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NON-CONFORMING INTERFACE FORMULATIONS FOR COUPLING VISCOUS COMPRESSIBLE FLUIDS AND ELASTIC SOLIDS

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ABSTRACT

Many studies have focused on the interaction between fluids and solids, ranging from non-compressible to compressible, inviscid to viscous fluids, with conforming and non-conforming interfaces. In many applications, such as microelectromechanical systems (MEMS), considering the interaction between fluid and solid is essential to simulate their behavior accurately. We model the fluid, e.g., usually air, as viscous and compressible flow by considering small acoustic perturbations in the linearized conservation equations for both mass and momentum. Similarly, the balance of momentum for the solid is linear when assuming small strains and linear elastic material behavior. This paper describes two non-conforming finite elements (FE) formulations for modeling the interaction between viscous acoustic and solid domains; a Nitsche-based and a Mortar FE formulations. In the Nitsche-based FE formulation, the continuity of velocity is enforced by a penalty factor selected by a scaling approach which makes the formulation dimensionally consistent. Alternatively, the Mortar formulation introduces a Lagrange multiplier (LM) to enforce the interface conditions. We present a performance comparison between these two formulations for a 2D wave propagation case study.

Keywords: *non-conforming interface, Nitsche-based FE formulation, Mortar FE formulation*

1. INTRODUCTION

In many industrial applications, e.g., MEMS loudspeakers, the interaction between the solid and the acoustic viscous domains has to be considered. Due to the small size of these devices, viscosity effects in the air strongly impact the device's behavior. Therefore, the viscous acoustic formulation must be used to accurately model the behavior of MEMS transducers.

These applications can be simulated using various models such as lumped or FE models. Additionally, the impact of the solid deformation on the flow is usually neglected as the mechanical displacement is small. For industrial applications flexibility in choosing mesh sizes for the solid and viscous acoustic domains independently, is advantageous, but leads to complications at the interface. A non-conforming interface FEM formulation using for example a Nitsche-based [1] or a Mortar [2] method can be subsequently employed.

This paper compares two non-conforming FE formulations for modeling compressible viscous fluid, elastic solid and their interaction. Details of the derivation of the coupled linearized formulations and their system matrices in FE formulations are discussed. Finally, a 2D wave propagation example is modeled and the results of these two methods are compared.

2. SOLID AND VISCOUS ACOUSTIC GOVERNING EQUATIONS

For modeling the solid domain, the linearized conservation of momentum is employed. The compressible viscous acoustic domain is modeled using the linearized conservation of mass and momentum. The derivation of these formulations, the necessary assumptions and the definition of solid and viscous acoustic stress tensors are described in [3, 4]. The coupling conditions between these two domains only affect the conservation of momentum at the boundary. Therefore, the viscous acoustic conservation of mass remains unchanged.

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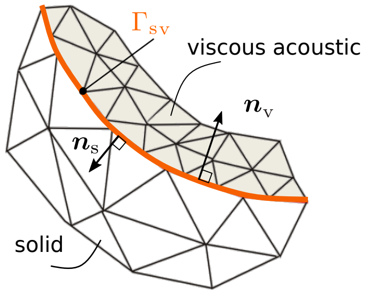

Figure 1. Simple sketch of a solid-viscous acoustic interaction problem including elastic solid Ω_s and viscous acoustic Ω_v domains with their interface Γ_{sv}.

2.1 Coupling conditions between viscous acoustic and solid mechanics

At the interface between solid and viscous acoustic domains (Γ_{sv}, see Fig. 1), the dynamic and kinematic conditions have to be enforced by applying the continuity of traction and velocity, respectively. Traction continuity, i.e., force equilibrium at the interface, is enforced by requiring

$$-\boldsymbol{\sigma} \cdot \boldsymbol{n}_{v} = \boldsymbol{\sigma}_{s} \cdot \boldsymbol{n}_{s} \quad \text{on } \Gamma_{sv}\,, \tag{1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_s$ are the viscous acoustic and solid stress tensors. The velocity continuity at the solid-viscous acoustic interface writes

$$i\omega \boldsymbol{u} = \boldsymbol{v} \quad \text{on } \Gamma_{sv}\,, \tag{2}$$

where $\boldsymbol{u}$ denotes the solid displacement, $\boldsymbol{v}$ the viscous acoustic velocity, and ω the angular velocity. To enforce these coupling conditions, two different methods are introduced: the Nitsche-based and the Mortar method. These methods have already been successfully implemented and validated in our open-source FE code [5].

