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Small ball probabilities and large deviations for grey Brownian motion

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Abstract

We show that the uniform norm of generalized grey Brownian motion over the unit interval has an analytic density, excluding the special case of fractional Brownian motion. Our main result is an asymptotic expansion for the small ball probability of generalized grey Brownian motion, which extends to other norms on path space. The decay rate is not exponential but polynomial, of degree two. For the uniform norm and the Hölder norm, we also prove a large deviations estimate.

Keywords: grey Brownian motion; fractional Brownian motion; small ball probabilities; small deviations; large deviations; Wright M-function.
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1 Introduction

Generalized grey Brownian motion (ggBm) is a two-parameter stochastic process $B_{\alpha,\beta}$, which is in general not Gaussian. Introduced in [17, 18], ggBm has been considered in the physics literature to model anomalous diffusions with non-Gaussian marginals, including both slow (variance grows slower than linearly) and fast diffusive behavior. The process $B_{\alpha,\beta}$ has stationary increments and is self-similar with parameter $H = \alpha/2$ [17, Proposition 3.2]. The marginal density of ggBm satisfies a fractional partial integro-differential equation [17]. Special cases of ggBm include fractional Brownian motion (fBm; $\beta = 1$), grey Brownian motion ([23]; $\alpha = \beta$), and Brownian motion ($\alpha = \beta = 1$). Our focus is mainly on the case $\beta < 1$. In [17], a generalized grey noise space is defined, motivated by white noise space, but with the Gaussian characteristic function replaced by the Mittag-Leffler function. The ggBm is then defined by evaluating generalized grey noise at the test function $1_{[0,t)}$. We do not go into details, because for our purposes, the representation

$$B_{\alpha,\beta}(t) = \sqrt{L_\beta}B_{\alpha/2}(t), \quad 0 < \alpha < 2,\ 0 < \beta < 1, \tag{1.1}$$

which was proved in [18], is more convenient. Here, $B_{\alpha/2}$ is a fBm with Hurst parameter $H = \alpha/2$, and L_β is an independent positive random variable whose density is the M-Wright function (see below). The representation (1.1) makes sense also in the limiting case $\beta = 1$, but we will not require this.

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The problem of small ball probabilities, also called small deviations, consists of estimating

$$\mathbb{P}\Big[\sup_{0\le t\le 1}|B_{\alpha,\beta}(t)|\le\varepsilon\Big],\quad \varepsilon\downarrow 0, \tag{1.2}$$

asymptotically. More generally, we can consider

$$\mathbb{P}\big[\|B_{\alpha,\beta}\|\le\varepsilon\big],\quad \varepsilon\downarrow 0,$$

where $\|\cdot\|$ is a norm on $C_0^\gamma[0,1]$, the space of γ-Hölder continuous functions starting at zero, with $0<\gamma<H=\alpha/2$. For ggBm with $\beta<1$, our main result (Theorem 2.3) shows that (1.2) is of order ε^2, and that this also holds for some other norms. For Gaussian processes, such as fBm ($\beta=1$), the small ball problem has been studied extensively [14], and exponential decay is typical. But there are also many works studying small ball probabilities for non-Gaussian processes; see, e.g., [1, 2] and the references therein. We refer to [12, 19] for other examples of processes with the small ball rate ε^2 of ggBm.

In Section 2, we will show that the known exponential small ball estimates for fBm can be used to deduce our quadratic small ball estimate for ggBm. As a byproduct, we show that the uniform norm (sup norm) of ggBm has a smooth, even analytic, pdf. In Section 3, we provide a large deviations estimate. The decay rate is exponential, but slower than Gaussian, depending on the parameter β.

Notation When we write $B_{\alpha,\beta}$, we always mean the process on the time interval $[0,1]$, i.e. $B_{\alpha,\beta}=(B_{\alpha,\beta}(t))_{0\le t\le 1}$. We write F_H for the cdf of $\|B_H\|$, assuming that the choice of the norm $\|\cdot\|$ is clear from the context. As usual, $\mathbb{R}^+=(0,\infty)$ denotes the positive reals. The letter C denotes various positive constants.

