# Dissertation

# Solution Stability in PDE-Constrained Optimization

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# Kurzfassung

Die mathematische optimale Kontrolltheorie, in den 1950er Jahren von Richard Bellman und Lev Pontryagin ins Leben gerufen, kann als eine Erweiterung der klassischen Theorie der Variationsrechnung angesehen werden. Sie befasst sich in ihrem ursprünglichen Sinne mit der Aufgabe der Optimierung (Maximierung oder Minimierung) eines Kostenfunktionals. Das Kostenfunktional hängt im Allgemeinen von Trajektorien ab, die durch Lösungen von gewöhnlichen Differentialgleichungen bestimmt und zusätzlich von Kontrollfunktionen beeinflusst werden und weiteren Einschränkungen unterliegen. Nach mathematischen Maßstäben ist die optimale Kontrolltheorie ein recht junges Gebiet, das aber durch seine erfolgreiche Anwendung im Ingenieurwesen und vielen anderen Bereichen der Industrie und Gesellschaft populär wurde. Mittlerweile ist sie ein aktives und stetig wachsendes Forschungsgebiet im Bereich der angewandten Mathematik. Ursprünglich beschäftigt mit Problemen, bei denen die zugrunde liegende Dynamik durch gewöhnliche Differentialgleichungen modelliert wird, wurde es in den letzten Jahrzehnten aufgrund des zunehmenden Interesses an Modellen, die sich mit physikalischen, chemischen und biologischen Fragen befassen, immer wichtiger, solche Probleme zu berücksichtigen bei welchen die zu Grunde liegende Dynamik durch partielle Differentialgleichungen bestimmt ist.

In dieser Arbeit betrachten wir solche PDG-beschränkten Optimalkontrollprobleme, bei denen die Kontrolle im Optimalkontrollproblem höchstens auf affine Weise auftritt. Für diese Probleme untersuchen wir die Eigenschaft der *Lösungsstabilität* lokaler optimaler Kontrollen und Zustände für eine gegebene Klasse von Kontrollproblemen, die bestimmte lokale Wachstumsbedingungen erfüllen. Unter *Lösungsstabilität* verstehen wir das Erhalten quantitativer Schätzungen des Abstands (in einer gegebenen Metrik) zwischen einer lokalen optimalen Kontrollfunktion eines gegebenen Problems, und einer Kontrollfunktion, die das System aus notwendigen Optimalitätsbedingungen erster Ordnung einer gestörten Version des Problems löst. Genauer, ist die Schätzung des Abstands der optimalen Kontrollen in Abhängigkeit von der Größe der Störung möglich, so spricht man von *Lösungsstabilität.* Diese Eigenschaft eines Kontrollproblems, ist eine wichtige Qualität, die ein optimales Steuerungsproblem charakterisiert. Sie ist beispielsweise für die Schätzung des theoretischen Fehlers einer numerischen Approximation eines Optimalsteuerungsproblems von Nutzen.

Für die Untersuchung der Lösungsstabilität, ist es hilfreich Bedingungen an das optimale Kontrollproblem zu finden welche Lösungsstabilität implizieren. Fündig werden wir unter den außreichenden Bedingungen für strikte lokale Optimalität. Es stellt sich heraus, dass einige dieser Bedingungen auch für die Lösungsstabilität ausreichend sind. Eine solche häufig verwendete Bedingungen besteht aus einer ausreichenden Optimalitätsbedingung zweiter Ordnung, die eine starke positive Definitheit der Zielfunktion in Bezug auf die Kontrolle erfordert. Da die Probleme in dieser Arbeit kontrollaffin sind, kann eine so starke Koerzivitätsbedingung nicht erfüllt werden. Um diese Schwierigkeit zu berücksichtigen, betrachten wir eine lokale Wachstumsannahme, die durch eine Bedingung motiviert ist, die in der affinen ODE-beschränkten optimalen Kontrolle auftritt, und gemeinsam die Variationen erster und zweiter Ordnung der Zielfunktion in Bezug auf die Kontrollfunktion umfasst. Um die Untersuchung der Lösungsstabilität zu systematisieren, verwenden wir Ansätze aus dem Bereich der Variationsanalyse, einem Bereich, der Methoden aus der Variationsrechnung und der konvexen Optimierung kombiniert und einen Rahmen für die Behandlung von Optimierungsproblemen bietet. Die Eigenschaft, der sogenannten *starken metrischen Subregularität*, hat sich als hilfreich für die Be-

handlung von Fehlerschätzungen für die numerische Approximation in ODE-beschränkten Optimalsteuerungsproblemen erwiesen. Der Begriff *starke metrische Subregularität* erscheint im Zusammenhang mit mengenwertigen Inklusionsabbildungen und beschreibt, eine bestimmte Regularitätseigenschaft der mengenwertigen Abbildung. Damit meinen wir, dass wir den Abstand einer Lösung der Abbildung zu einer Lösung der gestörten Abbildung anhand der Größe der Störung schätzen können, wenn die mengenwertige Inklusionsabbildung an einem bestimmten Punkt *stark metrisch subregulär* ist. Das Ziel besteht darin, diese Eigenschaft für die Optimalitätsabbildung des optimalen Kontrollproblems zu beweisen. Die Optimalitätsabbildung ist eine mengenwertige Inklusionsabbildung, welche die notwendigen Optimalitätsbedingungen erster Ordnung umfasst, ein System bestehend aus zwei Gleichungen und einer Variationsungleichung (die die Optimalitätsbedingung des zugehörigen Hamiltonians darstellt). Wenn die Optimalitätsabbildung die *starke metrische Subregularitätseigenschaft* erfüllt, ermöglicht dies, die Lösungsstabilitätseigenschaft des Problems in Bezug auf lineare und nichtlineare Störungen unter einer Vielzahl von Störungen auf strukturierte Weise zu untersuchen. Das erhaltene Subregularitätsergebnis bildet eine Grundlage für die Untersuchung zur Konvergenz- und Fehleranalyse für Diskretisierungsmethoden in zukünftigen Arbeiten.

Die wichtigsten Resultate in dieser Arbeit ist die Untersuchung der Subregularitätseigenschaft der Optimalitätsabbildung im Zusammenhang mit affinen semilinearen elliptischen und parabolischen PDE-beschränkten Optimalsteuerungsproblemen und die Einführung mehrerer neuer Annahmen, um den Schwierigkeiten des affinen Falles Rechnung zu tragen. Diese Annahmen sind durch Wachstumsbedingung an das Zielfunktional unter Einbeziehung der ersten und zweiten Variationen in Bezug auf die Steuerungen oder den linearisierten Zustand gegeben. Diese Annahmen sind schwächer als die bisher in der Literatur berücksichtigten und implizieren strikte lokale Optimalität. Alle Annahmen implizieren Lösungsstabilität für die optimalen Kontrollen oder Zustände und einer von ihnen ist stark genug, um die starke metrische Subregularität der vollständigen Optimalitätsabbildung zu implizieren. Als Konsequenz unserer Untersuchung erhalten wir Lösungsstabilitätsschätzungen vom Hölder- und Lipschitz-Typ für elliptische und parabolische PDG-beschränkte optimale Kontrollprobleme. Abschließend beweisen wir, dass diese Bedingungen ausreichend sind, um Fehlerschätzungen für die numerische Näherung zu erhalten, die durch ein Finite-Elemente-Approximationsschema erzeugt wird. Zudem verbessern wir auch einige der in der Literatur vorhandenen Fehlerabschätzungen.

Diese kumulative Arbeit behandelt die genannten Themen in fünf Veröffentlichungen.

## Abstract

Mathematical optimal control theory, an extension of the classical theory of calculus of variations, is concerned initially with the task of the optimization (maximization or minimization) of a cost functional that depends on trajectories corresponding to solutions of ordinary differential equations, influenced by control functions satisfying some given constraints. By mathematical standards, control theory is a relatively young field. It was initiated in the 1950s by mainly Richard Bellman and Lev Pontryagin and popularized by its successful application in engineering and many other areas of industry and society. By now, it is an active and growing aspect of applied mathematics. Traditionally concerned with problems where the underlying dynamics are modeled by ordinary differential equations, in recent decades, due to the increasing interest in models that are involved with specific physical, chemical, and biological questions, it has become more and more important to consider problems that are constrained by partial differential equations.

In this thesis, we consider such PDE-constrained optimal control problems where the control appears at most affinely in the optimal control problem. For these problems, we investigate the property of *solution stability* of local optimal controls and states for a given class of control problems that satisfy certain local growth conditions. By solution stability, we understand the obtainment of quantitative estimates on the distance (in some given metric) of a local optimal control corresponding to a given problem and a control solving the system of first-order necessary optimality conditions corresponding to a perturbed version of the problem. In other words, to obtain estimates on the distance of the controls with respect to the size of the perturbation, in the sense that the size of the perturbation dominates the distance of the controls, we speak of solution stability. This property is a crucial property characterizing an optimal control problem. For instance, it is of interest for estimating the theoretical error of a numerical approximation of an optimal control problem.

A way to investigate solution stability is to identify ”reasonable” conditions on the optimal control problem that imply local optimality and then prove that they are also sufficient for solution stability. One of these conditions consists of a commonly used second-order sufficient optimality condition for PDE-constrained optimal control problems, which requires the strong positive definiteness of the objective functional with respect to the control. Since the problems in this thesis are affine for the control, such a strong coercivity condition cannot be fulfilled. To account for this difficulty, we consider a local growth assumption motivated by a similar condition appearing in affine ODE-constrained optimal control, which jointly involves the first and second-order variations of the objective functional. We show that this local growth condition implies strict local optimality. Then, to systematize the study of solution stability, we use methods from the field of Variational Analysis. This field combines the Calculus of Variations and Convex Optimization, and provides a framework for the treatment of convex and nonconvex optimization problems, including a property that is closely connected to solution stability. This property, the so-called *strong metric subregularity*, has proven to help treat error estimates for the numerical approximation in ODE-constrained optimal control problems. Strong metric subregularity appears in the context of set-valued inclusion maps and, as the name implies, describes a certain regularity property of a set-valued map. By this, we mean that if the set-valued inclusion map is strong metric subregular at a given point, we can estimate the distance of that point to a solution of the perturbed map by the size of the perturbation. The goal is to prove this property for

the optimality map of the optimal control problem. The optimality map is a set-valued inclusion map involving the first-order necessary optimality conditions written as a system of two equations and one variational inequality (representing the optimality condition of the associated Hamiltonian), forming a *generalized equation*. If the optimality map satisfies the strong metric subregularity property, it allows us to study the solution stability property of the problem for linear and nonlinear perturbations under various perturbations and in a structured way. The obtained subregularity result provides a base for the study in future work of convergence and error analysis for discretization methods applied to the optimal control problem.

The main novelties in this thesis are the study of the subregularity property of the optimality map associated with affine semilinear elliptic and parabolic PDE-constrained optimal control problems, and to account for the difficulties of the affine case, the introduction of several novel assumptions jointly involving the first and second-order variations of the objective functional with respect to the control or the linearized state. These assumptions are weaker than the ones considered in the literature on affine problems but still imply strict local optimality. Besides, all of them imply solution stability for the optimal controls or states, and one is strong enough to imply the strong metric subregularity of the full optimality map. As a consequence of our investigation, we achieve solution stability estimates of Hölder and Lipschitz type for elliptic and parabolic PDE-constrained optimal control problems. Finally, we prove that these conditions are also sufficient for obtaining error estimates for the numerical approximation generated by a finite element approximation scheme. We also improved some of the error estimates existing in the literature.

This cumulative thesis treats the mentioned topics in five papers. We provide a short but further detailed summary of the papers' content on the introduction's first pages.

## Acknowledgement

First and foremost, I want to express my gratitude to my supervisor, Prof. Vladimir Veliov, for his guidance over the past three years. His skill, experience, continuous assistance and support have been fundamental to my growth both as an individual and as a mathematician. Also, I would like to thank Prof. Eduardo Casas for his interest in this work and his support. I wish to thank all the colleagues and co-authors that I met during these years and for the time we spent together. I would like to acknowledge the Austrian Science Fund for its financial support. My deepest gratitude is also to my family for all the support over the years.

# Declaration in lieu of oath

I declare in lieu of an oath that I have written this dissertation independently and without outside help, that I have not used any sources or aids other than those specified, and that I have marked the passages taken literally or analogously as such.

Vienna, 30.07.2023 Nicolai Alexander Jork

# Contents

## Introduction



## Publications

# Introduction

## Content of the thesis

This cumulative thesis consists of five papers. Below we list the individual papers and shortly comment on the obtained results.

[1] A. Domínguez Corella, N. Jork, V. M. Veliov: *Stability in Affine Optimal Control Problems Constrained by Semilinear Elliptic Partial Differential Equations*, ESAIM: COCV, 28(79) (2022).

[2] E. Casas, A. Domínguez Corella, N. Jork: *New assumptions for stability analysis in elliptic optimal control problems*, SIAM J. Control Optim. 61(3), (2023).

[3] N. Jork: *Finite Element Error Analysis and Solution Stability of Affine Optimal Control Problems*, submitted.

[4] A. Domínguez Corella, N. Jork, V. M. Veliov: *On the solution stability of parabolic optimal control problems*, Comput. Optim. Appl., (2023).

[5] A. Domínguez Corella, N. Jork, V. M. Veliov: *Solution stability of parabolic optimal control problems with fixed state-distribution of the controls*, Serdica, (2023).

In [1], the authors investigate the solution stability of affine optimal control problems constrained by semilinear elliptic partial differential equations with a Robin-type boundary condition. The underlying idea is to use a property from Variational Analysis, the so-called strong metric subregularity, and the formulation of the system of first-order necessary optimality conditions as a set-valued map to study the solution stability of the optimal control problem for strict local optimal controls that satisfy a specific local growth condition. Due to the smooth setting, the authors use the Fréchet differentiability of the control-to-state and the control-to-adjoint operator to establish an approach that provides the strong metric subregularity of the optimality map and subsequently obtain solution stability of the optimal controls and states under linear and nonlinear perturbation appearing in the objective functional and the constraining PDE. This is then applied to obtain error estimates for the Tikhonov regularization.

In [2], the authors introduce two new sufficient conditions for strict local optimality of an affine optimal control problem constrained by a semilinear elliptic non-monotone and non-coercive PDE with Dirichlet boundary condition. These assumptions are the weakest so far considered in the context of affine PDE-constrained optimization, which guarantee strong or weak strict local optimality and that allow for solution stability estimates for the optimal controls or states. The authors demonstrate this by discussing some related problems in the later sections. The proof of the main theorem uses a synthesis of the approach established in [1] and of the approach established in the papers by E. Casas, M. Mateos, A. Rösch, F. Tröltzsch, D. Wachsmuth and G.Wachsmuth.

In [3], the author considers an affine optimal control problem subject to semilinear elliptic PDEs with Dirichlet boundary condition. The main contribution of the paper is the proof of error estimates for the numerical approximation obtained by a finite element scheme under the sufficient optimality conditions introduced in [1] and [2]. The error estimates in the literature for the numerical approximation are improved in the case of an assumed Hölder-type growth of the joint first and second variation of the optimal control problem. Additionally, it is shown that the assumptions also suffice for the error estimates for a variational discretization scheme, where already existing error estimates are obtained under a simplified proof and weaker assumptions (than those being used before in the literature) introduced in [1] and [2]. Finally, the author answers a question raised in 2 on the relationship between a possible bang-bang structure of optimal controls and an assumption introduced in 2.

In [4], the authors introduced the strong metric subregularity approach of proving solution stability of the optimal controls and states for optimal control problems constrained by semilinear parabolic PDEs. The higher dimensionality of the parabolic setting (compared to the elliptic one), demands some adjustments of the proof strategy and relies on some technical new key lemmas that allow to overcome the obstacles of the parabolic setting. Additionally, the assumptions introduced in [2] are considered in the parabolic setting. The major contribution of this paper is the formulation of a modified strategy of proof of the strong metric subregularity property of the optimality map: This strategy consists of arguments of [1] and [2] and further improvements thereof. For instance, we drop some conditions on the feasible perturbations and provide a proof that does not rely on the linearization of the control-to-adjoint operator and that allows for a straightforward generalization of the optimal control problems. As a consequence, this will enable us to consider various other partial differential equations as constraints of the optimal control problem (see, for instance, the two-dimensional Navier-Stokes equation, [10]). To provide the reader with an overview of the restrictiveness of the used assumptions on the joint growth of the first and second variation in [1] and [2], we compare them to other assumptions that appeared in the context of solution stability of PDE-constrained optimal control problems. Finally, we prove the strongest Hölder-type stability estimates for the optimal controls and states so far (in the parabolic setting) and provide a Lipschitz stability result under an assumption introduced in [2].

In [5], a companion paper to [4], the authors consider a control problem where the spatial distribution of the controls is fixed. This allows for more general objective functionals, simplified proofs, and stronger Lipschitz-type solution stability estimates that resemble the results of the elliptic case.

At the beginning of each section that contains a paper, we comment shortly on the author's contribution.

# 1. Variational Analysis in Optimal Control Theory

Mathematical optimal control theory is a theory on the optimization of functionals. Usually, the functionals depend on a dynamical system, and the optimization is done under constraints on the involved variables. For instance, an optimal control problem often consists of several components: a control variable, a state equation, a cost functional to be minimized, and constraints that can be posed on both the control and the state. Initially, the state equation that appears in the problem was given by a possibly nonlinear ordinary differential equation. For the study of properties of optimal control problems, it is crucial to obtain necessary and sufficient conditions for optimality. In this regard, one of the central results in control theory is the famous Pontryagin maximum principle, a necessary condition for optimality. The principle states that the optimal control and the corresponding trajectory must satisfy coupled equations. These equations involve the system dynamics, the dynamics of the adjoint variables, and the Hamiltonian function. The intuition behind Pontryagin's maximum principle is that if $\bar{u}$ is an optimal control, then there exists a solution $\bar{p}$ of the adjoint equation, a kind of Lagrange multiplier associated with the state equation so that $\bar{u}$ maximizes point-wise the Hamiltonian. A vital notion used in optimization theory is the notion of a set-valued mapping. Set-valued mappings can be used to study systematically specific properties of optimization problems. The set-valued mapping relevant to this thesis is related to first-order necessary optimality conditions of affine PDE-constrained optimal control problems, which are given by the Pontryagin maximum principle.

The property of solution stability, the main interest of this thesis, is implied by a regularity property of the set-valued optimality mapping of the optimal control problem. This mapping is strongly related to the first-order necessary optimality conditions of the optimal control problem. We study the properties of this set-valued mapping in the framework of Variational Analysis. Variational Analysis concerns optimization problems arising in various areas, using fix-point methods and techniques from analysis and functional analysis, and further has deep connections to various branches of mathematics, such as ODEs and PDEs, optimal control theory, and functional analysis.
As an implication of a certain regularity property of the optimality mapping and the implied solution stability, we obtain error estimates for a finite element scheme. The numerical approximation can be considered a perturbation of the original problem. Thus, the error estimates are a consequence of the solution stability of the original problem. We expect additional error estimates for the numerical approximation in further work.

To discuss the topic of solution stability for ODE-constrained optimal control problems, which is the starting point and motivation of this thesis, we provide an overview of some notions and results from the field of Variational Analysis. Afterward, we provide an application to ODE-constrained optimal control in the subsequent section. For a comprehensive and self-contained introduction to control theory, we refer to [3] and [17].

## 1.1 Regularity of Set-Valued Maps

Let us overview some concepts and results of Variational Analysis that are relevant to the topic of this thesis. The presentation is based on the content and the structure provided in the excellent and self-contained introduction to Variational Analysis by Asen L. Dontchev in [14].

We begin with the formal definition of a variational problem.

**Definition 1.1.1** ([14] p.13)**.** *Let $X$ be a Banach space and $X^*$ it's dual. Given a function $g : X \to X^*$ and a set $C \subset X$, the problem to find $x \in X$ such that*

$$\langle g(x), z - x\rangle \geq 0 \quad \text{for all } z \in C, \tag{1.1}$$

*is called a variational problem.*

To introduce the type of set-valued mapping we are interested in, we need the normal cone, defined below.

**Definition 1.1.2.** *Let $C \subset X$ be a closed and convex. The normal cone to $C$ at $x$ is the set*

$$N_C(x) := \begin{cases} \left\{\Lambda \in X^* \middle| \langle \Lambda, z - x\rangle \leq 0 \text{ for all } z \in C\right\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{1.2}$$

The normal cone, defines the set-valued *normal cone mapping* $x \to N_C(x)$ on $X$ with respect to $C$. Using the normal cone, the variational inequality (1.1) can be equivalently written as

$$0 \in g(x) + N_C(x). \tag{1.3}$$

Thus, $g$ satisfying the variational problem (1.1), defines a set-valued map $\Psi(x) := g(x) + N_C(x)$ and the variational problem (1.1) can be seen as the problem of finding $x \in X$ such that $0 \in \Psi(x)$.
A motivation for considering a variational inequality as (1.1), is given by its connection to necessary first-order optimality conditions. To demonstrate this, let us take a Fréchet differentiable functional $J : X \to \mathbb{R}$ that is to be minimized over a closed and convex set $C$. By $DJ(x)$ we denote the Fréchet differential of $J$ at $x \in X$. For a local minimum $x$, it is necessary that

$$\langle DJ(x), z - x\rangle \geq 0 \quad \forall z \in C.$$

This can be written as $0 \in DJ(x) + N_C(x)$ or $0 \in \Psi(x)$, with $g := DJ$. Now, Variational Analysis provides a framework to study stability properties of (1.3) under perturbations appearing in the normal cone. Before stating further details, let us recall the meaning of Lipschitz continuity of a set-valued map. For this, we remember the definition of the domain, graph, and range of a set-valued mapping. The domain of a set-valued mapping F between the spaces $X$ and $Y$ is given by $\operatorname{dom} F = \{x \in X |\ F(x) \neq \emptyset\}$. The graph of $F$ is defined as $\operatorname{gph} F = \{(x, y) \in X \times Y |\ y \in F(x)\}$, and the range of F is given by $\operatorname{rge} F = \{y \in Y |$ there exists $x \in \operatorname{dom} F$ with $y \in F(x)\}$.

**Definition 1.1.3** (Lipschitz continuity). *Let $(X, h)$ and $(Y, \rho)$ be metric spaces. A set-valued mapping $F : X \twoheadrightarrow Y$ is called Lipschitz continuous with respect to a set $D \neq \emptyset$, $D \subset X$, if $D \subset \text{dom } F$, $F(x)$ is closed for all $x \in D$ and there exists a constant $\kappa \geq 0$ such that*

$$h(F(z), F(y)) \leq \kappa\rho(z, y) \quad \textit{for all } z, y \in D. \tag{1.4}$$

There are several notions of regularity considered in the context of Variational Analysis. We begin with metric regularity and introduce strong metric regularity afterward. Finally, we present a property called strong metric sub-regularity, a version of strong metric regularity that is important to this thesis.

## Metric Regularity

Metric regularity is a central property of set-valued maps in Variational Analysis. Its goal is to estimate the distance between approximate and exact solutions of inclusions with respect to a residual error.

**Definition 1.1.4** (Metric regularity, [14] p.37). *Let $(X, d)$ and $(Y, \rho)$ be metric spaces. A mapping $F : X \twoheadrightarrow Y$ is called metrically regular at $\bar{x}$ for $\bar{y}$, if $\bar{y} \in F(\bar{x})$, the graph of $F$ is locally closed at $(\bar{x}, \bar{y})$ and there is a constant $\kappa > 0$ and neighbourhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that*

$$d(x, F^{-1}(y)) \leq \kappa\rho(y, F(x)) \quad \textit{for all } (x, y) \in U \times V. \tag{1.5}$$

To rephrase, metric regularity states that if a set valued-map is metric regular at a point $y \in Y$, then one can estimate the distance of a point $x \in X$ of being a solution of $y \in F(x)$, by $\rho(y, F(x))$. It is well known that for linear bounded map $F$ between Banach spaces, metric regularity is equivalent to $F$ being surjective. Further, the metric regularity of a set-valued map is related to a Lipschitz-type property of its inverse (the so-called Aubin property).

**Definition 1.1.5** (Aubin property [14] p.39). *Let $(X, d)$ and $(Y, \rho)$ be metric spaces. A set-valued mapping $F : Y \twoheadrightarrow X$ satisfies the Aubin property at $\bar{y} \in Y$ for $\bar{x} \in X$, if $\bar{y} \in F(\bar{x})$, the graph of $F$ is locally closed at $(\bar{x}, \bar{y})$, there exists a constant $\kappa \geq 0$, and neighbourhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that*

$$d(G(\tilde{y}) \cap U, G(y)) \leq \kappa\rho(\tilde{y}, y) \quad \textit{for all } y, \tilde{y} \in V. \tag{1.6}$$

The following theorem connects the metric regularity of a set-valued function $F$ with the Aubin property of its inverse $F^{-1}$.

**Theorem 1.1.1** ([14] p.40). *A set-valued map $F : X \twoheadrightarrow Y$ is metrically regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa > 0$, iff its inverse has the Aubin property at $\bar{y}$ for $\bar{x}$ with constant $\kappa > 0$.*

Let us consider a property related to Lipschitz continuity, the so-called calmness property. A function $f : X \to Y$ between the metric spaces $(X, d)$ and $(Y, \rho)$ is called calm at $x$ for $D \subset X$ if $x \in D \cap \text{ dom } f$ and there exists a constant $\kappa \geq 0$ such that

$$\rho(f(x'), f(x)) \leq \kappa d(x', x) \text{ for all } x \in D \cap \text{ dom } f. \tag{1.7}$$

Given a function $f : X \to Y$ and a point $x \in \textit{dom} f$, the calmness modulus of $f$ at $x$, clm $(f; x)$, is defined as the infimum of the set of values $\kappa \geq 0$ for which there is a neighborhood $D$ of $x$ such that

(1.7) holds. The concept of calmness allows us to define the Fréchet differentiability of a function $f : X \to Y$ between Banach spaces $X$ and $Y$ at a point $x \in \operatorname{int dom} f$ as the existence of a linear mapping $Df(x) : X \to Y$ such that

$$\text{clm }(e;x) = 0 \text{ for } e(x') := f(x') - [f(x) + Df(x)(x'-x)].$$

We call a function $f : X \to Y$ strictly Fréchet differentiable at a point $x$ if there is a linear and bounded mapping $A : X \to Y$ such that

$$\text{Lip }(e;x) = 0 \text{ for } e(x') = f(x') - [f(x) + A(x'-x)].$$

Thus we have that for each $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ such that

$$\|f(x'') - [f(x') + Df(x)(x''-x')]\| \leq \varepsilon\|x''-x'\| \text{ for every } x'', x' \in U.$$

Strictly Fréchet differentiable functions are Fréchet differentiable and the continuously Fréchet differentiable functions are strictly Fréchet differentiable functions, satisfying Lip $(f;x) = \|Df(x)\|$. Having introduced the notion of metric regularity and the related terminology of calmness and strict Fréchet differentiability, we can now present a well-known sufficient condition for metric regularity.

**Theorem 1.1.2** (Lyusternik-Graves [14] p.47)**.** *Let $f : X \rightrightarrows Y$ be a single-valued map between Banach spaces $X$ and $Y$. Suppose that $f$ is strictly Fréchet differentiable at $\bar{x} \in \text{dom } f$ such that the differential $Df(\bar{x})$ is surjective. Then $f$ is metrically regular at $\bar{x}$ for $f(\bar{x})$.*

## Strong Metric Regularity

The notion of strong regularity of a set-valued mapping relates to invertibility properties in the context of set-valued maps between Banach spaces.

**Definition 1.1.6** (Strong metric regularity, [14] p.80)**.** *A mapping $F : X \rightrightarrows Y$ between Banach spaces $X$ and $Y$ is called strongly metrically regular at $\bar{x}$ for $\bar{y}$, if $(\bar{x}, \bar{y}) \in \text{graph } F$ and the inverse $F^{-1}$ admits a Lpischitz continuous single-valued localization around $\bar{y}$ for $\bar{x}$.*

As before we can characterize Lipschitz continuity of a localization in terms of the Aubin property.

**Proposition 1.1.3.** *Let $G : Y \rightrightarrows X$ be a set-valued mapping between Banach spaces $X$ and $Y$. Given a pair $(\bar{x}, \bar{y})$ in graph $G$, it is equivalent.*

1. *$G$ has a Lpischitz single-valued localisation $g$ around $\bar{y}$ for $\bar{x}$.*

2. *$G$ has the Aubin property at $\bar{y}$ for $\bar{x}$ and a nowhere multi-valued localisation around $\bar{y}$ for $\bar{x}$.*

The famous Robinson implicit function theorem gives a sufficient condition for strong metric regularity.

**Theorem 1.1.4** (Robinson, [14] p.81)**.** *Let $P$ and $X$ be Banach spaces and $X^*$ be the dual for $X$. For a function $f : P \times X \to X^*$ and a closed and convex set $C$ in $X$, consider the parameterized variational inequality*

$$\langle f(p,x), y - x\rangle \geq 0 \textit{ for all } y \in C,$$

*or equivalently,*

$$0 \in f(p,x) + N_C(x),$$

*where $p \in P$ is a parameter, and $N_C : X \to X^*$ is defined as the usual normal cone mapping. Denote by $S$ its solution mapping and let $\bar{x} \in S(\bar{p})$. Assume that:*

1. *$f(p,x)$ is Fréchet differentiable with respect to $x$ in a neighbourhood of the point $(x,p)$, and both $f(p,x)$ and $D_x f(p,x)$ depend continuously on $(p,x)$ in this neighbourhood. Also, $f$ is Lipschitz continuous in $p$ uniformly in $x$ around $(\bar{p}, \bar{x})$;*

2. *The inverse $G^{-1}$ of the set-valued mapping $G : X \twoheadrightarrow X^*$ defined as*

$$x \twoheadrightarrow G(x) = f(\bar{p}, \bar{x}) + D_x f(\bar{p}, \bar{x})(x - \bar{x}) + N_C(x), \quad \text{with } 0 \in G(\bar{x}),$$

*has a Lipschitz continuous single-valued localization around $0$ for $\bar{x}$.*

*Then $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$, which is Lipschitz continuous around $\bar{p}$.*

### Strong Metric Subregularity

Finally, we come to the property that plays a significant role in this thesis, the so-called strong metric subregularity. While strong metric regularity allows comparing two points around a given point at which the set-valued mapping is strongly metrically regular, the property of strong metric subregularity allows only the comparison of points with the point where the map is strongly metrically subregular, making it in a sense a *one-point* version of strong metric regularity and thus a weaker condition that applies to the affine problems we investigate in this thesis.

**Definition 1.1.7** (Strong metric subregularity, [14] p.114)**.** *A mapping $F : X \twoheadrightarrow Y$ acting between metric spaces $(X, \rho)$ and $(Y, d)$ is said to be strongly subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in gph\ F$, and there is a constant $\kappa > 0$ along with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that*

$$\rho(x, \bar{x}) \leq \kappa d(\bar{y}, F(x) \cap V) \quad \text{for all } x \in U.$$

The following subsection presents a sufficient condition for the strong metric subregularity of the optimality map of affine ODE-constrained optimal control problems. It turns out that this condition is also sufficient for the strong metric subregularity of the optimality map corresponding to PDE-constrained optimal control problems.

## 1.2 Strong Metric Subregularity in ODE-Constrained Optimal Control

We discuss an approach developed by Osmolovskii and Veliov in [20] for deriving strong metric subregularity of the optimality map for affine ODE constraint optimal control problems. These results are the starting point and motivation of this thesis.

