Planar L-Drawings of Directed Graphs

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Abstract

In this paper, we study drawings of directed graphs. We use the *L-drawing standard*, where each edge is represented by a polygonal chain that consists of a vertical line-segment incident to the source of the edge and a horizontal line-segment incident to the target.

First, we consider *planar L-drawings*. We provide necessary conditions for the existence of these drawings and show that testing for the existence of a planar L-drawing is an NP-complete problem. We also show how to decide in linear time whether there exists a planar L-drawing of a plane directed graph with a fixed assignment of the edges to the four sides (top, bottom, left, and right) of the vertices.

Second, we consider *upward- (resp. upward-rightward-) planar L-drawings*. We provide upper bounds on the maximum number of edges of graphs admitting such drawings. Moreover, we characterize the directed st-graphs admitting an upward- (resp. upward-rightward-) planar L-drawing as exactly those admitting an embedding supporting a bitonic (resp. monotonically decreasing) st-ordering.

Keywords and phrases Directed graphs, planar L-drawings, bitonic st-ordering, NP-complete

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1 Introduction

A (simple, finite) *directed graph* $G = (V, E)$ is a pair consisting of a finite set V of vertices and a finite set $E \subseteq \{(u, v) \in V \times V; u \neq v\}$ of ordered pairs of vertices. In an *L-drawing* of a directed graph each vertex v is assigned a point in the plane with exclusive integer x- and y-coordinates, and each directed edge (u, v) is assigned a polygonal chain consisting of a vertical segment starting at u and a horizontal segment ending at v [4]. Thus, an edge may leave a vertex to the top or the bottom and enter a vertex from the right or the left, respectively; we say that the edge uses the top, bottom, right, or left *port* of its end-vertices, respectively. The drawings of two edges may cross and partially overlap, in either their horizontal or vertical segments, following the model of Kornaropoulos and Tollis [22]. The ambiguity among crossings and bends is resolved by replacing bends with small rounded junctions. An L-drawing in which edges possibly overlap, but do not cross, is a *planar L-drawing* (see, e.g., Figure 1e). A planar L-drawing is *upward planar* if its edges are y-monotone (see, e.g., Figure 1b), and it is *upward-rightward planar* if its edges are simultaneously x-monotone and y-monotone.

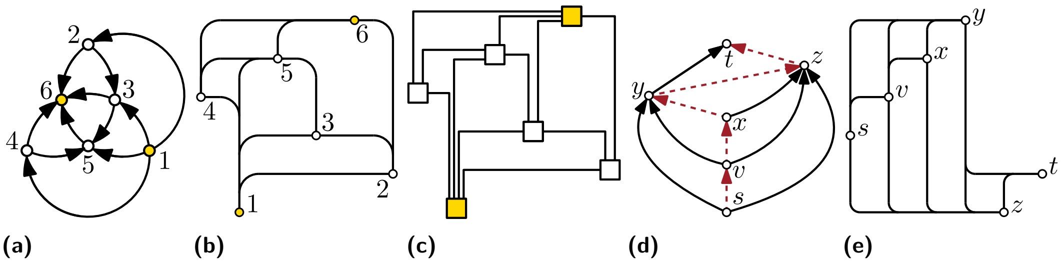


Figure 1 (a) A bitonic st-orientation that admits an upward-planar L-drawing (b). (c) The corresponding drawing in the Kandinsky model. (d) An upward-planar st-graph that does not admit an upward-planar L-drawing. (e) A planar L-drawing of the graph in (d).

Planar L-drawings correspond to drawings in the Kandinsky model [17] with *exactly one bend* per edge and with some restrictions on the angles around each vertex. See Theorem 1 for details and Figure 1c for an example of a Kandinsky drawing. It is NP-complete to decide whether a multigraph has a planar embedding that allows a Kandinsky drawing with at most one bend per edge [8, 9]. On the other hand, every simple planar graph has a Kandinsky drawing with at most one bend per edge [9]. Bend-minimization in the Kandinsky-model is NP-complete [8] even if a planar embedding is given, but can be approximated within a factor of two [5, 16]. Heuristics for drawings in the Kandinsky model with so-called empty faces and few bends have been discussed by Bekos et al. [6].

Planar L-drawings can also be seen as a directed version of *plus-contact-representations* [7, 15]: each vertex is represented by a "plus" consisting of a horizontal and vertical segment that intersect. Segments of two vertices may touch, but neither cross nor overlap. There is an edge (v, w) if and only if the vertical segment of v and the horizontal segment of w touch. Upward-planar L-drawings correspond to *T-contact-representations*, a degenerate version of plus-contact-representations in which the vertical segment of the plus representing a vertex is only above its horizontal segment. Each undirected planar graph admits a T-contact-representation [12], and, thus, can be oriented such that it admits an upward-planar L-drawing.

In a *k-modal embedding* of a directed planar graph each vertex is incident to at most k pairs of consecutive edges with opposite orientations. Planar L-drawings and upward-planar

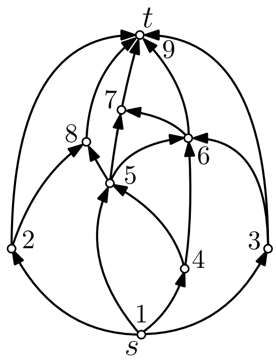


Figure 2 A bitonic *st*-ordering of a plane *st*-graph.

