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OPTIMAL CONVERGENCE RATES IN L^2 FOR A FIRST ORDER SYSTEM LEAST SQUARES FINITE ELEMENT METHOD

PART I: HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. We analyze a divergence based first order system least squares method applied to a second order elliptic model problem with homogeneous boundary conditions. We prove optimal convergence in the $L^2(\Omega)$ norm for the scalar variable. Numerical results confirm our findings.

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1. Introduction

Least Squares Finite Element Methods (LSFEM) are an important class of numerical methods for the solution of partial differential equations with a variety of applications. The main idea of the LSFEM is to reformulate the partial differential equation of interest as a minimization problem, for which a variety of tools is available. For example, even for non-symmetric or indefinite problems, the discretization with the least squares approach leads to symmetric, positive definite systems, which can be solved with well-established numerical technologies. Furthermore, the least squares technique is naturally quasi-optimal, albeit in a problem-dependent norm. For second order PDEs, which is the setting of the present work, the most common least squares approach is that of rewriting the equation as a First Order Least Squares System (FOSLS) that can be discretized with established finite element techniques. A benefit is that many quantities of interest are approximated directly without the need of postprocessing. We mention [5] as a classical monograph on the topic as well as the papers [4, 7, 9, 15].

The present work considers a Poisson-like second order model problem written as a system of first order equations. For the discretization, an $\boldsymbol{H}(\Omega, \operatorname{div}) \times H^1(\Omega)$-conforming least squares formulation is employed. Even though our model problem in its standard $H^1(\Omega)$ formulation is coercive our methods and lines of proof can most certainly be applied to other problems as well, see [2, 10] for an application to the Helmholtz equation. The LSFEM is typically quasi-optimal in some problem-dependent energy norm, which is, however, somewhat intractable; *a priori* error estimates in more familiar norms such as the $L^2(\Omega)$ norm of the scalar variable are thus desirable. Numerical examples in our previous work [2] suggested convergence rates in standard norms such

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as the $L^2(\Omega)$-norm which, to our best knowledge, are not fully explained by the current theory. In the present work, we develop a convergence theory with minimal assumptions on the regularity of the right-hand side.

1.1. Contribution of the present work

Our main contribution are optimal $L^2(\Omega)$ based convergence result for the least squares approximation u_h to the scalar variable u. Furthermore, we derive hp error estimates for the gradient of the scalar variable u as well as an hp error estimate for the vector variable $\boldsymbol{\varphi} = -\nabla u$ in the $L^2(\Omega)$ norm. Simultaneous error estimates of the gradient of the the scalar variable u and the vector variable $\boldsymbol{\varphi}$ in the $L^2(\Omega)$ norm are available in the literature, see Lemma 6.2 of [12]. We derive error estimates for these two quantities of interest separately. These optimality results are new in the sense that we achieve optimal convergence rates under minimal regularity assumptions on the data. Here, we call a method optimal in a certain norm, if the norm of the error made by the method is of the same order as the best approximation of the employed space.

1.1.1. Review of related results

In [15] the author considered the classical model problem $-\Delta u = f$ with inhomogeneous Dirichlet boundary condition $u = g$ in some smooth domain Ω. Unlike the present work the least squares formulation employs vector valued $H^1(\Omega)$ functions instead of $\boldsymbol{H}(\Omega, \mathrm{div})$ for the vector variable. The corresponding finite element spaces are chosen such that they satisfy simultaneous approximation properties in $L^2(\Omega)$ and $H^1(\Omega)$ for both the scalar variable u and the vector variable $\boldsymbol{\varphi}$. Using a duality argument akin to the one used in the present work the author arrived at the error estimate

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h\|(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, u - u_h)\|_b,$$

see Theorem 4.1 of [15], where $\|(\cdot,\cdot)\|_b$ denotes the corresponding energy norm. At this point higher order convergence rates are just a question of approximation properties in $\|(\cdot,\cdot)\|_b$, see Lemma 3.1 of [15] for a precise statement. As stated after the proof of Theorem 4.1 of [15], one can extract optimal convergence rates for sufficiently smooth data f and g. The smoothness of the data is important as the following considerations show: For the case of a smooth boundary Γ and $f \in L^2(\Omega)$ and $g \in H^{3/2}(\Gamma)$, elliptic regularity gives $u \in H^2(\Omega)$. Therefore u can be approximated by globally continuous piecewise polynomials of degree greater or equal to one with a error $O(h^2)$ in the $L^2(\Omega)$ norm, which is achieved by classical FEM, due to the Aubin-Nitsche trick. In contrast, the above least squares estimate does not give the desired rate: The norm $\|(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, u - u_h)\|_b$ contains a term of the form

$$\|\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{L^2(\Omega)} = \|f - \nabla \cdot \boldsymbol{\varphi}_h\|_{L^2(\Omega)},$$

from which no further convergence rate can be extracted, since f is only in $L^2(\Omega)$.

In [7] (see also [9]) the problem $-\nabla \cdot (A\nabla u) + Xu = f$ with uniformly elliptic diffusion matrix A and X a linear differential operator of order at most one together with homogeneous mixed Dirichlet and Neumann boundary conditions was considered. The least squares formulation presented therein employs the same spaces as the present work. Apart from nontrivial norm equivalence results, see Theorem 3.1 of [7], they also derived the following estimate of the least squares approximation

$$\|u - u_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{H(\mathrm{div},\Omega)} \lesssim h^s(\|u\|_{H^{s+1}(\Omega)} + \|\boldsymbol{\varphi}\|_{H^{s+1}(\Omega)}),$$

assuming $u \in H^{s+1}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^{s+1}(\Omega)$. This result is then optimal in the stated norm, however, the assumed regularity is somewhat unsatisfactory in the sense that if the solution $u \in H^{s+1}(\Omega)$ then the relation $\nabla u + \boldsymbol{\varphi} = 0$ merely provides the regularity $\boldsymbol{\varphi} \in \boldsymbol{H}^s(\Omega)$ and not the assumed regularity $\boldsymbol{\varphi} \in \boldsymbol{H}^{s+1}(\Omega)$.

In [4] the same model problem as well as the same least squares formulation is considered. The main goal of [4] is to establish $L^2(\Omega)$ error estimates for u and $\boldsymbol{\varphi}$. In Lemma 3.4 of [4] a result similar to Theorem 4.1 of [15] is obtained. This result, however, suffers from the same drawback as elaborated above. Furthermore, they prove optimality of the error of the vector variable $\boldsymbol{\varphi}$ in the $L^2(\Omega)$ norm, see Corollary 3.7 of [4].

Recently in Section 6.2 of [12] the authors also considered a first order system least squares formulation of $-\Delta u = f$ with homogeneous Dirichlet boundary condition. A simultaneous error estimate of the gradient of the the scalar variable u and the vector variable $\boldsymbol{\varphi}$ in the $L^2(\Omega)$ norm is derived in Lemma 6.2 of [12]. Note that the method of proof therein does not hinge on duality arguments. We point out that in our results we derive error estimates separately for these two quantities in the Theorems 4.8 and 4.10, which also manifests itself in the error estimates of Corollary 4.14.

The main tools for *a priori* error estimates in more tractable norms such as $L^2(\Omega)$ instead of the energy norm in a least squares setting are, as it is done in the present paper and the above literature, duality arguments, which lead to an estimate of the form

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h\|(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, u - u_h)\|_b.$$

As elaborated above it is not possible to extract the desired optimal rate from this estimate directly. In the proof of one of our main result (Thm. 4.12) we exploit the duality argument in a more delicate way, which allows us to lower the regularity requirements on $\boldsymbol{\varphi}$ to what could be expected from the regularity of the data f. Key components in the proof are the div-conforming approximation operators $\boldsymbol{I}_h^0$ and $\boldsymbol{I}_h$ (*cf.* Lems. 4.3, 4.6), which are also of independent interest.

1.1.2. Notation

Throughout this work, Ω denotes a bounded simply connected domain in $\mathbb{R}^d$, $d = 2, 3$ with connected boundary $\Gamma := \partial\Omega$ and outward unit normal vector $\boldsymbol{n}$. Let Γ consist of two disjoint parts Γ_D and Γ_N. For scalar functions in d variables the gradient ∇ and for $\mathbb{R}^d$-valued functions $\boldsymbol{\varphi}$ the divergence $\nabla\cdot$ are defined as usual as $\nabla u = (\partial_{x_1}, \cdots, \partial_{x_d})$ and $\nabla\cdot\boldsymbol{\varphi} = \sum_{i=1}^d \partial_{x_i}\boldsymbol{\varphi}_i$. For $d = 3$ the curl operator $\nabla\times$ of a vector field $\boldsymbol{\varphi}$ is given as usual by $\nabla\times\boldsymbol{\varphi} = (\partial_{x_2}\boldsymbol{\varphi}_3 - \partial_{x_3}\boldsymbol{\varphi}_2, -(\partial_{x_1}\boldsymbol{\varphi}_3 - \partial_{x_3}\boldsymbol{\varphi}_1), \partial_{x_1}\boldsymbol{\varphi}_2 - \partial_{x_2}\boldsymbol{\varphi}_1)^T$. In spatial dimension $d = 2$ the scalar-valued curl operator acting on vector fields is given by $\operatorname{curl}\boldsymbol{\varphi} = \partial_{x_1}\boldsymbol{\varphi}_2 - \partial_{x_2}\boldsymbol{\varphi}_1$ and the vector-valued curl operator acting on scalar functions by $\mathbf{curl}\, u = (\partial_{x_2}u, -\partial_{x_1}u)$. We remind the reader of the exact sequences in spatial dimension $d = 3$

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\Omega, \mathbf{curl}) \xrightarrow{\nabla\times} \boldsymbol{H}(\Omega, \operatorname{div}) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} \{0\},$$

as well as with zero boundary conditions

$$\{0\} \xrightarrow{\text{id}} H_0^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_0(\Omega, \mathbf{curl}) \xrightarrow{\nabla\times} \boldsymbol{H}_0(\Omega, \operatorname{div}) \xrightarrow{\nabla\cdot} L_0^2(\Omega) \xrightarrow{0} \{0\}.$$

In spatial dimension $d = 2$ there are two exact sequences which are isomorphic to each other *via* rotation

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\Omega, \operatorname{curl}) \xrightarrow{\operatorname{curl}} L^2(\Omega) \xrightarrow{0} \{0\}, \tag{1.1}$$

as well as

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\Omega, \operatorname{div}) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} \{0\}. \tag{1.2}$$

Note that *via* rotation the sequence in (1.2) can be obtained from the sequence in (1.1): With the matrix $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ we have $\operatorname{curl}\boldsymbol{\varphi} = \nabla\cdot(R\boldsymbol{\varphi})$ and $\mathbf{curl}\, u = R\nabla u$.

We consider the following spaces in spatial dimension $d = 2, 3$:

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega)\colon \nabla u \in \boldsymbol{L}^2(\Omega)\}, & \boldsymbol{H}(\Omega, \operatorname{div}) &= \{\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)\colon \nabla\cdot\boldsymbol{\varphi} \in L^2(\Omega)\}, \\ H_D^1(\Omega) &= \{u \in H^1(\Omega)\colon u = 0 \text{ on } \Gamma_D\}, & \boldsymbol{H}_N(\Omega, \operatorname{div}) &= \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega, \operatorname{div})\colon \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_N\}, \\ H_0^1(\Omega) &= \{u \in H^1(\Omega)\colon u = 0 \text{ on } \Gamma\}, & \boldsymbol{H}_0(\Omega, \operatorname{div}) &= \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega, \operatorname{div})\colon \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}. \end{aligned}$$

Additionally, we introduce the spaces corresponding to the curl operator in spatial dimension $d = 3$ as well as the scalar valued curl operator in spatial dimension $d = 2$:

$$\boldsymbol{H}(\Omega, \mathbf{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)\colon \nabla\times\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)\}, \qquad \boldsymbol{H}(\Omega, \operatorname{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)\colon \operatorname{curl}\boldsymbol{\varphi} \in L^2(\Omega)\},$$

$$\boldsymbol{H}_D(\Omega,\mathbf{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega,\mathbf{curl})\colon \boldsymbol{n}\times\boldsymbol{\varphi} = 0 \text{ on } \Gamma_D\}, \qquad \boldsymbol{H}_D(\Omega,\mathrm{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega,\mathrm{curl})\colon \boldsymbol{t}\cdot\boldsymbol{\varphi} = 0 \text{ on } \Gamma_D\},$$
$$\boldsymbol{H}_0(\Omega,\mathbf{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega,\mathbf{curl})\colon \boldsymbol{n}\times\boldsymbol{\varphi} = 0 \text{ on } \Gamma\}, \qquad \boldsymbol{H}_0(\Omega,\mathrm{curl}) = \{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega,\mathrm{curl})\colon \boldsymbol{t}\cdot\boldsymbol{\varphi} = 0 \text{ on } \Gamma\}.$$

where $\boldsymbol{t} = R^T\boldsymbol{n}$ is the corresponding tangential vector in spatial dimension $d = 2$. For further detail and references see [6, 11, 18, 19].

Since we also consider smooth boundaries we employ curved elements. We make the following assumptions on the triangulation.

Assumption 1.1 (Quasi-uniform regular meshes)**.** *Let $\widehat{K}$ be the reference simplex. Each element map $F_K\colon \widehat{K} \to K$ can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy, for constants $C_{\text{affine}}, C_{\text{metric}}, \rho > 0$ independent of K:*

$$\|A'_K\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h_K, \qquad \left\|(A'_K)^{-1}\right\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h_K^{-1},$$
$$\left\|(R'_K)^{-1}\right\|_{L^\infty(\tilde{K})} \le C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\tilde{K})} \le C_{\text{metric}} \rho^n n! \qquad \forall n \in \mathbb{N}_0.$$

Here, $\tilde{K} = A_K(\widehat{K})$ and $h_K > 0$ denotes the element diameter.

On the reference simplex $\widehat{K}$ we introduce the Raviart–Thomas **RT** and Brezzi–Douglas–Marini **BDM** elements:

$$\begin{aligned}
\mathcal{P}_p(\widehat{K}) &:= \operatorname{span}\{\boldsymbol{x}^{\boldsymbol{\alpha}}\colon |\boldsymbol{\alpha}| \le p\},\\
\mathbf{RT}_{p-1}(\widehat{K}) &:= \{\boldsymbol{p} + \boldsymbol{x}q\colon \boldsymbol{p} \in \mathcal{P}_{p-1}(\widehat{K})^d, q \in \mathcal{P}_{p-1}(\widehat{K})\},\\
\mathbf{BDM}_p(\widehat{K}) &:= \mathcal{P}_p(\widehat{K})^d.
\end{aligned}$$

Furthermore, we introduce the Nédélec type one and two elements in spatial dimension $d = 2$ and $d = 3$ by

$$\begin{aligned}
\mathbf{N}^I_{p-1}(\widehat{K}) &:= \{\boldsymbol{p} + q(y,-x)^T\colon \boldsymbol{p} \in \mathcal{P}_{p-1}(\widehat{K})^2, q \in \mathcal{P}_{p-1}(\widehat{K})\} &&\text{for } d = 2,\\
\mathbf{N}^I_{p-1}(\widehat{K}) &:= \{\boldsymbol{p} + \boldsymbol{x}\times\boldsymbol{p}\colon \boldsymbol{q},\boldsymbol{q} \in \mathcal{P}_{p-1}(\widehat{K})^3\} &&\text{for } d = 3,\\
\mathbf{N}^{II}_p(\widehat{K}) &:= \mathcal{P}_p(\widehat{K})^d.
\end{aligned}$$

We also recall the classical Piola transformation, which is the appropriate change of variables for $\boldsymbol{H}(\Omega,\mathrm{div})$. For a function $\boldsymbol{\varphi} : K \to \mathbb{R}^d$ and the element map $F_K\colon \widehat{K} \to K$ its Piola transform $\widehat{\boldsymbol{\varphi}} : \widehat{K} \to \mathbb{R}^d$ is given by

$$\widehat{\boldsymbol{\varphi}} = (\det F'_K)(F'_K)^{-1}\boldsymbol{\varphi} \circ F_K.$$

The spaces $S_p(\mathcal{T}_h)$, $\mathbf{BDM}_p(\mathcal{T}_h)$, and $\mathbf{RT}_{p-1}(\mathcal{T}_h)$ are given by standard transformation and (contravariant) Piola transformation of functions on the reference element:

$$\begin{aligned}
S_p(\mathcal{T}_h) &:= \{u \in H^1(\Omega)\colon u\big|_K \circ F_K \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\},\\
\mathbf{BDM}_p(\mathcal{T}_h) &:= \{\boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div},\Omega)\colon (\det F'_K)(F'_K)^{-1}\boldsymbol{\varphi}\big|_K \circ F_K \in \mathbf{BDM}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\},\\
\mathbf{RT}_{p-1}(\mathcal{T}_h) &:= \{\boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div},\Omega)\colon (\det F'_K)(F'_K)^{-1}\boldsymbol{\varphi}\big|_K \circ F_K \in \mathbf{RT}_{p-1}(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\}.
\end{aligned}$$

Similarly for Nédélec elements of type one in spatial dimension $d = 3$ with the aid of the covariant Piola transformation we introduce

$$\mathbf{N}^I_p(\mathcal{T}_h) := \left\{\boldsymbol{\varphi} \in \boldsymbol{H}(\Omega,\mathbf{curl})\colon (F'_K)^T\boldsymbol{\varphi}\big|_K \circ F_K \in \mathbf{N}^I_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\right\},$$

and analogously for Nédélec elements of type two and spatial dimension $d = 2$. Note that the Nédélec elements in spatial dimension $d = 2$ are just the rotated Raviart–Thomas and Brezzi–Douglas–Marini elements.

For the approximation properties of the $\boldsymbol{H}(\Omega, \text{div})$ conforming finite element spaces see Proposition 2.5.4 of [6] as a standard reference for non-curved elements and without the p-aspect. For an analysis of the hp-version under Assumption 1.1 we refer to Section 4 of [2].

Since we will look at a first order system formulation of a second order model problem we have two finite element spaces to choose, one for the scalar variable u, *i.e.*, the solution of the second order equation, and one for the vector variable $\boldsymbol{\varphi}$, which will be related to the scalar variable *via* $\boldsymbol{\varphi} = -\nabla u$. Summarizing, for the numerical discretization of the first order system we consider the following finite element spaces:

$$S_{p_s}(\mathcal{T}_h) \subseteq H^1(\Omega), \quad S^D_{p_s}(\mathcal{T}_h) \subseteq H^1_D(\Omega), \quad S^0_{p_s}(\mathcal{T}_h) \subseteq H^1_0(\Omega),$$
$$\mathbf{V}_{p_v}(\mathcal{T}_h) \subseteq \boldsymbol{H}(\Omega, \text{div}), \quad \mathbf{V}^N_{p_v}(\mathcal{T}_h) \subseteq \boldsymbol{H}_N(\Omega, \text{div}), \quad \mathbf{V}^0_{p_v}(\mathcal{T}_h) \subseteq \boldsymbol{H}_0(\Omega, \text{div}),$$

where the polynomial approximation of the scalar and vector variable is denoted by $p_s \geq 1$ and $p_v \geq 1$ respectively. For brevity denote by $\mathbf{V}_{p_v}(\mathcal{T}_h)$ either the Raviart–Thomas space $\mathbf{RT}_{p_v-1}(\mathcal{T}_h)$ or the Brezzi–Douglas–Marini space $\mathbf{BDM}_{p_v}(\mathcal{T}_h)$. The spaces $\mathbf{V}^N_{p_v}(\mathcal{T}_h)$ and $\mathbf{V}^0_{p_v}(\mathcal{T}_h)$ are denoted analogously. Furthermore, for brevity the Nédélec space $\mathbf{N}_{p_v}(\mathcal{T}_h)$ is either of type one or two, depending on the choice of $\mathbf{V}_{p_v}(\mathcal{T}_h)$. The same convention applies to the spaces with boundary conditions. See again [6, 11, 18, 19] for further details.

