DIPLOMARBEIT

Nicht-fallende Dividendenraten mit Rückversicherung für eine Brownsche Bewegung als Überschussprozess

ausgeführt am

Institut für
Finanz-und Verischerungsmathematik
TU Wien

unter der Anleitung von

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Wien, am 14.12.2023

Abstract

In this thesis, we deal with an optimal dividend payout from a surplus process regulated by a Brownian motion with drift with the additional restriction of ratcheting - the dividend rate must remain constant or increase but cannot decrease. An additional feature in this model is reinsurance. The aim is to create a dividend strategy that maximizes the expected total amount of discounted dividend payments until the potential ruin of a company. We focus on the model with only two dividend rates, where the optimality is achieved by using threshold strategies.
We also provide some examples to illustrate the optimality findings for the models with and without reinsurance.

Kurzfassung

In dieser Arbeit beschäftigen wir uns mit einer optimalen Dividendenauszahlung aus einem Überschussprozess, der durch eine Brownsche Bewegung mit Drift reguliert wird, mit der zusätzlichen Einschränkung des Ratchetings - die Dividendenzahlungsrate kann niemals reduziert werden. Ein weiteres Merkmal in diesem Modell ist die Rückversicherung. Das Ziel besteht nun darin, eine Dividendenstrategie zu finden, um die erwartete Summe der diskontierten Dividendenzahlungen bis zum möglichen Ruin eines Unternehmens zu maximieren. Wir konzentrieren uns auf das Modell mit nur zwei Dividendenraten, bei dem die Optimalität durch die Verwendung von Threshold Strategien erreicht wird.
Außerdem geben wir einige Beispiele an, um die Ergebnisse zur Optimalität für die Modelle mit und ohne Rückversicherung zu illustrieren.

Acknowledgments

I would like to extend my deepest appreciation to my mentor, Dr. Julia Eisenberg for her consistent support, invaluable guidance, and dedication of time towards this thesis. Her mentorship and insightful ideas significantly contributed to the completion of this work.
Additionally, I am grateful to my father, brother, and husband for their belief in me and my capabilities.
Last but certainly not least, my heartfelt thanks go to my mother and daughter. They are an integral part of my life, continuously inspiring me to strive for the best version of myself. Their presence and unwavering motivation have been a guiding force, driving me forward each day.

Statutory Declaration

I declare in lieu of an oath that I have written this master thesis myself and that I have not used any sources or resources other than stated for its preparation. This master thesis has not been submitted elsewhere for examination purposes.

Vienna, 14.12.2023

Jelena Trbovic

Contents

Nomenclature

$(\Omega, \mathcal{F}, \mathbb{P})$ Complete probability space

$(\mathcal{F}_t)_{t>0}$ Filtration

$\bar{c}$ The highest dividend rate

η Safety loading

$\Phi(s, X_s^u, u_s)$ Running costs or utility

$\Pi_{x,c}^S$ Set of all admissible dividend ratcheting strategies

$\Psi(T, X_T)$ Terminal costs of utility

τ Ruin time

b Retention level

$C = (C_t)_{t>0}$ Dividend ratcheting strategy

C_t The dividend payout rate at time t

d Premium income

$J(x; C)$ Expected payoff of strategy C

P Probability

S Set of dividend rates

$V_{(x,c)}^S$ Value function

W Brownian motion

X_t^C Controlled surplus process

X_t Surplus process

$X_0 = x$ Initial surplus

Y_i Claim

z Threshold function

$z(c_1)$ Threshold at dividend rate level c_1

1 Introduction

In actuarial science and quantitative finance, determining the best way to distribute dividends to shareholders from a surplus process is a well-known challenge. Such a problem can be formally described in a variety of ways in terms of objective functions and constraints, depending on risk preferences, boundary conditions and the specific circumstances. In 1957 De Finetti in [Fin57] first proposed to measure the performance of an insurance portfolio by the expected discounted sum of dividends payments until the time of ruin instead of the measure by the probability of ruin. Since then many studies have been written on maximizing projected discounted dividends, including modeling the total surplus of an insurance firm using a Brownian motion, a compound Poisson process, or a general Levy process with an infinite or finite time horizon.
Jeanblanc-Picqué and Shiryaev [M J95], along with Asmussen and Taksar [AT97], proposed a constrained dividend rate, meaning that the dividends paid per unit time should not exceed an upper limit. They demonstrate that the optimal dividend strategy under this condition is now a "generalized barrier strategy" referred to as a threshold strategy. In accordance with this approach, dividends are consistently paid at a constant rate when the adjusted surplus exceeds the threshold l, and no dividends are disbursed when the adjusted surplus falls below l.
In [HM22] was demonstrated that the value function represents the unique viscosity solution of a two-dimensional HJB equation. Furthermore, they established that it can be closely approximated through threshold strategies for a finite number of potential dividend rates. In this thesis we deal with the general ratcheting strategy, a dividend method created by Albrecher in [HB18] in which the dividend rate would not decrease over time but would instead rise if the underlying process reached a certain level and would then remain there until the point of ruin. The difference from the existing literature ([HM22]) lies in our approach, which assumes only two dividend rates and factors in the existence of reinsurance within the

insurance company.
The remaining sections of this thesis are structured as follows: We start by presenting fundamental concepts from stochastic control theory. Section 3 focuses on defining reinsurance terminologies essential for this thesis. Section 4 presents the optimal dividend strategy. The subsequent section 5 demonstrates numerical examples. For those interested, the R code employed for these examples is available in Section 7.

2 Stochastic control theory

The upcoming chapter is based on Professor Peter Grandits's "Stochastic Control Theory" lecture at Technical University Vienna and [Xue]. In this chapter, we will attempt to briefly explain the subject and goal of stochastic control theory. Stochastic control theory focuses on optimizing dynamic systems modeled through state processes. By utilizing external control in the form of a controlled process, the aim is to achieve an optimization goal defined by an objective functional.

2.1 The stochastic control problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{F}$ is a σ-algebra. Let's set a time horizon $T > 0$ and let $W = (W_t)_{t\in(0,T)}$ be a d-dimensional standard Brownian motion. The following components are necessary to describe the control problem:

- The control process $u = (u_t)_{t\in[0,T]}$ is a progressively measurable stochastic process taking values in $\mathcal{U} \subset \mathbb{R}^d$.

- The state process $X = (X_t)_{t\in[0,T]}$ is defined by a stochastic differential equation (or diffusion equation) in the form:

$$dX_t = a(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t \tag{2.1}$$

  with $X_0 = x$, where $a(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n\times d}$ are Borel-measurable functions. $b(\cdot,\cdot,\cdot)$ represents the drift coefficient, and $\sigma(\cdot,\cdot,\cdot)$ represents the diffusion coefficient of the diffusion process. Here, the notation X_t symbolizes the dependence of the state process on the control process u.

- The objective functional $J(t, x, u)$ is given by:

$$J(t, x, u) = \mathbb{E}\int_t^T \Phi(s, X_s^u, u_s)ds + \Psi(T, X_T) \tag{2.2}$$

where $\Phi(s, X_s^u, u_s)$ denotes the running costs or utility, and $\Psi(T, X_T)$ signifies the terminal costs of utility.

- The set of admissible control processes $\mathcal{A}(t,x) \subseteq (u_s)_{s\in[t,T]}$ for $(t,x) \in [0,T]$ consists of progressively measurable processes that have a unique strong solution on $[t,T]$ according to 2.1 and for which 2.2 is well-defined.

- The value function of the optimization problem is defined as:

$$V(t,x) = \sup_{u\in\mathcal{A}(t,x)} J(t,x,u)$$

The objective now is to compute the value $V(0,x_0)$ for a given initial value x_0 and to determine an optimal control process u^* such that the objective functional is maximized, hence satisfying $V(0,x_0) = J(0,x_0,u^*)$.

In solving stochastic control problems, the method used is dynamic programming, rooted in the Bellman principle, devised by mathematician Richard Bellman. This principle breaks down an optimization problem into smaller, similar sub-problems, whose solutions are then synthesized to form the overall solution. This approach yields the Hamilton-Jacobi-Bellman equation (HJB equation), a partial differential equation whose solution yields the optimum [R L88]
To derive the HJB (Hamilton-Jacobi-Bellman) equation, we utilize the Bellman principle. This principle states that behaving optimally within the interval $[t, t_i]$ and also acting optimally after t leads to a global optimum. The Bellman principle is formulated as follows:

$$V(t,x) = \sup_{u\in A(t,x)} \mathbb{E}[\int_t^{t_1} \Phi(s, X_s, u)ds + V(t_1, X_{t_1})] \tag{2.3}$$

We assume that the value function is sufficiently smooth, i.e., $V(t,x) \in \mathcal{C}^{1,2}$. Applying the Itô formula to $V(t_1, X_{t_1})$ gives:

$$\begin{aligned} V(t_1, X_{t_1}^u) = V(t,x) + \int_t^{t_1} (V_t(s, X_s^u) + b(s, X_s^u, u_s)D_x V(s, X_s^u) + \\ \frac{1}{2} tr(a(s, X_s^u, u_s)D_{xx}V(s, X_s^u)))ds + \int_t^{t_1} D_x V(s, X_s^u, u_s)dW_s \end{aligned}$$

Next, substituting $V(t,X)$ into 2.3 and assuming that

$$\int_t^{t_1} V_x(s,X_s^u)\sigma(s,X_s^u,u^s)dW_s$$

is a martingale for $t_1 > t$, and therefore has an expected value of zero, we get:

$$V(t,x) = \sup_{u\in A(t,x)} \mathbb{E}_{t,x}[\int_t^{t_1} \Phi(s,X_s,u) + V(t,x) +$$
$$\int_t^{t_1} V_t(s,X_s^u) + b(s,X_s^u,u_s)D_xV(s,X_s^u) + \frac{1}{2}tr(a(s,X_s^u,u_s)D_{xx}V(s,X_s^u)))ds$$

One divides by $(t_1 - t)$ and takes the limit as t_1 approaches t, assuming permissible interchanges. Then, we obtain the HJB equation:

$$0 = \sup_{u\in\mathcal{U}}\{\Phi(t,x,u) + V_t(t,x) + b(t,x,u)D_xV(t,x) + \frac{1}{2}tr(a(s,X_s^u,u_s)D_{xx}V(s,X_s^u))\}$$

With

$$\mathcal{L}^uV(t,x) := V_t(t,x) + b(t,x,u)D_xV(t,x) + \frac{1}{2}tr(a(t,x,u)D_{xx}V(t,x))$$

a simplified representation of the HJB equation follows:

$$0 = \sup_{u\in\mathcal{U}}\{\Phi(t,x,u) + \mathcal{L}^uV(t,x)\} \tag{2.4}$$

To determine an optimal control process u^*, the following procedure is followed:

1. Determine the maximum $u = \hat{u}(t,x)$ in 2.4 . If this exists, then it is dependent on V_t, V_x, and V_{xx}, hence $\hat{u}(t,x) = \tilde{u}(t,x,V_t(t,x),V_x(t,x),V_{xx}(t,x))$.