L-drawings determine 4-modal and 2-modal embeddings, respectively. In fact, all the edges incident to a vertex v and overlapping on their horizontal segment (resp., vertical segment) must all be incoming at v (resp., outgoing from v). Testing the existence of a 2-modal embedding can be easily done in linear time, via a straightforward reduction to planarity testing. On the other hand, the problem of testing the existence of a 4-modal embedding of a directed graph G is known to be NP-complete if the maximum degree Δ of G is larger than or equal to 7, and it is linear-time solvable if $\Delta = 6$ or if G is a partial 2-tree [24]. Angelini et al. [3] study planar L-drawings of plane bimodal graphs. In particular, they show that every bimodal graph without 2-cycles and, thus, every upward-plane graph admits a planar L-drawing respecting the given embedding. Moreover, planar L-drawings exist for every bimodal graph whose underlying undirected graph is a planar 3-tree, provided that each 2-cycle is replaced by a single edge. Planar L-drawings of subdivisions of plane 4-modal graphs with few subdivision vertices, so-called *planar confluent orthogonal drawings* have been discussed by Cornelsen and Diatzko [11].

Bitonic st-orderings were introduced by Gronemann for undirected planar graphs [19] as an alternative to canonical orderings, and have been recently extended to directed plane graphs [21]. Given a directed graph G, v is a *successor* of u and u is a *predecessor* of v if (u, v) is an edge of G. Intuitively, in a bitonic st-ordering of a plane st-graph, the successors of any vertex must form an increasing and then a decreasing sequence in the given embedding. See Figure 2 for an example. More precisely, a *planar st-graph* is a directed acyclic graph with a single source s and a single sink t that admits a planar embedding in which s and t lie on the boundary of the same face. A planar st-graph always admits an upward-planar straight-line drawing [13]. An upward-planar drawing Γ of a planar st-graph determines an *upward-planar embedding*, that is, the ordered lists $S(v)$ of the successors of each vertex v, as they appear from left-to-right in Γ. An *st-ordering* of a planar st-graph is an enumeration π of the vertices with distinct integers, such that $\pi(u) < \pi(v)$ for every edge (u, v). Given a *plane st-graph* $G = (V, E)$, i.e., a planar st-graph with a fixed upward-planar embedding $\mathcal{E}$, and an st-ordering π of G, consider the ordered list $S(v) = \langle v_1, v_2, \ldots, v_k \rangle$ of the successors of v determined by $\mathcal{E}$. The list $S(v)$ is *monotonically decreasing* with respect to π if, for every $i \in \{1, \ldots, k-1\}$, we have that $\pi(v_i) > \pi(v_{i+1})$. The list $S(v)$ is *monotonically increasing* with respect to π if, for every $i \in \{1, \ldots, k-1\}$, it holds that $\pi(v_i) < \pi(v_{i+1})$. The list $S(v)$ is *bitonic* with respect to π if there is a vertex v_h in $S(v)$ such that, for $i \in \{1, \ldots, h-1\}$ $\pi(v_i) < \pi(v_{i+1})$ and, for $i \in \{h, \ldots, k-1\}$, $\pi(v_i) > \pi(v_{i+1})$. E.g., in Figure 2, $S(2)$ and $S(5)$ are monotonically decreasing, $S(3)$, $S(4)$, and $S(6)$ are monotonically increasing, and the successor list of each vertex is bitonic. We say that the st-ordering π is *bitonic* or *monotonically decreasing*, respectively, if the successor list of each vertex is bitonic or monotonically decreasing, respectively. Also, the pair $\langle \mathcal{E}, \pi \rangle$ is called a *bitonic pair* or *monotonically decreasing pair*, respectively, of G.

Bitonic st-orderings turn out to be strongly related to upward-planar L-drawings of st-graphs. In fact, the y-coordinates of an upward-planar L-drawing yield a bitonic st-ordering; see, e.g., Figure 1b. Gronemann used bitonic st-orderings to obtain, on the one hand, upward-planar polyline grid drawings in quadratic area with at most $|V| - 3$ bends in total [21] and, on the other hand, contact representations with upside-down oriented T-shapes [20]. A bitonic st-ordering for biconnected undirected planar graphs can be computed in linear time [19] and the existence of a bitonic st-ordering for plane (directed) st-graphs can also be decided in linear time [21]. For the variable embedding setting, Chaplick et al. [10] presented a linear-time algorithm to decide whether an st-graph admits a bitonic pair. Gronemann [21] observed that if an st-planar graph does not admit a bitonic st-ordering, then it is always possible to split certain edges so that the resulting graph admits such an ordering. Generalizing the result of Chaplick et al. [10], Angelini et al. [1] described a linear-time algorithm to compute an embedding of a planar st-graph that allows for a bitonic st-ordering, after a minimum number of edge splits.

Our contribution. In this work, we initiate the investigation of planar and upward-planar L-drawings. In particular, our contributions are as follows.

- We establish the relationship between planar L-drawings and 1-bend Kandinsky drawings of the underlying undirected graph (Theorem 1).
- We prove that deciding whether a directed planar graph admits a planar L-drawing is NP-complete (Theorem 3).
- We show how to decide efficiently whether there is a planar L-drawing for a plane directed graph with a fixed assignment of the edges to the four ports of the vertices (Theorem 4).
- Finally, we study the density of graphs admitting an upward-planar (resp. upward-rightward-planar) L-drawing (Theorem 5), and we characterize the planar st-graphs admitting an upward-planar (resp. upward-rightward-planar) L-drawing as the st-graphs admitting a bitonic (resp. monotonic decreasing) pair (Theorem 6).

Building on our last result, Angelini et al. [2] have recently provided a characterization of general DAGs admitting an upward-planar L-drawing.

