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Interest Rate Modeling and Optimal Trading Portfolios with Dependence and Partial Information

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Kurzfassung der Dissertation

Thematisch befasst sich die vorliegende Dissertation mit zwei benachbarten Gebieten der Finanzmathematik. Im ersten Teil entwickeln wir ein analytisch nachvollziehbares Zinsstrukturmodell in reduzierter Form für ausfallgefährdete Anleihen. Unser Modell berücksichtigt sowohl empirisch stilisierte Fakten (negative momentane Korrelation zwischen Kreditspread und risikofreiem Zinssatz) als auch mathematische und ökonomische Erfordernisse (Ausfallintensität und risikofreier Zinssatz sind nicht-negativ), um ein besseres Verständnis der Kredit- und Zinsmärkte zu ermöglichen. In den verbleibenden Teilen analysieren wir optimale Handelsstrategien für Finanzmarktmodelle, bei welchen den Aktienkursen bestimmte Abhängigkeitsstrukturen auferlegt werden, für die nicht alle Modellparameter direkt beobachtbar sind. In diesem Zusammenhang analysieren wir dynamische Portfoliooptimierungsprobleme unter Teilinformationen, die sich auf den Paarhandel (Teil II) bzw. einen Großinvestor (Teil III) beziehen, der die Marktstimmung beeinflussen kann.

In Teil I präsentieren wir ein neuartiges Zinsstrukturmodell für ausfallgefährdete Anleihen, das in der Lage ist, negative momentane Korrelationen zwischen Kreditrisikozuschlag und risikofreiem Zinssatz zu erfassen, die in der empirischen Literatur dokumentiert sind, während die Positivität der Ausfallintensität und des risikofreien Zinssatzes erhalten bleibt. Für einen multivariaten Jacobi-Prozess (eindimensional auch als Wright-Fisher-Prozess bekannt) und ein geeignetes Funktional sind wir in der Lage, die Preise sowohl für ausfallfreie als auch ausfallbehaftete Nullkuponanleihen in relativ geschlossener Form zu berechnen, indem wir die Technik des exponentiellen Maßwechsels mit Hilfe des *carré-du-champ*-Operators sowie die Übergangsdichtefunktion aus der dualen Darstellung des Jacobi-Prozesses verwenden. Die resultierende Formel beinhaltet Reihen mit Quotienten von Gamma Funktionen und schnell abfallenden Exponentialfunktionen. Der Hauptvorteil des vorgeschlagenen Zinsstrukturmodells in reduzierter Form besteht darin, dass es eine flexiblere Korrelationsstruktur zwischen Zustandsvariablen bietet, die innerhalb eines relativ nachvollziehbaren Rahmens die zeitliche Zinsstruktur und somit die Preisentwicklung von Anleihen und Derivaten bestimmen. Darüber hinaus ist man in höheren Dimensionen nicht auf numerische Methoden für Differentialgleichungen angewiesen, die schwierig zu handhaben sein können (z.B. mehrdimensionale Riccati-Gleichungen in affinen und quadratischen Zinsstrukturmodellen), da die Übergangsdichtefunktion der Zustandsgrößen explizit bekannt ist. Dieser Teil basiert auf dem Preprint [10], das zur Veröffentlichung eingereicht werden soll.

In Teil II untersuchen wir ein dynamisches Problem der Portfoliooptimierung im Zusammenhang mit dem Paarhandel, also einer Anlagestrategie, die eine Long-Position in einem Wertpapier mit einer Short-Position in einem anderen Wertpapier mit ähnlichen Eigenschaften verbindet. Die Preisdifferenz bei derartigen Paaren, meist kurz als Spread bezeichnet, wird durch einen Gaußprozess modelliert, der stets zu seinem Mittelwert tendiert und dessen Driftrate durch einen nicht beobachtbaren, zeitkontinuierlichen Markovprozess mit endlich vielen Zuständen bestimmt wird. Mit Hilfe der klassischen stochastischen Fil-

tertheorie reduzieren wir dieses Problem mit Teilinformationen auf ein Äquivalentes mit vollständigen Informationen und lösen es für die logarithmische Nutzenfunktion. Hierbei wird das Endvermögen durch die Risikoexposition des Portfolios bestraft, welche wiederum durch die realisierte Volatilität des Vermögensprozesses gemessen wird. Wir charakterisieren optimale dollarneutrale Handelsstrategien sowie optimale Wertfunktionen unter Voll- und Teilinformationen und zeigen, dass das Prinzip der Sicherheitsäquivalenz für optimale Portfolio-Strategien gilt. Schließlich präsentieren wir eine numerische Analyse für einen einfachen Markovprozess mit zwei Zuständen. Dieser Teil stimmt im wesentlichen mit der Publikation [8] überein, die im International Journal of Theoretical and Applied Finance erschienen ist.

In Teil III analysieren wir ein Problem der Portfoliooptimierung für einen Investor, dessen Anlageentscheidungen einen indirekten Einfluss auf die Wertpapierpreise haben. Dazu betrachten wir ein Finanzmarktmodell mit einem zufallsbehafteten Preisprozess, dessen Dynamik nur aus Sprüngen besteht, deren Intensität durch einen nicht beobachtbaren kontinuierlichen Markovprozess mit endlichem Zutandsraum moduliert wird. Wir gehen davon aus, dass Entscheidungen des Investors den Generator des Markovprozesses beeinflussen, was zu einer indirekten Auswirkung auf den Preisprozess führt. Mit Hilfe der stochastischen Filtertheorie reduzieren wir dieses Problem mit Teilinformationen auf eines mit vollständigen Informationen und lösen es für logarithmische und Potenz-Nutzenfunktionen. Insbesondere wenden wir die stochastische Kontrolltheorie für stückweise deterministische Markovprozesse an, um die Optimalitätsgleichung abzuleiten und die Wertfunktion als eindeutige Viskositätslösung der zugehörigen dynamischen Hamilton-Jacobi-Bellman-Optimalitätsgleichung zu charakterisieren. Als Beispiel betrachten wir schließlich einen einfachen Markovprozess mit zwei Zuständen und diskutieren, wie sich die Befähigung des Investors, die Intensität des Markovprozesses zu beeinflussen, auf die optimalen Portfoliostrategien sowie den optimalen Vermögenswert unter vollständigen und partiellen Informationen auswirkt. Dieser Teil stimmt hauptsächlich mit dem Preprint [6] überein, das zur Veröffentlichung eingereicht wird (noch in Bearbeitung).

Abstract

This dissertation deals thematically with two neighboring areas of financial mathematics. In Part I we consider a tractable defaultable term-structure model in a reduced-form setting that takes care of empirical stylized facts (negative instantaneous correlation between credit spread and risk-free rate) coherent with the mathematical and economics-related facts (non-negative intensity and risk-free rate) with an aim towards better understanding of the dependence structure between credit and interest rate markets. In the remaining parts, we want to analyze optimal trading portfolios with specific dependence structures imposed on the stock prices with the view that not every model parameters are directly observable. In that respect, we analyze dynamic portfolio optimization problems under partial information related to a pairs trading in Part II and a market model with a large investor who can affect market sentiments in Part III.

In Part I, we provide a novel defaultable term structure model that is capable of capturing negative instantaneous correlation between credit spreads and risk-free rate documented in the empirical literature while sustaining the positivity of the default intensity and risk-free rate. Given a multivariate Jacobi (known also as Wright–Fisher in one-dimension) process and a certain functional, we are able to compute the zero-coupon bond prices, both defaultable and default-free, in a relatively tractable way by using the exponential change of measure technique with the help of *carré du champ* operator as well as by using the transition density function obtained from the dual representation of the multivariate Jacobi process. The resulting formula is represented by series involving ratios of gamma functions and fast converging exponential decay functions. The main advantage of the proposed reduced form model is that it provides a more flexible correlation structure between state variables governing the (defaultable) term structure within a relatively tractable framework for bond and derivative pricing. Moreover, in higher dimensions one does not need to rely on numerical schemes related to the differential equations, which may be difficult to handle (e.g., multi-dimensional Riccati equations in affine and quadratic term structure frameworks), because transition function of the state variables is known. This part is based on a preprint [10], which is to be submitted for publication.

In Part II, we study a dynamic portfolio optimization problem related to pairs trading, which is an investment strategy that matches a long position in one security with a short position in another security with similar characteristics. The relationship between pairs, called a spread, is modeled by a Gaussian mean-reverting process whose drift rate is modulated by an unobservable continuous-time, finite-state Markov chain. Using the classical stochastic filtering theory, we reduce this problem with partial information to an equivalent one with full information and solve it for the logarithmic utility function, where the terminal wealth is penalized by the riskiness of the portfolio according to the realized volatility of the wealth process. We characterize optimal dollar-neutral strategies as well as value functions under full and partial information and show that the certainty equivalence

principle holds for the optimal portfolio strategy. Finally, we provide a numerical analysis for a toy example with a two-state Markov chain. This part is mainly from [8] published in International Journal of Theoretical and Applied Finance.

In Part III, we analyze a portfolio optimization problem for an investor whose actions have an indirect impact on prices. We consider a market model with a risky asset price process following a pure-jump dynamics with an intensity modulated by an unobservable continuous-time finite-state Markov chain. We assume that decisions of the investor affect the generator of the Markov chain, which results in an indirect impact on the price process. Using stochastic filtering theory, we reduce this problem with partial information to one with full information and solve it for logarithmic and power utility preferences. In particular, we apply control theory for piecewise deterministic Markov processes (PDMP) to derive the optimality equation and characterize the value function as the unique viscosity solution of the Hamilton–Jacobi–Bellman (HJB) equation. Finally, we provide an example with a two-state Markov chain and discuss how investor's ability to control the intensity of it affects the optimal portfolio strategies as well as the optimal wealth under full and partial information. This part is mainly from a pre-print [6] that is submitted for publication (under revision).

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... ... "Altrove, c'è l'altrove. Io non mi occupo dell'altrove. Dunque, che questo romanzo abbia inizio. In fondo, è solo un trucco. Sì, è solo un trucco.", *Jep Gambardella.*

Contents

Chapter 1

Introduction

Understanding and quantifying dependence relations is crucial in the modeling of financial phenomena. Therefore, in this thesis, we would like to analyze certain dependence structures arising in reduced-form setting of defaultable term structure models as well as in dynamic portfolio optimization problems related to pairs trading and large investors. The dependence structure manifests itself as negative instantaneous correlation (or covariation) in the former, and as co-integration and dependence of optimal controls to an unobservable Markov chain in the latter. Below, we would like to give a synopsis of what we have studied in that respect by focusing on our contribution to the literature. More detailed introductions and motivations for the individual research topics can be found in respective parts of the thesis. We also provide an outline of the thesis at the end of this section.

In the first part, we introduce a novel dynamic defaultable yield curve (term structure of interest rates) model that can capture negative instantaneous correlation between credit spreads and risk-free rate documented in the empirical literature while sustaining the positivity of the default intensity and the risk-free rate. In the most simplistic case, the yield curve is a graph that represents the relationship between short and long-term interest rates, specifically in government securities (generally referred as default-free) or debt securities of companies (defaultable). By observing the yield curve, or more precisely the shape of the yield curve, economic agents can draw conclusions about the market expectation of the future of the economy. So the yield curve can be regarded as one of the most important tools for economic agents when they are making their decisions (i.e., when central bankers set their monetary policy, or when individuals deposit their savings to a bank or decide to take out a mortgage loan, or when insurance companies set their premiums etc.). *Credit spread* (or yield spread) of a given corporate bond is defined to be the difference between its yield and the yield of a government bond, or more generally (and more accurately after the recent debt crisis) a reference bond, which is assumed to be risk-free and has the same time to maturity. The *raison d'etre* of credit spreads is the risk of default inherent in corporate bonds, in which case the bond holders receive only the partial payment or no repayment at all. Therefore, to price corporate bonds, or more generally any defaultable bond and other credit sensitive instruments, it is necessary to consider the evolution of term-structure of credit spreads and the risk-free rate, as well as the correlation structure between two under the condition that both are given stochastically.

In a structural credit risk framework, as it is documented by [128] there is an unambiguous economic relation between the credit spread and the risk-free rate, manifesting itself as a negative correlation. On the other hand, in a reduced-form setting, this kind of negative correlation is captured by imposing negative instantaneous correlation between the state variables that drive the defaultable and non-defaultable term structures. However, in a setting where the risk-free rate and credit spread are given by affine diffusions although one has the analytical tractability of the bond prices, due to the admissibility conditions, one cannot simultaneously have the positivity of the spreads (or intensities) and the risk-free rate while sustaining negatively correlated increments of them. Non-negativity of the spread is a great concern due to the impossibility to construct a Cox process with a negative intensity rate. Similarly, it can be shown that in the presence of negative nominal interest rates, arbitrage opportunities arise if cash is available. Although, non-positivity of the interest rates and intensity process are ignored in the literature by assuming that its probability is close to zero, it might be a concern especially in term-structure modeling and complex derivative pricing. Motivated by the above discussion, we have these objectives :(a) to come up with a tractable (defaultable) term-structure model in a reduced-form setting that takes care of empirical stylized facts (negative instantaneous correlation between credit spreads and risk-free rate) coherent with the mathematical and theoretical facts (non-negative intensities and risk-free rate) with an aim towards better understanding of the credit markets and interest rate markets, (b) to understand better in a general setting the notion of instantaneous correlation in term-structure and credit risk models.

Our contribution in this part is the following. Given a multivariate Jacobi process, we are able to compute the zero-coupon bond prices (both defaultable and risk-free) in a tractable way by using the exponential change of measure technique as well as by using the transition density function obtained from the dual representation of the Jacobi process (see [69] or [67]). The resulting formula only involves series sum involving ratios of gamma functions and they are rapidly converging due to the terms involving functions with exponential decay. In the resulting model, instantaneous correlation between spreads of different credit classes and the risk-free term structure can be both negative and positive while sustaining the positivity assumption of the rates in contrast to the very well-established models of the affine framework. Moreover, in higher dimensions one does not need to rely on numerical schemes related to the differential equations, which may be difficult to handle (e.g., multi-dimensional Riccatti equations in affine and quadratic term-structure frameworks). Similarly, credit default-swaps and any general interest-rate derivatives can be easily priced since we have the relatively tractable bond prices from the procedure described above.

In the second part, we analyze a portfolio allocation problem related to pairs trading. Pairs trading is an investment strategy that attempts to capitalize on market inefficiencies arising from imbalances between two or more stocks. This kind of strategy involves a long position and a short position in a pair of similar stocks that have moved together historically. Examples of such pairs can be given: ExxonMobil and Royal Dutch and Shell for the oil industry, or Pfizer and GlaxoSmithKline for the pharmaceutical industry. The underlying rationale of pairs trading is to buy the underperformer, and sell the overperformer, in anticipation that the security that has performed badly will make up for loss in the coming periods, perhaps even overperform the other, and vice-versa. For this reason, it is also classified as a convergence or mean-reversion strategy. The pair of stocks is selected in a way that it forms a mean-reverting portfolio referred to as the *spread*. We consider the portfolio optimization problem of a trader with a logarithmic utility from risk penalized terminal wealth investing in a pair of assets whose dynamics have a certain dependence structure in a Markov regime-switching model. More precisely, we model the spread process (log-price differential) as an Ornstein–Uhlenbeck process with a partially observable Markov modulated drift. Our proposed model is an extended version of the model given by [135], who found the optimal pairs trading strategies in a dollar-neutral setting for an investor with power utility. Although investing equal dollar amount (as a proportion of wealth) in pairs seems to be restrictive, it is meaningful when CAPM (Capital Asset Pricing Model) betas of the selected stocks are very close to each other. Our model extends the work of [135] by allowing partially observed Markov-modulated drifts both for the price processes and the spread, hence enabling them to change with respect to different conditions. As the second extension, to find the optimal trading strategies, we use a risk-penalized terminal wealth as it is suggested in [146, Section 2.22].

To sum up, our key contributions in this part are the following. First, we characterize the optimal dollar-neutral strategies both in full and partial information settings with risk-penalized terminal wealth for a log-utility trader and show that optimal strategies depend on both the correlation between two assets and the mean-reverting spread. The effect of risk-penalization on optimal strategies is an increase in risk-aversion uniformly in a constant proportion that does not depend on time. Second, we characterize the optimal value function via Feynman–Kac formula. Third, using the innovations approach, we provide filtering equations that are necessary to reduce the problem with partial information to the one with full information. A nice feature of the solution in the partial information setting is that the optimal strategy is a linear function of the filtered state and hence it can be considered as a projection of the strategy in full information on the investor's information filtration. We also present numerical results for a toy example with a two-state Markov chain in both full and partial information settings. Our analysis shows that average data does not contain sufficient information to obtain the optimal value for the pairs trading problem for logarithmic utility preferences. This result is in contrast with the one for the classical portfolio optimization problem with Markov modulation; see [14, Section B]. Furthermore, our toy example suggests that there is always a gain from filtering due to the convexity arising from using filtered probabilities instead of constant ones.

In the third part, we analyze a market model that take care of the influence of large investors. The influence of large investors, such as hedge funds, mutual funds, and insurance companies, on prices of risky assets, can be studied from very different viewpoints ranging from direct price impact of order execution (selling or buying) to feedback effects from trading to hedge portfolios of derivatives written on the underlying. However, there is also an influence of large investors on the overall market sentiment that arises from their perceived informational superiority. That is, most of the time, the rest of the market takes large investors' portfolio decisions as signals revealing an important insider information not available to small or price-taking investors. Therefore in this study, we solve a finite-time utility maximization problem by considering a partially observable regime-switching environment, in which there is a large investor (or group of institutional investors) that has control over the intensity matrix of the continuous-time finite state Markov chain governing the state of the environment. We allow large investor's portfolio choices, as a fraction of the wealth invested in the risky asset, to have an indirect but persistent effect on the price process, through dependence on the controlled intensity of the Markov chain with next-neighborhood-type dynamics. We call this effect *market impact*. By taking the generator matrix of the unobservable Markov chain as a function of portfolio holdings of the large investor, and focusing on the price process with pure-jump dynamics affected by the unobservable Markov chain, we solve the problem of utility maximization from terminal wealth for logarithmic and power utility preferences. The idea to model market impact through an intensity-based framework is due to [28] where the authors deal with a control problem for optimal investment and consumption for a large investor in the full information case with asset prices following jump-diffusion dynamics and a market with two possible states.

To summarize our contributions in this part, firstly we solve the utility maximization problem for logarithmic and power utility preferences with indirect impact arising from controlling the intensity of the Markov chain both under full and partial information settings. For comparison purposes, we also give solutions to those problems without impact, that is, when there is no control of the intensity. Even for the simple logarithmic utility case, the presence of indirect impact makes pointwise maximization impossible and hence we need to rely on dynamic programming techniques. Secondly, we transform the partial information problem to a full information problem by using stochastic filtering and apply control theory for piecewise deterministic Markov processes (PDMP) to our problem to derive the optimality equation for the value function. We rely on the results given in [36] and characterize the value function as the unique viscosity solution of the associated dynamic programming equation. Thirdly, by focusing on a two-state Markov chain example, we show that there is always a gain for a large investor from controlling the intensity of the Markov chain both in full and partial settings albeit it is smaller in the latter one. In particular, the large investor can take advantage of the "bear" state of the market by short-selling. Also optimal strategies are more aggressive in the presence of market impact such that the large investor buys more in the "bull" state and short sells more in the "bear" state compared to the corresponding no-impact case. Also it is evident from numerical examples that, as time approaches to the maturity, optimal portfolio strategies with and without impact from intensity control converges to each other under both full and partial information settings.

Outline of the thesis

Part I is structured as follows. Chapter 1 introduces the problem and gives the related literature review. In Chapter 2, we try to formalize the notion of instantaneous correlation. Chapter 3 gives the general (defaultable) zero-coupon bond price formula, focusing on the exponential change of measure. In Chapter 4, we start modeling with multivariate Jacobi processes, by presenting different representations and discussing general properties of it. Then we give boundary unattainability conditions related to the multivariate Jacobi process. We continue then on computation of transition density functions via spectral methods and dual process representation. Then we state our bond pricing formula as well as certain applications such as credit default swap pricing. Finally in Chapter 5, as an Appendix, we provide a modest generalization of the Yamada–Watanabe condition that is needed to show pathwise uniqueness of certain stochastic differential equations (SDE) and we conclude with an example related to the trapped Jacobi process, which might be of independent interest on its own.

Part II is structured as follows. Chapter 1 introduces the pairs trading model that we analyze. In Chapter 2 we analyze the portfolio optimization problem in a full information setting. In Chapter 3 we solve the utility maximization problem under partial information. In Chapter 4, we provide the numerical analysis of our toy example with a two-state Markov chain. We give proofs related to the dynamic programming approach as an Appendix in Chapter 5.

Part III is structured as follows. In Chapter 1, we introduce the underlying framework and the main assumptions. We also provide a rigorous construction of the model. In Chapter 2, we study the optimization problem under full information. Chapter 3 contains the optimization problem under partial information and reduction of the problem to full information via stochastic filtering as well as the PDMP techniques to solve the problem and characterization of the optimal value function via unique viscosity solution of the related HJB equation. Finally, in Chapter 4, we present a two-state Markov chain example and discuss model implications for a large investor. We also provide an Appendix in Chapter 5, containing technical proofs.

Part I

Term Structure Modeling with Jacobi Processes

Chapter 1

Introduction

Credit spread (or yield spread) of a given corporate bond is defined to be the difference between its yield and the yield of a government bond, or more generally (and more accurately after the recent debt crisis) a reference bond, which is assumed to be risk-free and has the same time to maturity. The *raison d'etre* of credit spreads is the risk of default inherent in corporate bonds, in which case the bond holders receive only partial payment or no payment at all. Therefore, in order to price corporate bonds, or more generally any defaultable bond and other credit sensitive instruments, it is necessary to consider the evolution of credit spreads and the risk-free term structure, as well as the correlation structure between these assuming that both are given stochastically. The dynamic credit risk models that try to capture the dependence structure include but are not limited to [128], [44], [54], [57], [131] and [106] (in chronological order), in which different assumptions are in place regarding to the default event, recovery process or underlying state variables, etc. In what follows, we will briefly summarize the previous literature on defaultable term structure of interest rates with special emphasis on the correlation structure between the risk-free rate and the credit spreads.

Apart from the empirical studies (see for example, [53], [37], and recently [88] and references therein), there are mainly two strands of research for corporate bond valuation and defaultable term-structure modeling: structural models and reduced-form models. Structural models assume a stochastic process for the dynamics of the value of the firm and default occurs when the value process reaches a predetermined level, which might be determined endogenously in the model (see [20, Ch. 2 and Ch. 3] for a comprehensive review on structural models). On the other hand, reduced-form models treat the default as a random event following a hazard process with certain intensity, which may depend on the state variable process describing the economy, the default-free rate and other contingent claims (see [120], or [20] for the complete treatment of reduced-form models).

Longstaff and Schwartz [128] examine the dependence structure between default-free and defaultable term structures in a structural setting. In this setting, there are two effects at play determining the correlation between changes in credit spreads and changes in Treasury yields. The first effect is related to an increase in the risk-free rate, which leads to an increase in the drift of the firm value under the risk-neutral probability. This eventually leads to the lower default probability because the default event is defined to be the first time the value of the firm hits a predetermined barrier. The second effect is coming from the assumed correlation between the value of the firm and the risk-free interest rate. Of course when this correlation is negative, a positive interest rate change leads to an increase in the credit spread, and when it is positive it even amplifies the impact of the first effect, that is, more negative correlation between the credit spread and the risk-free rate. By applying 30-year government yield as a proxy of the default-free rate and the return on the S&P 500 index as a proxy for the asset value, they conclude that the correlation between changes in the asset value and changes in the credit spread is negative, and the stronger the negative relationship, the greater the correlation between the risk-free rate and the firm value. Therefore, there is an unambiguous economic relation between the credit spread and the risk-free rate, manifesting itself as the negative correlation in the structural framework. This observation is also evidenced by the empirical findings in [53], [37] and [88].

Contrary to the structural framework where modeling of the firm value is the primary concern, in a reduced-form setting default is modeled exogenously with the random time (mathematically by a totally inaccessible stopping time τ) and its hazard rate process possibly with a certain intensity. It is very well known that under the doubly stochastic framework for the default time, the instantaneous credit spread $(s_t)_{t\geq 0}$ coincides with the intensity of the default event given by a stochastic process $(\lambda_t)_{t\geq 0}$, provided that there is no recovery. More generally, $(s_t)_{t\geq 0}$ equals the product of hazard rate and percentage loss given default under various recovery schemes such as recovery of market value (RM), recovery of treasury value (RT) or recovery of face value (RF) (see [132, Ch. 9]). By using a reduced-form model, Duffee [53, 54], and Duffie and Singleton [57] documented negative correlation between changes in the credit spreads and the level as well as the slope of the term structure of risk-free interest rate. In those reduced-form models, the default-free short rate $(r_t)_{t\geq 0}$ and the credit spread $(s_t)_{t\geq 0}$ are given as correlated square-root processes providing analytically tractable bond prices due to affine properties (see [55] or [76, Ch. 10]). However, due to the implications of admissibility conditions on affine processes (see [43], [78]), one cannot simultaneously have a positive spread (or intensity) and risk-free rate while sustaining negatively correlated increments of $(r_t)_{t\geq 0}$ and $(s_t)_{t\geq 0}$ in these models. Non-positivity of the intensity is a great concern due to the impossibility to construct a Cox process with a negative intensity rate. Actually, in an affine diffusion models, in which the state is the canonical one, $\mathbb{R}_+^m \times \mathbb{R}^n$, there is a strict trade-off between the instantaneous correlation structure and the stochastic volatility components of the state variable processes used in modeling. To be more precise, let us assume that the default-free rate $(r_t)_{t\geq 0}$ and the credit spread $(s_t)_{t\geq 0}$ are modeled as correlated square-root processes

$$dr_t = \mu_r(\theta_r - r_t)\, dt + \sigma_r \sqrt{r_t}\, dW_t^1, \tag{1.1}$$

$$ds_t = \mu_s(\theta_s - s_t)\, dt + \sigma_s \sqrt{s_t}(\rho\, dW_t^1 + \sqrt{(1-\rho^2)}\, dW_t^2), \tag{1.2}$$

where $\rho \in [-1, 1]$ is interpreted as instantaneous correlation between r and s, and W^1 and W^2 are independent standard Brownian motions. By a direct computation, it is seen that the off-diagonal entry of the diffusion matrix $\sigma\sigma^\top$ is non-null unless $\rho = 0$, implying that even the processes $(r_t)_{t\geq 0}$ and $(s_t)_{t\geq 0}$ are Cox–Ingersoll–Ross (CIR) processes, which are typical examples of affine processes, they are not jointly affine (see [76, Ch. 10]). Hence the corresponding analytical tractability of the model is lost. The only way to include instantaneous correlation in an affine diffusion case (with standard canonical state space) is to choose one of the driving factor (state variable) processes as a Gaussian one such as a Vašíček process, however in that case the positivity of the interest rate or the intensity process can not be sustained anymore. For example, by using a three-dimensional affine diffusion on $\mathbb{R}_+^2 \times \mathbb{R}^1$ (which is a $\mathbb{A}_2(3)$ model in the terminology of Dai and Singleton [43]), Duffie and Singleton [57] propose a model with more flexible correlation structure by imposing restrictions on the parameters (other than the admissibility conditions), however due to the inclusion of a Gaussian factor this model still allows for negative risk-free short rates to occur with small probability. Although, non-positivity of the interest rates and intensity process are *disregarded* in the literature by assuming that its probability is close to zero, it might be a concern especially in term-structure modeling and complex derivative pricing. For example, in a recent study by Feldhütter [71], it is shown that in the Gaussian affine term-structure model, which offers a maximal flexibility in modeling correlations between the state variables, the probability of negative 1-year yields amounts to be a non-negligible 5.98 percent, which might become a more concerning issue during the current very low and near zero interest rates environment. For the implications of the negative interest rates in terms of pricing of zero-coupon bonds and interest rate derivatives, the reader is referred to an expository paper by L.C.G. Rogers [144]. In particular, L.C.G. Rogers showed that even if the parameters of an interest rate model that is governed by a Vašíček process are chosen in such a way that the probability of negative interest rate is close to zero, with the same set of parameters the bond price grows exponentially, which is counterfactual. It is also worth mentioning here the paper by F. Black [23], which explains the economic reasons of the impossibility of the negative interest rates and proposes to see nominal interest rates as options. For further studies addressing this issue in non-defaultable term-structure setting, we refer the reader to the works of [81], [145] and [91].

Of course one important issue while addressing the positivity of the risk-free rate and the intensity process is also not to restrict the domain of the processes, $(r_t)_{t\geq 0}$ and $(s_t)_{t\geq 0}$ in order to obtain negative correlation. Or to put it differently, one should postulate a model that does not lead to restrictions on the joint distribution of $(r_t)_{t\geq 0}$ and $(s_t)_{\geq 0}$ in such a way that they are not supported by the data. We choose to model the defaultable term structure in a setting where the default time τ is given by a doubly stochastic random time under the risk neutral pricing measure $\mathbb{Q}$, where all the uncertainty in the model is driven by an d-dimensional Markovian state process $X := (X_t)_{t\geq 0}$ taking values in a state space $E \subseteq \mathbb{R}^d$. This line of defaultable term-structure models can be traced back to [119], [130], [131] and [57]. Modeling the whole economy by some state variables can be justified by the empirical observation of Longstaff and Schwartz [128], since they found that changes in the level of interest rates are found to be more important for the variation in credit spreads than changes in the value of the firm. In other words, we can say that the general state of the economy is more important than the firm specific issues to capture the dependence structure between different term structures. In this respect, the economic background filtration represents the information generated by an arbitrage-free model for default-free security prices. More precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ denote a filtered probability space, where $\mathbb{Q}$ is a equivalent martingale measure and the default-free security prices are assumed to follow $(\mathcal{F}_t)_{t\geq 0}$-adapted processes as well as the instantaneous risk-free rate $(r_t)_{t\geq 0}$. If we let $H_t = \mathbb{I}_{\{\tau \leq t\}}$ be the associated default indicator process and set $\mathcal{H}_t = \sigma(H_s, s \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every $t \geq 0$, we can assume that default is observable and that investors have access to the information contained in the background filtration $(\mathcal{F}_t)_{t\geq 0}$, so that the information available to investors at time t is given by $\mathcal{G}_t$. Assuming zero recovery in case of default, at time t the valuation formula of the pre-default value of the promised payoff of a defaultable contingent claim with maturity $T \geq t$ represented by an $\mathcal{F}_T$-measurable bounded or non-negative random variable Y is given by the conditional expectation

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\Big\{-\int_t^T (r(X_u) + s(X_u))\,du\Big\}Y \,\Big|\, \mathcal{F}_t\right], \tag{1.3}$$

where the short rate process satisfies $r_t = r(X_t)$ for some measurable function $r : E \to \mathbb{R}_+$

and the intensity of the Cox process (credit spread), in which τ is the first jump time of it, satisfies $s_t = s(X_t)$ for some measurable $s : E \to \mathbb{R}_+$. In order to have an analytically tractable zero-coupon bond price formula or a derivative price depending on X_T, in which $Y \equiv 1$ in the former case and $Y \equiv g(X_T)$ for some measurable function g in the latter, one needs to impose additional conditions on the functions r and s and the state process X (and possibly on g for derivative pricing). Without including Gaussian factors, it is impossible to introduce negative correlation while sustaining the positivity of the short rate and the intensity process in a term-structure model, where the state processes are affine diffusions on a canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$ and r and s are linear functions of these. Even the inclusion of jump components in the state process does not alleviate the problem because of the non-negativity requirements of the processes r and s. One possible solution to this with a relatively tractable bond price formula is using quadratic term-structure models (see e.g., [2] and [125]), in which generally the state processes are given by quadratic functions of Gaussian affine processes (see [56] and [33] for applications in credit risk), however in these cases only in a part of the domain of the processes negative correlation is captured. That is, quadratic term-structure models might capture negative instantaneous correlation with certain probability, but not all the time (see the discussion in [120, Ch. 5]). Although, affine diffusions on the standard canonical state space have shortcomings in capturing correlation while sustaining the positivity, recent studies about affine processes on non-canonical state spaces such as conic, parabolic subspaces of $\mathbb{R}^d$ [152] or affine process on positive semi-definite matrices [40] are promising and worth to explore.

To circumvent problems mentioned above and propose a model capturing negative instantaneous correlation in a reduced-form setting, one starting point is to posit dependence between the default-free rate and the credit spread in a functional way. Before, we give our concrete modeling approach, in the next chapter we give what it means by instantaneous correlation and try to formalize it in a Markovian setting.

Chapter 2

Modeling Instantaneous Correlation

We introduce a time-homogeneous Markov process X as the state process and its extended generator $\mathcal{A}$. We also assume that paths of X are càdlàg. Let E be an open or closed subset of $\mathbb{R}^d$ (more generally a locally compact Hausdorff and second-countable regular space[1]) and denote the σ-algebra of Borel sets over E by $\mathcal{E}$. We denote the vector space of measurable functions $f : E \to \mathbb{R}$ by $\mathcal{M}(E)$ and the subset of (strictly) positive functions $f : E \to \mathbb{R}_{+(+)}$ by $\mathcal{M}_{+(+)}(E)$. Similarly, the subsets of bounded functions and (strictly) positive bounded functions are denoted by $\mathcal{M}^{\mathrm{b}}(E)$ and $\mathcal{M}^{\mathrm{b}}_{+(+)}(E)$, respectively, with $\|f\| = \sup_{x\in E} |f(x)|$. We will work in the canonical setup, in which the probability space Ω is the space of càdlàg functions $\omega : \mathbb{R}_+ \to E$, equipped with the Skorohod topology and the coordinate process $(X_t)_{t\geq 0}$ is defined by

$$X_t(\omega) := \omega(t), \quad t \geq 0.$$

The natural filtration is defined by $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\geq 0}$ where $\mathcal{F}^X_t = \sigma(X_s : 0 \leq s \leq t)$. We also take the right-continuous filtration, that is, $\mathcal{F}^X_t = \mathcal{F}^X_{t+}$ for $t \geq 0$. Obviously, X is adapted to its natural filtration. Also define that $\mathcal{F}^X_\infty = \bigvee_{t\geq 0} \mathcal{F}^X_t$. So the filtered probability space is given by the quadruple $(\Omega, \mathcal{F}^X_\infty, \mathbb{F}^X, \mathbb{Q})$. To be more precise on the probability measure $\mathbb{Q}$, for any initial distribution ν on $\mathcal{E}$, we denote it by $\mathbb{Q}_\nu$. If $\nu = \delta_{\{x\}}$ (unit mass at $x \in E$), we denote it by $\mathbb{Q}_x$.

Now, let $(P_t)_{t\geq 0}$ be the corresponding semigroup of X acting on a closed subspace $L \subset \mathcal{M}^{\mathrm{b}}(E)$ with an initial distribution ν, that is

$$\mathbb{E}_{\mathbb{Q}_\nu}[f(X_{s+t})|\mathcal{F}^X_s] \overset{\text{a.s.}}{=} P_t f(X_s)$$

for all $s, t \geq 0$ and $f \in L$. We know that if L is separating (or measure determining), then the semigroup $(P_t)_{t\geq 0}$ and ν determine the finite-dimensional distributions of the process ([70, Prop. 4.1.6]). Therefore, if we can identify the generator of the corresponding semigroup, the finite-dimensional distributions of a Markov process can be determined. Hence, determination of the generator(s) and their domains are crucial for characterizing Markov processes. However, our aim is to use the generator concept to define instantaneous correlation in a succinct way for certain Markov processes.

[1]By Urysohn's metrization theorem (see [82, Thm. 4.58], every Hausdorff, second-countable regular space is metrizable.

Sometimes we work with special Markov processes, called C_0-Feller processes[23], whose semigroup has the following properties; (1) $P_0 = \mathbb{1}$ (the identity operator), (2) $0 \le P_t f \le 1$ for all $t \ge 0$ and $f \in C_0(E)$ such that $0 \le f \le 1$, (3) $P_{s+t} = P_s P_t$, for $s,t \ge 0$, (4) $\lim_{t \searrow 0} P_t f(x) = f(x)$ for all $f \in C_0(E)$ and $x \in E$.

2.1 Extended generator and opérateur carré du champ

We begin with two definitions related to generators, namely the full and the extended generator of the Markov process X. In general, the full generator $\widehat{\mathcal{A}}$ of a measurable contraction semigroup[4] $(T_t)_{t\ge 0}$ on $L \subset \mathcal{M}^{\mathrm{b}}(E)$ is defined by

$$\widehat{\mathcal{A}} = \{(f,g) \in L \times L : T_t f - f = \int_0^t T_s g \, ds, \ t \ge 0\}. \tag{2.1}$$

Generally, the full generator is not single-valued (e.g., take right-shift operator, $T_t f(x) := f(x+t)$ on $\mathcal{M}^{\mathrm{b}}(\mathbb{R})$, since $(0,g) \in \widehat{\mathcal{A}}$ for every $g \in \mathcal{M}^{\mathrm{b}}(\mathbb{R})$). We can say that the full generator is single valued if $(0,g) \in \widehat{\mathcal{A}}$ implies that $g = 0$. In this case, one can interpret $\widehat{\mathcal{A}}$ as an operator on a subset of $\mathcal{M}^{\mathrm{b}}(E)$ that maps f to the unique function g, hence we can denote the domain of this operator by $\mathcal{D}(\widehat{\mathcal{A}})$, and write for a pair $(f,g) = (f, \widehat{\mathcal{A}}f)$ for $f \in \mathcal{D}(\widehat{\mathcal{A}})$. One important property of functions in the domain of the full generator is the following,

Proposition 2.2 ([70, Prop. 4.1.7])**.** *Let X be a (time-homogenous) Markov process taking values in E with the full generator $\widehat{\mathcal{A}}$. Then for $(f,g) \in \widehat{\mathcal{A}}$ the process defined by*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t g(X_s) \, ds, \quad t \ge 0, \tag{2.3}$$

is an $(\mathbb{F}^X, \mathbb{Q}_x)$-martingale for every $x \in E$.

Remark 2.4. The function g is uniquely defined up to sets of zero potential (see [143, Chp. VII]. Note that a Borel set $A \subset \mathcal{E}$ is said to be of zero potential if for every $x \in E$,

$$\mathbb{Q}_x \left[\int_0^\infty \mathbb{I}_A(X_s) \, ds = 0 \right] = 1.$$

This condition is equivalent to that the process spends almost surely no time in A when it starts at $x \in E$. We will identify all versions of the function g and we denote all these versions by $\widehat{\mathcal{A}}f$ if $(f,g) \in \widehat{\mathcal{A}}$.

From the above proposition, we can give the stochastic definition of the full generator of a Markov process and its domain by using the associated process M^f given by (2.3).

Definition 2.5 (Full generator)**.** The full generator $\widehat{\mathcal{A}}$ of an E-valued (time-homogeneous) Markov process $(X_t)_{t\ge 0}$ is defined by

$$\widehat{\mathcal{A}} = \{(f,g) \in \mathcal{M}^{\mathrm{b}}(E) \times \mathcal{M}^{\mathrm{b}}(E) : (M_t^f)_{t\ge 0} \text{ is an } (\mathbb{F}^X, \mathbb{Q}_x)\text{-martingale } \forall x \in E\}. \tag{2.6}$$

[2] $C_0(E)$ is the Banach space of continuous functions vanishing at infinity with the norm $\|f\| = \sup_{x\in E} |f(x)|$.

[3] See [70, pg. 164].

[4] See [70, pg. 23] for the definition of a measurable semigroup.

Definition 2.7 (Domain of the full generator)**.** A function $f \in \mathcal{M}^{\mathrm{b}}(E)$ belongs to the domain $\mathcal{D}(\widehat{\mathcal{A}})$ of the full generator $\widehat{\mathcal{A}}$ if there exists a function $g \in \mathcal{M}^{\mathrm{b}}(E)$ such that $(M_t^f)_{t\geq 0}$ is a $(\mathbb{F}^X, \mathbb{Q}_x)$-martingale for every $x \in E$.

Although the definitions above are enough for most of the applications, we should note that there may be some measurable functions f not in the domain of the full generator $\widehat{\mathcal{A}}$. A natural extension is to relax the conditions that the functions in $\mathcal{D}(\widehat{\mathcal{A}})$ be bounded and also to have a weaker requirement on M^f than to be a martingale. More precisely, we have,

Definition 2.8 (Extended generator)**.** The extended generator $\mathcal{A}$ of a E-valued (time-homogenous) Markov process $(X_t)_{t\geq 0}$ is defined by

$$\mathcal{A} = \{(f, g) \in \mathcal{M}(E) \times \mathcal{M}(E) : (M_t^f)_{t\geq 0} \text{ is a} \, (\mathbb{F}^X, \mathbb{Q}_x)\text{-local martingale} \, \forall x \in E\}, \qquad (2.9)$$

such that $\mathbb{Q}_x$ almost surely $\int_0^t |g(X_s)| \, ds < \infty$ for every $t \geq 0$, and $x \in E$.

Similarly, we can define the domain of the extended generator; see also [46, Def. 14.15].

Definition 2.10 (Domain of the extended generator)**.** A function $f \in \mathcal{M}(E)$ belongs to the domain of the extended generator $\mathcal{D}(\mathcal{A})$ if there exists a measurable function $g \in \mathcal{M}(E)$ such that $\mathbb{Q}_x$ almost surely $\int_0^t |g(X_s)| \, ds < \infty$ for every $t \geq 0$ and $(M_t^f)_{t\geq 0}$ is an $(\mathbb{F}^X, \mathbb{Q}_x)$-local martingale for every $x \in E$.

Remark 2.11. Similar to the case of a full generator, the function g is uniquely defined up to set of zero potential, and we denote all versions of g by $\mathcal{A}f := g$.

The advantage of the extended generator over the full generator, where bounded measurable functions are taken to be in the domain, is that the possible easy specification of its domain. Also, it can be a non-trivial concept for Markov processes that are also strict local martingales. Furthermore, as every martingale is a local martingale, it holds that $\widehat{\mathcal{A}} \subset \mathcal{A}$ and hence $\mathcal{D}(\widehat{\mathcal{A}}) \subset \mathcal{D}(\mathcal{A})$ showing that the extended generator do really "extend" the full generator. Similar to the martingale problem, the extended generator can also be used to characterize Markov processes uniquely, which is called the (local) martingale problem.

Now we give an example that we borrowed from [46] showing that why it might be convenient to work with the extended generator in terms of local martingales. Let's take a Poisson process $N := (N_t)_{t\geq 0}$. For $t > 0$, the random variable N_t has the Poisson distribution $P[N_t = n] = e^{-\lambda t}(\lambda t)^n/n!$, for some constant $\lambda > 0$. We let $(\mathcal{F}_t^N)_{t\geq 0}$ denote the natural filtration, $\mathcal{F}_t^N = \sigma\{N_s, s \leq t\}$. N has independent increments, implying that N is a Markov process. In particular, for any bounded function f and $s \leq t$,

$$\mathbb{E}[f(N_t)|\mathcal{F}_t^N] = \sum_{i=0}^{\infty} f(i + N_s) \frac{e^{-\lambda(t-s)}(\lambda(t-s))^i}{i!}.$$

One can also consider N as a Markov family on the state space $E = \{0, 1, \ldots\}$ with the measure $\mathbb{P}_x$ being such that $N_0 = x \in E$ and $N_t - x$ is a Poisson process. Now take the generator $\mathcal{A}^N$ of N. One can show that for a bounded f,

$$\mathbb{E}_x[f(N_t)] = f(x)(1 - \lambda t + o(t)) + f(x+1)(\lambda t + o(t)) + o(t),$$

where $o(t)$ refers to a function g with $\lim_{t\to 0} \frac{g(t)}{t} = 0$ and therefore,

$$\mathcal{A}^N = \lim_{t\to 0} \frac{\mathbb{E}_x[f(N_t)] - f(x)}{t} = \lambda(f(x+1) - f(x).$$

The domain $\mathcal{D}(\widehat{A}_S^N)$ of the *(strong) infinitesimal generator*[5] $\widehat{A}_S^N$, is the set of functions for which the above limit exists uniformly in x. However, it can be shown that for every function f such that $\mathbb{E}_x[|f(N_t)|] < \infty$ for all $x \in E$, $t \in \mathbb{R}_+$, the process

$$M_t^f = f(N_t) - f(N_0) - \lambda \int_0^t (f(N_s+1) - f(N_s))\, ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^N)_{t\geq 0}$ martingale. From the optional sampling theorem, the process $M^{f,n} := M^f_{t\wedge n \wedge T_n}$, where T_n, $n \geq 1$ are jump times of N, is a martingale, since $\tau_n := n \wedge T_n$ is a bounded stopping time. Note that $\mathbb{P}_x$ almost surely the process $M^{f,n}$ involves only the values of f on the set $A = \{x, x+1, \ldots, x+n\}$ and therefore $M^{f,n} = M^{\hat{f},n}$ with $\hat{f} = f(y)\mathbb{I}_A(y)$. Now since $\tau_n \to \infty$ as $n \to \infty$, $\mathbb{P}_x$-a.s, M^f is a local martingale for any finite-valued function f. This shows that $\mathcal{D}(\mathcal{A}^N)$, the domain of the extended generator, consists of all functions $f : E \to \mathbb{R}$ without any restrictions.

