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Semigroup topologies on endomorphism monoids of omega-categorical structures

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Abstract

Endomorphism monoids of ω-categorical structures are rich algebraic-topological objects, joining a semigroup (given by the composition operation) with a Polish topology (the topology of pointwise convergence defined via the evaluation at elements of the domain), with the additional compatibility property that the operation is continuous with respect to the topology. This thesis centers around the following question:

> How much information about the topology of pointwise convergence can be *reconstructed* from the algebraic semigroup structure?

More precisely, we ask in which aspects an arbitrary topology (potentially with some additional purely topological properties) for which the composition operation shall be continuous needs to resemble the topology of pointwise convergence.

For the first level of reconstruction, we consider another natural topology on endomorphism monoids, namely the so-called Zariski topology induced within the monoid by (non-)solutions to equations. For all concrete endomorphism monoids of ω-categorical structures on which the Zariski topology has been analysed thus far, it was shown to coincide with the topology of pointwise convergence. Regarding our central question, this yields that any Hausdorff semigroup topology on those endomorphism monoids can be *reconstructed* to be finer than the topology of pointwise convergence. We establish two systematic reasons for the two topologies to agree, formulated in terms of the model-complete core of the structure, as well as give an example of an ω-categorical structure on whose endomorphism monoid the topology of pointwise convergence and the Zariski topology differ.

For various endomorphism monoids of ω-categorical structures, an even higher level of reconstruction is attained: the topology of pointwise convergence sometimes turns out to be the unique Polish semigroup topology – in other words, the topology can be *uniquely reconstructed* from the semigroup structure if we restrict to Polish topologies. This problem was unsolved for the endomorphism monoid of the rational numbers with the non-strict order (so the semigroup of increasing maps on the rational numbers). We develop new techniques to prove that the topology of pointwise convergence is indeed the only Polish topology turning this semigroup into a topological one, and show why previous methods are insufficient for this matter.

Finally, we consider the class of countably infinite homogeneous graphs and analyse which of these have endomorphism monoids such that the topology of pointwise convergence is the unique Polish semigroup topology. We solve this problem for all previously untreated homogeneous graphs, with a single exception for which we provide partial results.

Kurzfassung

Endomorphismenmonoide von ω-kategorischen Strukturen bilden durch Kombination einer Halbgruppe (Verknüpfungsoperation) mit einer Polnischen Topologie (die Topologie der punktweisen Konvergenz, definiert durch Auswertung bei Elementen der Grundmenge) reichhaltige algebraisch-topologische Objekte, die die zusätzliche Kompatibilitätseigenschaft aufweisen, dass die Operation bezüglich der Topologie stetig ist. Diese Dissertation beschäftigt sich mit der folgenden Frage:

> Wieviel Information über die Topologie der punktweisen Konvergenz kann aus der algebraischen Halbgruppenstruktur *rekonstruiert* werden?

Genauer fragen wir, in welcher Hinsicht eine beliebige Topologie (von der wir möglicherweise zusätzliche rein topologische Eigenschaften annehmen), für die die Verknüpfungsoperation stetig ist, der Topologie der punktweisen Konvergenz ähneln muss.

Für die erste Stufe an Rekonstruktion betrachten wir eine andere natürliche Topologie auf Endomorphismenmonoiden, nämlich die sogenannte Zariski-Topologie, die innerhalb des Monoids durch (Nicht-)Lösungen von Gleichungen definiert wird. Für alle Endomorphismenmonoide ω-kategorischer Strukturen, für die die Zariski-Topologie bisher konkret analysiert wurde, hat sich gezeigt, dass sie mit der Topologie der punktweisen Konvergenz übereinstimmt. In Hinblick auf unsere zentrale Frage liefert dies die *Rekonstruktionsaussage*, dass auf diesen Endomorphismenmonoiden jede Hausdorffsche Halbgruppentopologie automatisch feiner sein muss als die Topologie der punktweisen Konvergenz. Wir geben zwei systematische, aus dem modellvollständigen Kern[2] der zugrundeliegenden Struktur abgeleitete Bedingungen an, unter denen die beiden Topologien stets übereinstimmen. Außerdem geben wir ein Beispiel einer ω-kategorischen Struktur an, sodass sich die beiden Topologien auf ihrem Endomorphismenmonoid unterscheiden.

Für verschiedenste Endomorphismenmonoide ω-kategorischer Strukturen gilt sogar eine höhere Stufe an Rekonstruktion: Manchmal stellt sich die Topologie der punktweisen Konvergenz nämlich als einzige Polnische Halbgruppentopologie heraus – anders formuliert kann die Topologie *eindeutig* aus der Halbgruppenstruktur *rekonstruiert* werden, wenn man sich auf Polnische Topologien einschränkt. Dieses Problem war für das Endomorphismenmonoid der rationalen Zahlen mit der schwachen Ordnung (also für die Halbgruppe der wachsenden Funktionen auf den rationalen Zahlen) noch ungelöst. Wir entwickeln neue Techniken, um zu zeigen, dass die Topologie der punktweisen Konvergenz die einzige Polnische Halbgruppentopologie auf diesem Monoid ist, und beweisen außerdem, wieso die bisherigen Methoden nicht zur Lösung geeignet waren.

Abschließend beschäftigen wir uns mit der Klasse der abzählbar unendlichen homogenen Graphen und analysieren, welche von ihnen ein Endomorphismenmonoid haben, auf dem

[2] englisch: model-complete core

die Topologie der punktweisen Konvergenz die einzige Polnische Halbgruppentopologie ist. Wir lösen dieses Problem für alle bisher unbehandelten homogenen Graphen mit einer einzigen Ausnahme, für die wir partielle Ergebnisse zeigen.

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Contents

1 Introduction

For an arbitrary set A, the space A^A of all self-maps of A is equipped with several interesting kinds of structure; we focus on an algebraic one and topological ones. On the one hand, A^A carries the composition operation $\circ$ of functions which forms a monoid structure (semigroup structure with a neutral element). On the other hand, a canonical topology on A^A is given by the product topology where each copy of A is endowed with the discrete topology. In this topology, a sequence $(f_n)_{n\in\mathbb{N}}$ converges to f if and only if for each $a \in A$, the sequence $(f_n(a))_{n\in\mathbb{N}}$ is eventually constant with value $f(a)$. It will be referred to as the *topology of pointwise convergence*, or *pointwise topology* for short, and has several interesting properties: if A is countable, it most importantly turns out to be a *Polish* (completely metrisable and second countable) topology. Furthermore, the two structures are compatible in the sense that the composition operation is continuous as a function $\circ\colon A^A \times A^A \to A^A$ with respect to the pointwise topology, where $A^A \times A^A$ carries the product topology. We say that the pointwise topology is a *semigroup topology* and that A^A together with the pointwise topology is a *topological semigroup*. Clearly, the algebraic structure restricts to all subsemigroups of A^A which in turn can be equipped with the respective subspace topologies. If the subsemigroup is a G_δ subset of A^A with respect to the pointwise topology, then it forms a Polish semigroup itself. The most prominent examples of such G_δ subsemigroups are $\mathrm{Sym}(A)$, the space of all permutations of A, as well as – more generally – the *automorphism group* $\mathrm{Aut}(\mathbb{A})$ and the *endomorphism monoid* $\mathrm{End}(\mathbb{A})$ of any given (model-theoretic) structure $\mathbb{A}$ with domain A. In the first two cases, the algebraic structure is even a group and the topology is a *group topology*, meaning that the inversion map on the group is also continuous. For an *ω-categorical structure* (defined e.g. in [Hod97]), the group $\mathrm{Aut}(\mathbb{A})$ and semigroup $\mathrm{End}(\mathbb{A})$ encode structural information about $\mathbb{A}$, in general, however, not from the abstract (semi-)group alone but only if additional data is used: if $\mathbb{A}'$ is another ω-categorical structure, then $\mathbb{A}$ and $\mathbb{A}'$ are *bi-interpretable* by first-order formulas if and only if $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{A}')$ are isomorphic as topological groups ([AZ86]); and – whenever neither $\mathbb{A}$ nor $\mathbb{A}'$ have constant endomorphisms – $\mathbb{A}$ and $\mathbb{A}'$ are *bi-interpretable* by the more restricted existential positive formulas if and only if the richer objects $\mathrm{End}(\mathbb{A})$ and $\mathrm{End}(\mathbb{A}')$ are isomorphic as topological semigroups ([BJ11]).

At this point, it is natural to ask how much topological information is already contained in the abstract algebraic structure, i.e. how much information about a topology can be *reconstructed* from the knowledge that it is compatible with a given algebraic structure. This problem has been studied from various angles and for many classes of algebraic structures over the years, using techniques from several areas of mathematics. Particular interest has been given to additional requirements on the topology, for instance that the topology be Polish or Hausdorff. We give two examples. Concerning vector spaces, it is a folklore result in functional analysis that the finite dimensional $\mathbb{R}$-vector space $\mathbb{R}^n$ carries a unique

Hausdorff vector space topology[1], namely the standard Euclidean topology. However, its additive group does carry multiple Hausdorff – even Polish – group topologies already for $n = 1$: the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$ are algebraically isomorphic (consider them as additive groups of vector spaces over $\mathbb{Q}$ of equal dimension) but $\mathbb{R}$ and $\mathbb{R}^2$ equipped with the Euclidean topologies are not homeomorphic. Hence, if we pull back the Euclidean topology on $\mathbb{R}^2$ to $\mathbb{R}$ via the isomorphism $(\mathbb{R}, +) \to (\mathbb{R}^2, +)$, we obtain a Polish group topology different from the Euclidean topology on $(\mathbb{R}, +)$. Note that this construction requires the axiom of choice (to find the algebraic isomorphism $(\mathbb{R}, +) \to (\mathbb{R}^2, +)$). This is not a coincidence since Solovay [Sol70] and Shelah [She84] showed the consistency of ZF (without choice) with the fact that *any* Polish group has a unique Polish group topology.

Returning to subsemigroups of A^A, one can ask for *reconstruction* results of the following shape:

> On the Polish (semi-)group $S \subseteq A^A$, the pointwise topology is the unique Polish (semi-)group topology.
>
> (hereafter: S has the Unique Polish Property or UPP for short)

For the class of groups, UPP has been extensively studied; examples include the full symmetric group $\mathrm{Sym}(A)$ ([Gau67] combined with [Las91]) and the automorphism group of the random (di-)graph ([HHLS93] combined with [KR07]). Additionally, $\mathrm{Aut}(\mathbb{Q}, \leq)$ – explicitly: the space of all increasing permutations of $\mathbb{Q}$ – also has UPP ([RS07] combined with [Las91]). Recent years brought results in the realm of semigroups as well; it was shown in [EJM$^+$] that the full transformation monoid A^A has UPP. Furthermore, the endomorphism monoids of the random graph, the random digraph and the equivalence relation with countably many equivalence classes of countably infinite size turn out to have UPP as well, see [EJM$^+$23]. One notices that the examples from these lists either contain only bijective functions (the groups) or contain both non-injective and non-surjective functions. This is essential for UPP to hold: by constructions given in [EJM$^+$], both the monoid $\mathrm{Inj}(A)$ of all injective functions on A and the monoid $\mathrm{Surj}(A)$ of all surjective functions on A carry multiple Polish semigroup topologies.

The papers [EJM$^+$] and subsequently [EJM$^+$23] show uniqueness of Polish semigroup topologies in two natural steps:

(1) Show that the pointwise topology is coarser than any Polish semigroup topology.

(2) Show that the pointwise topology is finer than any Polish semigroup topology.

Usually, the second step takes considerably more work than the first step.

For the purpose of Step (1), the authors of [EJM$^+$] transferred a notion from the theory of topological groups to the realm of semigroups, namely the so-called *Zariski topology* (or sometimes *verbal topology*), see [Bry77, DT18, Mar50]; roughly speaking, the closed sets in this topology are given by solution sets to identities in the language of semigroups. Hence, the Zariski topology is an object associated to the algebraic (semi-)group structure. Considering $\mathrm{End}(\mathbb{A})$ as an abstract semigroup, the Zariski topology can thus be regarded as an

[1] A topology $\mathcal{T}$ on $\mathbb{R}^n$ is called a *vector space topology* if it is a group topology with respect to addition and if the scalar multiplication $(\lambda, x) \mapsto \lambda x$ is continuous as a map $(\mathbb{R}, \mathcal{T}_{\mathrm{Eucl}}) \times (\mathbb{R}^n, \mathcal{T}) \to (\mathbb{R}^n, \mathcal{T})$; note that the scalar field $\mathbb{R}$ is to carry the standard Euclidean topology.

"internal" object. The pointwise topology, in contrast, is defined from the evaluations at elements of the domain of $\mathbb{A}$ and is thus an "external" object with respect to the abstract semigroup structure of $\mathrm{End}(\mathbb{A})$ – precisely speaking, the pointwise topology is associated to the semigroup action of $\mathrm{End}(\mathbb{A})$ on A. As it turns out, the Zariski topology is necessarily coarser than any Hausdorff semigroup topology on a given semigroup. In particular, the pointwise topology on $\mathrm{End}(\mathbb{A})$ is always finer than the Zariski topology. If one manages to show that the Zariski topology on $\mathrm{End}(\mathbb{A})$ even coincides with the pointwise topology for some structure $\mathbb{A}$, one can draw two conclusions: on an abstract level, the pointwise topology can also be understood as an "internal" object with respect to the abstract semigroup structure, thus giving another type of reconstruction statement; on a more concrete level, the pointwise topology then indeed is the coarsest (in particular) Polish semigroup topology on $\mathrm{End}(\mathbb{A})$, completing Step (1). This method was successfully used both in [EJM+] and in [EJM+23].

Step (2), on the other hand, is accomplished by means of lifting from a subset, usually (but not necessarily) the automorphism group, to the endomorphism monoid. To this end, the crucial instrument of *Property* $\boldsymbol{X}$ was introduced in [EJM+] which enables to understand the pointwise topology on $\mathrm{End}(\mathbb{A})$ by looking at the pointwise topology on e.g. $\mathrm{Aut}(\mathbb{A})$ as well as the composition operation: Property $\mathbf{X}$ means that a given endomorphism s can be decomposed in the form $s = gaf$ with fixed endomorphisms f, g and an automorphism a in such a way that for any neighbourhood V of a with respect to the pointwise topology within the *automorphism group*, the set of composites gVf is a neighbourhood of s with respect to the pointwise topology within the *endomorphism monoid*.

The main aim of this thesis is to extend and shed some more light on the techniques behind both Step (1) and Step (2).

To begin with, we consider Step (1) in more detail and remark that in each instance, the proof that the Zariski topology coincides with the pointwise topology has not been particularly systematic but tuned to the specific situation being considered, based on two sets of rather technical sufficient conditions established in [EJM+] and the ad hoc notion of so-called *arsfacere* structures introduced in [EJM+23] for which these conditions always hold. This raises the following problem:

Question A1. *Are there systematic reasons for equality of the topologies, in other words general and more structural properties to require for* $\mathbb{A}$ *which yield that our two topologies on* $\mathrm{End}(\mathbb{A})$ *coincide?*

Furthermore, for each ω-categorical structure $\mathbb{A}$ explicitly considered thus far, it was possible to show that the pointwise topology and the Zariski topology on $\mathrm{End}(\mathbb{A})$ coincide, leading to the authors of [EJM+23] asking the following question:

Question A2 ([EJM+23, Question 3.1])**.** *Is there an* ω*-categorical relational structure* $\mathbb{A}$ *such that the topology of pointwise convergence on* $\mathrm{End}(\mathbb{A})$ *is strictly finer than the Zariski topology?*

We establish two new sets of sufficient conditions on a structure $\mathbb{A}$ under which the Zariski topology and the pointwise topology on $\mathrm{End}(\mathbb{A})$ coincide – so, in particular, under which the pointwise topology is the coarsest Polish semigroup topology on $\mathrm{End}(\mathbb{A})$. To this end, we give a new application of so-called *model-complete cores* which have proved to be a helpful tool not only in the algebraic theory of constraint satisfaction problems [BOP18] but also – of independent purely mathematical interest – in the universal algebraic study of polymorphism clones of ω-categorical structures [BKO+17, BP16] as well as in the Ramsey-theoretic analysis of ω-categorical structures [Bod15].

We show in particular that for a *transitive* ω-categorical structure *without algebraicity* (standard properties of structures) such that the model-complete core of the structure is either finite or has no algebraicity itself, the Zariski topology and the pointwise topology on its endomorphism monoid coincide. These two cases leave a middle ground open – namely structures whose model-complete core is infinite but has algebraicity. Thus, this is where a positive answer to Question A2 could be found. And indeed, we give an example of an ω-categorical structure for which the pointwise topology on the endomorphism monoid is strictly finer than the Zariski topology. Being transitive as well as homogeneous in a finite relational language, this structure shows that even these additional standard *well-behavedness* assumptions are insufficient to guarantee that the two topologies coincide. This indicates that the structure of the model-complete core really contains the systematic reason for the two topologies to be equal.

Next, we turn to a more detailed investigation of Step (2): the technique exhibited in [EJM+] as well as [EJM+23] uses the notion of *automatic continuity* – this means that for a given fixed (semi-)group $(S, \mathcal{T})$ and any (semi-)group $(H, \mathcal{O})$ from a certain class of (semi-)groups, every algebraic homomorphism $S \to H$ is *automatically* continuous with respect to $\mathcal{T}$ and $\mathcal{O}$. As it turns out, if the endomorphism monoid in question equipped with the pointwise topology has automatic continuity with respect to the class of Polish semigroups, Step (2) can be immediately deduced to hold. As mentioned above, Property $\mathbf{X}$ is an instrument to connect the endomorphism monoid and e.g. the automorphism group; more precisely, it allows to lift automatic continuity from e.g. the automorphism group to the endomorphism monoid. This technique, however, encounters a road block when the endomorphism monoid does not satisfy automatic continuity (which does not necessarily imply that UPP fails). If in this situation the automorphism group of the structure *does* have automatic continuity, we can conclude that Property $\mathbf{X}$ *cannot* hold. An example for this behaviour is given by the endomorphism monoid of the rational numbers equipped with the usual non-strict order; explicitly, these are the increasing maps on $\mathbb{Q}$. We therefore ask the following question:

Question B. *Does the endomorphism monoid of* $\langle \mathbb{Q}, \leq \rangle$ *have UPP?*

As it turns out, Step (1) works smoothly; however, as described above, Step (2) cannot be performed by known techniques. Nevertheless, we show that the endomorphism monoid of $\langle \mathbb{Q}, \leq \rangle$ has UPP by developing a threefold generalisation of the technique involving Property $\mathbf{X}$ in order to tackle Step (2). Most importantly, we consider topologies that are finer than the pointwise topology in intermediate steps, showing that the monoid endowed with a finer topology does have a form of Property $\mathbf{X}$ – evidently, we subsequently have to reduce from that richer topology to the pointwise topology in an additional step; this is not necessary in the previous proofs using Property $\mathbf{X}$. Second, we also need to generalise Property $\mathbf{X}$ itself in two ways, leading to what we call *Pseudo-Property* $\overline{\mathbf{X}}$.

Finally, we apply our techniques to the following problem:

Question C ([EJM+23, Question 5.4])**.** *The endomorphism monoids of which homogeneous graphs have a unique Polish topology?*

By a classification result due to Lachlan and Woodrow [LW80], the class of homogeneous (countably infinite, loopless, symmetric) graphs consists of the random graph and four countably infinite families of graphs, namely the random $\mathbb{K}_n$-free graphs ($n \geq 3$); their dual graphs; the *irreflexive* equivalence relations with at most countably many equivalence classes of at most countably infinite but equal size; and their dual graphs (which turn out to be the complete k-partite graphs ($k \geq 1$ or k countably infinite) on parts of at most countably infinite but equal size). The random graph as well as the first and third family were already treated in [EJM+23]. We consider the second and fourth family and show for each of these graphs whether their endomorphism monoids have UPP or automatic continuity – with a single exception which has withstood our attacks, namely the complete infinite-partite graph on parts of countably infinite size with regard to UPP. Along the way, we also give an application of Pseudo-Property $\overline{\mathbf{X}}$ with respect to subsets other than the automorphism group.

This thesis is structured as follows: Chapter 2 contains the relevant definitions and known results which we will use in the sequel; in particular, we formally define the Zariski topology as well as our notion of *Pseudo-Property* $\overline{\mathbf{X}}$. In Chapter 3, we provide the sufficient conditions for the Zariski topology and the pointwise topology to coincide and give our counterexample with differing topologies. Chapter 4 is devoted to the proof that the endomorphism monoid of the rational numbers has UPP. Finally, we consider homogeneous graphs in Chapter 5. The chapters 3 and 4 are based on the articles [PS23a] and [PS23b], respectively.

2 General preliminaries

In the present chapter, we collect notions and known facts from various areas which will be necessary for this thesis. We consider structures and compatible functions in Section 2.1, the pointwise topology and Zariski topology in Section 2.2, standard model-theoretic notions in Section 2.3, graphs (in particular homogeneous ones) as a special case of these notions in Section 2.4, model-complete cores in Section 2.5, the Unique Polish Property and automatic continuity in Section 2.6, Pseudo-Property $\overline{\mathbf{X}}$ in Section 2.7, and finally the Back&Forth method in Section 2.8.

2.1 Structures, homomorphisms, embeddings, automorphisms

Notation 2.1.1. For a function $f\colon A \to B$ between arbitrary sets A, B and a tuple $\bar{a} = (a_1, \ldots, a_n)$ in A, we denote the tuple $(f(a_1), \ldots, f(a_n))$ of evaluations by $f(\bar{a})$ for notational simplicity. Note that in contrast to some related works (like [EJM+, EJM+23]), we denote the evaluation of the function f at the element a by $f(a)$ and write compositions of functions from right to left, i.e. $fg := f \circ g := (a \mapsto f(g(a)))$.

A *(relational) structure* $\mathbb{A} = \langle A, (R_i)_{i\in I}\rangle$ is a domain A (in the following always finite or countably infinite) equipped with m_i-ary relations $R_i \subseteq A^{m_i}$. If no misunderstandings can arise, we will not strictly distinguish between the structure $\mathbb{A}$ and its domain A. The indexed set $\{R_i : i \in I\}$ of relation symbols (where the symbol R_i shall also contain the information about the arity m_i) is called the *language* of $\mathbb{A}$. If $\mathbb{B}$ is another structure in the same language, i.e. $\mathbb{B} = \langle B, (S_i)_{i\in I}\rangle$ where S_i also has arity m_i, we call a function $f\colon A \to B$ a *homomorphism* and write $f\colon \mathbb{A} \to \mathbb{B}$ if f is compatible with all R_i and S_i, i.e. if $\bar{a} \in R_i$ implies $f(\bar{a}) \in S_i$. A homomorphism $f\colon \mathbb{A} \to \mathbb{A}$ is called an *endomorphism* of $\mathbb{A}$. We denote the set of all endomorphisms of $\mathbb{A}$ by $\mathrm{End}(\mathbb{A})$; it forms a monoid with the composition operation and the neutral element id_A. An *embedding* of $\mathbb{A}$ into $\mathbb{B}$ is an injective homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ which is additionally compatible with the complements of R_i and S_i, equivalently if $f(\bar{a}) \in S_i$ also implies $\bar{a} \in R_i$. The set of all *self-embeddings* of $\mathbb{A}$, i.e. of all embeddings of $\mathbb{A}$ into $\mathbb{A}$, is denoted by $\mathrm{Emb}(\mathbb{A})$; it also forms a monoid. An *isomorphism* between $\mathbb{A}$ and $\mathbb{B}$ is a surjective embedding from $\mathbb{A}$ into $\mathbb{B}$. The set of all *automorphisms* of $\mathbb{A}$, i.e. of all isomorphisms between $\mathbb{A}$ and itself, is denoted by $\mathrm{Aut}(\mathbb{A})$; it forms a group with the composition operation, the neutral element id_A and the inversion operation. In the special case that $\mathbb{A}$ is the structure without any relations, the endomorphism monoid is the full transformation monoid A^A, the self-embedding monoid is the set $\mathrm{Inj}(A)$ of all injective maps $A \to A$, and the automorphism group is the set $\mathrm{Sym}(A)$ of all permutations on A. A weakening of isomorphic structures is given by the following notion: Two structures $\mathbb{A}$ and $\mathbb{B}$ are called *homomorphically equivalent* if there exist homomorphisms $g\colon \mathbb{A} \to \mathbb{B}$ and $h\colon \mathbb{B} \to \mathbb{A}$.

If $C \subseteq A$, then the *induced substructure* $\mathbb{C}$ of $\mathbb{A}$ on C is the structure with domain C where each relation R_i is replaced by $R_i \cap C^{m_i}$. If $f \colon \mathbb{A} \to \mathbb{B}$ is a homomorphism, we will in a slight abuse of notation denote the substructure of $\mathbb{B}$ induced on $f(A)$ by $f(\mathbb{A})$.

2.2 Topologies

We endow A^A with the *pointwise topology*:

Definition 2.2.1. The product topology on A^A where each copy of A carries the discrete topology will be called the *topology of pointwise convergence* or *pointwise topology* for brevity, denoted by $\mathcal{T}_{pw}$.

On any subset of A^A, in particular on $\mathrm{End}(\mathbb{A}), \mathrm{Emb}(\mathbb{A}), \mathrm{Aut}(\mathbb{A}), \mathrm{Inj}(A), \mathrm{Sym}(A)$, a natural topology is given by the subspace topology of the pointwise topology, also denoted by $\mathcal{T}_{pw}$ (or $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{A})}$ et cetera if misunderstandings are possible).

The name ist derived from the following fact: A sequence $(f_n)_{n\in\mathbb{N}}$ in A^A converges to some $f \in A^A$ if and only if for every argument $a \in A$, the sequence of evaluations $(f_n(a))_{n\in\mathbb{N}}$ converges to $f(a)$ with respect to the discrete topology on A, i.e. if it is eventually constant with value $f(a)$. Explicitly, a basis for the pointwise topology is given by the sets

$$\left\{f \in A^A : f(\bar{a}) = \bar{b}\right\}, \qquad \bar{a}, \bar{b} \text{ finite tuples in } A.$$

If A is countable, it is a folklore fact that the pointwise topology on A^A (as well as on $\mathrm{End}(\mathbb{A}), \mathrm{Emb}(\mathbb{A}), \mathrm{Aut}(\mathbb{A}), \mathrm{Inj}(A), \mathrm{Sym}(A)$) is *Polish* (second countable and completely metrisable) as a (G_δ-subspace topology of a) countable product of Polish topologies. More precisely, $\mathrm{End}(\mathbb{A}), \mathrm{Emb}(\mathbb{A}), \mathrm{Inj}(A)$ are even closed in A^A while $\mathrm{Aut}(\mathbb{A})$ is closed in $\mathrm{Sym}(A)$ which in turn is a non-closed G_δ-subset of A^A. As can be easily seen, the composition operation $\circ \colon A^A \times A^A \to A$ is continuous with respect to this topology; hence, the pointwise topology is a *semigroup topology* and A^A (as well as $\mathrm{End}(\mathbb{A}), \mathrm{Emb}(\mathbb{A}), \mathrm{Aut}(\mathbb{A}), \mathrm{Inj}(A), \mathrm{Sym}(A)$) is a *topological semigroup*. On $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Sym}(A)$, the inversion operation is continous with respect to the pointwise topology, yielding a so-called *group topology* or *topological group*.

In Chapter 3, we will need to consider the topological closure of $\mathrm{Aut}(\mathbb{A})$ with respect to the pointwise topology within A^A (or, equivalently, within $\mathrm{End}(\mathbb{A})$ or $\mathrm{Emb}(\mathbb{A})$ since the latter are themselves closed in A^A) which we will call the "$\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{A})$" for brevity. We remark that for an ω-categorical structure $\mathbb{A}$ (see Definition 2.3.1), this closure consists precisely of the so-called *elementary self-embeddings* of $\mathbb{A}$ (see [Hod97]).

We will make frequent use of the left and right translations, defined on any semigroup S as follows:

Definition 2.2.2. Given a fixed $t \in S$, let

$$\lambda_t \colon S \to S, \quad \lambda_t(s) := ts$$
$$\rho_t \colon S \to S, \quad \rho_t(s) := st$$

denote the *left* and *right translation* on S by t, respectively.

If S is a topological semigroup, then λ_t and ρ_t are continuous maps for any $t \in S$.
Now we define the *Zariski topology*. For notational simplicity, we restrict to monoids.

Definition 2.2.3. Let S be a monoid.

(i) For $k, \ell \in \mathbb{N}$, $\ell < k$, and for $p_0, \dots, p_k, q_0, \dots, q_\ell \in S$ as well as $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \dots s q_0$ (if $\ell = 0$, then $\psi(s) = q_0$ for all $s \in S$), we define

$$M_{\varphi,\psi} := \{s \in S : \varphi(s) \neq \psi(s)\}.$$

(ii) The *Zariski topology* on S, denoted by $\mathcal{T}_{\text{Zariski}}$, is the topology generated by all sets $M_{\varphi,\psi}$. Explicitly, the basic open sets are the finite intersections of sets $M_{\varphi,\psi}$.

In general, the Zariski topology need not be a Hausdorff topology or a semigroup topology, but suitable weakenings do hold. On the one hand, it always satisfies the first separation axiom: every singleton set $\{s_0\}$ is $\mathcal{T}_{\text{Zariski}}$-closed (pick $\varphi(s) = s = 1s1$, where 1 denotes the neutral element of S, and $\psi(s) = s_0$). On the other hand, the left and right translations λ_t and ρ_t (where $t \in S$ is fixed) are continuous with respect to the Zariski topology: To see this, take arbitrary $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \dots s q_0$ as above and note that $\lambda_t^{-1}(M_{\varphi,\psi}) = M_{\tilde{\varphi},\tilde{\psi}}$ where $\tilde{\varphi}(s) := (p_k t)s(p_{k-1}t)s \dots s(p_1 t)s(p_0)$ and $\tilde{\psi}(s) := (q_\ell t)s(q_{\ell-1}t)s \dots s(q_1 t)s(q_0)$; similarly for ρ_t.

By a straightforward argument, the Zariski topology is coarser than any Hausdorff semigroup topology $\mathcal{T}$ on S: One has to show that $M_{\varphi,\psi}$ is $\mathcal{T}$-open. If $s \in M_{\varphi,\psi}$, then $\varphi(s) \neq \psi(s)$, so there exist $U, V \in \mathcal{T}$ with $\varphi(s) \in U$, $\psi(s) \in V$ and $U \cap V = \emptyset$ since $\mathcal{T}$ is Hausdorff. Then $O := \varphi^{-1}(U) \cap \psi^{-1}(V)$ is a $\mathcal{T}$-open set (by continuity of the semigroup operation) such that $s \in O \subseteq M_{\varphi,\psi}$.

Notation 2.2.4. In the sequel, we will have to distinguish multiple topologies on the same set; whenever the topology is not clear from the context, we will write $(S, \mathcal{T})$ for the space S endowed with the topology $\mathcal{T}$.

2.3 ω-categoricity, homogeneity, transitivity and algebraicity

Definition 2.3.1. A countably infinite structure $\mathbb{A}$ is *ω-categorical* if $\mathbb{A}$ is the only countably infinite model of its first-order theory, i.e. if any countably infinite structure (in the same language as $\mathbb{A}$) which satisfies the same first-order sentences as $\mathbb{A}$ is isomorphic to $\mathbb{A}$.

Several important properties of a structure $\mathbb{A}$ can be defined from the canonical group action of $\text{Aut}(\mathbb{A})$ by evaluation on A^n for $n \geq 1$ which we write as $\text{Aut}(\mathbb{A}) \curvearrowright A^n$. We will consider the (pointwise) *stabiliser* of a set $Y \subseteq A$ (usually finite), that is $\text{Stab}(Y) := \{\alpha \in \text{Aut}(\mathbb{A}) : \alpha(y) = y \text{ for all } y \in Y\}$. For a tuple $\bar{a} \in A^n$, we further define the *orbit* of $\bar{a}$ under the action, $\text{Orb}(\bar{a}) := \{\alpha(\bar{a}) : \alpha \in \text{Aut}(\mathbb{A})\}$, as well as the *Y-relative orbit* $\text{Orb}(\bar{a}; Y) := \{\alpha(\bar{a}) : \alpha \in \text{Stab}(Y)\}$ where $Y \subseteq A$.

By the characterisation theorem due to Engeler, Ryll-Nardzewski and Svenonius (see for instance [Hod97]), a countable structure $\mathbb{A}$ is ω-categorical if and only if for each $n \geq 1$, the action $\text{Aut}(\mathbb{A}) \curvearrowright A^n$ has only finitely many orbits. We say that $\mathbb{A}$ is a *transitive* structure if the action $\mathrm{Aut}(\mathbb{A}) \curvearrowright A$ has a single orbit. The structure $\mathbb{A}$ is said to have *no algebraicity* if for any finite $Y \subseteq A$ and any element $a \in A \setminus Y$, the Y-relative orbit $\mathrm{Orb}(a; Y)$ is infinite. Finally, we say that $\mathbb{A}$ is a *homogeneous* structure if any finite partial isomorphism $m\colon \bar{a} \mapsto \bar{b}$ on $\mathbb{A}$ can be extended to an automorphism $\alpha \in \mathrm{Aut}(\mathbb{A})$. It is easy to see that a countably infinite homogeneous structure in a *finite (relational) language*, i.e. $\mathbb{A} = \langle A, (R_i)_{i\in I}\rangle$ with I finite, is automatically ω-categorical.

In the sequel, an important property of ω-categorical structures without algebraicity will be the existence of "almost identical" embeddings/endomorphisms which can be obtained using a standard compactness argument.

Lemma 2.3.2 ([EJM+23, Lemma 3.6])**.** *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity. Then for every $a \in A$, there are f, g in the $\mathcal{T}_{pw}$-closure*[1] *of* $\mathrm{Aut}(\mathbb{A})$ *such that $f|_{A\setminus\{a\}} = g|_{A\setminus\{a\}}$ and $f(a) \neq g(a)$.*

If f and g are as in the previous lemma, then for any $s \in \mathrm{End}(\mathbb{A})$ we note that $a \in \mathrm{Im}(s)$ if and only if $fs \neq gs$. This yields the following fact which will be essential in Chapter 3.

Lemma 2.3.3 (contained in [EJM+, Proof of Lemma 5.3])**.** *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity. Then for every $a \in A$, the set $\{s \in \mathrm{End}(\mathbb{A}) : a \in \mathrm{Im}(s)\}$ is open in the Zariski topology on* $\mathrm{End}(\mathbb{A})$.

2.4 Graphs

We will use the following naming convention:

Definition 2.4.1. A relational structure $\mathbb{G} = (G, E)$ with a single binary relation (the so-called *edge relation* $E \subseteq G^2$) is called a *directed graph* (or *digraph* for short) if the relation is *loopless*, i.e. $(x, x) \notin E$ for all $x \in G$.

If the relation is *symmetric*, i.e. if $(x, y) \in E$ implies $(y, x) \in E$ for all $x, y \in G$, then $\mathbb{G}$ is plainly called a *graph*.

For a graph $\mathbb{G} = (G, E)$, the *dual graph* $\partial\mathbb{G}$ is the graph on the same domain obtained by "exchanging edges and non-edges", formally $\partial\mathbb{G} = (G, E')$ where

$$x\,E'\,y :\Leftrightarrow x\,\not{E}\,y \text{ and } x \neq y.$$

In the sequel, we will work with several special graphs:

Definition 2.4.2. Let $k, n \geq 1$.

(i) $\mathbb{K}_n$ denotes the *complete graph on n vertices*, explicitly $\mathbb{K}_n = (G, E)$ with $|G| = n$, and $x\,E\,y :\Leftrightarrow x \neq y$. Similarly, $\mathbb{K}_\omega = (\mathbb{N}, E)$ with $x\,E\,y :\Leftrightarrow x \neq y$ denotes the *complete graph on countably many vertices.*

(ii) $\mathbb{K}_{k,n}$ denotes the *complete k-partite graph on parts of size n*, explicitly $\mathbb{K}_{k,n} = (G, E)$ with $G := \{1, \ldots, k\} \times X$ for $|X| = n$, and $(i, x)\,E\,(j, y) :\Leftrightarrow i \neq j$. Similarly, $\mathbb{K}_{k,\omega}$, $\mathbb{K}_{\omega,n}$ and $\mathbb{K}_{\omega,\omega}$ are defined with domains $G := \{1, \ldots, k\} \times \mathbb{N}$, $G := \mathbb{N} \times X$ (for

[1] See the remarks preceding Definition 2.2.2.

$|X| = n$), and $G := \mathbb{N} \times \mathbb{N}$, respectively, called the *complete k-partite graph on parts of countably infinite size*, the *complete infinite-partite graph on parts of size n* and the *complete infinite-partite graph on parts of countably infinite size*.

We identify $\mathbb{K}_{k,1}$ with $\mathbb{K}_k$ as well as $\mathbb{K}_{\omega,1}$ with $\mathbb{K}_\omega$.

(iii) $\mathbb{E}_{k,n}$ denotes the *irreflexive equivalence relation with k equivalence classes of size n*, explicitly $\mathbb{E}_{k,n} = (G, E)$ with $G := \{1, \dots, k\} \times X$ for $|X| = n$, and $(i, x)\, E\, (j, y) :\Leftrightarrow i = j$ and $x \neq y$. Analogously to the above, we define $\mathbb{E}_{k,\omega}$, $\mathbb{E}_{\omega,n}$ and $\mathbb{E}_{\omega,\omega}$, the *irreflexive equivalence relation with k equivalence classes of countably infinite size*, the *irreflexive equivalence relation with countably many equivalence classes of size n* and the *irreflexive equivalence relation with countably many equivalence classes of countably infinite size*, respectively.

Note that $\mathbb{E}_{k,n} = \partial\mathbb{K}_{k,n}$, $\mathbb{E}_{k,\omega} = \partial\mathbb{K}_{k,\omega}$, $\mathbb{E}_{\omega,n} = \partial\mathbb{K}_{\omega,n}$ and $\mathbb{E}_{\omega,\omega} = \partial\mathbb{K}_{\omega,\omega}$.

(iv) The *random graph* is the *Fraïssé limit* (see [Hod97]) of the class of all finite graphs, so the unique (up to isomorphism) countably infinite homogeneous graph which contains all finite graphs as subgraphs.

(v) For $n \geq 3$, we say that a graph $\mathbb{G}$ is *$\mathbb{K}_n$-free* if it does not contain a copy of $\mathbb{K}_n$ as a subgraph. The random $\mathbb{K}_n$-free graph is the Fraïssé limit of the class of all $\mathbb{K}_n$-free graphs, so the unique (up to isomorphism) countably infinite homogeneous $\mathbb{K}_n$-free graph which contains as finite subgraphs precisely the finite $\mathbb{K}_n$-free graphs. We also write $\mathbb{F}_n$ for the random $\mathbb{K}_n$-free graph.

The random graph as well as the random $\mathbb{K}_n$-free graph can be characterised by so-called *extension properties* (see for instance [Hod97, Theorem 6.4.4] for the former): First, a countably infinite graph (G, E) is (isomorphic to) the random graph if for all disjoint sets $A, B \subseteq G$, there exists $y \notin A \cup B$ such that y has an edge to all elements of A and a non-edge to all elements of B – this also shows that the random graph is (isomorphic to) its own dual graph. Second, a countably infinite $\mathbb{K}_n$-free graph (G, E) is (isomorphic to) the random $\mathbb{K}_n$- free graph if for all disjoint sets $A, B \subseteq G$ such that the induced graph on A does not contain a copy of[2] $\mathbb{K}_{n-1}$, there exists $y \notin A \cup B$ such that y has an edge to all elements of A and a non-edge to all elements of B.

These special graphs essentially constitute *all* countably infinite homogeneous graphs:

Theorem 2.4.3 ([LW80]). *Any countably infinite homogeneous graph is isomorphic to one of the following:*

(1) the random graph

(2) the random $\mathbb{K}_n$-free graph $\mathbb{F}_n$ for some $n \geq 3$

(3) the dual of $\mathbb{F}_n$ for some $n \geq 3$

(4) $\mathbb{K}_{k,\omega}$, $\mathbb{K}_{\omega,n}$ or $\mathbb{K}_{\omega,\omega}$ for some $k, n \geq 1$ (a complete multipartite graph)

[2] This restriction ascertains that the conditions do not enforce a copy of $\mathbb{K}_n$.

(5) $\mathbb{E}_{k,\omega}$*,* $\mathbb{E}_{\omega,n}$ *or* $\mathbb{E}_{\omega,\omega}$ *for some* $k, n \geq 1$ *(an irreflexive equivalence relation with classes of equal size)*

In order to describe the automorphism group and endomorphism monoid of some of these graphs, the following notation will be useful in Chapters 3 and 5.

Notation 2.4.4. Let X be a set and let I be an index set. If $\tau\colon I \to I$ and if $s_i\colon X \to X$ for each $i \in I$, then $\bigsqcup^{\tau}_{i\in I} s_i$ shall denote the self-map of $I \times X$ defined by

$$\bigsqcup_{i\in I}^{\tau} s_i\colon \begin{cases} I \times X & \to I \times X \\ (i,x) & \mapsto (\tau(i), s_i(x)) \end{cases}$$

For $\tau\colon I \to I$ and $s\colon X \to X$, we further set $\tau \ltimes s := \bigsqcup^{\tau}_{i\in I} s$.

2.5 Model-complete cores

Definition 2.5.1. A structure $\mathbb{C}$ is called a *model-complete core* if the endomorphism monoid $\mathrm{End}(\mathbb{C})$ coincides with the $\mathcal{T}_{pw}$-closure of the automorphism group $\mathrm{Aut}(\mathbb{C})$.

In the case that $\mathbb{C}$ is ω-categorical, this means that any endomorphism of $\mathbb{C}$ is an elementary self-embedding; if $\mathbb{C}$ is finite, this means $\mathrm{End}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})$.

Every ω-categorical structure has a homomorphically equivalent model-complete core structure:

Theorem 2.5.2 (originally [Bod07, Theorem 16], alternative proof in [BKO+19, Theorem 5.7])**.** *Let* $\mathbb{A}$ *be an* ω*-categorical structure. Then there exists a model-complete core* $\mathbb{C}$ *such that* $\mathbb{A}$ *and* $\mathbb{C}$ *are homomorphically equivalent. Moreover,* $\mathbb{C}$ *is either* ω*-categorical or finite and uniquely determined (up to isomorphism).*

Because of the uniqueness result, $\mathbb{C}$ is commonly referred to as *the* model-complete core of $\mathbb{A}$. We will repeatedly use the following simple property of model-complete cores:

Lemma 2.5.3. *Let* $\mathbb{A}$ *be an* ω*-categorical structure and let* $\mathbb{C}$ *be its model-complete core. Then any homomorphism* $f\colon \mathbb{C} \to \mathbb{A}$ *is an embedding.*

Proof. If $g\colon \mathbb{A} \to \mathbb{C}$ denotes the homomorphism existing by homomorphic equivalence, then gf is an endomorphism of $\mathbb{C}$ and thus contained in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{C})$, in particular a self-embedding. This is only possible if f is an embedding. □

This lemma in particular applies to the homomorphism $h\colon \mathbb{C} \to \mathbb{A}$ yielded by homomorphic equivalence. Replacing $\mathbb{C}$ by its isomorphic copy $h(\mathbb{C})$, we will subsequently assume that $\mathbb{C}$ is a substructure of $\mathbb{A}$. Note that depending on the structure $\mathbb{A}$, it can but need not be possible to pick the homomorphism $g\colon \mathbb{A} \to \mathbb{C}$ to be surjective. For instance, the model-complete core of the random graph is the complete graph on countably many vertices, and any bijection from the random graph to the complete graph is a surjective homomorphism.

On the other hand, if $\mathbb{A}$ is given by the rational numbers $\mathbb{Q}$ extended by two elements $\pm\infty$, equipped with the canonical strict order, then the model-complete core of $\mathbb{A}$ is precisely

$\langle \mathbb{Q}, < \rangle$ which cannot coincide with any homomorphic image of $\mathbb{A}$ since such an image would have a greatest and a least element. If the model-complete core of $\mathbb{A}$ is finite, however, *any* homomorphism $g\colon \mathbb{A} \to \mathbb{C}$ is surjective, as can be seen by viewing g as an endomorphism of $\mathbb{A}$ and applying the following lemma we will also use later on:

Lemma 2.5.4. *If the model-complete core of an ω-categorical structure $\mathbb{A}$ is finite of size n, then the image of any endomorphism of $\mathbb{A}$ has size at least n.*

Proof. If $s \in \mathrm{End}(\mathbb{A})$, then $s(\mathbb{A})$ is homomorphically equivalent to $\mathbb{A}$. Hence, $s(\mathbb{A})$ and $\mathbb{A}$ have the same model-complete core which can therefore be regarded as a substructure of $s(\mathbb{A})$. □

2.6 Unique Polish Property and automatic continuity

Definition 2.6.1. Let $S \leq A^A$ be a sub(-semi-)group of A^A which is G_δ with respect to $\mathcal{T}_{pw}$ (so that $(S, \mathcal{T}_{pw})$ is a Polish (semi-)group). We say that S has the *Unique Polish Property* (abbreviated by *UPP*) if the pointwise topology is the only Polish (semi-)group topology on S.

We refer to the Introduction (Chapter 1) for examples of UPP and continue by considering a non-example in more detail, namely the monoid of self-embeddings of an ω-categorical structure $\mathbb{A}$ which turns out to carry an additional Polish semigroup topology. The following fact will be crucial in the proof of Polishness:

Lemma 2.6.2 ([Kec94, Proof of Theorem 1.2])**.** *Let $(X, \mathcal{T})$ be a Polish space and let $\{F_n : n \in \mathbb{N}\}$ be a countable set of closed subsets of X. Then the topology generated by $\mathcal{T} \cup \{F_n : n \in \mathbb{N}\}$ is again a Polish topology on X.*

Example 2.6.3 ([EJM+23, Proposition 5.1 and the discussion afterwards])**.** Let $\mathbb{A}$ be an ω-categorical structure with domain A. Then $\mathrm{Emb}(\mathbb{A})$ does not satisfy UPP (note that this encompasses the semigroup $\mathrm{Inj}(A)$ mentioned in the Introduction if $\mathbb{A}$ is taken to be the structure without relations). We define $\mathcal{T}'$ to be the topology generated by $\mathcal{T}_{pw}$ together with the sets $O_b := \{s \in \mathrm{Emb}(\mathbb{A}) : b \notin \mathrm{Im}(s)\}$, $b \in A$, and claim that $\mathcal{T}'$ is a Polish semigroup topology on $\mathrm{Emb}(\mathbb{A})$ different from $\mathcal{T}_{pw}$.

By Lemma 2.6.2, the topology $\mathcal{T}'$ is Polish – note that O_b is closed with respect to the pointwise topology since $\mathrm{Emb}(\mathbb{A}) \setminus O_b = \bigcup_{a \in A} \{s \in \mathrm{Emb}(\mathbb{A}) : s(a) = b\}$. To see that $\mathcal{T}'$ is a semigroup topology, it remains to show the following (since $\mathcal{T}_{pw}$ is a semigroup topology): if $s, t \in \mathrm{Emb}(\mathbb{A})$ such that $st \in O_b$, i.e. $b \notin \mathrm{Im}(st)$, there exist $\mathcal{T}'$-neighbourhoods V and W of s and t, respectively, such that $VW \subseteq O_b$. If $b \notin \mathrm{Im}(s)$, we set $V := O_b$ and $W := \mathrm{Emb}(\mathbb{A})$. If on the other hand $b \in \mathrm{Im}(s)$, say $b = s(a)$, then a cannot be contained in $\mathrm{Im}(t)$. Setting $V := \{\tilde{s} \in \mathrm{Emb}(\mathbb{A}) : \tilde{s}(a) = b\}$ and $W := O_a$, we claim that $VW \subseteq O_b$ – this follows from $\tilde{s}^{-1}\{b\} = \{a\}$ for all $\tilde{s} \in V$ which crucially uses that $\mathrm{Emb}(\mathbb{A})$ consists of injective functions.