Further notational conventions will be:

– lower case roman letters like u and v will be reserved for scalar valued functions;
– lower case boldface greek letters like $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ will be reserved for vector valued functions;
– K denotes the physical element and $\widehat{K}$ denotes the reference element;
– quantities like u_h and $\boldsymbol{\varphi}_h$ will be reserved for functions from the corresponding finite element space, again scalar and vector valued respectively;
– if not stated otherwise discrete functions without a $\tilde{\ }$ will be in some sense fixed, *e.g.*, resulting from a certain discretization scheme, whereas functions with a $\tilde{\ }$ will be arbitrary, *e.g.*, when dealing with quasi-optimality results;
– generic constants will either be denoted by C or hidden inside a $\lesssim$ and will be independent of the mesh size h and the polynomial degree p, if not otherwise stated. Note that the parameter γ in the definition of the model problem in (2.1) will not be explicitly tracked.

1.1.3. Outline

The outline of this paper is as follows. In Section 2 we introduce the model problem, the first order system least squares (FOSLS) method and prove norm equivalence results, which in turn guarantee unique solvability of the continuous as well as the discrete least squares formulation. Section 3 is devoted to the proof of duality results for the scalar variable, the gradient of the scalar variable as well as the vector variable. In the beginning of Section 4 we first exploit the duality result of Section 3 in order to prove $L^2(\Omega)$ error estimates for the scalar variable of the primal as well as the dual problem. We then argue first heuristically that these results are actually suboptimal and can be further improved. To that end we introduce an approximation operator that also satisfies certain orthogonality relations and prove best approximation results for this operator, which are then used to prove our main result (Thm. 4.12). Furthermore, we derive $L^2(\Omega)$ error estimates for the gradient of the scalar variable as well as the vector variable. In Section 5 we present numerical examples showcasing the proved convergence rates, focusing especially on the case of finite Sobolev regularity.

2. Model problem

Let $\Gamma = \partial\Omega$ consist of two disjoint parts Γ_D and Γ_N and let $f \in L^2(\Omega)$. (Later, we will focus on the special cases $\Gamma = \Gamma_D$ and $\Gamma = \Gamma_N$.) For $\gamma > 0$ fixed we consider the following model problem

$$\begin{aligned} -\Delta u + \gamma u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \partial_n u &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{2.1}$$

We formulate (2.1) a first order system. Introducing the new variable $\boldsymbol{\varphi} = -\nabla u$ we formally arrive at the system

$$\nabla \cdot \boldsymbol{\varphi} + \gamma u = f \quad \text{in } \Omega, \tag{2.2a}$$
$$\nabla u + \boldsymbol{\varphi} = \mathbf{0} \quad \text{in } \Omega, \tag{2.2b}$$
$$u = 0 \quad \text{on } \Gamma_D, \tag{2.2c}$$
$$\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_N. \tag{2.2d}$$

Introducing the differential operator $\mathcal{L}\colon \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega) \to L^2(\Omega) \times \boldsymbol{L}^2(\Omega)$, given by

$$\mathcal{L}\begin{pmatrix} \boldsymbol{\varphi} \\ u \end{pmatrix} = \begin{pmatrix} \nabla\cdot & \gamma \\ 1 & \nabla \end{pmatrix}\begin{pmatrix} \boldsymbol{\varphi} \\ u \end{pmatrix} = \begin{pmatrix} \nabla \cdot \boldsymbol{\varphi} + \gamma u \\ \nabla u + \boldsymbol{\varphi} \end{pmatrix},$$

we want to solve the equation

$$\mathcal{L}\begin{pmatrix} \boldsymbol{\varphi} \\ u \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{0} \end{pmatrix}.$$

The least squares approach to this problem is to find $(\boldsymbol{\varphi}, u) \in \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega)$ such that

$$\left(\mathcal{L}\begin{pmatrix} \boldsymbol{\varphi} \\ u \end{pmatrix}, \mathcal{L}\begin{pmatrix} \boldsymbol{\psi} \\ v \end{pmatrix}\right)_\Omega = \left(\begin{pmatrix} f \\ \mathbf{0} \end{pmatrix}, \mathcal{L}\begin{pmatrix} \boldsymbol{\psi} \\ v \end{pmatrix}\right)_\Omega \qquad \forall\, (\boldsymbol{\psi}, v) \in \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega),$$

where $(\cdot,\cdot)_\Omega$ denotes the usual $L^2(\Omega)$ scalar product. Introducing now the bilinear form b and the linear functional F by

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \coloneqq (\nabla \cdot \boldsymbol{\varphi} + \gamma u, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega + (\nabla u + \boldsymbol{\varphi}, \nabla v + \boldsymbol{\psi})_\Omega, \tag{2.3}$$
$$F((\boldsymbol{\psi}, v)) \coloneqq (f, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega, \tag{2.4}$$

we can state the mixed weak least squares formulation: Find $(\boldsymbol{\varphi}, u) \in \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega)$ such that

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) = F((\boldsymbol{\psi}, v)) \quad \forall\, (\boldsymbol{\psi}, v) \in \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega). \tag{2.5}$$

To see solvability of (2.5), let $u \in H^1_D(\Omega)$ be the unique solution of (2.1). In view of $f \in L^2(\Omega)$ the pair $(-\nabla u, u)$ is a solution of (2.5). Uniqueness follows if one can show that $b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) = 0$ for all $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_N(\Omega, \operatorname{div}) \times H^1_D(\Omega)$ implies $(\boldsymbol{\varphi}, u) = (\mathbf{0}, 0)$. To that end we introduce the (yet to be verified) norm $\|\cdot\|_b$ induced by b:

$$\|(\boldsymbol{\varphi}, u)\|_b \coloneqq \sqrt{b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u))}. \tag{2.6}$$

A general approach would be to show norm equivalence. In our case:

$$\|u\|_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|_{\boldsymbol{H}(\Omega,\operatorname{div})} \lesssim \|(\boldsymbol{\varphi}, u)\|_b \lesssim \|u\|_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|_{\boldsymbol{H}(\Omega,\operatorname{div})}.$$

We will employ methods similar to a duality argument in the following Theorem 2.1 to prove such a norm equivalence.

Theorem 2.1 (Norm equivalence)**.** *For all* $(\boldsymbol{\varphi}, u) \in \boldsymbol{H}_N(\Omega, \mathrm{div}) \times H^1_D(\Omega)$ *there holds the norm equivalence*

$$\|u\|^2_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\boldsymbol{H}(\Omega,\mathrm{div})} \lesssim b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u)) \lesssim \|u\|^2_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\boldsymbol{H}(\Omega,\mathrm{div})}. \tag{2.7}$$

Proof. First note that by definition

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u)) = \Big\|\underbrace{\nabla\cdot\boldsymbol{\varphi} + \gamma u}_{=:w}\Big\|^2_{L^2(\Omega)} + \Big\|\underbrace{\nabla u + \boldsymbol{\varphi}}_{=:\boldsymbol{\eta}}\Big\|^2_{L^2(\Omega)},$$

from which the second inequality in (2.7) is obvious. For the first inequality, we start by noting that $\boldsymbol{\varphi}$ and u are unique given w and $\boldsymbol{\eta}$. We now split $\boldsymbol{\varphi}$ and u as follows:

$$\begin{aligned} \nabla\cdot\boldsymbol{\varphi}_1 + \gamma u_1 &= w && \text{in } \Omega, & \nabla\cdot\boldsymbol{\varphi}_2 + \gamma u_2 &= 0 && \text{in } \Omega,\\ \nabla u_1 + \boldsymbol{\varphi}_1 &= \boldsymbol{0} && \text{in } \Omega, & \nabla u_2 + \boldsymbol{\varphi}_2 &= \boldsymbol{\eta} && \text{in } \Omega,\\ u_1 &= 0 && \text{on } \Gamma_D, & u_2 &= 0 && \text{on } \Gamma_D,\\ \boldsymbol{\varphi}_1\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_N, & \boldsymbol{\varphi}_2\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_N, \end{aligned}$$

with yet to be determined functions $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, u_1, and u_2. By summing up the two systems of equations we observe that $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$ and $u = u_1 + u_2$. Simply eliminating $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ in the above equations, we expect u_1 and u_2 to be solutions to

$$\begin{aligned} -\Delta u_1 + \gamma u_1 &= w && \text{in } \Omega, & -\Delta u_2 + \gamma u_2 &= -\nabla\cdot\boldsymbol{\eta} && \text{in } \Omega,\\ u_1 &= 0 && \text{on } \Gamma_D, & u_2 &= 0 && \text{on } \Gamma_D,\\ \partial_n u_1 &= 0 && \text{on } \Gamma_N, & \partial_n u_2 &= 0 && \text{on } \Gamma_N, \end{aligned}$$

where $-\nabla\cdot\boldsymbol{\eta}$ is to be understood as an element of $(H^1_D(\Omega))'$ given by $F : v \mapsto (\boldsymbol{\eta}, \nabla v)_\Omega$. Both equations are therefore uniquely solvable. This then determines the desired functions u_1, u_2 and consequently the functions $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, using the second equation in the first order systems.

Let us show that $(\boldsymbol{\varphi}_1, u_1)$ solves the above system. By construction it satisfies the differential equations and furthermore, since $\boldsymbol{\varphi}_1 = -\nabla u_1$, we have $\boldsymbol{\varphi}_1\cdot\boldsymbol{n} = -\nabla u_1\cdot\boldsymbol{n} = -\partial_n u_1 = 0$.

Let us show that $(\boldsymbol{\varphi}_2, u_2)$ satisfies the above system. Let $v \in C^\infty_0(\Omega)$ be arbitrary. Integration by parts and exploiting the weak formulation gives

$$(\nabla\cdot\boldsymbol{\varphi}_2, v)_\Omega = -(\boldsymbol{\varphi}_2, \nabla v)_\Omega = -(\boldsymbol{\eta}, \nabla v)_\Omega + (\nabla u_2, \nabla v)_\Omega = -(\gamma u_2, v)_\Omega.$$

Therefore the div-equation is satisfied. To verify the boundary conditions we calculate for any $v \in H^1_D(\Omega)$

$$\begin{aligned} \langle\boldsymbol{\varphi}_2\cdot\boldsymbol{n}, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} &= (\boldsymbol{\varphi}_2, \nabla v)_\Omega + (\nabla\cdot\boldsymbol{\varphi}_2, v)_\Omega\\ &= (-\nabla u_2 + \boldsymbol{\eta}, \nabla v)_\Omega + (\nabla\cdot\boldsymbol{\varphi}_2, v)_\Omega = 0, \end{aligned}$$

where we first used Green's theorem, then the equations of the first order system and at last the weak formulation for u_2. The *a priori* estimate of the Lax–Milgram theorem gives

$$\|u_1\|_{H^1(\Omega)} \lesssim \|w\|_{(H^1_D(\Omega))'} \le \|w\|_{L^2(\Omega)}, \qquad \|u_2\|_{H^1(\Omega)} \lesssim \|F\|_{(H^1_D(\Omega))'} \le \|\boldsymbol{\eta}\|_{L^2(\Omega)}.$$

Due to the splitting $u = u_1 + u_2$ it is now obvious that

$$\|u\|^2_{H^1(\Omega)} \lesssim \|w\|^2_{L^2(\Omega)} + \|\boldsymbol{\eta}\|^2_{L^2(\Omega)}.$$

We finally estimate the $\boldsymbol{H}(\Omega, \mathrm{div})$ norms of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ as follows

$$\begin{aligned} \|\boldsymbol{\varphi}_1\|^2_{\boldsymbol{H}(\Omega,\mathrm{div})} &= \|\boldsymbol{\varphi}_1\|^2_{L^2(\Omega)} + \|\nabla\cdot\boldsymbol{\varphi}_1\|^2_{L^2(\Omega)} = \|-\nabla u_1\|^2_{L^2(\Omega)} + \|w - \gamma u_1\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{L^2(\Omega)},\\ \|\boldsymbol{\varphi}_2\|^2_{\boldsymbol{H}(\Omega,\mathrm{div})} &= \|\boldsymbol{\varphi}_2\|^2_{L^2(\Omega)} + \|\nabla\cdot\boldsymbol{\varphi}_2\|^2_{L^2(\Omega)} = \|\boldsymbol{\eta} - \nabla u_2\|^2_{L^2(\Omega)} + \|-\gamma u_2\|^2_{L^2(\Omega)} \lesssim \|\boldsymbol{\eta}\|^2_{L^2(\Omega)}, \end{aligned}$$

which completes the proof. □

Remark 2.2. Theorem 2.1 (norm equivalence) does not hold on all of $\boldsymbol{H}(\Omega, \operatorname{div}) \times H^1(\Omega)$ since one can construct non-trivial solutions to the system

$$\nabla \cdot \boldsymbol{\varphi} + \gamma u = 0 \quad \text{in } \Omega, \qquad \nabla u + \boldsymbol{\varphi} = \mathbf{0} \quad \text{in } \Omega,$$

due to the missing boundary conditions, even though $\|(\boldsymbol{\varphi}, u)\|_b = 0$ by construction.

Remark 2.3. Theorem 2.1 (norm equivalence) is in fact much stronger than what we need to establish unique solvability of the system (2.5): The weaker coercivity estimate $\|u\|^2_{L^2(\Omega)} + \|\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \lesssim b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u))$ suffices to establish uniqueness.

Remark 2.4. In the literature there are two main ideas for showing unique solvability when working in a least squares setting concerning a first order system derived from a second order equation:

– The first one deduces solvability from the second order equation and uses a weaker coercivity estimates, as in Remark 2.3, to establish uniqueness. See also [2,10] for these kind of arguments for the Helmholtz equation.
– The second approach is to establish a stronger coercivity estimate as in Theorem 2.1 and directly apply the Lax–Milgram theorem to (2.5), where the right-hand side is a suitable continuous linear functional. See also [7,9] concerning the model problem in question and also [8] for the Stokes equation.

3. Duality argument

The current section is devoted to duality arguments that are later used for the analysis of the $L^2(\Omega)$ norms of $u - u_h$, $\nabla(u - u_h)$, and $\boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Since these duality arguments rely heavily on the elliptic shift theorem we restrict ourself to either the pure Neumann or Dirichlet boundary conditions, *i.e.*, $\Gamma = \Gamma_N$ or $\Gamma = \Gamma_D$. In contrast, when considering mixed boundary conditions one has to expect a singularity at the interface between the Dirichlet and Neumann condition, which has to be properly accounted for in the numerical analysis by graded meshes for both the primal and dual problem. This is beyond the scope of the present work. We introduce the following assumption regarding a shift theorem:

Assumption 3.1 ($\hat{s}$ shift property)**.** *Let $\hat{s} \geq -1$ be given. Then for every $f \in H^s(\Omega)$, $s \in [-1, \hat{s}]$ the problem*

$$-\Delta u + \gamma u = f \quad \text{in } \Omega,$$

with homogeneous Dirichlet or Neumann boundary conditions admits the regularity shift $u \in H^{s+2}(\Omega)$ with $\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{H^s(\Omega)}$ if $s \geq 0$ and, if $s < 0$,

$$\|u\|_{H^{s+2}(\Omega)} \lesssim \begin{cases} \|f\|_{H^s(\Omega)} & \text{for Dirichlet boundary conditions} \\ \|f\|_{\tilde{H}^s(\Omega)} & \text{for Neumann boundary conditions.} \end{cases}$$

Here, for $s \in (-1, 0)$, we set $H^s(\Omega) = (\tilde{H}^{-s}(\Omega))'$, $\tilde{H}^s(\Omega) = (H^{-s}(\Omega))'$ with the Sobolev spaces $H^{-s}(\Omega) = (L^2(\Omega), H^1(\Omega))_{-s,2}$ and $\tilde{H}^{-s}(\Omega) = (L^2(\Omega), H^1_0(\Omega))_{-s,2}$ defined by the real method of interpolation (see [16] for details).

Remark 3.2. For smooth, bounded domains Assumption 3.1 is satisfied for any $\hat{s} < \infty$, see *e.g.*, [13]. For convex domains Assumption 3.1 is satisfied for $\hat{s} = 0$ ([14], Chap. 2). For general bounded Lipschitz domains, Assumption 3.1 holds for any $\hat{s} < -1/2$, [21]. For general Lipschitz polygonals or polyhedra, Assumption 3.1 holds for some $\hat{s} > -1/2$, *e.g.*, for polygonals and Dirichlet boundary conditions, one has $\hat{s} = -1 + \pi/\omega - \varepsilon$, for any $\varepsilon > 0$, [1,14], where ω is the largest interior angle.

Our overall agenda is to derive regularity results for the dual solutions, always denoted by $(\boldsymbol{\psi}, v)$. For $w \in H^1(\Omega)$ and $\eta \in \boldsymbol{H}_0(\Omega, \operatorname{div})$ we prove the existence of dual solutions such that:

- $\|w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$, see Theorem 3.3,
- $\|\nabla w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$, see Theorem 3.4,
- $\|\boldsymbol{\eta}\|^2_{L^2(\Omega)} = b((\boldsymbol{\eta}, u), (\boldsymbol{\psi}, v))$, see Theorem 3.5.

These results are exploited in Section 4 with the special choices of $w = u - u_h$ and $\boldsymbol{\eta} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$, respectively.

Theorem 3.3 (Duality argument for the scalar variable)**.** *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq -1$. Then there holds:*

(i) *For $\Gamma = \Gamma_N$ and any $(\boldsymbol{\varphi}, w) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ such that $\|w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s},2)+1}(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^{\min(\hat{s}+2,2)}(\Omega)$, and $v \in H^{\min(\hat{s}+2,2)}(\Omega)$. Additionally the following estimates hold:*

$$\|v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} + \|\boldsymbol{\psi}\|_{H^{\min(\hat{s},2)+1}(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^{\min(\hat{s}+2,2)}(\Omega)} \lesssim \|w\|_{L^2(\Omega)}.$$

(ii) *For $\Gamma = \Gamma_D$ and any $(\boldsymbol{\varphi}, w) \in \boldsymbol{H}(\Omega, \operatorname{div}) \times H^1_0(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}(\Omega, \operatorname{div}) \times H^1_0(\Omega)$ such that $\|w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. The same regularity results and estimates as in* (i) *hold.*

Proof. We prove (i). Theorem 2.1 gives the existence of a unique $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ satisfying

$$(u, w)_\Omega = b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \quad \forall\, (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega). \tag{3.1}$$

For the regularity assertions, we introduce the auxiliary functions z and $\boldsymbol{\mu}$ by

$$\begin{aligned} \nabla \cdot \boldsymbol{\psi} + \gamma v &= z \quad \text{in } \Omega, \\ \nabla v + \boldsymbol{\psi} &= \boldsymbol{\mu} \quad \text{in } \Omega. \end{aligned} \tag{3.2}$$

Regularity properties of z and $\boldsymbol{\mu}$: regularity properties of z are inferred from a scalar elliptic equation satisfied by z. To that end, we note that (3.1) is equivalent to

$$(u, w)_\Omega = (\nabla u + \boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi} + \gamma u, z)_\Omega \qquad \forall\, (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega). \tag{3.3}$$

For $u = 0$ and integrating by parts we find

$$0 = (\boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi}, z)_\Omega = (\boldsymbol{\varphi}, \boldsymbol{\mu} - \nabla z)_\Omega \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{H}_0(\Omega, \operatorname{div}),$$

which gives $z \in H^1(\Omega)$ as well as $\boldsymbol{\mu} = \nabla z$. Inserting $\boldsymbol{\mu} = \nabla z$ and setting $\boldsymbol{\varphi} = 0$ in (3.3) we find

$$(u, w)_\Omega = (\nabla u, \nabla z)_\Omega + (\gamma u, z)_\Omega \quad \forall\, u \in H^1(\Omega).$$

Therefore z satisfies, in strong form,

$$\begin{aligned} -\Delta z + \gamma z &= w \quad \text{in } \Omega, \\ \partial_n z &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.4}$$

and Assumption 3.1 immediately give $z \in H^{\min(\hat{s}+2,2)}(\Omega)$ with the estimate $\|z\|_{H^{\min(\hat{s}+2,2)}(\Omega)} \lesssim \|w\|_{L^2(\Omega)}$.