One important property of the extended generator, its domain and the angle bracket processes (predictable quadratic variation) (see [103, pg. 38] or [141, pg. 124] for the definition) associated with the process is summarized as ;

Proposition 2.12. *Let $f, g \in \mathcal{D}(\mathcal{A})$ such that $f^2, g^2 \in \mathcal{D}(\mathcal{A})$. Then for local martingales M^f and M^g defined as in* (2.3)*, the signed measure related to the predictable quadratic covariation (angle bracket process) $\langle M^f, M^g\rangle$ is absolutely continuous with respect to Lebesgue measure if the quadratic covariation process $[M^f, M^g]$ is locally integrable.*

Proof. For $f \in \mathcal{D}(\mathcal{A})$, the process $\int_0^t \mathcal{A}f(X_s)\, ds$ is a continuous process of finite variation, hence $(f(X_t))_{t\geq 0}$ is a special semimartingale, being sum of a local martingale and a finite variation process. In particular, since the local martingale M^f is càdlàg, this process is càdlàg. By applying Itô's formula for the semimartingale $Z := f(X)$,

$$(f(X_t))^2 = (f(X_0))^2 + 2\left(\int_0^t f\mathcal{A}f(X_s)\, ds + \int_0^t Z_{s-} dM_s^f)\right) + [M^f, M^f]_t, \quad t \geq 0. \tag{2.13}$$

Since $[M^f, M^f]$ is locally integrable, we know that the predictable quadratic variation exists ([141, pg. 124] and one can define it as the dual predictable projection of $[M^f, M^f]$. Hence we can conclude that

$$(f(X_t))^2 - (f(X_0))^2 - 2\int_0^t f\mathcal{A}f(X_s)\, ds - \langle M^f, M^f\rangle_t, \quad t \geq 0, \tag{2.14}$$

is a local martingale. Now by assumption $f^2 \in \mathcal{D}(\mathcal{A})$,

$$(f(X_t))^2 - (f(X_0))^2 - \int_0^t \mathcal{A}f^2(X_s)\, ds, \quad t \geq 0, \tag{2.15}$$

is a local martingale. Since the difference of two local martingales is again local martingale, this implies that

$$\int_0^t \mathcal{A}f^2(X_s)\, ds - 2\int_0^t f\mathcal{A}f(X_s)\, ds - \langle M^f, M^f\rangle_t, \quad t \geq 0, \tag{2.16}$$

is a predictable local martingale of finite variation hence it is constant at 0 since $M_0^f = 0$. We can then conclude that $\langle M^f, M^f\rangle$ is absolutely continuous with respect to Lebesgue measure. Similar computation yields the same for $\langle M^g, M^g\rangle$ and the rest follows from the polarization identity. □

[5] See [46, Sec. 14.2] for the definition of a strong generator.

Remark 2.17. If the domain of the extended generator is an algebra, that is, if $f, g \in \mathcal{D}(\mathcal{A})$ then $fg \in \mathcal{D}(\mathcal{A})$, this naturally implies $f^2, g^2 \in \mathcal{D}(\mathcal{A})$ and hence;

Corollary 2.18. *Let $f, g \in \mathcal{D}(\mathcal{A})$, where $\mathcal{D}(\mathcal{A})$ is assumed to be an algebra. Then for the local martingales M^f and M^g defined as in (2.3), the signed measure $\langle M^f, M^g\rangle$ is absolutely continuous with respect to Lebesgue measure if $[M^f, M^g]$ is locally integrable.*

Remark 2.19. In the above, one can alternatively assume M^f and M^g be locally square-integrable since locally square-integrable local martingales have locally integrable quadratic covariation process.

One related result, which ensures that for every law ν on $\mathcal{E}$, for every $(\mathbb{Q}_v, \mathbb{F}^X)$ square-integrable martingales M and N, the signed measure $\langle M, N\rangle$ is absolutely continuous with respect to Lebesque measure if and only if the extended generator is an algebra when the given process is C_0-Feller process. This is first shown by [133].

Proposition 2.20. *Let X be a C_0-Feller process. Then for every law ν on $\mathcal{E}$ and for every $(\mathbb{Q}_v, \mathbb{F}^X)$ square-integrable martingales M and N, the signed measure $\langle M, N\rangle$ is absolutely continuous with respect to Lebesgue measure if and only if the domain of the extended generator $\mathcal{D}(\mathcal{A})$ is an algebra.*

Proof. See e.g., [26, Chp. 4] and references therein, especially [133], [75] and [24]). □

Now, we can define the opérateur carré du champ for functions $f, g \in \mathcal{D}(\mathcal{A})$ such that their product $fg \in \mathcal{D}(\mathcal{A})$.

Definition 2.21 (Opérateur carré du champ for a domain assumed to be an algebra)**.** Let X be an E-valued Markov process. If $\mathcal{D}(\mathcal{A})$ is an algebra, then the opérateur carré du champ is given by

$$\Gamma(f, g) := \mathcal{A}(fg) - f\mathcal{A}g - g\mathcal{A}f, \tag{2.22}$$

for $f, g \in \mathcal{D}(\mathcal{A})$.

For the computation of predictable quadratic variation (angle bracket) process related to certain Markov processes, we have,

Proposition 2.23. *Let X be an E-valued (time-homogenous) Markov process with its extended generator $\mathcal{A}$ and its domain $\mathcal{D}(\mathcal{A})$, which is assumed to be an algebra. For any $f \in \mathcal{D}(\mathcal{A})$ and the associated local martingale with $h = \mathcal{A}f$,*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t h(X_s)\, ds, \quad t \geq 0, \tag{2.24}$$

if the quadratic variation process $[M^f, M^f]$ is locally integrable then, M^f has the predictable quadratic variation (angle bracket) process

$$\langle M^f, M^f\rangle_t = \int_0^t \Gamma(f, f)(X_s)\, ds, \qquad t \geq 0. \tag{2.25}$$

Proof. Note that the since $[M^f, M^f]$ is locally integrable, $\langle M^f, M^f\rangle$ exists. By applying Itô's formula for the semimartingale $Z := f(X)$,

$$(f(X_t))^2 = (f(X_0))^2 + 2\left(\int_0^t f(X_s)h(X_s)\, ds + \int_0^t Z_{s-} dM_s^f)\right) + [M^f, M^f]_t, t \geq 0. \tag{2.26}$$

Since $\langle M^f, M^f\rangle$ is absolutely continuous with respect to Lebesgue measure as it is shown in Proposition 2.12, from the Motoo's theorem ([134] or [149, Thm. 66.2]) that an absolutely continuous additive functional is of the form $\int_0^t u(X_s)\,ds$ for some measurable function u, we can write $\langle M^f, M^f\rangle_t = \int_0^t u(X_s)\,ds$. Noting that $[M^f, M^f] - \langle M^f, M^f\rangle$ is a local martingale, the process

$$(f(X_t))^2 - (f(X_0))^2 - 2\int_0^t f(X_s)h(X_s)\,ds + \int_0^t u(X_s)\,ds \tag{2.27}$$

is a local martingale. Now, notice that $\mathcal{A}f^2 = u + 2fh$ and hence $u = \mathcal{A}f^2 - 2f\mathcal{A}f = \Gamma(f,f)$. □

Corollary 2.28. *Let X be an E-valued (time-homogenous) Markov process, in which the domain of the extended generator $\mathcal{D}(\mathcal{A})$ is assumed to be an algebra. For local martingales M^f and M^g (defined as in 2.3) for $f, g \in \mathcal{D}(\mathcal{A})$, if the quadratic variation process $[M^f, M^g]$ is locally integrable then the predictable quadratic covariation (angle bracket) process between M^f and M^g is given by*

$$\langle M^f, M^g\rangle_t = \int_0^t \Gamma(f,g)(X_s)\,ds, \qquad t \geq 0.$$

Proof. It follows from the polarization identity

$$\langle M^f, M^g\rangle = \frac{1}{2}(\langle M^f + M^g, M^f + M^g\rangle - \langle M^f, M^f\rangle - \langle M^g, M^g\rangle).$$

□

2.2 Instantaneous correlation

By the same reasoning, with the help of the "carré du champ" operator, we can also give the definition of instantaneous correlation between two processes $f(X)$ and $g(X)$ for $f, g \in \mathcal{D}(A)$.

Definition 2.29 (Instantaneous correlation)**.** Let X be a E-valued (time-homogenous) Markov process, in which the domain of the extended generator $\mathcal{D}(\mathcal{A})$ is assumed to be an algebra and assume that local martingales M^f and M^g (defined as in 2.3) for $f, g \in \mathcal{D}(\mathcal{A})$ have locally integrable quadratic variation process $[M^f, M^g]$. Then the instantaneous correlation between processes $f(X)$ and $g(X)$ for $f, g \in \mathcal{D}(\mathcal{A})$ is defined as

$$\rho_t(f(X), g(X)) := \frac{\Gamma(f,g)(X_t)}{\sqrt{\Gamma(f,f)(X_t)}\sqrt{\Gamma(g,g)(X_t)}}, \qquad t \geq 0.$$

Remark 2.30. If either $\Gamma(f,f)(X_t) = 0$ or $\Gamma(g,g)(X_t) = 0$ for some $t \geq 0$, we set

$$\rho_t(f(X), g(X)) := 0.$$

If the state process X is given by a d-dimensional diffusion process that satisfies the stochastic differential equation $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t$, where the functions μ and σ are assumed to satisfy certain conditions such that the SDE has a unique weak solution, then the extended generator $\mathcal{A}$ of X is equal on $C^2(E)$-functions to

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \mu_i(x)\frac{\partial}{\partial x_i},$$

where $a(x) = \sigma(x)\sigma^\top(x)$ ([143, Ch. VII]). Hence for $f, g \in C^2(E)$, the "carré du champ" operator is given by

$$\Gamma(f,g)(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x).$$

Therefore, the instantaneous correlation between $(f(X_t))_{t\geq 0}$ and $(g(X_t))_{t\geq 0}$ can be computed similarly by the help of that operator.

Remark 2.31. For a d-dimensional diffusion X, instantaneous correlation between components $X^{(i)}$ and $X^{(j)}$ is

$$\rho_t(X^{(i)}, X^{(j)}) = \frac{a_{ij}(X_t)}{\sqrt{a_{ii}(X_t)}\sqrt{a_{jj}(X_t)}}, \qquad t \geq 0.$$

As it is evident from the discussion above, the idea of the "carré du champ" operator arises from the generator of a Markov process. Actually, the origin of "carré du champ" operators traces back to the work of P.A. Meyer [133] and F. Hirsch [98]. The use of this tool in the mathematical finance literature is relatively scarce, see for example Davis [47] for its use in identifying the generator of the forward measure, Bouleau and Lamberton [25] for finding the hedging strategies in a Markovian setting, or De Waegenaere and Delbaen [48] for its use in dynamic insurance theory. It also deserves further examination because of its capability to reveal certain useful relationships in models where the state variables are represented in a Markovian setting. More precisely, we will utilize the "carré du champ" operator in computing the bond price formula in the next chapter.

Chapter 3

General Zero-Coupon Bond Price Formulation

We start with a summary of what we have done in this chapter. Suppose for simplicity assume we are at $t = 0$. By an *appropriate* equivalent change of measure from risk-neutral $\mathbb{Q}$ to some equivalent pricing measure $\mathbb{Q}^*$ one can show that, for $R(\cdot) := r(\cdot) + s(\cdot)$,

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{-\int_0^T R(X_u)\,du\right\} \,\Big|\, X_0 = x\right] = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{h(x)}{h(X_T)} \,\Big|\, X_0 = x\right] \tag{3.1}$$

for some $h \in \mathcal{D}(\mathcal{A})$, where the relation between R and h is given by

$$R = \frac{\mathcal{A}h}{h} = \mathcal{A}f + \frac{1}{2}\Gamma(f, f),$$

in which Γ is the "carré du champ" operator associated with the diffusion X and $f := \log h$. Moreover, the extended generator of the process X under $\mathbb{Q}^*$ is given by $\mathcal{A} + \Gamma(f, \cdot) := \mathcal{A} + \frac{\Gamma(h,\cdot)}{h}$.

The utmost important thing here is of course to justify the equivalent change of measure, which is dependent on the process X. More precisely, we don't know a priori whether we can find $f \in \mathcal{D}(\mathcal{A})$ (or $h \in \mathcal{D}(\mathcal{A})$) that justifies the formal statements above. Our major contribution in this work is to substantiate this idea and to put it in a clear setting with the aim of finding the pair of functions $(R = r + s, f)$ that allows flexible correlation structure and also enables us to compute the right-hand side of (3.1) easily. The method explained here is applied to find the Laplace transforms of stochastic integrals of certain one-dimensional diffusions by Hurd and Kuznetsov [101]. In particular, a formula similar to (3.1) is computed for the univariate Jacobi (Wright–Fisher) process where

$$R(x) = \alpha_1 \frac{x}{1-x} + \alpha_2 \frac{1-x}{x}$$

for α_1 and α_2 in a set determined by the parameters of the process. In this case, the function h turns out to be equal to $x^{\beta_1}(1-x)^{\beta_2}$, and enables the computation of the expectation by a fast converging series of hypergeometric functions. In the next chapter we will show how to extend this result to a multivariate Jacobi diffusion setting. The crucial point that leads to this result is that for appropriately chosen (R, h), the process remains a Jacobi process under the new measure $\mathbb{Q}^*$ and since the transition densities are known for (multivariate) Jacobi processes (see [94], [93], [154]) the computation of the expectation turns out to be a relatively tractable integral over a simplex including hypergeometric functions. This kind of exponential change of measures are investigated in a very general setting also by [35] for jump-diffusions and by [139] for Markov processes. Basically this kind of locally equivalent change of measures puts certain restrictions on the parameters of the model to prevent the process to reach the boundary of the state space in finite time (for example, in a simple CIR process setting, this condition is given by a Feller condition, see e.g. [35, Sect. 6]). Similar questions are also answered in terms of affine term-structure models (see e.g [34] and references therein), however the focus in those works is to specify the market price of risk (or equivalently the Girsanov kernel) so that under the risk-neutral measure the state processes are affine and hence allow tractable bond prices.

In this section, we assume $R \in \mathcal{M}_+(E)$ for defaultable and/or default-free short-rate process (or more precisely bank account process) to be well-defined (see [11]). This condition can be relaxed for computation of the bond price under certain parameter choice so that the related expectation is finite.

3.1 Exponential change of measure for bond pricing

Suppose that given a pair of functions $(R, h) \in \mathcal{M}_+(E) \times \mathcal{M}_{++}(E)$ such that

$$D_t = \frac{h(X_t)}{h(X_0)} \exp\left\{ -\int_0^t R(X_s)\, ds \right\}, \qquad t \geq 0, \tag{3.2}$$

is a $(\mathbb{F}^X, \mathbb{Q})$ martingale, then we can define a new probability $\mathbb{Q}^h$ on $\mathcal{F}_\infty^X$ by $\mathbb{Q}^h = D_t \cdot \mathbb{Q}$ on $\mathcal{F}_t^X$. One important problem is that whether after change of measure the process still stays Markovian. The answer to this question is positive and proved by [114] in a general case, (see also [143, Chp. VIII] for diffusion case or below). Remember that the shift operators θ_t for $t \geq 0$ are defined by $X_s(\theta_t(\omega)) = X_{s+t}(\omega)$ where X is the coordinate process, $\omega \in \Omega$ and these operators have the semi-group property $\theta_s \circ \theta_t = \theta_{s+t}$.

Theorem 3.3 (General bond pricing formula)**.** *Let X be a (time-homogeneous) E-valued Markov process (conservative, non-explosive and without killing) and $(R, h) \in \mathcal{M}_+(E) \times \mathcal{M}_{++}(E)$ and $(D_t)_{t\geq 0}$ given by* (3.2) *is a $(\mathbb{F}^X, \mathbb{Q}_x)$ martingale. Then X is Markov under $\mathbb{Q}_x^h$ and the price of the (defaultable) zero-coupon bond with fixed time of maturity $T > 0$ is given by,*

$$P(t,T) = \mathbb{E}_{\mathbb{Q}_x}\left[\exp\left\{ -\int_t^T R(X_u)\, du \right\} \,\Big|\, \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{h(X_t)}{h(X_T)} \,\Big|\, X_t \right] \qquad a.s., \tag{3.4}$$

for $t \in [0, T]$.

Proof. First the Markov property of X under $\mathbb{Q}^h$. Let g be a positive measurable function and Z an $\mathcal{F}_t^X$ measurable random variable. Since $D_{s+t} = D_t \cdot D_s \circ \theta_t$, we have

$$\begin{aligned} \mathbb{E}^x_{\mathbb{Q}^h}[Z\, g(X_{s+t})] &= \mathbb{E}^x_{\mathbb{Q}}[Z D_{s+t} g(X_{s+t})] && (3.5)\\ &= \mathbb{E}^x_{\mathbb{Q}}[Z\, D_t \mathbb{E}^{X_t}_{\mathbb{Q}}[D_s g(X_s)]] && (3.6)\\ &= \mathbb{E}^x_{\mathbb{Q}^h}[Z\, \mathbb{E}^{X_t}_{\mathbb{Q}^h}[Z g(X_s)]] && (3.7) \end{aligned}$$

implying the Markov property of X under $\mathbb{Q}^h$. The formula then follows from the martingale property of D, Bayes rule and the Markov property. $\square$

Now, we will show the relationships between functions R and h and the extended generator of X under $\mathbb{Q}^h$, if X is a diffusion process (without killing and explosion) by the help of “carré du champ” operator. However, before doing it so, we will briefly remind what we mean by a diffusion process (in literature there are various definitions) and local martingale problem associated with it.

In the sequel, a and μ will denote a matrix field and a vector field on E satisfying the conditions,

Assumption 3.8. *a and μ will denote a matrix field and a vector field on E satisfying the conditions*

i. the maps $x \mapsto a(x)$ and $x \mapsto \mu(x)$ are Borel measurable and locally bounded,

ii. for each $x \in E$, the matrix $a(x)$ is symmetric and non-negative definite.

Hence with such a pair (a, μ), we can associate the second order differential operator

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i(\cdot) \frac{\partial}{\partial x_i}. \tag{3.9}$$

Definition 3.10. A Markov process X with state space E is said to be a diffusion process with an (extended) generator $\mathcal{A}$ if all paths are continuous and for any $f \in C^2(E)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds, \qquad t \geq 0,$$

is an $\mathbb{F}^X$-local martingale for any $\mathbb{Q}_x$ (see Def. 2.1 and Prop. 2.2 in [143], pg. 294-295).

Remark 3.11. Note that the state space E is locally compact and separable, so there exists an increasing sequence of $\{U_n\}_{n\geq 1}$ of open relatively compact sets in E such that $\bar{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$ such that $E = \bigcup_n^\infty U_n$ and hence, since the life-time η of the process is infinity almost surely (remind our assumption that X is Markov without killing and explosion), there exist stopping times $\tau_n = \inf\{t \geq 0 \,:\, X_t \in \bar{U}_n^c\}$ for $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \tau_n = \infty$.

Remark 3.12. The process M^f for $f \in C^2(E)$ is continuous and locally bounded because a and μ are locally bounded and hence it is bounded on $[\tau_n \wedge n]$. One can also show that a Markov process X with state space E is a diffusion if it has continuous paths and for any $f \in C_c^\infty(E)$ [1],

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds, \qquad t \geq 0,$$

is an $\mathbb{F}^X$-martingale for any $\mathbb{Q}_x$ (see Prop. 2.2 in [143], pg. 295).

Remark 3.13. Note also that from the definition 3.10 of a diffusion process, it is evident that any function in $C_b^2(E)$ [2] , is in the domain of the extended generator of X.

Definition 3.14. A probability measure $\mathbb{P}$ on the canonical construction $(\Omega, \mathbb{F}^X)$ is a solution of the (local) martingale problem for $\mathcal{A}$ if for all $f \in C^2(E)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds, \qquad t \geq 0,$$

[1] $C_c^\infty(E)$ is the space of smooth (having continuous derivatives of all orders) functions on E with compact support.

[2] $C_b^2(E)$ is the space of bounded two times continuously differentiable functions on E.

is a $\mathbb{P}$-(local) martingale with respect to $\mathbb{F}^X$. We say that the (local) martingale problem for $\mathcal{A}$ is well posed if for every probability distribution ν on $\mathcal{E}$, there exists a unique solution $\mathbb{P}$ of the (local) martingale problem for $\mathcal{A}$ such that $\mathbb{P} \circ X_0^{-1} = \nu$.

Proposition 3.15. *Let $\mathcal{A}$ be the extended generator of a diffusion process X given by (3.9) that satisfies Assumption (3.8) and acts on the domain $\mathcal{D}(\mathcal{A}) = C^2(E)$. Moreover suppose for $(R,h) \in \mathcal{M}_+(E) \times \mathcal{M}_{++}(E)$, the process $(D_t)_{t\geq 0}$ given by (3.2) is a $(\mathbb{F}^X, \mathbb{Q}_x)$ martingale, then*

$$R = \frac{\mathcal{A}h}{h} = \mathcal{A}f + \frac{1}{2}\Gamma(f,f),$$

with $f := \log h$.

Proof. Since X is a diffusion with extended generator whose domain is $C^2(E)$, if $h \in \mathcal{D}(\mathcal{A})$ then $f \in \mathcal{D}(\mathcal{A})$ and therefore from the definition of the extended generator

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds, \qquad t \geq 0,$$

is a continuous local martingale. Using the integration by parts, one can show that

$$(M_t^f)^2 - \int_0^t \Gamma(f,f)(X_s)\,ds, \quad t \geq 0,$$

is a local martingale with

$$\langle M^f, M^f\rangle_t = \int_0^t \Gamma(f,f)(X_s)\,ds, \qquad t \geq 0.$$

Since $D_t = \mathcal{E}_t(M)$ for some local martingale M, we can write

$$\mathcal{E}_t(M^f) = \exp\left\{ f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds - \frac{1}{2}\int_0^t \Gamma(f,f)(X_s)\,ds \right\}, \tag{3.16}$$

$$= \frac{h(X_t)}{h(X_0)} \exp\left(-\int_0^t \left(\frac{\mathcal{A}h(X_s)}{h(X_s)} \right) ds \right). \tag{3.17}$$

□

Proposition 3.18. *Let $\mathcal{A}$ be the extended generator of a diffusion process X given by (3.9) that satisfies Assumption (3.8) and acts on the domain $\mathcal{D}(\mathcal{A}) = C^2(E)$. Moreover suppose for $(R,h) \in \mathcal{M}_+(E) \times \mathcal{M}_{++}(E)$, the process $(D_t)_{t\geq 0}$ given by (3.2) is a $(\mathbb{F}^X, \mathbb{Q}_x)$ martingale. Then the extended generator of the process under $\mathbb{Q}^h$ is equal on $C^2(E)$ to $\mathcal{A}^h = \mathcal{A} + \frac{\Gamma(h,\cdot)}{h}$.*

Proof. It follows from the Theorem 4.2 of [139]; see also [143, Chp. VIII, Sec. 3]. □

Now, generally we may not know that the process D, defined by

$$D_t = \frac{h(X_t)}{h(X_0)} \exp\left\{ -\int_0^t R(X_s)\,ds \right\}, \tag{3.19}$$

is a true martingale. That is, it may be a strict local martingale and hence we may not define the change of measure defined by $\mathbb{Q}^h = D_t \cdot \mathbb{Q}$ on $\mathcal{F}_t^X$. Now we will give sufficient conditions for D be true martingale. For the remaining, we assume $R = \frac{\mathcal{A}h}{h}$. That is,

$$D_t = \frac{h(X_t)}{h(X_0)} \exp\left\{ -\int_0^t \frac{\mathcal{A}h(X_s)}{h(X_s)}\,ds \right\}, \qquad t \geq 0.$$

Let $\mathcal{O}$ be an open subset of E, that is $\mathcal{O} = \widehat{\mathcal{O}}$ for some open subset $\widehat{\mathcal{O}}$ of $\mathbb{R}^N$. Let $\mathcal{O}^1 \subset \mathcal{O}^2 \subset \cdots$ be an increasing sequence of open subsets of E such that $\mathcal{O} = \cup_{n\geq 1}\mathcal{O}^n$ and define the following stopping times

$$R_n = \inf\{t : X_t \notin \mathcal{O}^n\}, \qquad n \geq 1.$$

and set $S_n = R_n \wedge n$ for $n \geq 1$. Then

$$\Lambda_n = \frac{1}{2}\int_0^{S_n} \nabla^T f(x)\alpha(x)\nabla f(x)\, ds, \tag{3.20}$$

where $f := \log h$ is well defined for $n \geq 1$.

Theorem 3.21. *Let $\mathbb{Q}$ be a solution of the martingale problem for $\mathcal{A}$ given by*

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i(x)\frac{\partial}{\partial x_i}.$$

Suppose that the martingale problem for $\mathcal{A}^h = \mathcal{A} + \frac{\Gamma(h,.)}{h}$ is well posed on $\mathbb{Q}^h$ such that $\mathbb{Q}^h_{|\mathcal{F}_0} \ll \mathbb{Q}_{|\mathcal{F}_0}$ and

$$\mathbb{E}_{\mathbb{Q}}[e^{\Lambda_n}] < \infty, \tag{3.22}$$

for all $n \geq 1$. Then for any $\mathbb{F}^X$-stopping time τ, if $\mathbb{Q}^h_{|\mathcal{F}_0} \sim \mathbb{Q}_{|\mathcal{F}_0}$ and $\mathbb{Q}[\tau < S_\infty] = \mathbb{Q}^h[\tau < S_\infty] = 1$, then $\mathbb{Q}^h_{|\mathcal{F}_\tau} \sim \mathbb{Q}_{|\mathcal{F}_\tau}$.

Proof. The proof is is based on a simplified version of [35, Theorem 2.4]. Take the localizing sequence of bounded stopping times $S_1 \leq S_2 \leq \ldots \nearrow \infty$ such that

$$\Lambda_n = \frac{1}{2}\int_0^{S_n} \nabla^T f(x)\alpha(x)\nabla f(x)\, ds \tag{3.23}$$

is uniformly bounded. From [126], $\mathcal{E}_{t\wedge S_n}(Z)$ is a martingale where

$$Z_t = \int_0^t \nabla^T f(X_s)\alpha(X_s)\nabla f(X_s) dW_s, \qquad t \geq 0.$$

Girsanov's theorem implies that for any $f \in C^2(E)$

$$f(X_t^{S_n}) - f(X_0) - \int_0^{t\wedge S_n} \mathcal{A}^h f(X_s^{S_n})\, ds \tag{3.24}$$

is a $\mathcal{E}_{S_n}(Z)\cdot\mathbb{Q}$-martingale. Uniqueness of the stopped martingale problem [70, Thm. 4.6.1] implies that $\mathcal{E}_{S_n}(Z)\cdot\mathbb{Q} = \mathbb{Q}^h$ on $\mathcal{F}^X_{S_n}$, where $\mathbb{Q}^h$ is the solution of the martingale problem for $\mathcal{A}^h$ with $\mathbb{Q}^h = \mathbb{Q}$ on $\mathcal{F}^X_0$. From monotone convergence theorem and $\{T < S_n\} \in \mathcal{F}_{T\wedge S_n}$,

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mathbb{Q}^h[T < S_n] \\ &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[\mathcal{E}_{T\wedge S_n}(Z)\mathbb{I}_{T<S_n}] \\ &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[\mathcal{E}_T(Z)\mathbb{I}_{T<S_n}] \\ &= \mathbb{E}_{\mathbb{Q}}[\mathcal{E}_T(Z)]. \end{aligned}$$

□

Remark 3.25. For (3.22) to hold, the following assumption is sufficient.

Assumption 3.26. *For every $n \geq 1$, there exists a finite constant C_n such that for all $x \in \mathcal{O}^n$,*

$$\nabla^T f(x)\alpha(x)\nabla f(x) < C_n.$$

Chapter 4

Modeling with Jacobi Processes

In this chapter, we apply the general bond pricing formula to a measure-valued diffusion processes, commonly known as Jacobi or Wright–Fisher processes, which are used in population genetics for the modeling of the gene frequencies in a certain population; see for example [109, Chp. 15] and [68]. Apart from the huge literature related to population genetics that studies these types of processes extensively, starting from the works of S. Wright [159], R. A. Fisher [79] and W. Feller [73], Jacobi processes are also featured as examples of polynomial processes recently; see [77] and references therein. However, we should mention here the first finance related works of [49] who used Jacobi process for interest rate modeling and [92] that studied smooth transitions in financial applications.

One can also interpret those processes as discrete probability distribution valued processes since these type of processes are random motions on the $(d-1)$-dimensional simplex in $\mathbb{R}^{d-1}$:

$$\Delta_d = \big\{(x_1,\dots,x_{d-1}) \in [0,1]^{d-1} \;\big|\; 1 - x_1 - \cdots - x_{d-1} \geq 0\big\}.$$

The corresponding diffusion operators act on functions on this simplex, however, sometimes it seems more natural to consider the following the standard $(d-1)$-dimensional simplex in $\mathbb{R}^d$

$$\Delta_d^+ = \big\{(x_1,\dots,x_d) \in [0,1]^d \;\big|\; x_1 + \cdots + x_d = 1\big\},$$

which can be interpreted as the set of all probability measures on $\{1,\dots,d\}$. This choice enables to treat the diffusion operators as degenerate elliptic differential operators acting on functions in a neighborhood of Δ_d^+. Moreover analysis becomes symmetric with respect to the coordinates. By projecting Δ_d^+ on Δ_d, one can have equivalent results.

Definition 4.1. Given $\alpha,\sigma > 0$ as well as $\gamma = (\gamma_1,\dots,\gamma_d)$ and $x = (x_1,\dots,x_d)$ in the interior of Δ_d^+, the multivariate Jacobi process $X = (X_t^{(1)},\dots,X_t^{(d)})_{t\geq 0}$ with the state space Δ_d^+ satisfies the system of coupled stochastic differential equations (SDE)

$$dX_t^{(i)} = \alpha(\gamma_i - X_t^{(i)})\,dt + \sigma\sqrt{X_t^{(i)}}\,dW_t^{(i)} - \sigma X_t^{(i)} \sum_{j=1}^{d} \sqrt{X_t^{(j)}}\,dW_t^{(j)}, \quad t > 0, \tag{4.2}$$

$$X_0^{(i)} = x_i, \tag{4.3}$$

where $W = (W_t^{(1)},\dots,W_t^{(d)})_{t\geq 0}$ is a d-dimensional standard Brownian motion.

Remark 4.4. The diffusion matrix of the multivariate Jacobi process $a(x) = (a_{ij}(x))_{i,j\in\{1,\ldots,d\}}$ is of the form

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1-x_i) & \text{if } i = j \\ -\sigma^2 x_i x_j & \text{otherwise.} \end{cases}$$

Remark 4.5. $X_t^{(1)} + \cdots + X_t^{(d)} = 1$ and the diffusion is degenerate.

For $d = 2$ the Jacobi process is also known as "Wright–Fisher" diffusion and (4.2) reduces to an one-dimensional SDE by using Pythagorean theorem.

Definition 4.6. For $d = 2$, Jacobi process is governed by a one-dimensional SDE,

$$dX_t = \alpha(\gamma - X_t)\,dt + \sigma\sqrt{X_t(1-X_t)}\,d\bar{W}_t, \qquad t \geq 0, \tag{4.7}$$

where $\bar{W}$ is a standard Brownian motion and $\alpha > 0$ is the mean reverting parameter, $\gamma \in [0,1]$ is the long-run mean of the process and $\sigma > 0$ is the parameter controlling the volatility.

By the Yamada–Watanabe theorem [160], there is a unique strong $[0,1]$-valued solution $(X_t)_{t\geq 0}$ of (4.7) for every starting value $x \in [0,1]$, and it is intuitively clear that increments of X_t and $1 - X_t$ are negatively correlated. Indeed they are even countermonotonic. From the Feller's classification of boundary points [109, Sect. 15.6], if $\alpha_1 := 2\alpha\gamma/\sigma^2 \geq 1$ and $\alpha_2 := 2\alpha(1-\gamma)/\sigma^2 \geq 1$, $\{0,1\}$ are entrance boundaries that cannot be reached from the interior of the state space, but can begin there and quickly move to the interior never to return back. Moreover, in this case it has the unique stationary distribution (has the strong ergodic property) given by the beta distribution with density

$$f_{\alpha_1,\alpha_2}(x) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1}(1-x)^{\alpha_2-1} & \text{for } x \in (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

where Γ denotes the gamma function. The stationary density has also meaningful when $0 < \alpha_1 < 1$ and $0 < \alpha_1 < 1$. In that case $\{0,1\}$ are regular boundary points that a diffusion process can both enter and leave from and hence the behavior at those points need to be specified.

For the general case, we can write the SDE that govern the multivariate Jacobi process taking values on Δ^d.

Remark 4.8. ϕ given below is a function that is defined on Δ_d° and is extendable continuously to Δ_d such that $\phi(x)\phi^T(x) = a(x)$ on Δ_d, we denote this extension by the same ϕ (see [148, Lemma 5.2]).

Definition 4.9. Given $\alpha, \sigma > 0$ as well as $\gamma = (\gamma_1, \ldots, \gamma_{d-1})$ and $x = (x_1, \ldots, x_{d-1})$ in Δ_d, the multivariate Jacobi process $X = (X_t^{(1)}, \ldots, X_t^{(d-1)})_{t\geq 0}$ with the state space Δ_d satisfies the system of stochastic differential equations (SDE)

$$dX_t^{(i)} = \alpha(\gamma_i - X_t^{(i)})\,dt + \sum_{j=1}^{d-1} \phi_{ij}(X_t)\,d\tilde{W}_t^{(j)}, \quad t > 0, \tag{4.10}$$

$$X_0^{(i)} = x_i, \tag{4.11}$$

where $\tilde{W} = (\tilde{W}_t^{(1)}, \dots, \tilde{W}_t^{(d-1)})_{t\geq 0}$ is a $(d-1)$-dimensional standard Brownian motion and $\phi(x) = (\phi_{ij}(x))_{i,j\in\{1,\dots,(d-1)\}}$ is the following lower triangular matrix,

$$\begin{aligned}
&\phi_{ij}(x) = 0, \quad j > i,\\
&\phi_{11}(x) = \sigma\sqrt{x_1(1-x_1)},\\
&\phi_{21}(x) = -\sigma\frac{x_2\sqrt{x_1}}{\sqrt{1-x_1}},\ \phi_{22}(x) = \sigma\frac{\sqrt{x_2\,(1-x_1-x_2)}}{\sqrt{1-x_1}},\\
&\phi_{31}(x) = -\sigma\frac{x_3\sqrt{x_1}}{\sqrt{1-x_1}},\ \phi_{32}(x) = -\sigma\frac{x_3\sqrt{x_2}}{\sqrt{(1-x_1-x_2)(1-x_1)}},\\
&\phi_{33}(x) = \sigma\frac{\sqrt{x_3\,(1-x_1-x_2-x_3)}}{\sqrt{1-x_1-x_2}},\\
&\dots
\end{aligned} \tag{4.12}$$

As it is evident from the related SDE (4.10), the second order differential operator defined by

$$\mathcal{J}_d = \frac{1}{2}\sum_{i,j=1}^{d-1} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1}\mu_i(x)\frac{\partial}{\partial x_i}, \tag{4.13}$$

where the diffusion matrix $a(x) = (a_{ij}(x))_{i,j\in\{1,\dots,(d-1)\}}$ is of the form

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1-x_i), & \text{if } i = j\\ -\sigma^2 x_i x_j, & \text{otherwise,}\end{cases}$$

and $\mu_i : \Delta_d \to \mathbb{R}$ for $i \in \{1\dots,(d-1)\}$ is Lipschitz continuous and satisfy suitable conditions so that X never leaves the simplex, generates the multivariate Jacobi process. More precisely,

Theorem 4.14 ([68, Theorem 1])**.** *Let Δ_d be an $(d-1)$-dimensional simplex and define $\mathcal{J}_d$ as in (4.13) where the diffusion matrix $a : \Delta_d \to S_d$ where $a(x) = (a_{ij}(x))_{i,j\in\{1,\dots,(d-1)\}}$ is of the form*

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1-x_i), & \textit{if } i = j\\ -\sigma^2 x_i x_j, & \textit{otherwise.}\end{cases}$$

and $\mu_i : \Delta_d \to \mathbb{R}$ for $i \in \{1,\dots,(d-1)\}$ is Lipschitz continuous and satisfy

$$\begin{aligned}
\mu_i(x) \geq 0 \qquad &\textit{if} \qquad x\in\Delta_d \qquad \textit{and} \qquad x_i = 0, \qquad i\in\{1,\dots,(d-1)\},\\
\sum_{i=1}^{d-1}\mu_i(x) \leq 0 \qquad &\textit{if} \qquad x\in\Delta_d \qquad \textit{and} \qquad \sum_{i=1}^{d-1} x_i = 1.
\end{aligned}$$

Then with $\mathcal{J}_d$ defined by (4.13), the closure of $\{(f, \mathcal{J}_d f) : f \in C^2(\Delta_d)\}$ is single-valued and generates a Feller semigroup on $C(\Delta_d)$. And the space of a polynomials on Δ_d is a core for the generator.

Proof. See [70, pg. 375] or [68]. □

Instead of providing the existence and uniqueness of the multivariate Jacobi diffusion via solution of a martingale problem as in [70, pg. 375] or [68], one can also follow the stochastic differential equations approach so that pathwise uniqueness of the solution to SDE (4.10)

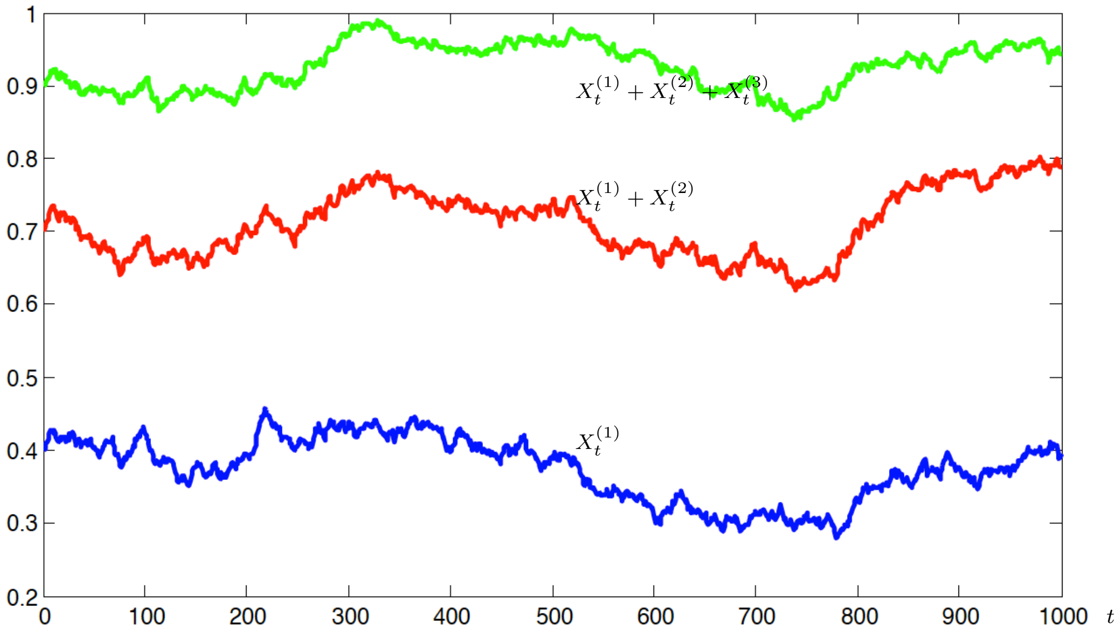

Figure 4.1: Random partition of the unit interval with a multivariate Jacobi process starting at its center $x = \gamma = (0.4, 0.3, 0.2, 0.1) \in \Delta_4^+$ with drift strength $\alpha = 0.5$ and volatility $\sigma = 0.3$.

leads to the strong uniqueness of the solution of (4.10). In order to do that, we can follow the representation of a multivariate Jacobi diffusion process as the unique solution of the SDE in Definition 4.9. Now, we state a theorem without giving the proof. It states the pathwise uniqueness of a multivariate Jacobi diffusion process, and its proof is based on an iterative application of Yamada–Watanaba condition. For Yamada–Watanabe condition related to the pathwise uniqueness of solutions to certain stochastic differential equations, we refer to the Appendix. In the following theorem, note that ϕ, given by (4.12), is a lower triangular matrix and drift coefficients are Lipschitz continuous, hence by a diagonalization argument one can apply the Yamada–Watanabe condition to the individual components starting from the first component. Since the drift μ_i for the i-th component is a function of only x_i and the diffusion matrix is lower triangular so that a_{ij} is dependent only on $x_1, \dots, x_i$ for $i \leq j$, an iterative application of Yamada–Watanabe condition leads to the result (see also [148]). A similar problem, which might have an independent importance involving a trapped Jacobi process, is tackled in the Appendix.

Theorem 4.15. *Let X be a multivariate Jacobi diffusion process governed by* (4.10)*. Then the solution to the* (4.10) *is pathwise unique.*

It is intuitively clear that for $i \neq j$ the increments of the components $X^{(i)}$ and $X^{(j)}$ are negatively correlated, indeed the covariation equals $-\sigma^2 X^{(i)} X^{(j)} < 0$. The Dirichlet distribution $\mathcal{D}(\alpha_1, \dots, \alpha_d)$ with density

$$\frac{\Gamma(\alpha_1 + \cdots + \alpha_d)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_d)} \prod_{i=1}^{d} x_i^{\alpha_i - 1}, \quad (x_1, \dots, x_{d-1}) \in \Delta_d^\circ,$$

where $x_d := 1 - x_1 - \cdots - x_{d-1}$ and $\alpha_i := 2\alpha\gamma_i/\sigma^2 > 1$ for $i \in \{1, \ldots, (d-1)\}$, is the stationary distribution.

4.1 Boundary classification of the multivariate Jacobi process

In this section, we give results related to the boundary conditions of a multivariate Jacobi process. Basically these conditions are needed to prevent the multivariate Jacobi process to reach the boundary of the simplex. Also they are needed for the equivalent change of measure in computation of the zero-coupon bond prices, since they prevent the newly defined state processes from exploding. We choose to examine the boundary conditions in a two-dimensional simplex since it is notationally and didactically more comprehensible. Generalization to an any finite dimension is straightforward. This part closely follows the lines of [148] and it is adapted to our setting.

Let us take a multivariate Jacobi process taking values in Δ_3. Denote Δ_3° as the interior and $\partial\Delta_3$ as the boundary of the simplex Δ_3. Since all the three sides of Δ_3 are similar, it is enough to study $\Sigma_1 = \{(x_1, 0) : 0 < x_1 < 1\}$. Let us define the regular and repulsive boundary points and unattainable and pure entrance boundary segments.

Suppose $x \in \Sigma_1$ and let $\mathcal{U}$ be a neighborhood in Δ_3 of x with $|x - z| > 0$ for every $z \in \Delta_3 - \Sigma_1$. For $\lambda > 0$, let $\mathcal{U}_\lambda = \mathcal{U} \cap \{(x_1, x_2) : x_2 > \lambda\}$.
Let us define the stopping time

$$T_\lambda = \inf\{t \geq 0 : X_t \notin \mathcal{U}_\lambda\}$$

and

$$T = \lim_{\lambda \searrow 0} T_\lambda.$$

Let $\Sigma_1^{T,\mathcal{U}}(\omega)$ be set of all accumulation points of $X_t(\omega)$ when $t \nearrow T$.

Definition 4.16 (Regular boundary points)**.** x is called regular if, for every $\mathcal{U}$ and for every neighborhood $\mathcal{N} \subset \Delta_3$, we have

$$\lim_{y \in \Delta_3^\circ,\, y \to x} \mathbb{Q}_y[\Sigma_1^{T,\mathcal{U}}(\omega) \subset \mathcal{N} \cap \Sigma_1] = 1$$

Let $\partial\mathcal{U}_\lambda = \mathcal{U} \cap \{y = (y_1, y_2) : y_2 < \lambda\}$, $\partial\mathcal{U}$ being the boundary in $\mathbb{R}^2$ of $\mathcal{U}$.

Definition 4.17 (Repulsive boundary points)**.** x is called repulsive if for some $\mathcal{U}$ and for some $\lambda > 0$, we have

$$\liminf_{y \in \Delta_3^\circ,\, y \to x} \mathbb{Q}_y[\Sigma_1^{T,\mathcal{U}}(\omega) \subset \partial\mathcal{U}_\lambda] < 1.$$

Definition 4.18 (Unattainable segment)**.** Let Σ be an open interval in Σ_1. Σ is called unattainable if for every $x \in \Sigma$, there exists a neighborhood $\mathcal{U}$ in Δ_3 such that

$$\mathbb{Q}_y[\Sigma_1^{T,\mathcal{U}}(\omega) \cap \Sigma = \emptyset] = 1$$

for every $y \in \mathcal{U} \cap \Delta_3^\circ$.

Definition 4.19 (Pure entrance boundary segment)**.** Let Σ be an open interval in Σ_1. Σ is called a pure entrance boundary segment if Σ is unattainable and of every $x \in \Sigma$ has a neighborhood $\mathcal{U}$ such that $\mathbb{Q}_y[T_\mathcal{U} < \infty] = 1$ for every $y \in \mathcal{U}$, where $T_\mathcal{U} = \inf\{t : X_t \notin \mathcal{U}\}$.

Proposition 4.20. *If $\alpha_2 > 1$, then Σ_1 is a pure entrance boundary segment and every point $x \in \Sigma_1$ is repulsive.*

Proof. See [148] that follows Khasminsky's results on boundary classification of degenerate diffusions [110]. □

Corollary 4.21. *Let X be a multivariate Jacobi process on Δ_d. If $\alpha_i := 2\alpha\gamma_i/\sigma^2 > 1$ for $i = 1, \dots d$ and X starts in the interior of Δ_d, i.e., $x \in \Delta_d^\circ$, then X almost surely never reaches $\partial\Delta_d$.*

Remark 4.22. Alternative to using Khasminsky's results on boundary classification of degenerate diffusions, one can also apply the one-dimensional Feller's classification to find boundary conditions due to the concatenation property of multivariate Jacobi processes. This is due to the special structure of the state space, Δ_d^+, which is a $(d-1)$-dimensional standard simplex. The concatenation property for Δ_d^+ states that the convex hull of any $d_1 < d$ points $(x_{d_1,1}, \dots, x_{d_1,d_1})$ in Δ_d^+ is again a simplex, called d_1-face. Hence, we can represent individual components of a multivariate Jacobi process on Δ_d^+ as one-dimensional diffusions on 1-faces (edges) of Δ_d^+. Therefore, once we find the (unattainability of boundary) conditions for each univariate Jacobi process living on a 1-face of Δ_d^+ by applying the one-dimensional result of Feller, we can characterize the boundary conditions that are sufficient for a multivariate Jacobi process not reaching the boundary of Δ_d^+. This gives exactly the same result as in Corollary 4.21.