2. Substitute the u from step 1 into 2.4, which yields a partial differential equation for $V(\cdot,\cdot)$ with the boundary condition $V(T,x) = \Psi(T,x)$. Solving this boundary value problem provides potential candidates for the value function of the optimization problem.

3 Reinsurance

The considerations in this chapter are based on [Car83], [Car13] and [Kil91]. We introduce now concepts which are frequently used in this thesis.

3.1 Definition of reinsurance

The shortest explanation of reinsurance can be found in [Kil91] who sees reinsurance as the insurance of an insurance company. Kiln explicitly points out that this does not include the insurance of the insurer's property, but only the underwriting business for sharing the liability assumed by an insurer.
Reinsurance, sometimes also known as cession, is distinct from other forms of insurance in three main ways:

1. In reinsurance, two insurance companies make a deal, which is different from normal insurance where it's usually just one person or company and an insurer. In reinsurance, one insurer is the reinsurer and the other is the reinsured, also called the cedant or ceding company. Reinsurers can also pass on the risk they've taken, called retrocession. Then, the original ceding company becomes the retrocedent, and the reinsurer becomes the retrocessionaire.
2. In regular insurance, what is being protected — like a person, property, or something you expect to gain — can face harm or loss. The insurance company gives direct protection for these risks. Then, if part of that risk moves from the main insurer to the reinsurer, the reinsurer starts being responsible for paying some of the claim, but only after the main insurer has paid a part of it first.

3. In insurance, there's a basic rule called the indemnity principle. It means that when something bad happens, the insured person gets money to cover the real financial loss, not more. But some regular insurance plans don't always stick to this rule. For example, life insurance might pay a set amount when something specific happens, like death, no matter the real loss.
But when it comes to reinsurers, they always follow this rule. They only pay a part of the money needed. The primary insurer is responsible for a certain percentage of the payment. This way, the insured person gets the right amount of money to cover their actual financial loss.

3.2 The role of reinsurance

There are various reasons why insurance companies fail. Some, like being inefficient or not charging enough, happen in other industries too. But the most common reason is when unexpected claim costs rise, which is the biggest expense for the insurance industry. Reinsurance mainly protects against these kinds of failures. At first, reinsurance was meant to help insurance companies deal with really big losses that could hurt their ability to handle claims. But as time went on, it started covering fluctuations in the overall amount of claims across their whole set of policies. Dr. F. L. Tuma described in [Tum33] that the fundamental aim of reinsurance is purely technical, involving the reduction of risks that an insurance company has assumed. R.L.Carter in [Car83] mentions that reinsurance availability offers additional advantages. He also summarizes the roles of reinsurance:

- The primary functions:
  - to safeguard insurers from underwriting losses that might compromise their solvency;
  - to stabilize underwriting outcomes;
  - to enhance an insurer's flexibility in terms of the types and sizes of risks and the volume of business he can underwrite;
  - to further divide the risk of loss.
- Due to the expansion of the reinsurance industry, prominent reinsurance companies and brokers are able to provide various secondary

services such as insurance underwriting, claims management, administrative support, and technical assistance.

- "Reinsurance is both a risk management and a financing decision" [Pla05]. It is the best method of risk sharing among risk-takers, working as a key tool for managing risks. When insurers buy reinsurance, it is like any regular company deciding to get insurance in the first place. Buying reinsurance specifically lowers the minimum capital requirement as a financing tool.

3.3 Forms of reinsurance

The following section is primarily based on [Car13]. Additional reference literature includes [Dum17],[Pfe99], and [Mar14]. There are several ways to categorize reinsurance.

- Type of business
  - Active reinsurance: refers to a situation where the reinsurer takes an active role in selecting and underwriting risks. In this type of reinsurance, the reinsurer has more control and involvement in choosing the risks they wish to cover. They actively participate in decision-making regarding the risks they want to reinsure.
  - Passive reinsurance: occurs when the primary insurer decides which risks to cede to the reinsurer. In this scenario, the reinsurer accepts risks that the primary insurer chooses to transfer without actively participating in the selection process. The reinsurer plays a more passive role and accepts risks as offered by the primary insurer.
- Technical form
  - Proportional reinsurance happens when the reinsurer takes on a predetermined percentage of a loss. Payments made to the primary insurer are determined by an agreed-upon sharing ratio. It's crucial to understand that not only losses but also liability and premiums are divided based on the agreed percentage in this type of reinsurance.

- Non-proportional or excess reinsurance involves the reinsurer covering a portion of the loss only when a specific threshold, known as the priority, is surpassed. In this type of reinsurance, there is no proportional distribution of the loss. Multiple non-proportional reinsurance contracts can be arranged for the same risk. This approach enables reinsurers to accurately anticipate the maximum claim amount that will be covered.

- Legal form
  - Facultative reinsurance works by the reinsurer evaluating each risk individually before deciding to take it on. It's commonly used for high-risk situations that could really affect the main insurer. With facultative reinsurance, the ceding company always looks for the right insurers for coverage in these cases.
  - Compulsory reinsurance differs from covering individual high-level risks by insuring a set of risks within a defined range. The primary insurer is required to pass on all business falling within this range to the reinsurer, who must provide reinsurance without the choice to refuse. Here, the principle of good faith becomes crucial because the reinsurer isn't aware of the specific individual risks being transferred.

Since we're using proportional reinsurance in this thesis, let's quickly talk about what it really means. "Quota share" and "Surplus share" reinsurance are the two most popular types of proportional reinsurance.
The quota share contract is a simple form of reinsurance. Here, the reinsurer agrees to cover a set percentage of every risk the main insurance company takes on. They share both the premiums and the losses accordingly. The reinsurer gets the same percentage of all the premiums, minus a set commission. It is straightforward and helps keep costs low for the main insurer. However, there are downsides. The reinsurer ends up taking on risks that the main insurer could handle on its own because the main insurer can't assess these risks.
The proportionate sharing of premiums and losses characterizes both surplus share and quota share reinsurance. The fundamental distinctions between the two are as follows:

- In surplus treaties, the reinsurer only assumes responsibility for the part of a risk that surpasses its own retention limit.

- Quota share reinsurance is applicable to any insurance category, while surplus treaties are limited to property and other insurance classes where the insurer's maximum potential liability is defined as a specified sum insured or policy limit.

4 Optimal dividend strategy in a Brownian model

The content of this chapter mainly follows the book "Stochastic Control in Insurance" [Sch08] and the papers "Optimal Ratcheting of Dividends in a Brownian Risk Model" [HM22] and "Optimality results for dividend problems in insurance" [AT09]. In this chapter, we aim to determine the optimal dividend strategy for an insurance company where the claims are reinsured by constant proportional reinsurance. The objective is to maximize dividend payouts until ruin occurs. We will determine the optimal strategy for a pair of dividend rates, demonstrating that a threshold strategy represents the most advantageous approach.

4.1 Model

We're examining a diffusion approximation of the classical risk model (detailed in [Sch08], appendix D), where claims are reinsured by proportional reinsurance with a retention level b. The premium income rate is denoted by $d = (1+\eta)\lambda\mu$, follows the expected value principle with a safety loading $\eta > 0$. Here, $\mathbb{E}[X_t]$ amounts to η times the expected outflow within a unit interval. For a claim Y_i, the cedent pays bY_i while the reinsurer pays $(1-b)Y_i$. The premium rate for reinsurance is $(1+\theta)(1-b)\lambda\mu$, and for the insurer, it's $d = (b(1+\theta) - (\theta - \eta))\lambda\mu$. Utilizing the diffusion approximation, we get for the post-reinsurance surplus:

$$x + (b\theta - (\theta - \eta))\lambda\mu t + b\sqrt{\lambda\mu_2} W_t$$

Here, W represents a standard Brownian motion. To prevent the problem from having a trivial solution, we assume $\theta > \eta$. For simplification, we denote $\lambda\mu = 1$ and $\sigma = \sqrt{\lambda\mu_2}$. The company's surplus process is expressed as:

$$X_t^b = x + (b\theta - (\theta - \eta))t + \sigma b W_t.$$

In a complete probability space defined by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathcal{P})$, a company utilizes a portion of its surplus to pay dividends to shareholders at rates within a subset S of the interval $[0, \bar{c}]$, where $0 \leq \bar{c} < S$ represents the highest allowed dividend rate. The dividend payout rate at time t is denoted by C_t. A dividend ratcheting strategy $C = (C_t)_{t>0}$ is defined based on an initial surplus $X_0 = x$ and a minimum dividend rate $c \in S$ at $t = 0$. A strategy is considered admissible if it is non-decreasing, right-continuous, adapted to the filtration $(\mathcal{F}_t)_{t>0}$, and satisfies $C_t \in S$ for all t. The controlled surplus process can be represented as:

$$X_t^{b,C} = X_t^b - \int_0^t C_s ds \tag{4.1}$$

We define $\Pi_{x,c}^S$ as the collection of all admissible dividend ratcheting strategies, where $x \geq 0$ represents the initial surplus, and $c \in S$ denotes the minimum initial dividend rate. For a given $C \in \Pi_x^S$, we define:

$$J(x; C) = \mathbb{E}\left[\int_0^\tau e^{-qs} C_s ds\right]$$

where $q > 0$ and $\tau = \inf\{t \geq 0 : X_t^C < 0\}$ stands for the ruin time. Thus, for any initial surplus $x \geq 0$ and initial dividend rate c, our objective is to maximize

$$V^S(x, c) = \sup_{C \in \Pi_{x,c}^S} J(x; C)$$

This represents the value function. As the company immediately goes to ruin if its initial capital is zero, it's important to clarify that $V^S(0, c) = 0$ for all $c \in S$.

Remark 1. *[HM22] Without the ratcheting constraint, the dividend optimization problem was thoroughly investigated in the published works (see, e.g., Shreve, Lehoczky, and Gaver [SG84], Asmussen and Taksar [AT97], and Gerber and Shiu [GS04]). This nonratcheting problem is one-dimensional in contrast to the ratcheting optimization problem. If* $V_{NR}(x, c)$ *represents* *the optimal value function, then* $V^S(x,c) \leq V_{NR}(x,c)$ *holds true for all* $x \geq 0$*. The function* V_{NR} *is increasing, concave, twice continuously differentiable with* $V_{NR}(0,c) = 0$ *and* $\lim_{x\to\infty} V_{NR}(x,c) = \frac{\bar{c}}{q}$*. Therefore, it satisfies the Lipschitz condition with the Lipschitz constant* $V_{NR}'(0,c)$*.*

Now, we gather some boundedness and monotonicity properties of the optimal value function of the dividend optimization problem with the ratcheting constraint. We'll restrict S to just two elements, specifically, c_1 and c_2, where $c_1 < c_2$ and $c_2 < \bar{c}$.

