Theoretical Computer Science 979 (2023) 114204

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Fractional covers of hypergraphs with bounded multi-intersection ☆,☆☆

Georg Gottlob [a], Matthias Lanzinger [a,*], Reinhard Pichler [b], Igor Razgon [c]

[a] University of Oxford, Wolfson Building, Parks Road, Oxford OX1 3AQ, United Kingdom
[b] TU Wien, Favoritenstrasse 9, 1040 Wien, Austria
[c] Birkbeck University of London, Malet Street, London WC1E 7HX, United Kingdom

A R T I C L E I N F O

Article history:
Received 18 March 2022
Received in revised form 4 August 2023
Accepted 15 September 2023
Available online 21 September 2023
Communicated by V. Bonifaci

Keywords:
Hypergraphs
Fractional hypertree width
Fractional graph theory
Fractional edge cover
Fractional hitting set

A B S T R A C T

Fractional (hyper-)graph theory is concerned with the specific problems that arise when fractional analogues of otherwise integer-valued (hyper-)graph invariants are considered. The focus of this paper is on fractional edge covers of hypergraphs. Our main technical result generalizes and unifies previous conditions under which the size of the support of fractional edge covers is bounded independently of the size of the hypergraph itself. We show how this combinatorial result can be used to extend previous tractability results for checking if the fractional hypertree width of a given hypergraph is $\leq k$ for some constant k. Moreover, we show a dual version of our main result for fractional hitting sets.

1. Introduction

Fractional (hyper-)graph theory [14] has evolved into a mature discipline in graph theory – building upon early research efforts that date back to the 1970s [2]. The crucial observation motivating this field is that many integer-valued (hyper-)graph invariants have a meaningful fractional analogue. Frequently, the integer-valued invariants are defined in terms of an integer linear program (ILP) and the fractional analogue is obtained by the fractional relaxation. Examples of problems which have been studied in fractional (hyper-)graph theory comprise matching problems, colouring problems, covering problem and many more.

Covering problems come in two principal flavours, namely *edge covers* (also referred to as set covers) and *hitting sets* (also referred to as vertex covers). We shall concentrate on edge covers in the first place, and afterwards show how our results translate to hitting sets. *Fractional edge covers* have attracted a lot of attention in recent times. On the one hand, this is due to a deep connection between information theory and database theory. Indeed, the famous "AGM bound" – named after

☆ This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.
☆☆ An extended abstract of this article was presented at the 45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020) [8].

* Corresponding author.
E-mail addresses: georg.gottlob@cs.ox.ac.uk (G. Gottlob), matthias.lanzinger@cs.ox.ac.uk (M. Lanzinger), pichler@dbai.tuwien.ac.at (R. Pichler), igor@dcs.bbk.ac.uk (I. Razgon).

https://doi.org/10.1016/j.tcs.2023.114204

Atserias, Grohe, and Marx [1] – establishes a tight upper bound on the number of result tuples of relational joins in terms of fractional edge covers. On the other hand, fractional hypertree width (*fhw*) is to date the most general width-notion that allows one to define tractable fragments of solving Constraint Satisfaction Problems (CSPs), answering Conjunctive Queries (CQs), and solving the Homomorphism Problem [11]. The fractional hypertree width of a hypergraph is defined in terms of the size of fractional edge covers of the bags in a tree decomposition.

Fractional (hyper-)graph invariants give rise to new challenges that do not exist in the integral case. Intuitively, if a fractional (hyper-)graph invariant is obtained by the relaxation of a linear program (LP), one would expect things to become easier, since we move from the intractable problem of ILPs to the tractable problem of LPs. However, also the opposite may happen, namely that the fractional relaxation introduces complications not present in the integral case. To illustrate such an effect, we first recall some basic definitions.

Definition 1. A *hypergraph* H is a tuple $H = (V, E)$, consisting of a set of vertices V and a set of hyperedges (or simply edges), which are non-empty subsets of V. Let γ be a function of the form $\gamma : E \to \mathbb{R}^+$, i.e., mapping edges to the non-negative reals. Then the set of vertices *covered* by γ is defined as $B(\gamma) = \{v \in V \mid \sum_{e \in E, v \in e} \gamma(e) \geq 1\}$. Intuitively, γ assigns weights to the edges and a vertex v is covered if the total weight of the edges containing v is at least 1.

A *fractional edge cover* of H is a function γ with $V \subseteq B(\gamma)$. An *integral edge cover* is obtained by restricting the function values of γ to $\{0, 1\}$. The *support* of γ is defined as $support(\gamma) = \{e \in E \mid \gamma(e) \neq 0\}$. The *weight* of γ is defined as $weight(\gamma) = \sum_{e \in E} \gamma(e)$. The minimum weight of a fractional (resp. integral) edge cover of a hypergraph H is referred to as the fractional (resp. integral) *edge cover number* of H.

The following example adapted from [6] illustrates which complications may arise if we move from the integral to the fractional case.

Example 1. Consider the family $(H_n)_{n \geq 2}$ of hypergraphs with $H_n = (V_n, E_n)$ defined as

$V_n = \{v_0, v_1, \ldots, v_n\}$
$E_n = \{e_0, e_1, \ldots, e_n\}$ with $e_0 = \{v_1, \ldots, v_n\}$ and $e_i = \{v_0, v_i\}$ for $i \in \{1, \ldots, n\}$.

The integral edge cover number of each H_n is 2 and an optimal integral edge cover can be obtained, e.g., by setting $\gamma_n(e_0) = \gamma_n(e_1) = 1$ and $\gamma_n(e) = 0$ for all other edges. In contrast, the fractional edge cover number is $2 - \frac{1}{n}$ and the unique optimal fractional edge cover is γ'_n with $\gamma'_n(e_0) = 1 - \frac{1}{n}$ and $\gamma'_n(e_i) = \frac{1}{n}$ for each $i \in \{1, \ldots, n\}$. For the support of these two covers, we have $|support(\gamma_n)| = 2$ and $|support(\gamma'_n)| = n + 1$. Hence, the support of the optimal edge covers is bounded in the integral case but unbounded in the fractional case. $\diamond$

As mentioned above, fractional hypertree width (*fhw*) is to date the most general width-notion that allows one to define tractable fragments of classical NP-complete problems, such as CSP solving and CQ answering. However, recognizing if a given hypergraph H has $fhw(H) \leq k$ for fixed $k \geq 2$ is itself an NP-complete problem [6], i.e., in the terminology of parameterised complexity, the problem is paraNP-hard. It has recently been shown that the problem of checking $fhw(H) \leq k$ becomes tractable if we can efficiently enumerate the fractional edge covers of weight $\leq k$ [9]. This fact can be exploited to show that, for classes of hypergraphs with bounded rank (i.e., max. size of edges), bounded degree (i.e., max. number of edges containing a particular vertex), or bounded intersection (i.e., max. number of vertices in the intersection of two edges), checking $fhw(H) \leq k$ becomes tractable. The size of the support has been recently [9] identified as a crucial parameter for the efficient enumeration of fractional edge covers of weight $\leq k$ for given $k \geq 1$.

The *overarching goal* of this work is to further extend and provide a uniform view of previously known structural properties of hypergraphs that guarantee a bound on the size of the support of fractional edge covers of a given weight. In particular, when looking at Example 1, we want to avoid the situation that the support of fractional edge covers with constantly bounded weight increases with the size of the hypergraph. Our *main combinatorial result* (Theorem 4) will be that the size of the support of a fractional edge cover does not depend on the number of vertices or edges of a hypergraph but instead only on the weight of the cover as well as the structure of its edge intersections.

Formally, the structure of the edge intersections is captured by the so-called *Bounded Multi-Intersection Property (BMIP)* [6]: a class $\mathcal{C}$ of hypergraphs has this property, if in every hypergraph $H \in \mathcal{C}$, the intersection of c edges of H has at most d elements, for constants $c \geq 2$ and $d \geq 0$. The BMIP thus generalizes all of the above mentioned hypergraph properties that ensure bounded support of fractional edge covers of given weight and, hence, also guarantee tractability of checking $fhw(H) \leq k$, namely bounded rank, bounded degree, and bounded intersection. Moreover, when considering the incidence graph G of H, the BMIP corresponds to G not containing large complete bipartite graphs. A notable result in the area of parameterized complexity [13] is the polynomial kernelizability of the Dominating Set Problem for graphs without $K_{c,d}$, i.e., without the complete bipartite graph on c and d edges. A minor tweaking of the results yields a polynomial kernelization for the Set Cover Problem if the corresponding incidence graph does not contain $K_{c,d}$. Our result thus reveals an interesting connection: it shows that a condition that enables efficient solving of the Set Cover problem also enables efficient checking of bounded fractional hypertree width.

In summary, the main results of this paper are as follows:

- First, we show that the size of the support of a fractional edge cover only depends on the weight of the cover and of the structure of its edge intersections (Theorem 4). More specifically, if the intersection of c edges of a hypergraph H has at most d elements, and H has a fractional edge cover of weight $\leq k$, then H also has a fractional edge cover of weight $\leq k$ with a support whose size only depends on c, d, and k.
- As an important consequence of this result, we show that the problem of checking if a given hypergraph H has $\mathit{fhw}(H) \leq k$ is tractable for hypergraph classes satisfying the BMIP (Theorem 30). In particular, BMIP generalizes all previously known hypergraph classes with tractable *fhw*-checking, namely bounded rank, bounded degree, and bounded intersection.
- We transfer our results on fractional edge covers to fractional hitting sets, where we again vastly generalize previously known hypergraph classes (such as hypergraphs of bounded rank [7]) that guarantee a bound on the size of the support of fractional hitting sets (Theorem 34).

