde{h_n^m}(u,v) := h_n^m(v,u)$, and obtain

$$\begin{aligned}
v_1 =& h_1^1\tilde{h_1^1},\\
v_2 =& 3\left(h_2^1\tilde{h_1^1} + h_1^1\tilde{h_2^1}\right) + 38h_1^1\tilde{h_1^1},\\
v_3 =& 18\left(h_3^1\tilde{h_1^3} + h_1^1\tilde{h_3^1}\right) + 45h_2^1\tilde{h_2^1} + 660\left(h_2^1\tilde{h_1^1} + h_1^1\tilde{h_2^1}\right) + \frac{10}{3}\left(h_1^2\tilde{h_1^1} + h_1^1\tilde{h_1^2}\right) + 14780h_1^1\tilde{h_1^1}.
\end{aligned}$$

Note that even though we have $v_3(k,l,1,1) \neq v_3(1,1,k,l)$ (i.e. we cannot just exchange starting and endpoint and hope to obtain the same number of paths), the resulting representation for v_3 in terms of products of the base functions is symmetrical. This is, as mentioned in the remark after Thm. 4.7, still due to a symmetry property, using the fact that the polyharmonic and adjoint polyharmonic functions are the same: the number of paths from a point x to a point y is forcibly the same as the number of paths from y to x with the reversed step set.

Appendix D

Table of polyharmonic functions

This section contains the first three asymptotic terms of the 19 orbit-summable examples from [31]. Models 1 to 16 correspond to the models in [31, Table 1], models 17 and 18 to the first two in [31, Table 2], and finally model 19 is the same as the first one in [31, Table 3]. The remaining for models do not satisfy (2.18). All these models have, according to Thm. 4.1, asymptotics of the form

$$q(k,l;n) = \frac{a}{\pi} \cdot \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{3} \frac{v_p(k,l) \sum_{i=1}^{m} \alpha_i^{-k} \beta_i^{-l} \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^4}\right) \right]. \tag{D.1}$$

Here, a is a multiplicative constant (which one could just as well move into the v_p, but is separate here to keep things shorter). The constants α, β, γ are defined as in Section 2.1.4, that is, given our dominant saddle point (x_0, y_0) with $x_0, y_0 > 0$, and another saddle point (x_i, y_i) we have $\alpha_i = x_i/x_0, \beta_i = y_i/y_0$ and $\zeta_i = S(x_i, y_i)/S(x_0, y_0)$. In particular, they all have modulus 1. The number of relevant saddle points is given by m. Also, it will be noted for each example if and when the cancellations occur that lead to the coefficients to vanish, i.e. for which constellations of n, k and l we will have $q(k,l;n) = 0$ (which is also a direct consequence of (D.1) and can be checked to agree with basic combinatorial considerations).

Model 1 (Simple walk)

$\mathcal{S} = x + y + x^{-1} + y^{-1}$.
Relevant saddle points: $(1,1), (-1,-1)$.
$\gamma = 4$,
$c = 2$,
$a = 16$.
Periodicity: $q(k,l,n)$ vanishes if $j + k \not\equiv n \mod 2$.

$$\begin{aligned}
v_1(k,l) &= (1+k)(1+l),\\
v_2(k,l) &= -\frac{1}{2}(1+k)(1+l)(15 + 2k(2+k) + 2l(2+l)),\\
v_3(k,l) &= \frac{1}{4}(1+k)(1+l)(317 + 168l + 4(k(2+k)(21 + k(2+k)) + 4k(2+k)l\\
&\quad + (25 + 2k(2+k))l^2 + 4l^3 + l^4)).
\end{aligned}$$

Model 2 (Diagonal walk)

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}$.
Relevant saddle points: $(1,1), (-1,-1), (1,-1), (-1,1)$.