Proposition 1. *The value function* $V^S(x,c)$ *is bounded by* $\bar{c}/q$*, nondecreasing in* x*, and nonincreasing in* c*.*

Proof. Firstly, we derive the boundedness result from the fact that the discounted value of paying the maximum rate $\bar{c}$ or in our case for $S = \{c_1, c_2\}$ up to infinity is $\frac{c_2}{q}$.
Secondly, for $c_1 < c_2$ we have $\Pi^S_{x,c_2} \subset \Pi^S_{x,c_1}$ and this proves non-increasing in c.
To show that the optimal value function is non-decreasing in x we will define similarly to [EK23] the strategy $\tilde{C}_s$ for $(t, x+h)$:

$$\tilde{C}_s = \begin{cases} C_s & \text{if } 0 \leq s < \tau^C; \\ c_2 & \text{otherwise.} \end{cases}$$

where $h > 0$, $\epsilon > 0$ and C is an ϵ-optimal strategy for (t,x).
Then, we have

$$V(t,x+h) - V(t,x) \geq V^{\tilde{C}}(t,x+h) - V^C(t,x) - \epsilon =$$

$$= \mathbb{E}_{(t,x+h)}[\int_0^{\tau^{\tilde{C}}} e^{-qs}\tilde{C}_s ds] - \mathbb{E}_{(t,x)}[\int_0^{\tau^C} e^{-qs} C_s ds] - \epsilon$$

$$\geq \mathbb{E}_{(t,x+h)}[\int_{\tau^C}^{\tau^{\tilde{C}}} \bar{c}\exp^{-qs} ds] - \epsilon \geq K_h - \epsilon$$

where K_h and ϵ can be selected separately of C.

$\square$

Proposition 2. *There exists a constant $K > 0$ such that*

$$0 \le V^S(x_2, c_1) - V^S(x_1, c_2) \le K[(x_2 - x_1) + (c_2 - c_1)]$$

for all $0 \le x_1 \le x_2$ and $c_1, c_2 \in S$ with $c_1 < c_2$.

Proof. According to Proposition 1, when we consider a set $S = \{c_1, c_2\}$, where $c_1 < c_2$:

$$0 \le V^S(x_2, c_1) - V^S(x_1, c_2) \tag{4.2}$$

for all $0 \le x_1 \le x_2$ and $c_1, c_2 \in S$ with $c_1 < c_2$.
Similarly as in [HM22] we aim to demonstrate the existence of a positive value K_1 such that the inequality below holds for any $C \in S$ and for all $0 \le x_1 \le x_2$:

$$V^S(x_2, c) - V^S(x_1, c) \le K_1(x_2 - x_1). \tag{4.3}$$

$$J(x_2; C) \ge V^S(x_2, c) - \varepsilon,$$

holds for a chosen $\varepsilon > 0$ and c belonging to the set of admissible strategies with initial capital x_2 $\Pi^S_{x_2,c}$. The corresponding control process is given by

$$X_t^{b,C} = x_2 + \int_0^t (b\theta - (\theta - \eta) - c_1 I_{t<\hat{\tau}} - c_2 I_{t\ge\hat{\tau}})\, ds + b\sigma W_t,$$

where $\hat{\tau}$ is the time, when the dividend increases to c_2. We consider τ as the ruin time of the process X_t^C and introduce $\widetilde{C} \in \Pi^S_{x_1,c}$ as $\widetilde{C}_t = C_t$ and the corresponding control process is now:

$$X_t^{b,\widetilde{C}} = x_1 + \int_0^t (b\theta - (\theta - \eta) - c_1 I_{t<\hat{\tau}} - c_2 I_{t\ge\hat{\tau}})\, ds + b\sigma W_t.$$

If $\widetilde{\tau} \le \tau$ represents the ruin time of the process $X_t^{\widetilde{C}}$ it holds that $X_t^C - X_t^{\widetilde{C}} = x_2 - x_1$ for $t \le \widetilde{\tau}$. Introducing $C_u^{\widetilde{\tau}} = C_{\widetilde{\tau}+u}$ allows us to express:

$$\begin{aligned}
J(x_2; C) - J(x_1; \widetilde{C}) &= \mathbb{E}\left[\int_{\widetilde{\tau}}^{\tau} e^{-qs}(c_1 I_{s<\hat{\tau}} + c_2 I_{s\ge\hat{\tau}})\, ds\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_{\widetilde{\tau}}^{\tau} e^{-qs}(c_1 I_{s<\hat{\tau}} + c_2 I_{s\ge\hat{\tau}})\, ds \middle| \mathcal{F}_{\widetilde{\tau}}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-q\widetilde{\tau}} \int_0^{\tau-\widetilde{\tau}} e^{-qu}(c_1 I_{\widetilde{\tau}+u<\hat{\tau}} + c_2 I_{\widetilde{\tau}+u\ge\hat{\tau}}) du \middle| \mathcal{F}_{\widetilde{\tau}}\right]\right] \\
&\le \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau-\widetilde{\tau}} e^{-qu}(c_1 I_{\widetilde{\tau}+u<\hat{\tau}} + c_2 I_{\widetilde{\tau}+u\ge\hat{\tau}}) du \middle| \mathcal{F}_{\widetilde{\tau}}\right]\right] \le V^S(x_2 - x_1, 0).
\end{aligned}$$

The final inequality arises from adjusting the stopping times and stems from Theorem 2 of Claisse, Talay, and Tan in [JT16]. Therefore, this leads us to:

$$\begin{aligned}
V^S(x_2, c) - V^S(x_1, c) &\leq J(x_2; C) - J(x_1; \widetilde{C}) + \varepsilon \\
&\leq V^S(x_2 - x_1, 0) + \varepsilon \\
&\leq V_{NR}(x_2 - x_1) + \varepsilon \\
&\leq K_1(x_2 - x_1) + \varepsilon.
\end{aligned}$$

This suggests that based on Proposition 1, 4.3 leads to the determination that $K_1 = V'_{NR}(0)$.
We will demonstrate that for c_1 and c_2 in S where $c_1 \leq c_2$, there exists a positive K_2 such that:

$$V^S(x, c_1) - V^S(x, c_2) \leq K_2(c_2 - c_1). \tag{4.4}$$

Consider a scenario where there is a positive value ε and a set C within $\Pi^S_{x_1,c}$ such that $J(x; C) \geq V^S(x, c1) - \varepsilon$. Now, define $\widehat{T}$ as the minimum time t when C_t surpasses c_2, and let τ represent the ruin time of the X^C_t process. We introduce $\widetilde{C}$ in Π^S_{x,c_2} such that $\widetilde{C}_t = c_2 I_{t<\widehat{T}} + C_t I_{t\geq\widehat{T}}$. $X^{\widetilde{C}}_t$ signifies the controlled surplus process associated with it and $\bar{\tau} \leq \tau$ represents the corresponding ruin time. When considering $\widetilde{C}_s - C_s \leq c_2 - c_1$, $X^C_\tau = X^C_{\bar{\tau}} - X^{\widetilde{C}}_{\bar{\tau}} \leq (c_2 - c_1)\bar{\tau}$. Following Remark 1, we observe that:

$$\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} C_s e^{-qs} ds\right] \leq \mathbb{E}\left[V^S(X^C_{\bar{\tau}}, 0)\right]$$

This inequality arises due to a modification or adjustment in the stopping times, influenced by Theorem 2 by Claisse, Talay, and Tan [JT16]. Consequently,

$$\begin{aligned}
\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} C_s e^{-qs} ds\right] &\leq \mathbb{E}\left[V_{NR}(X^C_{\bar{\tau}})\right] \\
&\leq \mathbb{E}\left[V_{NR}((c_2 - c_1)\bar{\tau})\right] \\
&\leq K_1 \mathbb{E}\left[e^{-q\bar{\tau}} \bar{\tau}(c_2 - c_1)\right].
\end{aligned}$$

Therefore, we can represent the inequality

$$
\begin{aligned}
V^S(x,c_1) - V^S(x,c_2) &\leq J(x;C) + \varepsilon - J(x;\widetilde{C}) \\
&= \mathbb{E}\Big[\int_0^{\bar{\tau}} \Big(C_s - \widetilde{C}_s\Big) e^{-qs} ds\Big] + \mathbb{E}\big[\int_{\bar{\tau}}^{\tau} C_s e^{-qs} ds\big] + \varepsilon \\
&\leq 0 + \mathbb{E}\big[\int_{\bar{\tau}}^{\tau} C_s e^{-qs} ds\big] + \varepsilon \\
&\leq K_1 \mathbb{E}[e^{-q\bar{\tau}} \bar{\tau}(c_2 - c_1)] + \varepsilon \\
&\leq K_2(c_2 - c_1) + \varepsilon.
\end{aligned}
$$

This inequality leads to a conclusion denoted by equation 4.4 if we take $K_2 = K_1 \max_{t \geq 0} e^{-qt} t$. The establishment of this relationship derives from the connections among equations 4.2, 4.3, and 4.4.

$\square$

4.2 Hamilton-Jacobi-Bellman equations

First of all, we motivate the Hamilton-Jacobi-Bellman equation. Let $\epsilon > 0$, then for each $x > 0$ exist a ϵ-optimal strategy c^x such that $J(x, c^x) \geq V(x, c) - \epsilon$. We choose a dividend rate c to be paid out until a point in time $h > 0$, or until the time of ruin τ if this occurs before. Hence we can write:

$$V(x,c) \geq J(x,c) = \mathbb{E}_x[\int_0^{\tau\wedge h} e^{-qs} c ds] + \mathbb{E}_x[I_{\{\tau>h\}} \int_h^\tau e^{-q(s+h)} u_s ds]$$

Since ϵ is arbitrary we find:

$$V(x,c) \geq \frac{c}{q}(1 - \mathbb{E}_x[e^{-q(\tau\wedge h)}] + e^{-qh}\mathbb{E}_x[V(X_{\tau\wedge h}, c)] \tag{4.5}$$

We assume that $V(x,c)$ is twice continuously differentiable. Using Ito's formula we get:

$$V(X_{\tau\wedge h}, c) = V(x,c) + \int_0^{\tau\wedge h} \sigma b V(x_s,c)' dW_s + \\ \int_0^{\tau\wedge h} ((b\theta - (\theta - \eta) - c)V(x_s,c)' + \frac{\sigma^2 b^2}{2} V(x)'', c) ds$$

We assume now that $\int_0^t V(x_s,c)' dW_s$ is a martingale. Combining the last equation and 4.5 and dividing by h gives

$$c\mathbb{E}[\frac{1-e^{-q(\tau\wedge h)}}{qh}] - \frac{1-e^{-qh}}{h} V(x,c) \\ + \mathbb{E}[\frac{1}{h}\int_0^{\tau\wedge h} ((b\theta - (\theta-\eta) - c)V(X_s,c)' + \frac{\sigma^2 b^2}{2} V(X_s,c)'') ds] \leq 0$$