The paper is organized as follows: after recalling some basic notions and results in Section 2, we will present our main technical result on fractional edge covers in Section 3. The detailed proof of a crucial lemma is separated in Section 3.2. In Section 4, we apply our result on the bounded support of fractional edge covers to fractional hypertree width and extend our main combinatorial result to fractional hitting sets. Finally, in Section 5, we summarize our results and give an extensive overview of interesting open questions in this area of research.

2. Preliminaries

Some general notation. It is convenient to use the following short-hand notation for various kinds of sets: we write $[n]$ for the set $\{1, \dots, n\}$ of natural numbers. Let S be a set of sets. Then we write $\bigcap S$ and $\bigcup S$ for the intersection and union, respectively, of the sets in S, i.e., $\bigcap S = \{x \mid x \in s \text{ for all } s \in S\}$ and $\bigcup S = \{x \mid x \in s \text{ for some } s \in S\}$.

Hypergraphs. We recall some basic notions on hypergraphs. We have already introduced in Section 1 hypergraphs as pairs (V, E) consisting of a set V of vertices and a set E of edges. W.l.o.g., we assume throughout this paper that a hypergraph contains no isolated vertices (i.e., vertices that do not occur in any edge), no pair of vertices that are incident to the exact same set of edges, and no empty edges. Such hypergraphs are typically referred to as *reduced* hypergraphs. Given a hypergraph $H = (V, E)$, the *dual hypergraph* $H^d = (W, F)$ is defined as $W = E$ and $F = \{\{e \in E \mid v \in e\} \mid v \in V\}$.

The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite graph (W, F) with $W = V \cup E$, such that, for every $v \in V$ and $e \in E$, there is an edge $\{v, e\}$ in F iff $v \in e$. Note that a hypergraph H and its dual hypergraph H^d have the same incidence graph.

In this work, we are particularly interested in the structure of the edge intersections of a hypergraph. To this end, recall the notion of (c, d)-hypergraphs for integers $c \geq 2$ and $d \geq 0$ from [9]: $H = (V, E)$ is a (c, d)-hypergraph if the intersection of any c distinct edges in E has at most d elements, i.e., for every subset $E' \subseteq E$ with $|E'| = c$, we have $|\bigcap E'| \leq d$. A class $\mathcal{C}$ of hypergraphs is said to satisfy the *bounded multi-intersection property (BMIP)* [6], if there exist $c \geq 2$ and $d \geq 0$, such that every H in $\mathcal{C}$ is a (c, d)-hypergraph. As a special case studied in [5,6], a class $\mathcal{C}$ of hypergraphs is said to satisfy the *bounded intersection property (BIP)*, if there exists $d \geq 0$, such that every H in $\mathcal{C}$ is a $(2, d)$-hypergraph. Hypergraphs with degree bounded by some constant $c \geq 1$ are $(c + 1, 0)$-hypergraphs. Moreover, bounded rank is clearly a special case of bounded intersection, that is, if the size of each hyperedge is bounded a constant d, also the intersection of any two hyperedges is of course bounded by d.

We now recall tree decompositions, which form the basis of various notions of width. A tuple $(T, (B_u)_{u \in T})$ is a *tree decomposition (TD)* of a hypergraph $H = (V, E)$, if T is a tree, every B_u is a subset of V and the following two conditions are satisfied:

(1) For every edge $e \in E$, there is a node u in T, such that $e \subseteq B_u$, and
(2) for every vertex $v \in V$, $\{u \in T \mid v \in B_u\}$ is connected in T.

Note that, by slight abuse of notation, we write $u \in T$ to express that u is a node in T.

For a function $f : 2^V \to \mathbb{R}^+$, the f*-width* of a TD $(T, (B_u)_{u \in T})$ is defined as $\sup\{f(B_u) \mid u \in T\}$ and the f-width of a hypergraph is the minimal f-width over all its TDs.

An edge weight function is a function $\gamma : E \to \mathbb{R}^+$. We call γ a *fractional edge cover* of a set $X \subseteq V$ by edges in E, if for every $v \in X$, we have $\sum_{\{e \mid v \in e\}} \gamma(e) \geq 1$. The weight of a fractional edge cover is defined as $\mathit{weight}(\gamma) = \sum_{e \in E} \gamma(e)$. For $X \subseteq V$, we write $\rho^*_H(X)$ to denote the minimal weight over all fractional edge covers of X. With respect to some edge weight function γ, we will say that the *weight of a vertex* is the sum of all the weights on edges that contain v. The *fractional hypertree width (fhw)* of a hypergraph H, denoted $\mathit{fhw}(H)$, is then defined as the f-width for $f = \rho^*_H$. Likewise, the *fhw* of a TD of H is its ρ^*_H-width.

The following technical lemma for weight-functions in (c, d)-hypergraphs will be important.

Lemma 2. *There is a function $f(c,d,k)$ with the following property: let H be a (c,d)-hypergraph and let γ be an edge weight function with weight$(\gamma) \leq k$. Moreover, let $0 < \epsilon \leq 1$ be dependent on c, d, k and assume that, for each $e \in E$, $\gamma(e) \leq \frac{\epsilon}{2c}$. Let $B^\epsilon(\gamma)$ be the set of all vertices of weight at least ϵ. Then $|B^\epsilon(\gamma)| \leq f(c,d,k)$ holds.*

The intuition of this lemma is as follows: suppose that we put rather little weight on each edge (namely $\leq \frac{\epsilon}{2c}$). Then, for a vertex v to be in $B^\epsilon(\gamma)$ (that is, to receive total weight $\geq \epsilon$ from all the edges containing v), v must be in the intersection of quite a big number of edges (namely $\geq 2c$ such edges). However, in a (c,d)-hypergraph, the intersection of c or more edges contains at most d vertices. So, intuitively, the "contribution" of each edge to covering a particular vertex is limited. Hence, a weight function γ with *weight*$(\gamma) \leq k$ can put the desired weight $\geq \epsilon$ only to a limited number of vertices, where this limit depends on c, d, and k.

Proof of Lemma 2. The proof is based on the following claim (which is Lemma 7.2 in [9]).

Claim A. Fix an integer $c \geq 1$. Let $X = (x_1, \ldots, x_n)$ be a sequence of positive numbers $\leq \delta$ and fix w such that $\sum_{j=1}^n x_j \geq w \geq \delta c$. Then we have $\sum x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_c} \geq (w - \delta c)^c$, where the sum is over all c-tuples $(i_1, \ldots, i_c)$ of *distinct* integers from $[n]$.

We proceed with a counting argument. Imagine a bipartite graph $G = (B^\epsilon(\gamma), T, E(G))$ where T is the set of all c-tuples of distinct edges from H. In G, there is an edge from $v \in B^\epsilon(\gamma)$ to $(e_1, \ldots, e_c) \in T$ iff v is in $e_1 \cap \cdots \cap e_c$. Furthermore, we assign weight $\prod_{j=1}^c \gamma(e_j)$ to every edge in $E(G)$ incident to a tuple $(e_1, \ldots, e_c) \in T$. To avoid confusion, in this proof, we write $E(G)$ and $E(H)$ to refer to the set of edges in the graph G and in the hypergraph H, respectively.

We now count the total weight in G from both sides. First observe that on the T side, we have degree at most d because H is a (c,d)-hypergraph. Therefore, the total weight in G is at most $d \cdot \sum_{(e_1,\ldots,e_c)\in T} \prod_{j=1}^c \gamma(e_j)$. Observe that $\sum_{(e_1,\ldots,e_c)\in T} \prod_{j=1}^c \gamma(e_j) \leq \left(\sum_{e_1 \in E(H)} \gamma(e_1)\right) \cdot \cdots \cdot \left(\sum_{e_c \in E(H)} \gamma(e_c)\right)$ as, by distributivity, all the terms of the sum on the left-hand side are also present on the right-hand side of the inequality. Furthermore, we have $\sum_{e \in E(H)} \gamma(e) \leq k$ and thus, by putting it all together, we see that the total weight in G is at most $k^c d$.

From the $B^\epsilon(\gamma)$ side, consider an arbitrary vertex $v \in B^\epsilon(\gamma)$ and let $e_1, \ldots, e_n$ be the edges in $E(H)$ containing v with nonzero weight. We have $\sum_{j=1}^n \gamma(e_j) \geq \epsilon$ and $\gamma(e_j) \leq \frac{\epsilon}{2c}$ for each $j \in [n]$. We can apply the above claim for $X = \{\gamma(e_1), \ldots, \gamma(e_n)\}$, $\delta = \frac{\epsilon}{2c}$, and $w = \epsilon$ to get the inequality $\sum \gamma(e_{j_1}) \cdot \cdots \cdot \gamma(e_{j_c}) \geq (\epsilon - \frac{\epsilon}{2c} \cdot c)^c = (\frac{\epsilon}{2})^c$, where the sum ranges over all c-tuples $(e_{j_1}, \ldots, e_{j_c})$ of distinct edges in $E(H)$ containing v.

