CANONICAL SYSTEMS WHOSE WEYL COEFFICIENTS HAVE DOMINATING REAL PART

By

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Abstract. For a two-dimensional canonical system $y'(t) = zJH(t)y(t)$ on the half-line $(0, \infty)$ whose Hamiltonian H is a.e. positive semi-definite, denote by q_H its Weyl coefficient. De Branges' inverse spectral theorem states that the assignment $H \mapsto q_H$ is a bijection between Hamiltonians (suitably normalised) and Nevanlinna functions.
The main result of the paper is a criterion when the singular integral of the spectral measure, i.e. $\operatorname{Re} q_H(iy)$, dominates its Poisson integral $\operatorname{Im} q_H(iy)$ for $y \to +\infty$. Two equivalent conditions characterising this situation are provided. The first one is analytic in nature, very simple, and explicit in terms of the primitive M of H. It merely depends on the relative size of the off-diagonal entries of M compared with the diagonal entries. The second condition is of geometric nature and technically more complicated. It involves the relative size of the off-diagonal entries of H, a measurement for oscillations of the diagonal of H, and a condition on the speed and smoothness of the rotation of H.

1 Introduction

We investigate the spectral theory of two-dimensional **canonical systems**

$$y'(t) = zJH(t)y(t), \quad t \in (a, b), \tag{1.1}$$

where $-\infty < a < b \leq \infty$, $z \in \mathbb{C}$ is the spectral parameter, J is the symplectic matrix $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and H is the Hamiltonian of the system. We deal with systems whose Hamiltonian satisfies

- ▷ $H(t) \in \mathbb{R}^{2\times 2}$ and $H(t) \geq 0$ a.e.;
- ▷ for all $c \in (a, b)$ we have $\int_a^c \operatorname{tr} H(s)\, ds < \infty$;
- ▷ $H(t) \not= 0$ a.e.

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We further assume that H is in the limit point case at the right endpoint b, i.e.,

$$\int_a^b \operatorname{tr} H(s)\,\mathrm{d}s = \infty. \tag{1.2}$$

A central role in the theory of such equations is played by the Weyl coefficient q_H associated with H. For Sturm–Liouville equations its construction goes back to H. Weyl [Wey10]. Let us recall the definition of q_H for canonical systems. To this end, let $W(t,z)$ be the (transpose of) the fundamental solution of the system (1.1), i.e., the unique 2×2-matrix-valued solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(t,z)J = zW(t,z)H(t), & t \in [a,b), \\ W(a,z) = I. \end{cases}$$

Note that the transposes of the rows of W are solutions of (1.1), and let us write

$$W(t,z) = \begin{pmatrix} w_{11}(t,z) & w_{12}(t,z) \\ w_{21}(t,z) & w_{22}(t,z) \end{pmatrix}.$$

If (1.2) is satisfied, then the following limit exists and is independent of ζ in the closed upper half-plane $\mathbb{C}^+ \cup \mathbb{R}$:

$$q_H(z) := \lim_{t \to b} \frac{w_{11}(t,z)\zeta + w_{12}(t,z)}{w_{21}(t,z)\zeta + w_{22}(t,z)}, \qquad z \in \mathbb{C} \setminus \mathbb{R};$$

the function q_H is called the **Weyl coefficient** associated with the Hamiltonian H. It is a Nevanlinna function or identically equal to ∞ (when $h_2(t) = 0$ for a.e. $t \in (a,b)$); a **Nevanlinna function**[1] is a function that is analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies $q_H(\overline{z}) = \overline{q_H(z)}$ and $\operatorname{Im} q_H(z) \cdot \operatorname{Im} z \geq 0$ for all z. The significance of the Weyl coefficient is that the measure μ in its Herglotz integral representation

$$q_H(z) = \alpha + \beta z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2} \Big)\,\mathrm{d}\mu$$

is a spectral measure for the differential operator constructed from the equation (1.1) (when $\beta > 0$, this differential operator is actually multi-valued and one can include a point mass at infinity with mass β).

A famous theorem by L. de Branges [Bra68] says that the assignment $H \mapsto q_H$ establishes a bijective correspondence between the set of all suitably normalised Hamiltonians on the one hand, and the set of all Nevanlinna functions on the other hand. In view of de Branges' correspondence, it is a natural task to translate properties from H to q_H (i.e., **direct spectral relations**) and vice versa from q_H

[1] Sometimes in the literature the terminology **Herglotz function** is used instead.

to H (i.e., **inverse spectral relations**). In the best case one can go both ways. For illustration, let us mention two examples of such theorems. It is possible to explicitly characterise those Hamiltonians H for which q_H has an analytic continuation to $\mathbb{C} \setminus [0, \infty)$, see [Win98], or those Hamiltonians for which q_H has a meromorphic continuation to all of $\mathbb{C}$, see [RW20]. The first result characterises that the differential operator associated with (1.1) is non-negative, the second one that it has discrete spectrum.

In the present paper we prove a direct and inverse spectral relation of a different kind. It belongs to a family of results which relate the behaviour of H locally at the left endpoint a with the behaviour of q_H when z tends to $+i\infty$; for physical reasons one also speaks of the **high-energy behaviour** of q_H. Recall that the behaviour of $\operatorname{Im} q_H(iy)$ at $+\infty$ is related to the behaviour of the spectral measure at $\pm\infty$; see, e.g., [LPW21, Section 4]. Our main result is Theorem 1.1 stated further below, where we characterise those Hamiltonians H for which[2]

$$\operatorname{Im} q_H(iy) \ll |q_H(iy)|, \qquad y \to +\infty, \tag{1.3}$$

i.e., those Hamiltonians for which the singular integral $\operatorname{Re} q_H(z)$ of the spectral measure strictly dominates the Poisson integral $\operatorname{Im} q_H(z)$.

In our theorem, where (1.3) is listed as item (i), we give two different conditions on H, called (ii) and (iii), which are both equivalent to (1.3). Condition (ii) is analytic in nature, very simple, and explicit in terms of the primitive

$$M(t) := \int_a^t H(s)\, \mathrm{d}s$$

of H, which is a non-negative and non-decreasing matrix function. It says that, locally at a, the off-diagonal entries of $M(t)$ should be as large as its diagonal entries. Condition (iii) is of geometric nature and somewhat more complicated. It involves the relative size of the off-diagonal entries of H compared with the diagonal entries, a measurement for oscillations of the diagonal of H, and a condition on the speed and smoothness of the "rotation" of H.

From a function-theoretic perspective, the behaviour exhibited by (1.3) is rather peculiar. For every Nevanlinna function q one has that for (in a measure-theoretic sense) most points on the boundary of the open upper half-plane (including $+i\infty$) condition (1.3) fails; see [Pol03] and recall that real and imaginary parts are comparable on approaching almost every point of the absolutely continuous spectrum. On the other hand, for a certain subclass of Nevanlinna functions it holds that for (in a topological sense) many boundary points (1.3) holds, cf. [Don01, Theorem 1]

[2] We use the notation "$f \ll g$" for $f/g \to 0$.

where one uses a curve that approaches the boundary tangentially. Neither of these statements has any implication for a single boundary point (in our case $+i\infty$). The condition (iii) in Theorem 1.1 is a very strong restriction on H. Hence, one message of Theorem 1.1 is that (1.3), i.e., strict dominance of the singular integral at a specific boundary point, is a rather rare phenomenon.

Our interest in the class of Hamiltonians with (1.3) originates from the recent result [LPW21, Theorem 1.1]. In this theorem we showed that, for every Hamiltonian H, the following estimates[3]

$$|q_H(iy)| \asymp A_H(y) \quad \text{and} \quad L_H(y) \lesssim \operatorname{Im} q_H(iy) \lesssim A_H(y) \quad \text{for } y \geq 1 \tag{1.4}$$

hold, where $L_H(y)$ and $A_H(y)$ are certain functions defined explicitly in terms of the primitive $M(t)$, and the constants in "$\asymp$" and "$\lesssim$" are independent of H; we recall details in Section 2.6. The question arises whether the lower bound $L_H(y)$ is sharp. The equivalence of (1.3) with Theorem 1.1 (ii) says that on a qualitative level the answer is affirmative: we have

$$\operatorname{Im} q_H(iy) \ll |q_H(iy)| \quad \Leftrightarrow \quad L_H(iy) \ll A_H(iy).$$

It is an open problem if there is a quantitative relation between $\operatorname{Im} q_H(iy)$ and $L_H(iy)$ (assuming that $\operatorname{Im} q_H(iy) \ll |q_H(iy)|$ and thinking up to universal multiplicative constants). This seems to be a rather involved question, and we expect that the equivalence of (1.3) with Theorem 1.1 (iii) will be of help to attack it.

Let us give a brief overview of the contents of the paper. In the remainder of the Introduction we formulate the main theorem, Theorem 1.1, and a sequence variant, Theorem 1.4, and provide an illustrative example. In Section 2 we provide some preliminaries and set up notation. Section 3 contains the proof of the equivalence of (i) and (ii) in our main results. Section 4 contains preparations for the proof of the equivalence with (iii), which is then carried out in Section 5. Finally, in Section 6 we consider the situation when the diagonal entries of H, or their primitives, are regularly varying.

Formulation of the main theorem. We formulate our main theorem for Hamiltonians that satisfy

▷ $a = 0, b = \infty$;

▷ neither of the diagonal entries of H vanishes a.e. on some interval starting at the left endpoint 0.

[3] We write "$f \lesssim g$" for $\exists c > 0.\quad f \leq cg$, and "$f \asymp g$" for $f \lesssim g \wedge g \lesssim f$.

Both assumptions are no loss in generality, and are only imposed for simplicity. The first one can always be achieved by a change of the independent variable in equation (1.1), and changes of variable do not alter the Weyl coefficient; see Section 2.2. The second condition excludes some exceptional cases where there is nothing to investigate: if it is not satisfied, then $\lim_{y\to\infty}\frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|}=1$; we provide more details in Sections 2.2 and 2.3.

Throughout the paper we write

$$H(t)=\begin{pmatrix} h_1(t) & h_3(t)\\ h_3(t) & h_2(t)\end{pmatrix},\qquad m_j(t):=\int_0^t h_j(s)\,\mathrm{d}s,\quad j=1,2,3; \tag{1.5}$$

sometimes we write $M(H,t)$ and $m_i(H,t)$ instead of $M(t)$ and $m_i(t)$ respectively to indicate the dependence on H. Moreover, λ denotes the Lebesgue measure.

Next, we have to introduce some notation which looks a bit technical on first sight, but actually is not. The intuition behind these quantities is discussed in Remark 1.3 below. The functions are well defined because $h_3(t)^2 \leq h_1(t)h_2(t)$ for a.e. $t>0$ and $m_1(t), m_2(t)>0$ for all $t>0$; the latter follows from the assumption that neither of the diagonal entries of H vanishes a.e. on an interval starting at 0. Set

$$\sigma_H(t):=\begin{cases}\frac{|h_3(t)|}{\sqrt{h_1(t)h_2(t)}} & \text{if } h_3(t)\neq 0,\\ 0 & \text{otherwise,}\end{cases} \tag{1.6}$$

$$\pi_{H,s}(t):=\begin{cases}\operatorname{sgn}(h_3(st))\frac{h_2(st)}{h_1(st)}\Big/\frac{m_2(s)}{m_1(s)} & \text{if } h_3(st)\neq 0,\\ 0 & \text{otherwise,}\end{cases} \tag{1.7}$$

$$\mathfrak{t}_s(t):=\frac{m_1(st)}{m_1(s)}+\frac{m_2(st)}{m_2(s)}, \tag{1.8}$$

where $s>0$ is a parameter.

Note that, for each fixed $s>0$, the function $\mathfrak{t}_s$ is absolutely continuous and its derivative

$$\mathfrak{t}_s'(t)=\frac{s}{m_1(s)}h_1(st)+\frac{s}{m_2(s)}h_2(st)$$

is positive a.e. Furthermore, $\mathfrak{t}_s(0)=0$ and $\lim_{t\to\infty}\mathfrak{t}_s(t)=\infty$; the latter follows from the relation $m_1(st)+m_2(st)=\int_0^{st}\operatorname{tr}H(x)\,\mathrm{d}x\to\infty$ as $t\to\infty$ by assumption. Thus $\mathfrak{t}_s$ is an increasing bijection from $[0,\infty)$ onto itself with absolutely continuous inverse function.

Now we are in position to state our main theorem.

1.1 Theorem. *Let H be a Hamiltonian defined on the interval $(0,\infty)$ such that* (1.2) *holds and neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint* 0*. Then the following statements are equivalent:*

(i) *Relation* (1.3) *holds, i.e.,*

$$\lim_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} = 0. \tag{1.9}$$

(ii) *We have*

$$\lim_{t\to 0} \frac{\det M(t)}{m_1(t)m_2(t)} = 0. \tag{1.10}$$

(iii) *For all* $T \in (0,\infty)$*, all* $\gamma \in [0,1)$*, and all open intervals* $I, J \subseteq \mathbb{R}\setminus\{0\}$ *with* $\overline{I}\cap\overline{J} = \emptyset$ *and at least one of* I *and* J *being bounded, the following limit relations hold:*

$$\lim_{s\to 0}\Big[\lambda\Big((0,T)\cap \mathfrak{t}_s\Big(\frac{1}{s}\sigma_H^{-1}([0,\gamma])\Big)\Big)\Big] = 0, \tag{1.11}$$

$$\lim_{s\to 0}[\lambda((0,T)\cap \mathfrak{t}_s(\pi_{H,s}^{-1}(I)))\cdot\lambda((0,T)\cap \mathfrak{t}_s(\pi_{H,s}^{-1}(J)))] = 0. \tag{1.12}$$

Under a certain additional assumption, the conditions in (iii) greatly simplify. This assumption is quite strong, and will, in many interesting cases, not be satisfied. Still, in order to understand the nature of (1.11) and (1.12) and the proof of their equivalence to (1.3), it is worth stating the following addition.

1.2 Addition to Theorem 1.1. *Assume that, in addition to the assumptions of Theorem 1.1, the following conditions hold:*

$$\operatorname{tr} H(t) = 1 \quad \textit{for a.e. } t\in(0,\infty), \tag{1.13}$$

$$\liminf_{t\to 0}\Big(\frac{m_1(t)}{t}\cdot\frac{m_2(t)}{t}\Big) > 0. \tag{1.14}$$

Then the equivalent properties (i)*,* (ii)*,* (iii) *in Theorem 1.1 are further equivalent to the following condition.*

(iv) *For all* γ *and* I, J *as in Theorem 1.1* (iii) *we have*

$$\lim_{t\to 0}\Big[\frac{1}{t}\lambda((0,t)\cap\sigma_H^{-1}([0,\gamma]))\Big] = 0, \tag{1.15}$$

$$\lim_{t\to 0}\Big[\frac{1}{t}\lambda((0,t)\cap\pi_H^{-1}(I))\cdot\frac{1}{t}\lambda((0,t)\cap\pi_H^{-1}(J))\Big] = 0, \tag{1.16}$$

where

$$\pi_H(t) := \begin{cases} \operatorname{sgn}(h_3(t))\frac{h_2(t)}{h_1(t)} & \textit{if } h_3(t)\neq 0,\\ 0 & \textit{otherwise.}\end{cases} \tag{1.17}$$

Note that (1.13) implies that $m_1(t)+m_2(t) = t$. Hence, by [LPW21, Theorem 1.1] (see also Proposition 2.9) we have

$$\liminf_{t\to 0}\Big(\frac{1}{t}m_1(t)\cdot\frac{1}{t}m_2(t)\Big) > 0 \quad\Leftrightarrow\quad \frac{m_1(t)}{m_2(t)} \asymp 1, \qquad t\to 0 \tag{1.18}$$
$$\Leftrightarrow\quad |q_H(ir)| \asymp 1, \quad r\to\infty.$$

We come to the promised explanation of the conditions in (iii) (and (iv)).

