Decidability of intuitionistic S4

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Logic S4 was among the first modal logics in the "modern" logical tradition, the fourth system of C. I. Lewis. Its common axiomatic formulation is due to Gödel. The commonly used semantics for it as the logic of all reflexive and transitive Kripke frames is also at least half a century old. Little remains unknown about it, and it enjoys most properties desirable of a well-behaved logic. In particular, its decidability was shown by Ladner (1977).

The propositional basis of S4 is classical, so it is natural to study what happens when it is replaced by intuitionistic propositional logic (IPL). While the transition is not entirely deterministic, we focus here on what eventually became known as *intuitionistic modal logics* in the tradition of Fischer Servi (1984) and Plotkin and Stirling (1986), which were investigated in detail by Simpson (1994). While it is reasonable to expect that intuitionistic reasoning makes things more complex compared to classical one, this is a priori more likely to cause the increase in complexity than to lead to an undecidable logic. Thus, it is all the more surprising that the problem of decidability of IS4, i.e., of intuitionistic S4, remained open since it was formulated by Simpson (1994). We finally solve this question positively: IS4 is decidable.

The langugage of ***logic*** IS4 is $A ::= \bot \mid a \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$ where $a \in \mathcal{A}$ is an atomic formula (note that, unlike for S4, modalities $\Box$ and $\Diamond$ are independent). Its axiom system is obtained by extending any standard axiom system for IPL with

k_1: $\Box(A \supset B) \supset (\Box A \supset \Box B)$ $\quad$ k_2: $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

k_3: $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$ $\quad$ k_4: $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$ $\quad$ k_5: $\Diamond\bot \supset \bot$

$\mathsf{4}$: $(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$ $\quad$ t: $(A \supset \Diamond A) \wedge (\Box A \supset A)$

and the standard necessitation rule. As classical S4, Kripke frames of IS4 are reflexive and transitive, but in the so-called *birelational semantics*:

A ***birelational model*** $\mathcal{M}$ for IS4 is a quadruple $\langle W, R, \leq, V\rangle$ of a set $W \neq \varnothing$ of ***worlds*** equipped with two preorders (i.e., reflexive and transitive relations) — an ***accessibility relation*** R and future relation $\leq$ — and a ***valuation function*** $V\colon W \to 2^{\mathcal{A}}$ satisfying:

(F_1) For all $x, y, z \in W$, if xRy and $y{\leq}z$, there exists $u \in W$ such that $x{\leq}u$ and uRz.

(F_2) For all $x, y, z \in W$, if $x{\leq}z$ and xRy, there exists $u \in W$ such that zRu and $y{\leq}u$.

(M) If $w{\leq}w'$, then $V(w) \subseteq V(w')$.

Forcing $\Vdash$ for atomic formulas is determined by the valuation function: $\mathcal{M}, w \Vdash a$ iff $a \in V(w)$, with $\mathcal{M}, w \nVdash \bot$. It is recursively extended to all formulas as follows:

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$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$	iff	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \supset B$	iff	for all w' with $w \leq w'$, if $\mathcal{M}, w' \Vdash A$, then $\mathcal{M}, w' \Vdash B$;
$\mathcal{M}, w \Vdash \Box A$	iff	for all w' and u with $w \leq w'$ and $w'Ru$, we have $\mathcal{M}, u \Vdash A$;
$\mathcal{M}, w \Vdash \Diamond A$	iff	there exists u such that wRu and $\mathcal{M}, u \Vdash A$.

Theorem (Fischer Servi (1984); Plotkin and Stirling (1986))**.** A formula A is a theorem of IS4 if and only if A is valid in every birelational model for IS4.

Our proof of decidability of IS4 is proof-theoretical. A proof search is performed in a suitable analytic sequent-like calculus for IS4. If the proof search is successful in finding a proof, the formula in question is derivable. Otherwise, a failed proof search provides sufficient information to construct a countermodel. The difficulties in applying this method to IS4 are not new either. It is typical that a naive proof search for a logic with transitive Kripke frames does not terminate. Thus, loop-checks are used for both S4 (w.r.t. transitive R) and IPL (w.r.t. transitive $\leq$) to stop the naive proof search. A non-terminating naive proof search is bound to enter into a loop due to the subformula property, which ensures a global bound on the number of sequents that can appear in a proof search. When that happens, a countermodel can be constructed by emulating the algorithm loop by an appropriate R-loop for S4 or $\leq$-loop for IPL.

The unique challenges of IS4 are due to the fact that the two sources of repetition can interact, creating a possibility of a proof search neither terminating nor repeating any sequents. To overcome this problem we use a fully labelled sequent calculus (see Maffeziolli et al. (2013); Marin et al. (2021)) with relational atoms for both relations R and $\leq$, where R-loops can be represented on a sequent level. Since labelled sequent rules do not ordinarily create such loops, we incorporate several loop-checks into the proof search algorithm by adding new rules for creating R-loops. This R-loop-enabled proof search still does not guarantee sequent repetition, forcing us to formulate a more complex loop-check condition with respect to $\leq$-loops: the proof search is stopped if the latest sequent can be *emulated* by an earlier sequent. The soundness of the new R-loop-creating rules is proved by a non-trivial *unfolding* algorithm that converts derivations with R-loops into proper loop-free derivations by creating multiple duplicates of each loop. Thus, this loop-augmented proof search provides a decision procedure for IS4.

Theorem. Logic IS4 is decidable.

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