We conclude that v (now considered as a vertex in G) is incident to edges whose total weight is $\geq (\frac{\epsilon}{2})^c$ in $E(G)$. Since we have seen above that the total weight of all edges in $E(G)$ is $\leq k^c d$, there can be no more than $d(\frac{2k}{\epsilon})^c$ vertices in $B^\epsilon(\gamma)$. □

Linear programs. We assume some familiarity with Linear Programs (LPs). Formally, we are dealing here with minimization problems of the form $\min \mathbf{c}^T \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{x}$ is a vector of n variables, $\mathbf{c}$ is a vector of n constants, $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ is a vector of m constants, and $\mathbf{0}$ stands for the n-dimensional zero-vector. More specifically, for a hypergraph $H = (V, E)$ and vertices $Y \subseteq V$, the fractional edge cover number $\rho_H^*(Y)$ of Y is obtained as the optimal value of the following LP: let $E = \{e_1, \ldots, e_n\}$ (note that in a slight departure from typical naming, n here is the number of edges in the hypergraph) and $Y = \{y_1, \ldots, y_m\}$, then c is the n-dimensional vector $(1, \ldots, 1)$, b is the m-dimensional vector $(1, \ldots, 1)$, and $\mathbf{A} \in \{0,1\}^{[m]\times[n]}$, such that $A_{ij} = 1$ if $y_i \in e_j$ and $A_{ij} = 0$ otherwise. In the sequel, we will refer to such LPs with $\mathbf{c} \in \{1\}^n$, $\mathbf{b} \in \{1\}^m$ and $\mathbf{A} \in \{0,1\}^{[m]\times[n]}$ as *unary* linear programs.

For given number n of edges, there are at most 2^n possible different inequalities of the form $\mathbf{A_i x} \geq 1$. We thus get the following property of unary LPs, which intuitively states that if the optimum is bigger than some threshold k, then it exceeds k by some distance.

Lemma 3. *For all positive integers n and k, there is an integer $D(n,k)$ such that for any unary LP Z of at most n variables if $OPT(Z) > k$ then $OPT(Z) - k > \frac{1}{D(n,k)}$, where $OPT(Z)$ denotes the minimum of the LP.*

3. Bounding the support of fractional edge covers

3.1. The main combinatorial result

In this section we establish our main combinatorial result, Theorem 4. Every set of vertices in a (c,d)-hypergraph can be covered in a way such that the size of the support depends only on c, d, and the set's fractional edge cover number. Recall that we denote by $B(\gamma)$ the set of all vertices v such that $\gamma(v) \geq 1$ where γ is a weight function on edges and the weight of a vertex is the sum of all incident edge weights. For sets S of hyperedges, it will also be convenient to write $\gamma(S)$ for $\sum_{e \in S} \gamma(e)$.

Theorem 4. *There is a function $h(c, d, k)$ such that the following is true. Let $H = (V, E)$ be a (c, d)-hypergraph and let γ be an assignment of weights to E. Let $k \in \mathbb{Q}^+$ such that $weight(\gamma) \le k$. Then there exists an assignment ν of weights to E such that $weight(\nu) \le k$, $B(\gamma) \subseteq B(\nu)$ and $|support(\nu)| \le h(c, d, k)$.*

The first step of our reasoning is to consider the situation where $|B(\gamma)|$ is bounded. In this case it is easy to transform γ into the desired ν. Partition all the hyperedges of H into equivalence classes corresponding to non-empty subsets of $B(\gamma)$ such that two edges e_1 and e_2 are equivalent if and only if $e_1 \cap B(\gamma) = e_2 \cap B(\gamma)$. Then let s_X be the total weight (under γ) of all the edges from the equivalence class where $e \cap B(\gamma) = X$. Identify one representative of each (non-empty) equivalence class and let e_X be the representative of the equivalence class corresponding to X. Then define ν as follows. For each X corresponding to a non-empty equivalence class, set $\nu(e_X) = s_X$. For each edge e whose weight has not been assigned in this way, set $\nu(e) = 0$. It is clear that $B(\gamma) \subseteq B(\nu)$ and that the support of ν is at most $2^{|B(\gamma)|}$, which is bounded by assumption.

Of course, in general we cannot assume that $|B(\gamma)|$ is bounded. Therefore, as the next step of our reasoning, we consider a more general situation where we have a bounded set $\mathbf{S} = \{S_1, \dots, S_r\}$ where each S_i is a set of at most c hyperedges such that the following conditions hold regarding $\mathbf{S}$:

(i) for each $1 \le i \le r$, $\gamma(S_i) \ge 1$, and
(ii) the set $U = B(\gamma) \setminus \bigcup_{i\in[r]} \bigcap S_i$ is of bounded size.

Then the assignment ν as in Theorem 4 can be defined as follows. For each $e \in \bigcup \mathbf{S}$, set $\nu(e) = \gamma(e)$. Next, we observe that for the subhypergraph $H' = H - \bigcup \mathbf{S}$, $|B_{H'}(\gamma)|$ is bounded, where the subscript H' means that we consider B for hypergraph H'. Therefore, we define ν on the remaining edges as in the paragraph above. It is not hard to see that the support of the resulting ν is of size at most $c \cdot r + 2^{|U|}$. We are going to show that such a family of sets of edges can always be found for (c, d)-hypergraphs (after a possible modification of γ).

Definition 5 *(Well-formed pair).* Let $H = (V, E)$ be a hypergraph and let γ be an edge weight function. We say $(\mathbf{S}, U)$ is a *well-formed pair* (with regard to γ) if it satisfies the following conditions:

1. $U \subseteq B(\gamma)$
2. $\mathbf{S} = \{S_1, \dots, S_r\}$ where each S_i is a set of at most c hyperedges of H.
3. $B(\gamma) \setminus U \subseteq \bigcup_{i\in[r]} \bigcap S_i$.

We denote $\sum_{i\in[r]} |S_i| + 2^{|U|}$ by $n(\mathbf{S}, U)$ and refer to it as the *size* of $(\mathbf{S}, U)$.

Definition 6. A well-formed pair $(\mathbf{S}, U)$ is *perfect* if there is an assignment $\nu : E \to [0, 1]$ with $weight(\nu) \le k$ and $|support(\nu)| \le n(\mathbf{S}, U)$ such that $\bigcup_{i\in[r]} \bigcap S_i \cup U \subseteq B(\nu)$.

Our aim now is to prove the existence of a perfect pair $(\mathbf{S}, U)$ of size bounded by a function depending on c, d, and k. Clearly, this will imply Theorem 4.

In particular, we will define the *initial pair* which is a well-formed pair but not necessarily perfect. Then we will define two transformations from one well-formed pair into another and prove existence of a function *transf* so that if $(\mathbf{S_1}, U_1)$ is transformed into $(\mathbf{S_2}, U_2)$, then $n(\mathbf{S_2}, U_2) \le transf(n(\mathbf{S_1}, U_1))$. We will then prove that if we form a sequence of well-formed pairs starting from the initial pair and obtain every next element by a transformation of the last one then, after a bounded number of steps we obtain a perfect well-formed pair. We start by defining the initial pair.

Definition 7. The *initial pair* is $(\mathbf{S_0}, U_0)$ where $\mathbf{S_0} = \{\{e\} \mid \gamma(e) \ge 1/(2c)\}$ and $U_0 = B(\gamma) \setminus \bigcup_{\{e\}\in \mathbf{S_0}} e$.

Lemma 8. *There is a function init such that $n(\mathbf{S_0}, U_0) \le init(c, d, k)$*

Proof. We can bound $|U_0|$ by applying Lemma 2 to all edges with weight less than $1/(2c)$, i.e., to the subhypergraph without the edges in $\mathbf{S_0}$. In particular, $|U_0| \le f(c, d, k)$ where f is as in Lemma 2 (for $\epsilon = 1$) and $|\bigcup \mathbf{S_0}| \le 2ck$ by construction. □

We now introduce our two kinds of transformations, *folding* and *extension*. A folding removes a set S^* of c edges from $\mathbf{S}$ and adds to U the vertices in the intersection of the edges of S^*. In the resulting well-ordered pair $(\mathbf{S}', U')$, $\mathbf{S}'$ has one less element than $\mathbf{S}$ and U', compared to U, has a bounded size increase of at most d vertices. Thus the action of folding gets the resulting well-formed pair closer to one with empty first component, which is a perfect pair according to the discussion in the beginning of this section.

Definition 9. Let $(\mathbf{S}, U)$ be a well-formed pair such that $\mathbf{S}$ contains elements of size c. Let $S^* \in \mathbf{S}$ such that $|S^*| = c$. Let $\mathbf{S}' = \mathbf{S} \setminus \{S^*\}$ and $U' = U \cup (\bigcap S^* \cap B(\gamma))$. We call $(\mathbf{S}', U')$ a *folding* of $(\mathbf{S}, U)$.

The folding, however, is possible only if $\mathbf{S}$ has an element of size c. Otherwise, we need a more complicated transformation called an *extension*. The extension takes an element $S \in \mathbf{S}$ of size $c' < c$ and expands it by replacing S with several subsets of $E(H)$ each containing all the edges of S plus one extra edge. This replacement may miss some of the elements v of $B(\gamma) \cap \bigcap S$ simply because v is not contained in any of these extra edges. This excess of missed elements is added to U and thus all the conditions of a well-formed pair are satisfied.