1.3 Remark. Let us first discuss the simpler conditions (1.15) and (1.16).

The role of σ_H is to quantify the relative size of the off-diagonal entries of H compared with the diagonal entries. Condition (1.15) can be written as

$$\lim_{t\to 0}\Big[\frac{1}{t}\lambda(\{x\in(0,t) : 1-\sigma_H(x)^2 \geq 1-\gamma^2\})\Big] = 0,$$

or, by rescaling, as

$$\lim_{t\to 0}\lambda(\{x\in(0,1) : 1-\sigma_H(tx)^2 \geq 1-\gamma^2\}) = 0.$$

The validity of this relation for all $\gamma \in [0,1)$ just says that the functions $x \mapsto 1-\sigma_H(tx)^2$ converge to 0 in measure as $t\to 0$. Since they are non-negative and bounded by 1, this is also equivalent to the fact that their integrals converge to 0. Note that

$$1-\sigma_H(x)^2 = \begin{cases} \frac{\det H(x)}{h_1(x)h_2(x)} & \text{if } h_3(x)\neq 0,\\ 1 & \text{otherwise.}\end{cases}$$

Hence the validity of (1.15) for all $\gamma\in[0,1)$ is (again by rescaling) equivalent to

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \frac{\det H(x)}{h_1(x)h_2(x)}\,\mathrm{d}x = 0,$$

where the integrand is understood as equal to 1 at points where its denominator vanishes; this means that the Hamiltonian should be almost of zero determinant in the vicinity of the left endpoint 0 in a measure-theoretic sense.

The role of π_H is not so obvious. It is related to what one may think of as "rotation" of H. To see this, write H in the form

$$H(t) = \begin{pmatrix} 1 & \sigma_H(t)\\ \sigma_H(t) & 1\end{pmatrix} \odot \left[\begin{pmatrix}\cos\varphi_H(t)\\ \sin\varphi_H(t)\end{pmatrix}\begin{pmatrix}\cos\varphi_H(t)\\ \sin\varphi_H(t)\end{pmatrix}^*\right], \tag{1.19}$$

where $\odot$ denotes the Hadamard, i.e., entry-wise, product of the 2×2-matrices. The first factor takes the relative size of the off-diagonal entries into account; the second factor has zero determinant and corresponds to some kind of rotation. The factorisation in (1.19) is possible, for instance, with

$$\varphi_H(t) := \begin{cases} \operatorname{Arccot}\sqrt{\frac{h_1(t)}{h_2(t)}} & \text{if } h_2(t) \neq 0,\, h_3(t) \geq 0, \\ \pi - \operatorname{Arccot}\sqrt{\frac{h_1(t)}{h_2(t)}} & \text{if } h_2(t) \neq 0,\, h_3(t) < 0, \\ 0 & \text{if } h_2(t) = 0, \end{cases} \tag{1.20}$$

where Arccot is the branch with values in $(0, \pi)$. Then

$$\pi_H(t) = \operatorname{sgn}\Big(\frac{\pi}{2} - \varphi_H(t)\Big) \cdot \tan^2 \varphi_H(t). \tag{1.21}$$

Now we map $\varphi_H(t) \in [0, \pi)$ onto the unit circle $\mathbb{T}$ by setting

$$\zeta_H(t) := e^{2i\varphi_H(t)}. \tag{1.22}$$

We may say—descriptively—that ζ_H is the rotation of H.

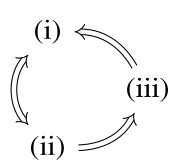
The statement (1.16) is equivalent to the following statement (see Section5): there are no two separated arcs on the unit circle, such that, in the vicinity of the left endpoint 0, $\zeta_H(t)$ often belongs to one arc and also often belongs to the other arc. In other words, the Hamiltonian should rotate so slowly that, on every interval close to 0, it looks—from a measure-theoretic viewpoint—as if its direction were constant; see also Example 1.7

The more complicated conditions (1.11) and (1.12) are weighted and rescaled variants of (1.15) and (1.16); see Section 2.7. The role of the function $\mathfrak{t}_s$ is to take care of heavy oscillations, and the purpose of the weight $\frac{m_2(s)}{m_1(s)}$ in the definition of $\pi_{H,s}$ is to level out the contributions of the two diagonal entries. Moreover, zooming into the vicinity of the left endpoint 0 is now achieved by sending the rescaling parameter s to 0.

Let us note that also the relation (1.11) can be rewritten in integral form, namely as

$$\lim_{s\to 0} \int_0^T \frac{\det H(s\mathfrak{t}_s^{-1}(t))}{(h_1h_2)(s\mathfrak{t}_s^{-1}(t))}\, \mathrm{d}t = 0.$$

To prove Theorem 1.1 we show the implications



Interestingly, very different methods enter in the proofs of the various implications.

- ▷ The implication "(i)⇒(ii)" is a direct consequence of [LPW21, Theorem 1.1] in the form of Proposition 2.9 below. We recall that this theorem is proved by directly studying Weyl discs and estimating the power series coefficients of the fundamental solution of the canonical system.
- ▷ The proof of "(ii)⇒(iii)" requires an elementary but elaborate analysis of the connection between H and its primitive M. In particular, estimates are proved where the constants are independent of the Hamiltonian. This is done in Section 4; see Propositions 4.1 and 4.2.
- ▷ To show "(iii)⇒(i)" and "(ii)⇒(i)" we use cluster sets and compactness arguments for Hamiltonians endowed with the inverse limit topology of weak L^1-topologies on finite intervals; see Section 2.1. Another necessary tool is provided in Section 2.5, and a crucial role is taken by a weighted variant of Y. Kasahara's rescaling trick [Kas75], which relates the behaviour of q_H towards $i\infty$ with weighted rescalings of H; see Section 2.7.

The proof of "(ii)⇒(i)" was included in order to decouple the equivalences between (i) and (ii), and between (i) and (iii), respectively. This enables reading the proof of "(i)⇔(ii)" without having to go into the technical details of Section 4. We thank a referee for suggesting an argument which makes this possible.

A sequence variant of the theorem. We can also give a variant of Theorem 1.1 where limits are replaced by limits inferior. It reads as follows.

1.4 Theorem. *Let H be a Hamiltonian defined on the interval* $(0,\infty)$ *such that* (1.2) *holds and neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint* 0. *Then the following statements are equivalent.*

(i) $\liminf_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} = 0$.

(ii) $\liminf_{t\to 0} \frac{\det M(t)}{m_1(t)m_2(t)} = 0$.

(iii) *For each* $T \in (0,\infty)$ *there exists a sequence* $(s_n)_{n\in\mathbb{N}}$ *with* $s_n \to 0$, *such that for all* $\gamma \in [0,1)$, *and all open intervals* $I, J \subseteq \mathbb{R}\setminus\{0\}$ *with* $\overline{I}\cap\overline{J} = \emptyset$ *and at least one of I and J being bounded, the following limit relations hold:*

$$\begin{aligned} &\lim_{n\to\infty} \Big[\lambda\Big((0,T)\cap \mathrm{t}_{s_n}\Big(\frac{1}{s_n}\sigma_H^{-1}([0,\gamma])\Big)\Big)\Big] = 0, \\ &\lim_{n\to\infty} [\lambda((0,T)\cap \mathrm{t}_{s_n}(\pi_{H,s_n}^{-1}(I)))\cdot\lambda((0,T)\cap \mathrm{t}_{s_n}(\pi_{H,s_n}^{-1}(J)))] = 0. \end{aligned} \tag{1.23}$$

Also in this case, the analogous addition holds.

1.5 Addition to Theorem 1.4. *Assume that, in addition to the assumptions of Theorem 1.4, relations* (1.13) *and* (1.14) *hold. Then the equivalent properties* (i), (ii), (iii) *in Theorem 1.4 are further equivalent to the following condition.*

(iv) *There exists a sequence* $(t_n)_{n\in\mathbb{N}}$ *with* $t_n \to 0$, *such that, for all* γ *and* I, J *as in Theorem 1.4* (iii), *we have*

$$\lim_{n\to\infty}\Big[\frac{1}{t_n}\lambda\big((0,t_n)\cap\sigma_H^{-1}([0,\gamma])\big)\Big]=0, \tag{1.24}$$

$$\lim_{n\to\infty}\Big[\frac{1}{t_n}\lambda\big((0,t_n)\cap\pi_H^{-1}(I)\big)\cdot\frac{1}{t_n}\lambda\big((0,t_n)\cap\pi_H^{-1}(J)\big)\Big]=0. \tag{1.25}$$

The conditions (1.23) and (1.24) can be rewritten in integral form in the very same way as before. Namely, (1.23) as

$$\lim_{n\to\infty}\int_0^T\frac{\det H(s_n\mathfrak{t}_{s_n}^{-1}(t))}{(h_1h_2)(s_n\mathfrak{t}_{s_n}^{-1}(t))}\,\mathrm{d}t=0,$$

and (1.24) as

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\frac{\det H(t)}{(h_1h_2)(t)}\,\mathrm{d}t=0.$$

Two examples. Let us illustrate Theorems 1.1 and 1.4 with two examples. The first one demonstrates a standard situation; it will be revisited in a more general form in Section 6 of the present paper, and in the forthcoming paper [LPW22]. The second example demonstrates a more peculiar situation, where $\frac{\operatorname{Im} q_H}{|q_H|}$ oscillates.

1.6 Example. Let $\alpha_1,\alpha_2>0$, $\beta_1,\beta_2\in\mathbb{R}$, set

$$\alpha_3:=\frac{\alpha_1+\alpha_2}{2},\qquad \beta_3:=\frac{\beta_1+\beta_2}{2},$$

and consider the Hamiltonian

$$H(t):=\begin{pmatrix} t^{\alpha_1-1}|\log t|^{\beta_1} & t^{\alpha_3-1}|\log t|^{\beta_3}\\ t^{\alpha_3-1}|\log t|^{\beta_3} & t^{\alpha_2-1}|\log t|^{\beta_2}\end{pmatrix},\qquad t\in(0,\infty).$$

For this example a computation shows the following facts (this is elementary and we skip details):

(i) For $y\to\infty$,

$$A_H(y)\asymp y^{\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}}(\log y)^{\frac{\beta_1\alpha_2-\beta_2\alpha_1}{\alpha_1+\alpha_2}}.$$

(ii) We have

$$L_H(y)\asymp\operatorname{Im} q_H(iy)\asymp|q_H(iy)|\asymp A_H(y)\quad\text{if }\alpha_1\neq\alpha_2,$$
$$L_H(y)\lesssim\operatorname{Im} q_H(iy)\ll|q_H(iy)|\asymp A_H(y)\quad\text{if }\alpha_1=\alpha_2.$$

(iii) The situation that $\lim_{y\to\infty}\frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|}=0$, equivalently that $\lim_{t\to 0}\frac{\det M(t)}{m_1(t)m_2(t)}=0$, appears only when $q_H(iy)$ grows very slowly. In fact, if $\alpha_1=\alpha_2$, then

$$A_H(y)\asymp(\log y)^{\frac{\beta_1-\beta_2}{2}},\qquad \frac{L_H(y)}{A_H(y)}=\frac{1}{(\log y)^2}.$$

1.7 Example. Let $(t_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that $\frac{t_{n+1}}{t_n}\to 0$ (and hence $t_n\to 0$), set $t_0:=\infty$ and consider the partition $(0,\infty)=I_+\cup I_-$ where

$$I_+:=\bigcup_{k=1}^{\infty}[t_{2k},t_{2k-1}),\qquad I_-:=\bigcup_{k=0}^{\infty}[t_{2k+1},t_{2k}).$$

Further, let $\varphi_+,\varphi_-\in(0,\pi)\setminus\{\frac{\pi}{2}\}$ with $\varphi_+\neq\varphi_-$ and define the Hamiltonian H by

$$H(t)=\begin{pmatrix}\cos^2\varphi(t) & \cos\varphi(t)\cdot\sin\varphi(t)\\ \cos\varphi(t)\cdot\sin\varphi(t) & \sin^2\varphi(t)\end{pmatrix}$$

where

$$\varphi(t)=\begin{cases}\varphi_+, & t\in I_+,\\ \varphi_-, & t\in I_-.\end{cases}$$

Clearly, (1.13) and (1.14) are satisfied, so that we can apply the Additions to Theorems 1.2 and 1.4. Since $\sigma_H(t)=1$ for $t>0$, the limit relation (1.15), and hence also (1.24), holds for every $\gamma\in[0,1)$. Let us now check whether (1.16) and (1.25) are satisfied. Since $\varphi(t)=\varphi_H(t)$, where $\varphi_H(t)$ is as in (1.20), it follows from (1.21) that

$$\pi_H(t)=\operatorname{sgn}\Big(\frac{\pi}{2}-\varphi(t)\Big)\cdot\tan^2\varphi(t)=\operatorname{sgn}\Big(\frac{\pi}{2}-\varphi_\pm\Big)\cdot\tan^2\varphi_\pm=:c_\pm\quad\text{when } t\in I_\pm.$$

The limit relations (1.16) and (1.25) hold trivially whenever $I\cap\{c_+,c_-\}=\emptyset$ or $J\cap\{c_+,c_-\}=\emptyset$. By symmetry, we only have to consider the case when $c_+\in I$ and $c_-\in J$, which we assume in the following. For $t>0$ we have

$$\begin{aligned}\lambda((0,t)\cap\pi_H^{-1}(I))&=\lambda((0,t)\cap I_+)\\&=\begin{cases}t-t_{2n}+\sum_{k=n+1}^{\infty}(t_{2k-1}-t_{2k}), & t\in[t_{2n},t_{2n-1}),\\ \sum_{k=n+1}^{\infty}(t_{2k-1}-t_{2k}), & t\in[t_{2n+1},t_{2n}),\end{cases}\\ \lambda((0,t)\cap\pi_H^{-1}(J))&=\lambda((0,t)\cap I_-)\\&=\begin{cases}\sum_{k=n}^{\infty}(t_{2k}-t_{2k+1}), & t\in[t_{2n},t_{2n-1}),\\ t-t_{2n+1}+\sum_{k=n+1}^{\infty}(t_{2k}-t_{2k+1}), & t\in[t_{2n+1},t_{2n}).\end{cases}\end{aligned}$$

Set $F(t):=\frac{1}{t}\lambda((0,t)\cap I_+)\cdot\frac{1}{t}\lambda((0,t)\cap I_-)$. Then

$$F(t_{2n})\leq\frac{1}{t_{2n}}\sum_{k=n+1}^{\infty}(t_{2k-1}-t_{2k})\leq\frac{t_{2n+1}}{t_{2n}}\to 0$$

as $n \to \infty$ and, similarly, $F(t_{2n+1}) \to 0$. This shows that (1.25) is satisfied and hence also (i) in Theorem 1.4. On the other hand, for $n \in \mathbb{N}$ such that $\frac{t_{2n}}{t_{2n-1}} \leq \frac{1}{2}$, we have

$$F(2t_{2n}) \geq \frac{1}{2t_{2n}}(2t_{2n} - t_{2n}) \cdot \frac{1}{2t_{2n}}(t_{2n} - t_{2n+1}) = \frac{1}{4}\Big(1 - \frac{t_{2n+1}}{t_{2n}}\Big) \to \frac{1}{4}$$

as $n \to \infty$. This implies that (1.16) is not fulfilled and hence neither is (i) in Theorem 1.1. To summarise, Theorems 1.1 and 1.4 show that

$$\liminf_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} = 0 \quad \text{and} \quad \limsup_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} > 0.$$

2 Preliminaries

2.1 Convergence of Hamiltonians. We use the following notation for Hamiltonians on a finite or infinite interval.

2.1 Definition. Let $T \in (0, \infty]$.

(i) $\mathbb{H}_T$ is the set of all measurable functions $H\colon (0, T) \to \mathbb{R}^{2\times 2}$ (up to equality a.e.) such that $H(t) \geq 0$ and $\operatorname{tr} H(t) > 0$ a.e.;

(ii) $\mathbb{H}^1_T$ is the set of all $H \in \mathbb{H}_T$ such that $\operatorname{tr} H(t) = 1$ a.e.;

(iii) $\mathbb{H}^{\mathrm{cs}}_T$ is the set of all $H \in \mathbb{H}^1_T$ that are constant and satisfy $\det H(t) = 0$ a.e.