Finally, $\mathcal{T}'$ differs from $\mathcal{T}_{pw}$: The automorphism group $\mathrm{Aut}(\mathbb{A})$ is clearly $\mathcal{T}'$-closed while there always exists a non-surjective embedding in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{A})$ (see [BPP17, Proof of Corollary 10]).

A crucial tool for verifying UPP turns out to be automatic continuity as discussed e.g. in [BPP21, EJM+, EJM+23, Her98, HHLS93, KR07, Las91, PS20, RS07, Tru89]:

Definition 2.6.4. Let S be a (semi-)group and let $\mathcal{T}$ be a topology on S (which need not be a (semi-)group topology). Given a class $\mathcal{K}$ of topological (semi-)groups, we say that $(S, \mathcal{T})$ has *automatic continuity* with respect to $\mathcal{K}$ if for any $(H, \mathcal{O}) \in \mathcal{K}$, all algebraic homomorphisms $S \to H$ are continuous as maps $(S, \mathcal{T}) \to (H, \mathcal{O})$.

This notion can be used to show that a given topology $\mathcal{T}$ on S is finer than any Polish semigroup topology on S: if $(S, \mathcal{T})$ has automatic continuity with respect to the class of Polish semigroups, then for any Polish semigroup topology $\mathcal{T}'$ on S, the identity map considered as a function id: $(S, \mathcal{T}) \to (S, \mathcal{T}')$ must be continuous which yields $\mathcal{T}' \subseteq \mathcal{T}$. In [EJM+] as well as [EJM+23], UPP is obtained for the examples mentioned in the Introduction (Chapter 1) by showing that the respective semigroup together with the pointwise topology has automatic continuity.

Given a topological group, one could apply Definition 2.6.4 in two different ways: one may wonder whether the topological group has automatic continuity with respect to the class of *groups* with some topological property (e.g. second countability), or whether it has automatic continuity with respect to the class of *semigroups* with the same property. As it turns out, we will be able to neglect this difference in the sequel because of the following fact (which we formulate in a slightly less general way for the sake of simplification):

Proposition 2.6.5 ([EJM+, Proposition 4.1])**.** *Let G be a topological group. Then G has automatic continuity with respect to the class of second countable topological semigroups if and only if G has automatic continuity with respect to the class of second countable topological groups.*

We will show in Chapter 4 that the endomorphism monoid of $\langle \mathbb{Q}, \leq \rangle$ with the pointwise topology *does not* have automatic continuity which will turn out to be the reason why we need a more sophisticated approach. At this point, we state the technical result we will use, reformulating it to match our terminology:

Proposition 2.6.6 ([BPP17, Proposition 9])**.** *Let M be a $\mathcal{T}_{pw}$-closed submonoid of A^A for a countable set A. Suppose that M contains a submonoid N such that*

(1) N is not $\mathcal{T}_{pw}$-closed in M;

(2) composing any element of M with an element outside N yields an element outside N.

Then $(M, \mathcal{T}_{pw})$ does not have automatic continuity with respect to the full transformation monoid A^A, equipped with the pointwise topology (in particular with respect to the class of Polish semigroups).

2.7 Pseudo-Property $\overline{\mathrm{X}}$

We now introduce our generalisation of Property **X** in its most general form so that all our applications of this method in Chapters 4 and 5 are suitable special cases of this “umbrella definition”.

Definition 2.7.1. Let S be a monoid with neutral element 1_S endowed with a topology[3] $\mathcal{T}$, let $m \geq 1$ and let $D^{(1)}, \ldots, D^{(m)} \subseteq S$ be subsets of S endowed with topologies $\mathcal{T}_{D^{(1)}}, \ldots, \mathcal{T}_{D^{(m)}}$. Then $(S, \mathcal{T})$ has *Pseudo-Property* $\overline{\mathbf{X}}$ (of length m) with respect to $(D^{(1)}, \mathcal{T}_{D^{(1)}}), \ldots, (D^{(m)}, \mathcal{T}_{D^{(m)}})$ if the following holds: For all $s \in S$ there exist elements $e_s, h_s^{(1)}, \ldots, h_s^{(m+1)} \in S$ and $a_s^{(1)} \in D^{(1)}, \ldots, a_s^{(m)} \in D^{(m)}$ such that

(i) e_s is *left-invertible* in S, i.e. there exists $p \in S$ such that $pe_s = 1_S$.

(ii) $e_s s = h_s^{(m+1)} a_s^{(m)} h_s^{(m)} a_s^{(m-1)} \ldots a_s^{(1)} h_s^{(1)}$.

(iii) For all $V^{(1)} \in \mathcal{T}_{D^{(1)}}, \ldots, V^{(m)} \in \mathcal{T}_{D^{(m)}}$ with $a_s^{(i)} \in V^{(i)}$, there exists $U \in \mathcal{T}$ with $s \in U$ such that
$$e_s U \subseteq h_s^{(m+1)} V^{(m)} h_s^{(m)} V^{(m-1)} \ldots V^{(1)} h_s^{(1)}.$$

If $D^{(1)} = \cdots = D^{(m)} =: D$ and $\mathcal{T}_{D^{(1)}} = \cdots = \mathcal{T}_{D^{(m)}} =: \mathcal{T}_D$, we say that $(S, \mathcal{T})$ has Pseudo-Property $\overline{\mathbf{X}}$ of length m with respect to $(D, \mathcal{T}_D)$.

Remark 2.7.2. Pseudo-Property $\overline{\mathbf{X}}$ of length m can thus be verified as follows: Given $s \in S$, we find suitable $e_s, h_s^{(1)}, \ldots, h_s^{(m+1)} \in S$ with e_s left-invertible and devise a method to write $e_s s = h_s^{(m+1)} a_s^{(m)} h_s^{(m)} a_s^{(m-1)} \ldots a_s^{(1)} h_s^{(1)}$ (where $a_s^{(i)} \in D^{(i)}$) in such a way that for arbitrary $\mathcal{T}_{D^{(i)}}$-neighbourhoods $V^{(i)}$ of $a_s^{(i)}$, there exists a $\mathcal{T}$-neighbourhood U of s such that our method applied to any $\tilde{s} \in U$ yields $\tilde{a}^{(i)} \in V^{(i)}$ (not just $\tilde{a}^{(i)} \in D^{(i)}$) with $e_s \tilde{s} = h_s^{(m+1)} \tilde{a}^{(m)} h_s^{(m)} \tilde{a}^{(m-1)} \ldots \tilde{a}^{(1)} h_s^{(1)}$. Thus, this neighbourhood U must be small enough to ensure two properties: first, it must encode enough information about s to make sure that the *same* auxiliary elements $e_s, h_s^{(1)}, \ldots, h_s^{(m+1)}$ can be used for $\tilde{s}$; second, it must ascertain that $\tilde{s}$ is "close enough" to s so that the resulting elements $\tilde{a}^{(i)}$ are "close enough" to $a_s^{(i)}$. Note the following equilibrium at the heart of Pseudo-Property $\overline{\mathbf{X}}$: Increasing the length m, it becomes easier to decompose a large class of elements s in the desired form. However, there are more conditions $\tilde{a}^{(i)} \in V^{(i)}$ to be taken care of, potentially interacting with each other and yielding a more complex situation.

The notation $\overline{\mathbf{X}}$ instead of $\mathbf{X}$ refers to the arbitrary number m of elements $a_s^{(i)} \in D^{(i)}$ on the right hand side, while the term "Pseudo" refers to the composition with the left-invertible element e_s on the left hand side, see [BP16, GJP19, BP20]. Thus, the "traditional" Property $\mathbf{X}$ mentioned in the Introduction (Chapter 1) corresponds in our terminology to Property $\overline{\mathbf{X}}$ of length 1 (without "Pseudo").

We will apply Pseudo-Property $\overline{\mathbf{X}}$ via the following proposition which generalises parts of [EJM+, Theorem 3.1]:

Proposition 2.7.3. *Let S be a monoid endowed with a topology $\mathcal{T}$ and let $D^{(1)}, \ldots, D^{(m)}$ be subsets of S endowed with topologies $\mathcal{T}_{D^{(1)}}, \ldots, \mathcal{T}_{D^{(m)}}$. If $(S, \mathcal{T})$ has Pseudo-Property $\overline{\mathbf{X}}$ with respect to $(D^{(1)}, \mathcal{T}_{D^{(1)}}), \ldots, (D^{(m)}, \mathcal{T}_{D^{(m)}})$, then the following statements hold:*

(i) If $(H, \mathcal{O})$ is a topological semigroup and $\varphi\colon S \to H$ is a homomorphism such that the restrictions $\varphi|_{D^{(i)}}$ are continuous as maps $\varphi|_{D^{(i)}}\colon (D^{(i)}, \mathcal{T}_{D^{(i)}}) \to (H, \mathcal{O})$, then φ is continuous as a map $\varphi\colon (S, \mathcal{T}) \to (H, \mathcal{O})$.

[3]Note: $(S, \mathcal{T})$ need not be a topological semigroup!

(ii) If $\mathcal{K}$ is a class of topological semigroups and if all $D^{(i)}$ are semigroups such that $(D^{(i)}, \mathcal{T}_{D^{(i)}})$ has automatic continuity with respect to $\mathcal{K}$, then $(S, \mathcal{T})$ also has automatic continuity with respect to $\mathcal{K}$.

Proof. Since (i) immediately implies (ii), we only prove the former.

We denote the neutral element of S by 1_S. Without loss of generality, φ is surjective. Therefore, H can be assumed to be a monoid with neutral element $\varphi(1_S)$. Let $O \in \mathcal{O}$ and $s \in S$ such that $\varphi(s) \in O$. We need to find $U \in \mathcal{T}$ such that $s \in U$ and $\varphi(U) \subseteq O$. By Pseudo-Property $\overline{\mathbf{X}}$, there exist $e_s, h_s^{(1)}, \ldots, h_s^{(m+1)} \in S$ with e_s left-invertible and $a_s^{(1)} \in D^{(1)}, \ldots, a_s^{(m)} \in D^{(m)}$ such that

$$e_s s = h_s^{(m+1)} a_s^{(m)} h_s^{(m)} a_s^{(m-1)} \ldots a_s^{(1)} h_s^{(1)}$$

and such that for arbitrary $V^{(1)} \in \mathcal{T}_{D^{(1)}}, \ldots, V^{(m)} \in \mathcal{T}_{D^{(m)}}$ with $a_s^{(i)} \in V^{(i)}$, there exists $U \in \mathcal{T}$ with $s \in U$ satisfying

$$e_s U \subseteq h_s^{(m+1)} V^{(m)} h_s^{(m)} V^{(m-1)} \ldots V^{(1)} h_s^{(1)}.$$

Denote the left inverse of e_s by p. The left translations

$$\lambda_{\varphi(e_s)} \colon (H, \mathcal{O}) \to (H, \mathcal{O}) \quad \text{and} \quad \lambda_{\varphi(p)} \colon (H, \mathcal{O}) \to (H, \mathcal{O})$$

are continuous (since $\mathcal{O}$ is a semigroup topology). Further,

$$\lambda_{\varphi(e_s)} \colon (H, \mathcal{O}) \to (\varphi(e_s)H, \mathcal{O}|_{\varphi(e_s)H}) \quad \text{and} \quad \lambda_{\varphi(p)} \colon (\varphi(e_s)H, \mathcal{O}|_{\varphi(e_s)H}) \to (H, \mathcal{O})$$

form inverse maps because $\varphi(p)$ is a left inverse of $\varphi(e_s)$ – here we use that H is a monoid with neutral element $\varphi(1_S)$. Thus, $\lambda_{\varphi(e_s)} \colon (H, \mathcal{O}) \to (\varphi(e_s)H, \mathcal{O}|_{\varphi(e_s)H})$ is a homeomorphism and we obtain $\varphi(e_s)O = \lambda_{\varphi(e_s)}(O) = P \cap \varphi(e_s)H$ for some $P \in \mathcal{O}$. Consequently,

$$\varphi(h_s^{(m+1)})\varphi(a_s^{(m)})\varphi(h_s^{(m)})\varphi(a_s^{(m-1)}) \ldots \varphi(a_s^{(1)})\varphi(h_s^{(1)}) = \varphi(e_s)\varphi(s) \in P \cap \varphi(e_s)H.$$

Using that the map $(b^{(1)}, \ldots, b^{(m)}) \mapsto \varphi(h_s^{(m+1)}) b^{(m)} \varphi(h_s^{(m)}) b^{(m-1)} \ldots b^{(1)} \varphi(h_s^{(1)})$ is continuous with respect to $\mathcal{O}$ (since $\mathcal{O}$ is a semigroup topology) yields sets $W^{(i)} \in \mathcal{O}$ such that $\varphi(a_s^{(i)}) \in W^{(i)}$ and

$$\varphi(h_s^{(m+1)}) W^{(m)} \varphi(h_s^{(m)}) W^{(m-1)} \ldots W^{(1)} \varphi(h_s^{(1)}) \subseteq P.$$

By the assumed continuity of $\varphi|_{D^{(i)}} \colon (D^{(i)}, \mathcal{T}_{D^{(i)}}) \to (H, \mathcal{O})$, the sets $V^{(i)} := \varphi|_{D^{(i)}}^{-1}(W^{(i)})$ are contained in $\mathcal{T}_{D^{(i)}}$. Thus, we can invoke Pseudo-Property $\overline{\mathbf{X}}$ to obtain a set $U \in \mathcal{T}$ such that $s \in U$ and

$$e_s U \subseteq h_s^{(m+1)} V^{(m)} h_s^{(m)} V^{(m-1)} \ldots V^{(1)} h_s^{(1)}.$$

Applying φ, we conclude

$$\varphi(e_s)\varphi(U) \subseteq \varphi(h_s^{(m+1)}) W^{(m)} \varphi(h_s^{(m)}) W^{(m-1)} \ldots W^{(1)} \varphi(h_s^{(1)}) \subseteq P,$$

and thus $\varphi(e_s)\varphi(U) \subseteq P \cap \varphi(e_s)H = \varphi(e_s)O$. Multiplying with $\varphi(p)$ from the left, we obtain $\varphi(U) \subseteq O$ as desired. □

2.8 Back&Forth

In Chapter 4, we will repeatedly use the "Back&Forth" method, see for instance [Hod97].

Definition 2.8.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be countably infinite structures in the same language and let $\mathcal{S}$ be a set of finite partial homomorphisms from $\mathbb{X}$ to $\mathbb{Y}$.

(i) $\mathcal{S}$ is a *Forth system* between $\mathbb{X}$ and $\mathbb{Y}$ if for all $m \in \mathcal{S}$ and all $x \in \mathbb{X}$ with $x \notin \operatorname{Dom}(m)$, there exists $m' \in \mathcal{S}$ such that m' extends m and $x \in \operatorname{Dom}(m')$.

(ii) $\mathcal{S}$ is a *Back system* between $\mathbb{X}$ and $\mathbb{Y}$ if for all $m \in \mathcal{S}$ and all $y \in \mathbb{Y}$ with $y \notin \operatorname{Im}(m)$, there exists $m' \in \mathcal{S}$ such that m' extends m and $y \in \operatorname{Im}(m')$.

(iii) $\mathcal{S}$ is a *Back&Forth system* between $\mathbb{X}$ and $\mathbb{Y}$ if it is both a Back system and a Forth system.

Iteratively extending finite partial homomorphisms so that their domains exhaust the entire structure $\mathbb{X}$ (Forth) or in an alternating fashion so that their domains and images exhaust $\mathbb{X}$ and $\mathbb{Y}$, respectively (Back&Forth), one obtains the following folklore result:

Lemma 2.8.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be countably infinite structures in the same language.*

(i) If $\mathcal{S}$ is a Forth system between $\mathbb{X}$ and $\mathbb{Y}$ which is closed under restriction, then any $m \in \mathcal{S}$ can be extended to a total homomorphism $s\colon \mathbb{X} \to \mathbb{Y}$ such that every finite restriction of s is contained in $\mathcal{S}$. In particular, if $\mathcal{S}$ consists of injective finite partial homomorphisms, then s can be picked to be injective as well.

(ii) If $\mathcal{S}$ is a Back&Forth system between $\mathbb{X}$ and $\mathbb{Y}$ which is closed under restriction, then any $m \in \mathcal{S}$ can be extended to a total and surjective homomorphism $s\colon \mathbb{X} \to \mathbb{Y}$ such that every finite restriction of s is contained in $\mathcal{S}$. In particular, if $\mathcal{S}$ consists of finite partial isomorphisms, then s can be picked to be an automorphism.

We will also employ the following variant.

Definition 2.8.3. Let $\mathbb{X}$ and $\mathbb{Y}$ be countably infinite structures in the same language and let $A \subseteq \mathbb{X}$ as well as $C \subseteq \mathbb{Y}$. Let further $\mathcal{S}$ be a set of finite partial homomorphisms from $\mathbb{X}$ to $\mathbb{Y}$.

(i) $\mathcal{S}$ is an (A, C)-*Back system* between $\mathbb{X}$ and $\mathbb{Y}$ if the following holds:

For all $m \in \mathcal{S}$ and all[4] $y \in C$, there exists $m' \in \mathcal{S}$ such that m' extends m and $\exists x \in A \cap \operatorname{Dom}(m')\colon m'(x) = y$.

(ii) $\mathcal{S}$ is an (A, C)-*Back&Forth system* between $\mathbb{X}$ and $\mathbb{Y}$ if it is both an (A, C)-Back system and a Forth system.

Lemma 2.8.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be countably infinite structures in the same language and let $A \subseteq \mathbb{X}$ as well as $C \subseteq \mathbb{Y}$. If $\mathcal{S}$ is an (A, C)-Back&Forth system between $\mathbb{X}$ and $\mathbb{Y}$, then any $m \in \mathcal{S}$ can be extended to a total homomorphism $s\colon \mathbb{X} \to \mathbb{Y}$ such that*

$$\forall y \in C\colon s^{-1}\{y\} \cap A \neq \emptyset.$$

[4] Note: Contrary to "Back" from above, $y \in \operatorname{Im}(m)$ is in general possible!

Proof. The argument proceeds in almost the same way as a standard Back&Forth construction: Instead of applying a Back step to all elements of $\mathbb{Y} \setminus \mathrm{Im}(m)$, one applies an (A, C)-Back step to all elements of C (even if they are contained in $\mathrm{Im}(m)$). □

3 Zariski topology on endomorphism monoids

We start by introducing *structures with mobile core*, a weakening of the standard notion of transitive structures, as well as formulating a technical condition from [EJM+] for the Zariski topology and the pointwise topology on a semigroup of transformations to be equal (Section 3.1). In Section 3.2, we address Question A1 and give two new sets of sufficient conditions on a structure $\mathbb{A}$, expressed in terms of the model-complete core, which imply that the Zariski topology and the pointwise topology on $\mathrm{End}(\mathbb{A})$ coincide. Section 3.3 provides an example of an ω-categorical structure such that the two topologies on its endomorphism monoid differ, thus answering Question A2.

3.1 Tools & Notions: structures with mobile core and a technical condition

Definition 3.1.1. Let $\mathbb{A}$ be an ω-categorical structure. Then $\mathbb{A}$ is said to have a *mobile core* if any element of $\mathbb{A}$ is contained in the image of an endomorphism into the model-complete core. Explicitly, for any $a \in \mathbb{A}$, there ought to exist a substructure $\mathbb{C}$ of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ with the following properties:

(i) $\mathbb{C}$ is a model-complete core homomorphically equivalent to $\mathbb{A}$,

(ii) $a \in g(A) \subseteq C$.

Note that structures with mobile core are a weakening of transitive structures (as introduced in Section 2.3): Let $\mathbb{A}$ be transitive, let $\mathbb{C}$ be its model-complete core with homomorphism $g\colon \mathbb{A} \to \mathbb{C}$, and let $a_0 \in A$ be a fixed element. If $a \in A$ is arbitrary, then transitivity yields $\alpha \in \mathrm{Aut}(\mathbb{A})$ such that $\alpha(g(a_0)) = a$. Hence, $\widetilde{\mathbb{C}} := \alpha(\mathbb{C})$ is an isomorphic copy of $\mathbb{C}$ with homomorphism $\tilde{g} := \alpha g\colon \mathbb{A} \to \widetilde{\mathbb{C}}$ such that $a \in \tilde{g}(A) \subseteq \widetilde{C}$. In fact, it suffices to assume that $\mathbb{A}$ is *weakly transitive*, i.e. that for all $a, b \in A$ there exists $s \in \mathrm{End}(\mathbb{A})$ with $s(a) = b$ – replacing α in the above argument by s, we still obtain that $s(\mathbb{C})$ is an isomorphic copy of $\mathbb{C}$ by Lemma 2.5.3.

On the other hand, there exist non-transitive structures which have a mobile core, for instance the disjoint union of two transitive structures where each part gets named by an additional unary predicate (to ascertain that the parts are invariant under any automorphism). Finally, the structure $\langle \mathbb{Q} \cup \{\pm\infty\}, < \rangle$ mentioned after Lemma 2.5.3 does not have a mobile core: The element $+\infty$ cannot be contained in any copy of the model-complete core $\langle \mathbb{Q}, < \rangle$.

Further, we will use the following technical condition from [EJM+]:

Lemma 3.1.2 ([EJM+, Lemma 5.3]). *Let X be an infinite set and let S be a subsemigroup of X^X such that for every $a \in X$ there exist $\alpha, \beta, \gamma_1, \ldots, \gamma_n \in S$ for some $n \in \mathbb{N}$ such that the following hold:*

(i) $\alpha|_{X\setminus\{a\}} = \beta|_{X\setminus\{a\}}$ *and* $\alpha(a) \neq \beta(a)$

(ii) $a \in \mathrm{Im}(\gamma_i)$ *for all* $i \in \{1, \ldots, n\}$*;*

(iii) for every $s \in S$ *and every* $x \in X \setminus \{s(a)\}$*, there is* $i \in \{1, \ldots, n\}$ *so that* $\mathrm{Im}(\gamma_i) \cap s^{-1}(x) = \emptyset$*.*

Then the Zariski topology of S is the pointwise topology.

We remark that (i) corresponds to Lemma 2.3.2 and that the proof proceeds by constructing the generating sets of the pointwise topology from the sets $\{s \in S : a \in \mathrm{Im}(s)\}$ exhibited in Lemma 2.3.3.

3.2 Two sets of sufficient conditions

This section is devoted to our answer to Question A1: we state and show our sufficient conditions for the pointwise topology and the Zariski topology to coincide.

3.2.1 Our results

Theorem A1. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity which has a mobile core. Then the Zariski topology on* $\mathrm{End}(\mathbb{A})$ *coincides with the pointwise topology if one of the following two conditions holds:*

(i) EITHER the model-complete core of $\mathbb{A}$ is finite,

(ii) OR the model-complete core of $\mathbb{A}$ is infinite and does not have algebraicity.

The cases (i) and (ii) will be treated separately in Subsections 3.2.2 and 3.2.3, respectively. Before we get to the proofs, we show how Theorem A1 can be used to easily verify that the Zariski topology and the pointwise topology coincide on the endomorphism monoids of a multitude of example structures. Some of them have been treated in [EJM+23], but our result applies to many other structures which have not yet been considered, e.g. the random $\mathbb{K}_n$-free graph or its dual graph as well as the complete multipartite graphs on parts of countably infinite size.

Corollary 3.2.1. *Let $\mathbb{A}$ be one of the following structures:*

(i) $\langle \mathbb{Q}, \leq \rangle$

(ii) the random reflexive partial order

(iii) the equivalence relation with either finitely or countably many equivalence classes of countable size

(iv) the random reflexive (di-)graph

(v) *the random reflexive* $\mathbb{K}_n$*-free graph for* $n \geq 3$

(vi) $\mathbb{K}_{k,\omega}$ *for* $k \geq 1$ *(the complete* k*-partite graph with parts of countably infinite size)*

(vii) $\langle \mathbb{Q}, < \rangle$

(viii) *the random strict partial order*

(ix) *the random tournament*

(x) $\mathbb{E}_{k,\omega}$ *and* $\mathbb{E}_{\omega,\omega}$ *(the irreflexive equivalence relation with either finitely or countably many equivalence classes of countably infinite size)*

(xi) *the random irreflexive (di-)graph*

(xii) *the random irreflexive* $\mathbb{K}_n$*-free graph*

(xiii) *the dual graph of the random irreflexive* $\mathbb{K}_n$*-free graph*

(xiv) $\mathbb{K}_{\omega,\omega}$ *(the complete infinite-partite graph with parts of countably infinite size)*

Then the pointwise topology and the Zariski topology on $\mathrm{End}(\mathbb{A})$ *coincide. In particular, the pointwise topology is the coarsest Hausdorff semigroup topology on* $\mathrm{End}(\mathbb{A})$*.*

Remark 3.2.2. With a different proof, the corollary also holds for $\mathbb{K}_{\omega,n}$ and $\mathbb{E}_{\omega,n}$ ($n \geq 1$) for which Theorem A1 is not applicable since these structures have algebraicity – see Proposition 5.1.2.

Proof. It is immediate that all structures in (i)-(xiv) are ω-categorical structures without algebraicity which are transitive (in particular, they have a mobile core); we note explicitly that this statement for (xiii) immediately follows from the corresponding fact for (xii) since a graph and its dual graph have the same automorphism group. For (i)-(v), the model-complete core of $\mathbb{A}$ is merely a single point with a loop. For (vi), the model-complete core of $\mathbb{A}$ is the complete graph on k vertices. Thus, the model-complete core is finite in (i)-(vi). For (vii) and (xii), the structure $\mathbb{A}$ is already a model-complete core, so the model-complete core of $\mathbb{A}$ is just $\mathbb{A}$ itself. For (viii) and (ix), the model-complete core of $\mathbb{A}$ is the structure $\langle \mathbb{Q}, < \rangle$. For (x), (xi), (xiii) and (xiv), the model-complete core of $\mathbb{A}$ is the complete graph on countably many vertices. Summarising, the model-complete core of $\mathbb{A}$ has no algebraicity in (vii)-(xiv).

In any case, Theorem A1 applies and yields the desired conclusion. □

3.2.2 Finite cores

First, we consider the case that $\mathbb{A}$ has a finite model-complete core.

Proposition 3.2.3. *Let* $\mathbb{A}$ *be an* ω*-categorical structure without algebraicity which has a mobile core. If the model-complete core of* $\mathbb{A}$ *is finite, then the Zariski topology on* $\mathrm{End}(\mathbb{A})$ *coincides with the pointwise topology.*

Proof. We show that the $\mathcal{T}_{pw}$-generating sets $\{s \in \mathrm{End}(\mathbb{A}) : s(a) = b\}$, $a, b \in A$, are $\mathcal{T}_{\mathrm{Zariski}}$-open by proving that they are $\mathcal{T}_{\mathrm{Zariski}}$-neighbourhoods of each element.

Let $s_0 \in \mathrm{End}(\mathbb{A})$ such that $s_0(a) = b$. Since $\mathbb{A}$ has a mobile core, there exist a copy $\mathbb{C}$ of the model-complete core of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ such that $a \in g(A) \subseteq C$. By Lemma 2.5.4, we know that $g(A) = C$. We set $n = |C|$ and write $g(\mathbb{A}) = \{a_1, \ldots, a_n\}$ where $a_1 = a$. Applying Lemma 2.3.3, we obtain that the set

$$V := \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_1), \ldots, s_0(a_n) \in \mathrm{Im}(s)\} = \bigcap_{j=1}^{n} \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_j) \in \mathrm{Im}(s)\}$$

is open in the Zariski topology. Since the translation $\rho_g \colon s \mapsto sg$ on $\mathrm{End}(\mathbb{A})$ is continuous with respect to the Zariski topology, the preimage

$$U := \rho_g^{-1}(V) = \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_1), \ldots, s_0(a_n) \in \mathrm{Im}(sg)\}$$

is $\mathcal{T}_{\mathrm{Zariski}}$-open as well. Again by Lemma 2.5.4, the images of the endomorphisms $s_0 g$ and sg (for arbitrary $s \in \mathrm{End}(\mathbb{A})$) must both have n elements. Hence, the images $s_0(a_i)$ are pairwise different and, further,

$$U = \{s \in \mathrm{End}(\mathbb{A}) : \mathrm{Im}(sg) = \{s_0(a_1), \ldots, s_0(a_n)\}\}.$$

The crucial observation is that $Ug = \{sg : s \in U\}$ is a finite set: Any element sg is determined by the ordered tuple $(s(a_1), \ldots, s(a_n))$. Since the unordered set $\{s(a_1), \ldots, s(a_n)\}$ is fixed for $s \in U$, there are only finitely many (at most $n!$, to be precise) possibilities for the ordered tuple.

Consequently, the set $M := \{sg : s \in U, s(a) \neq b\}$ is finite as well. We define

$$O := U \cap \bigcap_{t \in M} \{s \in \mathrm{End}(\mathbb{A}) : sg \neq t\} \in \mathcal{T}_{\mathrm{Zariski}}$$

and claim that $O = \{s \in \mathrm{End}(\mathbb{A}) : s \in U, s(a) = b\}$, subsequently giving

$$s_0 \in O \subseteq \{s \in \mathrm{End}(\mathbb{A}) : s(a) = b\}$$

as desired. If $s \in U$ with $s(a) = b$, then we take $z \in A$ with $g(z) = a$ and note $sg(z) = s(a) = b \neq t(z)$ for all $t \in M$. Conversely, if $s \in U$ but $s(a) \neq b$, then $t := sg \in M$, so $s \notin O$ – completing the proof. □

3.2.3 Cores without algebraicity

Now we consider structures $\mathbb{A}$ whose model-complete cores do not have algebraicity. This assumption will come into play via the following observation:

Lemma 3.2.4. *Let $\mathbb{B}$ be a countably infinite structure without algebraicity and let $b \in \mathbb{B}$. Then there exist f, h in the $\mathcal{T}_{pw}$-closure of* $\mathrm{Aut}(\mathbb{B})$ *such that $f(b) = b = h(b)$ and $f(B) \cap h(B) = \{b\}$ (so there exist two copies of $\mathbb{B}$ within $\mathbb{B}$ which only have b in common).*

Proof. We enumerate $B = \{b_n : n \in \mathbb{N}\}$ where $b_0 = b$. First, we recursively construct automorphisms $\alpha_n, \beta_n \in \mathrm{Aut}(\mathbb{B})$, $n \in \mathbb{N}$, such that

$$\alpha_{n+1}|_{\{b_0,\ldots,b_n\}} = \alpha_n|_{\{b_0,\ldots,b_n\}} \quad \text{and} \quad \beta_{n+1}|_{\{b_0,\ldots,b_n\}} = \beta_n|_{\{b_0,\ldots,b_n\}},$$
$$\text{as well as} \quad \alpha_n(\{b_0,\ldots,b_n\}) \cap \beta_n(\{b_0,\ldots,b_n\}) = \{b\}$$

for all $n \in \mathbb{N}$. We start by setting $\alpha_0 = \beta_0 := \mathrm{id}_B$. If α_n and β_n are already defined, we put $Y := \alpha_n(\{b_0,\ldots,b_n\})$ as well as $Z := \beta_n(\{b_0,\ldots,b_n\})$. Since $\mathbb{B}$ has no algebraicity, the relative orbits $\mathrm{Orb}(\alpha_n(b_{n+1}); Y)$ and $\mathrm{Orb}(\beta_n(b_{n+1}); Z)$ are infinite, so we can find an element $c_{n+1} \in \mathrm{Orb}(\alpha_n(b_{n+1}); Y)$ which is not contained in Z and then find an element $d_{n+1} \in \mathrm{Orb}(\beta_n(b_{n+1}); Z)$ which is not contained in $Y \cup \{c_{n+1}\}$. Taking $\gamma \in \mathrm{Stab}(Y)$ with $\gamma(\alpha_n(b_{n+1})) = c_{n+1}$ as well as $\delta \in \mathrm{Stab}(Z)$ with $\delta(\beta_n(b_{n+1})) = d_{n+1}$, and setting $\alpha_{n+1} := \gamma\alpha_n$ as well as $\beta_{n+1} := \delta\beta_n$ completes the construction. Finally, we set $f := \lim_{n\in\mathbb{N}} \alpha_n$ and $h := \lim_{n\in\mathbb{N}} \beta_n$; these maps are contained in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{B})$ and have the desired properties. □

Proposition 3.2.5. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity which has a mobile core. If the model-complete core of $\mathbb{A}$ is infinite and does not have algebraicity, then the Zariski topology on* $\mathrm{End}(\mathbb{A})$ *coincides with the pointwise topology.*

Proof. We check the assumptions of Lemma 3.1.2.

Since $\mathbb{A}$ is ω-categorical without algebraicity, property (i) follows from Lemma 2.3.2.

For properties (ii) and (iii), we fix $a \in A$, set $n = 2$ and construct $\gamma_1, \gamma_2 \in \mathrm{End}(\mathbb{A})$. Since $\mathbb{A}$ has a mobile core, there exist a copy $\mathbb{C}$ of the model-complete core of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ such that $a \in g(A) \subseteq C$. Since $\mathbb{C}$ has no algebraicity, there exist $f, h \colon \mathbb{C} \to \mathbb{C}$ in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{C})$ such that $f(a) = a = h(a)$ and $f(C) \cap h(C) = \{a\}$ by Lemma 3.2.4. Using the homomorphism $g \colon \mathbb{A} \to \mathbb{C}$, we set $\gamma_1 := fg$ and $\gamma_2 := hg$, considered as endomorphisms of $\mathbb{A}$. Then $a \in \mathrm{Im}(\gamma_i)$, i.e. (ii) holds. Suppose now that for some $s \in \mathrm{End}(\mathbb{A})$ and $x \in A$, we have $\mathrm{Im}(\gamma_i) \cap s^{-1}\{x\} \neq \emptyset$ for $i = 1, 2$. In order to prove (iii), the goal is to show $x = s(a)$. We rewrite to obtain the existence of $x_i \in A$ with $sfg(x_1) = s\gamma_1(x_1) = x = s\gamma_2(x_2) = shg(x_2)$. As a homomorphism from $\mathbb{C}$ to $\mathbb{A}$, the restriction $s|_C : \mathbb{C} \to \mathbb{A}$ is an embedding by Lemma 2.5.3, in particular injective. Hence,

$$fg(x_1) = hg(x_2) \in f(C) \cap h(C) = \{a\},$$

yielding $x = sfg(x_1) = s(a)$ as desired. □

Remark 3.2.6. Careful inspection of the proof of Proposition 3.2.5 shows that the assumption of $\mathbb{A}$ having no algebraicity is only used in order to apply Lemma 2.3.2 to obtain property (i) in Lemma 3.1.2. Thus, the proposition also applies to a structure with algebraicity that has a mobile core such that its model-complete core has no algebraicity, as long as one can – by some other means – verify property (i) (explicitly, for any element $a \in A$, there need to exist $f, g \in \mathrm{End}(\mathbb{A})$ such that $f|_{A\setminus\{a\}} = g|_{A\setminus\{a\}}$ and $f(a) \neq g(a)$). This observation will prove useful in Chapter 5.

3.3 Counterexample

In this section, we answer Question A2, giving an example of an ω-categorical (even homogeneous in a finite language) and transitive structure without algebraicity such that the Zariski topology on its endomorphism monoid does not coincide with the pointwise topology. By our results in Section 3.2, the model-complete core of this structure must be infinite and have algebraicity. Informally speaking, we take a complete graph on countably many vertices where each point has as fine structure a complete *bipartite* graph on countably many vertices, see Figure 3.1 below.

3.3.1 Definitions, notation and preliminary properties

We start by formally introducing our structure, henceforth called $\mathbb{G}$, and giving some notation. Differing slightly from the general Definition 2.4.2, it will turn out convenient to denote $\mathbb{K}_{2,\omega}$, the *complete bipartite graph on parts of countable size*, as follows: We write the domain as $K_{2,\omega} := A_{+1} \dot\cup A_{-1}$ where A_{+1} and A_{-1} are the (countably infinite) parts, so that the edge relation can be written as $E^{\mathbb{K}_{2,\omega}} := A_{-1} \times A_{+1} \cup A_{+1} \times A_{-1}$.

Definition 3.3.1. Let $\mathbb{G}$ denote the following structure over the language of two binary relations: We set $G := \mathbb{N} \times K_{2,\omega}$ (countably many copies of $K_{2,\omega}$) and define the relations as follows:

$$(i,x)\, E_1^{\mathbb{G}}\, (j,y) :\Leftrightarrow i \neq j,$$
$$(i,x)\, E_2^{\mathbb{G}}\, (j,y) :\Leftrightarrow i = j \text{ and } x\, E^{\mathbb{K}_{2,\omega}}\, y.$$

This means that the set of copies of $K_{2,\omega}$ forms a complete graph with respect to E_1 and that each copy $\{i\} \times K_{2,\omega}$ of $K_{2,\omega}$ is indeed a copy of the graph $\mathbb{K}_{2,\omega}$ (with respect to E_2); see Figure 3.1 below.

Note that an endomorphism s of $\mathbb{K}_{2,\omega}$ acts as a permutation on the set $\{A_{+1}, A_{-1}\}$ of parts since two ($E_2^{\mathbb{K}_{2,\omega}}$-connected) elements from different parts of $\mathbb{K}_{2,\omega}$ cannot be mapped to the same part of $\mathbb{K}_{2,\omega}$ – we either have $s(A_{+1}) \subseteq A_{+1}$ and $s(A_{-1}) \subseteq A_{-1}$ or $s(A_{+1}) \subseteq A_{-1}$ and $s(A_{-1}) \subseteq A_{+1}$.

Definition 3.3.2. For $s \in \mathrm{End}(\mathbb{K}_{2,\omega})$, we put $\mathrm{sgn}(s) \in \{+1,-1\}$ to be the sign of the permutation induced by s on $\{A_{+1}, A_{-1}\}$. Explicitly, this means that $s(A_e) \subseteq A_{e\cdot\mathrm{sgn}(s)}$ for $e = \pm 1$. As a slight abuse of notation, we will refer to $\mathrm{sgn}(s)$ as the *sign* of s.

Clearly, we have $\mathrm{sgn}(st) = \mathrm{sgn}(s)\,\mathrm{sgn}(t)$ for $s,t \in \mathrm{End}(\mathbb{K}_{2,\omega})$. As a tool, we define two very simple endomorphisms of $\mathbb{K}_{2,\omega}$.

Notation 3.3.3.

(i) In the sequel, $a_{+1} \in A_{+1}$ and $a_{-1} \in A_{-1}$ shall denote fixed elements.

(ii) We define $c_{+1} \in \mathrm{End}(\mathbb{K}_{2,\omega})$ and $c_{-1} \in \mathrm{End}(\mathbb{K}_{2,\omega})$ to be the unique endomorphisms of $\mathbb{K}_{2,\omega}$ with image $\{a_{+1}, a_{-1}\}$ and sign $+1$ and -1, respectively. So c_{+1} is constant on A_e with value a_e and c_{-1} is constant on A_e with value a_{-e} for $e = \pm 1$.

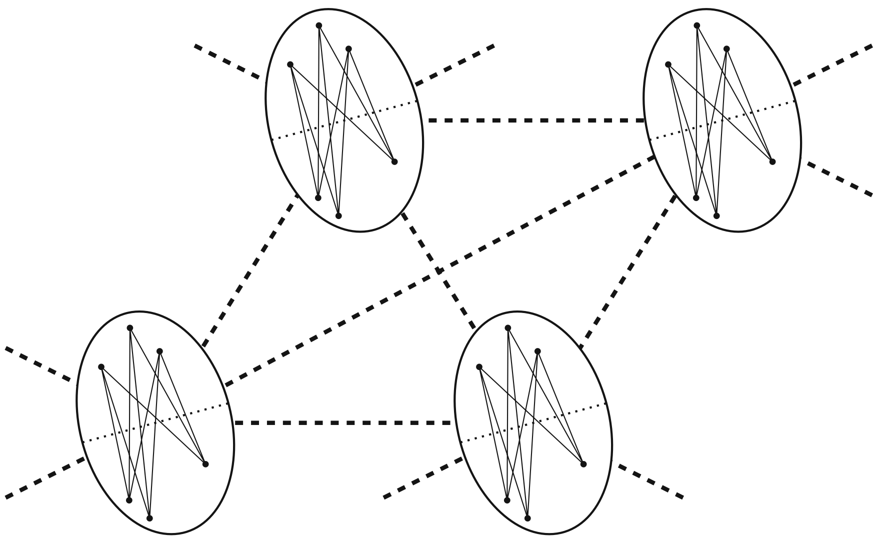

Figure 3.1: The structure $\mathbb{G}$: complete graph on countably many vertices (dashed) where each point has a complete bipartite graph on countably many vertices as fine structure (solid).

We now use Notation 2.4.4 to give an explicit description of the automorphisms and endomorphisms of $\mathbb{G}$.

Lemma 3.3.4.

(i) $\mathrm{End}(\mathbb{G}) = \{\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i : \tau \in \mathrm{Inj}(\mathbb{N}), s_i \in \mathrm{End}(\mathbb{K}_{2,\omega})\}$.

(ii) $\mathrm{Aut}(\mathbb{G}) = \{\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \alpha_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})\}$.

Remark 3.3.5. Lemma 3.3.4 exactly expresses that $\mathrm{End}(\mathbb{G})$ and $\mathrm{Aut}(\mathbb{G})$ are the (unrestricted) wreath products of $\mathrm{End}(\mathbb{K}_{2,\omega})$ with $\mathrm{Inj}(\mathbb{N})$ and $\mathrm{Aut}(\mathbb{K}_{2,\omega})$ with $\mathrm{Sym}(\mathbb{N})$, respectively, by the canonical actions of $\mathrm{Inj}(\mathbb{N})$ and $\mathrm{Sym}(\mathbb{N})$ on $\mathbb{N}$.

Proof (of Lemma 3.3.4). It is straightforward to see that the maps $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ in (i) and $\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i$ in (ii) form endomorphisms and automorphisms, respectively. Thus, (ii) follows immediately from (i) since $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ can only be bijective if $\tau \in \mathrm{Sym}(\mathbb{N})$ and $s_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})$.

To show (i), we first note that for any $s \in \mathrm{End}(\mathbb{K}_{2,\omega})$ and any two elements $(i,x),(i,y) \in G$ in the same copy of $K_{2,\omega}$, the images $s(i,x)$ and $s(i,y)$ are also contained in the same copy of $K_{2,\omega}$: Either x,y are connected in $\mathbb{K}_{2,\omega}$ in which case $s(i,x)$ and $s(i,y)$ are $E_2^{\mathbb{G}}$-connected and therefore contained in the same copy, or x,y are both connected in $\mathbb{K}_{2,\omega}$ to a common element z in which case $s(i,x)$ and $s(i,y)$ are both $E_2^{\mathbb{G}}$-connected to $s(i,z)$ and therefore contained in the same copy. Setting $\tau(i)$ to be the index of this copy, i.e. $s(i,x), s(i,y) \in \{\tau(i)\}\times K_{2,\omega}$, we obtain that s can be written as $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ for some functions $s_i\colon K_{2,\omega} \to K_{2,\omega}$. By compatibility of s with $E_1^{\mathbb{G}}$, the map τ needs to be injective. Further, the maps s_i are endomorphisms of $\mathbb{K}_{2,\omega}$ since s is compatible with $E_2^{\mathbb{G}}$. □

The representation in (ii) readily yields the following properties of $\mathbb{G}$ by means of lifting from $\mathrm{Sym}(\mathbb{N})$ and $\mathrm{Aut}(\mathbb{K}_{2,\omega})$:

Lemma 3.3.6. *$\mathbb{G}$ is ω-categorical, homogeneous, transitive and has no algebraicity.*

Proof. We start by showing that $\mathbb{G}$ is homogeneous which will also yield the ω-categoricity since $\mathbb{G}$ has a finite language. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be tuples in G and let $m\colon \bar{a} \mapsto \bar{b}$ be a finite partial isomorphism. Writing $a_k = (i_k, x_k)$ and $b_k = (j_k, y_k)$, we note that i_k and i_ℓ coincide if and only if j_k and j_ℓ coincide (for otherwise, either m or m^{-1} would not be compatible with $E_1^{\mathbb{G}}$). Hence, the map $i_k \mapsto j_k$ is a well-defined finite partial bijection and can thus easily be extended to some $\sigma \in \mathrm{Sym}(\mathbb{N})$ (in other words, the structure with domain $\mathbb{N}$ and without any relations is homogeneous). Further, if $i_{k_1} = \ldots = i_{k_N} =: i$, then $m_i\colon x_{k_1} \mapsto y_{k_1}, \ldots, x_{k_N} \mapsto y_{k_N}$ is a finite partial isomorphism of $\mathbb{K}_{2,\omega}$ since m is a finite partial isomorphism with respect to $E_2^{\mathbb{G}}$. The graph $\mathbb{K}_{2,\omega}$ is homogeneous, so m_i extends to $\alpha_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})$. Setting $\alpha_i = \mathrm{id}_{K_{2,\omega}}$ for all i such that no x_k is contained in the i-th copy of $K_{2,\omega}$ and putting $\alpha := \bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i \in \mathrm{Aut}(\mathbb{G})$, we obtain an extension of m.

Next, observe that $\mathbb{G}$ is transitive: given $a, b \in G$, the map $a \mapsto b$ is a finite partial isomorphism since neither $E_1^{\mathbb{G}}$ nor $E_2^{\mathbb{G}}$ contain any loops. Thus, homogeneity yields $\alpha \in \mathrm{Aut}(\mathbb{G})$ with $\alpha(a) = b$.

Finally, $\mathbb{G}$ does not have algebraicity since $\mathbb{K}_{2,\omega}$ does not have algebraicity: For a finite set $Y \subseteq G$ and $a = (i_0, x_0) \in G \setminus Y$, we set $Y_{i_0} := \{y \in K_{2,\omega} : (i_0, y) \in Y\} \not\ni x_0$ and note that $\mathrm{Orb}_{\mathbb{G}}(a; Y)$ encompasses the infinite set $\{i_0\} \times \mathrm{Orb}_{\mathbb{K}_{2,\omega}}(x_0; Y_{i_0})$ as witnessed by the automorphisms $\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} \alpha_i$ where $\alpha_{i_0} \in \mathrm{Stab}_{\mathbb{K}_{2,\omega}}(Y_{i_0})$ and $\alpha_i = \mathrm{id}_{K_{2,\omega}}$ for $i \neq i_0$. ☐

Remark 3.3.7. An alternative construction of $\mathbb{G}$ is as a first-order reduct of the *free superposition* (see [Bod15], this is a type of construction to combine two structures with different signatures in a "free" way) of $\mathbb{K}_{2,\omega}$ with $\mathbb{E}_{\omega,\omega}$. Since both structures are transitive and have no algebraicity, the superposition structure has the same properties which are then inherited by $\mathbb{G}$ since a first-order reduct can only have additional automorphisms.

To simplify the presentation, we additionally define a few notational shorthands concerning endomorphisms of $\mathbb{G}$:

Notation 3.3.8.

(i) For $p = \bigsqcup_{i\in\mathbb{N}}^{\xi} p_i \in \mathrm{End}(\mathbb{G})$, we define $\tilde{p} := \xi \in \mathrm{Inj}(\mathbb{N})$.

(ii) Given $p_0, \ldots, p_k \in \mathrm{End}(\mathbb{G})$ and $\varphi(s) := p_k s p_{k-1} s \ldots s p_0$, $s \in \mathrm{End}(\mathbb{G})$, we define $\tilde{\varphi}(\tau) := \tilde{p}_k \tau \tilde{p}_{k-1} \tau \ldots \tau \tilde{p}_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.

3.3.2 Proof strategy

The goal of Section 3.3 is to prove the following:

Theorem A2. *On the endomorphism monoid of the structure $\mathbb{G}$, the pointwise topology is strictly finer than the Zariski topology.*

Remark 3.3.9. Before we go into the details of the proof, let us remark that the structure $\mathbb{G}$ needs to have an infinite model-complete core which has algebraicity in order to have a chance of satisfying Theorem A2 – for otherwise, Theorem A1 would apply.