Consider the Lagrange-type optimal control problem

$$\min_{u \in \mathcal{U}} \Big\{ J(u) := \int_0^T L(t, x(t), u(t))\ \mathrm{d}t \Big\}, \tag{2.8}$$

subject to

$$\dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0, \tag{2.9}$$

$$u(t) \in U, \text{ a.e } t \in [0, T]. \tag{2.10}$$

The solutions to (2.9) are denoted by $x$ and take values in $\mathbb{R}^n$, the control $u$ has values $u(t)$ almost everywhere in a given set $U \subset \mathbb{R}^m$. The initial state $x_0$ and the final time $T > 0$ are fixed. The set of feasible controls $\mathcal{U}$ consists of all Lebesgue measurable and bounded functions $u : [0, T] \to U$. The states x, that are solutions of (2.8)-(2.10), are Lipschitz continuous functions. The investigated problem is assumed to be affine with respect to the control, that is, $L$ and $f$ have the following form:

$$f(t, x, u) := a(t, x) + B(t, x)u, \quad L(t, x, u) := w(t, x) + \langle s(t, x), u(t)\rangle, \tag{2.11}$$

where $B$ is a $n \times m$ matrix valued function, $s$ is a vector valued function and $w$ is a scalar valued function. We make the following assumption on the optimal control problem. The set $U$ is convex and compact, the functions $L$ and $f$ are two times differentiable in $x$, the second derivatives are continuous in $x$ locally uniformly in $t$; and for every $x \in \mathbb{R}^n$ and $u \in U$, the functions $L$ and $f$, and their first and second derivatives in $x$ are measurable and bounded in $t$.

The space $L^p([0, T], \mathbb{R}^r)$, with $p = 1, 2$ or $p = \infty$, consists of all Lebesgue measurable $r$-dimensional vector-functions defined on the interval $[0, T]$, for which the standard norm $\|\cdot\|_k$ is finite. As usual, $W^{1,1}([0, T], \mathbb{R}^r)$ denotes the space of absolutely continuous functions $x : [0, T] \to \mathbb{R}$ for which the first derivative belongs to $L^1([0, T], \mathbb{R}^r)$. The norm in $W^{1,1}([0, T], \mathbb{R}^r)$ is defined as $\|x\|_{W^{1,1}} := |x(0)| + \|\dot{x}\|_1$, so that $\|x\|_\infty \leq \|x\|_{W^{1,1}}$. The Hamiltonian associated with problem (2.8)-(2.10) as usual:

$$H(t, x, u, p) := L(t, x, u) + \langle p, f(t, x, u)\rangle \in \mathbb{R}^n. \tag{2.12}$$

For a certain convex and compact set $S$ (see [20, Remark 2.1]), we denote by $Lip$ a Lipschitz constant with respect to $x \in S$ (uniformly in $t \in [0, T]$, $u \in U, p \in S$) of the functions $f$, $L$ and $H$, their first derivatives in $x$, and $H_{ux}$ and $H_{up}$. By $M$ we denote a bound of the functions $f, f_u, f_x, f_{ux}, H_x, H_{xx}, H_{xu}, H_{xxu}$ and $H_{xxp}$ for $(t, x, u, p) \in [0, T] \times S \times U \times S$. Further, we denote by $\omega$ the modulus of continuity of $H_{xx}$, uniformly with respect to $(t, u, p) \in [0, T] \times U \times S$. Although the feasible controls $u \in \mathcal{U}$ are bounded, we consider the control-trajectory pairs $(x, u)$ as elements of the space $W^{1,1} \times L^1 := W^{1,1}([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R}^m)$. Applying Pontryagin's maximum principle for problem (2.8)-(2.10), and using the definition of the normal cone,

$$N_U(u) := \begin{cases} \left\{ y \in \mathbb{R}^n \middle| \langle y, v - u\rangle \leq 0 \text{ for all } v \in U \right\} & \text{if } u \in U, \\ \emptyset & \text{if } u \notin U, \end{cases} \tag{2.13}$$

we can formulate the following first-order optimality system for $(x, u)$ and the absolutely continuous function $p : [0, T] \to \mathbb{R}^n$, for a.e $t \in \mathbb{R}$, the so-called adjoint state, by

$$0 = -\dot{x}(t) + f(t, x(t), u(t)), \; x(0) - x0 = 0, \tag{2.14}$$

$$0 = \dot{p}(t) + \nabla_x H(t, x(t), u(t), p(t)), \tag{2.15}$$

$$0 = p(T), \tag{2.16}$$

$$0 \in \nabla_u H(t, x(t), u(t), p(t)) + N_U(u(t)). \tag{2.17}$$

In [20], the authors define the linearized equation for $(u-\bar{u}) \in U - \hat{u}(t)$ a.e. in $[0,T]$, of equation (2.9):

$$\delta x(t) = f_x(t,\hat{x}(t),\hat{u}(t))\delta x(t) + f_u(t,\hat{x}(t),\hat{u}(t))(u-\bar{u})(t), \ \ \delta x(0) = 0, t \in [0,T]. \tag{2.18}$$

Using the solution to the linearized state, we can calculate the first variation of the objective function at $\bar{u}$ in direction $u-\bar{u}$:

$$J'(\bar{u})(u-\bar{u}) = \int_0^T \langle \nabla_u H(t,\hat{x}(t),\hat{u}(t),\hat{p}(t)), (u-\bar{u})(t)\rangle \,\mathrm{d}t.$$

In what follows, let us abbreviate $\hat{H}(t) := H(t,\hat{x}(t),\hat{u}(t),\hat{p}(t))$. The quadratic functional of $(\delta x, u-\bar{u}) \in W^{1,1} \times L^1$, coming from the second variation reads in the affine case as

$$\Omega(\delta x, u-\bar{u}) := \int_0^T \left[1/2\langle \hat{H}_{xx}(t)\delta x(t), \delta x(t)\rangle + \langle \hat{H}_{ux}(t)\delta x(t), (u-\bar{u})(t)\rangle\right] \mathrm{d}t.$$

Let $\Gamma$ be the set of all $(\delta x, u-\hat{u}) \in W^{1,1} \times L^1$ such that $(u-\hat{u})(t) \in U - \hat{u}(t)$ a.e. in $[0,T]$ with $\delta x$ being the solution of the linearized equation. We make the following assumption on the joint growth of the first and second variation.

**Assumption 1.2.1** ([20])**.** *There exists a constant $c_0 > 0$ such that*

$$\int_0^T \langle \nabla_u \hat{H}(t), (u-\bar{u})(t)\rangle \ dt + \Omega(\delta x, u-\hat{u}) \geq c_0\|u-\bar{u}\|_1^2 \tag{2.19}$$

*for all $(\delta x, u-\bar{u}) \in \Gamma$.*

The authors show in [20], that assumption (1.2.1) is sufficient for strict local optimality; we prove in this thesis that this also applies to PDE-constrained problems.

**Corollary 1.2.2** ([20])**.** *Let $\hat{y} = (\hat{x},\hat{u},\hat{p}) \in W^{1,1} \times \mathcal{U} \times W^{1,1}$ be a solution of the part* (2.14)-(2.16) *of the optimality system* (2.14)-(2.17)*. Let, in addition, Assumption* (1.2.1) *be fulfilled. Then $(\hat{x},\hat{u})$ is a strict strong local solution of problem* (2.8)-(2.10)*. Consequently, inclusion* (2.17) *is also satisfied.*

By "strict strong local solution" the authors in [20] mean the following: there is a number $\varepsilon > 0$ such that for every $u \in \mathcal{U}$ with $\|u-\hat{u}\|_1 \leq \varepsilon$ and $u \neq \hat{u}$ (in the sense of $L^1$) it holds that if the corresponding solution of (2.9)) exists on $[0,T]$, then $J(u) > J(\hat{u})$.

Define the spaces $\mathcal{Y} := W_0^{1,1} \times \mathcal{U} \times W^{1,1}$, $\mathcal{Z} := L^1 \times L^1 \times \mathbb{R}^n \times L^\infty$. The optimality system corresponding to problem (2.8)-(2.10) can be recast as the generalized equation $0 \in \Psi(y)$, where $y = (x,u,p)$ and $\Psi : \mathcal{Y} \to \mathcal{Z}$,

$$\Psi(y) := \begin{pmatrix} -\dot{x} + f(\cdot,x,u) \\ p + \nabla_x H(\cdot,y) \\ p(T) \\ \nabla_u H(\cdot,y) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{2.20}$$

Here

$$N_{\mathcal{U}}(u) := \{v \in L^\infty(0,T) : |\ v(t) \in N_U(u(t)) \text{ for a.e. } t \in [0,T]\}.$$

Finally, to prove the strong metric subregularity of the optimality map, the authors in [20] need the following stronger version of Assumption (1.2.1).

**Assumption 1.2.3** ([20]). *Let* $\hat{y} = (\hat{x}, \hat{u}, \hat{p}) \in Y$ *satisfy* $0 \in \Psi(\hat{y})$ *and let there exists a constant* $c_0 > 0$ *such that*

$$\int_0^T \langle \nabla_u \hat{H}(t), (u - \bar{u})(t) \rangle \; dt + 2\Omega(\delta x, u - \bar{u}) \geq c_0 \|u - \hat{u}\|_1^2, \tag{2.21}$$

*for all* $(\delta x, u - \bar{u}) \in \Gamma$.

The main theorem of Osmolovskii and Veliov in [20] follows.

**Theorem 1.2.1** ([20]). *Let Assumption 1.2.3 be fulfilled. Then the optimality mapping, associated with problem* (2.8)-(2.10)*, is strongly metrically sub-regular at* $\hat{y} = (\hat{x}, \hat{u}, \hat{p})$ *for zero. Moreover, the parameters of SMsR can be chosen depending on the data of the problem* (2.8)-(2.10) *only through the constants* $Lip, M$ *and* $T$*, the modulus* $\omega$*, and the constant* $c_0$ *in Assumption 1.2.3.*

As an application of the achieved strong metric subregularity, Osmolovskii, and Veliov obtain error estimates for the Euler discretization; we refer to [20, Theorem 5.1] for the statement of their result. The approach outlined in [20] and the sufficient optimality assumptions introduced motivate us to consider strong metric subregularity and solution stability in a more general context. The content of this thesis, the investigation of solution stability properties of PDE-constraint optimal control problems, is inspired by their result, builds upon and further extends it. As seen in the next chapter, one of the main difficulties is the technical analysis needed for treating PDE-constrained problems and achieving "good" apriori estimates (especially for the parabolic case) and the right approach to capitalize on them.

# 2. Solution Stability in PDE-Constrained Optimal Control

There are many appearances of PDEs in the modeling of phenomena in physics, chemistry, and biology. Consequently, it has become increasingly important to consider optimal control problems where PDEs appear as one of the constraints. Due to the complex analytical structure of PDEs, especially if the data of the PDE is of low regularity, the investigation of existence, uniqueness, and regularity properties of solutions, and subsequently, the treatment of PDE-constrained problems has to overcome many difficulties. Often, tools used in the ODE setting do not apply to the PDE setting. Therefore, the first difficulties of PDE-constrained optimal control already arise with the concise analysis of the involved PDEs and the behavior of their solutions, building on apriori estimates and maximum principles.

Optimal control problems can be classified into convex and nonconvex ones. For an optimal control problem, the property of being convex is strong, and many interesting problems do not satisfy it. Indeed, if the constraining PDE is semilinear or, more generally, nonlinear, we cannot expect the convexity of the problem. For convex problems, the first-order necessary optimality condition is already sufficient, and under some properties of the objective functional, the existence of a unique global minimizer is easily achieved. This is not true for nonconvex problems; here, we may guarantee the existence of at least one global minimizer under some regularity assumptions on the problem, but further local minimizers may exist. To guarantee the local optimality of a given control, it is necessary to introduce additional conditions, so-called second-order sufficient conditions. Thus for the study of nonconvex problems, it is important to consider second-order optimality conditions.
One of the pillars in PDE-constrained optimal control is the formulation of sufficient second-order optimality conditions, which are as weak as possible and thus can be applied to many problems but are still feasible for numerical implementation. In this regard, much progress was made in the last decades, for instance by, among others, E. Casas, M. Mateos, A. Rösch, F. Tröltzsch and G. and D. Wachsmuth.

A central task in PDE-constrained optimal control is studying optimal controls' so-called solution stability property. This property quantifies the behavior of the optimal control problem at an optimal control with respect to disturbances. Indeed, if the optimal control problem does not satisfy some kind of solution stability property under a small perturbation of the data of the problem, the perturbation may steer the optimal solutions of the perturbed problem far away from the optimal control of the original problem. Of course, this behavior is bad for all numerical approximations as they can be seen as a perturbation of the continuous problem. It is, therefore, essential to identify sufficient conditions for solution stability and quantify their implication on the distance of optimal controls and states. The investigation of stability properties of optimization problems, in general, is based on the study of the system of necessary optimality conditions. The first-order necessary optimality conditions consist of two equations (primal and adjoint) and one variational inequality forming a *generalized equation*. The concept of *strong metric subregularity*, see [9, 15], of set-valued mappings, which we also employ in this thesis, has shown to be effective in many applications, especially ones related to error analysis, see [1].

We split the presentation of some results obtained in the publications encompassed by this thesis into two sections, one for elliptic problems and one for parabolic problems.

## 2.1 The Elliptic Optimal Control Problem

Let us begin with a short overview of the notation, definitions, and the PDEs involved. Afterward, we define the optimal control problem and present a general approach for obtaining solution stability and related results. Finally, we discuss the application of solution stability to obtain error estimates for the numerical approximation.

### 2.1.1 Facts on the involved PDEs

For a non-empty, bounded and Lebesgue measurable set $X \subset \mathbb{R}^n$, we denote by $L^p(\Omega)$, $1 \le p \le \infty$, the Banach space of all measurable functions $f : X \to \mathbb{R}$ for which the usual norm $\|f\|_{L^p(\Omega)}$ is finite. For a bounded Lipschitz domain $X \subset \mathbb{R}^n$, the Sobolev space $H_0^1(X)$ consists of functions that vanish on the boundary and that have weak first-order derivatives in $L^2(\Omega)$. The space $H_0^1(X)$ is equipped with its usual norm denoted by $\|\cdot\|_{H_0^1(\Omega)}$. By $H^{-1}(\Omega)$ we denote the topological dual of $H_0^1(\Omega)$, equipped with the standard norm $\|\cdot\|_{H^{-1}(\Omega)}$. We denote by $C(\bar{\Omega})$ the space of continuous functions on $\Omega$ that can be extended continuously to $\bar{\Omega}$ equipped with the $\|\cdot\|_{L^\infty(\Omega)}$-norm. The space $H_0^1(\Omega) \cap C(\bar{\Omega})$ is endowed with the norm

$$\|\psi\|_{H_0^1(\Omega)\cap C(\bar{\Omega})} := \|\psi\|_{H_0^1(\Omega)} + \|\psi\|_{C(\bar{\Omega})}.$$

A function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if $\psi(\cdot, y)$ is measurable for every $y \in \mathbb{R}$, and $\psi(x, \cdot)$ is continuous for a.e. $x \in \Omega$. A function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz, uniformly in the first variable, if for each $M > 0$ there exists $L > 0$ such that

$$|\psi(x, y_2) - \psi(x, y_1)| \le L|y_2 - y_1|$$

for a.e. $x \in \Omega$ and all $y_1, y_2 \in [-M, M]$.

We consider an optimal control problem subject to the PDE used in [4]. A similar PDE was considered in [11]. The approach presented here applies to both of them. To formulate the problem, let us define the differential operator $\mathcal{A} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$\mathcal{A}y = -\operatorname{div}\big(A(x)\nabla y\big) + b(x) \cdot \nabla y.$$

Throughout this section, the following assumptions stand.

**Assumption 2.1.1.** *The following statements are fulfilled.*

*(i) The set* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *is a bounded domain with a Lipschitz boundary* $\Gamma$. *The mapping* $A : \Omega \longrightarrow \mathbb{R}^{n\times n}$ *is measurable and bounded in* $\Omega$, *and there exists* $\Lambda_A > 0$ *such that* $\xi^\top A(x)\xi \geq \Lambda_A|\xi|^2$ *for a.e.* $x \in \Omega$ *and all* $\xi \in \mathbb{R}^n$. *The components* $a_{i,j} \in L^\infty(\Omega)$ *of* $A$ *are additionally assumed to satisfy* $a_{i,j} \in C^{0,1}(\bar{\Omega})$ *(i.e. Lipschitz continuous on* $\bar{\Omega}$*) in the subsection on error estimates for the numerical approximation.*

*(ii) We assume that* $b \in L^p(\Omega; \mathbb{R}^n)$ *with* $p \geq 3$ *if* $n = 3$ *and* $p > 2$ *arbitrary if* $n = 2$.

**Assumption 2.1.2.** *We assume that* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Carathéodory function of class* $C^2$ *with respect to the second variable satisfying:*

$$f(\cdot, 0) \in L^r(\Omega) \text{ with } r > \frac{n}{2} \text{ and } \frac{\partial f}{\partial y}(x, y) \geq 0 \ \forall y \in \mathbb{R}, \tag{1.1}$$

$$\forall M > 0\ \exists C_{f,M} > 0 \text{ such that } \left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \leq C_{f,M}\ \forall |y| \leq M, \tag{1.2}$$

$$\begin{cases} \forall M > 0 \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x, y_2) - \dfrac{\partial^2 f}{\partial y^2}(x, y_1)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M \text{ and } |y_2 - y_1| \leq \delta, \end{cases} \tag{1.3}$$

*for almost every* $x \in \Omega$.

Under the assumptions above it is known that $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism even though the operator is neither coercive nor monotone; see [7], [18, Theorem 8.3], [25]. Consider a weak solution $y$ to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x, y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.4}$$

That is $y \in H_0^1(\Omega)$ satisfies

$$\langle \mathcal{A}y, z\rangle := \int_\Omega A\nabla y \cdot \nabla z \,\mathrm{d}x + \int_\Omega b \cdot \nabla y z \,\mathrm{d}x = \int_\Omega (u - f(\cdot, y))z \,\mathrm{d}x \quad \forall z \in H_0^1(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

The next theorem regarding solutions to (1.4) by E. Casas, M. Mateos, and A. Rösch is central.

**Theorem 2.1.3** ([6]). *Let Assumptions 2.1.1 and 2.1.2 hold. If* $u$ *belongs to* $L^r(\Omega)$ *for some* $r > n/2$, *then there exists a unique weak solution* $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ *of (1.7). Moreover, there exists a constant* $K_{f,r}$ *independent of* $u$ *such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq K_{f,r}\big(\|u\|_{L^r(\Omega)} + \|f(\cdot, 0)\|_{L^r(\Omega)} + 1\big).$$

*Further, if* $\{u_k\}_{k=1}^\infty$ *is a sequence converging weakly to* $u$ *in* $L^r(\Omega)$, *then* $y_{u_k} \to y_u$ *strongly in* $H_0^1(\Omega) \cap C(\bar{\Omega})$.

For each $r > n/2$, we define the control-to-state operator $G_r : L^r(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$, $G_r(u) = y_u$. To calculate the first and second variation of the objective functional, we need the following theorem.

**Theorem 2.1.4.** *Let Assumptions 2.1.1 and 2.1.2 hold. For every $r > \frac{n}{2}$ the map $G_r$ is of class $C^2$, and the first and second derivatives at $u \in L^r(\Omega)$ in the directions $v, v_1, v_2 \in L^r(\Omega)$, denoted by $z_{u,v} = G_r'(u)v$ and $z_{u,v_1,v_2} = G_r''(u)(v_1, v_2)$, are the weak solutions of the equations*

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = v,$$
$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = -\frac{\partial^2 f}{\partial y^2}(x, y_u) z_{u,v_1} z_{u,v_2},$$

*respectively.*

The proof of Theorem (2.1.4) can be obtained by similar arguments as in [7].

### 2.1.2 The control problem

Given $-\infty < u_a < u_b < +\infty$ let us define the set of feasible controls

$$\mathcal{U} = \{u \in L^2(\Omega) : u_a \le u(x) \le u_b \ \text{ for a.a. } x \in \Omega\}. \tag{1.5}$$

We consider the following optimal control problem

$$\text{(P)} \ \min_{u \in \mathcal{U}} J(u) := \int_\Omega L(x, y_u(x), u(x))\, \mathrm{d}x, \tag{1.6}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x,y) = u \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{1.7}$$

The next assumption on the objective integrand $L$ is assumed to be satisfied for the remainder of this chapter.

**Assumption 2.1.1.** *The function $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Carathéodory and of class $C^2$ with respect to the second variable. In addition, we assume that*

$$\begin{cases} L(x,y,u) = L_a(x,y) + L_b(x,y)u \ \textit{ with } \ L_a(\cdot,0) \in L^1(\Omega), L_b(\cdot,0) \in L^\infty(\Omega), \\ \forall M > 0 \ \exists C_{L,M} > 0 \textit{ such that} \\ \left|\frac{\partial L}{\partial y}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial y^2}(x,y,u)\right| \le C_{L,M} \ \forall |y|, |u| \le M, \\ \forall \rho > 0 \textit{ and } M > 0 \ \exists \varepsilon > 0 \textit{ such that} \\ \left|\frac{\partial^2 L}{\partial y^2}(x,y_2,u) - \frac{\partial^2 L}{\partial y^2}(x,y_1,u)\right| < \rho \ |y_1|, |y_2| \le M \textit{ with } |y_2 - y_1| \le \varepsilon, \end{cases}$$

*for almost every $x \in \Omega$.*

By Theorem 2.1.3, the assumptions on $L$, and the boundedness of $\mathcal{U}$ in $L^\infty(\Omega)$, the existence of at least one global solution of (P) follows. The problem is nonconvex; thus, we consider the notion of local minimizers.

**Definition 2.1.2.** *We say that $\bar{u} \in \mathcal{U}$ is an $L^r(Q)$-weak local minimum of problem* (1.5)-(1.7)*, if there exists some $\varepsilon > 0$ such that*

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^r(\Omega)} \leq \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ a strong local minimum of the optimal control problem if there exists $\varepsilon > 0$ such that*

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar{u}}\|_{L^\infty(\Omega)} \leq \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.*

As a consequence of Theorem (2.1.4), Assumption 2.1.1, and the chain rule, we obtain the Fréchet differentiability of the objective functional.

**Theorem 2.1.5.** *Suppose that $r > \frac{n}{2}$. Then, the functional $J : L^r(\Omega) \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^r(\Omega)$ we have*

$$J'(u)v = \int_\Omega \Big[\varphi_u + \frac{\partial L}{\partial u}(x, y_u, u)\Big] v \, \mathrm{d}x,$$
$$J''(u)(v_1, v_2) = \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\Big] z_{u,v_1} z_{u,v_2} \, \mathrm{d}x,$$

*where $\varphi_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation*

$$\begin{cases} \mathcal{A}^*\varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u, u) \text{ in } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases}$$

We obtain the following classical Pontryagin-type necessary optimality condition.

**Theorem 2.1.6.** *Let $\bar{u}$ be a (strong or not) local minimizer of* (P)*, then there exist two unique elements $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{cases} \mathcal{A}\bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases} \tag{1.8}$$

$$\begin{cases} \mathcal{A}^*\bar{\varphi} + \dfrac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \dfrac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) \text{ in } \Omega, \\ \bar{\varphi} = 0 \text{ on } \Gamma, \end{cases} \tag{1.9}$$

$$\int_\Omega \Big[\bar{\varphi} + \frac{\partial L}{\partial u}(x, \bar{y}, \bar{u})\Big](u - \bar{u}) \, dx \geq 0 \quad \forall u \in \mathcal{U}. \tag{1.10}$$

### 2.1.3 Sufficient optimality conditions

Let us discuss sufficient conditions for weak or strong local optimality.

**Assumption 2.1.3** ([11, 19])**.** *Let $\bar{u} \in \mathcal{U}$, $\gamma \in (n/(2+n), 1]$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u-\bar{u}) + \beta J''(\bar{u})(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{1.11}$$

*for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$.*

In PDE-constrained optimization, Assumption 2.1.3, with $\beta = 1$ and $\gamma \in (n/(2+n), 1]$, was first introduced in [11]. It originated from nonlinear affine ODE-constrained optimal control theory and was first introduced in [21] ($\beta \in \{1/2, 1\}$ and $\gamma = 1$). We mention that Assumption (2.1.3) implies the optimal controls to be bang-bang, see [11]. In [4], two additional assumptions on the joint growth of the first and second variation were introduced. They present a weakening of Assumption 2.1.3.

**Assumption 2.1.4** ([4])**.** *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ with*

$$J'(\bar{u})(u-\bar{u}) + \beta J''(\bar{u})(u-\bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|u-\bar{u}\|_{L^1(\Omega)}$$

*for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$.*

It was conjectured that Assumption 2.1.4 may be applicable for controls that are not bang-bang. This was recently answered negatively in [19]. Still, if the optimal control is not bang-bang, we can apply the next assumption.

**Assumption 2.1.5** ([4])**.** *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ with*

$$J'(\bar{u})(u-\bar{u}) + \beta J''(\bar{u})(u-\bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}^2$$

*for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(\Omega)} < \alpha$.*

All of these assumptions imply strict local optimality.

**Theorem 2.1.6** ([12])**.** *The following holds.*

1. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.1.3. Then, there exist $\varepsilon > 0$ such that:*

   $$J(\bar{u}) + 1/2\|u-\bar{u}\|_{L^1(\Omega)}^{1+1/\gamma} \leq J(u) \tag{1.12}$$

   *for all $u \in \mathcal{U}$ such that $\|u-\bar{u}\|_{L^1(\Omega)} < \varepsilon$.*

2. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.1.4 and $\frac{\partial^2 L}{\partial uy} = 0$. Then, there exist $\varepsilon > 0$*

   $$J(\bar{u}) + 1/2\|y_u - y_{\bar{u}}\|_{L^2(\Omega)}\|u-\bar{u}\|_{L^1(\Omega)} \leq J(u) \tag{1.13}$$

   *for all $u \in \mathcal{U}$ such that $\|u-\bar{u}\|_{L^1(\Omega)} < \varepsilon$.*

3. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.1.5 and $\frac{\partial^2 L}{\partial uy} = 0$. Then, there exist $\varepsilon > 0$*

   $$J(\bar{u}) + 1/2\|y_u - y_{\bar{u}}\|_{L^2(\Omega)}^2 \leq J(u) \tag{1.14}$$

   *for all $u \in \mathcal{U}$ such that $\|y_u - y_{\bar{u}}\|_{L^\infty(\Omega)} < \varepsilon$.*

### 2.1.4 Strong metric Hölder subregularity

We begin with the system representing the necessary optimality conditions for problem (1.5)–(1.7):

$$\begin{cases} 0 &=& \mathcal{A}y - f(\cdot, y, u), \\ 0 &=& \mathcal{A}^* p - H_y(\cdot, y, p, u), \\ 0 &\in& H_u(\cdot, y, p) + N_{\mathcal{U}}(u), \end{cases} \tag{1.15}$$

If $u \in \mathcal{U}$ is a local solution of problem (1.5)–(1.7), then the triple $(y_u, p_u, u)$ is a solution of (1.15). The mapping defined by the right-hand side is called the *optimality mapping*. To formally introduce the optimality mapping, we fix a number $r > n/2$, define the set

$$D(\mathcal{A}) := \left\{ y \in H_0^1(\Omega) \cap L^\infty(\Omega) \middle| \; \mathcal{A}y \in L^r(\Omega), y = 0 \text{ on } \Gamma \right\}, \tag{1.16}$$

and consider $\mathcal{A} : D(\mathcal{A}) \to L^r(\Omega)$. Similarly, to address the adjoint equation, we define the mapping $\mathcal{A}^* : D(\mathcal{A}^*) \to L^r(\Omega)$, where

$$D(\mathcal{A}^*) := \left\{ p \in H_0^1(\Omega) \cap L^\infty(\Omega) \middle| \mathcal{A}^* p \in L^r(\Omega), p = 0 \text{ on } \Gamma \right\}.$$

Now, we can define some metric spaces:

$$\mathcal{Y} := D(\mathcal{A}) \times D(\mathcal{A}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega),$$

The spaces $\mathcal{Y}$ and $\mathcal{Z}$ are endowed with the following metrics. For $\psi_i = (y_i, p_i, u_i) \in \mathcal{Y}$ and $\zeta_i = (\xi_i, \eta_i, \rho_i) \in \mathcal{Z}$, $i \in \{1, 2\}$,

$$\begin{aligned} d_{\mathcal{Y}}(\psi_1, \psi_2) &:= \|y_1 - y_2\|_{L^2(\Omega)} + \|p_1 - p_2\|_{L^2(\Omega)} + \|u_1 - u_2\|_{L^1(\Omega)}, \\ d_{\mathcal{Z}}(\zeta_1, \zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(\Omega)} + \|\eta_1 - \eta_2\|_{L^2(\Omega)} + \|\rho_1 - \rho_2\|_{L^\infty(\Omega)}. \end{aligned}$$

We denote by $\mathbb{B}_{\mathcal{Y}}(\psi; \alpha)$ the closed ball in $\mathcal{Y}$, centered at $\psi$ and with radius $\alpha$. The notation for the ball $\mathbb{B}_{\mathcal{Z}}(\zeta; \alpha)$ is analogous. Let us remind the definition of the normal cone to the set $\mathcal{U}$ at $u \in L^1(\Omega)$:

$$N_{\mathcal{U}}(u) := \begin{cases} \left\{ \nu \in L^\infty(\Omega) \middle| \; \int_\Omega \nu(v - u)\,\mathrm{d}x \le 0 \;\; \forall v \in \mathcal{U} \right\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

Then the optimality mapping is defined as the set-valued mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$,

$$\Phi(y, p, u) := \begin{pmatrix} \mathcal{A}y - f(\cdot, y, u) \\ \mathcal{A}^* p - H_y(\cdot, y, p, u) \\ H_u(\cdot, y, p, u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{1.17}$$

The optimality system (1.15) can be recast as the inclusion

$$0 \in \Phi(y, p, u). \tag{1.18}$$

We denote $\bar{\psi} := (\bar{y}, \bar{p}, \bar{u}) = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ , where $\bar{u}$ is the fixed local solution of problem (1.5)–(1.7).

**Definition 2.1.1.** The optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called strongly Hölder subregular with exponent $\gamma > 0$ at $(\bar{\psi}, 0)$ if there exist positive numbers $\alpha_1, \alpha_2$ and $\kappa$ such that

$$d_{\mathcal{Y}}(\psi, \bar{\psi}) \le \kappa d_{\mathcal{Z}}(\zeta, 0)^{\gamma} \tag{1.19}$$

for all $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \alpha_1)$ and $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.

More explicitly, the inequality (1.19) reads as

$$\|y - y_{\bar{u}}\|_{L^2(\Omega)} + \|p - p_{\bar{u}}\|_{L^2(\Omega)} + \|u - \bar{u}\|_{L^1(\Omega)} \le \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)}\Big)^{\gamma}.$$

For technical reasons, let us make an assumption on the feasible perturbations of the optimality mapping that stands for the whole section.

**Assumption 2.1.7.** *For a fixed positive constant $C$, the admissible perturbation $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$ satisfy the restriction*

$$\|\xi\|_{L^r(\Omega)}, \|\eta\|_{L^r(\Omega)} \le C. \tag{1.20}$$

One of the main theorems in this section is the following.