If $T = \infty$, we often drop T from the notation and just write $\mathbb{H}$, $\mathbb{H}^1$ and $\mathbb{H}^{\mathrm{cs}}$ instead of $\mathbb{H}_\infty$, $\mathbb{H}^1_\infty$ and $\mathbb{H}^{\mathrm{cs}}_\infty$ respectively.

We recall how $\mathbb{H}^1$ can be topologised appropriately. This is already used in the work of L. de Branges. An explicit formulation is given in [Rem18]; for a more structural approach see [PW21], which we use as our main reference in the following.

For each $T < \infty$ the set $\mathbb{H}^1_T$ is a subset of $L^1((0, T), \mathbb{R}^{2\times 2})$, and hence naturally topologised with the $\|\llcorner\!\lrcorner\|_1$-topology or the weak L^1-topology. It turns out that the latter is more suitable because the weak L^1-topology on $\mathbb{H}^1_T$ is compact and metrisable; see [PW21, Lemma 2.3].

Now consider the family $(\mathbb{H}^1_T)_{T\in(0,\infty)}$ with the restriction maps $\rho^{T'}_T : \mathbb{H}^1_{T'} \to \mathbb{H}^1_T$ for $T \leq T'$. The set $\mathbb{H}^1$ can be naturally viewed as the inverse limit of this family: every function on $(0, \infty)$ can be identified with the family of all its restrictions to finite intervals. Endowed with the inverse limit topology (see, e.g., [Bou66, §I.4.4]), where we use the weak L^1-topology on $\mathbb{H}^1_T$, the set $\mathbb{H}^1$ becomes a compact metrisable space; see [PW21, Lemma 2.9]. The map that assigns to a Hamiltonian H its Weyl coefficient q_H is continuous when the set of Nevanlinna functions is endowed with the topology of locally uniform convergence; see [PW21, Theorem 2.12].

Throughout the remainder of the paper we often deal with limit points of families of Hamiltonians. In general, for a net $(x_i)_{i\in I}$ in some topological space X, we denote by $\mathrm{LP}(x_i)_{i\in I}$ the set of its limit points, i.e.,

$$\mathrm{LP}(x_i)_{i\in I} := \Big\{x \in X\colon\ \exists \text{ subnet } (x_{i(j)})_{j\in J}.\ \lim_{j\in J} x_{i(j)} = x\Big\}.$$

If there is a need to specify the topology, we shall add an index. For example, if X is a normed space, we write $\mathrm{LP}_{\|.\|}(x_i)_{i\in I}$ for limit points w.r.t. the norm topology, and $\mathrm{LP}_w(x_i)_{i\in I}$ for limit points w.r.t. the weak topology.

2.2 Remark. In our context the space X is usually metrisable, and the index set I is $\mathbb{N}$, $(0,1]$ or $[1,\infty)$, each endowed with the natural order (or the reverse order in the case of $(0,1]$). In these situations one can restrict attention to subsequences rather than subnets:

$$\mathrm{LP}(x_i)_{i\in I} = \{x \in X\colon\ \exists \text{ subsequence } (x_{i_n})_{n\in\mathbb{N}}.\ \lim_{n\to\infty} x_{i_n} = x\}.$$

Note that in the cases when $I = (0,1]$ or $I = [1,\infty)$, then $i_n \to 0$ or $i_n \to \infty$ respectively.

We need the following simple fact about constant singular limit points. It is proved using the compactness of $\mathbb{H}^1$, continuity of the restriction maps $\rho_T\colon \mathbb{H}^1 \to \mathbb{H}^1_T$, and the obvious fact that

$$\mathbb{H}^{\mathrm{cs}} = \{H \in \mathbb{H}^1\colon\ \forall T > 0.\ \rho_T(H) \in \mathbb{H}^{\mathrm{cs}}_T\}. \tag{2.1}$$

2.3 Lemma. *Let $(H_i)_{i\in I}$ be a net in $\mathbb{H}^1$. Then the following two equivalences hold.*

(i) $\mathrm{LP}(H_i)_{i\in I} \subseteq \mathbb{H}^{cs} \quad\Leftrightarrow\quad \forall T > 0.\quad \mathrm{LP}_w(\rho_T(H_i))_{i\in I} \subseteq \mathbb{H}^{cs}_T$.

(ii) $\mathrm{LP}(H_i)_{i\in I} \cap \mathbb{H}^{cs} \neq \emptyset \quad\Leftrightarrow\quad \forall T > 0.\quad \mathrm{LP}_w(\rho_T(H_i))_{i\in I} \cap \mathbb{H}^{cs}_T \neq \emptyset$.

Proof.
(i)"$\Leftarrow$": Assume that there exists $\mathring{H} \in \mathrm{LP}(H_i)_{i\in I} \setminus \mathbb{H}^{\mathrm{cs}}$. By (2.1) we find $T > 0$ such that $\rho_T(\mathring{H}) \notin \mathbb{H}^{\mathrm{cs}}_T$. Since ρ_T is continuous, we have $\rho_T(\mathring{H}) \in \mathrm{LP}_w(\rho_T(H_i))_{i\in I}$.
(i)"$\Rightarrow$": Assume that there exist $T > 0$ and $\widetilde{H}_T \in \mathrm{LP}_w(\rho_T(H_i))_{i\in I} \setminus \mathbb{H}^{\mathrm{cs}}_T$. Since $\mathbb{H}^1$ is compact and ρ_T is continuous, we find $\mathring{H} \in \mathrm{LP}(H_i)_{i\in I}$ such that $\rho_T(\mathring{H}) = \widetilde{H}_T$. Clearly, $\mathring{H} \notin \mathbb{H}^{\mathrm{cs}}$.
(ii)"$\Rightarrow$": Assume that there exists $\mathring{H} \in \mathrm{LP}(H_i)_{i\in I} \cap \mathbb{H}^{\mathrm{cs}}$. Continuity of ρ_T yields $\rho_T(\mathring{H}) \in \mathrm{LP}(H_i)_{i\in I} \cap \mathbb{H}^{\mathrm{cs}}_T$ for all $T > 0$.

(ii)"$\Leftarrow$": Assume that, for each $T > 0$, there exists $\widetilde{H}_T \in \mathrm{LP}_w(\rho_T(H_i))_{i\in I} \cap \mathbb{H}_T^{\mathrm{cs}}$. Since $\mathbb{H}^1$ is compact and ρ_T is continuous, we find $\mathring{H}_T \in \mathrm{LP}(H_i)_{i\in I}$ such that $\rho_T(\mathring{H}_T) = \widetilde{H}_T$. Again by compactness, there exists a limit point $\mathring{H} \in \mathrm{LP}(\mathring{H}_T)_{T>0}$, say $\mathring{H} = \lim_{n\to\infty} \mathring{H}_{t_n}$ with some sequence $t_n \to \infty$. Then $\mathring{H} \in \mathrm{LP}(H_i)_{i\in I}$, and for each $T > 0$ we have

$$\rho_T(\mathring{H}) = \lim_{n\to\infty}{}^{w} \rho_T(\mathring{H}_{t_n}) = \lim_{\substack{n\to\infty \\ t_n\geq T}}{}^{w} \rho_T(\rho_{t_n}(\mathring{H}_{t_n})) = \lim_{\substack{n\to\infty \\ t_n\geq T}}{}^{w} \underbrace{\rho_T(\widetilde{H}_{t_n})}_{\in\mathbb{H}_T^{\mathrm{cs}}} \in \mathbb{H}_T^{\mathrm{cs}}.$$

For the last inclusion recall that $\mathbb{H}_T^{\mathrm{cs}}$ is $\|\llcorner\|_1$-compact as a homeomorphic image of $\mathbb{R} \cup \{\infty\}$, see [PW21, §2.3], and hence also weakly closed. Again referring to (2.1) we obtain $\mathring{H} \in \mathbb{H}^{\mathrm{cs}}$. □

We also need the Weyl coefficients for constant Hamiltonians with zero determinant, which can be found by an elementary calculation; see [EKT18, Example 2.2(1)].[4]

2.4 Lemma. *Let H as in* (1.5) *be a constant Hamiltonian such that $h_3^2 = h_1 h_2$. Then*

$$q_H(z) = \begin{cases} \frac{h_3}{h_2} & \text{if } h_2 \neq 0, \\ \infty & \text{if } h_2 = 0. \end{cases}$$

2.2 Reparameterisation. Reparameterisation is the equivalence relation on the set of all Hamiltonians defined as follows.

2.5 Definition. Two Hamiltonians H and $\widehat{H}$, defined on respective intervals $[a, b)$ and $[\hat{a}, \hat{b})$, are called reparameterisations of each other if there exists a function $\varphi\colon [\hat{a}, \hat{b}) \to [a, b)$ that is strictly increasing, bijective and absolutely continuous with absolutely continuous inverse such that

$$\widehat{H}(x) = (H \circ \varphi)(x) \cdot \varphi'(x), \quad x \in [\hat{a}, \hat{b}) \text{ a.e.} \tag{2.2}$$

If H and $\widehat{H}$ are related as in (2.2) and y is a solution of (1.1), then $y \circ \varphi$ is a solution of (2.2) with H replaced by $\widehat{H}$. Similarly, the fundamental solutions satisfy $\widehat{W}(x, z) = W(\varphi(x), z)$ and hence

$$q_{\widehat{H}}(z) = q_H(z). \tag{2.3}$$

Moreover, the following obvious transformation rules hold:

$$\begin{aligned} \widehat{M} &= M \circ \varphi, & \operatorname{tr} \widehat{H}(s) &= \operatorname{tr} H(\varphi(s)) \cdot \varphi'(s), \\ \pi_{\widehat{H}} &= \pi_H \circ \varphi, & \sigma_{\widehat{H}} &= \sigma_H \circ \varphi. \end{aligned} \tag{2.4}$$

[4] In [EKT18] a different sign convention is used, namely the equation $y'(t) = -zJH(t)y(t)$ is studied instead of (1.1). The corresponding Weyl coefficient is $\widetilde{q}_H(z) = -q_H(-z)$.

Based on the transformation rule for the trace, we see that every equivalence class of Hamiltonians modulo reparameterisation contains exactly one element that is defined on the interval $(0,\infty)$ and is trace-normalised, i.e., whose trace is equal to 1 a.e. In fact, given a Hamiltonian H defined on some interval (a,b), we set $\mathfrak{t}(t) := \int_a^t \operatorname{tr} H(x)\,\mathrm{d}x$ and use $\varphi := \mathfrak{t}^{-1}$. This function is admissible to make a reparameterisation, since $\operatorname{tr} H(t) > 0$ a.e., and hence $\mathfrak{t}^{-1}$ is absolutely continuous.

Based on the transformation rule of the primitive M, we see that the quotient in (1.10) transforms correspondingly. Let us set

$$d(H,t) := \frac{\det M(t)}{m_1(t)m_2(t)}. \tag{2.5}$$

If H and $\widehat{H}$ are related as in (2.2), then

$$d(\widehat{H},s) = d(H,\varphi(s)). \tag{2.6}$$

2.3 Hamiltonians starting with a vanishing diagonal entry. If a Hamiltonian starts with an interval where a diagonal entry vanishes, then its Weyl coefficient has a simple, and extremal, asymptotics towards $+i\infty$.

Let H be a Hamiltonian defined on some interval (a,b). Recall the following classical facts; see, e.g., [KK68].

▷ Denote by $(a,\hat{a})$ the maximal interval starting at a such that $h_2(t)=0$ for $t\in(a,\hat{a})$ a.e., and assume that $\hat{a}>a$. Then

$$q_H(z) = \Big(\int_a^{\hat{a}} h_1(t)\,\mathrm{d}t\Big)\cdot z + q_{H|_{(\hat{a},b)}}(z).$$

The leading order term is the term that is linear in z:

$$\lim_{y\to+\infty} \frac{1}{y} q_H(iy) = i\int_a^{\hat{a}} \operatorname{tr} H(t)\,\mathrm{d}t,$$

The case $\hat{a}=b$ is formally included and corresponds to $q_H\equiv\infty$.

▷ Denote by $(a,\check{a})$ the maximal interval starting at a such that $h_1(t)=0$ for $t\in(a,\check{a})$ a.e., and assume that $\check{a}>a$. Then

$$q_H(z) = -\frac{1}{(\int_a^{\check{a}} h_2(t)\,\mathrm{d}t)\cdot z - \frac{1}{q_{H|_{(\check{a},b)}}(z)}}.$$

Again the linear term gives the leading order asymptotics:

$$\lim_{y\to+\infty} y q_H(iy) = \frac{i}{\int_a^{\check{a}} \operatorname{tr} H(t)\,\mathrm{d}t}.$$

The case $\check{a}=b$ is formally included and corresponds to $q_H\equiv 0$.

Translated to the spectral measure, $\hat{a} > a$ means that it should include a "point mass at infinity", and $\check{a} > a$ means that it has finite total mass.

In particular, the above relations show that, if $\hat{a} > a$ or $\check{a} > a$, then

$$\lim_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} = 1.$$

2.4 Representation of Hamiltonians by scalar functions. We study the representation of a Hamiltonian H by means of the functions σ_H and ζ_H, defined in (1.6) and (1.22) respectively, a bit more systematically. Denote by $\mathbb{T}$ the unit circle in the complex plane and, for $0 < T \leq \infty$, set

$$\mathscr{L}(T) := \{f : (0,T) \to [0,1] \times \mathbb{T} \colon f \text{ measurable}\}\big/_{\sim},$$

where $f \sim g$ means that f and g coincide almost everywhere. As usual, we suppress explicit notation of equivalence classes. Moreover, we write a function $f \in \mathscr{L}(T)$ generically as a pair $f = (\sigma, \zeta)$ with $\sigma\colon (0,T) \to [0,1]$ and $\zeta\colon (0,T) \to \mathbb{T}$.

The set $\mathscr{L}(T)$ is contained in $L^1((0,T), \mathbb{C}^2)$ if T is finite, and in $L^1_{\text{loc}}([0,\infty), \mathbb{C}^2)$ if $T = \infty$. In particular, for $T < \infty$, we may consider $\mathscr{L}(T)$ topologised with the $\|\text{\textvisiblespace}\|_1$-topology or the weak L^1-topology.