$\gamma = 4$,
$c = 2$,
$a = 2$.
Periodicity: $q(k,l,n)$ vanishes if $j + k \not\equiv n + k$ or $j + k \not\equiv \mod 2$.

$$\begin{aligned}
v_1(k,l) =& (1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{2}(1+k)(1+l)(9+k(2+k)+l(2+l)),\\
v_3(k,l) =& \frac{1}{4}(1+k)(1+l)(113+4k^3+k^4+4k(13+l(2+l))\\
&+2k^2(15+l(2+l))+l(2+l)(26+l(2+l))).
\end{aligned}$$

Model 3

$\mathcal{S} = xy + y + x^{-1}y + x^{-1}y + x^{-1} + x^{-1}y^{-1}$.
Relevant saddle points: $(1,1),(1,-1)$.
$\gamma = 6$,
$c = 2$,
$a = \frac{3\sqrt{6}}{2}$.
Periodicity: $q(k,l;n)$ vanishes if $l \not\equiv n \mod 2$.

$$\begin{aligned}
v_1(k,l) =& \frac{2\sqrt{6}}{\pi}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{16}(1+k)(1+l)(75+12k(2+k)+8l(2+l)),\\
v_3(k,l) =& \frac{1}{512}(1+k)(1+l)(7793+3424l+8(3k(2+k)(103+6k(2+k))+48k(2+k)l\\
&+6(41+4k(2+k))l^2+32l^3+8l^4)).
\end{aligned}$$

Model 4 (King's walk)

$\mathcal{S} = xy + y + x^{-1}y + x + x^{-1} + xy^{-1} + x^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $(1,1)$.
$\gamma = 8$,
$c = 2$,
$a = \frac{128}{27}$.

$$\begin{aligned}
v_1(k,l) =& (1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{3}(1+k)(1+l)(15+2k(2+k)+2l(2+l)),\\
v_3(k,l) =& \frac{1}{18}(1+k)(1+l)(307+168l+4(k(2+k)(21+k(2+k))\\
&+4k(2+k)l+(25+2k(2+k))l^2+4l^3+l^4)).
\end{aligned}$$

Model 5

$\mathcal{S} = xy + x^{-1}y + y^{-1}$.
Relevant saddle points: $\left(1, \frac{1}{\sqrt{2}}\right), \left(1, -\frac{1}{\sqrt{2}}\right), \left(-1, \frac{i}{\sqrt{2}}\right), \left(-1, \frac{-i}{\sqrt{2}}\right)$.
$\gamma = 2\sqrt{2}$,

$c = 2$,
$a = 4\sqrt{2}$.
Periodicity: $q(k,l;n)$ vanishes if $n+l \equiv 0 \mod 4 \wedge k \equiv 0 \mod 2$ or $n+l \equiv 2 \mod 4 \wedge k \equiv 1 \mod 2$.

$$\begin{aligned}
v_1(k,l) =& 2^{l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{4}2^{l/2}(1+k)(1+l)(27+8k+4k^2+10l+2l^2),\\
v_3(k,l) =& \frac{1}{32}2^{l/2}(1+k)(1+l)(1049+64k^3+16k^4+748l+264l^2+40l^3\\
&+4l^4+16k(43+14l+2l^2)+8k^2(51+14l+2l^2)).
\end{aligned}$$

Model 6

$\mathcal{S} = xy + x^{-1}y + y^{-1} + x + x^{-1}$.
Relevant saddle point: $\left(1, \frac{1}{\sqrt{2}}\right)$.
$\gamma = 2 + 2\sqrt{2}$,
$c = 2$,
$a = 2\sqrt{7+5\sqrt{2}}$.