The model-complete core of $\mathbb{K}_{2,\omega}$ is just the graph consisting of a single edge, as witnessed for instance by the substructure induced on $\{a_{+1}, a_{-1}\}$ and the homomorphism $c_{+1}\colon \mathbb{K}_{2,\omega} \to \{a_{+1}, a_{-1}\}$. We claim that the model-complete core of $\mathbb{G}$ is the complete graph on countably many vertices where each point has as fine structure a single edge, i.e. the substructure $\mathbb{C}$ of $\mathbb{G}$ induced on $C := \mathbb{N} \times \{a_{+1}, a_{-1}\} \subseteq G$; see Figure 3.2.

Similarly to the proof of Lemma 3.3.4, one easily checks that (here, $c_{\pm 1}$ are considered as self-maps of $\{a_{+1}, a_{-1}\}$)

$$\mathrm{End}(\mathbb{C}) = \left\{ \bigsqcup_{i\in\mathbb{N}}^{\tau} \gamma_i : \tau \in \mathrm{Inj}(\mathbb{N}), \gamma_i \in \{c_{+1}, c_{-1}\} \right\},$$

$$\mathrm{Aut}(\mathbb{C}) = \left\{ \bigsqcup_{i\in\mathbb{N}}^{\sigma} \gamma_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \gamma_i \in \{c_{+1}, c_{-1}\} \right\}.$$

Thus, any endomorphism is locally interpolated by an automorphism, and $\mathbb{C}$ is indeed a model-complete core. Additionally, $\mathbb{G}$ and $\mathbb{C}$ are homomorphically equivalent – an example of a homomorphism $\mathbb{G} \to \mathbb{C}$ is given by $\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_\mathbb{N}} c_{+1}$ (where c_{+1} is considered as a map defined on $K_{2,\omega}$).

Finally, $\mathbb{C}$ has algebraicity: any automorphism of $\mathbb{C}$ which stabilises $Y := \{(0, a_{+1})\}$ also stabilises $a := (0, a_{-1})$, so the Y-relative orbit of a is finite.

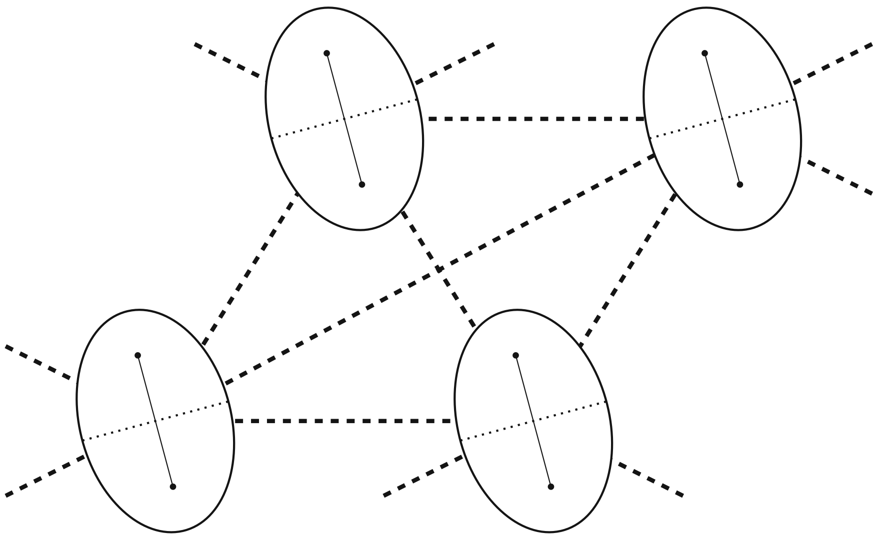

Figure 3.2: The model-complete core of $\mathbb{G}$: complete graph on countably many vertices (dashed) where each point has a single edge (solid) as fine structure.

In order to show Theorem A2, we will prove that $\mathcal{T}_{\mathrm{Zariski}}$-open sets on $\mathrm{End}(\mathbb{G})$ cannot determine the sign of the components s_i of $s = \bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$, in other words decide whether the functions s_i switch the two parts of $\mathbb{K}_{2,\omega}$ or not. On the other hand, $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$-open sets can determine the sign of finitely many components, thus showing $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})} \neq \mathcal{T}_{\mathrm{Zariski}}$. More precisely, we will prove that if a $\mathcal{T}_{\mathrm{Zariski}}$-generating set $M_{\varphi,\psi}$ contains $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$, then it also contains $\tau \ltimes c_{-1}$ for all elements τ of a "big" subset of $\mathrm{Inj}(\mathbb{N})$ – where "big" means either "$\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$" (if the terms φ and ψ have equal lengths; see Lemmas 3.3.10 and 3.3.11) or "$\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set" (if the terms φ and ψ have different lengths; see Lemma 3.3.12).

Our (almost trivial) first lemma analogously holds in a more general setting. Since we only apply it in case of terms of equal lengths, we formulate it in the present form.

Lemma 3.3.10.

(i) Let $k \geq 1$ and let $\xi_0, \ldots, \xi_k, \theta_0, \ldots, \theta_k \in \mathrm{Inj}(\mathbb{N})$ as well as $\tilde{\varphi}(\tau) := \xi_k \tau \xi_{k-1} \tau \ldots \tau \xi_0$ and $\tilde{\psi}(\tau) := \theta_k \tau \theta_{k-1} \tau \ldots \tau \theta_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.

If $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) \neq \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$, then $M_{\tilde{\varphi},\tilde{\psi}} = \left\{\tau \in \mathrm{Inj}(\mathbb{N}) : \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau)\right\}$ is a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$.

(ii) Let $k \geq 1$ and let $p_0, \ldots, p_k, q_0, \ldots, q_k \in \mathrm{End}(\mathbb{G})$ as well as $\varphi(s) := p_k s p_{k-1} s \ldots s p_0$ and $\psi(s) := q_k s q_{k-1} s \ldots s q_0$, $s \in \mathrm{End}(\mathbb{G})$. Assume $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) \neq \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$ (using the shorthand from Notation 3.3.8).

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood U of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes t \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ for all $\tau \in U$ and $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$. In particular, $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ for all $\tau \in U$.

The second lemma really requires the terms to be of equal length.

Lemma 3.3.11. *Let $k \geq 1$ and let $p_0, \ldots, p_k, q_0, \ldots, q_k \in \mathrm{End}(\mathbb{G})$ as well as $\varphi(s) := p_k s p_{k-1} s \ldots s p_0$ and $\psi(s) := q_k s q_{k-1} s \ldots s q_0$, $s \in \mathrm{End}(\mathbb{G})$.*
Assume $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}) \neq \psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})$ but $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) = \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$.

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood U of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ for all $\tau \in U$.

Finally, we formulate a result for terms of different lengths.

Lemma 3.3.12.

(i) Let $\ell < k$ and let $\xi_0, \ldots, \xi_k, \theta_0, \ldots, \theta_\ell \in \mathrm{Inj}(\mathbb{N})$ as well as $\tilde{\varphi}(\tau) := \xi_k \tau \xi_{k-1} \tau \ldots \tau \xi_0$ and $\tilde{\psi}(\tau) := \theta_\ell \tau \theta_{\ell-1} \tau \ldots \tau \theta_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.

Then $M_{\tilde{\varphi},\tilde{\psi}} = \left\{\tau \in \mathrm{Inj}(\mathbb{N}) : \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau)\right\}$ is $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open.

(ii) Let $\ell < k$ and let $p_0, \ldots, p_k, q_0, \ldots, q_\ell \in \mathrm{End}(\mathbb{G})$ as well as $\varphi(s) := p_k s p_{k-1} s \ldots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \ldots s q_0$, $s \in \mathrm{End}(\mathbb{G})$.

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set V such that $\tau \ltimes t \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ for all $\tau \in V$ and $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$. In particular, $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ for all $\tau \in V$.

We first demonstrate how these auxiliary statements are used and prove Theorem A2 before showing the statements themselves in Subsection 3.3.3.

Proof (of Theorem A2 given Lemmas 3.3.10, 3.3.11 and 3.3.12). Our goal is to show that any $\mathcal{T}_{\text{Zariski}}$-open set O containing $\text{id}_{\mathbb{N}} \ltimes c_{+1}$ also contains $\tau \ltimes c_{-1}$ for some $\tau \in \text{Inj}(\mathbb{N})$. This implies in particular that the $\mathcal{T}_{pw}$-open set $\{s \in \text{End}(\mathbb{G}) : s(0, a_{+1}) = (0, a_{+1})\}$ cannot be $\mathcal{T}_{\text{Zariski}}$-open, proving $\mathcal{T}_{\text{Zariski}} \neq \mathcal{T}_{pw}$.

It suffices to consider $\mathcal{T}_{\text{Zariski}}$-*basic* open sets O, i.e. $O = \bigcap_{h \in H} M_{\varphi_h, \psi_h} \ni \text{id}_{\mathbb{N}} \ltimes c_{+1}$ for some finite set H. If the terms φ_h and ψ_h have equal length, we apply Lemma 3.3.10 or 3.3.11 to find a $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-open neighbourhood U_h of $\text{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi_h, \psi_h}$ for all $\tau \in U_h$. If φ_h and ψ_h have different lengths, we instead apply[1] Lemma 3.3.12 to find a $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-dense and open set V_h such that $\tau \ltimes c_{-1} \in M_{\varphi_h, \psi_h}$ for all $\tau \in V_h$. Intersecting the respective sets U_h and V_h thus obtained yields a $\mathcal{T}_{pw}$-open neighbourhood U of $\text{id}_{\mathbb{N}}$ and a $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-dense and open set V such that $\tau \ltimes c_{-1} \in M_{\varphi_h, \psi_h}$ for all $\tau \in U$ whenever φ_h and ψ_h have equal length and such that $\tau \ltimes c_{-1} \in M_{\varphi_h, \psi_h}$ for all $\tau \in V$ whenever φ_h and ψ_h have different lengths. The intersection $U \cap V$ is nonempty; for any $\tau \in U \cap V$ we have $\tau \ltimes c_{-1} \in M_{\varphi_h, \psi_h}$ for all $h \in H$, i.e. $\tau \ltimes c_{-1} \in O$. This concludes the proof. □

Remark 3.3.13. A slight refinement of this proof even shows that the Zariski topology on $\text{End}(\mathbb{G})$ is not Hausdorff since $\text{id}_{\mathbb{N}} \ltimes c_{+1}$ and $\text{id}_{\mathbb{N}} \ltimes c_{-1}$ cannot be separated by open sets: By the proof, a given basic open set around $\text{id}_{\mathbb{N}} \ltimes c_{+1}$ contains $\tau \ltimes c_{-1}$ provided that τ is an element of the intersection of a certain $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-open neighbourhood of $\text{id}_{\mathbb{N}}$ and a $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-dense open set. The same idea similarly (but with an easier proof in the analogue of Lemma 3.3.11) yields that a given basic open set around $\text{id}_{\mathbb{N}} \ltimes c_{-1}$ contains $\tau' \ltimes c_{-1}$ provided that τ' is an element of the intersection of another $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-open neighbourhood of $\text{id}_{\mathbb{N}}$ and another $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$-dense open set. The intersection of these four sets is nonempty, so the basic open sets around $\text{id}_{\mathbb{N}} \ltimes c_{+1}$ and $\text{id}_{\mathbb{N}} \ltimes c_{-1}$ contain a common element (namely $\tau \ltimes c_{-1}$ for a certain $\tau \in \text{Inj}(\mathbb{N})$).

The preceding remark suggests the following refinement of Question A2:

Question 3.3.14. *Is there an ω-categorical (transitive?) relational structure $\mathbb{A}$ such that there exists a Hausdorff (even Polish?) semigroup topology on* $\text{End}(\mathbb{A})$ *which is* not *finer than the topology of pointwise convergence?*

3.3.3 Proof details

In this subsection, we prove Lemmas 3.3.10, 3.3.11 and 3.3.12 in sequence.

Proof (of Lemma 3.3.10).

(i). The set $M_{\tilde{\varphi}, \tilde{\psi}} \subseteq \text{Inj}(\mathbb{N})$ is open with respect to $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$ since $\tilde{\varphi}$ and $\tilde{\psi}$ are continuous with respect to $\mathcal{T}_{pw}|_{\text{Inj}(\mathbb{N})}$.

(ii). Set $U := M_{\tilde{\varphi}, \tilde{\psi}}$ and note that if $u := \varphi(\tau \ltimes t)$ and $v := \psi(\tau \ltimes t)$, then $\tilde{u} = \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau) = \tilde{v}$, so $u \neq v$. □

The second lemma requires more work.

[1] If ψ_h is longer than φ_h, we exchange these two terms.

Proof (of Lemma 3.3.11). We start by fixing some notation. We first write $p_j = \bigsqcup_{i\in\mathbb{N}}^{\xi_j} p_{j,i}$, $q_j = \bigsqcup_{i\in\mathbb{N}}^{\theta_j} q_{j,i}$ (so $\xi_j = \tilde{p}_j$, $\theta_j = \tilde{q}_j$) and $\delta := \tilde{\varphi}(\mathrm{id}_\mathbb{N}) = \tilde{\psi}(\mathrm{id}_\mathbb{N})$. Further, we define $\Xi_j := \xi_j\xi_{j-1}\dots\xi_0$ as well as $\Theta_j := \theta_j\theta_{j-1}\dots\theta_0$, $j = 0,\dots,k$. In particular, $\Xi_k = \Theta_k = \delta$. Let the two (distinct, by assumption) functions $\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{+1})$ and $\psi(\mathrm{id}_\mathbb{N} \ltimes c_{+1})$ differ at the point $(h,x) \in G$. Further, set $e \in \{-1,+1\}$ such that $x \in A_e$ and choose any $x' \in A_{-e}$.

In the course of the proof, we will require the explicit expansions of the compositions in $\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$:

$$\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1}) = \bigsqcup_{i\in\mathbb{N}}^{\delta} p_{k,\Xi_{k-1}(i)} c_{\pm 1} p_{k-1,\Xi_{k-2}(i)} \cdots c_{\pm 1} p_{0,i}$$
$$\psi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1}) = \bigsqcup_{i\in\mathbb{N}}^{\delta} q_{k,\Theta_{k-1}(i)} c_{\pm 1} q_{k-1,\Theta_{k-2}(i)} \cdots c_{\pm 1} q_{0,i}.$$

We proceed in two steps – first, we show that $\mathrm{id}_\mathbb{N} \ltimes c_{-1} \in M_{\varphi,\psi}$; second, we extend this to $\tau \ltimes c_{-1}$ for all τ in an appropriately constructed $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_\mathbb{N}$.

(1). $\mathrm{id}_\mathbb{N} \ltimes c_{-1} \in M_{\varphi,\psi}$: We compare $\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$ at (h,x) as well as (h,x'). In order to simplify notation, we define[2]

$$\begin{aligned} m &:= \mathrm{sgn}(p_{k-1,\Xi_{k-2}(h)}) \cdot \mathrm{sgn}(p_{k-2,\Xi_{k-3}(h)}) \cdot \ldots \cdot \mathrm{sgn}(p_{0,h}) \\ n &:= \mathrm{sgn}(q_{k-1,\Theta_{k-2}(h)}) \cdot \mathrm{sgn}(q_{k-2,\Theta_{k-3}(h)}) \cdot \ldots \cdot \mathrm{sgn}(q_{0,h}) \\ \widehat{p} &:= p_{k,\Xi_{k-1}(h)} \\ \widehat{q} &:= q_{k,\Xi_{k-1}(h)} \end{aligned}$$

and conclude

$$\begin{aligned} [\varphi\,(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x) &= \big(\delta(h), \widehat{p}(a_{me})\big), & [\varphi\,(\mathrm{id}_\mathbb{N} \ltimes c_{-1})](h,x) &= \big(\delta(h), \widehat{p}(a_{me(-1)^k})\big) \\ [\psi\,(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x) &= \big(\delta(h), \widehat{q}(a_{ne})\big), & [\psi\,(\mathrm{id}_\mathbb{N} \ltimes c_{-1})](h,x) &= \big(\delta(h), \widehat{q}(a_{ne(-1)^k})\big) \\ [\varphi\,(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x') &= \big(\delta(h), \widehat{p}(a_{-me})\big), & [\varphi\,(\mathrm{id}_\mathbb{N} \ltimes c_{-1})](h,x') &= \big(\delta(h), \widehat{p}(a_{-me(-1)^k})\big) \\ [\psi\,(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x') &= \big(\delta(h), \widehat{q}(a_{-ne})\big), & [\psi\,(\mathrm{id}_\mathbb{N} \ltimes c_{-1})](h,x') &= \big(\delta(h), \widehat{q}(a_{-ne(-1)^k})\big) \end{aligned}$$

If
$$\{\widehat{p}(a_{+1}), \widehat{p}(a_{-1})\} \neq \{\widehat{q}(a_{+1}), \widehat{q}(a_{-1})\},$$
then $\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{-1})$ and $\psi(\mathrm{id}_\mathbb{N} \ltimes c_{-1})$ cannot coincide on both (h,x) and (h,x'), so $\mathrm{id}_\mathbb{N} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed.

In case of
$$\{\widehat{p}(a_{+1}), \widehat{p}(a_{-1})\} = \{\widehat{q}(a_{+1}), \widehat{q}(a_{-1})\},$$
we distinguish further: If $m = n$, then $[\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x) \neq [\psi(\mathrm{id}_\mathbb{N} \ltimes c_{+1})](h,x)$ shows

$$\widehat{p}(a_{+1}) = \widehat{q}(a_{-1}) \quad \text{as well as} \quad \widehat{p}(a_{-1}) = \widehat{q}(a_{+1})$$

[2] m and n count how many times the fixed functions (except for the outermost ones) involved in evaluating $\varphi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_\mathbb{N} \ltimes c_{\pm 1})$ switch the parts of the h-th copy of $\mathbb{K}_{2,\omega}$.

which leads to[3] $[\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) \neq [\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x)$, so $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed. If on the other hand $m = -n$, then we analogously obtain

$$\widehat{p}(a_{+1}) = \widehat{q}(a_{+1}) \quad \text{as well as} \quad \widehat{p}(a_{-1}) = \widehat{q}(a_{-1})$$

and $[\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) \neq [\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x)$, so $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed.

(2). There exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood $U \subseteq \mathrm{Inj}(\mathbb{N})$ of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ for all $\tau \in U$: One immediately checks that for arbitrary $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$, the map $\chi_t \colon \mathrm{Inj}(\mathbb{N}) \to \mathrm{End}(\mathbb{G})$, $\chi_t(\tau) := \tau \ltimes t$ is continuous with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$ and[4] $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$. Since $M_{\varphi,\psi}$ is open with respect to $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$, the preimage $U := \chi_{c_{-1}}^{-1}(M_{\varphi,\psi}) \subseteq \mathrm{Inj}(\mathbb{N})$ is open with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$. By (1), the set U contains $\mathrm{id}_{\mathbb{N}}$ – completing the proof. □

Finally, we show the third lemma.

Proof (of Lemma 3.3.12).

(i). We have to prove that for two tuples $\bar{z}, \bar{w}$ of the same length such that $\bar{w}$ does not contain the same value twice (since we are working in $\mathrm{Inj}(\mathbb{N})$), the intersection

$$\{\tau \in \mathrm{Inj}(\mathbb{N}) : \tau(\bar{z}) = \bar{w}\} \cap M_{\tilde{\varphi},\tilde{\psi}}$$

is nonempty. The idea behind the proof is to find an element $x_0 \in \mathbb{N}$ and inductively construct a partial injection $\widehat{\tau}$ which extends $\bar{z} \mapsto \bar{w}$ such that the values

$$[\tilde{\varphi}(\widehat{\tau})](x_0) = \xi_k \widehat{\tau} \xi_{k-1} \widehat{\tau} \dots \widehat{\tau} \xi_0(x_0) \quad \text{and} \quad [\tilde{\psi}(\widehat{\tau})](x_0) = \theta_\ell \widehat{\tau} \theta_{\ell-1} \widehat{\tau} \dots \widehat{\tau} \theta_0(x_0)$$

are welldefined (i.e. $\xi_0(x_0) \in \mathrm{Dom}(\widehat{\tau})$, $\xi_1 \widehat{\tau} \xi_0(x_0) \in \mathrm{Dom}(\widehat{\tau})$ et cetera) and $[\tilde{\varphi}(\widehat{\tau})](x_0) \neq [\tilde{\psi}(\widehat{\tau})](x_0)$. This gives $\tau(\bar{z}) = \bar{w}$ and $\tau \in M_{\tilde{\varphi},\tilde{\psi}}$ for any $\tau \in \mathrm{Inj}(\mathbb{N})$ extending $\widehat{\tau}$.

More precisely, we will define (not necessarily distinct) elements $x_0, \dots, x_k, x_0', \dots, x_k' \in \mathbb{N}$ and $y_0 \dots, y_\ell, y_0', \dots, y_\ell' \in \mathbb{N}$ such that

(1) $x_0 = y_0$.

(2) $x_j' = \xi_j(x_j)$ for all $j = 0, \dots, k$.

(3) $y_j' = \theta_j(y_j)$ for all $j = 0, \dots, \ell$.

(4) $\widehat{\tau}$ defined by $\bar{z} \mapsto \bar{w}$, $(x_0', \dots, x_{k-1}') \mapsto (x_1, \dots, x_k)$, $(y_0', \dots, y_{\ell-1}') \mapsto (y_1, \dots, y_\ell)$ is a welldefined[5] partial injection.

(5) $x_k' \neq y_\ell'$. (This will crucially depend on the assumption $\ell < k$.)

We first pick $x_0 = y_0 \in \mathbb{N}$ such that $x_0' := \xi_0(x_0) \notin \bar{z}$ and $y_0' := \theta_0(y_0) \notin \bar{z}$; this is possible since the set $\xi_0^{-1}(\bar{z}) \cup \theta_0^{-1}(\bar{w})$ of forbidden points is finite by injectivity of ξ_0 and θ_0. Note that x_0' and y_0' are not necessarily different (in particular, $\xi_0 = \theta_0$ is possible).

Suppose $1 \leq i \leq \ell$ and that $x_0, \dots, x_{i-1}, x_0', \dots, x_{i-1}'$ as well as $y_0, \dots, y_{i-1}, y_0', \dots, y_{i-1}'$ are already defined such that (1)-(4) hold (with $i-1$ in place of both k and ℓ). We abbreviate $X_{i-1} := \{x_0, \dots, x_{i-1}\}$, $X_{i-1}' := \{x_0', \dots, x_{i-1}'\}$ and $Y_{i-1} := \{y_0, \dots, y_{i-1}\}$, $Y_{i-1}' := \{y_0', \dots, y_{i-1}'\}$. Pick $x_i, y_i \notin \bar{w} \cup X_{i-1} \cup Y_{i-1}$ such that $x_i' := \xi_i(x_i) \notin \bar{z} \cup X_{i-1}' \cup Y_{i-1}'$ and $y_i' := \theta_i(y_i) \notin \bar{z} \cup X_{i-1}' \cup Y_{i-1}'$ with the additional property that[6] x_i and y_i are chosen to be

[3] Here we use that φ and ψ have equal lengths (or more precisely: lengths of equal parity).

[4] Caution! We briefly consider the pointwise topology on $\mathrm{End}(\mathbb{G})$ instead of the Zariski topology.

[5] This means that if e.g. $x_0' = y_0'$, then $x_1 = y_1$.

[6] This ensures that $\widehat{\tau}$ is welldefined and injective.

distinct if and only if x'_{i-1} and y'_{i-1} are distinct (to obtain a welldefined partial injection in (4)). As with the construction of x_0 above, this is possible by finiteness of the forbidden sets.

If $\ell+1 \leq i \leq k$ and if $x_0, \dots, x_{i-1}, x'_0, \dots, x'_{i-1}$ as well as $y_0, \dots, y_\ell, y'_0, \dots, y'_\ell$ are already defined such that (1)-(4) hold (with $i-1$ in place of k), then we again abbreviate $X_{i-1} := \{x_0, \dots, x_{i-1}\}, X'_{i-1} := \{x'_0, \dots, x'_{i-1}\}$ and $Y_\ell := \{y_0, \dots, y_\ell\}, Y'_\ell := \{y'_0, \dots, y'_\ell\}$. Analogously to the previous step, we pick $x_i \notin \bar{w} \cup X_{i-1} \cup Y_\ell$ such that $x'_i := \xi_i(x_i) \notin \bar{z} \cup X'_{i-1} \cup Y'_\ell$. Note that in the final step $i = k$, we are picking x_k such that[7] $x'_k \notin \bar{z} \cup X'_{k-1} \cup Y'_\ell$. In particular, we require $x'_k \neq y'_\ell$, i.e. (5).

(ii). The set $V := M_{\tilde{\varphi},\tilde{\psi}} \subseteq \mathrm{Inj}(\mathbb{N})$ is $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense by the first statement and clearly $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open. For $\tau \in V$, we set $u := \varphi(\tau \ltimes c_{-1})$ as well as $v := \psi(\tau \ltimes c_{-1})$ and note that $\tilde{u} = \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau) = \tilde{v}$. This yields $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ as desired. □

[7] At this point, it is crucial that $\ell < k$ since we would never enter the second phase $\ell + 1 \leq i \leq k$ of the construction otherwise (more precisely, if ℓ were equal to k, we would have to determine x'_k at the same time as y'_ℓ and could not make sure that they are different).

4 The semigroup of increasing functions on the rational numbers has a unique Polish topology

This chapter is devoted to answering Question B positively: we show that the endomorphism monoid of $\langle \mathbb{Q}, \leq \rangle$ has UPP. In Section 4.1, we introduce some useful notation and revisit automatic continuity as well as the Back&Forth method in the context of $\langle \mathbb{Q}, \leq \rangle$. Section 4.2 provides an overview of the situation and relates our question to prior work. We proceed by presenting the proof strategy more thoroughly in Section 4.3. The details of the proof are then contained in Sections 4.4 and 4.5.

4.1 Tools & Notions: notation; more on automatic continuity and Back&Forth

We start by defining some notational shorthands.

Notation 4.1.1. We set

$$\mathcal{M}_{\mathbb{Q}} := \operatorname{End}(\mathbb{Q}, \leq) = \{f\colon \mathbb{Q} \to \mathbb{Q} \mid f \text{ increasing}\},$$
$$\mathcal{G}_{\mathbb{Q}} := \operatorname{Aut}(\mathbb{Q}, \leq) = \{f\colon \mathbb{Q} \to \mathbb{Q} \mid f \text{ bijective, (strictly) increasing}\}.$$

Additionally, it will be useful to embed $\mathbb{Q}$ into the real numbers $\mathbb{R}$. Consequently, we will allow intervals with irrational boundary points as well. Differing from standard notation, we only consider the *rational* points in this interval, unless explicitly mentioned otherwise:

Notation 4.1.2. For $\gamma_1, \gamma_2 \in \mathbb{R} \cup \{\pm\infty\}$, we put $(\gamma_1, \gamma_2) := \{q \in \mathbb{Q} : \gamma_1 < q < \gamma_2\}$. If $s \in \mathcal{M}_{\mathbb{Q}}$, we will denote $s((-\infty, \gamma))$ by $s(-\infty, \gamma)$ et cetera to avoid lengthy typesetting. In the same spirit, we will write $\sup \operatorname{Im}(s)$ as $\sup s$ and $\inf \operatorname{Im}(s)$ as $\inf s$.
Finally, we abbreviate $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$.

Intervals will be distinguished as follows:

Definition 4.1.3. An interval is called *rational* if its boundary points are contained in $\mathbb{Q} \cup \{\pm\infty\}$, and *irrational* if its boundary points are contained in $\mathbb{I} \cup \{\pm\infty\}$.

Next, we apply the notion of automatic continuity (see Definition 2.6.4) to $\mathcal{M}_{\mathbb{Q}}$ as well as $\mathcal{G}_{\mathbb{Q}}$. As it turns out, $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{pw})$ does not have automatic continuity while $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ satisfies a very strong version. We begin with the negative result which is a straightforward application of Proposition 2.6.6:

Proposition 4.1.4. *$(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{pw})$ does not have automatic continuity with respect to the class of Polish semigroups.*

Proof. We set $M := \mathcal{M}_\mathbb{Q}$ as well as

$$N := \{f \in \mathcal{M}_\mathbb{Q} : \inf f = -\infty \text{ and } \sup f = +\infty\}$$

and check the assumptions of Proposition 2.6.6. Clearly, N is a submonoid of M. If we define $f_n \in \mathcal{M}_\mathbb{Q}$ by

$$f_n(x) := \begin{cases} 0, & -n < x < n \\ x, & x \leq -n \text{ or } x \geq n \end{cases}$$

we have $f_n \in N$ but the sequence $(f_n)_{n\in\mathbb{N}}$ converges with respect to the pointwise topology, namely to the constant function with value 0 – which is not in N. Hence, N is not $\mathcal{T}_{pw}$-closed. Finally, if $g \in M$ and $f \notin N$, e.g. $\mathrm{Im}(f) \subseteq [u, +\infty)$, then $\mathrm{Im}(fg) \subseteq [u, +\infty)$ and $\mathrm{Im}(gf) \subseteq [g(u), +\infty)$, so $fg, gf \notin N$.

By Proposition 2.6.6, the topological semigroup $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{pw})$ does not have automatic continuity with respect to the class of Polish semigroups. □

On the other hand, $\mathcal{G}_\mathbb{Q}$ with the pointwise topology does have automatic continuity by the following result due to Rosendal and Solecki (which we reformulate to fit our notation).

Theorem 4.1.5 ([RS07, Corollary 5] combined with the remarks before [RS07, Corollary 3])**.** *$(\mathcal{G}_\mathbb{Q}, \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological groups.*

Explicitly, this means: If $(H, \mathcal{O})$ is a second countable topological group, then any group homomorphism $\varphi\colon (\mathcal{G}_\mathbb{Q}, \mathcal{T}_{pw}) \to (H, \mathcal{O})$ is continuous. By Proposition 2.6.5, the notions of automatic continuity with respect to the classes of second countable topological groups and second countable topological semigroups are equivalent, so we obtain:

Proposition 4.1.6. *$(\mathcal{G}_\mathbb{Q}, \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological semigroups, explicitly: If $(H, \mathcal{O})$ is a second countable topological semigroup, then any semigroup homomorphism $(\mathcal{G}_\mathbb{Q}, \mathcal{T}_{pw}) \to (H, \mathcal{O})$ is continuous.*

Finally, we use the technique of Back&Forth to find an answer to the following question: given $s, f \in \mathcal{M}_\mathbb{Q}$, under which conditions does there exist a map $s' \in \mathcal{M}_\mathbb{Q}$ such that $s = fs'$?

Lemma 4.1.7. *Let $s, f \in \mathcal{M}_\mathbb{Q}$ such that $\mathrm{Im}(s) \subseteq \mathrm{Im}(f)$.*

(i) Any finite partial increasing map m_0 from $\mathbb{Q}$ to $\mathbb{Q}$ satisfying $s(p) = fm_0(p)$ for all $p \in \mathrm{Dom}(m_0)$ can be extended to $s' \in \mathcal{M}_\mathbb{Q}$ with $s = fs'$.

(ii) Additionally suppose that for each $w \in \mathrm{Im}(f)$ the preimage $f^{-1}\{w\}$ is an irrational interval. Then any finite partial increasing injective *map m_0 from $\mathbb{Q}$ to $\mathbb{Q}$ satisfying $s(p) = fm_0(p)$ for all $p \in \mathrm{Dom}(m_0)$ can be extended to an* injective *$s' \in \mathcal{M}_\mathbb{Q}$ with $s = fs'$.*

Proof. The proofs of both statements are almost parallel: one verifies that the system $\mathcal{S}$ of all finite partial increasing [for (ii): strictly increasing] maps m from $\mathbb{Q}$ to $\mathbb{Q}$ satisfying $s(p) = fm(p)$ for all $p \in \mathrm{Dom}(m)$ is a Forth system and applies Lemma 2.8.2. □

4.2 Overview

As mentioned in the Introduction (Chapter 1), Question B cannot be answered using known techniques: since $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{pw})$ does not have automatic continuity but $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ does, the monoid $\mathcal{M}_{\mathbb{Q}}$ equipped with the pointwise topology cannot satisfy Property **X** (or even Pseudo-Property $\overline{\mathbf{X}}$) with respect to the group $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ by Proposition 2.7.3(ii). Further, the monoid does not fall into the regime of the known counterexamples either – in particular, Example 2.6.3 does not apply: $\mathcal{M}_{\mathbb{Q}}$ cannot be represented as a self-embedding monoid (it contains non-injective functions). On the other hand, Example 2.6.3 does encompass $\mathrm{End}(\mathbb{Q}, <) = \mathrm{Emb}(\mathbb{Q}, \leq)$, so if we were to replace $\mathcal{M}_{\mathbb{Q}} = \mathrm{End}(\mathbb{Q}, \leq)$ by $\mathrm{End}(\mathbb{Q}, <)$ in Question B, the resulting question would have a simple (negative) answer.

The main goal of the present chapter is to show the following theorem:

Theorem B. *The pointwise topology is the unique Polish semigroup topology on* $\mathcal{M}_{\mathbb{Q}}$.

It can be readily seen that the pointwise topology on $\mathcal{M}_{\mathbb{Q}}$ coincides with the Zariski topology, yielding that $\mathcal{T}_{pw}$ is coarser than any Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$ (Step (1) from the Introduction), see also Corollary 3.2.1. The essence of our strategy to prove the converse direction (Step (2) from the Introduction) and thus Theorem B is to find a finer topology (which turns out to be neither Polish nor a semigroup topology) on $\mathcal{M}_{\mathbb{Q}}$, the so-called *rich topology* $\mathcal{T}_{rich}$, with the following two properties: On the one hand, the rich topology needs to be sufficiently fine that $\mathcal{M}_{\mathbb{Q}}$ equipped with the rich topology satisfies Pseudo-Property $\overline{\mathbf{X}}$ with respect to $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ which gives that any Polish semigroup topology is contained in $\mathcal{T}_{rich}$; on the other hand, the rich topology needs to be sufficiently coarse that any Polish semigroup topology $\mathcal{T}$ with $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{rich}$ can be shown to indeed coincide with the pointwise topology. The latter reduction property is new and does not play a role in the previous proofs from [EJM+] and [EJM+23].

Examining our proof more closely, we in fact show a stronger result: We do not apply the full power of Polishness to prove that a Polish semigroup topology $\mathcal{T}$ on $\mathcal{M}_{\mathbb{Q}}$ coincides with the pointwise topology. Instead, we only use that $\mathcal{T}$ is a semigroup topology which is second countable, Hausdorff and regular, as well as that a countable intersection of dense open sets is dense, i.e. that $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T})$ is a *Baire space* (the conclusion of Baire's Category Theorem holds). By Urysohn's metrisation theorem (see e.g. [Wil70]), a second countable space is metrisable if and only if it is Hausdorff and regular, so we can reformulate our result in the following form: The pointwise topology is the unique second countable metrisable Baire semigroup topology on $\mathcal{M}_{\mathbb{Q}}$. Comparing with [EJM+23], most examples treated there (in particular, all examples listed in the Introduction) do not need full Polishness either; the results instead yield that the respective pointwise topologies are the unique second countable Hausdorff semigroup topologies. Hence, no regularity or completeness-type assumptions are necessary for these structures. For $\mathcal{M}_{\mathbb{Q}}$, we additionally need that the topology is regular and Baire, namely in the new reduction from the rich topology to the pointwise topology which does not occur in [EJM+23].

A notion related to UPP that has been considered over the years is *automatic homeomorphicity* as studied e.g. in [BEKP18, BPP17, BTVG17, EH90, PP16, PP18]:

Definition 4.2.1. Let $S \leq A^A$ be a sub(-semi-)group of $\mathrm{Sym}(A)$ (or A^A) which is closed with respect to the pointwise topology on $\mathrm{Sym}(A)$ (or A^A). We say that S has *automatic* *homeomorphicity* if any algebraic isomorphism from S to another closed sub(-semi-)group T of $\mathrm{Sym}(A)$ (or A^A) is indeed a homeomorphism between the respective pointwise topologies on S and T.

This property clearly is a weakening of UPP; it can be paraphrased as "unique *pointwise-like* semigroup topology". The paper [BTVG17] considered automatic homeomorphicity for transformations on the rational numbers; we quote the following:

Theorem 4.2.2 ([BTVG17, Theorems 2.6 and 4.5])**.** *Both* $\mathrm{End}(\mathbb{Q}, \leq)$ *and* $\mathrm{End}(\mathbb{Q}, <)$ *have automatic homeomorphicity.*

These results allow two interesting conclusions: On the one hand, they provide some evidence backing the conjecture that $\mathrm{End}(\mathbb{Q}, \leq)$ might have UPP. On the other hand, they show that this cannot be taken for granted – while $\mathrm{End}(\mathbb{Q}, <)$ has automatic homeomorphicity, we have already argued that it does not have UPP by an application of Example 2.6.3. Consequently, the topology exhibited via this example is a "proper" Polish topology which is not *pointwise-like*, i.e. it cannot be represented as a pointwise topology on some closed submonoid of A^A.

4.3 Proof strategy

As outlined in Section 4.2, we have to enrich the topology on $\mathcal{M}_\mathbb{Q}$ in order to make Pseudo-Property $\overline{\mathbf{X}}$ with respect to $(\mathcal{G}_\mathbb{Q}, \mathcal{T}_{pw})$ possible. To motivate, consider $s \in \mathcal{M}_\mathbb{Q}$ which is "unbounded on both sides", i.e. $\inf s = -\infty$ and $\sup s = +\infty$. Using the notation from Definition 2.7.1 and applying Remark 2.7.2, we strive for a representation of the form $e_s s = h_s^{(m+1)} a_s^{(m)} h_s^{(m)} a_s^{(m-1)} \dots a_s^{(1)} h_s^{(1)}$ with e_s left-invertible and $a_s^{(i)} \in \mathcal{G}_\mathbb{Q}$. In particular, e_s has to be unbounded on both sides as well, thus so too is the right hand side $h_s^{(m+1)} a_s^{(m)} h_s^{(m)} a_s^{(m-1)} \dots a_s^{(1)} h_s^{(1)}$ and consequently each $h_s^{(i)}$. For any $V^{(i)} \subseteq \mathcal{G}_\mathbb{Q}$, the set $h_s^{(m+1)} V^{(m)} h_s^{(m)} V^{(m-1)} \dots V^{(1)} h_s^{(1)}$ therefore only contains functions which are unbounded on both sides. Hence, any set U such that $e_s U \subseteq h_s^{(m+1)} V^{(m)} h_s^{(m)} V^{(m-1)} \dots V^{(1)} h_s^{(1)}$ must consist of such functions. Thus, in any topology on $\mathcal{M}_\mathbb{Q}$ which yields Pseudo-Property $\overline{\mathbf{X}}$, the set of all functions which are unbounded on both sides must have nonempty interior. Similar reasonings apply to the remaining kinds of "boundedness behaviour".

We define several types of subsets of $\mathcal{M}_\mathbb{Q}$:

Definition 4.3.1.

(0) $O^{(0)}_{x,y} := \{s \in \mathcal{M}_\mathbb{Q} : s(x) = y\}$; (pointwise)

$x, y \in \mathbb{Q}$

(1) $O^{(1)}_{I,J} := \{s \in \mathcal{M}_\mathbb{Q} : s(I) \subseteq J\}$; (generalised pointwise)

$I = (-\infty, p)$ and either $J = (-\infty, q]$ or $J = (-\infty, q)$ OR

$I = (p, +\infty)$ and either $J = [q, +\infty)$ or $J = (q, +\infty)$ for $p, q \in \mathbb{Q}$

(1^{cls}) $O^{(1)}_{I,J} := \{s \in \mathcal{M}_{\mathbb{Q}} : s(I) \subseteq J\}$; (generalised pointwise, closed image constraint)
$I = (-\infty, p)$ and $J = (-\infty, q]$ OR
$I = (p, +\infty)$ and $J = [q, +\infty)$ for $p, q \in \mathbb{Q}$

(2) $O^{(2)}_{LU} := \{s \in \mathcal{M}_{\mathbb{Q}} : \inf s \in L,\ \sup s \in U\}$; (boundedness types)
$L = \mathbb{R}$ or $L = \{-\infty\}$ AND $U = \mathbb{R}$ or $U = \{+\infty\}$
Explicitly, these are the following four sets:
$O^{(2)}_{\mathbb{R},\mathbb{R}} := \{s \in \mathcal{M}_{\mathbb{Q}} : \inf s \in \mathbb{R},\ \sup s \in \mathbb{R}\}$ (bounded-bounded)
$O^{(2)}_{-\infty,\mathbb{R}} := \{s \in \mathcal{M}_{\mathbb{Q}} : \inf s = -\infty,\ \sup s \in \mathbb{R}\}$ (unbounded-bounded)
$O^{(2)}_{\mathbb{R},+\infty} := \{s \in \mathcal{M}_{\mathbb{Q}} : \inf s \in \mathbb{R},\ \sup s = +\infty\}$ (bounded-unbounded)
$O^{(2)}_{-\infty,+\infty} := \{s \in \mathcal{M}_{\mathbb{Q}} : \inf s = -\infty,\ \sup s = +\infty\}$ (unbounded-unbounded)

(3) $O^{(3)}_{K} := \{s \in \mathcal{M}_{\mathbb{Q}} : \mathrm{Im}(s) \cap K = \emptyset\}$; (avoiding)
$K = [q_1, q_2]$ or $K = [q_1, q_2)$ or
$K = (q_1, q_2]$ or $K = (q_1, q_2)$ for $q_1, q_2 \in \mathbb{Q} \cup \{\pm\infty\}$, $q_1 \leq q_2$

(3^{opn}) $O^{(3)}_{K} := \{s \in \mathcal{M}_{\mathbb{Q}} : \mathrm{Im}(s) \cap K = \emptyset\}$; (avoiding, open constraint)
$K = (q_1, q_2)$ for $q_1, q_2 \in \mathbb{Q} \cup \{\pm\infty\}$, $q_1 < q_2$

We mention explicitly that the sets formed analogously to type 1 but with closed intervals I are already encompassed by type 0, i.e. the pointwise topology. For instance, if $I = (-\infty, p]$ and $J = (-\infty, q]$ or $J = (-\infty, q)$, then

$$\{s \in \mathcal{M}_{\mathbb{Q}} : s(I) \subseteq J\} = \bigcup_{y \in J} \{s \in \mathcal{M}_{\mathbb{Q}} : s(p) = y\} \in \mathcal{T}_{pw}.$$

We will make use of this fact in Section 4.5.

The types of sets defined above yield a template for constructing topologies.

Definition 4.3.2. If $M \subseteq \{0, 1, 1^{cls}, 2, 3, 3^{opn}\}$, then $\mathcal{T}_M$ is the topology generated by the sets of the types occurring in M. We further define the *rich topology* $\mathcal{T}_{rich} := \mathcal{T}_{0123}$; explicitly, this is the topology generated by

$$\begin{aligned}\mathcal{T}_{pw} &\cup \left\{O^{(1)}_{I,J} : I = (-\infty, p), J \in \{(-\infty, q], (-\infty, q)\}, p, q \in \mathbb{Q}\right\}\\ &\cup \left\{O^{(1)}_{I,J} : I = (p, +\infty), J \in \{[q, +\infty), (q, +\infty)\}, p, q \in \mathbb{Q}\right\}\\ &\cup \left\{O^{(2)}_{\mathbb{R},\mathbb{R}}, O^{(2)}_{-\infty,\mathbb{R}}, O^{(2)}_{\mathbb{R},+\infty}, O^{(2)}_{-\infty,+\infty}\right\}\\ &\cup \left\{O^{(3)}_{K} : K \in \{[q_1, q_2], [q_1, q_2), (q_1, q_2], (q_1, q_2)\}, q_1, q_2 \in \mathbb{Q} \cup \{\pm\infty\}, q_1 \leq q_2\right\}.\end{aligned}$$

If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{Q}$, it will be convenient to abbreviate $O^{(0)}_{\bar{x},\bar{y}} := \bigcap_{i=1}^{n} O^{(0)}_{x_i,y_i}$.

Remark 4.3.3. We note that $\mathcal{T}_{02}$, i.e. the topology generated by $\mathcal{T}_{pw}$ together with the boundedness types, "almost" is an alternative topology on $\mathcal{M}_{\mathbb{Q}}$ satisfying our requirements. It is immediate that $\mathcal{T}_{02}$ is a second countable Hausdorff semigroup topology. Further, it is regular since the boundedness types $O^{(2)}_{\mathbb{R},\mathbb{R}}, O^{(2)}_{-\infty,\mathbb{R}}, O^{(2)}_{\mathbb{R},+\infty}, O^{(2)}_{-\infty,+\infty}$ are clopen in $\mathcal{T}_{02}$, thus it is metrisable by Urysohn's metrisation theorem (see e.g. [Wil70]). However, the topology $\mathcal{T}_{02}$ is not *completely* metrisable, for otherwise the closed (in particular G_δ) subset $O^{(2)}_{\mathbb{R},\mathbb{R}}$ equipped with the subspace topology (which coincides with the pointwise topology) would be completely metrisable as well and thus a Baire space. As a contradiction, the sets $O_n := \left\{s \in O^{(2)}_{\mathbb{R},\mathbb{R}} : \sup s > n\right\}$, $n \in \mathbb{N}$, are dense and open in $O^{(2)}_{\mathbb{R},\mathbb{R}}$ but have empty intersection.

With this terminology, we can formulate our main technical results.

Proposition 4.3.4. *$(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{rich})$, the endomorphism monoid equipped with the rich topology, has Pseudo-Property $\overline{\mathbf{X}}$ of length 2 with respect to $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$.*

Let us note that we deem it unlikely that $\mathcal{M}_{\mathbb{Q}}$ equipped with any meaningful topology could have Pseudo-Property $\overline{\mathbf{X}}$ of length 1 (so Pseudo-Property $\mathbf{X}$) with respect to $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ since it is only the second automorphism which gives us enough flexibility and control over discontinuity points (see Definition 4.4.1 for this notion).

Proposition 4.3.5. *Let $\mathcal{T}$ be a Polish semigroup topology on the monoid $\mathcal{M}_{\mathbb{Q}}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{rich}$. Then $\mathcal{T} = \mathcal{T}_{pw}$.*

Before we get to their proofs, let us comment on how Theorem B follows from these results.

Proof (of Theorem B given Propositions 4.3.4 and 4.3.5). Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$. By Corollary 3.2.1(i), we know that the pointwise topology is the coarsest Hausdorff semigroup topology on $\mathcal{M}_{\mathbb{Q}} = \mathrm{End}(\mathbb{Q}, \leq)$, so we obtain $\mathcal{T}_{pw} \subseteq \mathcal{T}$. On the other hand, we note that $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological semigroups by Proposition 4.1.6. Combining Proposition 4.3.4 with Proposition 2.7.3(ii) yields that $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{rich})$ has automatic continuity with respect to the class of second countable topological semigroups as well. Since $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T})$ is second countable, the identity map $\mathrm{id}\colon (\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{rich}) \to (\mathcal{M}_{\mathbb{Q}}, \mathcal{T})$ is therefore continuous, in other words $\mathcal{T} \subseteq \mathcal{T}_{rich}$. By Proposition 4.3.5, we finally conclude $\mathcal{T} = \mathcal{T}_{pw}$. □

The proofs of Propositions 4.3.4 and 4.3.5 are the subject of Sections 4.4 and 4.5, respectively.

In the former section, we will find *generic* maps $e, f, g, h \in \mathcal{M}_{\mathbb{Q}}$ (with e left-invertible) so that the compositions $fahbg$ for $a, b \in \mathcal{G}_{\mathbb{Q}}$ exhaust the maps es for a great variety of $s \in \mathcal{M}_{\mathbb{Q}}$. Further, we will – roughly speaking – analyse how the compositions $fahbg$ change with varying $a, b \in \mathcal{G}_{\mathbb{Q}}$. Some of the complexity arises from the requirement that a, b be automorphisms.