2.6 Definition. Let $0 < T \leq \infty$. We define maps

$$\mathscr{L}(T) \underset{\Xi}{\overset{\Gamma}{\rightleftarrows}} \mathbb{H}^1_T$$

by

$$\Gamma[\sigma,\zeta](t) := \frac{1}{2} \begin{pmatrix} 1 + \operatorname{Re}\zeta(t) & \sigma(t)\operatorname{Im}\zeta(t) \\ \sigma(t)\operatorname{Im}\zeta(t) & 1 - \operatorname{Re}\zeta(t) \end{pmatrix}$$

and $\Xi[H](t) := (\sigma_H(t), \zeta_H(t))$, where σ_H and ζ_H are given by the formulae (1.6), (1.20), (1.22).

Let $(\sigma,\zeta) \in \mathscr{L}(T)$. Introducing the rotation angle $\varphi\colon (0,T) \to [0,\pi)$ by $\zeta(t) = e^{2i\varphi(t)}$ we can rewrite

$$\Gamma[\sigma,\zeta] = \begin{pmatrix} 1 & \sigma(t) \\ \sigma(t) & 1 \end{pmatrix} \odot \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix} \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}^{*}.$$

From this representation we see that Γ is a left-inverse of Ξ: given $H \in \mathbb{H}^1_T$, the matrices H and $\Gamma[\sigma_H, \zeta_H]$ both have trace 1, their quotients of diagonal entries coincide, and the relative size and sign of their off-diagonal entries coincide. Thus indeed

$$(\Gamma \circ \Xi)(H) = H.$$

Furthermore, observe the following continuity property, which holds since $\mathscr{L}(T)$ is uniformly bounded: for each $T < \infty$ we find a constant $C > 0$ such that, for all $(\sigma_1, \zeta_1), (\sigma_2, \zeta_2) \in \mathscr{L}(T)$,

$$\|\Gamma[\sigma_1, \zeta_1] - \Gamma[\sigma_2, \zeta_2]\|_1 \leq C\|(\sigma_1, \zeta_1) - (\sigma_2, \zeta_2)\|_1. \tag{2.7}$$

In particular, for each $T < \infty$, the map $\Gamma\colon \mathscr{L}(T) \to \mathbb{H}_T^1$ is $\|\lrcorner\|_1$-to-$\|\lrcorner\|_1$-continuous.

Note that the class of constant, singular, trace-normalised Hamiltonians can be represented as follows:

$$\mathbb{H}_T^{\mathsf{cs}} = \{\Gamma(1, \zeta) : \zeta \in \mathbb{T}\}$$

where we identify the constant $(1, \zeta)$ with the constant function in $\mathscr{L}(T)$.

2.5 Nets with constant limit points. In the proof of the implication (iii)⇒(i) in Theorems 1.1 and 1.4 we need the following fact about sequences in L^1-spaces which have only constant limit points. We do not know an explicit reference to the literature, and hence give a complete proof. In the formulation we tacitly identify $\mathbb{C}$ with the μ-a.e. constant functions in $L^1(\mu)$.

2.7 Proposition. *Let μ be a finite positive measure on a set Ω with $\mu \not\equiv 0$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^\infty(\mu)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. We consider $(f_n)_{n\in\mathbb{N}}$ as a sequence in $L^1(\mu)$. Then the following two statements are equivalent:*

$$\forall (f_{n_k})_{k\in\mathbb{N}} \text{ subsequence of } (f_n)_{n\in\mathbb{N}}. \quad \mathrm{LP}_{\|\lrcorner\|_1}(f_{n_k})_{k\in\mathbb{N}} \cap \mathbb{C} \neq \emptyset; \tag{2.8}$$

$$\forall A, B \subseteq \mathbb{C} \text{ compact, disjoint.} \quad \lim_{n\to\infty} [\mu(f_n^{-1}(A)) \cdot \mu(f_n^{-1}(B))] = 0. \tag{2.9}$$

If the equivalent conditions (2.8) *and* (2.9) *hold, then*

$$\mathrm{LP}_w(f_n)_{n\in\mathbb{N}} = \mathrm{LP}_{\|\lrcorner\|_1}(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}. \tag{2.10}$$

Proof. Let us first settle "(2.8)⇒(2.9)∧(2.10)", which is easy to see. Assume that (2.8) holds, and let $n_k \to \infty$. Then we find a further subsequence $(f_{n_{k(l)}})_{l\in\mathbb{N}}$ such that

$$f_{n_{k(l)}} \xrightarrow{\|\lrcorner\|_1} g$$

with some constant g. Since $\|\lrcorner\|_1$-convergence implies convergence in measure, we have

$$\lim_{l\to\infty} \mu(\{x \in \Omega\colon\ |f_{n_{k(l)}}(x) - g| \geq \varepsilon\}) = 0 \tag{2.11}$$

for every $\varepsilon > 0$. Now consider two compact disjoint subsets A, B of $\mathbb{C}$. Then the point g can belong to at most one of A and B. For definiteness, assume that $g \notin A$. Then the distance $\operatorname{dist}(A, g)$ is positive, and

$$f_n^{-1}(A) \subseteq \big\{x \in \Omega\colon\ |f_n(x) - g| \geq \operatorname{dist}(A, g)\big\}.$$

Relation (2.11) implies that

$$\lim_{l\to\infty} \mu\big(f_{n_{k(l)}}^{-1}(A)\big) = 0,$$

and hence also the limit in (2.9) along the subsequence $(n_{k(l)})_{l\in\mathbb{N}}$ is zero. Since we started with an arbitrary sequence $(n_k)_{k\in\mathbb{N}}$, the limit relation (2.9) follows.

Now let $f \in \mathrm{LP}_w(f_n)_{n\in\mathbb{N}}$ and choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k} \xrightarrow{w} f$. Then we find a further subsequence $(f_{n_{k(l)}})_{l\in\mathbb{N}}$ and a constant g such that $f_{n_{k(l)}} \xrightarrow{\|\text{\textvisiblespace}\|_1} g$. It follows that $f = g \in \mathrm{LP}_{\|\text{\textvisiblespace}\|_1}(f_n)_{n\in\mathbb{N}}$. We have thus shown that

$$\mathrm{LP}_w(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LP}_{\|\text{\textvisiblespace}\|_1}(f_n)_{n\in\mathbb{N}} \cap \mathbb{C},$$

and this implies (2.10).

We come to the converse implication "(2.9)⇒(2.8)". Assume from now on that (2.9) holds. Moreover, since (2.9) is inherited by subsequences, it is enough to prove (2.8) for the sequence $(f_n)_{n\in\mathbb{N}}$ itself. Further, let us set $M := \sup_{n\in\mathbb{N}} \|f_n\|_\infty$.

There exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and $a \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(\Omega)} \int_\Omega \operatorname{Re} f_{n_k}(x)\,\mathrm{d}\mu(x) = a. \tag{2.12}$$

Let $\varepsilon > 0$ be arbitrary and consider the compact, disjoint sets

$$A = \{z \in \mathbb{C}\colon\ \operatorname{Re} z \geq a + \varepsilon \wedge |z| \leq M\}, \quad B = \Big\{z \in \mathbb{C}\colon\ \operatorname{Re} z \leq a + \frac{\varepsilon}{2} \wedge |z| \leq M\Big\};$$

by assumption, (2.9) holds with these sets. Suppose that there exists a subsequence $(k(l))_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \mu\big(f_{n_{k(l)}}^{-1}(B)\big) = 0$. Then

$$\begin{aligned}
\int_\Omega \operatorname{Re} f_{n_{k(l)}}(x)\,\mathrm{d}\mu(x) &= \int_{f_{n_{k(l)}}^{-1}(B)} \operatorname{Re} f_{n_{k(l)}}(x)\,\mathrm{d}\mu(x) + \int_{\Omega\setminus f_{n_{k(l)}}^{-1}(B)} \operatorname{Re} f_{n_{k(l)}}(x)\,\mathrm{d}\mu(x) \\
&\geq -M\mu(f_{n_{k(l)}}^{-1}(B)) + \Big(a + \frac{\varepsilon}{2}\Big)\mu(\Omega \setminus f_{n_{k(l)}}^{-1}(B)) \\
&\to \Big(a + \frac{\varepsilon}{2}\Big)\mu(\Omega), \quad l \to \infty,
\end{aligned}$$

which is a contradiction to (2.12). Therefore (2.9) implies that

$$\lim_{k\to\infty} \mu(f_{n_k}^{-1}(A)) = 0,$$

which is equivalent to

$$\lim_{k\to\infty} \mu(\{x \in \Omega\colon\ \mathrm{Re} f_{n_k}(x) \geq a+\varepsilon\}) = 0. \tag{2.13}$$

In a similar way one shows that

$$\lim_{k\to\infty} \mu(\{x \in \Omega\colon\ \mathrm{Re} f_{n_k}(x) \leq a-\varepsilon\}) = 0,$$

which, together with (2.13), implies that $\mathrm{Re} f_{n_k} \to a$ in measure. Since $\mathrm{Re} f_{n_k}$ is uniformly integrable, it follows that $\mathrm{Re} f_{n_k} \xrightarrow{\|\llcorner\|_1} a$. In a completely analogous way one can find a subsequence such that $\mathrm{Im} f_{n_{k(l)}} \xrightarrow{\|\llcorner\|_1} b$ with some $b \in \mathbb{R}$. This proves (2.8). □

In the context of Hamiltonians on a finite interval, Proposition 2.7 implies the following fact.

2.8 Corollary. *Let $T < \infty$ and $(H_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{H}_T^1$, and denote by λ the Lebesgue measure on $(0,T)$. Assume that*

(i) $\forall \gamma \in [0,1),\ \lim_{n\in\infty} \lambda(\sigma_{H_n}^{-1}([0,\gamma])) = 0$,

(ii) $\forall A, B \subseteq \mathbb{T}$ *closed, disjoint*, $\lim_{n\in\infty}[\lambda(\zeta_{H_n}^{-1}(A)) \cdot \lambda(\zeta_{H_n}^{-1}(B))] = 0$.

Then

$$\mathrm{LP}_{\|\llcorner\|_1}(H_n)_{n\in\mathbb{N}} = \mathrm{LP}_w(H_n)_{n\in\mathbb{N}} \subseteq \mathbb{H}_T^{cs}.$$

Proof. We have to show that

$$\mathrm{LP}_w(H_n)_{n\in\mathbb{N}} \subseteq \mathrm{LP}_{\|\llcorner\|_1}(H_n)_{n\in\mathbb{N}} \cap \mathbb{H}_T^{cs}.$$

The condition (i) says that $\sigma_{H_n} \to 1$ in measure. Since $|\sigma_{H_n}(t)| \leq 1$ for a.e. t, σ_{H_n} tends to 1 also w.r.t. $\|\llcorner\|_1$. Consider a subsequence $(H_{n_k})_{k\in\mathbb{N}}$ of $(H_n)_{n\in\mathbb{N}}$ that converges weakly to some $\mathring{H} \in \mathbb{H}_T^1$. By (ii), we can apply Proposition 2.7 to the sequence $(\zeta_{H_{n_k}})_{k\in\mathbb{N}}$. This provides us with a constant $\zeta \in \mathbb{T}$ and a further subsequence $(\zeta_{H_{n_{k(l)}}})_{l\in\mathbb{N}}$ such that $\zeta_{H_{n_{k(l)}}} \to \zeta$ w.r.t. $\|\llcorner\|_1$. Recalling (2.7) we see that

$$\|H_{n_{k(l)}} - \Gamma(1,\zeta)\|_1 = \|\Gamma(\sigma_{H_{n_{k(l)}}}, \zeta_{H_{n_{k(l)}}}) - \Gamma(1,\zeta)\|_1 \to 0,$$

and hence $\mathring{H} = \Gamma(1,\zeta)$ and $\mathring{H} \in \mathrm{LP}_{\|\llcorner\|_1}(H_n)_{n\in\mathbb{N}}$. □

2.6 Estimates for imaginary part and modulus of the Weyl coefficient. In this subsection we recall lower and upper estimates for $\mathrm{Im}\, q_H$ and $|q_H|$ on the positive imaginary axis. This result is a special instance of [LPW21, Theorem 1.1] with $q = \frac{1}{4}$ and $\vartheta = \frac{\pi}{2}$ there and is used, in particular, in the proof of the implication (i)⇒(ii) in Theorem 1.1; the estimates for the modulus are also used in the proof of the implication (iii)⇒(i).

2.9 Proposition. *Let H be a Hamiltonian defined on the interval* $(0,\infty)$ *such that* (1.2) *holds and neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint* 0*, and let m_i be as in* (1.5) *and $d(H,t)$ as in* (2.5)*. For $r>0$, let $\mathring{t}(r)\in(0,\infty)$ be the unique number that satisfies*

$$(m_1m_2)\big(\mathring{t}(r)\big)=\frac{1}{(8r)^2}\,. \tag{2.14}$$

With

$$A_H(r):=\sqrt{\frac{m_1(\mathring{t}(r))}{m_2(\mathring{t}(r))}},\qquad L_H(r):=A_H(r)d\big(H,\mathring{t}(r)\big) \tag{2.15}$$

the inequalities

$$\frac{1}{44}A_H(r)\leq|q_H(ir)|\leq 44A_H(r),\qquad \frac{1}{64}L_H(r)\leq\operatorname{Im}q_H(ir)\leq\frac{79}{2}A_H(r)$$

hold for all $r>0$.

Note that the mapping $t\mapsto(m_1m_2)(t)$ is a strictly increasing bijection from $(0,\infty)$ onto itself, and therefore $\mathring{t}(r)$ is uniquely defined via (2.14). The mapping $r\mapsto\mathring{t}(r)$ is a strictly decreasing bijection from $(0,\infty)$ onto itself. It is the inverse of the function

$$\mathring{r}(t):=\frac{1}{8\sqrt{(m_1m_2)(t)}}.$$

2.7 A weighted rescaling transformation. In order to study the behaviour of q_H towards $i\infty$, we use a weighted rescaling transformation on the set of Hamiltonians. This is a variant of Y. Kasahara's rescaling trick invented in [Kas75] for Krein strings, and also used in slightly different forms in [KW10, EKT18, PW21, LPW22]. The main idea of the rescaling is to zoom into a neighbourhood of the left endpoint 0 when s in the following definition tends to 0.

2.10 Definition. Let $g_1,g_2:(0,\infty)\to(0,\infty)$ be continuous such that $g_1(s),g_2(s)\to\infty$ as $s\to 0$. Further, let $T\in(0,\infty]$ and set $g_3(s):=\sqrt{g_1(s)g_2(s)}$. For every $s>0$ define the map $\mathscr{A}_s\colon\mathbb{H}_T\to\mathbb{H}_{\frac{1}{s}T}$ by

$$(\mathscr{A}_sH)(t):=\begin{pmatrix}sg_1(s)h_1(st) & sg_3(s)h_3(st)\\ sg_3(s)h_3(st) & sg_2(s)h_2(st)\end{pmatrix},\quad t\in\Big(0,\frac{1}{s}T\Big).$$

In the following we shall use two special choices of g_1,g_2, namely

$$\text{Situation 1:}\quad g_1(s)=\frac{1}{m_1(s)},\quad g_2(s)=\frac{1}{m_2(s)} \tag{2.16}$$

or

$$\text{Situation 2:}\quad g_1(s)=g_2(s)=\frac{1}{s}\quad\text{and}\quad H\text{ satisfies (1.13) and (1.14);} \tag{2.17}$$

in both cases g_1, g_2 satisfy the assumptions in Definition 2.10. The functions in (2.16) are used in the proof of Theorems 1.1 and 1.4; the functions in (2.17) are used in the proof of the additions of these theorems.

In the following lemma we collect how the quantities defined in (1.6)–(1.8), (1.17) and (2.5) are transformed.