2 Preliminaries

In this section, we present definitions and preliminary results. A *directed (multi-)graph* $G = (V, E)$ consists of a finite set V of vertices and of a finite (multi-)set E of edges, which are ordered pairs of vertices. An edge (u, v) is directed from its *tail* u to its *head* v.

Planar drawings and embeddings. A graph is *planar* if it admits a drawing in the plane without edge crossings. A *planar drawing* Γ of a graph maps vertices to points and edges to internally-disjoint curves in the plane connecting the points of their end-vertices. Drawing Γ partitions the plane into topologically connected regions, called *faces*. The bounded faces are the *internal faces* and the unbounded face is the *outer face*. A planar drawing determines a circular ordering of the edges incident to each vertex. Two planar drawings of a connected planar graph are *equivalent* if they determine the same orderings. A *planar embedding* is an equivalence class of planar drawings. A *plane graph* is a planar graph with a fixed planar embedding and with a prescribed outer face.

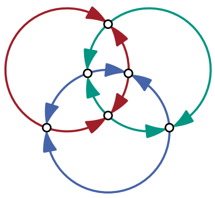


Figure 3 4-modal embedding of an orientation of the octahedron.

Vertex connectivity. A graph is *connected* if there is a (not necessarily directed) path between any two vertices and *disconnected* otherwise. A set of k vertices of a connected graph G is a k*-cut* if removing such vertices from G results in a disconnected graph. A connected graph is *biconnected* if it does not have any 1-cut, and a biconnected graph is *triconnected* if it does not have any 2-cut. A famous result by Whitney [25] states that a planar graph has a unique (up to a flip) planar embedding if and only if it is a subdivision of a triconnected planar graph.

Modality. Given a planar embedding $\mathcal{E}$ of a directed graph G and a vertex v of G, a pair of edges incident to v that are consecutive around v in $\mathcal{E}$ is *alternating* if they are not both incoming or both outgoing. We say that v is k*-modal*, if there exist exactly k alternating pairs of edges in the cyclic order around v. An embedding of a directed graph G is k*-modal*, if each vertex is at most k-modal. A 2-modal embedding is also called *bimodal*. An upward-planar drawing determines a bimodal embedding. However, the existence of a bimodal embedding is not a sufficient condition for the existence of an upward-planar drawing. Deciding whether a directed graph admits an upward-planar (straight-line) drawing is an NP-hard problem [18].

L-drawings. A planar L-drawing determines a 4-modal embedding. This implies that there exist planar directed graphs that do not admit planar L-drawings. A 6-wheel whose central vertex is incident to alternating incoming and outgoing edges is an example of a graph that does not admit any 4-modal embedding, and therefore any planar L-drawing.

On the other hand, the existence of a 4-modal embedding is not sufficient for the existence of a planar L-drawing. For instance, the octahedron depicted in Figure 3 does not admit a planar L-drawing. Since the octahedron is triconnected, it admits a unique combinatorial embedding (up to a flip). Each vertex is 4-modal. However, the rightmost vertex in a planar L-drawing must be 0-modal or 2-modal.

Any upward-planar L-drawing of an st-graph G can be modified to obtain an upward-planar drawing of G: Redraw each edge as a y-monotone curve arbitrarily close to the drawing of the corresponding 1-bend orthogonal polyline while avoiding crossings and edge–edge overlaps. However, not every upward-planar graph admits an upward-planar L-drawing. For instance, the graph in Figure 1d contains a subgraph that does not admit a bitonic st-ordering [21]. In Section 4 (Theorem 6), we show that this means it does not admit an upward-planar L-drawing.

The Kandinsky model. In the Kandinsky model [17], vertices are drawn as squares of equal size on a grid and edges—usually undirected—are drawn as orthogonal polylines on a finer grid; see Figure 1c. Two consecutive edges in the clockwise order around a vertex define a face and an angle in $\{0, \pi/2, \pi, 3\pi/2, 2\pi\}$ in that face. In order to avoid edges running through other vertices, the Kandinsky model requires the so-called *bend-or-end property*: There is an assignment of bends to vertices with the following three properties.

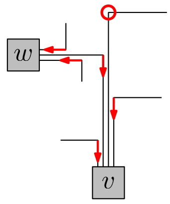


Figure 4 Assignment of bends to vertices for the bend-or-end property.

(a) Each bend is assigned to at most one vertex.

(b) A bend may only be assigned to a vertex to which it is connected by a segment (i.e., it must be the first bend on an edge).

(c) If e_1 and e_2 are two consecutive edges in the clockwise order around a vertex v that form a 0-angle inside face f, then a bend of e_1 or e_2 forming a $3\pi/2$ angle inside f must be assigned to v.

Further, the Kandinsky model requires that there are no *empty faces*. In an empty face, the distance of each point in the interior of the face to the boundary of the face is at most the smallest distance between two segments of any two edges. (Intuitively speaking, an empty face has area close to 0.)

Given a planar L-drawing, consider a vertex v and all edges incident to one of the four ports of v. By assigning to v all bends on these edges—except the bend furthest from v—we satisfy the bend-or-end property. We prove this in the following theorem.