Definition 10. Let $(\mathbf{S}, U)$ be a well-formed pair with $\mathbf{S} \neq \emptyset$ such that every element of $\mathbf{S}$ is of size at most $c - 1$. For the extension, let $S \in \mathbf{S}$ be an element called the *extended element* and let a set S' of hyperedges be called the *extending set*. We refer to $L = (\bigcap S \cap B(\gamma)) \setminus \bigcup S'$ as the set of *light vertices*. An *extension* of $(\mathbf{S}, U)$ is $(\mathbf{S}', U')$ where $\mathbf{S}' = (\mathbf{S} \setminus \{S\}) \cup \{S \cup \{e\} | e \in S'\}$ and $U' = U \cup L$.

Proposition 11. *With data as in Definition 10, $(\mathbf{S}', U')$ is a well-formed pair.*

At the first glance the transformation performed by the extension is radically opposite to the one done by the folding: the first component grows rather than shrinks. Note, however, that the new sets replacing the removed one contain a larger number of edges and thus they are closer to being of size c at which stage the folding can be applied to them. The intuition is that after a sufficiently large number of foldings and extensions, a well-formed pair with empty first component is eventually obtained.

For our overall goal, we then need to show that the size of the resulting perfect pair is indeed bounded by a function of c, d, and k. To that end, the following lemma first establishes that a single step in this process increases the size of the well-formed pair in a controlled manner. To streamline our path to the main result, the proof of the lemma is deferred to Section 3.2.

Lemma 12. *There is a function ext such that the following holds. Let $(\mathbf{S}, U)$ be a well-formed pair with $\mathbf{S} \neq \emptyset$ such that every of element of $\mathbf{S}$ is of size at most $c - 1$. Then one of the following two statements is true.*

1. *$(\mathbf{S}, U)$ is a perfect pair.*
2. *There is an extension $(\mathbf{S}', U')$ of $(\mathbf{S}, U)$ such that $n(\mathbf{S}', U') \leq ext(n(\mathbf{S}, U))$. We refer to $(\mathbf{S}', U')$ as a* bounded extension *of $(\mathbf{S}, U)$*

For the sake of syntactical convenience, we unify the notions of folding and bounded extension into a single notion of transformation and prove the related statement following from Lemma 12 and the definition of folding.

Definition 13. Let $(\mathbf{S}, U)$ and $(\mathbf{S}', U')$ be well-formed pairs. We say that $(\mathbf{S}', U')$ is a *transformation* of $(\mathbf{S}, U)$ if it is either a folding or a bounded extension of $(\mathbf{S}, U)$.

Lemma 14. *There is a monotone function transf with $transf(x) \geq x$ for any natural number x such that the following holds. If $(\mathbf{S}, U)$ is a well-formed pair, then one of the following two statements is true.*

1. *$(\mathbf{S}, U)$ is a perfect pair.*
2. *There exists a transformation $(\mathbf{S}', U')$ of $(\mathbf{S}, U)$ such that*
   $n(\mathbf{S}', U') \leq transf(n(\mathbf{S}, U))$.

Proof. Assume that $(\mathbf{S}, U)$ is not a perfect pair. Then $|\mathbf{S}|$ is not empty (see the discussion at the beginning of this section). Suppose that an element of $\mathbf{S}$ is of size c. Then we set $(\mathbf{S}', U')$ to be a folding of $(\mathbf{S}, U)$. By definition of the folding and of (c, d)-hypergraphs, $(\mathbf{S}', U')$ is obtained from $(\mathbf{S}, U)$ by removal of an element from $\mathbf{S}$ and adding at most d vertices to U. Hence the size of $(\mathbf{S}', U')$ is clearly bounded in the size of $(\mathbf{S}, U)$. If all elements of $\mathbf{S}$ are of size at most $c - 1$ then by Lemma 12, there is a bounded extension $(\mathbf{S}', U')$ of $(\mathbf{S}, U)$.

Clearly, we can specify a function $transf'$ so that in both cases $n(\mathbf{S}, U) \leq transf'(n(\mathbf{S}, U))$. In particular, to satisfy the requirement for $transf$, it suffices to set $transf(x) = \max(x, \max_{i \in [x]} transf'(i))$ for each natural number x. □

Now that we know that each individual step on our path to a perfect pair increases the size only in a bounded fashion, we need to establish that the number of steps is also bounded by a function of c, d, and k. The following auxiliary theorem states that such a bound exists.

Definition 15. A sequence of $(\mathbf{S_1}, U_1), \ldots, (\mathbf{S_q}, U_q)$ is a *sequence of transformations* if for each $i \in [q - 1]$ the following two statements hold

1. $(\mathbf{S_i}, U_i)$ is not a perfect pair.
2. $(\mathbf{S_{i+1}}, U_{i+1})$ is a transformation of $(\mathbf{S_i}, U_i)$ as in Lemma 14.

Theorem 16. *There is a monotone function sl such that the following is true. Let $(\mathbf{S_1}, U_1), \dots, (\mathbf{S_q}, U_q)$ be a sequence of transformations. Then*

$$q \le sl(n(\mathbf{S_1}, U_1)).$$

The proof of Theorem 16 is provided in Section 3.3.

In summary, we have shown that we can reach a perfect pair in a bounded number of transformations. Moreover, each transformation increases the size of a pair in a controlled manner. We are now ready to prove our main result.

Proof of Theorem 4. Consider the following algorithm.

1. Let $(\mathbf{S_0}, U_0)$ be the initial pair (see Definition 7).
2. $q \leftarrow 0$
3. While $(\mathbf{S_q}, U_q)$ is not a perfect pair
   (a) $q \leftarrow q+1$
   (b) Let $(\mathbf{S_q}, U_q)$ be a transformation of $(\mathbf{S_{q-1}}, U_{q-1})$, which exists by Lemma 14

By Theorem 16, the above algorithm stops with the final q being no higher than $sl(n(\mathbf{S_1}, U_1))$. It follows from the description of the algorithm that $(\mathbf{S_q}, U_q)$ is a perfect pair. It remains to show that its size is bounded by a function of c, d, k.

$$q \le sl(n(\mathbf{S_0}, U_0)) \le sl(init(c, d, k)) \tag{1}$$

the second inequality follows from Lemma 8 and the monotonicity of *sl*. Next, by the properties of *transf*, an inductive application of Lemma 14 and Lemma 8 yields

$$n(\mathbf{S_q}, U_q) \le transf^q(init(c, d, k)) \tag{2}$$

where superscript q means that function *transf* is composed with itself q-times, that is $transf(transf(transf(\dots)))$.

Let $h(c, d, k) = transf^{sl(init(c,d,k))}(init(c, d, k))$. It follows from combination of (1) and (2) that $n(\mathbf{S_q}, U_q) \le h(c, d, k)$. □

3.2. Proof of Lemma 12

The first step of the proof is to define a unary linear program of bounded size associated with $(\mathbf{S}, U)$. Then we will demonstrate that if the optimal value of this linear program is at most k, then $(\mathbf{S}, U)$ is perfect. Otherwise, we show that a bounded extension can be constructed.

In order to define the linear program, we first formally define equivalence classes of edges covering U (see the informal discussion in Section 3.1).

Definition 17 *(Working subset).* A set of vertices $U' \subseteq U$ is called *working subset* (for $(\mathbf{S}, U)$) if there is $e \in E(H) \setminus \bigcup \mathbf{S}$ such that $e \cap U = U'$. This e is called a *witnessing edge* of U' and the set of all witnessing edges of U' is denoted by $W_{U'}$.

Continuing on the previous definition, it is not hard to see that the sets $W_{U'}$ partition the set of edges of $E(H) \setminus \bigcup \mathbf{S}$ having a non-empty intersection with U. Choose an arbitrary but fixed representative of each $W_{U'}$ and let A_U be the set of these representatives which we also refer to as the set of *witnessing representatives*. Now, we are ready to define the linear program.

Definition 18 *($LP(\mathbf{S}, U)$).* The linear program $LP(\mathbf{S}, U)$ of $(\mathbf{S}, U)$ has the set of variables $X = \{x_e \mid e \in \bigcup \mathbf{S} \cup A_U\}$. The objective function is the minimization of $\sum_{x_e \in X} x_e$. The constraints are of the following three kinds.

1. $\{0 \le x_e \le 1 \mid x_e \in X\}$.
2. $\{One_S \mid S \in \mathbf{S}\}$ where One_S is $\sum_{e \in S} x_e \ge 1$.
3. $\{One_u \mid u \in U\}$ where One_u is $\sum_{e \in E_u} x_e \ge 1$ where E_u in turn is the subset of $\bigcup \mathbf{S} \cup A_U$ consisting of all the edges containing u.

Lemma 19. *Assume that the optimal solution of $LP(S, U)$ is at most k. Then $(\mathbf{S}, U)$ is a perfect pair.*

Proof. Each variable x_e of $LP(\mathbf{S}, U)$ corresponds to an edge e and this correspondence is injective. For each x_e, let $\nu(e)$ be the value of x_e in the optimal solution. For each edge e not having a corresponding variable, set $\nu(e) = 0$. Note that ν is an edge weight function with total weight at most k. It follows from a direct inspection that $U \cup \bigcup_{i\in[r]} \bigcap \mathbf{S_i} \subseteq B(\gamma)$ and the size of the support of ν is at most $n(\mathbf{S}, U)$. □

The proof of Lemma 19 establishes a correspondence between solutions of $LP(S, U)$ and edge weight functions. We will implicitly extend notions for edge weight functions (like their weight) to solutions of $LP(S, U)$ via this correspondence for the rest of this section.