Regularity properties of $\boldsymbol{v}$: eliminating $\boldsymbol{\psi}$ in (3.2), we discover that v satisfies

$$\begin{aligned} -\Delta v + \gamma v &= w + (1 - \gamma) z \quad \text{in } \Omega, \\ \partial_n v &= 0 \qquad\qquad\quad\;\; \text{on } \Gamma. \end{aligned} \tag{3.5}$$

By Assumption 3.1 we find $v \in H^{\min(\hat{s}+2,2)}(\Omega)$ with the *a priori* estimate

$$\|v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} \lesssim \|w + (1 - \gamma) z\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)}.$$

Regularity properties of $\boldsymbol{\psi}$: setting $\boldsymbol{\psi} = \nabla(z-v)$, we have found the desired pair $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$. Since $\boldsymbol{\psi} = \nabla(z-v)$ we first look at the regularity of $z - v$. Subtracting the equations (3.4), (3.5) satisfied by z and v respectively we obtain

$$\begin{aligned} -\Delta(z-v) + \gamma(z-v) &= (\gamma - 1)z \quad \text{in } \Omega, \\ \partial_n(z-v) &= 0 \quad \text{on } \Gamma, \end{aligned}$$

which gives $z - v \in H^{\min(\hat{s},2)+2}(\Omega)$ by Assumption 3.1 together with the estimate

$$\|z-v\|_{H^{\min(\hat{s},2)+2}(\Omega)} \lesssim \|w\|_{L^2(\Omega)}.$$

We can therefore conclude

$$\|\boldsymbol{\psi}\|_{H^{\min(\hat{s},2)+1}(\Omega)} = \|\nabla(z-v)\|_{H^{\min(\hat{s},2)+1}(\Omega)} \leq \|z-v\|_{H^{\min(\hat{s},2)+2}(\Omega)} \lesssim \|w\|_{L^2(\Omega)}.$$

Since $\nabla \cdot \boldsymbol{\psi} = z - \gamma v$, we have

$$\|\nabla \cdot \boldsymbol{\psi}\|_{H^{\min(\hat{s}+2,2)}(\Omega)} = \|z - \gamma v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} \lesssim \|w\|_{L^2(\Omega)},$$

which concludes the proof of (i). For the Dirichlet case (ii) the proof is completely analogous by replacing every Neumann boundary condition with a Dirichlet one. ☐

Theorem 3.4 (Duality argument for the gradient of the scalar variable). *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq -1$. Then there holds:*

(i) *For $\Gamma = \Gamma_N$ and any $(\boldsymbol{\varphi}, w) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ such that $\|\nabla w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s},1)+1}(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$, and $v \in H^1(\Omega)$. Additionally the following estimates hold:*

$$\|v\|_{H^1(\Omega)} + \|\boldsymbol{\psi}\|_{H^{\min(\hat{s},1)+1}(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} \lesssim \|\nabla w\|_{L^2(\Omega)}.$$

(ii) *For $\Gamma = \Gamma_D$ and any $(\boldsymbol{\varphi}, w) \in \boldsymbol{H}(\Omega, \mathrm{div}) \times H^1_0(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}(\Omega, \mathrm{div}) \times H^1_0(\Omega)$ such that $\|\nabla w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. The same regularity results and estimates as in* (i) *hold.*

Proof. We prove (i). Theorem 2.1 gives the existence of a unique $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ satisfying

$$(\nabla u, \nabla w)_\Omega = b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \quad \forall\, (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega). \tag{3.6}$$

For the regularity assertion, we introduce the auxiliary functions z and $\boldsymbol{\mu}$ by

$$\begin{aligned} \nabla \cdot \boldsymbol{\psi} + \gamma v &= z \quad \text{in } \Omega, \\ \nabla v + \boldsymbol{\psi} &= \boldsymbol{\mu} \quad \text{in } \Omega. \end{aligned} \tag{3.7}$$

Regularity properties of z and $\boldsymbol{\mu}$: we note that (3.6) is equivalent to

$$(\nabla u, \nabla w)_\Omega = (\nabla u + \boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi} + \gamma u, z)_\Omega \quad \forall\, (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega). \tag{3.8}$$

For $u = 0$ and integrating by parts we find

$$0 = (\boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi}, z)_\Omega = (\boldsymbol{\varphi}, \boldsymbol{\mu} - \nabla z)_\Omega$$

which gives $\boldsymbol{\mu} = \nabla z$. Inserting $\boldsymbol{\mu} = \nabla z$ and setting $\boldsymbol{\varphi} = 0$ in (3.8) we find

$$(\nabla u, \nabla w)_\Omega = (\nabla u, \nabla z)_\Omega + (\gamma u, z)_\Omega \quad \forall\, u \in H^1(\Omega),$$

which can be solved for $z \in H^1(\Omega)$ with the *a priori* estimate $\|z\|_{H^1(\Omega)} \lesssim \|\nabla w\|_{L^2(\Omega)}$. Formally, z satisfies

$$\begin{aligned} -\Delta z + \gamma z &= -\nabla \cdot \nabla w && \text{in } \Omega, \\ \partial_n z &= 0 && \text{on } \Gamma. \end{aligned} \tag{3.9}$$

where $-\nabla \cdot \nabla w \in (H^1(\Omega))'$ is to be understood as the mapping $u \mapsto (\nabla u, \nabla w)_\Omega$.
Regularity of $\boldsymbol{v}$: eliminating $\boldsymbol{\psi}$ from (3.7) and using $\boldsymbol{\mu} = \nabla z$, we discover that v satisfies

$$\begin{aligned} -\Delta v + \gamma v &= (1-\gamma) z - \nabla \cdot \nabla w && \text{in } \Omega, \\ \partial_n v &= 0 && \text{on } \Gamma. \end{aligned}$$

By the Lax–Milgram theorem we find that $v \in H^1(\Omega)$ as well as

$$\|v\|_{H^1(\Omega)} \lesssim \|(1-\gamma)z - \nabla \cdot \nabla w\|_{(H^1(\Omega))'} \lesssim \|\nabla w\|_{L^2(\Omega)}.$$

Regularity of $\boldsymbol{\psi}$: upon setting $\boldsymbol{\psi} = \nabla(z - v)$, we have found the solution $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ of (3.6). To prove the estimates and regularity results for $\boldsymbol{\psi}$ first note that

$$\begin{aligned} -\Delta(z - v) + \gamma(z - v) &= (1-\gamma) z && \text{in } \Omega, \\ \partial_n (z - v) &= 0 && \text{on } \Gamma, \end{aligned}$$

and therefore by Assumption 3.1 we find $z - v \in H^{\min(\hat{s},1)+2}(\Omega)$ with the estimate

$$\|z - v\|_{H^{\min(\hat{s},1)+2}(\Omega)} \lesssim \|\nabla w\|_{L^2(\Omega)}.$$

Finally since $\boldsymbol{\psi} = \nabla(z - v)$ the regularity assertion for $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s},1)+1}(\Omega)$ follows. For the Dirichlet case (ii) the proof is completely analogous by replacing every Neumann boundary condition with a Dirichlet one. □

Theorem 3.5 (Duality argument for the vector valued variable)**.** *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq -1$. Then there holds:*

(i) *For $\Gamma = \Gamma_N$ and any $(\boldsymbol{\eta}, u) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ such that $\|\boldsymbol{\eta}\|^2_{L^2(\Omega)} = b((\boldsymbol{\eta}, u), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$ and $v \in H^{\min(\hat{s},1)+2}(\Omega)$. Additionally the following estimates hold:*

$$\|v\|_{H^{\min(\hat{s},1)+2}(\Omega)} + \|\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} \lesssim \|\boldsymbol{\eta}\|_{L^2(\Omega)}.$$

(ii) *For $\Gamma = \Gamma_D$ and any $(\boldsymbol{\eta}, u) \in \boldsymbol{H}(\Omega, \operatorname{div}) \times H^1_0(\Omega)$ there exists $(\boldsymbol{\psi}, v) \in \boldsymbol{H}(\Omega, \operatorname{div}) \times H^1_0(\Omega)$ such that $\|\boldsymbol{\eta}\|^2_{L^2(\Omega)} = b((\boldsymbol{\eta}, u), (\boldsymbol{\psi}, v))$. The same regularity results and estimates as in* (i) *hold.*

Proof. We prove (i). Theorem 2.1 gives the existence of a unique $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega)$ such that

$$(\boldsymbol{\varphi}, \boldsymbol{\eta})_\Omega = b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega). \tag{3.10}$$

For the regularity assertions, we introduce the auxiliary functions z and $\boldsymbol{\mu}$ by

$$\begin{aligned} \nabla \cdot \boldsymbol{\psi} + \gamma v &= z && \text{in } \Omega, \\ \nabla v + \boldsymbol{\psi} &= \boldsymbol{\mu} && \text{in } \Omega. \end{aligned} \tag{3.11}$$

Regularity of z and $\boldsymbol{\mu}$: equation (3.10) is equivalent to

$$(\boldsymbol{\varphi}, \boldsymbol{\eta})_\Omega = (\nabla u + \boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi} + \gamma u, z)_\Omega \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \operatorname{div}) \times H^1(\Omega). \tag{3.12}$$

TABLE 1. Overview of regularity results of Theorems 3.3–3.5.

Ω	Duality for u	Duality for ∇u	Duality for $\boldsymbol{\varphi}$
Γ smooth	$v \in H^3$, $\boldsymbol{\psi} \in \boldsymbol{H}^2$, $\nabla \cdot \boldsymbol{\psi} \in H^2$	$v \in H^1$, $\boldsymbol{\psi} \in \boldsymbol{H}^2$, $\nabla \cdot \boldsymbol{\psi} \in H^1$	$v \in H^3$, $\boldsymbol{\psi} \in \boldsymbol{L}^2$, $\nabla \cdot \boldsymbol{\psi} \in H^1$
Ω convex polygonal/ polyhedral	$v \in H^2$, $\boldsymbol{\psi} \in \boldsymbol{H}^1$, $\nabla \cdot \boldsymbol{\psi} \in H^2$	$v \in H^1$, $\boldsymbol{\psi} \in \boldsymbol{H}^1$, $\nabla \cdot \boldsymbol{\psi} \in H^1$	$v \in H^2$, $\boldsymbol{\psi} \in \boldsymbol{L}^2$, $\nabla \cdot \boldsymbol{\psi} \in H^1$
Ω 2D polygonal	$v \in H^{3/2}$, $\boldsymbol{\psi} \in \boldsymbol{H}^{1/2}$, $\nabla \cdot \boldsymbol{\psi} \in H^{3/2}$	$v \in H^1$, $\boldsymbol{\psi} \in \boldsymbol{H}^{1/2}$, $\nabla \cdot \boldsymbol{\psi} \in H^1$	$v \in H^{3/2}$, $\boldsymbol{\psi} \in \boldsymbol{L}^2$, $\nabla \cdot \boldsymbol{\psi} \in H^1$

For $u = 0$ and integrating by parts we find

$$(\boldsymbol{\varphi}, \boldsymbol{\eta})_\Omega = (\boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi}, z)_\Omega = (\boldsymbol{\varphi}, \boldsymbol{\mu} - \nabla z)_\Omega$$

which gives $\boldsymbol{\mu} - \nabla z = \boldsymbol{\eta}$. Inserting $\boldsymbol{\mu} = \boldsymbol{\eta} + \nabla z$ and setting $\boldsymbol{\varphi} = 0$ in (3.10) we find

$$0 = (\nabla u, \boldsymbol{\eta} + \nabla z)_\Omega + (\gamma u, z)_\Omega \quad \forall\, u \in H^1(\Omega).$$

Hence, with the understanding that $\nabla \cdot \boldsymbol{\eta}$ means $u \mapsto (\nabla u, \boldsymbol{\eta})$, the function z solves

$$\begin{aligned} -\Delta z + \gamma z &= \nabla \cdot \boldsymbol{\eta} && \text{in } \Omega, \\ \partial_n z &= 0 && \text{on } \Gamma. \end{aligned} \tag{3.13}$$

Thus, $z \in H^1(\Omega)$ and setting $\boldsymbol{\mu} = \boldsymbol{\eta} + \nabla z$ we find (3.12) to be satisfied. Furthermore, note that by Lax–Milgram and our understanding of $\nabla \cdot \boldsymbol{\eta}$

$$\|z\|_{H^1(\Omega)} \lesssim \|\nabla \cdot \boldsymbol{\eta}\|_{(H^1(\Omega))'} \leq \|\boldsymbol{\eta}\|_{L^2(\Omega)}.$$

Regularity of $\boldsymbol{v}$: by eliminating $\boldsymbol{\psi}$ we find that v solves

$$\begin{aligned} -\Delta v + \gamma v &= (1 - \gamma) z && \text{in } \Omega, \\ \partial_n v &= 0 && \text{on } \Gamma. \end{aligned}$$

Again by Assumption 3.1 we find that $v \in H^{\min(\hat{s},1)+2}(\Omega)$ as well as

$$\|v\|_{H^{\min(\hat{s},1)+2}(\Omega)} \lesssim \|\eta\|_{L^2(\Omega)}.$$

Regularity of $\boldsymbol{\psi}$: we have $\boldsymbol{\psi} = \boldsymbol{\eta} + \nabla(z - v)$, and the regularity of $\boldsymbol{\psi}$ follows from that of z of v. For the Dirichlet case (ii) the proof is completely analogous by replacing every Neumann boundary condition with a Dirichlet one. □

Remark 3.6 (On the smoothness of Γ)**.** The results of Theorems 3.3–3.5 all hinge on a regularity shift for the Laplacian. We summarize the different regularity results for different assumptions on the domain and the boundary in Table 1, which follow from the observation $\hat{s} = \infty$ for smooth Γ, $\hat{s} = 0$ for convex Ω, and $\hat{s} = -1/2$ for general polygons (in fact, $\hat{s} > -1/2$ for polygons with the precise value given by the largest interior angle. In Section 4 we will only focus on the case where Assumption 3.1 is satisfied for some $\hat{s} \geq 0$. Hence, later results are also valid in convex polygonal/polyhedral domains. We track the parameter $\hat{s} \geq 0$ in the corresponding error estimates.

4. Error analysis

The goal of the present section is to establish optimal convergence rates for an hp version of the FOSLS method for the scalar variable, the gradient of the scalar variable as well as the vector variable, all measured in the $L^2(\Omega)$ norm, as long as the polynomial degree of the other variable is chosen appropriately.

4.1. Notation, assumptions, and road map of the current section

Throughout we denote by $(\boldsymbol{\varphi}_h, u_h)$ the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$ denote the corresponding error terms. For simplicity we also assume $\Gamma = \Gamma_N$, *i.e.*, $\Gamma_D = \emptyset$. Furthermore, p will denote the minimum of the two polynomial degrees p_s and p_v, *i.e.*, $p = \min(p_s, p_v)$. From here on we will only consider domains Ω satisfying Assumption 3.1 with $\hat{s} \geq 0$. Especially, we will not consider non-convex polygonal domains, where no full $H^2(\Omega)$ shift is satisfied. The overall agenda of the present section is as follows:

(1) We start by proving Lemma 3.4 of [4] in an hp setting using our duality argument, *i.e.*, the (in our sense) suboptimal $L^2(\Omega)$ estimate

$$\|e^u\|_{L^2(\Omega)} \lesssim h/p \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b.$$

This is done in Lemma 4.1. In Remark 4.2 we present heuristic arguments that suggest the possibility of optimal $L^2(\Omega)$ convergence rates. These arguments suggest to construct an $\boldsymbol{H}_0(\Omega, \operatorname{div})$ conforming approximation operator $\boldsymbol{I}_h^0$ with additional orthogonality properties.

(2) In Lemma 4.3 we prove that the operator $\boldsymbol{I}_h^0$ is in fact well defined. As a tool of independent interest we derive certain continuous and discrete Helmholtz decompositions in Lemmas 4.4 and 4.5. These decompositions are then used in Lemma 4.6 to analyze the $L^2(\Omega)$ error of the operator $\boldsymbol{I}_h^0$.

(3) Next we prove an hp version of Lemma 3.6 of [4] (an h analysis of $\boldsymbol{e}^{\boldsymbol{\varphi}}$ in the $L^2(\Omega)$ norm).

(4) In Theorem 4.10 we exploit the results of Lemma 4.9, which analyzes the convergence rate of the FOSLS approximation of the dual solution for the gradient of the scalar variable, in order to prove new optimal $L^2(\Omega)$ error estimates for ∇e^u.

(5) We analyze the convergence rate of the FOSLS approximation of the dual solution in various norms in Lemma 4.11. Finally we prove our main result, Theorem 4.12, which analyzes the convergence of e^u in the $L^2(\Omega)$ norm.

(6) Closing this section we derive Corollary 4.14, which summarizes the results for general right-hand side $f \in H^s(\Omega)$, by exploiting the estimates given by the Theorems 4.8, 4.10 and 4.12 together with the approximation properties of the employed finite element spaces.