The latter section has a different flavour in that we can allow maps to vary within $\mathcal{M}_{\mathbb{Q}}$, yielding less intricate constructions. Nonetheless, most (but not all) intermediate results can be reformulated as – albeit easier – Property $\overline{\mathbf{X}}$-type statements, namely with respect to the entire semigroup $\mathcal{M}_{\mathbb{Q}}$ (equipped with different topologies) instead of $\mathcal{G}_{\mathbb{Q}}$. One major exception (Proposition 4.5.20) crucially employs regularity of the topology $\mathcal{T}$ in combination with Polishness and cannot be reformulated as a (Pseudo-)Property $\overline{\mathbf{X}}$-type statement, for good reason: if the proof of Proposition 4.3.5 just consisted of a series of such statements, we could start from Proposition 4.3.4 and repeatedly apply Proposition 2.7.3 to show that $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological semigroups, contradicting Proposition 4.1.4.

4.4 The rich topology has Pseudo-Property $\overline{\mathrm{X}}$

This section is devoted to proving Proposition 4.3.4. With Remark 2.7.2 in mind, we want to find a decomposition $e_s s = f_s a_s h_s b_s g_s$ of a given $s \in \mathcal{M}_{\mathbb{Q}}$ with $e_s, f_s, g_s, h_s \in \mathcal{M}_{\mathbb{Q}}$ and $a_s, b_s \in \mathcal{G}_{\mathbb{Q}}$ as well as a $\mathcal{T}_{rich}$-neighbourhood U of s such that for any $\tilde{s} \in U$, we can similarly decompose $e_s\tilde{s} = f_s\tilde{a}h_s\tilde{b}g_s$ with $\tilde{a}, \tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ and the *same* maps e_s, f_s, g_s, h_s. Given $\mathcal{T}_{pw}$-neighbourhoods V and W of a_s and b_s, respectively, we additionally have to make sure that U can be taken small enough that for any $\tilde{s} \in U$, we can pick $\tilde{a} \in V$ and $\tilde{b} \in W$. This means that $\tilde{a}$ and a_s need to have the same behaviour on a given finite set, as do $\tilde{b}$ and b_s.

We will proceed in three steps. First, we will derive "compatibility conditions" such that $e_s s$ can be written in the form $f_s a_s \iota_s$. These conditions exhibit such a tight connection between s and ι_s that U can never force $e_s\tilde{s}$ to satisfy these conditions for all $\tilde{s} \in U$ and a fixed ι_s. In a second step, we will therefore expand ι_s in the form $\iota_s = h_s b_s g_s$ for fixed $g_s, h_s \in \mathcal{M}_{\mathbb{Q}}$ and varying $b_s \in \mathcal{G}_{\mathbb{Q}}$, yielding indeed $e_s s = f_s a_s h_s b_s g_s$. For $\tilde{s} \in U$, it turns out that we can pick $\tilde{\iota} = h_s\tilde{b}g_s$ which is compatible with $e_s\tilde{s}$ to obtain $e_s\tilde{s} = f_s\tilde{a}\tilde{\iota} = f_s\tilde{a}h_s\tilde{b}g_s$. All the while, we have to make sure that $\tilde{a}$ and a_s as well as $\tilde{b}$ and b_s coincide on given finite sets, resulting in a third major step.

4.4.1 Generic surjections, generic injections, sparse injections and basic formulas

Definition 4.4.1. Let $s \in \mathcal{M}_{\mathbb{Q}}$. We set

$$\begin{aligned} \mathrm{Cont}(s) &:= \{\gamma \in \mathbb{R} : \sup s(-\infty, \gamma) = \inf s(\gamma, +\infty)\} \\ \mathrm{Dc}(s) &:= \{\gamma \in \mathbb{R} : \sup s(-\infty, \gamma) < \inf s(\gamma, +\infty)\}, \end{aligned}$$

the sets of *continuity points* and *discontinuity points* of s, respectively. Additionally, we write $\mathrm{Dc}^{\mathbb{I}}(s) := \mathrm{Dc}(s) \cap \mathbb{I}$ for notational simplicity. Finally, we extend s to an increasing map $\bar{s} \colon \mathbb{R} \to \mathbb{R}$ by setting $\bar{s}(\gamma) := \sup s(-\infty, \gamma)$ for all $\gamma \in \mathbb{I}$.

We will frequently use the notion of limit points in the following sense:

Definition 4.4.2. Let $A \subseteq \mathbb{Q}$ and $\gamma \in \mathbb{R}$. We say that γ is a *limit point* of A if γ is contained in the closure of $A \setminus \{\gamma\}$ with respect to the standard topology on $\mathbb{R}$. The set of all limit points of A will be denoted by $\mathrm{LP}(A)$. If $s \in \mathcal{M}_{\mathbb{Q}}$, we will abbreviate $\mathrm{LP}(\mathrm{Im}(s))$ as $\mathrm{LP}(s)$ for better readability.

We collect a few easy facts:

Lemma 4.4.3. *Let $s \in \mathcal{M}_{\mathbb{Q}}$.*

(i) $\mathrm{Dc}(s)$ *is at most countable.*

(ii) If s is injective and $\mathrm{LP}(s) \subseteq \mathbb{I}$, then $\mathbb{Q} \subseteq \mathrm{Dc}(s)$ and $\bar{s}(\mathbb{I}) \subseteq \mathbb{I}$. In fact, $\sup s(-\infty, q) < s(q) < \inf s(q, +\infty)$ for all $q \in \mathbb{Q}$. Additionally, $(\mathbb{R} \setminus \mathrm{Im}(\bar{s})) \cap \mathbb{I}$ is topologically dense in $\mathbb{R}$.

(iii) If $b \in \mathcal{G}_{\mathbb{Q}}$, then $\mathrm{Dc}(b) = \emptyset$ and $\bar{b}\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection with $\bar{b}(\mathbb{I}) \subseteq \mathbb{I}$. Additionally, any increasing extension β of b to a set $M \subseteq \mathbb{R}$ coincides with $\bar{b}|_M$.

We define three kinds of *generic* maps in $\mathcal{M}_{\mathbb{Q}}$.

Definition 4.4.4.

(i) A map $f \in \mathcal{M}_{\mathbb{Q}}$ is called a *generic surjection* if it is surjective and for each $q \in \mathbb{Q}$, the preimage $f^{-1}\{q\}$ is an irrational interval, i.e. $f^{-1}\{q\} = (r_q, t_q)$ for $r_q, t_q \in \mathbb{I}$.

(ii) A map $g \in \mathcal{M}_{\mathbb{Q}}$ is called a *generic injection* if it is injective and unbounded-unbounded with $\mathrm{Dc}^{\mathbb{I}}(g) = \emptyset$ and $\mathrm{LP}(g) \subseteq \mathbb{I}$.

(iii) A map $h \in \mathcal{M}_{\mathbb{Q}}$ is called a *sparse injection* if it is injective, $\mathrm{Dc}^{\mathbb{I}}(h)$ is topologically dense in $\mathbb{R}$ and $\mathrm{LP}(h) \subseteq \mathbb{I}$.

It is an easy observation that such maps really exist.

Lemma 4.4.5.

(i) For every $A \subseteq \mathbb{Q}$, there exists a map $f \in \mathcal{M}_{\mathbb{Q}}$ with $\mathrm{Im}(f) = A$ such that the f-preimages of single elements are irrational intervals. In particular, there exists a generic surjection.

(ii) For every finite or countably infinite $A \subseteq \mathbb{I}$ and every boundedness type $O_{LU}^{(2)}$, there exists an injective map $\iota \in O_{LU}^{(2)}$ which satisfies $\mathrm{Dc}(\iota) = A \,\dot{\cup}\, \mathbb{Q}$ as well as $\mathrm{LP}(\iota) \subseteq \mathbb{I}$.

(iii) There exists a generic injection in $\mathcal{M}_{\mathbb{Q}}$.

(iv) There exists a sparse injection in $\mathcal{M}_{\mathbb{Q}}$ of any boundedness type.

Proof.
(i). We put $M := A \times \mathbb{Q}$ and set $<_M$ to be the lexicographic order on M where the first component is the significant one. Define $\pi\colon M \to \mathbb{Q}$ by $\pi(w, q) := w$. Since $(M, <_M)$ is countably infinite and densely ordered without greatest or least element, there exists an order isomorphism $\alpha\colon \mathbb{Q} \to M$. Setting $f := \pi \circ \alpha$, we obtain a map as desired.
(ii). We only consider the case $O_{LU}^{(2)} = O_{\mathbb{R},+\infty}$; the others are treated analogously. Put

$$M := (A \,\dot{\cup}\, \mathbb{Q} \,\dot{\cup}\, \{-\infty\}) \times \mathbb{Q}$$

and set $<_M$ to be the lexicographic order on M where the first component is the significant one. Define $j\colon \mathbb{Q} \to M$ by $j(x) := (x, 0)$. Since $(M, <_M)$ is countably infinite and densely ordered without greatest or least element, there exists an order isomorphism $\beta\colon M \to \mathbb{Q}$. Setting $\iota := \beta \circ j \in \mathcal{M}_{\mathbb{Q}}$, we obtain a map as desired.

(iii). Setting $A = \emptyset$ as well as $O^{(2)}_{LU} = O^{(2)}_{-\infty,+\infty}$ and using (ii), we obtain a generic injection.

(iv). Setting $A \subseteq \mathbb{I}$ to be a countably infinite topologically dense set and using (ii), we obtain a sparse injection with any boundedness type. □

Another useful notion is given by the *generalised inverse* of maps in $\mathcal{M}_{\mathbb{Q}}$.

Definition 4.4.6. Let $s \in \mathcal{M}_{\mathbb{Q}}$ and $y \in \mathbb{Q}$. Define[1] $s^L(y) := \sup s^{-1}(-\infty, y) \in \mathbb{R} \cup \{\pm\infty\}$ and $s^R(y) := \inf s^{-1}(y, +\infty) \in \mathbb{R} \cup \{\pm\infty\}$. If $s^L(y)$ and $s^R(y)$ coincide, we define $s^\dagger(y) := s^L(y) = s^R(y)$ (the *generalised inverse* of s at y).

The following observations are easily deduced directly from the definitions.

Lemma 4.4.7.

(i) Let $s \in \mathcal{M}_{\mathbb{Q}}$. If $y \in \operatorname{Im}(s)$ and if $x \in \mathbb{Q}$ is the only s-preimage of y, then $s^\dagger(y) = x$.

(ii) Let $s \in \mathcal{M}_{\mathbb{Q}}$ be injective. Then $s^\dagger(y)$ is a welldefined real number for all elements $y \in (\inf s, \sup s)$.

(iii) Let $g \in \mathcal{M}_{\mathbb{Q}}$ be a generic injection. Then $g^\dagger(y) \in \mathbb{Q}$ for all $y \in \mathbb{Q}$. In particular, g is left-invertible in $\mathcal{M}_{\mathbb{Q}}$ with left inverse $g^\dagger$. Moreover, for all rational intervals J with boundary points q_1 and q_2 in $\mathbb{Q} \cup \{\pm\infty\}$, the preimage $g^{-1}(J)$ is again a rational interval with boundary points[2] $g^\dagger(q_1)$ and $g^\dagger(q_2)$.

(iv) Let $g \in \mathcal{M}_{\mathbb{Q}}$ be a generic injection. Then the translation $\lambda_g\colon s \mapsto gs$ is continuous as a map[3] $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{rich}) \to (\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{rich})$.

(combine the $\mathcal{T}_{pw}$-continuity of λ_g with (iii) and the fact that s and gs have the same boundedness type since g is unbounded-unbounded)

As mentioned in the introduction of Section 4.4, we will derive compatibility conditions such that $e_s s = f_s a_s \iota_s$ for maps $s, e_s, f_s, \iota_s \in \mathcal{M}_{\mathbb{Q}}$ and $a_s \in \mathcal{G}_{\mathbb{Q}}$. It will be convenient to consider a slightly more general situation and aim for $\sigma = \pi a \iota$; this will then be applied for $\pi = f_s$ and $\sigma := e_s s$, and later on for $\tilde{\sigma} := e_s \tilde{s}$. Again by the introductory remarks, we will need to make sure that the function a maps $\bar{x} \mapsto \bar{y}$ for given tuples $\bar{x}, \bar{y}$.

First, we reformulate our problem in model-theoretic language. As a starting point, note that $\sigma = \pi a \iota$ is equivalent to the fact that $a(\iota(q)) \in \pi^{-1}\{\sigma(q)\}$ for all $q \in \mathbb{Q}$. Since π is increasing, the preimage $\pi^{-1}\{\sigma(q)\}$ is an interval. Thus, if $\sigma(q) = \sigma(q')$ for some $q, q' \in \mathbb{Q}$, then not only do $\iota(q)$ and $\iota(q')$ have to be mapped to the same interval, but all points *between* $\iota(q)$ and $\iota(q')$ have to be as well. This motivates the following definition:

Definition 4.4.8.

[1] We put $\sup \emptyset := -\infty$ and $\inf \emptyset := +\infty$.

[2] Note, however, that e.g. $g^{-1}[q_1, q_2]$ need not be closed.

[3] Since $\mathcal{T}_{rich}$ is not a semigroup topology – a fact on which most of Section 4.5 hinges – this cannot be taken for granted and depends on the genericity of g.

(i) Let $\{P_q : q \in \mathbb{Q}\}$ be a set of unary relation symbols and define the language L by[4] $L := \{<\} \cup \{P_q : q \in \mathbb{Q}\}$.

(ii) Let $\sigma, \pi, \iota \in \mathcal{M}_\mathbb{Q}$. For $q \in \mathbb{Q}$, we set

$$P_q^\mathbb{A} := \text{convex hull of } \iota(\sigma^{-1}\{\sigma(q)\})$$
$$P_q^\mathbb{B} := \pi^{-1}\{\sigma(q)\}$$

and define L-structures $\mathbb{A} = \langle \mathbb{Q}, <, (P_q^\mathbb{A})_{q\in\mathbb{Q}} \rangle$ and $\mathbb{B} = \langle \mathbb{Q}, <, (P_q^\mathbb{B})_{q\in\mathbb{Q}} \rangle$. If σ, π, ι are not clear from the context, we will write $\mathbb{A}(\sigma, \pi, \iota)$ and $\mathbb{B}(\sigma, \pi, \iota)$.

In the sequel, L, $\mathbb{A}$ and $\mathbb{B}$ will always denote the objects just defined. Note that a surjective L-homomorphism $a \colon \mathbb{A} \to \mathbb{B}$ is automatically contained in $\mathcal{G}_\mathbb{Q}$ and satisfies $\sigma = \pi a \iota$. Thus, our aim is to construct a surjective L-homomorphism extending a given map $\bar{x} \mapsto \bar{y}$. We will do so using the Back&Forth method, see Subsection 2.8.

Definition 4.4.9. A formula $\psi(\bar{z})$ over L is called *basic* if it is one of the formulas

(i) $P_q(z_i)$, $\quad q \in \mathbb{Q}$

(ii) $z_i < z_j$

(iii) $L_q(z_i) :\leftrightarrow \exists u \colon u < z_i \wedge P_q(u)$, $\quad q \in \mathbb{Q}$

(iv) $R_q(z_i) :\leftrightarrow \exists u \colon u > z_i \wedge P_q(u)$, $\quad q \in \mathbb{Q}$

For a basic formula $\psi(\bar{z})$ and a tuple $\bar{x}$ in $\mathbb{A}$, we write $\mathbb{A} \models \psi(\bar{x})$ if $\bar{x}$ satisfies the formula $\psi(\bar{z})$ in $\mathbb{A}$; we analogously define $\mathbb{B} \models \psi(\bar{y})$. If m is a (potentially partial) map from $\mathbb{A}$ to $\mathbb{B}$, then m is said to *preserve* $\psi(\bar{z})$ if $\mathbb{A} \models \psi(\bar{x})$ implies $\mathbb{B} \models \psi(m(\bar{x}))$ for all tuples $\bar{x}$ in the domain of m.

Note that basic formulas contain only existential and no universal quantifiers, so total homomorphisms $\mathbb{A} \to \mathbb{B}$ always preserve all basic formulas. In the following, we will work with partial maps from $\mathbb{A}$ to $\mathbb{B}$ preserving all basic formulas, either extending maps without losing that property or analysing when a given map indeed preserves all basic formulas.

4.4.2 Proving Proposition 4.3.4

The crucial technical results necessary for the proof of Proposition 4.3.4 are three lemmas, one for each of the steps mentioned in the introduction of Section 4.4: the Sandwich Lemma 4.4.11, the Preconditioning Lemma 4.4.12 and the Variation Lemma 4.4.13. In this subsection, we formulate them and demonstrate how they are used to show Proposition 4.3.4. For proofs of the three lemmas, we refer to the following subsections. We start by fixing some notation for the sake of brevity:

Definition 4.4.10. We say that $\sigma, \pi, \iota \in \mathcal{M}_\mathbb{Q}$ are *compatible* if

(a) $\sigma \in \mathcal{M}_\mathbb{Q}$ satisfies $\mathrm{LP}(\sigma) \subseteq \mathbb{I}$,

[4] Note: $<$ instead of $\leq$!

(b) $\pi \in \mathcal{M}_{\mathbb{Q}}$ is a generic surjection,

(c) $\iota \in \mathcal{M}_{\mathbb{Q}}$ is injective with $\mathrm{LP}(\iota) \subseteq \mathbb{I}$, has the same boundedness type as σ and satisfies $\mathrm{Dc}^{\mathbb{I}}(\iota) = \mathrm{Dc}^{\mathbb{I}}(\sigma)$.

Lemma 4.4.11 (Sandwich Lemma). *Let $\sigma, \pi, \iota \in \mathcal{M}_{\mathbb{Q}}$ be* compatible.
Then the following statements hold:

(i) The set of all finite partial L-homomorphisms m from $\mathbb{A}$ to $\mathbb{B}$ preserving all basic formulas is a Back&Forth system.

(ii) There exists $a \in \mathcal{G}_{\mathbb{Q}}$ such that $\sigma = \pi a \iota$. Indeed, if m is a finite partial L-homomorphism from $\mathbb{A}$ to $\mathbb{B}$ preserving all basic formulas, there exists $a \in \mathcal{G}_{\mathbb{Q}}$ extending m such that $\sigma = \pi a \iota$.

Referring back to the overview presented in the introduction of Section 4.4, we can now precisely state why our approach requires aiming for Pseudo-Property $\overline{\mathbf{X}}$ of length 2: to apply the Sandwich Lemma 4.4.11, we need that $\sigma = e_s s$ and ι have the same irrational discontinuity points, so the irrational discontinuity points of s and ι need to be closely connected. Since no $\mathcal{T}_{rich}$-neighbourhood U can encode $\mathrm{Dc}^{\mathbb{I}}(s)$, we cannot use a fixed map ι for all $\tilde{s}$ in U. Thus, we need to adapt ι to $\tilde{s}$. We will write $\iota = hbg$, where b varies in $\mathcal{G}_{\mathbb{Q}}$ and $g, h \in \mathcal{M}_{\mathbb{Q}}$ are fixed elements. As it will turn out, it is crucial that we are very free in stipulating finite pointwise behaviour not only of b on $\mathbb{Q}$ but also of the extension $\bar{b}$ on $\mathbb{I}$.

Lemma 4.4.12 (Preconditioning Lemma). *Let $g \in \mathcal{M}_{\mathbb{Q}}$ be a generic injection, let $h \in \mathcal{M}_{\mathbb{Q}}$ be a sparse injection and let $A \subseteq \mathbb{I}$ be finite or countably infinite*[5]*. Then there exists $b \in \mathcal{G}_{\mathbb{Q}}$ such that $\iota := hbg$ satisfies $\mathrm{Dc}^{\mathbb{I}}(\iota) = A$ as well as $\mathrm{LP}(\iota) \subseteq \mathbb{I}$, namely any $b \in \mathcal{G}_{\mathbb{Q}}$ with $\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}) = \bar{g}(A)$. The boundedness type of ι coincides with the boundedness type of h.*

Moreover, suppose that $\bar{z}$ and $\bar{w}$ are tuples in $\mathbb{Q}$, that $\bar{z}'$ and $\bar{w}'$ are tuples in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, and that $\bar{z}''$ and $\bar{w}''$ are tuples in $\bar{g}(A)$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively. If the partial map sending $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\bar{z}'' \mapsto \bar{w}''$ is strictly increasing, then $b \in \mathcal{G}_{\mathbb{Q}}$ can be picked so that $\bar{b}$ extends this map.

Combining the Preconditioning Lemma 4.4.12 (putting $A = \mathrm{Dc}^{\mathbb{I}}(e_s s)$, see the proof of Proposition 4.3.4 below) with the Sandwich Lemma 4.4.11, we can show that $\sigma := e_s s$ can be written in the form $\pi a_s \iota = f_s a_s h_s b_s g_s$ with $a_s, b_s \in \mathcal{G}_{\mathbb{Q}}$ if $\pi = f_s$ is a generic surjection, g_s and e_s are generic injections and h_s is a sparse injection with the same boundedness type as s – note in particular that the choice of the maps e_s, f_s, g_s, h_s only depends on the boundedness type of s. For the remaining part of Pseudo-Property $\overline{\mathbf{X}}$, we have to prove the following: If $a_s(\bar{x}) = \bar{y}$ as well as $b_s(\bar{z}) = \bar{w}$, there is a $\mathcal{T}_{rich}$-neighbourhood U of s such that for all $\tilde{s} \in U$ one can write $e_s \tilde{s} = f_s \tilde{a} h_s \tilde{b} g_s$, where $\tilde{a}, \tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ with $\tilde{a}(\bar{x}) = \bar{y}$ as well as $\tilde{b}(\bar{z}) = \bar{w}$. By the Preconditioning Lemma 4.4.12, we could find $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ with $\tilde{b}(\bar{z}) = \bar{w}$ such that $\tilde{\sigma} := e_s \tilde{s}, f_s, \tilde{\iota} := h_s \tilde{b} g_s$ are compatible. Thus, the Sandwich Lemma 4.4.11 would yield $\tilde{a} \in \mathcal{G}_{\mathbb{Q}}$ with $e_s \tilde{s} = f_s \tilde{a} h_s \tilde{b} g_s$ – however, this automorphism $\tilde{a}$ need not satisfy the condition

[5] When applying this lemma, we will put either $A = \mathrm{Dc}(e_s s)$ or $A = \mathrm{Dc}(e_s \tilde{s})$.

$\tilde{a}(\bar{x}) = \bar{y}$. To improve upon this strategy, the final statement of the Sandwich Lemma 4.4.11 suggests we construct $\tilde{b}$ in such a way that the finite partial map defined by $\bar{x} \mapsto \bar{y}$ preserves all basic formulas when considered as a map from $\mathbb{A}(\tilde{\sigma}, f_s, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f_s, \tilde{\iota})$.

Lemma 4.4.13 (Variation Lemma). *Let $\sigma, f, g, h \in \mathcal{M}_{\mathbb{Q}}$ and $a, b \in \mathcal{G}_{\mathbb{Q}}$ such that $\sigma = fahbg$, where $\mathrm{LP}(\sigma) \subseteq \mathbb{I}$, f is a generic surjection, g is a generic injection, h is a sparse injection with the same boundedness type as σ, and finally $\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}) = \bar{g}(\mathrm{Dc}^{\mathbb{I}}(\sigma))$. Let further $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ be tuples in $\mathbb{Q}$ such that $a(\bar{x}) = \bar{y}$ and $b(\bar{z}) = \bar{w}$.*
Then there exists a $\mathcal{T}_{rich}$-neighbourhood O of σ such that the following holds:

> *For any $\tilde{\sigma} \in O$ with $\mathrm{LP}(\tilde{\sigma}) \subseteq \mathbb{I}$, there exist tuples $\bar{z}^*$ and $\bar{w}^*$ in $\mathbb{Q}$ and tuples $\bar{z}'$ and $\bar{w}'$ in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, and tuples $\bar{z}''$ and $\bar{w}''$ in $\bar{g}(\mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma}))$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, such that*
>
> - *the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}^* \mapsto \bar{w}^*$, $\bar{z}' \mapsto \bar{w}'$, $\bar{z}'' \mapsto \bar{w}''$ is strictly increasing,*
> - *if $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ satisfies $\tilde{b}(\bar{z}) = \bar{w}$, $\tilde{b}(\bar{z}^*) = \bar{w}^*$, $\bar{\tilde{b}}(\bar{z}') = \bar{w}'$, $\bar{\tilde{b}}(\bar{z}'') = \bar{w}''$ and is such that $\tilde{\sigma}, f, \tilde{\iota} := h\tilde{b}g$ are compatible, then $\bar{x} \mapsto \bar{y}$ preserves all basic formulas when considered as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$.*

Combining these results, we can prove that $\mathcal{M}_{\mathbb{Q}}$ equipped with the rich topology has Pseudo-Property $\overline{\mathbf{X}}$ of length 2 with respect to $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$:

Proof (of Proposition 4.3.4 given Lemmas 4.4.11, 4.4.12 and 4.4.13). Let $s \in \mathcal{M}_{\mathbb{Q}}$. We follow the strategy outlined in Figure 4.1. First, we construct a decomposition $e_s s = f_s a_s h_s b_s g_s$.

We use Lemma 4.4.5 to find a generic injection $e_s \in \mathcal{M}_{\mathbb{Q}}$, a generic surjection $f_s \in \mathcal{M}_{\mathbb{Q}}$, a generic injection $g_s \in \mathcal{M}_{\mathbb{Q}}$ and a sparse injection $h_s \in \mathcal{M}_{\mathbb{Q}}$ with the same boundedness type as s. By Lemma 4.4.7(iii), the map e_s is left-invertible. Since e_s is unbounded-unbounded, $\sigma := e_s s$ has the same boundedness type as s (and as h_s) and satisfies $\mathrm{LP}(\sigma) \subseteq \mathrm{LP}(e_s) \subseteq \mathbb{I}$. Applying the Preconditioning Lemma 4.4.12 with $A = \mathrm{Dc}^{\mathbb{I}}(\sigma)$, we obtain $b_s \in \mathcal{G}_{\mathbb{Q}}$ such that $\iota_s := h_s b_s g_s$ is compatible with σ and f_s, namely $b_s \in \mathcal{G}_{\mathbb{Q}}$ with $\bar{b}_s^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}_s) = \bar{g}_s(\mathrm{Dc}^{\mathbb{I}}(\sigma))$. Using the Sandwich Lemma 4.4.11, we obtain $a_s \in \mathcal{G}_{\mathbb{Q}}$ such that

$$e_s s = \sigma = f_s a_s \iota_s = f_s a_s h_s b_s g_s.$$

This proves conditions (i) and (ii) in the definition of Pseudo-Property $\overline{\mathbf{X}}$.

For condition (iii), let $V, W \subseteq \mathcal{G}_{\mathbb{Q}}$ be open sets in the pointwise topology on $\mathcal{G}_{\mathbb{Q}}$ with $a_s \in V$ and $b_s \in W$. We need to find $U \in \mathcal{T}_{rich}$ with $s \in U$ such that $e_s U \subseteq f_s V h_s W g_s$. By shrinking the sets if necessary, we can assume that $V = \{\tilde{a} \in \mathcal{G}_{\mathbb{Q}} : \tilde{a}(\bar{x}) = \bar{y}\}$ and $W = \left\{\tilde{b} \in \mathcal{G}_{\mathbb{Q}} : \tilde{b}(\bar{z}) = \bar{w}\right\}$ for tuples $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ in $\mathbb{Q}$. We apply the Variation Lemma 4.4.13 for $\sigma = f_s a_s h_s b_s g_s$ to obtain a $\mathcal{T}_{rich}$-neighbourhood O of σ with the following property: If $\tilde{s} \in \mathcal{M}_{\mathbb{Q}}$ is such that $\tilde{\sigma} := e_s \tilde{s} \in O$, there exist tuples $\bar{z}^*$ and $\bar{w}^*$ in $\mathbb{Q}$ and tuples $\bar{z}'$ and $\bar{w}'$ in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, and tuples $\bar{z}''$ and $\bar{w}''$ in $\bar{g}(\mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma}))$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, such that $\bar{z} \mapsto \bar{w}$, $\bar{z}^* \mapsto \bar{w}^*$, $\bar{z}' \mapsto \bar{w}'$, $\bar{z}'' \mapsto \bar{w}''$ is strictly increasing and, additionally, if $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ satisfies $\tilde{b}(\bar{z}) = \bar{w}$, $\tilde{b}(\bar{z}^*) = \bar{w}^*$, $\bar{\tilde{b}}(\bar{z}') = \bar{w}'$, $\bar{\tilde{b}}(\bar{z}'') = \bar{w}''$ and is such that $\tilde{\sigma}, f, \tilde{\iota} := h\tilde{b}g$ are compatible, then $\bar{x} \mapsto \bar{y}$ preserves all basic formulas when considered as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f_s, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f_s, \tilde{\iota})$. Given such $\tilde{s} \in \mathcal{M}_{\mathbb{Q}}$, the Preconditioning Lemma 4.4.12 with $A = \mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma})$ as well as $\bar{z} \cup \bar{z}^*$ and $\bar{w} \cup \bar{w}^*$ in place of $\bar{z}$ and $\bar{w}$, respectively, yields $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ with the above properties. Hence, $\bar{x} \mapsto \bar{y}$ preserves all basic formulas when considered as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f_s, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f_s, \tilde{\iota})$, and the Sandwich Lemma 4.4.11 gives $\tilde{a} \in \mathcal{G}_{\mathbb{Q}}$ with $\tilde{a}(\bar{x}) = \bar{y}$ and $e_s\tilde{s} = f_s\tilde{a}\tilde{\iota} = f_s\tilde{a}h_s\tilde{b}g_s \in f_sVh_sWg_s$.

In other words, setting $U := \lambda_{e_s}^{-1}(O)$ gives $e_sU \subseteq f_sVg_sWh_s$ as desired. Noting that U is a $\mathcal{T}_{rich}$-neighbourhood of s by Lemma 4.4.7(iv) finishes the proof. □

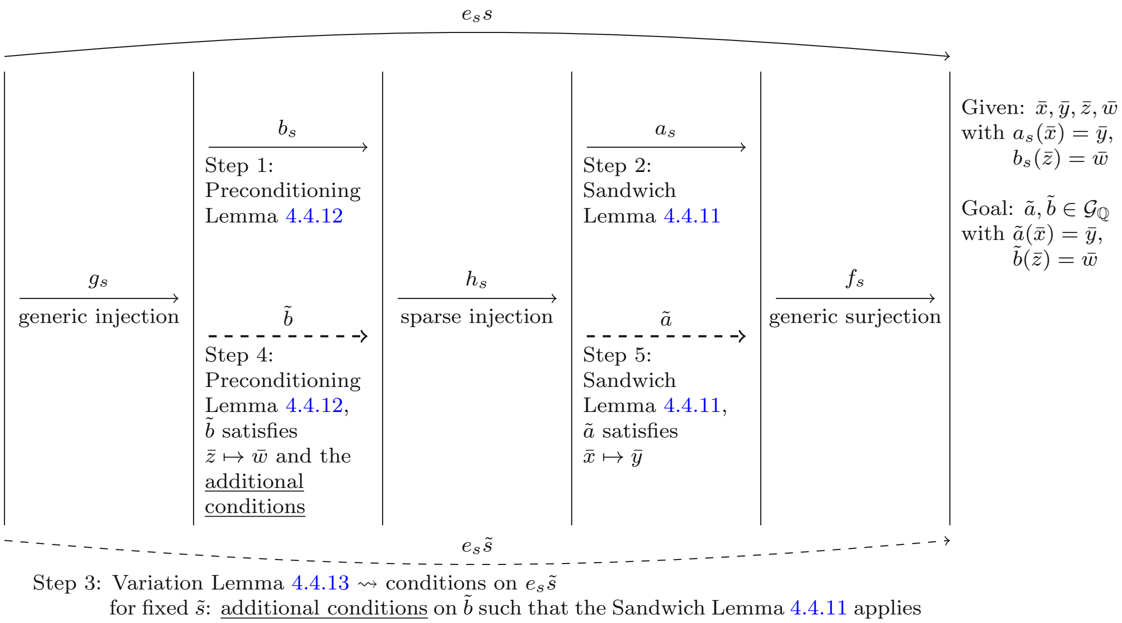

Figure 4.1: Illustration of the proof of Proposition 4.3.4.

4.4.3 Proving the Sandwich Lemma 4.4.11

The proof of the Sandwich Lemma 4.4.11 requires two additional auxiliary facts.

Since π is a generic surjection, the preimages $\pi^{-1}\{z\}$ have neither a greatest nor a least element. This implies the following simple yet crucial interpretation of the formulas $P_q(z)$, $L_q(z)$ and $R_q(z)$ in $\mathbb{B}$:

Lemma 4.4.14. *Let $\pi \in \mathcal{M}_{\mathbb{Q}}$ be a generic surjection. Then the following holds for all $q, y \in \mathbb{Q}$:*

(i) $\mathbb{B} \models P_q(y)$ *if and only if* $\sigma(q) = \pi(y)$.

(ii) $\mathbb{B} \models L_q(y)$ *if and only if* $\sigma(q) \leq \pi(y)$.

(iii) $\mathbb{B} \models R_q(y)$ *if and only if* $\sigma(q) \geq \pi(y)$.

In particular, $\mathbb{B} \models P_q(y)$ *implies* $\mathbb{B} \models L_q(y)$ *as well as* $\mathbb{B} \models R_q(y)$*.*

The following straightforward lemma intuitively means that our definition of $P_q^{\mathbb{A}}$ is the "correct" one:

Lemma 4.4.15. *Let* $\sigma, \pi, \iota \in \mathcal{M}_{\mathbb{Q}}$*. For all* $q, q' \in \mathbb{Q}$*, we have*

$$P_q^{\mathbb{A}} \cap P_{q'}^{\mathbb{A}} \neq \emptyset \Leftrightarrow P_q^{\mathbb{A}} = P_{q'}^{\mathbb{A}} \Leftrightarrow \sigma(q) = \sigma(q') \Leftrightarrow P_q^{\mathbb{B}} = P_{q'}^{\mathbb{B}} \Leftrightarrow P_q^{\mathbb{B}} \cap P_{q'}^{\mathbb{B}} \neq \emptyset.$$

Now we can prove the Sandwich Lemma 4.4.11:

Proof (of the Sandwich Lemma 4.4.11). Since the second statement follows by combining the first statement with Lemma 2.8.2 to obtain a surjective L-homomorphism $a\colon \mathbb{A} \to \mathbb{B}$ extending m, we only have to show (i). We will verify that the set of all finite partial L-homomorphisms m from $\mathbb{A}$ to $\mathbb{B}$ preserving all basic formulas has the Forth property and the Back property. Let m be such a homomorphism.

Forth. Given $x \in \mathbb{A} \setminus \mathrm{Dom}(m)$, we need to find $y \in \mathbb{B} \setminus \mathrm{Im}(m)$ such that the extension m' of m by $x \mapsto y$ is a finite partial L-homomorphism preserving all basic formulas. We will use the following general strategy: We first identify the desired position of y with respect to the predicates P_q, L_q, R_q, and then employ the fact that m preserves all basic formulas to find y such that m and $x \mapsto y$ are additionally order-compatible.

Let $\bar{a} = (a_1, \ldots, a_n)$ be an ascending enumeration of $\mathrm{Dom}(m)$ and let $\bar{b} := m(\bar{a})$. Since m is strictly increasing, $\bar{b}$ is an ascending enumeration of $\mathrm{Im}(m)$. Setting $a_0 := -\infty$ and $a_{n+1} := +\infty$ as well as $b_0 := -\infty$ and $b_{n+1} := +\infty$, there exists an index $i_0 \in \{0, \ldots, n\}$ such that $a_{i_0} < x < a_{i_0+1}$. We distinguish two cases:

Case 1 ($\exists q_0 \in \mathbb{Q}\colon x \in P_{q_0}^{\mathbb{A}}$): Since π is a generic surjection (property (b) of compatibility), it suffices to find y with

$$\sigma(q_0) = \pi(y) \text{ and } b_{i_0} < y < b_{i_0+1};$$

note that even though we do not know whether x satisfies L_{q_0} and R_{q_0} in $\mathbb{A}$, the element y certainly satisfies L_{q_0} and R_{q_0} in $\mathbb{B}$, see Lemma 4.4.14. Applying that m preserves R_{q_0} and L_{q_0}, one obtains $\pi(b_{i_0}) \leq \sigma(q_0) \leq \pi(b_{i_0+1})$ via Lemma 4.4.14 which yields the existence of y with the desired properties (by property (b) of compatibility, the preimage $\pi^{-1}\{\sigma(q_0)\}$ does not have a greatest or least element).

Case 2 ($\nexists q \in \mathbb{Q}\colon x \in P_q^{\mathbb{A}}$): In this case, we have $J_- := \{q \in \mathbb{Q} : \mathbb{A} \models L_q(x)\} = \iota^{-1}(-\infty, x)$ as well as $J_+ := \{q \in \mathbb{Q} : \mathbb{A} \models R_q(x)\} = \iota^{-1}(x, +\infty)$, and further $\mathbb{Q} = J_- \dot\cup J_+$ where the common boundary point of J_- and J_+ is $\iota^\dagger(x)$; note that $J_\pm$ could be empty, in which case $\iota^\dagger(x) = \pm\infty$. Similarly to Case 1, it suffices to find y with (for $J_- = \emptyset$, we put $\sup \sigma(J_-) = -\infty$; analogously for $J_+ = \emptyset$)

$$\sup \sigma(J_-) \leq \pi(y) \leq \inf \sigma(J_+) \text{ and } b_{i_0} < y < b_{i_0+1}.$$

This is accomplished by verifying

$$\sup \sigma(J_-) < \inf \sigma(J_+) \tag{4.1}$$
$$\pi(b_{i_0}) \leq \inf \sigma(J_+) \tag{4.2}$$
$$\sup \sigma(J_-) \leq \pi(b_{i_0+1}) \tag{4.3}$$
$$\exists u_0 \in \mathbb{Q}\colon \max(\sup \sigma(J_-), \pi(b_{i_0})) \leq u_0 \leq \min(\inf \sigma(J_+), \pi(b_{i_0+1})). \tag{4.4}$$

If u_0 is as in (4.4), there exists $y \in \pi^{-1}\{u_0\}$ with $b_{i_0} < y < b_{i_0+1}$. Any such y has the desired properties.

By our assumption for the current case, the element x is in a "gap" of $\mathbb{A}$; the inequality (4.1) expresses that there exists a matching "gap" of $\mathbb{B}$. To verify, one distinguishes by $\iota^\dagger(x)$ and applies convenient parts of the properties (a) and (c) of compatibility: If $\iota^\dagger(x) = -\infty$, i.e. $J_- = \emptyset$ and $J_+ = \mathbb{Q}$, then ι is bounded below, so σ is bounded below by property (c) of compatibility, yielding (4.1). For $\iota^\dagger(x) = +\infty$, one argues analogously. If J_- has a greatest element q, observe that $\sigma(J_+)$ consists of elements strictly greater than $\sigma(q)$. Use $\mathrm{LP}(\sigma) \subseteq \mathbb{I}$ (property (a) of compatibility) combined with Lemma 4.4.15 to see that $\inf \sigma(J_+)$ is either contained in $\sigma(J_+)$ or irrational. Conclude $\sigma(q) < \inf \sigma(J_+)$ which yields (4.1). If J_+ has a least element, one argues analogously. It remains to consider the case that $\iota^\dagger(x) \in \mathbb{R}$ and neither J_- nor J_+ has a greatest or least element, respectively. Then $\iota^\dagger(x) \in \mathbb{I}$ and, since $x \notin \mathrm{LP}(\iota)$ by property (c) of compatibility, also $\iota^\dagger(x) \in \mathrm{Dc}(\iota)$. Another application of property (c) of compatibility yields $\iota^\dagger(x) \in \mathrm{Dc}(\sigma)$ and thus (4.1).

The inequalities (4.2) and (4.3) are clear since m preserves all basic formulas, the inequality (4.4) is immediate from the previous ones.

Back. Given $y \in \mathbb{B} \setminus \mathrm{Im}(m)$, we need to find $x \in \mathbb{A} \setminus \mathrm{Dom}(m)$ such that the extension m' of m by $x \mapsto y$ is a finite partial L-homomorphism preserving all basic formulas. We proceed similarly to the Forth step.

As before, let $\bar{a} = (a_1, \dots, a_n)$ be an ascending enumeration of $\mathrm{Dom}(m)$ and let $\bar{b} := m(\bar{a})$ be the corresponding ascending enumeration of $\mathrm{Im}(m)$. We again set $a_0 := -\infty$ and $a_{n+1} := +\infty$ as well as $b_0 := -\infty$ and $b_{n+1} := +\infty$, and define the index $i_0 \in \{0, \dots, n\}$ such that $b_{i_0} < y < b_{i_0+1}$. We further set $I_- := \sigma^{-1}(-\infty, \pi(y))$, $I := \sigma^{-1}\{\pi(y)\}$ and $I_+ := \sigma^{-1}(\pi(y), +\infty)$. If x satisfies

$$\sup \iota(I_-) < x < \inf \iota(I_+) \text{ and } a_{i_0} < x < a_{i_0+1},$$

then m' extending m by $x \mapsto y$ preserves all basic formulas since $\sup_{q \in I_-} P_q^{\mathbb{A}} = \sup \iota(I_-)$ by Lemma 4.4.15 (and analogously for I_+). If $I \neq \emptyset$, note that even though we cannot predict whether x will be contained in $P_q^{\mathbb{A}}$ or will be below or above $P_q^{\mathbb{A}}$ for one (and thus for all) $q \in I$, the element y satisfies P_q as well as L_q, R_q in $\mathbb{B}$.

One finds the desired element x by verifying

$$\sup \iota(I_-) < \inf \iota(I_+) \tag{4.5}$$

$$a_{i_0} < \inf \iota(I_+) \tag{4.6}$$

$$\sup \iota(I_-) < a_{i_0+1} \tag{4.7}$$

and picking any x with $\max(\sup \iota(I_-), a_{i_0}) < x < \min(\inf \iota(I_+), a_{i_0+1})$. Using the properties (a) and (c) of compatibility, the inequality (4.5) follows just as (4.1) did in the Forth step. For the inequality (4.6), observe that $a_{i_0} < \iota(q)$ for all $q \in I_+$ (for otherwise, m would not preserve P_q and R_q) and that $\inf \iota(I_+)$ is either contained in $\iota(I_+)$ or irrational. The same argument yields the inequality (4.7). □

4.4.4 Proving the Preconditioning Lemma 4.4.12

To find b, we use a Back&Forth strategy.

Proof (of the Preconditioning Lemma 4.4.12). Combining the facts that g is continuous at all irrational points (by definition), that $\bar{g}(\mathbb{I}) \subseteq \mathbb{I}$ (by Lemma 4.4.3(ii)) and that any $\bar{b}$ for $b \in \mathcal{G}_{\mathbb{Q}}$ is continuous at all irrational points (by Lemma 4.4.3(iii)), we obtain that bg will always be continuous at all irrational points as well. Thus,

$$\mathrm{Dc}^{\mathbb{I}}(hbg) = \bar{g}^{-1}\Big(\bar{b}^{-1}(\mathrm{Dc}(h)) \cap \bar{g}(\mathbb{I})\Big) \cap \mathbb{I}.$$

If we use that $\bar{g}(\mathbb{I}), \bar{b}(\mathbb{I}) \subseteq \mathbb{I}$, we conclude

$$\mathrm{Dc}^{\mathbb{I}}(hbg) = \bar{g}^{-1}\left(\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g})\right) \cap \mathbb{I}.$$

Hence, it is sufficient to construct b in such a way that

$$\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}) = \bar{g}(A); \tag{4.8}$$

if we set $\iota := hbg$, then $\mathrm{LP}(\iota) \subseteq \mathrm{LP}(h) \subseteq \mathbb{I}$ and ι has the same boundedness type as h since both g and b are unbounded-unbounded.

To fulfil (4.8), note first that there exists a countable set $D \subseteq (\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ which is topologically dense in $\mathbb{R}$: by Lemma 4.4.3(ii), the set $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ is topologically dense in $\mathbb{R}$, so it suffices to pick D to be topologically dense in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ (which is possible since the latter is a subset of a separable metric space and therefore separable itself). In doing so, we can make sure that D contains all entries of $\bar{z}'$.

Instead of directly constructing a map $b\colon \mathbb{Q} \to \mathbb{Q}$ which satisfies (4.8), we will find an order isomorphism $\beta\colon \mathbb{Q} \,\dot{\cup}\, (\bar{g}(A) \cup D) \to \mathbb{Q} \,\dot{\cup}\, \mathrm{Dc}^{\mathbb{I}}(h)$ satisfying

$$\begin{aligned} \beta(\mathbb{Q}) &= \mathbb{Q} \quad \text{and} \quad \beta(\bar{g}(A) \cup D) = \mathrm{Dc}^{\mathbb{I}}(h) \qquad \text{as well as} \\ \beta(\bar{z}) &= \bar{w} \quad \text{and} \quad \beta(\bar{z}') = \bar{w}', \beta(\bar{z}'') = \bar{w}'' \end{aligned}$$

Setting $b := \beta|_{\mathbb{Q}}$ then yields the map as in (4.8) since $\bar{b}|_{\mathbb{Q}\dot{\cup}(\bar{g}(A)\cup D)} = \beta$ by uniqueness of the increasing extension (see Lemma 4.4.3(iii)) and therefore

$$\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}) = (\bar{g}(A) \cup D) \cap \mathrm{Im}(\bar{g}) = \bar{g}(A).$$

To obtain β, we show that the system $\mathcal{S}$ of all finite partial order isomorphisms m from $\mathbb{Q} \,\dot{\cup}\, (\bar{g}(A) \cup D)$ to $\mathbb{Q} \,\dot{\cup}\, \mathrm{Dc}^{\mathbb{I}}(h)$ such that

$$m\big(\mathbb{Q} \cap \mathrm{Dom}(m)\big) = \mathbb{Q} \cap \mathrm{Im}(m) \quad \text{and} \quad m\big((\bar{g}(A) \cup D) \cap \mathrm{Dom}(m)\big) = \mathrm{Dc}^{\mathbb{I}}(h) \cap \mathrm{Im}(m)$$

is a Back&Forth system – by Lemma 2.8.2, the finite partial order isomorphism defined by $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\bar{z}'' \mapsto \bar{w}''$ (which is a member of $\mathcal{S}$) can then be extended to a map β with the desired properties. For the Back step, suppose $m \in \mathcal{S}$ and $\gamma \notin \mathrm{Im}(m)$. Let γ'' be the greatest element of $\mathrm{Im}(m) \cap (-\infty, \gamma)$ (or $-\infty$ if no such element exists) and, dually, let γ' be the least element of $\mathrm{Im}(m) \cap (\gamma, +\infty)$ (or $+\infty$ if no such element exists). If $\gamma \in \mathbb{Q}$, pick[6] $\delta \in (m^{-1}(\gamma''), m^{-1}(\gamma'))_{\mathbb{R}} \cap \mathbb{Q}$; if $\gamma \in \mathrm{Dc}^{\mathbb{I}}(h)$, pick $\delta \in (m^{-1}(\gamma''), m^{-1}(\gamma'))_{\mathbb{R}} \cap (\bar{g}(A) \cup D)$ – by topological density of $\mathbb{Q}$ and $\bar{g}(A) \cup D \supseteq D$ in $\mathbb{R}$, this is always possible. Then the extension of m by $\delta \mapsto \gamma$ is an element of $\mathcal{S}$ as well. For the Forth step, one argues analogously. $\square$

[6] $(m^{-1}(\gamma''), m^{-1}(\gamma'))_{\mathbb{R}}$ refers to the interval of all *real* numbers between $m^{-1}(\gamma'')$ and $m^{-1}(\gamma')$.