2.2 Nitsche-based method

The Nitsche-based method is derived from the combination of the conservation of momentum of viscous acoustic and solid and adding the penalty and the symmetrization terms. The penalty term is added to ensure the continuity of velocities at the interface and defined as

$$\beta \int_{\Gamma_{sv}} (\boldsymbol{u}' - \boldsymbol{v}') \cdot (i\omega \boldsymbol{u} - \boldsymbol{v}) \, \mathrm{d}\Gamma\,, \tag{3}$$

where β is the user-defined penalty factor and $\bullet'$ denotes a test function. Employing a scaling approach, the penalty factor advantageously makes (3) dimensionally consistent [4]. The symmetrization term

$$\int_{\Gamma_{sv}} \boldsymbol{\sigma}_s(\boldsymbol{u}') \big(\boldsymbol{v} - i\omega \boldsymbol{u}\big) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma\,, \tag{4}$$

is added for symmetrizing the system matrices, where $\boldsymbol{n}$ is the normal direction at the interface, i.e., $\boldsymbol{n} = \boldsymbol{n}_s = -\boldsymbol{n}_v$. Finally, the coupled equations in the harmonic case are obtained as in [4]

$$i\omega \int_{\Omega_v} K p' p \, \mathrm{d}\Omega + \int_{\Omega_v} p' \boldsymbol{\nabla} \cdot \boldsymbol{v} \, \mathrm{d}\Omega = 0\,, \tag{5a}$$

$$\begin{aligned} i\omega \int_{\Omega_v} \rho_0 \boldsymbol{v}' \cdot \boldsymbol{v} \, \mathrm{d}\Omega - \int_{\Omega_v} \boldsymbol{\nabla} \boldsymbol{v}' : p \boldsymbol{I} \, \mathrm{d}\Omega \\ + \int_{\Omega_v} \mu \boldsymbol{\nabla} \boldsymbol{v}' : \big(\boldsymbol{\nabla} \boldsymbol{v} + (\boldsymbol{\nabla} \boldsymbol{v})^T\big) \, \mathrm{d}\Omega \\ + \int_{\Omega_v} (\mu_B - \frac{2}{3}\mu) \boldsymbol{\nabla} \boldsymbol{v}' : \boldsymbol{\nabla} \cdot \boldsymbol{v} \boldsymbol{I} \, \mathrm{d}\Omega + \int_{\Gamma_{sv}} \boldsymbol{v}' \cdot \boldsymbol{\sigma}_s \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\ - i\omega\beta \int_{\Gamma_{sv}} \boldsymbol{v}' \cdot \boldsymbol{u} \, \mathrm{d}\Gamma + \beta \int_{\Gamma_{sv}} \boldsymbol{v}' \cdot \boldsymbol{v} \, \mathrm{d}\Gamma = \boldsymbol{0}\,, \end{aligned} \tag{5b}$$

$$\begin{aligned} -\omega^2 \int_{\Omega_s} \boldsymbol{u}' \cdot \rho_s \boldsymbol{u} \Omega + \int_{\Omega_s} \boldsymbol{\nabla} \boldsymbol{u}' : \boldsymbol{C} : \boldsymbol{s} \, \mathrm{d}\Omega \\ - \int_{\Gamma_{sv}} \boldsymbol{u}' \cdot \boldsymbol{\sigma}_s \cdot \boldsymbol{n} \, \mathrm{d}\Gamma + i\omega\beta \int_{\Gamma_{sv}} \boldsymbol{u}' \cdot \boldsymbol{u} \, \mathrm{d}\Gamma - \beta \int_{\Gamma_{sv}} \boldsymbol{u}' \cdot \boldsymbol{v} \, \mathrm{d}\Gamma \\ + \int_{\Gamma_{sv}} \boldsymbol{\sigma}_s(\boldsymbol{u}') \, \boldsymbol{v} \cdot \boldsymbol{n} - i\omega \int_{\Gamma_{sv}} \boldsymbol{\sigma}_s(\boldsymbol{u}') \, \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{0}\,, \end{aligned} \tag{5c}$$

where p the viscous acoustic pressure, μ the shear, μ_B the bulk viscosity and K the comprehensibility coefficient.