**Theorem 2.1.2** ([11]). *Let Assumption 2.1.2 hold for some $\gamma \in (n/(2+n), 1]$ and $\beta = 1$. Then the optimality mapping $\Phi$ is strongly Hölder subregular at $(\bar{\psi}, 0)$ with exponent $\gamma$.*

To demonstrate the application of the obtained strong metric Hölder subregularity of the optimality mapping, let us consider the perturbed problem:

$$\min_{u \in \mathcal{U}} \Big\{ J_\zeta(u) := \int_\Omega L(x, y_u, u) + \rho u + \eta y_u \,\mathrm{d}x \Big\} \tag{1.21}$$

subject to (1.5) and

$$\begin{cases} \mathcal{A}y + f(\cdot, y) & = u + \xi \quad \text{in } \Omega, \\ y & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

The existence of a globally optimal solution to (1.21)-(1.22) is guaranteed by the assumptions on the optimal control problem and the direct method in the calculus of variations. Let the optimality mapping (1.17) be strongly metrically Hölder subregular at $\bar{u}$ with exponent $\gamma$ and numbers $\kappa, \alpha_1$ and $\alpha_2$. Given a solution $\bar{u}^\zeta$ of the system of first-order optimality conditions of the perturbed problem such that $\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)} < \alpha_1$, it holds

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} + \|\bar{y}^\zeta - \bar{y}\|_{L^2(\Omega)} + \|\bar{p}^\zeta - \bar{p}\|_{L^2(\Omega)} \le \kappa(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)})^\gamma.$$

Under condition (2.1.3) ($\beta = 1/2$), and the additional assumption that the second derivatives of $L$ and $f$ are Lipschitz with respect to the $y$-variable, we can prove another stability result as well.

**Theorem 2.1.8** ([19]). *Let $\bar{u}$ satisfy Assumption 2.1.3($\beta = 1/2$) for some $\gamma \in (n/(n+2), 1]$. There exist positive constants $c$ and $\alpha$ such that*

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} + \|\bar{y}^\zeta - \bar{y}\|_{L^2(\Omega)} + \|\bar{p}^\zeta - \bar{p}\|_{L^2(\Omega)} \le c(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)})^\gamma,$$

*for any minimizer $(\bar{y}^\zeta, \bar{p}^\zeta, \bar{u}^\zeta)$ of* (1.21)-(1.22) *with $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

We obtain solution stability of the optimal control problem with respect to perturbations also under the Assumptions 2.1.4 and 2.1.5. In the theorems below, we need to additionally assume that $L_1(x,y)u := gu$, for some function $g \in L^\infty(\Omega)$.

**Theorem 2.1.7** ([4]). *Let $\bar{u}$ be a local minimizer of* (P) *satisfying Assumption 2.1.4. Consider perturbations of the form $\zeta = (\xi, \eta, 0)$. Then, there exist positive constant $\kappa$ and $\alpha$ such that*

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} + \|\bar{y}^\zeta - \bar{y}\|_{L^2(\Omega)} + \|\bar{p}^\zeta - \bar{p}\|_{L^2(\Omega)} \leq \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}\Big),$$

*for all triple $(\bar{u}^\zeta, \bar{y}^\zeta, \bar{p}^\zeta)$ that solve the first-order necessary optimality condition of the perturbed problem and satisfy $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

If the objective integrand of the perturbed problem is of the form

$$L_\zeta(x,y,u) := L_0(x,y) + \eta y + \rho u + \frac{\varepsilon}{2}u^2, \ \varepsilon > 0,$$

we obtain the following state stability result.

**Theorem 2.1.8** ([4]). *Let $\bar{u}$ be a local minimizer of* (P) *satisfying Assumption 2.1.4. Then, there exist positive constant $\kappa$ and $\alpha$ such that*

$$\|\bar{y}^\zeta - \bar{y}\|_{L^2(\Omega)} + \|\bar{p}^\zeta - \bar{p}\|_{L^2(\Omega)} \leq \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)} + \varepsilon\Big),$$

*for all triple $(\bar{u}^\zeta, \bar{y}^\zeta, \bar{p}^\zeta)$ that solve the first-order necessary optimality condition of the perturbed problem and satisfy $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

**Theorem 2.1.9** ([4]). *Let $\bar{u}$ be a local minimizer of* (P) *satisfying Assumption 2.1.5. Consider perturbations of the form $\zeta = (\xi, \eta, 0)$. Then, there exist positive constant $\kappa$ and $\alpha$ such that*

$$\|\bar{y}^\zeta - \bar{y}\|_{L^2(\Omega)} + \|\bar{p}^\zeta - \bar{p}\|_{L^2(\Omega)} \leq \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}\Big)$$

*for all triple $(\bar{u}^\zeta, \bar{y}^\zeta, \bar{p}^\zeta)$ that solve the first-order necessary optimality condition of the perturbed problem and satisfy $\|\bar{y}^\zeta - \bar{y}\|_{L^\infty(\Omega)} < \alpha$.*

### 2.1.5 Error estimates for the numerical approximation

We consider the problem

$$\min_{u \in \mathcal{U}} \Big\{ J(u) := \int_\Omega L(x, y(x), u(x))\, \mathrm{d}x \Big\}, \tag{1.23}$$

subject to

$$\begin{cases} \mathcal{A}y + f(\cdot, y) = u & \text{in } \ \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.24}$$

In this section, we assume that the term $b$ in the operator $\mathcal{A}$ is zero.
We derive error estimates for the numerical approximation under Assumptions 2.1.3, 2.1.4, and 2.1.5.

The finite element scheme is close to the one in [6]. We also refer to [2] for an overview of the finite element method. In this section, $\Omega$ is convex and $\{\tau_h\}_{h>0}$ denotes a quasi-uniform family of triangulations of $\bar{\Omega}$. Denote $\bar{\Omega}_h = \cup_{T\in\tau_h} T$ and assume that every boundary node of $\Omega_h$ is a point of $\Gamma$. Further, suppose that there exists a constant $C_\Gamma > 0$ independent of $h$ such that the distance $d_\Gamma$ satisfies $d_\Gamma(x) < C_\Gamma h^2$ for every $x \in \Gamma_h = \partial\Omega_h$. The finite-dimensional space of discrete states is defined by

$$Y_h = \{z_h \in C(\bar{\Omega}) : z_{h|T} \in P_1(T) \ \forall T \in \tau_h \ \text{ and } z_h \equiv 0 \text{ on } \Omega \setminus \Omega_h\},$$

where $P_i(T)$ denotes the polynomials in $T$ of degree at most $i$. For $u \in L^2(\Omega)$, the associated discrete state is the unique element $y_h(u) \in Y_h$ that solves

$$a(y_h, z_h) + \int_{\Omega_h} f(x, y_h) z_h \,\mathrm{d}x = \int_{\Omega_h} u z_h \,\mathrm{d}x \quad \forall z_h \in Y_h, \tag{1.25}$$

where

$$a(y, z) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \partial_{x_i} y \partial_{x_j} z \,\mathrm{d}x \quad \forall y, z \in H^1(\Omega).$$

The existence and uniqueness of a solution for (1.25) follows standard arguments. We define the set of discrete controls for the discrete problem by

$$U_h := \{u_h \in L^\infty(\Omega_h) : u_{h|T} \in P_0(T) \ \forall T \in \tau_h\}.$$

By $\Pi_h$ we denote the linear projection onto $U_h$ in $L^2(\Omega_h)$, that is

$$(\Pi_h u)_{|T} = \frac{1}{|T|} \int_T u \,\mathrm{d}x, \quad \forall T \in \tau_h.$$

Altogether, defining $\mathcal{U}_h := U_h \cap \mathcal{U}$, we state the discrete problem:

$$\min_{u_h \in \mathcal{U}_h} \Big\{ J_h(u_h) := \int_{\Omega_h} L(x, y_h(u), u) \,\mathrm{d}x \Big\}. \tag{1.26}$$

The set $\mathcal{U}_h$ is compact and nonempty, and the existence of a global solution of (1.26) follows. For $u \in L^2(\Omega)$, the discrete adjoint state $p_h(u) \in Y_h$ is the unique solution of

$$a(z_h, p_h) + \int_{\Omega_h} \frac{\partial f}{\partial y}(x, y_h(u)) p_h z_h \,\mathrm{d}x = \int_{\Omega_h} \frac{\partial L}{\partial y}(x, y_h(u), u) z_h \,\mathrm{d}x \quad \forall z_h \in Y_h.$$

One can calculate that

$$J_h'(u)(v) = \int_{\Omega_h} (p_h(u) + \frac{\partial L}{\partial u}(x, y_h(u), u)) v \,\mathrm{d}x.$$

A local solution of (1.26) satisfies the variational inequality

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) \geq 0 \quad \forall u_h \in \mathcal{U}_h.$$

Now, we can formulate our main theorem on error estimates for the numerical approximation.

**Theorem 2.1.9** ([19]). *Let $\bar{u}$ be a local solution of* (P). *Consider the constant $\alpha$ corresponding to the Assumptions 2.1.3, 2.1.4 or 2.1.5. Consider a sequence of discrete optimal controls $\bar{u}_h \in \mathcal{U}_h$ of* (1.26) *that satisfy $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} < \alpha$. We recall that $\bar{y}$ is the solution of* (1.2) *and $y(\bar{u}_h)$ denotes the solution of* (1.25) *for $\bar{u}_h$.*

1. *Let $L_b = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption 2.1.5($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq c\sqrt{h}. \tag{1.27}$$

2. *Let $L_b = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption 2.1.4($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch. \tag{1.28}$$

3. *Let $\frac{\partial L_b}{\partial y} = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption 2.1.3($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^{\gamma}. \tag{1.29}$$

*If $\frac{\partial L_b}{\partial y} \neq 0$ we obtain the estimate*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^{\frac{(1+\min\{1/r,1/\gamma\})\gamma}{\gamma+1}}. \tag{1.30}$$

If a variational discretization scheme generates the discrete optimal control $u_h$, we obtain a stronger result.

**Theorem 2.1.10** ([19]). *Consider the constant $\alpha$ corresponding to the Assumptions 2.1.3, 2.1.4 or 2.1.5. Let $\{\bar{u}_h\}_h$ be a sequence of solutions to the first-order optimality condition of the discrete problems such that $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} < \alpha$.*

1. *Let Assumptions 2.1.5 be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq ch.$$

2. *Let Assumptions 2.1.4 be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq c(h|\log h|)^2.$$

3. *Let Assumptions 2.1.3 be satisfied by $\bar{u} \in \mathcal{U}$ for some $\gamma \in (n/(2+n), 1]$. There exists a positive constant $c$ independent of $h$ such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq c(h|\log h|)^{2\gamma}.$$

### 2.1.6 Solution stability and variational discretization

In this subsection, let us motivate a link between the property of solution stability and the obtainment of finite element error estimates. In this sense, Theorem 2.1.12 below shows that a property related to solution stability implies estimates for a finite element variational discretization scheme. The intuition may be, that once solution stability is achieved under certain growth conditions, we may expect error estimates for a variational discretization scheme under the same conditions and of similar strength.

Let us now define a property that we will call strong solution stability.

**Definition 2.1.11** (Strong solution stablility)**.** *We call the optimal control problem* (1.5)-(1.7) *strong solution stable at $\bar{u}$ for $V \subset \mathcal{U}$, with positive parameters $\kappa, \gamma$ and $\alpha$ if*

$$\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)} + \|\bar{y} - \bar{y}^\zeta\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}^\zeta\|_{L^\infty(\Omega)} \leq \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)}\Big)^\gamma \tag{1.31}$$

*for all triples $(\bar{u}^\zeta, \bar{y}^\zeta, \bar{p}^\zeta)$ related to the perturbed problem* (1.21)-(1.22) *that satisfy $\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)} < \alpha$ and*

$$J'_\zeta(\bar{u}^\zeta)(v - \bar{u}^\zeta) \geq 0 \quad \textit{for all } v \in V.$$

Assumption 2.1.3 implies strong solution stability. This can be observed in the proof of the strong metric subregularity property of the optimality map provided in [12].

**Theorem 2.1.12.** *Let the optimal control problem be strong solution stable at $\bar{u}$ for $\{\bar{u}\}$ with positive constants $\gamma, \kappa,$ and $\alpha$. Let $\{\bar{u}_h\}_h$ be a sequence of solutions to the discrete problems* (1.26) *with $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} < \alpha$. Then*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^\infty(\Omega)} \leq \kappa(h|\log h|)^{2\gamma}. \tag{1.32}$$

*Proof.* Given a minimizer $\bar{u}_h$ of the discrete problem (1.26), let $\zeta = (0, 0, \rho)$, with $\rho := p(\bar{u}_h) - p_{\bar{u}_h}$. Then we define the perturbed optimal control problem

$$\min_{u \in \mathcal{U}} \left\{ J_\zeta(u) := \int_\Omega L(x, y(x), u(x))\,\mathrm{d}x + \int_\Omega (p(\bar{u}_h) - p_{\bar{u}_h})u\,\mathrm{d}x \right\},$$

subject to

$$\begin{cases} Ay + f(\cdot, y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

It is easy to see that

$$J'_\zeta(\bar{u}_h)(\bar{u} - \bar{u}_h) = \int_\Omega p(\bar{u}_h)(\bar{u} - \bar{u}_h)\,\mathrm{d}x \geq 0. \tag{1.33}$$

But that is all we need of $\bar{u}_h$ to apply the strong solution stability at $\bar{u}$. That is, we obtain

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} \leq \kappa\|(p(\bar{u}_h) - p_{\bar{u}_h})\|^\gamma_{L^\infty(\Omega)}.$$

From here, the claim follows from estimates relating the discrete state and adjoint state with the continuous ones. $\square$

## 2.2 The Parabolic Optimal Control Problem

In this section, we introduce the strong metric Hölder-subregularity property of the optimality mapping of a parabolic problem. The higher dimensionality of the parabolic problem demands a more involved analysis.

### 2.2.1 Facts on the involved PDEs

First, we recall the standard definitions in the parabolic setting. Given a real Banach space $Z$, the space $L^p(0,T;\, Z)$ consist of all strongly measurable functions $y : [0,T] \to Z$ that satisfy

$$\|y\|_{L^p(0,T;\, Z)} := \Big( \int_0^T \|y(t)\|_Z^p \,\mathrm{d}t \Big)^{\frac{1}{p}} < \infty \qquad \text{if } \ 1 \leq p < \infty,$$

or, for $p = \infty$,

$$\|y\|_{L^\infty(0,T;\, Z)} := \inf\{M \in \mathbb{R} \mid \|y(t)\|_Z \leq M \text{ for a.e } t \in (0,T)\} < \infty.$$

The Hilbert space $W(0,T)$ consists of all of functions in $L^2(0,T;\, H_0^1(\Omega))$ that have a distributional derivative in $L^2(0,T;\, H^{-1}(\Omega))$, i.e.

$$W(0,T) := \left\{ y \in L^2(0,T;H_0^1(\Omega)) \Big| \ \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) \right\},$$

which is endowed with the norm

$$\|y\|_{W(0,T)} := \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial y/\partial t\|_{L^2(0,T;H^{-1}(\Omega))}.$$

The space $W(0,T)$ is compactly embedded in $L^2(Q)$ and continuously embedded in $C([0,T];\, L^2(\Omega))$, the Banach space of all continuous functions $y : [0,T] \to L^2(\Omega)$ equipped with $\max_{t\in[0,T]} \|y(t)\|_{L^2(\Omega)}$. The dual pairing between a Banach space $X$ and its dual is denoted by $\langle \cdot, \cdot \rangle_X$, for details on the spaces, see [26, 14, 16, 22].

Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Given a finite time $T > 0$, we denote by $Q := \Omega \times (0,T)$ the space-time cylinder, and its lateral boundary by $\Sigma := \partial\Omega \times (0,T)$. We define the elliptic operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ appearing in our parabolic PDE by

$$\mathcal{A}y = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

where $a_{i,j} \in L^\infty(\Omega)$ satisfy the uniform ellipticity condition

$$\exists \lambda_{\mathcal{A}} > 0 : \ \lambda_{\mathcal{A}} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \quad \text{for all } \xi \in \mathbb{R}^n \ \text{ and a.a. } x \in \Omega.$$

The functions $f$, $L_0 : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ of the variables $(x,t,y)$, and the "initial" function $y_0$ have the following properties.

1. For every $y \in \mathbb{R}$, $f(\cdot,\cdot,y) \in L^r(Q)$ with $r > 1 + 1/n$, $L_0(\cdot,\cdot,y) \in L^1(Q)$, and $y_0 \in L^\infty(\Omega)$.
2. For a.e. $(x,t) \in Q$ the first and the second derivatives of $f$ and $L_0$ with respect to $y$ exist and are locally bounded and locally Lipschitz continuous, uniformly with respect to $(x,t) \in Q$. Moreover, $\frac{\partial f}{\partial y}(x,t,y) \geq 0$ for a.e. $(x,t) \in Q$ and for all $y \in \mathbb{R}$.

Let $u \in L^r(Q)$, $r > 1 + 1/n$. We consider solutions of the following semilinear variational equality for $y \in W(0,T)$ with $y(\cdot,0) = y_0$:

$$\int_0^T \langle \frac{\partial y}{\partial t} + \mathcal{A}y, \psi \rangle_{H_0^1(\Omega)} \, dt = \int_0^T \langle u - f(\cdot; y), \psi \rangle_{L^2(\Omega)} \, dt \tag{2.34}$$

for all $\psi \in L^2(0,T,H_0^1(\Omega))$, that is, weak solutions of

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(\cdot,y) = u & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases}$$

The next theorem provides the existence and uniqueness of solutions to (2.34). The first part of the next theorem can be found in [4, Theorem 2.1], and the second in [5, Theorem 2.1].

**Theorem 2.2.1** ([4, 5]). *For every $u \in L^q(0,T;L^p(\Omega))$ with $\frac{1}{q} + \frac{n}{2p} < 1$ and $q, p \geq 2$ there exists a unique solution $y_u \in L^\infty(Q) \cap W(0,T)$ of (2.34). Moreover, the following estimates hold*

$$\|y_u\|_{L^\infty(Q)} \leq \eta(\|u\|_{L^q(0,T;L^p(\Omega))} + \|f(\cdot,\cdot,0)\|_{L^q(0,T;L^p(\Omega))} + \|y_0\|_{L^\infty(\Omega)}),$$

$$\|y_u\|_{C([0,T];L^2(\Omega))} + \|y_u\|_{L^2(0,T;H_0^1(\Omega))} \leq K(\|u\|_{L^2(Q)} + \|f(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}),$$

*for a monotone non-decreasing function $\eta : [0,\infty) \to [0,\infty)$ and some constant $K$ both independent of $u$. Finally, if $u_k \rightharpoonup u$ weakly in $L^q(0,T;L^p(Q))$, then*

$$\|y_{u_k} - y_u\|_{L^\infty(Q)} + \|y_{u_k} - y_u\|_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

The achievement of strong metric Hölder subregularity of the parabolic optimality mapping is possible due to the following lemma.

**Lemma 2.2.2.** *Let $u \in L^r(Q)$, $r > 1 + n/2$, and $0 \leq a_0 \in L^\infty(Q)$. Let $h_u$ be the unique solution of (2.34) with $f(\cdot,y) = a_0$ and let $p_u$ be a solution of the problem*

$$\begin{cases} -\frac{\partial p}{\partial t} + \mathcal{A}^*p + a_0 p = u & \text{in } Q, \\ p = 0 \text{ on } \Sigma, \; p(\cdot,T) = 0 & \text{on } \Omega. \end{cases}$$

*Then, for any $s_n \in [1, \frac{n+2}{n})$ there exists a constant $C_{s_n'} > 0$ independent of $u$ and $a_0$ such that*

$$\max\{\|h_u\|_{L^{s_n}(Q)}, \|p_u\|_{L^{s_n}(Q)}\} \leq C_{s_n'} \|u\|_{L^1(Q)}.$$

*Here $s_n'$ denotes the Hölder conjugate of $s_n$.*

We denote the control-to-state operator $G_r : L^r(Q) \to W(0,T) \cap L^\infty(Q)$, $r > 1 + n/2$, assigning to each $u$, the unique state $y_u$, by $G_r(v) := y_u$. The differentiability of the control-to-state operator under the assumptions in this section is well known, see [8, Theorem 2.4].

**Theorem 2.2.3.** *The control-to-state operator is of class $C^2$ and for every $u, v, w \in L^r(Q)$, it holds that $z_{u,v} := G_r'(u)v$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = v & in \;\; Q, \\ z = 0 \;\; on \;\; \Sigma, \;\; z(\cdot,0) = 0 \;\; on \; \Omega, \end{cases} \tag{2.35}$$

*and $\omega_{u,(v,w)} := G_r''(u)(v,w)$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = -f_{yy}(x,t,y_u)z_{u,v}z_{u,w} & in \;\; Q, \\ z = 0 \;\; on \;\; \Sigma, \;\; z(\cdot,0) = 0 \;\; on \; \Omega. \end{cases} \tag{2.36}$$

### 2.2.2 The control problem

For functions $u_a, u_b \in L^\infty(Q)$ with $u_a < u_b$ a.e in $Q$, the set of feasible controls is given by

$$\mathcal{U} := \{u \in L^\infty(Q) | \; u_a \le u \le u_b \;\; \text{for a.a. } (x,t) \in Q\}. \tag{2.37}$$

The optimal control problem reads

$$\text{(P)} \quad \min_{u \in \mathcal{U}} \left\{ J(u) := \int_Q L(x,t,y(x,t),u(x,t)) \, \mathrm{d}x \, \mathrm{d}t \right\}, \tag{2.38}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(\cdot,y) = u & \text{in} \;\; Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{2.39}$$

For a number $m$ and a function $g \in L^\infty(Q)$, the objective integrand in (2.38) is defined as

$$L(x,t,y,u) := L_0(x,t,y) + (my + g)u. \tag{2.40}$$

By the direct method of the Calculus of Variations, we can prove that there exists at least one global minimizer, see [24, Theorem 5.7]. As in the elliptic case, due to the nonconvexity of the control problem, we need to consider the notion of local minimizers.

**Definition 2.2.4.** *We say that $\bar{u} \in \mathcal{U}$ is an $L^r(Q)$-weak local minimum of problem* (2.37)-(2.39)*, if there exists some $\varepsilon > 0$ such that*

$$J(\bar{u}) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^r(Q)} \le \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ a strong local minimum of the optimal control problem if there exists $\varepsilon > 0$ such that*

$$J(\bar{u}) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \le \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.*

A relation between these notions of optimality is discussed in [5, Lemma 2.8]. Let us calculate the first and second variation of the objective functional.

**Theorem 2.2.5.** *The functional $J : L^r(Q) \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^r(Q)$ we have*

$$\begin{aligned} J'(u)v &= \int_Q \Big(\frac{dL_0}{dy}(x,t,y_u) + mu\Big) z_{u,v} + (my_u + g)v \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q (p_u + my_u + g)v \,\mathrm{d}x\,\mathrm{d}t, \\ J''(u)(v_1,v_2) &= \int_Q \Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) - p_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_Q m(z_{u,v_1} v_2 + z_{u,v_2} v_1) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

*Here, $p_u \in W(0,T) \cap C(\bar{Q})$ is the unique solution of the adjoint equation*

$$\begin{cases} -\dfrac{dp}{dt} + \mathcal{A}^* p + \dfrac{\partial f}{\partial y}(x,t,y_u)p = \dfrac{\partial L}{\partial y}(x,t,y_u,u) \text{ in } Q, \\ p = 0 \text{ on } \Sigma, \ p(\cdot,T) = 0 \text{ on } \Omega. \end{cases}$$

We introduce the Hamiltonian corresponding to our parabolic optimal control problem $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x,t,y,p,u) \mapsto H(x,t,y,p,u) \in \mathbb{R}$, by

$$H(x,t,y,p,u) := L(x,t,y,u) + p(u - f(x,t,y)).$$

The definition of the Hamiltonian allows us to write compactly the local form of the first-order Pontryagin-type necessary optimality conditions for problem (2.37)-(2.39), see e.g. [5, 8, 23].

**Theorem 2.2.6.** *If $\bar{u}$ is a weak local minimizer for problem* (2.37)-(2.39)*, then there exist unique elements $\bar{y}, \bar{p} \in W(0,T) \cap L^\infty(Q)$ such that*

$$\begin{cases} \frac{d\bar{y}}{dt} + \mathcal{A}\bar{y} + f(x,t,\bar{y}) = \bar{u} \text{ in } Q, \\ \bar{y} = 0 \text{ on } \Sigma, \ \bar{y}(\cdot,0) = y_0 \text{ on } \Omega. \end{cases} \tag{2.41}$$

$$\begin{cases} -\dfrac{d\bar{p}}{dt} + \mathcal{A}^*\bar{p} = \dfrac{\partial H}{\partial y}(x,t,\bar{y},\bar{p},\bar{u}) \text{ in } Q, \\ \bar{p} = 0 \text{ on } \Sigma, \ \bar{p}(\cdot,T) = 0 \text{ on } \Omega. \end{cases} \tag{2.42}$$

$$\int_Q \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})(u - \bar{u}) \,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall u \in \mathcal{U}. \tag{2.43}$$

### 2.2.3 Sufficient optimality conditions

We adapt the sufficient optimality conditions introduced in [4, 11] to the parabolic setting.

**Assumption 2.2.7** ([12])**.** *Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|^2_{L^1(Q)} \tag{2.44}$$

*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \alpha$.*

**Assumption 2.2.8** ([12])**.** *Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}\|u - \bar{u}\|_{L^1(Q)}$$

*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \alpha$.*

**Assumption 2.2.9** ([12])**.** *Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}$$

*for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha$.*

The next theorem states that the Assumptions 2.2.7, 2.2.8, and 2.2.9 are again sufficient for strict local optimality.

**Theorem 2.2.10** ([12])**.** *The following holds.*

1. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.2.7. Then, there exist $\varepsilon > 0$ such that:*

$$J(\bar{u}) + 1/2\|u - \bar{u}\|^{1+1/\gamma}_{L^1(Q)} \leq J(u) \tag{2.45}$$

   *for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon$.*

2. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.2.8 and $\frac{\partial^2 L}{\partial uy} = 0$. Then, there exist $\varepsilon > 0$*

$$J(\bar{u}) + 1/2\|y_u - y_{\bar{u}}\|_{L^2(\Omega)}\|u - \bar{u}\|_{L^1(Q)} \leq J(u) \tag{2.46}$$

   *for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(\Omega)} < \varepsilon$.*

3. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 2.2.9 and $\frac{\partial^2 L}{\partial uy} = 0$. Then, there exist $\varepsilon > 0$*

$$J(\bar{u}) + 1/2\|y_u - y_{\bar{u}}\|^2_{L^2(Q)} \leq J(u) \tag{2.47}$$

   *for all $u \in \mathcal{U}$ such that $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$.*

### 2.2.4 Strong metric Hölder subregularity and further stability results

We study the strong metric Hölder subregularity property (SMHSr) of the optimality map. Similar to the elliptic case, we need to make some preliminary definitions. Given the initial data $y_0$ appearing in (2.39), we define the set

$$D(\mathcal{L}) := \Big\{ y \in W(0,T) \cap L^\infty(Q) \Big| \Big(\frac{d}{dt} + \mathcal{A}\Big)y \in L^r(Q), y(\cdot,0) = y_0 \Big\}. \tag{2.48}$$

We define $\mathcal{L} : D(\mathcal{L}) \to L^r(Q)$, $\mathcal{L} := \frac{d}{dt} + \mathcal{A}$. By writing $\Big(\frac{d}{dt} + \mathcal{A}\Big)y \in L^r(Q)$ we understand that there exists a function $\phi \in L^r(Q)$ such that

$$\int_Q \Big(\frac{d}{dt}y + \mathcal{A}y\Big)\psi \,\mathrm{d}x\,\mathrm{d}t = \int_Q \phi\psi \,\mathrm{d}x\,\mathrm{d}t, \text{ for all } \psi \in L^2(0,T,H^1_0(\Omega)).$$

Of course, this is satisfied if $y$ solves a corresponding PDE with data $\phi$. To address the adjoint equation, we define the mapping $\mathcal{L}^* : D(\mathcal{L}^*) \to L^r(Q)$, $\mathcal{L}^* := (-\frac{d}{dt} + \mathcal{A}^*)$, where

$$D(\mathcal{L}^*) := \Big\{p \in W(0,T) \cap L^\infty(Q) \Big| \Big(-\frac{d}{dt} + \mathcal{A}^*\Big)p \in L^r(Q), p(\cdot,T) = 0\Big\}.$$

Using the mappings $\mathcal{L}$ and $\mathcal{L}^*$, the semilinear state equation (2.39) and the linear adjoint equation (2.42) can be written in a short way:

$$\mathcal{L}y = u - f(\cdot, y)$$

$$\mathcal{L}^*p = L_y(\cdot, y_u, u) - pf_y(\cdot, y_u) = \frac{\partial H}{\partial y}(\cdot, y_u, p, u).$$

We remind the definition of normal cone to the set $\mathcal{U}$ at $u \in L^1(Q)$:

$$N_{\mathcal{U}}(u) := \begin{cases} \{\nu \in L^\infty(Q) |\ \int_Q \nu(v-u)\,\mathrm{d}x\,\mathrm{d}t \le 0 \quad \forall v \in \mathcal{U}\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

The first order necessary optimality condition for problem (2.37)-(2.39) in Theorem 2.2.6 read as

$$\begin{cases} 0 &= \mathcal{L}y + f(\cdot, y) - u, \\ 0 &= \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ 0 &\in H_u(\cdot, y, p) + N_{\mathcal{U}}(u). \end{cases} \tag{2.49}$$

We define the sets

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(Q) \times L^2(Q) \times L^\infty(Q), \tag{2.50}$$

and the set-valued *optimality mapping* of problem ((2.37)-(2.39)), $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$, by

$$\Phi\begin{pmatrix} y \\ p \\ u \end{pmatrix} := \begin{pmatrix} \mathcal{L}y + f(\cdot, y) - u \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y, p, u) \\ \frac{\partial H}{\partial u}(\cdot, y, p, u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{2.51}$$

Using the abbreviation $\psi := (y, p, u)$, the system (2.49) can be rewritten as the inclusion $0 \in \Phi(\psi)$. To study the stability of the system (2.49) under perturbations $\xi, \eta \in L^r(Q)$ and $\rho \in L^\infty(Q)$, we consider the perturbed system

$$\begin{cases} \xi &= \mathcal{L}y + f(\cdot, y) - u, \\ \eta &= \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ \rho &\in \frac{\partial H}{\partial u}(\cdot, y, p) + N_{\mathcal{U}}(u), \end{cases} \tag{2.52}$$

which is equivalent to the inclusion $\zeta := (\xi, \eta, \rho) \in \Phi(\psi)$. Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$, we denote by $B_{\mathcal{X}}(c, \alpha)$ the closed ball of center $c \in \mathcal{X}$ and radius $\alpha > 0$. The spaces $\mathcal{Y}$ and $\mathcal{Z}$, introduced in (2.50), are endowed with the metrics

$$
\begin{aligned}
d_{\mathcal{Y}}(\psi_1, \psi_2) &:= \|y_1 - y_2\|_{L^2(Q)} + \|p_1 - p_2\|_{L^2(Q)} + \|u_1 - u_2\|_{L^1(Q)}, \\
d_{\mathcal{Z}}(\zeta_1, \zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(Q)} + \|\eta_1 - \eta_2\|_{L^2(Q)} + \|\rho_1 - \rho_2\|_{L^\infty(Q)},
\end{aligned}
\tag{2.53}
$$

where $\psi_i = (y_i, p_i, u_i)$ and $\zeta_i = (\xi_i, \eta_i, \rho_i)$, $i \in \{1, 2\}$. From now on, we denote $\bar{\psi} := (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ to simplify notation.

We remember the definition of strong metric Hölder subregularity.