▶ **Theorem 1.** *A graph has a planar L-drawing if and only if it admits a drawing in the Kandinsky model with the following properties*

1. *Each edge bends exactly once.*
2. *At each vertex, the angle between two outgoing (or between two incoming) edges is 0 or π.*
3. *At each vertex, the angle between an incoming edge and an outgoing edge is $\pi/2$ or $3\pi/2$.*

Proof. Given a drawing in the Kandinsky model that meets conditions (1)–(3), we can bundle the edges on the finer grid to lie on the coarser grid. It remains to perturb the coordinates such that the x- and y-coordinates, respectively, of the vertices are distinct: Assume that two vertices v and w have the same y-coordinate. Let $\delta > 0$ be the minimum difference in y-coordinates between v and any vertex or segment above v. Since all edges have one bend, we can shift v upward by $\delta/2$—changing only the drawing of edges incident to v. Doing this iteratively yields a planar L-drawing—or a rotation of $\pi/2$ of it.

Given a planar L-drawing, we can distribute the edges on the finer grid maintaining the embedding. Since all vertices have distinct x- and y-coordinates, there are no empty faces. It remains to assign the bends to the vertices in order to fulfill the bend-or-end property: For each port p (top, right, bottom, left) of a vertex v, and each edge incident to p, we assign its bend to v except for the bend furthest from p. In the example in Figure 4, the bend furthest from the top of v is marked with a red circle. Observe that if the bend b on an edge between two vertices v and w is not a furthest bend for v then it is a furthest bend for w. This follows by planarity: Namely, if bend b was neither furthest for v nor for w then there was an edge e incident to v with b in its interior and an edge e' incident to w with b in its interior. Thus, e and e' would cross at b. Hence, no bend will be assigned to two vertices. ◀

3 General planar L-drawings

We consider the problem of deciding whether a graph admits a planar L-drawing. In Section 3.1, we show that the problem is NP-complete if no planar embedding is given. In the fixed embedding setting (Section 3.2), the problem can be formulated as an ILP. If we additionally fix the ports, the problem can be solved in linear time.

3.1 Variable embedding setting

As a central building block for our hardness reduction we use a directed graph W that can be constructed starting from a 4-wheel with central vertex c and rim (u, v, w, z). We orient the edges of W so that v and z (the *V-ports* of W) are sinks and u and w (the *H-ports* of W) are sources. The two outgoing edges (c, v) and (c, z) of c pass vertically through c and are colored red; similarly, (u, c) and (w, c) pass horizontally through c and are colored blue. Finally, we add directed edges (v, c), (z, c), (c, w), and (c, u); see the black spoke edges in Figure 5.

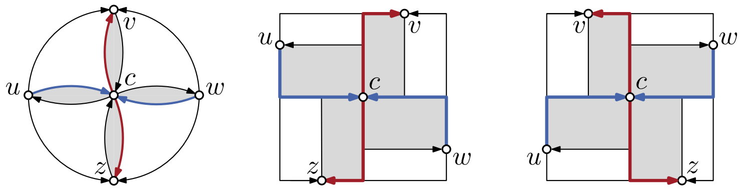


Figure 5 4-wheel graph W and two planar L-drawings of W.

We now describe the key property of the planar L-drawings of W.

▶ **Lemma 2.** *In any planar L-drawing of W with cycle (u, v, w, z) as the outer face, the edges of the outer face form a rectangle that contains vertex c.*

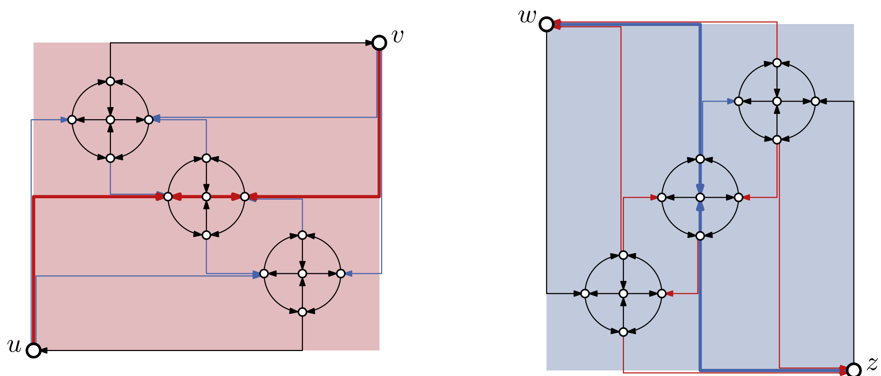
Proof. In any orthogonal drawing of W, the outer cycle (u, v, w, z) forms an orthogonal polygon P with at least four convex corners. Since any two consecutive edges on the outer cycle have the same direction with respect to their common vertex $r \in \{u, v, w, z\}$, i.e., they are either both incoming or outgoing at r, they must use the same port or two opposite ports of r. In fact, if they used the same port, they would form an angle of 2π in the outer face and force the edge (r, c) to use the very same port. This, however, would imply that all three edges incident to r have the same direction, which is a contradiction. Hence each of the four outer vertices has an angle of π in the outer face and cannot form a convex corner of P.

Since there are four edges on the outer cycle, each of which has exactly one bend, this immediately implies that P is a rectangle whose corners are formed by the bends of the four edges of the outer face and each of the four vertices of the outer face must lie on one of the rectangle sides. The remaining edges to c use the port inside P, consistently bend once (left or right) from the perspective of c, and then connect to c from all four sides. Figure 5 shows an example. ◀

We are now ready to present the main result of this section.