4.2. The standard duality argument

Before formulating various duality arguments, we recall that the conforming least squares approximation $(\boldsymbol{\varphi}_h, u_h)$ is the best approximation in the $\|\cdot\|_b$ norm:

$$\|(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, u - u_h)\|_b = \min_{\substack{\tilde{u}_h \in S_{p_s}(\mathcal{T}_h), \\ \tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)}} \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h)\|_b. \tag{4.1}$$

Lemma 4.1. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$,*

$$\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \lesssim \frac{h}{p}\|u - \tilde{u}_h\|_{H^1(\Omega)} + \frac{h}{p}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p}\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. Apply Theorem 3.3 (duality argument for the scalar variable) with $w = e^u$. For any $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, we find due to the Galerkin orthogonality and the Cauchy–Schwarz inequality:

$$\|e^u\|^2_{L^2(\Omega)} = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v)) = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)) \leq \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \left\|(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)\right\|_b. \tag{4.2}$$

Using Theorem 2.1 (norm equivalence), and exploiting the regularity results and estimates of Theorem 3.3 as well as the $H^1(\Omega)$ and $\boldsymbol{H}(\Omega, \mathrm{div})$ conforming operators in [17], we can find $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, such that

$$\begin{aligned}\|(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)\|_b &\lesssim \|v - \tilde{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{\boldsymbol{H}(\Omega,\mathrm{div})} \\ &\lesssim (h/p)^{\min(\hat{s}+1,1)}\|v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} + (h/p)^{\min(\hat{s}+1,2)}\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{\min(\hat{s}+1,2)}(\Omega,\mathrm{div})} \\ &\lesssim (h/p)^{\min(\hat{s}+1,1)}\|e^u\|_{L^2(\Omega)} = h/p\|e^u\|_{L^2(\Omega)},\end{aligned}$$

where we exploited the regularity for $(\boldsymbol{\psi}, v)$ and the *a priori* estimates of Theorem 3.3, which proves the first estimate. The second one follows by the fact that the least squares solution is the projection with respect to the scalar product b. Therefore $\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \leq \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h)\|_b$. The result follows by applying the norm equivalence given in Theorem 2.1. □

Remark 4.2 (Heuristic arguments for improved $L^2(\Omega)$ convergence)**.** We present an argument why improved convergence of the scalar variable u can be expected. Assume Γ to be smooth, so that Assumption 3.1 is satisfied with $\hat{s} = \infty$. We again start by applying our duality argument and exploit the Galerkin orthogonality as in (4.2) in the proof of Lemma 4.1. Instead of immediately applying the Cauchy–Schwarz inequality we investigate the terms in the b scalar product and analyze the best rate we can expect from the regularity of the dual problem:

$$\begin{aligned}\|e^u\|^2_{L^2(\Omega)} &= b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)) \\ &= (\underbrace{\nabla \cdot \boldsymbol{e}^{\boldsymbol{\varphi}} + \gamma e^u}_{☹}, \underbrace{\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)}_{\sim h^2} + \gamma \underbrace{(v - \tilde{v}_h)}_{\sim h^2})_\Omega + (\underbrace{\nabla e^u + \boldsymbol{e}^{\boldsymbol{\varphi}}}_{☹}, \underbrace{\nabla (v - \tilde{v}_h)}_{\sim h} + \underbrace{\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h}_{\sim h^3})_\Omega.\end{aligned}$$

Note that the terms are not equilibrated and we cannot expect any rate from the terms marked by ☹ for $u \in H^1(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$. Let $(\boldsymbol{\psi}_h, v_h)$ be the least squares approximation of $(\boldsymbol{\psi}, v)$. Choosing now $(\tilde{\boldsymbol{\psi}}_h, \tilde{v}_h) = (\boldsymbol{\psi}_h, v_h)$, *i.e.*, choosing the least squares approximation of the dual solution, and again exploiting the Galerkin orthogonality we have for any $(\tilde{\boldsymbol{\varphi}}_h, \tilde{u}_h)$:

$$\begin{aligned}\|e^u\|^2_{L^2(\Omega)} &= b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)) = b((\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)) \\ &= (\underbrace{\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)}_{☹} + \gamma \underbrace{(u - \tilde{u}_h)}_{\sim h^2}, \underbrace{\nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}}}_{\sim h} + \gamma \underbrace{e^v}_{\sim h^2})_\Omega + (\underbrace{\nabla (u - \tilde{u}_h)}_{\sim h} + \underbrace{\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h}_{\sim h}, \underbrace{\nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}}}_{\sim h})_\Omega.\end{aligned}$$

The improved convergence of the dual solution will be shown in Lemma 4.11. From a best approximation viewpoint the $\nabla \cdot$ term involving $\boldsymbol{\varphi}$ still has no rate. To be more precise, the second term has the right powers of h resulting in an overall h^2. Since the term $\gamma(u - \tilde{u}_h)$ already has order h^2 we have no problem with that one. The term with the worst rate is

$$(\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega \sim h.$$

Out of the box we cannot find an extra h to get optimal convergence. We note that $\boldsymbol{\psi}$ has significant regularity, which we did not exploit yet. To do so, we use an operator $\boldsymbol{I}^0_h$ mapping into the conforming finite element space of the vector variable. We add and subtract any $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ in the right argument of $(\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega$. We then find with $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}^0_h \boldsymbol{\varphi}$

$$(\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}^0_h \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega = (\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}^0_h \boldsymbol{\varphi}), \nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h))_\Omega + (\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}^0_h \boldsymbol{\varphi}), \nabla \cdot (\tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi}_h))_\Omega.$$

Note that $\tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi}_h$ is a discrete object. If we assume $\boldsymbol{I}_h^0$ to satisfy the orthogonality condition

$$(\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^0 \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{\chi}_h)_\Omega = 0, \qquad \forall \boldsymbol{\chi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)$$

we arrive in view of the smoothness of $\boldsymbol{\psi}$ at

$$(\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^0 \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega = (\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^0 \boldsymbol{\varphi}), \underbrace{\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)}_{h^2})_\Omega \sim h^2.$$

Therefore the operator $\boldsymbol{I}_h^0$ should satisfy the aforementioned orthogonality condition and have good approximation properties in $L^2(\Omega)$, as needed above. In the following we will construct operators $\boldsymbol{I}_h^0$ and $\boldsymbol{I}_h$ acting on $\boldsymbol{H}_0(\Omega, \operatorname{div})$ and $\boldsymbol{H}(\Omega, \operatorname{div})$ respectively.

4.3. The operators $\boldsymbol{I}_h^0$ and $\boldsymbol{I}_h$

In the spirit of Remark 4.2 a natural choice for the operator $\boldsymbol{I}_h^0$ is the following constrained minimization problem

$$\boldsymbol{I}_h^0 \boldsymbol{\varphi} = \underset{\boldsymbol{\varphi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)}{\operatorname{argmin}} \frac{1}{2} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{L^2(\Omega)}^2 \qquad \text{s.t.} \qquad (\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^0 \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{\chi}_h)_\Omega = 0 \qquad \forall \boldsymbol{\chi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h).$$

The corresponding Lagrange function is

$$L(\boldsymbol{\varphi}_h, \boldsymbol{\lambda}_h) = \frac{1}{2} \|\boldsymbol{\varphi}_h - \boldsymbol{\varphi}\|_{L^2(\Omega)}^2 + (\nabla \cdot (\boldsymbol{\varphi}_h - \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{\lambda}_h)_\Omega$$

and the associated saddle point problem is to find $(\boldsymbol{\varphi}_h, \boldsymbol{\lambda}_h) \in \mathbf{V}_{p_v}^0(\mathcal{T}_h) \times \mathbf{V}_{p_v}^0(\mathcal{T}_h)$ such that

$$(\boldsymbol{\varphi}_h - \boldsymbol{\varphi}, \boldsymbol{\mu}_h)_\Omega + (\nabla \cdot \boldsymbol{\mu}_h, \nabla \cdot \boldsymbol{\lambda}_h)_\Omega = 0 \quad \forall \boldsymbol{\mu}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h), \tag{4.3a}$$

$$(\nabla \cdot (\boldsymbol{\varphi}_h - \boldsymbol{\varphi}), \nabla \cdot \boldsymbol{\eta}_h)_\Omega = 0 \quad \forall \boldsymbol{\eta}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h). \tag{4.3b}$$

Uniqueness is not given since only the divergence of the Lagrange parameter appears. However, by focussing on the divergence of the Lagrange parameter, we can formulate it in the following way: Find $(\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{V}_{p_v}^0(\mathcal{T}_h) \times \nabla \cdot \mathbf{V}_{p_v}^0(\mathcal{T}_h)$ such that

$$(\boldsymbol{\varphi}_h, \boldsymbol{\mu}_h)_\Omega + (\nabla \cdot \boldsymbol{\mu}_h, \lambda_h)_\Omega = (\boldsymbol{\varphi}, \boldsymbol{\mu}_h)_\Omega \quad \forall \boldsymbol{\mu}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h), \tag{4.4a}$$

$$(\nabla \cdot \boldsymbol{\varphi}_h, \eta_h)_\Omega = (\nabla \cdot \boldsymbol{\varphi}, \eta_h)_\Omega \quad \forall \eta_h \in \nabla \cdot \mathbf{V}_{p_v}^0(\mathcal{T}_h). \tag{4.4b}$$

The construction of $\boldsymbol{I}_h$ is completely analogous, one just drops the zero boundary conditions everywhere. Problem (4.4) defining the operator $\boldsymbol{I}_h^0$ is in fact a classical mixed finite element discretization of the Laplacian, with both scalar valued right-hand side $\nabla \cdot \boldsymbol{\varphi}$ in (4.4b) and vector valued right-hand side $\boldsymbol{\varphi}$ in (4.4a). For unique solvability and stability of $\boldsymbol{I}_h^0$ and $\boldsymbol{I}_h$, we refer to Section 7.1 of [6]. Regarding the stability with respect to the polynomial degree p we refer to Section 4.3 of [3]. We therefore have

Lemma 4.3. *For any mesh $\mathcal{T}_h$ satisfying Assumption 1.1, the operators $\boldsymbol{I}_h^0 : \boldsymbol{H}_0(\Omega, \operatorname{div}) \to \mathbf{V}_{p_v}^0(\mathcal{T}_h)$ and $\boldsymbol{I}_h : \boldsymbol{H}(\Omega, \operatorname{div}) \to \mathbf{V}_{p_v}(\mathcal{T}_h)$ are well defined with bounds independent of the mesh size h and the polynomial degree p. They are projections.*

We now analyze the approximation properties of the operator $\boldsymbol{I}_h^0$ and $\boldsymbol{I}_h$ in the $L^2(\Omega)$ norm. To that end we need certain decompositions on the continuous as well as the discrete level.

Lemma 4.4 (Continuous and discrete Helmholtz-like decomposition – no boundary conditions)**.** *Let* Ω *satisfy Assumption 3.1 for some* $\hat{s} \geq 0$*. In spatial dimension* $d = 3$ *the operators* $\mathbf{\Pi}^{\text{curl}} \colon \boldsymbol{H}(\Omega, \text{div}) \to \nabla \times \boldsymbol{H}(\Omega, \mathbf{curl})$ *and* $\mathbf{\Pi}_h^{\text{curl}} \colon \mathbf{V}_{p_v}(\mathcal{T}_h) \to \nabla \times \mathbf{N}_{p_v}(\mathcal{T}_h)$ *given by*

$$(\mathbf{\Pi}^{\text{curl}} \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu})_\Omega = (\boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu})_\Omega \qquad \forall \boldsymbol{\mu} \in \boldsymbol{H}(\Omega, \mathbf{curl}), \tag{4.5}$$

$$(\mathbf{\Pi}_h^{\text{curl}} \boldsymbol{\varphi}_h, \nabla \times \boldsymbol{\mu})_\Omega = (\boldsymbol{\varphi}_h, \nabla \times \boldsymbol{\mu})_\Omega \quad \forall \boldsymbol{\mu} \in \mathbf{N}_{p_v}(\mathcal{T}_h) \tag{4.6}$$

are well defined. Furthermore, the remainder $\boldsymbol{r}$ *of the continuous decomposition* $\boldsymbol{\varphi} = \mathbf{\Pi}^{\text{curl}} \boldsymbol{\varphi} + \boldsymbol{r}$ *satisfies*

$$\begin{aligned} \nabla \cdot \boldsymbol{r} &= \nabla \cdot \boldsymbol{\varphi} && \text{in } \Omega, \\ \nabla \times \boldsymbol{r} &= 0 && \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{r} &= 0 && \text{on } \Gamma, \end{aligned}$$

as well as $\boldsymbol{r} \in \boldsymbol{H}^1(\Omega)$*. Additionally there exists* $R \in H^2(\Omega) \cap H_0^1(\Omega)$ *such that* $\boldsymbol{r} = \nabla R$*, where* R *satisfies*

$$\begin{aligned} \Delta R &= \nabla \cdot \boldsymbol{\varphi} && \text{in } \Omega, \\ R &= 0 && \text{on } \Gamma. \end{aligned} \tag{4.7}$$

Finally, the estimate $\|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \|R\|_{H^2(\Omega)} \lesssim \|\nabla \cdot \boldsymbol{\varphi}\|_{L^2(\Omega)}$ *holds. In spatial dimension* $d = 2$ *analogous results hold for the operators* $\mathbf{\Pi}^{\text{curl}} \colon \boldsymbol{H}(\Omega, \text{div}) \to \mathbf{curl}\, H^1(\Omega)$ *and* $\mathbf{\Pi}_h^{\text{curl}} \colon \mathbf{V}_{p_v}(\mathcal{T}_h) \to \mathbf{curl}\, S_{p_v+1}(\mathcal{T}_h)$ *given by*

$$(\mathbf{\Pi}^{\text{curl}} \boldsymbol{\varphi}, \mathbf{curl}\, \mu)_\Omega = (\boldsymbol{\varphi}, \mathbf{curl}\, \mu)_\Omega \qquad \forall \mu \in H^1(\Omega), \tag{4.8}$$

$$(\mathbf{\Pi}_h^{\text{curl}} \boldsymbol{\varphi}_h, \mathbf{curl}\, \mu)_\Omega = (\boldsymbol{\varphi}_h, \mathbf{curl}\, \mu)_\Omega \quad \forall \mu \in S_{p_v+1}(\mathcal{T}_h). \tag{4.9}$$

Proof. We prove the results for $d = 3$ first. For unique solvability of the variational definition of the operators, just note that they are the $L^2(\Omega)$ orthogonal projections on $\nabla \times \boldsymbol{H}(\Omega, \mathbf{curl})$ and $\nabla \times \mathbf{N}_{p_v}(\mathcal{T}_h)$ respectively. By construction we have

$$(\boldsymbol{r}, \nabla \times \boldsymbol{\mu})_\Omega = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{H}(\Omega, \mathbf{curl})$$

which by definition gives $\nabla \times \boldsymbol{r} = 0$. Furthermore, by the characterization of $\boldsymbol{H}_0(\Omega, \mathbf{curl})$ given in Theorem 3.33 of [19] we have $\boldsymbol{n} \times \boldsymbol{r} = 0$. Since $\mathbf{\Pi}^{\text{curl}} \boldsymbol{\varphi} \in \nabla \times \boldsymbol{H}(\Omega, \mathbf{curl})$ we immediately have $\nabla \cdot \boldsymbol{r} = \nabla \cdot \boldsymbol{\varphi}$. Exploiting the exact sequence property of the following de Rahm complex

$$\{0\} \xrightarrow{\text{id}} H_0^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_0(\Omega, \mathbf{curl}) \xrightarrow{\nabla \times} \boldsymbol{H}_0(\Omega, \text{div}) \xrightarrow{\nabla \cdot} L_0^2(\Omega) \xrightarrow{0} \{0\}$$

in the case that both Ω and Γ are simply connected, we can find $R \in H_0^1(\Omega)$ such that $\boldsymbol{r} = \nabla R$. Therefore R solves (4.7). The Friedrichs inequality and elliptic regularity theory then give the desired estimate. In spatial dimension $d = 2$ the results and lines of proof stay the same. The only difference is the use of the two dimensional exact sequence (1.1): As in the case $d = 3$ we have

$$(\boldsymbol{r}, \mathbf{curl}\, \mu)_\Omega = 0 \quad \forall \mu \in H^1(\Omega).$$

By partial integration we again find $\operatorname{curl} \boldsymbol{r} = 0$ as well as $\boldsymbol{t} \cdot \boldsymbol{r} = 0$. Hence, by the exact sequence (1.1) (with zero boundary conditions), we can find $R \in H_0^1(\Omega)$ such that $\boldsymbol{r} = \nabla R$. The remainder of the proof is completely analogous to the case $d = 3$. □

By nearly the same arguments we also have a version for zero boundary conditions:

Lemma 4.5 (Continuous and discrete Helmholtz-like decomposition - zero boundary conditions)**.** *Let* Ω *satisfy Assumption 3.1 for some* $\hat{s} \geq 0$*. The operators* $\mathbf{\Pi}^{\text{curl},0} \colon \boldsymbol{H}_0(\Omega, \text{div}) \to \nabla \times \boldsymbol{H}_0(\Omega, \mathbf{curl})$ *and* $\mathbf{\Pi}_h^{\text{curl},0} \colon \mathbf{V}_{p_v}^0(\mathcal{T}_h) \to \nabla \times \mathbf{N}_{p_v}^0(\mathcal{T}_h)$ *given by*

$$(\mathbf{\Pi}^{\text{curl},0} \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu})_\Omega = (\boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu})_\Omega \qquad \forall \boldsymbol{\mu} \in \boldsymbol{H}_0(\Omega, \mathbf{curl}) \tag{4.10}$$

$$(\mathbf{\Pi}_h^{\mathrm{curl},0}\boldsymbol{\varphi}_h, \nabla\times\boldsymbol{\mu})_\Omega = (\boldsymbol{\varphi}_h, \nabla\times\boldsymbol{\mu})_\Omega \quad \forall\boldsymbol{\mu}\in\mathbf{N}_{p_v}^0(\mathcal{T}_h) \tag{4.11}$$

are well defined. Furthermore, the remainder $\boldsymbol{r}$ *of the continuous decomposition* $\boldsymbol{\varphi} = \mathbf{\Pi}^{\mathrm{curl},0}\boldsymbol{\varphi} + \boldsymbol{r}$ *satisfies*

$$\begin{aligned} \nabla\cdot\boldsymbol{r} &= \nabla\cdot\boldsymbol{\varphi} && \text{in } \Omega, \\ \nabla\times\boldsymbol{r} &= 0 && \text{in } \Omega, \\ \boldsymbol{r}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma, \end{aligned}$$

as well as $\boldsymbol{r}\in\boldsymbol{H}^1(\Omega)$*. Additionally there exists an* $R\in H^2(\Omega)\cap H^1(\Omega)/\mathbb{R}$ *such that* $\boldsymbol{r}=\nabla R$*, where* R *satisfies*

$$\begin{aligned} \Delta R &= \nabla\cdot\boldsymbol{\varphi} && \text{in } \Omega, \\ \partial_n R &= 0 && \text{on } \Gamma. \end{aligned} \tag{4.12}$$

Finally, the estimate $\|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \|R\|_{H^2(\Omega)} \lesssim \|\nabla\cdot\boldsymbol{\varphi}\|_{L^2(\Omega)}$ *holds. Analogous results hold in spatial dimension* $d=2$ *as in Lemma 4.4.*

Proof. We only sketch the proof for $d=3$. The results for $d=2$ follow as in Lemma 4.4. Unique solvability as well as $\nabla\times\boldsymbol{r}=0$ and $\nabla\cdot\boldsymbol{r}=\nabla\cdot\boldsymbol{\varphi}$ follows by the same arguments as in the proof of Lemma 4.4. Since $\boldsymbol{\varphi}\in\boldsymbol{H}_0(\Omega,\mathrm{div})$ and $\mathbf{\Pi}^{\mathrm{curl},0}\boldsymbol{\varphi}\in\nabla\times\boldsymbol{H}_0(\Omega,\mathbf{curl})\subset\boldsymbol{H}_0(\Omega,\mathrm{div})$ we find

$$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{\varphi}\cdot\boldsymbol{n} - \mathbf{\Pi}^{\mathrm{curl},0}\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0.$$

Again by the exact sequence

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\Omega,\mathbf{curl}) \xrightarrow{\nabla\times} \boldsymbol{H}(\Omega,\mathrm{div}) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} \{0\}$$

we can find $R\in H^1(\Omega)$ such that $\boldsymbol{r}=\nabla R$. Finally since $\partial_n R = \nabla R\cdot\boldsymbol{n} = \boldsymbol{r}\cdot\boldsymbol{n} = 0$, we find that R solves (4.12). The Poincaré inequality and elliptic regularity theory then give the desired results. □