**Definition 2.2.11.** *Let $\bar{\psi}$ satisfy $0 \in \Phi(\bar{\psi})$. We say that the optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is* strongly metrically Hölder subregular *(SMHSr) at $(\bar{\psi}, 0)$ with exponent $\theta > 0$ if there exist positive numbers $\alpha_1, \alpha_2$ and $\kappa$ such that*

$$
d_{\mathcal{Y}}(\psi, \bar{\psi}) \leq \kappa d_{\mathcal{Z}}(\zeta, 0)^{\theta}
$$

*for all $\psi \in B_{\mathcal{Y}}(\bar{\psi}; \alpha_1)$ and $\zeta \in B_{\mathcal{Z}}(0; \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.*

We make a restriction on the set of admissible perturbations, call it $\Gamma$, which is valid for the remaining part of this section.

**Assumption 2.2.12.** *For a fixed positive constant $C$, the admissible perturbation $\zeta = (\xi, \eta, \rho) \in \Gamma \subset \mathcal{Z}$ satisfy the restriction*

$$
\|\xi\|_{L^r(Q)}, \|\eta\|_{L^r(Q)} \leq C. \tag{2.54}
$$

The main result in this section on parabolic optimal control problems is the following theorem.

**Theorem 2.2.13** ([12])**.** *Let Assumption 2.2.7 be fulfilled for the reference solution $\bar{\psi} = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ of $0 \in \Phi(\psi)$. Then the mapping $\Phi$ is strongly metrically Hölder subregular at $(\bar{\psi}, 0)$. More precisely, for every $\varepsilon \in (0, 1/2]$ there exist positive constants $\alpha_n$ and $\kappa_n$ (with $\alpha_1$ and $\kappa_1$ independent of $\varepsilon$) such that for all $\psi \in \mathcal{Y}$ with $\|u - \bar{u}\|_{L^1(Q)} \leq \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$, the following inequalities are satisfied.*

1. *In the case $m = 0$ in* (2.40)*:*

$$
\|u - \bar{u}\|_{L^1(Q)} \leq \kappa_n \Big( \|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\theta_0},
$$
$$
\|y_u^\zeta - y_{\bar{u}}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar{u}}\|_{L^2(Q)} \leq \kappa_n \Big( \|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\theta},
$$

*where*

$$
\begin{aligned}
&\theta_0 = \theta = 1 && \text{if } n = 1, \\
&\theta_0 = \theta = 1 - \varepsilon && \text{if } n = 2, \\
&\theta_0 = \frac{10}{11} - \varepsilon, \quad \theta = \frac{9}{11} - \varepsilon && \text{if } n = 3.
\end{aligned}
$$

*2. In the general case $m \in \mathbb{R}$:*

$$\|u-\bar{u}\|_{L^1(Q)} \le \kappa_n \Big( \|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)} \Big),$$
$$\|y_u^\zeta - y_{\bar{u}}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar{u}}\|_{L^2(Q)} \le \kappa_n \Big( \|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)} \Big)^{\theta_0}.$$

To obtain stability results under the assumption for $k \in \{1,2\}$, we either don't allow perturbations $\rho$ (appearing in the inclusion in (2.52)) or they need to satisfy

$$\rho \in D(\mathcal{L}^*). \tag{2.55}$$

**Theorem 2.2.14** ([12])**.** *Let $m = 0$ and let some of the Assumptions 2.2.8, 2.2.9 be fulfilled for the reference solution $\bar{\psi} = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ of $0 \in \Phi(\psi)$. Let, in addition, the set $\Gamma$ of feasible perturbations be restricted to such $\zeta \in \Gamma$ for which the component $\rho$ is either zero or satisfies* (2.55)*. The numbers $\alpha_n$, $\kappa_n$ and $\varepsilon$ are as in Theorem 2.2.13. Then the following statements hold for $n \in \{1,2,3\}$:*

*1. Under Assumption 2.2.8, the estimations*

$$\|u-\bar{u}\|_{L^1(Q)} \le \kappa_n \Big( \|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),$$
$$\|y_u^\zeta - y_{\bar{u}}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar{u}}\|_{L^2(Q)} \le \kappa_n \Big( \|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\theta_0},$$

*with $\theta_0$ as in Theorem 2.2.13, hold for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(Q)} < \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

*2. Under Assumption 2.1.5, the estimation*

$$\|y_u^\zeta - y_{\bar{u}}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar{u}}\|_{L^2(Q)} \le \kappa_n \Big( \|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)$$

*hold for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

### 2.2.5 Strong metric subregularity for OCPs with fixed-spacial distributed controls

If the controls appear in the PDE with a fixed spacial distribution, we can improve the estimate and recover estimates resembling the elliptic case. To state the result, we define shortly the optimal control problem.

The set of admissible controls is

$$\mathcal{U} := \{u \in L^\infty(0,T)^m |\ u_{a,j} \le u_j \le u_{b,j} \ \text{ for a.e. } t \in [0,T],\ 1 \le j \le m\}, \tag{2.56}$$

where $u_a, u_b \in L^\infty(0,T)^m$ and $u_{a,j}(t) < u_{b,j}(t)$ a.e. in $[0,T]$, $1 \le j \le m$. We define the optimal control problem by

$$\text{(P)} \quad \min_{u \in \mathcal{U}} \Big\{ J(u) := \int_Q [L_0(x,t,y(x,t)) + \langle L_1(x,t,y(x,t)), u(t)\rangle]\, \mathrm{d}x\, \mathrm{d}t \Big\}, \tag{2.57}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(x,t,y) = \langle g(x), u(t)\rangle & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{2.58}$$

As before, $y : Q \to \mathbb{R}$ is the state, $u : [0,T] \to \mathbb{R}^m$, is the control, $m \in \mathbb{N}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^m$, the functions $L_0, L_1, f, g$ are of corresponding dimensions, $\mathcal{A}$ is an uniformly elliptic operator. Moreover, $g := (g_1, ..., g_m)$ with $g_j \in L^\infty(\Omega)$ satisfies $\operatorname{supp}(g_j) \cap \operatorname{supp}(g_i) = \emptyset$ for all $i, j = 1, ..., m$, $i \neq j$ and $\operatorname{meas}(\operatorname{supp}(g_i)) > 0$ for at least one $i$.

We make the following adaption of the sufficient optimality conditions if the controls appear with a fixed spacial distribution.

**Assumption 2.2.15** ([13]). *Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|^2_{L^1(0,T)^m} \tag{2.59}$$

*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha$.*

**Assumption 2.2.16** ([13]). *Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}\|u - \bar{u}\|_{L^1(0,T)^m}$$

*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha$.*

Under condition $(B_0)$ we can formulate the version of Theorem 2.2.13 for problem (2.56)-(2.58).

**Theorem 2.2.17** ([13]). *Let Assumption 2.2.15 be fulfilled for the reference solution $\bar{\psi} = (\bar{y}, \bar{p}, \bar{u})$ of $0 \in \Phi(\psi)$. Then, the mapping $\Phi$ is strongly metrically subregular at $(\bar{\psi}, 0)$. More precisely, there exist $\alpha_n, \kappa_n > 0$ such that for all $\psi \in \mathcal{Y}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} \leq \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$, the following inequality is satisfied:*

$$\begin{aligned} &\|\bar{u} - u\|_{L^1(0,T)^m} + \|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \\ &\qquad \leq \kappa_n \Big( \max_{1 \leq j \leq m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big). \end{aligned}$$

To obtain results under the assumption for $k \in \{1, 2\}$, we need the restrictions that the perturbations $\rho$ appearing in the inclusion satisfy

$$\rho = \mu\sigma \tag{2.60}$$

where $\mu = \int_\Omega g \, \mathrm{d}x \in \mathbb{R}^m$ and $\sigma \in W^{1,2}(0,T)$ with $\sigma(T) = 0$.

**Theorem 2.2.18** ([13]). *Let some of the Assumptions 2.2.16 and 2.2.9 be fulfilled for the reference solution $\bar{\psi} = (\bar{y}, \bar{p}, \bar{u})$ of $0 \in \Phi(\psi)$. Further, let $L_1$ be affine with respect to $y$. In addition, the set $\Gamma$ of feasible perturbations is restricted to such $\zeta \in \Gamma$ for which the component $\rho$ is either zero or satisfies* (2.60)*. The numbers $\alpha_n$ and $\kappa_n$ are as in Theorem 2.2.17. Then the following statements hold for $n \in \{1, 2, 3\}$:*

1. *Under Assumption 2.2.16, the estimation*

$$\|\bar{u} - u\|_{L^1(0,T)^m} + \|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \leq \kappa_n \Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),$$

*hold for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

2. *Under Assumption 2.2.9, the estimation*

$$\|\bar{y} - y_u^\zeta\|_{L^2(Q)} + \|\bar{p} - p_u^\zeta\|_{L^2(Q)} \leq \kappa_n \Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)$$

*hold for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

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# Publications

# I. Stability in affine Optimal Control Problems Constrained by Semilinear Elliptic PDEs

## Outline



## Bibliographic Information


The content of this chapter was published by the author together with co-authors in:

*Domínguez Corella, A., Jork, N., Veliov, V.M., Stability in Affine Optimal Control Problems Constrained by Semilinear Elliptic Partial Differential Equations, ESAIM: COCV.*

## Author’s contribution

The publication is a joint work of the three authors, each contributing substantially. The author of this thesis contributed mainly to the proofs and content of sections I.1-I.3, I.4.1, I.5.1, parts of I.5.3, I.6.2, and I.6.3.

## I.1 Abstract

This paper investigates stability properties of affine optimal control problems constrained by semilinear elliptic partial differential equations. This is done by studying the so-called metric subregularity of the set-valued mapping associated with the system of first-order necessary optimality conditions. Preliminary results concerning the differentiability of the functions involved are established, especially the so-called switching function. Using this ansatz, more general nonlinear perturbations are encompassed, and under weaker assumptions than the ones previously considered in the literature on control-constrained elliptic problems. Finally, the applicability of the results is illustrated with some error estimates for the Tikhonov regularization.

## I.2 Introduction

We consider the following optimal control problem

$$\min_{u\in\mathcal{U}} \left\{ \int_\Omega \Big[ w(x,y) + s(x,y)u \Big]\, dx \right\}, \tag{2.1}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + d(x,y) &= \beta(x)u \quad \text{in} \quad \Omega \\ A(x)\nabla y\cdot \nu + b(x)y &= 0 \qquad\ \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

The set $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, where $n \in \{2,3\}$. The unit outward normal vector field on the boundary $\partial\Omega$, which is single-valued a.e. in $\partial\Omega$, is denoted by $\nu$. The control set is given by

$$\mathcal{U} := \{u : \Omega \to \mathbb{R} \ \text{measurable}: \ \ b_1(x) \le u(x) \le b_2(x) \ \text{for a.e.}\ x \in \Omega\},$$

where $b_1$ and $b_2$ are bounded measurable functions satisfying $b_1(x) \le b_2(x)$ for a.e. $x \in \Omega$. The functions $w : \Omega\times\mathbb{R} \to \mathbb{R}$, $s : \Omega\times\mathbb{R}\to\mathbb{R}$, $d : \Omega\times\mathbb{R}\to\mathbb{R}$, $\beta : \Omega\to\mathbb{R}$ and $b : \partial\Omega\to\mathbb{R}$ are real-valued and measurable, and $A : \Omega \to \mathbb{R}^{n\times n}$ is a measurable matrix-valued function.

There are many motivations for studying stability of solutions, in particular for error analysis of numerical methods, see e.g., [30, 31]. Most of the stability results for elliptic control problems are obtained under a second-order growth condition (analogous to the classical Legendre-Clebsch condition). For literature concerning this type of problems, the reader is referred to [19, 22, 23, 25, 26, 35] and the references therein. In optimal control problems like (2.1)–(2.2), where the control appears linearly (hence, called affine problems) this growth condition does not hold. The so-called bang-bang solutions are ubiquitous in this case, see [4, 10, 11]. To give an account of the state of the art in stability of bang-bang problems, we mention the works [1, 28, 29, 33, 37] on optimal control of ordinary differential equations. Associated results for optimization problems constrained by partial differential equations have been gaining relevance in recent years, see [5, 9, 10, 11, 13, 34]. However, its stability has been only investigated in a handful of papers, see e.g., [13, 32, 34]. From these works, we mention here particularly [34], where the authors consider linear perturbations in the state and adjoint equations for a similar problem with Dirichlet boundary condition. They use the so-called structural assumption (a growth assumption satisfied near the jumps of the control) on the adjoint variable. This assumption has been widely used in the literature on bang-bang control of ordinary differential equations in a somewhat different form, see, e.g., [1, 28, 33, 37].

The investigations of stability properties of optimization problems, in general, are usually based on the study of similar properties of the corresponding system of necessary optimality conditions. The first order necessary optimality conditions for problem (2.1)–(2.2) can be recast as a system of two elliptic equations (primal and adjoint) and one variational inequality (representing the minimization condition of the associated Hamiltonian), forming together a *generalized equation*, that is, an inclusion involving a set-valued mapping called *optimality mapping*. The concept of *strong metric subregularity*, see [12, 17], of set-valued mappings has shown to be efficient in many applications especially ones related to error analysis, see [2]. This also applies to optimal control problems of ordinary differentials equations, see e.g., [16, 28].

In the present paper, we investigate the strong metric subregularity property of the optimality mapping associated with problem (2.1)–(2.2). We present sufficient conditions for strong subregularity of this mapping on weaker assumptions than the ones used in literature, see Section I.7 for precise details. The structural assumption in [34] is weakened and more general perturbations are considered. Namely, perturbations in the variational inequality, appearing as a part of the first-order necessary optimality conditions, are considered; which are important in the numerical analysis of ODE and PDE-constrained optimization problems. Moreover, nonlinear perturbations are investigated, which provides a framework for applications, as illustrated with an estimate related to the Tikhonov regularization. The concept of linearization is employed in a functional framework in order to deal with nonlinearities. The needed differentiability of the control-to-adjoint mapping and the switching function (see Section I.4) is proved, and the derivatives are used to obtain adequate estimates needed in the stability results. Finally, we consider nonlinear perturbations in a general framework. We propose the use of the compact-open topology to have a notion of "closeness to zero" of the perturbations. In our particular case this topology can me metrized, providing a more "quantitative" notion. Estimates in this metric are obtained in Section I.6.

## I.3 Preliminaries

The Euclidean space $\mathbb{R}^s$ is considered with its usual norm, denoted by $|\cdot|$. As usual, for $p \in [1, \infty)$, we denote by $L^p(\Omega)$ the space of all measurable $p$-integrable functions $\psi : \Omega \to \mathbb{R}^s$ with the norm

$$|\psi|_{L^p(\Omega)} := \Big( \sum_{i=1}^{s} \int_\Omega |\psi_i(x)|^p \, dx \Big)^{\frac{1}{p}}.$$

The space $L^\infty(\Omega)$ consists of all measurable essentially bounded functions $\psi : \Omega \to \mathbb{R}^s$ with the norm

$$|\psi|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |\psi(x)|.$$

We denote by $C(\bar{\Omega})$ the space of continuous functions on $\Omega$ that can be extended continuously to $\bar{\Omega}$ equipped with the $L^\infty$-norm. We denote by $H^1(\Omega)$ the space of functions $\psi \in L^2(\Omega)$ having all first order weak derivatives in $L^2(\Omega)$ endowed with its usual norm. The space $H^1(\Omega) \cap C(\bar{\Omega})$ is endowed with the norm

$$|\psi|_{H^1(\Omega)\cap C(\bar\Omega)} := |\psi|_{H^1(\Omega)} + |\psi|_{C(\bar\Omega)}.$$

A function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if $\psi(\cdot, y)$ is measurable for every $y \in \mathbb{R}$, and $\psi(x, \cdot)$ is continuous for a.e. $x \in \Omega$. A function $\psi : \Omega\times\mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz, uniformly in the first variable, if for each $M > 0$ there exists $L > 0$ such that

$$|\psi(x, y_2) - \psi(x, y_1)| \le L|y_2 - y_1|$$

for a.e. $x \in \Omega$ and all $y_1, y_2 \in [-M, M]$. In order to abbreviate notation, we define $f, g : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$f(x, y, u) := \beta(x)u - d(x, y) \quad \text{and} \quad g(x, y, u) := w(x, y) + s(x, y)u.$$

The following assumption is supposed to hold throughout the remainder of the paper. It ensures that the mathematical objects related to problem (2.1)–(2.2) that we consider are well defined. Assumption I.3.1 is quite standard in the literature, see the book [39].

**Assumption I.3.1.** *The following statements are assumed to hold.*

*(i) The set $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. The matrix $A(x)$ is symmetric for a.e. $x$ in $\Omega$, and there exists $\alpha > 0$ such that $\xi \cdot A(x)\xi \ge \alpha|\xi|^2$ for a.e. $x$ in $\Omega$ and all $\xi \in \mathbb{R}^n$.*

*(ii) The functions $w, s$ and $d$ are Carathéodory, twice differentiable with respect to the second variable and their second derivatives are locally Lipschitz, uniformly in the first variable.*

*(iii) The functions $A, \beta, b, d(\cdot, 0), d_y(\cdot, 0), w_y(\cdot, 0)$ and $s_y(\cdot, 0)$ are measurable and bounded.*

*(iv) The function $d_y(\cdot, y)$ is nonnegative a.e. in $\Omega$ for all $y \in \mathbb{R}$. The function $b$ is nonnegative a.e. in $\partial\Omega$ and $|b|_{L^\infty(\partial\Omega)} > 0$.*

Items $(i)$ and $(iv)$ of Assumption I.3.1 ensure that the partial differential equations appearing in this paper have unique solutions in the space $H^1(\Omega) \cap L^\infty(\Omega)$.

### I.3.1 The elliptic operator

We consider the set $D(\mathcal{L})$ of all functions $y \in H^1(\Omega) \cap L^\infty(\Omega)$ for which there exists $h \in L^2(\Omega)$ such that

$$\int_\Omega A(x)\nabla y \cdot \nabla\varphi \, dx + \int_{\partial\Omega} b(x)y\varphi \, ds(x) = \int_\Omega h\varphi \, dx \quad \forall\varphi \in H^1(\Omega). \tag{3.3}$$

As usual, $ds$ denotes the Lebesgue surface measure. It is easy to see that for each $y \in D(\mathcal{L})$ there exists a unique element $h \in L^2(\Omega)$ such that (3.3) holds. We define the operator $\mathcal{L} : D(\mathcal{L}) \to L^2(\Omega)$ by assigning each $y \in D(\mathcal{L})$ to the function $h \in L^2(\Omega)$ satisfying (3.3). By definition, a function $y \in H^1(\Omega) \cap L^\infty(\Omega)$ belongs to $D(\mathcal{L})$ if, and only if, it is the weak solution of the linear elliptic partial differential equation

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) &= h \quad \text{in} \quad \Omega, \\ A(x)\nabla y\cdot\nu + b(x)y &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

for some $h \in L^2(\Omega)$. The following lemma is of trivial nature.

**Lemma I.3.2.** *The set $D(\mathcal{L})$ is a linear subspace of $H^1(\Omega) \cap L^\infty(\Omega)$. Moreover, the operator $\mathcal{L} : D(\mathcal{L}) \to L^2(\Omega)$ is a well-defined linear mapping.*

If $D(\mathcal{L})$ is endowed with the norm of $L^2(\Omega)$, then $\mathcal{L}$ is an unbounded operator from $D(\mathcal{L})$ to $L^2(\Omega)$. Since $A(x)$ is symmetric for a.e. $x \in \Omega$, by (3.3) we have

$$\int_\Omega \mathcal{L}y\varphi \, dx = \int_\Omega y\mathcal{L}\varphi \, dx \tag{3.4}$$

for all $y, \varphi \in D(\mathcal{L})$, the so-called integration by parts formula.

**Remark I.3.3.** If $\partial\Omega$ is of class $C^{1,1}$, $A$ is Lipschitz in $\bar{\Omega}$, and $b$ is Lipschitz and positive in $\partial\Omega$, then

$$D(\mathcal{L}) = \{y \in H^2(\Omega) : A(\cdot)\nabla y \cdot \nu + b(\cdot)y = 0\},$$

and $\mathcal{L}y = -\operatorname{div}\big(A(\cdot)\nabla y\big)$ for all $y \in D(\mathcal{L})$, see [20, Theorem 2.4.2.6].

The following lemma shows the inclusion $D(\mathcal{L}) \subset C(\bar{\Omega})$. Its proof can be found in [39, Theorem 4.7] and follows the arguments in [4, 38].

**Lemma I.3.4.** *Let $\alpha \in L^\infty(\Omega)$ be nonnegative and $h \in L^2(\Omega)$. There exists a unique function $y \in D(\mathcal{L})$ such that*

$$\mathcal{L}y + \alpha(\cdot)y = h \tag{3.5}$$

*and this function belongs to $C(\bar{\Omega})$. Moreover, for each $r > n/2$ there exists a positive number $c$ such that*

$$|y|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c|h|_{L^r(\Omega)}$$

*for all $\alpha \in L^\infty(\Omega)$ nonnegative, $y \in D(\mathcal{L})$, and $h \in L^2(\Omega) \cap L^r(\Omega)$ satisfying (3.5).*

The following technical lemma can be deduced from Lemma I.3.4, see the proof of [11, Lemma 3.4]. Its use in optimal control of elliptic partial differential equations dates from the paper [10, Lemma 2.6]. It has shown to be useful for diverse estimates, see [10, 34].

**Lemma I.3.5.** *There exists a positive number $c$ such that*

$$|y|_{L^2(\Omega)} \leq c|h|_{L^1(\Omega)}$$

*for all nonnegative $\alpha \in L^\infty(\Omega)$, $y \in D(\mathcal{L})$ and $h \in L^2(\Omega)$ satisfying (3.5).*

The proof of the next result can be found in [8, Theorem 2.11] in the case of a Dirichlet problem, see also [21, Lemma 6.8]. Here we adapt the argument below Theorem 2.1 in [7, p. 618].

**Lemma I.3.6.** *Let $\alpha \in L^\infty(\Omega)$ be nonnegative, $\{h_m\}_{m=1}^\infty$ be a sequence in $L^2(\Omega)$ and $h \in L^2(\Omega)$. For each $m \in \mathbb{N}$, let $y_m \in C(\bar{\Omega})$ be the unique function satisfying $\mathcal{L}y_m + \alpha(\cdot)y_m = h_m$, and let $y \in C(\bar{\Omega})$ be the unique function satisfying of $\mathcal{L}y + \alpha(\cdot)y = h$. If $h_m \rightharpoonup h$ weakly in $L^2(\Omega)$, then $y_m \to y$ in $C(\bar{\Omega})$.*

*ProofL.* et $p \in (2n/(n+2), n/(n-1))$. Then $W^{1,p}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and consequently, by Schauder's Theorem, $L^2(\Omega)$ is compactly embedded in $W^{1,p}(\Omega)^*$. By the latter compact embedding, every weakly convergent sequence in $L^2(\Omega)$ converges also in $W^{1,p}(\Omega)^*$ to the same limit. Define $\mathcal{K} : L^2(\Omega) \to C(\bar{\Omega})$ by $\mathcal{K}h := y$, where $y \in C(\bar{\Omega})$ is the unique function satisfying $\mathcal{L}y + \alpha(\cdot) = h$. The result follows from [27, Theorem 3.14], since that theorem asserts that the linear operator $\mathcal{K}$ is continuous from $L^2(\Omega)$ endowed with the norm of $W^{1,p}(\Omega)^*$ to $C(\bar{\Omega})$. □

**Remark I.3.7.** Using the definitions of the set $D(\mathcal{L})$ and the operator $\mathcal{L}$, we can write in a shorter way the partial differential equations involved in this paper. For example, given $u \in \mathcal{U}$, to say that $y$ belongs to $D(\mathcal{L})$ and satisfies $\mathcal{L}y + d(\cdot, y) = \beta(\cdot)u$ is equivalent to say that $y$ belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ and satisfies the weak formulation of (2.2), that is

$$\int_\Omega A(x)\nabla y \cdot \nabla\varphi \, dx + \int_\Omega d(x, y)\varphi \, dx + \int_{\partial\Omega} b(x)y\varphi \, ds(x) = \int_\Omega \beta(x)u\varphi \, dx$$

for all $\varphi \in H^1(\Omega)$. This weak formulation makes sense since, by $(ii)$ and $(iii)$ of Assumption I.3.1, for any $y \in L^\infty(\Omega)$, the function $d(\cdot, y)$ belongs to $L^\infty(\Omega)$.

### I.3.2 The control model

Having in mind Remark I.3.7, given a function $u \in \mathcal{U}$ we say that $y_u \in D(\mathcal{L})$ is the associated state to $u \in \mathcal{U}$ if

$$\mathcal{L}y_u = f(\cdot, y_u, u). \tag{3.6}$$

The following proposition shows that the mapping $u \to y_u$ from $\mathcal{U}$ to $D(\mathcal{L})$ is well-defined. Its proof can be found in the standard literature; it follows from [39, Theorem 4.8], see also [39, p. 212].

**Proposition I.3.8.** *For each $u \in \mathcal{U}$ there exists a unique state $y_u \in D(\mathcal{L})$ associated with $u \in \mathcal{U}$. Moreover, $\{y_u : u \in \mathcal{U}\}$ is a bounded subset of $H^1(\Omega) \cap C(\bar{\Omega})$ and for each $r > n/2$ there exists $c > 0$ such that*

$$|y_{u_2} - y_{u_1}|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c|u_2 - u_1|_{L^r(\Omega)}$$

*for all $u_1, u_2 \in \mathcal{U}$.*

We call the function $\mathcal{G} : \mathcal{U} \to H^1(\Omega) \cap C(\bar{\Omega})$ given by $\mathcal{G}(u) := y_u$ the control-to-state mapping. The functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ given by

$$\mathcal{J}(u) := \int_\Omega g(x, y_u, u) \, dx$$

is called the objective functional of problem (2.1)–(2.2).

**Definition I.3.9.** Let $\bar{u}$ belong to $\mathcal{U}$.

(i) We say that $\bar{u}$ is a global solution of problem (2.1)–(2.2) if $\mathcal{J}(\bar{u}) \leq \mathcal{J}(u)$ for all $u \in \mathcal{U}$.

(ii) We say that $\bar{u}$ is a local solution of problem (2.1)–(2.2) if there exists $\varepsilon_0 > 0$ such that $\mathcal{J}(\bar{u}) \leq \mathcal{J}(u)$ for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \leq \varepsilon_0$.

(iii) We say that $\bar{u}$ is a strict local solution of problem (2.1)–(2.2) if there exists $\varepsilon_0 > 0$ such that $\mathcal{J}(\bar{u}) < \mathcal{J}(u)$ for all $u \in \mathcal{U}$ with $u \neq \bar{u}$ and $|u - \bar{u}|_{L^1(\Omega)} \leq \varepsilon_0$.

Under Assumption I.3.1, problem (2.1)–(2.2) has at least one global solution. The proof is routine and can be obtained by standard arguments; namely, taking a minimizing sequence and using the weak compactness of $\mathcal{U}$ in $L^2(\Omega)$.

**Lemma I.3.10.** *Problem (2.1)–(2.2) has at least one global solution.*

In order to make notation simpler, from now on we fix a local solution $\bar{u} \in \mathcal{U}$ of problem (2.1)–(2.2). We call the function $H : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$H(x, y, p, u) := g(x, y, u) + pf(x, y, u),$$

the Hamiltonian of problem (2.1)–(2.2). Given $u \in \mathcal{U}$, we say that $p_u \in D(\mathcal{L})$ is the costate associated with $u \in \mathcal{U}$ if

$$\mathcal{L}p_u = H_y(\cdot, y_u, p_u, u).$$

The following proposition shows that the mapping $u \to p_u$ from $\mathcal{U}$ to $D(\mathcal{L})$ is well defined. We give the proof of this elementary result because it seems not to be explicitly stated in the literature.

**Proposition I.3.11.** *For each $u \in \mathcal{U}$ there exists a unique costate $p_u \in D(\mathcal{L})$ associated with $u \in \mathcal{U}$. Moreover, $\{p_u : u \in \mathcal{U}\}$ is a bounded subset of $H^1(\Omega) \cap C(\bar{\Omega})$ and for each $r > n/2$ there exist $c > 0$ such that*

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega) \cap C(\bar{\Omega})} \leq c|u_2 - u_1|_{L^r(\Omega)}$$

*for all $u_1, u_2 \in \mathcal{U}$.*

*ProofT.* he existence and uniqueness follows from Lemma I.3.4. Given $u \in \mathcal{U}$, the function $p_u$ satisfies

$$\mathcal{L}p_u + d_y(\cdot, y_u)p_u = g_y(\cdot, y_u, u).$$

By $(ii)$, $(iii)$ and $(iv)$ of Assumption I.3.1, for each $u \in \mathcal{U}$, the function $d_y(\cdot, y_u)$ is nonnegative and belongs to $L^\infty(\Omega)$. By $(ii)$ and $(iii)$ of Assumption I.3.1, for each $u \in \mathcal{U}$ the function $g_y(\cdot, y_u, u)$ belongs to $L^\infty(\Omega)$. Furthermore, since by Proposition I.3.8 the set $\{y_u : u \in \mathcal{U}\}$ is bounded in $C(\bar{\Omega})$, there exists $M_1 > 0$ such that

$$|g_y(\cdot, y_u, u)|_{L^\infty(\Omega)} \leq M_1$$

for all $u \in \mathcal{U}$. By Lemma I.3.4, there exists a positive number $c_1$ such that for all $u \in \mathcal{U}$

$$|p_u|_{H^1(\Omega) \cap C(\bar{\Omega})} \leq c_1 |g_y(\cdot, y_u, u)|_{L^\infty(\Omega)}.$$

Thus, $M_2 := c_1 M_1$ is a bound for the set $\{p_u : u \in \mathcal{U}\}$ in $H^1(\Omega) \cap C(\bar{\Omega})$. Let $u_1, u_2 \in \mathcal{U}$ and $r > n/2$. We have then

$$\mathcal{L}(p_{u_2} - p_{u_1}) + d_y(\cdot, y_{u_2})(p_{u_2} - p_{u_1}) = H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1).$$

By Lemma I.3.4, there exists a positive number $c_2$ (independent of $u_1$ and $u_2$) such that

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c_2 |H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1)|_{L^r(\Omega)}.$$

By $(ii)$ of Assumption I.3.1 and the boundedness of the set $\{p_u : u \in \mathcal{U}\}$ in $C(\bar{\Omega})$, there exists $L > 0$ such that

$$|H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1)| \le L\Big(|y_{u_2} - y_{u_1}| + |u_2 - u_1|\Big) \quad \text{a.e. in } \Omega.$$

Consequently,

$$\begin{aligned}|p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar{\Omega})} &\le c_2 L\big(|y_{u_1} - y_{u_2}|_{L^r(\Omega)} + |u_1 - u_2|_{L^r(\Omega)}\big)\\ &\le c_2 L\Big((\operatorname{meas}\Omega)^{\frac{1}{r}} |y_{u_2} - y_{u_1}|_{L^\infty(\Omega)} + |u_2 - u_1|_{L^r(\Omega)}\Big).\end{aligned}$$

By Proposition I.3.8, there exists a constant $c_3 > 0$ (independent of $u_1$ and $u_2$) such that

$$|y_{u_2} - y_{u_1}|_{C(\bar{\Omega})} \le c_3 |u_2 - u_1|_{L^r(\Omega)}.$$

Thus,

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c_2 L\big(1 + c_3(\operatorname{meas}\Omega)^{\frac{1}{r}}\big)|u_2 - u_1|_{L^r(\Omega)}.$$

The estimate follows defining $c := c_2 L\big(1 + c_3(\operatorname{meas}\Omega)^{\frac{1}{r}}\big)$. □

We call the function $\mathcal{S} : \mathcal{U} \to H^1(\Omega) \cap C(\bar{\Omega})$ given by $\mathcal{S}(u) := p_u$ the control-to-adjoint mapping. The following proposition gives us another useful estimate; it can be easily proved employing Lemma I.3.5 and the argument in the proof of [39, Theorem 4.16].