2 Analyticity of the cdf and small ball probability

The M-Wright function, which is the pdf of L_β in (1.1), is defined by

$$M_\beta(x)=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\Gamma(1-\beta-\beta n)},\quad x\ge 0,\ 0<\beta<1. \tag{2.1}$$

It is not obvious that M_β is a pdf; for this, and more information on M_β and its generalizations, we refer to [16]. For later use, we note that it follows from Euler's reflection formula that

$$\frac{1}{\Gamma(1-\beta-\beta n)}=\frac{\sin\big(\pi(\beta+\beta n)\big)}{\pi}\Gamma(\beta+\beta n) \tag{2.2}$$

(cf. [24, p. 41] and [16, (3.8)]), which shows, by Stirling's formula for the gamma function, that the series in (2.1) defines an entire function. For this, the crude version

$$\Gamma(x)=x^{x+o(x)},\quad x\uparrow\infty, \tag{2.3}$$

of Stirling's formula suffices. We will also need the asymptotic behavior of M_β at infinity [16, (4.5)],

$$M_\beta(x)=\exp\Big(-\frac{1-\beta}{\beta}(\beta x)^{\frac{1}{1-\beta}}+O(\log x)\Big),\quad x\uparrow\infty. \tag{2.4}$$

Our main assumption is that fBm satisfies an exponential small ball estimate w.r.t. to the chosen norm $\|\cdot\|$.

Assumption 2.1. *For $0<H<1$, there are $\theta,C_1,C_2>0$ such that*

$$-C_1\varepsilon^{-\theta}\le\log\mathbb{P}\big[\|B_H\|\le\varepsilon\big]\le -C_2\varepsilon^{-\theta},\quad \varepsilon\in(0,1].$$

For the uniform norm, it is known that this holds with $\theta = 1/H$,

$$-C_1\varepsilon^{-1/H} \le \log \mathbb{P}\Big[\sup_{0\le t\le 1}|B_H(t)| \le \varepsilon\Big] \le -C_2\varepsilon^{-1/H}. \tag{2.5}$$

Assumption 2.1 also holds for the γ-Hölder norm, where $0<\gamma<H$, and for the L^2-norm. See [4, 14, 15] for the corresponding values of θ, and for much more information on small ball probabilities for fBm and other Gaussian processes.

The examples we just mentioned are norms in the classical sense, and so we stick to this terminology in our statements. From our proofs, it is clear that it would suffice throughout to assume that $\|\cdot\|$ is a measurable non-negative homogeneous functional.

Proposition 2.2. *Let $0<\alpha<2$, $0<\beta<1$. If the norm $\|\cdot\|$ satisfies Assumption 2.1, then the cdf of $\|B_{\alpha,\beta}\|$ is an analytic function on $\mathbb{R}^+$. In particular, this holds for the cdf of $\|B_{\alpha,\beta}\|_\infty = \sup_{0\le t\le 1}|B_{\alpha,\beta}(t)|$.*

Proof. Recall that F_H denotes the cdf of $\|B_H\|$. From (1.1) we find

$$\begin{aligned}\mathbb{P}\big[\|B_{\alpha,\beta}\|\le\varepsilon\big] &= \int_0^\infty F_H(\varepsilon x^{-1/2})M_\beta(x)dx\\ &= 2\varepsilon^2\int_0^\infty F_H(y)M_\beta(\varepsilon^2/y^2)y^{-3}dy.\end{aligned} \tag{2.6}$$

As M_β extends to an entire function (see above), the last integrand clearly is an entire function of ε for any fixed $y>0$. The function M_β is bounded on $\mathbb{R}^+$, as follows, e.g., from (2.1) and (2.4). Thus, the integrand in (2.6) can be bounded by an integrable function of y, independently of ε. Hence, the conditions of a standard criterion for complex differentiation under the integral sign [8, Theorem IV.5.8] are satisfied, which yields the assertion. □

Note that fBm, i.e. $\beta=1$, is not covered by Proposition 2.2. In [13], it is shown by Malliavin calculus that $\sup_{0\le t\le 1}B_H(t)$ (without the absolute value) has a C^∞ density.

