Nonlinear Anal. TMA 241 (2024) 113493

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Research paper

On the uniqueness of solutions to the isotropic L_p dual Minkowski problem

Yingxiang Hu [a,b], Mohammad N. Ivaki [b,*]

[a] School of Mathematical Sciences, Beihang University, Beijing 100191, China
[b] Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10, 1040 Wien, Austria

ARTICLE INFO

Communicated by Enrico Valdinoci

Keywords:
Lp dual Minkowski problem
Uniqueness

ABSTRACT

We prove that the unit sphere is the only smooth, strictly convex solution to the isotropic L_p dual Minkowski problem $h^{p-1}|Dh|^{n+1-q}\mathcal{K}=1$, provided $(p,q)\in(-n-1,-1]\times[n,n+1)$.

1. Introduction

An important question in convex geometry is the uniqueness or non-uniqueness of origin-centered spheres as solutions to the isotropic L_p dual Minkowski problem:

$$h^{p-1}|Dh|^{n+1-q}\mathcal{K}=c,\quad c\in(0,\infty). \tag{1.1}$$

The L_p dual Minkowski problem was first introduced by Lutwak, Yang, and Zhang [34], acting as a bridge that connects the L_p-Minkowski problem to the dual Minkowski problem. The former, the L_p-Minkowski problem, was introduced by Lutwak in his influential paper [31] three decades ago and has since been extensively investigated; see, e.g., [3,6,7,9,11,16,17,22,24–26,32,33,35,38–40,43]. The latter, the dual Minkowski problem, was proposed recently by Huang et al. in [20] and further studied in [5,13,18,19,28,41,42]. There has been significant progress on the L_p dual Minkowski problem after the paper [34], such as [4,12,14,21,27]; however, the complete answer to the uniqueness and non-uniqueness question, as stated above, has been elusive in the most interesting case: without the *origin-symmetry* assumption.

Here are the known uniqueness and non-uniqueness results for the isotropic L_p dual Minkowski problem:

- [8], uniqueness of solutions for $-(n+1)\le p<1$ and $q=n+1$ (see also [1,2,37]);
- [21], uniqueness of solutions for $p>q$;
- [12], uniqueness of origin-symmetric solutions for

  $$-(n+1)\le p<q\le\min\{n+1,n+1+p\};$$

- [14], uniqueness of solutions for $1<p<q\le n+1$, or $-(n+1)\le p<q<-1$, or the uniqueness of solutions up to rescaling for $p=q$;
- [30], complete classification for $n=1$;
- [10], non-uniqueness of solutions under any of the following assumptions:

* Corresponding author.
E-mail addresses: huyingxiang@buaa.edu.cn, yingxiang.hu@tuwien.ac.at (Y. Hu), mohammad.ivaki@tuwien.ac.at (M.N. Ivaki).

https://doi.org/10.1016/j.na.2024.113493
Received 20 October 2023; Accepted 11 January 2024
Available online 22 January 2024

(i) $q - 2(n+1) > p \geq 0$;
(ii) $q > 0$ and $-q^* < p < \min\{0, q - 2n - 2\}$, where

$$q^* := \begin{cases} \frac{q}{q-n}, & \text{if } q \geq n+1 \\ \frac{nq}{q-1}, & \text{if } 1 < q < n+1 \\ +\infty, & \text{if } 0 < q \leq 1; \end{cases}$$

(iii) $p + 2(n+1) < q \leq 0$.

In the recent work [23], employing the local Brunn–Minkowski inequality, the following uniqueness was proved.

Theorem. *Let $n \geq 2$ and assume $-(n+1) \leq p$ and $q \leq n+1$, with at least one being strict. Suppose $\mathcal{M}^n$ is a smooth, strictly convex, origin-centered hypersurface such that $h^{p-1}|Dh|^{n+1-q}\mathcal{K} = c$ with $c > 0$. Then $\mathcal{M}^n$ is an origin-centered sphere.*

Here, we also employ the local Brunn–Minkowski inequality as our main tool to establish the following uniqueness result.