If the limit and expectation can be interchanged, for $h \to 0$ it follows

$$\frac{\sigma^2 b^2}{2} V''(x,c) + (b\theta - (\theta - \eta) - c)V'(x,c) - qV(x,c) + c \leq 0$$

Since this inequality has to be true for all $0 \leq c \leq \bar{c}$ and it is equality at least for one c, we have the HJB equation with boundary condition:

$$V(0) = 0$$

$$\sup_{0\le c\le \bar{c}}\left\{\frac{\sigma^2 b^2}{2}V''(x,c)+(b\theta-(\theta-\eta)-c)V'(x,c)-qV(x,c)+c\right\}=0$$

First to find a Hamilton-Jacobi-Bellman equation and then to show that the value function is the unique viscosity solution of the corresponding HJB equation with boundary condition $\bar{c}/q$ when x goes to infinity, where $\bar{c} = \max S$ will be our approach to solve this optimisation problem for $S \subset [0, \infty]$ where S is a finite set. The scenario in which an explicit solution can be found is the most practical. The equation must be solved numerically if an explicit solution cannot be found. However, it is only logical to use a numerical solution after confirming that a solution is indeed present (see [HM22] for more optimisations problems and different techniques to show that the value function really solves the equation).
Let us now analyze the scenario $S = \{c\}$ - paying a constant dividend rate c up to the ruin time is the unique admissible strategy in this specific case. The value function $V^c_{(x,c)}$ is the unique solution of the second-order differential equation

$$\mathcal{L}^c(V) := \frac{\sigma^2 b^2}{2}\frac{\partial}{\partial x^2}V + ((b\theta-(\theta-\eta))-c)\frac{\partial}{\partial x}V - qV + c = 0 \tag{4.6}$$

with boundary conditions $V^c(0,c) = 0$ and $\lim_{x\to\infty} V^c(x,c) = c/q$. The particular solution of this differential equation 4.6 is $\frac{c}{q}$. The homogenous solution of 4.6 is of the form:

$$a_1 e^{\lambda_1(c)x} + a_2 e^{\lambda_2(c)x}$$

where a_1,$a_2 \in \mathbb{R}$ and $\lambda_1(c) > 0$ and $\lambda_2(c) < 0$ are the roots of the characteristic equation

$$\frac{\sigma^2 b^2}{2}z^2 + ((b\theta-(\theta-\eta))-c)z - q = 0$$

associated to the operator $\mathcal{L}^c$, that is,

$$\lambda_1(c) := \frac{c-(b\theta-(\theta-\eta))+\sqrt{(c-(b\theta-(\theta-\eta)))^2+2q\sigma^2 b^2}}{\sigma^2 b^2},$$

$$\lambda_2(c) := \frac{c-(b\theta-(\theta-\eta))-\sqrt{(c-(b\theta-(\theta-\eta)))^2+2q\sigma^2 b^2}}{\sigma^2 b^2}$$

Here, we list some fundamental characteristics of λ_1 and λ_2.
1. $\lambda_1(c) = -\lambda_2(c)$ if $c = (b\theta - (\theta - \eta))$ and $\lambda_1^2(c) \leq \lambda_2^2(c)$ if and only if $c - (b\theta - (\theta - \eta)) \geq 0$
2. $\lambda_1'(c) = \frac{1}{\sigma^2 b^2}(1 + \frac{c-(b\theta-(\theta-\eta))}{\sqrt{(c-(b\theta-(\theta-\eta)))^2+2q\sigma^2 b^2}})$ and
$\lambda_2'(c) = \frac{1}{\sigma^2 b^2}(1 - \frac{c-(b\theta-(\theta-\eta))}{\sqrt{(c-(b\theta-(\theta-\eta)))^2+2q\sigma^2 b^2}})$, so $\lambda_1'(c)$,$\lambda_2'(c)$ $\in(0, \frac{2}{\sigma^2 b^2})$.

The general solution of the differential equation 4.6 is:

$$\frac{c}{q} + a_1 e^{\lambda_1(c)x} + a_2 e^{\lambda_2(c)x}$$

The solutions of $\mathcal{L}^c(V) = 0$ with boundary condition $V(0) = 0$ follow this form:

$$\frac{c}{q}(1 - e^{\lambda_2(c)x}) + a(e^{\lambda_1(c)x} - e^{\lambda_2(c)x})$$

with a $\in \mathbb{R}$. And finally, the unique solution of $\mathcal{L}^c(V) = 0$ with boundary condition $V(0) = 0$ and $\lim_{x\to\infty} V(x) = \frac{c}{q}$ corresponds to $a = 0$, so that

$$V^c(x, c) = \frac{c}{q}(1 - e^{\lambda_2(c)x}) \tag{4.7}$$

We also have that $V^c(\cdot, c)$ is increasing and concave. Given a set $S \subset [0, \infty]$ with $\bar{c} = \max S < \infty$, we have that

$$V^S(x, c) \leq V^{\bar{c}}(x, \bar{c}) = \frac{\bar{c}}{q}(1 - e^{\lambda_2(\bar{c})x})$$

and so, by Remark 1, we conclude that $\lim_{x\to\infty} V^S(x, c) = \frac{\bar{c}}{q}$ for any $c \in S$.

4.3 Hamilton-Jacobi-Bellman equations for $S = \{c_1, c_2\}$

Initially, our aim is to derive the HJB equation for the generalized set S - when S would consist of finitely many dividend rates. Subsequently, we'll demonstrate how this equation would manifest if the set S were reduced to just two elements: c_1 and c_2. Hence, we're examining the scenario:

$$S = \{c_1, c_2\},$$

where $0 < c_1 < c_2 < \bar{c}$. We note $V^S(x, c_i) = V^{\{c_1,c_2\}}(x, c_i)$ and simplify the notation

$$V^{c_i}(x) = V^S(x, c_i). \tag{4.8}$$

So now we have

$$V^{c_1}(x) \geq V^{c_2}(x) \geq 0,$$

where $V^{c_2}(x) = V^{\{c_2\}}(x, c_2)$ as defined in (4.7). The HJB equation linked to (4.8) is given by:

$$\max\{\mathcal{L}^{c_i}(V^{c_i}(x)), V^{c_{i+1}}(x) - V^{c_i}(x)\} = 0 \tag{4.9}$$

for $x \geq 0$ and $i = 1$.

We demonstrate that V^{c_i} represents the viscosity solution of the respective HJB equation.

Definiton 1. *(a) A locally Lipschitz function $\overline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity supersolution to (4.9) at $x \in (0, \infty)$ if any twice continuously differentiable function $\varphi : [0, \infty) \to \mathbb{R}$ with $\varphi(x) = \overline{u}(x)$ such that $\overline{u} - \varphi$ reaches the minimum at x satisfies*

$$\max\{\mathcal{L}^{c_i}\varphi(x), V^{c_{i+1}}(x) - \varphi(x)\} \leq 0$$

The function φ is called a test function for supersolution at x.

(b) A function $\underline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity subsolution to (4.9) at $x \in (0, \infty)$ if any twice continuously differentiable function $\psi : [0, \infty) \to \mathbb{R}$ with $\psi(x) = \underline{u}(x)$ such that $\underline{u} - \psi$ reaches the maximum at x satisfies

$$\max\{\mathcal{L}^{c_i}\varphi(x), V^{c_{i+1}}(x) - \varphi(x)\} \geq 0$$

The function ψ is called a test function for subsolution at x.

(c) A function $u : [0, \infty) \to \mathbb{R}$ which is both a supersolution and subsolution at $x \in [0, \infty)$ is called a viscosity solution to (4.9) at x.

Theorem 1. *The optimal value function $V^{c_i}(x)$ for $1 \leq i < n$ is the unique viscosity solution of the associated HJB equation (4.9) with boundary condition $V^{c_i}(0) = 0$ and $\lim_{x\to\infty} V^{c_i}(x) = \bar{c}/q$.*

Theorem 2. *The optimal value function $V^{c_i}(x)$ for $1 \leq i < n$ is the smallest viscosity supersolution of the the associated HJB equation (4.9) with boundary condition $V^{c_i}(0) = 0$ and $\lim_{x\to\infty} V^{c_i}(x) \geq \bar{c}/q$.*

In our case where $S = \{c_1, c_2\}$ and $0 < c_1 < c_2 < \bar{c}$, we note $V^S(x, c_i) = V^{\{c_1,c_2\}}(x, c_i)$ and simplify the notation $V^{c_i}(x) = V^S(x, c_i)$. So now we have $V^{c_1}(x) \geq V^{c_2}(x) \geq 0$. The HJB equation linked to (4.8) is given by $\max\{\mathcal{L}^{c_1}(V^{c_1}(x)), V^{c_2}(x) - V^{c_1}(x)\} = 0$ for $x \geq 0$ and $i = 1$. In a similar manner as previously mentioned, we show that V^{c_i} denotes the viscosity solution for the corresponding HJB equation.

4.4 Comparison of the value functions for $b = 0$ and $b = 1$ and $S = \{c\}$

In this part, we aim to compare the value functions when there's no reinsurance ($b = 1$) against complete reinsurance ($b = 0$) using the same constant strategy c. We assume that $c > \eta$. Several examples and proofs cited in [Eis09] and in [Eis10] were beneficial for this capital. We'll demonstrate that, in certain scenarios with specific values of q, the value function appears more favorable without reinsurance compared to the one with full reinsurance. In conclusion, we'll find that this particular value of q remains consistent across all q values. Hence, we take $S = \{c\}$. In this scenario, the only permissible strategy involves maintaining a constant dividend rate, denoted as c until the occurrence of the ruin time. Next, we'll illustrate two methods for computing the value function. The initial approach involves calculation based on the expectation value, while the second method relies on the time of ruin. Let's begin with the initial approach:
The value function $V^{\{c\}}(x, c)$ for $b = 0$ is the unique solution of the differential equation:

$$\mathcal{L}^c(V) := (\eta - \theta - c)\frac{\partial}{\partial x}V - qV + c = 0 \tag{4.10}$$

with boundary conditions $V^{\{c\}}(0, c) = 0$ and $\lim_{x\to\infty} V^{\{c\}}(x, c) = \frac{c}{q}$. The solutions from equation 4.10 follow this form:

$$\frac{c}{q} + a_1 e^{\lambda_1(c)x}$$

with $a_1 \in \mathbb{R}$, where $\theta_1(c) < 0$ is the solution of the equation

$$(\eta - \theta - c)z - q = 0$$

linked to the operator $\mathcal{L}^c$, that is,

$$\lambda_1(c) := \frac{q}{\eta - \theta - c}$$

The solutions of equation $\mathcal{L}^c(V) = 0$ with boundary condition $V(0) = 0$ follow now this form

