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# All centralising monoids given by conservative majority operations on $\{0, 1, 2, 3\}$

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# A Galois correspondence

$\mathcal{O}_A^{(n)} = A^{A^n} = \{ f \mid f \colon A^n \longrightarrow A \}$ $\qquad$ $\mathcal{O}_A = \bigcup_{n \in \mathbb{N}_+} A^{A^n}$

**Commutation of $f \in \mathcal{O}_A^{(m)}$ with $g \in \mathcal{O}_A^{(n)}$**

$$g \perp f \quad :\Longleftrightarrow \quad g \in \operatorname{Hom}(\langle A; f \rangle^n; \langle A; f \rangle)$$

**Centraliser of $F \subseteq \mathcal{O}_A$**

$$\begin{aligned} F^* &= \{ g \in \mathcal{O}_A \mid \forall f \in F \colon g \perp f \} \\ &= \bigcup_{n \in \mathbb{N}_+} \operatorname{Hom}(\langle A; F \rangle^n; \langle A; F \rangle) \end{aligned}$$

(that's a clone)

$$F^{**} = (F^*)^* \supseteq \langle F \rangle_{\mathcal{O}_A} \supseteq F$$

(bicentraliser)

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# Unary commuting operations

$F \subseteq \mathcal{O}_A$

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unary part of a centraliser
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**$s \in \mathcal{O}_A^{(1)}$**

$$s \in F^{*(1)} \iff \forall f \in F\ \underbrace{\forall \mathrm{x} \in A^{\operatorname{ar} f}\colon s(f(\mathrm{x})) = f(s \circ \mathrm{x})}_{s \perp f}$$

# Centralising monoids + observations

**For $M \subseteq \mathcal{O}_A^{(1)}$ TFAE**

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# Maximal centralising monoids

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...coatom in the lattice of centralising monoids on $A$

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## Minimal functions

= a minimum arity generator $f$ of a minimal clone;
by Rosenberg's Theorem, $f$ is

1. special unary function $f \in \mathcal{O}_A^{(1)}$
2. $f \in \mathcal{O}_A^{(2)}$, idempotent: $f(x,x) \approx x$
3. $f \in \mathcal{O}_A^{(3)}$ minority $f(x,y,z) \approx x \oplus y \oplus z$
4. $f \in \mathcal{O}_A^{(3)}$ majority $f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx x$
5. $f \in \mathcal{O}_A^{(k)}$ proper semiprojection, $3 \leq k \leq |A|$

# (Maximal) centralising monoids for $|A| \geq 3$

**All centralising monoids for $|A| = 3$**

- ISMVL 2011 Machida, Rosenberg
- ISMVL 2015 Goldstern, Machida, Rosenberg
- 192 monoids identified, 10 maximal ones
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## Next step: maximal ones for $|A| = 4$

…**much** harder (combinatorial explosion)

- witnesses $\leftrightarrow$ types of minimal functions, one at a time
- MB, 2020–22: all centralising monoids with majority witn.
  1 715 monoids, 147 maximal ones (among the 1 715)

# In this talk...

**Again majority operations on $\{0, 1, 2, 3\}$ as witnesses**

...all centralising monoids with conservative majority witn.

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$$f \in \mathcal{O}_A \text{ conservative} \iff \forall B \subseteq A\colon \quad B \leq \langle A; f\rangle$$

# In this talk. . .

## Again majority operations on $\{0, 1, 2, 3\}$ as witnesses

. . . all centralising monoids with conservative majority witn.

$f \in \mathcal{O}_A$ conservative $\iff$ $\forall B \subseteq A\colon \quad B \leq \langle A; f\rangle$

## Why conservative?

- maximal centralising monoids $\leftrightarrow$ minimal functions
- minimal clones gen. by majority operations:
  described for $|A| \leq 4$ Csákány 1983, Waldhauser 2000
- minimal clones gen. by conservative majority operations:
  described for $|A| < \aleph_0$ ! Csákány 1986

  (all subsets are subuniverses, restriction is a clone hom.)

# Results for $A = \{0,1,2,3\}$

## Centralising monoids $M$ with conservative majority witnesses

- $\mathcal{O}_A^{(1)}$ + 720 proper centralising monoids (cons. maj. witn.)
  42 % of the 1 715
- all 720: $M = \{f\}^{*(1)}$ (∃ single cons. maj. witness)
  242 conjugacy classes of witnesses
- more efficiently: $\exists G$ of 226 cons. maj. operations:
  $$\forall \text{monoid } M\, \exists F \subseteq G: \quad M = F^{*(1)}$$
  (all witnesses drawn from a set $G$ of 226 cons. maj. ops., with 52 conjugacy types)

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## Maximal centralising monoids with cons. maj. witn.