As stated above, in case the optimal value of $LP(\mathbf{S}, U)$ is greater than k, we are going to demonstrate the existence of a bounded extension of $(\mathbf{S}, U)$. The first step towards identifying such an extension is to identify the extended element of $\mathbf{S}$. Combining Lemma 3 from Section 2 with Lemma 20 below, we observe that $\mathbf{S}$ has an element S^* such that $\gamma(S^*)$ is bounded away from 1. This S^* will be the extended element.

Lemma 20. *Let $(\mathbf{S}, U)$ be a well-formed pair. Let $\mathbf{S}^*$ be the subset of $\mathbf{S}$ consisting of all S such that $\gamma(S) < 1$. Let α be an optimal solution for $LP(\mathbf{S}, U)$. Then $weight(\alpha) \leq weight(\gamma) + \sum_{S\in\mathbf{S}^*}(1 - \gamma(S))$.*

Proof. Let β be an arbitrary assignment of weights to the hyperedges of H. We say that β satisfies a constraint One_S for $S \in \mathbf{S}$ if $\beta(S) \geq 1$ and that β satisfies the constraint One_u for $u \in U$ if $\beta(E_u) \geq 1$.

We are going to demonstrate an assignment of weights whose total weight exceeds that of γ by at most $\sum_{S\in\mathbf{S}^*}(1-\gamma(S))$ and that satisfies all the constraints One_S and One_v. Clearly, this will imply correctness of this theorem.

For each $S \in \mathbf{S}^*$, choose an arbitrary edge $e_S \in S$ and let $INCR$ be the set of all such edges. For each $e \in INCR$, let $incr_e = \max\{1 - \gamma(S) \mid e = e_S\}$. That is, $e \in INCR$ can be the representative of multiple $S \in \mathbf{S}^*$ and $incr_e$ represents the maximal $1 - \gamma(S)$ over all the sets S for which e is the representative e_S.

Let γ' be obtained from γ as follows. If $e \in INCR$ then $\gamma'(e) = \gamma(e) + incr_e$. Otherwise, $\gamma'(e) = \gamma(e)$. It is not hard to see that γ' satisfies the constraints One_S for each $S \in \mathbf{S}$, and that $weight(\gamma') \leq weight(\gamma) + \sum_{S\in\mathbf{S}^*}(1 - \gamma(S))$. Since γ' does not decrease the weight of any edge, we also observe $U \subseteq B(\gamma')$.

Let $\{U_1, \ldots, U_a\}$ be all the working subsets of U and let $e_1, \ldots, e_a$ be the respective witnessing representatives. The set of edges $e_1, \ldots, e_a$ corresponds to the set A_U from Definition 18. Then the assignment γ'' of weights is defined as follows.

1. If there is an $i \in [a]$ such that $e \in W_{U_i}$ and $e = e_i$, then $\gamma''(e) = \gamma'(W_{U_i}) = \gamma(W_{U_i})$.
2. If $e \in \bigcup \mathbf{S}$, then $\gamma''(e) = \gamma'(e)$.
3. Otherwise, $\gamma''(e) = 0$.

Let $W = \bigcup_{i\in[a]} W_{U_i}$. Note that, by construction, $\gamma'(W) = \gamma''(W)$ and the weights of edges outside W are the same under γ' and γ'' and thus, $weight(\gamma') = weight(\gamma'')$. Moreover since $\bigcup \mathbf{S}$ does not intersect with W, γ'' satisfies the constraints One_S for all $S \in \mathbf{S}$.

It remains to show that γ'' satisfies the constraints One_u for each $u \in U$. Let $e_1, \ldots, e_r$ be the edges of $\bigcup \mathbf{S}$ containing u, let $\{U_1, \ldots, U_b\}$ be the working subsets of U containing u, and let $e'_1, \ldots, e'_b$ be the respective witnessing representatives. As $u \in B(\gamma')$, it follows that $\sum_{i\in[r]} \gamma'(e_i) + \sum_{i\in[b]} \gamma'(W_{U_i}) \geq 1$. By construction, $\gamma''(e_i) = \gamma'(e_i)$ for each $1 \leq i \leq r$ and $\gamma''(e'_i) = \gamma'(W_{U_i})$ for each $1 \leq i \leq b$. Consequently, $\sum_{i\in[r]} \gamma''(e_i) + \sum_{i\in[b]} \gamma''(e'_i) \geq 1$. We conclude that γ'' satisfies One_u. □

Recall that Lemma 3 states that for integers n and k, and unary LP Z with at most n variables and $OPT(Z) > k$, there is an integer $D(n, k)$ such that $OPT(Z) - k > \frac{1}{D(n,k)}$. Together with Lemma 20 this implies the following corollary.

Corollary 21. *Let $(\mathbf{S}, U)$ be a well-formed pair. Assume that $weight(\gamma) \leq k$ while $OPT(LP(\mathbf{S}, U)) > k$. Let $n = n(\mathbf{S}, U)$. Then there is an $S^* \in \mathbf{S}$ with $1 - \gamma(S^*) > \frac{1}{D(n,k)\cdot|\mathbf{S}|}$. In particular this means that $\mathbf{S}^*$ is not empty where $\mathbf{S}^*$ is as in Lemma 20.*

Proof. Note that the number of variables of $LP(\mathbf{S}, U)$ is at most n. It follows from the combination of Lemma 3 and Lemma 20 that $weight(\gamma) + \sum_{S\in\mathbf{S}^*}(1 - \gamma(S)) > k + 1/D(n, k)$ and, since $weight(\gamma) \leq k$, $\sum_{S\in\mathbf{S}^*}(1 - \gamma(S)) > 1/D(n, k)$ and hence there is $S^* \in \mathbf{S}^*$ with $(1 - \gamma(S^*)) > \frac{1}{D(n,k)\cdot|\mathbf{S}^*|} \geq \frac{1}{D(n,k)\cdot|\mathbf{S}|}$. □

We are now ready to prove Lemma 12. Let us first recall the lemma and note that the first component of the well-formed pair in the statement is non-empty and its elements contain at most $c - 1$ edges.

Lemma 12. *There is a function ext such that the following holds. Let $(\mathbf{S}, U)$ be a well-formed pair with $\mathbf{S} \neq \emptyset$ such that every of element of $\mathbf{S}$ is of size at most $c - 1$. Then one of the following two statements is true.*

1. *$(\mathbf{S}, U)$ is a perfect pair.*
2. *There is an extension $(\mathbf{S}', U')$ of $(\mathbf{S}, U)$ such that $n(\mathbf{S}', U') \leq ext(n(\mathbf{S}, U))$. We refer to $(\mathbf{S}', U')$ as a* bounded extension *of $(\mathbf{S}, U)$*

Proof of Lemma 12. If the value of the optimal solution of $LP(\mathbf{S}, U)$ is at most k, we are done by Lemma 19.

Otherwise, let $S^* \in \mathbf{S}$ be as in Corollary 21. Let $\epsilon = (D(n,k) \cdot |\mathbf{S}|)^{-1}$. It follows from Corollary 21 that vertices of $B(\gamma) \cap \bigcap S^*$ need weight contribution of at least ϵ from hyperedges of H other than S^*. We define the extending set S' as the set of all hyperedges of H other than S^* whose weight is at least $\epsilon/2c$ and therefore $|S'| \leq 2ck/\epsilon$. We observe that the set L of light vertices (cf., Definition 10) is the subset of $B(\gamma) \cap \bigcap S^*$ consisting of all vertices x that, besides S^* are contained only in hyperedges of weight smaller than $\epsilon/2c$. By Lemma 2, $|L| \leq f(c,d,k)$ and the size of S^* is at most $c-1$ by assumption. It is not hard to see that the size of the resulting extension is bounded as well. □

3.3. Proof of Theorem 16

For this theorem, rather than considering a well-formed pair $(\mathbf{S}, U)$ itself we consider the pair (A, b) where A is the multiset of sizes of the sets of $\mathbf{S}$ and $|U| = b$. We call (A, b) a *bare bones c-pair* (c-BBP). A transformation of $(\mathbf{S}, U)$ is translated into a bounded size transformation of (A, b). In the next five definitions we formalize this intuition. Then we state Theorem 27 claiming that a sufficiently long sequence of bounded transformations of c-BBPs results in one where the first component is empty. This will imply Theorem 16 because a c-BBP with the empty first components is translated back into well-formed pair with the empty first component which is perfect. Finally, we prove Theorem 27.

Definition 22. A *bare bones c-pair*, abbreviated as c-BBP is a pair (A, b) where A is a multiset of integers in the range $[1, c]$ and b is just a non-negative integer. We denote $2^b + \sum_{x \in A} x$ by $n(A, b)$. Note that the number of occurrences of each $x \in A$ in the sum is its multiplicity in A.

Definition 23. Let (A, b) be a c-BBP and assume that $c \in A$. Let $A' = A \setminus \{c\}$ (that is, the multiplicity of c in A is reduced by one) and let $b' = b + d$ where d is a non-negative integer. Clearly (A', b') is a c-BBP, we refer to it as a *folding* of (A, b).