We now show that, for $\beta<1$, the small ball probability of ggBm is of order ε^2 as $\varepsilon\downarrow 0$. For $2/\theta+\beta<1$ ($\alpha+\beta<1$ for the uniform norm), we express it as a power series, which yields a full asymptotic expansion. We write

$$\eta_k(H) := \mathbb{E}\big[\|B_H\|^{-k}\big], \quad k\in\mathbb{N},$$

for the negative moments of the norm of fBm, omitting the dependence on the norm $\|\cdot\|$ in the notation $\eta_k(H)$. By integration by parts, it is easy to see that $\eta_k(H)$ is finite under Assumption 2.1.

Theorem 2.3. *Let $0<\alpha<2$, $0<\beta<1$, and define $H=\alpha/2$. Under Assumption 2.1, the small ball probability of ggBm satisfies*

$$\mathbb{P}\big[\|B_{\alpha,\beta}\|\le\varepsilon\big] \sim \frac{\eta_2(H)\varepsilon^2}{\Gamma(1-\beta)}, \quad \varepsilon\downarrow 0. \tag{2.7}$$

If, additionally, $2/\theta+\beta<1$, then it has the convergent series representation

$$\mathbb{P}\big[\|B_{\alpha,\beta}\|\le\varepsilon\big] = \sum_{n=0}^\infty \frac{(-1)^n\eta_{2n+2}(H)}{(n+1)n!\Gamma(1-\beta-\beta n)}\varepsilon^{2n+2}, \quad \varepsilon\ge 0. \tag{2.8}$$

In particular, if $\|\cdot\|=\|\cdot\|_\infty$, then (2.8) *holds for $\alpha+\beta<1$.*

Proof. By integration by parts, we have

$$\int_0^\infty \frac{F_H(y)}{y^{2n+3}}dy = \frac{1}{2n+2}\int_0^\infty y^{-2n-2}F_H(dy) = \frac{\eta_{2n+2}(H)}{2n+2}. \tag{2.9}$$

The assertion (2.7) follows from (2.6), (2.1) for $x=0$, (2.9) for $n=0$, and dominated convergence, because M_β is a bounded function. For the next statement, define

$$G_N(\varepsilon,y) := \sum_{n=N+1}^\infty \frac{(-1)^n\varepsilon^{2n-2N+1}}{y^{2n}n!\Gamma(1-\beta-\beta n)}, \quad y>0,\ \varepsilon\in[0,1],$$

so that (2.6) yields, for $N\in\mathbb{N}$,

$$\begin{aligned}&\mathbb{P}\big[\|B_{\alpha,\beta}\|\le\varepsilon\big]\\ &\quad = 2\varepsilon^2\int_0^\infty \frac{F_H(y)}{y^3}\sum_{n=0}^N \frac{(-\varepsilon^2/y^2)^n}{n!\Gamma(1-\beta-\beta n)}dy + 2\varepsilon^{2N+1}\int_0^\infty \frac{F_H(y)}{y^3}G_N(\varepsilon,y)dy. \end{aligned}\tag{2.10}$$

For the finite sum, we can use (2.9) to rewrite the summands as in (2.8). We now provide an integrable bound for the last integrand in (2.10) that does not depend on $\varepsilon\in[0,1]$. It is clear that

$$|G_N(\varepsilon,y)| \le \sum_{n=N+1}^\infty \frac{1}{y^{2n}n!\Gamma(1-\beta-\beta n)}, \quad y>0,\ \varepsilon\in[0,1]. \tag{2.11}$$