4.4.5 Proving the Variation Lemma 4.4.13, special cases

In a series of lemmas, we first consider the cases that can occur in the special situation that $\bar{x}$ and $\bar{y}$ consist of a single element. In Subsection 4.4.6, we will then amalgamate these special cases to a full proof. We will always consider the same setup:

Notation 4.4.16. We say that $(*)$ holds if we are in the following situation:

(a) $\sigma, \tilde{\sigma} \in \mathcal{M}_{\mathbb{Q}}$ satisfy $\mathrm{LP}(\sigma), \mathrm{LP}(\tilde{\sigma}) \subseteq \mathbb{I}$ and have the same boundedness type,

(b) $f \in \mathcal{M}_{\mathbb{Q}}$ is a generic surjection,

(c) $g \in \mathcal{M}_{\mathbb{Q}}$ is a generic injection,

(d) $h \in \mathcal{M}_{\mathbb{Q}}$ is a sparse injection with the same boundedness type as σ and $\tilde{\sigma}$,

(e) $a \in \mathcal{G}_{\mathbb{Q}}$,

(f) $b \in \mathcal{G}_{\mathbb{Q}}$ satisfies $\bar{b}^{-1}(\mathrm{Dc}^{\mathbb{I}}(h)) \cap \mathrm{Im}(\bar{g}) = \bar{g}(\mathrm{Dc}^{\mathbb{I}}(\sigma))$,

(g) $\sigma = fahbg$, so a preserves all basic formulas as a map from $\mathbb{A}(\sigma, f, \iota)$ to $\mathbb{B}(\sigma, f, \iota)$, where $\iota := hbg$.

To simplify the arguments, we reformulate the property of preserving all basic formulas central to the Sandwich Lemma 4.4.11.

Lemma 4.4.17. *Let $\sigma, \pi, \iota \in \mathcal{M}_{\mathbb{Q}}$ such that π is a generic surjection. Then the map $x \mapsto y$ preserves all basic formulas when considered as a finite partial map from $\mathbb{A}$ to $\mathbb{B}$ if and only if the following two conditions hold:*

$$\sigma\left(\iota^{-1}(-\infty, x]\right) \subseteq (-\infty, \pi(y)] \quad \textit{and} \quad \sigma\left(\iota^{-1}[x, +\infty)\right) \subseteq [\pi(y), +\infty). \tag{4.9}$$

Proof. Assume first that (4.9) holds. Since the finite partial map in question has a one-element domain and image, we do not have to consider formulas of the form $z_i < z_j$. If $\mathbb{A} \models P_q(x)$ for some $q \in \mathbb{Q}$, then there exist $q', q'' \in \mathbb{Q}$ such that $\iota(q'') \leq x \leq \iota(q')$ and $\sigma(q'') = \sigma(q) = \sigma(q')$. By (4.9), we obtain $\sigma(q) = \sigma(q'') \leq \pi(y) \leq \sigma(q') = \sigma(q)$, so $y \in P_q^{\mathbb{B}}$. If $\mathbb{A} \models L_q(x)$ for some $q \in \mathbb{Q}$, there exists $u \in P_q^{\mathbb{A}}$ such that $u < y$. By the previous argument we have $\sigma(q) = \pi(u) \leq \pi(y)$, so $\mathbb{B} \models L_q(y)$ (see Lemma 4.4.14). Finally, if $\mathbb{A} \models R_q(x)$ for some $q \in \mathbb{Q}$, we argue analogously.

Now assume that $x \mapsto y$ preserves all basic formulas as a finite partial map from $\mathbb{A}$ to $\mathbb{B}$. We only show $\sigma\left(\iota^{-1}(-\infty, x]\right) \subseteq (-\infty, \pi(y)]$. Let $q \in \iota^{-1}(-\infty, x]$, i.e. $\iota(q) \leq x$. If $\iota(q) = x$, then $x \in P_q^{\mathbb{A}}$, so $y \in P_q^{\mathbb{B}}$ by assumption and thus $\sigma(q) = \pi(y)$. If $\iota(q) < x$, then $x \in L_q^{\mathbb{A}}$, so $y \in L_q^{\mathbb{B}}$ and thus $\sigma(q) \leq \pi(y)$ (see Lemma 4.4.14). ☐

After these preparations, we can state and prove the series of auxiliary lemmas.

Lemma 4.4.18 (see Figure 4.2)**.** *Let $\sigma, \tilde{\sigma}, f, g, h, a, b$ such that $(*)$ holds. Let further $x, y \in \mathbb{Q}$ such that $a(x) = y$.*

Suppose that $x \notin (\inf h, \sup h)$. Then one of the following two cases occurs:

(1) *(i)* $\mathrm{Im}(h) \subseteq (-\infty, x)$ *and* $\mathrm{Im}(\sigma) \subseteq (-\infty, f(y)]$.

(ii) *If*

$$\mathrm{Im}(\tilde{\sigma}) \subseteq (-\infty, f(y)], \quad \textit{i.e.}\ \mathrm{Im}(\tilde{\sigma}) \cap (f(y), +\infty) = \emptyset,$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$*, the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

(2) *(i)* $\mathrm{Im}(h) \subseteq (x, +\infty)$ *and* $\mathrm{Im}(\sigma) \subseteq [f(y), +\infty)$.

(ii) *If*

$$\mathrm{Im}(\tilde{\sigma}) \subseteq [f(y), +\infty), \quad \textit{i.e.}\ \mathrm{Im}(\tilde{\sigma}) \cap (-\infty, f(y)) = \emptyset,$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$*, the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

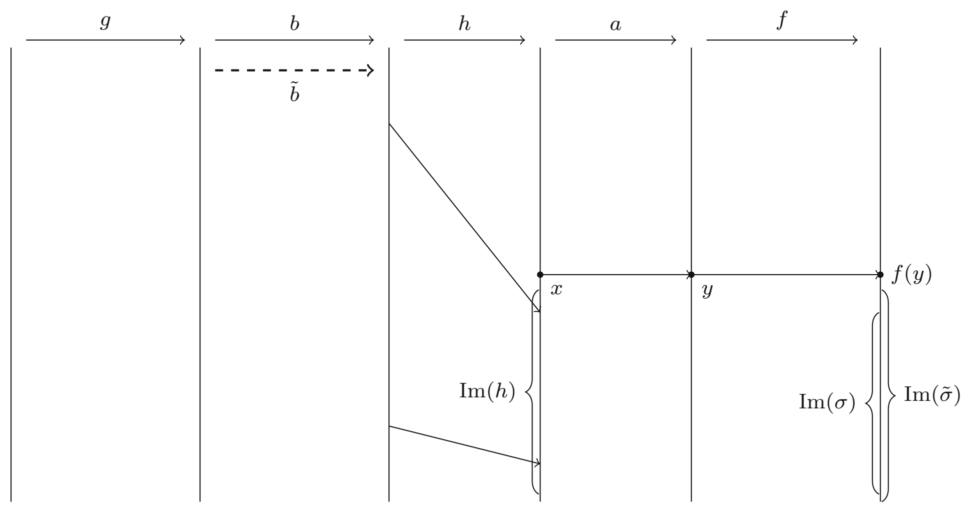

Figure 4.2: Illustration of Lemma 4.4.18, Case (1).

Proof. Our assumption $x \notin (\inf h, \sup h)$ implies that either $h(r) \leq x$ for all $r \in \mathbb{Q}$ or $h(r) \geq x$ for all $r \in \mathbb{Q}$. Since h is injective, $\mathrm{Im}(h)$ cannot have a greatest or least element. Thus, either $\mathrm{Im}(h) \subseteq (-\infty, x)$ or $\mathrm{Im}(h) \subseteq (x, +\infty)$. We only treat the former case which corresponds to (1).

(i). We have to show that $\mathrm{Im}(\sigma) \subseteq (-\infty, f(y)]$ – this follows directly from Lemma 4.4.17 by noting $\iota^{-1}(-\infty, x] = g^{-1}(b^{-1}(h^{-1}(-\infty, x])) = \mathbb{Q}$.

(ii). With the same argument as in (i), observe $\tilde{\iota}^{-1}(-\infty, x] = \mathbb{Q}$ and $\tilde{\iota}^{-1}[x, +\infty) = \emptyset$. Thus, the statement follows by another application of Lemma 4.4.17. □

Lemma 4.4.19 (see Figure 4.3)**.** *Let* $\sigma, \tilde{\sigma}, f, g, h, a, b$ *such that* $(*)$ *holds. Let further* $x, y \in \mathbb{Q}$ *such that* $a(x) = y$.

Suppose that $x \in (\inf h, \sup h)$ *with*[7] $r := h^{\dagger}(x) \in \mathbb{Q}$*, and set* $p := b^{-1}(r)$ *as well as* $I_{-} := \iota^{-1}(-\infty, x]$ *and* $I_{+} := \iota^{-1}[x, +\infty)$*. Then the following holds:*

[7] Note that this encompasses the case $x \in \mathrm{Im}(h)$ and in particular $x \in \mathrm{Im}(\iota)$.

(i) I_- *and* I_+ *are rational intervals with* $\sigma(I_-) \subseteq (-\infty, f(y)]$ *and* $\sigma(I_+) \subseteq [f(y), +\infty)$.

(ii) If

$$\tilde{\sigma}(I_-) \subseteq (-\infty, f(y)] \quad \text{and} \quad \tilde{\sigma}(I_+) \subseteq [f(y), +\infty),$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *with* $\tilde{b}(p) = b(p)\ (= r)$, *the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

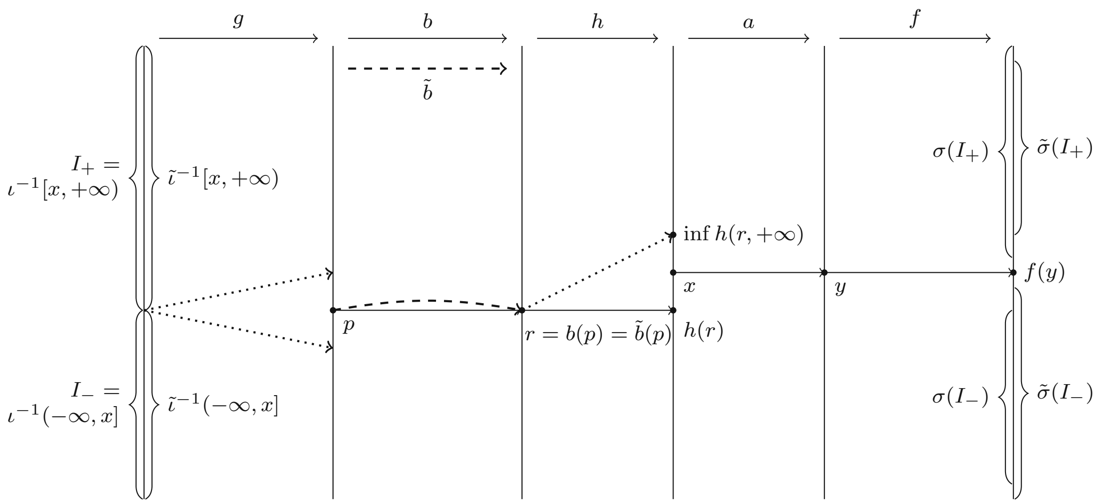

Figure 4.3: Illustration of Lemma 4.4.19.

Proof.

(i). Since $r = h^\dagger(x) \in \mathbb{Q}$, we know that $h^{-1}(-\infty, x)$ and $h^{-1}(x, +\infty)$ are rational intervals, both with boundary point r. Hence, the intervals $h^{-1}(-\infty, x]$ and $h^{-1}[x, +\infty)$ also have boundary point r (if $x \in \mathrm{Im}(h)$, the intervals become closed; otherwise, they do not change). By $b \in \mathcal{G}_{\mathbb{Q}}$, the preimages $b^{-1}(h^{-1}(-\infty, x])$ and $b^{-1}(h^{-1}[x, +\infty))$ are rational intervals as well, both with boundary point $p = b^{-1}(r)$. Finally, $I_- = g^{-1}(b^{-1}(h^{-1}(-\infty, x]))$ and $I_+ = g^{-1}(b^{-1}(h^{-1}[x, +\infty)))$ are rational intervals by applying Lemma 4.4.7(iii).

The inclusions $\sigma(I_-) \subseteq (-\infty, f(y)]$ and $\sigma(I_+) \subseteq [f(y), +\infty)$ follow from Lemma 4.4.17.

(ii). We claim that $\tilde{\iota}^{-1}(-\infty, x] = \iota^{-1}(-\infty, x] = I_-$ and $\tilde{\iota}^{-1}[x, +\infty) = \iota^{-1}[x, +\infty) = I_+$; the statement then follows by another application of Lemma 4.4.17. To this end, it suffices to note that $\tilde{b}^{-1}(h^{-1}(-\infty, x])$ coincides with $b^{-1}(h^{-1}(-\infty, x])$ since they have the same structure (open/closed) as $h^{-1}(-\infty, x]$ and the same boundary point, namely $\tilde{b}^{-1}(r) = p = b^{-1}(r)$. Analogously, $\tilde{b}^{-1}(h^{-1}[x, +\infty))$ coincides with $b^{-1}(h^{-1}[x, +\infty))$. □

Lemma 4.4.20 (see Figures 4.4 and 4.5)**.** *Let* $\sigma, \tilde{\sigma}, f, g, h, a, b$ *such that* $(*)$ *holds. Let further* $x, y \in \mathbb{Q}$ *such that* $a(x) = y$.

Suppose that $x \in (\inf h, \sup h)$ *with* $\gamma := h^\dagger(x) \in \mathbb{I}$ *and* $q := \iota^\dagger(x) \in \mathbb{Q}$. *Then one of the following four cases occurs:*

(1) *(i)* $g(q) < \bar{b}^{-1}(\gamma) < \inf g(q,+\infty)$. *Additionally, there exist* $u, v \in \mathbb{Q}$ *such that* $g(q) < u < v < \inf g(q,+\infty)$ *and* $hb(u) < x < hb(v)$. *Finally,* $\sigma(q) \leq f(y)$ *and* $\sigma(q,+\infty) \subseteq [f(y),+\infty)$.

(ii) *If*

$$\tilde{\sigma}(q) = \sigma(q) \quad \text{and} \quad \tilde{\sigma}(q,+\infty) \subseteq [f(y),+\infty),$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *with* $\tilde{b}(u) = b(u)$ *and* $\tilde{b}(v) = b(v)$, *the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

(2) *(i)* $g(q) < \bar{b}^{-1}(\gamma) = \inf g(q,+\infty)$. *Additionally, there exists* $u \in \mathbb{Q}$ *such that* $g(q) < u < \inf g(q,+\infty)$ *and* $hb(u) < x$. *Finally,* $\sigma(q) \leq f(y)$ *and* $\sigma(q,+\infty) \subseteq [f(y),+\infty)$ *as well as* $\bar{b}^{-1}(\gamma) \in (\mathbb{R} \setminus \operatorname{Im}(\bar{g})) \cap \mathbb{I}$ *and* $\gamma \in \operatorname{Dc}^{\mathbb{I}}(h)$.

(ii) *If*

$$\tilde{\sigma}(q) = \sigma(q) \quad \text{and} \quad \tilde{\sigma}(q,+\infty) \subseteq [f(y),+\infty),$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *with* $\tilde{b}(u) = b(u)$ *and* $\bar{\tilde{b}}(\bar{b}^{-1}(\gamma)) = \bar{b}(\bar{b}^{-1}(\gamma)) = \gamma$ *(so* $\bar{\tilde{b}}^{-1}(\gamma) = \bar{b}^{-1}(\gamma)$*), the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

(3) *(i)* $\sup g(-\infty,q) < \bar{b}^{-1}(\gamma) < g(q)$. *Additionally, there exist* $u, v \in \mathbb{Q}$ *such that* $\sup g(-\infty,q) < u < v < g(q)$ *and* $hb(u) < x < hb(v)$. *Finally,* $\sigma(-\infty,q) \subseteq (-\infty, f(y)]$ *and* $\sigma(q) \geq f(y)$.

(ii) *If*

$$\sigma(-\infty,q) \subseteq (-\infty, f(y)] \quad \text{and} \quad \tilde{\sigma}(q) = \sigma(q),$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *with* $\tilde{b}(u) = b(u)$ *and* $\tilde{b}(v) = b(v)$, *the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

(4) *(i)* $\sup g(-\infty,q) = \bar{b}^{-1}(\gamma) < g(q)$. *Additionally, there exists* $v \in \mathbb{Q}$ *such that* $\sup g(-\infty,q) < v < g(q)$ *and* $x < hb(v)$. *Finally,* $\sigma(-\infty,q) \subseteq (-\infty, f(y)]$ *and* $\sigma(q) \geq f(y)$ *as well as* $\bar{b}^{-1}(\gamma) \in (\mathbb{R} \setminus \operatorname{Im}(\bar{g})) \cap \mathbb{I}$ *and* $\gamma \in \operatorname{Dc}^{\mathbb{I}}(h)$.

(ii) *If*

$$\sigma(-\infty,q) \subseteq (-\infty, f(y)] \quad \text{and} \quad \tilde{\sigma}(q) = \sigma(q),$$

then for any $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *with* $\tilde{b}(v) = b(v)$ *and* $\bar{\tilde{b}}(\bar{b}^{-1}(\gamma)) = \bar{b}(\bar{b}^{-1}(\gamma)) = \gamma$ *(so* $\bar{\tilde{b}}^{-1}(\gamma) = \bar{b}^{-1}(\gamma)$*), the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

Proof. First of all, note that $\iota^{\dagger}(x)$ is welldefined since $x \in (\inf h, \sup h)$ which coincides with $(\inf \iota, \sup \iota)$ by the unboundedness on either side of both g and b. Since $\gamma = h^{\dagger}(x)$ is irrational and h is injective, x cannot be contained in $\operatorname{Im}(h)$, in particular not in $\operatorname{Im}(\iota)$. In any case, we have $h^{-1}(-\infty, x] = h^{-1}(-\infty, x) = (-\infty, \gamma)$ and $h^{-1}[x, +\infty) = h^{-1}(x, +\infty) = (\gamma, +\infty)$. The cases (1) and (2) correspond to $\iota(q) < x$, in other words $\iota^{-1}(-\infty, x] = (-\infty, q]$ and $\iota^{-1}[x, +\infty) = (q, +\infty)$, while the cases (3) and (4) correspond to $\iota(q) > x$, in

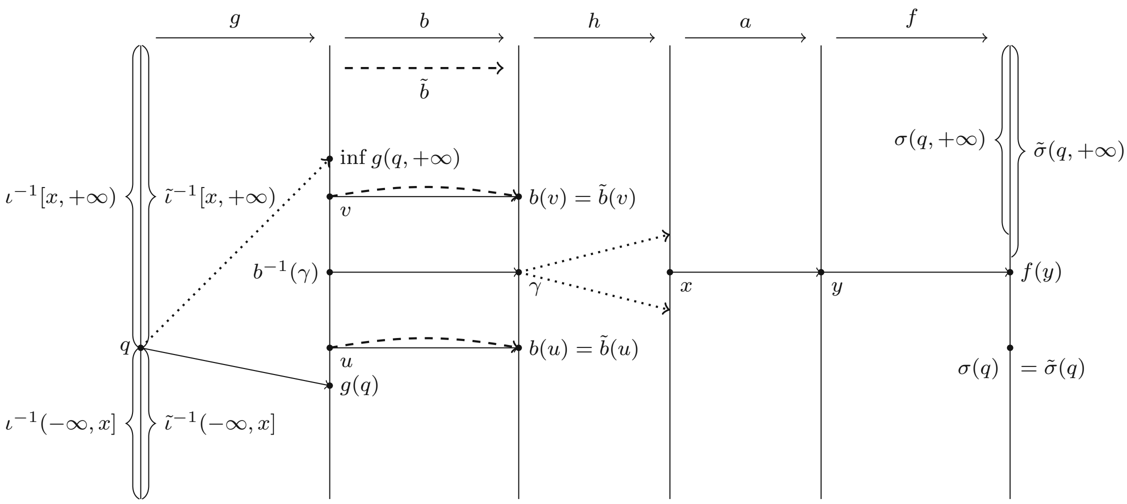

Figure 4.4: Illustration of Lemma 4.4.20, Case (1).

other words $\iota^{-1}(-\infty,x]=(-\infty,q)$ and $\iota^{-1}[x,+\infty)=[q,+\infty)$. We only treat the former cases. Since $\mathrm{LP}(h)\subseteq\mathbb{I}$, the point x cannot be a limit point of h, so we have $\sup h(-\infty,\gamma)<x<\inf h(\gamma,+\infty)$ and $\gamma\in\mathrm{Dc}^{\mathbb{I}}(h)$. Additionally, $x<\inf\iota(q,+\infty)$ by the same argument. If $\iota(q)<x$, we obtain $bg(q)<\gamma\leq\inf bg(q,+\infty)$, i.e. $g(q)<\bar{b}^{-1}(\gamma)\leq\inf g(q,+\infty)$. The first two cases are distinguished by checking whether the latter inequality is strict or not.

(1) $g(q)<\bar{b}^{-1}(\gamma)<\inf g(q,+\infty)$.

(i). Take any $u,v\in\mathbb{Q}$ with $g(q)<u<\bar{b}^{-1}(\gamma)<v<\inf g(q,+\infty)$ to satisfy $g(q)<u<v<\inf g(q,+\infty)$ and $hb(u)<x<hb(v)$. The remaining statements follows from Lemma 4.4.17: $\sigma(q)\in\sigma(-\infty,q]=\sigma\left(\iota^{-1}(-\infty,x]\right)\subseteq(-\infty,f(y)]$ and $\sigma(q,+\infty)=\sigma\left(\iota^{-1}[x,+\infty)\right)\subseteq[f(y),+\infty)$.

(ii). We use $\tilde{b}(u)=b(u)$ and $\tilde{b}(v)=b(v)$ to verify the conditions in Lemma 4.4.17. Note that $h^{-1}(-\infty,x]\subseteq(-\infty,b(v))$ and $h^{-1}[x,+\infty)\subseteq(b(u),+\infty)$, so that

$$\tilde{\iota}^{-1}(-\infty,x]=g^{-1}(\tilde{b}^{-1}(h^{-1}(-\infty,x]))\subseteq g^{-1}(-\infty,v)=(-\infty,q]\qquad\text{and}$$
$$\tilde{\iota}^{-1}[x,+\infty)=g^{-1}(\tilde{b}^{-1}(h^{-1}[x,+\infty)))\subseteq g^{-1}(u,+\infty)=(q,+\infty)$$

which yields

$$\tilde{\sigma}\left(\tilde{\iota}^{-1}(-\infty,x]\right)\subseteq\tilde{\sigma}(-\infty,q]\subseteq(-\infty,\tilde{\sigma}(q)]=(-\infty,\sigma(q)]\subseteq(-\infty,f(y)]\quad\text{and}$$
$$\tilde{\sigma}\left(\tilde{\iota}^{-1}[x,+\infty)\right)\subseteq\tilde{\sigma}(q,+\infty)\subseteq[f(y),+\infty).$$

(2) $g(q)<\bar{b}^{-1}(\gamma)=\inf g(q,+\infty)$.

(i). Take any $u\in\mathbb{Q}$ with $g(q)<u<\bar{b}^{-1}(\gamma)$ to satisfy $g(q)<u<\inf g(q,+\infty)$ and $hb(u)<x$. The statements $\sigma(q)\leq f(y)$ and $\sigma(q,+\infty)\subseteq[f(y),+\infty)$ follow just as in (1). We have already argued that $\gamma\in\mathrm{Dc}^{\mathbb{I}}(h)$, so it remains to show

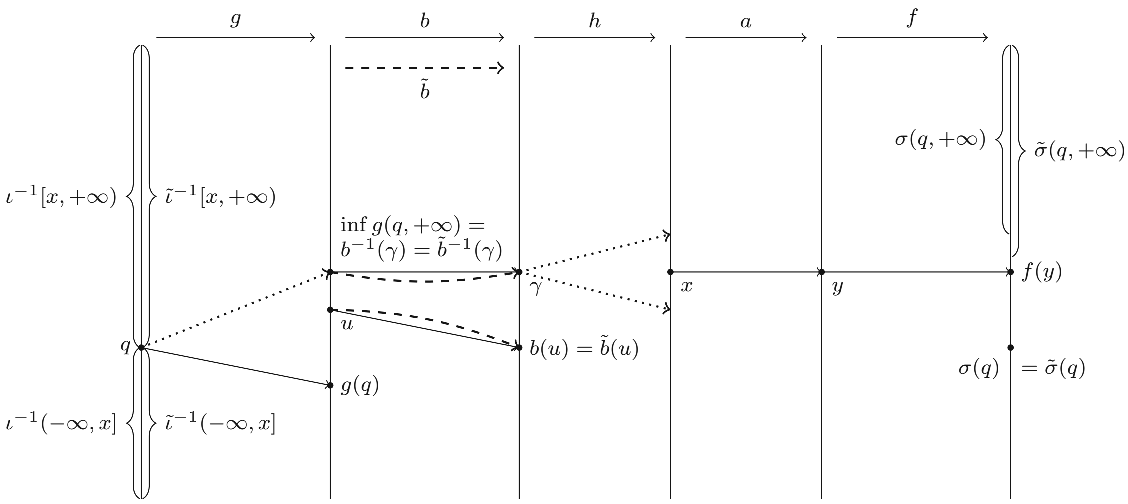

Figure 4.5: Illustration of Lemma 4.4.20, Case (2).

$\bar{b}^{-1}(\gamma) \in (\mathbb{R} \setminus \operatorname{Im}(\bar{g})) \cap \mathbb{I}$. We know that γ is irrational, so $\bar{b}^{-1}(\gamma)$ is as well by Lemma 4.4.3(iii). Additionally, $\bar{b}^{-1}(\gamma) = \inf g(q,+\infty)$ cannot be contained in $\operatorname{Im}(\bar{g})$ since g is injective.

(ii). Similarly to (1), we use $\tilde{b}(u) = b(u)$ and $\bar{\tilde{b}}^{-1}(\gamma) = \bar{b}^{-1}(\gamma)$ to verify the conditions in Lemma 4.4.17. Observe $h^{-1}(-\infty,x] = (-\infty,\gamma)$ and $h^{-1}[x,+\infty) \subseteq (b(u),+\infty)$, so that

$$\tilde{\iota}^{-1}(-\infty,x] = g^{-1}(-\infty,\bar{\tilde{b}}^{-1}(\gamma)) = \iota^{-1}(-\infty,x] = (-\infty,q] \quad \text{and}$$
$$\tilde{\iota}^{-1}[x,+\infty) \subseteq g^{-1}(u,+\infty) = (q,+\infty),$$

which yields $\tilde{\sigma}(\tilde{\iota}^{-1}(-\infty,x]) \subseteq (-\infty,f(y)]$ and $\tilde{\sigma}(\tilde{\iota}^{-1}[x,+\infty)) \subseteq [f(y),+\infty)$ as in (1).

□

In the remaining case that $x \in (\inf h, \sup h)$ and both $h^{\dagger}(x)$ and $\iota^{\dagger}(x)$ are irrational, we take a similar but somewhat more involved route in that the automorphisms $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ we are picking do not simply mimic the behaviour of b on sufficiently many elements. Instead, we redefine $\tilde{b}$ on certain crucial points which are tuned to the specific $\tilde{\sigma}$ being considered. In doing so, we have to make sure that our desired redefinition does not violate the condition for $\tilde{b}$ on finitely many points given by Pseudo-Property $\overline{\mathbf{X}}$ and the previous auxiliary lemmas. We split our treatment of this problem into two subcases.

Lemma 4.4.21 (see Figure 4.6)**.** *Let $\sigma, \tilde{\sigma}, f, g, h, a, b$ such that $(*)$ holds. Let further $x, y \in \mathbb{Q}$ such that $a(x) = y$.*

Suppose that $x \in (\inf h, \sup h)$ with $\gamma := h^{\dagger}(x) \in \mathbb{I}$ and $\delta := \iota^{\dagger}(x) \in \mathbb{I}$. Additionally, suppose that $f(y) \in \operatorname{Im}(\sigma)$. Let $\bar{z}$ and $\bar{w}$ be tuples in $\mathbb{Q}$ and let $\bar{z}'$ and $\bar{w}'$ be tuples in $(\mathbb{R} \setminus \operatorname{Im}(\bar{g})) \cap \mathbb{I}$ and $\operatorname{Dc}^{\mathbb{I}}(h)$, respectively, such that $b(\bar{z}) = \bar{w}$ and $\bar{b}(\bar{z}') = \bar{w}'$. Assume *that $\bar{z} \cup \bar{z}'$ contains both an element greater and less than $\bar{g}(\delta)$. Put z_- and z_+ to be the greatest entry of $\bar{z} \cup \bar{z}'$ less than $\bar{g}(\delta)$ and the least entry of $\bar{z} \cup \bar{z}'$ greater than $\bar{g}(\delta)$, respectively, and put w_- and w_+ to be the corresponding entries of $\bar{w} \cup \bar{w}'$. Then one of the following two cases occurs*[8]*:*

(1) *(i)* *There exist $q, q' \in \mathbb{Q}$ such that $q < q' < \delta$ and $z_- < g(q) < g(q') < \bar{g}(\delta) < z_+$ as well as $\sigma(q) = \sigma(q') = f(y)$; further, $\gamma = \bar{b}(\bar{g}(\delta))$ and $w_- < \gamma < w_+$.*

(ii) *If*

$$\tilde{\sigma}(q) = \sigma(q) = f(y) \quad \text{and} \quad \tilde{\sigma}(q') = \sigma(q') = f(y),$$

and if $\tilde{u}, \tilde{v}, \hat{u}, \hat{v} \in \mathbb{Q}$ satisfy $g(q) < \tilde{u} < \tilde{v} < g(q')$ as well as $w_- < \hat{u} < \gamma < \hat{v} < w_+$, then the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\tilde{u} \mapsto \hat{u}, \tilde{v} \mapsto \hat{v}$ is strictly increasing. Additionally, for any $b \in \mathcal{G}_{\mathbb{Q}}$ such that $\bar{b}(\bar{z}) = \bar{w}$, $\bar{b}(\bar{z}') = \bar{w}'$ and $b(\tilde{u}) = \hat{u}$, $b(\tilde{v}) = \hat{v}$, the map $x \mapsto y$ preserves all basic formulas as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ where $\tilde{\iota} := h\tilde{b}g$.

(2) *(i)* *There exist $q, q' \in \mathbb{Q}$ such that $\delta < q < q'$ and $z_- < \bar{g}(\delta) < g(q) < g(q') < z_+$ as well as $\sigma(q) = \sigma(q') = f(y)$; further, $\gamma = \bar{b}(\bar{g}(\delta))$ and $w_- < \gamma < w_+$.*

(ii) *If*

$$\tilde{\sigma}(q) = \sigma(q) = f(y) \quad \text{and} \quad \tilde{\sigma}(q') = \sigma(q') = f(y),$$

and if $\tilde{u}, \tilde{v}, \hat{u}, \hat{v} \in \mathbb{Q}$ satisfy $g(q) < \tilde{u} < \tilde{v} < g(q')$ as well as $w_- < \hat{u} < \gamma < \hat{v} < w_+$, then the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\tilde{u} \mapsto \hat{u}, \tilde{v} \mapsto \hat{v}$ is strictly increasing. Additionally, for any $b \in \mathcal{G}_{\mathbb{Q}}$ such that $\bar{b}(\bar{z}) = \bar{w}$, $\bar{b}(\bar{z}') = \bar{w}'$ and $b(\tilde{u}) = \hat{u}$, $b(\tilde{v}) = \hat{v}$, the map $x \mapsto y$ preserves all basic formulas as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ to $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ where $\tilde{\iota} := h\tilde{b}g$.

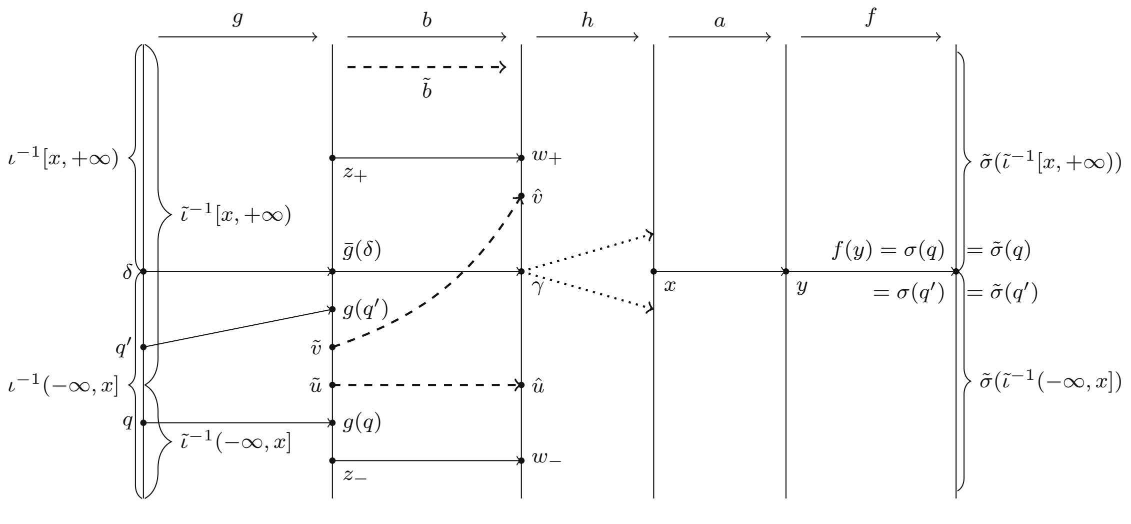

Figure 4.6: Illustration of Lemma 4.4.21, Case (1).

[8] It is possible that both cases occur simultaneously; it this happens, pick one of them arbitrarily.

Proof. As in the proof of Lemma 4.4.20, the generalised inverse $\iota^\dagger(x)$ is welldefined and x cannot be contained in $\mathrm{Im}(h)$, in particular $\mathrm{Im}(\iota)$. Additionally, we have $h^{-1}(-\infty, x] = h^{-1}(-\infty, x) = (-\infty, \gamma)$ and $h^{-1}[x, +\infty) = h^{-1}(x, +\infty) = (\gamma, +\infty)$ as well as $\iota^{-1}(-\infty, x] = \iota^{-1}(-\infty, x) = (-\infty, \delta)$ and $\iota^{-1}[x, +\infty) = \iota^{-1}(x, +\infty) = (\delta, +\infty)$. This also yields that $\bar{b}(\bar{g}(\delta)) = \gamma$. By Lemma 4.4.17, we conclude[9]

$$\begin{aligned} \sigma(-\infty, \delta) &= \sigma\left(\iota^{-1}(-\infty, x]\right) \subseteq (-\infty, f(y)] \quad \text{and} \\ \sigma(\delta, +\infty) &= \sigma\left(\iota^{-1}[x, +\infty)\right) \subseteq [f(y), +\infty). \end{aligned}$$

Since $f(y) \in \mathrm{Im}(\sigma)$ and since δ is irrational, this is only possible if σ is locally constant with value $f(y)$ either below or above δ (or both). These two situations form the cases (1) and (2), respectively. We only treat the former option.

(i). By our preparatory reasoning, there exist $q, q' \in \mathbb{Q}$ with $q < q' < \delta$ and $\sigma(q) = \sigma(q') = f(y)$. The number $\bar{g}(\delta)$ is irrational by Lemma 4.4.3(ii), and obviously not an element of $\mathbb{R} \setminus \mathrm{Im}(\bar{g})$. Thus, $\bar{g}(\delta)$ cannot be contained in $\bar{z} \cup \bar{z}'$. Consequently, $\gamma = \bar{b}(\bar{g}(\delta))$ cannot be contained in $\bar{w} \cup \bar{w}'$, and we conclude $w_- < \gamma < w_+$ from $z_- < \bar{g}(\delta) < z_+$. Since $\delta \in \mathbb{I} = \mathrm{Cont}(g)$, we can pick q, q' close enough to δ to ascertain $z_- < g(q) < g(q') < \bar{g}(\delta) < z_+$.

(ii). By the definitions of z_- and z_+ and the fact that $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ is strictly increasing, the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\tilde{u} \mapsto \hat{u}, \tilde{v} \mapsto \hat{v}$ is strictly increasing. For the second statement, we check the assumptions of Lemma 4.4.17. Note that

$$\begin{aligned} \tilde{\iota}^{-1}(-\infty, x] &= g^{-1}(\tilde{b}^{-1}(-\infty, \gamma)) \subseteq g^{-1}(-\infty, \tilde{v}) \subseteq (-\infty, q'] \qquad \text{and} \\ \tilde{\iota}^{-1}[x, +\infty) &= g^{-1}(\tilde{b}^{-1}(\gamma, +\infty)) \subseteq g^{-1}(\tilde{u}, +\infty) \subseteq [q, +\infty), \end{aligned}$$

so

$$\begin{aligned} \tilde{\sigma}\left(\tilde{\iota}^{-1}(-\infty, x]\right) &\subseteq \tilde{\sigma}(-\infty, q'] \subseteq (-\infty, \tilde{\sigma}(q')] = (-\infty, \sigma(q')] = (-\infty, f(y)] \quad \text{and} \\ \tilde{\sigma}\left(\tilde{\iota}^{-1}[x, +\infty)\right) &\subseteq \tilde{\sigma}[q, +\infty) \subseteq [\tilde{\sigma}(q), +\infty) = [\sigma(q), +\infty) = [f(y), +\infty). \end{aligned}$$ □

Our final auxiliary lemma treats the second subcase of the situation that $x \in (\inf h, \sup h)$ and $h^\dagger(x), \iota^\dagger(x) \in \mathbb{I}$.

Lemma 4.4.22 (see Figure 4.7)**.** *Let $\sigma, \tilde{\sigma}, f, g, h, a, b$ such that $(*)$ holds. Let further $x, y \in \mathbb{Q}$ such that $a(x) = y$.*

Suppose that $x \in (\inf h, \sup h)$ with $\gamma := h^\dagger(x) \in \mathbb{I}$ and $\delta := \iota^\dagger(x) \in \mathbb{I}$. Additionally, suppose that $f(y) \notin \mathrm{Im}(\sigma)$. Let $\bar{z}$ and $\bar{w}$ be tuples in $\mathbb{Q}$ and let $\bar{z}'$ and $\bar{w}'$ be tuples in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, such that $b(\bar{z}) = \bar{w}$ and $\bar{b}(\bar{z}') = \bar{w}'$. Assume that $\bar{z} \cup \bar{z}'$ contains both an element greater and less than $\bar{g}(\delta)$. Put z_- and z_+ to be the greatest entry of $\bar{z} \cup \bar{z}'$ less than $\bar{g}(\delta)$ and the least entry of $\bar{z} \cup \bar{z}'$ greater than $\bar{g}(\delta)$, respectively, and put w_- and w_+ to be the corresponding entries of $\bar{w} \cup \bar{w}'$. Then the following holds:

[9] Note that we cannot express $\tilde{\sigma}(-\infty, \delta) \subseteq (-\infty, f(y)]$ and $\tilde{\sigma}(\delta, +\infty) \subseteq [f(y), +\infty)$ using the rich topology from Definition 4.3.2 since δ is irrational. This is one of the reasons why we need to redefine $\tilde{b}$ instead of transferring the behaviour of b at sufficiently many points.

(i) $\sigma^\dagger(f(y)) = \delta$ *and* $\delta \in \mathrm{Dc}^{\mathbb{I}}(\sigma)$. *Additionally,* $\gamma \in \mathrm{Dc}^{\mathbb{I}}(h)$ *and* $\gamma = \bar{b}(\bar{g}(\delta))$. *Further,* $z_- < \bar{g}(\delta) < z_+$ *as well as* $w_- < \gamma < w_+$. *Finally,* $\sigma\left(g^{-1}(-\infty, z_-]\right) \subseteq (-\infty, f(y))$ *and* $\sigma\left(g^{-1}[z_+, +\infty)\right) \subseteq (f(y), +\infty)$.

If $z_\pm$ *and* $w_\pm$ *are rational, then* $I_- := g^{-1}(-\infty, z_-]$ *and* $I_+ := g^{-1}[z_+, +\infty)$ *are rational intervals.*

(ii) *If* $z_\pm$ *and* $w_\pm$ *are rational and if*

$$\mathrm{Im}(\tilde{\sigma}) \cap \{f(y)\} = \emptyset \quad \text{and} \quad \tilde{\sigma}(I_-) \subseteq (-\infty, f(y)) \quad \text{and} \quad \tilde{\sigma}(I_+) \subseteq (f(y), +\infty),$$

then – setting $\tilde{H}_- := \tilde{\sigma}^{-1}(-\infty, f(y)] = \tilde{\sigma}^{-1}(-\infty, f(y))$ *and* $\tilde{H}_+ := \tilde{\sigma}^{-1}[f(y), +\infty) = \tilde{\sigma}^{-1}(f(y), +\infty)$ *– we have* $\sup g(\tilde{H}_-) < z_+$ *as well as* $z_- < \inf g(\tilde{H}_+)$. *Further, there exists* $\tilde{\rho} \in ((\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}) \cup \bar{g}(\mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma}))$ *such that* $z_- < \tilde{\rho} < z_+$ *and* $g(\tilde{H}_-) \subseteq (-\infty, \tilde{\rho})$ *as well as* $g(\tilde{H}_+) \subseteq (\tilde{\rho}, +\infty)$. *The finite partial map* $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ *and* $\tilde{\rho} \mapsto \gamma$ *is strictly increasing and, additionally, for any* $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *such that* $\bar{\tilde{b}}(\bar{z}) = \bar{w}$, $\bar{\tilde{b}}(\bar{z}') = \bar{w}'$ *and* $\bar{\tilde{b}}(\tilde{\rho}) = \gamma$, *the map* $x \mapsto y$ *preserves all basic formulas as a finite partial map from* $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ *to* $\mathbb{B}(\tilde{\sigma}, f, \tilde{\iota})$ *where* $\tilde{\iota} := h\tilde{b}g$.

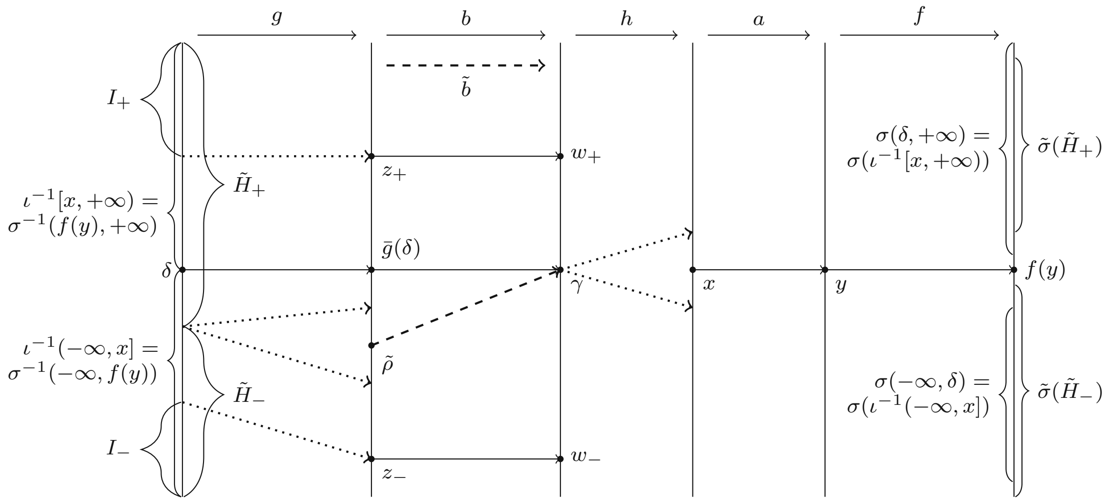

Figure 4.7: Illustration of Lemma 4.4.22.

Proof. As in the proof of Lemma 4.4.21, the generalised inverse $\iota^\dagger(x)$ is welldefined and x cannot be contained in $\mathrm{Im}(h)$, in particular $\mathrm{Im}(\iota)$.

Additionally, $h^{-1}(-\infty, x] = h^{-1}(-\infty, x) = (-\infty, \gamma)$ and $h^{-1}[x, +\infty) = h^{-1}(x, +\infty) = (\gamma, +\infty)$ as well as $\iota^{-1}(-\infty, x] = \iota^{-1}(-\infty, x) = (-\infty, \delta)$ and $\iota^{-1}[x, +\infty) = \iota^{-1}(x, +\infty) = (\delta, +\infty)$. We again conclude $\bar{b}(\bar{g}(\delta)) = \gamma$. Combining Lemma 4.4.17 with $f(y) \notin \mathrm{Im}(\sigma)$, we obtain $\sigma(-\infty, \delta) \subseteq (-\infty, f(y))$ as well as $\sigma(\delta, +\infty) \subseteq (f(y), +\infty)$ which yields $\delta = \sigma^\dagger(f(y))$ and

$$\sigma\left(g^{-1}(-\infty, z_-]\right) \subseteq \sigma\left(g^{-1}(-\infty, \bar{g}(\delta))\right) \subseteq (-\infty, f(y)) \quad \text{as well as}$$
$$\sigma\left(g^{-1}[z_+, +\infty)\right) \subseteq \sigma\left(g^{-1}(\bar{g}(\delta), +\infty)\right) \subseteq (f(y), +\infty).$$

(i). We know that $f(y) \in \mathbb{Q}$ cannot be a limit point of σ, so $\sup \sigma(-\infty, \delta) < f(y) < \inf \sigma(\delta, +\infty)$, in particular $\delta \in \mathrm{Dc}^{\mathbb{I}}(\sigma)$. Using $\mathrm{LP}(h) \subseteq \mathbb{I}$ in the same fashion, we obtain $\gamma \in \mathrm{Dc}^{\mathbb{I}}(h)$. The remaining statements $z_- < \bar{g}(\delta) < z_+$ as well as $w_- < \gamma < w_+$ follow just as in the proof of Lemma 4.4.21.

Finally, if $z_\pm$ and $w_\pm$ are rational, then the intervals $I_- := g^{-1}(-\infty, z_-]$ and $I_+ := g^{-1}[z_+, +\infty)$ are rational by Lemma 4.4.7(iii).

(ii). We have $\mathbb{Q} = \tilde{H}_- \dot{\cup} \tilde{H}_+$, so $\sup \tilde{H}_- = \inf \tilde{H}_+$. By our assumption on $\tilde{\sigma}$, we know that $I_- \cap \tilde{H}_+ = \emptyset$, so $\inf g(\tilde{H}_+) \geq z_-$. In fact, this inequality is strict since $\inf g(\tilde{H}_+)$ is either contained in $g(\tilde{H}_+)$ or irrational by $\mathrm{LP}(g) \subseteq \mathbb{I}$. One argues analogously to show $\sup g(\tilde{H}_-) < z_+$. To find $\tilde{\rho}$, we distinguish whether $\sup \tilde{H}_- = \inf \tilde{H}_+$ is rational or irrational.

Case 1 ($\tilde{q} := \tilde{\sigma}^\dagger(f(y)) = \sup \tilde{H}_- = \inf \tilde{H}_+ \in \mathbb{Q}$): We conclude $\sup g(\tilde{H}_-) < \inf g(\tilde{H}_+)$ from Lemma 4.4.3(ii). Combined with $\sup g(\tilde{H}_-) < z_+$ and $z_- < \inf g(\tilde{H}_+)$, this implies $\max\left(\sup g(\tilde{H}_-), z_-\right) < \min\left(\inf g(\tilde{H}_+), z_+\right)$. Any irrational $\tilde{\rho}$ between these two numbers satisfies the requirements – note that $\tilde{\rho}$ is contained in $\mathbb{R} \setminus \mathrm{Im}(\bar{g})$ by injectivity of g.

Case 2 ($\tilde{\delta} := \tilde{\sigma}^\dagger(f(y)) = \sup \tilde{H}_- = \inf \tilde{H}_+ \in \mathbb{I}$): We obtain $\sup g(\tilde{H}_-) = \bar{g}(\tilde{\delta}) = \inf g(\tilde{H}_+)$ since $\tilde{\delta} \in \mathrm{Cont}(g)$, so $z_- < \bar{g}(\tilde{\delta}) < z_+$. Since $\tilde{\sigma}(-\infty, \tilde{\delta}) \subseteq (-\infty, f(y))$ and $\tilde{\sigma}(\tilde{\delta}, +\infty) \subseteq (f(y), +\infty)$ and since $f(y) \in \mathbb{Q}$ cannot be a limit point of $\tilde{\sigma}$, we conclude $\tilde{\delta} \in \mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma})$. Hence, we set $\tilde{\rho} := \bar{g}(\tilde{\delta})$.