Lemma 4.6. *Let* Ω *satisfy Assumption 3.1 for some* $\hat{s}\geq 0$*. The operator* $\boldsymbol{I}_h^0$ *satisfies for arbitrary* $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)$ *the estimates*

$$\left\|\boldsymbol{\varphi}-\boldsymbol{I}_h^0\boldsymbol{\varphi}\right\|_{L^2(\Omega)} \lesssim \|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p_v}\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}, \tag{4.13}$$

$$\left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)} \leq \|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}. \tag{4.14}$$

The same estimates hold true for the operator $\boldsymbol{I}_h$ *for arbitrary* $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$*.*

Proof. We prove the result for $d=3$ first. Let $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)$ be arbitrary. Due to the orthogonality relation satisfied by the operator $\boldsymbol{I}_h^0$ the estimate (4.14) is obvious. We have with $\boldsymbol{e}=\boldsymbol{\varphi}-\boldsymbol{I}_h^0\boldsymbol{\varphi}$

$$\|\boldsymbol{e}\|_{L^2(\Omega)}^2 = (\boldsymbol{e}, \boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)_\Omega + (\boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})_\Omega.$$

In order to treat the second term we apply Lemma 4.5 and split the discrete object $\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^0\boldsymbol{\varphi}\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)$ on the discrete and the continuous level. That is, we write

$$\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi} = \nabla\times\boldsymbol{\mu}+\boldsymbol{r}, \qquad\qquad \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi} = \nabla\times\boldsymbol{\mu}_h+\boldsymbol{r}_h$$

for certain $\boldsymbol{\mu}\in\boldsymbol{H}_0(\Omega,\mathbf{curl})$, $\boldsymbol{r}\in\boldsymbol{H}_0(\Omega,\mathrm{div})$, $\boldsymbol{\mu}_h\in\mathbf{N}_{p_v}^0(\mathcal{T}_h)$, and $\boldsymbol{r}_h\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)$. Since $\nabla\cdot\nabla\times = 0$ we have

$$(\boldsymbol{\varphi}-\boldsymbol{I}_h^0\boldsymbol{\varphi}, \nabla\times\boldsymbol{\mu}_h)_\Omega = 0$$

by definition of the operator $\boldsymbol{I}_h^0$ and consequently

$$(\boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})_\Omega = (\boldsymbol{e}, \nabla \times \boldsymbol{\mu}_h + \boldsymbol{r}_h)_\Omega = (\boldsymbol{e}, \boldsymbol{r}_h)_\Omega = (\boldsymbol{e}, \boldsymbol{r}_h - \boldsymbol{r})_\Omega + (\boldsymbol{e}, \boldsymbol{r})_\Omega =: T_1 + T_2.$$

Treatment of T_1: to estimate T_1 we first need one of the commuting projection based interpolation operators defined in [17]. Specifically, the global operator $\mathbf{\Pi}_p^{\mathrm{div}}$ given in Remark 2.9 of [17], see also [20]. Let therefore $\mathbf{\Pi}_{p_v}^{\mathrm{div},\star}$ denote either the operator $\mathbf{\Pi}_{p_v-1}^{\mathrm{div}}$ if $\mathbf{V}_{p_v}^0(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}^0(\mathcal{T}_h)$ or the analogous operator $\mathbf{\Pi}_{p_v}^{\mathrm{div}}$ in the case $\mathbf{V}_{p_v}^0(\mathcal{T}_h) = \mathbf{BDM}_{p_v}^0(\mathcal{T}_h)$. First note that $\nabla \cdot \boldsymbol{r} = \nabla \cdot \boldsymbol{r}_h \in \nabla \cdot \mathbf{V}_{p_v}^0(\mathcal{T}_h)$. By the commuting diagram property of the operator $\mathbf{\Pi}_{p_v}^{\mathrm{div},\star}$ as well as the projection property we therefore have

$$\nabla \cdot (\mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r} - \boldsymbol{r}_h) = \mathbf{\Pi}_{p_v}^{L^2}(\nabla \cdot \boldsymbol{r}) - \nabla \cdot \boldsymbol{r}_h = \mathbf{\Pi}_{p_v}^{L^2}(\nabla \cdot \boldsymbol{r}_h) - \nabla \cdot \boldsymbol{r}_h = 0.$$

By the exact sequence property we therefore have $\mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r} - \boldsymbol{r}_h \in \nabla \times \mathbf{N}_{p_v}^0(\mathcal{T}_h)$. Furthermore, the definition of $\boldsymbol{r}$ and $\boldsymbol{r}_h$ in Lemma 4.5 gives the orthogonality relation $\boldsymbol{r} - \boldsymbol{r}_h \perp \nabla \times \mathbf{N}_{p_v}^0(\mathcal{T}_h)$. Putting it all together we have

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}^2 = (\boldsymbol{r} - \boldsymbol{r}_h, \boldsymbol{r} - \mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r})_\Omega + (\boldsymbol{r} - \boldsymbol{r}_h, \mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r} - \boldsymbol{r}_h)_\Omega = (\boldsymbol{r} - \boldsymbol{r}_h, \boldsymbol{r} - \mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r})_\Omega,$$

which, by the Cauchy–Schwarz inequality, gives

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \leq \|\boldsymbol{r} - \mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r}\|_{L^2(\Omega)}.$$

Since $\nabla \cdot \boldsymbol{r} = \nabla \cdot \boldsymbol{r}_h$ is discrete we may apply ([17], Thm. 2.10(vi)) as well as perform a simple scaling argument to arrive at

$$\left\|\boldsymbol{r} - \mathbf{\Pi}_{p_v}^{\mathrm{div},\star}\boldsymbol{r}\right\|_{L^2(\Omega)} \lesssim \frac{h}{p_v}\|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \frac{h}{p_v}\left\|\nabla \cdot (\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)},$$

where the last estimate is due to the *a priori* estimate of Lemma 4.5. Summarizing we have

$$T_1 \lesssim \frac{h}{p_v}\|\boldsymbol{e}\|_{L^2(\Omega)}\left\|\nabla \cdot (\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)} \lesssim \frac{h}{p_v}\|\boldsymbol{e}\|_{L^2(\Omega)}\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)},$$

where the last estimate follows by adding and subtracting $\boldsymbol{\varphi}$, the triangle inequality as well as the second inequality of the present lemma.

Treatment of T_2: the term T_2 is treated with a duality argument. We select $\boldsymbol{\psi} \in \boldsymbol{H}(\Omega, \mathrm{div})$ such that

$$(\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{\psi})_\Omega = (\boldsymbol{v}, \boldsymbol{r})_\Omega \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\Omega, \mathrm{div}).$$

To that end, we note that by Lemma 4.5 we have $\boldsymbol{r} = \nabla R$ for some $R \in H^2(\Omega)$. Therefore for $\boldsymbol{v} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$ we have

$$(\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{\psi})_\Omega = (\boldsymbol{v}, \boldsymbol{r})_\Omega = (\boldsymbol{v}, \nabla R)_\Omega = -(\nabla \cdot \boldsymbol{v}, R)_\Omega$$

so that a desired $\boldsymbol{\psi}$ can be found as $\boldsymbol{\psi} = \nabla w$ with w solving

$$\begin{aligned} -\Delta w &= R \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Furthermore, since $R \in H^2(\Omega)$, we have by construction $-R = \nabla \cdot \boldsymbol{\psi} \in H^2(\Omega)$. Finally, the following estimates hold

$$\|\nabla \cdot \boldsymbol{\psi}\|_{H^2(\Omega)} = \|R\|_{H^2(\Omega)} \lesssim \left\|\nabla \cdot (\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}, \tag{4.15}$$

due to elliptic regularity and the results of Lemma 4.5. We therefore have for any $\boldsymbol{\psi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)$

$$T_2 = (\boldsymbol{e}, \boldsymbol{r})_\Omega = (\nabla \cdot \boldsymbol{e}, \nabla \cdot \boldsymbol{\psi})_\Omega = (\nabla \cdot \boldsymbol{e}, \nabla \cdot (\boldsymbol{\psi} - \boldsymbol{\psi}_h))_\Omega \leq \|\nabla \cdot \boldsymbol{e}\|_{L^2(\Omega)}\|\nabla \cdot (\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{L^2(\Omega)},$$

where we used the definition of T_2, the duality argument elaborated above, the orthogonality relation of $\boldsymbol{I}_h^0$ to insert any $\boldsymbol{\psi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)$, and the Cauchy–Schwarz inequality. Finally exploiting the *a priori* estimate of $\boldsymbol{\psi}$ in (4.15) we find for $p_v > 1$ that

$$T_2 \le \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)} \cdot \inf_{\boldsymbol{\psi}_h\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)} \|\nabla\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\|_{L^2(\Omega)} \lesssim \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}(h/p_v)^2\|\nabla\cdot\boldsymbol{\psi}\|_{H^2(\Omega)}$$
$$\lesssim \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}(h/p_v)^2\big\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\big\|_{L^2(\Omega)}.$$

In the lowest order case $p_v = 1$ we cannot fully exploit the regularity. However, we find

$$\|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)} = \|R\|_{H^1(\Omega)} \lesssim \big\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\big\|_{(H^1(\Omega))'}. \tag{4.16}$$

Proceeding as above and using estimate (4.16) we find

$$T_2 \le \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)} \cdot \inf_{\boldsymbol{\psi}_h\in\mathbf{V}_{p_v}^0(\mathcal{T}_h)} \|\nabla\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_h)\|_{L^2(\Omega)} \lesssim \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}h/p_v\|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)}$$
$$\lesssim \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}h/p_v\big\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\big\|_{(H^1(\Omega))'} \lesssim \|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}h/p_v\big\|\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}\big\|_{L^2(\Omega)}.$$

The last last estimate is due to integration by parts and the boundary condition of $\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}$; in fact

$$\big\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\big\|_{(H^1(\Omega))'} = \sup_{v\in H^1(\Omega)} \frac{|(\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}), v)_\Omega|}{\|v\|_{H^1(\Omega)}} = \sup_{v\in H^1(\Omega)} \frac{|(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}, \nabla v)_\Omega|}{\|v\|_{H^1(\Omega)}}$$
$$\le \big\|\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}\big\|_{L^2(\Omega)}$$

holds. Putting everything together we have for $p_v > 1$

$$\begin{aligned}\|\boldsymbol{e}\|_{L^2(\Omega)}^2 &= (\boldsymbol{e}, \boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)_\Omega + (\boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})_\Omega = (\boldsymbol{e}, \boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)_\Omega + T_1 + T_2 \\ &\lesssim \|\boldsymbol{e}\|_{L^2(\Omega)}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p_v}\|\boldsymbol{e}\|_{L^2(\Omega)}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} \\ &\quad + \frac{h^2}{p_v^2}\|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}\big\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi})\big\|_{L^2(\Omega)} \\ &\lesssim \|\boldsymbol{e}\|_{L^2(\Omega)}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p_v}\|\boldsymbol{e}\|_{L^2(\Omega)}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} + \frac{h^2}{p_v^2}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}^2,\end{aligned}$$

where the last estimate again follows from inserting $\boldsymbol{\varphi}$ and using the second estimate of the present lemma. Young's inequality then yields the result for the operator $\boldsymbol{I}_h^0$. The lowest order case is treated analogous. For the operator $\boldsymbol{I}_h$ the only difference is that one applies Lemma 4.4 instead of Lemma 4.5 and perform the duality argument on all of $\boldsymbol{H}(\Omega, \mathrm{div})$ instead of $\boldsymbol{H}_0(\Omega, \mathrm{div})$. Here it is important to note that the potential R given by Lemma 4.4 satisfies homogeneous boundary conditions, so that the boundary term vanishes in the partial integration. In spatial dimension $d = 2$ there are two adjustments to be made: First, in the continuous and discrete splitting of $\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}$ one utilizes the results of Lemma 4.5 for $d = 2$. Second, the treatment of T_1 needs to be adjusted. In spatial dimension $d = 2$ in [17] the exact sequence

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\Omega, \mathrm{curl}) \xrightarrow{\mathrm{curl}} L^2(\Omega) \xrightarrow{0} \{0\}, \tag{4.17}$$

is considered and a corresponding operator $\boldsymbol{\Pi}_p^{\mathrm{curl}}$ is constructed, see Definition 2.6 and Remark 2.9 of [17]. However, since, as discussed in Section 1, the exact sequence (4.17) can be obtained from the exact sequence

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^1(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\Omega, \mathrm{div}) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} \{0\}. \tag{4.18}$$

via rotation the operator $\boldsymbol{\Pi}_p^{\mathrm{curl}}$ is applicable and one employs ([17], Thm. 2.13 (iv)) instead of Theorem 2.10(vi) of [17] in the treatment of T_1. The remainder of the proof stays the same and is therefore omitted. □

Remark 4.7. $\boldsymbol{H}(\Omega, \mathrm{div})$-conforming approximation operators similar to $\boldsymbol{I}_h$ and $\boldsymbol{I}_h^0$ are presented in [12], where the focus is on a patchwise construction rather than the (global) orthogonalities (4.3b), (4.4b). We stress, however, that we do not use the operator of [12] since the operator constructed and analyzed in Theorem 3.2 of [12] is not p robust, *i.e.*, the constants in the error estimates may depend on p.

Theorem 4.8. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$,*

$$\begin{aligned}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} &\lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p}\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} \\ &\lesssim \frac{h}{p}\|u - \tilde{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p}\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.\end{aligned}$$

Proof. We prove the result for $d = 3$ first. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ denote the dual solution given by Theorem 3.5 applied to $\boldsymbol{\eta} = \boldsymbol{e}^{\boldsymbol{\varphi}}$. Theorem 3.5 gives $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$, and $v \in H^{\min(\hat{s},1)+2}(\Omega)$. Due to the Galerkin orthogonality we have for any $(\tilde{\boldsymbol{\psi}}_h, \tilde{v}_h)$

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|^2_{L^2(\Omega)} = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v)) = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)). \tag{4.19}$$

We now estimate all terms in the above equation (4.19):

$$\begin{aligned}(\nabla e^u + \boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla(v - \tilde{v}_h))_\Omega &\leq \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \|\nabla(v - \tilde{v}_h)\|_{L^2(\Omega)}, \\ (\nabla \cdot \boldsymbol{e}^{\boldsymbol{\varphi}} + \gamma e^u, \nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h) + \gamma(v - \tilde{v}_h))_\Omega &\lesssim \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \Big[\|\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\|_{L^2(\Omega)} + \|v - \tilde{v}_h\|_{L^2(\Omega)}\Big], \\ (\nabla e^u, \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_\Omega = -(e^u, \nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h))_\Omega &\leq \|e^u\|_{L^2(\Omega)} \|\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, we conclude that

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|^2_{L^2(\Omega)} \lesssim \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \left[\left\|\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\right\|_{L^2(\Omega)} + \|v - \tilde{v}_h\|_{H^1(\Omega)}\right] + (\boldsymbol{e}^{\boldsymbol{\varphi}}, \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_\Omega, \tag{4.20}$$

the limiting term being for now the last one. To overcome the lack of regularity of $\boldsymbol{\psi}$ we perform a Helmholtz decomposition. In fact, since $\boldsymbol{\psi} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$ as well as $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$ there exist $\boldsymbol{\rho} \in \boldsymbol{H}_0(\Omega, \mathbf{curl})$ and $z \in H^2(\Omega)$ such that $\boldsymbol{\psi} = \nabla \times \boldsymbol{\rho} + \nabla z$. The construction is as follows: Let $z \in H^1(\Omega)$ solve

$$\begin{aligned}-\Delta z &= -\nabla \cdot \boldsymbol{\psi} \quad &&\text{in } \Omega, \\ \partial_n z &= 0 &&\text{on } \Gamma.\end{aligned}$$

Since $\nabla \cdot (\boldsymbol{\psi} - \nabla z) = 0$ as well as $(\boldsymbol{\psi} - \nabla z) \cdot \boldsymbol{n} = 0$ by construction, the exact sequence property of the employed spaces allows for the existence of $\boldsymbol{\rho} \in \boldsymbol{H}_0(\Omega, \mathbf{curl})$ such that $\boldsymbol{\psi} - \nabla z = \nabla \times \boldsymbol{\rho}$. Finally the following estimates hold due to the *a priori* estimate of the Lax–Milgram theorem and partial integration for the first estimate, Assumption 3.1 for the second, and the triangle inequality together with the first estimate for the third one:

$$\begin{aligned}\|z\|_{H^1(\Omega)} &\lesssim \|\nabla \cdot \boldsymbol{\psi}\|_{(H^1(\Omega))'} \leq \|\boldsymbol{\psi}\|_{L^2(\Omega)}, \\ \|z\|_{H^2(\Omega)} &\lesssim \|\nabla \cdot \boldsymbol{\psi}\|_{L^2(\Omega)}, \\ \|\nabla \times \boldsymbol{\rho}\|_{L^2(\Omega)} &\leq \|\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla z\|_{L^2(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{L^2(\Omega)}.\end{aligned}$$

We now continue estimating (4.20) by applying the Helmholtz decomposition. For any $\tilde{\boldsymbol{\psi}}^c_h$, $\tilde{\boldsymbol{\psi}}^g_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ we have with $\tilde{\boldsymbol{\psi}}_h = \tilde{\boldsymbol{\psi}}^c_h + \tilde{\boldsymbol{\psi}}^g_h$

$$(\boldsymbol{e}^{\boldsymbol{\varphi}}, \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_\Omega = (\boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla \times \boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}^c_h)_\Omega + (\boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla z - \tilde{\boldsymbol{\psi}}^g_h)_\Omega =: T^c + T^g.$$

Treatment of T^g: by the Cauchy–Schwarz inequality we have

$$T^g = (\boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla z - \tilde{\boldsymbol{\psi}}_h^g)_\Omega \le \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \left\|\nabla z - \tilde{\boldsymbol{\psi}}_h^g\right\|_{L^2(\Omega)}.$$

Treatment of T^c: for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ we have

$$T^c = (\boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c)_\Omega = (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, \nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c)_\Omega + (\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h, \nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c)_\Omega =: T_1^c + T_2^c.$$

Treatment of T_1^c: by the Cauchy–Schwarz inequality we have

$$T_1^c = (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, \nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c)_\Omega \le \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} \left\|\nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c\right\|_{L^2(\Omega)}.$$

Treatment of T_2^c: in order to treat T_2^c we proceed as in the proof of Lemma 4.6 and apply Lemma 4.5 to split the discrete object $\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ on the discrete and the continuous level:

$$\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h = \nabla\times\boldsymbol{\mu} + \boldsymbol{r}, \qquad\qquad \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h = \nabla\times\boldsymbol{\mu}_h + \boldsymbol{r}_h,$$

for certain $\boldsymbol{\mu} \in \boldsymbol{H}_0(\Omega, \mathbf{curl})$, $\boldsymbol{r} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$, $\boldsymbol{\mu}_h \in \mathbf{N}^0_{p_v}(\mathcal{T}_h)$, and $\boldsymbol{r}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$. We now choose $\tilde{\boldsymbol{\psi}}_h^c = \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho}$ given by Lemma 4.5. Exploiting the definition of the operator $\boldsymbol{\Pi}_h^{\mathrm{curl},0}$ we find

$$\begin{aligned} T_2^c &= (\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h, \nabla\times\boldsymbol{\rho} - \tilde{\boldsymbol{\psi}}_h^c)_\Omega = \underbrace{(\nabla\times\boldsymbol{\mu}_h, \nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho})_\Omega}_{=0} + (\boldsymbol{r}_h, \nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho})_\Omega \\ &= (\boldsymbol{r}_h - \boldsymbol{r}, \nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho})_\Omega + (\boldsymbol{r}, \nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho})_\Omega \\ &=: T_1 + T_2. \end{aligned}$$

Treatment of T_1: with the same notation as in the proof of Lemma 4.6 and with exactly the same arguments we have

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \lesssim \frac{h}{p_v}\|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \frac{h}{p_v}\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h)\|_{L^2(\Omega)}.$$

By the Cauchy–Schwarz inequality we have

$$T_1 \lesssim \frac{h}{p_v}\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h)\|_{L^2(\Omega)} \left\|\nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho}\right\|_{L^2(\Omega)} \lesssim \frac{h}{p_v}\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{\varphi}_h)\|_{L^2(\Omega)}\|\nabla\times\boldsymbol{\rho}\|_{L^2(\Omega)},$$

where the last estimate follows from the fact that

$$\left\|\nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho}\right\|_{L^2(\Omega)} \le \|\nabla\times\boldsymbol{\rho} - \nabla\times\tilde{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}$$

for any $\tilde{\boldsymbol{\rho}}_h \in \mathbf{N}^0_{p_v}(\mathcal{T}_h)$ since it is a projection. Finally inserting $\boldsymbol{\varphi}$ and applying the triangle inequality as well as estimating $\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\|_{L^2(\Omega)}$ by $\|(e^u, \boldsymbol{e}^{\boldsymbol{\varphi}})\|_b$ we find

$$T_1 \lesssim \frac{h}{p_v}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}\|\nabla\times\boldsymbol{\rho}\|_{L^2(\Omega)} + \frac{h}{p_v}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b\|\nabla\times\boldsymbol{\rho}\|_{L^2(\Omega)}.$$

Treatment of T_2: note again that $\boldsymbol{\rho} \in \boldsymbol{H}_0(\Omega, \mathbf{curl})$ and the fact that $\boldsymbol{\Pi}_h^{\mathrm{curl},0}$ maps into $\nabla\times\mathbf{N}^0_{p_v}(\mathcal{T}_h)$. Therefore, we can write $\nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},0}\nabla\times\boldsymbol{\rho} = \nabla\times\widehat{\boldsymbol{\rho}}$ for some $\widehat{\boldsymbol{\rho}} \in \boldsymbol{H}_0(\Omega, \mathbf{curl})$ and the boundary terms consequently vanish in the following integration by parts

$$T_2 = (\boldsymbol{r}, \nabla\times\widehat{\boldsymbol{\rho}})_\Omega = (\nabla\times\boldsymbol{r}, \widehat{\boldsymbol{\rho}})_\Omega.$$

Finally, $T_2 = 0$, since $\nabla \times \boldsymbol{r} = 0$ by Lemma 4.5.