Theorem 1.1. *Let $n \geq 2$. Suppose $\mathcal{M}^n$ is a smooth, strictly convex hypersurface with $h > 0$, such that $h^{p-1}|Dh|^{n+1-q}\mathcal{K} = 1$. Suppose either*

1. *$-(n+1) < p \leq -1$ and $n \leq q \leq n+1$,*
2. *or $-(n+1) \leq p \leq -n$ and $1 \leq q < n+1$.*

Then $\mathcal{M}^n$ is the unit sphere.

2. Background

2.1. Convex geometry

Let $(\mathbb{R}^{n+1}, \delta := \langle\, ,\rangle, D)$ denote the Euclidean space with its standard inner product and flat connection, and let $(\mathbb{S}^n, \bar{g}, \bar{\nabla})$ denote the unit sphere equipped with its standard round metric and Levi-Civita connection.

Suppose K is a smooth, strictly convex body in $\mathbb{R}^{n+1}$ with the origin in its interior. Write $\mathcal{M} = \mathcal{M}^n = \partial K$ for the boundary of K. The Gauss map of $\mathcal{M}$, denoted by ν, takes the point $p \in \mathcal{M}$ to its unique unit outward normal $x = \nu(p) \in \mathbb{S}^n$. The support function of K is defined by

$$h(x) := \max\{\langle x, y\rangle : \ y \in K\}, \quad x \in \mathbb{S}^n.$$

The inverse Gauss map $X = \nu^{-1} : \mathbb{S}^n \to \mathcal{M}$ is given by

$$X(x) = Dh(x) = \bar{\nabla}h(x) + h(x)x, \quad x \in \mathbb{S}^n.$$

The support function can also be expressed as $h(x) = \langle X(x), x\rangle = \langle \nu^{-1}(x), x\rangle$, for $x \in \mathbb{S}^n$. The radial function of K is defined by

$$r(x) := |X(x)| = (|\bar{\nabla}h(x)|^2 + h^2(x))^{\frac{1}{2}}.$$

Moreover, the Gauss curvature of $\mathcal{M}$ is defined as

$$\frac{1}{\mathcal{K}(x)} := \frac{\det(\bar{\nabla}^2 h + \bar{g}h)}{\det(\bar{g})}\Big|_x, \quad x \in \mathbb{S}^n.$$

Note that the matrix $A[h] := \bar{\nabla}^2 h + h\bar{g} = D^2 h|_{T\mathbb{S}^n}$ is positive-definite. The eigenvalues of the matrix $A[h]|_x$ with respect to the metric $\bar{g}|_x$, denoted by $\{\lambda_i\}_{i=1}^n$ are the principal radii of curvature at the point $X(x) \in \mathcal{M}$. Then $\sigma_n = \mathcal{K}^{-1} = \Pi_i \lambda_i$. The curvature equation (1.1) can be reformulated as the following Monge–Ampére equation:

$$h^{1-p}|Dh|^{q-n-1}\det(\bar{\nabla}^2 h + \bar{g}h) = c.$$

The polar body of K is defined by

$$K^* := \{y \in \mathbb{R}^{n+1} : \langle x, y\rangle \leq 1 \quad \forall x \in K\}.$$

It is well-known that K^* is also a smooth, strictly convex body in $\mathbb{R}^{n+1}$ with the origin in its interior. Moreover, the following identity holds

$$\frac{h^{n+2}(x)(h^*(x^*))^{n+2}}{\mathcal{K}(x)\mathcal{K}^*(x^*)} = 1. \tag{2.1}$$

Here h^* and $\mathcal{K}^*$ denote respectively the support function and Gauss curvature of K^*, and $x^* := X(x)/|X(x)|$.