**Proposition I.3.12.** *There exists $c > 0$ such that*

$$|y_{u_2} - y_{u_1}|_{L^2(\Omega)} + |p_{u_2} - p_{u_1}|_{L^2(\Omega)} \le c|u_2 - u_1|_{L^1(\Omega)}$$

*for all $u_1, u_2 \in \mathcal{U}$.*

We close this subsection with the following result.

**Proposition I.3.13.** *Let $\{u_m\}_{m=1}^\infty$ be a sequence in $\mathcal{U}$ and $u \in \mathcal{U}$. If $u_m \rightharpoonup u$ weakly in $L^2(\Omega)$, then $y_{u_m} \to y_u$ and $p_{u_m} \to p_u$ in $C(\bar{\Omega})$.*

*ProofW.* e prove only the convergence $p_{u_m} \to p_u$ in $C(\bar{\Omega})$, the convergence $y_{u_m} \to y_u$ in $C(\bar{\Omega})$ is analogous. Let $\{p_{u_{m_k}}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{p_{u_m}\}_{m=1}^\infty$. By the compact embedding

$H^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence of $\{p_{u_{m_k}}\}_{k=1}^\infty$, denoted in the same way, and $p \in L^2(\Omega)$ such that $p_{u_{m_k}} \to p$ in $L^2(\Omega)$. Since $y_{u_{m_k}} \to y_u$ in $C(\bar\Omega)$, one can deduce that

$$H_y(\cdot, y_{u_{m_k}}, p_{u_{m_k}}, u_{m_k}) \rightharpoonup H_y(\cdot, y_u, p, u) \quad \text{weakly in } L^2(\Omega).$$

By Lemma I.3.6, we have $p_{u_{m_k}} \to p_u$ in $C(\bar\Omega)$. The result follows, since every subsequence of $\{p_{u_m}\}_{m=1}^\infty$ has a further subsequence that converges to $p_u$ in $C(\bar\Omega)$. □

## I.4 Differentiability of the mappings involved

In this section, we prove some preliminary results concerning the differentiability of the control-to-state mapping, the control-to-adjoint mapping and the switching mapping (to be defined later). Some of these properties are well known for the control-to-state mapping; see, e.g., [5, 10, 11, 34, 39]. Nevertheless, we require more specific estimates than the ones in the literature. The differentiability of the control-to-adjoint mapping and the switching mapping has not been studied before in the literature on elliptic control-constrained problems, therefore we devote this section to obtain appropriate estimates needed in the study of stability in the next section.

### I.4.1 The state and adjoint mappings

We begin this subsection recalling the definition of directional derivative, see [18, pp.2-4] or [24, p.171]. Let $Y$ be a normed space and $\mathcal{F} : \mathcal{U} \to Y$ a mapping. Given $u \in \mathcal{U}$ and $v \in \mathcal{U} - u$, if the limit

$$d\mathcal{F}(u; v) := \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u + \varepsilon v) - \mathcal{F}(u)}{\varepsilon}$$

exists in $Y$, we say that $\mathcal{F}(u; v)$ is the (Gâteaux) differential of $\mathcal{F}$ at $u$ in the direction $v$. Note that by convexity of $\mathcal{U}$, $u + \varepsilon v$ belongs to $\mathcal{U}$ for every $u \in \mathcal{U}$, $v \in \mathcal{U} - u$ and $\varepsilon \in [0, 1]$. We will restrict ourselves to this simple definition of directional derivative, as further differentiability properties are not needed in our analysis of stability.

Recall that $\bar u \in \mathcal{U}$ is a fixed solution of problem (2.1)–(2.2). As it is well-known, the differential of the control-to-state mapping at $\bar u$ is related to the linearization of the system equation around $\bar u$. Bearing this in mind, given $v \in L^2(\Omega)$, we denote by $z_v$ the unique[1] solution of the equation

$$\mathcal{L} z_v = f_y(\cdot, y_{\bar u}, \bar u) z_v + f_u(\cdot, y_{\bar u}, \bar u) v. \tag{4.7}$$

The proof of the following estimate can be found in the standard literature, see the proof of [39, Theorem 4.17] for the case of a Neumann boundary problem (the proof is the same for Robin or Dirichlet boundary). It can also be deduced by the same arguments given in the proof of Proposition I.4.2.

[1]The uniqueness follows from Lemma I.3.4, and the fact that equation (4.7) can be rewritten as

$$\mathcal{L} z_v + d_y(\cdot, y_{\bar u}) z_v = \beta(\cdot) v.$$

**Proposition I.4.1.** *For each $r > n/2$ there exists $c > 0$ such that*

$$|y_u - y_{\bar{u}} - z_{u-\bar{u}}|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c|u - \bar{u}|^2_{L^r(\Omega)} \qquad \forall u \in \mathcal{U}.$$

One of the first things that can be deduced from Proposition I.4.1 is the differentiability of the control-to-state mapping $\mathcal{G}$. Given $v \in L^2(\Omega)$ satisfying $\bar{u} + v \in \mathcal{U}$, the differential of the control-to-state mapping $\mathcal{G}$ at $\bar{u}$ in the direction $v$ exists and is given by $d\mathcal{G}(\bar{u}; v) = z_v$. For further differentiability properties of the control-to-state mapping, we refer the reader to [8, Theorem 2.12].

In order to study the differential of the control-to-adjoint mapping we introduce the following notations. Given $v \in L^2(\Omega)$, we denote by $q_v$ the unique[2] solution of the equation

$$\mathcal{L}q_v = H_{yy}(\cdot, y_{\bar{u}}, p_{\bar{u}}, \bar{u})z_v + H_{yp}(\cdot, y_{\bar{u}}, p_{\bar{u}}, \bar{u})q_v + H_{yu}(\cdot, y_{\bar{u}}, p_{\bar{u}}, \bar{u})v. \tag{4.8}$$

The following estimate is concerned with the differentiability of the control-to-adjoint mapping. To the best of our knowledge, this result does not appear in the literature; therefore we present its proof, although it is standard.

**Proposition I.4.2.** *For each $r > n/2$ there exists $c > 0$ such that*

$$|p_u - p_{\bar{u}} - q_{u-\bar{u}}|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c|u - \bar{u}|^2_{L^r(\Omega)} \qquad \forall u \in \mathcal{U}.$$

*ProofG.* iven $u \in \mathcal{U}$, we define $\psi_u : \Omega \to \mathbb{R}^4$ by $\psi_u(x) := (x, y_u(x), p_u(x), u(x))$. For each $u \in \mathcal{U}$, we denote by $\tilde{q}_{u-\bar{u}}$ the unique solution of the equation

$$\mathcal{L}\tilde{q}_{u-\bar{u}} = H_{yy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{yp}(\psi_{\bar{u}})\tilde{q}_{u-\bar{u}} + H_{yu}(\psi_{\bar{u}})(u - \bar{u}).$$

Let $u \in \mathcal{U}$ and $r > n/2$ be arbitrary. Using the Taylor Theorem (integral form of the remainder) and *(ii)*-*(iii)* of Assumption I.3.1, one can find $\alpha_1, \alpha_2, \alpha_3 \in L^\infty(\Omega)$ such that

$$\begin{aligned} H_y(\psi_u) =& H_y(\psi_{\bar{u}}) + H_{yy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{yp}(\psi_{\bar{u}})(p_u - p_{\bar{u}}) + H_{yu}(\psi_{\bar{u}})v \\ &+ \alpha_1(\cdot)(y_u - y_{\bar{u}})^2 + \alpha_2(\cdot)(y_u - y_{\bar{u}})(p_u - p_{\bar{u}}) + \alpha_3(\cdot)(y_u - y_{\bar{u}})v, \end{aligned}$$

where $v = u - \bar{u}$. Hence

$$\mathcal{L}(p_u - p_{\bar{u}} - \tilde{q}_v) = H_{yp}(\psi_{\bar{u}})(p_u - p_{\bar{u}} - \tilde{q}_v) + \Big[\alpha_1(\cdot)(y_u - y_{\bar{u}}) + \alpha_2(\cdot)(p_u - p_{\bar{u}}) + \alpha_3(\cdot)v\Big](y_u - y_{\bar{u}}).$$

By Lemma I.3.4, Proposition I.3.8 and Proposition I.3.11, there exists $c_1 > 0$ such that

$$|p_u - p_{\bar{u}} - \tilde{q}_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c_1|v|^2_{L^r(\Omega)}.$$

Now,

$$\mathcal{L}(\tilde{q}_v - q_v) = H_{yy}(\psi_{\bar{u}})(y_u - y_{\bar{u}} - z_v) + H_{yp}(\psi_{\bar{u}})(\tilde{q}_v - q_v).$$

[2] The uniqueness follows from Lemma I.3.4, and the fact that equation (4.8) can be rewritten as

$$\mathcal{L}q_v + d_y(\cdot, y_{\bar{u}})q_v = H_{yy}(\cdot, y_{\bar{u}}, p_{\bar{u}}, \bar{u})z_v + H_{yu}(\cdot, y_{\bar{u}}, p_{\bar{u}}, \bar{u})v.$$

By Lemma I.3.4 and Proposition I.4.1, there exists $c_2 > 0$ such that

$$|\tilde{q}_v - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c_2 |v|^2_{L^r(\Omega)}.$$

Finally, by the triangle inequality

$$|p_u - p_{\bar{u}} - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq |p_u - p_{\bar{u}} - \tilde{q}_v|_{H^1(\Omega)\cap C(\bar{\Omega})} + |\tilde{q}_v - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})}.$$

The result follows taking $c := c_1 + c_2$. □

Given $v \in L^\infty(\Omega)$ satisfying $\bar{u} + v \in \mathcal{U}$, the differential of the control-to-adjoint mapping $\mathcal{S}$ at $\bar{u}$ in the direction $v$ exists and is given by $d\mathcal{S}(\bar{u}; v) = q_v$.

We now state further properties concerning the mappings $v \to z_v$ and $v \to q_v$.

**Proposition I.4.3.** *The following statements hold.*

*(i) For each $r > n/2$ there exists a positive number $c$ such that*

$$|z_v|_{H^1(\Omega)\cap C(\bar{\Omega})} + |q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c|v|_{L^r(\Omega)} \quad \forall v \in L^2(\Omega) \cap L^r(\Omega).$$

*(ii) There exists a positive number $c$ such that*

$$|z_v|_{L^2(\Omega)} + |q_v|_{L^2(\Omega)} \leq c|v|_{L^1(\Omega)} \quad \forall v \in L^2(\Omega).$$

*(iii) Let $\{v_k\}_{k=1}^\infty$ be a sequence in $L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $z_{v_k} \to z_v$ and $q_{v_k} \to q_v$ in $C(\bar{\Omega})$.*

*ProofI.* tems $(i)$ and $(ii)$ follow from Lemma I.3.4 and I.3.5, respectively. Item $(iii)$ follows from Lemma I.3.6. □

### I.4.2 The switching mapping

Let us begin this subsection by recalling the first-order necessary condition (Pontryagin principle in integral form) for problem (2.1)–(2.2). If $u \in \mathcal{U}$ is a local solution of problem (2.1)–(2.2), then

$$\int_\Omega \Big[s(x, y_u) + \beta(x) p_u\Big](w - u)\, dx \geq 0 \quad \forall w \in \mathcal{U}. \tag{4.9}$$

The variational inequality (4.9) motivates the following definition. For each $u \in \mathcal{U}$, define

$$\sigma_u := s(\cdot, y_u) + \beta(\cdot) p_u.$$

Observe that $\sigma_u = H_u(\cdot, y_u, p_u)$. The mapping $\mathcal{Q} : \mathcal{U} \to L^\infty(\Omega)$ given by $\mathcal{Q}(u) := \sigma_u$ is called the switching mapping. Given $v \in L^2(\Omega)$, we define

$$\pi_v := H_{uy}(\cdot, y_{\bar{u}}, p_{\bar{u}}) z_v + H_{up}(\cdot, y_{\bar{u}}, p_{\bar{u}}) q_v.$$

This definition is justified by the following estimate.

**Proposition I.4.4.** *For each $r > n/2$ there exists $c > 0$ such that*

$$|\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} \leq c|u - \bar{u}|^2_{L^r(\Omega)} \qquad \forall u \in \mathcal{U}.$$

*ProofG.* iven $u \in \mathcal{U}$, we define $\psi_u : \Omega \to \mathbb{R}^3$ by $\psi_u(x) := (x, y_u(x), p_u(x))$. For each $u \in \mathcal{U}$, we denote

$$\tilde{\pi}_{u-\bar{u}} := H_{uy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{up}(\psi_{\bar{u}})(p_u - p_{\bar{u}}).$$

Let $u \in \mathcal{U}$ and $r > n/2$ be arbitrary, and abbreviate $v = u - \bar{u}$. Using the Taylor Theorem (integral form of the remainder) and *(ii)*-*(iii)* of Assumption I.3.1, one can find $\alpha \in L^\infty(\Omega)$ such that

$$H_u(\psi_u) = H_u(\psi_{\bar{u}}) + H_{uy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{up}(\psi_{\bar{u}})(p_u - p_{\bar{u}}) + \alpha(\cdot)(y_u - y_{\bar{u}})^2.$$

Therefore, by Proposition I.3.8, there exists $c_1 > 0$ such that

$$|\sigma_u - \sigma_{\bar{u}} - \tilde{\pi}_v|_{L^\infty(\Omega)} \leq c_1|v|^2_{L^r(\Omega)}.$$

Now,

$$|\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)} \leq |H_{uy}(\cdot, y_{\bar{u}}, p_{\bar{u}})(y_u - y_{\bar{u}} - z_v) + H_{up}(\cdot, y_{\bar{u}}, p_{\bar{u}})(q_u - q_{\bar{u}} - q_v)|_{L^\infty(\Omega)}.$$

Hence, by Proposition I.4.1 and I.4.2, there exists $c_2 > 0$ such that

$$|\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)} \leq c_2|v|^2_{L^r(\Omega)}.$$

Finally, by the triangle inequality,

$$|\sigma_u - \sigma_{\bar{u}} - \pi_v|_{L^\infty(\Omega)} \leq |\sigma_u - \sigma_{\bar{u}} - \tilde{\pi}_v|_{L^\infty(\Omega)} + |\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)}.$$

The result follows defining $c := c_1 + c_2$. □

Proposition I.4.4 yields immediately that the differential of the switching mapping $\mathcal{Q}$ at $\bar{u}$ in any direction $v \in \mathcal{U} - \bar{u}$ exists and is given by $d\mathcal{Q}(\bar{u}; v) = \pi_v$.

One of the important features of the mapping $v \to \pi_v$ is the following.

**Proposition I.4.5.** *For all $v \in L^2(\Omega)$, we have*

$$\int_\Omega \pi_v v \, dx = \int_\Omega \Big[ H_{yy}(x, y_{\bar{u}}, p_{\bar{u}}, \bar{u}) z_v^2 + 2H_{uy}(x, y_{\bar{u}}, p_{\bar{u}}, \bar{u}) z_v v \Big] \, dx.$$

*ProofI.* n order to simplify notation, we write $\psi_{\bar{u}}(x) := (x, y_{\bar{u}}(x), p_{\bar{u}}(x), \bar{u}(x))$ for each $x \in \Omega$. Let $v \in L^2(\Omega)$ be arbitrary. By the integration by parts formula (3.4) and the concrete form of the Hamiltonian, we get

$$\begin{aligned}\int_\Omega H_{up}(\psi_{\bar{u}}) q_v v \, dx &= \int_\Omega \big(\mathcal{L} z_v + d_y(x, y_{\bar{u}}) z_v\big) q_v \, dx = \int_\Omega \big(\mathcal{L} q_v + d_y(x, y_{\bar{u}}) q_v\big) z_v \, dx \\ &= \int_\Omega \big(H_{yy}(\psi_{\bar{u}}) z_v + H_{uy}(\psi_{\bar{u}}) v\big) z_v = \int_\Omega \Big[ H_{yy}(\psi_{\bar{u}}) z_v^2 + H_{uy}(\psi_{\bar{u}}) z_v v \Big] \, dx.\end{aligned}$$

The result follows since

$$\int_\Omega \pi_v v\,dx = \int_\Omega H_{uy}(\psi_{\bar u}) z_v v\,dx + \int_\Omega H_{up}(\psi_{\bar u}) q_v v\,dx.$$

□

We give further properties of the mapping $v \to \pi_v$ in the next proposition, its proof follows trivially from Proposition I.4.3.

**Proposition I.4.6.** *The following statements hold.*

*(i) For each $r > n/2$ there exists a positive number $c$ such that*

$$|\pi_v|_{L^\infty(\Omega)} \le c|v|_{L^r(\Omega)} \quad \forall v \in L^2(\Omega) \cap L^r(\Omega).$$

*(ii) There exists a positive number $c$ such that*

$$|\pi_v|_{L^2(\Omega)} \le c|v|_{L^1(\Omega)} \quad \forall v \in L^2(\Omega).$$

*(iii) Let $\{v_k\}_{k=1}^\infty$ be a sequence in $L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $\pi_{v_k} \to \pi_v$ in $L^\infty(\Omega)$.*

Proposition I.4.5 motivates the following definition. For each $v \in L^2(\Omega)$, define

$$\Lambda(v) := \int_\Omega \Big[H_{yy}(x, y_{\bar u}, p_{\bar u}, \bar u) z_v^2 + 2H_{uy}(x, y_{\bar u}, p_{\bar u}, \bar u) z_v v\Big]\,dx. \tag{4.10}$$

**Remark I.4.7.** We mention that the quadratic form $\Lambda : L^2(\Omega) \to \mathbb{R}$ is the second variation of the objective functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ at $\bar u$. By Proposition I.4.5, we also have the following representation

$$\Lambda(v) = \int_\Omega \pi_v v\,dx \quad \forall v \in L^2(\Omega).$$

We close this section with a result concerning the quadratic form (4.10).

**Proposition I.4.8.** *Let $\{v_k\}_{k=1}^\infty \subset L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $\Lambda(v_k) \to \Lambda(v)$.*

*Proof.* By Proposition I.4.6, $\pi_{v_k} \to \pi_v$ in $L^\infty(\Omega)$, therefore

$$\Lambda(v_k) = \int_\Omega (\pi_{v_k} - \pi_v) v_k\,dx + \int_\Omega \pi_v v_k\,dx \to \int_\Omega \pi_v v\,dx.$$

□

# I.5 Stability

In this section, we study the stability of the optimal solution of problem (2.1)–(2.2) with respect to perturbations. As usual in optimization, the stability of the solution is derived from stability of the system of necessary optimality conditions. The investigated stability property of the latter is the so-called strong metric Hölder subregularity (SMHSr), see e.g., [17, Section 3I] or [12, Section 4]. After introducing the assumptions we study the SMHSr property of the variational inequality (9). Then the result is used to obtain this property for the whole system of necessary optimality conditions

## I.5.1 The main assumption

We begin the section recalling that $\bar{u} \in \mathcal{U}$ is a local minimizer of problem (2.1)–(2.2), and the definition of the quadratic form $\Lambda : L^2(\Omega) \to \mathbb{R}$ in (4.10).

**Assumption I.5.1.** *There exist positive numbers $\alpha_0, \gamma_0$ and $k^* \in [1, 4/n)$ such that*

$$\int_\Omega \sigma_{\bar{u}}(u - \bar{u})\, dx + \Lambda(u - \bar{u}) \geq \gamma_0 |u - \bar{u}|_{L^1(\Omega)}^{k^*+1}, \tag{5.11}$$

*for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \leq \alpha_0$.*

Assumption I.5.1 resembles the well-known $L^2$-coercivity condition in optimal control, with two substantial differences: $(i)$ the left-hand side of (5.11) involves a linear term (not only the quadratic form in the $L^2$-coercivity condition); $(ii)$ the $L^1$-norm appears in the right-hand side of (5.11). We mention that the standard $L^2$-coercivity condition cannot hold in affine problems. Assumption I.5.1 in the particular case $k^* = 1$ has been used before in the literature on optimal control problems constrained by ordinary differential equations, see [28, Assumption A2'] or [29, Assumption A2]. A similar assumption was used in [15, Assumption 2].We first point out that if $\bar{u}$ satisfies Assumption I.5.1, then it must be bang-bang. A control $u \in \mathcal{U}$ is bang-bang if $u(x) \in \{b_1(x), b_2(x)\}$ for a.e. $x$ in $\Omega$. The proof of this result follows the arguments given in the proof of [11, Theorem 2.1].

**Proposition I.5.2.** *If $\bar{u} \in \mathcal{U}$ satisfies Assumption I.5.1, then $\bar{u}$ is bang-bang.*

*ProofL.* et $\alpha_0$ and $\gamma_0$ be the positive numbers in Assumption I.5.1. Suppose that there exists $\varepsilon > 0$ and a measurable set $E \subset \Omega$ of positive measure such that

$$\bar{u}(x) \in [b_1(x) + \varepsilon, b_2(x) - \varepsilon] \quad \text{for a.e. } x \in E.$$

Define $\varepsilon^* := \min\{\alpha_0(\operatorname{meas} E)^{-1}, \varepsilon\}$. Let $\{v_m\}_{m=1}^\infty \subset L^2(\Omega)$ be a sequence converging to zero weakly in $L^2(\Omega)$ such that for each $m \in \mathbb{N}$, $v_m(x) \in \{-\varepsilon^*, \varepsilon^*\}$ for a.e. $x \in \Omega$. For each $m \in \mathbb{N}$, define

$$u_m(x) := \begin{cases} \bar{u}(x) & if \quad x \notin E \\ \bar{u}(x) + v_m(x) & if \quad x \in E. \end{cases}$$

Clearly, for each $m \in \mathbb{N}$, $u_m$ belongs to $\mathcal{U}$ and

$$|u_m - \bar{u}|_{L^1(\Omega)} = \varepsilon^* \operatorname{meas} E.$$

Hence, by Assumption I.5.1

$$\int_\Omega \sigma_{\bar u}(u_m - \bar u)\, dx + \Lambda(u_m - \bar u) \ge \gamma_0 \Big(\varepsilon^* \operatorname{meas} E\Big)^{k^*+1} \tag{5.12}$$

for all $m \in \mathbb{N}$. Since $u_m \rightharpoonup \bar u$ weakly in $L^2(\Omega)$, we have by Proposition I.4.8 that the left hand side of (5.12) converges to 0; a contradiction. □

Proposition I.5.2 makes the following lemma relevant. The proof follows the argument used in the proof of [6, Theorem 4.4]. Alternatively, as argued in the proof of [34, Theorem 4.3], one can also use [40, Theorem 1] and the fact that for a.e. $x \in \Omega$, $u(x)$ is an extremal point of $\overline{\operatorname{conv}}(\{u_k(x)\}_{k=1}^\infty \cup u(x))$ if $u \in \mathcal{U}$ is bang-bang.

**Lemma I.5.3.** *Let $u \in \mathcal{U}$ be bang-bang, and $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ be a sequence. If $u_k \rightharpoonup u$ weakly in $L^1(\Omega)$, then $u_k \to u$ in $L^1(\Omega)$.*

*ProofL.* et $\Omega_i := \{x \in \Omega : u(x) = b_i(x)\}$, $i = 1, 2$. Let $\chi_{\Omega_i} : \Omega \to \{0, 1\}$ denote the characteristic function of the set $\Omega_i$, $i = 1, 2$. Now, by definition of weak convergence

$$\int_\Omega |u_k - u|\, dx = \int_\Omega \chi_{\Omega_1}(u_n - \bar u)\, dx - \int_\Omega \chi_{\Omega_2}(u_n - \bar u)\, dx \to 0.$$

□

The next proposition shows that the switching mapping satisfies a growth condition. The proof consists of two steps. The first one is to show that Assumption I.5.1 implies this growth condition for the linearization of the switching mapping. The second step is to adequately use the linearization as an approximation of the switching mapping.

**Proposition I.5.4.** *Let Assumption 2 be fulfilled. Then there exist positive numbers $\alpha$ and $\gamma$ such that*

$$\int_\Omega \sigma_u(u - \bar u)\, dx \ge \gamma |u - \bar u|_{L^1(\Omega)}^{k^*+1}$$

*for all $u \in \mathcal{U}$ with $|u - \bar u|_{L^1(\Omega)} \le \alpha$.*

*Proof.* Let $\alpha_0, \gamma_0$ and $k^*$ be the positive numbers in Assumption I.5.1. Fix $r \in (n/2, 2/k^*)$. Using Proposition I.4.4, a constant $c > 0$ can be found such that

$$|\sigma_u - \sigma_{\bar u} - \pi_{u-\bar u}|_{L^\infty(\Omega)} \le c|u - \bar u|_{L^1(\Omega)}^{2/r} \qquad \forall u \in \mathcal{U}. \tag{5.13}$$

From Proposition I.4.5 and Assumption I.5.1, we have

$$\int_\Omega \Big[\sigma_{\bar u} + \pi_{u-\bar u}\Big](u - \bar u)\, dx = \int_\Omega \sigma_{\bar u}(u - \bar u)\, dx + \Lambda(u - \bar u) \ge \gamma_0 |u - \bar u|_{L^1(\Omega)}^{k^*+1} \tag{5.14}$$

for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \le \alpha_0$. Define $\gamma := \gamma_0/2$ and

$$\alpha := \min\left\{\alpha_0, \gamma^{\frac{r}{2-k^* r}} c^{-\frac{r}{2-k^* r}}\right\}.$$

Then, by (5.13)

$$|\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} \le c|u - \bar{u}|_{L^1(\Omega)}^{\frac{2}{r}} = c|u - \bar{u}|_{L^1(\Omega)}^{\frac{2}{r}-k^*} |u - \bar{u}|_{L^1(\Omega)}^{k^*} \le \gamma |u - \bar{u}|_{L^1(\Omega)}^{k^*} \tag{5.15}$$

for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \le \alpha$. We have for all $u \in \mathcal{U}$

$$\int_\Omega \sigma_u (u - \bar{u})\, dx = \int_\Omega \Big[\sigma_{\bar{u}} + \pi_{u-\bar{u}}\Big](u - \bar{u})\, dx + \int_\Omega \Big[\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}\Big](u - \bar{u})\, dx.$$

Consequently, by (5.14) and (5.15),

$$\begin{aligned} \int_\Omega \sigma_u (u - \bar{u})\, dx &\ge \gamma_0 |u - \bar{u}|_{L^1(\Omega)}^{k^*+1} - |\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} |u - \bar{u}|_{L^1(\Omega)} \\ &\ge (\gamma_0 - \gamma)|u - \bar{u}|_{L^1(\Omega)}^{k^*+1} = \gamma |u - \bar{u}|_{L^1(\Omega)}^{k^*+1} \end{aligned}$$

for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \le \alpha$. □

### I.5.2 Some existence and stability results

We now pass to some preparatory lemmas concerning the existence of solutions of inclusions (also called generalized equations, see [36]) related to the first order necessary condition of problem (2.1)–(2.2). Given $r \in [1, \infty]$, we denote by $\mathbb{B}_{L^r}(c; \alpha)$ the closed ball in $L^r(\Omega)$ with center $c \in L^r(\Omega)$ and radius $\alpha > 0$.

The variational inequality (4.9) can be written as the inclusion

$$0 \in \sigma_u + N_{\mathcal{U}}(u), \tag{5.16}$$

where the normal cone at $u$ to the set $\mathcal{U}$ is given by

$$N_{\mathcal{U}}(u) = \left\{\sigma \in L^\infty(\Omega) : \int_\Omega \sigma(w - u)\, dx \le 0 \quad \forall w \in \mathcal{U}\right\}.$$

**Lemma I.5.5.** *For all $\rho \in L^\infty(\Omega)$ and $\varepsilon > 0$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon)$ satisfying*

$$\rho \in \sigma_u + N_{\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon)}(u).$$

*ProofL.* et $\rho \in L^\infty(\Omega)$ and $\varepsilon > 0$. Consider the functional $\mathcal{J}_\rho : \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon) \to \mathbb{R}$ given by

$$\mathcal{J}_\rho(u) := \int_\Omega \big[g(y_u, u) - \rho u\big]\, dx = \mathcal{J}(u) - \int_\Omega \rho u\, dx.$$

The functional $\mathcal{J}_\rho$ has at least one global minimizer $u_\rho \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon)$ since $\mathcal{U} \cap \mathbb{B}_{L_1}(\bar{u};\varepsilon)$ is a weakly sequentially compact subset of $L^2(\Omega)$ and $\mathcal{J}_\rho$ is weakly sequentially continuous. By the Pontryagin principle,

$$\int_\Omega \left[\sigma_{u_\rho} - \rho\right](u - u_\rho)\, dx \geq 0 \quad \forall u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon).$$

We have then that $u_\rho$ satisfies $\rho \in \sigma_{u_\rho} + N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon)}(u_\rho)$. □

**Lemma I.5.6.** *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be closed and convex subsets of $L^1(\Omega)$ such that $\mathcal{V}_1 \cap int\, \mathcal{V}_2 \neq \emptyset$. Then*

$$N_{\mathcal{V}_1\cap\mathcal{V}_2}(u) = N_{\mathcal{V}_1}(u) + N_{\mathcal{V}_2}(u) \tag{5.17}$$

*for all $u \in \mathcal{V}_1 \cap \mathcal{V}_2$.*

*ProofG.* iven a set $\mathcal{W} \subset L^1(\Omega)$, let $s_\mathcal{W} : L^\infty(\Omega) \to \mathbb{R} \cup \{+\infty\}$ denote the support function to $\mathcal{W}$, that is

$$s_\mathcal{W}(h) := \sup_{w\in\mathcal{W}} \int_\Omega hw\, dx.$$

By [3, Proposition 3.1], the set $\operatorname{Epi} s_{\mathcal{V}_1} + \operatorname{Epi} s_{\mathcal{V}_2}$ is a weakly$^*$ closed subset of $L^\infty(\Omega)$. Then the representation (5.17) holds according to [3, Theorem 3.1]. □

We can now prove existence of solutions of the inclusion $\rho \in \sigma_u + N_\mathcal{U}(u)$ that are close (in the $L^1$-norm) to $\bar{u}$ whenever $\rho$ is close to zero (in the norm $L^\infty$-norm). The proof follows the arguments in [14, p. 1127].