4.2 Equivalent measure change for multivariate Jacobi process

Theorem 4.23. *Let $\mathcal{J}_d$ be the second order differential operator acting on $C^2(\Delta_d)$*

$$\mathcal{J}_d = \frac{1}{2}\sum_{i,j=1}^{d-1} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1}\mu_i(x)\frac{\partial}{\partial x_i}, \tag{4.24}$$

where the diffusion matrix $a(x) = (a_{ij}(x))_{i,j\in\{1,\dots,(d-1)\}}$ is

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1-x_i), & \text{if } i = j \\ -\sigma^2 x_i x_j, & \text{otherwise,} \end{cases}$$

with $\sigma > 0$ and the drift is $\mu_i(x) = \alpha(\gamma_i - x_i)$ where $\alpha > 0$, $1 > \gamma_i > 0$ for $i \in \{1, \dots, d\}$ with $\gamma_d := 1 - \sum_{i=1}^{d-1}\gamma_i$. Moreover assume that the conditions for the unattainability of the boundary of $\partial\Delta_d$ are satisfied (given by the Corollary 4.21), that is, $\alpha_i > 1$ for $i \in \{1, \dots, d\}$ and $(x_1, \dots, x_{d-1}) \in \Delta_d^\circ$. Let the function $h \in C^2(\Delta_d^\circ)$ be given by

$$h(x) = \prod_{i=1}^{d} x_i^{a_i}$$

where $a_i = \frac{1}{2}\Big(1 - \alpha_i + \sqrt{(\alpha_i - 1)^2 + \frac{8k_i}{\sigma^2}}\Big)$ such that $k_i > -\frac{(\alpha_i-1)^2\sigma^2}{8}$ where we set $x_d := 1 - \sum_1^{d-1} x_i$. Then $\mathbb{F}^X$-adapted process defined by

$$D_t = \frac{\prod_{i=1}^{d}(X_t^{(i)})^{a_i}}{\prod_{i=1}^{d} x_i^{a_i}} \exp\left\{-\int_0^t \left(\sum_{i=1}^{d} k_i \frac{1 - X_s^{(i)}}{X_s^{(i)}} - C_{\alpha,\gamma,\sigma}^{a_1,\dots,a_d}\right) ds\right\}, \qquad t \geq 0, \tag{4.25}$$

with

$$C^{a_1,\ldots,a_d}_{\alpha,\gamma,\sigma} = \alpha \sum_{i=1}^{d} a_i(1-\gamma_i) + \frac{\sigma^2}{2}\left(\left(\sum_{i=1}^{d} a_i\right)^2 - \sum_{i=1}^{d} a_i^2\right) \tag{4.26}$$

is a true $\mathbb{F}^X$-martingale and $\mathbb{Q}_x$ is equivalent to $\mathbb{Q}^h_x$.

Proof. By direct computation

$$\frac{\mathcal{J}_d h}{h} = \sum_{i=1}^{d} k_i \frac{1-x_i}{x_i} - C^{a_1,\ldots,a_d}_{\alpha,\gamma,\sigma} \tag{4.27}$$

where $k_i = a_i\alpha\gamma_i + \frac{1}{2}\sigma^2 a_i(a_i-1)$ for $i \in \{1,\ldots,d\}$ and the constant $C^{a_1,\ldots,a_d}_{\alpha,\gamma,\sigma}$ given above. From Proposition 3.18 (note that when we are using this theorem, we still do not know D is a true martingale, hence we formally calculate the generator) the extended generator under $\mathbb{Q}^h$ is given by

$$\mathcal{J}^h_d = \frac{1}{2}\sum_{i,j=1}^{d-1} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1} \hat{\mu}_i(x)\frac{\partial}{\partial x_i}, \tag{4.28}$$

where the drift function is given by $\hat{\mu}_i(x) = \hat{\alpha}(\hat{\gamma}_i - x_i)$ with

$$\hat{\alpha} = \alpha + \sigma^2 \sum_{i=1}^{d} a_i \qquad \text{and} \qquad \hat{\gamma}_i = \frac{\alpha\gamma_i + \sigma^2 a_i}{\alpha + \sigma^2\sum_{i=1}^{d} a_i},$$

for $i \in \{1,\ldots,d\}$. Hence $\mathcal{J}^h_d$ solves the $\mathbb{Q}^h_x$ martingale problem (it is again a Jacobi process on Δ_d) and this problem is well posed (solution is unique) due to Theorem 4.14. Moreover, from assumptions on k_i observe that a_i can be chosen as real numbers while fixing k_i on a domain determined by the parameters of the process $(\sigma, \alpha, \gamma_1, \ldots, \gamma_d)$ for $i \in \{1,\ldots,d\}$. Hence the conditions for unattainability of the boundary of Δ_d under the new measure are satisfied. In fact $2\hat{\alpha}\hat{\gamma}_i > \sigma^2$ since $2\hat{\alpha}\hat{\gamma}_i - \sigma^2 = \sigma^2(\alpha_i - 1 + 2a_i) > 0$ for $i \in \{1,\ldots,d\}$. Therefore $\partial\Delta_d$ is the union of unattainable segments, that is 2-faces of a d-dimensional simplex. Moreover Assumption 3.26 holds since drift function μ_i for $i = i \in \{1,\ldots,d\}$ become linear under $\mathbb{Q}^h_x$ therefore bounded on compacts. Then from the Theorem 3.21, D is a true martingale and hence $\mathbb{Q}_x$ is equivalent to $\mathbb{Q}^h_x$. □

4.3 (Defaultable) zero-coupon bond pricing formula

Now, we will give the the zero-coupon bond price formula for a multivariate Jacobi process. $X^{(1)}, \ldots, X^{(d)}$ are bounded by 1, however, we can remove the bound by choosing new state process as

$$Y^{(i)} := \frac{1 - X^{(i)}}{X^{(i)}}$$

for $i \in \{1,\ldots,d\}$, instead. Notice that this choice keeps the negative instantaneous correlation.

If we consider the risk-free interest rate as well as the credit spreads (risk-free → AAA, AAA → AA, AA → A, …) as linear combinations of $Y^{(1)}, \ldots, Y^{(d)}$, (see Figure 4.2 and Figure 4.3) then given

$$k_i > -\frac{1}{8}(\alpha_i - 1)^2\sigma^2$$

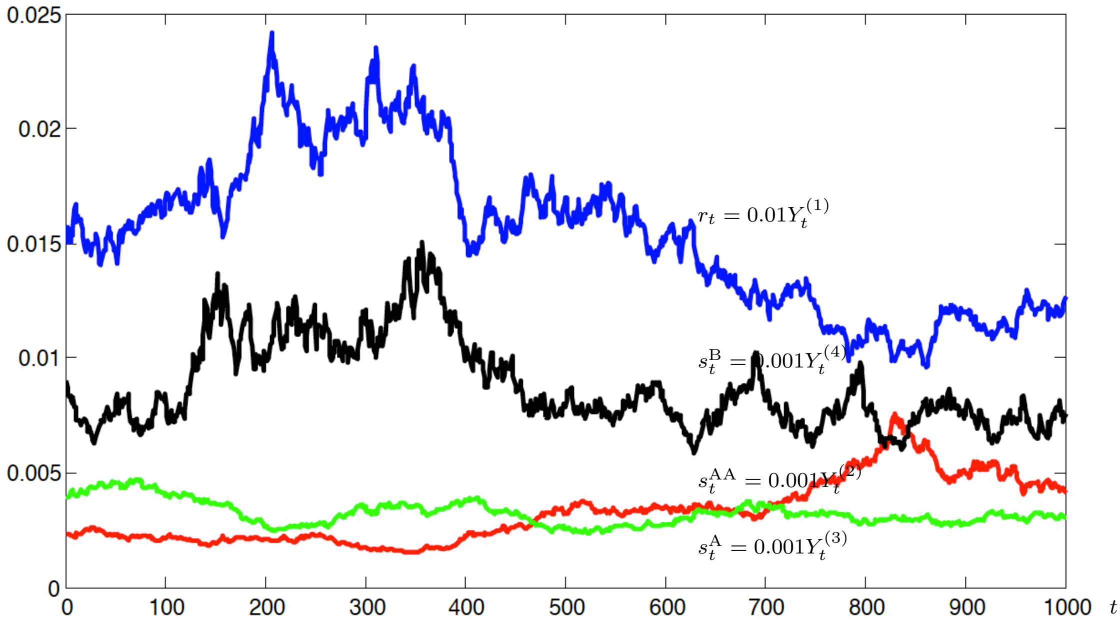

Figure 4.2: Default-free short rate process r and additive credit spread processes s^{AA}, s^{A}, s^{B} given by $k_i Y^{(i)} := k_i(1 - X^{(i)})/X^{(i)}$ with $k_1 = 0.01$ and $k_2 = k_3 = k_4 = 0.001$, other parameters as in Figure 4.1

for $i \in \{1, \ldots, d\}$, initial value $x \in \Delta_d^\circ$ and fixed time to maturity $T > 0$, we can define the defaultable zero-coupon bond price process by (the continuous version of)

$$P(t,T) = \mathbb{E}_{\mathbb{Q}_x}\left[\exp\left(-\int_t^T \sum_{i=1}^d k_i \frac{1 - X_s^{(i)}}{X_s^{(i)}}\, ds\right) \,\middle|\, \mathcal{F}_t\right], \qquad t \in [0,T]. \tag{4.29}$$

Remark 4.30. Since the instantaneous correlation $\rho(Y^i, Y^j)$ is negative for $i, j \in \{1, \ldots, d\}$ and for $i \in \{1, \ldots, d\}$ k_i can take negative values, our model has a flexible correlation structure. That is, not only negative instantaneous correlation between state variables but also positive instantaneous correlation can be captured.

Theorem 4.31. *Let the state process X is given by a multivariate Jacobi process with the extended generator acting on $C^2(\Delta_d)$ and given by*

$$\mathcal{J}_d = \frac{1}{2}\sum_{i,j=1}^{d-1} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1} \mu_i(x)\frac{\partial}{\partial x_i}, \tag{4.32}$$

where the diffusion matrix $a(x) = (a_{ij}(x))_{i,j\in\{1,\ldots,(d-1)\}}$ is

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1 - x_i), & \text{if } i = j \\ -\sigma^2 x_i x_j, & \text{otherwise,} \end{cases}$$

with $\sigma > 0$ and the drift is $\mu_i(x) = \alpha(\gamma_i - x_i)$ where $\alpha > 0$, $1 > \gamma_i > 0$ for $i \in \{1, \ldots, d\}$ with $\gamma_d := 1 - \sum_{i=1}^{d-1} \gamma_i$. Moreover assume that the conditions for the unattainability of the boundary of $\partial\Delta_d$ are satisfied (given by the Corollary 4.21), that is, $\alpha_i > 1$ for $i \in \{1, \ldots, d\}$

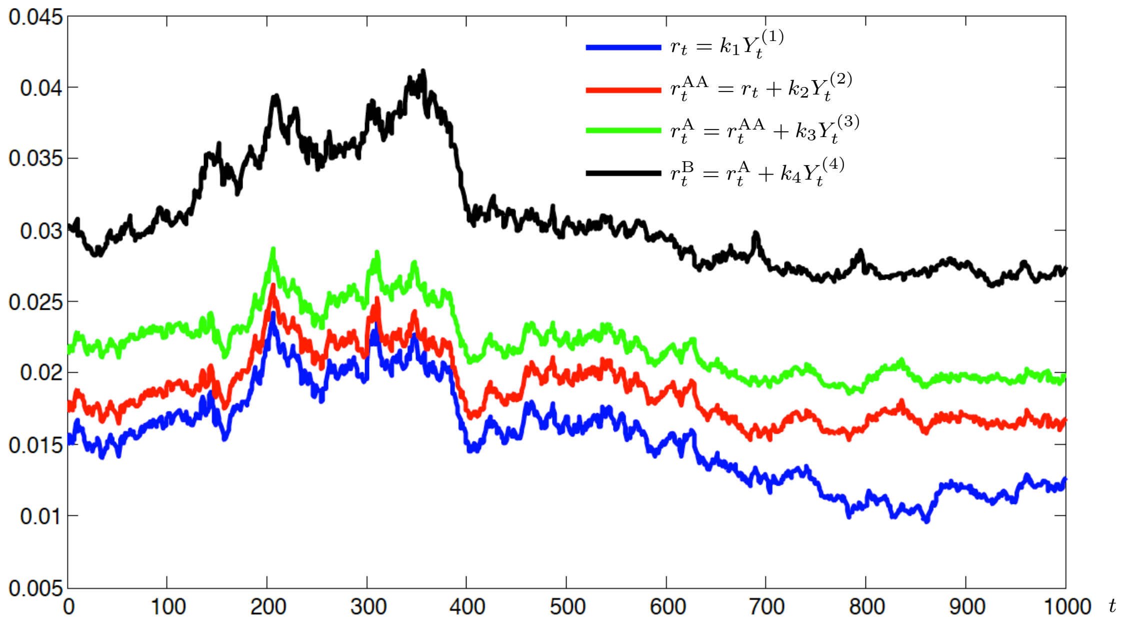

Figure 4.3: Default-free short rate process r and defaultable short rate processes r^{AA}, r^{A}, r^{B} given by $\sum_{j=1}^{i} k_j(1-X^{(j)})/X^{(j)}$ with $k_1 = 0.01$ and $k_2 = k_3 = k_4 = 0.001$, other parameters as in Figure 4.1.

and $(x_1, \dots, x_{d-1}) \in \Delta_d^\circ$. *If the short-rate is given by* $(R(X_t))_{t\geq 0}$ *where* $R(x) = \sum_{i=1}^{d} k_i \frac{1-x_i}{x_i}$ *with* $k_i > -\frac{(\alpha_i-1)^2\sigma^2}{8}$ *for* $i \in \{1, \dots, d\}$, *then the price of a (defaultable) zero-coupon with fixed time to maturity* $T > 0$ *is for* $t \in [0, T]$ *by,*

$$\mathbb{E}_{\mathbb{Q}_x}\left[\exp\left\{-\int_t^T \left(\sum_{i=1}^{d} k_i \frac{1-X_s^{(i)}}{X_s^{(i)}}\right) ds\right\} \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{e^{-(T-t)C_{\alpha,\gamma,\sigma}^{a_1,\dots,a_d}} \prod_{i=1}^{d}(X_t^{(i)})^{a_i}}{\prod_{i=1}^{d}(X_T^{(i)})^{a_i}} \,\Big|\, \mathcal{F}_t\right], \tag{4.33}$$

where $a_i = \frac{1}{2}\left(1 - \alpha_i + \sqrt{(\alpha_i-1)^2 + \frac{8k_i}{\sigma^2}}\right)$ *for* $i \in \{1, \dots, d\}$ *and*

$$C_{\alpha,\gamma,\sigma}^{a_1,\dots,a_d} = \alpha \sum_{i=1}^{d} a_i(1-\gamma_i) + \frac{\sigma^2}{2}\left(\left(\sum_{i=1}^{d} a_i\right)^2 - \sum_{i=1}^{d} a_i^2\right).$$

Proof. It is immediate from the general change of measure Theorem 4.23. □

As it is seen from the right hand side of (4.33), in order to compute the bond price we have to compute the mixed power moments of X under $\mathbb{Q}_x^h$. Note that due to the measure change X stays as a Jacobi process. Hence computation of the right hand side is achieved by an integral over simplex Δ_d if we know the probability transition density of the process X. Luckily, the transition density function of a Jacobi process is available and can be characterized by multi-dimensional orthogonal polynomials. Alternatively, it can also be characterized by using a dual process approach that simplifies the computation of the related bond price in dimensions $d > 2$. In the following, first we will give a brief overview and results related to the transition density of Jacobi processes by following the spectral expansion approach. Although dual process approach gives more tractable transition density for our purposes, we still prefer to give results related to spectral approach utilizing multi-dimensional orthogonal polynomials since it is easy to see the relation between orthogonal polynomials (Jacobi polynomials) and the transition density. Also this method can be seen as a natural generalization of the previous literature, specifically the work of [101]. Similar accounts on the spectral approach can be found in [94], [93], [19] and more recently in [153] and [150], from which we will refer to certain results below.

In what follows, for notational simplicity, we reparametrize multivariate Jacobi process by setting $\theta_i = \alpha_i + 2a_i$ for $i \in \{1, \dots, d\}$ and $|\theta| = \sum_{i=1}^{d} \theta_i$ leading to the the drift function $\mu_i(x) = \frac{|\theta|}{2}(\frac{\theta_i}{|\theta|} - x_i)$, for $i \in \{1, \dots, d\}$, and the new stationary distribution given by the Dirichlet distribution $\mathcal{D}(\theta_1, \dots, \theta_d)$. Note that this parametrization makes drift pointing to relative interior of the simplex.

4.4 Transition density function by spectral expansion

Let $p(t; x, y)\, dy = \mathbb{Q}(X_t \in dy | X_0 = x)$ for $x, y \in \Delta_d$ be the transition density function of multivariate Jacobi process. It satisfies the Kolmogorov backward equation

$$\frac{\partial}{\partial t} p(t; x, y) = \mathcal{J}_d p(t; x, y)$$

with the initial condition $\hat{\delta}_{x=y}$, where $\mathcal{J}_d$ is the extended generator associated with X and $\hat{\delta}$ is the Dirac delta. Also it can be shown shown by using integration by parts that for all $f, g \in C^2(\Delta_d)$, the operator $\mathcal{J}_d$ is symmetric with respect to their stationary distribution Π, which is given by the Dirichlet distribution. By [65, Theorem 1.4.4], $\mathcal{J}_d$ has countably many non-positive eigenvalues $\{-\Lambda_0, -\Lambda_1, \Lambda_2, \cdots\}$ such that $0 < \Lambda_0 \leq \Lambda_1 \leq \Lambda_2 \leq \cdots$ and $\Lambda_n \nearrow \infty$ as $n \to \infty$. Let ψ_n denote the eigenfunctions with eigenvalue $-\Lambda_n$. An eigenfunction ψ_n satisfies

$$\mathcal{J}_d \psi_n = -\Lambda_n \psi_n.$$

Moreover eigenfunctions are elements of $L^2(\Delta_d, \Pi)$, Hilbert space of the square integrable functions with respect to Π that has a inner product $\langle \cdot, \cdot \rangle$ and they satisfy

$$\langle \psi_n, \psi_m \rangle = \int_{\Delta_d} \psi_n(x) \psi_m(x) \Pi(dx) = \delta_{n,m} \kappa_n$$

where κ_n is some constant and δ is the Kronecker delta. Now since $\exp(-\Lambda_n t)\psi_n(x)$ is a solution to the Kolmogorov backward equation, one can reach the spectral representation of the transition density function as,

$$p(t; x, y) = \sum_{n=0}^{\infty} \frac{1}{\kappa_n} e^{-\Lambda_n t} \psi_n(x) \psi_n(y) \Pi(y). \tag{4.34}$$

Now we briefly remind univariate Jacobi polynomials (known also as hypergeometric polynomials), which are class of orthogonal polynomials that generalize Gegenbauer, Legendre and Chebyshev polynomials (see [1, Chp.22]).

Univariate Jacobi polynomials

The Jacobi polynomials $p_n^{(\alpha,\beta)}(z)$, for $z \in [0, 1]$ satisfy the differential equation

$$(1 - z^2)\frac{d^2 f(z)}{dz^2} + [\beta - \alpha - (\alpha + \beta + 2)z]\frac{df(z)}{dz} + n(n + \alpha + \beta + 1)f(z) = 0.$$

If $\alpha, \beta > -1$, the set $\{p_n\}_{n=0}^{\infty}$ forms an orthogonal system on $[-1, 1]$ with respect to the weight function $(1-z)^\alpha(1+z)^\beta$. For our purposes, we can use the following modification of these polynomials on Δ_2. For $x \in [0, 1]$ and $\alpha, \beta > 0$

$$P_n^{(\alpha,\beta)}(x) = p_n^{(\beta-1,\alpha-1)}(2x-1).$$

Modified Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $x \in [0, 1]$ satisfy the differential equation

$$x(1-x)\frac{d^2 f(x)}{dx^2} + [\alpha - (\alpha+\beta)x]\frac{df(x)}{dx} + n(n+\alpha+\beta-1)f(x) = 0,$$

and for fixed $\alpha, \beta > 0$ $\{P_n^{(\alpha,\beta))}\}_{n=0}^{\infty}$ forms an orthogonal system on [0,1] with respect to the $x^{\alpha-1}(1-x)^{\beta-1}$. In fact,

$$\int_0^1 P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)x^{\alpha-1}(1-x)^{\beta-1}\,dx = \delta_{m.n}c_n^{(\alpha,\beta)}$$

where

$$c_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(2n+\alpha+\beta-1)\Gamma(n+\alpha+\beta-1)(\Gamma(n+1)}. \tag{4.35}$$

Multivariate Jacobi polynomials

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. Also denote $\mathbf{n} = (n_1, \ldots, n_{d-1}) \in \mathbb{N}_0^{d-1}$ and $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}_{++}^d$. Moreover define the norm of $\mathbf{n}$ as $|\mathbf{n}| := \sum_{i=1}^{d-1} n_i$. One can define the multivariate Jacobi polyomials as in [153] or [96]. We mainly follow [153] for this section.

Definition 4.36. For every $\mathbf{n}$ and θ, the multivariate Jacobi polynomial $R_{\mathbf{n}}^{\theta}$ is defined by

$$R_{\mathbf{n}}^{\theta}(x) = \prod_{i=1}^{d-1}\left[\left(1 - \frac{x_i}{1-\sum_{j=1}^{i-1}x_j}\right)^{N_i} P_{n_i}^{(\theta_i,\Theta_i+2N_i)}\left(\frac{x_i}{1-\sum_{j=1}^{i-1}x_j}\right)\right], \tag{4.37}$$

where $N_i = \sum_{j=i+1}^{d-1} n_j$ and $\Theta_i = \sum_{j=i+1}^{d} \theta_i$.

Proposition 4.38. *For all $\mathbf{n} \in \mathbb{N}_0^{d-1}$ the multivariate Jacobi polynomials $R_{\mathbf{n}}^{\theta}(x)$ satisfy*

$$\mathcal{J}_d R_{\mathbf{n}}^{\theta}(x) = -\lambda_{|\mathbf{n}|}^{\theta} R_{\mathbf{n}}^{\theta}(x), \tag{4.39}$$

where

$$\lambda_{|\mathbf{n}|}^{\theta} = \frac{1}{2}|\mathbf{n}|(|\mathbf{n}| - 1 + |\theta|).$$

Proof. See [153] or [94]. □

Proposition 4.40. *Transition density function of the multivariate dimensional Jacobi process with parameters $\theta = (\theta_1, \ldots, \theta_d) \in (0, \infty)^d$ is given by*

$$p_\theta(t; x, y) = \sum_{\mathbf{n}\in\mathbb{N}_0^{d-1}} \frac{1}{K_{\mathbf{n}}^{\theta}} e^{-\lambda_{|\mathbf{n}|}^{\theta} t} R_{\mathbf{n}}^{\theta}(x) R_{\mathbf{n}}^{\theta}(y) \Pi_{i=1}^d y_i^{\theta_i - 1}, \tag{4.41}$$

where $K_{\mathbf{n}}^{\theta} = \prod_{i=1}^{d-1} c_{n_i}^{(\theta_i,\Theta_i+2N_i)}$, and $x_d = 1 - |x|$.

Proof. The transition density function $p_\theta(t;x,y)$ is obtained by substituting the eigenvalues and eigenfunctions obtained in Proposition (4.38) to the spectral expansion (4.34). $\square$

So the right hand side of (4.33) is computed via

$$\mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{e^{-(T-t)C_{\alpha,\sigma}^{\gamma_1,\dots,\gamma_d}}\prod_{i=1}^d (X_t^{(i)})^{a_i}}{\prod_{i=1}^d (X_T^{(i)})^{a_i}} \,\Big|\, \mathcal{F}_t\right]$$
$$= e^{-(T-t)C_{\alpha,\sigma}^{\gamma_1,\dots,\gamma_d}}\left(\prod_{i=1}^d (X_t^{(i)})^{a_i}\right)\int_{\Delta_d}\frac{\sum_{\mathbf{n}\in\mathbb{N}_0^{d-1}}\frac{1}{K_{\mathbf{n}}^\theta}e^{-\lambda_{|\mathbf{n}|}^\theta(T-t)}R_{\mathbf{n}}^\theta(X_t)R_{\mathbf{n}}^\theta(y)}{\prod_{i=1}^d y_i^{a_i-\theta_i+1}}dy, \quad (4.42)$$

One can compute the integral on the right hand side of (4.42) and hence bond price explicitly in the case of $d=2$. Let

$$H(t,T,x,k_1,k_2) := \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{1}{X_T^{k_1}(1-X_T)^{k_2}} \,\Big|\, X_t = x\right]$$

Lemma 4.43. *Let X be a Jacobi process on Δ_1 (or Δ_2^+), for the parameter set and conditions given as in 4.31. For $0\le t\le T$, $T>0$, $0<x<1$ and $k_i > -\frac{(\alpha_i-1)^2\sigma^2}{8}$ for $i=1,2$, $H(t,T,x,k_1,k_2)<\infty$.*

Proof.

$$\mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{1}{X_T^{a_1}(1-X_T)^{a_2}} \,\Big|\, X_t = x\right] = \int_0^1 y^{-a_1}(1-y)^{-a_2}p_\theta(T-t;X_t,y)dy \quad (4.44)$$

The transition density function given by (4.41) under $\mathbb{Q}_x^h$ reduces to

$$p_\theta(t;x,y) = \frac{y^{\theta_1-1}(1-y)^{\theta_2-1}}{\mathcal{B}(\theta_1,\theta_2)}\sum_{n=0}^\infty \frac{e^{-n(n+\theta_1+\theta_2-1)\frac{t}{2}}}{h_n^2}P_n^{\theta_1-1,\theta_2-1}(2x-1)P_n^{\theta_1-1,\theta_2-1}(2y-1), \quad (4.45)$$

where

$$h_n^2 = \frac{(\theta_1)_n(\theta_2)_n}{(\theta_1+\theta_2)_{n-1}(\theta_1+\theta_2-1+2n)n!}$$

with

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \frac{(x+n-1)!}{(x-1)!}$$

denoting the Pochhammer symbol and Γ and $\mathcal{B}$ is the gamma and beta functions, respectively. Now it is easy to see that the integral in (4.44) is finite if and only if $\theta_1 > a_1$ and $\theta_2 > a_2$, and it can be computed by the fact that

$$\int_0^1 y^{\theta_1-a_1-1}(1-y)^{\theta_2-a_2-1}P_n^{\theta_1-1,\theta_2-1}(2y-1)\,dy = \mathcal{B}(\theta_1-a_1,\theta_2-a_2)\frac{(\theta_2)_n}{n!}r_n, \quad (4.46)$$

where r_n are hypergeometric functions of matrix arguments and computed by the a three term recurrence relation pertaining to the Hahn polynomials (see [101, Theorem 4.1]). Note that $\theta_i > a_i$ for $i=\{1,2\}$, by the boundary conditions of the multivariate Jacobi process on the new measure $\mathbb{Q}_x^h$. $\square$

Corollary 4.47. *Let X be a Jacobi process on Δ_1 (or Δ_2^+), for the parameter set and conditions given as in 4.31. Then the zero-coupon bond price $P(t,T)$ for $0 \le t \le T$, $T > 0$, $0 < x < 1$ and $k_i > -\frac{(\alpha_i-1)^2\sigma^2}{8}$ for $i = \{1,2\}$ is given by,*

$$P(t,T) = e^{-(T-t)C_{\alpha,\gamma,\sigma}} X_t^{a_1}(1-X_t)^{a_2} H(t,T,X_t,k_1,k_2), \tag{4.48}$$

where $C_{\alpha,\gamma,\sigma} = (a_1\alpha(1-\gamma) + a_2\alpha\gamma + \sigma^2 a_1 a_2)$ and $a_i = \frac{1}{2}\Big(1 - \alpha_i + \sqrt{(\alpha_i-1)^2 + \frac{8k_i}{\sigma^2}}\Big)$ for $i = 1,2$.

4.5 Transition density function by dual process representation

Alternative and more convenient way to compute the transition density of multivariate Jacobi process is proposed by [69] and uses the (moment) dual process representation of the Jacobi processes (see page 188 and thereon for dual process representation of a Markov process in [70]). In fact, the transition density function of the Jacobi process is given by the mixture of (dual) pure-death process with sampling distribution given by a multinominal distribution and where the observations are taken from a Dirichlet distribution. As it is well known in population genetics literature, the process X can be seen as proportions evolving in time of d traits, types or etc. in a population. Therefore, probabilistically one can interpret those proportions $(X_t^{(1)},\dots,X_t^{(d)})$ to be equivalent to first sampling from the dual process L that starts from infinity and characterize the number of line of descents at time $t > 0$, then given the value of L_t sampling from a multinomial distribution characterizing different types of individuals and lastly generating an observation (proportions used in multinomial distribution) from a Dirichlet distribution. To be complete in our exposition we now give the derivation of the transition density function of X. Consider a multivariate Jacobi process X taking its values on the $(d-1)$-dimensional standard simplex Δ_d^+ and characterized by the second order differential operator,

$$\mathcal{J}_d^+ = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\mu_i(x)\frac{\partial}{\partial x_i}, \tag{4.49}$$

acting on $C^2(\Delta_d^+)$. The diffusion matrix $a(x) = (a_{ij}(x))_{i,j\in\{1,\dots,d\}}$ is of the form

$$a_{ij}(x) = \begin{cases} \sigma^2 x_i(1-x_i), & \text{if } i = j \\ -\sigma^2 x_i x_j, & \text{otherwise,} \end{cases}$$

with $\sigma > 0$ and the drift is given by

$$\mu_i(x) = \alpha(\gamma_i - x_i),$$

with $\alpha > 0$, $0 < \gamma_i < 1$ for $i = \{1,\dots,d\}$, such that $\sum_{i=1}^{d}\gamma_i = 1$.

We denote the transition function of X with parameter set $\theta = (\theta_1,\dots,\theta_d)$ as $P_\theta(t,x,y)$. It is known that the reversible stationary distribution of X is given by a Dirichlet distribution whose density with respect to $(d-1)$ dimensional Lebesgue measure on Δ_d^+,

$$\mathcal{D}(x,\theta) = \frac{\Gamma(|\theta|)}{\Gamma(\theta_1)\dots\Gamma(\theta_d)} x_1^{\theta_1-1}\dots x_d^{\theta_d-1}.$$

Also we use the multi-index notation,

$$|m| = \sum_{i=1}^{d} m_i, \qquad \binom{|m|}{m} = \frac{|m|}{m_1!\dots,m_d!}, \qquad x^m = \prod_{i=1}^{d} x_i^{m_i}$$

for $m \in \mathbb{N}_0^d$ and $x \in \Delta_d$, with $0^0 := 1$ and e_i is the unit basis vector, whose i-th entry is 1. Furthermore,

$$m_{(n)} = m(m+1)\dots(m+n-1), \qquad n \geq 1; \quad m_{(0)} = 1,$$

$$m_{[n]} = m(m-1)\dots(m-n+1), \qquad n \geq 1; \quad m_{[0]} = 1.$$

Let us define test functions $f_l \in C^2(\Delta_d^+)$ for each $l \in \mathbb{N}_0^d$ by $f_l(x) = x^l$. Applying the generator to test these functions, we obtain

$$\mathcal{J}_d^+ f_l = \frac{\sigma^2}{2} \sum_{i=1}^{d} l_i(l_i - 1 + \theta_i) f_{l-\epsilon_i} - \frac{\sigma^2}{2} |l|(|l| - 1 + |\theta|) f_l, \qquad l \in \mathbb{N}_0^d. \tag{4.50}$$

By setting for all $l, m \in \mathbb{N}_0^d$,

$$\beta(l,m) = \begin{cases} \frac{\sigma^2}{2} l_i(l_i - l + \theta_i), & \text{if } m = l - e_i, \\ -\frac{\sigma^2}{2} |l|(|l| - 1 + |\theta|), & \text{if } m = l, \\ 0, & \text{otherwise} \end{cases} \tag{4.51}$$

we can write (4.50) as,

$$\mathcal{J}_d^+ f_l = \sum_{m \in \mathbb{N}_0^d} \beta(l,m) f_m, \qquad l \in \mathbb{N}_0^d. \tag{4.52}$$

Note that $\beta(l,m) \geq 0$ for all $l \neq m$ and $\beta(l,l) \leq 0$ for all l. Also define mixed moments of Dirichlet distribution

$$\mu_l := \int_{\Delta_d} f_l \, d\mathcal{D}(\theta) = \frac{\Gamma(|\theta|)}{\Gamma(\theta_1)\dots\Gamma(\theta_d)} \frac{\Gamma(l_1 + \theta_1)\dots\Gamma(l_d + \theta_d)}{\Gamma(|l| + |\theta|)}. \tag{4.53}$$

By (4.53) $\mu_l > 0$ for all $l \in \mathbb{N}_0^d$. Now, if we take the expectation of $\mathcal{J}_d^+ f_l$ for $l \in \mathbb{N}_0^d$ with respect to the stationary distribution of X, we have

$$0 = \int_{\Delta_d} \mathcal{J}_d \, f_l \, d\mathcal{D}(\theta) = \sum_{m \in \mathbb{N}_0^d} \beta(l,m) \mu_m, \tag{4.54}$$

due to the fact that the we can change the integration and summation since every summand in (4.52) except one of them is non-negative. If we define

$$q(l,m) := \frac{\beta(l,m)\mu_m}{\mu_l},$$

with

$$q(l, l - \epsilon_i) = \frac{\sigma^2}{2} l_i(|l_i| - 1 + |\theta|),$$

then by (4.54)

$$\sum_{m\in\mathbb{N}_0^d} q(l,m) = 0, \qquad l \in \mathbb{N}_0^d.$$

Therefore, $\{q(l,m)\}$ is the infinitesimal generator matrix for a pure death process $L := (L_t)_{t\geq 0}$, starting at infinity and taking values in $\mathbb{N}_0^d$ with transition probabilities $p_{lm}(t)$. $p_{lm}(t)$ can be found by the help of a related one-dimensional death process $K := (K_t)_{t\geq 0}$ that takes values in $\mathbb{N}_0$ and has death rates $\rho(k,k-1) := \frac{\sigma^2}{2}k(k-1+|\theta|)$. For the process K, the distribution

$$\mathbb{P}[K_t = k \,|\, K_0 = n] = d_{nk}^{|\theta|}(t)$$

is given by ([74, Thm. 4.3]),

$$\begin{aligned} d_{nk}^{|\theta|}(t) &= \sum_{m=k}^{l} \rho_m^{|\theta|}(t) \frac{(-1)^{m-k}(2m+|\theta|-1)(k+|\theta|)_{(m-1)} n_{[m]}}{k!(m-k)!(n+|\theta|)_{(m)}}, \qquad 1 \leq k \leq n \\ d_{n0}^{|\theta|}(t) &= 1 + \sum_{m=1}^{l} \rho_m^{|\theta|}(t) \frac{(-1)^m |\theta|_{(m-1)} n_{[m]}(2m+|\theta|-1)}{m!(n+|\theta|)_{(m)}}, \qquad k = 0, \end{aligned} \tag{4.55}$$

where $\rho_m^{|\theta|}(t) = e^{-\rho(m,m-1)t}$. By taking the limit $n \to \infty$, one can also compute

$$\mathbb{P}[K_t = k] = d_k^{|\theta|}(t), \qquad k \geq 0,\ t > 0, \tag{4.56}$$

where

$$d_k^{|\theta|}(t) = \begin{cases} 1 - \sum_{m=1}^{\infty} \rho_m^{|\theta|}(t)(-1)^{m-1} \frac{(2m+|\theta|-1)|\theta|_{(m-1)}}{m!}, & \text{if } k = 0, \\ \sum_{m=k}^{\infty} \rho_m^{|\theta|}(t)(-1)^{m-k} \frac{(2m+|\theta|-1)(k+|\theta|)_{(m-1)}}{k!(m-k)!}, & \text{if } k \geq 1. \end{cases} \tag{4.57}$$

From the probabilities given by (4.55), one can find transition probabilities $p_{lm}(t)$ of d-dimensional pure death process $(L_t)_{t\geq 0}$, since there is an hypergeometric sampling in selecting, that is, for $l, m \in \mathbb{N}_0^d$,

$$p_{lm}(t) := \mathbb{P}[L_t = m \,| L_0 = l] = d_{|l|\,|m|}^{|\theta|}(t) \frac{\binom{l_1}{m_1} \cdots \binom{l_d}{m_d}}{\binom{|l|}{|m|}}, \qquad l \geq m, \quad t \geq 0. \tag{4.58}$$

Furthermore, if we define the function

$$g_l = \frac{f_l}{\mu_l},$$

(4.50) becomes

$$\mathcal{J}_d^+ g_l = \sum_{m\in\mathbb{N}_0^d} q(l,m) g_m, \qquad l \in \mathbb{N}_0^d.$$

Hence we have shown that

Proposition 4.59. *The pure death process $(L_t)_{t\geq 0}$ taking values in $\mathbb{N}_0^d$ with infinitesimal generator matrix given by $\{q(l,m)\}_{l,m\in\mathbb{N}_0^d}$ is dual to the multivariate Jacobi process with the duality relation*

$$\mathbb{E}_x[g_l(X_t)] = \mathbb{E}_l[g_{L_t}(x)], \qquad (l,x) \in \mathbb{N}_0^d \times \Delta_d^+, \quad t \geq 0. \tag{4.60}$$

Remark 4.61. On the left hand side of (4.60), expectation is taken with respect to the distribution of X_t, and on the right hand side with respect to the distribution of L_t. More precisely,

$$\frac{1}{\mu_l}\int_{\Delta_d^+} f_l(y)P_\theta(t,x,dy) = \sum_{m\in\mathbb{N}_0^d} \frac{p_{lm}(t)}{\mu_m} f_m(x), \qquad (x,k)\in\Delta_d^+\times\mathbb{N}_0^d, \quad t\geq 0. \tag{4.62}$$

Now, we show how to represent $P_\theta(t,x,\cdot)$ in terms of transition probabilities $p_{lm}(t)$ of dual process L and related one-dimensional process K. In order to achieve this, we need the following lemma showing that how we can use moments to define a sampling distribution converging weakly to the distribution we are looking for.

Lemma 4.63. *Let λ be a probability measure on Δ_d^+ and define*

$$\lambda_n := \sum_{l\in\mathbb{N}_0^d\colon |l|=n} \binom{n}{l} \int_{\Delta_d^+} f_l\, d\lambda\, \delta_{l/n}, \tag{4.64}$$

then $\lambda_n \Rightarrow_w \lambda$(converges weakly).

Proof. For an open set $\mathcal{O}\subset\Delta_d^+$,

$$\begin{aligned}
\liminf_{n\to\infty} \lambda_n(\mathcal{O}) &= \liminf_{n\to\infty} \int_{\Delta_d} \sum_{l\in\mathbb{N}_0^d\colon |l|=n} \binom{l}{k} f_l(y)\delta_{l/n}(\mathcal{O})d\lambda(y) \\
&\geq \int_{\Delta_d^+} \liminf_{n\to\infty} \sum_{l\in\mathbb{N}_0^d\colon |l|=n} \binom{n}{l} f_l(y)\delta_{l/n}(\mathcal{O})d\lambda(y) \\
&\geq \int_{\Delta_d^+} \delta_y(\mathcal{O})d\lambda(y) \\
&= \lambda(\mathcal{O}),
\end{aligned} \tag{4.65}$$

where the first inequality is due to Fatou's Lemma and the second is due to weak law of large numbers for an i.i.d sequence of multinomial random vectors. ☐

Now notice that

$$\lim_{(|l|,l/|l|)\to(\infty,x)} p_{lm}(t) = d^{|\theta|}_{|m|}(t)\binom{|m|}{m}x^m =: d_m^{|\theta|}(t,x)$$

by (4.58). One can also show that $d_m^{|\theta|}(t,x)$ defines a probability distribution on $\mathbb{N}_0^d$ for each $t>0$ and $x\in\Delta_d^+$; see [74, Sec. 4]. Fix $t>0$ and $x\in\Delta_d^+$, then by (4.62), we have for $y\in\Delta_d^+$,

$$\begin{aligned}
\lim_{(|l|,l/|l|)\to(\infty,y)} \frac{1}{\mu_l}\int_{\Delta_d} f_l(z)P_\theta(t,x,dz) &= \lim_{(|l|,l/|l|)\to(\infty,y)} \sum_{m\in\mathbb{N}_0^d} \frac{p_{lm}(t)f_m(x)}{\mu_m} \\
&= \sum_{m\in\mathbb{N}_0^d} \frac{d_m^{|\theta|}(t,y)f_m(x)}{\mu_m} \\
&=: \phi(t,x,y).
\end{aligned} \tag{4.66}$$

The following lemma is the consequence of the fact that $d_m^{|\theta|}(t,x)$ defines a probability distribution on $\mathbb{N}_0^d$ for each $t>0$ and $x\in\Delta_d^+$.

Lemma 4.67. *ϕ is a probability density with respect to $\mathcal{D}(\theta)$. That is,*

$$\int_{\Delta_k} \phi(t,x,y)\mathcal{D}(\theta,dy) = 1, \quad \text{for } \mathcal{D}(\theta) \text{ a.e, } x \in \Delta_d^+.$$

Let us define

$$\phi_n(x) := \begin{cases} \frac{\int_{\Delta_d^+} f_l P(t,x,dy)}{\int_{\Delta_d^+} f_l\, \mathcal{D}(\theta,dy)}, & \text{if } x = \frac{l}{n},\ |l| = n,\ l \in \mathbb{N}_0^d \\ 0, & \text{otherwise .} \end{cases} \tag{4.68}$$

For the following, $P_{\theta,n}(t,x,.)$ and $\mathcal{D}_n(\theta)$ are defined as δ_n in Lemma 4.63. Then, for $g \in C(\Delta_d^+)$ with $g \geq 0$,

$$\begin{aligned} \int_{\Delta_d^+} g\, P_\theta(t,x,dy) &= \lim_{n\to\infty} \int_{\Delta_d^+} g\, P_{\theta,n}(t,x,dy) \\ &= \lim_{n\to\infty} \sum_{l\in\mathbb{N}_0^d:|l|=n} g\left(\frac{l}{n}\right)\binom{n}{l} \int_{\Delta_d^+} f_l\, P_\theta(t,x,dy) \\ &= \lim_{n\to\infty} \sum_{l\in\mathbb{N}_0^d:|l|=n} g\left(\frac{l}{n}\right)\phi_n\left(\frac{l}{n}\right)\binom{n}{l} \int_{\Delta_d^+} f_l\, \mathcal{D}(\theta,dy) \\ &= \lim_{n\to\infty} \int_{\Delta_d^+} g\, \phi_n\, \mathcal{D}_n(\theta,dy) \geq \int_{\Delta_d^+} g\, \phi\, d\mathcal{D}(\theta). \end{aligned} \tag{4.69}$$

This implies that $P_\theta(t,x,dy) \geq \phi(t,x,y)\,\mathcal{D}(\theta,dy)$ and since by Lemma 4.67, ϕ is a probability density with respect to the stationary distribution Π of X,

$$P(t,x,dy) = \phi(t,x,y)\,\mathcal{D}(\theta,dy), \qquad \mathcal{D}(\theta) \text{ almost everywhere, } x \in \Delta_d^+.$$

Now, because X is a reversible process, that is, $P_\theta(t,x,.)$ is reversible with respect to $\mathcal{D}(\theta)$, we conclude that

$$\phi(t,x,y) = \phi(t,y,x) = \sum_{m\in\mathbb{N}_0^d} \frac{d_m^{|\theta|}(t,x) f_m(y)}{\mu_m}$$

and hence we can write

$$\begin{aligned} P_\theta(t,x,dy) &= \sum_{m\in\mathbb{N}_0^d} \frac{d_m^{|\theta|}(t,x) f_m(y)}{\mu_m}\, \mathcal{D}(\theta,dy) \\ &= \sum_{m\in\mathbb{N}_0^d} d_m^{|\theta|}(t,x)\mathcal{D}(k+\theta,dy). \end{aligned} \tag{4.70}$$

for $\mathcal{D}(\theta)$ almost everywhere, $x \in \Delta_d^+$. Also it can be shown that $P_\theta(t,x,dy)$ satisfy the Feller property and weakly continuous in $y \in \Delta_d^+$. Therefore (4.70) holds for all $x \in \Delta_d^+$.