Collecting all the terms: collecting the terms together with the estimate $\|\nabla \times \boldsymbol{\rho}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}$ from the Helmholtz decomposition and the regularity estimates of Lemma 3.5 we find

$$(\boldsymbol{e^\varphi}, \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_\Omega \lesssim \left[\left\| \nabla z - \tilde{\boldsymbol{\psi}}_h^g \right\|_{L^2(\Omega)} + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p_v} \|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} + \frac{h}{p_v} \|(\boldsymbol{e^\varphi}, e^u)\|_b \right] \times \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}. \tag{4.21}$$

For the choice $\tilde{\boldsymbol{\psi}}_h^c = \boldsymbol{\Pi}_h^{\mathrm{curl},0} \nabla \times \boldsymbol{\rho} \in \nabla \times \mathbf{N}^0_{p_v}(\mathcal{T}_h)$ we have

$$\left\| \nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h) \right\|_{L^2(\Omega)} = \left\| \nabla \cdot (\nabla z - \tilde{\boldsymbol{\psi}}_h^g) \right\|_{L^2(\Omega)}.$$

Due to the regularity of $z \in H^2(\Omega)$ as well as $\nabla \cdot \nabla z = \Delta z = \nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$ we can find $\tilde{\boldsymbol{\psi}}_h^g \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ such that

$$\left\| \nabla z - \tilde{\boldsymbol{\psi}}_h^g \right\|_{\boldsymbol{H}(\Omega,\mathrm{div})} \lesssim \frac{h}{p_v} \|\nabla z\|_{\boldsymbol{H}^1(\Omega,\mathrm{div})} \lesssim \frac{h}{p_v} \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} \lesssim \frac{h}{p_v} \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)} \lesssim \frac{h}{p_v} \|(\boldsymbol{e^\varphi}, e^u)\|_b.$$

Therefore, estimate (4.21) can be summarized as follows:

$$(\boldsymbol{e^\varphi}, \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_\Omega \lesssim \left[\frac{h}{p_v} \|(e^u, \boldsymbol{e^\varphi})\|_b + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p_v} \|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} \right] \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}. \tag{4.22}$$

Again due to the regularity of $v \in H^{\min(\hat{s},1)+2}(\Omega)$ we can find $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$ such that

$$\|v - \tilde{v}_h\|_{H^1(\Omega)} \lesssim \frac{h}{p_s} \|v\|_{H^2(\Omega)} \lesssim \frac{h}{p_s} \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}.$$

Finally, summarizing the estimates (4.20) and (4.22) and again using

$$\left\| \nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h) \right\|_{L^2(\Omega)} = \left\| \nabla \cdot (\nabla z - \tilde{\boldsymbol{\psi}}_h^g) \right\|_{L^2(\Omega)} \lesssim \frac{h}{p_v} \|(\boldsymbol{e^\varphi}, e^u)\|_b$$

we find

$$\|\boldsymbol{e^\varphi}\|^2_{L^2(\Omega)} \lesssim \left[\frac{h}{p} \|(\boldsymbol{e^\varphi}, e^u)\|_b + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p} \|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} \right] \|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}.$$

Canceling one power of $\|\boldsymbol{e^\varphi}\|_{L^2(\Omega)}$ then yields the first estimate. The second one follows again by the fact that the least squares approximation is the projection with respect to b and the norm equivalence given in Theorem 2.1. In spatial dimension $d = 2$ there is one important adjustments to be made: The Helmholtz decomposition of $\boldsymbol{\psi}$ after equation (4.20) now reads $\boldsymbol{\psi} = \mathbf{curl}\,\rho + \nabla z$, for some $\rho \in H^1_0(\Omega)$, *i.e.*, a scalar potential instead of a vector field in three dimensions, and $z \in H^2(\Omega)$. The construction stays exactly the same. The remainder of the proof is adjusted analogously to the proof of Lemma 4.6. □

Lemma 4.9. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e^\varphi} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ be the solution of the dual problem given by Theorem 3.4 with $w = e^u$. Additionally, let $(\boldsymbol{\psi}_h, v_h)$ be the least squares approximation of $(\boldsymbol{\psi}, v)$ and denote $e^v = v - v_h$ and $\boldsymbol{e^\psi} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then,*

$$\left\| (\boldsymbol{e^\psi}, e^v) \right\|_b \lesssim \|\nabla e^u\|_{L^2(\Omega)} \quad \text{and} \quad \|e^v\|_{L^2(\Omega)} \lesssim \frac{h}{p} \|\nabla e^u\|_{L^2(\Omega)} \quad \text{and} \quad \left\| \boldsymbol{e^\psi} \right\|_{L^2(\Omega)} \lesssim \frac{h}{p} \|\nabla e^u\|_{L^2(\Omega)}.$$

Proof. Theorem 3.4 provides $\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{\min(\hat{s},1)+1}(\Omega)} + \|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \lesssim \|\nabla e^u\|_{L^2(\Omega)}$. Stability of the least squares method (*cf.* (4.1)) yields

$$\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim \|\nabla e^u\|_{L^2(\Omega)}.$$

By Lemma 4.1 we have

$$\|e^v\|_{L^2(\Omega)} \lesssim h/p\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b,$$

which together with the above gives the second estimate. By Theorem 4.8 we have

$$\left\|\boldsymbol{e}^{\boldsymbol{\psi}}\right\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|v-\tilde{v}_h\|_{H^1(\Omega)} + \left\|\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\right\|_{L^2(\Omega)} + \frac{h}{p}\left\|\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\right\|_{L^2(\Omega)}$$

for any $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$. The result follows immediately by again exploiting the regularity of the dual solution and the approximation properties of the employed spaces. □

Theorem 4.10. *Let Ω satisfy Assumption 3.1 for some $\hat{s}\geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$. Then, for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$,*

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \|u-\tilde{u}_h\|_{H^1(\Omega)} + \frac{h}{p}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. As in Remark 4.2 with $(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)$ denoting the error of the FOSLS approximation of the dual solution given by Theorem 3.4 (duality argument for the gradient of the scalar variable) applied to $w = e^u$ we have for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$

$$\begin{aligned}\|\nabla e^u\|^2_{L^2(\Omega)} &= b((\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h, u-\tilde{u}_h), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v))\\ &= (\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h) + \gamma(u-\tilde{u}_h), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}} + \gamma e^v)_\Omega + (\nabla(u-\tilde{u}_h) + \boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h, \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega.\end{aligned}$$

We specifically choose $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}^0_h\boldsymbol{\varphi}$. In the following we heavily use the properties of the operator $\boldsymbol{I}^0_h$ given in Lemma 4.6. First we exploit the regularity of the dual solution using Lemma 4.9 as well as the estimates of Theorem 3.4:

$$\begin{aligned}(\gamma(u-\tilde{u}_h), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}} + \gamma e^v)_\Omega &\lesssim \|u-\tilde{u}_h\|_{L^2(\Omega)}\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim \|u-\tilde{u}_h\|_{H^1(\Omega)}\|\nabla e^u\|_{L^2(\Omega)},\\ (\nabla(u-\tilde{u}_h), \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|\nabla(u-\tilde{u}_h)\|_{L^2(\Omega)}\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim \|u-\tilde{u}_h\|_{H^1(\Omega)}\|\nabla e^u\|_{L^2(\Omega)},\\ (\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}, \nabla e^v)_\Omega &= -(\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}), e^v)_\Omega \leq \left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^v\|_{L^2(\Omega)}\\ &\lesssim h/p\left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|\nabla e^u\|_{L^2(\Omega)},\\ (\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}), \gamma e^v)_\Omega &\leq \left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^v\|_{L^2(\Omega)} \lesssim h/p\left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|\nabla e^u\|_{L^2(\Omega)},\\ (\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}, \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \left\|\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}\right\|_{L^2(\Omega)}\left\|\boldsymbol{e}^{\boldsymbol{\psi}}\right\|_{L^2(\Omega)} \lesssim h/p\left\|\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}\right\|_{L^2(\Omega)}\|\nabla e^u\|_{L^2(\Omega)},\\ (\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &= (\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi}), \nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))_\Omega \leq \left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\left\|\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\right\|_{L^2(\Omega)}\\ &\lesssim h/p\left\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^0_h\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|\nabla e^u\|_{L^2(\Omega)}.\end{aligned}$$

Canceling one power of $\|\nabla e^u\|_{L^2(\Omega)}$, collecting the terms, and using the estimate for $\boldsymbol{I}^0_h$ we arrive at the asserted estimate. □

As a tool in the proof of our main theorem (Thm. 4.12) we need to analyze the error of the FOSLS approximation of the dual solution. This is summarized in

Lemma 4.11. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ be the solution of the dual problem given by Theorem 3.3 with $w = e^u$. Additionally, let $(\boldsymbol{\psi}_h, v_h)$ be the least squares approximation of $(\boldsymbol{\psi}, v)$ and denote $e^v = v - v_h$ and $\boldsymbol{e}^{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then,*

$$\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim \frac{h}{p}\|e^u\|_{L^2(\Omega)} \quad \text{and} \quad \|e^v\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^2 \|e^u\|_{L^2(\Omega)}.$$

Furthermore,

$$\left\|\boldsymbol{e}^{\boldsymbol{\psi}}\right\|_{L^2(\Omega)} \lesssim \begin{cases} h\|e^u\|_{L^2(\Omega)} & \text{if} \quad \mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_0(\mathcal{T}_h), \\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,2)} \|e^u\|_{L^2(\Omega)} & \text{else.} \end{cases}$$

Proof. Theorem 3.3 gives $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s},2)+1}(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^{\min(\hat{s}+2,2)}(\Omega)$ and $v \in H^{\min(\hat{s}+2,2)}(\Omega)$ with norms bounded by $\|e^u\|_{L^2(\Omega)}$. Therefore we have in view of optimality of the FOSLS method in the b-norm

$$\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \overset{(4.1)}{\leq} \left\|(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h, v - \tilde{v}_h)\right\|_b \lesssim h/p\|e^u\|_{L^2(\Omega)},$$

where the first estimate holds for any $\tilde{v}_h \in S_{p_s+1}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ and the second one follows with the same arguments as in the proof of Lemma 4.1. By Lemma 4.1 we have

$$\|e^v\|_{L^2(\Omega)} \lesssim h/p\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b,$$

which together with the above gives the second estimate. By Theorem 4.8 we have

$$\left\|\boldsymbol{e}^{\boldsymbol{\psi}}\right\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|v - \tilde{v}_h\|_{H^1(\Omega)} + \left\|\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\right\|_{L^2(\Omega)} + \frac{h}{p}\left\|\nabla \cdot (\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\right\|_{L^2(\Omega)}$$

for any $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$. The result follows immediately by again exploiting the regularity of the dual solution and the approximation properties of the employed spaces. □

Theorem 4.12. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$ and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$. Then, for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$,*

$$\|e^u\|_{L^2(\Omega)} \lesssim \begin{cases} h\|u - \tilde{u}_h\|_{H^1(\Omega)} + h\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + h\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} & \text{for } \mathbf{RT}^0_0(\mathcal{T}_h), \\ h\|u - \tilde{u}_h\|_{H^1(\Omega)} + h^{\min(\hat{s}+1,2)}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + h\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} & \text{for } \mathbf{BDM}^0_1(\mathcal{T}_h), \\ \frac{h}{p}\|u - \tilde{u}_h\|_{H^1(\Omega)} + (\frac{h}{p})^{\min(\hat{s}+1,2)}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + (\frac{h}{p})^2\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} & \text{else.} \end{cases}$$

Proof. As in Remark 4.2 with $(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)$ denoting the FOSLS approximation of the dual solution given by Theorem 3.3 applied to $w = e^u$ we have for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$

$$\begin{aligned} \|e^u\|^2_{L^2(\Omega)} &= b((\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)) \\ &= (\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h) + \gamma(u - \tilde{u}_h), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}} + \gamma e^v)_\Omega + (\nabla(u - \tilde{u}_h) + \boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega. \end{aligned}$$

We specifically choose $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}^0_h \boldsymbol{\varphi}$. In the following we heavily use the properties of the operator $\boldsymbol{I}^0_h$ given in Lemma 4.6. First we exploit the regularity of the dual solution using Lemma 4.11 as well as the estimates of Theorem 3.3:

$$(\gamma(u - \tilde{u}_h), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}} + \gamma e^v)_\Omega \lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)}\left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim h/p\|u - \tilde{u}_h\|_{H^1(\Omega)}\|e^u\|_{L^2(\Omega)},$$

$$
\begin{aligned}
(\nabla(u-\tilde{u}_h), \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|\nabla(u-\tilde{u}_h)\|_{L^2(\Omega)} \left\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\right\|_b \lesssim h/p\|u-\tilde{u}_h\|_{H^1(\Omega)}\|e^u\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}, \nabla e^v)_\Omega &= -(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}), e^v)_\Omega \leq \left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^v\|_{L^2(\Omega)}\\
&\lesssim (h/p)^2\left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)},\\
(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}), \gamma e^v)_\Omega &\leq \left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^v\|_{L^2(\Omega)}\\
&\lesssim (h/p)^2\left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}, \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \left\|\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}\right\|_{L^2(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)}\\
&\lesssim \begin{cases} h\left\|\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)} & \text{if } \mathbf{V}_{p_v}^0(\mathcal{T}_h) = \mathbf{RT}_0^0(\mathcal{T}_h),\\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,2)}\left\|\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)} & \text{else,}\end{cases}\\
(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &= (\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}), \nabla\cdot(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h))_\Omega \leq \left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\left\|\nabla\cdot(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\right\|_{L^2(\Omega)}\\
&\lesssim \begin{cases} h\left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)} & \text{if } p_v = 1,\\ \left(\frac{h}{p}\right)^2\left\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi})\right\|_{L^2(\Omega)}\|e^u\|_{L^2(\Omega)} & \text{else.}\end{cases}
\end{aligned}
$$

Canceling one power of $\|e^u\|_{L^2(\Omega)}$, collecting the terms, and using the estimate for $\boldsymbol{I}_h^0$ we arrive at the asserted estimate. ☐

Remark 4.13. Before stating the general corollary with prescribed right-hand side $f \in H^s(\Omega)$ we highlight the improved convergence result. Consider $f \in L^2(\Omega)$ and let Assumption 3.1 be satisfied for some $\hat{s} \geq 0$, which is satisfied for example for a convex polygonal or polyhedral domain Ω or a smooth boundary Γ. For the classical conforming finite element method one observes convergence $O(h^2)$ due to the Aubin-Nitsche trick. More precisely, for the classical FEM approximation u_h^{FEM} to the model problem there holds

$$\left\|u - u_h^{\mathrm{FEM}}\right\|_{L^2(\Omega)} \lesssim h^2\|u\|_{H^2(\Omega)} \lesssim h^2\|f\|_{L^2(\Omega)}.$$

As elaborated in Section 1 this rate could not be obtained for the FOSLS method by previous results, since further regularity of the vector variable $\boldsymbol{\varphi}$ would be necessary. Results like Lemma 3.4 of [4] and Theorem 4.1 of [15] are essentially a duality argument like Theorem 3.3 and the strategy of Lemma 4.1. Without further analysis the estimate of Lemma 4.1, does not give any further powers of h, since the b-norm is equivalent to the $\boldsymbol{H}(\Omega, \mathrm{div}) \times H^1(\Omega)$ norm. Theorem 4.12 ensures, at least if the space $\mathbf{V}_{p_v}^0(\mathcal{T}_h)$ is not of lowest order, *i.e.* $p_v > 1$, that the FOSLS method converges also as $O(h^2)$. More precisely, the estimate in Theorem 4.12 together with the approximation properties of the employed finite element spaces and $p_v > 1$ and $p_s \geq 1$ gives

$$\|e^u\|_{L^2(\Omega)} \lesssim h^2\|u\|_{H^2(\Omega)} + h^{\min(\hat{s}+1,2)}h\|\boldsymbol{\varphi}\|_{H^1(\Omega)} + h^2\|\nabla\cdot\boldsymbol{\varphi}\|_{L^2(\Omega)} \lesssim h^2\|f\|_{L^2(\Omega)}.$$

So in fact the optimal rate in the sense of the beginning of Section 4 is achieved. If the lowest order case $p_v = 1$ also achieves optimal order is yet to be answered. Numerical experiments in Section 5, however, indicate it to be true.

We summarize the results for general right-hand side $f \in H^s(\Omega)$. This summary is essentially the estimates given by the Theorems 4.8, 4.10, and 4.12 together with the approximation properties of the employed finite element spaces. For brevity and readability we only focus on the case of smooth boundary Γ, *i.e.*, where Assumption 3.1 is satisfied for any $\hat{s} > 0$.