Furthermore, we apply the standard Galerkin method by substituting the approximations of velocity, pressure and displacement into the conservation of mass (5a), conservation of momentum for both viscous acoustic (5b) and solid (5c). The resulting system of equations for the viscous acoustic formulation coupled to the solid domain is

$$\begin{pmatrix} \boldsymbol{S}_{pp} & \boldsymbol{S}_{p\boldsymbol{v}} & 0 \\ \boldsymbol{S}_{\boldsymbol{v}p} & \boldsymbol{S}_{\boldsymbol{v}\boldsymbol{v}} & \boldsymbol{S}_{\boldsymbol{v}\boldsymbol{u}} \\ 0 & \boldsymbol{S}_{\boldsymbol{u}\boldsymbol{v}} & \boldsymbol{S}_{\boldsymbol{u}\boldsymbol{u}} \end{pmatrix} \begin{pmatrix} \{p\} \\ \{\boldsymbol{v}\} \\ \{\boldsymbol{u}\} \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{f\boldsymbol{v}\} \\ \{f\boldsymbol{u}\} \end{pmatrix},$$

where $\boldsymbol{S}_{ij} = -\omega^2 \boldsymbol{M}_{ij} + i\omega \boldsymbol{C}_{ij} + \boldsymbol{K}_{ij}$ with the following mass matrix

$$\boldsymbol{M}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{M}_{\boldsymbol{uu}} \end{pmatrix},$$

damping matrix

$$\boldsymbol{C}_{ij} = \begin{pmatrix} \boldsymbol{C}_{pp} & 0 & 0 \\ 0 & \boldsymbol{C}_{\boldsymbol{vv}} & \boldsymbol{C}_{\Gamma_{\boldsymbol{vu}}} \\ 0 & 0 & \boldsymbol{C}_{\Gamma_{\boldsymbol{uu}}} \end{pmatrix}$$

and stiffness matrix

$$\boldsymbol{K}_{ij} = \begin{pmatrix} 0 & \boldsymbol{K}_{p\boldsymbol{v}} & 0 \\ \boldsymbol{K}_{\boldsymbol{v}p} + \boldsymbol{K}_{\Gamma_{\boldsymbol{v}_p}} & \boldsymbol{K}_{\boldsymbol{vv}} + \boldsymbol{K}_{\Gamma_{\boldsymbol{vv}}} & \boldsymbol{K}_{\Gamma_{\boldsymbol{vu}}} \\ 0 & \boldsymbol{K}_{\boldsymbol{uv}} + \boldsymbol{K}_{\Gamma_{\boldsymbol{uv}}} & \boldsymbol{K}_{\boldsymbol{uu}} \end{pmatrix}. \quad (6)$$

2.3 Mortar method

For modeling the coupling between solid and viscous acoustic domains using the Mortar method an additional unknown $\boldsymbol{t}$ is used. This unknown represents the traction at the interface, as in the LM, and verifies

$$\boldsymbol{t} = -\boldsymbol{\sigma} \cdot \boldsymbol{n}_{\mathrm{v}} = \boldsymbol{\sigma}_{\mathrm{s}} \cdot \boldsymbol{n}_{\mathrm{s}} . \quad (7)$$