Proposition 4.71. *The transition probability measure of the multivariate Jacobi process with parameters $\theta = (\theta_1,\ldots,\theta_d) \in (0,\infty)^d$ is given by*

$$p_\theta(t;x,\cdot) = \sum_{k\in\mathbb{N}_0} d_k^{|\theta|}(t) \sum_{\substack{l\in\mathbb{N}_0^d \\ |l|=k}} \mathcal{M}(l;k,x)\mathcal{D}(\theta+l), \quad t>0,\ x\in\Delta_d^+,$$

where $\mathcal{D}(\theta+l)$ are Dirichlet distributions, $\mathcal{M}(\,\cdot\,;k,x)$ are multinomial distributions, and $d_k^{|\theta|}(t)$ with $k\in\mathbb{N}_0$ are the transition probabilities of the (dual-related) pure death process K "starting from infinity" with death rates $\varrho(k,k-1)=\frac{1}{2}k(k-1+|\theta|)\sigma^2$ for $k\in\mathbb{N}$, and given by

$$d_k^{|\theta|}(t)=\begin{cases}1-\sum_{m=1}^{\infty}\rho_m^{|\theta|}(t)(-1)^{m-1}\frac{(2m+|\theta|-1)|\theta|_{(m-1)}}{m!}, & \text{if } k=0,\\ \sum_{m=k}^{\infty}\rho_m^{|\theta|}(t)(-1)^{m-k}\frac{(2m+|\theta|-1)(k+|\theta|)_{(m-1)}}{k!(m-k)!}, & \text{if } k\geq 1,\end{cases}$$

with $\rho_m^{|\theta|}(t)=e^{-m(m+|\theta|-1)t/2}$.

Corollary 4.72. *The transition function of the bivariate Jacobi process with parameters $(\theta_1,\theta_2)\in(0,\infty)^2$ is given by*

$$p_\theta(t;x,y)=\sum_{n\in\mathbb{N}^2}d_{|n|}^{|\theta|}(t)\binom{|n|}{n_1}x^{n_1}(1-x)^{n_2}\mathcal{B}(\theta_1+n_1,\theta_2+n_2)^{-1}y^{n_1+\theta_1-1}(1-y)^{n_2+\theta_2-1},$$

where $\mathcal{B}$ denotes the Beta function.

Remark 4.73. To have a meaning of the transition density, one can think of the infinite number of individuals that make up L_0 as the leaves in a forest of trees. Each tree either grows from a founder at time t (corresponding to time zero in the diffusion process) or its root arose through a new mutation.This subdivides the leaves into families and leads to the Dirichlet mixture. If there are k founder lineages, then their types are determined by sampling k individuals from the diffusion at time zero, and hence the probability that the numbers of founder lineages of types $\{1,\dots,d\}$ are given by $\mathcal{M}(l;k,p)$ with $|l|=k$.

Now we are ready to give the bond price formula.

Theorem 4.74. *Given $\alpha,\sigma>0$ and $\gamma=(\gamma_1,\dots,\gamma_d)\in(\Delta_d^+)^\circ$, assume that $\alpha_i:=2\alpha\gamma_i/\sigma^2>1$ for $i=1,\dots,d$ (boundary unattainable). Take any $k_i>-\frac{1}{8}(\alpha_i-1)^2\sigma^2$ for $i\in\{1,\dots,d\}$. Define $a_i=\frac{1}{2}\big(1-\alpha_i+\sqrt{(\alpha_i-1)^2+8k_i/\sigma^2}\,\big)$ and $\theta_i=\alpha_i+2a_i$ for $i\in\{1,\dots,d\}$. Then, for $t\in[0,T)$,*

$$\begin{aligned}P(t,T)=\exp\big(-(T-t)C_{\alpha,\gamma,\sigma}^{a_1,\dots,a_d}\big)\Bigg(\prod_{i=1}^{d}(X_t^{(i)})^{a_i}\Bigg)\sum_{k\in\mathbb{N}_0}d_k^{|\theta|}(T-t)\\ \times\sum_{\substack{l\in\mathbb{N}_0^d\\|l|=k}}\mathcal{M}(l;k,X_t)\frac{\Gamma(|\theta+l|)}{\prod_{i=1}^{d}\Gamma(\theta_i+l_i)}\frac{\prod_{i=1}^{d}\Gamma(\alpha_i+a_i+l_i)}{\Gamma\big(\sum_{i=1}^{d}(\alpha_i+a_i+l_i)\big)},\end{aligned}$$

where $|\cdot|$ denotes the sum of components and

$$C_{\alpha,\gamma,\sigma}^{a_1,\dots,a_d}=\alpha\sum_{i=1}^{d}a_i(1-\gamma_i)+\frac{\sigma^2}{2}\Bigg(\Bigg(\sum_{i=1}^{d}a_i\Bigg)^2-\sum_{i=1}^{d}a_i^2\Bigg),$$

$\mathcal{M}(\,\cdot\,;k,p)$ the multinomial distribution with $k\in\mathbb{N}_0$ trails and probabilities $p=(p_1,\dots,p_d)\in\Delta_d^+$,

$$\mathcal{M}(l;k,p)=\frac{k!}{l_1!\dots l_d!}p_1^{l_1}\dots p_d^{l_d},\qquad l\in\mathbb{N}_0^d,\ |l|=k,$$

and $d_k^{|\theta|}(\tau)$ *for* $k \in \mathbb{N}_0$ *are the transition probabilities of a (dual) related total population pure death process and given by*

$$d_k^{|\theta|}(\tau) = \begin{cases} 1 - \sum_{m=1}^{\infty} \rho_m^{|\theta|}(\tau)(-1)^{m-1}\frac{(2m+|\theta|-1)|\theta|_{(m-1)}}{m!}, & \text{if } k = 0, \\ \sum_{m=k}^{\infty} \rho_m^{|\theta|}(\tau)(-1)^{m-k}\frac{(2m+|\theta|-1)(k+|\theta|)_{(m-1)}}{k!(m-k)!}, & \text{if } k \geq 1, \end{cases}$$

with $\rho_m^{|\theta|}(\tau) = e^{-m(m+|\theta|-1)\sigma^2\tau/2}$ *and the rising factorial.*

Proof. Zero-coupon bond price $P(t,T)$ boils down to the computation of the integral given by,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{1}{\prod_{i=1}^d (X_T^{(i)})^{a_i}} \;\Big|\; \mathcal{F}_t\right] &= \int_{\Delta_d} \prod_{i=1}^d y_i^{-a_i} \sum_{k=0}^{\infty} d_k^{|\theta|}(T-t) \sum_{|l|=k} \mathcal{M}(l;k,X_t)\mathcal{D}(y,\theta+l)dy \\ &= \sum_{k=0}^{\infty} d_k^{|\hat{\theta}|}(T-t) \sum_{|l|=k} \mathcal{M}(l;k,X_t) \int_{\Delta_d} \frac{\Gamma(|\theta+l|)}{\prod_{i=1}^d \Gamma(\theta_i+l_i)} \prod_{i=1}^d y_i^{\theta_i+l_i-1-a_i} dy \end{aligned} \tag{4.75}$$

where in the second equality due to Fubini–Tonelli theorem, one can change the order of sum and integral. Since $\theta_i > a_i$ for $i \in \{1,\dots,d\}$, the integral is finite and can be computed as,

$$\int_{\Delta_d} \frac{\Gamma(|\theta+l|)}{\prod_{i=1}^d \Gamma(\theta_i+l_i)} \prod_{i=1}^d y_i^{\alpha_i+a_i+l_i}\, dy = \frac{\Gamma(|\theta+l|)}{\prod_{i=1}^d \Gamma(\theta_i+l_i)} \frac{\prod_{i=1}^d \Gamma(\alpha_i+a_i+l_i)}{\Gamma(\sum_{i=1}^d(\alpha_i+a_i+l_i))}$$

□

Remark 4.76. As it is seen from Theorem 4.74, computation of the bond price involves just series sum involving ratios of gamma functions. The series are rapidly converging due to the terms involving $d_k^{|\theta|}$.

Simulation of the multivariate Jacobi process

The dual process representation has a natural interpretation in terms of Kingman's coalescent, which is the moment dual to the Jacobi diffusion. The ancestral process $K := |L|$ represents the number of lineages surviving a time t back in an infinite-leaf coalescent tree, when lineages are lost both by coalescence and by mutation. As it is also explained in Remark 4.73, this leads to a simulation algorithm

Algorithm 4.77 (Simulation of the multivariate Jacobi process)**.**

i. Simulate the random variable K_t *from the transition probabilities* $d_k^{|\theta|}(t)$ *with* $k \in \mathbb{N}_0$.

ii. Given $K_t = k$*, simulate* $L_t \sim \mathcal{M}(\cdot, k, x)$.

iii. Given $L_t = l = (l_1, \dots, l_d)$*, simulate* $X_t \sim \mathcal{D}(\theta + l)$.

iv. Return X_t.

For the algorithm, the crucial part is the first step, simulation the dual-related process K_t, since the transition function of the dual process is explicit but rather complicated. One can use the numerical approximation as in [95]. Another possibility is to use an alternating series method as in [107]. This approach gives exact sampling.

4.6 Credit default swap (CDS) pricing

A credit default swap is a credit derivative that offers insurance against default, in which two counter-parties agree to exchange cash flows according to the following. An investor, who wants to be protected against default, will make fixed premium payments in arrears proportional to the fixed rate of s_{CDS}, which is called CDS spread, at dates $T_1, \ldots, T_n$ to the protection seller as long as default has not occurred. On the other hand, in case of default before T_n, the protection seller will make a payment to the protection buyer depending on the recovery scheme. Those cash flows are denoted premium leg and protection leg, respectively. Like in the classical swap agreements, the CDS spread s_{CDS} is found by equating the discounted premium and protection leg payments. More precisely, in a reduced-form setting s_{CDS} is given by assuming $T_0 = 0$ [118, Ch. 8],

$$s_{\mathrm{CDS}} = \frac{(1-V)\int_0^{T_n} \mathbb{E}_{\mathbb{Q}}\left[\lambda(X_s)\exp\left\{-\int_0^s (r(X_u)+\lambda(X_u))\,du\right\}\right]\,ds}{\sum_{i=1}^n (T_i - T_{i-1})\,\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{-\int_0^{T_i}(r(X_u)+\lambda(X_u))\,du\right\}\right]}, \tag{4.78}$$

where $(\lambda(X_t))_{t\geq 0}$ is the intensity of the default and $1-V$ is the payment obligation of the protection seller, where V is the deterministic recovery rate that might be different than the actual recovery rate. As it is seen from (4.78), in order to have a tractable formula for s_{CDS}, expectations both in the numerator and the denominator should be computable in a tractable way. Since, the expectation in the denominator is similar to the bond price formula, it can be computed by Theorem 4.74. Similarly, expectations such as

$$\mathbb{E}_{\mathbb{Q}}\left[\lambda(X_t)\exp\Big\{-\int_0^t (r(X_u)+\lambda(X_u))\,du\Big\}\right]$$

can be computed by modifying

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\Big\{-\int_0^T R(X_u)\,du\Big\}g(X_T)\;\Big|\;X_0 = x\right] = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{h(x)g(X_T)}{h(X_T)}\;\Big|\;X_0 = x\right],$$

which can be computed easily for appropriately chosen g. More precisely, let $R(\cdot)$ be given by $R(\cdot) = r(\cdot) + \lambda(\cdot)$, where

$$r(x) = \sum_{i=1}^{d_1} k_i \frac{1-x_i}{x_i} \qquad \text{and} \qquad \lambda(x) = \sum_{i=d_1+1}^{d} k_i \frac{1-x_i}{x_i},$$

where d_1 denotes the number of factors used for the risk-free rate. The rest of the factors, $d - d_1$ many, are used for the credit spread. Therefore assuming $t = 0$, we need to calculate

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_x}\left[\exp\left\{-\int_0^t \left(\sum_{i=1}^d k_i \frac{1-X_s^{(i)}}{X_s^{(i)}}\right)ds\right\}\left(\sum_{i=d_1+1}^d k_i \frac{1-X_t^{(i)}}{X_t^{(i)}}\right)\;\Big|\;X_0 = x\right] \\
&\qquad = \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{e^{-(T-t)C_{\alpha,\sigma}^{\gamma_1,\ldots,\gamma_d}}\prod_{i=1}^d x^{a_i}}{\prod_{i=1}^d (X_t^{(i)})^{a_i}}\left(\sum_{i=d_1+1}^d k_i \frac{1-X_t^{(i)}}{X_t^{(i)}}\right)\;\Big|\;X_0 = x\right]
\end{aligned} \tag{4.79}$$

Right-hand side can be computed explicitly by using the transition density function as in the case of bond-price formula. Let

$$G(0,T,x,a_1,\ldots,a_d) = \mathbb{E}_{\mathbb{Q}_x^h}\left[\frac{\prod_{i=1}^d x_i^{a_i}}{\prod_{i=1}^d (X_T^{(i)})^{a_i}}\;\Big|\;X_0 = x\right]$$

Then the right-hand side of (4.79) is finite and can be given by

$$\overline{G(T)} := \sum_{i=d_1+1}^{d} G(0,T,x,a_1,\ldots,a_i+1,\ldots,a_d) - G(0,T,x,a_1,\ldots,a_d) - \sum_{i=d_1+1}^{d} k_i \tag{4.80}$$

Hence the CDS spread is

$$s_{\text{CDS}} = \frac{(1-V)\int_0^{T_n} e^{-C^{a_1,\ldots,a_d}_{\alpha,\gamma,\sigma} s}\overline{G(s)}\, ds}{\sum_{i=1}^{n}(T_i - T_{i-1})\, P(0,T_i)}. \tag{4.81}$$

Chapter 5

Appendix: Yamada–Watanabe Condition for Pathwise Uniqueness

Here we briefly explain the Yamada–Watanabe condition[1], which relaxes the Lipschitz condition for the pathwise uniqueness of solutions of stochastic differential equations (SDEs) of the type

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t\,, \quad t \geq 0.$$

Hence, this condition can be used to show the strong uniqueness of solutions of a SDE with certain non-Lipschitz coefficients. The main references for this part are [58], [102], [160] and [157]. In mathematical finance, this is of particular interest for the Cox–Ingersoll–Ross model (CIR model for short), which describes the stochastic evolution of interest rates $(r_t)_{t\geq 0}$ by the SDE

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t, \quad t \geq 0,$$

with $r_0 \geq 0$, where α, μ with $\alpha\mu \geq 0$ and σ denote real constants.

Before stating the main theorem, we start with some definitions necessary for the sequel.

Definition 5.1. Given two jointly Borel measurable functions b: $[0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and σ: $[0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and a probability measure μ on $(\mathbb{R}^n, \mathcal{B}_n)$, a solution of the stochastic differential equation

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \quad t \geq 0, \tag{5.2}$$

with initial distribution μ is a pair (W, X) of continuous adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ such that

1. $W = (W_t)_{t\geq 0}$ is a standard $(\mathcal{F}, \mathbb{P})$-Brownian motion with values in $\mathbb{R}^d$,
2. the initial value X_0 has distribution μ,
3. the integrals implicitly given by (5.2) are well defined, i.e., for all $t \geq 0$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$, the corresponding component functions of b and σ satisfy

$$\int_0^t \sigma_{ij}^2(s, X_s)\,ds \overset{\text{a.s.}}{<} \infty \qquad \text{and} \qquad \int_0^t |b_i(s, X_s)|\,ds \overset{\text{a.s.}}{<} \infty,$$

[1] The extended version of this appendix can be found as lecture notes under [9].

4. for every $i \in \{1, 2, \ldots, n\}$, the i-th component process of

$$X = \big((X_t^{(1)}, \ldots, X_t^{(n)})^\top\big)_{t\ge 0}$$

satisfies, up to indistinguishability,

$$X_t^{(i)} = X_0^{(i)} + \int_0^t b_i(s, X_s)\, ds + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(s, X_s)\, dW_s^j, \quad t \ge 0. \tag{5.3}$$

Definition 5.4. We say that there is pathwise uniqueness for the SDE (5.2) with initial distribution μ, if whenever (W, X) and $(\tilde{W}, \tilde{X})$ are two solutions of (5.2) defined on the same filtered probability space with $W = \tilde{W}$ (same Brownian motion) and $X_0 \overset{\text{a.s.}}{=} \tilde{X}_0$ (same $\mathcal{F}_0$-measurable initial condition with distribution μ), then X and $\tilde{X}$ are indistinguishable, that is, there exists a set $N \in \sigma(\bigcup_{t\ge 0} \mathcal{F}_t)$ with $\mathbb{P}[N] = 0$ such that $\{X_t \ne \tilde{X}_t\} \subset N$ for all $t \in [0, \infty)$.

In this section, we give the main result, which combines the one- and multi-dimensional case. However, we mention that the Yamada–Watanabe condition is essentially a one-dimensional result (see Remarks 2 and 3 in [157]). For the one-dimensional setting, there is also an approach to pathwise uniqueness using local times, see [122].

The following theorem is the main result of this appendix; for its proof we assume that the filtration is right-continuous. We use $|\cdot|$ for the n-dimensional Euclidean norm and $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm.

Theorem 5.5. *Consider the stochastic differential equation* (5.2). *Assume that there exist a constant* $\gamma > 0$ *and functions* $\kappa, \varrho\colon [0,\gamma] \to [0,\infty)$ *satisfying* $\kappa(0) = 0$,

$$|b(t,x) - b(t,y)| \le \kappa(|x-y|), \tag{5.6}$$

and

$$\|\sigma(t,x) - \sigma(t,y)\|_{\mathrm{F}} \le \varrho(|x-y|) \tag{5.7}$$

for all $t \in [0,\infty)$ *and* $x, y \in \mathbb{R}^n$ *with* $|x-y| \le \gamma$. *Furthermore, assume that* ϱ *is non-decreasing,* $\varrho(u) > 0$ *for all* $u \in (0,\gamma]$ *and its square satisfies the Osgood condition*[2], *i.e.,*

$$\int_0^\gamma \frac{du}{\varrho^2(u)} = \infty. \tag{5.8}$$

In addition, assume that there exists a non-decreasing, concave and continuous function $G\colon [0,\gamma] \to [0,\infty)$ *with* $G(0) = 0$, *strictly positive on* $(0,\gamma]$, *such that*

$$G(u) \ge \kappa(u) + \frac{n-1}{2u}\varrho^2(u) \qquad \forall\, u \in (0,\gamma] \tag{5.9}$$

and it also satisfies the Osgood condition

$$\int_0^\gamma \frac{du}{G(u)} = \infty. \tag{5.10}$$

Then the pathwise uniqueness of solutions of (5.2) *holds for every initial distribution* μ.

[2] Named after William Fogg Osgood, cf. [137]

Remark 5.11. Note that for $n = 1$ with the choice $G(u) = \kappa(u)$ for $u \in [0, \gamma]$, the conditions on G are actually conditions on κ, in particular (5.10) reduces to the Osgood condition

$$\int_0^\gamma \frac{du}{\kappa(u)} = \infty. \tag{5.12}$$

For $n \geq 2$ and vanishing drift b, we can choose κ to be the zero function. With the choice $G(u) = \frac{n-1}{2u}\varrho^2(u)$ for $u \in [0, \gamma]$, the condition (5.10) is equivalent to

$$\int_0^\gamma \frac{u}{\varrho^2(u)}\, du = \infty, \tag{5.13}$$

which is substantially more restrictive than (5.8).

Proof of Theorem 5.5. The main idea of the proof is to construct a sequence $(f_k)_{k\in\mathbb{N}}$ of C^2-functions $f_k\colon \mathbb{R}^n \to [0, \infty)$ approximating the Euclidean norm $\mathbb{R}^n \ni z \mapsto |z|$, such that Itō's multi-dimensional formula can be applied to $f_k(X_t - Y_t)$, where X and Y are two solutions of the SDE (5.2) with $X_0 \stackrel{\text{a.s.}}{=} Y_0$. By then passing to the limit $k \to \infty$, the aim is to show that $\mathcal{E}[|X_t - Y_t|] = 0$ for all $t \geq 0$, which implies pathwise uniqueness.

The first step is to construct such approximations. Due to assumption (5.8), there exists a sequence

$$\gamma = a_0 > a_1 > a_2 > \cdots > a_k \searrow 0$$

such that

$$\int_{a_k}^{a_{k-1}} \frac{du}{\varrho^2(u)} = k, \quad k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$, we can construct a continuous function $\phi_k\colon [0, \infty) \to [0, \infty)$ such that

$$\phi_k(u) \begin{cases} \leq \frac{2}{k\varrho^2(u)} & \text{for } u \in (a_k, a_{k-1}), \\ = 0 & \text{otherwise,} \end{cases} \tag{5.14}$$

and

$$\int_{a_k}^{a_{k-1}} \phi_k(u)\, du = 1,$$

because the upper bound (5.14) of ϕ_k integrates to 2 over (a_k, a_{k-1}). Next we define the auxiliary function $\varphi_k\colon [0, \infty) \to [0, \infty)$ by

$$\varphi_k(w) = \int_0^w \int_0^v \phi_k(u)\, du\, dv, \quad w \geq 0.$$

Note that φ_k is a twice continuously differentiable function with $\varphi_k(w) = 0$ for $w \in [0, a_k]$. Furthermore,

$$\varphi_k'(w) = \int_0^w \phi_k(u)\, du \begin{cases} = 0 & \text{for } w \in [0, a_k], \\ \leq 1 & \text{for } w \in (a_k, a_{k-1}), \\ = 1 & \text{for } w \in [a_{k-1}, \infty). \end{cases} \tag{5.15}$$

Therefore, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is monotone increasing with $w - a_{k-1} \leq \varphi_k(w) \leq w$ for all $w \in [a_{k-1}, \infty)$. Finally, we can define the approximating sequence $(f_k)_{k\in\mathbb{N}}$ by

$$f_k(z) := \varphi_k(|z|), \quad k \in \mathbb{N},\ z \in \mathbb{R}^n.$$

It follows that each f_k is a twice continuously differentiable function on $\mathbb{R}^n$ and that $f_k(z) \nearrow |z|$ uniformly in $z \in \mathbb{R}^n$ as $k \to \infty$.

Now, let X and Y be two solutions of (5.2) with $X_0 \overset{\text{a.s.}}{=} Y_0$, driven by the same d-dimensional Brownian motion, and define the difference process Z by

$$Z_t := X_t - Y_t = \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s, \quad t \geq 0,$$

where we simplified the notation by defining the $\mathbb{R}^n$-valued stochastic process

$$b_s = b(s, X_s) - b(s, Y_s), \quad s \geq 0,$$

and the matrix-valued stochastic process

$$\sigma_s = \sigma(s, X_s) - \sigma(s, Y_s), \quad s \geq 0.$$

Define $\tau = \inf\{t \geq 0 : |Z_t| \geq \gamma\}$. Since $\{z \in \mathbb{R}^n : |z| \geq \gamma\}$ is closed and Z has continuous paths, τ is a stopping time. By assumption (5.6),

$$|b_{s\wedge\tau}| \leq \kappa(|Z_{s\wedge\tau}|), \quad s \geq 0. \tag{5.16}$$

We note that the definition of the Frobenius matrix norm and assumption (5.7) imply

$$\operatorname{tr}[\sigma_{s\wedge\tau}\sigma_{s\wedge\tau}^\top] = \|\sigma_{s\wedge\tau}\|_{\mathrm{F}}^2 \leq \varrho^2(|X_{s\wedge\tau} - Y_{s\wedge\tau}|) = \varrho^2(|Z_{s\wedge\tau}|), \quad s \geq 0. \tag{5.17}$$

Fix $k \in \mathbb{N}$. Applying Itō's multi-dimensional formula to $f_k(Z_t)$, we obtain up to indistinguishability,

$$f_k(Z_t) = I_k(t) + J_k(t), \quad t \geq 0, \tag{5.18}$$

with

$$I_k(t) := \int_0^t \nabla f_k(Z_s)\, \sigma_s\, dW_s, \quad t \geq 0, \tag{5.19}$$

and

$$J_k(t) := \int_0^t \left(\nabla f_k(Z_s)\, b_s + \frac{1}{2}\operatorname{tr}[H_k(Z_s)\sigma_s\sigma_s^\top]\right) ds, \quad t \geq 0, \tag{5.20}$$

where $\nabla f_k(z)$ and $H_k(z)$ denote the gradient vector and the Hessian matrix of f_k at $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, respectively. We will now fix $t \geq 0$ and define suitable stopping times, so that we can treat the expectation of these two terms.

The stochastic process I_k is a local martingale starting at zero, hence there exists an increasing sequence $(T_{k,l})_{l\in\mathbb{N}}$ of stopping times with $T_{k,l} \to \infty$ as $l \to \infty$ such that, for every $l \in \mathbb{N}$, the process $M_{k,l}(s) := I_k(s \wedge T_{k,l})$ with $s \geq 0$ is a uniformly integrable martingale. By Doob's optional stopping theorem [,],

$$\mathbb{E}[I_k(t \wedge \tau \wedge T_{k,l})] = 0, \quad l \in \mathbb{N}. \tag{5.21}$$

Note that $\nabla f_k(z) = 0$ for $|z| \leq a_k$ and $\nabla f_k(z) = \varphi_k'(|z|)z/|z|$ for $|z| \geq a_k$, hence $|\nabla f_k(z)| \leq 1$ for all $z \in \mathbb{R}^n$ by (5.15). Therefore, by the Cauchy–Schwarz inequality and estimate (5.16),

$$|\nabla f_k(Z_{s\wedge\tau})\, b_{s\wedge\tau}| \leq |\nabla f_k(Z_{s\wedge\tau})|\, |b_{s\wedge\tau}| \leq \kappa(|Z_{s\wedge\tau}|), \quad s \geq 0. \tag{5.22}$$

The diagonal components of the Hessian matrix are given by

$$(H_k(z))_{i,i} = \frac{\partial^2 f_k(z)}{\partial z_i^2} = \phi_k(|z|)\frac{z_i^2}{|z|^2} + \varphi_k'(|z|)\frac{|z|^2 - z_i^2}{|z|^3}, \quad i \in \{1,\dots,n\},$$

(remember that φ_k' and ϕ_k are zero in a neighborhood of the origin). Since φ_k' is uniformly bounded by one, see (5.15), the above equation implies that

$$\operatorname{tr}[H_k(z)] = \phi_k(|z|) + \varphi_k'(|z|)\frac{n-1}{|z|} \le \phi_k(|z|) + \frac{n-1}{|z|}\mathbb{I}_{\{z\neq 0\}}, \quad z \in \mathbb{R}^n. \tag{5.23}$$

Note that φ_k is convex on $[0,\infty)$ by construction, also the Euclidean norm is convex on $\mathbb{R}^n$, hence f_k is convex. Therefore, the Hessian H_k of f_k is positive semi-definite everywhere. Since also $\sigma_{s\wedge\tau}\sigma_{s\wedge\tau}^\top$ is positive semi-definite,

$$0 \le \operatorname{tr}[H_k(Z_{s\wedge\tau})\sigma_{s\wedge\tau}\sigma_{s\wedge\tau}^\top] \le \operatorname{tr}[H_k(Z_{s\wedge\tau})]\operatorname{tr}[\sigma_{s\wedge\tau}\sigma_{s\wedge\tau}^\top], \quad s \ge 0. \tag{5.24}$$

Combining this inequality with (5.17) and (5.23) in the first step and using (5.14) in the second one implies that

$$\begin{aligned} 0 \le \operatorname{tr}[H_k(Z_{s\wedge\tau})\sigma_{s\wedge\tau}\sigma_{s\wedge\tau}^\top] &\le \phi_k(|Z_{s\wedge\tau}|)\varrho^2(|Z_{s\wedge\tau}|) + \frac{n-1}{|Z_{s\wedge\tau}|}\varrho^2(|Z_{s\wedge\tau}|)\mathbb{I}_{\{Z_{s\wedge\tau}\neq 0\}} \\ &\le \frac{2}{k} + \frac{n-1}{|Z_{s\wedge\tau}|}\varrho^2(|Z_{s\wedge\tau}|)\mathbb{I}_{\{Z_{s\wedge\tau}\neq 0\}}, \quad s \ge 0. \end{aligned} \tag{5.25}$$

Inserting the estimates (5.22) and (5.25) into (5.20) and using the upper bound (5.9) given by G, it follows that

$$|J_k(t\wedge\tau)| \le \frac{t}{k} + \int_0^{t\wedge\tau} G(|Z_{s\wedge\tau}|)\,ds, \quad t \ge 0.$$

It follows from (5.18) that, for all $l \in \mathbb{N}$ and $t \ge 0$,

$$\mathbb{E}[f_k(Z_{t\wedge\tau\wedge T_{k,l}})] = \mathbb{E}[I_k(t\wedge\tau\wedge T_{k,l})] + \mathbb{E}[J_k(t\wedge\tau\wedge T_{k,l})].$$

The first expectation on the right-hand side vanishes due to (5.21). Noting that G is non-negative, it follows that

$$\mathbb{E}[f_k(Z_{t\wedge\tau\wedge T_{k,l}})] \le \frac{t}{k} + \int_0^t \mathbb{E}[G(|Z_{s\wedge\tau}|)]\,ds, \quad l \in \mathbb{N}, t \ge 0.$$

Since by assumption G is concave on $[0,\gamma]$, Jensen's inequality implies that

$$\mathbb{E}[G(|Z_{s\wedge\tau}|)] \le G(\mathbb{E}[|Z_{s\wedge\tau}|]), \quad s \ge 0. \tag{5.26}$$

Letting $l \to \infty$, using Fatou's lemma, it follows that

$$\mathbb{E}[f_k(Z_{t\wedge\tau})] \le \frac{t}{k} + \int_0^t G(\mathbb{E}[|Z_{s\wedge\tau}|])\,ds.$$

Letting $k \to \infty$ and using monotone converge theorem, we obtain for the difference process Z the estimate

$$\mathbb{E}[|Z_{t\wedge\tau}|] \le \int_0^t G(\mathbb{E}[|Z_{s\wedge\tau}|])\,ds, \quad t \ge 0. \tag{5.27}$$

The stopping at τ makes sure that $[0,\infty) \ni s \mapsto \mathbb{E}[|Z_{s\wedge\tau}|]$ is $[0,\gamma]$-valued and continuous (apply the dominated convergence theorem). Due to (5.10) and Bihari's inequality (see Theorem 5.28(2) below with $\beta \equiv 1$, $u(s) = \mathbb{E}[|Z_{s\wedge\tau}|]$ and $w(x) = G(x \wedge \gamma)$ for all $x \geq 0$), estimate (5.27) implies that $\mathbb{E}[|Z_{t\wedge\tau}|] = 0$. Since $Z_{t\wedge\tau} = Z_t 1_{\{\tau>t\}} + Z_\tau 1_{\{\tau\leq t\}}$ and $|Z_\tau| = \gamma > 0$ on $\{\tau < \infty\}$, it follows that $\mathbb{P}[\tau \leq t] = 0$, hence $\mathbb{E}[|Z_t|] = 0$, therefore $X_t \overset{\text{a.s.}}{=} Y_t$. Since this holds for all rational $t \geq 0$ and since the processes X and Y have continuous paths, they are indistinguishable. □

Bihari's inequality

Bihari's inequality [21, 138], proved by Hungarian mathematician Imre Bihari (1915–1998), is a nonlinear generalization of the Grönwall–Bellman inequality. It is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equations.

Theorem 5.28. *Let I denote an interval of the real line of the form $[a,\infty)$, $[a,b]$ or $[a,b)$ with $a < b$. Let β, u: $I \to [0,\infty)$ and w: $[0,\infty) \to [0,\infty)$ be three functions, where u and w are continuous on I, β is continuous on the interior I° of I with $\int_a^t \beta(s)\,ds < \infty$ for all $t \in I$, and w is non-decreasing and strictly positive on $(0,\infty)$.*

1. *If, for some $\alpha > 0$, the function u satisfies the inequality*

$$u(t) \leq \alpha + \int_a^t \beta(s)\,w(u(s))\,ds, \qquad t \in I, \tag{5.29}$$

then

$$u(t) \leq F^{-1}\biggl(\int_a^t \beta(s)\,ds\biggr), \qquad t \in [a,T), \tag{5.30}$$

where F^{-1} is the inverse function of

$$F(x) := \int_\alpha^x \frac{dy}{w(y)}, \qquad x > 0,$$

and

$$T := \sup\biggl\{t \in I \biggm| \int_a^t \beta(s)\,ds < \int_\alpha^\infty \frac{dy}{w(y)}\biggr\}.$$

2. *If the function u satisfies the inequality* (5.29) *with $\alpha = 0$ and*

$$\int_0^x \frac{dy}{w(y)} = \infty \qquad \text{for all } x > 0, \tag{5.31}$$

then $u(t) = 0$ for all $t \in I$.

Remark 5.32. If $\int_\alpha^\infty \frac{dy}{w(y)} = \infty$, then (5.30) is valid on $[0,\infty)$. An example of such a function is $w(y) = y$ for $y \in [0,\infty)$.

Remark 5.33. The assumptions on the function β allow for a singularity at the left end point a of the interval I, for example $\beta(s) = (s-a)^{-\gamma}$ for $s > a$ with $\gamma \in (0,1)$. The integrability assumption for β ensures that $T > a$ in (5.30).

Proof of Theorem 5.28. (1) Denoting the right-hand side of (5.29) by

$$v(t) := \alpha + \int_a^t \beta(s)\, w(u(s))\, ds, \qquad t \in I,$$

we have $u \le v$ on I by (5.29), which implies that $w(u(s)) \le w(v(s))$ for all $s \in I$ since w is non-decreasing. Using $\alpha > 0$, the definitions of F and v as well as this inequality, it follows that

$$\frac{dF(v(s))}{ds} = \frac{v'(s)}{w(v(s))} = \frac{\beta(s)\, w(u(s))}{w(v(s))} \le \beta(s), \qquad s \in I^{\circ}.$$

Integrating this between a and t and using $F(v(a)) = F(\alpha) = 0$,

$$F(v(t)) = F(v(t)) - F(v(a)) \le \int_a^t \beta(s)\, ds, \qquad t \in I.$$

Since F is strictly increasing,

$$v(t) \le F^{-1}\Big(\int_a^t \beta(s)\, ds\Big), \qquad t \in [a, T).$$

Since $u(t) \le v(t)$, the inequality (5.30) follows.

(2) Consider any $t \in I$ and $x > 0$. Due to (5.31) there exists $\alpha \in (0, x]$ such that

$$\int_\alpha^x \frac{dy}{w(y)} = \int_a^t \beta(s)\, ds.$$

Since u also satisfies (5.29) with this α, (5.30) implies that

$$u(t) \le F^{-1}\Big(\int_a^t \beta(s)\, ds\Big) = x.$$

Since $x > 0$ was arbitrary, $u(t) = 0$ follows. □

5.1 Trapped Jacobi process

Here, we give an example where we can apply an iterative procedure to obtain the pathwise uniqueness of the trapped Jacobi process.

Definition 5.34. The trapped Jacobi process $X = (X_t^1, \ldots, X_t^d)_{t \ge 0}$ with the state space Δ_d^+ satisfies the stochastic differential equations (SDE) with $\sigma > 0$ for $i = \{1, \ldots, d\}$

$$dX_t^i = m_i(X_t)\, dt + \sigma \sum_{j=1}^d \Big\{ (\delta_{ij} - X_t^i) \sqrt{X_t^j} \Big\}\, dW_t^j, \quad t > 0, \tag{5.35}$$

$$X_0^i = x_0^i, \tag{5.36}$$

where $W = (W_t^1, \ldots, W_t^d)_{t \ge 0}$ is a d-dimensional standard Brownian motion, $x = (x_0^1, \ldots, x_0^d) \in \Delta_d^+$ and drift functions m_i's are Lipschitz continuous on Δ_d^+ satisfying:

1. $\sum_{i=1}^d m_i(x) = 0, \quad x \in \Delta_d^+$,
2. $m_i(x) \le K x^i$ for $i = \{1, \ldots, d\}$, and $x \in \Delta_d^+$.

Theorem 5.37. *The pathwise uniqueness of the solution of the d-dimensional stochastic differential equation (SDE)* (5.35) *holds. Hence there is strong existence and uniqueness for* (5.35).

Before giving the detailed proof of the Theorem (5.37), the crucial steps that lead to the argument are worth mentioning. First of all, since the (local) Lipschitz continuity holds for the diffusion part of the equation (5.35) in the interior of the Δ_d^+, it is fairly standard to show that the pathwise uniqueness holds until the process hits the boundary of Δ_d^+, which we denote as $\partial\Delta_d^+$. On the other hand, conditions given for the drift part allows us to keep the process in the Δ_d^+ (condition i.) or stick to the $\partial\Delta_d^+$ (condition ii.) once one of the components of the process hits to it. Although the diffusion part is not a symmetric or a triangular matrix, its form allows us to use induction as the main tool in the proof. That is, once one of the components hits to the boundary, the remaining components move on a simplex with dimension reduced by one until the process end up at the unit vectors.

Proof. The first step is to show that the pathwise uniqueness holds up to the first hitting time of $\partial\Delta_d^+$ of two solutions, X and Y, of (5.35).

Let us define $\tau_X := \inf\{t \geq 0,\ X_t \in \partial\Delta_d^+\}$ and $\tau_Y := \inf\{t \geq 0,\ Y_t \in \partial\Delta_d^+\}$ as $(\mathcal{F}_t)_{t\geq 0}$ stopping times.

Also define $\tau_X^\epsilon := \inf\{t \geq 0,\ X_t \in \mathcal{N}(\partial\Delta_d^+, \epsilon)\}$ and $\tau_Y^\epsilon := \inf\{t \geq 0,\ Y_t \in \mathcal{N}(\partial\Delta_d^+, \epsilon)\}$, where $\mathcal{N}(\partial\Delta_d^+, \epsilon)\}$ is the $\epsilon > 0$ open neighborhood of $\partial\Delta_d^+$. Moreover, $\tau^\epsilon := \tau_X^\epsilon \wedge \tau_Y^\epsilon$, is again a stopping time and hence we can write for any $i \in \{1, \ldots, d\}$,

$$\begin{aligned} X^i_{t\wedge\tau^\epsilon} - Y^i_{t\wedge\tau^\epsilon} = &\int_0^{t\wedge\tau^\epsilon} (m_i(X_s) - m_i(Y_s))\, ds \\ &+ \sigma \sum_{j=1}^d \int_0^{t\wedge\tau^\epsilon} \left\{ (\delta_{ij} - X_s^i)\sqrt{X_s^j} - (\delta_{ij} - Y_s^j)\sqrt{Y_s^i} \right\} dW_s^j \end{aligned} \tag{5.38}$$

Since the drift and diffusion functions are Lipschitz on $\Delta_d^+ \backslash \mathcal{N}(\partial\Delta_d^+, \epsilon)$, we have the estimate

$$\mathbb{E}\left[|X_{t\wedge\tau^\epsilon} - Y_{t\wedge\tau^\epsilon}|^2\right] \leq A^{T,\epsilon} \int_0^t \mathbb{E}\left[|X_{s\wedge\tau^\epsilon} - Y_{s\wedge\tau^\epsilon}|\right] ds \tag{5.39}$$

for $t \leq T < \infty$ where $A^{T,\epsilon}$ is a constant depending on T and ϵ. From this we can conclude that $X_{t\wedge\tau^\epsilon} = Y_{t\wedge\tau^\epsilon}$ almost surely for all $t \leq T$ and $\tau_X^\epsilon = \tau_Y^\epsilon$ almost surely. By letting $\epsilon \searrow 0$, we can conclude that $\tau_X \overset{\text{a.s.}}{=} \tau_Y =: \tau$ and $X_t \overset{\text{a.s.}}{=} Y_t$ for $t < \tau$. Also, $X_\tau = Y_\tau =: Z = (Z^1, \ldots, Z^d)$ on the event $\{\tau < \infty\}$.

Now, the next step is to show that once one component of the process hits the associate boundary, remaining components are still on the simplex yet the component that hits to boundary sticks to it. Mathematically, it is sufficient to show that

$$X_t^i = Y_t^i \overset{\text{a.s.}}{=} 0,\ t \geq \tau \tag{5.40}$$

on the event $\{\tau < \infty,\ Z^i = 0\} := \mathcal{S}_i$ for $i \in \{1, \ldots, d\}$. It can be seen that if (5.40) holds, then $X_t^i = Y_t^i \in \Delta_d^{+,i}$ almost surely for $t \geq \tau$ on the event $\{\tau < \infty, Z \in \Delta_d^{+,i}\}$ where $\Delta_d^{i,+}$ is the $(d-2)$-dimensional simplex generated by setting $x^i = 0$ on Δ_d^+. In order to show (5.40), write the dynamics of X as if it starts at Z at τ, i.e,

$$X^i_{\tau+t} = Z^i + \int_\tau^{\tau+t} m_i(X_s)\, ds + \sigma \sum_{j=1}^d \int_\tau^{\tau+t} \left\{ (\delta_{ij} - X_s^i)\sqrt{X_s^j} \right\} dW_s^j. \tag{5.41}$$

for all $i \in \{1, \ldots, d\}$ and $t \geq 0$. Now let's define a d-dimensional standard Brownian motion by $(\tilde{W}_t)_{t\geq 0}) = (W_{\tau+t})_{t\geq 0}$ and rewrite (5.41) as

$$X^i_{\tau+t} = Z^i + \int_0^t m_i(X_{\tau+s})\, ds + \sigma \sum_{j=1}^{d} \int_0^t \left\{ (\delta_{ij} - X^i_{\tau+s}) \sqrt{X^j_{\tau+s}} \right\} d\tilde{W}^j_s. \tag{5.42}$$

for all $i \in \{1, \ldots, d\}$ and $t \geq 0$. Taking the expectation on the event $\mathcal{S}_i$, by the second property on the drift function, that is, $m_i(x) \leq Kx^i$, we have,

$$0 \leq \mathbb{E}\left[X^i_{\tau+t}; \mathcal{S}_i\right] \leq K \int_0^t \mathbb{E}\left[X^i_{\tau+s}; \mathcal{S}_i\right] ds \tag{5.43}$$

implying that $X^i_t \overset{\text{a.s.}}{=} 0$ on $\mathcal{S}_i$ for $t \geq \tau$. Since above computations are same for Y, we have $X^i_t = Y^i_t \in \Delta^{+,i}_d$ almost surely for $t \geq \tau$ on the event $\{\tau < \infty, Z \in \Delta^{+,i}_d\}$. Now we show by induction that

Lemma 5.44. $X_t \overset{a.s.}{=} Y_t$ *for* $t \geq \tau$ *on* $\{\tau < \infty\}$.

Proof of the Lemma 5.44. First of all assume that the (5.35) holds for $(d-1)$-dimensional system. If we can show that $X_t \overset{\text{a.s.}}{=} Y_t$ for $t \geq \tau$ on every $\mathcal{S}_i$ for $i \in \{1, \ldots, d\}$, it is immediate that the claim holds. Now take $i = d$, since $x \in \Delta^{+,d}_d$ the diffusion part of (5.35) has the same form but the last row and the column vanishes, hence (5.42) can be written with one-dimension less (pay attention to the summation)

$$X^i_{\tau+t} = Z^i + \int_0^t m_i(X_{\tau+s})\, ds + \sigma \sum_{j=1}^{d-1} \int_0^t \left\{ (\delta_{ij} - X^i_{\tau+s}) \sqrt{X^i_{\tau+s}} \right\} d\tilde{W}^j_s. \tag{5.45}$$

However, by the induction hypothesis (5.45) holds for $i \in \{1, \ldots, (d-1)\}$, hence the argument holds for $\mathcal{S}_d$. For the other $\mathcal{S}_i$'s, the argument also holds for similarly, the only difference being the dimension of the simplex reduced more than one. Hence $X_t \overset{\text{a.s.}}{=} Y_t$ for $t \geq \tau$ on every $\mathcal{S}_i$ for $i = 1, \ldots, d$ and the claim holds. □

From Lemma 5.45 holds and the arguments given above are valid for $d = 2$ (for the initialization of the induction hypothesis), we have the pathwise uniqueness and by the Yamada-Watanabe theorem [160, Prop. 1], strong existence and the uniqueness of the solution of (5.35) holds. □

Part II

Pairs Trading under Drift Uncertainty and Risk Penalization

Chapter 1

Introduction and the Underlying Framework

Pairs trading is an investment strategy that attempts to capitalize on market inefficiencies arising from imbalances between two or more stocks. This kind of strategy involves a long position and a short position in a pair of similar stocks that have moved together historically. Examples of such pairs can be given: ExxonMobil and Royal Dutch and Shell for the oil industry, or Pfizer and GlaxoSmithKline for the pharmaceutical industry. The underlying rationale of pairs trading is to buy the underperformer, and sell the overperformer, in anticipation that the security that has performed badly will make up for loss in the coming periods, perhaps even overperform the other, and vice-versa. For this reason, it is also classified as a convergence or mean-reversion strategy. The pair of stocks is selected in a way that it forms a mean-reverting portfolio referred to as the *spread*. By forming an appropriate spread, pairs traders try to limit the directional risk that arises from the market's up or down movements by simultaneously going long on one stock and short in another. Since market risk is mitigated, profits depend only on the price changes between the two stocks and they can be realized through a net gain on the spread. Therefore,one can also see pairs trading in the class of *market-neutral* trading strategies. To achieve market neutrality, traders can choose corresponding strategies so that the resulting portfolio has zero (CAPM) beta, hence it is beta-neutral. Alternatively, one can use a *dollar-neutral* strategy, which is investing an equal dollar amount in each stock. However, we should remark that market neutrality does not imply either risk-free return or arbitrage in the classical sense. The risk inherited in pair strategies is different from the risk in investment strategies involving only a long or short position in a specific stock or market. Indeed, pairs trading is a form of statistical arbitrage, which can be defined broadly as a long-horizon trading strategy that generates riskless profits asymptotically (see [99] for the definition of the statistical arbitrage and [90] for the existence of statistical arbitrage for pairs trading strategies). As it is empirically documented by [87], coupled with a simple pairs selection algorithm, such statistical arbitrage strategies may yield average annualized excess returns of up to 11 percent, which still remains profitable after compensated by the most conservative transaction costs.