$$\frac{c}{q}(1 - e^{\lambda_1(c)x})$$

with $a \in \mathbb{R}$. And finally, the unique solution of $\mathcal{L}^c(V) = 0$ is

$$V^0(x, c) = \frac{c}{q}(1 - e^{\lambda_1(c)x})$$

$$V^0(x, c) = \frac{c}{q}(1 - e^{\frac{q}{\eta-\theta-c}x}) \tag{4.11}$$

Now, considering a retention level of $b = 1$ (no reinsurance) in our model, the value function $V^c_{(x,c)}$ for $b = 1$ is the unique solution of the differential equation:

$$\mathcal{L}^c(V) := \frac{\sigma^2}{2}\frac{\partial}{\partial x^2}V + (\eta - c)\frac{\partial}{\partial x}V - qV + c = 0 \tag{4.12}$$

with boundary conditions $V^c(0, c) = 0$ and $\lim_{x\to\infty} V^c(x, c) = c/q$. The solutions of equation 4.6 follow this form:

$$\frac{c}{q} + a_1^{\lambda_1(c)x} + a_2^{\lambda_2(c)x}$$

with a_1,$a_2 \in \mathbb{R}$, where $\lambda_1(c) > 0$ and $\lambda_2(c) < 0$ are the roots of the characteristic equation

$$\frac{\sigma^2}{2}z^2 + (\eta - c)z - q = 0$$

linked to the operator $\mathcal{L}^c$, that is,

$$\lambda_1(c) := \frac{c - \eta + \sqrt{(\eta - c)^2 + 2q\sigma^2}}{\sigma^2},$$

$$\lambda_2(c) := \frac{c - \eta - \sqrt{(\eta - c)^2 + 2q\sigma^2}}{\sigma^2}$$

The solutions of equation $\mathcal{L}^c(V) = 0$ with boundary condition $V(0) = 0$ follow now this form

$$\frac{c}{q}(1-e^{\lambda_2(c)x})+a(e^{\lambda_1(c)x}-e^{\lambda_2(c)x})$$

with $a \in \mathbb{R}$.

And finally, the unique solution to $\mathcal{L}^c(V)=0$ with boundary condition $V(0)=0$ and $\lim_{x\to\infty} V(x)=\frac{c}{q}$ corresponds to $a=0$, so that

$$V^1(x,c)=\frac{c}{q}(1-e^{\lambda_2(c)x})$$

$$V^1(x,c)=\frac{c}{q}(1-e^{-\frac{(\eta-c)+\sqrt{(\eta-c)^2+2q\sigma^2}}{\sigma^2}}) \tag{4.13}$$

Now, we're utilizing the second method and show for some special values of the rate c with a positive probability that the surplus process for $b=1$ remains positive up to infinity.
The choice of the admissible strategy c for the case $b=0$ which corresponds to buying continuously full reinsurance until the time of ruin leads to a X^{b_0}:

$$X^0_t = x+(\eta-\theta-c)t,$$

with negative drift. As a consequence, the time of ruin τ^{c_1,b_0} can be explicitly computed and that is:

$$\tau=\inf\{t\geq 0 : X^{b_0}_t<0\}$$

$$\tau=\inf\{t\geq 0 : x+(\eta-\theta-c)t<0\}$$

$$\tau^{c,0}=\frac{x}{\theta-\eta+c}$$

$$J(x;c)=\mathbb{E}[\int_0^{\tau^{c,0}} e^{-qs}cds]=c\mathbb{E}[\int_0^{\tau^{c,0}} e^{-qs}cds]$$

$$V^0(x,c) = \frac{c}{q}(1 - e^{\frac{q}{\eta-\theta-c}x})$$

Since $\theta > \eta$ (if not, the insurer could gain a risk-free profit by purchasing full reinsurance and still collecting a positive premium) it follows that the drift $\eta - \theta - c$ is negative. Hence the ruin will occur almost surely and we have:

$$\mathbb{P}_x(\tau^{c,0} = \infty) = 0 \tag{4.14}$$

The choice of the admissible strategy c for the case $b = 1$ which corresponds to no reinsurance until the time of ruin leads to a X^1, which is:

$$X_t^1 = x + (\eta - c)t + \sigma W_t$$

and

$$\tau^{c,1} = \inf\{t \geq 0 : X_t^{b_1} < 0\}$$

$$J(x;c) = \mathbb{E}[\int_0^{\tau^{c,b_1}} e^{-qs} c ds] = \frac{c}{q}\mathbb{E}[1 - e^{-q\tau^{c,b_1}}]$$

Calculating the Laplace transform of $\tau^{c,1}$ (refer to [Sch08], page 216) we get that

$$\mathbb{E}[e^{-q\tau^{c,b_1}}] = e^{\frac{(c-\eta)-\sqrt{(c-\eta)^2+2q\sigma^2}}{\sigma^2}x}$$

It follows:

$$V^1(x,c) = \frac{c}{q}(1 - e^{\frac{(c-\eta)-\sqrt{(c-\eta)^2+2q\sigma^2}}{\sigma^2}x})$$

We refer now to the example (1) in [BS10], page 295.
For $\eta > c$:

$$\mathbb{P}_x(\tau^{c,1} = \infty) = 1 - e^{-(\eta-c)x-|\eta-c|x} = 1 - e^{-(\eta-c)x-(\eta-c)x} = 1 - e^{2x(c-\eta)} > 0$$

We have now from 4.14 and the last equation that

$$\mathbb{P}_x(\tau^{c,0} = \infty) = 0$$

and for $\eta > c$:

$$\mathbb{P}_x(\tau^{c,1} = \infty) > 0$$

We show that in certain situations, with particular values of q, the value function seems better without reinsurance when compared to the function with complete reinsurance. We compare:

$$V^1(x, c) = \frac{c}{q}(1 - e^{\frac{(c-\eta)-\sqrt{(c-\eta)^2+2q\sigma^2}}{\sigma^2}x})$$

and

$$V^0(x, c) = \frac{c}{q}(1 - e^{\frac{q}{\eta-\theta-c}x}).$$

Because the functions appear quite similar and differ only in their exponents, we'll focus our comparison on these exponents. We have now this function:

$$f(q) = \frac{(c-\eta) - \sqrt{(c-\eta)^2 + 2q\sigma^2}}{\sigma^2} - \frac{q}{\eta - \theta - c}$$

The first derivative of $f(q)$ would be:

$$f'(q) = -\frac{1}{\eta - \theta - c} - \frac{1}{\sqrt{(c-\eta)^2 + 2q\sigma^2}}$$

For $q = 0$ it applies:

$$f'(0) = \frac{1}{\theta - \eta + c} - \frac{1}{|c - \eta|}$$

Since $c > \eta$ we have that $|c - \eta| = c - \eta$. This implies that there is a unique minimum q^* of $f(q)$. Now, we know that there is a q_1 so that $f(q_1) = 0$. For each $q \leq q_1$ it holds:

$$\frac{(c-\eta) - \sqrt{(c-\eta)^2 + 2q\sigma^2}}{\sigma^2} \leq \frac{q}{\eta - \theta - c},$$

which implies

$$V^1(x, c) \geq V^0(x, c),$$

what means that the value function without reinsurance looks better when compared to the function with full reinsurance for these particular values of q.
For $q > q_1$ we have:

$$V^1(x, c) < V^0(x, c).$$

4.5 The optimal strategies

In [HM22] is showed for $b = 1$ that, regardless of whether S is finite or an interval with max $S = \bar{c}$, the optimal strategy for sufficiently small $\bar{c}$ is to immediately start paying dividends at the maximum rate $\bar{c}$.

4.6 The optimal dividend strategies

We take $S = \{c_1, c_2\}$, where $c_1 < c_2$. Similarly as in [HB18] we will describe the optimal dividend strategy. Given that the optimal value function V^{c_i} is a viscosity solution to (4.9), there are values of x where $V^{c_i} = V^{c_{i+1}}$, which in our case means $V^{c_1} = V^{c_2}$ and also values of x where $\mathcal{L}^{c_i}(V^{c_i})(x) = 0$, in our case $\mathcal{L}^{c_1}(V^{c_1})(x) = 0$. So now we can partition $(0, \infty)$ into the closed set $D_1 = \{x : V^{c_1}(x) = V^{c_2}(x)\}$ and the open set $E_1 = \{x : V^{c_1}(x) > V^{c_2}(x)\}$. The optimal strategy would be to pay dividend at rate c_1, when the current surplus is in E_1 and to increase dividend rate to c_2 when the current surplus is in D_1.
Since $S = \{c_1, c_2\}$ the optimal strategy of this restricted optimization problem has just three possibilities. The first one is to pay dividends at a rate of c_2 until ruin; the second one is to pay dividends at a rate c_1 until ruin; the third option is to pay dividends at a rate of c_1 and then increase to c_2. We will describe this optimal strategy mathematically.
For $\tilde{S} = S \setminus \{c_2\} = \{c_1\}$ we will define a function $z : \tilde{S} \to [0, \infty)$ and a threshold strategy:

$$\pi^z = (C_{x,c_1})_{(x,c_1)\in[0,\infty)\times S}$$

where $C_{x,c_1} \in \Pi^S_{x,c_1}$ as follow:

- if $i = 2$ dividends will be paid out at a rate of c_2 up to the time of ruin
- if $i = 1$ and $x < z(c_1)$ dividends will be paid out at a rate of c_1 as long as the surplus is less than $z(c_1)$ up to the ruin time; if the current surplus hits $z(c_1)$ before the time of ruin, proceed as follows $C_{x,c_2} \in \Pi^S_{x,c_2}$. We can write

$$(C_{x,c_1})_t = c_1 I_{t<\tilde{\tau}<\tau} + c_1 I_{t<\tau<\tilde{\tau}} + (C_{X_{\tilde{\tau}},c_2}) I_{\tilde{\tau}<t<\tau}$$

where $\tilde{\tau}$ is the first time at which the surplus reaches $z(c_1)$ and τ is the time of ruin.

We call $z(c_1)$ the threshold at dividend rate level c_1 and $z : \tilde{S} \to [0, \infty)$ the threshold function.
The expected payoff of the stationary strategy π^z is defined as:

$$W^z(x, c_1) := J(x; C_{x,c_1})$$

for $i = 1, 2$.