- 107, all among the 147 maximal monoids with maj. witn.,
- 12 maximal monoids up to conjugacy,
- 10–12 up to isomorphism

# How to get there

## Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$
$4^4 = 256$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | × | | × | × | × | × | $\cdots$ | | × | × |
| | | × | × | × | × | × | $\cdots$ | | | |
| | × | × | × | × | × | × | $\cdots$ | | | × |
| | | | | | $\vdots$ | | | | | |
| all cons. maj. | | | × | × | × | × | | | × | |
| | × | | × | × | | | | × | | |
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| | | | × | | × | × | | × | | × |

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$\forall M$

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\overbrace{\hphantom{\times}}$ | $\overbrace{\hphantom{\times}}$ | $\overbrace{\hphantom{\times}}^{M}$ | $\overbrace{\hphantom{\times}}$ | $\overbrace{\hphantom{\cdots}}$ | | | |
| | $\times$ | | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | $\times$ | $\times$ |
| | | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | | |
| | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | | $\times$ |
| | | | | | $\vdots$ | | | | | |
| all cons. maj. | | | $\times$ | $\times$ | $\times$ | $\times$ | | | $\times$ | |
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| | $\times$ | $\times$ | | $\times$ | | | | | $\times$ | |
| | | | $\times$ | | $\times$ | $\times$ | | $\times$ | | $\times$ |

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## Using the Galois connection (aka formal context)

$\forall\, M\, \exists F \subseteq \mathrm{Maj}_A \cap \mathrm{Pol}_A\, \mathfrak{P}(A)$:

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| | | | $M$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $F$ | $\times$ | | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | $\times$ | $\times$ |
| | | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | | |
| | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | | | $\times$ |
| | | | | | $\vdots$ | | | | | |
| all cons. maj. | | | $\times$ | $\times$ | $\times$ | $\times$ | | | $\times$ | |
| | $\times$ | | $\times$ | $\times$ | | | | $\times$ | | |
| | $\times$ | $\times$ | | $\times$ | | | | | $\times$ | |
| | | | $\times$ | | $\times$ | $\times$ | | $\times$ | | $\times$ |

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$$
\begin{array}{r|cccccccccc}
 & & & \multicolumn{5}{c}{\overbrace{\hphantom{\times \quad \times \quad \times \quad \times \quad \cdots}}^{M}} & & & \\
\hline
\multirow{4}{*}{$F \left\{ \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right.$} & \times & & \times & \times & \times & \times & \cdots & & \times & \times \\
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 & & & & & \vdots & & & & & \\
\text{all cons. maj.} & & & \times & \times & \times & \times & & & \times & \\
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 & & & \times & & \times & \times & & \times & & \times
\end{array}
$$

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$$\forall M \, \exists F \subseteq \mathrm{Maj}_A \cap \mathrm{Pol}_A \mathfrak{P}(A): \quad M = F^{*(1)} = \bigcap_{f \in F} \{f\}^{*(1)}$$

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| | | | | | $M$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | × | | × | × | × | × | ⋯ | | × | × |
| $F$ | | × | × | × | × | × | ⋯ | | | |
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| all cons. maj. | | | × | × | × | × | | | × | |
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| | | | × | | × | × | | × | | × |

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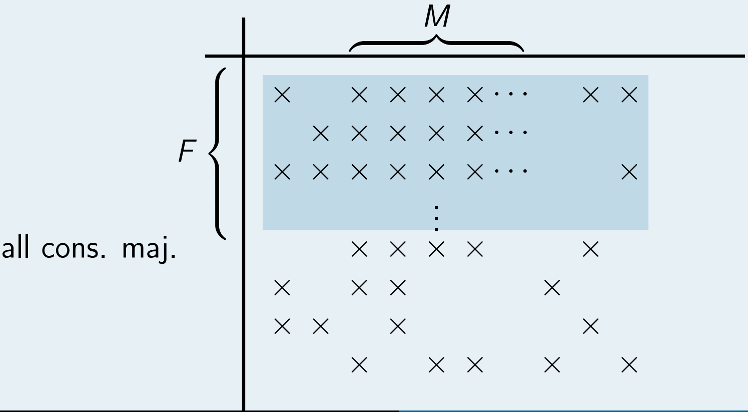