Continuity of the velocities (2), is enforced in a weak sense at the interface, i.e., we require

$$\int_{\Gamma_{\mathrm{sv}}} \boldsymbol{t}' \cdot (i\omega \boldsymbol{u} - \boldsymbol{v}) \, \mathrm{d}\Gamma = \boldsymbol{0}, \quad (8)$$

which also provides the additional equations necessary for the determination of the introduced unknown at the interface. The continuity of traction is ensured thanks to the additional unknown $\boldsymbol{t}$ at the interface. In this method, the conservation of mass (5a) remains unchanged, and the solid and viscous acoustic conservation of momentum are obtained as

$$i\omega \int_{\Omega_{\mathrm{v}}} \rho_0 \boldsymbol{v}' \cdot \boldsymbol{v} \, \mathrm{d}\Omega - \int_{\Omega_{\mathrm{v}}} \boldsymbol{\nabla} \boldsymbol{v}' : p\boldsymbol{I} \, \mathrm{d}\Omega$$
$$+ \int_{\Omega_{\mathrm{v}}} \mu \boldsymbol{\nabla} \boldsymbol{v}' : \left(\boldsymbol{\nabla} \boldsymbol{v} + (\boldsymbol{\nabla} \boldsymbol{v})^T\right) \mathrm{d}\Omega + \int_{\Omega_{\mathrm{v}}} (\mu_{\mathrm{B}} - \frac{2}{3}\mu) \boldsymbol{\nabla} \boldsymbol{v}' : \boldsymbol{\nabla} \cdot \boldsymbol{v} \boldsymbol{I} \, \mathrm{d}\Omega$$
$$+ \int_{\Gamma_{\mathrm{sv}}} \boldsymbol{v}' \cdot \boldsymbol{t} \, \mathrm{d}\Gamma = \boldsymbol{0}, \quad (9a)$$

$$-\omega^2 \int_{\Omega_{\mathrm{s}}} \boldsymbol{u}' \cdot \rho_{\mathrm{s}} \boldsymbol{u} \Omega + \int_{\Omega_{\mathrm{s}}} \boldsymbol{\nabla} \boldsymbol{u}' : \boldsymbol{C} : \boldsymbol{s} \, \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{sv}}} \boldsymbol{u}' \cdot \boldsymbol{t} \, \mathrm{d}\Gamma = \boldsymbol{0} . \quad (9b)$$

The resulting system matrix for the Mortar method is obtained as

$$\begin{pmatrix} \boldsymbol{S}_{pp} & \boldsymbol{S}_{p\boldsymbol{v}} & 0 & 0 \\ \boldsymbol{S}_{\boldsymbol{v}p} & \boldsymbol{S}_{\boldsymbol{vv}} & \boldsymbol{S}_{\boldsymbol{vu}} & \boldsymbol{S}_{\boldsymbol{vt}} \\ 0 & \boldsymbol{S}_{\boldsymbol{uv}} & \boldsymbol{S}_{\boldsymbol{uu}} & \boldsymbol{S}_{\boldsymbol{ut}} \\ 0 & \boldsymbol{S}_{\boldsymbol{tv}} & \boldsymbol{S}_{\boldsymbol{tu}} & 0 \end{pmatrix} \begin{pmatrix} \{p\} \\ \{\boldsymbol{v}\} \\ \{\boldsymbol{u}\} \\ \{\boldsymbol{t}\} \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{f\boldsymbol{v}\} \\ \{f\boldsymbol{u}\} \\ \{0\} \end{pmatrix},$$

where the mass matrix only includes $\boldsymbol{M}_{\boldsymbol{uu}}$. The damping and stiffness matrices are defined as

$$\boldsymbol{C}_{ij} = \begin{pmatrix} \boldsymbol{C}_{pp} & 0 & 0 & 0 \\ 0 & \boldsymbol{C}_{\boldsymbol{vv}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{C}_{\Gamma_{\boldsymbol{tu}}} & 0 \end{pmatrix},$$

$$\boldsymbol{K}_{ij} = \begin{pmatrix} 0 & \boldsymbol{K}_{p\boldsymbol{v}} & 0 & 0 \\ \boldsymbol{K}_{\boldsymbol{v}p} + \boldsymbol{K}_{\Gamma_{\boldsymbol{v}_p}} & \boldsymbol{K}_{\boldsymbol{vv}} & 0 & \boldsymbol{K}_{\Gamma_{\boldsymbol{vt}}} \\ 0 & \boldsymbol{K}_{\boldsymbol{uv}} & \boldsymbol{K}_{\boldsymbol{uu}} & \boldsymbol{K}_{\Gamma_{\boldsymbol{ut}}} \\ 0 & \boldsymbol{K}_{\Gamma_{\boldsymbol{tv}}} & 0 & 0 \end{pmatrix}. \quad (10)$$

The primary disadvantage of this approach stems from the presence of a zero on the diagonal of the system matrix, resulting in issues with saddle points.