$$\begin{aligned}
v_1(k,l) =& 2^{l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{-239+169\sqrt{2}}{2}2^{l/2}(1+k)(1+l)(116592+82443\sqrt{2}\\
&+4(3363+2378\sqrt{2})k(2+k)+2l(26327+18616\sqrt{2}+(8119+5741\sqrt{2})l)),\\
v_3(k,l) =& \frac{-47321+33461\sqrt{2}}{16}2^{l/2}(787537656+556873217\sqrt{2}\\
&+32(941664+665857\sqrt{2})k^3+8(941664+665857\sqrt{2})k^4\\
&+16k(24218939+17125376\sqrt{2}+l(9255076+6544327\sqrt{2}(2273378+1607521\sqrt{2})l))\\
&+8k^2(27985595+19788804\sqrt{2}+l(9255076+6544327\sqrt{2}+(2273378+1607521\sqrt{2})l))\\
&+2l(345292494+244158664\sqrt{2}+l(168998590+119500049\sqrt{2}\\
&+l(35593948+25168722\sqrt{2}+(5488420+3880899\sqrt{2})l)))).
\end{aligned}$$

Model 7

$\mathcal{S} = y + y^{-1} + xy + x^{-1}y$.
Relevant saddle points: $\left(1, \frac{1}{\sqrt{3}}\right), \left(1, -\frac{1}{\sqrt{3}}\right)$.
$\gamma = 2\sqrt{3}$,
$c = 2$,
$a = 6\sqrt{3}$.
Periodicity: $q(k,l;n)$ vanishes if if $n+l \not\equiv 0 \mod 2$.

$$\begin{aligned}
v_1(k,l) =& 3^{l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{8}3^{l/2}(1+k)(1+l)(57+24k+12k^2+20l+4l^2),\\
v_3(k,l) =& \frac{1}{128}3^{l/2}(1+k)(1+l)(4649+576k^3+144k^4+3160l+1080l^2+160l^3\\
&+16l^4+48k(85+28l+4l^2)+24k^2(109+28l+4l^2)).
\end{aligned}$$

Model 8

$\mathcal{S} = xy + y + x^{-1}y + x + x^{-1} + y^{-1}$.
Relevant saddle point: $\left(1, \frac{1}{\sqrt{3}}\right)$.
$\gamma = 2 + 2\sqrt{3}$,
$c = 2$,
$a = 2\sqrt{10 + 6\sqrt{3}}$.

$$
\begin{aligned}
v_1(k,l) =& 3^{l/2}(1+k)(1+l),\\
v_2(k,l) =& \frac{-332313 + 191861\sqrt{3}}{48} 3^{l/2}(1+k)(1+l)(16850187 + 9728460\sqrt{3}\\
&+ 12(191861 + 110771\sqrt{3})k(2+k) + 4l(1623931 + 937577\sqrt{3} + (524174 + 302632\sqrt{3})l)),
\end{aligned}
$$

$$
\begin{aligned}
v_3(k,l) =& \frac{897909603 - 518408351\sqrt{3}}{4608} 3^{l/2}(1+k)(1+l)\\
&\cdot (3(6408751361721 + 3700094323859\sqrt{3}) + 1152(897909603 + 518408351\sqrt{3})k^3\\
&+ 288(897909603 + 518408351\sqrt{3})k^4 + 16(925894850856 + 534565641383\sqrt{3})l\\
&+ 16(434596264689 + 250914270407\sqrt{3})l^2 + 128(10381790502 + 5993929541\sqrt{3})l^3\\
&+ 64(3351044259 + 1934726305\sqrt{3})l^4 + 96k(113649411684 + 65615518429\sqrt{3}\\
&+ 4(9395817327 + 5424677663\sqrt{3})l + 8(1226567328 + 708158977\sqrt{3})l^2)\\
&+ 48k^2(135199242156 + 78057318853\sqrt{3} + 4(9395817327 + 5424677663\sqrt{3})l\\
&+ 8(1226567328 + 708158977\sqrt{3})l^2)).
\end{aligned}
$$

Model 9

$\mathcal{S} = xy + y + x^{-1}y + x^{-1}y^{-1} + xy^{-1}$.
Relevant saddle points: $\left(1, \sqrt{\frac{2}{3}}\right), \left(1, -\sqrt{\frac{2}{3}}\right)$.
$\gamma = 2\sqrt{6}$,
$c = 2$,
$a = \frac{12\sqrt{6}}{5\sqrt{5}}$.
Periodicity: $q(k,l;n)$ is non-zero if $l + n \equiv 0 \mod 2$.