By construction, the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}' \mapsto \bar{w}'$ and $\tilde{\rho} \mapsto \gamma$ is strictly increasing. For the preservation statement, we verify the conditions in Lemma 4.4.17. Note that

$$\tilde{\iota}^{-1}(-\infty, x] = g^{-1}(-\infty, \tilde{\rho}) = \mathbb{Q} \setminus g^{-1}(\tilde{\rho}, +\infty) \subseteq \mathbb{Q} \setminus g^{-1}(g(\tilde{H}_+)) = \tilde{H}_- \quad \text{and}$$
$$\tilde{\iota}^{-1}[x, +\infty) = g^{-1}(\tilde{\rho}, +\infty) = \mathbb{Q} \setminus g^{-1}(-\infty, \tilde{\rho}) \subseteq \mathbb{Q} \setminus g^{-1}(g(\tilde{H}_-)) = \tilde{H}_+,$$

so

$$\tilde{\sigma}\left(\tilde{\iota}^{-1}(-\infty, x]\right) \subseteq \tilde{\sigma}(\tilde{H}_-) \subseteq (-\infty, f(y)) \quad \text{and}$$
$$\tilde{\sigma}\left(\tilde{\iota}^{-1}[x, +\infty)\right) \subseteq \tilde{\sigma}(\tilde{H}_+) \subseteq (f(y), +\infty). \qquad \square$$

Remark 4.4.23. Examining the last proof more closely, one observes that we never used $a(x) = y$ other than via the inclusion of intervals from Lemma 4.4.17. Hence, we in fact proved the following slightly stronger statement which will be useful when amalgamating the auxiliary lemmas:

Let $\sigma, \tilde{\sigma}, f, g, h, a, b$ such that $(*)$ holds. Let further $x, y \in \mathbb{Q}$ such that

$$\sigma\left(\iota^{-1}(-\infty, x]\right) \subseteq (-\infty, f(y)] \quad \text{and} \quad \sigma\left(\iota^{-1}[x, +\infty)\right) \subseteq [f(y), +\infty).$$

Suppose that $x \in (\inf h, \sup h)$ with $\gamma := h^\dagger(x) \in \mathbb{I}$ and $\delta := \iota^\dagger(x) \in \mathbb{I}$. Additionally, suppose that $f(y) \notin \mathrm{Im}(\sigma)$. Let $\bar{z}$ and $\bar{w}$ be tuples in $\mathbb{Q}$ and let $\bar{z}'$ and $\bar{w}'$ be tuples in $(\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $\mathrm{Dc}^{\mathbb{I}}(h)$, respectively, such that $b(\bar{z}) = \bar{w}$ and $\bar{b}(\bar{z}') = \bar{w}'$. Assume that $\bar{z} \cup \bar{z}'$ contains both an element greater and less than $\bar{g}(\delta)$. Put z_- and z_+ to be the greatest entry of $\bar{z} \cup \bar{z}'$ less than $\bar{g}(\delta)$ and the least entry of $\bar{z} \cup \bar{z}'$ greater than $\bar{g}(\delta)$, respectively, and put w_- and w_+ to be the corresponding entries of $\bar{w} \cup \bar{w}'$. Then the following holds:

(i) $\sigma^\dagger(f(y)) = \delta$ *and* $\delta \in \mathrm{Dc}^{\mathbb{I}}(\sigma)$*. Additionally,* $\gamma \in \mathrm{Dc}^{\mathbb{I}}(h)$ *and* $\gamma = \bar{b}(\bar{g}(\delta))$*. Further,* $z_- < \bar{g}(\delta) < z_+$ *as well as* $w_- < \gamma < w_+$*. Finally,* $\sigma\left(g^{-1}(-\infty, z_-]\right) \subseteq (-\infty, f(y))$ *and* $\sigma\left(g^{-1}[z_+, +\infty)\right) \subseteq (f(y), +\infty)$*.*

If $z_\pm$ *and* $w_\pm$ *are rational, then* $I_- := g^{-1}(-\infty, z_-]$ *and* $I_+ := g^{-1}[z_+, +\infty)$ *are rational intervals.*

(ii) *If* $z_\pm$ *and* $w_\pm$ *are rational and if*

$$\mathrm{Im}(\tilde{\sigma}) \cap \{f(y)\} = \emptyset \quad \textit{and} \quad \tilde{\sigma}(I_-) \subseteq (-\infty, f(y)) \quad \textit{and} \quad \tilde{\sigma}(I_+) \subseteq (f(y), +\infty),$$

then – setting $\tilde{H}_- := \tilde{\sigma}^{-1}(-\infty, f(y)] = \tilde{\sigma}^{-1}(-\infty, f(y))$ *and* $\tilde{H}_+ := \tilde{\sigma}^{-1}[f(y), +\infty) = \tilde{\sigma}^{-1}(f(y), +\infty)$ *– we have* $\sup g(\tilde{H}_-) < z_+$ *as well as* $z_- < \inf g(\tilde{H}_+)$*. Further, there exists* $\tilde{\rho} \in ((\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}) \cup \bar{g}(\mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma}))$ *such that* $z_- < \tilde{\rho} < z_+$ *and* $g(\tilde{H}_-) \subseteq (-\infty, \tilde{\rho})$ *as well as* $g(\tilde{H}_+) \subseteq (\tilde{\rho}, +\infty)$*. The finite partial map* $\bar{z} \mapsto \bar{w}$*,* $\bar{z}' \mapsto \bar{w}'$ *and* $\tilde{\rho} \mapsto \gamma$ *is strictly increasing and, additionally, for any* $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ *such that* $\tilde{b}(\bar{z}) = \bar{w}$*,* $\bar{\tilde{b}}(\bar{z}') = \bar{w}'$ *and* $\bar{\tilde{b}}(\tilde{\rho}) = \gamma$*, we have*

$$\tilde{\sigma}\left(\tilde{\iota}^{-1}(-\infty, x]\right) \subseteq (-\infty, f(y)] \quad \textit{and} \quad \tilde{\sigma}\left(\tilde{\iota}^{-1}[x, +\infty)\right) \subseteq [f(y), +\infty),$$

where $\tilde{\iota} := h\tilde{b}g$*.*

4.4.6 Proving the Variation Lemma 4.4.13, full

Finally, we amalgamate the special cases.

Proof (of the Variation Lemma 4.4.13). We construct O as an intersection of $\mathcal{T}_{rich}$-subbasic open sets, i.e. of sets of the types 0, 1, 2, 3.

By adding to the intersection O the condition that $\tilde{\sigma}$ has the same boundedness type as h (type 2), we can ascertain that $(*)$ holds. Considering that $\bar{x} \mapsto \bar{y}$ automatically preserves the formulas $z_i < z_j$ since $a(\bar{x}) = \bar{y}$ and that all the other basic formulas are unary, it suffices to pick the automorphism $\tilde{b} \in \mathcal{G}_{\mathbb{Q}}$ in such a way that the map $x \mapsto y$ preserves all basic formulas for each corresponding pair x, y in $\bar{x}, \bar{y}$.

First, we treat those corresponding pairs x, y in $\bar{x}, \bar{y}$ for which

(a) $x \notin (\inf h, \sup h)$ OR

(b) $x \in (\inf h, \sup h)$ with $h^\dagger(x) \in \mathbb{Q}$ OR

(c) $x \in (\inf h, \sup h)$ with $h^\dagger(x) \in \mathbb{I}$ and $\iota^\dagger(x) \in \mathbb{Q}$.

Applying Lemmas 4.4.18, 4.4.19 and 4.4.20 each yields a finite intersection of sets of types 0, 1, 2, 3 and additional conditions of the form $\tilde{b}(z) = w = b(z)$ for $z, w \in \mathbb{Q}$ or $\bar{\tilde{b}}(z') = w' = \bar{b}(z')$ for $z' \in (\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}$ and $w' \in \mathrm{Dc}^{\mathbb{I}}(h)$ under which $x \mapsto y$ always preserves all basic formulas as a finite partial map from $\mathbb{A}(\tilde{\sigma}, f, \tilde{\iota})$ to $\mathbb{B}(\sigma, f, \iota)$ where $\tilde{\iota} := h\tilde{b}g$. We add the sets of types 0, 1, 2, 3 to the intersection O, we add the points z and w to $\bar{z}^*$ and $\bar{w}^*$, respectively, and we add the points z' and w' to $\bar{z}'$ and $\bar{w}'$, respectively. Summarising, we obtain that if $\tilde{\sigma}$ is contained in the set O constructed thus far and if $\tilde{b}(\bar{z}^*) = \bar{w}^*$

and $\tilde{b}(\bar{z}') = \bar{w}'$, then $x \mapsto y$ preserves all basic formulas for each corresponding pair x, y with one of the three properties (a)-(c).

It remains to consider those corresponding pairs x, y in $\bar{x}, \bar{y}$ for which

(d) $x \in (\inf h, \sup h)$ with $\gamma := h^\dagger(x) \in \mathbb{I}$ and $\delta := \iota^\dagger(x) \in \mathbb{I}$.

Put z_- and z_+ to be the greatest entry of $\bar{z} \cup \bar{z}^* \cup \bar{z}'$ less than $\bar{g}(\delta)$ and the least entry of $\bar{z} \cup \bar{z}^* \cup \bar{z}'$ greater than $\bar{g}(\delta)$, respectively, and put w_- and w_+ to be the corresponding entries of $\bar{w} \cup \bar{w}^* \cup \bar{w}'$. As a first step, Lemmas 4.4.21 and 4.4.22 yield that $z_- < \bar{g}(\delta) < z_+$ (as well as $w_- < \gamma < w_+$) and $\gamma = \bar{b}(\bar{g}(\delta))$. Hence, we can find *rationals* $\hat{z}_-, \hat{z}_+ \in \mathbb{Q}$ such that $z_- < \hat{z}_- < \bar{g}(\delta) < \hat{z}_+ < z_+$. We add $\hat{z}_\pm$ to $\bar{z}^*$ and $\hat{w}_\pm := b(\hat{z}_\pm)$ to $\bar{w}^*$. In this way, we can assume that $z_\pm$ and $w_\pm$ are always rational for each corresponding pair x, y.

If x_1, y_1 and x_2, y_2 are two such pairs (without loss of generality, let $x_1 < x_2$) and if

$$\gamma_1 := h^\dagger(x_1) < \gamma_2 := h^\dagger(x_2),$$

we enrich $\bar{z}^*$ and $\bar{w}^*$ even further: putting $\delta_1 := \iota^\dagger(x_1)$ and $\delta_2 := \iota^\dagger(x_2)$, we know that

$$\bar{b}(\bar{g}(\delta_1)) = \gamma_1 < \gamma_2 = \bar{b}(\bar{g}(\delta_2))$$

by Lemmas 4.4.21 and 4.4.22, and hence $\bar{g}(\delta_1) < \bar{g}(\delta_2)$. If we pick $\tilde{z}, \tilde{w} \in \mathbb{Q}$ such that

$$\bar{g}(\delta_1) < \tilde{z} < \bar{g}(\delta_2) \text{ and } \tilde{w} := b(\tilde{z}), \tag{4.10}$$

then $\gamma_1 < \tilde{w} < \gamma_2$. We add $\tilde{z}$ to $\bar{z}^*$ and $\tilde{w}$ to $\bar{w}^*$. If $z_{\pm,1}, w_{\pm,1}, z_{\pm,2}, w_{\pm,2}$ denote the values[10] $z_\pm, w_\pm$ for x_1, y_1 and x_2, y_2, respectively, we obtain $z_{+,1} \leq \tilde{z} \leq z_{-,2}$ and $w_{+,1} \leq \tilde{w} \leq w_{-,2}$. Distinguishing cases, we conclude that whichever combination of Lemmas 4.4.21 and 4.4.22 applies to x_1, y_1 and x_2, y_2, the resulting conditions on $\tilde{b}$ will be compatible, i.e. strictly increasing. By way of example, consider the case that x_1, y_1 fall into the scope of Lemma 4.4.21 and x_2, y_2 fall into the scope of Lemma 4.4.22. Then we are required to pick

$$\tilde{u}, \tilde{v}, \hat{u}, \hat{v} \in \mathbb{Q} \quad \text{and} \quad \tilde{\rho} \in ((\mathbb{R} \setminus \mathrm{Im}(\bar{g})) \cap \mathbb{I}) \cup \bar{g}(\mathrm{Dc}^{\mathbb{I}}(\tilde{\sigma}))$$

with (in particular)

$$z_{-,1} < \tilde{u} < \tilde{v} < z_{+,1} \leq z_{-,2} < \tilde{\rho} < z_{+,2} \quad \text{and} \quad w_{-,1} < \hat{u} < \hat{v} < w_{+,1} \leq w_{-,2} < \gamma_2 < w_{+,2}.$$

Thus, for any $\tilde{u}, \tilde{v}, \hat{u}, \hat{v}, \tilde{\rho}$ we could pick, the finite partial map $\bar{z} \mapsto \bar{w}$, $\bar{z}^* \mapsto \bar{w}^*$, $\bar{z}' \mapsto \bar{w}'$, $\tilde{u} \mapsto \hat{u}$, $\tilde{v} \mapsto \hat{v}$ and $\tilde{\rho} \mapsto \gamma_2$ is automatically strictly increasing.

Finally, we treat the possibility that

$$h^\dagger(x_1) = h^\dagger(x_2).$$

We will show that we can reduce to a single application of Lemma 4.4.21, Lemma 4.4.22 or Remark 4.4.23. First, let x_1, y_1 and x_2, y_2 and x_3, y_3 be three corresponding pairs with $x_1 < x_2 < x_3$ (and consequently $y_1 < y_2 < y_3$) but $h^\dagger(x_1) = h^\dagger(x_2) = h^\dagger(x_3)$. Then

$$h^{-1}(-\infty, x_1] = h^{-1}(-\infty, x_2] = h^{-1}(-\infty, x_3] \quad \text{and}$$
$$h^{-1}[x_1, +\infty) = h^{-1}[x_2, +\infty) = h^{-1}[x_3, +\infty),$$

[10] They are necessarily rational!

so

$$\tilde\iota^{-1}(-\infty, x_1] = \tilde\iota^{-1}(-\infty, x_2] = \tilde\iota^{-1}(-\infty, x_3] \quad \text{and}$$
$$\tilde\iota^{-1}[x_1, +\infty) = \tilde\iota^{-1}[x_2, +\infty) = \tilde\iota^{-1}[x_3, +\infty)$$

for *all* $\tilde\iota = h\tilde b g$ we could pick in the sequel. It is immediate from Lemma 4.4.17 that we can drop x_2, y_2 from $\bar x, \bar y$; more precisely: if $x_1 \mapsto y_1$ and $x_3 \mapsto y_3$ preserve all basic formulas, then so does $x_2 \mapsto y_2$. Hence, we can assume that $\bar x, \bar y$ contains only two corresponding pairs x_1, y_1 and x_2, y_2 with $h^\dagger(x_1) = h^\dagger(x_2)$. If additionally $f(y_1) = f(y_2)$, we can drop one of the pairs from $\bar x, \bar y$ and apply Lemma 4.4.21 or 4.4.22 to the remaining one. If on the other hand $f(y_1) < f(y_2)$, we apply Lemma 4.4.17 to $x_1 \mapsto y_1$ and $x_2 \mapsto y_2$ as finite partial maps from $\mathbb{A}$ to $\mathbb{B}$ to obtain

$$\sigma\left(\iota^{-1}(-\infty, x_1]\right) \subseteq (-\infty, f(y_1)] \quad \text{and} \quad \sigma\left(\iota^{-1}[x_1, +\infty)\right) \subseteq [f(y_2), +\infty).$$

Since $\iota^{-1}(-\infty, x_1]$ and $\iota^{-1}[x_1, +\infty)$ partition the whole of $\mathbb{Q}$ (note that $x_1 \notin \operatorname{Im}(\iota)$), this implies $\operatorname{Im}(\sigma) \cap (f(y_1), f(y_2)) = \emptyset$. We add the condition

$$\operatorname{Im}(\tilde\sigma) \cap (f(y_1), f(y_2)) = \emptyset \quad (\text{type } 3)$$

to the intersection O and pick $\hat y \in \mathbb{Q}$ such that $f(y_1) < f(\hat y) < f(y_2)$. Then

$$\sigma\left(\iota^{-1}(-\infty, x_1]\right) \subseteq (-\infty, f(\hat y)] \quad \text{and} \quad \sigma\left(\iota^{-1}[x_1, +\infty)\right) \subseteq [f(\hat y), +\infty).$$

Applying Remark 4.4.23 to the pair $x_1, \hat y$ one obtains

$$\tilde\sigma\left(\tilde\iota^{-1}(-\infty, x_1]\right) \subseteq (-\infty, f(\hat y)] \quad \text{and} \quad \tilde\sigma\left(\tilde\iota^{-1}[x_1, +\infty)\right) \subseteq [f(\hat y), +\infty)$$

under suitable conditions on $\tilde\sigma$ and $\tilde b$ (see below). By our choice of $\hat y$, since $\tilde\iota^{-1}[x_2, +\infty) = \tilde\iota^{-1}[x_1, +\infty)$ and since $\operatorname{Im}(\tilde\sigma) \cap (f(y_1), f(y_2)) = \emptyset$, Lemma 4.4.17 yields that this is equivalent to $x_1 \mapsto y_1$ and $x_2 \mapsto y_2$ both preserving all basic formulas.

To complete the proof, we apply either Lemma 4.4.21, Lemma 4.4.22 or Remark 4.4.23 (the latter only if we use the reduction from two instances of Lemmas 4.4.21 or 4.4.22 to a single instance of Remark 4.4.23 as derived above) to each corresponding pair x, y in $\bar x, \bar y$ satisfying (d). This yields additional sets of types 0, 1, 2, 3 and additional tuples $\bar\zeta^*, \bar\eta^*$ in $\mathbb{Q}$, $\bar\zeta'$ in $(\mathbb{R} \setminus \operatorname{Im}(\bar g)) \cap \mathbb{I}$, $\bar\zeta''$ in $\bar g(\operatorname{Dc}^{\mathbb{I}}(\tilde\sigma))$ and $\bar\eta', \bar\eta''$ in $\operatorname{Dc}^{\mathbb{I}}(h)$ such that for all these pairs x, y, the map $x \mapsto y$ preserves all basic formulas as a finite partial map from $\mathbb{A}(\tilde\sigma, f, \tilde\iota)$ to $\mathbb{B}(\tilde\sigma, f, \tilde\iota)$ where $\tilde\iota := h\tilde b g$, whenever $\tilde\sigma$ is contained in the the additional sets and $\tilde b \in \mathcal{G}_{\mathbb{Q}}$ satisfies $\tilde b(\bar z) = \bar w$, $\tilde b(\bar z^*) = \bar w^*$, $\bar{\tilde b}(\bar z') = \bar w'$ as well as $\tilde b(\bar\zeta^*) = \bar\eta^*$, $\bar{\tilde b}(\bar\zeta') = \bar\eta'$, $\bar{\tilde b}(\bar\zeta'') = \bar\eta''$. We add the additional sets to the intersection O and add the tuples $\bar\zeta^*, \bar\eta^*$ to $\bar z^*, \bar w^*$, the tuples $\bar\zeta', \bar\eta'$ to $\bar z', \bar w'$ and the tuples $\bar\zeta'', \bar\eta''$ to $\bar z'', \bar w''$, respectively. Note that the resulting finite partial map $\bar z \mapsto \bar w$, $\bar z^* \mapsto \bar w^*$, $\bar z' \mapsto \bar w'$, $\bar z'' \mapsto \bar w''$ is strictly increasing: different entries of the new tuples $\bar\zeta^*, \bar\zeta', \bar\zeta'', \bar\eta^*, \bar\eta', \bar\eta''$ cannot interfere with each other since the generalised inverses $h^\dagger(x)$ are pairwise distinct and since we added the elements $\tilde z$ and $\tilde w$ from (4.10) to $\bar z^*$ and $\bar w^*$.

If $\tilde\sigma \in O$ and if $\tilde b \in \mathcal{G}_{\mathbb{Q}}$ satisfies $\tilde b(\bar z) = \bar w$, $\tilde b(\bar z^*) = \bar w^*$, $\bar{\tilde b}(\bar z') = \bar w'$, $\bar{\tilde b}(\bar z'') = \bar w''$, then by our previous construction of $\bar z^*, \bar z', \bar w^*, \bar w'$, the finite partial map $x \mapsto y$ preserves all basic formulas as a map from $\mathbb{A}(\tilde\sigma, f, \tilde\iota)$ to $\mathbb{B}(\tilde\sigma, f, \tilde\iota)$ not only for for each pair x, y with property (d) but also for each pair x, y with one of the properties (a)-(c) – thus completing the proof. □

4.5 Reduction of the rich to the pointwise topology

The aim of this section is to prove Proposition 4.3.5. We will argue in several steps, each having the following general form:

Notation 4.5.1. If $\mathcal{T}_a$ and $\mathcal{T}_b$ are topologies on $\mathcal{M}_\mathbb{Q}$ with $\mathcal{T}_{pw} \subseteq \mathcal{T}_a, \mathcal{T}_b$, then $\mathcal{T}_a \rightsquigarrow \mathcal{T}_b$ shall denote the following statement[11]:

> Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_\mathbb{Q}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_a$.
>
> Then $\mathcal{T} \subseteq \mathcal{T}_b$.

We will require an additional auxiliary type of subsets of $\mathcal{M}_\mathbb{Q}$ which encompasses type 3 (see Definition 4.3.1).

Definition 4.5.2.

(4) $O_A^{(4)} := \{s \in \mathcal{M}_\mathbb{Q} : \mathrm{Im}(s) \subseteq A\}$ for $A \subseteq \mathbb{Q}$ (restricting)

The proof will proceed along the following route:

$$\mathcal{T}_{rich} = \mathcal{T}_{0123} \overset{4.5.6}{\rightsquigarrow} \mathcal{T}_{01^{cls}23^{opn}} \overset{4.5.9}{\rightsquigarrow} \mathcal{T}_{024} \overset{4.5.12}{\rightsquigarrow} \mathcal{T}_{023^{opn}} \overset{4.5.20}{\rightsquigarrow} \mathcal{T}_{03^{opn}} \overset{4.5.23}{\rightsquigarrow} \mathcal{T}_0 = \mathcal{T}_{pw}$$

Proof (of Proposition 4.3.5 given Lemmas 4.5.6, 4.5.9, 4.5.12, 4.5.20 and 4.5.23).
Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_\mathbb{Q}$ with $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{rich} = \mathcal{T}_{0123}$. By Lemma 4.5.6, we know that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{01^{cls}23^{opn}}$. Analogously, we apply Lemmas 4.5.9, 4.5.12, 4.5.20 and 4.5.23 in sequence to finally conclude $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{pw}$, i.e. $\mathcal{T} = \mathcal{T}_{pw}$ as claimed. □

4.5.1 Reductions $\mathcal{T}_{0123} \rightsquigarrow \mathcal{T}_{01^{cls}23^{opn}} \rightsquigarrow \mathcal{T}_{024}$

For the first two reductions, we will need to determine the image of $\mathcal{T}_{0123}$-basic open (and in particular $\mathcal{T}_{01^{cls}23^{opn}}$-basic open) sets under a suitable left translation λ_f. This requires a canonical representation of basic open sets in $\mathcal{T}_{0123}$ and $\mathcal{T}_{01^{cls}23^{opn}}$ which will later be also applied to $\mathcal{T}_{023^{opn}}$-basic open sets.

Definition 4.5.3. Let $O \neq \emptyset$ be a $\mathcal{T}_{0123}$-basic open (or $\mathcal{T}_{01^{cls}23^{opn}}$-basic open or $\mathcal{T}_{023^{opn}}$-basic open) set, i.e.

$$O = \bigcap_{i=1}^{n} O^{(0)}_{x_i,y_i} \cap \bigcap_{j=1}^{m} O^{(1)}_{I_j,J_j} \cap \bigcap_{k=1}^{\widetilde{m}} O^{(1)}_{\tilde{I}_k,\tilde{J}_k} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{K_\ell} \tag{4.11}$$

where $I_j = (-\infty, p_j)$ and $\tilde{I}_k = (\tilde{p}_k, +\infty)$ for $p_j, \tilde{p}_k \in \mathbb{Q}$. We call the representation (4.11) *stratified* if

(S1) $\forall i = 1, \ldots, n-1$: $x_i < x_{i+1}$ (then automatically, $y_i \leq y_{i+1}$ since $O \neq \emptyset$)

(S2) $\forall j = 1, \ldots, m-1$: $p_j < p_{j+1}$ and $J_j \subseteq J_{j+1}$

[11] In many (but not all!) applications of this notation, we will have $\mathcal{T}_b \subseteq \mathcal{T}_a$.

(S3) $\forall k = 1, \dots, \widetilde{m} - 1\colon \tilde{p}_k < \tilde{p}_{k+1}$ and $\tilde{J}_k \supseteq \tilde{J}_{k+1}$

(S4) $\forall \ell = 1, \dots, N - 1\colon\ \sup K_\ell \le \inf K_{\ell+1}$ and $(\inf K_\ell, \sup K_{\ell+1}) \setminus (K_\ell \cup K_{\ell+1}) \neq \emptyset$

(S5) $\forall j = 1, \dots, m\, \forall i = 1, \dots, n\colon p_j \le x_i \Rightarrow y_i \notin J_j$

(S6) $\forall k = 1, \dots, \widetilde{m}\, \forall i = 1, \dots, n\colon \tilde{p}_k \ge x_i \Rightarrow y_i \notin \tilde{J}_k$

(S7) $\forall j = 1, \dots, m\, \forall \ell = 1, \dots, N\colon (J_j \cap K_\ell \neq \emptyset \Rightarrow \exists t \in J_j\colon t > K_\ell)$

(S8) $\forall k = 1, \dots, \widetilde{m}\, \forall \ell = 1, \dots, N\colon (\tilde{J}_k \cap K_\ell \neq \emptyset \Rightarrow \exists t \in \tilde{J}_k\colon t < K_\ell)$

Lemma 4.5.4. *Any $\mathcal{T}_{0123}$-basic open set O has a stratified representation.*

The same holds for a $\mathcal{T}_{01^{cls}23^{opn}}$-basic open set, where the resulting representation again consists of sets of types 0, 1^{cls}, 2 and 3^{opn}.

The same holds for a $\mathcal{T}_{023^{opn}}$-basic open set, where the resulting representation again consists of sets of types 0, 2 and 3^{opn}.

Proof. We start with any representation

$$O = \bigcap_{i=1}^{n} O^{(0)}_{x_i,y_i} \cap \bigcap_{j=1}^{m} O^{(1)}_{I_j,J_j} \cap \bigcap_{k=1}^{\widetilde{m}} O^{(1)}_{\tilde{I}_k,\tilde{J}_k} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{K_\ell}$$

and turn it into a stratified one in several steps, one for each item in Definition 4.5.3.

(S1). Rearrange the x_i in increasing order.

(S2). Rearrange the p_j in increasing order; if $p_j = p_{j+1}$, drop the larger set of J_j and J_{j+1}. If J_j is not a subset of J_{j+1}, then $J_{j+1} \subseteq J_j$ and $O^{(1)}_{I_j,J_j} \cap O^{(1)}_{I_{j+1},J_{j+1}}$ can be replaced by $O^{(1)}_{I_{j+1},J_{j+1}}$.

(S3). Analogously to (S2).

(S4). Rearrange the K_ℓ by increasing order of $\inf K_\ell$.
If $\sup K_\ell > \inf K_{\ell+1}$ or if $(\inf K_\ell, \sup K_{\ell+1}) \subseteq K_\ell \cup K_{\ell+1}$, then $K_\ell \cup K_{\ell+1}$ is again an interval and $O^{(3)}_{K_\ell} \cap O^{(3)}_{K_{\ell+1}}$ can be replaced by $O^{(3)}_{K_\ell \cup K_{\ell+1}}$.

(S5). If $p_j \le x_i$ and $y_i \in J_j$, then $O^{(0)}_{x_i,y_i} \cap O^{(1)}_{I_j,J_j}$ can be replaced by $O^{(0)}_{x_i,y_i}$.

(S6). Analogously to (S5).

(S7). If $J_j \cap K_\ell \neq \emptyset$ but no element of J_j is greater than K_ℓ, then $J_j \setminus K_\ell$ is again a rational interval and $O^{(1)}_{I_j,J_j} \cap O^{(3)}_{K_\ell}$ can be replaced by $O^{(1)}_{I_j,J_j \setminus K_\ell} \cap O^{(3)}_{K_\ell}$.

(S8). Analogously to (S7).

If all intervals J_j and $\tilde{J}_k$ are closed and all intervals K_ℓ are open, then so are the respective intervals in the representation we obtain after going through (S1)-(S8), proving the second statement.

If additionally no sets of type 1^{cls} occur, then the representation will only contains sets of types 0, 2 and 3^{opn} since the above procedure never generates sets of type 1 if there are none in the original set. □

Now we can provide the result about images of basic open sets under certain left translations (which can be seen as a generalisation of Lemma 4.1.7). Its proof is somewhat technical, but the main idea is very straightforward: if $f, s' \in \mathcal{M}_{\mathbb{Q}}$ and if s' satisfies $s'(-\infty, p) \subseteq (-\infty, q)$, then one might be led to believe that necessarily $fs'(-\infty, p) \subseteq (-\infty, f(q))$. However, in general only the conclusion $fs'(-\infty, p) \subseteq (-\infty, f(q)]$ is true, namely if the preimage $f^{-1}\{f(q)\}$ contains not only q but also elements less than q. This can be ensured by requiring that $f^{-1}\{f(q)\}$ is an irrational interval. Indeed, if f is also surjective, then *any* s with $s(-\infty, p) \subseteq (-\infty, f(q)]$ can be rewritten as $s = fs'$ where $s'(-\infty, p) \subseteq (-\infty, q)$. An analogous fact holds for sets of type 3 – if s' avoids $[u, v]$, then fs' in general only avoids $(f(u), f(v))$ and, conversely, if f is surjective with $f^{-1}\{f(u)\}$, $f^{-1}\{f(v)\}$ irrational and if s avoids $(f(u), f(v))$, then s can be rewritten as $s = fs'$ where s' avoids $[u, v]$. Combining these facts for the building blocks of (stratified representations of) basic open sets requires thorough bookkeeping.

Lemma 4.5.5. *Let $O \neq \emptyset$ be a nonempty $\mathcal{T}_{0123}$-basic open set with stratified representation*

$$O = \bigcap_{i=1}^{n} O^{(0)}_{x_i,y_i} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),J_j} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),\tilde{J}_k} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{K_\ell}.$$

Define

$$q_j := \sup J_j, \quad \tilde{q}_k := \inf \tilde{J}_k, \quad u_\ell := \inf K_\ell \quad \text{and} \quad v_\ell := \sup K_\ell.$$

Let further $f \in \mathcal{M}_{\mathbb{Q}}$ be unbounded-unbounded such that for all $w \in \operatorname{Im}(f)$, the preimage $f^{-1}\{w\}$ is an irrational interval. Then (putting $f(\pm\infty) := \pm\infty$) we have

$$\lambda_f(O) = \{s : \operatorname{Im}(s) \subseteq \operatorname{Im}(f)\} \cap \bigcap_{i=1}^{n} O^{(0)}_{x_i,f(y_i)} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),(-\infty,f(q_j)]} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),[f(\tilde{q}_k),+\infty)} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(f(u_\ell),f(v_\ell))}.$$

Proof. The inclusion "$\subseteq$" is immediate, so we deal only with "$\supseteq$".

Take $s \in \mathcal{M}_{\mathbb{Q}}$ such that $\operatorname{Im}(s) \subseteq \operatorname{Im}(f)$ and

$$s \in \bigcap_{i=1}^{n} O^{(0)}_{x_i,f(y_i)} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),(-\infty,f(q_j)]} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),[f(\tilde{q}_k),+\infty)} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(f(u_\ell),f(v_\ell))}. \tag{4.12}$$

We want to find $s' \in O$ such that $s = fs'$. The latter statement is equivalent to the assertion $s'(s^{-1}\{w\}) \subseteq f^{-1}\{w\}$ for all $w \in \operatorname{Im}(s)$. Since $\operatorname{Im}(s) \subseteq \operatorname{Im}(f)$, we have $f^{-1}\{w\} \neq \emptyset$ for all $w \in \operatorname{Im}(s)$. Note that if one takes $s'|_{s^{-1}\{w\}}$ to be an increasing map $s^{-1}\{w\} \to f^{-1}\{w\}$ independently for each $w \in \operatorname{Im}(s)$, their union will be increasing as well since $s^{-1}\{w_1\} < s^{-1}\{w_2\}$ and $f^{-1}\{w_1\} < f^{-1}\{w_2\}$ for all $w_1 < w_2$. Additionally requiring $s' \in O$ amounts to the following properties:

(i) $\forall i = 1, \ldots, n \colon s'(x_i) = y_i$

(ii) $s' \in O_{LU}^{(2)}$

(iii) $\forall j = 1, \ldots, m \, \forall w \in \mathrm{Im}(s) \colon s'\left(s^{-1}\{w\} \cap (-\infty, p_j)\right) \subseteq J_j \cap f^{-1}\{w\}$

(iv) $\forall k = 1, \ldots, \widetilde{m} \, \forall w \in \mathrm{Im}(s) \colon s'\left(s^{-1}\{w\} \cap (\tilde{p}_k, +\infty)\right) \subseteq \tilde{J}_k \cap f^{-1}\{w\}$

(v) $\forall \ell = 1, \ldots, N \, \forall w \in \mathrm{Im}(s) \colon s'\left(s^{-1}\{w\}\right) \cap K_\ell = \emptyset$

To simplify the proof, we replace (i) by:

(vi) $\forall i = 1, \ldots, n \, \forall w \in \mathrm{Im}(s) \colon s'\left(s^{-1}\{w\} \cap (-\infty, x_i)\right) \subseteq (-\infty, y_i] \cap f^{-1}\{w\}$

(vii) $\forall i = 1, \ldots, n \, \forall w \in \mathrm{Im}(s) \colon s'\left(s^{-1}\{w\} \cap (x_i, +\infty)\right) \subseteq [y_i, +\infty) \cap f^{-1}\{w\}$

If we find s' satisfying (ii)-(vii), then we can redefine $s'(x_i) := y_i$ to obtain $s' \in O$ – by (vi) and (vii), the resulting map will still be an element of $\mathcal{M}_{\mathbb{Q}}$; and since $O \neq \emptyset$, mapping $x_i \mapsto y_i$ cannot contradict (ii)-(v).

As a first step, we show that the statements in (ii)-(vii) are already implied by the requirement $s'(s^{-1}\{w\}) \subseteq f^{-1}\{w\}$ for many values w. If $w < f(q_j)$, then $f^{-1}\{w\} \subseteq J_j$, so (iii) automatically holds for $w < f(q_j)$ and (vi) for $w < f(y_i)$; a dual argument yields (iv) for $w > f(\tilde{q}_k)$ and (vii) for $w > f(y_i)$. Finally, if $w < f(u_\ell)$ or $w > f(v_\ell)$, then (v) is automatically satisfied as well since $f^{-1}\{w\} \cap K_\ell = \emptyset$.

Using (4.12), we obtain that (ii)-(vii) hold for many more values w. For instance, (iii) holds for $w > f(q_j)$: since $s(-\infty, p_j) \subseteq (-\infty, f(q_j)]$, we have $s^{-1}\{w\} \cap (-\infty, p_j) = \emptyset$. Similarly, (vi) holds for $w > f(y_i)$, (iv) holds for $w < f(\tilde{q}_k)$ and (vii) holds for $w < f(y_i)$. In (v), we do not have to consider $f(u_\ell) < w < f(v_\ell)$ since $\mathrm{Im}(s) \cap (f(u_\ell), f(v_\ell)) = \emptyset$. Finally, since f is unbounded-unbounded, any function s' with $s = fs'$ has the same boundedness type as s, i.e. (ii) is automatically satisfied as well.

Collecting the previous arguments and additionally reformulating (v), it suffices to ascertain the following properties (instead of (ii)-(vii)):

(iii') $\forall j = 1, \ldots, m \colon s'\left(s^{-1}\{f(q_j)\} \cap (-\infty, p_j)\right) \subseteq J_j \cap f^{-1}\{f(q_j)\}$

(iv') $\forall k = 1, \ldots, \widetilde{m} \colon s'\left(s^{-1}\{f(\tilde{q}_k)\} \cap (\tilde{p}_k, +\infty)\right) \subseteq \tilde{J}_k \cap f^{-1}\{f(\tilde{q}_k)\}$

(v') $\forall \ell = 1, \ldots, N \colon s'\left(s^{-1}\{f(u_\ell)\}\right) \subseteq f^{-1}\{f(u_\ell)\} \setminus K_\ell$ and
$s'\left(s^{-1}\{f(v_\ell)\}\right) \subseteq f^{-1}\{f(v_\ell)\} \setminus K_\ell$

(vi') $\forall i = 1, \ldots, n \colon s'\left(s^{-1}\{f(y_i)\} \cap (-\infty, x_i)\right) \subseteq (-\infty, y_i] \cap f^{-1}\{f(y_i)\}$

(vii') $\forall i = 1, \ldots, n \colon s'\left(s^{-1}\{f(y_i)\} \cap (x_i, +\infty)\right) \subseteq [y_i, +\infty) \cap f^{-1}\{f(y_i)\}$

We replace O by

$$\bigcap_{i=1}^{n} O^{(1)}_{(-\infty,x_i),(-\infty,y_i]} \cap O^{(1)}_{(x_i,+\infty),[y_i,+\infty)} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),J_j} \cap \bigcap_{k=1}^{\widetilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),\tilde{J}_k} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{K_\ell}, \tag{4.13}$$

observing that this representation is still stratified (up to adding the elements x_i to the sets $\{p_1, \ldots, p_m\}$ as well as $\{\tilde{p}_1, \ldots, \tilde{p}_{\widetilde{m}}\}$ and rearranging).

Since we have

$$s \in \left\{s' : \mathrm{Im}(s') \subseteq \mathrm{Im}(f)\right\} \cap \bigcap_{i=1}^{n} O^{(1)}_{(-\infty,x_i),(-\infty,f(y_i)]} \cap O^{(1)}_{(x_i,+\infty),[f(y_i),+\infty)} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),(-\infty,f(q_j)]} \cap \bigcap_{k=1}^{\widetilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),[f(\tilde{q}_k),+\infty)} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(f(u_\ell),f(v_\ell))},$$

we can, without loss of generality, subsume (vi') and (vii') in (iii') and (iv') so that we only have to deal with (iii')-(v').

If we can find an increasing map satisfying (iii')-(v'), then any extension s' of that map satisfying $s'(s^{-1}\{w\}) \subseteq f^{-1}\{w\}$ for $w \neq f(q_j), f(\tilde{q}_k), f(u_\ell), f(v_\ell)$ ($j = 1,\ldots,m$; $k = 1,\ldots,\widetilde{m}$; $\ell = 1,\ldots,N$) will be an element of $\mathcal{M}_{\mathbb{Q}}$ for which (ii)-(vii) hold – thus completing the proof.

Since the f-preimages of single elements are assumed to be irrational intervals, the right hand sides in (iii')-(v') are nonempty[12]: the preimage $f^{-1}\{f(q_j)\}$ in (iii') is an irrational interval which contains q_j, so q_j must be contained in the interior of $f^{-1}\{f(q_j)\}$; we conclude $J_j \cap f^{-1}\{f(q_j)\} \neq \emptyset$. For the other items, we argue analogously, noting in (v') that u_ℓ and v_ℓ are limit points not only of K_ℓ but also of $\mathbb{Q} \setminus K_\ell$.

In the remainder of the proof, we will use that the representation of O is stratified to show that combinations of (iii')-(v') are not contradictory, either. This could only happen if they are making statements about the same s-preimage, i.e. if the f-images of some of the points $q_j, \tilde{q}_k, u_\ell, v_\ell$ coincide. For each

$$w \in \{f(q_1),\ldots,f(q_m), f(\tilde{q}_1),\ldots,f(\tilde{q}_{\widetilde{m}}), f(u_1),\ldots,f(u_N), f(v_1),\ldots,f(v_N)\},$$

we will define s' on $s^{-1}\{w\}$.

We first show that we can find an image $s'(z)$ satisfying (iii')-(v') for each indivdual $z \in s^{-1}\{w\}$. If $f(q_j) = f(q_{j+1})$, then $J_j \cap f^{-1}\{f(q_j)\} \subseteq J_{j+1} \cap f^{-1}\{f(q_{j+1})\}$ by (S2). Therefore, it suffices to consider the *least* j such that $z \in (-\infty, p_j)$ (if such a j exists) and fulfil $s'(z) \in J_j$ – the other conditions of the same type will then be automatically satisfied. Analogously, it is enough to consider the *greatest* k such that $z \in (\tilde{p}_k, +\infty)$ (if it exists). For given z, we can thus reduce (iii') and (iv') to a *single* condition of the respective types (if they occur at all). Therefore, we need to map z to the intersection of $f^{-1}\{w\}$ and a combination of J_j and $\tilde{J}_k$ and $\bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell$ which respectively occur if $z \in (-\infty, p_j) \wedge f(q_j) = w$ and $z \in (\tilde{p}_k, +\infty) \wedge f(\tilde{q}_k) = w$ and $f(v_{\ell_1}) = f(u_{\ell_1+1}) = f(v_{\ell_1+1}) = \ldots, f(u_{\ell_2}) = w$ (and possibly $f(u_{\ell_1}) = w$ or $f(v_{\ell_2}) = w$ as well – we do not know whether the chain $f^{-1}\{w\} \cap (\{u_\ell : \ell = 1,\ldots,N\} \cup \{v_\ell : \ell = 1,\ldots,N\})$ begins and ends with an element u_* or v_*!). We distinguish cases by the types of sets actually occurring and show that this intersection is always nonempty[13]:

- We have already argued that $f^{-1}\{w\} \cap J_j \neq \emptyset \neq f^{-1}\{w\} \cap \tilde{J}_k$ and $f^{-1}\{w\} \cap \mathbb{Q} \setminus K_\ell \neq \emptyset$.

[12] This is the first half of the main observation behind the lemma!
[13] This is the second half of the main observation behind the lemma!

- If the sets $\mathbb{Q} \setminus K_\ell$ occur for $\ell = \ell_1, \ldots, \ell_2$, then $f^{-1}\{w\} \cap \bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell \neq \emptyset$, since $f^{-1}\{w\} \subseteq \bigcup_{\ell=\ell_1}^{\ell_2} K_\ell$ combined with (S4) would yield $f^{-1}\{w\} \subseteq K_\ell$ for some ℓ, contradicting the previous item.

- If both J_j and $\tilde{J}_k$ occur, then $J_j \cap \tilde{J}_k \neq \emptyset$ since $s(z) \in J_j \cap \tilde{J}_k$ by $s \in O$. Therefore, $\tilde{q}_k \leq q_j$ and $\sup(J_j \cap \tilde{J}_k) = q_j$ as well as $\inf(J_j \cap \tilde{J}_k) = \tilde{q}_k$. We know that both q_j and $\tilde{q}_k$ are contained in the interval $f^{-1}\{w\}$, whence $f^{-1}\{w\} \cap J_j \cap \tilde{J}_k = J_j \cap \tilde{J}_k \neq \emptyset$.

- If J_j and $\bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell$ occur, then $f^{-1}\{w\} \cap J_j \cap \bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell = \emptyset$ would imply $f^{-1}\{w\} \cap J_j \subseteq K_\ell$ for some ℓ, again via (S4). We pick any $r \in f^{-1}\{w\} \cap J_j \neq \emptyset$, and thus $r \in K_\ell$. By (S7), there exists $t \in J_j$ such that $t > K_\ell$. In particular, $t \in [r, q_j] \subseteq f^{-1}\{w\}$ which yields the contradiction $t \in f^{-1}\{w\} \cap J_j$ but $t \notin K_\ell$.

- If $\tilde{J}_k$ and $\bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell$ occur, one argues analogously.

- If both J_j and $\tilde{J}_k$ as well as $\bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell$ occur, we again derive a contradiction from $f^{-1}\{w\} \cap J_j \cap \tilde{J}_k \cap \bigcap_{\ell=\ell_1}^{\ell_2} \mathbb{Q} \setminus K_\ell = \emptyset$. As in the previous cases, (S4) yields $\emptyset \neq f^{-1}\{w\} \cap J_j \cap \tilde{J}_k \subseteq K_\ell$ for some ℓ. By (S7), there exists $t \in J_j$ such that $t > K_\ell$. Increasing t if necessary, one obtains the contradiction $t \in J_j \cap \tilde{J}_k = f^{-1}\{w\} \cap J_j \cap \tilde{J}_k$ but $t \notin K_\ell$.

Finally, we combine our arguments for each individual $z \in s^{-1}\{w\}$ to a definition of s' on the whole of $s^{-1}\{w\}$. We define an equivalence relation $\sim$ on $s^{-1}\{w\}$ by putting $z \sim z'$ if and only if z and z' are contained in the same intervals of the shapes $(-\infty, p_j)$ and $(\tilde{p}_k, +\infty)$. Clearly, there are only finitely many $\sim$-equivalence classes. Let $z_1, \ldots, z_M$ be a system of representatives of these equivalence classes which we assume to be arranged in increasing order. By the previous part of our proof, we can pick images $s'(z_1), \ldots, s'(z_M) \in f^{-1}\{w\}$ such that (iii')-(v') hold, where $s'(z_1), \ldots, s'(z_M)$ are in increasing order – the latter is possible by (S2) and (S3). Defining s' on the equivalence class represented by z_h to be the constant function with value $s'(z_h)$, we obtain an increasing function $s' \colon s^{-1}\{w\} \to f^{-1}\{w\}$ such that (iii')-(v') hold. □

Now we can prove the first two reductions.

Lemma 4.5.6. *It holds that* $\mathcal{T}_{0123} \rightsquigarrow \mathcal{T}_{01^{cls}23^{opn}}$.

Proof. Let $O \in \mathcal{T} \subseteq \mathcal{T}_{0123}$. We show that O is a $\mathcal{T}_{01^{cls}23^{opn}}$-neighbourhood of every element of O.

Take $s \in O$ and, using Lemma 4.4.5, pick a generic surjection f. Since $\operatorname{Im}(s) \subseteq \mathbb{Q} = \operatorname{Im}(f)$, there exists $s' \in \mathcal{M}_\mathbb{Q}$ such that $s = fs' = \lambda_f(s')$ by Lemma 4.1.7(i). Therefore, $s' \in \lambda_f^{-1}(O)$ where this set is $\mathcal{T}$-open by continuity of λ_f. In particular, $\lambda_f^{-1}(O) \in \mathcal{T}_{0123}$, so there exists

$$O' = O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{j=1}^{m} O^{(1)}_{I_j,J_j} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{\tilde{I}_k,\tilde{J}_k} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{K_\ell} \tag{4.14}$$

such that $s' \in O' \subseteq \lambda_f^{-1}(O)$. We conclude $s = \lambda_f(s') \in \lambda_f(O') \subseteq O$. By Lemma 4.5.4, we can assume the representation (4.14) to be stratified. Since $\operatorname{Im}(f) = \mathbb{Q}$, Lemma 4.5.5

asserts that $\lambda_f(O')$ is a $\mathcal{T}_{01^{cls}23^{opn}}$-basic open set, so O is indeed a $\mathcal{T}_{01^{cls}23^{opn}}$-neighbourhood of s. □

Remark 4.5.7. We can reformulate the proof of Lemma 4.5.6 as follows: We show that $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{01^{cls}23^{opn}})$ has Property **X** with respect to $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{0123})$, using the decomposition $s = fs' \operatorname{id}_\mathbb{Q}$ for a fixed generic surjection f, the fixed map $\operatorname{id}_\mathbb{Q}$ and varying s'. Applying Proposition 2.7.3(i) to the map $\operatorname{id}\colon (\mathcal{M}_\mathbb{Q}, \mathcal{T}_{0123}) \to (\mathcal{M}_\mathbb{Q}, \mathcal{T})$ – which is continuous since $\mathcal{T} \subseteq \mathcal{T}_{0123}$, note also that $(\mathcal{M}_\mathbb{Q}, \mathcal{T})$ is a topological semigroup – yields the continuity of $\operatorname{id}\colon (\mathcal{M}_\mathbb{Q}, \mathcal{T}_{01^{cls}23^{opn}}) \to (\mathcal{M}_\mathbb{Q}, \mathcal{T})$, so $\mathcal{T} \subseteq \mathcal{T}_{01^{cls}23^{opn}}$.