Corollary 4.14. *Let Γ be smooth and $f \in H^t(\Omega)$ for some $t \geq 0$. Then the solution to (2.2) satisfies $u \in H^{t+2}(\Omega)$, $\boldsymbol{\varphi} \in \boldsymbol{H}^{t+1}(\Omega)$ and $\nabla\cdot\boldsymbol{\varphi} \in H^t(\Omega)$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Furthermore, let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for the lowest order case $p_v = 1$,*

$$\|e^u\|_{L^2(\Omega)} \lesssim h^{\min(t+1,2)}\|f\|_{H^t(\Omega)}.$$

For $p_v > 1$ there holds

$$\|e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(t+1,p_s,p_v+1)+1} \|f\|_{H^t(\Omega)}.$$

Furthermore, the estimate

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(t+1,p_s,p_v+1)} \|f\|_{H^t(\Omega)}.$$

holds. Finally, we have

$\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$	$\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$
$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(t+1,p_s+1,p_v)} \|f\|_{H^t(\Omega)}$	$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(t+1,p_s+1,p_v+1)} \|f\|_{H^t(\Omega)}.$

Proof. The regularity result follows immediately by the shift theorem discussed in Remark 3.2 and the method of real interpolation, see *e.g.*, [16, 24], together with the fact that $\boldsymbol{\varphi} = -\nabla u$. We now analyze the quantities in the estimates of the Theorems 4.8, 4.10 and 4.12:

$$\|u - \tilde{u}_h\|_{H^1(\Omega)} \lesssim (h/p)^{\min(t+1,p_s)} \|u\|_{H^{t+2}(\Omega)} \lesssim (h/p)^{\min(t+1,p_s)} \|f\|_{H^t(\Omega)},$$

$$\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} \lesssim \begin{cases} (h/p)^{\min(t+1,p_v)} \|\boldsymbol{\varphi}\|_{H^{t+1}(\Omega)} \lesssim (h/p)^{\min(t+1,p_v)} \|f\|_{H^t(\Omega)} & \text{for } \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h), \\ (h/p)^{\min(t+1,p_v+1)} \|\boldsymbol{\varphi}\|_{H^{t+1}(\Omega)} \lesssim (h/p)^{\min(t+1,p_v+1)} \|f\|_{H^t(\Omega)} & \text{for } \mathbf{BDM}^0_{p_v}(\mathcal{T}_h), \end{cases}$$

$$\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} \lesssim (h/p)^{\min(t,p_v)} \|\nabla \cdot \boldsymbol{\varphi}\|_{H^t(\Omega)} \lesssim (h/p)^{\min(t,p_v)} \|f\|_{H^t(\Omega)}.$$

The estimates of the Theorems 4.8, 4.10, and 4.12 together with the above estimates give, after straightforward calculations, the asserted rates. ☐

We close this section with some remarks concerning sharpness of the estimates of Corollary 4.14:

Remark 4.15. Let the assumptions of Corollary 4.14 be satisfied. From a best approximation point of view, since $u \in H^{s+2}(\Omega)$, we have

$$\inf_{\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)} \|u - \tilde{u}_h\|_{L^2(\Omega)} = O(h^{\min(s+1,p_s)+1})$$

$$\inf_{\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)} \|\nabla(u - \tilde{u}_h)\|_{L^2(\Omega)} = O(h^{\min(s+1,p_s)})$$

$$\inf_{\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)} \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} = \begin{cases} O(h^{\min(s+1,p_v)}) & \text{if } \mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h), \\ O(h^{\min(s+1,p_v+1)}) & \text{if } \mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h). \end{cases}$$

Excluding the lowest order case $p_v = 1$ we have, choosing $p_v \geq p_s - 1$, sharpness of the estimates for e^u and ∇e^u. This can be easily seen, since the rates guaranteed by Corollary 4.14 for $\|e^u\|_{L^2(\Omega)}$ and $\|\nabla e^u\|_{L^2(\Omega)}$ are the same as the ones from a best approximation argument. The estimates are therefore sharp. The lowest order case $p_v = 1$ seems to be suboptimal, as the numerical examples in Section 5 suggest. In all other cases, *i.e.*, $p_v > 1$ and $p_v < p_s - 1$, our numerical examples suggest sharpness of the estimates, in both the setting of a smooth solution as well as one with finite Sobolev regularity, but not achieving the best approximation rate. Since in the least squares functional the term $\|\nabla u_h + \boldsymbol{\varphi}_h\|_{L^2(\Omega)}$ enforces ∇u_h and $\boldsymbol{\varphi}_h$ to be *close*, it is to be expected that an insufficient choice of p_v limits the convergence rate. A theoretical justification concerning the observed rates in the cases $p_v = 1$ as well as $p_v > 1$ and $p_v < p_s - 1$ is yet to be provided. In conclusion, when the application in question is concerned with approximation of u or ∇u in the $L^2(\Omega)$ norm, the best possible rate with the smallest number of degrees of freedom is achieved with the choice $p_v = p_s - 1$ regardless of the choice of $\mathbf{V}^0_{p_v}(\mathcal{T}_h)$. Therefore, it is computationally favorable to choose Raviart–Thomas elements over Brezzi–Douglas–Marini elements. Turning now to $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$

similar arguments guarantee sharpness of the estimates. In this case when $p_s + 1 \geq p_v$ and $p_s + 1 \geq p_v + 1$, for the choice of Raviart–Thomas elements and Brezzi–Douglas–Marini elements respectively. Again the other cases are open for theoretical justification. However, both theoretical as well as the numerical examples in Section 5 suggest the choice of Brezzi–Douglas–Marini elements over Raviart–Thomas elements, when application is concerned with approximation of $\boldsymbol{\varphi}$ in the $L^2(\Omega)$ norm.

5. Numerical examples

All our calculations are performed with the *hp*-FEM code NETGEN/NGSOLVE by J. Schöberl [22,23]. The curved boundaries are implemented using second order rational splines.

In the following we will perform two different numerical experiments:

In Example 5.1 we consider the case $f \in C^\infty(\overline{\Omega})$. The suboptimal estimate $\|e^u\|_{L^2(\Omega)} \lesssim h/p\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b$ of Lemma 4.1 suffices to deduce optimal rates. Hence we only highlight two aspects of the least squares approach: On the one hand the optimal choice of the employed polynomial degrees p_s and p_v. On the other hand the superiority of Brezzi–Douglas–Marini elements over Raviart–Thomas elements when approximating the vector valued variable.

In Example 5.2 we showcase our new convergence result by choosing $f \in \cap_{\varepsilon>0} H^{1/2-\varepsilon}(\Omega)$, but $f \notin H^{1/2}(\Omega)$ with $u \in \cap_{\varepsilon>0} H^{5/2-\varepsilon}(\Omega)$ and $\boldsymbol{\varphi} \in \cap_{\varepsilon>0} \boldsymbol{H}^{3/2-\varepsilon}(\Omega)$.

In all graphs, the actual numerical results are given by red dots. The rate that is guaranteed by Corollary 4.14 is plotted in black together with the number written out near the bottom right. Furthermore, in blue the reference line for the best rate possible with the employed space $S_{p_s}(\mathcal{T}_h)$ or $\mathbf{V}^0_{p_v}(\mathcal{T}_h)$ is plotted, depending on the quantity of interest, *i.e.*, for $\|e^u\|_{L^2(\Omega)}$ the blue reference line corresponds to $h^{\min(s+1,p_s)+1}$, for $\|\nabla e^u\|_{L^2(\Omega)}$ the blue reference line corresponds to $h^{\min(s+1,p_s)}$ and for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ the blue reference line corresponds to $h^{\min(s+1,p_v)}$ for $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$ and $h^{\min(s+1,p_v+1)}$ for $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

Example 5.1. We consider as the domain Ω the unit sphere in $\mathbb{R}^2$. The exact solution is the smooth function $u(x, y) = \cos(2\pi(x^2 + y^2))$. The numerical results are plotted in Figures 1 and A.1 for $\|e^u\|_{L^2(\Omega)}$, in Figures A.2 and A.3 for $\|\nabla e^u\|_{L^2(\Omega)}$, and in Figures 2 and 3 for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. There are some remarks to be made:

- Consider Figure 1 depicting $\|e^u\|_{L^2(\Omega)}$ using Raviart–Thomas elements. The rates guaranteed by Corollary 4.14 are achieved in the numerical experiment. The important subfigures are the ones in the subdiagonal of the discussed figure, *i.e.*, corresponding to the choice $p_v = p_s - 1$. Here, apart from the lowest order case, the best possible rate with the smallest number of degrees of freedom is achieved. Above this subdiagonal, *i.e.*, $p_v \geq p_s$, additional degrees of freedom will not increase the rate of convergence, since by pure best approximation arguments the rate of convergence is limited by the polynomial degree p_s of the scalar variable. Below this subdiagonal, *i.e.*, $p_v < p_s - 1$, we notice that the rate of convergence is also limited by the polynomial degree p_v of the vector variable. Note that the results for $\|e^u\|_{L^2(\Omega)}$ in Corollary 4.14 are independent of the choice of the vector valued finite element space, which is also confirmed by our experiments. Additional convergence plots can be found in Appendix A.
- Consider Figures 2 and 3 depicting $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. Apart from similar observations as for the scalar variable, it is notable that a difference in the approximation properties of the different spaces for the vector variable is observed, as predicted by Corollary 4.14. Consider wanting to achieve a rate of $O(h^5)$. The combination of spaces with the smallest number of degrees of freedom corresponds to $\mathbf{BDM}^0_4(\mathcal{T}_h) \times S_4(\mathcal{T}_h)$ and $\mathbf{RT}^0_4(\mathcal{T}_h) \times S_4(\mathcal{T}_h)$ respectively, highlighting the superiority of the Brezzi–Douglas–Marini elements when approximating $\boldsymbol{\varphi}$. For further discussion see again Remark 4.15. Finally, consider the subfigures above the diagonal in Figures 2 and 3. Here, as discussed above for the scalar variable, the rate of convergence is also limited by the polynomial degree p_s. As an example consider the case $\mathbf{BDM}^0_5(\mathcal{T}_h) \times S_1(\mathcal{T}_h)$, *i.e.*, the upper right corner of Figure 3: The rate of convergence when approximating the vector valued variable (with $p_v = 5$) is limited by the insufficient choice of the polynomial degree in the scalar variable ($p_s = 1$).

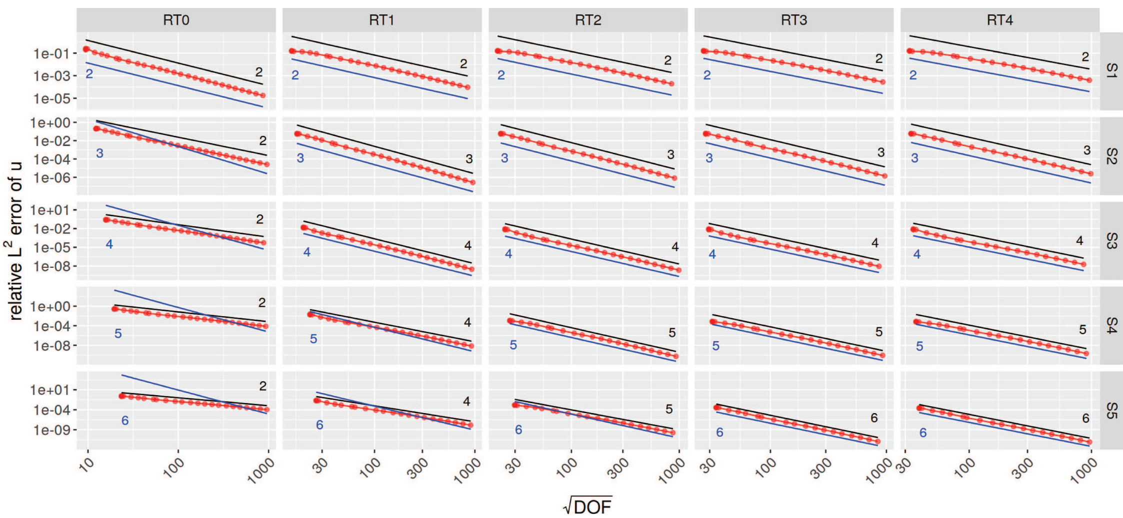

FIGURE 1. (*cf.* Example 5.1) Convergence of $\|e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

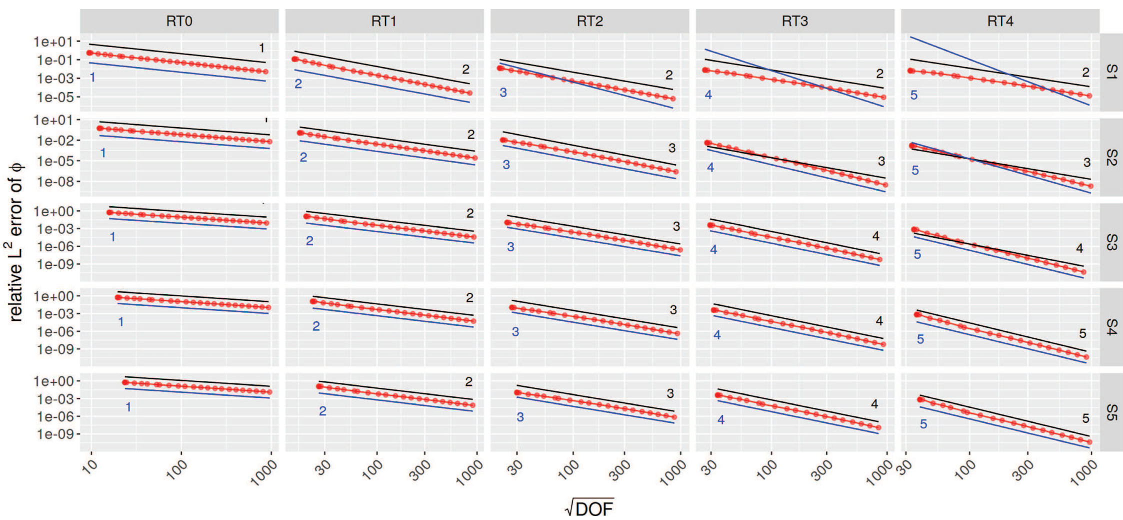

FIGURE 2. (*cf.* Example 5.1) Convergence of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

– We close the discussion of Example 5.1 showcasing the separate error estimates for $\|\nabla e^u\|_{L^2(\Omega)}$ and $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ derived in the Theorems 4.8 and 4.10. Consider the case $p_s = 1$ and the use of Brezzi–Douglas–Marini elements. Corollary 4.14 then predicts $h^{\min(p_s,p_v+1)} = h$ as a rate of convergence for $\|\nabla e^u\|_{L^2(\Omega)}$. For $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ a rate of convergence as $h^{\min(p_s+1,p_v+1)} = h^2$ is predicted. This effect can be seen in the first rows of Figures 3 and A.3.

Example 5.2. For our second example we consider again the case of Ω being the unit sphere in $\mathbb{R}^2$. The exact solution $u(x,y)$ is calculated corresponding to the right-hand side $f(x,y) = \mathbb{1}_{[0,1/2]}(\sqrt{x^2+y^2})$. Therefore

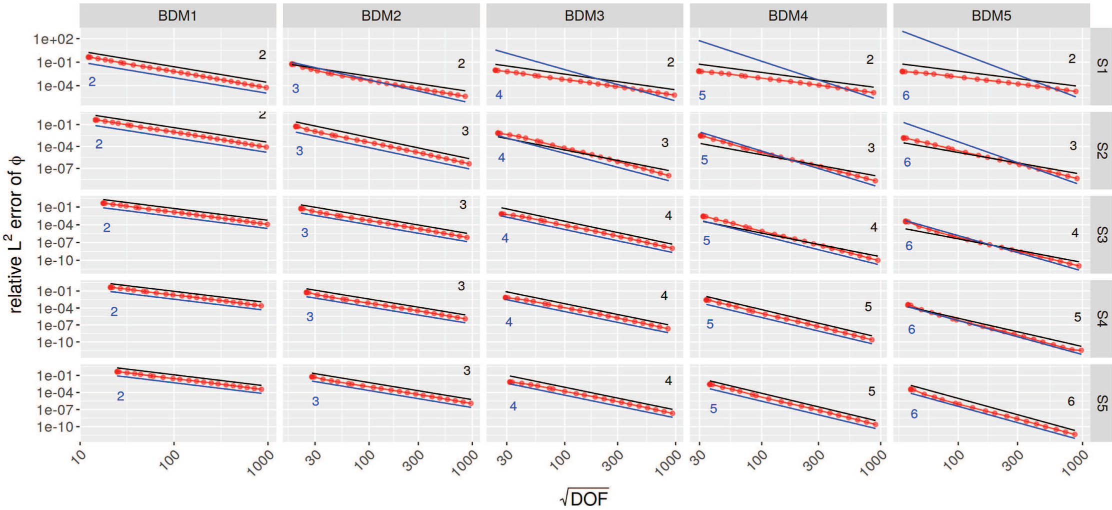

FIGURE 3. (*cf.* Example 5.1) Convergence of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

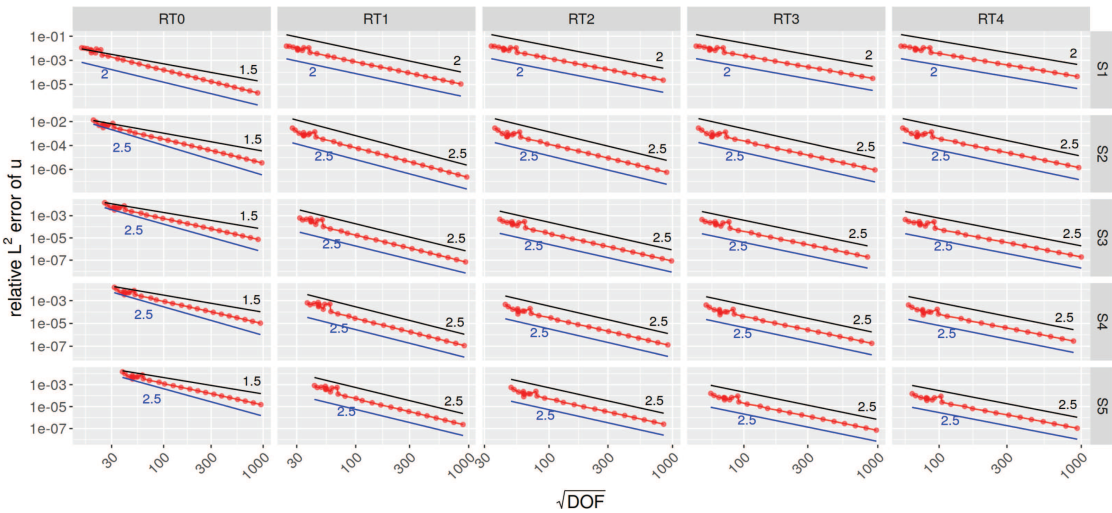

FIGURE 4. (*cf.* Example 5.2) Convergence of $\|e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

$u \in \cap_{\varepsilon>0} H^{5/2-\varepsilon}(\Omega)$. The numerical results for the choice of Raviart–Thomas elements are plotted in Figure 4 for $\|e^u\|_{L^2(\Omega)}$, in Figure 5 for $\|\nabla e^u\|_{L^2(\Omega)}$ and in Figure 6 for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. Apart from the remarks already made in Example 5.1 we note that we observe the improved convergence result when dealing with limited Sobolev regularity of the data. Furthermore, in the lowest order case $p_v = 1$ the guaranteed rate seems to be suboptimal. The plots for the choice of Brezzi–Douglas–Marini elements are presented in Appendix A.