Proposition 3. *We have the following formula for W^z:*

$$W^z(x, c_2) = \frac{c_2}{q}(1 - e^{\lambda_2(c_2)x})$$

$$W^z(x, c_1) = \begin{cases} W^z(x, c_2) & \text{if } x \geq z(c_1); \\ \frac{c_1}{q}(1 - e^{\lambda_2(c_1)x}) + a^z(c_1)(e^{\lambda_1(c_1)x} - e^{\lambda_2(c_1)x}) & \text{otherwise.} \end{cases}$$

where $a^z(c_1) := \frac{W^z(z(c_1),c_2) - \frac{c_1}{q}(1-e^{\lambda_2(c_1)z(c_1)})}{e^{\lambda_1(c_1)z(c_1)} - e^{\lambda_2(c_1)z(c_1)}}$ and $\lambda_1(c_i) > 0$ and $\lambda_2(c_i) < 0$ for $i = 1, 2$ and for $b > 0$, which means if reinsurance is bought, are the roots of the characteristic equation

$$\frac{\sigma^2 b^2}{2} z^2 + ((b\theta - (\theta - \eta)) - c_i)z - q = 0$$

that is,

$$\lambda_1(c_i) := \frac{c_i - (b\theta - (\theta - \eta)) + \sqrt{(c_i - (b\theta - (\theta - \eta)))^2 + 2q\sigma^2 b^2}}{\sigma^2 b^2},$$

$$\lambda_2(c_i) := \frac{c_i - (b\theta - (\theta - \eta)) - \sqrt{(c_i - (b\theta - (\theta - \eta)))^2 + 2q\sigma^2 b^2}}{\sigma^2 b^2}.$$

Proof. We refer to [HB18], proof of Proposition 5.2. □

Let us now maximize the expected payoff $W^z(x, c_i)$ among all threshold functions $z : \tilde{S} \to [0, \infty)$. From now on, we only consider $c_2 > \frac{q\sigma^2}{2\mu}$ (otherwise $z^* = 0$). It's important to note that in this context, we are seeking the

highest expected payoff among all possible threshold strategies, not among all possible admissible strategies. Since $W^z(x, c_2)$ is known, we will solve this optimization problem using a backward recursion. This approach consists of looking for the optimal threshold strategy as a one one-dimensional optimization problem.
Since in our case S consists of only c_1 and c_2, our approach is to find the maximum of $a^z(c_i)$ among all the possible threshold functions, where i = 1. From Proposition 3, we can obtain $W^{z^*}(x, c_1)$ and $z^*(c_1)$. We will now define the continuous function $G : [0, \infty) \to \mathbb{R}$:

$$G_1(y) := \frac{W^z(y, c_2) - \frac{c_1}{q}(1 - e^{\theta_2(c_1)y})}{e^{\theta_1(c_1)y} - e^{\theta_2(c_1)y}} \tag{4.15}$$

for $y > 0$. Since

$$\lim_{y\to\infty} W^z(y, c_2) = \frac{c_2}{q} > \frac{c_1}{q}$$

then for y large enough, $G_1(y) > 0$ and $\lim_{y\to\infty} G_1(y) = 0$. G_1 reaches its maximum in $[0, \infty)$ and

$$z^*(c_1) = \min(arg \max_{y\to\infty} G_1(y))$$

exist because of the continuity of G_1. If $z^*(c_1) > 0$ for $x \in [0, z^*(c_1))$, the function $W^{z^*}(x, c_1)$ fulfills $\mathcal{L}^{c_1}(W^{c_1})(x) = 0$ and for $x \geq z^*(c_1)$, the function $W^{z^*}(x, c_1)$ fulfills $W^{z^*}(x, c_1) = W^{z^*}(x, c_2)$ (this also holds for $z^*(c_1) = 0$).

Theorem 3. *If z^* is the optimal threshold function, then $W^{z^*}(x, c_i)$ is the optimal function $V^{c_i}(x)$ defined in the section 4.1 for $i = 1, 2$.*

Proof. Similarly, as in [HB18]: by Definition 3: $W^{z^*}(x, c_2) = V^{c_2}(x)$. If $i = n - 1 = 1$ (in our case), by definition, $W^{z^*}(x, c_{n-1}) = W^{z^*}(x, c_n) = W^{z^*}(x, c_2) = \frac{c_2}{q}(1 - e^{\lambda_2(c_2)x})$ for $x \geq z^*(c_1)$. □

5 Numerical examples

Now, let's examine a numerical example for $S = \{c_1, c_2\}$ with the parameters $\theta = 0.5$, $\eta = 0.4$, $\sigma = 0.2$, $c_1 = 0.2$, $c_2 = 0.3$ and $q = 0.04$ for $b = 0.01, 0.5$ and 1. To determine the optimal value function V associated with these values, we follow these steps:

- Firstly, we define $\lambda_i(c_1)$ and $\lambda_i(c_2)$, for i=1,2.
- Next, we establish the continuous function $G_i(y)$, initially defined as described in (4.15), as the function $a^z(c_i)$ to determine $z(c_i)$ for $i = 1$. Following this, we determine the value of y representing $z(c_i)$ within $a^z(c_i)$, so that y maximizes G_i, again for $i = 1$.
- After that we redefine $a^z(c_1)$ and use $z(c_1)$ to determine the optimal value function V.

In (4.4) we showed, for $S = c$ with specific values of q (see (2.2)), the value function without reinsurance appears more favorable compared to the one with full reinsurance. In our numerical case we take $c = 0.3$ and demonstrate that this holds true across all $0 \leq q \leq 1$ values (Figure 5.1). Figure 5.2 shows the relationship between value function and the starting capital x for different b values in the classical dividend problem with a ratcheting constraint - dividend can never be decreased. The $z(c_1)$ values for $b = 0.01$, $b = 0.5$ and $b = 1$ are 0.1010101, 3.838384 and 0.8080808, respectively. In this scenario, the best strategy is a threshold strategy detailed in (4.6).
The efficiency loss when decreasing b is relatively small for large initial capital x, but the resulting expected discounted dividends are significantly smaller for both small and intermediate values of x. The relative efficiency loss is not enormous for $b = 0.5$ (see Figures 5.3 and 5.4 for the plots of this difference).

In Figures 5.4 to 5.6, we present similar plots considering the scenario where $c_1 = 0.8$, $c_2 = 0.9$ causing the maximum dividend rate to be at

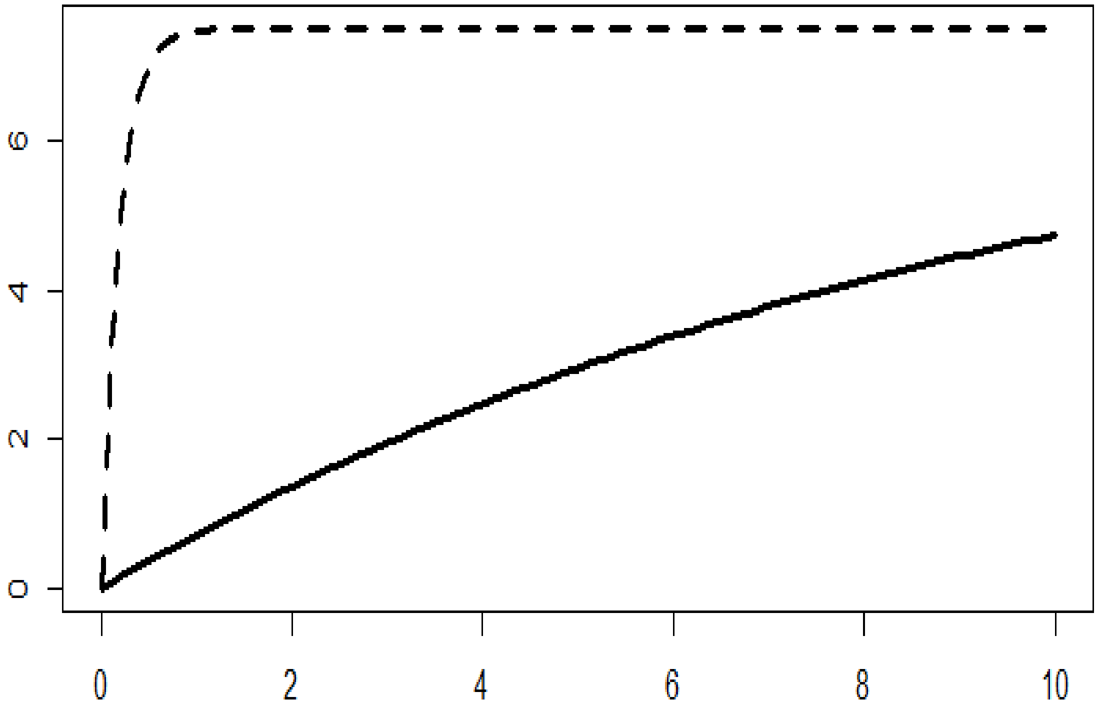

Figure 5.1: Value function for $b = 1$ (dashed) and $b = 0$ (black) for a constant c

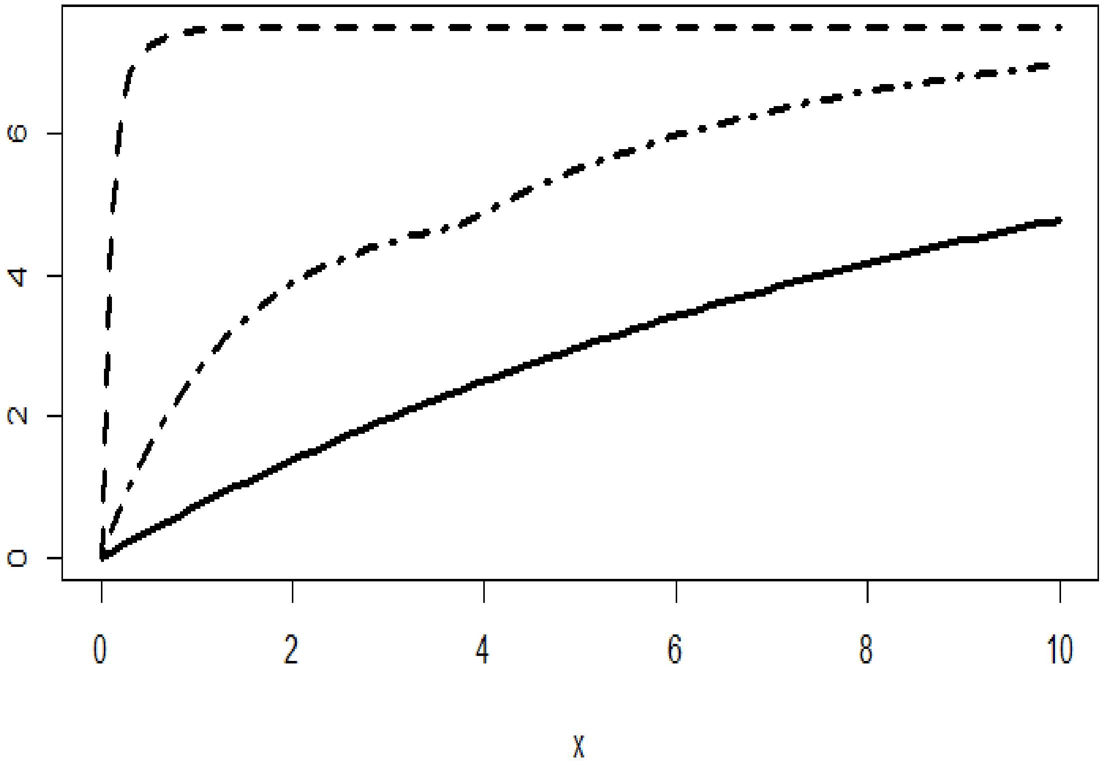

Figure 5.2: Value function for $b = 1$ (dashed), $b = 0.5$ (dotdashed) and $b = 0.01$ (black)