By (2.2) and Stirling's formula,

$$\frac{1}{n!|\Gamma(1-\beta-\beta n)|} \le n^{-(1-\beta)n+o(n)} \le Cn^{-(1-\hat\beta)n}, \quad n\in\mathbb{N}, \tag{2.12}$$

for any $\hat\beta>\beta$; we will fix $\hat\beta$ later. From (2.11), (2.12), and Stirling's formula, we conclude

$$\begin{aligned}|G_N(\varepsilon,y)| &\le C\sum_{n=N+1}^\infty \frac{1}{y^{2n}\Gamma((1-\hat\beta)n)}\\ &= y^{-2}E_{1-\hat\beta,1-\hat\beta}(y^{-2}) - \sum_{n=1}^N \frac{1}{y^{2n}\Gamma((1-\hat\beta)n)},\end{aligned}\tag{2.13}$$

where

$$E_{u,v}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(un+v)}, \quad u,v>0,\ z\in\mathbb{C},$$

denotes the two-parameter Mittag-Leffler function. (Alternatively, we could also argue with the less common function $z\mapsto\sum_{n\ge1}n^{-un}z^n$; see the asymptotic formula on p. 55 of [11] and the references given there.) We now use the uniform bound (2.13) in (2.10). Integrability at ∞ is obvious, and we now show integrability at zero. By [10, Theorem 4.3],

$$E_{1-\hat\beta,1-\hat\beta}(y^{-2}) = \exp\Big(y^{-\frac{2}{1-\hat\beta}}\big(1+o(1)\big)\Big), \quad y\downarrow 0.$$

We see, using Assumption 2.1 for F_H, that the last integrand in (2.10) satisfies

$$\frac{F_H(y)}{y^3}G_N(\varepsilon,y) \le \exp\Big(-C_2y^{-\theta} + y^{-\frac{2}{1-\hat\beta}} + o\big(y^{-\left(\theta\wedge\frac{2}{1-\hat\beta}\right)}\big)\Big), \quad y\downarrow 0,$$

uniformly w.r.t. $\varepsilon\in[0,1]$. This is integrable if $\theta>2/(1-\hat\beta)$, i.e., $2/\theta+\hat\beta<1$. Clearly, our assumption that $2/\theta+\beta<1$ allows us to chose such a $\hat\beta>\beta$. By the following lemma, $\eta_{2n+2}(H)=2^{2n/\theta+o(n)}$. Using (2.12), we can thus take the limit $N\uparrow\infty$ in (2.10) for fixed $\varepsilon\in[0,1]$, which proves (2.8) for these ε. The extension to any $\varepsilon\ge0$ follows by analytic continuation, using Proposition 2.2. □

In the preceding proof, we applied the following estimate for negative moments of the supremum of fBm. Note that moments with *positive* exponent are estimated in [22]; see also [21].

Lemma 2.4. *Under Assumption 2.1, for $k \uparrow \infty$, we have $\eta_k(H) = k^{k/\theta + o(k)}$.*

Proof. We show only the upper estimate, as the lower one can be proven analogously. By (2.5), there is $\varepsilon_0 > 0$ such that

$$F_H(y) \le 2\exp(-C_2 y^{-\theta}), \quad 0 < y \le \varepsilon_0.$$

Define $\tilde{K} := 2 \vee \exp(C_2 \varepsilon_0^{-\theta})$. Then,

$$F_H(y) \le \tilde{K} \exp(-C_2 y^{-\theta}), \quad y > 0;$$

note that the right hand side is ≥ 1 for $y \ge \varepsilon_0$. This implies

$$\begin{aligned} \eta_k(H) &= \int_0^\infty y^{-k} F_H(dy) = k \int_0^\infty y^{-k-1} F_H(y) dy \\ &\le k\tilde{K} \int_0^\infty \exp(-C_2 y^{-\theta}) y^{-k-1} dy \\ &= e^{O(k)} \int_0^\infty e^{-w} w^{k/\theta - 1} dw \\ &= e^{O(k)} \Gamma(k/\theta - 1) = k^{k/\theta + o(k)}, \end{aligned}$$

by Stirling's formula (2.3) for the gamma function. □

If $2/\theta + \beta > 1$, then the series in (2.8) diverges for any $\varepsilon > 0$. Indeed, there is an increasing sequence (n_j) in $\mathbb{N}$ such that the lower bound

$$\operatorname{dist}(1 - \beta - \beta n_j, \mathbb{Z}) \ge C > 0, \quad j \in \mathbb{N},$$

