RATE-OPTIMAL GOAL-ORIENTED ADAPTIVE FEM FOR SEMILINEAR ELLIPTIC PDES

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ABSTRACT

The talk presents some results of our recent work [1]: Let $\mathcal{X} := H_0^1(\Omega)$. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and given $f, g \in L^2(\Omega)$, we aim to approximate the linear goal quantity

$$G(u) := \int_\Omega gu \, \mathrm{d}x$$

where $u \in \mathcal{X}$ is the weak solution of the semilinear elliptic PDE

$$(1) \qquad -\mathrm{div}(\boldsymbol{A}\nabla u) + b(u) = f \quad \text{in } \Omega \quad \text{subject to} \quad u = 0 \quad \text{on } \partial\Omega.$$

Here, the diffusion matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is uniformly positive definite, and the smooth nonlinearity $b(\cdot)$ is monotone and satisfies certain growth conditions. For a FEM subspace $\mathcal{X}_H \subseteq \mathcal{X}$, the discrete formulation of the *primal problem* (1) reads: Find $u_H \in \mathcal{X}_H$ such that

$$(2) \qquad \langle \boldsymbol{A}\nabla u_H, \nabla v_H\rangle + \langle b(u_H), v_H\rangle = \langle f, v_H\rangle \quad \text{for all } v_H \in \mathcal{X}_H,$$

where $\langle v, w\rangle := \int_\Omega vw \, \mathrm{d}x$ denotes the $L^2(\Omega)$-scalar product.

We approximate $G(u)$ by means of the computable quantity $G(u_H)$. The optimal error control of the goal error $G(u) - G(u_H)$ involves the *(practical) dual problem*: Find $z_H[u_H] \in \mathcal{X}_H$ such that

$$\langle \boldsymbol{A}\nabla z_H[u_H], \nabla v_H\rangle + \langle b'(u_H) z_H[u_H], v_H\rangle = G(v_H) \quad \text{for all } v_H \in \mathcal{X}_H.$$

We prove the goal error estimate

$$C^{-1}|G(u) - G(u_H)| \;\le\; \|u - u_H\|_{\mathcal{X}} \, \|z[u_H] - z_H[u_H]\|_{\mathcal{X}} + \|u - u_H\|_{\mathcal{X}}^2.$$

Based on residual error estimators, we formulate a goal-oriented adaptive algorithm (GOAFEM), which guarantees convergence and, as the main contribution, optimal algebraic convergence rates.

References

[1] Roland Becker, Maximilian Brunner, Michael Innerberger, Jens Markus Melenk, and Dirk Praetorius, *Goal-oriented adaptive finite element method for semilinear elliptic PDEs*, arXiv:2112.06687, 2021.

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