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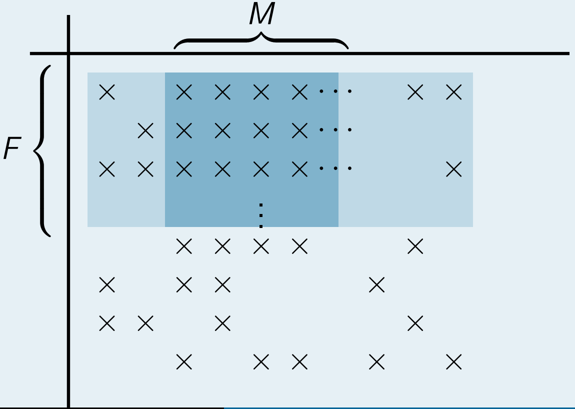


all cons. maj.

## Using the Galois connection (aka formal context)

$$\forall\, M\, \exists F \subseteq \mathrm{Maj}_A \cap \mathrm{Pol}_A\, \mathfrak{P}(A):\quad M = F^{*(1)} = \bigcap_{f\in F}\{f\}^{*(1)}$$

Just compute arbitrary intersections!

all unary ops. $\mathcal{O}_4^{(1)}$

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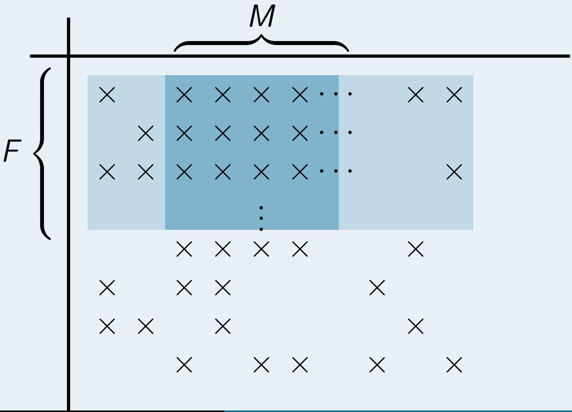


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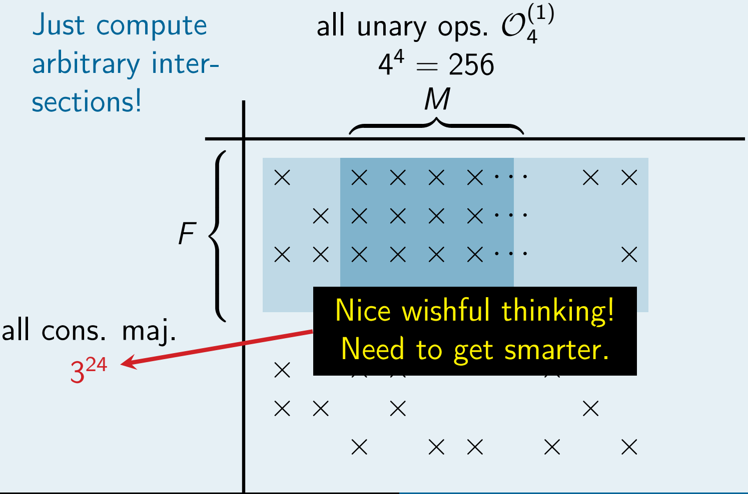

## Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

| | | | | | | | | | | | trivial $s$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| | × | × | × | × | × | × | $\cdots$ | | | × | × | × |
| | | | | | $\vdots$ | | | | | | × | × |
| all cons. maj. $3^{24}$ | | | × | × | × | × | | | × | | × | × |
| | × | | × | × | | | | × | | | × | × |
| | × | × | | × | | | | | × | | × | × |
| | | | × | | × | × | | × | | × | × | × |

# How to get there

## Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$
$4^4 = 256$

all cons. maj.
$3^{24}$

| | | | | | | | | | | trivial | $s$ |
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|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\times$ | | $\times$ | $\times$ | $\times$ | $\times \cdots$ | $\times$ | $\times$ | $\times$ | $\times$ | |
| | | $\times$ | $\times$ | $\times$ | $\times$ | $\times \cdots$ | | | $\times$ | $\times$ | |
| | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times \cdots$ | | $\times$ | $\times$ | $\times$ | |
| | | | | | $\vdots$ | | | | $\times$ | $\times$ | |
| all cons. maj. | | | | | | | | | $\times$ | $\times$ | |
| $3^{24}$ | | | | | | | | | $\times$ | $\times$ | |
| many $f$ | (empty) | | | | | | | | $\times$ | $\times$ | |
| (exclude) | | | | | | | | | $\times$ | $\times$ | |