2.11 Lemma. *Let g_1, g_2 be as in Definition 2.10 and $H \in \mathbb{H}$. Then*

$$M(\mathscr{A}_s H, t) = \int_0^t (\mathscr{A}_s H)(x)\,\mathrm{d}x = \begin{pmatrix} g_1(s)m_1(st) & g_3(s)m_3(st) \\ g_3(s)m_3(st) & g_2(s)m_2(st) \end{pmatrix}, \tag{2.18}$$

$$\sigma_{\mathscr{A}_s H}(t) = \sigma_H(st), \quad \pi_{\mathscr{A}_s H}(t) = \frac{g_2(s)}{g_1(s)}\pi_H(st), \quad d(\mathscr{A}_s H, t) = d(H, st). \tag{2.19}$$

If, in addition, (1.2) *holds, then*

$$q_{\mathscr{A}_s H}(z) = \frac{g_3(s)}{g_2(s)} q_H(g_3(s)z). \tag{2.20}$$

Proof. Relations (2.18) and the first two equalities in (2.19) follow easily from the definitions. Further, (2.18) implies the third equality in (2.19). Finally, (2.20) follows from [EKT18, Lemma 2.7]. □

If the functions g_1, g_2 are as in (2.16), the relation (2.20) yields

$$q_{\mathscr{A}_{\mathring{t}(r)} H}\Big(\frac{z}{8}\Big) = \frac{1}{A(r)} q_H(rz). \tag{2.21}$$

In the following lemma we prove an a priori estimate for the modulus of the Weyl coefficient of $\mathscr{A}_s H$ at a particular point, which is used in the proof of Theorems 1.1 and 1.4. This property follows from the choice of g_1, g_2 in (2.16) in the general case or from the assumption (1.14) in the additions to the main theorems.

2.12 Lemma. *Let $H \in \mathbb{H}$ such that* (1.2) *holds, let g_1, g_2, g_3 be as in Definition 2.10, and assume that* (2.16) *or* (2.17) *is satisfied. Then*

$$\Big|q_{\mathscr{A}_s H}\Big(\frac{i}{8}\Big)\Big| \asymp 1, \quad s \in (0, 1].$$

Proof. If g_1, g_2 are as in (2.16), the assertion is clear from (2.21) and Proposition 2.9.

Assume that (2.17) holds. Set $x_s := \mathring{t}(\frac{1}{8})$ where $\mathring{t}(r)$ is the unique number that satisfies (2.14) for $\mathscr{A}_s H$ instead of H. Then

$$g_1(s)m_1(sx_s)g_2(s)m_2(sx_s) = 1.$$

This is equivalent to $(m_1 m_2)(sx_s) = s^2$. The latter relation implies that $sx_s \to 0$ as $s \to 0$. Assumptions (1.13) and (1.14) yield $m_i(t) \asymp t$, $i = 1, 2$ and hence

$$A_{\mathscr{A}_s H}\Big(\frac{1}{8}\Big) = \sqrt{\frac{m_1(sx_s)}{m_2(sx_s)}} \asymp 1, \qquad s \in (0, 1].$$

We obtain from Proposition 2.9 that

$$\Big|q_{\mathscr{A}_s H}\Big(\frac{i}{8}\Big)\Big| \asymp A_{\mathscr{A}_s H}\Big(\frac{1}{8}\Big) \asymp 1, \qquad s \in (0, 1].$$

□

In the proof of Theorems 1.1 and 1.4 in Section 5 we also need the trace of the primitive of the rescaled Hamiltonian. Let g_1, g_2 be as in Definition 2.10 and $H \in \mathbb{H}$. For $s > 0$ set

$$\tau_s(t) := \int_0^t \operatorname{tr}(\mathscr{A}_s H)(x)\, dx = g_1(s)m_1(st) + g_2(s)m_2(st), \qquad t \in (0, \infty). \tag{2.22}$$

Since $\tau_s'(t) = sg_1(s)h_1(st) + sg_2(s)h_2(st) > 0$ a.e., the function τ_s is strictly increasing. If, in addition, H is in the limit point case at ∞, then τ_s is a bijection from $(0, \infty)$ onto itself. Note that for the choice (2.16) we have $\tau_s = \mathfrak{t}_s$.

3 Proof of "(i)⇔(ii)" in Theorems 1.1 and 1.4

We use the following fact which also plays a role later.

3.1 Lemma. *Let H_s, $s > 0$, be the trace-normalised reparameterisation of $\mathscr{A}_s H$, i.e., the Hamiltonian that satisfies*

$$(\mathscr{A}_s H)(t) = H_s(\tau_s(t)) \cdot \tau_s'(t), \tag{3.1}$$

where $\tau_s(t)$ is defined in (2.22). *Moreover, let $T \in (0, \infty)$. Then*

$$\lim_{t\to 0} d(H, t) = 0 \quad \Leftrightarrow \quad \lim_{s\to 0} d(H_s, T) = 0, \tag{3.2}$$

$$\liminf_{t\to 0} d(H, t) = 0 \quad \Leftrightarrow \quad \liminf_{s\to 0} d(H_s, T) = 0. \tag{3.3}$$

Proof. Let $T \in (0, \infty)$ be arbitrary. By (2.6), (2.19) and (2.22) we have

$$d(H_s, T) = d(\mathscr{A}_s H, \tau_s^{-1}(T)) = d(H, s\tau_s^{-1}(T)). \tag{3.4}$$

Set

$$u(s) := s\tau_s^{-1}(T). \tag{3.5}$$

The explicit form of $\tau_s(t)$ and the continuity of g_1 and g_2 show that the function $(s,t) \mapsto \tau_s(t)$ is continuous from $(0,\infty)^2$ to $(0,\infty)$. Moreover, for every $s > 0$, the mapping $t \mapsto \tau_s(t)$ is a homeomorphism from $(0,\infty)$ onto itself. By the implicit function theorem as in, e.g., [Kum80], the function $s \mapsto \tau_s^{-1}(T)$, and with it also $s \mapsto u(s)$, is continuous. Moreover, we have

$$T = \tau_s\Big(\frac{u(s)}{s}\Big) = g_1(s)m_1\big(u(s)\big) + g_2(s)m_2\big(u(s)\big) \tag{3.6}$$

for all s. Since $g_i(s) \to \infty$ as $s \to 0$ by assumption (see Definition 2.10), it follows that $\lim_{s\to 0} u(s) = 0$. The assertions now follow from (3.4). ☐

Proof of "(i)⇔(ii)" in Theorems 1.1 and 1.4. Let H be as in the formulation of the theorems.

① The implications "(i)⇒(ii)" in Theorems 1.1 and 1.4 are a direct consequence of [LPW21, Theorem 1.1] in the form of Proposition 2.9 since this result implies

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} \geq \frac{\frac{1}{64}L_H(r)}{44A_H(r)} = \frac{1}{2816}d(H, \mathring{t}(r))$$

for every $r > 0$. It remains to recall that $\mathring{t}$, defined via (2.14), is a strictly decreasing bijection from $(0,\infty)$ onto itself.

② In this step we show that

$$\lim_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \quad\Leftrightarrow\quad \operatorname{LP}(\mathcal{A}_s)_{s\in(0,1]} \subseteq \mathbb{H}^{\mathsf{cs}}, \tag{3.7}$$

$$\liminf_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \quad\Leftrightarrow\quad \operatorname{LP}(\mathcal{A}_s)_{s\in(0,1]} \cap \mathbb{H}^{\mathsf{cs}} \neq \emptyset. \tag{3.8}$$

Let $r_n \to \infty$. Then we have the equivalences

$$\lim_{n\to\infty} \frac{\operatorname{Im} q_H(ir_n)}{|q_H(ir_n)|} = 0 \quad\Leftrightarrow\quad \lim_{n\to\infty} \operatorname{Im} q_{\mathscr{A}_{\mathring{t}(r_n)}H}\Big(\frac{i}{8}\Big) = 0 \quad\Leftrightarrow\quad \operatorname{LP}(\mathcal{A}_{\mathring{t}(r_n)})_{n\in\mathbb{N}} \subseteq \mathbb{H}^{\mathsf{cs}}$$

The first one holds because of (2.21) and Lemma 2.12, and the second by the maximum principle and compactness of $\mathbb{H}$. Remembering that $\mathring{t}$ is a decreasing bijection we obtain (3.7) and (3.8).

③ To prove the implication "(ii)⇒(i)" in Theorem 1.1, assume that

$$\lim_{t\to 0} d(H,t) = 0.$$

By Lemma 3.1 we have $\lim_{s\to 0} d(H_s, T) = 0$ for all $T > 0$. Since $\operatorname{tr} H_s = 1$ a.e., it follows that also $\lim_{s\to 0} \det M_s(T) = 0$ for all $T > 0$ where M_s is the primitive of H_s.

Let $\hat{H} \in \mathrm{LP}(\mathscr{A}_s H)_{s\in(0,1]}$, and denote its primitive by $\hat{M}$. Then $\det \hat{M}(T) = 0$ for all T. This means that the whole interval $(0, \infty)$ is indivisible for $\hat{H}$, i.e., $\hat{H} \in \mathbb{H}^{\mathrm{cs}}$. Now (3.7) yields the required assertion.

④ For "(ii)⇒(i)" in Theorem 1.4 assume that $\liminf_{t\to 0} d(H, t) = 0$. Then for each $T > 0$ we have $\liminf_{s\to 0} d(H_s, T) = 0$ and, arguing as above, obtain a limit point $H_T \in \mathrm{LP}(\mathscr{A}_s H)_{s\in(0,1]}$ for which the interval $(0, T)$ is indivisible. Let $\phi_T \in [0, \pi)$ be the type of this indivisible interval. Choose a sequence $(T_n)_{n\in\mathbb{N}}$ such that $(\phi_{T_n})_{n\in\mathbb{N}}$ converges, say, $\phi_{T_n} \to \phi$. Then $(H_{T_n})_{n\in\mathbb{N}}$ converges to the Hamiltonian for which $(0, \infty)$ is indivisible of type ϕ. Since $\mathrm{LP}(\mathscr{A}_s H)_{s\in(0,1]}$ is closed, we can refer to (3.8) to finish the proof. □

4 Bounds for the off-diagonal entries and the rotation

In this section we show that the relative size, $\sigma_H(t)$, of the off-diagonal entries of a Hamiltonian and its rotation, $\zeta_H(t)$, can be estimated from above by $d(H, t)$; recall that the latter is defined in (2.5). These estimates are used in the proof of the implication (ii)⇒(iii) in Theorems 1.1 and 1.4.

We start with an estimate for the off-diagonal entry. As usual, λ denotes the Lebesgue measure.

4.1 Proposition. *Let $H \in \mathbb{H}^1$, and assume that neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint 0. For each $\gamma \in (0, 1)$ we have*

$$\forall t > 0. \quad \frac{1}{t}\lambda\big((0, t) \cap \sigma_H^{-1}([0, \gamma])\big) \le \frac{1}{1-\gamma^2} d(H, t). \tag{4.1}$$

Proof. Throughout the proof we fix $\gamma \in (0, 1)$ and $t > 0$ and often suppress t notationally. To shorten notation, we set $I_\gamma := (0, t) \cap \sigma_H^{-1}([0, \gamma])$ and $I'_\gamma := (0, t) \setminus I_\gamma$. Further, set

$$\xi_1 := \Big[\int_{I'_\gamma} h_1(s)\,\mathrm{d}s\Big]^{\frac{1}{2}}, \quad \xi_2 := \Big[\int_{I_\gamma} h_1(s)\,\mathrm{d}s\Big]^{\frac{1}{2}},$$

$$\eta_1 := \Big[\int_{I'_\gamma} h_2(s)\,\mathrm{d}s\Big]^{\frac{1}{2}}, \quad \eta_2 := \Big[\int_{I_\gamma} h_2(s)\,\mathrm{d}s\Big]^{\frac{1}{2}},$$

define the vectors

$$\mathfrak{x} := \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \mathfrak{y} := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

and let $\|\llcorner\!\lrcorner\|$ and $\langle \llcorner\!\lrcorner, \llcorner\!\lrcorner \rangle$ be the Euclidian norm and the inner product in $\mathbb{R}^2$. Since neither h_1 nor h_2 vanishes a.e. on $(0, t)$ by assumption, we have $\mathfrak{x} \neq 0$, $\mathfrak{y} \neq 0$, and we can write

$$\xi = \|\xi\| \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad \eta = \|\eta\| \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

with $\theta_1, \theta_2 \in \left[0, \frac{\pi}{2}\right]$. Moreover, we set $\theta := \max\{\theta_1, \theta_2\}$.

The relation $I_\gamma = \{s \in (0,t)\colon\ |h_3(s)| \leq \gamma\sqrt{h_1(s)h_2(s)}\}$ implies that

$$\begin{aligned} |m_3(t)| &\leq \int_{I'_\gamma} |h_3(s)|\,\mathrm{d}s + \int_{I_\gamma} |h_3(s)|\,\mathrm{d}s \\ &\leq \int_{I'_\gamma} \sqrt{h_1(s)h_2(s)}\,\mathrm{d}s + \int_{I_\gamma} \gamma\sqrt{h_1(s)h_2(s)}\,\mathrm{d}s \\ &\leq \xi_1\eta_1 + \gamma\xi_2\eta_2 = \langle A\xi, \eta\rangle \end{aligned}$$

with $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)$. For the diagonal terms in $M(t)$ we have

$$m_1(t) = \int_{I'_\gamma} h_1(s)\,\mathrm{d}s + \int_{I_\gamma} h_1(s)\,\mathrm{d}s = \xi_1^2 + \xi_2^2 = \|\xi\|^2,$$

and, similarly, $m_2(t) = \|\eta\|^2$, which leads to

$$\frac{m_3(t)^2}{m_1(t)m_2(t)} \leq \frac{[\langle A\xi, \eta\rangle]^2}{\|\xi\|^2\,\|\eta\|^2} \leq \min\left\{\frac{\|A\xi\|^2}{\|\xi\|^2}, \frac{\|A\eta\|^2}{\|\eta\|^2}\right\}.$$

The latter quotients can be rewritten as follows:

$$\frac{\|A\xi\|^2}{\|\xi\|^2} = \frac{\xi_1^2 + \gamma^2\xi_2^2}{\|\xi\|^2} = \cos^2\theta_1 + \gamma^2\sin^2\theta_1 = 1 - (1-\gamma^2)\sin^2\theta_1$$

and analogously, $\|A\eta\|^2/\|\eta\|^2 = 1 - (1-\gamma^2)\sin^2\theta_2$, which yields

$$\begin{aligned} \frac{m_3(t)^2}{m_1(t)m_2(t)} &\leq \min\{1 - (1-\gamma^2)\sin^2\theta_1, 1 - (1-\gamma^2)\sin^2\theta_2\} \\ &= 1 - (1-\gamma^2)\sin^2\theta. \end{aligned} \tag{4.2}$$

Since H is trace-normalised, we have

$$\xi_2^2 + \eta_2^2 = \int_{I_\gamma} h_1(s)\,\mathrm{d}s + \int_{I_\gamma} h_2(s)\,\mathrm{d}s = \lambda(I_\gamma), \qquad \|\xi\|^2 + \|\eta\|^2 = t,$$

which implies that

$$\frac{\lambda(I_\gamma)}{t} = \frac{\|\xi\|^2\sin^2\theta_1 + \|\eta\|^2\sin^2\theta_2}{\|\xi\|^2 + \|\eta\|^2} \leq \sin^2\theta. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$d(H,t) = 1 - \frac{m_3(t)^2}{m_1(t)m_2(t)} \geq (1-\gamma^2)\sin^2\theta \geq (1-\gamma^2)\frac{\lambda(I_\gamma)}{t},$$

which proves (4.1). □

Now we come to an estimate for the rotation of H.