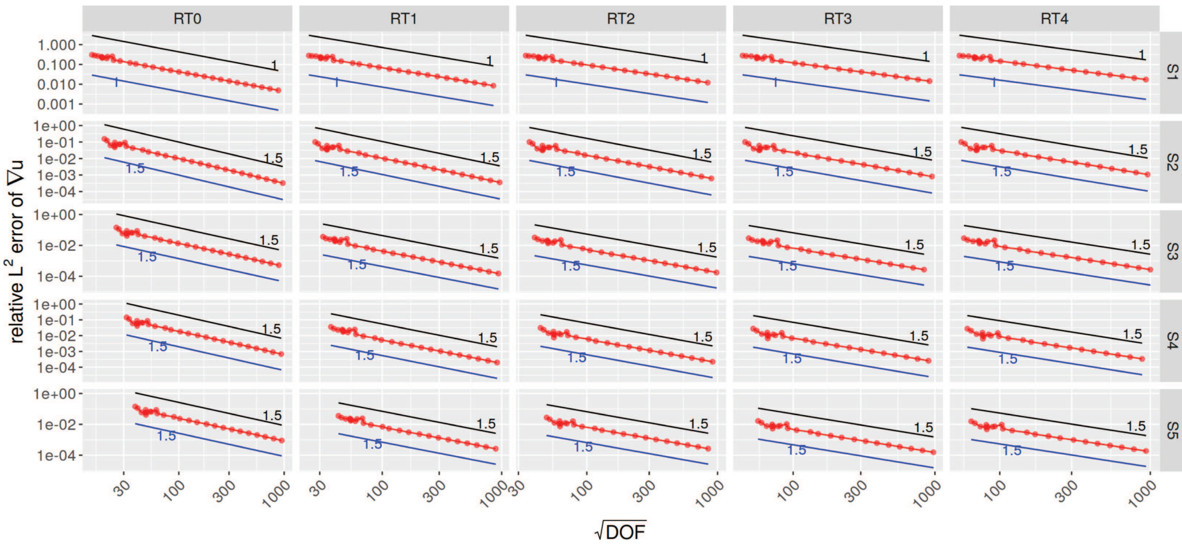

FIGURE 5. (*cf.* Example 5.2) Convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

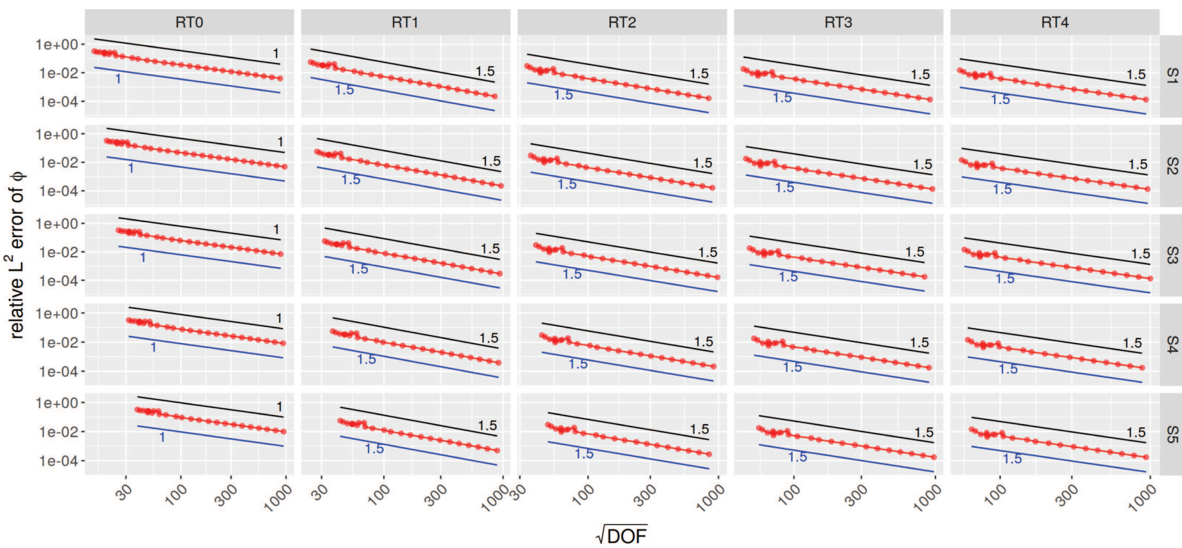

FIGURE 6. (*cf.* Example 5.2) Convergence of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

APPENDIX A.

For completeness we present additional convergence plots below. In Figure A.1 we plot $\|e^u\|_{L^2(\Omega)}$ employing Brezzi–Douglas–Marini elements for the problem considered in Example 5.1. The Figures A.2 and A.3 depicting $\|\nabla e^u\|_{L^2(\Omega)}$ are essentially the same just one order less than $\|e^u\|_{L^2(\Omega)}$. The numerical results for the finite regularity solution considered in Example 5.2 are plotted in Figure A.4 for $\|e^u\|_{L^2(\Omega)}$, in Figure A.5 for $\|\nabla e^u\|_{L^2(\Omega)}$ and in Figure A.6 for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$.

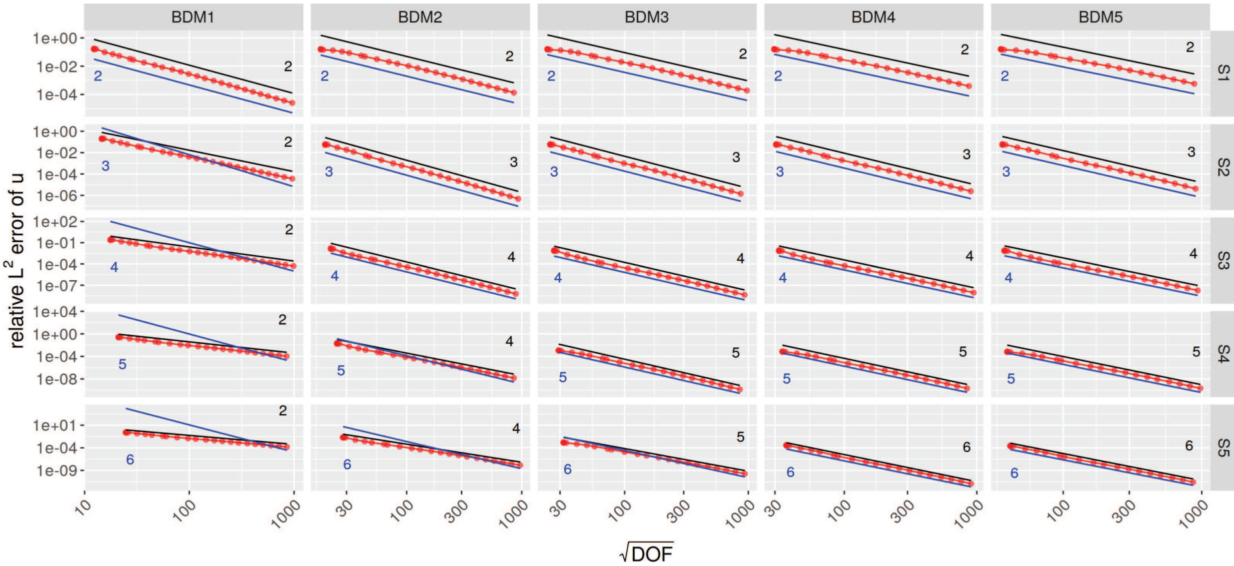

FIGURE A.1. (*cf.* Example 5.1) Convergence of $\|e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

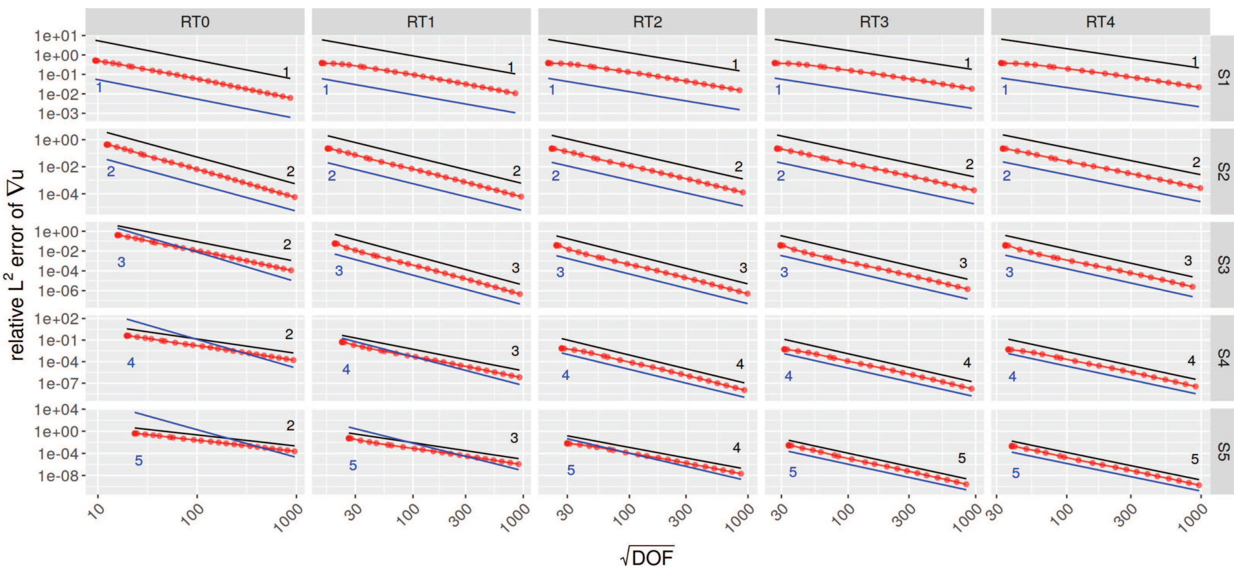

FIGURE A.2. (*cf.* Example 5.1) Convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\text{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$.

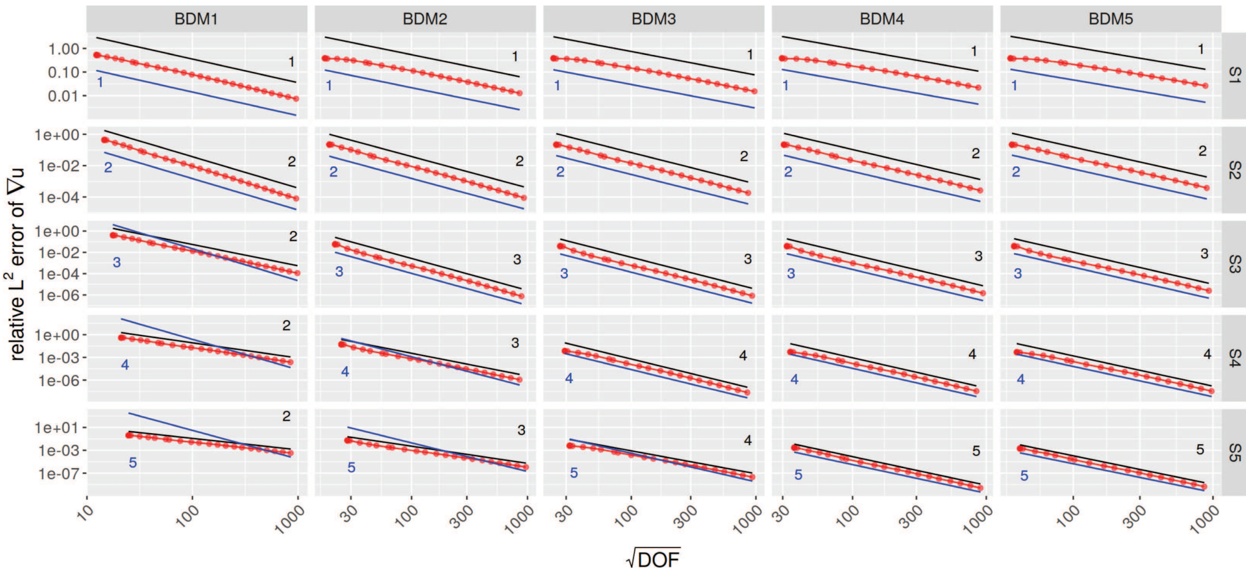

FIGURE A.3. (*cf.* Example 5.1) Convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

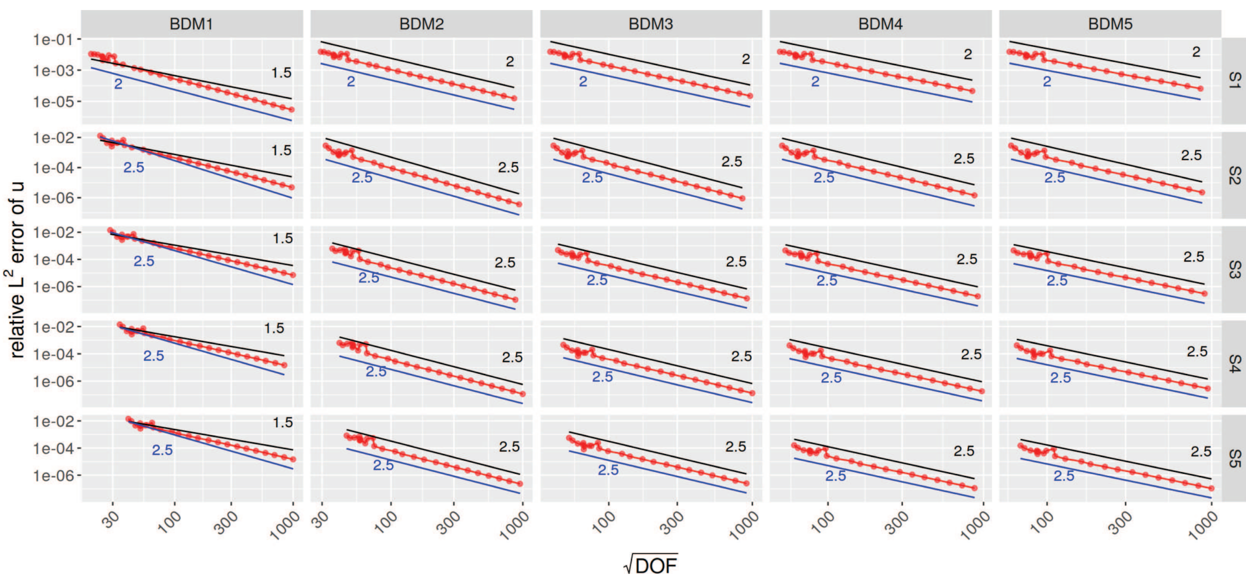

FIGURE A.4. (*cf.* Example 5.2) Convergence of $\|e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

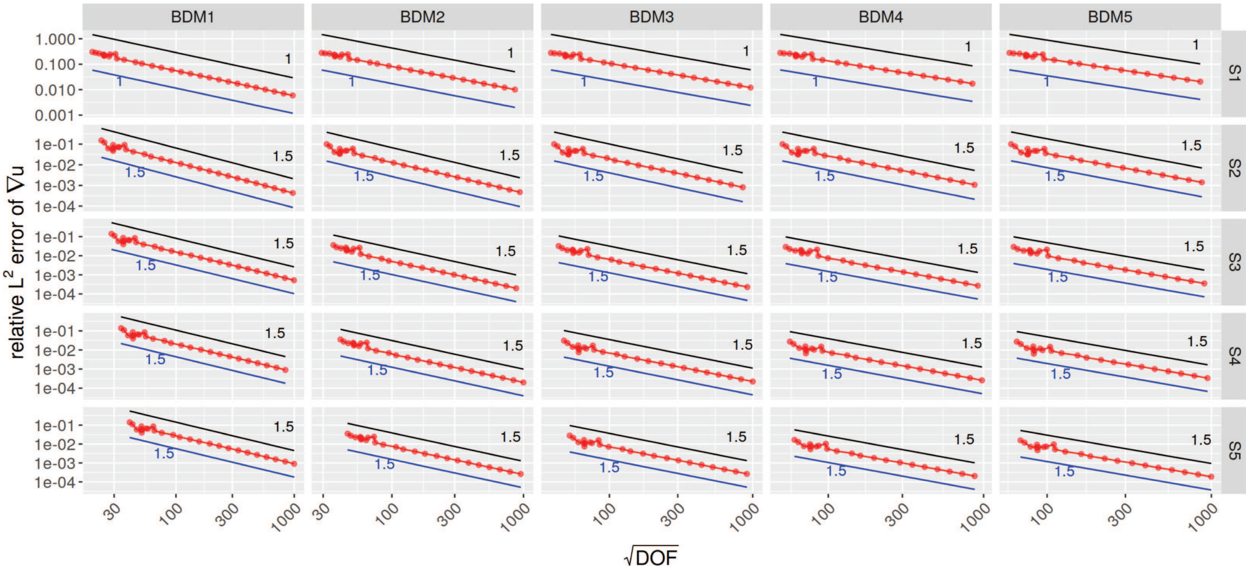

FIGURE A.5. (*cf.* Example 5.2) Convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

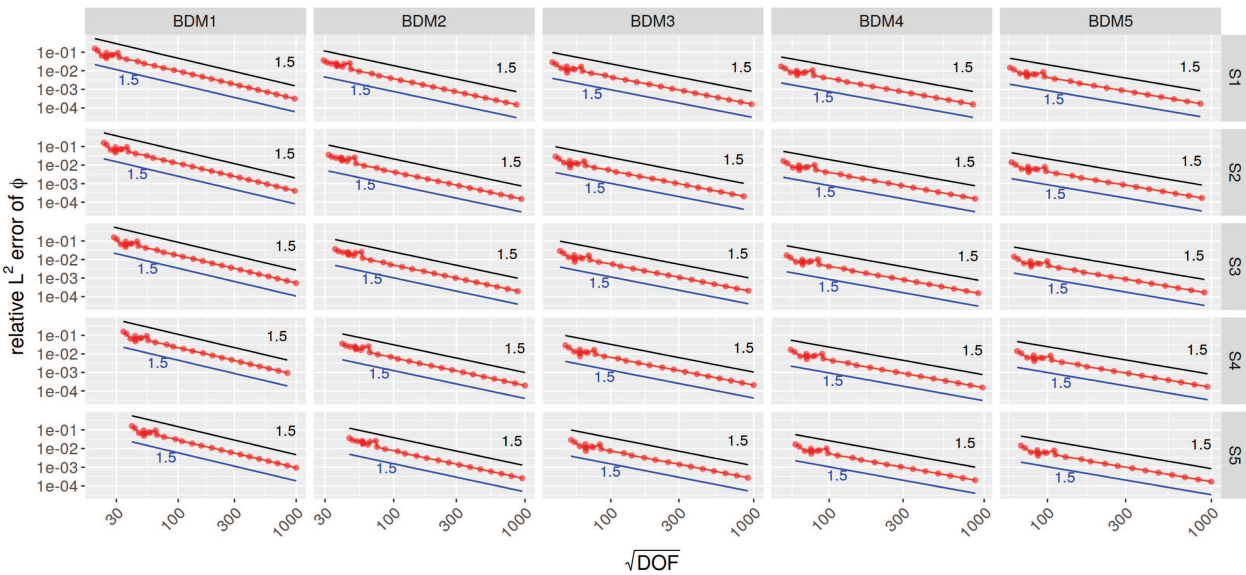

FIGURE A.6. (*cf.* Example 5.2) Convergence of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ *vs.* $\sqrt{\mathrm{DOF}} \sim 1/h$ employing $\mathbf{V}^0_{p_v}(\mathcal{T}_h) = \mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$.

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