▶ **Theorem 3.** *It is NP-complete to decide whether a directed graph admits a planar L-drawing.*

Proof. We reduce from HV-rectilinear planarity testing, which is NP-hard even for biconnected graphs [14]. An instance of this problem is an undirected planar graph G of vertex



(a) for a V-edge

(b) for an H-edge

Figure 6 Edge gadgets

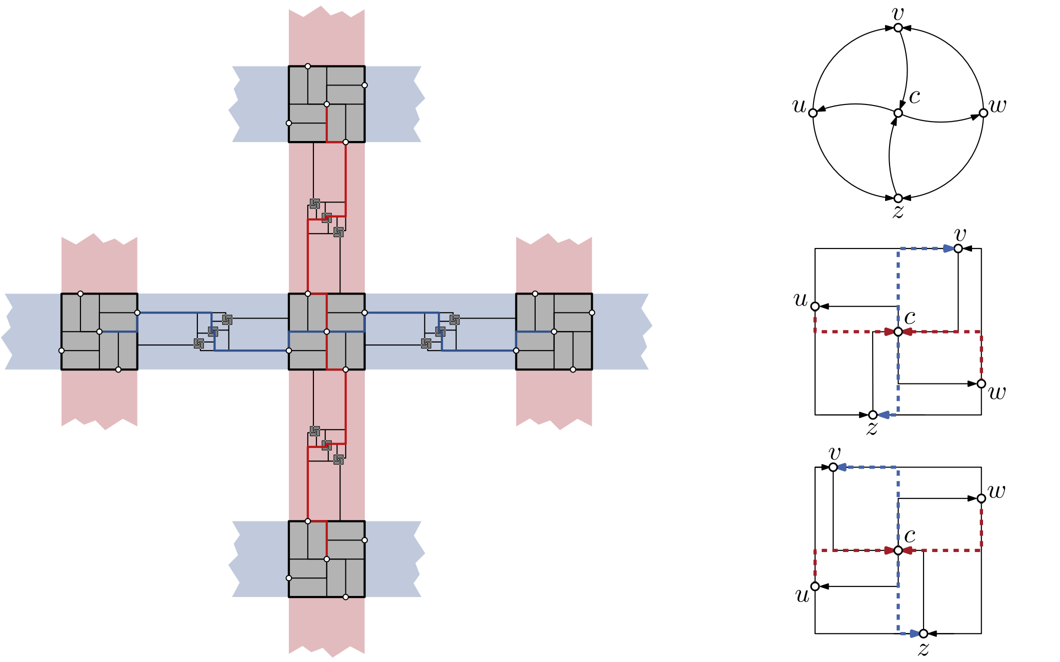
degree at most 4, where each edge is labeled either H or V. The task is to decide whether G admits a planar orthogonal drawing (without bends) such that H-edges are drawn horizontally and V-edges are drawn vertically. We call such a drawing a planar *HV-drawing*.

Given a biconnected HV-graph G, we construct an instance G' of planar L-drawing by replacing each vertex by a 4-wheel as in Figure 5, each edge (u, v) labeled V (*V-edge*) with the red gadget shown in Figure 6a and each edge (w, z) labeled H (*H-edge*) with the blue gadget shown in Figure 6b. For a V-edge (u, v), the two vertices of the edge gadget labeled u and v are identified with the V-ports of the vertex gadgets of u and v. Symmetrically, for an H-edge (w, z), the two vertices of the edge gadget labeled w and z are identified with the H-ports of the vertex gadgets of w and z. From there, the ports connect by an edge in the corresponding color to the central vertex of their vertex gadget. Obviously, this reduction is polynomial in the size of G.

Our high-level construction is somewhat similar to Brückner's NP-completeness proof for 1-Embeddability in the Kandinsky model [9, Theorem 3] in that we define gadgets that have a very limited flexibility in terms of their embeddings to realize horizontal and vertical edges. Yet, the internals of the gadgets themselves and the reduction are quite different.

We claim that G' has a planar L-drawing if and only if G has a planar HV-drawing. So first assume that G' admits a planar L-drawing Γ'. We transform Γ' into a planar HV-drawing. In a first step, we draw each vertex v of G at the position of the central vertex of the vertex gadget for v. Due to Lemma 2, the edge gadgets are attached to the bounding boxes of the vertex gadgets. Hence, for each edge (u, v) of G, we can draw an orthogonal path from u to v by tracing the thick edges (red for a V-edge, blue for an H-edge) in its edge gadget and the two incident vertex gadgets (see Figures 5 and 6). This intermediate drawing (which is a subdrawing of Γ') is a planar orthogonal drawing of G, where each edge is an 8-bend orthogonal staircase path with total rotation 0. Using Tamassia's network flow model for orthogonal graph drawings [23], we can argue that an edge with rotation 0 is equivalent to a rectilinear edge without bends. In fact, the flow corresponding to the eight bends is cyclic and can be reduced to a flow of value 0, which implies no bends. We refer to Brückner [9, Lemma 7] for the details of this argument.

Now, conversely, assume that G admits a planar HV-drawing Γ. In order to show that Γ can be transformed into an L-drawing of G', we first "thicken" Γ by inflating vertices at grid points to squares and edges to corresponding rectangles, see Figure 7a. To avoid crossings, we simply refine the grid on which Γ is drawn. Since each vertex gadget in G' can be drawn



(a) Connections among edge gadgets with the orthogonal edge paths highlighted as thick polylines.

(b) Modifying the wheel gadget into an oriented graph.

Figure 7 Gadgets of the hardness reduction

inside a square (Figure 5) and each edge gadget inside a rectangle (Figure 6), we can insert their drawings into the thickened drawing of G as illustrated in Figure 7a. This produces an L-drawing of G'.