Definition 24. Let (A, b) be a c-BBP and let $x \in A$ such that $x < c$. Let A' be obtained from A by removal of one occurrence of x and adding d_1 occurrences of $x+1$ for some non-negative integer d_1. Let $b' = b + d_2$ for some non-negative integer d_2. Clearly (A', b') is a c-BBP, we refer to it as an *extension* of (A, b)

Definition 25. Let (A, b) and (A', b') be c-BBPs such that (A', b') is either a folding or an extension of $(A.b)$. We then say that (A', b') is a *transformation* of (A, b). Let $n = n(A, b)$ and $n' = n(A', b')$ and suppose that $n' \leq g(n)$ for some function g. We then say that (A', b') is a *g-transformation* of (A, b).

Definition 26. Let g be a function of one argument and let $(A_1, b_1), \dots, (A_r, b_r)$ be a sequence of c-BBPs such that for each $2 \leq i \leq r$, the c-BPP (A_i, b_i) is a g-transformation of (A_{i-1}, b_{i-1}). We call $(A_1, b_1), \dots, (A_r, b_r)$ a g-transformation sequence. Note that for each $1 \leq i < r$, A_i is not empty for otherwise, it is impossible to apply a transformation to (A_i, b_i).

Theorem 27. *Let g be a function of one argument. Then there is a function $h[g]$ such that if $(A_1, b_1), \dots, (A_r, b_r)$ is a g-transformation sequence then $r \leq h[g](n)$ where $n = n(A_1, b_1)$.*

We first show how to prove Theorem 16 using Theorem 27 and then we will prove Theorem 27 itself. We first recall the theorem.

Theorem 16. *There is a monotone function sl such that the following is true. Let $(\mathbf{S_1}, U_1), \dots, (\mathbf{S_q}, U_q)$ be a sequence of transformations. Then*

$$q \leq sl(n(\mathbf{S_1}, U_1)).$$

Proof of Theorem 16. Let $(\mathbf{S}, U)$ be a well-formed pair and let $bbp(\mathbf{S}, U)$ be (A, b) where A is the multiset of sizes of elements of $\mathbf{S}$ (each x occurs in A exactly the number of times as there are sets of size x in $\mathbf{S}$) and $b = |U|$. It is not hard to see that (A, b) is a c-BBP. Moreover,

$$n(\mathbf{S}, U) = n(A, b) \tag{3}$$

Let $(\mathbf{S_1}, U_1), \dots, (\mathbf{S_r}, U_r)$ be a transformation sequence. Let $(A_1, b_1), \dots, (A_r, b_r)$ be a sequence of c-BBPs such that $(A_i, b_i) = bbp(\mathbf{S_i}, U_i)$ for each $1 \leq i \leq r$.

We are going to show that $(A_1, b_1), \dots, (A_r, b_r)$ is a *transf*-transformation sequence. By Theorem 27, this will imply that $r \leq h[transf](n)$ where $n = n(A_1, b_1) = n(\mathbf{S_1}, U_1)$ by (3) thus implying the theorem.

So, consider two arbitrary consecutive elements (A_i, b_i) and (A_{i+1}, b_{i+1}).

Assume first that $(\mathbf{S_{i+1}}, U_{i+1})$ is obtained from $(\mathbf{S_i}, U_i)$ by folding. It is not hard to see that (A_{i+1}, b_{i+1}) is obtained from (A_i, b_i) by removing one occurrence of c and adding $b_{i+1} = b_i + (|U_{i+1}| - |U_i|)$. That is (A_{i+1}, b_{i+1}) is obtained from

(A_i, b_i) as result of folding. As $n(\mathbf{S_{i+1}}, U_{i+1}) \le \mathit{transf}(n(\mathbf{S_i}, U_i))$, it follows from (3) that $n(A_{i+1}, b_{i+1}) \le \mathit{transf}(n(A_i, b_i))$. We conclude that (A_{i+1}, b_{i+1}) is obtained from (A_i, b_i) as a result of a *transf*-transformation.

Assume now that $(\mathbf{S_{i+1}}, U_{i+1})$ is obtained from $(\mathbf{S_i}, U_i)$ by extension. This means that $\mathbf{S_{i+1}}$ is obtained from $\mathbf{S_i}$ by removal of some S^* of size less than c and replacing it with d_1 sets of size $c+1$ for some integer $d_1 \ge 0$. Also U_{i+1} is obtained from U_i by adding d_2 new elements for some integer $d_2 \ge 0$. It follows by construction that (A_{i+1}, b_{i+1}) is an extension of (A_i, b_i). By the same argumentation as in the end of the previous paragraph, we conclude that (A_{i+1}, b_{i+1}) is obtained from (A_i, b_i) by *transf*-transformation. □

Proof of Theorem 27. We assume w.l.o.g. that g is monotone that is for $n_1 < n_2$ $g(n_1) \le g(n_2)$. Indeed, otherwise, since g is defined over non-negative integer, we can define $g^*(n)$ as the maximum over $g(0), \dots, g(n)$ and use g^* instead of g. The monotonicity allows us to derive the following inequality.

Suppose that $(A_1, b_1), \dots, (A_x, b_x)$ is a g-transformation sequence and $x \le y$. Then

$$n(A_x, b_x) \le g^{(y)}(n(A_1, b_1)) \tag{4}$$

For $i \in \{0, \dots, c-1\}$, a g-transformation is subset of q-transformations with an additional property recursively defined as follows.

1. (A', b') is a $(g, 0)$-transformation of (A, b) if (A', b') is obtained from (A, b) by folding.
2. Suppose $i > 0$ and $(g, i-1)$-transformation has been defined Then (A', b') is a (g, i)-transformation of (A, b) if it is either a $(g, i-1)$-transformation or an extension where the element removed from A is an occurrence of $c - i$.

A (g, i)-transformation sequence $(A_1, b_1), \dots, (A_r, b_r)$ where for each $2 \le j \le r$, (A_j, b_j) is obtained from (A_{j-1}, b_{j-1}) by (g, i)-transformation. The sequence is *final* if all elements of A_r are smaller than $c - i$, that is no further extension of the sequence is possible.

We prove by induction that for each $i \in \{0, \dots, c-1\}$, there is a function $h_i[g]$ such that r as above is at most $h_i[g](n(A_1, b_1))$. Then $h_{c-1}[g]$ will be the desired function $h[g]$. For the sake of simplicity, we will omit g in the square brackets and refer to these functions as $h_0, \dots, h_{c-1}$.

It is important to observe that our induction is from above, in the sense that we with an increase in i we allow for the removal of occurrences of lower values. As a consequence, a (g, i)-transformation of (A, b) can never increase the number of occurrences $c - i$ in A: a folding only removes values from A and an extension can only introduce $c - i$ by removing occurrences of $c - i - 1$, which is not permitted in a (g, i)-transformation. It follows that in any (g, i)-transformation sequence starting at (A_1, b_1), there can be at most m extensions that remove an occurrence of $c - i$, where m is the number of $c - i$'s in A_1. This observation is key to the following argument.

The existence of function h_0 is easy to observe. Indeed, the number of consecutive foldings is at most the multiplicity of c in A_1. So, we can put $h_0 = n(A_1, b_1)$.

Assume now that $i > 0$ and that $(A_1, b_1), \dots, (A_r, b_r)$ is a (g, i)-transformation sequence. If it is in fact a $(g, i-1)$-transformation sequence then $r \le h_{i-1}(A_1, b_1)$ by the induction assumption. Otherwise, let $1 < x_1 < \dots < x_a \le r$ be all the indices such that for each $1 \le j \le a$, (A_{x_j}, b_{x_j}) is obtained from $(A_{x_j - 1}, b_{x_j - 1})$ by extension removing an element $c - i$.

For the sake of succinctness, denote $n(A_1, b_1)$ by n and for each $1 \le j \le a$, we denote $n(A_{x_j}, b_{x_j})$ by n_j.

For each integer $j \ge 1$, define function f_j as follows. $f_1(x) = h_{i-1}(x) + 1$. Suppose that $j > 1$ and that f_{j-1} has been defined. Then $f_j(x) = f_{j-1}(x) + h_{i-1}(g^{(f_{j-1}(n))}(x))$.

We show that for each $1 \le j \le a$, $x_j \le f_j(n)$. Note that $(A_1, b_1), \dots, (A_{x_1 - 1}, b_{x_1 - 1})$ is a $(g, i-1)$-transformation sequence. Hence, by the induction assumption, $x_1 - 1 \le h_{i-1}(n)$ and $x_1 \le f_1(n)$.