4.2 Proposition. *Let $H \in \mathbb{H}^1$, and assume that neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint 0. For each pair of closed, disjoint subsets $A, B \subseteq \mathbb{T}$ there exists a constant $c(A,B) > 0$, which is independent of H, such that*

$$\forall t > 0. \quad \frac{1}{t}\lambda((0,t) \cap \zeta_H^{-1}(A)) \cdot \frac{1}{t}\lambda((0,t) \cap \zeta_H^{-1}(B)) \leq c(A,B) \cdot d(H,t).$$

Heading towards the proof of this proposition, we present two lemmata. The first one is an easy observation, which shows how information about the Hamiltonian H on an interval $I \subseteq (0,\infty)$ can be used to estimate $d(H,t)$. In these two lemmata we use the following notation, which extends the notation of the primitive to functions that may vanish on sets of positive measure: for a Hamiltonian H, $I \subseteq (0,\infty)$ and $t > 0$, set

$$M(H\mathbb{1}_I, t) \equiv \begin{pmatrix} m_1(H\mathbb{1}_I, t) & m_3(H\mathbb{1}_I, t) \\ m_3(H\mathbb{1}_I, t) & m_2(H\mathbb{1}_I, t) \end{pmatrix} := \int_{I \cap (0,t)} H(s)\,\mathrm{d}s.$$

4.3 Lemma. *Let $H \in \mathbb{H}^1$ and $I \subseteq (0,\infty)$. For $t > 0$, we have*

$$d(H,t) \geq \frac{\det M(H\mathbb{1}_I, t)}{t^2}.$$

Proof. The fact that H is positive semi-definite gives $M(H,t) \geq M(H\mathbb{1}_I, t) \geq 0$, and in turn

$$\det M(H,t) \geq \det M(H\mathbb{1}_I, t) \geq 0.$$

Together with $m_i(H,t) \leq t$, which is a consequence of $\operatorname{tr} H = 1$ a.e., we obtain

$$d(H,t) = \frac{\det M(H,t)}{m_1(H,t)m_2(H,t)} \geq \frac{\det M(H\mathbb{1}_I, t)}{m_1(H,t)m_2(H,t)} \geq \frac{1}{t^2}\det M(H\mathbb{1}_I, t). \qquad \square$$

The second lemma contains the crucial estimates. For $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ we denote the corresponding arc on $\mathbb{T}$ by

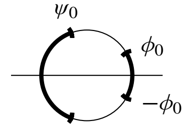
$$A[\alpha,\beta] := \{\exp(it) \colon \ \alpha \leq t \leq \beta\}.$$

4.4 Lemma. *The following estimates hold.*

(i) *Let ϕ_0, ψ_0 satisfy $0 \leq \phi_0 < \psi_0 \leq \pi$ and set*

$$I_1 := \zeta_H^{-1}\big(A[-\phi_0, \phi_0]\big),$$

$$I_2 := \zeta_H^{-1}\big(A[\psi_0, 2\pi - \psi_0]\big).$$



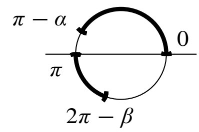
Then, for all H and $t > 0$, we have

$$d(H,t) \geq \sin^2\Big(\frac{\psi_0 - \phi_0}{2}\Big) \cdot \frac{1}{t}\lambda(I_1 \cap (0,t)) \cdot \frac{1}{t}\lambda(I_2 \cap (0,t)).$$

(ii) *Let* $\alpha, \beta \in (0, \pi]$ *and set*

$$I_1 := \zeta_H^{-1}(A[0, \pi - \alpha]),$$

$$I_2 := \zeta_H^{-1}(A[\pi, 2\pi - \beta]).$$



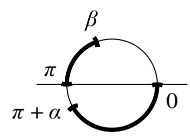
Then, for all H *and* $t > 0$, *we have*

$$d(H, t) \geq \sin^2\Big(\frac{\alpha}{2}\Big) \sin^2\Big(\frac{\beta}{2}\Big) \cdot \frac{1}{t}\lambda(I_1 \cap (0, t)) \cdot \frac{1}{t}\lambda(I_2 \cap (0, t)).$$

The same holds for

$$I_1 := \zeta_H^{-1}(A[\pi + \alpha, 2\pi]),$$

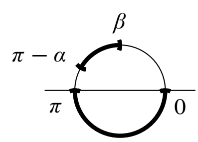
$$I_2 := \zeta_H^{-1}(A[\beta, \pi]).$$



(iii) *Let* $\alpha, \beta \in (0, \pi]$ *satisfy* $\alpha + \beta \leq \pi$ *and set*

$$I_1 := \zeta_H^{-1}(A[\beta, \pi - \alpha]),$$

$$I_2 := \zeta_H^{-1}(A[\pi, 2\pi]).$$



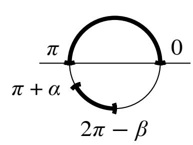
Then, for all H *and* $t > 0$, *we have*

$$d(H, t) \geq \sin^2\Big(\frac{\min\{\alpha, \beta\}}{2}\Big) \cdot \frac{1}{t}\lambda(I_1 \cap (0, t)) \cdot \frac{1}{t}\lambda(I_2 \cap (0, t)).$$

The same holds for

$$I_1 := \zeta_H^{-1}(A[\pi + \alpha, 2\pi - \beta]),$$

$$I_2 := \zeta_H^{-1}(A[0, \pi]).$$



Proof.

① We start with a general calculation. Let $K_1, K_2 \subseteq [0, t]$ be disjoint and set $K := K_1 \cup K_2$. We can use the inequality $|h_3| \leq \sqrt{h_1 h_2}$ and the Cauchy–Schwarz inequality in the last step to obtain

$$
\begin{aligned}
&\det M(H\mathbb{1}_K, t) \\
&\quad = m_1(H\mathbb{1}_K, t) m_2(H\mathbb{1}_K, t) - m_3^2(H\mathbb{1}_K, t) \\
&\quad = \Big(\int_{K_1} h_1(x)\,\mathrm{d}x + \int_{K_2} h_1(x)\,\mathrm{d}x\Big)\Big(\int_{K_1} h_2(x)\,\mathrm{d}x + \int_{K_2} h_2(x)\,\mathrm{d}x\Big) \\
&\qquad - \Big(\int_{K_1} h_3(x)\,\mathrm{d}x + \int_{K_2} h_3(x)\,\mathrm{d}x\Big)^2 \\
&\quad \geq \int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x - \Big(\int_{K_1} \sqrt{h_1(x)h_2(x)}\,\mathrm{d}x\Big)^2 \\
&\qquad + \int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x - \Big(\int_{K_2} \sqrt{h_1(x)h_2(x)}\,\mathrm{d}x\Big)^2 \qquad (4.4) \\
&\qquad + \int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x + \int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x \\
&\qquad - 2\int_{K_1} h_3(x)\,\mathrm{d}x \int_{K_2} h_3(x)\,\mathrm{d}x \\
&\quad \geq \int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x + \int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x \\
&\qquad - 2\int_{K_1} h_3(x)\,\mathrm{d}x \int_{K_2} h_3(x)\,\mathrm{d}x.
\end{aligned}
$$

Using once more $|h_3| \leq \sqrt{h_1 h_2}$ and the Cauchy–Schwarz inequality we arrive at a complete square:

$$
\begin{aligned}
&\det M(H\mathbb{1}_K, t) \\
&\quad \geq \int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x + \int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x \\
&\qquad - 2\Big[\int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x \int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x\Big]^{\frac{1}{2}} \qquad (4.5) \\
&\quad = \Big[\Big(\int_{K_1} h_1(x)\,\mathrm{d}x \int_{K_2} h_2(x)\,\mathrm{d}x\Big)^{\frac{1}{2}} - \Big(\int_{K_2} h_1(x)\,\mathrm{d}x \int_{K_1} h_2(x)\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big]^2.
\end{aligned}
$$

② Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ and set $J := \zeta_H^{-1}(A[\alpha, \beta])$. Then

$$
\begin{aligned}
x \in J \quad &\Leftrightarrow \quad \zeta_H(x) \in A[\alpha, \beta] \\
&\Leftrightarrow \quad \exists\, n \in \mathbb{Z}.\quad \varphi_H(x) - n\pi \in \Big[\frac{\alpha}{2}, \frac{\beta}{2}\Big] \qquad (4.6) \\
&\Rightarrow \quad \exists\, \varphi \in \Big[\frac{\alpha}{2}, \frac{\beta}{2}\Big].\quad h_1(x) = \cos^2\varphi, \quad h_2(x) = \sin^2\varphi.
\end{aligned}
$$

For the rest of the proof set $K_i := I_i \cap (0, t)$ for $i = 1, 2$, and $K := K_1 \cup K_2$. We consider the three cases in (i), (ii), (iii) separately.

③ Let us first consider the situation in item (i). It follows from (4.6) that

$$h_1(x) \geq \cos^2\Big(\frac{\phi_0}{2}\Big), \quad h_2(x) \leq \sin^2\Big(\frac{\phi_0}{2}\Big), \quad x \in K_1,$$
$$h_1(x) \leq \cos^2\Big(\frac{\psi_0}{2}\Big), \quad h_2(x) \geq \sin^2\Big(\frac{\psi_0}{2}\Big), \quad x \in K_2.$$

This, together with Lemma 4.3 and (4.5), implies

$$\begin{aligned} &d(H,t) \\ &\geq \frac{1}{t^2}\det M(H\mathbb{1}_K,t) \\ &\geq \frac{1}{t^2}\bigg[\sqrt{\cos^2\Big(\frac{\phi_0}{2}\Big)\lambda(K_1)\sin^2\Big(\frac{\psi_0}{2}\Big)\lambda(K_2)} - \sqrt{\cos^2\Big(\frac{\psi_0}{2}\Big)\lambda(K_2)\sin^2\Big(\frac{\phi_0}{2}\Big)\lambda(K_1)}\bigg]^2 \\ &= \bigg[\cos\Big(\frac{\phi_0}{2}\Big)\sin\Big(\frac{\psi_0}{2}\Big) - \cos\Big(\frac{\psi_0}{2}\Big)\sin\Big(\frac{\phi_0}{2}\Big)\bigg]^2 \cdot \frac{\lambda(K_1)\lambda(K_2)}{t^2} \\ &= \sin^2\Big(\frac{\psi_0-\phi_0}{2}\Big)\cdot\frac{\lambda(K_1)}{t}\cdot\frac{\lambda(K_2)}{t}, \end{aligned}$$

which is the asserted statement in (i).

④ Next, we consider the situation in item (ii). Here h_3 is non-negative on I_1 and non-positive on I_2, or vice versa. Thus, Lemma 4.3 and inequality (4.4) yield

$$\begin{aligned} d(H,t) &\geq \frac{1}{t^2}\det M(H\mathbb{1}_K,t) \\ &\geq \frac{1}{t^2}\bigg[\int_{K_1} h_1(x)\,\mathrm{d}x\int_{K_2} h_2(x)\,\mathrm{d}x + \int_{K_2} h_1(x)\,\mathrm{d}x\int_{K_1} h_2(x)\,\mathrm{d}x\bigg]. \end{aligned} \tag{4.7}$$

By (4.6) we have the estimates $h_1(x) \geq \cos^2((\pi \pm \alpha)/2) = \sin^2(\alpha/2)$ for $x \in K_1$, and $h_2(x) \geq \sin^2(\beta/2)$ for $x \in K_2$, and hence

$$d(H,t) \geq \sin^2\Big(\frac{\alpha}{2}\Big)\sin^2\Big(\frac{\beta}{2}\Big)\cdot\frac{\lambda(K_1)}{t}\cdot\frac{\lambda(K_2)}{t}.$$

⑤ Finally, we consider the situation in item (iii). Again h_3 is non-negative on I_1 and non-positive on I_2, or vice versa, and therefore we obtain (4.7). Further, for $x \in I_1$ we have the estimates $h_1(x) \geq \cos^2((\pi \pm \alpha)/2) = \sin^2(\alpha/2)$ and $h_2(x) \geq \sin^2(\beta/2)$ by (4.6). With Lemma 4.3 we obtain

$$\begin{aligned} d(H,t) &\geq \sin^2\Big(\frac{\alpha}{2}\Big)\cdot\frac{\lambda(K_1)}{t^2}\int_{K_2} h_2(x)\,\mathrm{d}x + \sin^2\Big(\frac{\beta}{2}\Big)\cdot\frac{\lambda(K_1)}{t^2}\int_{K_2} h_1(x)\,\mathrm{d}x \\ &\geq \min\Big\{\sin^2\Big(\frac{\alpha}{2}\Big),\sin^2\Big(\frac{\beta}{2}\Big)\Big\}\cdot\frac{\lambda(K_1)}{t^2}\int_{K_2}(h_1(x)+h_2(x))\,\mathrm{d}x \\ &= \sin^2\Big(\frac{\min\{\alpha,\beta\}}{2}\Big)\cdot\frac{\lambda(K_1)}{t}\cdot\frac{\lambda(K_2)}{t}. \end{aligned}$$

□

Proof of Prop 4.2. Let $\hat{d}$ denote the intrinsic metric on $\mathbb{T}$ which assigns to a pair of points the length of the shortest arc connecting them.

① As a first step we settle the case when $A, B \subseteq \mathbb{T}$ are two closed, disjoint arcs with lengths strictly less than π. Set $\delta := \hat{d}(A, B)$, fix $t > 0$, and set

$$\nu_A := \frac{1}{t}\lambda(\zeta_H^{-1}(A) \cap (0, t)), \quad \nu_B := \frac{1}{t}\lambda(\zeta_H^{-1}(B) \cap (0, t)).$$

Based on Lemma 4.4 we are going to show that

$$d(H, t) \geq \frac{\sin^4(\delta/4)}{4} \cdot \nu_A \nu_B. \tag{4.8}$$

To this end, we distinguish four cases.

▷ Assume that one of A and B is contained in $A[0, \pi]$, the other one is contained in $A[\pi, 2\pi]$, and either $\hat{d}(A, 1) \leq \hat{d}(B, 1)$ and $\hat{d}(B, -1) \leq \hat{d}(A, -1)$, or $\hat{d}(B, 1) \leq \hat{d}(A, 1)$ and $\hat{d}(A, -1) \leq \hat{d}(B, -1)$. Then Lemma 4.4 (ii) with the choice $\alpha = \beta = \delta/2$ yields

$$d(H, t) \geq \sin^4(\delta/4) \cdot \nu_A \nu_B, \tag{4.9}$$

which is even stronger than (4.8).

▷ Assume that one of A and B is contained in $A[0, \pi]$, the other one is contained in $A[\pi, 2\pi]$, and either $\hat{d}(A, 1) \leq \hat{d}(B, 1)$ and $\hat{d}(A, -1) \leq \hat{d}(B, -1)$, or $\hat{d}(B, 1) \leq \hat{d}(A, 1)$ and $\hat{d}(B, -1) \leq \hat{d}(A, -1)$. Then Lemma 4.4 (iii) with the choice $\alpha = \beta = \delta/2$ yields $d(H, t) \geq \sin^2(\delta/4)\nu_A \nu_B$, which implies (4.9).

▷ Assume that both A and B are contained $A[0, \pi]$, or both are contained in $A[\pi, 2\pi]$. Then Lemma 4.4 (i) yields $d(H, t) \geq \sin^2(\delta/2)\nu_A \nu_B$, which implies 4.9.