In this work, we consider the portfolio optimization problem of a trader with a logarithmic utility from risk penalized terminal wealth investing in a pair of assets whose dynamics have a certain dependence structure in a Markov regime-switching model. More precisely, we model the spread process (log-price differential) as an Ornstein–Uhlenbeck process with a partially observable Markov modulated drift. Our motivation for modeling the drift of the spread and drifts of both assets as a function of an unobservable finite-state Markov chain has certain advantages. Firstly, drifts of financial assets are hardly constant and observable, especially if we think of the convergence-type investment strategies that are usually valid for longer periods. Secondly, although pairs are selected in such a way that they have similar characteristics, the dynamics of the spread between them might be prone to different regimes. For example, if one leg of the pair is selected to be listed in an index such as the S&P 500 while the other is not, this might increase the demand for the one that is listed. Hence, that would eventually increase the level of the spread, at least until the one listed in the index is deleted from the index or the other leg of the pair is also added. Moreover, in reality, it is difficult to observe or characterize both microstructure (market-based) or macrostructure (economy-wide) state variables changing with respect to different regimes. That would necessitate using a partial information framework to model such state processes.

Numerous studies analyze portfolio selection problems in a full or partial information and/or Markov regime-switching framework, see, for example, [162], [14], and [151] for the full information case with Markov regime switching or [15], [84], and [22] for the partial information case. However, to the best of our knowledge, identification of optimal pairs trading strategies in a Markov-modulated setting under partial information is new.

Our proposed model is an extended version of the model given by [135], who found the optimal pairs trading strategies in a dollar-neutral setting for an investor with power utility. Although investing equal dollar amount (as a proportion of wealth) in pairs seems to be restrictive, it is meaningful when CAPM betas of the selected stocks are very close to each other. Our model extends the work of [135] by allowing partially observed Markov-modulated drifts both for the price processes and the spread, hence enabling them to change with respect to different conditions. As the second extension, to find the optimal trading strategies, we use a risk penalized terminal wealth as it is suggested in Section 2.22 of [146]. By penalizing the terminal wealth according to the realized volatility of the wealth process, the investor hopes to prevent the pairs trader pursuing risky strategies. Using risk penalization seems to be appropriate in pairs trading as most such strategies are executed by hedge funds and proprietary trading houses, which engage in high-risk transactions on behalf of investors. Risk penalization effectively increases the risk aversion of the trader and makes her take a less risky position. Apart from certain mathematical convenience, our choice of logarithmic utility function can be justified on several financial grounds. Firstly, although an investor can choose any utility function, representing her risk tolerance, a repetitive situation such as the one reflected in mean-reversion type trading strategies tends to force the utility function into the one that is close to a logarithmic one. For instance, in the power utility case it can be shown in a very simple example that too aggressive or too conservative choices for the risk-aversion parameter imply unrealistic preferences such as betting on strategies that have large losses with high probability and hence not suitable if the investor is focused in a long sequence of repeated trials, see e.g., Chapter 15 of [129]). This can only be alleviated when the risk-aversion parameter γ, in power utility [1], is close to zero, behaving more like the logarithmic utility. Therefore, we can argue that utility functions that are close to the logarithmic ones are appropriate for our setting. Secondly, by penalizing the terminal wealth with the realized volatility of the portfolio and using logarithmic utility, we can capture the intertemporal risk factor in our model more easily with just one parameter.

Although both the empirical and theoretical literature on pairs trading has been growing,

[1] $U(x) = \frac{x^\gamma}{\gamma}$ for $\gamma \leq 1$.

published research on optimal portfolio problem is rather limited. [135] solve the stochastic control problem for pairs trading with power utility for terminal wealth. [156] develop an optimal portfolio strategy to invest in two risky assets and the money market account, assuming that log-prices are co-integrated, as in the option pricing model of [52]. [29] extend [156] to allow the investor to trade in multiple co-integrated assets and provide an explicit closed-form solution of the dynamic trading strategy while assuming that the drift of asset returns consists of an idiosyncratic and common drift component. [123] solve the optimal pairs trading problem within a power utility setting, where the drift uncertainty is modeled by a continuous mean-reverting process. Here we should also remark that the work of [7] that characterizes the optimal delta-neutral and beta-neutral strategies in a converge trading model with regime-switching under full and partial information. It is also worth mentioning here the work of [64], which proposes a pairs trading strategy based on stochastic filtering of a mean-reverting Gaussian Markov chain for the spread, which is observed in Gaussian noise.

Apart from identification of optimal trading strategies through utility maximization from terminal wealth, there is also recent literature on optimal liquidation and optimal (entry-exit) timing strategies related to pairs trading. For example, studies by [60], [121], and [161] focus on how to liquidate optimally a pairs trade by incorporating stop-loss thresholds. Moreover, [127] study an optimal double-stopping problem to analyze the timing for starting and subsequently liquidating the position, subject to transaction costs, and [124] analyze a multiple entry-exit problem of a pair of co-integrated assets. An extensive list of references and a literature review on pairs trading and statistical arbitrage can be found in the recent survey paper by [113].

To sum up, our contributions in this part can be stated as follows. First, we characterize the optimal dollar-neutral strategies both in full and partial information settings with risk-penalized terminal wealth for a log-utility trader and show that optimal strategies are dependent on both the correlation between two assets and the mean-reverting spread. The effect of risk-penalization on optimal strategies is an increase in risk-aversion uniformly in a constant proportion that is not dependent on time. Second, we characterize the optimal value function via Feynman–Kac formula. Third, using the innovations approach, we provide filtering equations that are necessary to reduce the problem with partial information to the one with full information. A nice feature of the solution in the partial information setting is that the optimal strategy is a linear function of the filtered state and hence it can be considered as a projection of the full information one on the investor's information filtration.

We also present numerical results for a toy example with a two-state Markov chain in both full and partial information settings. Our analysis shows that average data does not contain sufficient information to obtain the optimal value for the pairs trading problem for logarithmic utility preferences. This result is in contrast with the one for the classical portfolio optimization problem with Markov modulation, see Section B in [14]). Furthermore, our toy example suggests that there is always a gain from filtering due to the convexity arising from using filtered probabilities instead of constant ones.

1.1 The pairs trading model

We consider a finite time interval $[0, T]$ and a continuous-time finite-state Markov chain Y defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbf{P})$, where $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the global filtration that satisfies the usual conditions; all processes we consider here are assumed to be $\mathbb{G}$-adapted. Suppose Y has the state space $\mathcal{E} = \{e_1, e_2, ..., e_K\}$ where, without loss of generality, we assume that e_k is the basis column vector of $\mathbb{R}^K$. Y has the intensity matrix $Q = (q^{ij})_{i,j\in\{1,\dots,K\}}$ and its initial distribution is denoted by $\Pi = (\Pi^1, \cdots, \Pi^K)$. The semimartingale decomposition of Y is given by

$$Y_t = Y_0 + \int_0^t Q^\top Y_s ds + M_t, \tag{1.1}$$

for every $t \in [0, T]$, where M is a $(\mathbb{G}, \mathbf{P})$-martingale.

We consider a market with a risk-free asset and two stocks. We assume that the dynamics of the risk-free asset is given by

$$dS_t^{(0)} = rS_t^{(0)}\, dt, \quad S_0^{(0)} > 0, \tag{1.2}$$

where $r \in \mathbb{R}$ is the risk-free interest rate. The stocks have prices $S^{(1)}$ and $S^{(2)}$, and the price process of the first stock is assumed to follow a Markov-modulated diffusion given by

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu(Y_t)\, dt + \sigma\, dW_t^{(1)}, \quad S_0^{(1)} > 0, \tag{1.3}$$

with $\sigma > 0$ and where $W^{(1)}$ is a $\mathbb{G}$-Brownian motion independent of Y. Since the Markov chain takes values in a finite state space we have that for every $t \in [0, T]$, $\mu(Y_t) = \boldsymbol{\mu} Y_t$ with $\boldsymbol{\mu} = (\mu_1, \dots, \mu_K)^\top$ and $\mu_i = \mu(e_i) \in \mathbb{R}$ for every $i \in \{1, \dots, K\}$.

It is assumed that the spread $S_t = \log S_t^{(1)} - \log S_t^{(2)}$, $t \in [0, T]$, follows a Markov-modulated Ornstein–Uhlenbeck process:

$$dS_t = \kappa(\theta(Y_t) - S_t)\, dt + \eta\, dW_t, \quad S_0 \in \mathbb{R}, \tag{1.4}$$

where $\kappa > 0$ and $\eta > 0$, W is a $\mathbb{G}$-Brownian motion with $\langle W^{(1)}, W\rangle_t = \rho t$, $\rho \in (-1, 1)$, and $\theta(Y_t) = \boldsymbol{\theta} Y_t$, $t \in [0, T]$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_K)^\top$ and $\theta_i = \theta(e_i) \in \mathbb{R}$ for every $i \in \{1, \dots, K\}$. It follows from (1.3) and (1.4) that

$$\frac{dS_t^{(2)}}{S_t^{(2)}} = \left(\mu(Y_t) - \kappa(\theta(Y_t) - S_t) + \frac{1}{2}\eta^2 - \rho\sigma\eta\right) dt + \sigma\, dW_t^{(1)} - \eta\, dW_t, \quad S_0^{(2)} > 0. \tag{1.5}$$

Let X be the value of a self-financing portfolio and let $h^{(1)}$ and $h^{(2)}$ denote fractions of the wealth invested in $S^{(1)}$ and $S^{(2)}$, respectively.

Admissible Investment Strategies. We consider *dollar-neutral* pairs trading strategies. This corresponds to take $h^{(1)}$ and $h^{(2)}$ such that

$$h_t^{(1)} = -h_t^{(2)}, \quad t \in [0, T]. \tag{1.6}$$

In the sequel we are going to use the notation $h = h^{(1)}$. Note that $h_t \in \mathbb{R}$ for every $t \in [0, T]$ and the portfolio weight on the risk-free asset is always 1. In order to ensure that the wealth process is well defined, we consider investment strategies that satisfy

$$\mathbb{E}\left[\int_0^T h_u^2\, du\right] < \infty. \tag{1.7}$$

Definition 1.8. A $\mathbb{G}$-progressive self-financing investment strategy which satisfy (1.6) and (1.7) is called an *admissible investment strategy.* We denote the set of admissible strategies by $\mathcal{A}$.

For every $h \in \mathcal{A}$, the dynamics of the pairs-trading portfolio is given by

$$\frac{dX_t^h}{X_t^h} = \left(h_t \left(\kappa(\theta(Y_t) - S_t) - \frac{\eta^2}{2} + \rho\sigma\eta\right) + r\right) dt + h_t\eta \, dW_t, \quad X_0^h > 0. \tag{1.9}$$

Notice that for a given $h \in \mathcal{A}$, X^h is a controlled process. In what follows, for the sake of notational simplicity we suppress h dependency and write X instead of X^h. The objective of the trader is to maximize expected utility from terminal wealth. However, in the risk-penalized setting, the goal is to prevent the trader from pursuing risky strategies at the expense of the investor; see Section 2.22 of [146]. The investor agrees to pay the trader at time T the risk-penalized amount

$$Z_T = X_T \exp\left(-\frac{1}{2}\varepsilon \int_0^T \eta^2 h_s^2 \, ds\right), \quad \varepsilon \geq 0. \tag{1.10}$$

Hence the terminal value of the wealth process is 'discounted' by its realized volatility. It follows from Itô's formula that the dynamics of Z is given by:

$$\frac{dZ_t}{Z_t} = \left(h_t \left(\kappa(\theta(Y_t) - S_t) - \frac{\eta^2}{2} + \rho\sigma\eta\right) + r - \frac{\varepsilon\eta^2 h_t^2}{2}\right) dt + h_t\eta \, dW_t, \quad Z_0 > 0. \tag{1.11}$$

In what follows, we study the optimization problem for a trader who is endowed with a logarithmic utility in case of regime switching and risk penalization. First, we consider the situation where the trader may observe the Markov chain Y that influences the dynamics of price processes and the spread. Subsequently, we assume that the Markov chain is not observable and solve the optimization problem under partial information.

Chapter 2

Optimization Problem under Full Information

In this section, we suppose that the trader can observe all sources of randomness in the market. Her penalized wealth at time T is given by

$$Z_T = z \exp\left\{ \int_t^T \left(h_u \left(\kappa(\theta(Y_u) - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta \right) - \frac{h_u^2 \eta^2 (1+\varepsilon)}{2} + r \right) du \right. \\ \left. + \int_t^T h_u \eta \, dW_u \right\}, \tag{2.1}$$

for every $h \in \mathcal{A}$. Note that, condition (1.7) guarantees that the stochastic integral in the above expression is a true martingale and hence has a zero expected value.

Formally the trader faces the following optimization problem

$$\max \; \mathbb{E}^{t,z,s,i}[\log Z_T], \tag{2.2}$$

where $\mathbb{E}^{t,z,s,i}$ denotes the conditional expectation given $Z_t = z$, $S_t = s$ and $Y_t = e_i$. We define the value function of the trader by

$$V(t, z, s, i) := \sup_{h \in \mathcal{A}} \mathbb{E}^{t,z,s,i} \left[\log Z_T \right]. \tag{2.3}$$

From now on, we use the following notation for the partial derivatives: for every function $g : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, we write, for instance, g_t for the partial derivative with respect to time.

In the following theorem we characterize the optimal strategy and the corresponding value function.

Theorem 2.4. *Consider a trader with a logarithmic utility function with risk penalization parameter $\varepsilon \geq 0$. Then the optimal portfolio strategy $h^* \in \mathcal{A}$ is*

$$h^*(t, s, i) = \frac{1}{1+\varepsilon} \left(\frac{\kappa\,(\theta_i - s)}{\eta^2} + \frac{\rho\sigma}{\eta} - \frac{1}{2} \right). \tag{2.5}$$

The value function is of the form

$$V(t, z, s, i) = \log(z) + r(T - t) + d(t)s^2 + c(t, i)s + f(t, i), \tag{2.6}$$

where the function $d(t)$ is given by

$$d(t) = \frac{\kappa}{4\eta^2(1+\varepsilon)}\left(1 - e^{-2\kappa(T-t)}\right), \tag{2.7}$$

and the functions $c(t,i)$ and $f(t,i)$ for $i \in \{1,\dots,K\}$ solve the following system of ordinary differential equations

$$c_t(t,i) - \kappa c(t,i) + 2\kappa\theta_i d(t) - \frac{\kappa^2\theta_i - \kappa\frac{\eta^2}{2} + \kappa\rho\sigma\eta}{\eta^2(1+\varepsilon)} + \sum_{j=1}^{K} c(t,j)q^{ij} = 0, \tag{2.8}$$

$$f_t(t,i) + d(t)\eta^2 + \kappa\theta_i c(t,i) + \frac{\left(\kappa\theta_i - \frac{1}{2}\eta^2 + \rho\sigma\eta\right)^2}{2\eta^2(1+\varepsilon)} + \sum_{j=1}^{K} f(t,j)q^{ij} = 0 \tag{2.9}$$

with terminal conditions $c(T,i) = 0$ and $f(T,i) = 0$ for all $i \in \{1,\dots,K\}$.

Proof. We first apply pointwise optimization to obtain the optimal portfolio strategy. By computing the expectation in (2.2), we get

$$\begin{aligned}\mathbb{E}^{t,z,s,i}[\log Z_T] = &\log(z) + r(T-t) - \mathbb{E}^{t,s,i}\left[\int_t^T \frac{h_u^2\eta^2(1+\varepsilon)}{2}\,du\right] \\ &+ \mathbb{E}^{t,s,i}\left[\int_t^T h_u\left(\kappa(\theta(Y_u) - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta\right)du\right],\end{aligned} \tag{2.10}$$

where, according to the previous notation, $\mathbb{E}^{t,s,i}$ denotes the conditional expectation given $S_t = s$ and $Y_t = e_i$. The first order condition given by

$$-h_t^*\eta^2(1+\varepsilon) + \kappa(\theta(Y_t) - S_t) - \frac{\eta^2}{2} + \rho\sigma\eta = 0, \tag{2.11}$$

provides the following candidate for the optimal strategy

$$h^*(t,s,i) = \frac{1}{1+\varepsilon}\left(\frac{\kappa(\theta_i - s)}{\eta^2} + \frac{\rho\sigma}{\eta} - \frac{1}{2}\right). \tag{2.12}$$

The second order condition, $-\eta^2(1+\varepsilon) < 0$, ensures that h^* is the well defined maximizer and hence the optimal portfolio strategy. By inserting the optimal strategy into (2.10), we get a stochastic representation for the optimal value, that is,

$$\log(z) + r(T-t) + \mathbb{E}^{t,s,i}\left[\int_t^T \frac{(\kappa(\theta(Y_u) - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)}\,du\right]. \tag{2.13}$$

Next, we characterize the value function by means of Feynman-Kac formula for Markov-modulated diffusion processes; see [13] and [66]. To this, for every $i \in \{1,\dots,K\}$ we define functions $u(\cdot,\cdot,i) : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ by

$$u(t,s,i) = \mathbb{E}^{t,s,i}\left[\int_t^T \frac{(\kappa(\theta(Y_u) - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)}\,du\right]. \tag{2.14}$$

Then for every $i \in \{1, \dots, K\}$, functions $u(\cdot, \cdot, i)$, satisfy

$$u_t(t,s,i) + \kappa(\theta_i - s)u_s(t,s,i) + \frac{\eta^2}{2}u_{ss}(t,s,i) + \sum_{j=1}^{K} u(t,s,j)q^{ij} + \frac{(\kappa(\theta_i - s) + \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)} = 0, \tag{2.15}$$

with the terminal condition $u(T, s, i) = 0$. Suppose that function $u(t, s, i)$ is of the form $u(t, s, i) = d(t)s^2 + c(t, i)s + f(t, i)$. By using this ansatz, we get the following equation

$$0 = c_t(t,i)s + d_t(t)s^2 + f_t(t,i) + \eta^2 d(t) + \frac{(\kappa(\theta_i - s) - \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)} + \kappa(\theta_i - s)(c(t,i) + 2d(t)s) + \sum_{j=1}^{K}(c(t,j)s + f(t,j))q^{ij}. \tag{2.16}$$

Collecting together the terms with s^2, s and the remaining ones we get that the function $d(t)$ solves

$$d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0, \quad d(T) = 0, \tag{2.17}$$

and for every $i \in \{1, \dots, K\}$, $c(t, i)$ and $f(t, i)$ solve the system of ODEs in (2.8) and (2.9), respectively; see, e.g., Theorem 3.9 in [155]. □

Remark 2.18.

i) Note that the optimal value is always positive provided that $z > 1$, and the expectation in (2.13) can also be evaluated by computing the first and second moments of the Markov-modulated Ornstein–Uhlenbeck process. This can be achieved, for example as given in [100], by solving a non-homogeneous linear system of differential equations.

ii) In the current setting the market is in general incomplete implying that, for instance, we can not rely on the martingale approach; see, for example, [22].

The optimal portfolio strategy h^* has three components. The component related to dollar-neutrality is given by $1/2(1 + \varepsilon)$. This is intuitively clear considering "non-pairs" in the sense that there is no correlation ($\rho = 0$) and no-cointegration ($\kappa = 0$). The other two components are arising from the dependence structure between two stocks. Namely, the first component $\kappa\,(\theta_i - s)\,/(1 + \varepsilon)\eta^2$ is related to the co-integration between two stocks, whereas the second component $\rho\sigma/\eta(1 + \varepsilon)$ is related to the correlation structure. To wit, suppose now that the current spread is equal to the long-term mean of the current regime, that is $(\theta_i - s) = 0$ or $\kappa = 0$, then the optimal strategy for a given $\varepsilon > 0$ is determined by only the correlation ρ between first stock and spread scaled by the ratio of volatilities of both. One can interpret this case as the dollar-neutral investment strategy in assets with correlated returns. On the other hand, if ρ is zero, the optimal strategy is determined only by the spread dynamics.

Remark 2.19. Suppose that, instead of a *dollar-neutral* strategy, the trader wants to use a *beta-neutral* strategy, that is a strategy of the form $\beta_1 h^{(1)} + \beta_2 h^{(2)} = 0$, where β_1 and β_2

denote CAPM betas of $S^{(1)}$ and $S^{(2)}$, respectively. Then the optimal strategy is given by

$$h^*(t,s,i) = \frac{1}{1+\varepsilon}\left(\frac{\mu_i\beta_2(\beta_2-\beta_1)+\beta_1\beta_2\kappa\left(\theta_i-s\right)-\beta_1\beta_2\frac{\eta^2}{2}+\beta_1\beta_2\rho\sigma\eta}{(\sigma(\beta_2-\beta_1)-\beta_1\eta)^2}\right), \tag{2.20}$$

and the value function has the similar structure as in the dollar-neutral case given above.

Chapter 3

Optimization Problem under Partial Information

We assume now that the state process Y is not directly observable by the trader. Instead, she observes the price processes $S^{(1)}$ and $S^{(2)}$ and she knows the model parameters. Hence, information available to the trader is carried by the natural filtration of $S^{(1)}$ and $S^{(2)}$. This is equivalent to the set of information carried by $S^{(1)}$ and the spread S, that is,

$$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0},\ \mathcal{F}_t = \sigma\{S_u, S_u^{(1)}, 0 \leq u \leq t\},\ \mathcal{F}_t \subset \mathcal{G}_t. \tag{3.1}$$

In the sequel we assume that filtration $\mathbb{F}$ satisfies the usual hypotheses.

Admissible Investment Strategies. Decisions of the trader should depend only on the information available to her at time t. That is, we consider self-financing investment strategies such that h is $\mathbb{F}$-progressive. Then we have the following definition of admissible strategies under partial information.

Definition 3.2. An $\mathbb{F}$-progressive self-financing investment strategy h that satisfies (1.6) and (1.7) is an $\mathbb{F}$-*admissible investment strategy.* We denote the set of $\mathbb{F}$-admissible strategies by $\mathcal{A}^{\mathbb{F}}$.

The partially informed trader aims to maximize the expected utility $\mathbb{E}[\log Z_T]$, over the class $\mathcal{A}^{\mathbb{F}}$. In this case, we naturally end up with an optimal control problem under partial information. In the next part, to solve such a problem we will derive an equivalent control problem under full information via the so-called reduction approach; see, e.g., [80]. This requires the derivation of the filtering equation for the unobservable state variable. After reduction, the corresponding control problem can be interpreted as one with smooth transitions governed by the dynamics of filtered probabilities. We discuss this aspect in Chapter 4 for the case of a two-state Markov chain.

3.1 The filtering equation

In this section we address the problem of characterizing the conditional distribution of the unobservable Markov chain Y, given the observation. In our setting, the observations process is given by the pair

$$(dR_t, dS_t)^\top = A(t, Y_t, S_t)dt + \Sigma dB_t, \tag{3.3}$$

where process R is the log-return of $S^{(1)}$, i.e., $dR_t = dS_t^{(1)}/S_t^{(1)}$ with $R_0 = 0$, $B = (W^{(1)}, W^{(2)})^\top$ is a 2-dimensional $\mathbb{G}$-Brownian motion independent of Y and

$$A(t, Y_t, S_t) = \begin{pmatrix} \mu(Y_t) \\ \kappa(\theta(Y_t) - S_t) \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ \rho\eta & \sqrt{1-\rho^2}\eta \end{pmatrix}, \quad t \in [0, T]. \tag{3.4}$$

Note that processes R and $S^{(1)}$ generate the same information.

For any function f, we denote by $\widehat{f(Y)}$ the optional projection with respect to filtration $\mathbb{F}$, that is $\widehat{f(Y_t)} = \mathbb{E}\left[f(Y_t)|\mathcal{F}_t\right]$, a.s., for every $t \in [0, T]$. Process $\widehat{f(Y)}$, for every function f, provides the filter. By the finite state property of the Markov chain we get that

$$\widehat{f(Y_t)} = \sum_{j=1}^{K} f(e_j) p_t^j, \quad t \in [0, T], \tag{3.5}$$

where $p_t^j = \mathbf{P}(Y_t = e_j|\mathcal{F}_t)$, $t \in [0, T]$. Then, in order to characterize the conditional distribution of Y, it is sufficient to derive the dynamics of the processes p^j, $j \in \{1, \dots, K\}$. To this, we will use the so-called *innovations approach*. This method is based on finding a suitable $\mathbb{F}$-progressive process that drives the dynamics of the filter; see, e.g., [158] and [61] for more details. We define the 2-dimensional process $I = (I^{(1)}, I^{(2)})^\top$ by

$$I_t = B_t + \int_0^t \Sigma^{-1}(A(u, Y_u, S_t) - \widehat{A(u, Y_u, S_t)})\, du, \quad t \in [0, T]. \tag{3.6}$$

Explicitly we have

$$I_t^{(1)} = W_t^{(1)} + \int_0^t \frac{\mu(Y_u) - \widehat{\mu(Y_u)}}{\sigma}\, du, \tag{3.7}$$

$$I_t^{(2)} = W_t^{(2)} + \int_0^t \frac{\sigma\kappa(\theta(Y_u) - \widehat{\theta(Y_t)}) - \rho\eta(\mu(Y_u) - \widehat{\mu(Y_u)})}{\sigma\eta\sqrt{1-\rho^2}}\, du, \tag{3.8}$$

for every $t \in [0, T]$.

Remark 3.9. The process I is called innovation process and it is well known that I is an $(\mathbb{F}, \mathbf{P})$-Brownian motion; see Proposition 2.30 in [12].

Note that, since the signal Y and the Brownian motion B driving the observation process are assumed to be independent, the filtration $\mathbb{F}$ coincides with the natural filtration of the innovation process; see Theorem 1 in [3]. Then, by Theorem III.4.34-(a) in [103] every $(\mathbf{P}, \mathbb{F})$-local martingale M admits the following representation:

$$M_t = M_0 + \int_0^t H_u\, dI_u, \quad t \in [0, T], \tag{3.10}$$

for some $\mathbb{F}$-predictable 2-dimensional process H such that

$$\int_0^T \|H_u\|^2\, du < \infty \quad \mathbf{P} - \text{a.s.} \tag{3.11}$$

We recall the notation $\boldsymbol{\mu} = (\mu_1, \dots, \mu_K)^\top$, where $\mu_i = \mu(e_i) \in \mathbb{R}$, and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_K)^\top$, where $\theta_i = \theta(e_i) \in \mathbb{R}$. Also introduce $\mathbf{f} = (f_1, \dots, f_K)^\top$, where $f_i = f(e_i) \in \mathbb{R}$. The next theorem provides the filter dynamics.

Theorem 3.12. *For every $i \in \{1, \ldots, K\}$, the filter process p^i satisfies*

$$p_t^i = p_0^i + \int_0^t \sum_{j=1}^K q^{ji} p_u^j \, du + \frac{1}{\sigma} \int_0^t p_u^i (\mu^i - \boldsymbol{\mu}^\top p_u) \, dI_u^{(1)}$$
$$+ \frac{1}{\sigma\eta\sqrt{1-\rho^2}} \int_0^t p_u^i \left(\sigma\kappa \left(\theta_i - \boldsymbol{\theta}^\top p_u \right) - \eta\rho \left(\mu_i - \boldsymbol{\mu}^\top p_u \right) \right) dI_u^{(2)}, \quad p_0^i = \Pi^i, \tag{3.13}$$

for every $t \in [0, T]$.

Proof. Consider the semimartingale decomposition of $f(Y)$ given by

$$f(Y_t) = f(Y_0) + \int_0^t \langle Q\mathbf{f}, Y_{u^-} \rangle \, du + M_t^{(1)}, \quad t \in [0, T], \tag{3.14}$$

where $M^{(1)}$ is a $(\mathbb{G}, \mathbf{P})$-martingale. Now, projecting over $\mathbb{F}$ leads to

$$\widehat{f(Y_t)} - \widehat{f(Y_0)} - \int_0^t \langle Q\mathbf{f}, \widehat{Y}_{u^-} \rangle \, du = M_t^{(2)}, \quad t \in [0, T], \tag{3.15}$$

where $M^{(2)}$ is an $(\mathbb{F}, \mathbf{P})$-martingale. Using the martingale representation in (3.10) we get

$$\widehat{f(Y_t)} - \widehat{f(Y_0)} - \int_0^t \langle Q\mathbf{f}, \widehat{Y}_{u^-} \rangle \, du = \int_0^t H_u \, dI_u, \quad t \in [0, T]. \tag{3.16}$$

Let $m_t = I_t + \int_0^t \Sigma^{-1} \widehat{A(u, Y_u)} \, du$, for every $t \in [0, T]$. Computing the product $f(Y) \cdot m$ and projecting on $\mathbb{F}$, we obtain

$$\widehat{f(Y_t) \cdot m_t} = \int_0^t m_u \langle Q\mathbf{f}, \widehat{Y}_u \rangle \, du + \int_0^t \Sigma^{-1} \widehat{f(Y_u) A(u}, Y_u) \, du + M_t^{(3)}, \ t \in [0, T], \tag{3.17}$$

for some $(\mathbb{F}, \mathbf{P})$-martingale $M^{(3)}$. We now compute the product $\widehat{f(Y)} \cdot m$ as

$$\widehat{f(Y_t)} \cdot m_t = \int_0^t m_u \langle Q\mathbf{f}, \widehat{Y}_u \rangle \, du + \int_0^t \Sigma^{-1} \widehat{f(Y_u)} \widehat{A(u, Y_u)} \, du + \int_0^t H_u \, du + M_t^{(4)}, \tag{3.18}$$

for every $t \in [0, T]$, where $M^{(4)}$ is an $(\mathbb{F}, \mathbf{P})$-martingale. Comparing the finite variation terms in (3.17) and (3.18), we get

$$H_t^{(1)} = \frac{\widehat{f(Y_t)\mu(Y_t)} - \widehat{f(Y_t)}\widehat{\mu(Y_t)}}{\sigma}, \tag{3.19}$$

$$H_t^{(2)} = \frac{\sigma\kappa(\widehat{f(Y_t)\theta(Y_t)} - \widehat{f(Y_t)}\widehat{\theta(Y_t)}) - \eta\rho(\widehat{f(Y_t)\mu(Y_t)} - \widehat{f(Y_t)}\widehat{\mu(Y_t)})}{\sigma\eta\sqrt{1-\rho^2}}, \tag{3.20}$$

for every $t \in [0, T]$. Finally choosing $f(Y_t) = \mathbf{1}_{\{Y_t = e_i\}}$, we obtain the result. □

Remark 3.21. Here notice that the drift and diffusion coefficients in (3.13) are continuous, bounded and locally Lipschitz. This implies that p is the unique strong solution of the filtering equation (3.13).

3.2 Reduction of the optimal control problem

The semimartingale decompositions of Z and S with respect to the observation filtration are given by

$$\begin{aligned} Z_t = & Z_0 + \int_0^t Z_u \left(h_u \left(\kappa(\boldsymbol{\theta}^\top p_u - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta \right) + r - \frac{\varepsilon\eta^2 h_u^2}{2} \right) du \\ & + \eta \int_0^t h_u Z_u \left(\rho \, dI_u^{(1)} + \sqrt{1-\rho^2} \, dI_u^{(2)} \right), \quad t \in [0,T], \end{aligned} \tag{3.22}$$

and

$$S_t = S_0 + \int_0^t \kappa(\boldsymbol{\theta}^\top p_u - S_u) \, du + \eta \int_0^t \left(\rho \, dI_u^{(1)} + \sqrt{1-\rho^2} \, dI_u^{(2)} \right), \quad t \in [0,T]. \tag{3.23}$$

Thanks to uniqueness of the solution of the filtering equation we can consider the $(K+2)$-dimensional process (Z, S, p) as the state process and introduce the equivalent optimal control problem under full information; see, e.g., [80]. We have

$$\max \ \mathbb{E}^{t,z,s,\mathbf{p}}[\log Z_T], \tag{3.24}$$

where $\mathbb{E}^{t,z,s,\mathbf{p}}$ denotes the conditional expectation given $Z_t = z$, $S_t = s$ and $p_t = \mathbf{p}$, where $(z, s, \mathbf{p}) \in \mathbb{R}_+ \times \mathbb{R} \times \Delta_K$, with Δ_K denoting the $(K-1)$-dimensional simplex. We define the value function of the trader by

$$V(t, z, s, \mathbf{p}) := \sup_{h \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}^{t,z,s,\mathbf{p}} \left[\log Z_T \right]. \tag{3.25}$$

To obtain the optimal strategy it is possible to apply pointwise maximization, which also leads to an explicit characterization for the value function. This is given in the next theorem.

Theorem 3.26. *Consider a trader with a logarithmic utility function with risk penalization parameter $\varepsilon \geq 0$. Then the optimal portfolio strategy $h^* \in \mathcal{A}^{\mathbb{F}}$ under partial information is*

$$h^*(t, s, \mathbf{p}) = \frac{1}{1+\varepsilon} \left(\frac{\kappa\left(\boldsymbol{\theta}^\top \mathbf{p} - s\right)}{\eta^2} + \frac{\rho\sigma}{\eta} - \frac{1}{2} \right). \tag{3.27}$$

The value function is of the form

$$V(t, z, s, \mathbf{p}) = \log(z) + r(T-t) + d(t)s^2 + c(t, \mathbf{p})s + f(t, \mathbf{p}), \tag{3.28}$$

where the function $d(t)$ is given by

$$d(t) = \frac{\kappa}{4\eta^2(1+\varepsilon)} \left(1 - e^{-2\kappa(T-t)} \right), \tag{3.29}$$

and the functions $c(t, \mathbf{p})$ and $f(t, \mathbf{p})$ solve the following system of partial differential equations:

$$c_t(t, \mathbf{p}) + \frac{1}{2} \sum_{i,j=1}^K \widetilde{\alpha}^{ij}(\mathbf{p}) c_{p^i p^j}(t, \mathbf{p}) + \sum_{i,j=1}^K c_{p^i}(t, \mathbf{p}) q^{ji} p^j + \kappa\big(2d(t)\boldsymbol{\theta}^\top \mathbf{p} - c(t, \mathbf{p})\big) - \gamma(\mathbf{p}) = 0, \tag{3.30}$$

$$\begin{aligned} & f_t(t, \mathbf{p}) + \frac{1}{2} \sum_{i,j=1}^K \widetilde{\alpha}^{ij}(\mathbf{p}) f_{p^i p^j}(t, \mathbf{p}) + \sum_{i,j=1}^K f_{p^i}(t, \mathbf{p}) q^{ji} p^j + \eta \sum_{i=1}^K c_{p^i}(t, \mathbf{p}) \widetilde{\beta}^i(\mathbf{p}) \\ & \quad + c(t, \mathbf{p}) \kappa \boldsymbol{\theta}^\top \mathbf{p} + \eta^2 d(t) + \frac{\left(\kappa\boldsymbol{\theta}^\top \mathbf{p} - \frac{1}{2}\eta^2 + \rho\sigma\eta\right)^2}{2\eta^2(1+\varepsilon)} = 0, \end{aligned} \tag{3.31}$$

with terminal conditions $c(T,\mathbf{p}) = 0$ *and* $f(T,\mathbf{p}) = 0$ *for every* $\mathbf{p} \in \Delta_K$*, and where*

$$\widetilde{\alpha}^{i,j}(\mathbf{p}) = H^{(i,1)}(\mathbf{p})H^{(j,1)}(\mathbf{p}) + H^{(i,2)}(\mathbf{p})H^{(j,2)}(\mathbf{p}), \quad i,j \in \{1,\dots,K\}, \tag{3.32}$$

$$\widetilde{\beta}^{i}(\mathbf{p}) = \rho H^{(i,1)}(\mathbf{p}) + \sqrt{1-\rho^2}H^{(i,2)}(\mathbf{p}), \quad i \in \{1,\dots,K\}, \tag{3.33}$$

$$H^{(i,1)}(\mathbf{p}) = p^i\frac{(\mu_i - \boldsymbol{\mu}^\top\mathbf{p})}{\sigma}, \qquad H^{(i,2)}(\mathbf{p}) = p^i\frac{\sigma\kappa\left(\theta_i - \boldsymbol{\theta}^\top\mathbf{p}\right) - \eta\rho\left(\mu_i - \boldsymbol{\mu}^\top\mathbf{p}\right)}{\sigma\eta\sqrt{1-\rho^2}}, \tag{3.34}$$

$$\gamma(\mathbf{p}) = \frac{\kappa}{\eta^2(1+\varepsilon)}\left(\boldsymbol{\theta}^\top\mathbf{p} - \frac{1}{2}\eta^2 + \rho\sigma\eta\right). \tag{3.35}$$

Proof. The proof of Theorem 3.26 follows the same lines of that of Theorem 2.4 and it is provided here for completeness. Here, as in the case of full information we maximize pointwisely. We first write

$$\begin{aligned}\mathbb{E}^{t,z,s,\mathbf{p}}[\log Z_T] = {}&\log(z) + r(T-t) - \mathbb{E}^{t,s,\mathbf{p}}\left[\int_t^T \frac{h_u^2\eta^2(1+\varepsilon)}{2}\,du\right]\\ &+ \mathbb{E}^{t,s,\mathbf{p}}\left[\int_t^T h_u\left(\kappa(\boldsymbol{\theta}^\top p_u - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta\right)du\right],\end{aligned} \tag{3.36}$$

where, $\mathbb{E}^{t,s,\mathbf{p}}$ denotes the conditional expectation given $S_t = s$ and $p_t = \mathbf{p}$. The first and second order conditions imply that the optimal strategy is given by

$$h^*(t,s,\mathbf{p}) = \frac{1}{1+\varepsilon}\left(\frac{\kappa\left(\boldsymbol{\theta}^\top\mathbf{p} - s\right)}{\eta^2} + \frac{\rho\sigma}{\eta} - \frac{1}{2}\right). \tag{3.37}$$

This leads to the following stochastic representation for the optimal value,

$$\log(z) + r(T-t) + \mathbb{E}^{t,s,\mathbf{p}}\left[\int_t^T \frac{(\kappa(\boldsymbol{\theta}^\top p_u - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)}\,du\right]. \tag{3.38}$$

We define the function $u : [0,T]\times\mathbb{R}\times\Delta_K \to \mathbb{R}_+$ by

$$u(t,s,\mathbf{p}) = \mathbb{E}^{t,s,\mathbf{p}}\left[\int_t^T \frac{(\kappa(\boldsymbol{\theta}^\top p_u - S_u) - \frac{\eta^2}{2} + \rho\sigma\eta)^2}{2\eta^2(1+\varepsilon)}\,du\right]. \tag{3.39}$$

By applying the Feynman–Kac formula and plugging the ansatz $u(t,s,\mathbf{p}) = d(t)s^2 + c(t,\mathbf{p})s + f(t,\mathbf{p})$ in the resulting equation leads to the system of linear partial differential equations in (3.30)-(3.31) and the following linear ordinary differential equation

$$d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0, \quad d(T) = 0. \tag{3.40}$$

Note that the system (3.30) and (3.31) admits a unique solution; see Chapter 9 of [86]. □

Comments and discussion. By Theorem 2.4 and Theorem 3.26, optimal strategies depend on both the correlation between two assets and the mean-reverting spread. Moreover, they do not depend on the risk-free rate r because a priori we restrict ourselves to the dollar-neutral pairs trading strategies. Comparing optimal strategies under full and partial information, we can say that the so-called *certainty equivalence principle* holds[1], i.e., the optimal portfolio strategy in the latter case can be obtained by replacing the unobservable state variable with its filtered estimate.

The effect of risk-penalization on optimal strategies is to increase the risk-aversion uniformly in a constant proportion that is not dependent on time. It effectively decreases the proportion of wealth invested in pairs and increases the proportion of wealth invested in the risk-free asset. Considering the optimal value functions, in both cases, they are quadratic functions of the current value of the spread. However, in both cases, coefficients (factor loadings) on the quadratic term, s^2, depend only on time. This result is worth to mention since it means that the trader does not really consider the effect of the partial information on the quadratic level of the current spread. Finally, note that similar results hold true for beta-neutral strategies.

[1]This unorthodox definition of certainty equivalence principle is due to [115] and used in literature related to partial information models; see, e.g., [14].

Chapter 4

Toy Example: Two-State Markov Chain

In this chapter, we give a toy example of our proposed model, where the unobservable Markov chain has only two states. Here our main aim is to demonstrate certain qualitative features of the model that are difficult to verify analytically. During our analysis, we set $z = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\mu_1 = 0.2$ and $\mu_2 = 1$. In the first step, we consider the full information case, where the trader knows the state of the Markov chain. Then, we investigate the case with partial information.

4.1 The full information case

In this part, we employ Theorem 2.4 where we solve the corresponding system of ODEs numerically. In the following, since we set $z = 1$ we suppress the dependence of the value function on z and write $V(t, s, i)$ for $i \in \{1, 2\}$.

In Figure 4.1, we illustrate optimal values with respect to time to maturity for a given initial state and for different values of initial spread ($s = 0.1$, $s = 0.3$ and $s = 0.7$). It suggests that for all initial states and for all values of the initial spread, the optimal value increases in time to maturity since trading possibilities increase as there would be more time to trade. Moreover, as it is expected from a pairs trading strategy, the wider the gap between the initial spread and the long-run mean of the initial state's spread, the higher the optimal value provided that there is enough time to have the spread close with high probability. For example, in Figure 4.1 (left panel), we observe $V(t, 0.1, 2) > V(t, 0.1, 1)$ for all t . This corresponds to the case where the trader could exploit the wide enough gap between initial spread, $s = 0.1$, and the long-run mean of the second state, $\theta_2 = 0.6$. A similar behaviour is observed in Figure 4.1 (right panel), where in this case the gap between the initial spread, $s = 0.7$ and the long-run mean of the first state, $\theta_1 = 0.1$, is large enough for $V(t, 0.7, 1) > V(t, 0.7, 2)$ for all t.

However in Figure 4.1 (middle panel), there is no clear dominance between optimal values corresponding to different initial states. This can be explained by the following observation. The initial spread, which is 0.3, is approximately at the same distance to both states' long-run means hence the intersection point of the two functions $V(\cdot, 0.3, 1)$ and $V(\cdot, 0.3, 2)$ depends more on the transition intensities of the Markov chain q^{12} and q^{21}. In particular, for this example, fixing all other parameters, the intersection point moves to the right as q^{12} gets larger. Overall we can conclude that the main determinants of the observed dominance are the gap between initial spread and the long-run mean of states, transition probabilities as well as remaining time to maturity.

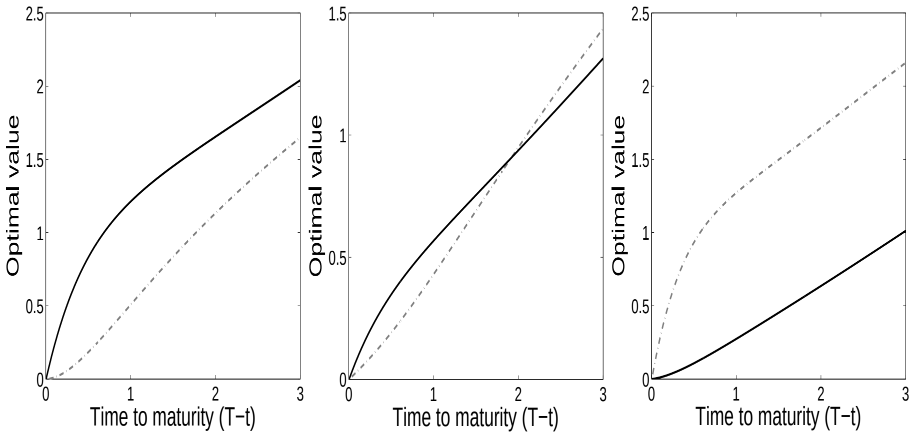

Figure 4.1: Optimal value as a function of time to maturity for different values of initial spread, s, when the initial state is e_1 (dashed line) or e_2 (solid line). Left panel $s = 0.1$. Middle panel $s = 0.3$. Right panel $s = 0.7$. Other parameters: $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\kappa = 1$, $\rho = 0.9$, $\sigma = 0.2$, $\eta = 0.2$, $\varepsilon = 0.3$, $q^{12} = 0.7$ and $q^{21} = 0.2$.

Next in Figure 4.2, we compare the value of the current optimal portfolio problem with the optimal value computed using the *averaged data*. Let $(\pi, 1-\pi)$ denotes the stationary distribution of the Markov chain Y. Suppose we have two traders, one of which ignores the Markov modulated nature of the underlying spread and considers the *averaged data* $\overline{\theta} = \pi\theta_1 + (1-\pi)\theta_2$ as the long-run mean spread. On the other hand, the second trader assumes our proposed Markov modulated model, that is, she acts in line with what Theorem 2.4 suggests. We want to compare the value function $\overline{V}(t,s)$ obtained in the model assuming averaged data with the value function in the Markov-modulated case. In this way, we intend to see whether the knowledge of averaged data is sufficient to obtain the optimal value for the current pairs trading problem. To this, we set $q^{12} = 1$ and $q^{21} = 2$, and compute $\pi = q^{21}/(q^{12} + q^{21}) = 0.67$. Then, we get $\overline{\theta} = 0.27$. In Figure 4.2 we plot $\overline{V}(t,s)$ versus $\mathbb{E}^{\pi}[V(t,s,Y_t)] = \pi V(t,s,1) + (1-\pi)V(t,s,2)$. We observe that $\mathbb{E}^{\pi}[V(t,s,Y_t)] > \overline{V}(t,s)$. This implies that the averaged data does not contain sufficient information to obtain the optimal value for the pairs trading problem and hence on the average, the second trader performs better than the first one. This result is in contrast with the one for the classical portfolio optimization problem with Markov modulation in the case of logarithmic utility preferences; see Section B of [14]. We attribute this to the mean-reverting nature of the underlying state variable.

Figure 4.3 depicts the behavior of the value function with respect to the mean-reversion speed κ, for correlation values $\rho = 0.1$ and $\rho = 0.9$. In the case without Markov switching one would expect higher values of κ to yield higher optimal values since that would imply more visits to the long-run mean generating profit opportunities from pairs trading more frequently. Here, we observe that higher values of κ not necessarily lead to larger portfolio values since there is the risk of a regime switch which would result in a sudden change in the long-run mean value.