holds. For rational $\beta \in (0,1)$, this is clear by periodicity. For irrational β, it follows from the classical fact that the sequence of fractional parts $\{n\beta\}$ is dense in $[0,1]$ (Kronecker's approximation theorem). Hence, again by (2.2) and Stirling's formula,

$$\frac{1}{|\Gamma(1 - \beta - \beta n_j)|} \ge n_j^{\beta n_j + o(n_j)},$$

which, together with Lemma 2.4, shows divergence. We leave it as an open problem if (2.8) still holds in the sense of an asymptotic expansion of the small ball probability, if $2/\theta + \beta \ge 1$.

3 Large deviations

For fractional Brownian motion, it is well known that

$$\mathbb{P}\Big[\sup_{0 \le t \le 1} |B_H(t)| \ge y\Big] = \exp\big(-\tfrac{1}{2} y^2 + o(y^2)\big), \quad y \uparrow \infty. \tag{3.1}$$

Indeed, the upper estimate follows from

$$\mathbb{P}\Big[\sup_{0 \le t \le 1} |B_H(t)| \ge y\Big] \le 2\,\mathbb{P}\Big[\sup_{0 \le t \le 1} B_H(t) \ge y\Big]$$

and the Borell-TIS inequality [20, Theorem 4.2], and the lower one is clear from $\sup_{0 \le t \le 1} |B_H(t)| \ge B_H(1)$. The following result gives a large deviation estimate for ggBm. For $\beta = 1$, the distribution has a Gaussian upper tail, of course. For $0 < \beta < 1$, the decay is between exponential and Gaussian, which is sometimes called compressed exponential.

Theorem 3.1. *Let $0 < \alpha < 2$ and $0 < \beta \le 1$, and assume that $\|\cdot\|$ is a norm on the Hölder space $C_0^\gamma[0,1]$, where $0 < \gamma < H = \alpha/2$, such that*

$$\mathbb{P}\big[\|B_H\| \ge y\big] = \exp\big(-\kappa y^2 + o(y^2)\big), \quad y \uparrow \infty, \tag{3.2}$$

for some $\kappa > 0$. Then there are constants $K_1, K_2 > 0$ such that

$$\exp\Big(-K_1 y^{\frac{2}{2-\beta}}\big(1+o(1)\big)\Big) \le \mathbb{P}\big[\|B_{\alpha,\beta}\| \ge y\big] \tag{3.3}$$

$$\le \exp\Big(-K_2 y^{\frac{2}{2-\beta}}\big(1+o(1)\big)\Big), \quad y \uparrow \infty. \tag{3.4}$$

Proof. We may assume $\beta < 1$, because for $\beta = 1$ we have $B_{\alpha,1} = B_H$, and the assumption (3.2) makes the statement trivial. With $\bar{F}_H = 1 - F_H$ the tail distribution function of $\|B_H\|$, we have, from (1.1),

$$\mathbb{P}\big[\|B_{\alpha,\beta}\| \ge y\big] = \int_0^\infty \bar{F}_H(yx^{-1/2}) M_\beta(x)dx. \tag{3.5}$$

If $\kappa = \frac{1}{2}$, then $\bar{F}_H$ satisfies

$$\bar{F}_H(y) = \exp\big(-\tfrac{1}{2}y^2 + o(y^2)\big), \quad y \uparrow \infty, \tag{3.6}$$

by (3.2). We assume $\kappa = \frac{1}{2}$ for rest of the proof, as $\kappa > 0$ is a trivial extension. In what follows, the integral (3.5) is estimated by dividing it into several parts. Let $0 < \hat{\kappa} < \frac{1}{2}$ be arbitrary. Since M_β is bounded, we obtain

$$\begin{aligned}
\int_0^1 \bar{F}_H(yx^{-1/2}) M_\beta(x)dx &\le C \int_0^1 e^{-\hat{\kappa}y^2/x} M_\beta(x)dx \\
&\le C \int_0^1 e^{-\hat{\kappa}y^2/x} dx \\
&= C\big(e^{-\hat{\kappa}y^2} - \hat{\kappa}y^2\Gamma(0, \hat{\kappa}y^2)\big),
\end{aligned}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function. Using a well-known expansion of that function [7, §8.11], we conclude

$$\int_0^1 \bar{F}_H(yx^{-1/2}) M_\beta(x)dx \le \exp\big(-\hat{\kappa}y^2 + o(y^2)\big), \quad y \uparrow \infty. \tag{3.7}$$