To see that the problem is in NP, we note that for an embedding of a graph and a given orthogonal representation (see Tamassia [23]) of that embedding, one can check in polynomial time whether all edges are represented by valid L-shapes. ◀

We remark that the graph G' that we construct in our reduction is a simple directed graph. With the exception of the four double-spoke edges of the wheel graph W (see Figure 5), each underlying undirected graph would not have multi-edges. We can, however, modify our reduction by removing the red and blue edges in Figure 5 so that the entire graph G' becomes an oriented graph, i.e., a graph without 2-cycles. Figure 7b shows the modified wheel gadget W. In that case, however, when we construct the intermediate orthogonal paths for the edges of the HV-drawing, we still use the removed "mirrored" L-shapes for the first and last two segments of each edge path (indicated as dashed edges in Figure 7b), which is always possible without crossings in any L-drawing of W. Otherwise the orthogonal path between the two central vertices of the connected wheels would not have rotation 0.

3.2 Fixed embedding and port assignment

In this section, we show how to decide efficiently whether there is a planar L-drawing for a plane directed graph with a fixed assignment of the edges to the four ports of the vertices. Using Theorem 1 and the ILP formulation of Barth et al. [5], we first set up linear inequalities that describe whether a plane 4-modal graph has a planar L-drawing. Using these inequalities, we then transform our decision problem into a matching problem that can be solved in linear time. We call a vertex v an *in/out-vertex* on a face f if v is incident to both an incoming edge and an outgoing edge on f.

The ILP formulation is as follows. For each vertex v and each face f incident to v, we introduce a variable $x_{vf} \in \{0,1,2\}$ that is meant to measure the angle in f at v: the angle between two edges entering v or two edges leaving v is $x_{vf} \cdot \pi$ and the angle between an edge entering v and an edge leaving v is $x_{vf} \cdot \pi + \pi/2$. For each edge e that is incident to v and f, we introduce a variable $x^v_{fe} \in \{0,1\}$. The intended meaning of $x^v_{fe} = 1$ is that there is a convex bend in face f on edge e assigned to vertex v to fulfill the bend-or-end property. Using these variables, we now formulate four conditions that are satisfied if and only if a given plane directed graph admits is a planar L-drawing.

(1) The angles around a vertex v sum up to 2π:

$$\sum_{f \text{ incident to } v} x_{vf} = \begin{cases} 2 & \text{if } v \text{ is 0-modal} \\ 1 & \text{if } v \text{ is 2-modal} \\ 0 & \text{if } v \text{ is 4-modal} \end{cases}$$

(2) All edges are bent exactly once, i.e., for each edge $e = \{v,w\}$ separating faces f and h, we have

$$x^v_{fe} + x^v_{he} + x^w_{fe} + x^w_{he} = 1.$$

(3) The number of convex angles minus the number of concave angles is 4 in each inner face and -4 in the outer face, i.e., for each face f, we have

$$\sum_{\substack{e=\{v,w\} \\ \text{separates} \\ f \text{ and } h}} (x^v_{fe} - x^v_{he} + x^w_{fe} - x^w_{he}) + \sum_{\substack{v \text{ incident to } f, \\ v \text{ not in/out}}} (2 - 2x_{vf}) + \sum_{\substack{v \text{ incident to } f, \\ v \text{ in/out}}} (2 - (2x_{vf} + 1)) = \pm 4.$$

(4) The bend-or-end property is fulfilled, i.e., for any two edges e_1 and e_2 that are consecutive around a vertex v and that are both incoming or both outgoing, and for the faces f_1, f, and f_2 that are separated by e_1 and e_2 (in the cyclic order around v), it holds that

$$x_{vf} + x^v_{f_1 e_1} + x^v_{f_2 e_2} \geq 1.$$

Note that (2) implies $-x^v_{he} - x^w_{he} = x^v_{fe} + x^w_{fe} - 1$. Hence, (3) yields

(3′) $$\sum_{\substack{e=\{v,w\} \\ \text{incident to } f}} (x^v_{fe} + x^w_{fe}) - \sum_{v \text{ incident to } f} x_{vf} = \pm 2 + (\# \text{ in/out-vertices on } f - \deg(f))/2.$$

Observe that the number of in/out-vertices on a face f is odd if and only if $\deg(f)$ is odd. Moreover, if we omit the bend-or-end property, we can formulate the remaining conditions as an uncapacitated network flow problem. The network has three types of nodes: one for each vertex, face, and edge of the graph. It has two types of edges: from vertices to incident faces and from faces to incident edges. The supplies are $(4-k)/2$ for the k-modal vertices, $\pm 2 + (\#\text{in/out-vertices} - \deg(f))/2$ for a face f, and -1 for the edges.

▶ **Theorem 4.** *Given a directed plane graph G and, for each edge e, labels out$(e) \in \{\text{top}, \text{bottom}\}$ and in$(e) \in \{\text{right}, \text{left}\}$, we can decide in linear time whether G admits a planar L-drawing in which each edge e leaves its tail at out(e) and enters its head at in(e).*

Proof. Observe that the labeling determines the bends, i.e., the value $x^v_{fe} + x^w_{fe}$ for each edge $e = (v, w)$ and each incident face f. First, we have to check whether the cyclic order of the edges around a vertex is compatible with the labels, i.e., in clockwise order we have outgoing edges labeled (top, $\cdot$), incoming edges labeled ($\cdot$, left), outgoing edges labeled (bottom, $\cdot$), and incoming edges labeled ($\cdot$, right). For a fixed port, edges bending to the left must precede edges bending to the right. We call an edge a *middle edge* of a port if it is the last edge bending to the left or the first edge bending to the right. Observe that each port has zero, one, or two middle edges. For example, the top port of vertex v in Figure 4 has two middle edges, namely the second edge and the third edge from the left.