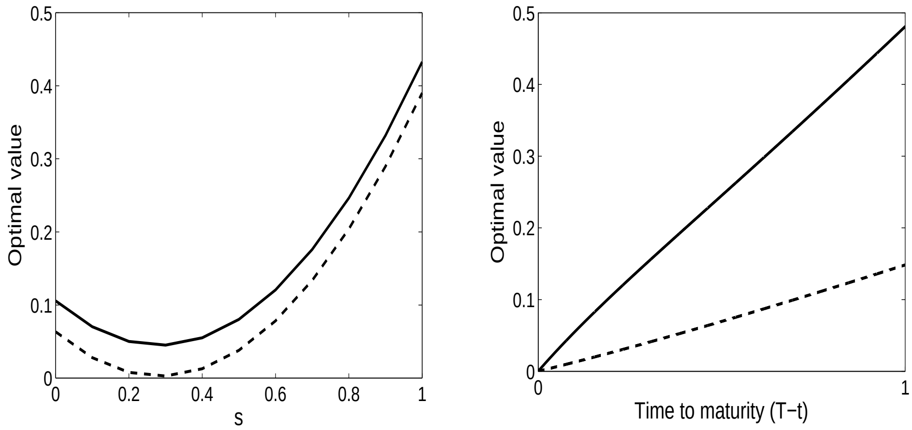

Figure 4.2: Left panel: Optimal value as a function of initial spread for time to maturity $T - t = 0.1$ years. Right panel: Optimal value as a function of time to maturity for initial spread $s = 0.3$. The solid line (resp. dashed line) indicates the optimal value corresponding to Markov switching case (resp. averaged data case). Other parameters: $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\kappa = 1$, $\rho = 0.9$, $\sigma = 0.2$, $\eta = 0.2$, $\varepsilon = 0.5$, $q^{12} = 1$ and $q^{21} = 2$.

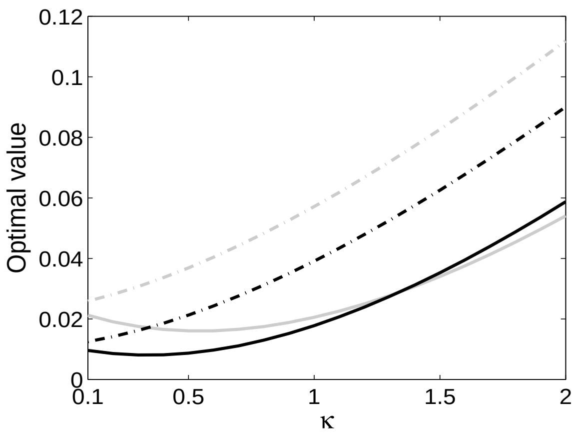

Figure 4.3: Impact of mean reversion speed κ on optimal value. Dashed line (resp. solid line) corresponds to the optimal value when the initial state is e_1 (resp. e_2). Grey line: $\rho = 0.1$, black line: $\rho = 0.9$. Other parameters: $T - t = 3$ years, $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $s = 0.3$, $\eta = 0.9$, $\sigma = 0.2$, $\varepsilon = 0.3$, $q^{12} = 0.7$ and $q^{21} = 0.2$.

4.2 The partial information case

In the partially observable setting, having only two states enables us to reduce the number of state variables for our filtered control problem since $p := p^1 = 1 - p^2$. Then we only need the dynamics of p, given, after arrangement, by

$$dp_t = (q^{12} + q^{21}) \left(\frac{q^{21}}{q^{12} + q^{21}} - p_t \right) dt + \sqrt{\nu_1^2 + \nu_2^2} p_t (1 - p_t)\, dI_t^{(3)}, \tag{4.1}$$

where

$$\nu_1 = \frac{(\mu_1 - \mu_2)}{\sigma}, \tag{4.2}$$

$$\nu_2 = \frac{\sigma\kappa(\theta_1 - \theta_2) - \eta\rho(\mu_1 - \mu_2)}{\sigma\eta\sqrt{1 - \rho^2}}, \tag{4.3}$$

and

$$I^{(3)} = \frac{\nu_1}{\sqrt{\nu_1^2 + \nu_2^2}} I^{(1)} + \frac{\nu_2}{\sqrt{\nu_1^2 + \nu_2^2}} I^{(2)} \tag{4.4}$$

is an $\mathbb{F}$-Brownian motion. We can write the semimartingale decomposition of wealth and spread processes with respect to filtration $\mathbb{F}$ as

$$\begin{aligned} dZ_t =& Z_t \left(h_t \left(\kappa(\theta_2 + (\theta_1 - \theta_2) p_t - S_t) - \frac{\eta^2}{2} + \rho\sigma\eta \right) + r - \frac{\varepsilon \eta^2 h_t^2}{2} \right) dt \\ &+ \eta h_t Z_t\, d\tilde{I}_t, \end{aligned} \tag{4.5}$$

and

$$dS_t = \kappa \left(\theta_2 + (\theta_1 - \theta_2) p_t - S_t \right) dt + \eta\, d\tilde{I}_t, \tag{4.6}$$

where $\tilde{I}$ is a $\mathbb{F}$-Brownian motion with

$$\langle \tilde{I}, I^{(3)} \rangle_t = \frac{\nu_1 \rho + \nu_2 \sqrt{1 - \rho^2}}{\sqrt{\nu_1^2 + \nu_1^2}} t. \tag{4.7}$$

Note that one can interpret the reduced control problem with state variables (Z, S, p) given by (4.5), (4.6) and (4.1) as a pairs trading model with smooth transitions. More precisely, one can see p as a state variable process governing smooth transitions between two regimes with different long-term means for the spread, that is, θ_1 and θ_2. The dynamics of p is also very similar to a mean-reverting Jacobi-type (or Wright–Fisher) diffusion used in population genetics to model allele frequencies[1]; see, e.g., [68], [148] or Chapter 4 of Part I.

In this case the value function can be written as $V(t, z, s, p^1, p^2) = \widetilde{V}(t, z, s, p)$, and, as in Theorem 3.26, the optimal value is given by $\widetilde{V}(t, z, s, p) = \log(z) + r(T - t) + d(t)s^2 + \widetilde{c}(t, p)s + \widetilde{f}(t, p)$, where the function $d(t)$ is given by

$$d(t) = \frac{\kappa}{4\eta^2(1 + \varepsilon)} \left(1 - e^{-2\kappa(T - t)} \right), \tag{4.8}$$

and the functions $\widetilde{c}(t, p)$ and $\widetilde{f}(t, p)$ solve the following system of partial differential equations:

[1]For the Jacobi or Wright–Fisher diffusion, the diffusion coefficient is given by $\sqrt{p(1 - p)}$.

$$\widetilde{c}_t(t,p) - \frac{\kappa^2(\theta_2+(\theta_1-\theta_2)p) - \kappa(\frac{-\eta^2}{2}+\rho\sigma\eta)}{\eta^2(1+\varepsilon)} - \kappa\widetilde{c}(t,p) + 2\kappa(\theta_2+(\theta_1-\theta_2)p)d(t)$$
$$+ (q^{12}+q^{21})\left(\frac{q^{21}}{q^{12}+q^{21}} - p\right)\widetilde{c}_p(t,p) + \frac{1}{2}(\nu_1^2+\nu_2^2)p^2(1-p)^2\widetilde{c}_{pp}(t,p) = 0, \tag{4.9}$$

$$\widetilde{f}_t(t,p) + \frac{\kappa^2(\theta_2+(\theta_1-\theta_2)p)^2 + (\rho\sigma\eta-\frac{\eta^2}{2})^2 + 2\kappa(\theta_2+(\theta_1-\theta_2)p)(\rho\sigma\eta-\frac{\eta^2}{2})}{2\eta^2(1+\varepsilon)}$$
$$+ \eta^2 d(t) + \kappa(\theta_2+(\theta_1-\theta_2)p)\widetilde{c}(t,p) + (q^{12}+q^{21})\left(\frac{q^{21}}{q^{12}+q^{21}} - p\right)\widetilde{f}_p(t,p)$$
$$+ \frac{1}{2}(\nu_1^2+\nu_2^2)p^2(1-p)^2\widetilde{f}_{pp}(t,p) + \kappa(\theta_1-\theta_2)p(1-p)\widetilde{c}_p(t,p) = 0, \tag{4.10}$$

with terminal conditions $\widetilde{c}(T,p) = 0$ and $\widetilde{f}(T,p) = 0$ for every $p \in [0,1]$,

We use an explicit finite-difference method to solve the system of PDEs given in (4.10) numerically. In order to guarantee the positivity of the scheme we use forward-backward approximation for the first order derivatives. The value function in the partial information case has a similar behavior with respect to the parameters as the one in the full information case. However, we stress that in the partial information setting, also the drift parameters μ_1 and μ_2 play a role. In particular, the relative values of μ_1, μ_2 and the noise parameters σ and η control for the precision of the filtered probability estimates.

In Figure 4.4 we illustrate that the trader benefits from using filtered estimates instead of average data. As it can be seen clearly, gains from filtering increase in time to maturity. On the other hand, gains get smaller as p moves towards 0.5, which represent the most uncertain situation.

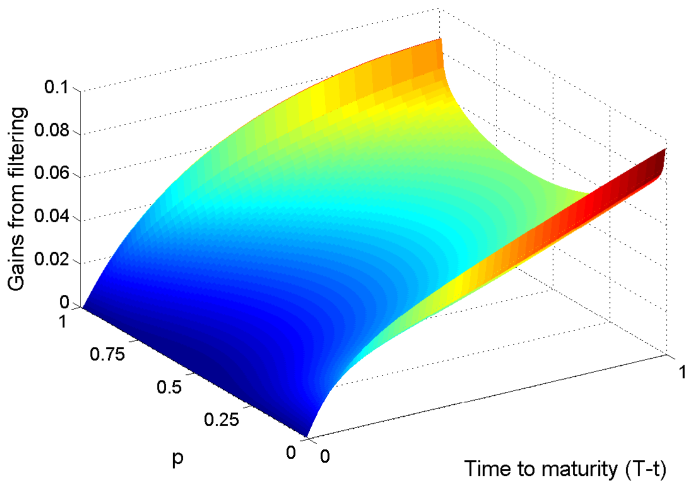

Figure 4.4: Gains from filtering as a function of p and time to maturity. Other parameters: $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\mu_1 = 0.2$, $\mu_2 = 1$, $\kappa = 1$, $\rho = 0.9$, $\sigma = 0.2$, $\eta = 0.2$, $\varepsilon = 0.5$, $s = 0.3$, $q^{12} = 1$ and $q^{21} = 2$.

We can summarize the findings of this section as follows: (a) the wider the gap between the initial spread and the long-run mean of the initial state's spread, the higher the optimal value provided that there is enough time to have the spread close with high probability, (b) the average data does not contain sufficient information to obtain the optimal value for the current pairs trading problem, (c) higher values of the mean reversion speed κ does not necessarily imply higher optimal values, and (d) in the partial information setting, there is a gain from filtering due to the convexity originating from using filtered probabilities.

Chapter 5

Appendix

In this appendix, we give the proofs of Theorem 2.4 and Theorem 3.26 via dynamic programming approach, alternative to the pointwise optimization given before. In particular, we give the existence and verification results for the corresponding problems under full and partial information.

5.1 Existence and verification for the problem under full information

Define $b(s,i) = \kappa(\theta_i - s) - \frac{\eta^2}{2} + \rho\sigma\eta$ and denote by $\mathcal{L}^h$ the generator of the process (t, Z, S, Y), that is

$$\begin{aligned}\mathcal{L}^h F(t,z,s,i) &= F_t(t,z,s,i) + \sum_{j=1}^{K} F(t,z,s,j) q^{ij} \\ &+ \frac{1}{2}\left(h^2\eta^2 z^2 F_{zz}(t,z,s,i) + \eta^2 F_{ss}(t,z,s,i) + 2h\eta^2 z F_{zs}(t,z,s,i)\right) \\ &+ \left(b(s,i)h - \frac{\varepsilon\eta^2 h^2}{2} + r\right) z F_z(t,z,s,i) + \kappa(\theta_i - s) F_s(t,z,s,i),\end{aligned} \tag{5.1}$$

for every function $F(\cdot, i) \in C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R})$, for every $i \in \{1, \dots, K\}$. Suppose that the value function $V(\cdot, i) \in C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R})$ (i.e. bounded, differentiable with respect to t and twice differentiable with respect to z and s) for every $i \in \{1, \dots, K\}$, then it solves the Hamilton–Jacobi–Bellman (HJB) equation given by

$$0 = \sup_{h \in \mathcal{A}} \mathcal{L}^h V(t,z,s,i) \tag{5.2}$$

for every $i \in \{1, \dots, K\}$, subject to the terminal condition $V(T,z,s,i) = \log(z)$, for all $(z,s) \in \mathbb{R}_+ \times \mathbb{R}$ and $i \in \{1, \dots, K\}$.

Proof. (Theorem 2.4). *Existence:* Consider the HJB equation (5.2). By the first order condition, the candidate for an optimal strategy is given in the feedback form

$$h^*(t,z,s,i) = -\frac{\eta^2 V_{zs}(t,z,s,i) + b(s,i) V_z(t,z,s,i)}{\eta^2 (z V_{zz}(t,z,s,i) - \varepsilon V_z(t,z,s,i))}. \tag{5.3}$$

It follows from the form of the utility function that for all $i \in \{1, \dots, K\}$ the value function can be rewritten as $V(t, z, s, i) = \log(z) + u(t, s, i)$, for some function $u(t, s, i)$ such that $u(T, s, i) = 0$. Since $V(t, z, s, i)$ is concave and increasing in z, the second order condition, given by $zV_{zz} - \varepsilon V_z < 0$, holds true for $\varepsilon > 0$ and therefore (5.3) is the well defined maximizer and hence the optimal portfolio strategy.

Inserting the ansatz for the value function in equations (5.3) and (5.2) leads to

$$0 = u_t(t, s, i) + \frac{1}{2}\eta^2 u_{ss}(t, s, i) + \kappa(\theta_i - s)u_s(t, s, i) + \sum_{j=1}^{K} u(t, s, j)q^{ij} + \frac{1}{2}\frac{b^2(s, i)}{\eta^2(1+\varepsilon)} + r, \tag{5.4}$$

and the optimal strategy becomes

$$h^*(t, z, s, i) = \frac{b(s, i)}{\eta^2(1+\varepsilon)}. \tag{5.5}$$

Suppose now that the function $u(t, s, i)$ is of the form $u(t, s, i) = d(t)s^2 + c(t, i)s + f(t, i)$. Computing the derivatives and plugging into (5.4) we get the following equation

$$\begin{aligned} 0 =& r + c_t(t, i)s + d_t(t)s^2 + f_t(t, i) + \eta^2 d(t) + \frac{\Big(\kappa(\theta_i - s) - \frac{\eta^2}{2} + \rho\sigma\eta\Big)^2}{\eta^2(1+\varepsilon)} \\ &+ \kappa(\theta_i - s)(c(t, i) + 2d(t)s) + \sum_{j=1}^{K}(c(t, j)s + f(t, j))q^{ij}. \end{aligned} \tag{5.6}$$

Collecting together the terms with s^2, s and the remaining ones we get that the function $d(t)$ solves

$$d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0, \quad d(T) = 0, \tag{5.7}$$

and for every $i \in \{1, \dots, K\}$, $c(t, i)$ and $f(t, i)$ solve the system of ODEs in (2.8) and (2.9), respectively (see e.g., [155, Theorem 3.9]). Therefore $V(\cdot, i) \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R})$ for every $i \in \{1, \dots, K\}$.

Verification: In order to conclude that V is the value function, we need to show a verification result. Let $v(t, z, s, i)$ be a solution of the HJB equation (5.2). Given an admissible control $h \in \mathcal{A}$, let Z^h be the solution to equation (1.11). Applying Itô's formula we get

$$\begin{aligned} v(T, Z_T^h, S_T, Y_T) &= v(t, z, s, i) + \int_t^T \mathcal{L}^h v(r, Z_r^h, S_r, Y_r)\, dr \\ &+ \int_t^T \Big(v_z(r, Z_r^h, S_r, Y_r)Z_r^h h_r \eta + v_s(r, Z_r^h, S_r, Y_r)\eta\Big)\, dW_r \\ &+ \int_t^T \sum_{j=1}^{K} \Big(v(r, Z_r^h, S_r, j) - v(r, Z_r^h, S_r, Y_{r^-})\Big)(m - \nu)(dr \times \{j\}), \end{aligned} \tag{5.8}$$

Since v satisfies equation (5.2) we get

$$\begin{aligned} v(T, Z_T^h, S_T, Y_T) &\le v(t, z, s, i) \\ &+ \int_t^T \Big(v_z(r, Z_r^h, S_r, Y_r)Z_r^h h_r \eta + v_s(r, Z_r^h, S_r, Y_r)\eta\Big)\, dW_r \\ &+ \int_t^T \sum_{j=1}^{K} \Big(v(r, Z_r^h, S_r, j) - v(r, Z_r^h, S_r, Y_{r^-})\Big)(m - \nu)(dr \times \{j\}). \end{aligned} \tag{5.9}$$

The form of v, together with the fact that h is an admissible strategy provides that integrals with respect to Brownian motion and the compensated jump measure are indeed true martingale. Taking the expectation on both sides of the inequality (5.9) we get that

$$V(t,z,s,i) \leq v(t,z,s,i). \tag{5.10}$$

If h is a maximizer of equation (5.2) then we obtain the equality in the expression above. □

5.2 Existence and verification for the problem under partial information

Proof. (Theorem 3.26). *Existence:* Denote by $b(s,\mathbf{p}) = \kappa(\boldsymbol{\theta}^\top \mathbf{p} - s) - \frac{\eta^2}{2} + \rho\sigma\eta$. For the current setting we have the following HJB equation

$$\begin{aligned}
0 =& \mathcal{L}^h_{\mathbb{F}} V(t,z,s,\mathbf{p}) \\
=& \sup_{h \in \mathcal{A}^{\mathbb{F}}} \Bigg\{ V_t(t,z,s,\mathbf{p}) + \left(-\frac{\varepsilon\eta^2 h^2}{2} + b(s,\mathbf{p})h + r \right) z V_z(t,z,s,\mathbf{p}) \\
&+ \kappa \left(\boldsymbol{\theta}^\top \mathbf{p} - s \right) V_s(t,z,s,\mathbf{p}) + \sum_{i,j=1}^K q^{ji} p^j V_{p^i}(t,z,s,\mathbf{p}) \\
&+ \frac{1}{2} \left(h^2\eta^2 z^2 V_{zz}(t,z,s,\mathbf{p}) + \eta^2 V_{ss}(t,z,s,\mathbf{p}) + 2h\eta^2 z V_{zs}(t,z,s,\mathbf{p}) \right) \\
&+ \sum_{i=1}^K V_{sp^i}(t,z,s,\mathbf{p}) \eta \widetilde{\beta}^i(\mathbf{p}) + \sum_{i=1}^K h\eta z \widetilde{\beta}^i(\mathbf{p}) V_{zp^i}(t,z,s,\mathbf{p}) \\
&+ \frac{1}{2} \sum_{i,j=1}^K \widetilde{\alpha}^{ij}(\mathbf{p}) V_{p^i p^j}(t,z,s,\mathbf{p}) \Bigg\},
\end{aligned} \tag{5.11}$$

subject to the terminal condition $V(T,z,s,\mathbf{p}) = \log(z)$, for all $z > 0$, $s \in \mathbb{R}$ and for every $\mathbf{p} \in \Delta_K$, where $\mathcal{L}^h_{\mathbb{F}}$ the $(\mathbb{F}, \mathbf{P})$-generator of the process (Z, S, p).

From the first order condition, the candidate for an optimal strategy is given in the feedback form

$$h^*(t,z,s,\mathbf{p}) = -\frac{\eta^2 V_{zs}(t,z,s,\mathbf{p}) + b(s,\mathbf{p}) V_z(t,z,s,\mathbf{p}) + \sum_{i=1}^K \eta \widetilde{\beta}^i(\mathbf{p}) V_{zp^i}(t,z,s,\mathbf{p})}{\eta^2 (z V_{zz}(t,z,s,\mathbf{p}) - \varepsilon V_z(t,z,s,\mathbf{p}))}. \tag{5.12}$$

It follows from the form of the utility function that the value function can be rewritten as $V(t,z,s,\mathbf{p}) = \log(z) + u(t,s,\mathbf{p})$, for some function $u(t,s,\mathbf{p})$ such that $u(T,s,\mathbf{p}) = 0$ for all $(s,\mathbf{p}) \in (\mathbb{R} \times \Delta_K)$. Since $V(t,z,s,\mathbf{p})$ is concave and increasing in z, the second order condition, given by $zV_{zz} - \varepsilon V_z < 0$, holds true for $\varepsilon > 0$ and therefore (5.12) is the maximizer and the optimal portfolio strategy.

Here, we choose u of the form $u(t,s,\mathbf{p}) = d(t)s^2 + c(t,\mathbf{p})s + f(t,\mathbf{p})$. Inserting this ansatz in equations (5.11) and (5.12) leads to the system of linear partial differential equations in (3.30)-(3.31) and the following linear ordinary differential equation

$$d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0, \quad d(T) = 0. \tag{5.13}$$

Note that the system (3.30) and (3.31) admits a unique solution (see e.g., [86, Chp. 9]). This implies that $V \in C^{1,2,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \Delta_K)$.

Verification: To conclude that V is the value function, we show a verification result. Let $v(t,z,s,\mathbf{p})$ be a solution of (5.11) with the boundary condition $v(T,z,s,\mathbf{p}) = \log(z)$. Let $h \in \mathcal{A}^{\mathbb{F}}$ be an $\mathbb{F}$-admissible control, let Z^h the solution to equation (3.22). Applying Itô's formula we get

$$\begin{aligned} v(T, Z_T^h, S_T, p_T) &= v(t,z,s,\mathbf{p}) + \int_t^T \mathcal{L}_{\mathbb{F}}^h v(u, Z_u^h, S_u, p_u)\, du \\ &+ \int_t^T \rho\eta \left(v_z(u, Z_u^h, S_u, p_u) Z_u^h h_u + v_s(u, Z_u^h, S_u, p_u) \right) dI_u^{(1)} \\ &+ \int_t^T \sqrt{1-\rho^2}\eta \left(v_z(u, Z_u^h, S_u, p_u) Z_u^h h_u + v_s(u, Z_u^h, S_u, p_u) \right) dI_u^{(1)} \\ &+ \int_t^T \sum_{i=1}^K H^{(i,1)}(p_u) v_{p^i}(u, Z_u^h, S_u, p_u)\, dI_u^{(1)} + \int_t^T \sum_{i=1}^K H^{(i,2)}(p_u) v_{p^i}(u, Z_u^h, S_u, p_u)\, dI_u^{(2)}. \end{aligned} \tag{5.14}$$

By equation (5.11) we get

$$\begin{aligned} v(T, Z_T^h, S_T, p_T) &\leq v(t,z,s,\mathbf{p}) \\ &+ \int_t^T \rho\eta \left(v_z(u, Z_u^h, S_u, p_u) Z_u^h h_u + v_s(u, Z_u^h, S_u, p_u) \right) dI_u^{(1)} \\ &+ \int_t^T \sqrt{1-\rho^2}\eta \left(v_z(u, Z_u^h, S_u, p_u) Z_u^h h_u + v_s(u, Z_u^h, S_u, p_u) \right) dI_u^{(1)} \\ &+ \int_t^T \sum_{i=1}^K H^{(i,1)}(p_u) v_{p^i}(u, Z_u^h, S_u, p_u)\, dI_u^{(1)} + \int_t^T \sum_{i=1}^K H^{(i,2)}(p_u) v_{p^i}(u, Z_u^h, S_u, p_u)\, dI_u^{(2)}. \end{aligned} \tag{5.15}$$

Note that stochastic integrals with respect to $I^{(1)}$ and $I^{(2)}$ are true martingales. Indeed, by the form of the solution of the HJB equation $v(t,z,s,\mathbf{p}) = \log(z) + d(t)s^2 + c(t,\mathbf{p})s + f(t,\mathbf{p})$, the fact that h is an $\mathbb{F}$-admissible strategy and boundedness of the functions $d(t), c(t,\mathbf{p}), f(t,\mathbf{p})$ and their derivatives over the compact interval $[0,T] \times \Delta_K$, we get that

$$\mathbb{E}\left[\int_t^T \left(h_u^2 + c^2(u, p_u) + d^2(u) S_u^2 \right) du \right] < \infty, \tag{5.16}$$

$$\mathbb{E}\left[\sum_{i=1}^K \left(c_{p^i}^2(u, p_u) S_u^2 + f_{p^i}^2(u, p_u) \right) \left(\left(H^{(i,1)}(p_u) \right)^2 + \left(H^{(i,2)}(p_u) \right)^2 \right) du \right] < \infty. \tag{5.17}$$

Then taking the expectation on both sides of inequality (5.15) implies that $V(t,z,s,\mathbf{p}) \leq v(t,z,s,\mathbf{p})$. Moreover if h^* is a maximizer of equation (5.11), then we obtain the equality $V(t,z,s,\mathbf{p}) = v(t,z,s,\mathbf{p})$. $\square$

Part III

Portfolio Optimization for a Large Investor Controlling Market Sentiments

Chapter 1

Introduction and Problem Formulation

The influence of large investors, such as hedge funds, mutual funds, and insurance companies, on prices of risky assets, can be studied from very different viewpoints ranging from direct price impact of order execution (selling or buying) to feedback effects from trading to hedge portfolios of derivatives written on the underlying. However, there is also an influence of large investors on the overall market sentiment that arises from their perceived informational superiority. That is, most of the time, the rest of the market takes large investors' portfolio decisions as signals revealing important insider information not available to small or price-taking investors. Moreover, due to the herding behavior, this effect can be intensified when markets are caught up in certain extreme situations like speculative bubbles or market downturns. Of course, by knowing that they have such an influence on the market, large investors can exploit this fact by changing their portfolio and consumption choices during those times and try to gain an advantage[1]. However, it is difficult, even for a large investor, to observe the exact state of the overall market and its effect on the price of the risky asset and hence to act accordingly. Not knowing the exact state of the environment naturally necessitates a partial information setting, in which the large investor only observes the price process of the risky asset.

Therefore in this study, we solve a finite-time utility maximization problem by considering a partially observable regime-switching environment, in which there is a large investor (or group of institutional investors) that has control over the intensity matrix of the continuous-time finite state Markov chain governing the state of the environment. We allow large investor's portfolio choices, as a fraction of the wealth invested in the risky asset, to have an indirect but persistent effect on the price process, through dependence on the controlled intensity of the Markov chain with next-neighborhood-type dynamics. We call this effect *market impact*. By taking the generator matrix of the unobservable Markov chain as a function of portfolio holdings of the large investor, and focusing on the price process with pure-jump dynamics affected by the unobservable Markov chain, we solve the problem of utility maximization from terminal wealth for logarithmic and power utility preferences. The idea to model market impact through an intensity-based framework is due to [28] where

[1]For example in the US large institutional investor needs to fill the SEC Form 13F, a form with the Securities and Exchange Commission (SEC) also known as the Information Required of Institutional Investment Managers Form. It is a required form from institutional investment managers with over 100 million in qualifying assets. It contains information about the investment manager and a list of their recent investment holdings.

the authors deal with a control problem for optimal investment and consumption for a large investor in the full information case with asset prices following jump-diffusion dynamics and a market with two possible states.

We start with a rigorous construction of the underlying setting by using a change of measure argument. This provides existence of the model under partial information. As a first step towards the characterization of the optimal strategy in the partial information setting, we solve the corresponding filtering problem via reference probability approach. Then, we reduce the optimal control problem under partial information to a full information one, as e.g., in [16, 31] where unobservable variables are replaced by their filtered estimates. Since the state of the resulting optimal control problem is piecewise deterministic, we resort to the theory of optimal control for piecewise deterministic Markov processes (PDMP) given in [45] (see also [46] for more details). The idea is that a control problem for PDMPs can be recast as a Markov decision process, in which the value function is characterized by a fixed point argument. Here we should note that, although identifying the optimal control problem for PDMP with a Markov decision process is well studied (see [46, 51, 5, 83, 17, 39] and references therein), to the best of our knowledge, a concrete application of optimal control of PDMP that covers the control of the intensity of an unobservable Markov chain is novel. To this extent, we use suitable modifications of certain results from [36], where the main motivation is to study optimal liquidation in a partial information setting with asset prices having pure-jump dynamics directly affected by the control (see also Section 3.2 for a deeper discussion). This allows us to write the value function as the unique fixed point of the reward operator and represent it as the unique viscosity solution of the Hamilton–Jacobi–Bellman (HJB) equation.

In our setting, the state of the environment with regime-switches can be interpreted in various ways. One natural interpretation, for example in a two-state case, is that states can be characterized as "bear" or "bull" market sentiments so that the large investor try to change the direction (uptrend or downtrend) of the market by her portfolio choices. Similarly, one may also explain those states as different levels of market "liquidity" in a market microstructure framework, or different stages of a business cycle in a more general macrostructure framework. In the former one, a large investor can be seen as a liquidity provider or a market maker, whereas in the latter, she can be considered as a central planner such as a central bank or government[2].

We should also remark that manipulation-type strategies pursued by large investors, in which there is an uncertainty coming from a market reaction against those manipulation attempts, can be modeled in this partially observed control framework. In a similar vein to the credit risk modeling, our framework can be considered as a reduced-form modeling of market manipulation since the impact of large investor on prices is indirect via her influence on market sentiments as opposed to models with direct impact on structural variables such as drift or volatility of the asset price process (see, e.g., [104, 105] for market manipulation models with large investors having a direct impact on asset price dynamics in discrete and continuous time). In particular, our setting allows for a large investor to change the probability of being in a "bull" or "bear" market by her actions. For example, by short-selling, a large investor may prevent the market going to a "bull" state and hence gain advantage of a "bear" market sentiment. Similarly, one can use the proposed model to analyze herding and momentum like behavioural effects on stock prices arising from large

[2]Certain central banks (Japan and Swiss) around the word have recently invested heavily in stock markets. Although their objective is different than utility maximization from terminal wealth, the same setting (indirectly influencing the economy to give a boost) can be analyzed in the same way.

investors' portfolio choices, since for example, in our proposed setting with pure-jump type asset price dynamics, one can mimic a market situation, in which a large investor try to influence the market sentiment by changing her portfolio and hence switch it from "bear" to "bull", where upward jumps are observed more likely, or vice-versa.

Considering the high-frequency nature of the markets that large investors are involved with (see [111, 63] for asset prices with Markov modulated pure-jump dynamics), we choose to work with a pure-jump process modulated by a Markov chain. Here we emphasize that our analysis covers the case of finite-activity price dynamics. As we strongly use the properties of PDMPs to solve the optimal control problem, generalizing the current setting into one with infinite-activity jump processes or continuous dynamics is not straightforward and would require a different theory.

There is ample amount of literature related to the optimal decision of a large investor, analyzed in various settings. The most related work to ours is [28] that studies the optimal consumption-portfolio choice problem, in which the asset price dynamics are given by a jump-diffusion affected by the regime-switching environment controlled by a large investor in a full information setting. They show that optimal strategies have significant deviations from the strategies obtained in the classical Merton problem. More importantly, they show that there can be situations (market manipulations) such that the large investor can consume even though she has no gain in utility from consumption. Generally, in the literature, the effect of large investors on asset prices are direct in the sense that decision variables (such as portfolio holdings, the speed of trading, etc.) have shown up in the drift or the volatility of the risky asset price process). For instance, the models of [42], [41], [112] and [59] examine optimal consumption and investment problem of a large investor with portfolio choices affecting the instantaneous expected returns in various settings. In the context of optimal order execution problems where the stock price process is driven by a diffusion, investors impact is modeled by volume or speed of trading affecting directly the drift (see, e.g., Almgren–Chriss model [4] and its variants).

There is also a large strand of literature concerning the portfolio optimization problems with Markov modulated price dynamics under partial information. [116] and [117] coonsder the case in which the drift uncertainty is modeled by a linear Gaussian process. [108] has studied the similar problem with a constant but unknown drift. [147] and [97] have treated the portfolio optimization problem in a multi-asset setting under partial information, and found the optimal portfolio strategy with martingale approach. On the other hand, [15] have addressed the portfolio optimization problem with unobservable Markov chain modulated drift process by using a dynamic programming approach. [22] considers a general setting and provides explicit representations of the optimal wealth and investment processes for the utility maximization problem under partial information by using the martingale approach. [84] solves the portfolio optimization problem under partial information by including expert opinions. Regarding portfolio optimization problems under partial information, one can finally refer to [140] giving a very broad overview of previous studies on the subject. For the full information case, there are also studies analyzing portfolio selection problems in a Markov regime-switching framework (see for example [162], [14], and [151]).

To summarize our contributions, firstly we solve the utility maximization problem for logarithmic and power utility preferences with indirect impact arising from controlling the intensity of the Markov chain both under full and partial information settings. For comparison purposes, we also give solutions to those problems without impact, that is, when there is no control of the intensity. Even for the simple logarithmic utility case, the presence of indirect impact makes point-wise maximization impossible and hence we need to rely on dynamic programming techniques. Secondly, we transform the partial information problem to a full information problem by using stochastic filtering and apply control theory for piecewise deterministic Markov processes (PDMP) to our problem to derive the optimality equation. We rely on the results given in [36] and characterize the value function as the unique viscosity solution of the associated dynamic programming equation. Thirdly, by focusing on a two-state Markov chain example, we show that there is always a gain for a large investor from controlling the intensity of the Markov chain both in full and partial settings albeit it is smaller in the latter one. In particular, the large investor can take advantage of the "bear" state of the market by short-selling. Also optimal strategies are more aggressive in the presence of market impact such that the large investor buys more in the "bull" state and short sells more in the "bear" state compared to the corresponding no-impact case. Also it is evident from numerical examples that, as time approaches to the maturity, optimal portfolio strategies with and without impact from intensity control converges to each other under both full and partial information settings.

1.1 Underlying framework

We consider a finite time interval $[0,T]$ and continuous trading in the market. We are given the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ satisfies the usual conditions; all processes we consider here are assumed to be $\mathbb{F}$-adapted.

We have an investor with a given initial wealth $w \in \mathbb{R}_{>0}$, and whose objective is to form a self-financing portfolio over the finite period $[0,T]$ in order to maximize the expected utility from terminal wealth by investing in a risky asset and in a risk-free bond. Let $h = \{h_t,\ t \in [0,T]\}$ be the $\mathbb{F}$-predictable process denoting the fraction of wealth invested in a risky asset. Then, $1 - h_t$ gives the fraction of the wealth invested in the bond at time $t \in [0,T]$. We allow for the short-selling of the risky asset and the risk-free bond. That is, $h_t \in \mathbb{R}$ for every $t \in [0,T]$. We work under the following assumption.

Assumption 1.1. *$h_t \in [-L, L]$, for some $L > 0$, for every $t \in [0,T]$.*

This assumption implies that controls take values in a compact space. In section 4 we will discuss an example where Assumption 1.1 is naturally satisfied. We denote by $Y^{(h)}$ a continuous-time finite-state Markov chain representing the state of the market. $Y^{(h)}$ takes values in the canonical state space $\mathcal{E} = \{e_1, e_2, ..., e_K\}$ where e_k is the k^{th} basis column vector of $\mathbb{R}^K$. The initial distribution of the Markov chain is given by $\pi_0 = (\pi_0^1, \cdots, \pi_0^K)$. The notation $Y^{(h)}$ stands for the fact that we assume that the action of the investor has an impact on the state of the market. Formally, we have that the infinitesimal generator of $Y^{(h)}$ is of the form $Q(h_t) = (q^{i,j}(h_t))_{i,j \in \{1,\ldots,K\}}$[3]. To keep the notation simple, in the following we restrict to the case with next-neighbour dynamics that is $q^{i,j}(\cdot) = 0$ if $|i - j| > 1$. However, results can be extended to the general case, see, e.g. Remark 2.16. This implies the following structure for the generator

$$Q(h_t) = \begin{pmatrix} -q^{1,2}(h_t) & q^{1,2}(h_t) & 0 & \ldots & 0 & 0 \\ q^{2,1}(h_t) & -q^{2,1}(h_t) - q^{2,3}(h_t) & q^{2,3}(h_t) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & q^{K,K-1}(h_t) & -q^{K,K-1}(h_t) \end{pmatrix} \quad (1.2)$$

[3]Note that the generator is well defined since h is assumed to be predictable.

where $q^{i,j} : [-L, L] \to \mathbb{R}_{\geq 0}$ is a nonnegative continuous function for $i \neq j$ and $i, j \in \{1, \ldots, K\}$.

We consider a risk-free bond and a risky asset as available instruments in the market, with price processes $B = \{B_t,\ t \geq 0\}$ and $S = \{S_t,\ t \geq 0\}$, respectively. The bond price is assumed to follow

$$dB_t = \rho B_t dt, \quad B_0 \in \mathbb{R}_{>0},$$

where $\rho > 0$ is the instantaneous risk-free rate. The risky asset price process has pure-jump dynamics that is affected from the state of the market. This gives a way to model risky asset price process with an indirect impact arising from the dependency of the switching intensity of the Markov chain on the trading strategy. Formally, it evolves according to the following equation

$$dS_t^{(h)} = S_{t^-}^{(h)} \int_{\mathcal{Z}} G(t, Y_{t^-}^{(h)}, \zeta) \mathcal{N}(dt, d\zeta), \quad S_0 \in \mathbb{R}_{>0}, \tag{1.3}$$

where $\mathcal{N}(dt, d\zeta)$ is a Poisson random measure on $\mathbb{R}_{\geq 0} \times \mathcal{Z}$, with $\mathcal{Z} \subseteq \mathbb{R}$, having an intensity $\varsigma(d\zeta)dt$ independent of the Markov chain $Y^{(h)}$ such that $\varsigma(\mathcal{Z}) < \infty$ and $G : [0, T] \times \mathcal{E} \times \mathcal{Z} \to \mathbb{R}$ is a measurable function, continuous in time and satisfying

$$\mathbb{E}\left[\int_0^T \int_{\mathcal{Z}} G^2(t, Y_{t^-}^{(h)}, \zeta) \varsigma(d\zeta) dt\right] < \infty.$$

In order to ensure the non-negativity of the stock price process $S^{(h)}$ we further assume that $1 + G(t, e_i, \zeta) > 0$ for every $(t, \zeta) \in [0, T] \times \mathcal{Z}$ and $i \in \{1, \ldots, K\}$ and moreover we assume that equation (1.3) has a unique solution. A set of sufficient conditions for uniqueness of the solution is given, for example, in [136, Theorem 1.19].

Let $R^{(h)} := \{R_t^{(h)},\ t \in [0, T]\}$ be the return process,

$$dR_t^{(h)} = \int_{\mathcal{Z}} G(t, Y_{t^-}^{(h)}, \zeta) \mathcal{N}(dt, d\zeta),$$

and introduce the random measure $\mu(dt, dz)$ associated to its jumps

$$\mu(dt, dz) := \sum_{s : \Delta R_s^{(h)} \neq 0} \mathbf{1}_{\{s, \Delta R_s^{(h)}\}}(dt, dz). \tag{1.4}$$

Then the following equality holds

$$R_t^{(h)} = \int_0^t \int_{\mathcal{Z}} G(t, Y_{t^-}^{(h)}, \zeta)\ \mathcal{N}(dt, d\zeta) = \int_0^t \int_{\mathbb{R}} z\ \mu(dt, dz),$$

for every $t \in [0, T]$. For every $A \in \mathcal{B}(\mathbb{R})$, we denote

$$\eta^{\mathbf{P}}(t, Y_{t^-}^{(h)}, A) = \varsigma(D_t^A),$$

where $D_t^A := \{\zeta \in \mathcal{Z} :\ G(t, Y_{t^-}^{(h)}, \zeta) \in A \setminus \{0\}\}$. Then $\eta^{\mathbf{P}}(t, Y_{t^-}^{(h)}, dz)dt$ provides the $(\mathbb{F}, \mathbf{P})$-dual predictable projection of the measure μ see, e.g. [27, Chapter 8]. Assumptions on G and ς imply that $\eta^{\mathbf{P}}(t, e_i, z)$ is continuous in time,

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} z^2 \eta^{\mathbf{P}}(t, Y_{t^-}^{(h)}, dz) dt\right] < \infty, \tag{1.5}$$

the counting process associated to the jumps of R is $\mathbf{P}$-nonexplosive and R has a finite mean and variance. Let $W^{(h)} = \{W_t^{(h)},\ t \in [0,T]\}$ be the wealth process corresponding to a self-financing strategy $h = \{h_t, t \in [0,T]\}$. The dynamics of $W^{(h)}$ is given by

$$dW_t^{(h)} = W_{t^-}^{(h)} \left((1-h_t)\rho dt + h_t \int_{\mathbb{R}} z\mu(dt,dz) \right), \quad W_0 \in \mathbb{R}_{>0}. \tag{1.6}$$

In order to ensure that the wealth process is positive we make the assumption

Assumption 1.7. $1 + h_t z > 0$ $\mathbf{P}$-*a.s. for every* $(t,z) \in [0,T] \times supp(\eta^{\mathbf{P}})$*, where* $supp(\eta^{\mathbf{P}}) \subset \mathbb{R}$ *is the support of the measure* $\eta^{\mathbf{P}}(t,e,dz)$*.*

This implies we can write the solution for (1.6) as,

$$\begin{aligned} W_t^{(h)} = W_0 \exp\Bigg\{ \int_0^t \left((1-h_s)\rho + \int_{\mathbb{R}} \log(1+h_s z)\eta^{\mathbf{P}}(s, Y_{s^-}^{(h)} dz) \right) ds \\ + \int_0^t \int_{\mathbb{R}} \log(1+h_s z)\nu(ds,dz) \Bigg\}, \end{aligned}$$

for every $t \in [0,T]$, where

$$\nu(dt,dz) := \mu(dt,dz) - \eta^{\mathbf{P}}(t, Y_{t^-}^{(h)}, dz)dt \tag{1.8}$$

indicates the compensated jump measure associated with μ.
In the following, we always work under the *standing assumptions* made in Section 1.1.

1.2 Existence of the model

There is a non-trivial dependence between the stock price process $S^{(h)}$ given in (1.3) and Markov chain $Y^{(h)}$ with the generator (1.2). Hence it is not a priori clear that the pair $(Y^{(h)}, S^{(h)})$ exists. A rigorous construction of such a setting under full information is provided in [28, Section 2] by using a change of measure argument. In the case of partial information there is an additional type of circularity through the observation filtration. Precisely, the stock price process is affected by the decision of the trader which is adapted to the filtration generated by the price process itself. In the following part to tackle with this problem we also use a change of measure argument and provide the existence of the model in the case of partial information.

We start with a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, that supports a Markov chain $Y^{(h)}$ with generator $Q(h_t)$ of the form (1.2) and a random measure μ as in (1.4) associated with the jumps of the return process. We assume that μ has $(\mathbb{F}, \mathbf{Q})$-dual predictable projection $\eta^{\mathbf{Q}}(t,dz)dt$, independent of the Markov chain, that is for every $\mathbb{F}$-predictable process $H(\cdot, z) = \{H(t,z), t \in [0,T]\}$, indexed by z, we have that

$$\int_0^t \int_{\mathbb{R}} H(s,z)(\mu(ds,dz) - \eta^{\mathbf{Q}}(s,dz)ds), \quad t \in [0,T]$$

is an $(\mathbb{F}, \mathbf{Q})$-martingale. Using the typical terminology from filtering we refer to $\mathbf{Q}$ as the *reference probability.* Then we define $\mathbb{F}^S := \{\mathcal{F}_t^S, t \in [0,T]\}$ as the right-continuous filtration generated by the stock price process, augmented with $\mathbf{Q}$-null sets. Note that, since the compensator $\eta^{\mathbf{Q}}$ does not depend on the trading strategy h, so does $\mathbb{F}^S$. Thus the filtration is exogeneously defined. We assume that for every $(t, e) \in [0, T] \times \mathcal{E}$, the measure $\eta^{\mathbf{Q}}(t, dz)$ is equivalent to $\eta^{\mathbf{P}}(t, e, dz)$, with Radon-Nikodym derivative

$$\Psi(t, e, z) + 1 := \frac{d\eta^{\mathbf{P}}(t, e, z)}{d\eta^{\mathbf{Q}}(t, z)}, \tag{1.9}$$

that satisfies

$$\mathbb{E}^{\mathbf{Q}}\left[\exp \int_0^T \int_{\mathbb{R}} \Psi^2(t, Y_{t^-}^{(h)}, z)\eta^{\mathbf{Q}}(t, dz)dt\right] < \infty, \tag{1.10}$$

where $\mathbb{E}^{\mathbf{Q}}[\cdot]$ indicates expectation under the measure $\mathbf{Q}$. For every strategy h, we now define the process $Z^{(h)}$ by

$$Z_t^{(h)} = 1 + \int_0^t \int_{\mathbb{R}} Z_{s^-}^{(h)} \Psi(s, Y_{s^-}^{(h)}, z)(\mu(ds, dz) - \eta^{\mathbf{Q}}(s, dz)ds), \quad t \in [0, T]. \tag{1.11}$$

Condition (1.10) guarantees that $Z^{(h)}$ is an $(\mathbb{F}, \mathbf{Q})$-martingale with $\mathbb{E}^{\mathbf{Q}}[Z_T^{(h)}] = 1$, see, e.g. [142, Theorem 9]. Then we set

$$\left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathcal{F}_t} = Z_t^{(h)}, \quad t \in [0, T].$$

Similarly to [36, Lemma 3.1], it can be proved that $\mathbf{P}$ and $\mathbf{Q}$ are equivalent and that for every admissible strategy h, a pair $(Y^{(h)}, S^{(h)})$ exists and it is unique in law. Note that the measure $\mathbf{P}$ depends on h, and that for any choice of h the resulting measures are equivalent. Therefore we suppress the dependence.