Furthermore, let $j > 1$. Then $(A_{x_{j-1}}, b_{x_{j-1}}), \dots, (A_{x_j - 1}, b_{x_j - 1})$ is also a $(q, i-1)$-transformation sequence. Therefore, by the induction assumption, $x_j \le x_{j-1} + h_{i-1}(n_{j-1})$. By the induction assumption, $x_{j-1} \le f_{j-1}(n)$ and, by (4), $n_{j-1} \le g^{(f_{j-1}(n))}(n)$. Therefore, $x_j \le f_{j-1}(n) + h_{j-1}(g^{(f_{j-1}(n))}(n)) = f_j(n)$ as required. Applying the same argumentation to the sequence following (A_{x_a}, b_{x_a}), we conclude that $r \le f_{a+1}$. As noted above, (g, i)-transformations cannot introduce new occurrences of $c - i$ and thus a is at most the number of occurrences of $c - i$ in (A_1, b_1). We can generously bound a by n and conclude that $r \le f_{n+1}$. Hence, we can set $h_i = f_{n+1}$. □

4. Applications and extensions

4.1. Checking fractional hypertree width

Now that our main combinatorial result has been established we move our attention to an algorithmic application of the support bound. In particular, we are interested in the problem of deciding whether for an input hypergraph H and constant k we have $\mathit{fhw}(H) \le k$. The problem is known to be NP-hard even for $k = 2$ [6]. However, as noted in the introduction, it has recently been shown that for hypergraph classes which enjoy bounded intersection or bounded degree, it is indeed tractable to check $\mathit{fhw}(H) \le k$ for constant k [9].

Here we show that our main combinatorial result reveals a large class of instances, that subsumes and extends all previously known cases, for which checking fhw is tractable. To establish the result we make use of the framework for tractable width checking developed in [9]. We will only recall the necessary key components here and use them in a black-box fashion.

Definition 28. Let $\rho^*_q(U)$ be the minimal weight of an assignment γ such that $U \subseteq B(\gamma)$ and $|support(\gamma)| \leq q$. We define the *q-limited* fractional hypertree width of a hypergraph H as its ρ^*_q-width.

Lemma 29 *(Theorem 4.5 & Lemma 6.5 in [9]). Fix c, d, and q as constant integers. There is a polynomial-time algorithm testing whether a given (c, d)-hypergraph has q-limited fractional hypertree width at most k.*

The underlying intuition of q-limited *fhw* is that the bounded support allows for a polynomial-time enumeration of all the (inclusion) maximal covers of sufficient weight. For (c, d)-hypergraphs, it is then possible to compute a set of candidate bags such that a fitting tree decomposition, if one exists, uses bags only from this set. Deciding whether a tree decomposition can be created from a given set of candidate bags is tractable under some minor restrictions to the structure of the resulting decomposition (not of any concern to the case discussed here).

Recall, a class $\mathcal{C}$ of hypergraphs is said to satisfy the *bounded multi-intersection property (BMIP)* if there exist $c \geq 2$ and $d \geq 0$, such that every H in $\mathcal{C}$ is a (c, d)-hypergraph. We now apply our main result and show that, under BMIP, there exists a constant q such that the q-limited fractional hypertree width always equals fractional hypertree width. From the previous lemma it is then straightforward to arrive at the desired tractability result.

Theorem 30. *There is a polynomial-time algorithm for testing whether the fhw of the given (c, d)-hypergraph H is at most k (the degree of the polynomial is upper bounded by a fixed function depending on c, d, k).*

Proof. It follows from Theorem 4 that if $fhw(H) \leq k$ for a (c, d)-hypergraph H then the $h(c, d, k)$-limited *fhw* of H is also at most k.

Indeed, let $(T, (B_u)_{u\in T})$ be a tree decomposition with *fhw* at most k. Then, according to Theorem 4, for each node u in T there is an edge weight function γ with $|support(\gamma)| \leq h(c, d, k)$ such that $B_u \subseteq B(\gamma)$. In other words, it follows that $(T, (B_u)_{u\in T})$ has ρ^*_q-width at most k where q is $h(c, d, k)$. Thus, H also has $h(c, d, k)$-limited fractional hypertree width at most k. For completeness of the procedure, note that the $h(c, d, k)$-limited fractional hypertree width can never be lower than $fhw(H)$.

To test whether $fhw(H) \leq k$ it is therefore enough to test whether the $h(c, d, k)$-limited *fhw* of H is at most k. This can be done in a polynomial time according to Lemma 29. □

4.2. Extension to fractional hitting set

There are two natural dual concepts of fractional edge covers. One is the notion of *fractional hitting sets* which is dual in the sense that it is equivalent to the fractional edge cover on the dual hypergraph. The other, *fractional independent sets*, corresponds to the dual linear program of a linear programming formulation of fractional edge covers. Here we discuss how our results extend to hitting sets.

We start by giving a formal definition of the fractional hitting set problem. Let $H = (V, E)$ be a hypergraph and $\beta : V \to [0, 1]$ be an assignment of weights to the vertices of H. Analogous to the definition of fractional edge covers we define

- $B_V(\beta) = \{e \in E \mid \sum_{v\in e} \beta(v) \geq 1\}$,
- $vsupport(\beta) = \{v \in V \mid \beta(v) > 0\}$,
- and $weight(\beta) = \sum_{v\in V} \beta(v)$.

A fractional hitting set is also called a fractional transversal in some contexts (cf. [14]). For a set of edges E', we denote the weight of the minimal fractional hitting set β such that $E' \subseteq B_V(\beta)$ as $\tau^*(E')$. For hypergraph $H = (V, E)$, we say $\tau^*(H) = \tau^*(E)$. Recall, that we assume reduced hypergraphs and therefore there is a one-to-one correspondence of vertices in H and edges in H^d. We will make use of the following straightforward observations about the connection of what we will call *dual weight assignments*.

Proposition 31. *Let $H = (V, E)$ be a (reduced) hypergraph and let $H^d = (W, F)$ be its dual. We write f_v to identify the edge in F that corresponds to the vertex v in V. The following two statements hold:*

- *For every $\gamma : E \to [0, 1]$ and the function $\beta : W \to [0, 1]$ with $\beta(e) = \gamma(e)$ it holds that $B_V(\beta) = \{f_v \mid v \in B(\gamma)\}$.*[1]

[1] Recall that the edges E of H are the vertices W of H^d.

- *For every $\beta : V \to [0,1]$ and the function $\gamma : F \to [0,1]$ with $\gamma(f_v) = \beta(v)$ it holds that $B(\gamma) = \{v \mid f_v \in B_v(\beta)\}$.*

For the hitting set, a more specific version of our main result is already known. This result is due to Zoltán Füredi [7], who extended earlier results by Chung et al. [4]. Recall that a hypergraph H with rank r is also a $(1,r)$-hypergraph, i.e., this can be considered a special case of our setting. Furthermore, note that the statement holds only for weight minimal hitting sets.

Proposition 32 *([7], page 152, Proposition 5.11.(iii)). For every hypergraph H of rank (i.e., maximal edge size) r, and every fractional hitting set w for H satisfying weight$(w) = \tau^*(H)$, the property $|vsupport(w)| \le r \cdot \tau^*(H)$ holds.*

In the following we will extend Theorem 4 to an analogous statement for fractional hitting sets thereby generalizing the previous proposition significantly. To derive the result we need a final observation about (c,d)-hypergraphs. In a sense, we show that bounded multi-intersection is its own dual property.

Lemma 33. *Let H be a (c,d)-hypergraph. Then the dual hypergraph H^d is a $(d+1, c-1)$-hypergraph.*[2]

Proof. Let $G = (V \cup E, A)$ be the incidence graph of H. H being a (c,d)-hypergraph is equivalent to G not having $K_{c,d+1}$ as a subgraph, with c vertices taken from E and $d+1$ vertices taken from V. As the incidence graph $G^d = (W \cup F, B)$ of H^d is isomorphic to G, with vertices and edges changing sides, we conclude that G^d does not have $K_{d+1,c}$ as a subgraph with $d+1$ vertices taken from F and c vertices taken from W. This is equivalent to saying that H^d is a $(d+1, c-1)$-hypergraph. □

Theorem 34. *There is a function $h(c,d,k)$ such that the following is true. Let c, d, k be constants. Let H be a (c,d)-hypergraph and β be an assignment of weights to $V(H)$. Assume that $weight(\beta) \le k$. Then there is an assignment ν of weights to $V(H)$ such that $weight(\nu) \le k$, $B_v(\beta) \subseteq B_v(\nu)$ and $|vsupport(\nu)| \le h(c,d,k)$.*

Proof. Let γ be the dual weight assignment of β as in Proposition 31. That is, $\gamma : F \to [0,1]$ is an edge weight assignment in the dual hypergraph $H^d = (W, F)$ with $|support(\gamma)| = |vsupport(\beta)|$ and $weight(\gamma) = weight(\beta)$.

From Lemma 33 we have that H^d is a $(d+1, c-1)$-hypergraph and thus by Theorem 4 there is an edge weight function ν' with $B(\gamma) \subseteq B(\nu')$ and $|support(\nu')| \le h'(d+1, c-1, k)$. Let ν now be the dual weight assignment of ν'. By Proposition 31 we then see that also $B_v(\beta) \subseteq B_v(\nu)$ and $|vsupport(\nu)| = |support(\nu')| \le h'(d+1, c-1, k)$. □

5. Conclusion & open questions

5.1. Conclusion

We have proved novel upper bounds on the size of the support of fractional edge covers and vertex covers. These bounds have then been fruitfully applied to the problem of checking $fhw(H) \le k$ for given hypergraph H. Recall that, without imposing any restrictions on the hypergraph H, this problem is NP-complete even for $k = 2$ [6], thus ruling out even XP-membership. In contrast, for hypergraph classes that exhibit bounded multi-intersection, we have managed to establish XP-membership, that is, checking $fhw(H) \le k$ for hypergraphs in such a class is feasible in polynomial time for any constant k. Beyond the application to checking fractional hypertree width, our main result reveals completely new and far-reaching connections between fractional covers in hypergraphs and hypergraph structure which may be of independent interest in a wide variety of fields.