As $\beta < 1$, this is negligible compared to the decay rate claimed in (3.3) and (3.4). Now define $h(y) := y^2/(\log y)$. Since $\bar{F}_H \le 1$, and using (2.4), we have

$$\begin{aligned}
\int_{h(y)}^\infty \bar{F}_H(yx^{-1/2}) M_\beta(x)dx &\le \int_{h(y)}^\infty M_\beta(x)dx \\
&\le \int_{h(y)}^\infty \exp\big(-Cx^{\frac{1}{1-\beta}}\big)dx \\
&\le \exp\big(-Ch(y)^{\frac{1}{1-\beta}}\big).
\end{aligned}$$

Since $h(y) = y^{2+o(1)}$ and $2/(1-\beta) > 2/(2-\beta)$, this is of faster decay than (3.4).

It remains to show that the integral $\int_1^{h(y)} \bar{F}_H(yx^{-1/2}) M_\beta(x)dx$ has the claimed growth order (3.4). By dividing the exponent in (2.4) by 2, which makes the decay slower, we obtain

$$M_\beta(x) \le C \exp\Big(-\frac{1-\beta}{2\beta}(\beta x)^{\frac{1}{1-\beta}}\Big), \quad x \ge 1.$$

Similarly, (3.6) implies

$$\bar{F}_H(y) \le Ce^{-y^2/3}, \quad y \ge 1.$$

Analogously, we can *increase* the constants in the exponents to find *lower* estimates, for which the following reasoning is analogous, and yields (3.3). Therefore, we only discuss the upper estimate for $\int_1^{h(y)}$. This is a straightforward application of the Laplace method [5, Chapter 4] to the integral

$$\int_1^{h(y)} \exp\Big(-\frac{y^2}{3x} - \frac{1-\beta}{2\beta}(\beta x)^{\frac{1}{1-\beta}}\Big)dx,$$

which results from the two preceding estimates. The integrand is a strictly concave function with a maximum at

$$x_0(y) = cy^{\frac{2(1-\beta)}{2-\beta}} \in (1, h(y))$$

for some constant $c > 0$. As we are not concerned with lower order terms, it suffices to evaluate the integrand at $x_0(y)$ to conclude

$$\int_1^{h(y)} \bar{F}(yx^{-1/2})M_\beta(x)dx \le \exp\Big(-Cy^{\frac{2}{2-\beta}}(1+o(1))\Big).$$

This completes the proof. □

We now comment on applying Theorem 3.1 to other norms than the sup norm, which requires verifying (3.2). As mentioned above, for the sup norm, this follows from the Borell-TIS inequality. For an arbitrary norm $\|\cdot\|$ on Hölder space, we have

$$\mathbb{P}\big[\|B_H\| \ge y\big] = \mathbb{P}\big[y^{-1}B_H \in \{\|f\| \ge 1\}\big], \quad y > 0.$$

In principle, this is in the scope of the general LDP (large deviation principle) for Gaussian measures [6, Theorem 3.4.12], [15, Theorem 8.3], but it may not be trivial to verify the assumptions. For $H = \frac{1}{2}$ and the Hölder norm, this was done in [3], extending Schilder's theorem. Note that choosing a stronger topology than the uniform one enlarges the dual space of path space, making it harder to verify the defining property of a Gaussian measure. For the Hölder topology, we are on safe grounds, though, by another approach: Using the double sum method for Gaussian fields, Fatalov has shown that (3.1) holds for the γ-Hölder norm [9, Theorem 1.3], and so Theorem 3.1 is applicable to this norm (with $0 < \gamma < H$, of course).

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