If the compatibility check does not fail then the labels also determine the angles around the vertices, i.e., the variables x_{vf} for each vertex v and each incidence to a face f. Now, we check whether these values fulfill Conditions 1, 2, and 3'.

Finally, we have to check, whether Condition 4, i.e., the bend-or-end property can be fulfilled. To this end, we have to assign edges with concave bends to zero angles at an incident vertex in the same face. We must assign, for each port of a vertex v, all but the middle edges to v. If at this stage an edge is assigned to two vertices, then G does not admit a planar L-drawing with the given port assignment.

Otherwise, it remains to deal with the zero angles between two middle edges of a port. To this end, consider the following graph B that has two types of nodes; a *port node* for each port with two middle edges and a *middle-edge node* for each edge that is a middle edge of at least one port and that is not yet assigned to a vertex. A port node corresponding to a port of a vertex v and a middle-edge node corresponding to an edge e are adjacent in B if and only if e is a middle edge of v. Observe that B is a bipartite graph of maximum degree two and, thus, consists of paths, even length cycles, and isolated nodes. We have to test whether B admits a matching in which every port node is matched. This is true if and only if no port node is isolated and there is no maximal path starting and ending at a port node. ◀

4 Upward- and upward-rightward planar L-drawings

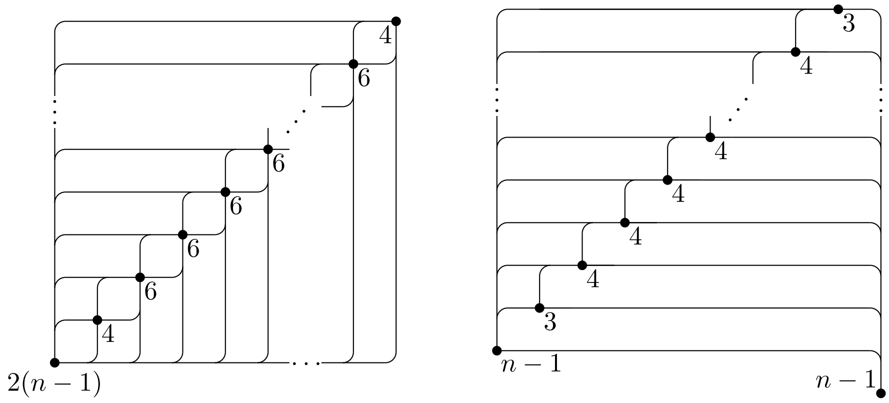
In this section, we study the density (Theorem 5) and provide a characterization (Theorem 6) of the graphs that admit upward-planar and upward-rightward planar L-drawings.

We start by showing an upper bound on the maximum number of edges of a graph that admits an (upward-, upward-rightward-planar) L-drawing.

▶ **Theorem 5.** *A directed graph with n vertices that admits a planar, upward-planar, or upward-rightward-planar L-drawing has at most $4n - 6$, $3n - 6$, or $2n - 3$ edges and these bounds are tight.*

Proof. In the following, let n denote the number of vertices of the given graph.

planar: Consider for each port of a vertex the furthest bend. Recall that the bend on any edge is the furthest bend of at least one of its end-vertices. On the other hand each vertex has at most four furthest bends. Thus there can be at most $4n$ edges. Consider now the outer face. The topmost (bottommost, rightmost, leftmost) vertex doesn't have a furthest bend at its top (bottom, right, left) port. Moreover in a maximal L-planar drawing there are at least two edges e_1 and e_2 on the outer face such that its bend is a furthest bend of both end-vertices: Consider the bottommost vertex v. If v is neither the leftmost nor the rightmost vertex, let u_1 and u_2 be the leftmost and rightmost vertex such that there is an edge $e_1 = (u_1, v)$ and $e_2 = (u_2, v)$, respectively. If v is the leftmost (rightmost) vertex, let u be the rightmost (leftmost) vertex such that there is an edge $e_1 = (u, v)$ and let



(a) maximum L-planar **(b)** maximum upward L-planar

Figure 8 A graph with n vertices and (a) $4n - 6$ edges with a planar L-drawing (b) $3n - 6$ edges with an upward-planar L-drawing. Labels indicate vertex degrees.

w be the topmost vertex such that there is an edge $e_2 = (v, w)$. This yields the $4n - 6$ bound. Finally, Figure 8a shows a graph with a planar L-drawing and $4n - 6$ edges.

upward-planar: Since upward-planar graphs must be acyclic, they cannot contain 2-cycles. Thus, there are at most $3n - 6$ edges. Figure 8b shows a graph with an upward-planar L-drawing and $3n - 6$ edges.

upward-rightward-planar: Each vertex has at most two furthest bends. The bottommost vertex has no furthest bend to the left, the rightmost vertex has no furthest bend to the top and in a maximal upward-rightward planar L-drawing there is at least one bend that is furthest for both end-vertices. Hence, there are at most $2n - 3$ edges. Omitting all but the upward-rightward edges in Figure 8a yields a graph with $2n - 3$ edges.

◀

Although characterizing the plane directed graphs that admit a planar L-drawing seems to be an elusive goal, we can characterize two natural subclasses of planar L-drawings by means of bitonic st-orderings.

▶ **Theorem 6.** *A planar st-graph admits an upward-planar (resp. upward-rightward-planar) L-drawing if and only if it admits a bitonic (resp. monotonically decreasing) pair.*

Proof. Let $G = (V, E)$ be a planar st-graph with n vertices.