Chapter 2

Optimization Problem under Full Information

In this part we assume that the investor has the full knowledge of the market. Formally this means that the available information is given by the filtration $\mathbb{F}$. This leads to the following definition of admissible strategies.

Definition 2.1. A portfolio strategy h is $\mathbb{F}$-admissible if it is $\mathbb{F}$-predictable and Assumptions 1.1 and 1.7 hold. We denote the set of $\mathbb{F}$-admissible strategies by $\mathcal{H}$.

Suppose we are given a strictly increasing, strictly concave and continuously differentiable utility function $U : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying Inada conditions, i.e. $\lim_{w\to 0} \frac{\partial U}{\partial w}(w) = \infty$ and $\lim_{w\to\infty} \frac{\partial U}{\partial w}(w) = 0$. The goal of the investor is to solve the following optimization problem

$$\max \mathbb{E}^{t,w,i}\left[U(W_T^{(h)})\right], \tag{2.2}$$

over all admissible strategies, subject to the initial value of the wealth $W_t^{(h)} = w$ and initial state $Y_t^{(h)} = e_i$ for some $i \in \{1, \dots, K\}$.
The value function for the current optimization problem is

$$V(t, w, e_i) = \sup_{h\in\mathcal{H}} \mathbb{E}^{t,w,i}[U(W_T^{(h)})].$$

If V is continuous and differentiable with respect to the first two arguments, i.e. $V \in \mathcal{C}_b^{1,1}([0,T] \times \mathbb{R}_{>0} \times \mathcal{E})$, then it can be characterized as the unique classical solution of the HJB equation given by

$$0 = \sup_{h\in[-L,L]} \left\{\mathcal{L}^{(h)} V(t, w, e_i)\right\},$$

with $\mathcal{L}^{(h)}$ being the $(\mathbb{F}, \mathbf{P})$-Markov generator of the pair $(W^{(h)}, Y^{(h)})$. Explicitly, we have

$$\begin{aligned} 0 = \sup_{h\in[-L,L]} \Bigg\{ &\frac{\partial V}{\partial t}(t, w, e_i) + \frac{\partial V}{\partial w}(t, w, e_i) w(1 - h)\rho \\ &+ \sum_{j=1}^{K} (V(t, w, e_j) - V(t, w, e_i)) q^{i,j}(h) \\ &+ \int_{\mathbb{R}} \left[V\left(t, w(1 + hz), e_i\right) - V(t, w, e_i)\right] \eta^{\mathbf{P}}(t, e_i, dz) \Bigg\}, \end{aligned} \tag{2.3}$$

with the final condition $V(T, w, e_i) = U(w)$, for every $w \in \mathbb{R}_{>0}$ and $i \in \{1, \dots, K\}$. A similar problem including a diffusion part under full information has been studied in [28] in detail. In the sequel, for comparison purposes we provide results for the logarithmic and power utility. For the verification result we refer to [28, Theorem 3.1].

2.1 Logarithmic utility

We consider the portfolio optimization problem for a large investor with logarithmic utility preference. That is, we have $U(w) = \log(w)$. For comparison purposes, we first study the degenerate case, where the generator of the Markov chain does not depend on the actions of the investor. This is the case where the investor has no market impact. Then we move to our primary interest, the case with market impact. Normally the logarithmic utility is the simplest case and can be solved by pointwise maximization. However, with the inclusion of the market impact this is not possible anymore since the current actions of the investor have an influence on the future states of the market and therefore may change the jump intensity of the asset price process.

Logarithmic utility - No market impact

To begin with we provide a characterization of the optimal strategy and a stochastic representation for the value function in the setting where the intensity of the Markov chain does not depend on the portfolio strategy, that is, $q^{i,j}(h) \equiv q^{i,j}$, for $i, j \in \{1, \dots, K\}$ and every control h. In this case the optimal control problem can be solved directly. First note that by applying the Itô's formula we get

$$V(t, w, e_i) = \log(w) + \sup_{h \in \mathcal{H}} \widetilde{\beta}(t, e_i; h),$$

where

$$\widetilde{\beta}(t, e_i; h) = \mathbb{E}^{t,i}\left[\int_t^T \left((1 - h_s)\rho + \int_R \log(1 + h_s z)\eta^{\mathbf{P}}(s, Y_{s-}, dz)\right) ds \right.$$
$$\left. + \int_t^T \int_{\mathbb{R}} \log(1 + h_s z)\nu(ds, dz)\right].$$

Proposition 2.4. *Suppose $U(w) = \log(w)$ for $w > 0$.*

i) Let $h^*(t, e_i)$ satisfy either

$$\int_{\mathbb{R}} \frac{z}{1 + h^*(t, e_i)z} \eta^{\mathbf{P}}(t, e_i, dz) = \rho \tag{2.5}$$

or $h^*(t, e_i) \in \{-L, L\}$, for every $i \in \{1, \dots, K\}$. Then the optimal strategy $h_t^* = h^*(t, e_i)$ for every $t \in [0, T]$ and $i \in \{1, \dots, K\}$.

ii) The value function is of the form

$$V(t, w, e_i) = \log(w) + \mathbb{E}^{t,i}\left[\int_t^T \left((1 - h_s^*)\rho + \int_R \log(1 + h_s^* z)\eta^{\mathbf{P}}(s, Y_{s-}, dz)\right) ds\right].$$

Proof. The result follows by computing directly $\log(W_T^{(h)})$, taking expectation and maximizing pointwisely. □

We will see that for the case study of Section 4, equation (2.5) always admits a solution $h^*(t, e_i) \in (-L, L)$. An extensive study of the utility maximization with logarithmic preferences in the classical case (i.e. without market impact) is given by [89], where the optimal strategy is characterized in terms of the local characteristics (drift, volatility and jump intensity) of the semimartingale driving the asset price process (see, e.g., [89, Theorem 3.1]).

Logarithmic utility - Market impact

In the case with market impact the above procedure does not apply. This is due to the fact that at any point in time the decision of the investor may change the future state of the Markov chain. Therefore here we address the problem via dynamic programming. Precisely, we study the solution to equation (2.3) with the terminal condition $V(T, w, e_i) = \log(w)$. We consider the ansatz $V(t, w, e_i) = \log(w) + \beta(t, e_i)$. Then we have the following system of equations for (t, e_i)

$$-\frac{\partial \beta}{\partial t}(t, e_i) = \sup_{h \in [-L,L]} \Big\{(1-h)\rho + (\beta(t, e_{i+1}) - \beta(t, e_i))q^{i,i+1}(h) \tag{2.6}$$

$$+ (\beta(t, e_{i-1}) - \beta(t, e_i))q^{i,i-1}(h) + \int_{\mathbb{R}} \log(1+hz)\eta^{\mathbf{P}}(t, e_i, dz)\Big\}. \tag{2.7}$$

for every $t \in [0, T]$ and $i \in \{2, \dots, K-1\}$, and

$$\frac{d\beta}{dt}(t, e_1) = - \sup_{h \in [-L,L]} \Big\{(1-h)\rho + (\beta(t, e_2) - \beta(t, e_1))\, q^{1,2}(h) \tag{2.8}$$

$$+ \int_{\mathbb{R}} \log(1+hz)\eta^{\mathbf{P}}(t, e_1, dz)\Big\}, \tag{2.9}$$

$$\frac{d\beta}{dt}(t, e_K) = - \sup_{h \in [-L,L]} \Big\{(1-h)\rho + (\beta(t, e_{K-1}) - \beta(t, e_K))\, q^{K,K-1}(h) \tag{2.10}$$

$$+ \int_{\mathbb{R}} \log(1+hz)\eta^{\mathbf{P}}(t, e_K, dz)\Big\}, \tag{2.11}$$

respectively, with boundary conditions $\beta(T, e_i) = 0$ for $i \in \{1, \dots, K\}$. Equations (2.7),(2.9) and (2.11) imply that given an optimizer h^*, $\beta(t, e_i)$, $i \in \{1, \dots, K\}$, is the unique solution of this system of ordinary differential equations (ODEs). This follows from the continuity of the coefficients [155, Theorem 3.9]. In principle, one can solve the system numerically using, for instance, backward Euler method. In particular, as pointed out in [28], at each time step t_n of the numerical procedure one should find the maximizer $h^*(t_n)$, and then solve the resulting ODE.

From (1.5) and the boundedness of $q^{i,j}(h)$ the verification result in [28, Theorem 3.1] applies.

2.2 Power utility

In this part we work under the assumption of power utility, that is, $U(w) = \frac{1}{\theta} w^\theta$, $\theta < 1$, $\theta \neq 0$. We address the corresponding optimization problem by dynamic programming technique.

In what follows we investigate the solution to the equation (2.3) with the terminal condition $V(T, w, e_i) = \frac{w^\theta}{\theta}$. To this, we suggest the following ansatz for the value function

$$V(t, w, e_i) = \frac{w^\theta}{\theta} e^{\theta\gamma(t, e_i)}, \quad i \in \{1, \dots, K\}. \tag{2.12}$$

Inserting (2.12) into (2.3) leads to equations

$$\begin{aligned}\frac{d\gamma}{dt}(t, e_i) = &- \sup_{h \in [-L, L]} \left\{ (1-h)\rho + \frac{1}{\theta}\left(e^{\theta(\gamma(t, e_{i-1}) - \gamma(t, e_i))} - 1\right) q^{i,i-1}(h) \right. \\ &\left. + \frac{1}{\theta}\left(e^{\theta(\gamma(t, e_{i+1}) - \gamma(t, e_i))} - 1\right) q^{i,i+1}(h) + \frac{1}{\theta}\int_{\mathbb{R}} \left((1+hz)^\theta - 1\right) \eta^{\mathbf{P}}(t, e_i, dz) \right\}\end{aligned} \tag{2.13}$$

for every $t \in [0, T]$ and $i \in \{2, \dots, K-1\}$, and

$$\begin{aligned}\frac{d\gamma}{dt}(t, e_1) = &- \sup_{h \in [-L, L]} \left\{ (1-h)\rho + \frac{1}{\theta}\left(e^{\theta(\gamma(t, e_2) - \gamma(t, e_1))} - 1\right) q^{1,2}(h) \right. \\ &\left. + \frac{1}{\theta}\int_{\mathbb{R}} \left((1+hz)^\theta - 1\right) \eta^{\mathbf{P}}(t, e_1, dz) \right\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\frac{d\gamma}{dt}(t, e_K) = &- \sup_{h \in [-L, L]} \left\{ (1-h)\rho + \frac{1}{\theta}\left(e^{\theta(\gamma(t, e_{K-1}) - \gamma(t, e_K))} - 1\right) q^{K,K-1}(h) \right. \\ &\left. + \frac{1}{\theta}\int_{\mathbb{R}} \left((1+hz)^\theta - 1\right) \eta^{\mathbf{P}}(t, e_K, dz) \right\},\end{aligned} \tag{2.15}$$

respectively, with final conditions $\gamma(T, e_i) = 0$ for $i \in \{1, \dots, K\}$. Given an optimizer h^*, $\gamma(t, e_i)$, every $i \in \{1, \dots, K\}$, is the unique solution of the system of first order ODEs given by equations (2.13),(2.14) and (2.15). Note that a simple transformation, i.e., $F(t, e_i) = e^{\theta\gamma(t, e_i)}$, yields to a system of linear ODEs. One can follow the same procedure as in the case of logarithmic utility and solve the system numerically.

Moreover by boundedness of $q^{i,j}(h)$ and condition (1.5), the verification result in [28, Theorem 3.1] applies.

Remark 2.16. Suppose that matrix Q has a general form, that is entries $q^{i,j}(h)$ are not necessarily null for $|i-j| > 1$. Then, with the same procedure we get that the value function is of the form (2.12) where functions $\gamma(t, e_1)$ solve the system

$$\begin{aligned}\frac{d\gamma}{dt}(t, e_i) = &- \sup_{h \in [-L, L]} \left\{ (1-h)\rho + \frac{1}{\theta}\sum_{j=1}^{K}\left(e^{\theta(\gamma(t, e_j) - \gamma(t, e_i))} - 1\right) q^{i,j}(h) \right. \\ &\left. + \frac{1}{\theta}\int_{\mathbb{R}} \left((1+hz)^\theta - 1\right) \eta^{\mathbf{P}}(t, e_i, dz) \right\}\end{aligned} \tag{2.17}$$

for every $t \in [0, T]$ and $i \in \{1, \dots, K\}$.

Chapter 3

Optimization Problem under Partial Information

In the current section we assume that the state process Y is not directly observable by the investor. Instead, she observes the price process S and knows the model parameters. Hence, the available information is represented by the natural filtration generated by the risky asset price process, $\mathbb{F}^S$.

At any time $t \in [0, T]$ the decision of the investor depends only on the available information. Accordingly, we define the set of admissible strategies as follows.

Definition 3.1. A portfolio strategy h is $\mathbb{F}^S$-admissible if it is $\mathbb{F}^S$-predictable and assumptions 1.1 and 1.7 hold. We denote the set of $\mathbb{F}^S$-admissible strategies by $\widetilde{\mathcal{H}}$.

Considering $\mathbb{F}^S$-predictable investment strategies results in an optimal control problem under partial information. In a Markovian setting as the one outlined here we can reduce the control problem under partial information to an equivalent control problem under full information where the unobservable state variable, namely the Markov chain $Y^{(h)}$, is replaced by the filtered estimates, see, for example, [16, 31]. This requires to solve a filtering problem where the unobservable signal is given by the Markov chain $Y^{(h)}$ and the observation process is the pure jump process $S^{(h)}$. The literature on filtering problem with pure jump process observation is relatively large. A brief list of results includes for instance [27, 32, 62, 85, 30]. In the upcoming part, we deal with the filtering problem corresponding to our setting by using the so called *reference probability approach*. The idea is to consider a probability measure equivalent to $\mathbf{P}$ under which, the dynamics of the observation process $S^{(h)}$ and the unobservable Markov chain $Y^{(h)}$ become independent, and then determine the dynamics of the *unnormalized filter*. The construction of the model provided in section 1.2 offers a natural candidate to be the reference probability. This approach has also been used in [36] to break the circularity of information, due to a direct dependence of the jump intensity of the stock price on the control. In the current setting the circularity arises from an indirect dependence on the strategy h.

3.1 Filtering and reduction to full information

For every function $f : \mathcal{E} \to \mathbb{R}$ and every control h we define the filter $\pi^{(h)}(f) := \{\pi_t^{(h)}(f), t \in [0, T]\}$ by

$$\pi_t^{(h)}(f) = \mathbb{E}\left[f(Y_t^{(h)})|\mathcal{F}_t^S\right], \quad t \in [0, T],$$

and denote by $\pi_{t-}^{(h)}(f)$ its predictable version.

By the Kallinapur-Striebel formula, we get that

$$\pi_t^{(h)}(f) = \frac{\mathbb{E}^{\mathbf{Q}}\left[Z_t f(Y_t^{(h)})|\mathcal{F}_t^S\right]}{\mathbb{E}^{\mathbf{Q}}\left[Z_t|\mathcal{F}_t^S\right]}, \quad t \in [0,T]. \tag{3.2}$$

Denote by $p_t^{(h)}(f) := \mathbb{E}^{\mathbf{Q}}\left[Z_t f(Y_t^{(h)})|\mathcal{F}_t^S\right]$ for every $t \in [0,T]$. The process $p^{(h)}(f) = \{p_t^{(h)}(f), t \in [0,T]\}$ is called the unnormalized filter.

In the sequel we compute the dynamics of the process $p^{(h)}(f)$, and then derive the filtering equation by applying (3.2). The main results are stated in Proposition 3.3 and Proposition 3.7.

Proposition 3.3 (The Zakai equation)**.** *Let $f : \mathcal{E} \to \mathbb{R}$. Then, for every $t \in [0,T]$, the unnormalized filter solves the equation*

$$p_t^{(h)}(f) = \pi_0(f) + \int_0^t p_s^{(h)}(Qf)ds + \int_0^t \int_{\mathbb{R}} p_{s^-}^{(h)}\left(\Psi(z)f\right)\left(\mu(ds,dz) - \eta^{\mathbf{Q}}(s,dz)ds\right), \tag{3.4}$$

where

$$p_{t^-}^{(h)}\left(\Psi(z)f\right) := \mathbb{E}^{\mathbf{Q}}\left[f(Y_{t-})Z_{t-}\Psi(t, Y_{t^-}^{(h)}, \nu_{t-}, z) \mid \mathcal{F}_t^S\right],$$
$$p_t^{(h)}(Qf) := \mathbb{E}^{\mathbf{Q}}\left[Z_t\langle Q\mathbf{f}, Y_t^{(h)}\rangle \mid \mathcal{F}_t^S\right],$$

and $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors on $\mathbb{R}^K$.

The proof of this result is provided in Appendix 5.1.

Process $\pi^{(h)}$ is infinite dimensional, in general. However, when the unobservable signal is a Markov chain with finite state space we have that

$$\pi_t^{(h)}(f) = \sum_{j=1}^K f(e_j)\pi_t^j, \quad t \in [0,T],$$

where for every $i \in \{1, \dots, K\}$ and every fixed strategy h,

$$\pi_t^i := \mathbb{E}\left[\mathbf{1}_{\{Y_t^{(h)} = e_i\}}|\mathcal{F}_t^S\right], \quad t \in [0,T],$$

denote the conditional state probability of the Markov chain $Y^{(h)}$. Here for the ease of notation we suppress the dependence of processes π^i, for $i \in \{1, \dots, K\}$ on the trading strategy h. Then, clearly, conditional state probabilities π^i, for $i \in \{1, \dots, K\}$ fully characterize the filter, which becomes finite dimensional.

In order to obtain dynamics of π^i, for $i \in \{1, \dots, K\}$, we introduce the following notation

$$\pi_{t^-}^{(h)}(\eta^{\mathbf{P}}(dz))dt := \sum_{i=1}^K \pi_{t-}^i \eta^{\mathbf{P}}(t, e_i, dz)dt. \tag{3.5}$$

It is not difficult to show that for every nonnegative $(\mathbb{F}^S, \mathbf{P})$-predictable process indexed by z, $\Phi := \{\Phi(t,z),\ t \in [0,T]\}$ such that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\Phi(s,z)|\pi_{t^-}^{(h)}(\eta^{\mathbf{P}}(dz))dt\right] < \infty,$$

the following holds (see, [50, V T28])

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \Phi(t,z)\mu(dt,dz)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \Phi(t,z) \sum_{i=1}^K \pi^i_{t-} \eta^{\mathbf{P}}(t,e_i,dz)dt\right],$$

which implies that (3.5) provides the $(\mathbb{F}^S, \mathbf{P})$-dual predictable projection of the measure μ and that the process $\int_0^t \int_{\mathbb{R}} \Phi(s,z) \left(\mu(ds,dz) - \pi^{(h)}_{s^-}(\eta^{\mathbf{P}}(dz))ds\right)$, $t \in [0,T]$, is an $(\mathbb{F}^S, \mathbf{P})$-martingale.

Let $\nu^\pi(dt,dz)$ denote the $(\mathbb{F}^S, \mathbf{P})$-compensated measure, that is

$$\nu^\pi(dt,dz) := \mu(dt,dz) - \pi^{(h)}_{t^-}(\eta^{\mathbf{P}}(dz))dt. \tag{3.6}$$

Proposition 3.7. *The process π^i, for all $i \in \{1,\dots,K\}$ solves the equation*

$$d\pi^i_t = \sum_{j=1}^K q^{j,i}(h_t)\pi^j_t dt + \int_{\mathbb{R}} \pi^i_{t^-} u^i(t, \pi^{(h)}_{t^-}, z)\nu^\pi(dt,dz), \tag{3.8}$$

where $u^i(t,\pi^{(h)}_t, z) := \dfrac{1}{\sum_{j=1}^K \pi^j_t \frac{d\eta^{\mathbf{P}}(t,e_j,z)}{d\eta^{\mathbf{P}}(t,e_i,z)}} - 1$ and $\dfrac{d\eta^{\mathbf{P}}(t,e_j,z)}{d\eta^{\mathbf{P}}(t,e_i,z)}$ denotes the Radon-Nikodym derivative of the measure $\eta^{\mathbf{P}}(t,e_j,dz)$ with respect to $\eta^{\mathbf{P}}(t,e_i,dz)$ and $\pi^{(h)}$ is the vector process $(\pi^1,\dots,\pi^K)$ which takes values on the $(K-1)$-dimensional simplex Δ_K.

Proof. First, note that $p^{(h)}(1)$ satisfies

$$p^{(h)}_t(1) = 1 + \int_0^t \int_{\mathbb{R}} p^{(h)}_{s^-}(\Psi(z)))(\mu(ds,dz) - \eta^{\mathbf{Q}}(s,dz)ds), \quad t \in [0,T].$$

By applying Kallianpur-Striebel formula (3.2) we get that

$$d\pi^{(h)}_t(f) = \pi^{(h)}_t(Qf)dt + \int_{\mathbb{R}} \left(\frac{\pi^{(h)}_{t^-}(f\Psi(z))}{\pi^{(h)}_{t^-}(\Psi(z)} - \pi^{(h)}_{t^-}(f)\right)(\mu(dt,dz) - \pi^{(h)}_{t^-}(\eta^{\mathbf{P}}(dz))dt),$$

where $\pi^{(h)}_t(Qf) = \mathbb{E}\left[\langle Q\mathbf{f}, Y^{(h)}_t\rangle \mid \mathcal{F}^S_t\right]$. Finally, the result follows by choosing $f(y) = \mathbf{1}_{\{y=e_i\}}$. ☐

Uniqueness of the solution of the filtering equation is necessary to transform the optimal control problem stated in (3.10) into an equivalent one involving only observable processes. Therefore, in the rest of this section we assume that the system of equations (3.8) has a unique solution.

Remark 3.9. A sufficient condition for the uniqueness of the solution of the system of equations (3.8) is, for example,

$$\sup_{t\in[0,T]} \eta^{\mathbf{P}}(t,e_i,\mathbb{R}) < \infty,$$

for every $i \in \{1,\dots,K\}$, see, e.g., [30]. This is satisfied in our model since ς is a finite measure.

Note that the asset price process $S^{(h)}$ as well as the wealth process $W^{(h)}$ have a representation with respect to investor's information, given by

$$
\begin{aligned}
S_t^{(h)} =& S_0 + \int_0^t \sum_{i=1}^K \int_{\mathbb{R}} z S_s^{(h)} \pi_s^i \eta^{\mathbf{P}}(s, e_i, dz) ds + + \int_0^t \int_{\mathbb{R}} z S_s^{(h)} \nu^{\pi}(dt, dz), \\
W_t^{(h)} =& W_0 + \int_0^t W_s^{(h)} \left((1-h_s)\rho + \sum_{i=1}^K \int_{\mathbb{R}} z \eta^{\mathbf{P}}(s, e_i, dz) \right) dt + W_{s^-}^{(h)} h_s \int_{\mathbb{R}} z \nu^{\pi}(dt, dz),
\end{aligned}
$$

for every $t \in [0, T]$. In the partial information framework we can write the objective of the investor as

$$
\max \mathbb{E}^{t,w,\boldsymbol{\pi}} \left[U(W_T^{(h)}) \right], \tag{3.10}
$$

over the set of $\mathbb{F}^S$-admissible controls, where $\mathbb{E}^{t,w,\boldsymbol{\pi}}$ denotes the conditional expectation given $W_t^{(h)} = w$ and $\pi_t^{(h)} = \boldsymbol{\pi}$ for $(w, \boldsymbol{\pi}) \in \mathbb{R}_{>0} \times \Delta_K$. The control problem is characterized by the $(K+1)$-dimensional state process $(W^{(h)}, \pi^{(h)})$. We define the reward and the value functions as

$$
\begin{aligned}
J(t, w, \boldsymbol{\pi}; h) &= \mathbb{E}^{t,w,\boldsymbol{\pi}} \left[U(W_T^{(h)}) \right], \\
V(t, w, \boldsymbol{\pi}) &= \sup_{h \in \widetilde{\mathcal{H}}} J(t, w, \boldsymbol{\pi}; h).
\end{aligned}
$$

3.2 Solution via piecewise deterministic Markov processes approach

The state process of the optimization problem, consisting of the wealth process and the filter, augmented by the time variable, is a piecewise deterministic Markov process (PDMP), see [46]. A PDMP is a combination of a deterministic flow, characterized as the solution of an ordinary differential equation, and random jumps.

To identify the proper structure of the problem and the appropriate conditions to apply the theory of control for PDMP, we start by introducing some notation. Let $\mathcal{X} = \mathbb{R}_{>0} \times \Delta_K$ be the state space and $\widetilde{\mathcal{X}} = [0, T] \times \mathbb{R}_{>0} \times \Delta_K$ be the augmented one and denote the state process and the augmented state process by $X^{(h)} := (W^{(h)}, \pi^{(h)})$ and $\widetilde{X}^{(h)} := (t, W^{(h)}, \pi^{(h)})$, respectively. Denote by $\{T_n\}_{n \in \mathbb{N}}$ the sequence of jump times of the state process $\widetilde{X}^{(h)}$, with $T_0 = 0$. Then between two consecutive jump times before time T, i.e. $t \in [T_n \wedge T, T_{n+1} \wedge T)$, the state process $\widetilde{X}^{(h)}$ is described by the ODE $d\widetilde{X}_t^{(h)} = g(\widetilde{X}_t^{(h)}, h_t) dt$, where the vector field $g : \widetilde{\mathcal{X}} \times [-L, L] \to \mathbb{R}$ is given by

$$
\begin{aligned}
g^{(1)}(\widetilde{x}, h) = 1, \quad g^{(2)}(\widetilde{x}, h) = w(1-h)\rho, \\
g^{(i+2)}(\widetilde{x}, h) = \sum_{j=1}^K \pi^j \left(q^{j,i}(h) + \int_{\mathbb{R}} \pi^i u^i(t, \boldsymbol{\pi}, z) \eta^{\mathbf{P}}(t, e_j, dz) \right), \quad i \in \{1, \dots, K\}.
\end{aligned}
$$

The jump rate of the state process is given by $\lambda(\widetilde{x})$, $\widetilde{x} \in \widetilde{\mathcal{X}}$ where

$$
\lambda(\widetilde{x}) = \lambda(t, w, \boldsymbol{\pi}) = \sum_{i=1}^K \pi^i \eta^{\mathbf{P}}(t, e_i, \mathbb{R}),
$$

and it is independent of w. According to [46], the transition kernel that governs the jumps of the state process is described by the operator $Q_{\widetilde{X}} : \widetilde{\mathcal{X}} \times [-L, L] \to \widetilde{\mathcal{X}}$ such that for any continuous and bounded function f defined on $\widetilde{\mathcal{X}}$ we have

$$\begin{aligned} Q_{\widetilde{X}} f(\widetilde{x}, h) &:= \int_{\widetilde{\mathcal{X}}} f(\widetilde{y}) Q_{\widetilde{X}}(d\widetilde{y} \mid \widetilde{x}, h) \\ &= \lambda(\widetilde{x}) \sum_{j=1}^{K} \pi^j \int_{\mathbb{R}} f\left(t, w(1+hz), \pi^1(1+u^1(t, \boldsymbol{\pi}, z)), \dots, \pi^K(1+u^K(t, \boldsymbol{\pi}, z))\right) \eta^{\mathbf{P}}(t, e_j, dz) . \end{aligned}$$

The state process of the optimal liquidation problem is a PDMP with characteristics given by the vector field g, the jump rate λ and the transition kernel $Q_{\widetilde{X}}$. It is standard in control theory for PDMPs to work with so-called open-loop controls.

Denote by $\mathcal{A}$ the set of measurable mappings $\alpha : [0, T] \to [-L, L]$ and define an *admissible open loop portfolio strategy* as a sequence of measurable functions $\{h^n\}_{n \geq 0} : \widetilde{\mathcal{X}} \to \mathcal{A}$, such that the portfolio weight at time t is given by

$$h_t = h^n\left(t - T_n, I_n\right), \quad \text{for} \quad t \in (T_n \wedge T, T_{n+1} \wedge T], \tag{3.11}$$

where $I_n := X^{(h)}_{T_n}$ denotes the post jump state. Since at any jump time the evolution of the state process is known up to the next jump time, the idea is that an optimal investment strategy consists of a sequence of choices h^n taken at each jump time $T_n < T$ and to be followed up to $T_{n+1} \wedge T$. Note that jump times and the post jump state depend on the strategy via the intensity of jump arrivals. However there is no ambiguity or ill-defined notion, since the pair (T_n, I_n) depends on the decision h^{n-1}, made at time T_{n-1}. Although in the most general form of admissible strategies h^n should depend on the whole past history, it is possible to show that this larger class of policies does not increase the value of the control problem,, see, e.g. [18, Section 2.2]. Therefore in the sequel we consider admissible strategies of the form (3.11).

From now on, we use the notation without emphasizing the dependence on the admissible strategy h, that is, for example we write $\widetilde{X}$ instead of $\widetilde{X}^{(h)}$. Given an admissible strategy $\{h^n\}_{n \geq 0}$ and an initial state $x \in \mathcal{X}$, we denote by $\mathbf{P}^{\{h^n\}}_{(t,x)}$ (equiv. $\mathbf{P}^{\{h^n\}}_{\widetilde{x}}$) the law of the state process provided that $X_t = x$ and that the investor uses the strategy $\{h^n\}_{n \in \mathbb{N}}$. The reward function associated to an admissible strategy $\{h^n\}_{n \in \mathbb{N}}$ is given by

$$J\left(t, x, \{h^n\}\right) = \mathbb{E}^{\{h^n\}}_{(t,x)}\left[U(W_T)\right],$$

and the value function of the optimization problem under partial information is

$$V(t, x) = V(\widetilde{x}) = \sup\left\{J\left(t, x, \{h^n\}\right) : \{h^n\}_{n \in \mathbb{N}} \text{ admissible strategy}\right\} . \tag{3.12}$$

The corresponding Markov decision model

The optimization problem in (3.12) can be reduced to an optimization problem in an infinite horizon Markov decision model (MDM). Here we use the same techniques as in [36], to solve the utility maximization problem from terminal wealth. To give an idea, we show that the value function of the piecewise deterministic control problem can be identified as the value function of a certain Markov decision problem that can be solved by a fixed point argument, see [18, Chapter 8] for details.

Although the main technique used to handle the optimization problem is similar to that developed in [36], there are a few differences which are worth explaining. In [36], the authors study an optimal liquidation problem for an investor whose actions directly affect the stock price dynamics by increasing the intensity of downward jumps in a partial information setting. The stock price dynamics is given by a pure-jump process. The goal is to maximize the expected total reward represented by a functional consisting in a combination of running profits, which linearly depend on the liquidation rate (that is the control), and terminal value representing the price of a block transaction at the final time. The first difference with our setup is the model, as here we consider an indirect effect on prices through the generator of the unobservable Markov chain. Secondly, objectives are different, because we aim to maximize the expected utility from terminal wealth. We stress that, on the other hand, unfortunately we cannot directly rely on the results in [17, 18], since the jump intensity of the PDMP is stochastic. By mimicking the argument in [36, Section 5], we characterize the optimal value function as the unique viscosity solution of the (generalized) HJB equation. This notion also has the advantage to permit a numerical study, compared to [18] where optimal strategies and optimal value functions are obtained by a policy iteration procedure, which has a fast convergence rate.

We recall that, in for a piecewise deterministic control problem, decisions are made only at the jump times of the state process. With this idea in mind, the infinite horizon Markov decision model corresponding to the PDMP can be introduced as follows. We consider the sequence $\{L_n\}_{n\in\mathbb{N}}$ of random variables defined by

$$L_n = (T_n, X_{T_n}) = \widetilde{X}_{T_n} \text{ for } \; T_n < T, \quad n \in \mathbb{N}\,,$$

and set $L_n = \Delta$ for $T_n \geq T$ where Δ is some cemetery state. In other words a state $\widetilde{x} = (t, x) = (t, w, \boldsymbol{\pi})$ represents a jump time t and the wealth w and filter $\boldsymbol{\pi}$ just after the jump.

For a function $\alpha \in \mathcal{A}$, we denote by $\widetilde{\varphi}_t^\alpha(\widetilde{x})$ the flow of the initial value problem $\frac{d}{ds}\widetilde{X}(s) = g\big(\widetilde{X}(s), \alpha_s\big)$ with initial condition $\widetilde{X}(0) = \widetilde{x}$. Equivalently, the piecewise deterministic process $\widetilde{X}$ is given by $\widetilde{X}_t = \widetilde{\varphi}^\alpha_{t-T_n}(\widetilde{X}_{T_n})$, for every $t \in [T_n, T_{n+1})$ before time T. We use the notation $\widetilde{\varphi}^\alpha_t = (t, \varphi^\alpha)$ to stress the dependence on time.

In the sequel we define two fundamental quantities for the MDM $\{L_n\}_{n\geq 0}$, which are the jump intensity $\lambda^\alpha(\widetilde{x})$ and the transition kernel $Q_L f(\widetilde{x}, \alpha)$, for every $\alpha \in \mathcal{A}$ and $\widetilde{x} \in \widetilde{\mathcal{X}}$. Precisely we have

$$\begin{aligned} \lambda_s^\alpha(\widetilde{x}) &= \lambda(\widetilde{\varphi}_s^\alpha(\widetilde{x}), \alpha_s) := \lambda((t+s, \varphi_s^\alpha), \alpha_s), \\ \Lambda_s^\alpha(\widetilde{x}) &= \Lambda^\alpha(s; \widetilde{x}) := \int_0^s \lambda_u^\alpha(\widetilde{x}) du. \end{aligned} \tag{3.13}$$

The distribution of the interarrival times $T_{n+1} - T_n$ given $L_n = (t, x)$ and $h^n = \alpha$ is equal to $\lambda^\alpha(\widetilde{x}) e^{-\Lambda_u^\alpha(\widetilde{x})} du$, where $\widetilde{x} = (t, x)$. Then, for any bounded measurable function $f : \widetilde{X} \cup \{\Delta\} \to \mathbb{R}$, the transition kernel of the MDM is given by

$$Q_L f\big((t, x), \alpha\big) = \int_0^{T-t} \lambda_u^\alpha(\widetilde{x}) e^{-\Lambda_u^\alpha(\widetilde{x})} Q_{\widetilde{X}} f\big(u + t, \varphi_u(\widetilde{x}), \alpha_u\big) du + e^{-\Lambda^\alpha_{T-t}(\tilde{x})} f(\bar{\Delta}),$$

with $Q_L \mathbf{1}_{\{\Delta\}}(\Delta, \alpha) = 1$.

We indicate by w_t, the wealth component of the flow $\widetilde{\varphi}^\alpha$. A one-stage reward function $r \colon \widetilde{\mathcal{X}} \times \mathcal{A} \to \mathbb{R}_{\geq 0}$ is defined by

$$r(\widetilde{x}, \alpha) = e^{-\Lambda^\alpha_{T-t}(\widetilde{x})} U(w_{T-t}), \quad r(\Delta) = 0.$$

It is clear that the expected reward of a policy $\{h^n\}_{n\geq 0}$ is given by

$$J_{\infty}^{\{h^n\}}(\widetilde{x}) = \mathbb{E}_{\widetilde{x}}^{\{h^n\}}\left[\sum_{n=0}^{\infty} r\left(L_n, h^n(L^n)\right)\right],$$

and

$$J_{\infty}(\widetilde{x}) := \sup\left\{J_{\infty}^{\{h^n\}}(\widetilde{x}) : \{h^n\}\ \mathbb{F}^S - \text{admissible strategy}\right\}. \tag{3.14}$$

The next step is to verify that this construction of an infinite-stage Markov decision model leads to an optimal control problem which is equivalent to the original PDP control problem. Lemma 3.15 below shows that the value functions corresponding to the MDM and the control problem for PDMP coincide. The proof uses a similar argument to [18, Theorem 9.3.1] and is provided in Appendix 5.2.

Lemma 3.15. *It holds for all $\mathbb{F}^S$-admissible strategies $\{h^n\}_{n\in\mathbb{N}}$ that $V^{\{h^n\}} = J_{\infty}^{\{h^n\}}$ and hence $V = J_{\infty}$, that is, control problems* (3.12) *and* (3.14) *are equivalent.*

Define the operator $\mathcal{T}$ of the Markov decision model as

$$\mathcal{T}v(\widetilde{x}) := \sup_{\alpha\in\mathcal{A}}\Big\{e^{-\Lambda^{\alpha}_{T-t}(\widetilde{x})}U(w_{T-t}) + \int_0^{T-t}\lambda_u^{\alpha}(\widetilde{x})e^{-\Lambda_u^{\alpha}(\widetilde{x})}Q_{\widetilde{X}}v(t+u,\varphi_u(\widetilde{x}),\alpha_u)du\Big\}.$$

Then the idea is to characterize the value function as the unique fixed point of the operator $\mathcal{T}$. This is provided in the next theorem. In the sequel we assume the following.

Assumption 3.16. *Mappings $(\widetilde{x},\alpha)\mapsto r(\widetilde{x},\alpha)$ and $(\widetilde{x},\alpha)\mapsto Q_L v(\widetilde{x},\alpha)$ for every $v\in\mathcal{B}_b$, are continuous on $\widetilde{\mathcal{X}}\times\widetilde{\mathcal{A}}$ with respect to the Young topology on $\widetilde{\mathcal{A}}$.*

A sufficient condition is given by Assumption 5.5 in Appendix 5.2.

Theorem 3.17. *Suppose that Assumption 3.16 holds, let $b(\widetilde{x}) = b(t,x) := e^{c(T-t)}s$, for some $c\geq 0$, and $b(\bar{\Delta}) = 0$ and define the set $\mathcal{B}_b$ of functions $v:\widetilde{\mathcal{X}}\to\mathbb{R}$ such that $v(\widetilde{x})\leq Cb(\widetilde{x})$, $C\geq 0$. Then we have that*

- *i) the value function V is continuous on $\widetilde{\mathcal{X}}$ and satisfies the boundary conditions $V(T,w,\boldsymbol{\pi}) = U(w)$.*
- *ii) V is the unique fixed point of the operator $\mathcal{T}$ in $\mathcal{B}_b$.*

Proof. First note that by Lemma 5.4 in Appendix 5.2 the MDM is contracting and by Proposition 5.6 reward function r and transition kernel Q_L are continuous. By applying [18, Theorem 7.3.6] we obtain that V is the fixed point of the maximal reward operator extended to the class of the relaxed controls and finally the result of the theorem follows from the second part of [36, Theorem 4.10]. □

In order to characterize the optimal value function in terms of the solution of a suitable HJB equation, we resort to the viscosity solution analysis. This also legitimates the numerical study which will be done in the next chapter.

As a first step we want to reduce the problem to the case where the state process takes values in a compact set. Since the case of logarithmic utility is a limiting case of the power utility we only write the reduction for the latter. By using positive homogeneity we have that

$$V(t,w,\boldsymbol{\pi}) = \frac{w^\theta}{\theta}\overline{V}(t,\boldsymbol{\pi}).$$

Define the compact set $\widetilde{\mathcal{Y}} := [0,T]\times\Delta_K$.

We now define $\overline{g}\colon \widetilde{\mathcal{Y}}\times[-L,L]\to\mathbb{R}^{K+2}$ by identifying

$$\overline{g}^{(1)} = g^{(1)}, \text{ and } \overline{g}^{(k+1)} = g^{(k+2)}\,,\ k=1,\dots,K\,.$$

and denote by $\overline{\varphi}_u(\alpha,\widetilde{y})$ the flow of $\overline{g}$.

Since the jump intensity λ introduced in (3.13) is independent of w, by Theorem 3.17, the optimality equation for $\overline{V}$ is given by

$$\overline{V}(\widetilde{y}) = \sup_{\alpha\in A}\Big\{\int_0^{T-t}\lambda_u^\alpha(\widetilde{y})e^{-\Lambda_u^\alpha(\widetilde{y})}\overline{Q}\ \overline{V}\big(u+t,\overline{\varphi}_u(\alpha,\widetilde{y}),\alpha_u\big)du + \frac{1}{\theta}e^{-\Lambda_{T-t}^\alpha(\widetilde{y})}\Big\},$$

where, for $h\in[-L,L]$, $\widetilde{y}\in\widetilde{\mathcal{Y}}$, and any measurable function $\Psi\colon\widetilde{\mathcal{Y}}\to\mathbb{R}_{\geq 0}$, $\overline{Q}$ defines the new transition kernel

$$\overline{Q}\Psi(\widetilde{y},h) := \lambda(\widetilde{y})\sum_{j=1}^K \pi^j\int_{\mathbb{R}}(1+hz)^\theta\Psi\left(t,(\pi^i(1+u^i(t,\boldsymbol{\pi},z)))_{i=1,\dots,K}\right)\eta^{\mathbf{P}}(t,e_j,dz).$$

This, in turn, implies that the value function $\overline{V}$ satisfies $\overline{V} = \overline{\mathcal{T}V}$, with the reward operator $\overline{\mathcal{T}}$ given by

$$\overline{\mathcal{T}}\Psi(\widetilde{y}) = \sup_{\alpha\in A}\Big\{\int_0^{T-t}\lambda_u^\alpha(\widetilde{y})e^{-\Lambda_u^\alpha(\widetilde{y})}\overline{Q}\Psi\big(u+t,\overline{\varphi}_u(\alpha,\widetilde{y}),\alpha_u\big)du + \frac{1}{\theta}e^{-\Lambda_{T-t}^\alpha(\widetilde{y})}\Big\}.$$

In the sequel we aim to show that $\overline{V}$ solves, in the viscosity sense, the equation

$$F_{\overline{V}}\big(\widetilde{y},\overline{V}(\widetilde{y}),\nabla\overline{V}(\widetilde{y})\big) = 0, \text{ for } \widetilde{y}\in\widetilde{\mathcal{Y}}^0, \quad \overline{V}(\widetilde{y}) = \frac{1}{\theta} \text{ for } \widetilde{y}\in\partial\widetilde{\mathcal{Y}}, \tag{3.18}$$

where, for $\Psi\colon\widetilde{\mathcal{Y}}\to\mathbb{R}_{\geq 0}$, the function $F_\Psi\colon\widetilde{\mathcal{Y}}\times\mathbb{R}_{>0}\times\mathbb{R}^{K+1}\to\mathbb{R}$ if given by

$$F_\Psi(\widetilde{y},v,p) = -\sup_{\nu\in[-L,L]}\big\{-\lambda(\widetilde{y},\nu)v + \overline{g}(\widetilde{y},\nu)p + \overline{Q}\Psi(\widetilde{y},\nu)\big\}\,.$$

The following result, proven in [36, Theorem 4.10] applies.

Theorem 3.19. *Suppose that Assumption 3.16 holds. Then the value function $\overline{V}$ is the unique continuous viscosity solution of* (3.18) *in $\widetilde{\mathcal{Y}}$ and a comparison principle holds.*

Explicitly, the HJB equation for the value function in the partial information setting is given by

$$\begin{aligned} 0 = \sup_{h\in[-L,L]}\Bigg\{&\frac{\partial V}{\partial t}(t,w,\boldsymbol{\pi}) + w(1-h)\rho\frac{\partial V}{\partial w}(t,w,\boldsymbol{\pi}) \\ &+\sum_{k,j=1}^K\frac{\partial V}{\partial\pi^k}(t,w,\boldsymbol{\pi})\pi^j\left(q^{jk}(h) - \int_{\mathbb{R}}\pi^k u^k(t,\boldsymbol{\pi},z)\eta^{\mathbf{P}}(t,e_j,dz)\right) \\ &+\sum_{j=1}^K\pi^j\int_{\mathbb{R}}\left[V\left(t,w(1+hz),(\pi^i(1+u^i(t,\boldsymbol{\pi},z)))_{i\in\{1,\dots,K\}}\right) - V(t,w,\boldsymbol{\pi})\right]\eta^{\mathbf{P}}(t,e_j,dz)\Bigg\}. \end{aligned} \tag{3.20}$$

In the next sections we analyze the case of logarithmic and power utility functions in detail, in the partial information framework.

3.3 Logarithmic utility under partial information

According to analysis conducted in the full information framework, we study the optimization problem with and without impact. For the logarithmic utility preferences this leads to two different approaches: in the first case pointwise maximization applies, while in the second one we need to use a dynamic programming approach. We also provide a comparison between the optimal strategies under full and partial information.

Logarithmic utility - No market impact

We first assume that the investor has no impact, that is entries in the generator of the Markov chain do not depend on the trading strategy, and we solve the optimal control problem directly. For a fixed strategy $h \in \widetilde{\mathcal{H}}$ by applying the Itô formula we get

$$V(t, w, \boldsymbol{\pi}) = \log(w) + \sup_{h \in \widetilde{\mathcal{H}}} \widetilde{B}(t, \boldsymbol{\pi}; h),$$

where

$$\widetilde{B}(t, \boldsymbol{\pi}; h) = \mathbb{E}^{t,\boldsymbol{\pi}} \left[\int_t^T \left((1 - h_s)\rho + \sum_{i=1}^K \pi_s^i \int_{\mathbb{R}} \log(1 + h_s z) \eta^{\mathbf{P}}(s, e_i, dz) \right) ds \right. \\ \left. + \int_t^T \int_{\mathbb{R}} \log(1 + h_s z) \nu^{\pi}(ds, dz) \right].$$

The following result follows from [16, Lemma 4.1].