Finally, let us introduce the measure $dV := h\sigma_n d\mu = (h/\mathcal{K})d\mu$, where μ is the spherical Lebesgue measure of the unit sphere $\mathbb{S}^n$. Then the measure $\sigma_n d\mu$ is the surface-area measure of K, and V is a constant multiple of the cone-volume measure of K. We refer to [38] for additional background.

2.2. Centro-affine geometry

In this section, we recall some basics from centro-affine geometry. For the related concepts, we refer the reader to [29,36] and, in particular, to the excellent paper by Milman [35].

Let $X : \mathbb{S}^n \to \mathcal{M}$ be a smooth embedding of $\mathcal{M}$ (which we consider it to be Dh as in the previous section), and consider the transversal normal field $\xi(x) := X(x)$ (the centro-affine normal). The transversal vector ξ induces the volume form V (as in the previous section),

$$V(e_1, \dots, e_n) = \det(dX(e_1), \dots, dX(e_n), \xi), \quad e_i \in T\mathbb{S}^n,$$

a connection ∇, as well as a metric g^ξ on $\mathbb{S}^n$ as follows:

$$D_u dX(v) = dX(\nabla_u v) - g^\xi(u, v)\xi, \quad u, v \in T\mathbb{S}^n. \tag{2.2}$$

Note that g^ξ is symmetric and positive-definite. Moreover, while ∇ is not the Levi-Civita connection of g^ξ, it is torsion-free and

$$\nabla V \equiv 0. \tag{2.3}$$

The conormal field $\xi^* : \mathbb{S}^n \to (\mathbb{R}^{n+1})^* \sim \mathbb{R}^{n+1}$ is the unique smooth vector field to the dual space of $\mathbb{R}^{n+1}$, such that $\langle \xi^*, dX\rangle = 0$ and $\langle \xi, \xi^*\rangle = 1$. Moreover, ξ^* is an immersion and transversal to its image, and it induces a bilinear form g^{ξ^*} and a torsion-free connection ∇^* on $\mathbb{S}^n$,

$$D_u d\xi^*(v) = d\xi^*(\nabla^*_u v) - g^{\xi^*}(u, v)\xi^*, \quad u, v \in T\mathbb{S}^n.$$

We furnish all geometric quantities associated with ξ^* with $*$.

It is known that $g^\xi = g^{\xi^*}$ and that the two connections ∇^* and ∇ are conjugate with respect to g^ξ:

$$ug^\xi(v_1, v_2) = g^\xi(\nabla_u v_1, v_2) + g^\xi(v_1, \nabla^*_u v_2), \quad u, v_1, v_2 \in T\mathbb{S}^n.$$

Moreover, by [35, Proposition 4.2] (or taking the inner product of (2.2) with ν), we find

$$g^\xi = g^{\xi^*} = \frac{A[h]}{h} =: g.$$

For a smooth function $f : \mathbb{S}^n \to \mathbb{R}$, the Hessian and Laplacian with respect to (∇, g) are defined as

$$\operatorname{Hess} f(u, v) = \nabla df(u, v) = v(uf) - df(\nabla_v u)$$

and $\Delta f = \operatorname{div}_g(\nabla f) = \sum_i g(\nabla_{e_i}\nabla f, e_i)$, where $\{e_i\}_{i=1}^n$ is a local g-orthonormal frame of $T\mathbb{S}^n$. We write Hess^* and Δ^* respectively for the Hessian and Laplacian with respect to (∇^*, g). Since ∇, ∇^* are conjugate, we have

$$v(uf) = vg(\nabla f, u) = g(\nabla_v \nabla f, u) + df(\nabla^*_v u).$$

Therefore, we obtain

$$\Delta f = \operatorname{tr}_g \operatorname{Hess}^* f, \quad \Delta^* f = \operatorname{tr}_g \operatorname{Hess} f.$$

By [35, Proposition 4.2], we have

$$\operatorname{Hess}^* f + gf = \frac{1}{h}\left(\bar\nabla^2(hf) + \bar g hf\right) = \frac{A[hf]}{h}.$$