▷ Assume that neither of the above three cases takes place, and set

$$A_1 := A \cap A[0, \pi], \quad A_2 := A \cap A[\pi, 2\pi],$$
$$B_1 := B \cap A[0, \pi], \quad B_2 := B \cap A[\pi, 2\pi].$$

Then A_i and B_j are contained in $A[0, \pi]$ or $A[\pi, 2\pi]$, and satisfy $\hat{d}(A_i, B_j) \geq \delta$. Moreover, since the lengths of A and B are strictly less than π, the sets A_i and B_j are again closed arcs. The already settled cases can be applied to A_i and B_j, which yields

$$d(H, t) \geq \sin^4(\delta/4) \cdot \nu_{A_i} \nu_{B_j}, \quad i, j \in \{1, 2\};$$

cf. (4.9). There is at least one choice of $i, j \in \{1, 2\}$ such that $\nu_{A_i} \geq \nu_A/2$ and $\nu_{B_j} \geq \nu_B/2$. Using this choice we obtain (4.8).

② The general case, namely when A and B are arbitrary closed, disjoint subsets of $\mathbb{T}$, is deduced by appropriately covering A and B with arcs.

Set $\delta := \hat{d}(A, B)$ and consider the open cover of $\mathbb{T}$ consisting of all open arcs with length $\delta/3$. Since A is compact, there exist finitely many of these arcs whose union covers A, say $A_1, \dots, A_N$. In addition, we may assume that $A \cap A_i \neq \emptyset$ for all $i \in \{1, \dots, N\}$. In the same way we obtain arcs $B_1, \dots, B_{N'}$ whose union covers B and such that each of them intersects B.

We have $\hat{d}\big(\overline{A_i}, \overline{B_k}\big) \geq \delta/3$ for all i and k by construction, and (4.8) tells us that

$$d(H, t) \geq \frac{\sin^4(\delta/12)}{4} \cdot \nu_{\overline{A_i}} \nu_{\overline{B_k}}.$$

For each $t > 0$ there is at least one choice of $i \in \{1, \dots, N\}$ and $k \in \{1, \dots, N'\}$ such that $\nu_{\overline{A_i}} \geq \nu_A/N$ and $\nu_{\overline{B_k}} \geq \nu_B/N'$. Using this choice we arrive at

$$d(H, t) \geq \frac{\sin^4(\delta/12)}{4NN'} \cdot \nu_A \nu_B.$$

Note that the constants δ, N, N' only depend on A and B, but not on H or t. □

5 Proof of equivalence with (iii) in Theorems 1.1 and 1.4

We have now collected all necessary tools to carry out the proof of equivalence with condition (iii) in our main theorems. Our plan to proceed is to first work with a modified variant of (iii), namely "(iii′)" stated below, and prove that "(ii)⇒(iii′)⇒(i)". After that we show "(iii′)⇔(iii)", which is elementary.

In the following we consider the weighted rescalings $\mathscr{A}_s H$ of H from Definition 2.10. For most part of the proof, g_1 and g_2 are arbitrary functions that satisfy the assumptions in Definition 2.10. Only at the very end of the proof of (iii′)⇔(iii) we choose g_1, g_2 as in (2.16) for the proof of Theorems 1.1 and 1.4, and we use g_1, g_2 as in 2.17 for the additions to these theorems. Again let H_s, $s > 0$, be the trace-normalised reparameterisation of $\mathscr{A}_s H$, cf. Lemma 3.1. Moreover, recall that λ denotes the Lebesgue measure.

The modified variant of (iii) reads as follows.

▷ In Theorem 1.1:

(iii′) For all $T \in (0, \infty)$, all $\gamma \in [0, 1)$ and all closed, disjoint sets $A, B \subseteq \mathbb{T}$, the following limit relations hold:

$$\lim_{s \to 0} \lambda\big((0, T) \cap \sigma_{H_s}^{-1}([0, \gamma])\big) = 0, \tag{5.1}$$

$$\lim_{s \to 0} \big[\lambda\big((0, T) \cap \zeta_{H_s}^{-1}(A)\big) \cdot \lambda\big((0, T) \cap \zeta_{H_s}^{-1}(B)\big)\big] = 0. \tag{5.2}$$

▷ In Theorem 1.4:

(iii′) For each $T \in (0,\infty)$ there exists a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to 0$ such that, for all $\gamma \in [0,1)$ and all closed, disjoint sets $A, B \subseteq \mathbb{T}$, the following limit relations hold:

$$\lim_{n\to\infty} \lambda\big((0,T)\cap \sigma_{H_{s_n}}^{-1}([0,\gamma])\big) = 0, \tag{5.3}$$

$$\lim_{n\to\infty} [\lambda((0,T)\cap \zeta_{H_{s_n}}^{-1}(A))\cdot \lambda((0,T)\cap \zeta_{H_{s_n}}^{-1}(B))] = 0. \tag{5.4}$$

Note that the statement of (iii′) depends on the choice of the functions g_1, g_2 in Definition 2.10 because the family $(H_s)_{s>0}$ depends on g_1 and g_2.

The implication "(ii)⇒(iii′)" is a consequence of Propositions 4.1 and 4.2.

Proof of (ii)⇒(iii′) in Theorems 1.1 and 1.4. Let us first consider the situation in Theorem 1.1. Assume that $\lim_{t\to 0} d(H,t) = 0$. Then $\lim_{s\to 0} d(H_s,T) = 0$ by (3.2). Hence, Propositions 4.1 and 4.2 applied to H_s yield (iii′).

Now let us consider the situation in Theorems 1.4. Assume that

$$\liminf_{t\to 0} d(H,t) = 0.$$

By (3.3) there exist $s_n > 0$ with $s_n \to 0$ such that $\lim_{n\to\infty} d(H_{s_n},T) = 0$. We can apply Propositions 4.1 and 4.2 to H_{s_n} to obtain (iii′). □

The implication "(iii′)⇒(i)" is a consequence of Corollary 2.8.

Proof of (iii′)⇒(i) in Theorems 1.1 and 1.4. By (2.3) and (2.20) we have

$$q_{H_s}(z) = q_{\mathscr{A}_s H}(z) = \frac{g_3(s)}{g_2(s)} q_H(g_3(s)z). \tag{5.5}$$

It follows from Lemma 2.12 that

$$\Big|q_{H_s}\Big(\frac{i}{8}\Big)\Big| = \Big|q_{\mathscr{A}_s H}\Big(\frac{i}{8}\Big)\Big| \asymp 1, \quad s \in (0,1]. \tag{5.6}$$

Thus the constant Hamiltonians $\Gamma(0,1)$ and $\Gamma(0,-1)$, where Γ is defined in Definition 2.6, cannot be limit points of $(H_s)_{s\in(0,1]}$ since $q_{\Gamma(0,1)}(z) = \infty$ and $q_{\Gamma(0,-1)}(z) = 0$ by Lemma 2.4. Relations (5.5) and (5.6) imply that

$$\frac{\operatorname{Im} q_H(g_3(s)\frac{i}{8})}{|q_H(g_3(s)\frac{i}{8})|} = \frac{\operatorname{Im} q_{H_s}(\frac{i}{8})}{|q_{H_s}(\frac{i}{8})|} \asymp \operatorname{Im} q_{H_s}\Big(\frac{i}{8}\Big), \quad s \in (0,1]. \tag{5.7}$$

If $T \in (0,\infty)$ and $s_n \in (0,1]$ with $s_n \to 0$ are such that (5.3) and (5.4) hold for all $\gamma \in [0,1)$ and all closed, disjoint sets $A, B \subseteq \mathbb{T}$, then

$$\mathrm{LP}_w(\rho_T(H_{s_n}))_{n\in\mathbb{N}} \subseteq \mathbb{H}_T^{\mathsf{cs}} \tag{5.8}$$

by Corollary 2.8; recall that $\rho_T : \mathbb{H}^1 \to \mathbb{H}_T^1$ is the restriction map

▷ First assume that (iii′) in the sequence variant for Theorem 1.4 holds. Then, for each $T \in (0, \infty)$, we can choose a sequence $(s_n)_{n\in\mathbb{N}}$ that satisfies (5.3) and (5.4) for all $\gamma \in [0, 1)$ and all closed, disjoint sets $A, B \subseteq \mathbb{T}$, and hence (5.8). Lemma 2.3 (ii) implies that $\mathrm{LP}(H_s)_{s\in(0,1]} \cap \mathbb{H}^{\mathrm{cs}} \neq \emptyset$. Since $\Gamma(0, \pm 1) \notin \mathrm{LP}(H_s)_{s\in(0,1]}$, we find $\zeta \in \mathbb{T} \setminus \{\pm 1\}$ and some sequence $s_n \to 0$ such that $\lim_{n\to\infty} H_{s_n} = \Gamma(1, \zeta)$. By the continuity of the mapping $H \mapsto q_H$ and Lemma 2.4 this implies that

$$\lim_{n\to\infty} q_{H_{s_n}}\Big(\frac{i}{8}\Big) = q_{\Gamma(1,\zeta)}\Big(\frac{i}{8}\Big) = \frac{\mathrm{Im}\,\zeta}{1 - \mathrm{Re}\,\zeta} \in \mathbb{R},$$

and hence $\lim_{n\to\infty} \mathrm{Im}\, q_{H_{s_n}}(i/8) = 0$. By the assumptions in Definition 2.10, g_3 is continuous, and $g_3(s) \to \infty$ as $s \to 0$. With $r_n := g_3(s_n)/8$ it follows from (5.7) that

$$\lim_{n\to\infty} \frac{\mathrm{Im}\, q_H(ir_n)}{|q_H(ir_n)|} = 0,$$

which shows (i) in Theorem 1.4.

▷ Assume that (iii′) in the continuous variant for Theorem 1.1 holds. We start with an arbitrary sequence $r_n \to \infty$. Since $g_3(s) \to \infty$ as $s \to 0$ and g_3 is continuous, we find $s_n > 0$ for large enough n such that $s_n \to 0$ and $r_n = g_3(s_n)/8$. By (5.1) and (5.2) the relations (5.3) and (5.4) hold for every $T \in (0, \infty)$, every $\gamma \in [0, 1)$ and all closed, disjoint $A, B \subseteq \mathbb{T}$ for the sequence $(s_n)_{n\in\mathbb{N}}$. Thus (5.8) holds for all $T \in (0, \infty)$, and Lemma 2.3 (i) gives $\mathrm{LP}(H_{s_n})_{n\in\mathbb{N}} \subseteq \mathbb{H}^{\mathrm{cs}}$. Using that $\mathbb{H}^1$ is compact and that $\Gamma(0, \pm 1)$ cannot occur as a limit point, we find a subsequence $(H_{s_{n_k}})_{k\in\mathbb{N}}$ and $\zeta \in \mathbb{T} \setminus \{\pm 1\}$, such that $\lim_{k\to\infty} H_{s_{n_k}} = \Gamma(1, \zeta)$ and hence $\lim_{k\to\infty} q_{H_{s_{n_l}}}(i/8) \in \mathbb{R}$ as above. Now relation (5.7) implies that

$$\lim_{k\to\infty} \frac{\mathrm{Im}\, q_H(ir_{n_k})}{|q_H(ir_{n_k})|} = 0.$$

Since the sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n \to \infty$ was arbitrary, the desired relation (1.9) follows.

This finishes the proof of (iii′)⇒(i). □

Showing that (iii) and (iii′) are equivalent is elementary.

Proof of (iii)⇔(iii′) and of (iv)⇔(iii′) under the assumption of (1.13), (1.14). We prove the asserted equivalences for the continuous variant in Theorem 1.1. The proof for the sequence variant in Theorem 1.4 is—word by word—the same. We proceed in several steps. In the first two steps we show that (iii′) is equivalent to (iii‴) stated below. In the last step we prove that (iii‴) is equivalent to (iii) and—under the additional assumptions (1.13), (1.14)—also equivalent to (iv).

① We show that (iii′) is equivalent to an analogous condition, say (iii″), where the limit relation (5.2) is required to hold for all open arcs $V, W \subseteq \mathbb{T}\setminus\mathbb{R}$ with $\overline{V}\cap\overline{W} = \emptyset$ and lengths at most $\frac{\pi}{2}$, instead of all closed disjoint sets $A, B \subseteq \mathbb{T}$.

The implication (iii′)⇒(iii″) is of course trivial. To show the converse, assume we know (5.1) and (5.2) for arcs as above. Let $A, B \subseteq \mathbb{T}$ be closed and disjoint. Then we can choose open arcs $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ of lengths at most $\frac{\pi}{2}$ such that

$$
\begin{gathered}
A \subseteq \bigcup_{i=1}^n V_i, \quad B \subseteq \bigcup_{j=1}^m W_j, \\
\bigcup_{i=1}^n \overline{V_i} \cap \bigcup_{j=1}^m \overline{W_j} = \emptyset, \quad \mathbb{R}\cap A = \mathbb{R}\cap \bigcup_{i=1}^n \overline{V_i}, \quad \mathbb{R}\cap B = \mathbb{R}\cap \bigcup_{j=1}^m \overline{W_j}.
\end{gathered} \tag{5.9}
$$

If an arc V_i intersects $\mathbb{R}$, we can split it into the two arcs $V_i \cap \mathbb{C}^+$ and $V_i \cap \mathbb{C}^-$, and the singleton $V_i \cap \mathbb{R}$; here $\mathbb{C}^+$ and $\mathbb{C}^-$ are the open upper and lower half-planes respectively. Hence, we may assume that our arcs V_i, W_j do not intersect the real axis on the cost of adding the set $\{1, -1\}$ to the covering, i.e., we can write

$$
A \subseteq \{1, -1\} \cup \bigcup_{i=1}^n V_i, \quad B \subseteq \{1, -1\} \cup \bigcup_{j=1}^m W_j \tag{5.10}
$$

instead of (5.9).