Proposition 3.21. *Suppose $U(w) = \log(w)$ for $w > 0$.*

i) Let $h^*(t, \boldsymbol{\pi})$ satisfy either

$$\sum_{j=1}^K \pi_t^j \int_{\mathbb{R}} \frac{z}{1 + h^*(t, \boldsymbol{\pi}) z} \eta^{\mathbf{P}}(t, e_j, dz) = \rho \tag{3.22}$$

or $h^*(t, \boldsymbol{\pi}) \in \{-L, L\}$. Then the optimal strategy $h_t^* = h^*(t, \boldsymbol{\pi})$ for every $t \in [0, T]$ and $\boldsymbol{\pi} \in \Delta_K$.

ii) The value function is of the form

$$V(t, w, \boldsymbol{\pi}) = \log(w) + \mathbb{E}^{t,\boldsymbol{\pi}} \left[\int_t^T \left((1 - h_s^*)\rho + \sum_{i=1}^K \pi_s^j \int_R \log(1 + h_s^* z) \eta^{\mathbf{P}}(s, e_i, dz) \right) ds \right].$$

Remark 3.23. Comparing the results in Proposition 2.4 and Proposition 3.21 we observe similar structures for the optimal strategies. Precisely in the partial information case the optimal strategy solves an equation of the form (3.22) where $(\mathbb{F}, \mathbf{P})$-compensator of the jump measure is replaced by the $(\mathbb{F}^S, \mathbf{P})$-compensator. Intuitively this is due to the myopic property of the logarithmic utility; the agent replaces the unobserved local characteristics of the return process by their filtered estimates ignoring the extra risk associated with the information uncertainty (see, for example, [72]).

Logarithmic utility - Market impact

According to full information, when there is an impact on the state of the Markov chain we cannot apply pointwise maximization, but we can characterize the value function as the solution of the HJB equation, in the viscosity sense. Here we propose the following ansatz $V(t, w, \boldsymbol{\pi}) = \log(w) + B(t, \boldsymbol{\pi})$, for some function B with the terminal condition $B(T, \boldsymbol{\pi}) = 0$, for all $\boldsymbol{\pi} \in \Delta_K$. Substituting this form of the value function into (3.20), we obtain the following equation

$$0 = \sup_{h \in [-L, L]} \left\{ \frac{\partial B}{\partial t}(t, \boldsymbol{\pi}) + (1-h)\rho \right. \tag{3.24}$$

$$+ \sum_{k,j=1}^{K} \frac{\partial B}{\partial \pi^k}(t, \boldsymbol{\pi}) \pi^j \left(q^{jk}(h) - \int_{\mathbb{R}} \pi^k u^k(t, \boldsymbol{\pi}, z) \eta^{\mathbf{P}}(t, e_j, dz) \right)$$

$$\left. + \sum_{j=1}^{K} \pi^j \int_{\mathbb{R}} \log(1+hz) + \left[B\left(t, (\pi^i(1+u^i(t, \boldsymbol{\pi}, z)))_{i \in \{1,\dots,K\}}\right) - B(t, \boldsymbol{\pi}) \right] \eta^{\mathbf{P}}(t, e_j, dz) \right\}. \tag{3.25}$$

By Theorem 3.19, the value function is the unique viscosity solution of problem (3.25). Given the form of the compensator $\eta^{\mathbf{P}}$, the equation can be solved, for instance using a numerical scheme.

3.4 Power utility under partial information

In this part we will work under the assumption of power utility, that is, $U(w) = \frac{1}{\theta} w^\theta$, $\theta < 1$, $\theta \neq 0$. Then the value function of the investor is

$$V(t, w, \boldsymbol{\pi}) = \sup_{h \in \widetilde{\mathcal{H}}} \mathbb{E}^{t, w, \boldsymbol{\pi}} \left[\frac{1}{\theta} (W_T)^\theta \right],$$

where $\mathbb{E}^{t,w,\boldsymbol{\pi}}[\cdot]$ denotes the conditional expectation given $W_t = w$ and $\pi_t = \boldsymbol{\pi}$. By positive homogeneity, the value function can be rewritten as $V(t, w, \boldsymbol{\pi}) = \frac{1}{\theta} w^\theta \Gamma(t, \boldsymbol{\pi})$, for some function $\Gamma : [0, T] \times \Delta_K \to \mathbb{R}_{>0}$ with $\Gamma(T, \boldsymbol{\pi}) = 1$, for all $\boldsymbol{\pi} \in \Delta_K$. Substituting this form of the value function into (3.20), we obtain the equation

$$0 = \sup_{h \in [-L, L]} \left\{ \frac{\partial \Gamma}{\partial t}(t, \boldsymbol{\pi}) + \Gamma(t, \boldsymbol{\pi}) \theta (1-h) \rho \right.$$

$$+ \sum_{k,j=1}^{K} \frac{\partial \Gamma}{\partial \pi^k}(t, \boldsymbol{\pi}) \pi^j \left(q^{jk}(h) - \int_{\mathbb{R}} \pi^k u^k(t, \boldsymbol{\pi}, z) \eta^{\mathbf{P}}(t, e_j, dz) \right)$$

$$\left. + \sum_{j=1}^{K} \pi^j \int_{\mathbb{R}} \left[(1+hz)^\theta \Gamma\left(t, (\pi^i(1+u^i(t, \boldsymbol{\pi}, z)))_{i \in \{1,\dots,K\}}\right) - \Gamma(t, \boldsymbol{\pi}) \right] \eta^{\mathbf{P}}(t, e_j, dz) \right\}.$$

Therefore, after reduction, we deal with a problem having a bounded state space $[0, T] \times \Delta_K$. Theorem 3.19 ensures existence and uniqueness of a viscosity solution for this problem. We solve it numerically in case of a two-state Markov chain in the next section.

Chapter 4

A Model with a Two-State Markov Chain

Suppose that we have a state process Y described by a Markov chain with the state space $\mathcal{E} = \{e_1, e_2\}$. Without loss of generality we may assume that e_1 represents the good (bull) state of the market and e_2 is representing a bad (bear) state. We consider the situation where the investor's assets holdings are taken as a signal for the rest of the market that tends to behave accordingly. Then for the market, intensities of switching between the bull and the bear state depend on the portfolio weights of the reference "large" investor. In the current setting, we also assume that the impact of the portfolio choices on the Markov chain is linear, and assume that the infinitesimal generator has the form

$$q^{12}(h_t) = a_1 - b_1 h_t, \quad q^{21}(h_t) = a_2 + b_2 h_t.$$

To guarantee that the entries $q^{1,2}$ and $q^{2,1}$ of the matrix stay positive we take $a_1, a_2 > 0$, $b_1 \in (0, a_1/L)$ and $b_2 \in (0, a_2/L)$.

This choice for the generator has the following motivation. If the investor buys, then she tends to increases the probability for the market to stay in (resp. switch to) the bull state, provided that the current state of the market is bull (resp. bear). Conversely, when the investor sells, the probability to stay in or jump to the bear state increases. This mechanism reflects certain real world situations such as manipulation and herding, which are frequently observed in markets where large investors are involved.

We assume that the return process may have two possible jump sizes, $\Delta R^{(h)} \in \{-\vartheta, +\vartheta\}$. Formally, it is given by

$$R_t^{(h)} := N_t^{-(h)} + N_t^{+(h)}, \quad t \in [0,T],$$

where

$$dN_t^{-(h)} = \int_{\mathbb{R}} -\vartheta \mathbf{1}_{[-\lambda^-(Y_{t^-}^{(h)}),0]}(\zeta)\mathcal{N}(dt,d\zeta), \quad dN_t^{+(h)} = \int_{\mathbb{R}} \vartheta \mathbf{1}_{[0,\lambda^+(Y_{t^-}^{(h)})]}(\zeta)\mathcal{N}(dt,d\zeta)$$

are two Poisson processes with jump sizes ϑ and intensities $\lambda^+(e_i) = \lambda_i^+, \lambda^-(e_i) = \lambda_i^-$, $i \in \{1,2\}$, for some constants $\lambda_1^+, \lambda_2^+, \lambda_1^-, \lambda_2^- > 0$ and such that $\lambda_1^+ > \max\{\lambda_1^-, \lambda_2^+\}$ and $\lambda_2^- > \max\{\lambda_1^-, \lambda_2^+\}$. This conditions imply that the intensity of an upward jump is larger in the bull state of the market compared the bear one. Moreover in the bull state it is more likely to observe an upward jump then a downward jump. In this example we take the Poisson random measure $\mathcal{N}(dt, d\zeta)$ with intensity $\varsigma(d\zeta)dt = \mathbf{1}_{[-\lambda_2^-, \lambda_1^+]} d\zeta dt$. Then the compensator has the form

$$\eta^{\mathbf{P}}(t, e_i, dz) = \lambda_i^+ \delta_{\{\vartheta\}}(dz) + \lambda_i^- \delta_{\{-\vartheta\}}(dz),$$

where $\delta_{\{x\}}(dz)$ is the Dirac mass at point x. Notice that here Assumption 1.7 is satisfied for $-\frac{1}{\vartheta} < h_t < \frac{1}{\vartheta}$.

In the reminder of this section we are going to compare the results for logarithmic and power utility choices for the full and the partial information settings.

4.1 Logarithmic utility

First we consider a investor with full information on the market. Starting with the case of no market impact, applying Proposition 2.4 and checking the first and second order conditions we obtain the optimal strategy

$$h^*(t, e_i) = \frac{\lambda_i^+ + \lambda_i^-}{2\rho} - \sqrt{\left(\frac{\lambda_i^+ + \lambda_i^-}{2\rho} + \frac{1}{\vartheta}\right)^2 - 2\frac{\lambda_i^+}{\vartheta\rho}}.$$

In particular, if $\lambda_i^+ = \lambda_i^- = \lambda_i$ we get that $h^*(t, e_i) = \frac{\lambda_i}{\rho} - \sqrt{\frac{\lambda_i^2}{\rho^2} + \frac{1}{\vartheta^2}}$. Finally we can characterize the value function as

$$\begin{aligned} V(t, w, e_i) = &\log(w) + \mathbb{E}^{t,i}\Bigg[\int_t^T \big((1 - h^*(s, e_1))\mathbf{1}_{\{Y_s = e_1\}}\rho + (1 - h^*(s, e_2))\mathbf{1}_{\{Y_s = e_2\}}\rho\big)\, ds \\ &+ \int_t^T \int_R \log(1 + h^*(s, e_1)z)\mathbf{1}_{\{Y_{s^-} = e_1\}}\eta^{\mathbf{P}}(ds, e_1, dz) \\ &+ \int_t^T \int_R \log(1 + h^*(s, e_2)z)\mathbf{1}_{\{Y_{s^-} = e_2\}}\eta^{\mathbf{P}}(ds, e_2, dz)\Bigg]. \end{aligned}$$

For the case where the impact is non-zero, the value function can be characterized as $V(t, w, e_i) = \log(w) + \beta(t, e_i)$, $i \in \{1, 2\}$ with the functions $\beta(t, e_1)$ and $\beta(t, e_2)$ solving

$$\begin{aligned} \frac{d\beta}{dt}(t, e_1) = &- \sup_{h\in[-L,L]} \Bigg\{(1 - h)\rho + (\beta(t, e_2) - \beta(t, e_1))\,(a_1 - b_1 h) \\ &+ \int_{\mathbb{R}} \log(1 + hz)\eta^{\mathbf{P}}(t, e_1, dz)\Bigg\}, \\ \frac{d\beta}{dt}(t, e_2) = &- \sup_{h\in[-L,L]} \Bigg\{(1 - h)\rho + (\beta(t, e_1) - \beta(t, e_2))\,(a_2 + b_2 h) \\ &+ \int_{\mathbb{R}} \log(1 + hz)\eta^{\mathbf{P}}(t, e_2, dz)\Bigg\}, \end{aligned}$$

respectively, with boundary conditions $\beta(T, e_i) = 0$, for $i = \{1, 2\}$.

Assume now that the investor's information is given by the filtration $\mathbb{F}^S$. There, by Proposition 3.21, the optimal strategy in case of no market impact turns out to be

$$h^*(t, \boldsymbol{\pi}) = \frac{\boldsymbol{\pi}^\top\Lambda^+ + \boldsymbol{\pi}^\top\Lambda^-}{2\rho} - \sqrt{\left(\frac{\boldsymbol{\pi}^\top\Lambda^+ + \boldsymbol{\pi}^\top\Lambda^-}{2\rho} + \frac{1}{\vartheta}\right)^2 - 2\frac{\boldsymbol{\pi}^\top\Lambda^+}{\vartheta\rho}}.$$

where $(\Lambda^+)^\top = (\lambda_1^+, \lambda_2^+)$ and similarly $(\Lambda^-)^\top = (\lambda_1^-, \lambda_2^-)$. This is the classical case where the optimal strategy has the same structure of that under full information in which the unobserved components are replaced by their filtered estimates. The stochastic representation of the value function is given by

$$\begin{aligned} V(t, w, \boldsymbol{\pi}) = &\log(w) + \mathbb{E}^{t,\pi}\Bigg[\int_t^T (1 - h^*(s, \pi_s))\rho ds \\ &+ \int_t^T \pi_s^1 \int_R \log(1 + h^*(s, \pi_s)z)\eta^{\mathbf{P}}(s, e_1, dz)ds \\ &+ \int_t^T \pi_s^2 \int_R \log(1 + h^*(s, \pi_s)z)\eta^{\mathbf{P}}(s, e_2, dz)ds\Bigg]. \end{aligned}$$

In the partial information case, the value function has the form $V(t, w, \pi) = \log(w) + B'(t, \pi)$ where $B'(t, \pi) = B(t, \pi, (1 - \pi))$ is the solution of the HJB equation

$$\begin{aligned} 0 = \sup_{h\in[-L,L]} &\Bigg\{ \frac{\partial B'}{\partial t}(t, \pi) + (1 - h)\rho + \frac{\partial B'}{\partial \pi}(t, \pi)\left(\pi q^{11}(h) + (1 - \pi)q^{21}(h)\right) \\ &- \frac{\partial B'}{\partial \pi}\pi(1 - \pi)(\lambda_1^+ + \lambda_1^- - \lambda_2^+ - \lambda_2^-) \\ &+ (\pi\lambda_1^+ + (1 - \pi)\lambda_2^+)\log(1 + hz) + (\pi\lambda_1^- + (1 - \pi)\lambda_2^-)\log(1 - hz) \\ &+ \pi(\lambda_1^+ + \lambda_1^-)\left[B'\left(t, \frac{\pi\lambda_1^+}{\pi\lambda_1^+ + (1 - \pi)\lambda_2^+}\right) - B'(t, \pi)\right] \\ &+ (1 - \pi)(\lambda_2^+ + \lambda_2^-)\left[B'\left(t, \frac{\pi\lambda_1^+}{\pi\lambda_1^+ + (1 - \pi)\lambda_2^+}\right) - B'(t, \pi)\right]\Bigg\}. \end{aligned}$$

An explicit solution of the above equation is difficult to find. In general, it is possible to apply numerical experiments to get the qualitative behavior of both the value function and the optimal strategy. Since the logarithmic utility case do not provide any simplification, for numerical study we only consider the power utility case.

4.2 Power utility

In the power utility case, when the investor has a full information on the state of the market, analysis of the optimization problem leads to solving the system

$$\begin{aligned} \frac{d\gamma}{dt}(t, e_1) = &- \sup_{h\in[-L,L]} \Bigg\{(1 - h)\rho + \frac{1}{\theta}\left(e^{\theta(\gamma(t,e_2)-\gamma(t,e_1))} - 1\right)(a_1 - b_1 h) \\ &+ \frac{\lambda_1^+}{\theta}\left((1 + h\vartheta)^\theta - 1\right) + \frac{\lambda_1^-}{\theta}\left((1 - h\vartheta)^\theta - 1\right)\Bigg\}, \\ \frac{d\gamma}{dt}(t, e_2) = &- \sup_{h\in[-L,L]} \Bigg\{(1 - h)\rho + \frac{1}{\theta}\left(e^{\theta(\gamma(t,e_1)-\gamma(t,e_2))} - 1\right)(a_2 + b_2 h) \\ &+ \frac{\lambda_2^+}{\theta}\left((1 + h\vartheta)^\theta - 1\right) + \frac{\lambda_2^-}{\theta}\left((1 - h\vartheta)^\theta - 1\right)\Bigg\}, \end{aligned}$$

with the final condition $\gamma(T, e_1) = \gamma(T, e_2) = 0$.

For the solution we use the following algorithm. Let $(t_0, \ldots, t_N)$ be the sequence of discretized time points with $t_0 = 0$ and $t_N = T$. Knowing the final conditions allows to compute easily the control h_T^* at time T. Then, using a backward scheme we solve the corresponding ODE at t_{N-1}. Given the value at t_{N-1}, now we can compute the control $h_{t_{N-1}}^*$ and we proceed until $t_0 = 0$.

In the numerical analysis we use the set of parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = -0.1$, $a_2 = 5$, $b_2 = 0.1$.

In Figure 4.1 we plot the optimal investment strategies for cases where the initial state of the Markov chain is bull (lighter line) or bear (darker line) both with (solid line) and without (dashed line) market influence. Firstly, we observe that in all cases the optimal strategies never reach the values $\{-L, L\}$ corresponding to $L = 50$, meaning that there is always an interior solution. Secondly, we can see that as time approaches to maturity, the optimal strategy in the case with impact converges to the one in the no-impact case. Moreover, actions of the investor are very different when we compare cases with and without impact. Consider for instance the situation where the initial state is bull. We observe that in the no-impact case the strategy is constant and always positive, meaning that the investor always buys. On the other hand, in the case with impact the investor short-sells if time to maturity is large. This kind of an action might be interpreted in the following way. The investor tries to produce a jump in the Markov chain and make advantage of lower prices that would prevail in a future time. Clearly, she switches her behavior as time to maturity becomes shorter, since there is not enough time to make such a change. For the case of initial bear state, we see that the investor always short-sells. This is reasonable for the current parameter choice as on average the prices tend to go down. For the case with impact, the strategy turns out to be more aggressive.

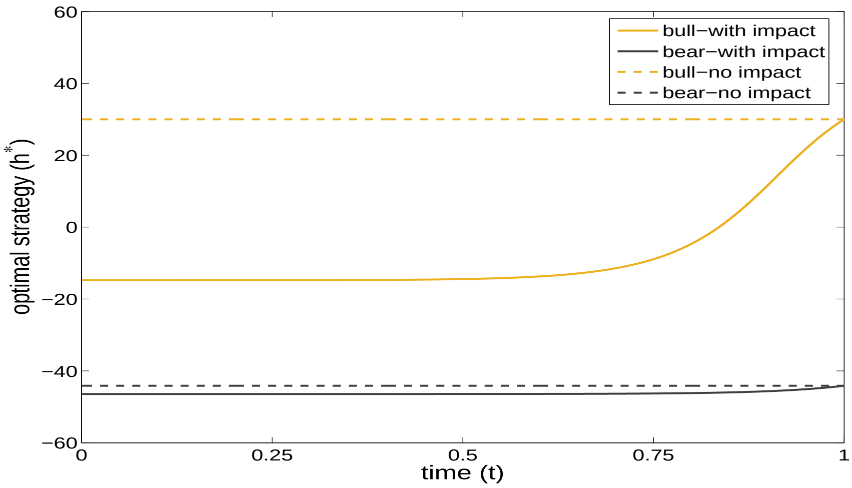

Figure 4.1: Optimal strategy under full information. The solid line (resp. dashed line) corresponds to the case with (resp. without) impact. Other parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = 0.1$, $a_2 = 5$, $b_2 = 0.1$.

The different behavior for investors with market influence results in positive gains from utility maximization. Indeed, as we see in Figure 4.2, the value functions corresponding to the impact cases are sensibly larger than those corresponding to no-impact cases. The optimal value corresponding to the bad state is larger than the optimal value for the initial good state. This is a consequence of the fact that the investor is allowed to short-sell, and clearly this also depends on our choice of the intensities of upward and downward jumps. In other words, we see that there is no absolute good and bad state.

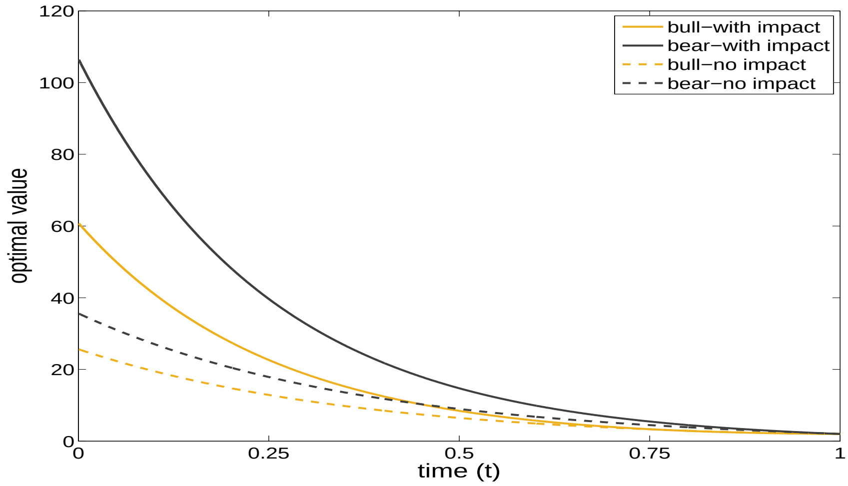

Figure 4.2: Optimal value under full information. The solid line (resp. dashed line) corresponds to the case with (resp. without) impact. Other parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = 0.1$, $a_2 = 5$, $b_2 = 0.1$.

Suppose now that the available information for the investor is given by $\mathbb{F}^S$. Note that, for a two-state Markov chain we have $\pi^1 + \pi^2 = 1$, then denote π^1 with π, we can define $\Gamma'(t, \boldsymbol{\pi}) := \Gamma(t, \pi, (1-\pi))$ and reduce the dimension of the optimization problem. In this case the function Γ' can be characterized as the solution of the HJB

$$
\begin{aligned}
0 = \sup_{h \in [-L,L]} \bigg\{ & \frac{\partial \Gamma'}{\partial t}(t,\pi) + \Gamma'(t,\pi)\theta(1-h)\rho + \frac{\partial \Gamma'}{\partial \pi}(t,\pi)\left(\pi q^{11}(h) + (1-\pi)q^{21}(h)\right) \\
& - \frac{\partial \Gamma'}{\partial \pi}\pi(1-\pi)(\lambda_1^+ + \lambda_1^- - \lambda_2^+ - \lambda_2^-) \\
& + (\pi\lambda_1^+ + (1-\pi)\lambda_2^+)\left((1+h\vartheta)^\theta \Gamma'\left(t, \frac{\pi\lambda_1^+}{\pi\lambda_1^+ + (1-\pi)\lambda_2^+}\right) - \Gamma'(t,\pi)\right) \\
& + (\pi\lambda_1^- + (1-\pi)\lambda_2^-)\left((1-h\vartheta)^\theta \Gamma'\left(t, \frac{\pi\lambda_1^-}{\pi\lambda_1^- + (1-\pi)\lambda_2^-}\right) - \Gamma'(t,\pi)\right)\bigg\}.
\end{aligned}
$$

Since, in general it is not possible to find an explicit solution to the above maximization problem we deepen our analysis through numerical experiments. In the case of partial information, we use an explicit finite difference method to solve the corresponding partial integro-differential equation. In order to guarantee the positivity of the scheme we use forward-backward approximation for the first order derivatives (see, for instance, [38]). Also, to ensure the convergence of the scheme we verify the usual consistency and stability conditions.

As in the full information case we study both the optimal strategy and the value function, and obtain results that are consistent with those in the full information setting. We observe in Figure 4.3 that optimal strategies in the impact case converge, for values of time close to maturity, to optimal strategies in the no-impact cases. Moreover the interesting behavior of the investor with an impact is preserved: for the initial bull state she short-sells when the time is far from maturity and in the initial bear state the strategy is always more aggressive.

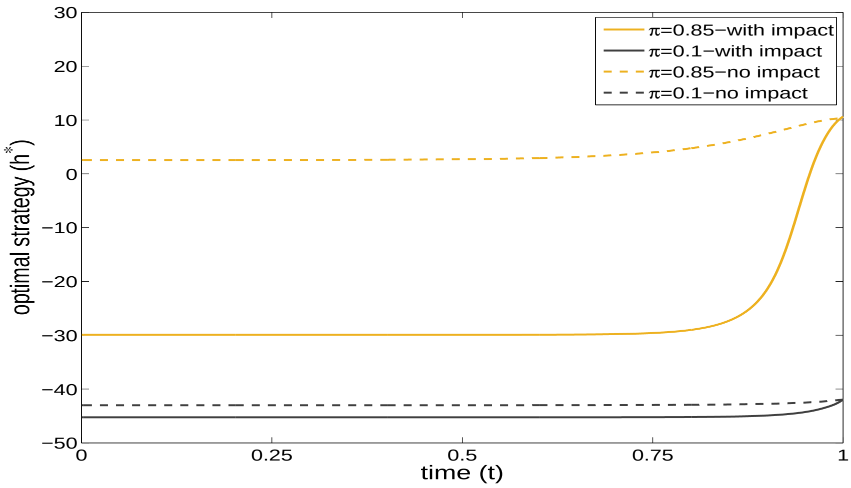

Figure 4.3: Optimal strategy under partial information for different values of the conditional probability of being in the bull state π. The solid line (resp. dashed line) corresponds to the case with (resp. without) impact. Other parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = 0.1$, $a_2 = 5$, $b_2 = 0.1$.

The value function is consistently larger for the investor with an impact, see Figure 4.4.

Finally, we analyze the gains from filtering. In order to do that we compare the value functions corresponding to two investors. The first one uses the optimal strategy obtained in the partial information setting, while the second one ignores the presence of two different regimes in the market. Instead the second one uses the average parameters, $\lambda^+ = \lambda_1^+ \bar{\pi} + \lambda_2^+(1-\bar{\pi})$, $\quad \lambda^- = \lambda_1^- \bar{\pi} + \lambda_2^-(1-\bar{\pi})$, where $\bar{\pi} = \frac{a_2}{a_1+a_2}$. In Figure 4.5, we observe that the investor's gains from using filtered estimates, instead of the average parameters, are always non-negative. Those profits justify the additional complexity induced by partial information.

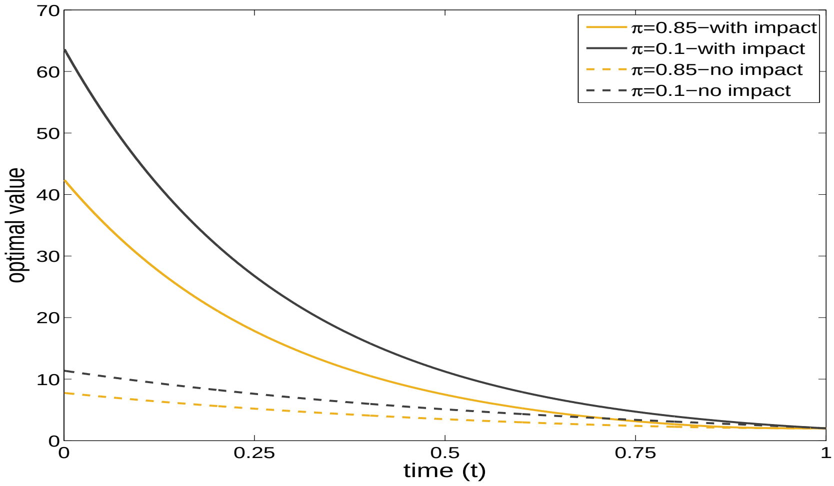

Figure 4.4: Optimal value under partial information for different values of the conditional probability of being in the bull state π. The solid line (resp. dashed line) corresponds to the case with (resp. without) impact. Other parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = 0.1$, $a_2 = 5$, $b_2 = 0.1$.

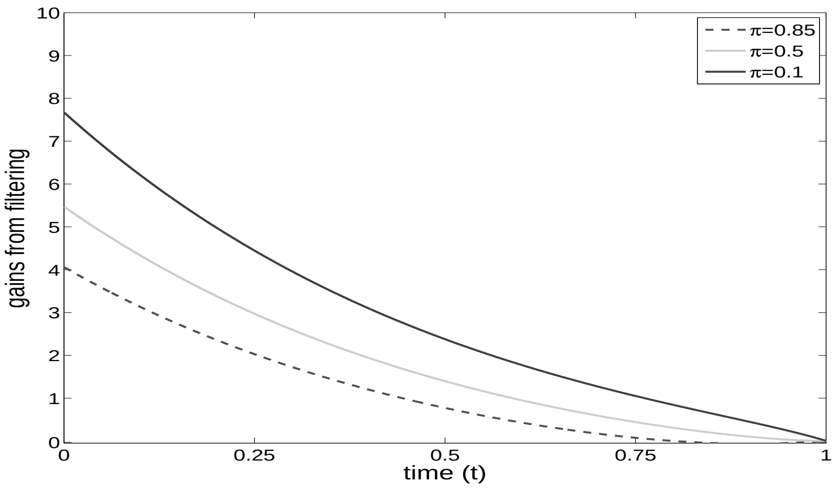

Figure 4.5: Gains from filtering as a function of time for different values of the conditional probability of being in the bull state π. Parameters: $T = 1$ year, $w = 1$, $\rho = 0$, $\vartheta = 0.02$, $\lambda_1^+ = 10$, $\lambda_1^- = 5$, $\lambda_2^+ = 5$, $\lambda_2^- = 20$, $\theta = 0.5$, $a_1 = 5$, $b_1 = 0.1$, $a_2 = 5$, $b_2 = 0.1$.

Chapter 5

Appendix: Technical Results and Proofs

5.1 Filtering

The following result is needed in the proof of Proposition 3.3 given below.

Lemma 5.1. *For $t \in [0,T]$ let U be an integrable $\mathcal{F}_t$-measurable random variable and let $\mathcal{F}^S := \bigvee_{t\geq 0} \mathcal{F}^S_t$. Then $\mathbb{E}^{\mathbf{Q}}\left[U \mid \mathcal{F}^S_t\right] = \mathbb{E}^{\mathbf{Q}}\left[U \mid \mathcal{F}^S\right]$.*

Proof. The result can be obtained by applying the same idea of [12, Proposition 3.15] to jump processes. □

Proof of Proposition 3.3. This proof follows the same argument of that of [36, Theorem 3.2] and it is given here for completeness. The idea is to compute the product $Z^{(h)}_t f(Y^{(h)}_t)$, for every $t \in [0,T]$, and then its projection onto $\mathcal{F}^S_t$. However, in order to be rigorous we need some technical steps. Consider a function $f : \mathcal{E} \to \mathbb{R}$. For every $\mathbb{F}^S$- predictable control h, using the semimartingale decomposition of $Y^{(h)}$ and applying the Itô's formula we get

$$df(Y^{(h)}_t) = Q^\top(h_t) f(Y^{(h)}_t)dt + dM^{(f)}_t,$$

where $M^{(f)}$ is an $(\mathbb{F}, \mathbf{P})$-martingale and $Q^\top$ denotes the transpose of the generator matrix Q. For every $t \in [0,T]$ we introduce the process $Z^\epsilon = \{Z^\epsilon_t,\ t \in [0,T]\}$, where

$$Z^\epsilon_t := \frac{Z^{(h)}_t}{1 + \epsilon Z^{(h)}_t}, \quad t \in [0,T].$$

Note that $0 < Z^\epsilon < 1/\epsilon$ and that as $\epsilon \to 0$, Z^ϵ_t converges to $Z^{(h)}_t$ for every $t \in [0,T]$. The dynamics of Z^ϵ is given by

$$dZ^\epsilon_t = Z^\epsilon_{t^-} \int_{\mathbb{R}} \frac{\Psi(t, Y^{(h)}_{t^-}, z)}{1 + \epsilon Z^{(h)}_t (1 + \Psi(t, Y^{(h)}_{t^-}, z))} \mu(dt, dz) - Z^\epsilon_{t^-} \int_{\mathbb{R}} \frac{\Psi(t, Y^{(h)}_{t^-}, \nu_{t-}, z)}{1 + \epsilon Z^{(h)}_{t^-}} \eta^{\mathbf{Q}}(t, dz)dt.$$

We apply Itô's product rule to compute $Z^\epsilon f(Y^{(h)})$. Precisely, we get

$$
\begin{aligned}
d\big(Z^\epsilon_t f(Y^{(h)}_t)\big) =& Z^\epsilon_{t^-} Q^\top(h_t) f(Y^{(h)}_t)\rangle dt + Z^\epsilon_{t^-} dM^f_t \\
&- f(Y^{(h)}_{t^-}) Z^\epsilon_{t^-} \int_{\mathbb{R}} \frac{\Psi(t, Y^{(h)}_{t^-}, z)}{1+\epsilon Z^{(h)}_{t^-}} \eta^{\mathbf{Q}}(t, dz) dt \\
&+ f(Y^{(h)}_{t^-}) Z^\epsilon_{t^-} \int_{\mathbb{R}} \frac{\Psi(t, Y^{(h)}_{t^-}, z)}{1+\epsilon Z^{(h)}_{t^-}(1+\Psi(t, Y^{(h)}_{t^-}, z))} \mu(dt, dz).
\end{aligned}
\tag{5.2}
$$

Taking conditional expectation with respect to $\mathcal{F}^S_t$ from (5.2) and applying Lemma 5.1 and Fubini theorem we get for every $t \in [0, T]$, that

$$
\begin{aligned}
\mathbb{E}^{\mathbf{Q}}\left[Z^\epsilon_t f(Y^{(h)}_t) \mid \mathcal{F}^S_t\right] &= \frac{\pi_0(f)}{1+\epsilon} + \int_0^t \mathbb{E}^{\mathbf{Q}}\left[Z^\epsilon_{s^-} Q^\top(h_s) f(Y^{(h)}_s) \mid \mathcal{F}^S\right] ds \\
+ \int_0^t \int_{\mathbb{R}} \mathbb{E}^{\mathbf{Q}}&\left[f(Y^{(h)}_{s^-}) Z^\epsilon_{s^-} \frac{\Psi(s, Y^{(h)}_{s^-}, z)}{1+\epsilon Z^{(h)}_{s^-}(1+\Psi(s, Y^{(h)}_{s^-}, z))} \mid \mathcal{F}^S\right] \mu(ds, dz) \\
- \int_0^t \int_{\mathbb{R}} \mathbb{E}^{\mathbf{Q}}&\left[f(Y^{(h)}_{s^-}) Z^\epsilon_{s^-} \frac{\Psi(s, Y^{(h)}_{s^-}, z)}{1+\epsilon Z^{(h)}_{s^-}} \mid \mathcal{F}^S\right] \eta^{\mathbf{Q}}(s, dz) ds\,.
\end{aligned}
$$

Note that here we also use the fact that $\mathbb{E}^{\mathbf{Q}}\left[\int_0^t Z^\epsilon_{s^-} dM^f_s \mid \mathcal{F}^S_t\right] = 0$, which follows from the definition of conditional expectation and the fact that S and $Y^{(h)}$ have no common jump times. Finally, for $\epsilon \to 0$, by applying dominated convergence theorem we obtain

$$
\begin{aligned}
\mathbb{E}^{\mathbf{Q}}\left[Z^{(h)}_t f(Y^{(h)}_t) \mid \mathcal{F}^S\right] &= \pi_0(f) + \int_0^t \mathbb{E}^{\mathbf{Q}}\left[Z^{(h)}_{s^-} Q^\top(h_s) f(Y^{(h)}_s) \mid \mathcal{F}^S\right] ds \\
+ \int_0^t \int_{\mathbb{R}} \mathbb{E}^{\mathbf{Q}}&\left[f(Y^{(h)}_{s^-}) Z^{(h)}_{s^-} \Psi(s, Y^{(h)}_{s^-}, z) \mid \mathcal{F}^S\right] \big(\mu(ds, dz) - \eta^{\mathbf{Q}}(s, dz)\big)\ ,
\end{aligned}
$$

for every $t \in [0, T]$. Expression 3.4 follows by Lemma 5.1. □

5.2 Markov decision models

This section contains proofs and some additional results on Markov decision models.

Proof of Lemma 3.15. The proof follows the same lines of [18, Theorem 9.3.1]. Let $\{T_n\}_{n\geq 0}$ be the sequence of jump times of the PDMP $\widetilde{X}$. Then we have, for every admissible strategy $\{h^n\}_{n\geq 0}$ that

$$
\begin{aligned}
V^{\{h^n\}} &= \mathbb{E}^{\{h^n\}}\left[U(W_T)\right] = \mathbb{E}^{\{h^n\}}\left[\sum_{n=0}^{\infty} \mathbf{1}_{T_n < T < T_{n+1}} U(W_T)\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}^{\{h^n\}}\left[\mathbb{E}^{\{h^n\}}\left[\mathbf{1}_{T_n < T < T_{n+1}} U(W_T) | T_n < T, X_{T_n \wedge T}\right]\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}^{\{h^n\}}\left[\mathbb{E}^{\{h^n\}}\left[e^{-\Lambda^{h^n}_{T-T_n}(L_n)} U(w_{T-T_n})\right] \mathbf{1}_{T_n < T}\right] \\
&= \mathbb{E}^{\{h^n\}}\left[\sum_{n=0}^{\infty} \mathbf{1}_{T_n < T} r(L_n, h^n)\right] = J^{\{h^n\}}_\infty.
\end{aligned}
$$

□

In the following we want to ensure continuity for the reward function and the transition kernel over a class of admissible controls which is compact. Therefore, according to the general theory, we enlarge the action space introducing the set of relaxed controls, and define a suitable topology, called the *Young Topology.* We refer to [46, 18] for more details.

The set of *relaxed controls* is given by

$$\widetilde{\mathcal{A}} := \{\alpha : [0,T] \to \mathcal{M}^1([-L,L])\ \},$$

where $\mathcal{M}^1([-L,L])$ is the set of probability measures on $[-L,L]$.

In the context of relaxed control, we define an admissible relaxed strategy as a sequence of mappings $\{\nu^n\} : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{A}}$.

To make the set $\widetilde{\mathcal{A}}$ compact, we introduce the Young topology as the coarsest topology such that all mappings of the form

$$\alpha \to \int_0^T \int_{-L}^L f(t,u)\alpha_t(du)dt$$

are continuous for all functions $f : [0,T] \times [-L,L] \to \mathbb{R}$ that are continuous in the second argument, measurable in the first one and $\int_0^T \max_{u\in[-L,L]} |f(t,u)|dt < \infty$ (see, e.g [18, Chapter 8]).

We remark that the set of non-relaxed controls is a dense subspace of relaxed controls, see e.g. [18, 36],.

For a measurable function $v : [-L,L] \to \mathbb{R}$ and some measure $\xi \in \mathcal{M}^1([-L,L])$, we define $\langle \xi, v\rangle := \int_{-L}^L v(\nu)\xi(d\nu)$. In order to use the properties of the set $\widetilde{\mathcal{A}}$ we now extend some definitions for $\alpha \in \widetilde{\mathcal{A}}$. First, the vector fields g of the PDMP becomes

$$g(\widetilde{x},\alpha) = \langle \alpha, g(\widetilde{x},\cdot)\rangle = \int_{-L}^L g(\widetilde{x},\nu)\alpha_s(d\nu),$$

the jump intensity is given by $\lambda_s^\alpha(\widetilde{x}) = \langle \alpha_s(d\nu), \lambda(t+s,\varphi_s^\alpha,\nu)\rangle$, and $\Lambda_s^\alpha = \Lambda_s^\alpha(\widetilde{x}) = \int_0^s \lambda_u^\alpha(\widetilde{x})du$, the reward function

$$r(\widetilde{x},\alpha) = e^{-\Lambda_{T-t}^\alpha} U(w_{T-t}),$$

and finally the transition kernel is

$$Q_L v\big(\widetilde{x},\alpha\big) = \int_0^{T-t} \lambda_u^\alpha(\widetilde{x}) e^{-\Lambda_u^\alpha} \langle \alpha_u(d\nu), Q_{\widetilde{X}} v(t+u,\varphi_u(\widetilde{x}),\nu)\rangle du + e^{-\Lambda_{T-t}^\alpha} v(\bar{\Delta}),$$

for every measurable function $v : [-L,L] \to \mathbb{R}$.

Moreover we have the following extension of the operator $\mathcal{T}$

$$\mathcal{T}\phi(\widetilde{x}) = \sup_{\alpha\in\widetilde{\mathcal{A}}} \big(r(\widetilde{x},\alpha) + Q_L\phi\big(\widetilde{x},\alpha\big)\big).$$

In the next lemma we show that there exists a bounding function for the MDM and the MDM is contracting. This is essential to prove that the value function is the unique fixed point of the operator $\mathcal{T}$.

Definition 5.3. A function $b : \widetilde{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ is called a *bounding function* for a MDM, if there are constants $c_r, c_b > 0$ such that $|r(\widetilde{x}, \alpha)| \leq c_r b(\widetilde{x})$ and $Q_L b(\widetilde{x}, \alpha) \leq c_b b(\widetilde{x})$ for all $(\widetilde{x}, \alpha) \in \widetilde{\mathcal{X}} \times \mathcal{A}$. If moreover $c_b < 1$, the MDM is *contracting.*

We define for a bounding function b the set $\mathcal{B}_b$ of functions $v : \widetilde{\mathcal{X}} \to \mathbb{R}$ such that $v(\widetilde{x}) \leq Cb(\widetilde{x})$.

Lemma 5.4. *Let $b(\widetilde{x}) = b(t,x) = e^{c(T-t)}s,\ c \geq 0$, and $b(\bar{\Delta}) = 0$. Then $b(\widetilde{x})$ is a bounding function and the MDM with the kernel Q_L is contracting for sufficiently large c.*

Proof. Since $e^{-\Lambda_u^\alpha(\widetilde{x})} < 1$ we get $r(\widetilde{x}, \alpha) \leq w$. Next we turn to estimating $Q_L b(\widetilde{x}, \alpha) = \int_{\widetilde{\mathcal{X}}} b(x') Q_L(dx'|\widetilde{x}, \alpha)$. It holds that

$$
\begin{aligned}
&\int_{\widetilde{\mathcal{X}}} b(x') Q_L(dx'|\widetilde{x}, \alpha) \\
&\quad = \int_0^{T-t} e^{c(T-s-t)} e^{-\Lambda_s^\alpha(\widetilde{x})} \int_{-L}^{L} \int_{\mathbb{R}} w(1+hz) \sum_{j=1}^{K} \pi_j \eta^j(t+s, dz) \alpha_s(dh) ds \\
&\quad \leq b(\widetilde{x}) c_\eta \int_0^T e^{-cr} dr = b(\widetilde{x}) c_\eta \frac{1}{\gamma}(1 - e^{-cT}) \leq \frac{c_\eta}{\gamma} b(\widetilde{x}),
\end{aligned}
$$

where we define

$$
c_\eta = \sup_{\substack{h \in [-L,L] \\ j \in \{1,\ldots,K\} \\ t \in [0,T]}} \left\{ \int_{\mathbb{R}} (1+hz) \eta^{\mathbf{P}}(t, e_j, dz) \right\} < \infty.
$$

Clearly $\frac{c_\eta}{c} < 1$ for sufficiently large c, so that the MDM is contracting. $\square$

We now make an assumption that ensures the continuity of the reward function and the transition kernel. This guarantees that the optimality operator $\mathcal{T}$ maps continuous functions into continuous functions.

Assumption 5.5. *For any real sequence $\{(t_n, \sigma_n)\}_{n\in\mathbb{N}}$, with $(t_n, \sigma_n) \in [0,T) \times \Delta^K$, such that $(t_n, \sigma_n) \xrightarrow[n\to\infty]{} (t, \sigma)$, the functions $u^i(t, \sigma, z)$ given in Proposition 3.7 satisfy*

$$
\lim_{n\to\infty} \sup_{z \in supp(\eta^{\mathbf{P}})} |u^i(t_n, \sigma_n, z) - u^i(t, \sigma, z)| = 0\,,
$$

where $supp(\eta^{\mathbf{P}})$ indicates the set $\{z \in \mathbb{R} :\ \eta^{\mathbf{P}}(t, e_i, z) \neq 0,\ t \in [0,T],\ i \in \{1, \ldots, K\}\}$.

Proposition 5.6. *Under Assumption 5.5, the mappings $(\widetilde{x}, \alpha) \mapsto r(\widetilde{x}, \alpha)$ and $(\widetilde{x}, \alpha) \mapsto Q_L v(\widetilde{x}, \alpha)$ for every $v \in \mathcal{B}_b$, are continuous on $\widetilde{\mathcal{X}} \times \widehat{\mathcal{A}}$ with respect to the Young topology on $\widehat{\mathcal{A}}$.*

Proof. Let $(\widetilde{x}_n, \alpha_n)$ be a sequence converging to $(\widetilde{x}, \alpha)$ as $n \to \infty$. Then by [45, Theorem 43.5] we have that

$$
\lim_{n\to\infty} \sup_{u \in [0,T]} |\widetilde{\varphi}_u^{\alpha_n}(\widetilde{x}_n) - \widetilde{\varphi}_u^{\alpha}(\widetilde{x})| = 0.
$$

This implies the continuity of the reward function r. Moreover the continuity of the mapping $(\widetilde{x}, \alpha) \mapsto Q_L v(\widetilde{x}, \alpha)$ follows from the fact that, for every function $v \in \mathcal{B}_b$, by Assumption 5.5 the mapping

$$
(\widetilde{x}, \alpha) \mapsto \int_{\mathbb{R}} v(t, w(1+hz), \pi^1(1 + u^1(t, \boldsymbol{\pi}, z)), \ldots, \pi^1(1 + u^1(t, \boldsymbol{\pi}, z))) \eta^{\mathbf{P}}(t, e_i, dz)
$$

is continuous. To prove this we can follow the same lines of [36, Lemma B.1] since, in our setting, $t \mapsto \eta^{\mathbf{P}}(t, e_i, z)$ is continuous and $\lambda^{max} := \sup_{\substack{i \in \{1,\ldots,K\} \\ t \in [0,T]}} \eta^{\mathbf{P}}(t, e_i, dz) < \infty$. $\square$

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