The second reduction is a slightly more involved application of Lemma 4.5.5, picking both f and s' in a more thoughtful way (tuned to the specific s being considered) by the following construction.

Lemma 4.5.8. *Let $s, f \in \mathcal{M}_\mathbb{Q}$ with $\operatorname{Im}(f) = (-\infty, \inf s) \cup \operatorname{Im}(s) \cup (\sup s, +\infty)$ and such that the preimages $f^{-1}\{w\}$ are irrational intervals, i.e. $f^{-1}\{w\} = (r_w, t_w)$ for all $w \in \operatorname{Im}(f)$, where $r_w, t_w \in \mathbb{I}$. Then there exists $s' \in \mathcal{M}_\mathbb{Q}$ such that $s = fs'$ and the following hold for all $p \in \mathbb{Q}$:*

(i) If $\sup s'(-\infty, p) < s'(p)$ then $\sup s'(-\infty, p) = r_{s(p)}$.

(ii) If $\inf s'(p, +\infty) > s'(p)$ then $\inf s'(p, +\infty) = t_{s(p)}$.

Proof. Defining s' as the union of order isomorphisms between $s^{-1}\{w\}$ and either $[z, z']$ or $(r_w, z']$ or $[z, t_w)$ or (r_w, t_w) where z and z' are fixed elements of $f^{-1}\{w\}$ – depending on the order type of $s^{-1}\{w\}$ – we obtain a map with the following properties:

(a) $s = fs'$

(b) $\forall w \in \operatorname{Im}(s)\colon \big(s^{-1}\{w\}$ has no greatest element $\Rightarrow \sup s'(s^{-1}\{w\}) = t_w\big)$ and[14]
$\forall w \in \operatorname{Im}(s)\colon \big(s^{-1}\{w\}$ has no least element $\Rightarrow \inf s'(s^{-1}\{w\}) = r_w\big)$

(c) $\forall w \in \operatorname{Im}(s)\colon s'|_{s^{-1}\{w\}}$ is continuous[15]

We only show (i), the second assertion follows analogously. Assuming $\sup s'(-\infty, p) < s'(p)$, we distinguish two cases:

Case 1 ($s(-\infty, p)$ has a greatest element): We set $w := \max s(-\infty, p)$. Then there exists $p_0 < p$ such that $s|_{(p_0,p)} \equiv w$. Observe first that $s(p) > w$, i.e. p is the supremum of $s^{-1}\{w\}$ but not a greatest element – for otherwise $(p_0, p] \subseteq s^{-1}\{w\}$, so (c) would yield $\sup s'(-\infty, p) = s'(p)$. By (b), we have $\sup s'(s^{-1}\{w\}) = t_w$. Since $\sup s'(-\infty, p) = \sup s'(s^{-1}\{w\})$, it remains to show $t_w = r_{s(p)}$, equivalently $(w, s(p)) \cap \operatorname{Im}(f) = \emptyset$. It suffices to note that $(w, s(p)) \cap \operatorname{Im}(s) = \emptyset$ and that $\operatorname{Im}(f) \setminus \operatorname{Im}(s)$ and the convex hull of $\operatorname{Im}(s)$ are disjoint by choice of $\operatorname{Im}(f)$.

Case 2 ($s(-\infty, p)$ does not have a greatest element): For each $p' < p$, there exists p'' such that $p' < p'' < p$ and $s(p') < s(p'') \le s(p)$. We have $s'(p'') \in f^{-1}\{s(p'')\}$ and thus

$$\sup s'(-\infty, p) \ge s'(p'') \ge \inf f^{-1}\{s(p'')\} = r_{s(p'')} \ge t_{s(p')}.$$

[14] In other words: s' exhausts $f^{-1}\{w\}$ whenever possible.

[15] Note: If $s^{-1}\{w\}$ has e.g. a greatest element, this does *not* mean that s' is continuous at that point but rather that s' is left-continuous there.

Hence, $\sup s'(-\infty,p) \geq \sup_{p'<p} t_{s(p')}$. We claim that $\sup_{p'<p} t_{s(p')} \geq r_{s(p)}$. The opposite would yield $f(q) \in \big(\sup s(-\infty,p), s(p)\big)$ for any $q \in \big(\sup_{p'<p} t_{s(p')}, r_{s(p)}\big)$. However, $\big(\sup s(-\infty,p), s(p)\big) \cap \operatorname{Im}(f) = \emptyset$ with the same reasoning as in Case 1.

On the other hand, $s'(p') \leq t_{s(p')} < r_{s(p)}$ for each $p' < p$ since $s(p') < s(p)$, so $\sup s'(-\infty,p) \leq r_{s(p)}$. □

For our reduction, we take into account the following two observations: on the one hand, if $s' \in O^{(1)}_{(-\infty,p),(-\infty,q]}$ and $s'(p) \leq q$, then $O^{(1)}_{(-\infty,p),(-\infty,q]}$ can be replaced by $O^{(0)}_{p,s'(p)}$; compare with (S5). On the other hand, if $r := \sup s'(-\infty,p) \in \mathbb{I}$, then no set of the form $O^{(1)}_{(-\infty,p),(-\infty,q]}$ with $q \in \mathbb{Q}$ (!) containing s' can prohibit that $\sup \tilde{s}'(-\infty,p) > r$ for some $\tilde{s}' \in O^{(1)}_{(-\infty,p),(-\infty,q]}$.

Lemma 4.5.9. *It holds that* $\mathcal{T}_{01^{cls}23^{opn}} \rightsquigarrow \mathcal{T}_{024}$.

Proof. Let $O \in \mathcal{T} \subseteq \mathcal{T}_{01^{cls}23^{opn}}$. We show that O is a $\mathcal{T}_{024}$-neighbourhood of every element of O.

Take $s \in O$ and, using Lemma 4.4.5, pick $f \in \mathcal{M}_{\mathbb{Q}}$ such that

$$\operatorname{Im}(f) = (-\infty, \inf s) \cup \operatorname{Im}(s) \cup (\sup s, +\infty)$$

and all the preimages $f^{-1}\{w\}$ are irrational intervals, i.e. $f^{-1}\{w\} = (r_w, t_w)$ for all $w \in \operatorname{Im}(f)$, where $r_w, t_w \in \mathbb{I}$ (note that $r_w = -\infty$ or $t_w = +\infty$ is impossible since f is unbounded-unbounded). By Lemma 4.5.8, there exists $s' \in \mathcal{M}_{\mathbb{Q}}$ satisfying $s = fs'$ and the following for all $p \in \mathbb{Q}$:

(i) If $\sup s'(-\infty,p) < s'(p)$ then $\sup s'(-\infty,p) = r_{s(p)}$.

(ii) If $\inf s'(p,+\infty) > s'(p)$ then $\inf s'(p,+\infty) = t_{s(p)}$.

Similarly to the proof of Lemma 4.5.6, we use $s = fs' = \lambda_f(s')$, the $\mathcal{T}$-continuity of λ_f and the assumption $\mathcal{T} \subseteq \mathcal{T}_{01^{cls}23^{opn}}$ to obtain a $\mathcal{T}_{01^{cls}23^{opn}}$-basic open set

$$O' = O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),(-\infty,q_j]} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),[\tilde{q}_k,+\infty)} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)} \tag{4.15}$$

such that $s \in \lambda_f(O') \subseteq O$. We additionally use Lemma 4.5.4 and assume that the representation (4.15) is stratified. If we have $s'(p_j) \leq q_j$ for some $j \in \{1,\ldots,m\}$, then $s' \in O^{(0)}_{p_j,s'(p_j)} \subseteq O^{(1)}_{(-\infty,p_j),(-\infty,q_j]}$ and we replace $O^{(1)}_{(-\infty,p_j),(-\infty,q_j]}$ in (4.15) by $O^{(0)}_{p_j,s'(p_j)}$. We proceed analogously if $s'(\tilde{p}_k) \geq \tilde{q}_k$. By rerunning the stratification procedure from Lemma 4.5.4, we again obtain a stratified representation. In our situation of (S7) and (S8) already holding, the proof of Lemma 4.5.4 never adds new sets of type 1. Hence, we can assume that

$$s'(p_j) > q_j \quad \text{for all } j \qquad \text{and} \qquad s'(\tilde{p}_k) < \tilde{q}_k \quad \text{for all } k. \tag{4.16}$$

Lemma 4.5.5 yields

$$\lambda_f(O') = \{\tilde{s} : \operatorname{Im}(\tilde{s}) \subseteq \operatorname{Im}(f)\} \cap O^{(0)}_{\bar{x},f(\bar{y})} \cap \bigcap_{j=1}^{m} O^{(1)}_{(-\infty,p_j),(-\infty,f(q_j)]} \cap \bigcap_{k=1}^{\tilde{m}} O^{(1)}_{(\tilde{p}_k,+\infty),[f(\tilde{q}_k),+\infty)} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(f(u_\ell),f(v_\ell))}. \quad (4.17)$$

From (4.16) and $s' \in O'$, we obtain $\sup s'(-\infty, p_j) \leq q_j < s'(p_j)$ for all j as well as $\inf s'(\tilde{p}_k, +\infty) \geq \tilde{q}_k > s'(\tilde{p}_k)$ for all k. By (i) and (ii), we conclude $\sup s'(-\infty, p_j) = r_{s(p_j)}$ for all j and $\inf s'(\tilde{p}_k, +\infty) = t_{s(\tilde{p}_k)}$ for all k. Therefore, $q_j \geq r_{s(p_j)}$ for all j and $\tilde{q}_k \leq t_{s(\tilde{p}_k)}$ for all k. Since the left hand sides of these inequalities are rational numbers while the right hand sides are irrational, we even obtain $q_j > r_{s(p_j)}$ for all j and $\tilde{q}_k < t_{s(\tilde{p}_k)}$ for all k. In other words, we have $f(q_j) \geq s(p_j)$ for all j and $f(\tilde{q}_k) \leq s(\tilde{p}_k)$ for all k. Consequently, we can replace the sets of type 1 in (4.17) by sets of type 0, similarly to the above: we set

$$P := \{\tilde{s} : \operatorname{Im}(\tilde{s}) \subseteq \operatorname{Im}(f)\} \cap O^{(0)}_{\bar{x},f(\bar{y})} \cap O^{(0)}_{\bar{p},s(\bar{p})} \cap O^{(0)}_{\widetilde{\bar{p}},s(\widetilde{\bar{p}})} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(f(u_\ell),f(v_\ell))}$$

where $\bar{p} = (p_1, \ldots, p_m), \widetilde{\bar{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_{\widetilde{m}})$ to obtain $s \in P \subseteq \lambda_f(O') \subseteq O$.

Putting

$$A := \operatorname{Im}(f) \cap \mathbb{Q} \setminus \bigcup_{\ell=1}^{N} (f(u_\ell), f(v_\ell)),$$

we see that

$$P = O^{(0)}_{\bar{x},f(\bar{y})} \cap O^{(0)}_{\bar{p},s(\bar{p})} \cap O^{(0)}_{\widetilde{\bar{p}},s(\widetilde{\bar{p}})} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}$$

is a $\mathcal{T}_{024}$-(basic) open set. Hence, O is indeed a $\mathcal{T}_{024}$-neighbourhood of s, as claimed. □

Remark 4.5.10. We can reformulate the proof of Lemma 4.5.9 as follows: We show that $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{024})$ has Property **X** with respect to $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{01^{cls}23^{opn}})$, again using the decomposition $s = fs'\mathrm{id}_{\mathbb{Q}}$ – this time for a fixed map f with $\operatorname{Im}(f) = (-\infty, \inf s) \cup \operatorname{Im}(s) \cup (\sup s, +\infty)$ whose preimages of single points are irrational intervals, the fixed map $\mathrm{id}_{\mathbb{Q}}$ and varying s'. As in Remark 4.5.7, we apply Proposition 2.7.3(i) to the continous map $\mathrm{id}\colon (\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{01^{cls}23^{opn}}) \to (\mathcal{M}_{\mathbb{Q}}, \mathcal{T})$ to obtain $\mathcal{T} \subseteq \mathcal{T}_{024}$.

4.5.2 Reduction $\mathcal{T}_{024} \rightsquigarrow \mathcal{T}_{023^{opn}}$

For the next statement, we again aim at showing that a $\mathcal{T}$-open set is a $\mathcal{T}_{023^{opn}}$-neighbourhood of its elements. Instead of directly doing this for *all* elements, we start by restricting to *injective* elements – this special case contains the bulk of the work. The main observation behind it is an analysis of “products” of the form $O^{(0)}_{\bar{z},\bar{z}} \circ O^{(4)}_{A}$ if $\bar{z} = (z_1, \ldots, z_n)$ is a tuple in $\mathbb{Q}$ and A is densely ordered (which is connected to g being injective). Clearly, if $(z_i, z_j) \cap A = \emptyset$, then no element of $O^{(0)}_{\bar{z},\bar{z}} \circ O^{(4)}_{A}$ can hit (z_i, z_j) – yielding a condition of type 3^{opn} instead of 4. As it turns out, this is the only obstruction to points in the image.

Lemma 4.5.11. *Let $\mathcal{T}$ be a semigroup topology on $\mathcal{M}_\mathbb{Q}$ such that $\mathcal{T} \subseteq \mathcal{T}_{024}$. Then any injective endomorphism $g \in \mathcal{M}_\mathbb{Q}$ has a neighbourhood basis consisting of $\mathcal{T}_{023^{opn}}$-open sets.*

Proof. Given an injective endomorphism g, let O be any $\mathcal{T}$-open neighbourhood of g.

We use continuity of the composition map. Since $g = \mathrm{id}_\mathbb{Q} \circ g \in O$, there exist $\mathcal{T}$-neighbourhoods V_1 of $\mathrm{id}_\mathbb{Q}$ and V_2 of g such that $V_1 \circ V_2 \subseteq O$. By assumption, $\mathcal{T} \subseteq \mathcal{T}_{024}$, hence there exist $\mathcal{T}_{024}$-basic open sets U_1, U_2 such that $\mathrm{id}_\mathbb{Q} \in U_1 \subseteq V_1$ and $g \in U_2 \subseteq V_2$. Note that a $\mathcal{T}_{024}$-basic open set containing $\mathrm{id}_\mathbb{Q}$ has the form $O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}$ for a tuple $\bar{z}$ in $\mathbb{Q}$ – sets of type 4 cannot occur. We can assume that U_2 has the form $U_2 = O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}$, where A is a densely ordered set (for otherwise, replace A by $\mathrm{Im}(g)$).

We obtain

$$g \in \left(O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}\right) \circ \left(O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}\right) \subseteq V_1 \circ V_2 \subseteq O. \tag{4.18}$$

The lemma will be proved once we find a $\mathcal{T}_{023^{opn}}$-open set P with

$$g \in P \subseteq \left(O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}\right) \circ \left(O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}\right).$$

Since (4.18) remains valid if we expand the tuple $\bar{z}$, we can assume that the elements listed in $\bar{y}$ are contained in $\bar{z}$. We write $\bar{z} = (z_1, \ldots, z_n)$ where the elements z_i shall be sorted in ascending order. Adding additional elements $z_\pm$ to $\bar{z}$ if necessary, we can assume that $z_1 = z_- < \inf A$ if A is bounded below and that $z_n = z_+ > \sup A$ if A is bounded above. To simplify notation, we set $z_0 := -\infty$ as well as $z_{n+1} := +\infty$. Further, we define

$$\mathcal{M}_0 := \left\{(i,j) \in \{0, \ldots, n+1\}^2 : i < j,\ (z_i, z_j) \cap A = \emptyset\right\},$$
$$\mathcal{M}_1 := \left\{(i,j) \in \{0, \ldots, n+1\}^2 : i < j,\ |(z_i, z_j) \cap A| = 1\right\}.$$

Note that $\{(z_i, z_j) : (i,j) \in \mathcal{M}_0\}$ always contains $(-\infty, z_-)$ if A is bounded below and $(z_+, +\infty)$ if A is bounded above. For $(i,j) \in \mathcal{M}_1$, define $w_{i,j}$ such that $(z_i, z_j) \cap A = \{w_{i,j}\}$. Expanding the tuple $\bar{z}$ once more, we can assume that the elements $w_{i,j}$ are also contained in $\bar{z}$. Defining $\mathcal{M}_0$ and $\mathcal{M}_1$ from this expanded tuple, we obtain $\mathcal{M}_1 = \emptyset$. For each pair (i,j), the set $(z_i, z_j) \cap A$ is thus either empty or it contains at least two elements – in which case it contains an infinite densely ordered set. We claim that

$$g \in P := O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{(i,j) \in \mathcal{M}_0} O^{(3)}_{(z_i,z_j)} \subseteq \left(O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}\right) \circ \left(O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}\right); \tag{4.19}$$

note that $\mathrm{Im}(g) \cap (z_i, z_j) \subseteq A \cap (z_i, z_j) = \emptyset$ for all $(i,j) \in \mathcal{M}_0$.

To prove the set inclusion in (4.19), the crucial step is to find $f \in O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}$ such that

$$\forall q \in \mathbb{Q} \setminus \bigcup_{(i,j) \in \mathcal{M}_0} (z_i, z_j) \colon f^{-1}\{q\} \cap A \neq \emptyset. \tag{4.20}$$

This will be accomplished via a Back&Forth strategy, distinguishing whether A is bounded or unbounded above and below. In the following, we will consider the case that A is bounded below and unbounded above; the other cases are treated analogously. We will first find an increasing map $\varphi\colon [z_-,+\infty)\to[z_-,+\infty)$ such that (4.20) holds with φ in place of f (note that $\mathbb{Q}\setminus\bigcup_{(i,j)\in\mathcal{M}_0}(z_i,z_j) = [z_-,+\infty)\setminus\bigcup_{(i,j)\in\mathcal{M}_0}(z_i,z_j)$). To this end, we consider the following property of a finite partial increasing map m from $[z_-,+\infty)$ to $[z_-,+\infty)$:

(+) For any $q\in\mathbb{Q}\setminus\bigcup_{(i,j)\in\mathcal{M}_0}(z_i,z_j)$ and all $u,u'\in\operatorname{Dom}(m)$ with $u<u'$, if $m(u)<q<m(u')$, then $(u,u')\cap A$ is an infinite densely ordered set[16].

Setting $C:=\mathbb{Q}\setminus\bigcup_{(i,j)\in\mathcal{M}_0}(z_i,z_j)$, we claim that the system of all finite partial increasing maps m from $[z_-,+\infty)$ to $[z_-,+\infty)$ satisfying (+) is an (A,C)-Back&Forth system (see Definition 2.8.3). In order to simplify notation, we formally add the elements $+\infty,-\infty$ to both $\operatorname{Dom}(m)$ and $\operatorname{Im}(m)$.

$(\mathbf{A},\mathbf{C})$-Back: Given $q\in C$, we set $u_-:=\max\{u\in\operatorname{Dom}(m): m(u)<q\}$ and further $u_+:=\min\{u\in\operatorname{Dom}(m): q<m(u)\}$; since we added $\pm\infty$ to the domain and image of m, these elements are welldefined[17]. We claim that $(u_-,u_+)\cap A$ is an infinite densely ordered set. Note that $u_-\neq-\infty$ since $q\in C\subseteq[z_-,+\infty)$. If u_+ is finite as well, our claim follows from condition (+), and if $u_+=+\infty$, it follows from A being unbounded above. Taking $p\in(u_-,u_+)\cap A$ such that both $(u_-,p)\cap A$ and $(p,u_+)\cap A$ are infinite densely ordered sets, we obtain that the extension m' of m by $m'(p):=q$ is an increasing map which still satisfies condition (+).

Forth: Given $p\in\mathbb{Q}\setminus\operatorname{Dom}(m)$, we set $u_-:=\max\{u\in\operatorname{Dom}(m): u<p\}$ and[18] $u_+:=\min\{u\in\operatorname{Dom}(m): p<u\}$. We distinguish cases:

Case 1 (both $(u_-,p)\cap A$ and $(p,u_+)\cap A$ are infinite densely ordered): Pick *any* $q\in\mathbb{Q}$ with $m(u_-)\le q\le m(u_+)$.

Case 2 ($(u_-,p)\cap A$ is infinite densely ordered, but $(p,u_+)\cap A$ is not): Pick $q:=m(u_+)$.

Case 3 ($(u_-,p)\cap A$ is not infinite densely ordered, but $(p,u_+)\cap A$ is): Pick $q:=m(u_-)$.

Case 4 (neither $(u_-,p)\cap A$ nor $(p,u_+)\cap A$ are infinite densely ordered): Pick *any* $q\in\mathbb{Q}$ with $m(u_-)\le q\le m(u_+)$.

Observe that the extension m' of m by $m'(p):=q$ is an increasing map which still satisfies condition (+); for Case 4, we see that $(u_-,u_+)\cap A$ is not infinite densely ordered – since (+) holds, $(u_-,u_+)\cap A$ is never considered as a set of the form $(u,u')\cap A$ in condition (+), even less so $(u_-,p)\cap A$ and $(p,u_+)\cap A$.

Since $\mathcal{M}_1=\emptyset$, we know that m defined by $\bar z\mapsto\bar z$ satisfies (+). Thus, Lemma 2.8.4 yields $\varphi\colon[z_-,+\infty)\to[z_-,+\infty)$ with $\varphi(\bar z)=\bar z$ and

$$\forall q\in C=\mathbb{Q}\setminus\bigcup_{(i,j)\in\mathcal{M}_0}(z_i,z_j)\colon \varphi^{-1}\{q\}\cap A\neq\emptyset.$$

Extending φ to a total map f by setting $f(q):=q$ for $q\in(-\infty,z_-)$, this finishes the definition of f; by design, $f\in O^{(0)}_{\bar z,\bar z}$. Since C is unbounded above, f must be as well. Moreover, f is obviously unbounded below, yielding $f\in O^{(0)}_{\bar z,\bar z}\cap O^{(2)}_{-\infty,+\infty}$ as desired.

[16] equivalently: This intersection contains at least two elements.

[17] Note: There might exist elements u of $\operatorname{Dom}(m)$ in between u_- and u_+, necessarily with $m(u)=q$. As can be seen from the proof, these elements can be neglected since they cannot prevent that an extension of m with image q is increasing.

[18] Here, there can never exist elements u of $\operatorname{Dom}(m)$ in between u_- and u_+!

Using f, we can finally prove (4.19). Let $s \in O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{(i,j)\in\mathcal{M}_0} O^{(3)}_{(z_i,z_j)}$. We will prove $s \in \left(O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}\right) \circ \left(O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}\right)$ by finding $h \in O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}$ such that $s = fh$. The latter equality can be certainly satisfied by picking, for each

$$q \in \operatorname{Im}(s) \subseteq \mathbb{Q} \setminus \bigcup_{(i,j)\in\mathcal{M}_0} (z_i, z_j) \subseteq \operatorname{Im}(f),$$

any element $p_q \in f^{-1}\{q\} \neq \emptyset$ and defining h by $h(c) := p_{s(c)}$. Because of (4.20), the elements p_q can be chosen in A, thus yielding $\operatorname{Im}(h) \subseteq A$. For the entries y_i of $\bar{y}$, we can pick $p_{y_i} = y_i$ since $f(\bar{y}) = \bar{y}$ – note that $\bar{y}$ has been added to $\bar{z}$, that the entries are pairwise different since g is injective and that $y_i \in A$ (by $g(\bar{x}) = \bar{y}$). Thus, $s(\bar{x}) = \bar{y}$ implies $h \in O^{(0)}_{\bar{x},\bar{y}}$. Since $f \in O^{(2)}_{-\infty,+\infty}$, the boundedness type of h is the same as the boundedness type of s which in turn is the same as the boundedness type of g. Hence, $h \in O^{(2)}_{LU}$ and we conclude $h \in O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}$. Therefore

$$s = fh \in \left(O^{(0)}_{\bar{z},\bar{z}} \cap O^{(2)}_{-\infty,+\infty}\right) \circ \left(O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap O^{(4)}_{A}\right),$$

thus proving (4.19) and, consequently, the lemma. □

Lemma 4.5.12. *It holds that $\mathcal{T}_{024} \rightsquigarrow \mathcal{T}_{023^{opn}}$.*

Proof. Let $O \in \mathcal{T}$. We show that O is a $\mathcal{T}_{023^{open}}$-neighbourhood of every element of O.

Take $s \in O$. We claim that for any generic surjection $f \in \mathcal{M}_\mathbb{Q}$ (which exists by Lemma 4.4.5), there is some injective $g \in \mathcal{M}_\mathbb{Q}$ such that $s = fg$: Since $\operatorname{Im}(s) \subseteq \mathbb{Q} = \operatorname{Im}(f)$ and since the preimages $f^{-1}\{w\}$ are irrational intervals, Lemma 4.1.7(ii) applies and yields an injective $g \in \mathcal{M}_\mathbb{Q}$ as desired.

We use continuity of the translation map λ_f. Since $\lambda_f(g) = s \in O$, there exists a $\mathcal{T}$-neighbourhood V of g such that $\lambda_f(V) \subseteq O$. By Lemma 4.5.11, there exists a $\mathcal{T}_{023^{opn}}$-basic open set U such that $g \in U \subseteq V$; we assume U to be stratified via Lemma 4.5.4. Hence, $s \in \lambda_f(U) \subseteq O$. Using Lemma 4.5.5, we obtain that $\lambda_f(U)$ is a $\mathcal{T}_{023^{opn}}$-basic open set which proves the lemma. □

Remark 4.5.13. We can combine Lemmas 4.5.11 and 4.5.12 and reformulate the proof of $\mathcal{T}_{024} \rightsquigarrow \mathcal{T}_{023^{opn}}$ as follows: We show that $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{023^{opn}})$ has Property $\overline{\mathbf{X}}$ of length 2 with respect to $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{024})$, using the decomposition $s = f \operatorname{id}_\mathbb{Q} \operatorname{id}_\mathbb{Q} g \operatorname{id}_\mathbb{Q}$ where the first, third and fifth position are fixed and the second and fourth position are varying, subsequently yielding $\tilde{s} = f \tilde{f} \operatorname{id}_\mathbb{Q} \tilde{h} \operatorname{id}_\mathbb{Q}$. As in Remarks 4.5.7 and 4.5.10, we apply Proposition 2.7.3(i) to the continous map $\operatorname{id}\colon (\mathcal{M}_\mathbb{Q}, \mathcal{T}_{024}) \to (\mathcal{M}_\mathbb{Q}, \mathcal{T})$ to obtain $\mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$.

4.5.3 Reduction $\mathcal{T}_{023^{opn}} \rightsquigarrow \mathcal{T}_{03^{opn}}$

In our next reduction, we eliminate the sets of type 2, i.e. the boundedness types, from the upper bound. Compared to our previous reductions, this requires a different approach; we use the regularity of the given topology $\mathcal{T}$ in a crucial way. The main observation is the following: if O is $\mathcal{T}$-open and $s \in O$, there exists a $\mathcal{T}$-open set P such that $s \in P \subseteq \overline{P}^{\mathcal{T}} \subseteq O$, where $\overline{P}^{\mathcal{T}}$ denotes the topological closure of P with respect to $\mathcal{T}$. Our proof essentially amounts to showing that taking this topological closure eliminates the sets $O^{(2)}_{LU}$ from P – this corresponds to $O^{(2)}_{LU}$ being topologically dense. It is easy to see that $O^{(2)}_{LU}$ is dense with respect to the pointwise topology; however, this set is obviously not dense with respect to $\mathcal{T}_{023^{opn}}$. Hence, independently of the above sketch, it can also be seen as an important step in showing $\mathcal{T} = \mathcal{T}_{pw}$ that indeed $O^{(2)}_{LU}$ is dense with respect to $\mathcal{T}$ as well. This will depend on the Polishness of $\mathcal{T}$. We start with a variant of Lemma 4.5.8.

Lemma 4.5.14. *Let $s, f \in \mathcal{M}_{\mathbb{Q}}$ and $q \in \mathbb{Q} \setminus \mathrm{Im}(s)$ such that $\mathrm{Im}(f) = \mathrm{Im}(s) \dot{\cup} \{q\}$ where the preimages $f^{-1}\{w\}$ are irrational intervals, i.e. $f^{-1}\{w\} = (r_w, t_w)$ for all $w \in \mathrm{Im}(f)$, where $r_w, t_w \in \mathbb{I} \cup \{\pm\infty\}$. Then the following hold:*

(1) Suppose there is $p \in \mathbb{Q}$ such that $\sup s(-\infty, p) = \max s(-\infty, p) < q < s(p)$. Then there exists $s' \in \mathcal{M}_{\mathbb{Q}}$ such that $s = fs'$ and $\sup s'(-\infty, p) = r_q \in \mathbb{I}$.

(2) Suppose that $\sup s = \max s < q$. Then there exists $s' \in \mathcal{M}_{\mathbb{Q}}$ such that $s = fs'$ and $\sup s' = r_q \in \mathbb{I}$.

(3) Suppose that $q < \min s = \inf s$. Then there exists $s' \in \mathcal{M}_{\mathbb{Q}}$ such that $s = fs'$ and $\inf s' = t_q \in \mathbb{I}$.

Proof. One picks $s' \in \mathcal{M}_{\mathbb{Q}}$ with

(a) $s = fs'$

(b) $\forall w \in \mathrm{Im}(s)\colon \big(s^{-1}\{w\}$ has no greatest element $\Rightarrow \sup s'(s^{-1}\{w\}) = t_w\big)$ and
$\forall w \in \mathrm{Im}(s)\colon \big(s^{-1}\{w\}$ has no least element $\Rightarrow \inf s'(s^{-1}\{w\}) = r_w\big)$

(c) $\forall w \in \mathrm{Im}(s)\colon s'|_{s^{-1}\{w\}}$ is continuous

and argues as in Case 1 of the proof of Lemma 4.5.8 with q in place of $s(p)$. The boundary points r_q (for (1),(2)) and t_q (for (3)) are finite since there exist elements in $\mathrm{Im}(f)$ which are below q and above q, respectively. □

Next, we show that the set $\mathrm{Surj}(\mathbb{Q})$ of all surjective elements of $\mathcal{M}_{\mathbb{Q}}$ is dense with respect to our given Polish semigroup topology $\mathcal{T}$ with $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$ – this uses Polishness in an essential way and is another step in matching $\mathcal{T}$ to $\mathcal{T}_{pw}$.

Lemma 4.5.15. *Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$.*

(i) For each $q \in \mathbb{Q}$, the set $M_q := \{s \in \mathcal{M}_{\mathbb{Q}} : q \in \mathrm{Im}(s)\}$ is $\mathcal{T}$-dense.

(ii) The set $\mathrm{Surj}(\mathbb{Q})$ of surjective endomorphisms on $\mathbb{Q}$ is $\mathcal{T}$-dense.

Proof.
(i). Let $O \in \mathcal{T}$ be open and nonempty; we have to show $O \cap M_q \neq \emptyset$. Since $\mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$, the set O contains a nonempty $\mathcal{T}_{023^{opn}}$-basic open set; we write

$$\emptyset \neq O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)} \subseteq O$$

which we assume to be a stratified representation, see Lemma 4.5.4. If q is contained in $\bar{y}$, then *any* $s \in O$ has q in its image, so we assume the contrary. Distinguishing by the position of q relative to $\bar{y} = (y_1, \dots, y_n)$ and $(u_1, v_1, \dots, u_N, v_N)$ and by the required boundedness type $O^{(2)}_{LU}$, one easily constructs (by a piecewise definition) a map $s \in O$ possibly together with a rational $p \in \mathbb{Q}$ such that $\sup s(-\infty, p) = \max s(-\infty, p) < q < s(p)$ (if $\mathbb{Q} \setminus \bigcup_{\ell=1}^{N}(u_\ell, v_\ell)$ contains elements less and elements greater than q) or $\sup s = \max s < q$ (if $\mathbb{Q} \setminus \bigcup_{\ell=1}^{N}(u_\ell, v_\ell)$ contains only elements less than q) or $q < \min s = \inf s$ (if $\mathbb{Q} \setminus \bigcup_{\ell=1}^{N}(u_\ell, v_\ell)$ contains only elements greater than q). We use Lemma 4.4.5 to find $f \in \mathcal{M}_\mathbb{Q}$ with $\operatorname{Im}(f) = \operatorname{Im}(s) \dot\cup \{q\}$ and $f^{-1}\{w\} = (r_w, t_w)$ for all $w \in \operatorname{Im}(f)$, where $r_w, t_w \in \mathbb{I} \cup \{\pm\infty\}$. By Lemma 4.5.14, there exists $s' \in \mathcal{M}_\mathbb{Q}$ such that $s = fs'$ and $\sup s'(-\infty, p) = r_q \in \mathbb{I}$ or $\sup s' = r_q \in \mathbb{I}$ or $\inf s' = t_q \in \mathbb{I}$. Applying continuity of the translation map λ_f at s' as well as $\mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$, we obtain a $\mathcal{T}_{023^{opn}}$-basic open set

$$O' = O^{(0)}_{\bar{x}', \bar{y}'} \cap O^{(2)}_{L'U'} \cap \bigcap_{\ell=1}^{N'} O^{(3)}_{(u'_\ell, v'_\ell)}$$

such that $s' \in O'$ and $s \in \lambda_f(O') \subseteq O$. In particular, $\operatorname{Im}(s') \subseteq \mathbb{Q} \setminus \bigcup_{\ell=1}^{N}(u'_\ell, v'_\ell) =: A'$, so either r_q or t_q is a limit point of A'. Since r_q and t_q are irrational while the boundary points of A' are rational, either r_q or t_q must in fact be contained in the interior of A'. Thus, $A' \cap f^{-1}\{q\} = A' \cap (r_q, t_q) \neq \emptyset$; we pick z' in this intersection.

Similarly to our construction of s, we distinguish by the positition of z' relative to $\bar{y}' = (y'_1, \dots, y'_{n'})$ and $(u'_1, v'_1, \dots, u'_{N'}, v'_{N'})$ and by the required boundedness type $O^{(2)}_{L'U'}$ to find a map $\tilde{s}' \in O'$ with $z' \in \operatorname{Im}(\tilde{s}')$. We obtain $\tilde{s} := \lambda_f(\tilde{s}') = f\tilde{s}' \in O$ and $q \in \operatorname{Im}(\tilde{s})$, i.e. $\tilde{s} \in O \cap M_q \neq \emptyset$.

(ii). For each $q \in \mathbb{Q}$, the set $M_q = \{s \in \mathcal{M}_\mathbb{Q} : q \in \operatorname{Im}(s)\}$ is $\mathcal{T}$-open since $\mathcal{T}_{pw} \subseteq \mathcal{T}$. By (i), it is also $\mathcal{T}$-dense. Since $\mathcal{T}$ is a Polish topology, Baire's Category Theorem applies and yields the $\mathcal{T}$-density of $\operatorname{Surj}(\mathbb{Q}) = \bigcap_{q \in \mathbb{Q}} M_q$. □

By definition, any $\mathcal{T}_{023^{opn}}$-open set can be represented as a union of sets of the form

$$O^{(0)}_{\bar{x}, \bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell, v_\ell)}.$$

If we rearrange to separate the $\mathcal{T}_{02}$-interior from the "proper" type 3^{opn} portion, we obtain the following alternative notation which will prove to be very helpful:

Notation 4.5.16. Setting

$$A := O^{(2)}_{-\infty,+\infty} \qquad B := O^{(2)}_{-\infty,\mathbb{R}} \qquad C := O^{(2)}_{\mathbb{R},+\infty} \qquad D := O^{(2)}_{\mathbb{R},\mathbb{R}},$$

we can rewrite any $\mathcal{T}_{023^{opn}}$-open set O as

$$O = (O_A \cap A) \cup (O_B \cap B) \cup (O_C \cap C) \cup (O_D \cap D) \cup \bigcup_{i \in I} \left(O^{(0)}_{\bar{x}^{(i)}, \bar{y}^{(i)}} \cap O^{(2)}_{L^{(i)}, U^{(i)}} \cap \bigcap_{\ell=1}^{N^{(i)}} O^{(3)}_{(u^{(i)}_\ell, v^{(i)}_\ell)} \right)$$

where $O_A, O_B, O_C, O_D \in \mathcal{T}_{pw}$, $\bar{x}^{(i)}, \bar{y}^{(i)}$ are tuples in $\mathbb{Q}$, $N^{(i)} \geq 1$ and $u^{(i)}_\ell, v^{(i)}_\ell \in \mathbb{Q}$.

Note that the sets O_A, O_B, O_C, O_D could in general be empty even if O is nonempty. However, for $O \in \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$, one uses the previous lemma to prove:

Lemma 4.5.17. *Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$, and let $O \in \mathcal{T}$ be nonempty. Then $O \subseteq \overline{O_A}^{\mathcal{T}}$. In particular, $O_A \neq \emptyset$.*

Proof. Aiming for a contradiction, we assume $O \nsubseteq \overline{O_A}^{\mathcal{T}}$. Thus, denoting the complement of $\overline{O_A}^{\mathcal{T}}$ by $\left(\overline{O_A}^{\mathcal{T}}\right)^{\mathrm{c}}$, we know that $O \cap \left(\overline{O_A}^{\mathcal{T}}\right)^{\mathrm{c}}$ is a nonempty $\mathcal{T}$-open set. However, $O^{(3)}_{(u_\ell^{(i)}, v_\ell^{(i)})} \cap \mathrm{Surj}(\mathbb{Q}) = \emptyset$ and $(B \cup C \cup D) \cap \mathrm{Surj}(\mathbb{Q}) = \emptyset$ imply

$$O \cap \left(\overline{O_A}^{\mathcal{T}}\right)^{\mathrm{c}} \cap \mathrm{Surj}(\mathbb{Q}) = O_A \cap A \cap \left(\overline{O_A}^{\mathcal{T}}\right)^{\mathrm{c}} \cap \mathrm{Surj}(\mathbb{Q}) \subseteq O_A \cap \left(\overline{O_A}^{\mathcal{T}}\right)^{\mathrm{c}} = \emptyset,$$

which contradicts Lemma 4.5.15(ii). □

With this result, we can attain an important intermediate step already hinted at in our proof outline in the introductory remarks to Subsection 4.5.3.

Lemma 4.5.18. *Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$. Then any nonempty $O \in \mathcal{T}$ has nonempty $\mathcal{T}_{pw}$-interior. Consequently, a subset of $\mathcal{M}_{\mathbb{Q}}$ is $\mathcal{T}$-dense if and only if it is $\mathcal{T}_{pw}$-dense; in particular, every boundedness type $O^{(2)}_{LU}$ is $\mathcal{T}$-dense.*

Proof. By regularity, there exists a nonempty $P \in \mathcal{T}$ such that $\overline{P}^{\mathcal{T}} \subseteq O$ and therefore $\overline{P_A \cap A}^{\mathcal{T}} \subseteq O$. Since $A \supseteq \mathrm{Surj}(\mathbb{Q})$ is $\mathcal{T}$-dense by Lemma 4.5.15(ii) and P_A is $\mathcal{T}$-open, we obtain $\overline{P_A \cap A}^{\mathcal{T}} = \overline{P_A}^{\mathcal{T}}$ from elementary topology. Thus, $P_A \subseteq \overline{P_A \cap A}^{\mathcal{T}} \subseteq O$ and the $\mathcal{T}_{pw}$-interior of O contains the set P_A which is nonempty by Lemma 4.5.17.

That any $\mathcal{T}$-dense set is $\mathcal{T}_{pw}$-dense follows from $\mathcal{T}_{pw} \subseteq \mathcal{T}$. For the converse, assume that M is $\mathcal{T}_{pw}$-dense and let O be nonempty and $\mathcal{T}$-open. Since the $\mathcal{T}_{pw}$-interior of O is nonempty, it has nonempty intersection with M, in particular $M \cap O \neq \emptyset$. □

Next, we use the previous results to show that taking the topological closure with respect to $\mathcal{T}$ eliminates the boundedness types from open sets. This is the crucial technical step in the proof of our reduction.

Lemma 4.5.19. *Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_{\mathbb{Q}}$ such that $\mathcal{T}_{pw} \subseteq \mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$. Let further $O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)} \neq \emptyset$ be a nonempty $\mathcal{T}_{023^{opn}}$-basic open set. Then*

$$\overline{O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}}^{\mathcal{T}} = O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}.$$

Proof. The inclusion "$\subseteq$" follows from $O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}$ being $\mathcal{T}_{pw}$-closed, in particular $\mathcal{T}$-closed.

For the other inclusion "$\supseteq$", take $s \in O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}$ and consider a $\mathcal{T}$-open set O containing s. We have to show $O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)} \cap O \neq \emptyset$. Pick $f \in \mathcal{M}_{\mathbb{Q}}$ such that

$\mathrm{Im}(f) = \mathbb{Q}\setminus\left(\bigcup_{\ell=1}^{N}(u_\ell, v_\ell)\right) \supseteq \mathrm{Im}(s)$. By continuity of the translation map λ_f, the preimage $\lambda_f^{-1}(O)$ is $\mathcal{T}$-open. Lemma 4.1.7(i) yields a map $s' \in \mathcal{M}_{\mathbb{Q}}$ such that $s = fs'$. We conclude from $\mathcal{T}_{pw} \subseteq \mathcal{T}$ that the intersection $\emptyset \neq \lambda_f^{-1}(O) \cap O^{(0)}_{\bar{x}, s'(\bar{x})} \ni s'$ is $\mathcal{T}$-open. By Lemma 4.5.18, the boundedness type $O^{(2)}_{LU}$ is $\mathcal{T}$-dense, therefore there exists $\tilde{s}' \in \lambda_f^{-1}(O) \cap O^{(0)}_{\bar{x}, s'(\bar{x})} \cap O^{(2)}_{LU}$. We define $\tilde{s} := f\tilde{s}' = \lambda_f(\tilde{s}')$ and claim

$$\tilde{s} \in O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell, v_\ell)} \cap O$$

which will complete the proof. We only argue $\tilde{s} \in O^{(2)}_{LU}$, the rest is straightforward. If $-\infty$ occurs among the u_ℓ, then $L = \mathbb{R}$ since $O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)} \neq \emptyset$. Further, $\tilde{s}$ is bounded below since $\tilde{s} \in \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}$. If on the other hand $-\infty$ is not contained among the u_ℓ, then f is unbounded below, so $\tilde{s}$ is unbounded below if and only if $\tilde{s}'$ is unbounded below which occurs if and only if $L = \{-\infty\}$. Arguing analogously for upper bounds, we conclude $\tilde{s} \in O^{(2)}_{LU}$. □

Lemmas 4.5.18 and 4.5.19 finally enable us to show our reduction:

Lemma 4.5.20. *It holds that* $\mathcal{T}_{023^{opn}} \rightsquigarrow \mathcal{T}_{03^{opn}}$.

Proof. Let $O \in \mathcal{T}$. We show that O is a $\mathcal{T}_{03^{opn}}$-neighbourhood of every element of O.

Take $s \in O$. By regularity, there exists $P \in \mathcal{T}$ such that $s \in P \subseteq \overline{P}^{\mathcal{T}} \subseteq O$. Since $\mathcal{T} \subseteq \mathcal{T}_{023^{opn}}$, there exists a $\mathcal{T}_{023^{opn}}$-basic open set

$$U = O^{(0)}_{\bar{x},\bar{y}} \cap O^{(2)}_{LU} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell,v_\ell)}$$

such that $s \in U \subseteq P$, in particular $s \in \overline{U}^{\mathcal{T}} \subseteq O$. By Lemma 4.5.19, the $\mathcal{T}$-closure of U is $O^{(0)}_{\bar{x},\bar{y}} \cap \bigcap_{\ell=1}^{N} O^{(3)}_{(u_\ell^{(i)},v_\ell^{(i)})}$. Hence, O is indeed a $\mathcal{T}_{03^{opn}}$-neighbourhood of s. □

Remark 4.5.21. As already stated in the concluding remarks of Section 4.3, the reduction $\mathcal{T}_{023^{opn}} \rightsquigarrow \mathcal{T}_{03^{opn}}$ is the only one whose proof cannot be reformulated as a (Pseudo-)Property $\overline{\mathbf{X}}$-type statement. Starting from Proposition 4.3.4 and applying Proposition 2.7.3 along the route $\mathcal{T}_{rich} = \mathcal{T}_{0123} \rightsquigarrow \mathcal{T}_{01^{cls}23^{opn}} \rightsquigarrow \mathcal{T}_{024} \rightsquigarrow \mathcal{T}_{023^{opn}}$, we obtain that $(\mathcal{M}_{\mathbb{Q}}, \mathcal{T}_{023^{opn}})$ has automatic continuity with respect to the class of second countable topological semigroups. However, we cannot continue on to $\mathcal{T}_{03^{opn}}$. Thus, the reduction $\mathcal{T}_{023^{opn}} \rightsquigarrow \mathcal{T}_{03^{opn}}$ is indeed fundamentally different.

4.5.4 Reduction $\mathcal{T}_{03^{opn}} \rightsquigarrow \mathcal{T}_0 = \mathcal{T}_{pw}$

In our final reduction, we eliminate the sets of type 3^{opn}. The technique resembles those of Subsections 4.5.1 and 4.5.2, albeit with crucial involvement of the $\mathcal{T}$-density of $\mathrm{Surj}(\mathbb{Q})$ shown in Subsection 4.5.3. The following easy observation gives an idea as to why this is important.

Lemma 4.5.22. *Let $\mathcal{T}$ be a Polish semigroup topology on $\mathcal{M}_\mathbb{Q}$ such that $\mathcal{T} \subseteq \mathcal{T}_{03^{opn}}$. Let further $O \in \mathcal{T}$ and let $f \in O$ be surjective. Then O is a $\mathcal{T}_{pw}$-neighbourhood of f, in other words, there exists $P \in \mathcal{T}_{pw}$ such that $f \in P \subseteq O$.*

Proof. With the same spirit as in Notation 4.5.16, we can write

$$O = O_{pw} \cup \bigcup_{i \in I} \left(O^{(0)}_{\bar{x}^{(i)}, \bar{y}^{(i)}} \cap \bigcap_{\ell=1}^{N^{(i)}} O^{(3)}_{(u^{(i)}_\ell, v^{(i)}_\ell)} \right),$$

where $O_{pw} \in \mathcal{T}_{pw}$, $\bar{x}^{(i)}, \bar{y}^{(i)}$ are tuples in $\mathbb{Q}$, $N^{(i)} \geq 1$ and $u^{(i)}_\ell, v^{(i)}_\ell \in \mathbb{Q}$. Since none of the sets $O^{(0)}_{\bar{x}^{(i)}, \bar{y}^{(i)}} \cap \bigcap_{\ell=1}^{N^{(i)}} O^{(3)}_{(u^{(i)}_\ell, v^{(i)}_\ell)}$ can contain surjective functions, f has to be contained in $P := O_{pw}$. □

Lemma 4.5.23. *It holds that $\mathcal{T}_{03^{opn}} \rightsquigarrow \mathcal{T}_0 = \mathcal{T}_{pw}$.*

Proof. Let $O \in \mathcal{T}$. We show that O is a $\mathcal{T}_{pw}$-neighbourhood of every element of O.

Take $s \in O$. By $\mathcal{T}$-continuity of the composition map $\circ$ and since $s \circ \mathrm{id}_\mathbb{Q} \in O$, there exist $\mathcal{T}$-open sets U and V such that $s \in U$, $\mathrm{id}_\mathbb{Q} \in V$ and $U \circ V \subseteq O$. Using Lemma 4.5.22, we can shrink V and assume that V is $\mathcal{T}_{pw}$-open; shrinking further we can even take V to be $\mathcal{T}_{pw}$-basic open, so $V = O^{(0)}_{\bar{x},\bar{x}}$. The set $U \cap O^{(0)}_{\bar{x},s(\bar{x})}$ is a nonempty $\mathcal{T}$-open set. By Lemma 4.5.15(ii), the surjective functions form a $\mathcal{T}$-dense set, so there exists a function $f \in U \cap O^{(0)}_{\bar{x},s(\bar{x})} \cap \mathrm{Surj}(\mathbb{Q})$.