$$
\begin{aligned}
v_1(k,l) =& \left(\frac{3}{2}\right)^{l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{2}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)(93 + 24k + 12k^2 + 14l + 10l^2),\\
v_3(k,l) =& \frac{1}{20000}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)(59581 + 2880k^3 + 720k^4 + 20540l + 14400l^2 + 1400l^3\\
&+ 500l^4 + 240k(133 + 10l + 10l^2) + 120k^2(157 + 10l + 10l^2)).
\end{aligned}
$$

Model 10

$\mathcal{S} = xy + y + x^{-1}y + x + x^{-1} + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $\left(1, \sqrt{\frac{2}{3}}\right)$.

$\gamma = 2 + 2\sqrt{6}$,
$c = 2$,
$a = \frac{2\sqrt{(15893+4923\sqrt{6}}}{19\sqrt{19}}$.

$$
\begin{aligned}
v_1(k,l) &= \left(\frac{3}{2}\right)^{l/2} (1+k)(1+l), \\
v_2(k,l) &= -\frac{\sqrt{1/5(-19+9\sqrt{6})}}{(21660(1+\sqrt{6})^{3/2}(4027973401873 + 1644413252328\sqrt{6})} \left(\frac{3}{2}\right)^{l/2} (1+k)(1+l) \\
&\quad \cdot (45(18505016772606410202221 + 7554641462421834757881\sqrt{6}) \\
&\quad + 114(8498794022835323792 58 + 3469618130827117730 63\sqrt{6})k(2+k) \\
&\quad + 19l(10112697572977267931562 + 4128491496146047169957\sqrt{6} \\
&\quad + 19(3332193626270490807 72 + 1360362351419521707 67\sqrt{6})l)),
\end{aligned}
$$

$$
\begin{aligned}
v_3(k,l) =& \frac{\sqrt{-95+45\sqrt{6}}}{312770400(1+\sqrt{6})^{39/2}(7+2\sqrt{6})^5(2390878567 + 976071762\sqrt{6})} \left(\frac{3}{2}\right)^{l/2} (1+k)(1+l) \\
& \cdot (227854743198721887169974924310262 19 + 93021309384960729148277350953754 59\sqrt{6} \\
& + 190(5871497127223435343063362325843 8 + 2397028664652450295227100063879 3\sqrt{6})l \\
& + 38(2736(86640690655716693926703034 09 + 35370913844804731165412950 49\sqrt{6})k^3 \\
& + 684(86640690655716693926703034 09 + 35370913844804731165412950 49\sqrt{6})k^4 \\
& + 19l^2(7(15000641943020052698954122627 01 + 61239864290984569707201676228 6\sqrt{6}) \\
& + 19l(808415333998817599308122479 18 + 330034178089790122266143184 48\sqrt{6} \\
& + 19(13318881554059560613996041 79 + 54374106253354947517862376 9\sqrt{6})l)) \\
& + 12k(2421838485513162257281543352588 6 + 9887114214903395698238202393071\sqrt{6} \\
& + 38l(857654419416240243618470576 33 + 350135950535457043680470820 63\sqrt{6} \\
& + 19(339699440182848345592627828 1 + 138681715726179131449351466 6\sqrt{6})l)) \\
& + 6k^2(5(56338400698064607631746183760 78 + 23000055772452982878762065870 83\sqrt{6}) \\
& + 38l(857654419416240243618470576 33 + 350135950535457043680470820 63\sqrt{6} \\
& + 19(339699440182848345592627828 1 + 138681715726179131449351466 6\sqrt{6})l)))).
\end{aligned}
$$

Model 11

$\mathcal{S} = y + xy^{-1} + y^{-1} + x^{-1}y^{-1}$.
Relevant saddle points: $\left(1, \sqrt{3}\right), \left(1, -\sqrt{3}\right)$.
$\gamma = 2\sqrt{3}$,
$c = 2$,
$a = 6\sqrt{3}$.