Let us define $Q(u, v) = \nabla^*_v u - \nabla_v u$ for all $u, v \in T\mathbb{S}^n$. Then by [29, (6.2)],

$$\operatorname{tr}_g Q = -\nabla \log\left(\frac{h^{n+2}}{\mathcal{K}}\right).$$

In particular, we have

$$(\Delta - \Delta^*)f = -\sum_i Q(e_i, e_i)f = d\log\frac{h^{n+2}}{\mathcal{K}}(\nabla f). \tag{2.4}$$

We conclude this section by recalling the local Brunn–Minkowski inequality, reformulated in the language of centro-affine geometry (cf. [35]): Let $f \in C^1(\mathbb{S}^n)$. Then

$$n\int f^2 dV \leq \int |\nabla f|_g^2 dV + n\frac{(\int f dV)^2}{\int dV}. \tag{2.5}$$

The equality holds if and only if for some $w \in \mathbb{R}^{n+1}$,

$$f(x) = \langle \frac{x}{h(x)}, w\rangle, \quad \forall x \in \mathbb{S}^n.$$

Moreover, by [35, (5.9)] we also have

$$n\int |\nabla f|_g^2 dV \leq \int (\Delta f)^2 dV, \quad \forall f \in C^2(\mathbb{S}^n).$$

3. Uniqueness

The following identity is at the heart of our approach to employing the local Brunn–Minkowski inequality.

Theorem 3.1. *There holds*

$$\Delta X + nX = h\bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}}. \tag{3.1}$$

In particular,

$$n\int X dV = \int h\bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}} dV.$$

Proof. Let $w \in \mathbb{R}^{n+1}$ be a fixed vector. By the centro-affine Gauss equation for $\xi = X$ (cf. (2.2) and [35, Section 3.8]), we have

$$\Delta^*\langle X, w\rangle + n\langle X, w\rangle = 0.$$

Now let $\{v_i\}_{i=1}^n$ be a local $\bar{g}$-orthonormal frame of $T\mathbb{S}^n$ that diagonalizes $A[h]$ at x_0 and $A[h]|_{x_0}(v_i, v_j) = \delta_{ij}\lambda_i$. Define $e_i = \sqrt{\frac{h}{\lambda_i}}v_i$, $i = 1, \dots, n$. Then we have $g|_{x_0}(e_i, e_j) = \delta_{ij}$. Hence, using $e_i X|_{x_0} = \lambda_i e_i$ (cf. [15, (4.15)]) and (2.4), at x_0 we obtain

$$\begin{aligned}
\Delta\langle X, w\rangle + n\langle X, w\rangle &= (\Delta - \Delta^*)\langle X, w\rangle \\
&= g(\nabla \log \frac{h^{n+2}}{\mathcal{K}}, \nabla\langle X, w\rangle) \\
&= \sum_i g(\nabla \log \frac{h^{n+2}}{\mathcal{K}}, \lambda_i\langle e_i, w\rangle e_i) \\
&= \sum_i \lambda_i\langle e_i, w\rangle d \log \frac{h^{n+2}}{\mathcal{K}}(e_i) \\
&= \sum_i \langle \bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}}, hw\rangle.
\end{aligned}$$

The second identity follows from integrating (3.1) against dV. □

Lemma 3.2. *Let $0 < f \in C^2(\mathbb{S}^n)$. Then*

$$\int f^2 \left(\langle \bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}}, hX\rangle - |X|^2 |\nabla \log f|_g^2\right) dV \leq n\frac{|\int fX dV|^2}{\int dV}.$$

Proof. Let $\{E_k\}_{k=1}^{n+1}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. We define

$$f_k = f\langle X, E_k\rangle, \quad k = 1, \dots, n+1.$$

In view of Theorem 3.1, we have

$$\Delta f_k + nf_k = f\langle \bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}}, hE_k\rangle + \langle X, E_k\rangle \Delta f + 2g(\nabla f, \nabla\langle X, E_k\rangle).$$