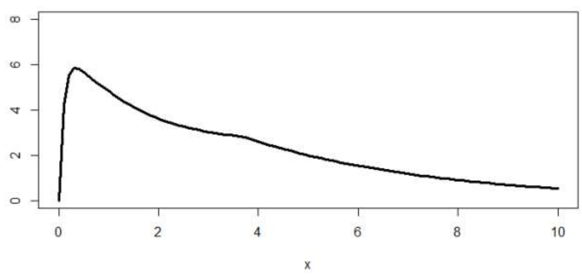

Figure 5.3: Difference in value functions for b=1 and b=0.5

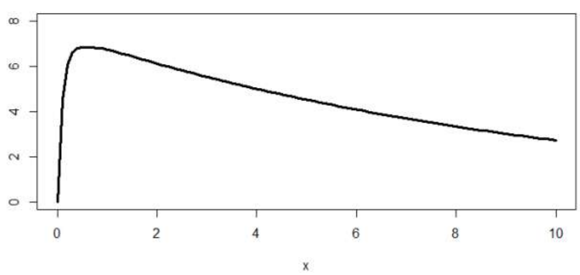

Figure 5.4: Difference in value functions for b=1 and b=0.01

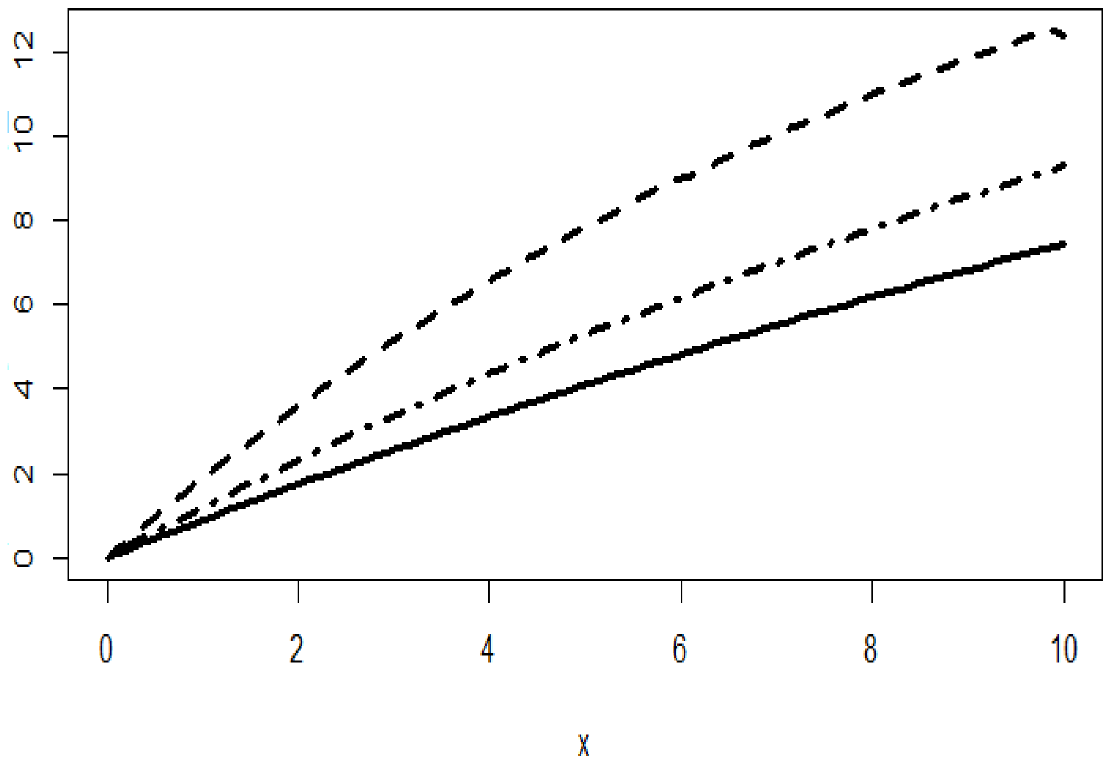

Figure 5.5: Value function for $b = 1$ (dashed), $b = 0.5$ (dotdashed) and $b = 0.01$ (black)

least double the drift $b\theta - (\theta - \eta)$ of the uncontrolled risk process. The overall picture remains similar, the impact of decreasing b, which means buying reinsurance at a lower retention level, becomes more noticeable. Additionally, for larger initial capital x, the inefficiency brought about by buying reinsurance is evident.

Figures 5.8 and 5.9 represent value function V as a function of the initial capital x and retention level b. Figure 5.10 illustrates the value function in relation to b while keeping x constant at 5. It confirms that opting out of reinsurance results in a higher value function.

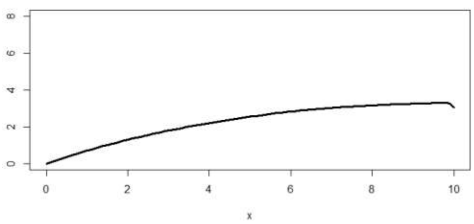

Figure 5.6: Difference in value functions for b=1 and b=0.5

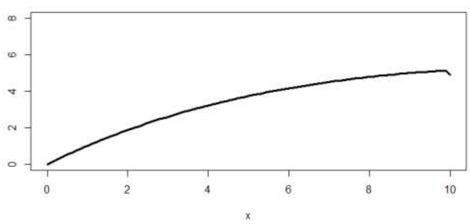

Figure 5.7: Difference in value functions for b=1 and b=0.01

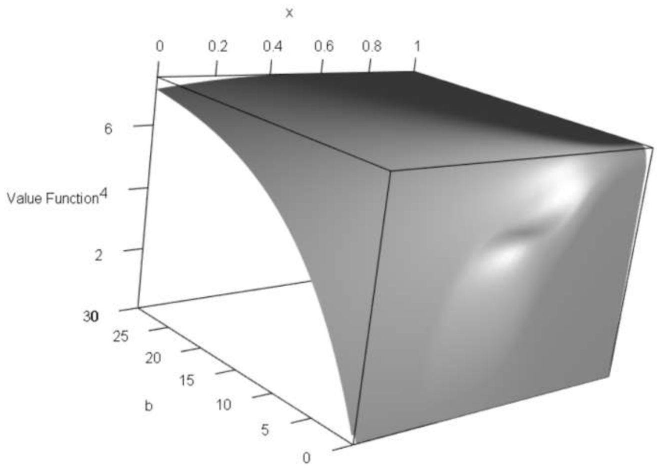

Figure 5.8: Value function of x and b

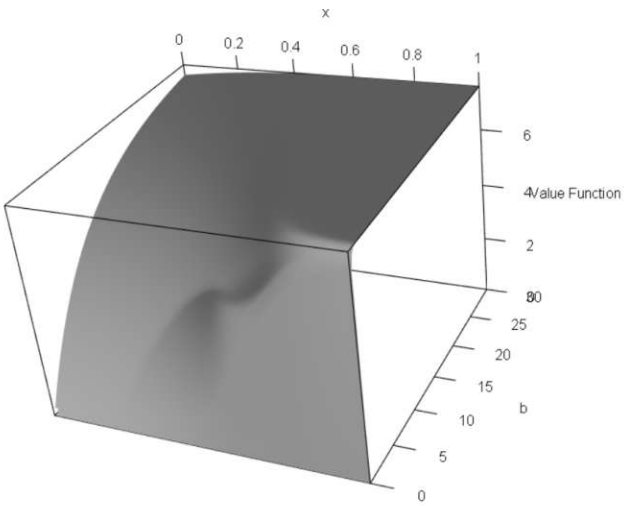

Figure 5.9: Value function of x and b

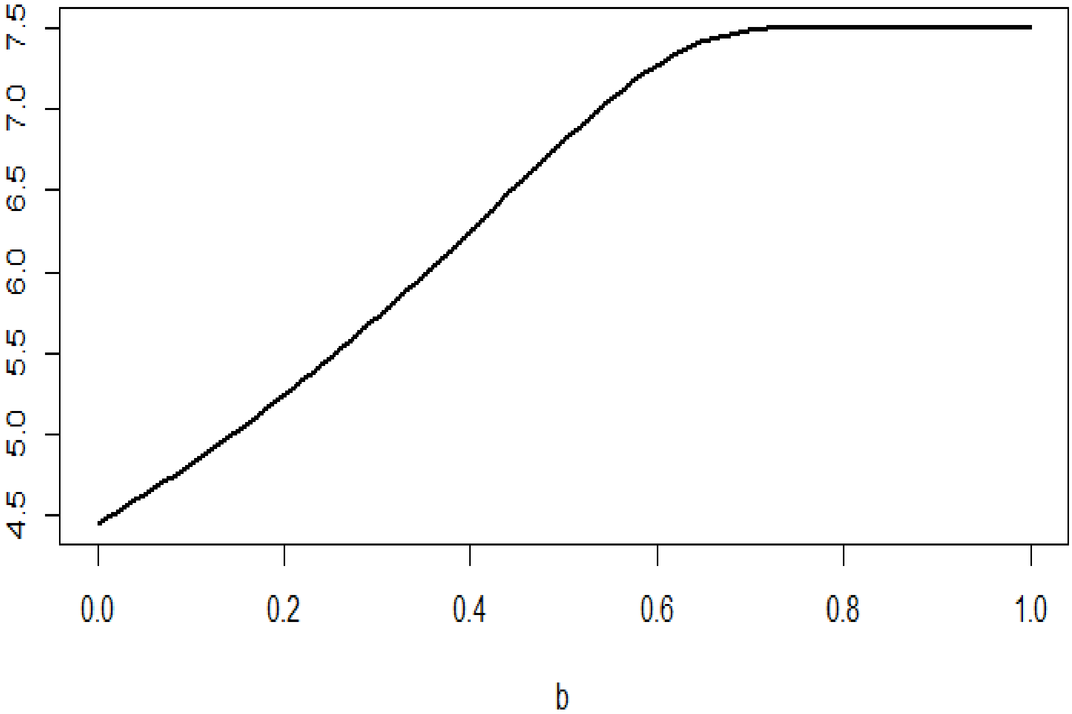

Figure 5.10: Value function of b for x=5

6 Conclusion

In this master thesis, we explored a specific approach to managing dividends in insurance risk theory. Our focus was on a ratcheting dividend strategy combined with reinsurance, where the dividend rate remains constant or can only increase once throughout the surplus process. We developed mathematical formulas to calculate the expected discounted dividend payouts until insolvency. Through numerical demonstrations across various retention levels represented by b, we observed that this particular ratcheting strategy performed optimally when b was set at 1, indicating a scenario without reinsurance. Additionally, we visually highlighted differences in value functions and created a three-dimensional plot illustrating the behavior of the value function in our analysis.