Periodicity: $q(k,l;n)$ is non-zero if $l+n \equiv 0 \mod 2$.

$$\begin{aligned}
v_1(k,l) =& 3^{-l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{8}3^{-l/2}(1+k)(1+l)(57+12k(2+k)+4(-1+l)l),\\
v_3(k,l) =& \frac{1}{128}3^{-l/2}(1+k)(1+l)(4649-824l+8(3k(2+k)(85+6k(2+k))-36k(2+k)l\\
&+3(29+4k(2+k))l^2-4l^3+2l^4)).
\end{aligned}$$

Model 12

$\mathcal{S} = y + xy^{-1} + y^{-1} + x^{-1}y^{-1} + x + x^{-1}$.
Relevant saddle point: $\left(1, \sqrt{3}\right)$.
$\gamma = 2 + 2\sqrt{3}$,
$c = 2$,
$a = 2\sqrt{10+6\sqrt{3}}$.

$$\begin{aligned}
v_1(k,l) =& 3^{-l/2}(1+k)(1+l),\\
v_2(k,l) =& \frac{191861\sqrt{3}-332313}{48}3^{-l/2}(1+k)(1+l)(16850187+9728460\sqrt{3}+12(191861\\
&+110771\sqrt{3})k(2+k)+4l(472765+272951\sqrt{3}+(524174+302632\sqrt{3})l)),\\
v_3(k,l) =& \frac{897909603-518408351\sqrt{3}}{4608}3^{-l/2}(1+k)(1+l)(3(6408751361721+3700094323859\sqrt{3})\\
&+1152(897909603+518408351\sqrt{3})k^3+288(897909603+518408351\sqrt{3})k^4\\
&+96k(113649411684+65615518429\sqrt{3}+4l(416721297+240594153\sqrt{3}\\
&+2(1226567328+708158977\sqrt{3})l))+48k^2(135199242156+78057318853\sqrt{3}\\
&+4l(416721297+240594153\sqrt{3}+2(1226567328+708158977\sqrt{3})l))\\
&+16l(254773295988+147093431021\sqrt{3}+l(316845801201+182931008615\sqrt{3}\\
&+4l(6044773068+3489951358\sqrt{3}+(3351044259+1934726305\sqrt{3})l)))).
\end{aligned}$$

Model 13

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1} + y^{-1}$.
Relevant saddle points: $\left(1, \sqrt{\frac{3}{2}}\right), \left(1, -\sqrt{\frac{3}{2}}\right)$.
$\gamma = 2\sqrt{6}$,
$c = 2$,
$a = \frac{12\sqrt{6}}{5\sqrt{5}}$.

Periodicity: $q(k,l;n)$ is non-zero if $n+l \equiv 0 \mod 2$.

$$
\begin{aligned}
v_1(k,l) &= \left(\frac{2}{3}\right)^{l/2}(1+k)(1+l),\\
v_2(k,l) &= -\frac{1}{20}\left(\frac{2}{3}\right)^{l/2}(1+k)(1+l)(93+12k(2+k)+2l(13+5l)),\\
v_3(k,l) &= \frac{1}{4000}\left(\frac{2}{3}\right)^{l/2}(1+k)(1+l)(59581+32660l+20(6k(2+k)(133+6k(2+k))\\
&\quad +180k(2+k)l+60(14+k(2+k))l^2+130l^3+25l^4)).
\end{aligned}
$$

Model 14

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1} + x + x^{-1} + y^{-1}$.
Relevant saddle point: $\left(1, \sqrt{\frac{3}{2}}\right)$.
$\gamma = 2 + 2\sqrt{6}$,
$c = 2$,
$a = \frac{15893+4923\sqrt{6}}{19\sqrt{19}}$.