Therefore,

$$\sum_k f_k(\Delta f_k + nf_k) = f^2\langle \bar{\nabla} \log \frac{h^{n+2}}{\mathcal{K}}, hX\rangle + f|X|^2\Delta f + fg(\nabla f, \nabla|X|^2). \tag{3.2}$$

Moreover, by integration by parts (cf. (2.3)), there holds

$$\int |X|^2 f\Delta f + fg(\nabla f, \nabla|X|^2) dV = -\int |X|^2 |\nabla f|_g^2 dV. \tag{3.3}$$

By the local Brunn–Minkowski inequality (see (2.5)), we have

$$\sum_k \int f_k(\Delta f_k + n f_k) dV \le n \sum_k \frac{\langle \int f X dV, E_k\rangle^2}{\int dV}.$$

Thus the claim follows from (3.2) and (3.3). □

Lemma 3.3. *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is C^1-smooth and $f = \varphi(r)$. Then we have*

$$\int f^2 \langle \bar\nabla \log \frac{h^{n+2}}{\mathcal{K}} - (r(\log\varphi)')^2 \bar\nabla \log r, hX\rangle dV \le n \frac{|\int f X dV|^2}{\int dV}.$$

Proof. Let $\{v_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ be as in the proof of Theorem 3.1. We calculate

$$e_i(\log f) = (\log\varphi)' e_i r = \frac{(\log\varphi)'}{r}\lambda_i \langle e_i, X\rangle = \frac{(\log\varphi)'}{r}\sqrt{h\lambda_i}\langle v_i, X\rangle,$$

and

$$r^2 |\nabla \log f|_g^2 = \sum_i ((\log\varphi)')^2 h\lambda_i (dh(v_i))^2 = \sum_i (r(\log\varphi)')^2 \langle \bar\nabla \log r, hX\rangle.$$

Now the inequality follows from Lemma 3.2. □

Proof of Theorem 1.1. Let $\alpha = q - n - 1$. Due to Lemma 3.3 with $\varphi(r) = r^{q-n-1}$, our assumption $h^{n+2}\mathcal{K}^{-1} = h^{n+1+p} r^{n+1-q}$, we obtain

$$(n+1+p)\int r^{2\alpha}|\bar\nabla h|^2 dV \le \alpha(\alpha+1)\int r^{2\alpha}\langle \bar\nabla \log r, h\bar\nabla h\rangle dV + n\frac{|\int r^\alpha X dV|^2}{\int dV}.$$

Assuming $\alpha^2 + \alpha \le 0$ (i.e. $n \le q \le n+1$) we obtain

$$(n+1+p)\int r^{2\alpha}|\bar\nabla h|^2 dV \le n\frac{|\int r^\alpha X dV|^2}{\int dV}. \tag{3.4}$$

Using the identity $\bar\Delta x + nx = 0$ and integration by parts, we have

$$\begin{aligned}\int h^p X d\mu &= \int h^{p+1} x + h^p \bar\nabla h d\mu \\ &= \int -\frac{h^{p+1}}{n}\bar\Delta x + h^p\bar\nabla h d\mu \\ &= \frac{n+p+1}{n}\int h^p \bar\nabla h d\mu.\end{aligned}$$

Therefore, due to $r^\alpha dV = r^\alpha \frac{h}{\mathcal{K}} d\mu = h^p d\mu$, we obtain

$$\begin{aligned}\int r^\alpha X dV = \int h^p X d\mu &= \frac{n+1+p}{n}\int h^p\bar\nabla h d\mu \\ &= \frac{n+1+p}{n}\int r^\alpha \bar\nabla h dV.\end{aligned}$$

Since $n+1+p > 0$, inequality (3.4) yields

$$\int r^{2\alpha}|\bar\nabla h|^2 dV \le \frac{n+1+p}{n}\frac{|\int r^\alpha\bar\nabla h dV|^2}{\int dV}.$$

We may rewrite this inequality as

$$\int \left| r^\alpha \bar\nabla h - \frac{\int r^\alpha \bar\nabla h dV}{\int dV}\right|^2 dV \le \frac{p+1}{n}\frac{|\int r^\alpha\bar\nabla h dV|^2}{\int dV}.$$

Thus h is constant, provided $-(n+1) < p \le -1$ and $n \le q \le n+1$.