Below we identify a number of interesting open problem that are closely related to our main results.

5.2. Precise computation of $fhw(H)$

We have shown that for any (c,d)-hypergraph H, the question $fhw(H) \overset{?}{\le} k$ can be answered in polynomial time with the degree of the polynomial depending on c, d, and k. Suppose we are given a constant k such that $fhw(H) \le k$. Is it possible to compute the optimal (precise) value of $fhw(H)$ in polynomial time with the degree of the polynomial depending on c, d, and k?

We know that $1 \le fhw(H) \le k$ and that for each $1 \le k' \le k$ we can test $fhw(H) \le k'$ in time polynomial in c, d, k'. It might seem that $fhw(k)$ can be efficiently computed by repeated binary-search like querying $fhw(H) \le k'$ for values of k' getting closer and closer to the actual value of $fhw(H)$.

[2] Note that the superscript of H^d only signifies that it is the dual of H. It is not connected to the integer constant d used for the multi-intersection width of H.

Unfortunately, this method does not work. More specifically, there is no function $h^*(c,d,k)$ upper bounding the degree of the polynomial for the runtime of the resulting algorithm. Indeed, if such a function existed then it would hold that $h(c,d,k') \le h^*(c,d,k)$ for any $1 \le k' \le k$. The proposition below demonstrates that this is not the case.

Proposition 35. *$h(2,1,x)$ tends to infinity as x approaches 2 from below.*

Proof. Recall the hypergraph family $(H_r)_{r\ge 2}$ from Example 1 that we defined as follows. $H_r = (V_r, E_r)$ with $V_r = \{v_0,\dots,v_r\}$ and $E = \{e_0,\dots,e_r\}$ with $e_0 = \{v_1,\dots,v_r\}$ and $e_i = \{v_0, v_i\}$ for $1 \le i \le r$.

It is known that the size of the smallest fractional edge cover of H_r is $2-1/r$ and the cover is witnessed by the unique assignment of weights where the weight of e_0 is $1-1/r$ and the weight of the rest of the hyperedges is $1/r$. Clearly the support of this assignment of weights is $r+1$ and hence $h(2,1,2-1/r) \ge r+1$ for each integer $r \ge 2$. In fact, H_r witnesses that for any $2-1/r \le x < 2$, $h(2,1,x) \ge r+1$. Indeed, a direct inspection shows that any edge cover of H_r of a support of size smaller than $r+1$ needs to have weight at least 2. It follows that even if we set x larger than $2-1/r$ but still smaller than 2, the support of size $r+1$ is needed anyway. The rest of the proof is an elementary calculus exercise. □

The impossibility to efficiently compute *fhw*(H) for (c,d)-hypergraphs by the method as above, of course, does not mean that the parameter cannot be efficiently computed. We leave the possibility of such a computation as an interesting open question.

Open question 1. Let H be a (c,d) hypergraph such that *fhw*$(H) \le k$ for some integral constant k. Is it possible to compute *fhw*(H) in a polynomial time with the degree of the polynomial depending on c,d,k?

It seems that a positive answer to Question 1 requires a new algorithmic approach for the computation of fractional hypertree decompositions of small width where bags do not necessarily have bounded support. This will require a deeper insight into the structure of hypertree decompositions of hypergraphs.

5.3. From bounded multi-intersection to bounded VC dimension

It is known that (c,d)-hypergraphs have VC dimension at most $c+d$ [9]. Therefore, it is natural to ask whether it is possible to generalize Theorem 4 from bounded multi-intersection to bounded VC dimension. More precisely, is there a function f such that for any constants d and k, any hypergraph H of VC dimension at most d and fractional edge cover of weight at most k, has a fractional edge cover γ of weight at most k and the support of γ is of size at most $f(d,k)$? We conjecture that the answer to this question is negative and that there is a class of hypergraphs witnessing the negative answer.

Conjecture 1. *There are constants d,k and an infinite class $\mathcal{H}$ of hypergraphs whose VC dimension is at most d, the fractional edge cover is at most k and the set $\{minsupport_k(H) | H \in \mathcal{H}\}$ is unbounded where $minsupport_k(H)$ is the smallest size of support of an edge cover of H of weight at most k.*

Let us discuss the reason why we stated the above conjecture. A notable result [13] in the area of parameterized complexity implies that the (non-fractional) set cover problem is FPT for (c,d)-hypergraphs parameterized by the size k. (The result is stated for dominating sets but can be reformulated in terms of set covers through a minor modification.) On the other hand, the problem becomes W[1]-hard already for VC dimension 2 [3]. Thus the set cover problem for bounded VC dimension is notably harder than for bounded multi-intersection.

5.4. Fixed-parameter tractability

Recall that fractional hypertree width (*fhw*) is defined as the f-width where f is the *fractional* edge cover number of the bags of a tree decomposition. Analogously, the generalized hypertree width (*ghw*) is defined as the f-width where f is the *integral* edge cover number of the bags of a tree decomposition. The computation of both, *fhw* and *ghw* is hard for hypergraphs in general [10,6]. However, our recent results demonstrate that the generally intractable problems for the computation of these notions of width admit XP-algorithms for restricted classes of hypergraphs. It is therefore natural to ask whether even more efficient algorithms, and in particular FPT-algorithms, are possible.

We believe that the $(2,1)$-hypergraphs are the right class to start this investigation with. Even more specifically, we propose to first look at the situation for generalized hypertree width. The parameterized intractability, if established for this class of hypergraphs, will extend to parameterized intractability for (c,d)-hypergraphs in general. Moreover, the methods used in Section 4.1 to show tractability of checking *fhw* rely on the tractability of checking *ghw* for the respective fragments. On the other hand, if an FPT-algorithm for generalized hypertree width is obtained, it is likely to be based on a novel insight, thus inspiring further research regarding a possibility of its generalization. We believe that the case of $(2,1)$-hypergraphs is a critical starting point for such considerations as the general case for (c,d)-hypergraphs may involve significant additional combinatorial challenges which are not directly relevant for the key observations. We therefore propose the following future research question.

Open question 2. Is there an FPT-algorithm parameterized by k that tests $ghw(H) \leq k$ for $(2, 1)$-hypergraphs?

Recent work has shown that, analogously to treewidth in graphs, *ghw* can be characterised in terms of forbidden substructures in degree-2 hypergraphs (i.e., $(3, 0)$-hypergraphs) [12]. Such a characterisation can provide an alternative path towards fixed-parameter tractable checking of $ghw \leq k$ (in the degree 2 case) through deciding whether certain substructures (whose size depends on k) are present in the hypergraph.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors acknowledge support by the Vienna Science and Technology Fund (WWTF) [10.47379/ICT2201] and the Austrian Science Fund (FWF): Project P30930. Georg Gottlob is a Royal Society Research Professor and acknowledges support by the Royal Society in this role through the “RAISON DATA” project (Reference No. RP\R1\201074). Matthias Lanzinger acknowledges support by the Royal Society “RAISON DATA” project (Reference No. RP\R1\201074).

References

[1] A. Atserias, M. Grohe, D. Marx, Size bounds and query plans for relational joins, SIAM J. Comput. 42 (4) (2013) 1737–1767.
[2] C. Berge, Fractional Graph Theory, ISI Lecture Notes, vol. 1, Macmillan of India, 1978.
[3] K. Bringmann, L. Kozma, S. Moran, N.S. Narayanaswamy, Hitting set for hypergraphs of low VC-dimension, in: Proc. ESA, 2016, 23.
[4] F.R.K. Chung, Z. Fueredi, M. Garey, R.L. Graham, On the fractional covering number of hypergraphs, SIAM J. Discrete Math. 1 (1) (1988) 45–49.
[5] W. Fischl, G. Gottlob, D.M. Longo, R. Pichler, Hyperbench: a benchmark and tool for hypergraphs and empirical findings, in: Proc. PODS, ACM, 2019, pp. 464–480.
[6] W. Fischl, G. Gottlob, R. Pichler, General and fractional hypertree decompositions: hard and easy cases, in: Proc. PODS, 2018, pp. 17–32.
[7] Z. Füredi, Matchings and covers in hypergraphs, Graphs Comb. 4 (1) (1988) 115–206.
[8] G. Gottlob, M. Lanzinger, R. Pichler, I. Razgon, Fractional covers of hypergraphs with bounded multi-intersection, in: Proc. MFCS, in: LIPIcs, vol. 170, 2020, 41.
[9] G. Gottlob, M. Lanzinger, R. Pichler, I. Razgon, Complexity analysis of generalized and fractional hypertree decompositions, J. ACM 68 (5) (Sept. 2021).
[10] G. Gottlob, Z. Miklós, T. Schwentick, Generalized hypertree decompositions: NP-hardness and tractable variants, J. ACM 56 (6) (2009) 30.
[11] M. Grohe, D. Marx, Constraint solving via fractional edge covers, ACM Trans. Algorithms 11 (1) (2014) 4.
[12] M. Lanzinger, The complexity of conjunctive queries with degree 2, in: Proc. PODS, ACM, 2022, pp. 91–102.
[13] G. Philip, V. Raman, S. Sikdar, Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond, ACM Trans. Algorithms 9 (1) (2012) 11.
[14] E. Scheinerman, D. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs, Dover Publications, Inc., 2011.