We claim that $f \circ O^{(0)}_{\bar{x},\bar{x}} = O^{(0)}_{\bar{x},f(\bar{x})}$ $(= O^{(0)}_{\bar{x},s(\bar{x})})$. The inclusion "$\subseteq$" is clear; for the converse inclusion "$\supseteq$", we argue as follows: given $\tilde{s} \in O^{(0)}_{\bar{x},f(\bar{x})}$, the finite partial map m defined by $\bar{x} \mapsto \bar{x}$ satisfies $\tilde{s}(p) = fm(p)$ for all $p \in \mathrm{Dom}(m)$. Since f is surjective, we can apply Lemma 4.1.7(i) to find $\tilde{s}' \in \mathcal{M}_\mathbb{Q}$ such that $\tilde{s}'(\bar{x}) = \bar{x}$ and $\tilde{s} = f\tilde{s}'$, thus proving the claim.

We obtain

$$s \in O^{(0)}_{\bar{x},s(\bar{x})} = f \circ O^{(0)}_{\bar{x},\bar{x}} \subseteq U \circ V \subseteq O,$$

showing that O is indeed a $\mathcal{T}_{pw}$-neighbourhood of s, as desired. □

Remark 4.5.24. We can reformulate the proof of Lemma 4.5.23 as follows: We show that $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{pw})$ has Property $\overline{\mathbf{X}}$ of length 2 with respect to $(\mathcal{M}_\mathbb{Q}, \mathcal{T}_{03^{opn}})$, using the decomposition $s = \mathrm{id}_\mathbb{Q}\, s\, \mathrm{id}_\mathbb{Q}\, \mathrm{id}_\mathbb{Q}\, \mathrm{id}_\mathbb{Q}$ where the first, third and fifth position are fixed and the second and fourth position are varying, subsequently yielding $\tilde{s} = \mathrm{id}_\mathbb{Q}\, f\, \mathrm{id}_\mathbb{Q}\, \tilde{s}'\, \mathrm{id}_\mathbb{Q}$. As in Remarks 4.5.7, 4.5.10 and 4.5.13, we apply Proposition 2.7.3(i) to the continous map $\mathrm{id}\colon (\mathcal{M}_\mathbb{Q}, \mathcal{T}_{03^{opn}}) \to (\mathcal{M}_\mathbb{Q}, \mathcal{T})$ to obtain $\mathcal{T} \subseteq \mathcal{T}_{pw}$. Observe that the existence of f requires the density statements from the previous reduction (which were shown using Polishness).

5 Endomorphism monoids of homogeneous graphs

We continue the programme of studying Question C, i.e. analysing homogeneous graphs with regard to whether their endomorphism monoids have UPP and arrive at an almost complete answer to this question. In Section 5.1, we present the previous status of this question as well as formulate our results. The subsequent sections are devoted to the respective proofs, considering the dual of the random $\mathbb{K}_n$-free graph $\partial\mathbb{F}_n$ as well as the complete infinite-partite graph with parts of finite (equal) size $\mathbb{K}_{\omega,n}$ in Section 5.2, the complete k-partite graph with parts of countably infinite size $\mathbb{K}_{k,\omega}$ in Section 5.3 and, finally, discussing partial results about the complete infinite-partite graph with parts of countably infinite size $\mathbb{K}_{\omega,\omega}$ in Section 5.4.

5.1 Overview

To recap the classification theorem of homogeneous graphs (Theorem 2.4.3), a homogeneous graph $\mathbb{G}$ is isomorphic to one of the following:

(1) the random graph

(2) the random $\mathbb{K}_n$-free graph $\mathbb{F}_n$ for some $n \geq 3$

(3) the dual of $\mathbb{F}_n$ for some $n \geq 3$

(4) $\mathbb{K}_{k,\omega}$, $\mathbb{K}_{\omega,n}$ or $\mathbb{K}_{\omega,\omega}$ for some $k, n \geq 1$ (a complete multipartite graph)

(5) $\mathbb{E}_{k,\omega}$, $\mathbb{E}_{\omega,n}$ or $\mathbb{E}_{\omega,\omega}$ for some $k, n \geq 1$ (an irreflexive equivalence relation with classes of equal size)

We first collect some facts about the endomorphism monoids and automorphism groups for a selection of these structures. In this chapter, we always work with the representations from Definition 2.4.2 and use Notation 2.4.4.

Lemma 5.1.1.

(i) $\operatorname{End}(\mathbb{K}_{k,\omega}) = \left\{\bigsqcup^{\sigma}_{i=1,\dots,k} s_i : \sigma \in \operatorname{Sym}(k), s_i\colon \mathbb{N} \to \mathbb{N}\right\}$,

$\operatorname{Aut}(\mathbb{K}_{k,\omega}) = \left\{\bigsqcup^{\sigma}_{i=1,\dots,k} \alpha_i : \sigma \in \operatorname{Sym}(k), \alpha_i \in \operatorname{Sym}(\mathbb{N})\right\}$

(ii) $\operatorname{End}(\mathbb{K}_{\omega,n}) \supsetneq \left\{\bigsqcup^{\tau}_{i\in\mathbb{N}} s_i : \tau \in \operatorname{Inj}(\mathbb{N}), s_i\colon \{1,\dots,n\} \to \{1,\dots,n\}\right\}$,

$\operatorname{Aut}(\mathbb{K}_{\omega,n}) = \left\{\bigsqcup^{\sigma}_{i\in\mathbb{N}} \alpha_i : \sigma \in \operatorname{Sym}(\mathbb{N}), \alpha_i \in \operatorname{Sym}(n)\right\}$

(iii) $\mathrm{End}(\mathbb{K}_{\omega,\omega}) \supsetneq \{\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i : \tau \in \mathrm{Inj}(\mathbb{N}), s_i\colon \mathbb{N}\to\mathbb{N}\}$,
$\mathrm{Aut}(\mathbb{K}_{\omega,\omega}) = \{\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \alpha_i \in \mathrm{Sym}(\mathbb{N})\}$

Proof. We proceed very similarly to the proof of Lemma 3.3.4. First, all the maps from above are easily seen to be endomorphisms and automorphisms of the respective structures. Second, we note that since the graph relation connects the parts of the graph, any endomorphism has to map elements from different parts (i.e. $a \in \{i\}\times X$ and $b \in \{j\}\times X$ for $i \neq j$ where $X = \{1,\dots,n\}$ or $X = \mathbb{N}$) to different parts. For the graph $\mathbb{K}_{k,\omega}$ with finitely many parts, this implies that elements from the same part need to be mapped to the same part as well which yields (i) with the same argument as in Lemma 3.3.4.

In general, however, this conclusion is not true; for instance, the map $s\colon \mathbb{N}\times X \to \mathbb{N}\times X$ defined by

$$s(i,x) := \begin{cases} (0,x), & i=0, x \text{ even} \\ (1,x), & i=0, x \text{ odd} \\ (i+1,x), & i \geq 1 \end{cases}$$

where $X = \{1,\dots,n\}$ or $X = \mathbb{N}$ is an endomorphism of $\mathbb{K}_{\omega,n}$ and $\mathbb{K}_{\omega,\omega}$, respectively. Finally, automorphisms α of $\mathbb{K}_{\omega,n}$ and $\mathbb{K}_{\omega,\omega}$ do need to preserve the parts since α^{-1} is an endomorphism as well. All in all, this gives (ii) as well as (iii). □

In order to analyse which of these graphs have endomorphism monoids satisfying UPP, we start by checking whether the pointwise topology on these monoids is the coarsest Polish semigroup topology – this turns out to be true for all of them.

Proposition 5.1.2. *Let $\mathbb{G}$ be a homogeneous graph. Then the Zariski topology on* $\mathrm{End}(\mathbb{G})$ *coincides with the pointwise topology. In particular, the pointwise topology is the coarsest Polish semigroup topology on* $\mathrm{End}(\mathbb{G})$.

Proof. The graphs from (1)-(3) as well as $\mathbb{K}_{k,\omega}, \mathbb{K}_{\omega,\omega}$ and $\mathbb{E}_{k,\omega}, \mathbb{E}_{\omega,\omega}$ are contained in Corollary 3.2.1. For $\mathbb{K}_{\omega,n}$, Theorem A1 is not directly applicable since this structure has algebraicity. Using Lemma 5.1.1(ii), we can, however, find for arbitrary $a = (i_0, x_0)$ in the domain $K_{\omega,n} := \mathbb{N}\times\{1,\dots,n\}$ of $\mathbb{K}_{\omega,n}$ endomorphisms f and g such that $f|_{K_{\omega,n}\setminus\{a\}} = g|_{K_{\omega,n}\setminus\{a\}}$ and $f(a) \neq g(a)$: we pick $f = \bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_\mathbb{N}} f_i$ and $g = \bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_\mathbb{N}} g_i$ where $f_i = g_i = \mathrm{id}_{\{1,\dots,n\}}$ for $i \neq i_0$ and[1] $f_{i_0}|_{\{1,\dots,n\}\setminus\{x_0\}} = g_{i_0}|_{\{1,\dots,n\}\setminus\{x_0\}}$ as well as $f_{i_0}(x_0) \neq g_{i_0}(x_0)$. Since $\mathbb{K}_{\omega,n}$ is clearly transitive such that its model-complete core (the complete graph on countably many vertices) has no algebraicity, Remark 3.2.6 applies and yields the desired conclusion.

Finally, $\mathbb{E}_{\omega,n}$ was treated in [EJM+23, Proposition 3.10] with a tailoured approach – note that $\mathbb{E}_{\omega,n}$ also has algebraicity but has a very specific form, namely the disjoint union of countably many copies of a finite structure. □

The previous proposition implies that the only way for UPP to fail on the endomorphism monoid of a homogeneous graph is the existence of a Polish semigroup topology which is strictly finer than the pointwise topology. In this case, the endomorphism monoid cannot have automatic continuity with respect to the class of Polish semigroups (in particular with respect to the class of second countable topological semigroups). Combining the

[1] Note that f_{i_0} and g_{i_0} need not be injective.

results from [EJM+23] with our contributions, we almost arrive at a complete answer to Question C, that is a classification of all endomorphism monoids of homogeneous graphs with regard to UPP; see also Table 5.1.

Theorem C. *The following homogeneous graphs have an endomorphism monoid which satisfies UPP:*

(1) the random graph

(4a) $\mathbb{K}_{k,\omega}$ *for some* $k \geq 1$

(5a) $\mathbb{E}_{\omega,n}$ *for some* $n \geq 1$

In fact, the respective endomorphism monoids have automatic continuity with respect to the class of second countable topological semigroups.

The following homogeneous graphs have an endomorphism monoid which violates UPP (in particular, does not have automatic continuity with respect to the class of second countable topological semigroups):

(2) the random $\mathbb{K}_n$*-free graph* $\mathbb{F}_n$ *for some* $n \geq 3$

(3) the dual of $\mathbb{F}_n$ *for some* $n \geq 3$

(4b) $\mathbb{K}_{\omega,n}$ *for some* $n \geq 1$

(5b) $\mathbb{E}_{k,\omega}$ *for some* $k \geq 1$ *as well as* $\mathbb{E}_{\omega,\omega}$

Finally, the endomorphism monoid of $\mathbb{K}_{\omega,\omega}$ *does not have automatic continuity with respect to the class of second countable topological semigroups.*

Structure	random graph	$\mathbb{F}_n$	$\partial\mathbb{F}_n$	$\mathbb{K}_{k,\omega}$	$\mathbb{K}_{\omega,n}$	$\mathbb{K}_{\omega,\omega}$	$\mathbb{E}_{k,\omega}$	$\mathbb{E}_{\omega,n}$	$\mathbb{E}_{\omega,\omega}$
UPP	✓	✗	✗	✓	✗	???	✗	✓	✗
AC	✓	✗	✗	✓	✗	✗	✗	✓	✗

Table 5.1: Summary of Theorem C
(AC means "automatic continuity with respect to the class of second countable topological semigroups")

Proof. The random graph, the random $\mathbb{K}_n$-free graph $\mathbb{F}_n$ for some $n \geq 3$ as well as the graphs $\mathbb{E}_{k,\omega}, \mathbb{E}_{\omega,n}, \mathbb{E}_{\omega,\omega}$ for some $k, n \geq 1$ have been treated in [EJM+23]. We will consider the remaining graphs in the next subsections, starting with $\partial\mathbb{F}_n$ and $\mathbb{K}_{\omega,n}$ in Proposition 5.2.1, continuing with $\mathbb{K}_{k,\omega}$ in Proposition 5.3.1 and ending on $\mathbb{K}_{\omega,\omega}$ in Proposition 5.4.1. □

5.2 $\partial\mathbb{F}_n$ and $\mathbb{K}_{\omega,n}$ do not have UPP

Proposition 5.2.1. *The endomorphism monoids* $\mathrm{End}(\partial\mathbb{F}_n)$ *and* $\mathrm{End}(\mathbb{K}_{\omega,n})$ *do not satisfy UPP.*

We exhibit Polish semigroup topologies on $\mathrm{End}(\partial\mathbb{F}_n)$ and $\mathrm{End}(\mathbb{K}_{\omega,n})$ which are strictly finer than the respective pointwise topologies. Our construction generalises Example 2.6.3 (which was tailoured to injective maps) to functions whose preimages of single points have bounded size.

Definition 5.2.2. Let $m \geq 1$.

(i) A map $s\colon A \to A$ is called *m-almost injective* if $\left|s^{-1}\{b\}\right| \leq m$ for all $b \in A$.

(ii) Let $S \leq A^A$ be a semigroup of A^A consisting of m-almost injective maps. For $D \subseteq A$ with $|D| \leq m$ and $b \in A$, we define

$$O(D,b) := \left\{s \in S : s^{-1}\{b\} = D\right\}.$$

(In the case $D = \emptyset$, this means $O(\emptyset, b) := \{s \in S : b \notin \mathrm{Im}(s)\}$.)

Further, we set $\mathcal{T}_m$ to be the topology on S generated by $\mathcal{T}_{pw}$ as well as all the sets $O(D,b)$ for $D \subseteq A$ with $|D| \leq m$ and $b \in A$.

Note that it would suffice to consider $|D| \leq m-1$: since S consists of m-almost injective functions, the set $O(D,b)$ for $D = \{a_1,\dots,a_m\}$ is already contained in $\mathcal{T}_{pw}$ as $O(D,b) = \{s \in S : s(a_1) = \dots = s(a_m) = b\}$. In particular, this implies that if S consists of injective maps (so if $m = 1$), then $\mathcal{T}_m$ coincides with the topology $\mathcal{T}'$ constructed in Example 2.6.3. With similar arguments to this example, we can obtain the following statements which will readily yield Proposition 5.2.1:

Proposition 5.2.3. *Let $m \geq 1$ and let $S \leq A^A$ be a $\mathcal{T}_{pw}$-closed subsemigroup of A^A consisting of m-almost injective maps. Then $\mathcal{T}_m$ is a Polish semigroup topology on S. Further, if S contains the automorphism group of an ω-categorical structure, then $\mathcal{T}_m$ is strictly finer than $\mathcal{T}_{pw}$, so S does not have UPP.*

Proof. We first show that $\mathcal{T}_m$ is a semigroup topology. Since $\mathcal{T}_{pw}$ is a semigroup topology, it suffices to verify the following: if $s,t \in S$ such that $st \in O(D,b)$ for some $D \subseteq A$ with $|D| \leq m$ and $b \in A$, there exist $\mathcal{T}_m$-neighbourhoods V and W of s and t, respectively, such that $VW \subseteq O(D,b)$. This is straightforward by putting $V := O(s^{-1}\{b\}, b)$ and $W := \bigcap_{a \in s^{-1}\{b\}} O(t^{-1}\{a\}, a)$ (if $s^{-1}\{b\} = \emptyset$, we set $W := S$); note that $\bigcup_{a \in s^{-1}\{b\}} t^{-1}\{a\} = (st)^{-1}\{b\} = D$.

Next, we show that $\mathcal{T}_m$ is Polish by inductive application of Lemma 2.6.2. Setting $\mathcal{T}_m[-1] := \mathcal{T}_{pw}$ for notational convenience, we recursively define $\mathcal{T}_m[k]$ for $k = 0,\dots,m$ to be the topology generated by

$$\mathcal{T}_m[k-1] \cup \{O(D,b) : D \subseteq A \text{ with } |D| = m-k \text{ and } b \in A\}.$$

Then $\mathcal{T}_m = \mathcal{T}_m[m]$. By the argument directly preceding this proposition, we have $\mathcal{T}_m[0] = \mathcal{T}_{pw}$ which is Polish since S is assumed to be $\mathcal{T}_{pw}$-closed. If we already know that $\mathcal{T}_m[k-1]$

is Polish, then it suffices to argue that the (countably many) sets $O(D,b)$ for $D \subseteq A$ with $|D| = m-k$ and $b \in A$ are $\mathcal{T}_m[k-1]$-closed and apply Lemma 2.6.2. Writing $D = \{a_1, \ldots, a_{m-k}\}$, we have

$$O(D,b) = \left\{s \in S : s(a_1) = \cdots = s(a_{m-k}) = b \text{ and } \left|s^{-1}\{b\}\right| \leq m-k\right\}.$$

Observing that $\{s \in S : s(a_1) = \cdots = s(a_{m-k}) = b\}$ is closed (even clopen) in $\mathcal{T}_{pw}$ and that

$$\left\{s \in S : \left|s^{-1}\{b\}\right| > m-k\right\} = \bigcup_{\substack{D' \subseteq A \\ |D'| \geq m-(k-1)}} O(D',b)$$

is open in $\mathcal{T}_m[k-1]$ yields the desired closedness of $O(D,b)$ with respect to $\mathcal{T}_m[k-1]$.

Finally, by the same argument as in Example 2.6.3, the topology $\mathcal{T}_m$ is strictly finer than $\mathcal{T}_{pw}$ if S contains the automorphism group of an ω-categorical structure $\mathbb{A}$: while the group $\mathrm{Aut}(\mathbb{A})$ is $\mathcal{T}_m$-closed, there exists a non-surjective map in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{A})$ (see [BPP17, Proof of Corollary 10]); since S is closed, this map is also contained in S (and therefore in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{A})$ *within* S). □

Proof (of Proposition 5.2.1). Given Proposition 5.2.3, we only have to find a number m such that $\mathrm{End}(\partial\mathbb{F}_n)$ and $\mathrm{End}(\mathbb{K}_{\omega,n})$ consist of m-almost injective maps. Since both graphs have no loops, the preimage of a single point under an endomorphism must be an *independent set*, i.e. a set whose elements are not connected to each other by any edges. An independent set of size k in $\partial\mathbb{F}_n$ corresponds to a copy of $\mathbb{K}_k$ in $\mathbb{F}_n$, so we know that $k \leq n-1 =: m$. Similarly, an independent set in $\mathbb{K}_{\omega,n}$ must be contained in a single part of the graph, so we know that its size is at most $n =: m$. □

5.3 $\mathbb{K}_{k,\omega}$ has UPP

Proposition 5.3.1. *The endomorphism monoid* $\mathrm{End}(\mathbb{K}_{k,\omega})$ *has UPP. Moreover, it has automatic continuity with respect to the class of second countable topological semigroups.*

Our proof consists of an application of Pseudo-Property $\overline{\mathbf{X}}$ with a different flavour than in [EJM+23] or in Chapter 4, namely with respect to several (different) subsemigroups of $\mathrm{End}(\mathbb{K}_{k,\omega})$ rather than with respect to the automorphism group. To this end, we will use the description $\mathrm{End}(\mathbb{K}_{k,\omega}) = \left\{\bigsqcup^{\sigma}_{i=1,\ldots,k} s_i : \sigma \in \mathrm{Sym}(k), s_i \colon \mathbb{N} \to \mathbb{N}\right\}$ exhibited in Lemma 5.1.1(i).

Definition 5.3.2. For $j = 1, \ldots, k$, we set

$$D^{(j)} := \left\{\bigsqcup_{i=1,\ldots,k}^{\mathrm{id}_{\{1,\ldots,k\}}} s_i : s_j \colon \mathbb{N} \to \mathbb{N} \text{ and } s_i = \mathrm{id}_{\mathbb{N}} \text{ for all } i \neq j\right\}.$$

Further, we define the *canonical map* $\chi^{(j)} \colon \mathbb{N}^{\mathbb{N}} \to D^{(j)}$ by $h \mapsto \bigsqcup^{\mathrm{id}_{\{1,\ldots,k\}}}_{i=1,\ldots,k} s_i$ where $s_j := h$ and $s_i := \mathrm{id}_{\mathbb{N}}$ for $i \neq j$.

Lemma 5.3.3. *The canonical map $\chi^{(j)}$ is a homeomorphism $(\mathbb{N}^{\mathbb{N}}, \mathcal{T}_{pw}) \to (D^{(j)}, \mathcal{T}_{pw})$. In particular, $(D^{(j)}, \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological semigroups.*

Proof. It is straightforward that $\chi^{(j)}$ is a semigroup isomorphism, so it remains to consider the topologies.

We first show that $\chi^{(j)}$ is continuous. Let O be a nonempty basic open subset in the pointwise topology on $D^{(j)}$, i.e.

$$O = \left\{ s \in D^{(j)} : s(\ell_m, x_m) = (\ell'_m, y_m) \text{ for } m = 1, \ldots, n \right\}.$$

Since $D^{(j)}$ consists of maps of the form $\bigsqcup_{i=1,\ldots,k}^{\mathrm{id}_{\{1,\ldots,k\}}} s_i$ and since $O \neq \emptyset$, we have $\ell_m = \ell'_m$ for all $m = 1, \ldots, n$. For the same reason, $x_m = y_m$ for all m such that $\ell_m \neq j$. Therefore, we can write

$$O = \left\{ s \in D^{(j)} : s(j, x_m) = (j, y_m) \text{ for } m = 1, \ldots, n \right\}$$

because the condition $s(\ell, x) = (\ell, x)$ for $x \in \mathbb{N}$ and $\ell \neq j$ automatically holds for all $s \in D^{(j)}$ and can thus be dropped from the definition of O. We obtain

$$(\chi^{(j)})^{-1}(O) = \left\{ h \in \mathbb{N}^{\mathbb{N}} : h(x_m) = y_m \text{ for } m = 1, \ldots, n \right\}$$

which is (basic) open in the pointwise topology on $\mathbb{N}^{\mathbb{N}}$. This proves the continuity of $\chi^{(j)}$.

To show that $(\chi^{(j)})^{-1}$ is continuous as well, we proceed similarly (but more simply): If

$$P = \{ h \in \omega^{\omega} : h(x_m) = y_m \text{ for } m = 1, \ldots, n \}$$

is basic open in the pointwise topology on $\mathbb{N}^{\mathbb{N}}$, then

$$\chi^{(j)}(P) = \left\{ s \in D^{(j)} : s(j, x_m) = (j, y_m) \text{ for } m = 1, \ldots, n \right\}$$

which is (basic) open in the pointwise topology on $D^{(j)}$.

Finally, the full transformation monoid $\mathbb{N}^{\mathbb{N}}$ equipped with the pointwise topology has automatic continuity with respect to the class of second countable topological semigroups by [EJM+, Theorem 5.4]. This translates to $D^{(j)}$ via $\chi^{(j)}$. □

After this preparation, we can show that $\mathrm{End}(\mathbb{K}_{k,\omega})$ has UPP.

Proof (of Proposition 5.3.1). We prove that $(\mathrm{End}(\mathbb{K}_{k,\omega}), \mathcal{T}_{pw})$ satisfies Property $\overline{\mathbf{X}}$ with respect to $(D^{(1)}, \mathcal{T}_{pw}|_{D^{(1)}}), \ldots, (D^{(k)}, \mathcal{T}_{pw}|_{D^{(k)}})$. By Proposition 2.7.3(ii) and Lemma 5.3.3, we then obtain that $(\mathrm{End}(\mathbb{K}_{k,\omega}), \mathcal{T}_{pw})$ has automatic continuity with respect to the class of second countable topological semigroups. In particular, the pointwise topology is the finest Polish semigroup topology on $\mathrm{End}(\mathbb{K}_{k,\omega})$. On the other hand, it is the coarsest Polish semigroup topology on $\mathrm{End}(\mathbb{K}_{k,\omega})$ by Proposition 5.1.2. All in all, the pointwise topology is the unique Polish semigroup topology on $\mathrm{End}(\mathbb{K}_{k,\omega})$ which will complete the proof.

To obtain Property $\overline{\mathbf{X}}$, we use the following decomposition: Any $s = \bigsqcup_{i=1,\dots,k}^{\sigma} s_i \in \mathrm{End}(\mathbb{K}_{k,\omega})$ can be rewritten as

$$s = \left(\bigsqcup_{i=1,\dots,k}^{\sigma} \mathrm{id}_{\mathbb{N}} \right) \circ \left(\bigsqcup_{i=1,\dots,k}^{\mathrm{id}_{\{1,\dots,k\}}} s_{in} \right) \circ \dots \circ \left(\bigsqcup_{i=1,\dots,k}^{\mathrm{id}_{\{1,\dots,k\}}} s_{i1} \right)$$

where $s_{ij} = s_i$ for $i = j$ and $s_{ij} = \mathrm{id}_{\mathbb{N}}$ otherwise. We put $g_s^{(k+1)} := \bigsqcup_{i=1,\dots,k}^{\sigma} \mathrm{id}_{\mathbb{N}}$ as well as $g_s^{(j)} := \mathrm{id}_{K_{k,\omega}}$ and $a_s^{(j)} := \bigsqcup_{i=1,\dots,k}^{\mathrm{id}_{\{1,\dots,k\}}} s_{ij} \in D^{(j)}$ for $j = 1, \dots, k$ which gives

$$s = g_s^{(k+1)} a_s^{(k)} g_s^{(k)} a_s^{(k-1)} \dots a_s^{(1)} g_s^{(1)}.$$

It remains to consider open subsets $V^{(1)} \in \mathcal{T}_{pw}|_{D^{(1)}}, \dots, V^{(k)} \in \mathcal{T}_{pw}|_{D^{(k)}}$ of $D^{(1)}, \dots, D^{(k)}$ with $a_s^{(j)} \in V^{(j)}$ and find an open subset $U \in \mathcal{T}_{pw}$ of $\mathrm{End}(\mathbb{K}_{k,\omega})$ with the property that $s \in U \subseteq g_s^{(k+1)} V^{(k)} g_s^{(k)} V^{(k-1)} \dots V^{(1)} g_s^{(1)}$. We claim that

$$U := g_s^{(k+1)} V^{(k)} g_s^{(k)} V^{(k-1)} \dots V^{(1)} g_s^{(1)}$$

is already an open set with respect to the pointwise topology on $\mathrm{End}(\mathbb{K}_{k,\omega})$. Assuming that the sets $V^{(j)}$ are basic open in the pointwise topology, we can write

$$V^{(j)} = \left\{ \tilde{s} \in D^{(j)} : \tilde{s}(j, x_m^{(j)}) = (j, y_m^{(j)}) \text{ for } m = 1, \dots, n^{(j)} \right\}$$

by the same argument as in the proof of Lemma 5.3.3 where, without loss of generality, $n^{(j)} \geq 1$. Then

$$\begin{aligned} &V^{(k)} g_s^{(k)} V^{(k-1)} \dots V^{(1)} g_s^{(1)} \\ &\quad = \left\{ \tilde{s} = \bigsqcup_{i=1,\dots,n}^{\mathrm{id}_{\{1,\dots,n\}}} \tilde{s}_i : \tilde{s}(j, x_m^{(j)}) = (j, y_m^{(j)}) \text{ for } m = 1, \dots, n^{(j)}, j = 1, \dots, k \right\} \\ &\quad = \left\{ \tilde{s} \in \mathrm{End}(\mathbb{K}_{k,\omega}) : \tilde{s}(j, x_m^{(j)}) = (j, y_m^{(j)}) \text{ for } m = 1, \dots, n^{(j)}, j = 1, \dots, k \right\} \end{aligned}$$

where the second equality follows from the fact that $n^{(j)} \geq 1$ for all $j = 1, \dots, k$. Finally, we obtain

$$\begin{aligned} U &= g_s^{(k+1)} V^{(k)} g_s^{(k)} V^{(k-1)} \dots V^{(1)} g_s^{(1)} \\ &= \left\{ \tilde{s} \in \mathrm{End}(\mathbb{K}_{k,\omega}) : \tilde{s}(j, x_m^{(j)}) = (\sigma(j), y_m^{(j)}) \text{ for } m = 1, \dots, n^{(j)}, j = 1, \dots, k \right\} \end{aligned}$$

which yields that U is open in the pointwise topology as desired. □

5.4 Open problem: $\mathbb{K}_{\omega,\omega}$

We close this chapter with a discussion of $\mathbb{K}_{\omega,\omega}$.

Proposition 5.4.1. *The automorphism group* $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$ *has automatic continuity with respect to the class of second countable topological semigroups.*

However, the endomorphism monoid $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ *does not have automatic continuity with respect to the class of Polish semigroups.*

The core of our proof for the automorphism group is another application of Property $\overline{\mathbf{X}}$, this time within $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$ with respect to suitable subsets, while the proof for the endomorphism monoid is a direct application of Proposition 2.6.6. For the former, we will use the description $\mathrm{Aut}(\mathbb{K}_{\omega,\omega}) = \left\{\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \alpha_i \in \mathrm{Sym}(\mathbb{N})\right\}$ from Lemma 5.1.1(iii).

Definition 5.4.2. We define the following subsets of $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$:
$D_{fix} := \left\{\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} \alpha_i : \alpha_i \in \mathrm{Sym}(\mathbb{N})\right\}$, $D_{perm} := \left\{\bigsqcup_{i\in\mathbb{N}}^{\sigma} \mathrm{id}_{\mathbb{N}} : \sigma \in \mathrm{Sym}(\mathbb{N})\right\}$.

Similar arguments as in Lemma 5.3.3 yield the following:

Lemma 5.4.3. *The* canonical maps $\chi_{fix}\colon (\mathrm{Sym}(\mathbb{N}), \mathcal{T}_{pw}) \to (D_{fix}, \mathcal{T}_{pw})$, $\sigma \mapsto \bigsqcup_{i\in\mathbb{N}}^{\sigma} \mathrm{id}_{\mathbb{N}}$ *as well as* $\chi_{perm}\colon (\mathrm{Sym}(\mathbb{N})^{\mathbb{N}}, \mathcal{T}_{pw}^{\mathbb{N}}) \to (D_{perm}, \mathcal{T}_{pw})$, $(\alpha_i)_{i\in\mathbb{N}} \mapsto \bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} \alpha_i$ *(where* $\mathcal{T}_{pw}^{\mathbb{N}}$ *denotes the product topology on* $\mathrm{Sym}(\mathbb{N})^{\mathbb{N}}$*) are homeomorphisms.*

While automatic continuity is readily seen to be preserved under finite products[2] of monoids (in particular finite powers), this is unclear for the case of infinite powers. However, we can make use of a stronger concept called *ample generics* that is preserved under countable products.

Definition 5.4.4. Let $(G, \mathcal{T})$ be a topological group and let $n \geq 1$. We say that $\bar{g} = (g_1, \ldots, g_n) \in G^n$ is a *generic n-tuple* if the orbit of $\bar{g}$ with respect to the diagonal conjugation action of G on G^n is comeagre in G^n with respect to the product topology on G^n, explicitly if the set

$$\left\{(hg_1h^{-1}, \ldots, hg_nh^{-1}) : h \in G\right\} \subseteq G^n$$

is comeagre.

The group $(G, \mathcal{T})$ is said to have *ample generics* if there exist generic n-tuples in G for all $n \geq 1$.

Having ample generics is a sufficient condition for automatic continuity:

Theorem 5.4.5 ([KR07, Theorem 1.10] combined with the remarks before [RS07, Corollary 3]). *If* $(G, \mathcal{T})$ *is a Polish group which has ample generics, then it has automatic continuity with respect to the class of second countable topological groups.*

The automorphism groups of some ω-categorical structures were shown to have automatic continuity via ample generics, for instance the random graph [HHLS93, Hru92]. On the other hand, the automorphism group of the rational numbers does not have ample generics [KR07] (even though it has automatic continuity, see Theorem 4.1.5). Another positive example is the full symmetric group; for later reference:

[2]The argument is essentially contained in the proof of Proposition 5.3.1 – one shows that a product of m monoids satisfies Property $\overline{\mathbf{X}}$ of length m with respect to the canonical copies of the single monoids within the product.

Theorem 5.4.6 ([HHLS93, Section 3]). *The symmetric group* $\mathrm{Sym}(A)$ *on a countably infinite set* A *has ample generics.*

As mentioned above, having ample generics is preserved under countable products:

Lemma 5.4.7. *Let* $(G_i, \mathcal{T}_i)$, $i \in \mathbb{N}$, *be topological groups each containing generic* n*-tuples. Then* $(\prod_{i\in\mathbb{N}} G_i, \prod_{i\in\mathbb{N}} \mathcal{T}_i)$ *also contains a generic* n*-tuple. In particular, if all the groups* $(G_i, \mathcal{T}_i)$ *have ample generics, then* $(\prod_{i\in\mathbb{N}} G_i, \prod_{i\in\mathbb{N}} \mathcal{T}_i)$ *does as well.*

Proof. For the sake of notational simplicity, set $G := \prod_{i\in\mathbb{N}} G_i$ and $\mathcal{T} := \prod_{i\in\mathbb{N}} \mathcal{T}_i$.

Let $\bar{g}_i = (g_{i1}, \dots, g_{in})$ be a generic n-tuple in G_i, i.e. its orbit under the diagonal conjugation action

$$O_i := \{(hg_{i1}h^{-1}, \dots, hg_{in}h^{-1}) : h \in G_i\}$$

is comeagre in G_i^n. We write

$$G_i^n \setminus O_i = \bigcup_{k\in\mathbb{N}} M_{ik}$$

for nowhere dense sets $M_{ik} \subseteq G_i^n$. Putting $\bar{g} := ((g_{i1})_{i\in\mathbb{N}}, \dots, (g_{in})_{i\in\mathbb{N}}) \in G^n$, we claim that $\bar{g}$ is a generic n-tuple in G. The orbit of $\bar{g}$ under the diagonal conjugation action of G is $O := \prod_{i\in\mathbb{N}} O_i$, so we have to show that O is comeagre in G^n. For $A_j \subseteq G_j^n$, we define

$$X_j(A_j) := \{((g'_{i1})_{i\in\mathbb{N}}, \dots, (g'_{in})_{i\in\mathbb{N}}) \in G^n : (g'_{j1}, \dots, g'_{jn}) \in A_j\}$$

and note that

$$G^n \setminus O = \bigcup_{i\in\mathbb{N}} X_i(G_i^n \setminus O_i) = \bigcup_{i,k\in\mathbb{N}} X_i(M_{ik}).$$

Thus, the proof is completed if the sets $X_i(M_{ik})$ are shown to be nowhere dense in $(G^n, \mathcal{T}^n)$. Writing $\overline{P}^{\mathcal{O}}$ for the topological closure of a set P with respect to a topology $\mathcal{O}$, it suffices to observe that $\overline{X_i(M_{ik})}^{\mathcal{T}^n} = X_i(\overline{M_{ik}}^{\mathcal{T}_i^n})$ has empty $\mathcal{T}^n$-interior since the set $\overline{M_{ik}}^{\mathcal{T}_i^n}$ of its i-th components has empty $\mathcal{T}_i^n$-interior. □

Combining the previous results, we obtain automatic continuity for the subsets D_{fix} and D_{perm}.

Lemma 5.4.8. *Both* D_{fix} *and* D_{perm} *have automatic continuity with respect to the class of second countable topological groups.*

Proof. By Theorem 5.4.6, the symmetric group $\mathrm{Sym}(\mathbb{N})$ has ample generics, so the product $\mathrm{Sym}(\mathbb{N})^{\mathbb{N}}$ does as well by Lemma 5.4.7.

Theorem 5.4.5 yields that both $\mathrm{Sym}(\mathbb{N})^{\mathbb{N}}$ and $\mathrm{Sym}(\mathbb{N})$ have automatic continuity with respect to the class of second countable topological groups which translates to D_{fix} and D_{perm} via χ_{fix} and χ_{perm} by Lemma 5.4.3. □

After this preparation, we can show that automatic continuity holds for $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$ and fails for $\mathrm{End}(\mathbb{K}_{\omega,\omega})$.

Proof (of Proposition 5.4.1). For the first statement, we show that $(\mathrm{Aut}(\mathbb{K}_{\omega,\omega}), \mathcal{T}_{pw})$ satisfies Property $\overline{\mathbf{X}}$ of with respect to $(D_{fix}, \mathcal{T}_{pw}|_{D_{fix}}), (D_{perm}, \mathcal{T}_{pw}|_{D_{perm}})$ via the following decomposition: Any $\alpha = \bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i \in \mathrm{Aut}(\mathbb{K}_{\omega,\omega})$ can be rewritten as

$$\alpha = \left(\bigsqcup_{i\in\mathbb{N}}^{\sigma} \mathrm{id}_{\mathbb{N}}\right) \circ \left(\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} \alpha_i\right) = g_\alpha^{(3)} a_\alpha^{(2)} g_\alpha^{(2)} a_\alpha^{(1)} g_\alpha^{(1)}$$

where $g_\alpha^{(1)} = g_\alpha^{(2)} = g_\alpha^{(3)} := \mathrm{id}_{\mathrm{Aut}(\mathbb{K}_{\omega,\omega})}$ as well as $a_\alpha^{(1)} := \bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} \alpha_i \in D_{fix}$ and $a_\alpha^{(2)} := \bigsqcup_{i\in\mathbb{N}}^{\sigma} \mathrm{id}_{\mathbb{N}} \in D_{perm}$. It remains to consider open subsets $V^{(1)} \in \mathcal{T}_{pw}|_{D_{fix}}$ and $V^{(2)} \in \mathcal{T}_{pw}|_{D_{perm}}$ of D_{fix} and D_{perm} with $a_\alpha^{(1)} \in V^{(1)}, a_\alpha^{(2)} \in V^{(2)}$ and find an open subset $U \in \mathcal{T}_{pw}$ of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ such that $\alpha \in U \subseteq g_\alpha^{(3)} V^{(2)} g_\alpha^{(2)} V^{(1)} g_\alpha^{(1)}$. As in the proof of Proposition 5.3.1, the set $U := g_\alpha^{(3)} V^{(2)} g_\alpha^{(2)} V^{(1)} g_\alpha^{(1)} = V^{(2)} V^{(1)}$ turns out to be open: we can assume $V^{(1)}$ and $V^{(2)}$ to be basic open in the pointwise topology, so we can write

$$V^{(1)} = \left\{\tilde{\alpha} \in D_{fix} : \tilde{\alpha}(j_m, z_m) = (j_m, y_m) \text{ for } m = 1, \ldots, n^{(1)}\right\}$$
$$V^{(2)} = \left\{\tilde{\alpha} \in D_{perm} : \tilde{\alpha}(\ell_m, x_m) = (\ell'_m, x_m) \text{ for } m = 1, \ldots, n^{(2)}\right\}$$

which yields

$$U = V^{(2)} V^{(1)} = \Big\{\tilde{\alpha} \in \mathrm{Aut}(\mathbb{K}_{\omega,\omega}) : \tilde{\alpha}(j_m, z_m) = \alpha(j_m, z_m) \text{ for } m = 1, \ldots, n^{(1)} \text{ and }$$
$$\tilde{\alpha}(\ell_m, x_m) = \alpha(\ell_m, x_m) \text{ for } m = 1, \ldots, n^{(2)}\Big\}$$

as desired. Finally, Proposition 2.7.3(ii) combined with Lemma 5.4.8 yields that $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$ has automatic continuity with respect to the class of second countable topological groups which is equivalent to automatic continuity with respect to the class of second countable topological semigroups by Proposition 2.6.5.

For the second statement, we apply Proposition 2.6.6. Thus, we have to find a submonoid N of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ which is not closed with respect to the pointwise topology such that composing an element of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ with an element outside N yields an element outside N. Setting $N := \left\{\bigsqcup_{i\in\mathbb{N}}^{\sigma} s_i : \sigma \in \mathrm{Sym}(\mathbb{N}), s_i \colon \mathbb{N} \to \mathbb{N}\right\}$, i.e. N shall contain the endomorphisms which preserve the parts of $\mathbb{K}_{\omega,\omega}$ and whose images additionally intersect *all* parts, it is immediate that N is a submonoid of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$. Further, N is not closed with respect to the pointwise topology: if $\sigma_n \in \mathrm{Sym}(\mathbb{N})$, $n \in \mathbb{N}$, are permutations converging to a non-surjective map $\tau \notin \mathrm{Sym}(\mathbb{N})$, then the endomorphisms $\bigsqcup_{i\in\mathbb{N}}^{\sigma_n} \mathrm{id}_{\mathbb{N}} \in N$ converge to $\bigsqcup_{i\in\mathbb{N}}^{\tau} \mathrm{id}_{\mathbb{N}} \notin N$. Finally, composing an element of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ with an element outside N yields an element outside N which can be seen by distinguishing cases as follows. Let $s \notin N$ and $t \in \mathrm{End}(\mathbb{K}_{\omega,\omega})$. If s misses one of the parts, say the m-th, then st clearly misses the same part while ts misses the part (or parts) to which t maps the m-th part (since t has to map elements from different parts to elements from different parts). On the other hand, if s does not preserve one of the parts, say $s(\{m\} \times \mathbb{N})$ intersects both $\{i\} \times \mathbb{N}$ and $\{j\} \times \mathbb{N}$, then ts does not preserve the m-th part either, while st admits two possibilities: either $\mathrm{Im}(t)$ intersects both $s^{-1}(\{i\} \times \mathbb{N})$ and $s^{-1}(\{j\} \times \mathbb{N})$ (then there necessarily exists a single k such that $t(\{k\} \times \mathbb{N})$ intersects both preimages) in which case st does not preserve the k-th part, or $\mathrm{Im}(t)$ does not intersect $s^{-1}(\{i\}\times\mathbb{N})$ or $s^{-1}(\{j\}\times\mathbb{N})$ in which case st misses the i-th or the j-th part, respectively. Proposition 2.6.6 now yields the conclusion. □

The only missing item toward a complete answer of Question C is thus given by the following problem:

Question 5.4.9. *Does the endomorphism monoid of $\mathbb{K}_{\omega,\omega}$ have UPP?*

We note that the difficulty in this question arises from the fact that endomorphisms of $\mathbb{K}_{\omega,\omega}$ need not preserve the parts of the graph, in other words that they can "split" parts of the graph to multiple parts. Just like $\mathbb{N}^{\mathbb{N}}$ has Property $\mathbf{X}$ with respect to $\mathrm{Sym}(\mathbb{N})$, the submonoid $M := \left\{\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}_{\mathbb{N}}} s_i : \sigma \in \mathrm{Sym}(\mathbb{N}), s_i\colon \mathbb{N}\to\mathbb{N}\right\} \leq \mathrm{End}(\mathbb{K}_{\omega,\omega})$ has Property $\mathbf{X}$ with respect to $\mathrm{Aut}(\mathbb{K}_{\omega,\omega})$; by Proposition 5.4.1, it thus has automatic continuity with respect to the class of second countable topological semigroups. Further, the submonoid $N := \left\{\bigsqcup_{i\in\mathbb{N}}^{\sigma} s_i : \sigma \in \mathrm{Sym}(\mathbb{N}), s_i\colon \mathbb{N}\to\mathbb{N}\right\} \leq \mathrm{End}(\mathbb{K}_{\omega,\omega})$ of non-splitting endomorphisms which hit all the parts of the graph (used already in the proof of the second part of Proposition 5.4.1) can also be seen to have automatic continuity with respect to the class of second countable topological semigroups as it has Property $\overline{\mathbf{X}}$ with respect to $(D_{perm}, \mathcal{T}_{pw}|_{D_{perm}}), (M, \mathcal{T}_{pw}|_M)$. On the other hand, the finer Polish topology on $\mathrm{Inj}(\mathbb{N})$ exhibited in Example 2.6.3 can be lifted to the submonoid $\left\{\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i : \tau \in \mathrm{Inj}(\mathbb{N}), s_i\colon \mathbb{N}\to\mathbb{N}\right\} \leq \mathrm{End}(\mathbb{K}_{\omega,\omega})$ of all non-splitting endomorphisms which not necessarily hit all the parts of the graph. However, this finer topology does *not* yield a suitable finer topology on all of $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ since it loses the property of being a semigroup topology as soon as splitting endomorphisms come into play.

By Proposition 5.4.1, Question 5.4.9 bears many similarities to Question B about the rational numbers $\langle\mathbb{Q},\leq\rangle$. Equipped with the pointwise topology, $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ *cannot* satisfy Pseudo-Property $\overline{\mathbf{X}}$ with respect to $(\mathcal{G}_{\mathbb{Q}}, \mathcal{T}_{pw})$. If the answer to Question 5.4.9 is positive, any proof involving Pseudo-Property $\overline{\mathbf{X}}$ thus requires an auxiliary *rich* topology as in Chapter 4.

Finally, in analogy to Remark 4.3.3, we have discovered a topology which "almost" is an alternative topology satisfying our requirements:

Example 5.4.10. Using the projection map $\pi_1\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $(i,x)\mapsto i$, we define, for every $s\in\mathrm{End}(\mathbb{K}_{\omega,\omega})$ and $i\in\mathbb{N}$, the set $\mathcal{S}_i(s) := \pi_1(s(\{i\}\times\mathbb{N}))$ of parts of the graph hit by s from the i-th part. Further, we define the *splitting number* $p_s := \sum_{i\in\mathbb{N}}(|\mathcal{S}_i(s)|-1) \in \mathbb{N}\cup\{\infty\}$ (so p_s counts how many "surplus" parts are hit by s via splitting; for example if s splits one part into two parts, another one into five and maps all the other parts to a single part, then $p_s = 1+4 = 5$) as well as the *missing number* $q_s := |\mathbb{N}\setminus\pi_1(\mathrm{Im}(s))| \in \mathbb{N}\cup\{\infty\}$ (so q_s counts how many parts of the graph are missed by s). It is straightforward to verify that the sum $r_s := p_s + q_s \in \mathbb{N}\cup\{\infty\}$ is compatible with the composition in the sense that $r_{st} = r_s + r_t$ for all $s,t\in\mathrm{End}(\mathbb{K}_{\omega,\omega})$.

Thus, setting $P_k := \{s\in\mathrm{End}(\mathbb{K}_{\omega,\omega}) : r_s = k\}$ for all $k\in\mathbb{N}\cup\{\infty\}$, the topology $\mathcal{T}'$ generated by $\mathcal{T}_{pw}$ together with the sets P_k, $k\in\mathbb{N}\cup\{\infty\}$, is a semigroup topology. It clearly is second countable Hausdorff and, since all P_k are clopen in $\mathcal{T}'$, also regular. Hence, $\mathcal{T}'$ is metrisable, again by Urysohn's metrisation theorem. However, $\mathcal{T}'$ is not *completely* metrisable by a similar argument to Remark 4.3.3: The set $X := \{s\in\mathrm{End}(\mathbb{K}_{\omega,\omega}) : p_s = 0, q_s = \infty\}$ is

$\mathcal{T}_{pw}$-closed in P_∞, so $\mathcal{T}'$-closed in $\mathrm{End}(\mathbb{K}_{\omega,\omega})$. Therefore, equipped with the subspace topology (which coincides with the pointwise topology), it would be completely metrisable as well and thus a Baire space. As a contradiction, the sets $O_n := \{s \in X : s \text{ hits the } n\text{-th part}\}$ are dense and open in X but have empty intersection.

Just like the boundedness types used in Remark 4.3.3 were part of the rich topology for $\mathrm{End}(\mathbb{Q}, \leq)$, we suspect that because of their compatibility with the composition operation, the sets P_k exhibited in Example 5.4.10 could play an important role in either a rich topology for $\mathrm{End}(\mathbb{K}_{\omega,\omega})$ or in a hypothetical finer topology on $\mathrm{End}(\mathbb{K}_{\omega,\omega})$.

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Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 29. November 2023

Clemens Schindler