For any Hamiltonian we have $\zeta_H^{-1}(\{1, -1\}) \subseteq \sigma_H^{-1}(\{0\})$ by the definition of σ_H. Hence, (5.1) guarantees that $\lim_{s\to 0} \lambda((0, T) \cap \zeta_{H_s}^{-1}(\{1, -1\})) = 0$. We know that, for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$,

$$
\lim_{s\to 0}[\lambda((0, T) \cap \zeta_{H_s}^{-1}(V_i)) \cdot \lambda((0, T) \cap \zeta_{H_s}^{-1}(W_j))] = 0,
$$

and we obtain from (5.10) that also

$$
\lim_{s\to 0}[\lambda((0, T) \cap \zeta_{H_s}^{-1}(A)) \cdot \lambda((0, T) \cap \zeta_{H_s}^{-1}(B))] = 0.
$$

② We make a transformation to pass from the unit circle to the real line. Consider the function

$$
\phi_+ : \begin{cases} (0, \infty) \to \mathbb{T}\cap\mathbb{C}^+ \\ x \mapsto \frac{1-x}{1+x} + i\sqrt{1 - \left(\frac{1-x}{1+x}\right)^2}. \end{cases}
$$

This is a differentiable homeomorphism from $(0, \infty)$ onto $\mathbb{T}\cap\mathbb{C}^+$, and open intervals in $(0, \infty)$ correspond to open arcs in $\mathbb{T}\cap\mathbb{C}^+$. Moreover, for an interval $I \subseteq (0, \infty)$ we have

$$
\inf I = 0 \iff 1 \in \overline{\phi_+(I)} \quad \text{and} \quad \sup I = \infty \iff -1 \in \overline{\phi_+(I)}.
$$

For any $z \in \mathbb{T} \cap \mathbb{C}^+$ the relation

$$\operatorname{Re} \phi_+ \Big(\frac{1 - \operatorname{Re} z}{1 + \operatorname{Re} z} \Big) = \operatorname{Re} z$$

holds and hence also

$$\phi_+ \Big(\frac{1 - \operatorname{Re} z}{1 + \operatorname{Re} z} \Big) = z.$$

Let $t \in (0, \infty)$ and assume that $\operatorname{Im} \zeta_H(t) > 0$. Then $\varphi_H(t) \in (0, \frac{\pi}{2})$, and by (1.21) we have $\pi_H(t) = \tan^2 \varphi_H(t)$. Since $\zeta_H(t) \in \mathbb{T} \cap \mathbb{C}^+$, we have

$$\zeta_H(t) = \phi_+ \Big(\frac{1 - \operatorname{Re} \zeta_H(t)}{1 + \operatorname{Re} \zeta_H(t)} \Big) = \phi_+ \Big(\frac{1 - \cos(2\varphi_H(t))}{1 + \cos(2\varphi_H(t))} \Big) = \phi_+(\tan^2 \varphi_H(t)) = \phi_+(\pi_H(t)).$$

Thus, for every open arc $V \subseteq \mathbb{T} \cap \mathbb{C}^+$,

$$\zeta_H^{-1}(V) = \pi_H^{-1}(\phi_+^{-1}(V)).$$

For the lower half-plane we proceed analogously. Consider the function

$$\phi_- : \begin{cases} (-\infty, 0) \to \mathbb{T} \cap \mathbb{C}^- \\ x \mapsto \frac{1-|x|}{1+|x|} - i\sqrt{1 - (\frac{1-|x|}{1+|x|})^2}, \end{cases}$$

which is a differentiable homeomorphism from $(-\infty, 0)$ onto $\mathbb{T} \cap \mathbb{C}^-$ such that open intervals in $(0, \infty)$ correspond to open arcs in $\mathbb{T} \cap \mathbb{C}^-$, and that, for an interval $I \subseteq (-\infty, 0)$, we have

$$\sup I = 0 \iff 1 \in \overline{\phi_-(I)} \quad \text{and} \quad \inf I = -\infty \iff -1 \in \overline{\phi_-(I)}.$$

As above one shows that, for an arc $V \subseteq \mathbb{T} \cap \mathbb{C}^-$,

$$\zeta_H^{-1}(V) = \pi_H^{-1}(\phi_-^{-1}(V)).$$

Now we combine the mappings ϕ_+ and ϕ_-; let $\phi \colon \mathbb{R} \setminus \{0\} \to \mathbb{T} \setminus \mathbb{R}$ be defined by $\phi|_{(0,\infty)} = \phi_+$ and $\phi|_{(-\infty,0)} = \phi_-$. The above considerations show that (iii′) is equivalent to the following condition (iii‴).

(iii‴) For all $T \in (0, \infty)$, all $\gamma \in [0, 1)$ and all open intervals $I, J \subseteq \mathbb{R} \setminus \{0\}$ with $\overline{I} \cap \overline{J} = \emptyset$ and at least one of them bounded the following limit relations hold:

$$\lim_{s \to 0} \lambda((0, T) \cap \sigma_{H_s}^{-1}([0, \gamma])) = 0, \tag{5.11}$$

$$\lim_{s \to 0} [\lambda((0, T) \cap \pi_{H_s}^{-1}(I)) \cdot \lambda((0, T) \cap \pi_{H_s}^{-1}(J))] = 0. \tag{5.12}$$

③ It follows from (2.4), (2.22) and (2.19) that

$$\sigma_{H_s}(t) = \sigma_{\mathscr{A}_s H}(\tau_s^{-1}(t)) = \sigma_H(s\tau_s^{-1}(t)), \tag{5.13}$$

$$\pi_{H_s}(t) = \pi_{\mathscr{A}_s H}(\tau_s^{-1}(t)) = \frac{g_2(s)}{g_1(s)}\pi_H(s\tau_s^{-1}(t)). \tag{5.14}$$

▷ To show (iii)⇔(iii‴), let us choose g_1, g_2 as in (2.16). Then $\tau_s(t) = \mathfrak{t}_s(t)$ for all $t \in (0,\infty)$, and (5.14) can be simplified to $\pi_{H_s}(t) = \pi_{H,s}(\mathfrak{t}_s^{-1}(t))$. This and (5.13) show that the following equivalences hold:

$$x \in \sigma_{H_s}^{-1}([0,\gamma]) \quad \Leftrightarrow \quad x \in \mathfrak{t}_s\Big(\frac{1}{s}\sigma_H^{-1}([0,\gamma])\Big),$$
$$x \in \pi_{H_s}^{-1}(I) \quad \Leftrightarrow \quad x \in \mathfrak{t}_s(\pi_{H,s}(I)).$$

This settles the equivalence (iii)⇔(iii‴).

▷ Finally, assume that (1.13) and (1.14) in the addition to Theorem 1.1 hold. Let us choose g_1, g_2 as in (2.17). Then $\tau_s(t) = \frac{1}{s}\big(m_1(st) + m_2(st)\big) = t$ by (1.13) and hence $\sigma_{H_s}(t) = \sigma_H(st)$. For fixed $T, s \in (0,\infty)$ and $\gamma \in [0,1)$ we have

$$\begin{aligned}\lambda((0,T) \cap \sigma_{H_s}^{-1}([0,\gamma])) &= \lambda(\{x \in (0,T) : sx \in \sigma_H^{-1}([0,\gamma])\}) \\ &= \int_{(0,T)} \mathbb{1}_{\sigma_H^{-1}([0,\gamma])}(sx)\,\mathrm{d}x = \frac{1}{s}\int_{(0,sT)} \mathbb{1}_{\sigma_H^{-1}([0,\gamma])}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{s}\lambda((0,sT) \cap \sigma_H^{-1}([0,\gamma])).\end{aligned}$$

Hence, for fixed $\gamma \in [0,1)$, the following equivalences hold:

$$\begin{aligned}\forall T \in (0,\infty),\ (5.11) \text{ is true } &\Leftrightarrow \ \forall T \in (0,\infty),\ \lim_{s\to 0}\Big[\frac{1}{s}\lambda((0,sT) \cap \sigma_H^{-1}([0,\gamma]))\Big] = 0 \\ &\Leftrightarrow \ \lim_{t\to 0}\Big[\frac{1}{t}\lambda((0,t) \cap \sigma_H^{-1}([0,\gamma]))\Big] = 0.\end{aligned}$$

In a similar way one shows that (5.12) is true for every $T \in (0,\infty)$ if and only if (1.16) holds. This establishes the equivalence of (iii‴) and (iv) and finishes the proof of Theorems 1.1 and 1.4 and their additions. □

6 Hamiltonians with regularly varying diagonal

As a class of examples we consider Hamiltonians whose primitive M has regularly varying diagonal entries. Recall that a function $f\colon (0,\infty) \to (0,\infty)$ is called **regularly varying with index ρ at** 0 if

$$\forall t > 0. \quad \lim_{s\to 0}\frac{f(st)}{f(s)} = t^\rho;$$

see, e.g., [BGT89, §1.4.2]. Typical examples of regularly varying functions are $f(t) = t^\rho \cdot |\log t|^{\beta_1} \cdot (\log|\log t|)^{\beta_2}$ with $\rho, \beta_1, \beta_2 \in \mathbb{R}$, where higher iterates of logarithms can be added. In the theorem below we show that a Hamiltonian with regularly varying diagonal primitives is well behaved in the sense that $d(H,t) \gtrsim 1$ unless its diagonal entries are of the same size on the power scale, i.e., their indices coincide. This is closely related to our forthcoming paper [LPW22], where we investigate Hamiltonians whose Weyl coefficients have regularly varying asymptotics towards $+i\infty$.

6.1 Theorem. *Let H be a Hamiltonian defined on the interval $(0,\infty)$ and assume that neither h_1 nor h_2 vanishes a.e. on some neighbourhood of the left endpoint 0. Assume that m_1 and m_2 are regularly varying at 0 with positive indices ρ_1 and ρ_2 respectively. Then*

$$\liminf_{t\to 0} d(H,t) \geq 1 - \Big(\frac{\sqrt{\rho_1\rho_2}}{\frac{1}{2}(\rho_1+\rho_2)}\Big)^2.$$

Proof. Let $(\mathscr{A}_s H)_{s>0}$ be the family of rescaled Hamiltonians as in Definition 2.10 with g_1, g_2 from (2.16), and let $(H_s)_{s>0}$ be the corresponding trace-normalised family as in (3.1).

① In the first step of the proof we show that every accumulation point of $(H_s)_{s>0}$, for $s \to 0$, is of a special form. It follows from (2.18) that

$$\Big[\int_0^t (\mathscr{A}_s H)(x)\,\mathrm{d}x\Big]_{ii} = \frac{m_i(st)}{m_i(s)}, \quad i \in \{1,2\}, \tag{6.1}$$

where $[C]_{ii}$ denotes the ith entry on the diagonal of a matrix C, and hence

$$\mathfrak{t}_s(t) = \tau_s(t) = \int_0^t \operatorname{tr}(\mathscr{A}_s H)(x)\,\mathrm{d}x = \frac{m_1(st)}{m_1(s)} + \frac{m_2(st)}{m_2(s)}$$

where $\mathfrak{t}_s$ and τ_s are defined in (1.8) and (2.22) respectively. Set $\mathfrak{t}(t) := t^{\rho_1} + t^{\rho_2}$ for $t \in (0,\infty)$. The assumptions about m_1 and m_2 and the Uniform Convergence Theorem for regularly varying functions (see, e.g., [BGT89, Theorem 1.5.2]) imply that $\lim_{s\to 0}\mathfrak{t}_s(t) = \mathfrak{t}(t)$ locally uniformly for $t \in (0,\infty)$. The functions $\mathfrak{t}_s$ and $\mathfrak{t}$ are continuous and increasing bijections from $(0,\infty)$ onto itself, and it follows that also $\lim_{s\to 0}\mathfrak{t}_s^{-1}(T) = \mathfrak{t}^{-1}(T)$ for all $T \in (0,\infty)$.

Let $s_n \to 0$ be a sequence such that the limit $\widetilde{H} := \lim_{n\to\infty} H_{s_n}$ exists, and let $\widehat{H}$ be the reparameterisation defined by $\widehat{H} := (\widetilde{H} \circ \mathfrak{t}) \cdot \mathfrak{t}'$. Using (6.1) we find, for $T \in (0, \infty)$ and $i \in \{1, 2\}$, that

$$\Big[\int_0^{\mathfrak{t}^{-1}(T)} \widehat{H}(t)\,\mathrm{d}t\Big]_{ii} = \Big[\int_0^{\mathfrak{t}^{-1}(T)} \widetilde{H}(\mathfrak{t}(t))\mathfrak{t}'(t)\,\mathrm{d}t\Big]_{ii} = \Big[\int_0^{T} \widetilde{H}(x)\,\mathrm{d}x\Big]_{ii}$$
$$= \lim_{n\to\infty}\Big[\int_0^{T} H_{s_n}(x)\,\mathrm{d}x\Big]_{ii} = \lim_{n\to\infty}\Big[\int_0^{\mathfrak{t}_{s_n}^{-1}(T)} (\mathscr{A}_{s_n}H)(t)\,\mathrm{d}t\Big]_{ii}$$
$$= \lim_{n\to\infty}\frac{m_1(s_n\mathfrak{t}_{s_n}^{-1}(T))}{m_1(s_n)} = \mathfrak{t}^{-1}(T)^{\rho_1},$$

again by the Uniform Convergence Theorem. Hence $\widehat{H}$ is of the form

$$\widehat{H}(t) = \begin{pmatrix} \rho_1 t^{\rho_1-1} & ❄ \\ ❄ & \rho_2 t^{\rho_2-1} \end{pmatrix} \tag{6.2}$$

where the off-diagonal entries are unknown.

② For Hamiltonians $\widehat{H}$ of the form (6.2) an estimate for $d(\widehat{H}, t)$ holds. With $\hat{h}_j$ being the entries of $\widehat{H}$ we have

$$|\hat{h}_3(t)| \le \sqrt{\hat{h}_1(t)\hat{h}_2(t)} = \sqrt{\rho_1\rho_2}\, t^{\frac{1}{2}(\rho_1+\rho_2)-1}$$

and hence

$$|\widehat{m}_3(t)| \le \int_0^t |\hat{h}_3(x)\,\mathrm{d}x| \le \frac{\sqrt{\rho_1\rho_2}}{\frac{1}{2}(\rho_1+\rho_2)} t^{\frac{1}{2}(\rho_1+\rho_2)},$$

from which we find that, for all $t > 0$,

$$d(\widehat{H}, t) \ge 1 - \Big(\frac{\sqrt{\rho_1\rho_2}}{\frac{1}{2}(\rho_1+\rho_2)}\Big)^2. \tag{6.3}$$

③ We make a limiting argument to complete the proof. Let $(t_n)_{n=1}^{\infty}$ be a sequence of positive numbers with $t_n \to 0$. Fix $T > 0$ and let again $u(s)$ be the function in (3.5). For large enough n, choose $s_n \to 0$ such that $u(s_n) = t_n$, and extract a subsequence $(s_{n(k)})_{k\in\mathbb{N}}$ such that the limit $\widehat{H} := \lim_{k\to\infty} H_{s_{n(k)}}$ exists. Using (2.19), (2.6) and (6.3) we obtain

$$d(H, t_{n(k)}) = d(H, u(s_{n(k)})) = d(\mathscr{A}_{s_{n(k)}}H, \mathfrak{t}_{s_{n(k)}}^{-1}(T))$$
$$= d(H_{s_{n(k)}}, T) \xrightarrow{k\to\infty} d(\widehat{H}, T) \ge 1 - \Big(\frac{\sqrt{\rho_1\rho_2}}{\frac{1}{2}(\rho_1+\rho_2)}\Big)^2.$$

Since the (t_n) was arbitrary, the claim follows. □

As a consequence, if $\rho_1 \neq \rho_2$ in Theorem 6.1, then (ii) in Theorem 1.4 is not satisfied and hence neither is (i) (under the assumption that (1.2) holds), i.e., one has $\liminf_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} > 0$. If, on the other hand, the diagonal entries themselves (and not just their primitives) are regularly varying with the same index, then the situation is different.

6.2 Proposition. *Assume that $h_1(t), h_2(t) > 0$ a.e., that h_1, h_2 are regularly varying with the same index $\alpha > -1$, and set $h_3(t) := \sqrt{h_1(t)h_2(t)}$, $t \in (0,\infty)$. Then $\lim_{t\to 0} d(H,t) = 0$ and hence $\lim_{y\to\infty} \frac{\operatorname{Im} q_H(iy)}{|q_H(iy)|} = 0$.*

Proof. The off-diagonal entry h_3 is also regularly varying with index α. It follows from Karamata's Theorem (e.g. [BGT89, Theorem 1.5.10] transformed from the asymptotics at ∞ to the asymptotics at 0 by a change of variable) that $m_i(t) = \frac{1}{1+\alpha} t h_i(t)(1 + o(1))$ as $t \to 0$ for $i = 1, 2, 3$. Hence

$$d(H,t) = \frac{m_1(t)m_2(t) - m_3(t)^2}{m_1(t)m_2(t)} = \frac{h_1(t)h_2(t)(1+o(1)) - h_3(t)^2(1+o(1))}{h_1(t)h_2(t)(1+o(1))}$$
$$= \frac{h_1(t)h_2(t)o(1)}{h_1(t)h_2(t)(1+o(1))} \to 0$$

as $t \to 0$. The last statement follows from Theorem 1.1. □

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