**Lemma I.5.7.** *Let Assumption I.5.1 hold. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\rho \in \mathbb{B}_{L^\infty}(0;\delta)$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon)$ satisfying $\rho \in \sigma_u + N_\mathcal{U}(u)$.*

*ProofL.* et $\alpha$ and $\gamma$ be the numbers in Proposition I.5.4. Define $\varepsilon_0 := \min\{\varepsilon, \alpha\}$ and $\delta := \varepsilon_0^{k^*}\gamma/2$. Let $\rho \in L^\infty(\Omega)$ with $|\rho|_{L^\infty(\Omega)} \leq \delta$. By Lemma I.5.5, there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)$ such that

$$\rho \in \sigma_u + N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u).$$

Since trivially $\bar{u} \in \mathcal{U} \cap \operatorname{int}\mathbb{B}_{L^1}(\bar{u}, \varepsilon_0)$, by Lemma I.5.6 we have

$$N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = N_\mathcal{U}(u) + N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u). \tag{5.18}$$

Thus there exists $\nu \in N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u)$ such that

$$\rho - \sigma_u - \nu \in N_\mathcal{U}(u).$$

By definition of the normal cone,

$$0 \geq \int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx - \int_\Omega \nu(\bar{u} - u)\, dx. \tag{5.19}$$

As $\bar{u} \in \mathbb{B}_{L^1}(\bar{u}; \varepsilon_0)$ and $\nu \in N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u)$, we have

$$\int_\Omega \nu(\bar{u} - u)\, dx \leq 0.$$

Consequently, by (5.19) and Proposition I.5.4

$$0 \geq \int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx \geq -|\rho|_{L^\infty(\Omega)}|u - \bar{u}|_{L^1(\Omega)} + \gamma|u - \bar{u}|_{L^1(\Omega)}^{k^*+1},$$

which implies

$$|u - \bar{u}|_{L^1(\Omega)} \leq \gamma^{-\frac{1}{k^*}}|\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \leq 2^{-\frac{1}{k^*}}\varepsilon_0 < \varepsilon_0.$$

As $u \in \operatorname{int} \mathbb{B}_{L^1}(\bar{u}; \varepsilon_0)$, we have $N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = \{0\}$. Thus by (5.18),

$$\rho \in \sigma_u + N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = \sigma_u + N_{\mathcal{U}}(u). \tag{5.20}$$

□

The following lemma shows how Proposition I.5.4 (and consequently Assumption I.5.1) is related to Hölder- stability.

**Lemma I.5.8.** *Let Assumption I.5.1 hold. There exist positive numbers $\alpha, \delta$ and $c$ such that for every $\rho \in \mathbb{B}_{L^\infty}(0; \delta)$ there exists $u \in \mathbb{B}_{L^1}(\bar{u}, \alpha)$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$. Moreover,*

$$|u - \bar{u}|_{L^1(\Omega)} \leq c|\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \tag{5.21}$$

*for all $\rho \in L^\infty(\Omega)$ and $u \in \mathbb{B}_{L^1}(\bar{u}; \alpha)$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$.*

*ProofT.* he existence part follows from Lemma I.5.7. Let $\alpha$ and $\gamma$ be the positive numbers in Proposition I.5.4. Since $\rho - \sigma_u \in N_{\mathcal{U}}(u)$, we have

$$\int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx \leq 0.$$

By Proposition I.5.4,

$$\begin{aligned} 0 \geq \int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx &= \int_\Omega \sigma_u(u - \bar{u})\, dx + \int_\Omega \rho(\bar{u} - u)\, dx \\ &\geq \gamma\left(\int_\Omega |u - \bar{u}|\, dx\right)^{k^*+1} - |\rho|_{L^\infty(\Omega)} \int_\Omega |u - \bar{u}|\, dx. \end{aligned}$$

Hence

$$\int_\Omega |u - \bar{u}|\, dx \leq \left(\frac{1}{\gamma}|\rho|_{L^\infty(\Omega)}\right)^{1/k^*} = \gamma^{-\frac{1}{k^*}}|\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}}.$$

The result follows defining $c = \gamma^{-\frac{1}{k^*}}$. □

For inequality (5.21) to hold, Lemma I.5.8 requires that the controls are close in the $L^1$-norm to the reference solution (by Lemma I.5.7, the existence of such controls is guaranteed). This closeness assumption on the controls can be removed if the solution of inclusion (5.16) is unique. In particular, if (5.16) has a unique solution, then problem (2.1)-(2.2) has unique optimal control (minimizer).

**Lemma I.5.9.** *Let Assumption I.5.1 hold, and suppose additionally $0 \in \sigma_u + N_{\mathcal{U}}(u)$ has a unique solution $\bar{u} \in \mathcal{U}$. There exist positive numbers $\delta$ and $c$ such that*

$$|u - \bar{u}|_{L^1(\Omega)} \leq c|\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}}.$$

*for all $\rho \in \mathbb{B}_{L^\infty}(0;\delta)$ and $u \in \mathcal{U}$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$.*

*ProofL.* et $\alpha$ and $c$ be the positive numbers in Lemma I.5.8. First we prove that there exists $\delta > 0$ such that if $u \in \mathcal{U}$ and $\rho \in L^\infty(\Omega)$ satisfy $\rho \in \sigma_u + N_{\mathcal{U}}(u)$ and $|\rho|_{L^\infty(\Omega)} \leq \delta$, then $u \in \mathbb{B}_{L^1}(\bar{u};\alpha)$. Suppose not, then there exist sequences $\{\rho_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ and $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ such that $\rho_k \in \sigma_{u_k} + N_{\mathcal{U}}(u_k)$, $\rho_k \to 0$ in $L^\infty(\Omega)$, and $|u_k - \bar{u}|_{L^1(\Omega)} > \alpha$. Since $\mathcal{U}$ is weakly sequentially compact in $L^2(\Omega)$, there exists a subsequence of $\{u_k\}_{k=1}^\infty$, denoted in the same way, and $u^* \in \mathcal{U}$ such that $u_k \rightharpoonup u^*$ weakly in $L^2(\Omega)$. Using Proposition I.3.13, one can see that $\rho_k - \sigma_{u_k} \to \sigma_{u^*}$ in $L^\infty(\Omega)$. Consequently, as $\rho_k \in \sigma_{u_k} + N_{\mathcal{U}}(u_k)$ for all $n \in \mathbb{N}$, we obtain $0 \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. Then, by assumption, $u^* = \bar{u}$, so $u^*$ is bang-bang. By Lemma I.5.3, we have $u_k \to u^*$ in $L^1(\Omega)$; a contradiction. The result follows from Lemma I.5.8. □

### I.5.3 Strong metric subregularity

Let us begin considering the following system representing the necessary optimality conditions (Pontryagin principle) for problem (2.1)–(2.2):

$$\begin{cases} 0 &= \mathcal{L}y - f(\cdot, y, u), \\ 0 &= \mathcal{L}p - H_y(\cdot, y, p, u), \\ 0 &\in H_u(\cdot, y, p) + N_{\mathcal{U}}(u), \end{cases} \tag{5.22}$$

If $u \in \mathcal{U}$ is a local solution of problem (2.1)–(2.2), then the triple $(y_u, p_u, u)$ is a solution of (4.66). Therefore, the mapping that defines the right-hand side is referred to as the *optimality mapping*. In order to give a strict definition and recast system (4.66) in a functional frame, we introduce the metric spaces

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega),$$

endowed with the following metrics. For $\psi_i = (y_i, p_i, u_i) \in \mathcal{Y}$ and $\zeta_i = (\xi_i, \eta_i, \rho_i) \in \mathcal{Z}$, $i \in \{1, 2\}$,

$$d_{\mathcal{Y}}(\psi_1, \psi_2) := |y_1 - y_2|_{L^2(\Omega)} + |p_1 - p_2|_{L^2(\Omega)} + |u_1 - u_2|_{L^1(\Omega)},$$
$$d_{\mathcal{Z}}(\zeta_1, \zeta_2) := |\xi_1 - \xi_2|_{L^2(\Omega)} + |\eta_1 - \eta_2|_{L^2(\Omega)} + |\rho_1 - \rho_2|_{L^\infty(\Omega)}.$$

Both metrics are shift-invariant. We denote by $\mathbb{B}_{\mathcal{Y}}(\psi;\alpha)$ the closed ball in $\mathcal{Y}$, centered at $\psi$ and with radius $\alpha$. The notation for the ball $\mathbb{B}_{\mathcal{Z}}(\zeta;\alpha)$ is analogous. Then the optimality mapping is defined as the set-valued mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ given by

$$\Phi(y,p,u) = \begin{pmatrix} \mathcal{L}y - f(\cdot,y,u) \\ \mathcal{L}p - H_y(\cdot,y,p,u) \\ H_u(\cdot,y,p,u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{5.23}$$

Then the optimality system (4.66) can be recast as the inclusion

$$0 \in \Phi(y,p,u). \tag{5.24}$$

Our purpose is to study the stability of system (4.66), or equivalently of inclusion (5.24), with respect to perturbations on the left-hand side. From now on, we denote $\bar{\psi} := (\bar{y},\bar{p},\bar{u}) = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ , where $\bar{u}$ is the fixed local solution of problem (2.1)–(2.2).

**Definition I.5.10.** The optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called strongly Hölder subregular with exponent $\lambda > 0$ at $(\bar{\psi}, 0)$ if there exist positive numbers $\alpha_1, \alpha_2$ and $\kappa$ such that

$$d_{\mathcal{Y}}(\psi,\bar{\psi}) \le \kappa d_{\mathcal{Z}}(\zeta,0)^{\lambda} \tag{5.25}$$

for all $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi};\alpha_1)$ and $\zeta \in \mathbb{B}_{\mathcal{Z}}(0;\alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.

More explicitly, the inequality (5.25) reads as

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_{\bar{u}}|_{L^2(\Omega)} + |u - \bar{u}|_{L^1(\Omega)} \le \kappa\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} + |\rho|_{L^\infty(\Omega)}\Big)^{\lambda}. \tag{5.26}$$

Hence, if the optimality mapping is strongly Hölder subregular, all solutions of the system

$$\begin{cases} \xi &= \mathcal{L}y - f(\cdot,y,u), \\ \eta &= \mathcal{L}p - H_y(\cdot,y,p,u), \\ \rho &\in H_u(\cdot,y,p) + N_{\mathcal{U}}(u). \end{cases} \tag{5.27}$$

that are near $(y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ satisfy the Hölder estimate (5.26) with respect to the perturbations $\zeta = (\xi,\eta,\rho)$, provided they are small enough. The subregularity property is weaker than the well known strong regularity (see [17, pp. 178-179]); this allows to relax the assumptions to prove stability.

**Remark I.5.11.** If $\Phi$ is strongly Hölder subregular at $(\bar{\psi}, 0)$, then from (5.25) applied with $\zeta = 0$ we obtain that $\bar{\psi}$ is the unique solution of (5.24) in $B_{\mathcal{Y}}(\bar{\psi};\alpha_1)$, hence $\bar{u}$ is the unique local solution of problem (2.1)–(2.2) in this ball. In particular, $\bar{u}$ is a strict local minimizer.

We are now ready to state our main result.

**Theorem I.5.12.** *Let Assumption I.5.1 hold. Then the optimality mapping $\Phi$ is strongly Hölder subregular at $(\bar{\psi}, 0)$ with exponent $\lambda = 1/k^*$.*

*ProofL.* et $\alpha$ and $c$ be the positive numbers in Lemma I.5.8. Let $\zeta = (\xi, \eta, \rho) \in B_{\mathcal{Z}}(0;1)$ and $\psi = (y, p, u) \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \alpha)$ such that $\zeta \in \Phi(\psi)$. By a standard argument, we can find $c_1 > 0$ (independent of $\psi$ and $\zeta$) such that

$$|y - y_u|_{L^\infty(\Omega)} + |p - p_u|_{L^\infty(\Omega)} \leq c_1\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)}\Big). \tag{5.28}$$

Since $H_u$ is locally Lipschitz uniformly in the first variable, and the sets $\{y_u : u \in \mathcal{U}\}$, $\{p_u : u \in \mathcal{U}\}$ are bounded in $C(\bar{\Omega})$, there exists $c_2 > 0$ (independent of $\psi$) such that

$$|H_u(\cdot, y, p) - H_u(\cdot, y_u, p_u)|_{L^\infty(\Omega)} \leq c_2\Big(|y - y_u|_{L^\infty(\Omega)} + |p - p_u|_{L^\infty(\Omega)}\Big) \tag{5.29}$$

Define $\nu := \rho + H_u(\cdot, y_u, p_u) - H_u(\cdot, y, p)$. By (4.81) and (5.29), there exists $c_3 > 0$ (independent of $\psi$ and $\zeta$) such that

$$|\nu|_{L^\infty(\Omega)} \leq c_3\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} + |\rho|_{L^\infty(\Omega)}\Big) = c_3|\zeta|_{\mathcal{Z}}.$$

As $\rho \in H_u(\cdot, y, p) + N_{\mathcal{U}}(u)$, we have $\nu \in H_u(\cdot, y_u, p_u) + N_{\mathcal{U}}(u)$. Then by Lemma I.5.8,

$$|u - \bar{u}|_{L^1(\Omega)} \leq c|\nu|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \leq c c_3^{\frac{1}{k^*}} |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} := c_4|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}. \tag{5.30}$$

Now, by Proposition I.3.12, there exists $c_5 > 0$ (independent of $\psi$) such that $|y_u - y_{\bar{u}}|_{L^2(\Omega)} \leq c_5|u - \bar{u}|_{L^1(\Omega)}$. Consequently, by (5.30)

$$\begin{aligned} |y - y_{\bar{u}}|_{L^2(\Omega)} &\leq |y - y_u|_{L^2(\Omega)} + |y_u - y_{\bar{u}}|_{L^2(\Omega)} \\ &\leq c_1 \text{meas}\, \Omega^{\frac{1}{2}}\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)}\Big) + c_5|u - \bar{u}|_{L^1(\Omega)} \\ &\leq (c_1 \text{meas}\, \Omega^{\frac{1}{2}} + c_5 c_4)|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} =: c_6|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}. \end{aligned}$$

Analogously, there exists $c_7 > 0$ (independent of $\psi$ and $\zeta$) such that

$$|p - p_{\bar{u}}|_{L^2(\Omega)} \leq c_7|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}.$$

Putting all together,

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_{\bar{u}}|_{L^2(\Omega)} + |u - \bar{u}|_{L^1(\Omega)} \leq (c_4 + c_6 + c_7)|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}.$$

Finally, let $\alpha_1 := \alpha$, $\alpha_2 := 1$ and $\kappa := c_4 + c_6 + c_7$. Since the constants $c_4, c_6$ and $c_7$ are independent of $\psi$ and $\zeta$, so is $\kappa$. Thus we have (5.25) for all $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \alpha_1)$ and $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$. □

The strong subregularity property defined above does not require existence of solutions of the perturbed inclusion (4.61) in a neighborhood of the reference solution $\bar{\psi}$. The next theorem answers the existence question.

**Theorem I.5.13.** *Let Assumption I.5.1 hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$ there exists $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \varepsilon)$ satisfying the inclusion $\zeta \in \Phi(\psi)$.*

*ProofF.* or each $u \in \mathcal{U}$ and $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$, define $\nu_{u,\zeta} := \rho + H_u(\cdot, y_u, p_u) - H_u(\cdot, y_{u,\zeta}, p_{u,\zeta})$, where $y_{u,\zeta}$ and $p_{u,\zeta}$ are the unique solutions of

$$\begin{cases} \mathcal{L}y &= f(\cdot, y, u) + \xi, \\ \mathcal{L}p &= H_y(\cdot, y, p, u) + \eta. \end{cases} \tag{5.31}$$

By a standard argument, one can find positive numbers $c_1$ and $c_2$ such that

$$|y_{u,\zeta} - y_u|_{L^2(\Omega)} + |p_{u,\zeta} - p_u|_{L^2(\Omega)} \le c_1\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)}\Big), \tag{5.32}$$

and $|\nu_{u,\zeta}|_{L^\infty(\Omega)} \le c_2|\zeta|_{\mathcal{Z}}$ for all $u \in \mathcal{U}$ and $\zeta \in \mathcal{Z}$. Let $\varepsilon > 0$ be arbitrary. By Lemma I.5.7, the exists $\delta_0 > 0$ such that for each $\nu \in \mathbb{B}_{L^\infty}(0; \delta_0)$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ satisfying $\nu \in \sigma_u + N_{\mathcal{U}}(u)$. Define $\delta := \min\{c_2^{-1}\delta_0, (2c_1)^{-1}\varepsilon\}$ and let $\zeta^* \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$ be arbitrary; we will prove that there exists $u^* \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ such that $\nu_{u^*,\zeta^*} \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. First, observe that

$$|\nu_{u,\zeta^*}|_{L^\infty(\Omega)} \le c_2|\zeta^*|_{\mathcal{Z}} \le \delta_0 \quad \forall u \in \mathcal{U}.$$

Therefore, by Lemma I.5.7, we can inductively define a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ such that $\nu_{u_k,\zeta^*} \in \sigma_{u_{k+1}} + N_{\mathcal{U}}(u_{k+1})$ and $|u_k - \bar{u}|_{L^1(\Omega)} \le \varepsilon/2$ for all $k \in \mathbb{N}$. Since $\mathcal{U}$ is weakly compact in $L^2(\Omega)$, we may assume that $u_k \rightharpoonup u^*$ weakly in $L^2(\Omega)$ for some $u^* \in \mathcal{U}$. Weak convergence in $L^2(\Omega)$ implies weak convergence in $L^1(\Omega)$ and $\mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ is weakly sequentially closed in $L^1(\Omega)$, therefore $u^* \in \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$. Using Proposition I.3.13, one can see that $\nu_{u_k,\zeta^*} - \sigma_{u_{k+1}} \to \nu_{u^*,\zeta^*} - \sigma_{u^*}$ in $L^\infty(\Omega)$, and consequently that $\nu_{u^*,\zeta^*} \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. We conclude then that $\zeta^* \in \Phi(\psi^*)$, where $\psi^* := (y_{u^*,\zeta^*}, p_{u^*,\zeta^*}, u^*)$. Finally, by definition of $\delta$ and (5.32)

$$|\psi^* - \bar{\psi}|_{\mathcal{Y}} \le c_1|\zeta|_{\mathcal{Z}} + \varepsilon/2 \le \varepsilon.$$

Thus, $\zeta^* \in \Phi(\psi^*)$ and $\psi^* \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \varepsilon)$, which completes the proof. □

The next theorem claims that *all* solutions of the perturbed optimality system (4.61) are arbitrarily close to the solution of the unperturbed optimality system, provided that the solution of the latter is globally unique, Assumption I.5.1 holds, and the perturbation is sufficiently small.

**Theorem I.5.14.** *Let Assumption I.5.1 hold and suppose additionally that $\bar{\psi}$ is the unique element of $\mathcal{Y}$ that satisfies $0 \in \Phi(\bar{\psi})$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$ and $\psi \in \mathcal{Y}$ satisfy $\zeta \in \Phi(\psi)$, then $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \varepsilon)$.*

*ProofL.* et $\delta_0$ and $c_0$ be the positive numbers in Lemma I.5.9. Let $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$ and $\psi = (y, p, u) \in \mathcal{Y}$ be such that $\zeta \in \Phi(\psi)$. Define $\nu := \rho + H_u(\cdot, y_u, p_u) - H_u(\cdot, y, p)$. Arguing as in the proof of Theorem IV.4.9, we can find positive numbers $c_1$ and $c_2$ (independent of $\psi$ and $\zeta$) such that $|\nu|_{L^\infty(\Omega)} \le c_1|\zeta|_{\mathcal{Z}}$ and

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_{\bar{u}}|_{L^2(\Omega)} \le c_2\Big(|\zeta|_{\mathcal{Z}} + |u - \bar{u}|_{L^1(\Omega)}\Big).$$

Let $\delta := \min\{c_1^{-1}\delta_0, (2c_0c_2)^{-k^*}c_1^{-1}\varepsilon^{k^*}, (2c_2)^{-1}\varepsilon\}$ and suppose that $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$. As $\rho \in H_u(\cdot, y, p) + N_{\mathcal{U}}(u)$, we have $\nu \in H_u(\cdot, y_u, p_u) + N_{\mathcal{U}}(u)$. By Lemma I.5.9,

$$|u - \bar{u}|_{L^1(\Omega)} \le c_0|\nu|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \le c_0 c_1^{\frac{1}{k^*}}|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} \le c_2^{-1}\varepsilon/2.$$

Thus,

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_{\bar{u}}|_{L^2(\Omega)} + |u - \bar{u}|_{L^1(\Omega)} \leq c_2\Big(\delta + c_2^{-1}\varepsilon/2\Big) \leq \varepsilon.$$

□

## I.6 Nonlinear Perturbations

In this section, we apply the subregularity results in Section I.5 for studying the effect of certain nonlinear perturbations on the optimal solution. We consider the following family of problems

$$\min_{u\in\mathcal{U}} \left\{ \int_\Omega \Big[g(x, y, u) + \eta(x, y, u)\Big]\, dx \right\}, \tag{6.33}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + d(x, y) + \xi(x, y) &= \beta(x)u \quad \text{in} \quad \Omega \\ A(x)\nabla y \cdot \nu + b(x)y &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{6.34}$$

In order to specify the perturbations $\xi$ and $\eta$ under consideration and their topology, we begin the section by recalling some elementary notions of functional analysis.

As usual, $C(\mathbb{R}^s)$ denotes the space of all continuous functions $\omega : \mathbb{R}^s \to \mathbb{R}$. For each $m \in \mathbb{N}$, let $K_m$ denote the closed ball in $\mathbb{R}^s$ centered at zero with radius $m$. Consider the metric on $C(\mathbb{R}^s)$ given by

$$d_C(\omega_1, \omega_2) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|\omega_1 - \omega_2|_{L^\infty(K_m)}}{1 + |\omega_1 - \omega_2|_{L^\infty(K_m)}}.$$

This metric induces the compact-convergence topology on $C(\mathbb{R}^s)$. In this topology, a sequence $\{\omega_m\}_{m=1}^\infty \subset C(\mathbb{R}^s)$ converges to $\omega \in C(\mathbb{R}^s)$ if and only if $|\omega - \omega_m|_{L^\infty(K)} \to 0$ for every compact set $K \subset \mathbb{R}^s$. This topology is also known as the compact-open topology, see [**?**, Chapter 7] The following lemma is straightforward and follows from the definition of $d_C$.

**Lemma I.6.1.** *For each compact set $K \subset \mathbb{R}^s$ there exists $m \in \mathbb{N}$ such that*

$$|\omega_1 - \omega_2|_{L^\infty(K)} \leq 2^m d_C(\omega_1, \omega_2)$$

*for all $\omega_1, \omega_2 \in C(\mathbb{R}^s)$ such that $d_C(\omega_1, \omega_2) \leq 2^{-m}$.*

*ProofL.* et $K$ be a compact subset of $\mathbb{R}^s$. There exists $i \in \mathbb{N}$ such that $K \subset K_i$, where $K_i$ denotes the closed ball in $\mathbb{R}^s$ centered at zero with radius $i$. Now, by definition of the metric $d_C$,

$$\frac{|\omega|_{L^\infty(K_i)}}{1 + |\omega|_{L^\infty(K_i)}} \leq 2^i d_C(\omega, 0) \quad \forall \omega \in C(\mathbb{R}^s).$$

Hence,

$$|\omega|_{L^\infty(K_i)} \leq \frac{2^i d_C(\omega,0)}{1-2^i d_C(\omega,0)} \leq 2^{i+1} d_C(\omega,0)$$

for all $\omega \in C(\mathbb{R}^s)$ with $d_C(\omega,0) \leq 2^{-(i+1)}$. Let $m = i+1$. Then

$$|\omega_2-\omega_1|_{L^\infty(K)} \leq |\omega_2-\omega_1|_{L^\infty(K_i)} \leq 2^m d_C(\omega_2-\omega_1,0) = 2^m d_C(\omega_2,\omega_1)$$

for all $\omega_1, \omega_2 \in C(\mathbb{R}^s)$ with $d_C(\omega_1,\omega_2) \leq 2^{-m}$. □

### I.6.1 The perturbations

We begin describing the space of perturbations appearing in equation (6.34). Let $\Upsilon_s$ be the set of all continuously differentiable functions $\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $d_y(x,y) + \xi_y(x,y) \geq 0$ for all $x \in \Omega$ and $y \in \mathbb{R}$. The set $\Upsilon_s$ does not constitute a linear space, but it allows to have well-defined states for each perturbation.

**Proposition I.6.2.** *For each $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ there exists a unique function $y_u^\xi \in D(\mathcal{L})$ satisfying*

$$\mathcal{L}y_u^\xi + d(\cdot, y_u^\xi) + \xi(\cdot, y_u^\xi) = \beta(\cdot)u.$$

*Moreover, there exist positive numbers $M$ and $\delta$ such that $|y_u^\xi|_{L^\infty(\Omega)} \leq M$ for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ with $d_C(\xi,0) \leq \delta$.*

*ProofT.* he existence follows from [39, Theorem 4.8]. Moreover, also from this theorem, there exists $c > 0$ such that

$$|y_u^\xi|_{L^\infty(\Omega)} \leq c\big|\beta(\cdot)u - d(\cdot,0) - \xi(\cdot,0)\big|_{L^\infty(\Omega)}$$

for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$. Let $K := \bar{\Omega} \times \{0\}$, then by Lemma I.6.1 there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} |y_u^\xi|_{L^\infty(\Omega)} &\leq c\Big(|\beta|_{L^\infty(\Omega)}|u|_{L^\infty(\Omega)} + |d(\cdot,0)|_{L^\infty(\Omega)} + |\xi|_{L^\infty(K)}\Big) \\ &\leq c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u\in\mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot,0)|_{L^\infty(\Omega)} + 2^m d_C(\xi,0)\Big) \\ &\leq c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u\in\mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot,0)|_{L^\infty(\Omega)} + 1\Big) \end{aligned}$$

for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ with $d_C(\xi,0) \leq 2^{-m}$. The result follows defining $\delta := 2^{-m}$ and

$$M := c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u\in\mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot,0)|_{L^\infty(\Omega)} + 1\Big).$$

□

We now proceed to describe the perturbations appearing in the cost functional (6.33). Consider the set $\Upsilon_c$ of all continuously differentiable functions $\eta : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\eta(x,y,\cdot)$ is convex for all $x \in \Omega$ and $y \in \mathbb{R}$. We have the following result concerning the adjoint variable of the perturbed problem. Its proof is similar to the one of Proposition I.6.2.

**Proposition I.6.3.** *For each $u \in \mathcal{U}$, $\xi \in \Upsilon_s$ and $\eta \in \Upsilon_c$ there exists a unique function $p_u^{\xi,\eta} \in D(\mathcal{L})$ satisfying*

$$\mathcal{L}p_u^{\xi,\eta} + \left[d_y(\cdot, y_u^\xi) + \xi_y(\cdot, y_u^\xi)\right]p_u^{\xi,\eta} = g_y(\cdot, y_u^\xi, u) + \eta_y(\cdot, y_u^\xi, u).$$

*Moreover, there exist positive numbers $M$ and $\delta$ such that $|p_u^{\xi,\eta}|_{L^\infty(\Omega)} \le M$ for all $u \in \mathcal{U}$, $\xi \in \Upsilon_s$ and $\eta \in \Upsilon_c$ with $d_C(\xi, 0) + d_C(\xi_y, 0) + d_C(\eta_y, 0) \le \delta$.*

We denote $\Upsilon := \Upsilon_s \times \Upsilon_c$, and write $\zeta := (\xi, \eta)$ for a generic element of $\Upsilon$. We endow $\Upsilon$ with the pseudometric $d_\Upsilon : \Upsilon \times \Upsilon \to [0, \infty)$ given by

$$d_\Upsilon(\zeta, \zeta') := d_C(\xi, \xi') + d_C(\xi_y, \xi'_y) + d_C(\eta_y, \eta'_y) + d_C(\eta_u, \eta'_u).$$

## I.6.2 The stability result

We are now ready to state the problem (6.33)-(6.34) in a precise way. Given $\zeta \in \Upsilon$, problem $\mathcal{P}_\zeta$ is given by

$$\min_{u \in \mathcal{U}} \left\{ \mathcal{J}_\zeta(u) := \int_\Omega \left[ g(x, y_u^\xi, u) + \eta(x, y_u^\xi, u) \right] dx \right\}. \tag{6.35}$$

Due to the convexity of the cost in the control variable, each problem $\mathcal{P}_\zeta$ has at least one global solution. For each $\zeta \in \Upsilon$, we fix a local minimizer $\hat{u}_\zeta \in \mathcal{U}$ of problem $\mathcal{P}_\zeta$. By the local minimum principle, for each $\zeta = (\xi, \eta) \in \Upsilon$, the triple $(\hat{y}_\zeta, \hat{p}_\zeta, \hat{u}_\zeta) := (y^\xi_{\hat{u}_\zeta}, p^{\xi,\eta}_{\hat{u}_\zeta}, \hat{u}_\zeta)$ satisfies the system

$$\begin{cases} 0 = \mathcal{L}y - f(\cdot, y, u) - \xi(\cdot, y), \\ 0 = \mathcal{L}p - H_y(\cdot, y, p, u) + \eta_y(\cdot, y, u) - \xi_y(\cdot, y)p, \\ 0 \in H_u(\cdot, y, p) + \eta_u(\cdot, y, u) + N_\mathcal{U}(u). \end{cases} \tag{6.36}$$

As a consequence of Theorem IV.4.9, we have the following result.

**Theorem I.6.4.** *Let Assumption I.5.1 hold. There exist positive numbers $\alpha, \alpha'$ and $c$ such that*

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le c d_\Upsilon(\zeta, 0)^{1/k^*}$$

*for all $\zeta \in \Upsilon$ such that $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le \alpha$ and $d_\Upsilon(\zeta, 0) \le \alpha'$.*

*ProofB.* y Theorem IV.4.9, the mapping $\Phi$ is strongly Hölder subregular at $(\bar{\psi}, 0)$ with exponent $1/k^*$. Let $\alpha_1, \alpha_2$ and $\kappa$ be the positive numbers in the definition of strong subregularity. By Proposition I.6.2 and I.6.3 there exist positive numbers $M$ and $\delta_0$ such that

$$|y_u^\xi|_{L^\infty(\Omega)} + |p_u^{\xi,\eta}|_{L^\infty(\Omega)} \le M$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \le \delta_0$. Let $K := \bar{\Omega} \times [-M, M]$. By Lemma I.6.1, there exists $m \in \mathbb{N}$ such that

$$|\xi(\cdot, y_u^\xi)|_{L^2(\Omega)} \le \text{meas}\,\Omega^{\frac{1}{2}} |\xi|_{L^\infty(K)} \le 2^m \text{meas}\,\Omega^{\frac{1}{2}} d_C(\xi, 0) \le 2^m \text{meas}\,\Omega^{\frac{1}{2}} d_\Upsilon(\zeta, 0)$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \leq \min\{2^{-m}, \delta_0\}$. Repeating this argument, we can find positive numbers $\delta$ and $c_0$ such that

$$|\xi(\cdot, y_u^\xi)|_{L^2(\Omega)} + |\xi_y(\cdot, y_u^\xi)p_u^{\xi,\eta}|_{L^2(\Omega)} + |\eta_y(\cdot, y_u^\xi, u)|_{L^2\Omega} + |\eta_u(\cdot, y_u^\xi, u)|_{L^\infty} \leq c_0 d_\Upsilon(\zeta, 0) \tag{6.37}$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \leq \delta$. Using Proposition I.3.12 and Lemma I.6.1, one can find positive numbers $\alpha$ and $\delta'$ such that

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \leq \alpha_1$$

for all $\zeta \in \Upsilon$ with $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \leq \alpha$ and $d_\Upsilon(\zeta, 0) \leq \delta'$. Observe that by (6.36), we have

$$\begin{pmatrix} \xi(\cdot, \hat{y}_\zeta) \\ -\eta_y(\cdot, \hat{y}_\zeta, \hat{u}_\zeta) + \xi_y(\cdot, \hat{y}_\zeta)\hat{p}_\zeta \\ -\eta_u(\cdot, \hat{y}_\zeta, \hat{u}_\zeta) \end{pmatrix} \in \Phi(\hat{y}_\zeta, \hat{p}_\zeta, \hat{u}_\zeta)$$

for all $\zeta \in \Upsilon$. Let $\alpha' := \min\{c_0^{-1}\alpha_2, \delta, \delta'\}$. Then by Hölder subregularity of $\Phi$ and (6.37),

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \leq \kappa c_0^{\frac{1}{k^*}} d_\Upsilon(\zeta, 0)^{\frac{1}{k^*}}$$

for all $\zeta \in \Upsilon$ such that $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \leq \alpha$ and $d_\Upsilon(\zeta, 0) \leq \alpha'$. The result follows defining $c := \kappa c_0^{\frac{1}{k^*}}$. $\square$

### I.6.3 An application: Tikhonov regularization

In what follows we present an application of the theory derived in the previous chapters, namely the so-called Tikhonov regularization. For a more detailed description and an account of the state of the art, the reader is referred to [32, 42, 41]. We derive estimates on the convergence rate of the solution of the regularized problem when the regularization parameter tends to zero. The results that appear in the literature require the so-called structural assumption and positive-definiteness (in some sense) of the second derivative of the objective functional. Using Theorem IV.4.9, we can obtain these results under weaker assumptions than those used in the literature so far. One can compare these results with [32, Theorem 4.4] (where a tracking problem with semilinear elliptic equation is considered) when it comes to the stability of the controls. In Section I.7, we give more details on how the assumptions in the literature interplay with Assumption I.5.1.