# How to get there

## Using the Galois connection (aka formal context)

$\forall s$ nontrivial

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

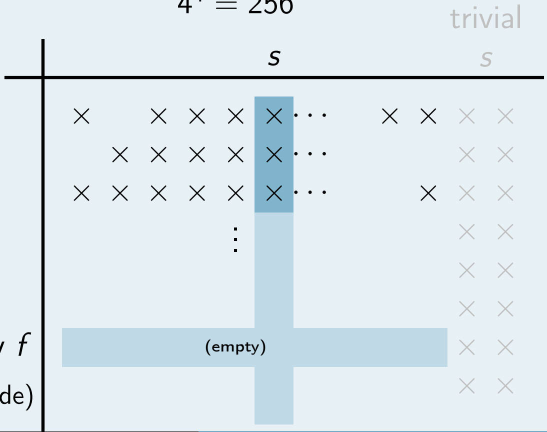
| | | | | | | | | | | | trivial | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $s$ | | | | | $s$ | |
| | × | | × | × | × | × | $\cdots$ | | × | × | × | × |
| | | × | × | × | × | × | $\cdots$ | | | | × | × |
| | × | × | × | × | × | × | $\cdots$ | | | × | × | × |
| | | | | | $\vdots$ | | | | | | × | × |
| all cons. maj. | | | | | | | | | | | × | × |
| $3^{24}$ | | | | | | | | | | | × | × |
| many $f$ | | | | | | (empty) | | | | | × | × |
| (exclude) | | | | | | | | | | | × | × |

## Using the Galois connection (aka formal context)

$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$:

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$




all cons. maj.

$3^{24}$

many $f$

(exclude)

## Using the Galois connection (aka formal context)

$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$: store $\{f\}^{*(1)}$

all unary ops. $\mathcal{O}_4^{(1)}$

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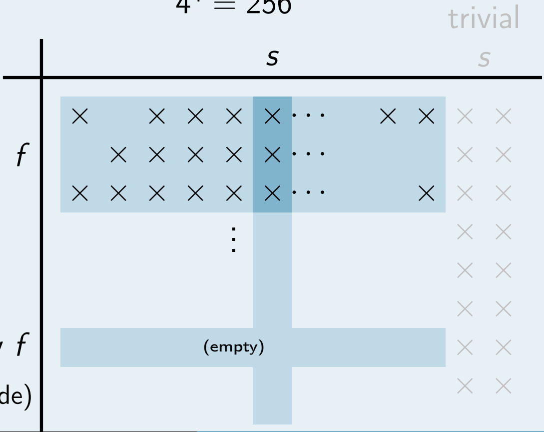



all cons. maj.

$3^{24}$

many $f$

(exclude)

# How to get there

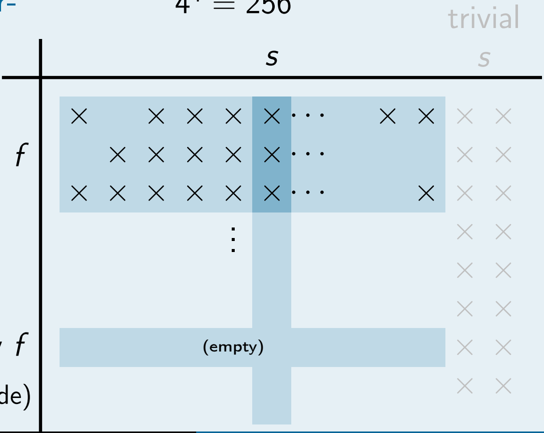
## Using the Galois connection (aka formal context)

$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$: store $\{f\}^{*(1)}$

Now compute arbitrary intersections!

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$




all cons. maj.

$3^{24}$

many $f$

(exclude)

# The end. . .

# The end...

...of the COVID pandemic hopefully comes soon.

# The end...

...of the COVID pandemic hopefully comes soon.

**This is just the end of my talk.**

- Any remarks / questions are welcome.
- Thank you for your attention.