In view of (2.1), the polar body K^* satisfies the following isotropic L_{-q} dual Minkowski problem:

$$(h^*)^{-1-q}|Dh^*|^{n+1+p}\mathcal{K}^* = 1.$$

Hence, the uniqueness result also holds when $n \le -p \le (n+1)$ and $-(n+1) < -q \le -1$. □

Data availability

No data was used for the research described in the article.

Acknowledgments

We thank the referee for their comments. The work of the first author was supported by the National Key Research and Development Program of China 2021YFA1001800, the National Natural Science Foundation of China 12101027, and the Fundamental Research Funds for the Central Universities, China. Both authors were supported by the Austrian Science Fund (FWF) under Project P36545.

References

[1] B. Andrews, Gauss curvature flow: the fate of the rolling stones, Invent. Math. 138 (1999) 151–161.
[2] B. Andrews, Classification of limiting shapes for isotropic curve flows, J. Amer. Math. Soc. 16 (2003) 443–459.
[3] G. Bianchi, K.J. Böröczky, A. Colesanti, D. Yang, The L_p-Minkowski problem for $-n < p < 1$, Adv. Math. 341 (2019) 493–535.
[4] K. Böröczky, F. Fodor, The L_p dual Minkowski problem for $p > 1$ and $q > 0$, J. Differential Equations 266 (2019) 7980–8033.
[5] K. Böröczky, M. Henk, H. Pollehn, Subspace concentration of dual curvature measures of symmetric convex bodies, J. Differential Geom. 109 (2018) 411–429.
[6] K. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn–Minkowski inequality, Adv. Math. 231 (2012) 1974–1997.
[7] K. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. 26 (2013) 831–852.
[8] S. Brendle, K. Choi, P. Daskalopoulos, Asymptotic behavior of flows by powers of the Gaussian curvature, Acta Math. 219 (2017) 1–16.
[9] P. Bryan, M.N. Ivaki, J. Scheuer, A unified flow approach to smooth, even L_p-Minkowski problems, Anal. PDE 12 (2019) 259–280.
[10] H. Chen, S. Chen, Q.-R. Li, Variations of a class of Monge–Ampére-type functionals and their applications, Anal. PDE 14 (2021) 689–716.
[11] S. Chen, Y. Huang, Q.-R. Li, J. Liu, The L_p-Brunn–Minkowski inequality for $p < 1$, Adv. Math. 368 (2020) 107166.
[12] C. Chen, Y. Huang, Y. Zhao, Smooth solutions to the L_p dual Minkowski problem, Math. Ann. 373 (2019) 953–976.
[13] S. Chen, Q.-R. Li, On the planar dual Minkowski problem, Adv. Math. 333 (2018) 87–117.
[14] H. Chen, Q.-R. Li, The L_p dual Minkowski problem and related parabolic flows, J. Funct. Anal. 281 (2021) 108139.
[15] S.Y. Cheng, S.T. Yau, On the regularity of the solution of the n-dimensional Minkowski problem, Comm. Pure Appl. Math. 29 (1976) 495–516.
[16] K.-S. Chou, X.-J. Wang, The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006) 33–83.
[17] Y. He, Q.-R. Li, X.-J. Wang, Multiple solutions of the L_p-Minkowski problem, Calc. Var. Partial Differential Equations 55 (2016) 117.
[18] M. Henk, H. Pollehn, Necessary subspace concentration conditions for the even dual Minkowski problem, Adv. Math. 323 (2018) 114–141.