$$
\begin{aligned}
v_1(k,l) &= \left(\frac{2}{3}\right)^{l/2}(1+k)(1+l),\\
v_2(k,l) &= -\frac{\sqrt{5}}{4332(1+\sqrt{6})^{23/2}(119287+48682\sqrt{6})}\left(\frac{2}{3}\right)^{l/2}(1+k)(1+l)\\
&\quad \cdot (45(923103972550017581+376855618721953841\sqrt{6})\\
&\quad +114(423953710538905 38+173078377568431 43\sqrt{6})k(2+k)\\
&\quad +19l(758833876782694710+309792632946741935\sqrt{6}+19(166223095689742 92\\
&\quad +6786029465163487\sqrt{6})l)),\\
v_3(k,l) &= \frac{\sqrt{5}(847+342\sqrt{6})}{12510816(1+\sqrt{6})^{39}2(7+2\sqrt{6})^5(2390878567+976071762\sqrt{6})}\left(\frac{2}{3}\right)^{l/2}(1+k)(1+l)\\
&\quad (227854743198721887169974924310 26219+930213093849607291482773509537 5459\sqrt{6}\\
&\quad +38(4072786415407560526455994506114 74+1662708091514581230216415732136 39\sqrt{6})l\\
&\quad +38(2736(866406906557166939267030340 9+353709138448047311654129504 9\sqrt{6})k^3\\
&\quad +684(866406906557166939267030340 9+353709138448047311654129504 9\sqrt{6})k^4\\
&\quad +19l^2(12049478807347825345170268735043+4919179124080296718913160023698\sqrt{6}\\
&\quad +19l(1216054662218235614019275872 90+4964522369612050800053649444 0\sqrt{6}\\
&\quad +19(133188815540595606139960417 9+543741062533549475178623769\sqrt{6})l))\\
&\quad +12k(2421838485513162257281543352 5886+988711421490339569823820239 3071\sqrt{6}\\
&\quad +38l(172406132597340718288550091723+70384508898350435533460032553\sqrt{6}\\
&\quad +19(339699440182848345592627828 1+138681715726179131449351466 6\sqrt{6})l))\\
&\quad +6k^2(5(56338400698064607631746183760 78+230000557724529828787620658708 3\sqrt{6})\\
&\quad +38l(172406132597340718288550091723+70384508898350435533460032553\sqrt{6}\\
&\quad +19(339699440182848345592627828 1+138681715726179131449351466 6\sqrt{6})l)))).
\end{aligned}
$$

Model 15

$\mathcal{S} = y + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle points: $\left(1,\sqrt{2}\right), \left(1,-\sqrt{2}\right), \left(-1,i\sqrt{2}\right), \left(-1,-i\sqrt{2}\right)$.
$\gamma = 2\sqrt{2}$,
$c = 2$,
$a = 4\sqrt{2}$.
Periodicity: $q(k,l;n)$ is non-zero if $n+l \equiv 0 \mod 4 \wedge k \equiv 1 \mod 2$ or $n+l \equiv 2 \mod 3 \wedge k \equiv 0 \mod 2$.

$$\begin{aligned}
v_1(k,l) =& 2^{-l/2}(1+k)(1+l),\\
v_2(k,l) =& -\frac{1}{4}2^{-l/2}(1+k)(1+l)27+4k(2+k)+2(-1+l)l),\\
v_3(k,l) =& \frac{1}{32}2^{-l/2}(1+k)(1+l)(1049+8k(2+k)(43+2k(2+k))-188l-48k(2+k)l\\
&+8(21+2k(2+k))l^2-8l^3+4l^4).
\end{aligned}$$

Model 16

$\mathcal{S} = y + x + x^{-1} + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $\left(1,\sqrt{2}\right)$.
$\gamma = 2+2\sqrt{2}$,
$c = 2$,
$a = 2\sqrt{7+5\sqrt{2}}$.