"$\Rightarrow$": The y-coordinates of an upward- (upward-rightward-) planar L-drawing of G yield a bitonic (monotonically decreasing) st-ordering π with respect to the embedding $\mathcal{E}$ given by the L-drawing.

"$\Leftarrow$": Given a bitonic (monotonically decreasing) st-ordering π of G, we construct an upward- (upward-rightward-) planar L-drawing of G using an idea of Gronemann [21]. For $i = 1, \ldots, n$, let $v_i \in V$ be the vertex with $\pi(v_i) = i$, set the y-coordinate of v_i to i, and let G_i be the subgraph of G induced by $V_i = \{v_1, \ldots, v_i\}$.

For the x-coordinates we construct a partial order $\prec$ in such a way that, for $i = 2, \ldots, n$, all vertices on the outer face of G_i are comparable and the L-drawing of G_i is planar, embedding preserving, and has the property that any edge from V_i to $V \setminus V_i$ can be added upward and in an embedding-preserving way, no matter how we choose the x-coordinates of $v_{i+1}, \ldots, v_n$.

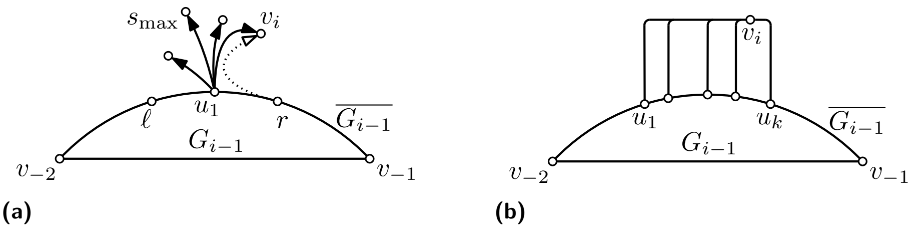


Figure 9 How to turn a bitonic st-ordering into a planar L-drawing.

During the construction, we augment G_i to $\overline{G}_i$ in such a way that the outer face $f_{\overline{G}_i}$ of $\overline{G}_i$ is a simple cycle. We start by adding two artificial vertices v_{-1} and v_{-2} with y-coordinates -1 and -2, respectively, that are connected to v_1 and to each other. We set $v_{-2} \prec v_1 \prec v_{-1}$. Now let $i \in \{2, \dots, n\}$ and assume that we have already fixed the relative coordinates of G_{i-1}. Let $u_1, \dots, u_k$ be the predecessors of v_i in ascending order with respect to $\prec$.

If π is monotonically decreasing or if $k = 1$, we first augment the graph. In the former case, we add to G an edge between v_i and the right neighbor of u_k on $f_{\overline{G}_{i-1}}$. In the latter case, let ℓ be the left neighbor and let r be the right neighbor of u_1 on $f_{\overline{G}_{i-1}}$, respectively; see Figure 9a. Following Gronemann [21], we add a dummy edge from either ℓ or r to v_i: Let $s_{\max}$ be the successor of u_1 that is maximum with respect to the st-ordering. We go in counter-clockwise circular order of the edges around u_1 from (u_1, v_i) to the left. If we hit $(u_1, s_{\max})$ before we hit (u_1, ℓ), we insert the edge (r, v_i) into G, otherwise we insert the edge (ℓ, v_i). E.g., when processing the graph in Figure 1a, we insert the edges $(v_{-1}, 2)$ and $(v_{-2}, 4)$. Note that inserting the dummy edge does not violate planarity since, on that side, u_k does not have any outgoing edge between (u_k, v_i) and $f_{\overline{G}_{i-1}}$.

We now extend $\prec$. Let $u_1, \dots, u_k$ be the $k \geq 2$ predecessors of v_i in the possibly augmented graph; see Figure 9b. Since G has a sink only on the outer face, we can place v_i anywhere between u_1 and u_k. Adding the two conditions $u_{k-1} \prec v_i \prec u_k$ also ensures that all edges except (u_k, v_i) are rightward. But (u_k, v_i) was introduced only as a dummy edge for the case of a monotonically decreasing π.

Any linear order that is compatible with $\prec$ yields unique x-coordinates in $\{1, \dots, n\}$ for the vertices of G. Together with the y-coordinates that we fixed above, we now have positions for the vertices in an upward- (upward-rightward-) planar L-drawing of G. Finally, we remove the dummy edges that we inserted earlier. ◀

We have thus also obtained a new proof of the following statement.

▶ **Corollary 7.** *Any undirected planar graph can be oriented such that it admits an upward-planar L-drawing.*

Proof. Triangulate the graph G and construct a bitonic st-ordering for undirected graphs [19]. Orient the edges from smaller to larger st-numbers. ◀

5 Conclusions and open problems

In this work, we initiated the investigation of planar and upward-planar L-drawings of directed graphs. In particular, we established interesting connections between the problem of computing drawings respecting the above drawing styles on one hand, and Kandinsky drawings and bitonic st-orderings on the other hand. Several interesting questions arise from our research.

1. Can we efficiently test whether a directed plane graph admits a planar L-drawing?
2. Can we efficiently test whether an upward-plane graph with multiple sources and sinks admits an upward-planar L-drawing?
3. Can we efficiently recognize the directed graphs that are edge maximal subject to having a planar L-drawing?
4. Does every bimodal graph have a planar L-drawing? Recall that the existence of planar L-drawings of bimodal graphs has been confirmed only if the input does not contain 2-cycles [3].

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