[19] Y. Huang, Y. Jiang, Variational characterization for the planar dual Minkowski problem, J. Funct. Anal. 277 (2019) 2209–2236.
[20] Y. Huang, E. Lutwak, D. Yang, G. Zhang, Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems, Acta Math. 216 (2016) 325–388.
[21] Y. Huang, Y. Zhao, On the L_p dual Minkowski problem, Adv. Math. 332 (2018) 57–84.
[22] M.N. Ivaki, E. Milman, L^p-Minkowski problem under curvature pinching, Int. Math. Res. Not. IMRN (2023) arXiv:2307.16484 (in press).
[23] M.N. Ivaki, E. Milman, Uniqueness of solutions to a class of isotropic curvature problems, Adv. Math. 435 (2023) No. 109350.
[24] H. Jian, J. Lu, X.-J. Wang, Nonuniqueness of solutions to the L_p-Minkowski problem, Adv. Math. 281 (2015) 845–856.
[25] A.V. Kolesnikov, E. Milman, Local L^p-Brunn–Minkowski inequalities for $p < 1$, Mem. Amer. Math. Soc. 277 (1360) (2022).
[26] Q.-R. Li, Infinitely many solutions for centro-affine Minkowski problem, Int. Math. Res. Not. IMRN 18 (2019) 5577–5596.
[27] Q.-R. Li, J. Liu, J. Lu, Nonuniqueness of solutions to the L_p dual Minkowski problem, Int. Math. Res. Not. IMRN 12 (2022) 9114–9150.
[28] Q.-R. Li, W.-M. Sheng, X.-J. Wang, Flow by Gauss curvature to the aleksandrov and dual Minkowski problems, J. Eur. Math. Soc. (JEMS) 22 (2020) 893–923.
[29] A.-M. Li, U. Simon, G. Zhao, Z. Hu, Global Affine Differential Geometry of Hypersurfaces, de Gruyter, 2015.
[30] H. Li, Y. Wan, Classification of solutions for the planar isotropic L_p dual Minkowski problem, 2022, arXiv:2209.14630.
[31] E. Lutwak, The brunn–Minkowski-firey theory I: mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993) 131–150.
[32] E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem, J. Differential Geom. 41 (1995) 227–246.
[33] E. Lutwak, D. Yang, G. Zhang, On the L_p-Minkowski problem, Trans. Amer. Math. Soc. 356 (2004) 4359–4370.
[34] E. Lutwak, D. Yang, G. Zhang, L_p Dual curvature measure, Adv. Math. 329 (2018) 85–132.
[35] E. Milman, Centro-affine differential geometry and the log-Minkowski problem, J. Eur. Math. Soc. (2023) http://dx.doi.org/10.4171/JEMS/1386.
[36] K. Nomizu, T. Sasaki, Affine Differential Geometry, Cambridge Univ. Press, 1994.
[37] C. Saroglou, On a non-homogeneous version of a problem of firey, Math. Ann. 382 (2022) 1059–1090.
[38] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, second ed., in: Encyclopedia Math. Appl., Cambridge Univ. Press, New York, 2014.
[39] A. Stancu, The discrete planar L_0-Minkowski problem, Adv. Math. 167 (2002) 160–174.
[40] A. Stancu, On the number of solutions to the discrete two-dimensional L_0-Minkowski problem, Adv. Math. 180 (2003) 290–323.
[41] Y. Zhao, The dual Minkowski problem for negative indices, Calc. Var. Partial Differential Equations 58 (2017) 18.
[42] Y. Zhao, Existence of solutions to the even dual Minkowski problem, J. Differential Geom. 110 (2018) 543–572.
[43] G. Zhu, The centro-affine Minkowski problem for polytopes, J. Differential Geom. 101 (2015) 159–174.