We consider the following family of problems $\{\mathcal{P}_\varepsilon\}_{\varepsilon \geq 0}$.

$$\min_{u \in \mathcal{U}} \left\{ \int_\Omega g(x, y, u)\, dx + \frac{\varepsilon}{2} \int_\Omega u^2\, dx \right\}, \tag{6.38}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + d(x, y) = \beta(x)u & \text{in} \quad \Omega \\ A(x)\nabla y \cdot \nu + b(x)y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.39}$$

**Lemma I.6.5.** *Let Assumption I.5.1 be fulfilled. For every $\alpha > 0$ there exists $\varepsilon_\alpha > 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ problem $\mathcal{P}_\varepsilon$ has a local solution $\hat{u}_\varepsilon \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \alpha)$.*

*ProofL.* et $\alpha > 0$ be arbitrary. By Remark I.5.11, $\bar{u}$ is a strict local minimizer, hence there exists $\alpha^* \leq \alpha$ such that $\mathcal{J}(\bar{u}) < \mathcal{J}(u)$ for all $\bar{u} \neq u \in \mathcal{U} \cap \mathbb{B}_{L^1(\Omega)}(\bar{u}; \alpha^*)$. Consider the family of problems $\mathcal{P}_\varepsilon^*$ given by

$$\min_{\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \alpha^*)} \left\{ \mathcal{J}(u) + \frac{\varepsilon}{2} |u|^2_{L^2(\Omega)} \right\}. \tag{6.40}$$

Each problem $\mathcal{P}_\varepsilon^*$ has a global solution $\hat{u}_\varepsilon$. There exists $\varepsilon^* > 0$ such that $|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha^*/2$ for all $\varepsilon \in (0, \varepsilon^*)$. Suppose the opposite. Then there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to zero such that $|\hat{u}_{\varepsilon_k} - \bar{u}|_{L^1(\Omega)} > \alpha^*/2$ for all $k \in \mathbb{N}$. Since $\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \alpha^*)$ is weakly compact in $L^2(\Omega)$, we may assume without loss of generality that $u_{\varepsilon_k} \rightharpoonup u^*$ for some $u^* \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \alpha^*)$. Since $y_{\hat{u}_{\varepsilon_k}} \to y_{u^*}$ in $C(\bar{\Omega})$, we obtain that

$$\mathcal{J}(u^*) \leq \liminf_{k \to \infty} \left[ \mathcal{J}(\hat{u}_{\varepsilon_k}) + \frac{\varepsilon_k}{2} |\hat{u}_{\varepsilon_k}|^2_{L^2(\Omega)} \right] \leq \liminf_{k \to \infty} \left[ \mathcal{J}(\bar{u}) + \frac{\varepsilon_k}{2} |\bar{u}|^2_{L^2(\Omega)} \right] = \mathcal{J}(\bar{u}).$$

Therefore $u^* = \bar{u}$ since $u^* \in \mathcal{U} \cap \mathbb{B}_{L^1(\Omega)}(\bar{u}; \alpha^*)$ and $\bar{u}$ is strict local minimum. By Proposition I.5.2, $u^* = \bar{u}$ is bang-bang. Weak convergence in $L^2(\Omega)$ implies weak convergence in $L^1(\Omega)$; consequently, by Lemma I.5.3, $\hat{u}_{\varepsilon_k} \to u^*$ in $L^1(\Omega)$, which is a contradiction. We can see that for all $\varepsilon \leq \varepsilon^*$, $\hat{u}_\varepsilon$ is a local solution of problem $\mathcal{P}_\varepsilon$. Indeed, if $u \in \mathcal{U} \cap B_{L^1(\Omega)}(\hat{u}_\varepsilon; \alpha^*/2)$, then

$$|u - \bar{u}|_{L^1(\Omega)} \leq |u - \hat{u}_\varepsilon|_{L^1(\Omega)} + |\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha^*,$$

and consequently, as $\hat{u}_\varepsilon$ is a global solution of problem $\mathcal{P}_\varepsilon^*$,

$$\mathcal{J}(\hat{u}_\varepsilon) + \frac{\varepsilon}{2} |\hat{u}_\varepsilon|_{L^2(\Omega)} \leq \mathcal{J}(u) + \frac{\varepsilon}{2} |u|_{L^2(\Omega)}.$$

The result follows defining $\varepsilon_\alpha := \varepsilon^*$.

□

**Theorem I.6.6.** *Let Assumption I.5.1 be fulfilled. Then there exist positive numbers $\alpha, \kappa$ and $\varepsilon_0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem $\mathcal{P}_\varepsilon$ has a local solution $\hat{u}_\varepsilon \in \mathbb{B}_{L^1}(\bar{u}; \alpha)$. Moreover,*

$$|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \kappa \varepsilon^{1/k^*} \tag{6.41}$$

*for every local solution $\hat{u}_\varepsilon$ of problem $\mathcal{P}_\varepsilon$ such that $\varepsilon \in (0, \varepsilon_0)$ and $|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha$.*

*ProofT.* he first claim follows from Lemma I.6.5. Let $\alpha, \alpha'$and $c$ be the positive numbers in Theorem I.6.4. Define $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $\eta_\varepsilon(u) := \varepsilon u^2/2$ and $\zeta_\varepsilon := (0, \eta_\varepsilon) \in \Upsilon$ for each $\varepsilon > 0$. Note that

$$d_C(\eta_\varepsilon, 0) := \sum_{m=1}^\infty \frac{1}{2^m} \frac{\varepsilon m^2/2}{1 + \varepsilon m^2/2} = \varepsilon \sum_{m=1}^\infty \frac{1}{2^m} \frac{m^2}{2 + \varepsilon m^2} \leq \varepsilon \sum_{m=1}^\infty \frac{m^2}{2^{m+1}} = 3\varepsilon$$

for all $\varepsilon > 0$. Analogously,

$$d_C(\frac{\partial \eta_\varepsilon}{\partial u}, 0) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\varepsilon m}{1 + \varepsilon m} \leq \varepsilon \sum_{m=1}^{\infty} \frac{m}{2^m} = 2\varepsilon$$

for all $\varepsilon > 0$. We conclude that $d_\Upsilon(\zeta_\varepsilon, 0) \leq 5\varepsilon \leq \alpha'$ for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \alpha'/5$. By Theorem I.6.4,

$$|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq 5^{\frac{1}{k^*}} c \varepsilon^{\frac{1}{k^*}}$$

for all $\varepsilon \in (0, \varepsilon_0)$ such that $|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha$. □

## I.7 Assumptions related to subregularity

In this section, we gather some results concerning Assumption I.5.1, in order to provide sufficient conditions under which it is fulfilled. Furthermore, we analyze related assumptions and their relationship between themselves. Recall that $\bar{u} \in \mathcal{U}$ is a local solution of problem (2.1)–(2.2). Since $\bar{u} \in \mathcal{U}$ satisfies the variational inequality (4.9), we have

$$\bar{u}(x) = \begin{cases} b_1(x) & if \quad \sigma_{\bar{u}}(x) > 0 \\ b_2(x) & if \quad \sigma_{\bar{u}}(x) < 0. \end{cases}$$

We introduce the following extended cone suggested in [5]. For a fixed $\tau > 0$ define

$$C_{\bar{u}}^\tau = \left\{ v \in L^2(\Omega) : v(x) \begin{cases} = 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| > \tau \text{ or } \bar{u}(x) \in (b_1(x), b_2(x)) \\ \geq 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \tau \text{ and } \bar{u}(x) = b_1(x) \\ \leq 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \tau \text{ and } \bar{u}(x) = b_2(x) \end{cases} \right\}.$$

We introduce the following modification of Assumption I.5.1.

**Assumption 2′.** *There exist positive numbers $\alpha_0$ and $\gamma_0$ such that*

$$\int_\Omega \sigma_{\bar{u}}(u - \bar{u})\, dx + \Lambda(u - \bar{u}) \geq \gamma_0 |u - \bar{u}|_{L^1(\Omega)}^{k^*+1},$$

*for all $u \in \mathcal{U}$ with $u - \bar{u} \in C_{\bar{u}}^\tau \cap \mathbb{B}_{L^1(\Omega)}(\bar{u}; \alpha_0)$.*

This assumption is seemingly weaker than Assumption I.5.1. However, we will prove that the two assumptions are equivalent. Before that, for technical purposes, we introduce the bilinear form $\Gamma : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ given by

$$\Gamma(v_1, v_2) := \frac{1}{2} \int_\Omega \left[ \pi_{v_1} v_2 + \pi_{v_2} v_1 \right] dx. \tag{7.42}$$

The bilinear form is particularly useful because of the following property.

$$\Lambda(v_1 + v_2) = \Gamma(v_1, v_1) + 2\Gamma(v_1, v_2) + \Gamma(v_2, v_2) \quad \forall v_1, v_2 \in L^2(\Omega). \tag{7.43}$$

We will require the following technical lemma.

**Lemma I.7.1.** *For every positive number $M$, there exists a positive number $c$ such that*

$$|\Gamma(v_1, v_2)| \le c|v_1|^{1/2}_{L^1(\Omega)}|v_2|_{L^1(\Omega)}$$

*for all $v_1, v_2 \in \mathbb{B}_{L^\infty}(0; M)$.*

*ProofB.* y Proposition I.4.6, there exist $c_1, c_2 > 0$ such that $|\pi_v|_{L^\infty(\Omega)} \le c_1|v|_{L^2(\Omega)}$ and $|\pi_v|_{L^2(\Omega)} \le c_2|v|_{L^1(\Omega)}$ for all $v \in L^2(\Omega)$. Let $M > 0$ be arbitrary. Observe that

$$\Big|\int_\Omega \pi_{v_1} v_2 \, dx\Big| \le |\pi_{v_1}|_{L^\infty(\Omega)}|v_2|_{L^1(\Omega)} \le c_1 M^{\frac{1}{2}}|v_1|^{\frac{1}{2}}_{L^1(\Omega)}|v_2|_{L^1(\Omega)},$$

and that

$$\Big|\int_\Omega \pi_{v_2} v_1 \, dx\Big| \le |\pi_{v_2}|_{L^2(\Omega)}|v_1|_{L^2(\Omega)} \le c_2 M^{\frac{1}{2}}|v_1|^{\frac{1}{2}}_{L^1(\Omega)}|v_2|_{L^1(\Omega)}$$

for all $v_1, v_2 \in \mathbb{B}_{L^\infty}(0; M)$. There result follows defining $c := 2^{-1}(c_1 + c_2)M^{\frac{1}{2}}$. □

**Proposition I.7.2.** *Assumptions I.5.1 and 2′ are equivalent.*

*Proof.* Clearly Assumption I.5.1 implies 2′. Let $\alpha_0$ and $\gamma_0$ be the numbers in Assumption 2′. Let $u \in \mathcal{U}$ and define

$$v_1(x) := \begin{cases} u(x) - \bar{u}(x) & if \quad |\sigma_{\bar{u}}(x)| \le \tau \\ 0 & if \quad |\sigma_{\bar{u}}(x)| > \tau, \end{cases}$$

and

$$v_2(x) := \begin{cases} 0 & if \quad |\sigma_{\bar{u}}(x)| \le \tau \\ u(x) - \bar{u}(x) & if \quad |\sigma_{\bar{u}}(x)| > \tau. \end{cases}$$

Clearly $v_1 \in C^\tau_{\bar{u}}$ and $v_1 + v_2 = u - \bar{u}$. Let $M$ be a bound for $\mathcal{U}$ in $L^\infty(\Omega)$, and let $c$ be the positive number in Lemma I.7.1 corresponding to $2M$. By Assumption 2′,

$$\begin{aligned} \int_\Omega \sigma_{\bar{u}}(u - \bar{u}) \, dx &= \int_\Omega \sigma_{\bar{u}} v_1 \, dx + \int_{|\sigma_{\bar{u}}|>\tau} \sigma_{\bar{u}} v_2 \, dx \\ &= \int_\Omega \sigma_{\bar{u}} v_1 \, dx + \Lambda(v_1) - \Lambda(v_1) + \int_{|\sigma_{\bar{u}}|>\tau} \sigma_{\bar{u}} v_2 \, dx \\ &\ge \gamma_0 |v_1|^{k+1} + \tau|v_2|_{L^1(\Omega)} - \Lambda(v_1), \end{aligned}$$

and

$$\begin{aligned}\Lambda(u-\bar u) &= \Lambda(v_1) + 2\Gamma(v_1, v_2) + \Lambda(v_2)\\ &\geq \Lambda(v_1) - 2c|v_1|_{L^1(\Omega)}^{1/2}|v_2|_{L^1(\Omega)} - c|v_2|_{L^1(\Omega)}^{1/2}|v_2|_{L^1(\Omega)}\\ &\geq \Lambda(v_1) - 3c|v_2|_{L^1(\Omega)}|u-\bar u|_{L^1(\Omega)}^{1/2}\end{aligned}$$

for $u \in \mathcal{U}$ with $|u-\bar u|_{L^1(\Omega)} \leq \alpha_0$. Thus

$$\begin{aligned}\int_\Omega \sigma_{\bar u}(u-\bar u)\,dx + \Lambda(u-\bar u) &\geq \gamma_0|v_1|^{k+1} + \tau|v_2|_{L^1(\Omega)} - 3c|v_2|_{L^1(\Omega)}|u-\bar u|_{L^1(\Omega)}^{1/2}\\ &= \gamma_0|v_1|^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar u|_{L^1(\Omega)}^{1/2}\Big)\end{aligned}$$

for $u \in \mathcal{U}$ with $|u-\bar u|_{L^1(\Omega)} \leq \alpha_0$. Now, by the reverse triangle inequality and Bernoulli's inequality (consider without loss of generality $u \neq \bar u$)

$$\begin{aligned}|v_1|_{L^1(\Omega)}^{k+1} &= |(u-\bar u) - v_2|_{L^1(\Omega)}^{k+1} \geq \Big(|u-\bar u|_{L^1(\Omega)} - |v_2|_{L^1(\Omega)}\Big)^{k+1}\\ &= |u-\bar u|_{L^1(\Omega)}^{k+1}\Big(1 - \frac{|v_2|_{L^1(\Omega)}}{|u-\bar u|_{L^1(\Omega)}}\Big)^{k+1} \geq |u-\bar u|_{L^1(\Omega)}^{k+1}\Big(1-(k+1)\frac{|v_2|_{L^1(\Omega)}}{|u-\bar u|_{L^1(\Omega)}}\Big)\\ &= |u-\bar u|_{L^1(\Omega)}^{k+1} - (k+1)|u-\bar u|_{L^1(\Omega)}^{k}|v_2|_{L^1(\Omega)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega \sigma_{\bar u}(u-\bar u)\,dx + \Lambda(u-\bar u) &\geq \gamma_0|v_1|^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar u|_{L^1(\Omega)}^{1/2}\Big)\\ &\geq \gamma_0|u-\bar u|_{L^1(\Omega)}^{k+1} - \gamma_0(k+1)|u-\bar u|_{L^1(\Omega)}^{k}|v_2|_{L^1(\Omega)} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar u|_{L^1(\Omega)}^{1/2}\Big)\\ &\geq \gamma_0|u-\bar u|_{L^1(\Omega)}^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - \gamma_0(k+1)|u-\bar u|_{L^1(\Omega)}^{k} - 3c|u-\bar u|_{L^1(\Omega)}^{1/2}\Big).\end{aligned}$$

Choosing $\alpha$ small enough, one can ensure

$$\begin{aligned}\int_\Omega \sigma_{\bar u}(u-\bar u)\,dx + \Lambda(u-\bar u) &\geq \gamma_0|u-\bar u|_{L^1(\Omega)}^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - \gamma_0(k+1)|u-\bar u|_{L^1(\Omega)}^{k} - 3c|u-\bar u|_{L^1(\Omega)}^{1/2}\Big)\\ &\geq \gamma_0|u-\bar u|_{L^1(\Omega)}^{k+1} + \frac{\tau}{2}|v_2|_{L^1(\Omega)} \geq \gamma_0|u-\bar u|_{L^1(\Omega)}^{k+1}\end{aligned}$$

for all $u \in \mathcal{U}$ with $|u-\bar u|_{L^1(\Omega)} \leq \alpha$.

□

Proposition I.7.2 allows splitting Assumption I.5.1 into two parts, as it follows in the next theorem.

**Theorem I.7.3.** *Let there exist numbers $\mu_1, \mu_2 \in \mathbb{R}$ and $\alpha > 0$ such that*

$$\int_\Omega \sigma_{\bar u} v\,dx \geq \mu_1|v|_{L^1(\Omega)}^{k^*+1} \tag{7.44}$$

*and*

$$\Lambda(v) \geq \mu_2 |v|_{L^1(\Omega)}^{k^*+1} \tag{7.45}$$

*for every $v \in (\mathcal{U} - \bar{u}) \cap C_{\bar{u}}^{\tau} \cap \mathbb{B}_{L^1(\Omega)}(\bar{u}; \alpha)$. If $\mu_1 + \mu_2 > 0$, then Assumption 2 is fulfilled, hence the optimality mapping $\Phi$ (see (4.60)) of problem (2.1)–(2.2) is strongly Hölder subregular with exponent $\lambda = 1/k^*$ at the reference point $(\bar{y}, \bar{p}, \bar{u})$.*

The proof consists of summation of (7.44) and (7.45) and utilization of Proposition I.7.2 and Theorem IV.4.9.

The splitting of Assumption 2 has the advantage that the inequalities in (7.44) and (7.45) can be analyzed separately. The next proposition is related to (7.44).

The following assumption has become standard in the literature on PDE optimal control problems with bang-bang controls, see, e.g., [**?**]

**Assumption I.7.4.** *There exists a positive number $\mu_0$ such that*

$$meas\{x \in \Omega : |\sigma_{\bar{u}}(x)| \leq \varepsilon\} \leq \mu_0 \varepsilon^{\frac{1}{k^*}} \quad \forall \varepsilon > 0.$$

**Proposition I.7.5.** *The following statements hold.*

*(i) If Assumption I.7.4 is fulfilled then there exists $\mu_1 > 0$ such that (7.44) holds for every $v \in \mathcal{U} - \bar{u}$.*

*(ii) Suppose there exists $\nu > 0$ such that $b_2(x) - b_1(x) \geq \nu$ for a.e. $x \in \Omega$. If (7.44) holds for every $v \in \mathcal{U} - \bar{u}$ then Assumption I.7.4 is fulfilled.*

*ProofT.* he proof of the first claim follows [34, Proposition 3.1], see also [10, Proposition 2.7]. It has been also proved several times in the literature on ordinary differential equations in a somewhat stronger form; see, e.g., [1, 28, 33, 37].

Let us prove the second claim. For each $\varepsilon > 0$, define

$$u_\varepsilon(x) := \begin{cases} \bar{u}(x) & \text{if} \qquad\qquad\qquad\qquad |\sigma_{\bar{u}}(x)| > \varepsilon \\[2ex] b_1(x) & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \varepsilon \quad \text{and} \quad \bar{u}(x) \in \Big[\dfrac{b_1(x) + b_2(x)}{2}, b_2(x)\Big] \\[2ex] b_2(x) & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \varepsilon \quad \text{and} \quad \bar{u}(x) \in \Big[b_1(x), \dfrac{b_1(x) + b_2(x)}{2}\Big). \end{cases}$$

Clearly each $u_\varepsilon$ belongs to $\mathcal{U}$, and

$$|u_\varepsilon(x) - \bar{u}(x)| \geq \frac{1}{2}|b_2(x) - b_1(x)| \tag{7.46}$$

for a.e $x \in \{s \in \Omega : |\sigma_{\bar{u}}(s)| \leq \epsilon\}$. From (7.44) we have

$$\mu_1 \Big( \int_{|\sigma_{\bar{u}}| \leq \varepsilon} |u_\varepsilon - \bar{u}| \, dx \Big)^{k+1} \leq \int_{|\sigma_{\bar{u}}| \leq \varepsilon} \sigma_{\bar{u}}(u_\varepsilon - \bar{u}) \, dx \leq \varepsilon \int_{|\sigma_{\bar{u}}| \leq \varepsilon} |u_\varepsilon - \bar{u}| \, dx.$$

This implies

$$\int_{|\sigma_{\bar{u}}|\le\varepsilon} |u_\varepsilon - \bar{u}|\,dx \le \mu_1^{-\frac{1}{k}}\varepsilon^{\frac{1}{k}}. \tag{7.47}$$

Using (7.46) and (7.47) we obtain that

$$\begin{aligned}\text{meas}\,\{x\in\Omega : |\sigma_{\bar{u}}(x)| \,\le\, \varepsilon\} &= \frac{1}{\nu}\int_{|\sigma_{\bar{u}}|\le\varepsilon} \nu\,dx \,\le\, \frac{1}{\nu}\int_{|\sigma_{\bar{u}}|\le\varepsilon} |b_2 - b_1|\,dx \,\le\, \frac{2}{\nu}\int_{|\sigma_{\bar{u}}|\le\varepsilon} |u_\varepsilon - \bar{u}|\,dx \\ &\le 2(\mu_1)^{-\frac{1}{k}}\nu^{-1}\,\varepsilon^{\frac{1}{k}}.\end{aligned}$$

Thus Assumption I.7.4 is fulfilled with $\mu_0 := 2(\mu_1)^{-\frac{1}{k}}\nu^{-1}$. □

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# II. New Assumptions for Stability Analysis in Elliptic Optimal Control Problems

## Outline



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# Abstract

This paper is dedicated to the stability analysis of the optimal solutions of a control problem associated with a semilinear elliptic equation. The linear differential operator of the equation is neither monotone nor coercive due to the presence of a convection term. The control appears only linearly, or even it can not appear in an explicit form in the objective functional. Under new assumptions, we prove Lipschitz stability of the optimal controls and associated states with respect to perturbations in the equation and the objective functional as well as with respect to the Tikhonov regularization parameter.

## II.1 Introduction

In this paper, we study the following optimal control problem

$$\text{(P)} \quad \min_{u \in \mathcal{U}} J(u) := \int_\Omega L(x, y_u(x), u(x))\,\mathrm{d}x,$$

where $\mathcal{U} = \{u \in L^2(\Omega) : u_a \leq u(x) \leq u_b \text{ for a.a. } x \in \Omega\}$, $-\infty < u_a < u_b < +\infty$. Here, $y_u$ denotes the solution of the semilinear elliptic equation:

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x,y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Assumptions on the data of the control problem (P) will be given below. The aim of this paper is to prove stability results for the local minimizers of (P) with respect to perturbations in the data of the control problem. There are quite a few previous papers devoted to this issue [14], [15], [16], [17], just to mention some of them. In all these cases, the second derivative of $L$ with respect to $u$ is bounded from below by a positive constant. This is the case where the Tikhonov term is involved in the objective functional. Under this condition and assuming sufficient second-order optimality conditions (SSOC), the Lipschitz stability of the optimal controls is proved. Here, we assume that $u$ appears linearly in $L(x, y, u)$ or even it does not appear at all. Therefore, the previous results do not apply. In this case, under (SSOC) for optimality, Lipschitz stability of the optimal states can be proved; see [7]. In Section II.4, we obtain analogous estimates for the optimal states replacing (SSOC) by a weaker condition; see (3.29). It is weaker because (SSOC) implies our assumption, but they are not equivalent. In addition, our assumption implies strict local optimality of the control; see Theorem II.3.5.

In order to prove stability of the optimal controls when they are not explicitly involved in the objective functional, besides (SSOC) an additional structural hypothesis is usually assumed. This situation was studied in [21], where the authors proved Lipschitz stability of the control with respect to linear perturbations simultaneously appearing in the state equation and the objective functional. The drawback is that the additional hypothesis is satisfied only by bang-bang controls. Here, we obtain analogous estimates changing the mentioned assumption by a weaker one, see (5.47). Though this second assumption (5.47) is stronger than (3.29), it can be satisfied by optimal controls independently if they are bang-bang or not. Moreover our assumption (5.47) is satisfied if the (SSOC) and the additional hypothesis are assumed.

Finally, under the assumption (5.47), Lipschitz stability is established for the optimal states with respect to simultaneous perturbations in the equations and in the objective functional with respect to the state and the control, and with respect to the Tikhonov regularization parameter. The stability with respect to the Tikhonov regularization has been studied in [7] and [20]. In [7], Hölder stability of the states is proved. In [20], stability of the control is proved under (SSOC) and the structural assumption. The reader is also referred to [23], [24], [25] for the case of linear partial differential equations.

In this paper, besides providing some new sufficient conditions for Lipschitz stability for the optimal control and associated states, we deal with a semilinear elliptic state equation that is neither monotone nor coercive. Though some crucial results for this state equation are taken from [6], some estimates have been proved that are not available in the literature.

The plan of this paper is as follows. In Section II.2, we analyze the state equation. First, we establish some properties of the linear differential operator of the state equation, and the full semilinear equation is analyzed in the second part of the section. The control problem (P) is studied in Section II.3. We prove that our assumption (3.29) is a sufficient condition for strong local optimality. Section II.4 is dedicated to the proof of Lipschitz stability of the optimal states. In Section II.5 we introduce the stronger condition (5.47) replacing (3.29) that allows us to establish the Lipschitz stability of the optimal controls. Finally, in Section II.6, the Tikhonov regularization is considered.

## II.2 Analysis of the partial differential equation

In this section we analyze the equation (1.1). We split the section in two parts. In the first part, we establish the results concerning the linear operator of the elliptic equation. In the second subsection, the nonlinear equation will be studied.

### Analysis of the linear differential operator

We define the differential operator $\mathcal{A} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$\mathcal{A}y = -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y.$$

The following assumptions are supposed to hold throughout the paper. They ensure that the mathematical objects under consideration are well-defined.

**Assumption II.2.1.** *The following statements are fulfilled.*

*(i) The set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with a Lipschitz boundary $\Gamma$. The mapping $A : \Omega \longrightarrow \mathbb{R}^{n\times n}$ is measurable and bounded in $\Omega$, and there exists $\Lambda_A > 0$ such that $\xi^\top A(x)\xi \geq \Lambda_A|\xi|^2$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.*

*(ii) We assume that $b \in L^p(\Omega;\mathbb{R}^n)$ with $p \geq 3$ if $n = 3$ and $p > 2$ arbitrary if $n = 2$.*

Under these assumptions it is known that $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism despite the fact that the operator is neither coercive nor monotone; see [6], [13, Theorem 8.3], [22]. The following identity is satisfied

$$\langle \mathcal{A}y, z\rangle = \int_\Omega A\nabla y\cdot\nabla z\,\mathrm{d}x + \int_\Omega b\cdot\nabla y z\,\mathrm{d}x \quad \forall y, z \in H_0^1(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Along this paper, we will set

$$\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla y(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The next lemma states some properties of $\mathcal{A}$ that will be used later.

**Lemma II.2.2.** *The following statements are fulfilled:*

*(i) There exists a constant $C_{\Lambda_A,b}$ such that Gårding's inequality holds*

$$\langle \mathcal{A}y, y\rangle \geq \frac{\Lambda_A}{4}\|y\|^2_{H_0^1(\Omega)} - C_{\Lambda_A,b}\|y\|^2_{L^2(\Omega)} \quad \forall y \in H_0^1(\Omega). \tag{2.2}$$

*(ii) Let $a \in L^\infty(\Omega)$ be a nonnegative function and $h \in H^{-1}(\Omega)$. If $y \in H_0^1(\Omega)$ satisfies $\mathcal{A}y + ay = h$ and $h$ is a nonnegative linear form, then $y$ is a nonnegative function as well.*

*(iii) Let $a$ be as above and $h \in L^r(\Omega)$ with $r > \frac{n}{2}$. Then, the solution $y$ of the above equation belongs to $H_0^1(\Omega) \cap C(\bar\Omega)$. Moreover, there exists a constant $C_r$ independent of $a$ and $h$ such that*

$$\|y\|_{H_0^1(\Omega)} + \|y\|_{C(\bar\Omega)} \leq C_r\|h\|_{L^r(\Omega)}. \tag{2.3}$$

*Proof.* The proof of (2.2) can be found in [6]; see also [13, Lemma 8.4]. For the proof of *(ii)* the reader is referred again to [6] and [13, Theorem 8.1]. The $H_0^1(\Omega) \cap C(\bar\Omega)$ regularity of $y$ for functions $h \in L^r(\Omega)$ is well known; see [13, Lemma 8.31]. It remains to prove the estimates (2.3) for a constant $C_r$ independent of $h$ and $a$. Let us denote by $y_{a,h} \in H_0^1(\Omega) \cap C(\bar\Omega)$ the solution of $\mathcal{A}y + ay = h$. With $y_{0,h}$ we denote the solution corresponding to $a \equiv 0$. Then, the estimate $\|y_{0,h}\|_{C(\bar\Omega)} \leq C\|h\|_{L^r(\Omega)}$ is well known for a constant $C$ depending on $r$, but independent of $h$. Let us write $h = h^+ - h^-$. From *(ii)* we know that $y_{a,h^+} \geq 0$ and $y_{a,h^-} \geq 0$. Now, since $\mathcal{A}(y_{a,h^+} - y_{0,h^+}) + a(y_{a,h^+} - y_{0,h^+}) = -ay_{0,h^+}$, again by item *(ii)*, we obtain $0 \leq y_{a,h^+} \leq y_{0,h^+}$ and consequently $\|y_{a,h^+}\|_{C(\bar\Omega)} \leq \|y_{0,h^+}\|_{C(\bar\Omega)}$. Analogously, by the same argument $0 \leq y_{a,h^-} \leq y_{0,h^-}$ and consequently $\|y_{a,h^-}\|_{C(\bar\Omega)} \leq \|y_{0,h^-}\|_{C(\bar\Omega)}$. Therefore,

$$\begin{aligned}\|y_{a,h}\|_{C(\bar\Omega)} &\leq \|y_{a,h^+}\|_{C(\bar\Omega)} + \|y_{a,h^-}\|_{C(\bar\Omega)} \leq \|y_{0,h^+}\|_{C(\bar\Omega)} + \|y_{0,h^-}\|_{C(\bar\Omega)}\\ &\leq C\Big(\|h^+\|_{L^r(\Omega)} + \|h^-\|_{L^r(\Omega)}\Big) \leq 2C\|h\|_{L^r(\Omega)},\end{aligned}$$

where $C$ is independent of $a$ and $h$. To prove the corresponding estimate in $H_0^1(\Omega)$ we use Gårding's inequality (2.2) and the above estimate:

$$\begin{aligned}&\frac{\Lambda_A}{4}\|y_{a,h}\|^2_{H_0^1(\Omega)} \leq \langle \mathcal{A}y_{a,h}, y_{a,h}\rangle + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)} \leq \langle \mathcal{A}y_{a,h}, y_{a,h}\rangle + \int_\Omega a y_{a,h}^2 \,\mathrm{d}x + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)}\\ &= \int_\Omega h y_{a,h} \,\mathrm{d}x + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)} \leq |\Omega|^{\frac{r-1}{r}}\|h\|_{L^r(\Omega)}\|y_{a,h}\|_{C(\bar\Omega)} + C_{\Lambda_A,b}|\Omega|\|y_{a,h}\|^2_{C(\bar\Omega)}\\ &\leq 2C\Big(|\Omega|^{\frac{r-1}{r}} + 2CC_{\Lambda_A,b}|\Omega|\Big)\|h\|^2_{L^r(\Omega)},\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. Since the above constants are independent of $a$ and $h$, the inequality completes the proof of (2.3). □

Now, we consider the adjoint operator $\mathcal{A}^* : H_0^1(\Omega) \to H^{-1}(\Omega)$ of $\mathcal{A}$. Since $\mathcal{A}$ is an isomorphism, $\mathcal{A}^*$ is also an isomorphism. It is obvious that $\mathcal{A}^*\varphi = -\operatorname{div}\left(A^\top \nabla\varphi\right) - \operatorname{div}\left(\varphi b\right)$. The operator $\mathcal{A}^*$ satisfies the same properties established in Lemma II.2.2. Indeed, the Gårding's inequality is a consequence of (2.2) and the identity $\langle \mathcal{A}^*\varphi, \varphi\rangle = \langle \mathcal{A}\varphi, \varphi\rangle$. The proof of the estimate (2.3) is the same for the operator $\mathcal{A}^*$. We only prove the statement $(ii)$. Let $h \in H^{-1}(\Omega)$ be a nonnegative linear form. This means that $\langle h, y\rangle \geq 0$ for every nonnegative function $y \in H_0^1(\Omega)$. Let $\varphi \in H_0^1(\Omega)$ satisfy $\mathcal{A}^*\varphi + a\varphi = h$. Now, given a nonnegative function $w \in L^2(\Omega)$ we take $y \in H_0^1(\Omega)$ satisfying $\mathcal{A}y + ay = w$. By Lemma II.2.2-$(ii)$ we have that $y \geq 0$. Then, we obtain

$$\int_\Omega w\varphi\, dx = \langle \mathcal{A}y + ay, \varphi\rangle = \langle \mathcal{A}^*\varphi + a\varphi, y\rangle = \langle h, y\rangle \geq 0.$$

Since $w$ is an arbitrary nonnegative function of $L^2(\Omega)$, this inequality yields $\varphi \geq 0$.