$$\begin{aligned}
v_1(k,l) =& 2^{-l/2}(1+k)(1+l),\\
v_2(k,l) =& \frac{4756-3363\sqrt{2}}{8}2^{-l/2}(1+k)(1+l)(116592+82443\sqrt{2}\\
&+4(3363+2378\sqrt{2})k(2+k)+2l(6149+4348\sqrt{2}+(8119+5741\sqrt{2})l)),\\
v_3(k,l) =& \frac{-941664+665857\sqrt{2}}{64}2^{-l/2}(1+k)(1+l)(787537656+556873217\sqrt{2}\\
&+32(941664+665857\sqrt{2})k^3+8(941664+665857\sqrt{2})k^4+16k(24218939\\
&+17125376\sqrt{2}+l(-161564-114243\sqrt{2}+(2273378+1607521\sqrt{2})l))\\
&+8k^2(27985595+19788804\sqrt{2}+l(-161564-114243\sqrt{2}+(2273378+1607521\sqrt{2})l))\\
&+2l(77265162+54634720\sqrt{2}+l(114437518+80919545\sqrt{2}+l(8313412+5878470\sqrt{2}\\
&+(5488420+3880899\sqrt{2})l)))).
\end{aligned}$$

Model 17 (Tandem walk)

$\mathcal{S} = y + x^{-1} + xy^{-1}$.
Relevant saddle points: $(1,1), \left(e^{4\pi i/3}, e^{2\pi i/3}\right), \left(e^{2\pi i/3}, e^{4\pi i/3}\right)$.
$\gamma = 3$,
$c = 3$,
$a = \frac{27\sqrt{3}}{2}$.

Periodicity: $q(k,l;n)$ is non-zero if $2k+l \equiv n \mod 3$.

$$\begin{aligned}
v_1(k,l) =&(1+k)(1+l)(2+k+l),\\
v_2(k,l) =&-\frac{1}{3}(1+k)(1+l)(2+k+l)(38+3k^2+9l+3l^2+3k(3+l)),\\
v_3(k,l) =&\frac{1}{18}(1+k)(1+l)(2+k+l)(1930+9k^4+954l+417l^2+60l^3+9l^4\\
&+6k^3(8+3l)+3k^2(121+33l+9l^2)+3k(300+157l+39l^2+6l^3)).
\end{aligned}$$

Model 18

$\mathcal{S} = y + x + xy^{-1} + y^{-1} + x^{-1} + x^{-1}y$.
Relevant saddle point: $(1,1)$.
$\gamma = 6$,
$c = 3$,
$a = \frac{27\sqrt{3}}{2}$.

$$\begin{aligned}
v_1(k,l) =&(1+k)(1+l)(2+k+l),\\
v_2(k,l) =&-(1+k)(1+l)(2+k+l)(11+k^2+k(3+l)+l(3+l)),\\
v_3(k,l) =&\frac{9}{104}(1+k)(1+l)(2+k+l)(940+6k^4+12k^3(3+l)+3l(3+l)(57+2l(3+l))\\
&+3k(3+l)(57+4l(3+l))+9k^2(25+2l(4+l))).
\end{aligned}$$

Model 19 (Gouyou-Beauchamps)

$\mathcal{S} = x + x^{-1} + xy^{-1} + x^{-1}y$.
Relevant saddle points: $(1,1), (-1,1)$.
$\gamma = 4$,
$c = 4$,
$a = 64$.
Periodicity: $q(k,l;n)$ is non-zero if $k+n \equiv 0 \mod 2$.

$$\begin{aligned}
v_1(k,l) =&(1+k)(1+l)(2+k+l)(3+k+2l),\\
v_2(k,l) =&-\frac{1}{2}(1+k)(1+l)(2+k+l)(3+k+2l)(35+2k^2+4k(2+l)+4l(3+l)),\\
v_3(k,l) =&\frac{1}{8}(1+k)(1+l)(2+k+l)(3+k+2l)(25+2k^2+4k(2+l)\\
&+4l(3+l))(61+2k^2+4k(2+l)+4l(3+l)).
\end{aligned}$$