Moving forward, there are several potential directions to expand this study. Future research could explore scenarios involving multiple barriers, where the implementation of a ratcheting strategy coupled with reinsurance could provide valuable insights into the dynamics and outcomes associated with gradual increases in dividend rates across multiple thresholds.

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7 Attachment

7.1 Code in R

This section lists all codes that are used for simulations in R.

```
theta2_c1 <- function(c1, theta, eta, b, q, sigma) {
  ifelse(b == 0,
         q / (eta - theta - c1),
         (c1 - (b * theta - (theta - eta)) -
         sqrt((c1 - (b * theta - (theta - eta)))^2 +
         2 * q * sigma^2 * b^2)) / (sigma^2 * b^2)
  )
}

theta2_c2 <- function(c2, theta, eta, b, q, sigma) {
  ifelse(b == 0,
         q / (eta - theta - c2),
         (c2 - (b * theta - (theta - eta)) -
         sqrt((c2 - (b * theta - (theta - eta)))^2 +
         2 * q * sigma^2 * b^2)) / (sigma^2 * b^2)
  )
}

theta1_c1 <- function(c1, theta, eta, b, q, sigma) {
  ifelse(b == 0,
         q / (eta - theta - c1),
         (c1 - (b * theta - (theta - eta)) +
         sqrt((c1 - (b * theta - (theta - eta)))^2 +
         2 * q * sigma^2 * b^2)) / (sigma^2 * b^2)
  )
```

```
}
```

```
# Define G(y) function
G <- function(theta, c1, c2, eta, b, q, sigma, y) {
  num <- (c2/q) * (1 - exp(theta2_c2(c2, theta,
  eta, b, q, sigma) * y)) -
    (c1/q) * (1 - exp(theta2_c1(c1, theta,
    eta, b, q, sigma) * y))
  denom <- exp(theta1_c1(c1, theta, eta, b, q, sigma) * y) -
    exp(theta2_c1(c1, theta, eta, b, q, sigma) * y)
  num / denom
}
```

```
# Find the value of y that maximizes G(y) with respect to b
find_d <- function(b, theta, c1, c2, eta, q, sigma, y_values) {
  max_y_index <- which.max(G(theta, c1, c2, eta, b, q,
  sigma, y_values))
  d <- y_values[max_y_index]
  return(d)
}
```

```
# Redefine a_c1 and value_function using the calculated
d with respect to b
a_c1 <- function(b, theta, c1, c2, eta, q, sigma, y_values) {
  d <- find_d(b, theta, c1, c2, eta, q, sigma, y_values)
  num <- (c2/q) * (1 - exp(theta2_c2(c2, theta,
  eta, b, q, sigma) * d)) -
    (c1/q) * (1 - exp(theta2_c1(c1, theta, eta,
    b, q, sigma) * d))
  denom <- exp(theta1_c1(c1, theta, eta, b, q, sigma) * d) -
```

```
    exp(theta2_c1(c1, theta, eta, b, q, sigma) * d)
  num / denom
}

value_function <- function(b, c1, c2, theta, eta,
q, sigma, x, y_values) {
  d <- find_d(b, theta, c1, c2, eta, q, sigma, y_values)
  ifelse(x >= d,
         (c2 / q) * (1 - exp(theta2_c2(c2, theta,
         eta, b, q, sigma) * x)),
         (c1 / q) * (1 - exp(theta2_c1(c1, theta,
         eta, b, q, sigma) * x)) +
           a_c1(b, theta, c1, c2, eta, q, sigma, y_values) *
           (exp(theta1_c1(c1, theta, eta, b, q, sigma) * x) -
              exp(theta2_c1(c1, theta, eta, b, q, sigma) * x))
  )
}

value_function_1 <- function(b, c2, theta, eta, q,
sigma, x, y_values) {

  (c2 / q) * (1 - exp(theta2_c2(c2, theta, eta, b,
  q, sigma) * x))


}

value_function_2 <- function(b, c2, theta, eta, q,
sigma, x, y_values) {

  (c2 / q) * (1 - exp(theta2_c2(c2, theta, eta, b,
  q, sigma) * x))


}
```

```
# value function for the same c2=0.3 and b=1 and b=0.01
#curve(value_function_1(b=1, c2 = 0.3, theta =
0.5, eta = 0.4, q = 0.04, sigma = 0.2, x),
#   from = 0, to = 30, n = 100, xlab = "x",
#   ylab = "Value Function", col = "red", lwd = 3,
#   main = "Value Function for b=1 (red), b=0.5 (blue)
and b=0.01 (black)")


#curve(value_function_1(b=0, c2 = 0.3, theta = 0.5,
eta = 0.4, q = 0.04, sigma = 0.2, x),
#     from = 0, to = 30, n = 100, add=TRUE, xlab = "x",
#      ylab = "Value Function", col = "black", lwd = 3,
#      main = "Value Function for b=1 (red), b=0.5 (blue)
and b=0.01 (black)")

# Specify y_values for the curve function
y_values <- seq(0, 10, length.out = 100)

curve(value_function(b=1, c1 = 0.2, c2 = 0.3, theta = 0.5,
eta = 0.4, q = 0.04, sigma = 0.2, x, y_values),
     from = 0, to = 10, n = 100, xlab = "x",
     col = "red", lwd = 3)


curve(value_function(b=0.5, c1 = 0.2, c2 = 0.3, theta = 0.5,
eta = 0.4, q = 0.04, sigma = 0.2, x, y_values),
     from = 0, to = 10, n = 100, add=TRUE, xlab = "x",
     col = "blue", lwd = 3)


curve(value_function(b=0.1, c1 = 0.2, c2 = 0.3, theta = 0.5,
eta = 0.4, q = 0.04, sigma = 0.2, x, y_values),
      from = 0, to = 10, n = 100, add=TRUE, xlab = "x",
     col = "black", lwd = 3)

#difference_plot <- function(x, c1, c2, theta, eta, q, sigma, b1, b2)
```

```
{
# d_b1 <- find_d(b1, theta, c1, c2, eta, q, sigma, y_values)
# d_b2 <- find_d(b2, theta, c1, c2, eta, q, sigma, y_values)

# value_function_b1 <- value_function(b1, c1, c2, theta, eta,
q, sigma, x, y_values)
# value_function_b2 <- value_function(b2, c1, c2, theta, eta,
q, sigma, x, y_values)

#difference <- value_function_b1 - value_function_b2

#  return(difference)
#}

#curve(difference_plot(x, c1 = 0.8, c2 = 0.9, theta = 0.5,
eta = 0.4, q = 0.04, sigma = 0.2, b1 = 1, b2 = 0.01),
#     from = 0, to = 10, n = 100, xlab = "x", ylab =
"Difference in Value Functions",
#    col = "green", lwd = 3,
#     main = "Difference in Value Functions for b=1 and b=0.5",
#     ylim = c(0, 8)
#)




# Generating values for b and x
 b_values <- seq(0, 1, length.out = 100)
 x_values <- seq(0, 10, length.out = 100)

# Creating a meshgrid of b and x values
 grid <- expand.grid(b = b_values, x = x_values)

# Evaluating the value_function for each combination of b and x
 grid$value <- mapply(function(b, x) value_function(b, c1 = 0.2,
 c2 = 0.3, theta = 0.5, eta = 0.4, q = 0.04, sigma = 0.2,
```

```
 x, y_values),
                     grid$b, grid$x)

# Reshape data for plotting
 z_matrix <- matrix(grid$value, nrow = length(b_values),
 ncol = length(x_values))

# Define custom tick labels for x and y axes
 # custom_x_labels <- seq(0, 1, by = 2)  # Adjust according to
 your preference
 #custom_y_labels <- seq(0, 10, length.out = 5)  # Adjust
 according to your preference

# Create a 3D surface plot with custom axis labels
persp3D(z = z_matrix, x = b_values, y = x_values, theta = 30,
phi = 30,
         col = "skyblue", border = "black",
        xlab = "x", ylab = "b", zlab = "Value Function",
         main = "Value Function Surface Plot",
        xticklabs = custom_x_labels, yticklabs = custom_y_labels)


x <- 0:10
y <- 0:1

# To illustrate simple
z_values <- function(x, b){
  value_function(b, c1 = 0.2, c2 = 0.3, theta = 0.5, eta = 0.4,
  q = 0.04, sigma = 0.2, x, y_values)
}

# prepare variables.
z <- outer(x, b, z_values)

# plot the 3D surface
persp(x, b, z, main='3D Plot', shade = 0.6, col = gray(seq(0.2, 0.8,
length = 100)), theta = 30, phi = 15, ticktype='detailed',
```

```
cex.axis = 0.7)
```

```
library(rgl)

b_values <- seq(0, 1, length.out = 100)
x_values <- seq(0, 30, length.out = 100)

# Creating a meshgrid of b and x values
grid <- expand.grid(b = b_values, x = x_values)

# Evaluating the value_function for each combination of b and x
grid$value <- mapply(function(b, x) value_function(b, c1 = 0.2,
c2 = 0.3, theta = 0.5, eta = 0.4, q = 0.04, sigma = 0.2,
x, y_values),
                     grid$b, grid$x)

# Reshape data for plot3d
z_matrix <- matrix(grid$value, nrow = length(x_values),
ncol = length(b_values))


# Prepare variables.

# Plot the 3D surface with smaller axis labels
#persp(x, b, z, shade = 0.6, col = "yellow", theta = 30,
phi = 15, ticktype='detailed', cex.axis = 0.7,
aspect = c(1, 1, 0.7))
rgl::persp3d(b_values, x_values, z_matrix, col = "gray",
shade = 0.6, theta = 30, phi = 15, xlab = "x", ylab = "b",
zlab = "Value Function", ticktype='detailed', cex.axis = 0.7,
aspect = c(1, 1, 0.7))

b_values <- seq(0, 1, length.out = 100)
```

```
# Calculate the corresponding value function for each 'b'
value_curve <- sapply(b_values, function(b) {
  value_function(b, c1 = 0.2, c2 = 0.3, theta = 0.5,
  eta = 0.4, q = 0.04, sigma = 0.2, x = 9, y_values)
})

# Plot the curve for the value function with respect to 'b'
plot(b_values, value_curve, type = "l",
     xlab = "b", ylab = "Value Function",
     col = "black", lwd = 2,
     main = "Value Function as a function of b")
```