3. RESULTS

To compare the described methods, various test cases were implemented and validated. This section presents a 2D wave propagation in a channel excited by solid displacements. The geometry and boundary conditions are depicted in Fig. 2, and the material properties of these domains are chosen to have equal wavelengths. Density, Poisson ratio, and elasticity of the solid are 1.225 $\mathrm{kg\,m^{-3}}$, 0.33 and 9.5e-4 $\mathrm{N\,m^{-2}}$, respectively, and the fluid material properties are described in Tab. 1.

Figure 3 and 4 show the solid and viscous acoustic velocity and pressure distribution along the channel using the Nitsche-based and Mortar formulations. The velocity field in the solid domain shows the standing wave behavior, whereas the velocity field in the viscous domain shows the decaying waves, where the velocity reaches zero at the end of the channel. Both methods exhibit these behaviors and produce identical numerical results. The main

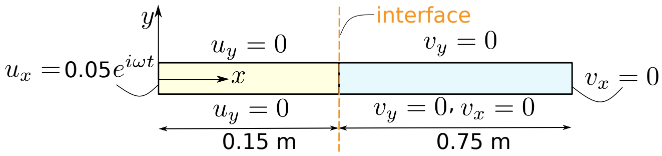

Figure 2. Boundary conditions for 2D wave propagation. Yellow and blue colors show the solid and fluid regions, respectively

Table 1. Fluid material properties

Properties	Value
Density in $\mathrm{kg\,m^{-3}}$	1.225
Bulk viscosity in $\mathrm{N\,s\,m^{-2}}$	1.22e-1
Shear viscosity in $\mathrm{N\,s\,m^{-2}}$	1.5e-2
Compression modulus in $\mathrm{N\,m^{-2}}$	1.427e9

advantage of these methods is their ability to use non-conforming interfaces, making these methods beneficial for modeling complex geometries where specific meshes are required for different domains.

4. CONCLUSION

This paper compares two methods for modeling solid and viscous acoustic interaction: the Nitsche-based and the Mortar methods. These methods were thoroughly explained and their FE formulations and associated system matrices were obtained. The main difference between these methods is that the Nitsche-based method enforces the continuity of velocities at the solid-viscous acoustic interface through a penalty term, which requires a user-

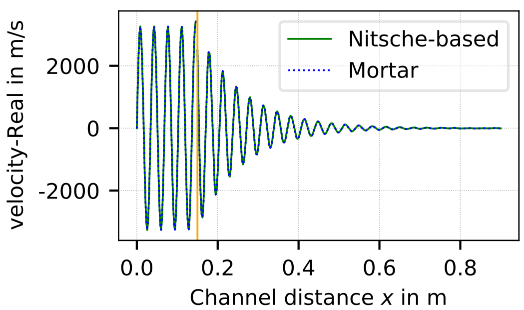

Figure 3. Velocity distribution along the channel

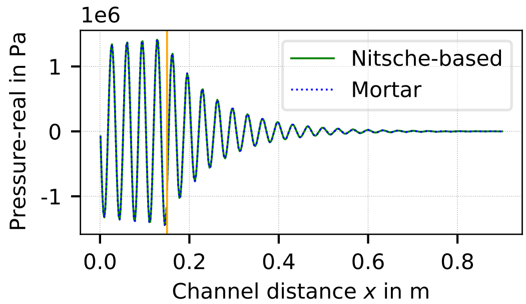

Figure 4. Pressure distribution along the channel

defined penalty factor. Whereas the Mortar method introduces a new unknown at the interface called the LM to enforce coupling conditions. However, the new unknown causes saddle point problems due to a zero on the diagonal of the system matrix. Both methods were implemented and validated using various test cases, including a 2D wave propagation with a solid excitation, where both methods were shown to produce identical results.

5. REFERENCES

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