We finish this subsection by proving an $L^s(\Omega)$ estimate.

**Lemma II.2.3.** *Assume that $s \in [1, \frac{n}{n-2})$, $s'$ is its conjugate, and let $a \in L^\infty(\Omega)$ be a nonnegative function. Then, there exists a constant $C_{s'}$ independent of $a$ such that*

$$\begin{cases} \|y_h\|_{L^s(\Omega)} \leq C_{s'}\|h\|_{L^1(\Omega)}, \\ \|\varphi_h\|_{L^s(\Omega)} \leq C_{s'}\|h\|_{L^1(\Omega)}, \end{cases} \quad \forall h \in H^{-1}(\Omega) \cap L^1(\Omega), \tag{2.4}$$

*where $y_h$ and $\varphi_h$ satisfy the equations $\mathcal{A}y_h + ay_h = h$ and $\mathcal{A}^*\varphi_h + a\varphi_h = h$, respectively, and $C_{s'}$ is given by* (2.3) *with $r = s'$.*

*Proof.* We prove the estimate (2.4) for $\varphi_h$ and $n = 3$, the proof being identical for $y_h$ and analogous for $n = 2$ with minor modifications. First we observe that $H_0^1(\Omega) \subset L^6(\Omega) \subset L^3(\Omega)$, hence $\varphi_h \in L^s(\Omega)$. As a consequence we obtain that $|\varphi_h|^{s-1}\operatorname{sign}(\varphi_h) \in L^{s'}(\Omega)$. Moreover, $s < 3$ implies that $s' > \frac{3}{2}$. According to Lemma II.2.2-$(iii)$, the solution of $\mathcal{A}y + ay = |\varphi_h|^{s-1}\operatorname{sign}(\varphi_h)$ belongs to $H_0^1(\Omega) \cap C(\bar{\Omega})$ and satisfies $\|y\|_{C(\bar{\Omega})} \leq C_{s'}\||\varphi_h|^{s-1}\operatorname{sign}(\varphi_h)\|_{L^{s'}(\Omega)} = C_{s'}\|\varphi_h\|_{L^s(\Omega)}^{s-1}$, where $C_{s'}$ is independent of $a$ and $h$. Using these facts we infer

$$\begin{aligned}\|\varphi_h\|_{L^s(\Omega)}^s &= \int_\Omega |\varphi_h|^s\, \mathrm{d}x = \langle \mathcal{A}y + ay, \varphi_h\rangle = \langle \mathcal{A}^*\varphi_h + a\varphi_h, y\rangle \\ &= \int_\Omega hy\, \mathrm{d}x \leq \|h\|_{L^1(\Omega)}\|y\|_{C(\bar{\Omega})} \leq C_{s'}\|h\|_{L^1(\Omega)}\|\varphi_h\|_{L^s(\Omega)}^{s-1}.\end{aligned}$$

This proves (2.4) for $\varphi_h$. □

### Analysis of the semilinear equation

In this subsection, we formulate some results concerning the semilinear equation (1.1). For this purpose we make the following assumptions on the nonlinear term of the equation.

**Assumption II.2.4.** *We assume that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class $C^2$ with respect to the second variable satisfying:*

$$f(\cdot, 0) \in L^r(\Omega) \text{ with } r > \frac{n}{2} \text{ and } \frac{\partial f}{\partial y}(x, y) \geq 0 \ \forall y \in \mathbb{R}, \tag{2.5}$$

$$\forall M > 0 \ \exists C_{f,M} > 0 \text{ such that } \left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \leq C_{f,M} \ \forall |y| \leq M, \tag{2.6}$$

$$\begin{cases} \forall M > 0 \text{ and } \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \left|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M \text{ and } |y_2 - y_1| \leq \delta, \end{cases} \tag{2.7}$$

*for almost every $x \in \Omega$.*

**Theorem II.2.5.** *Let Assumptions II.2.1 and II.2.4 hold. If $u$ belongs to $L^r(\Omega)$ for some $r > n/2$, then there exists a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ of (1.1). Moreover, there exists a constant $K_{f,r}$ independent of $u$ such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq K_{f,r}\big(\|u\|_{L^r(\Omega)} + \|f(\cdot, 0)\|_{L^r(\Omega)} + 1\big). \tag{2.8}$$

*Further, if $\{u_k\}_{k=1}^{\infty}$ is a sequence converging weakly to $u$ in $L^r(\Omega)$, then $y_{u_k} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$.*

The reader is referred to [6] for the proof of this result. As a consequence of (2.8) we get

$$\exists K_U > 0 \text{ such that } \|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq K_U \quad \forall u \in \mathcal{U}. \tag{2.9}$$

For each $r > n/2$, we define the map $G_r : L^r(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$ by $G_r(u) = y_u$.

**Theorem II.2.6.** *Let Assumptions II.2.1 and II.2.4 hold. For every $r > \frac{n}{2}$ the map $G_r$ is of class $C^2$, and the first and second derivatives at $u \in L^r(\Omega)$ in the directions $v, v_1, v_2 \in L^r(\Omega)$, denoted by $z_{u,v} = G_r'(u)v$ and $z_{u,v_1,v_2} = G_r''(u)(v_1, v_2)$, are the solutions of the equations*

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = v, \tag{2.10}$$

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = -\frac{\partial^2 f}{\partial y^2}(x, y_u) z_{u,v_1} z_{u,v_2}, \tag{2.11}$$

*respectively.*

The proof of this theorem is an easy application of the implicit function theorem; see [6].

**Lemma II.2.7.** *The following statements are fulfilled.*

*(i) Suppose that $r > \frac{n}{2}$ and $s \in [1, \frac{n}{n-2})$. Then, there exist constants $K_r$ depending on $r$ and $M_s$ depending on $s$ such that for every $u, \bar{u} \in \mathcal{U}$*

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{C(\bar{\Omega})} \leq K_r \|y_u - y_{\bar{u}}\|^2_{L^{2r}(\Omega)}, \tag{2.12}$$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(\Omega)} \leq M_s \|y_u - y_{\bar{u}}\|^2_{L^2(\Omega)}. \tag{2.13}$$

*(ii) Taking $C_X = K_2\sqrt{|\Omega|}$ if $X = C(\bar\Omega)$ and $C_X = M_2$ if $X = L^2(\Omega)$, the following inequality holds*

$$\|z_{u,v} - z_{\bar u,v}\|_X \le C_X \|y_u - y_{\bar u}\|_X \|z_{\bar u,v}\|_X \ \forall u, \bar u \in \mathcal{U} \text{ and } \forall v \in L^2(\Omega). \tag{2.14}$$

*(iii) Let $X$ be as in (ii). There exists $\varepsilon > 0$ such that for all $\bar u, u \in \mathcal{U}$ with $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le \varepsilon$ the following inequalities are satisfied*

$$\frac{1}{2}\|y_u - y_{\bar u}\|_X \le \|z_{\bar u, u-\bar u}\|_X \le \frac{3}{2}\|y_u - y_{\bar u}\|_X, \tag{2.15}$$

$$\frac{1}{2}\|z_{\bar u,v}\|_X \le \|z_{u,v}\|_X \le \frac{3}{2}\|z_{\bar u,v}\|_X \quad \forall v \in L^2(\Omega). \tag{2.16}$$

*Proof.* Let us set $\phi = y_u - y_{\bar u} - z_{\bar u, u-\bar u} \in H_0^1(\Omega) \cap C(\bar\Omega)$. From the equations satisfied by the three functions and using the mean value theorem we get

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_{\bar u})\phi = \Big[\frac{\partial f}{\partial y}(x, y_{\bar u}) - \frac{\partial f}{\partial y}(x, y_\theta)\Big](y_u - y_{\bar u}),$$

where $y_\theta(x) = y_{\bar u}(x) + \theta(x)(y_u(x) - y_{\bar u}(x))$ with $\theta : \Omega \longrightarrow [0,1]$ measurable. Using again the mean value theorem we deduce

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_{\bar u})\phi = -\theta \frac{\partial^2 f}{\partial y^2}(x, y_\vartheta)(y_u - y_{\bar u})^2$$

with $y_\vartheta(x) = y_{\bar u}(x) + \vartheta(x)(y_\theta(x) - y_{\bar u}(x))$ and $\vartheta : \Omega \longrightarrow [0,1]$ measurable. By Lemma II.2.2-($iii$) and taking into account (2.6) and (2.9) we infer the existence of $C_r$ independent of $u, \bar u \in \mathcal{U}$ such that

$$\|\phi\|_{C(\bar\Omega)} \le C_r C_{f,K_U} \|(y_u - y_{\bar u})^2\|_{L^r(\Omega)} = C_r C_{f,K_U} \|y_u - y_{\bar u}\|^2_{L^{2r}(\Omega)},$$

which proves (2.12) with $K_r = C_r C_{f,K_U}$. To prove (2.13) we use Lemma II.2.3 to obtain

$$\|\phi\|_{L^s(\Omega)} \le C_{s'} C_{f,K_U} \|(y_u - y_{\bar u})^2\|_{L^1(\Omega)} = C_{s'} C_{f,K_U} \|y_u - y_{\bar u}\|^2_{L^2(\Omega)}.$$

Taking $M_s = C_{s'} C_{f,K_U}$, (2.13) follows.

Now we prove (2.14) for $X = C(\bar\Omega)$. Setting $\psi = z_{u,v} - z_{\bar u,v}$ and subtracting the corresponding equations we infer with the mean value theorem

$$\mathcal{A}\psi + \frac{\partial f}{\partial y}(x, y_u)\psi = \Big[\frac{\partial f}{\partial y}(x, y_{\bar u}) - \frac{\partial f}{\partial y}(x, y_u)\Big] z_{\bar u,v} = \frac{\partial^2 f}{\partial y^2}(x, y_\theta)(y_{\bar u} - y_u) z_{\bar u,v}.$$

Taking $r = 2$ in (2.3) and using (2.6) and (2.9) it follows from the above equation

$$\|\psi\|_{C(\bar\Omega)} \le C_2 C_{f,K_U} \|(y_{\bar u} - y_u) z_{\bar u,v}\|_{L^2(\Omega)} \le K_2 \sqrt{|\Omega|}\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \|z_{\bar u,v}\|_{C(\bar\Omega)},$$

which proves (2.14) for $X = C(\bar\Omega)$. The proof for $X = L^2(\Omega)$ is analogous, we use the estimate (2.4) for $s = 2$ instead of (2.3).

To prove (2.15) for $X = C(\bar\Omega)$ we use (2.12) with $r = 2$ to get

$$\begin{aligned}\|y_u - y_{\bar u}\|_{C(\bar\Omega)} &\le \|\phi\|_{C(\bar\Omega)} + \|z_{\bar u,u-\bar u}\|_{C(\bar\Omega)} \le K_2\|y_u - y_{\bar u}\|^2_{L^4(\Omega)} + \|z_{\bar u,u-\bar u}\|_{C(\bar\Omega)}\\ &\le K_2\sqrt{|\Omega|}\|y_u - y_{\bar u}\|^2_{C(\bar\Omega)} + \|z_{\bar u,u-\bar u}\|_{C(\bar\Omega)}.\end{aligned}$$

Choosing $\varepsilon_1 = [2K_2\sqrt{|\Omega|}]^{-1}$ the first inequality of (2.15) follows if $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon_1$. To deal with the case $X = L^2(\Omega)$ we use (2.13) with $s = 2$ and obtain

$$\begin{aligned}\|y_u - y_{\bar u}\|_{L^2(\Omega)} &\le \|\phi\|_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)} \le M_2\|y_u - y_{\bar u}\|^2_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\ &\le M_2\sqrt{|\Omega|}\|y_u - y_{\bar u}\|_{C(\bar\Omega)}\|y_u - y_{\bar u}\|_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}.\end{aligned}$$

Hence, taking $\varepsilon_2 = [2M_2\sqrt{|\Omega|}]^{-1}$ we obtain the first inequality of (2.15) with $X = L^2(\Omega)$ if $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon_2$.

To prove the second inequality of (2.15) for $X = C(\bar\Omega)$, we proceed as follows

$$\begin{aligned}\|z_{\bar u,u-\bar u}\|_{C(\bar\Omega)} &\le \|\phi\|_{C(\bar\Omega)} + \|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le K_2\sqrt{|\Omega|}\|y_u - y_{\bar u}\|^2_{C(\bar\Omega)} + \|y_u - y_{\bar u}\|_{C(\bar\Omega)}\\ &\le \frac{3}{2}\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \quad \text{if } \|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon_1.\end{aligned}$$

Similarly the second inequality of (2.15) follows if $X = L^2(\Omega)$ with $\varepsilon_2$ replacing $\varepsilon_1$.

Finally, we prove (2.16). Using (2.14) we obtain

$$\begin{aligned}\|z_{u,v}\|_X &\le \|z_{u,v} - z_{\bar u,v}\|_X + \|z_{\bar u,v}\|_X \le C_X\|y_u - y_{\bar u}\|_X\|z_{\bar u,v}\|_X + \|z_{\bar u,v}\|_X,\\ \|z_{\bar u,v}\|_X &\le \|z_{u,v} - z_{\bar u,v}\|_X + \|z_{u,v}\|_X \le C_X\|y_u - y_{\bar u}\|_X\|z_{\bar u,v}\|_X + \|z_{u,v}\|_X.\end{aligned}$$

Therefore, selecting $\varepsilon = [2C_2]^{-1}$ for $X = C(\bar\Omega)$ and $\varepsilon = [2C_2\sqrt{|\Omega|}]^{-1}$ for $X = L^2(\Omega)$, (2.16) follows if $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le \varepsilon$. □

## II.3 The Control Problem

In this section, we make assumptions on the objective functional $J$ so that (P) has at least one solution and the first and second-order conditions for local optimality can be established. Since the problem is not convex, we will consider not only global minimizers, but also local minimizers. Throughout this paper, we will say that $\bar u$ is a local minimizer of (P) if $\bar u \in \mathcal{U}$ and there exists a ball $B_\rho(\bar u) \subset L^2(\Omega)$ such that $J(\bar u) \le J(u)$ for every $u \in \mathcal{U} \cap B_\rho(\bar u)$. We will also say that $\bar u$ is a strong local minimizer of (P) if $\bar u \in \mathcal{U}$ and there exists $\varepsilon > 0$ such that $J(\bar u) \le J(u)$ for every $u \in \mathcal{U}$ with $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon$. If the previous inequalities are strict whenever $u \neq \bar u$, then we say that $\bar u$ is a strict (strong) local minimizer. As far as we know, the notion of strong local minimizers in the framework of control of partial differential equations was introduced for the first time in [1]; see also [2].

We make the following assumptions on $L$.

**Assumption II.3.1.** *The function $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Carathéodory and of class $C^2$ with respect to the second variable. In addition, we assume that*

$$L(x, y, u) = L_0(x, y) + g(x)u \quad \text{with} \quad L_0(\cdot, 0) \in L^1(\Omega) \quad \text{and} \quad g \in L^\infty(\Omega), \tag{3.17}$$

$$\begin{cases} \forall M > 0 \ \exists \psi_M \in L^2(\Omega) \text{ and } C_{L,M} > 0 \text{ such that} \\ \left|\frac{\partial L}{\partial y}(x, y, u)\right| \leq \psi_M(x) \quad \text{and} \quad \left|\frac{\partial^2 L}{\partial y^2}(x, y, u)\right| \leq C_{L,M} \ \forall |y| \leq M, \end{cases} \tag{3.18}$$

$$\begin{cases} \forall M > 0 \text{ and } \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \left|\frac{\partial^2 L}{\partial y^2}(x, y_2, u) - \frac{\partial^2 L}{\partial y^2}(x, y_1, u)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M, \ |y_2 - y_1| \leq \delta, \end{cases} \tag{3.19}$$

*for almost every $x \in \Omega$.*

Using Theorem II.2.5, the assumptions on $L$, and the boundedness of $\mathcal{U}$ in $L^\infty(\Omega)$, the existence of at least one solution of (P) follows. Indeed, if we take a minimizing sequence $\{u_k\}_{k=1}^\infty$, we can assume that $u_k \stackrel{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$. Then Theorem II.2.5 implies that $y_{u_k} \to y_{\bar{u}}$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$. Further, using (2.9) and (3.18) with $M = K_U$ we infer with the mean value theorem

$$|L_0(x, y_{u_k}(x))| \leq |L_0(x, 0)| + \psi_{K_U}(x) K_U.$$

Then we can apply Lebesgue's dominated convergence theorem to pass to the limit in the objective functional and to obtain $J(u_k) \to J(\bar{u})$.

In order to derive the first-order optimality conditions satisfied by a local minimizer we address the issue of the differentiability of the objective functional $J$.

**Theorem II.3.2.** *Suppose that $r > \frac{n}{2}$. Then, the functional $J : L^r(\Omega) \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^r(\Omega)$ we have*

$$J'(u)v = \int_\Omega (\varphi_u + g) v \, \mathrm{d}x, \tag{3.20}$$

$$J''(u)(v_1, v_2) = \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\Big] z_{u,v_1} z_{u,v_2} \, \mathrm{d}x, \tag{3.21}$$

*where $\varphi_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation*

$$\begin{cases} \mathcal{A}^* \varphi + \frac{\partial f}{\partial y}(x, y_u)\varphi = \frac{\partial L}{\partial y}(x, y_u, u) \text{ in } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases} \tag{3.22}$$

This is a straightforward consequence of Theorem II.2.6, Assumption II.3.1, and the chain rule. The only critical issue is the existence, uniqueness, and regularity of $\varphi_u$. But this is an immediate consequence of Lemma II.2.2-(iii) that, as already mentioned, applies to the operator $\mathcal{A}^*$ as well. From this theorem, the optimality conditions follow in the classical way.

**Theorem II.3.3.** *Let $\bar{u}$ be a (strong or not) local minimizer of* (P), *then there exist two unique elements $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{cases} \mathcal{A}\bar{y} + f(x, \bar{y}) = \bar{u} \ \text{in } \Omega, \\ \bar{y} = 0 \ \text{on } \Gamma, \end{cases} \tag{3.23}$$

$$\begin{cases} \mathcal{A}^*\bar{\varphi} + \frac{\partial f}{\partial y}(x, \bar{y})\bar{\varphi} = \frac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) \ \text{in } \Omega, \\ \bar{\varphi} = 0 \ \text{on } \Gamma, \end{cases} \tag{3.24}$$

$$\int_\Omega (\bar{\varphi} + g)(u - \bar{u})\, dx \geq 0 \quad \forall u \in \mathcal{U}. \tag{3.25}$$

The derivation of sufficient second-order conditions for local optimality is more delicate. First, we introduce the cone of critical directions on which we formulate the necessary second-order conditions for optimality: if $\bar{u} \in \mathcal{U}$ is a local minimizer of (P) we define

$$C_{\bar{u}} = \{v \in L^2(\Omega) : J'(\bar{u})v = 0 \ \text{ and } v \text{ satisfies the sign conditions (3.26)}\},$$

$$v(x) \begin{cases} \geq 0 & \text{if } \bar{u}(x) = u_a, \\ \leq 0 & \text{if } \bar{u}(x) = u_b. \end{cases} \tag{3.26}$$

As usual, from (3.25) we deduce that $(\bar{\varphi} + g)(x)v(x) \geq 0$ for almost all $x \in \Omega$ if $v \in L^2(\Omega)$ satisfies (3.26). Therefore, the condition $J'(\bar{u})v = 0$ for $v$ satisfying (3.26) is only possible if $v(x) = 0$ for almost every $x \in \Omega$ such that $(\bar{\varphi} + g)(x) \neq 0$. Therefore, $C_{\bar{u}}$ can be written

$$C_{\bar{u}} = \{v \in L^2(\Omega) : \text{satisfying (3.26) and } v(x) = 0 \text{ if } |(\bar{\varphi} + g)(x)| > 0\}.$$

It is well known that every local minimizer $\bar{u}$ satisfies the second-order necessary optimality condition: $J''(\bar{u})v^2 \geq 0$ for all $v \in C_{\bar{u}}$; see, for instance, [8]. However, based on $C_{\bar{u}}$ it is not possible to get sufficient second-order conditions for local optimality. The reader is referred to [12] for a counterexample. A procedure suggested by several authors consists in extending the cone of critical directions $C_{\bar{u}}$; see [10, 11, 18, 19]. Two possible extensions of $C_{\bar{u}}$ seem natural after the above comments: for $\tau > 0$ we define the extended cones

$$D_{\bar{u}}^\tau = \{v \in L^2(\Omega) : \text{satisfying (3.26) and } v(x) = 0 \text{ if } |(\bar{\varphi} + g)(x)| > \tau\},$$
$$G_{\bar{u}}^\tau = \{v \in L^2(\Omega) : \text{satisfying (3.26) and } J'(\bar{u})v \leq \tau \|z_v\|_{L^1(\Omega)}\}.$$

On any of these cones, we can formulate sufficient second-order conditions for local optimality. Obviously, both are extensions of $C_{\bar{u}}$. In [3], the authors introduced the cone $C_{\bar{u}}^\tau = D_{\bar{u}}^\tau \cap G_{\bar{u}}^\tau$, which is also an extension of $C_{\bar{u}}$. They proved that the first order optimality conditions (3.23)–(3.25) along with the condition

$$\exists \delta > 0 \text{ such that } J''(\bar{u})v^2 \geq \delta \|z_v\|_{L^2(\Omega)}^2 \quad \forall v \in C_{\bar{u}}^\tau \tag{3.27}$$

imply the existence of $\kappa > 0$ and $\varepsilon > 0$ such that

$$J(\bar{u}) + \frac{\kappa}{2}\|y_u - \bar{y}\|_{L^2(\Omega)}^2 \leq J(u) \ \ \forall u \in \mathcal{U} \text{ such that } \|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon. \tag{3.28}$$

Actually, the proof of [3] was carried out for a parabolic control problem with $g = 0$. However, the same proof works for the elliptic case and $g \neq 0$. Here, we formulate a new assumption leading to the same result (3.28) as (3.27) does.

**Assumption II.3.4.** *There exist numbers $\alpha > 0$ and $\gamma > 0$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)} \; \forall u \in \mathcal{U} \text{ with } \|y_u - \bar{y}\|_{C(\bar{\Omega})} < \alpha. \tag{3.29}$$

It was proved in [4] that (3.27) implies (3.29). Therefore, (3.29) appears as a weaker assumption. However, the next theorem proves that it is sufficient to imply (3.28).

**Theorem II.3.5.** *Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (3.23)–(3.25) *and Assumption II.3.4. Then, there exist $\varepsilon > 0$ and $\kappa > 0$ such that* (3.28) *holds.*

Before proving this theorem we establish some lemmas.

**Lemma II.3.6.** *Let $\bar{u} \in \mathcal{U}$ be fixed with associated state $\bar{y}$. Then, the following inequality holds for all $\theta \in [0,1]$ and $u \in \mathcal{U}$*

$$\|y_{\bar{u}+\theta(u-\bar{u})} - \bar{y}\|_{C(\bar{\Omega})} \leq (C_2 C_{f,K_U} \sqrt{|\Omega|} \|y_u - \bar{y}\|_{C(\bar{\Omega})} + 1) \|y_u - \bar{y}\|_{C(\bar{\Omega})}, \tag{3.30}$$

*where $C_2$ is the constant of* (2.3) *with $r = 2$ and $C_{f,K_U}$ is the one deduced from* (2.6) *and* (2.9).

*Proof.* The proof of this lemma is based on the analogous result for parabolic control problems established in [5]. We take $\theta \in [0,1]$ and $u \in \mathcal{U}$. We set $\phi = y_{\bar{u}+\theta(u-\bar{u})} - [\bar{y} + \theta(y_u - \bar{y})]$. Then, we have

$$\mathcal{A}\phi + f(x, y_{\bar{u}+\theta(u-\bar{u})}) - [f(x,\bar{y}) + \theta(f(x,y_u) - f(x,\bar{y}))] = 0.$$

Applying the mean value theorem, we obtain measurable functions $\theta_i : \Omega \longrightarrow [0,1]$, $i = 1,2$, such that

$$f(x, y_{\bar{u}+\theta(u-\bar{u})}) - f(x,\bar{y}) = \frac{\partial f}{\partial y}(x, y_1)(y_{\bar{u}+\theta(u-\bar{u})} - \bar{y}) \text{ and } y_1 = \bar{y} + \theta_1(y_{\bar{u}+\theta(u-\bar{u})} - \bar{y}),$$
$$f(x, y_u) - f(x,\bar{y}) = \frac{\partial f}{\partial y}(x, y_2)(y_u - \bar{y}) \text{ with } y_2 = \bar{y} + \theta_2(y_u - \bar{y}).$$

Inserting these identities in the above partial differential equation we infer

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_1)(y_{\bar{u}+\theta(u-\bar{u})} - \bar{y}) - \theta \frac{\partial f}{\partial y}(x, y_2)(y_u - \bar{y}) = 0.$$

Noting that $y_{\bar{u}+\theta(u-\bar{u})} - \bar{y} = \phi + \theta(y_u - \bar{y})$, the above equality and a new application of the mean value theorem lead to

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_1)\phi = \theta\Big[\frac{\partial f}{\partial y}(x, y_2) - \frac{\partial f}{\partial y}(x, y_1)\Big](y_u - \bar{y}) = \theta \frac{\partial^2 f}{\partial y^2}(x, y_3)(y_u - \bar{y})^2,$$

where $y_3 = y_1 + \theta_3(y_2 - y_1)$. Using (2.3) with $r = 2$, (2.6), and (2.9) we infer

$$\|\phi\|_{C(\bar\Omega)} \le C_2 C_{f,K_U} \|(y_u - \bar y)^2\|_{L^2(\Omega)} \le C_2 C_{f,K_U} \sqrt{|\Omega|} \|y_u - \bar y\|^2_{C(\bar\Omega)}.$$

This implies

$$\|y_{\bar u+\theta(u-\bar u)} - \bar y\|_{C(\bar\Omega)} = \|\phi + \theta(y_u - \bar y)\|_{C(\bar\Omega)} \le (C_2 C_{f,K_U} \sqrt{|\Omega|} \|y_u - \bar y\|_{C(\bar\Omega)} + 1)\|y_u - \bar y\|_{C(\bar\Omega)}.$$

□

**Lemma II.3.7.** *There exists a constant $M_U > 0$ such that*

$$\|\varphi_u\|_{C(\bar\Omega)} \le M_U \quad \forall u \in \mathcal{U}. \tag{3.31}$$

*Moreover, given $\bar u \in \mathcal{U}$ with associated state $\bar y$ and adjoint state $\bar\varphi$, we have*

$$\|\varphi_{\bar u+\theta(u-\bar u)} - \bar\varphi\|_{C(\bar\Omega)} \le C\|y_u - \bar y\|_{C(\bar\Omega)} \quad \forall\theta \in [0,1] \text{ and } \forall u \in \mathcal{U}, \tag{3.32}$$

*where $C$ depends only on $f$, $L$, $\mathcal{U}$, and $\Omega$.*

*Proof.* For the proof of (3.31) we use (2.3) with $r = 2$, (2.9), and (3.18) as follows

$$\|\varphi_u\|_{C(\bar\Omega)} \le C_2 \Big\|\frac{\partial L}{\partial y}(x, y_u, u)\Big\|_{L^2(\Omega)} \le M_U = C_2\|\psi_{K_U}\|_{L^2(\Omega)}.$$

Let us prove (3.32). Given $u \in \mathcal{U}$ and $\theta \in [0,1]$ let us denote $u_\theta = \bar u + \theta(u - \bar u)$, $y_\theta = y_{u_\theta}$, and $\varphi_\theta = \varphi_{u_\theta}$. Subtracting the equations satisfied by $\varphi_\theta$ and $\bar\varphi$ we get with the mean value theorem

$$\begin{aligned}\mathcal{A}^*(\varphi_\theta - \bar\varphi) + \frac{\partial f}{\partial y}(x,\bar y)(\varphi_\theta - \bar\varphi) &= \frac{\partial L}{\partial y}(x, y_\theta, u_\theta) - \frac{\partial L}{\partial y}(x, \bar y, \bar u)\\ + \Big[\frac{\partial f}{\partial y}(x,\bar y) - \frac{\partial f}{\partial y}(x, y_\theta)\Big]\varphi_\theta &= \Big[\frac{\partial^2 L}{\partial y^2}(x, y_\vartheta, u_\vartheta) - \varphi_\theta \frac{\partial^2 f}{\partial y^2}(x, y_\vartheta)\Big](y_\theta - \bar y),\end{aligned}$$

where $y_\vartheta = \bar y + \vartheta(y_\theta - \bar y)$ for some measurable function $\vartheta : \Omega \longrightarrow [0,1]$. Now, we apply (2.3) with $r = 2$, (2.9), (3.31), (2.6), and (3.18) to get from the above equation

$$\|\varphi_\theta - \bar\varphi\|_{C(\bar\Omega)} \le C_2(C_{L,K_U} + M_U C_{f,K_U})\sqrt{|\Omega|}\|y_\theta - \bar y\|_{C(\bar\Omega)}.$$

Then, (3.32) follows from Lemma II.3.6. □

**Lemma II.3.8.** *For every $\rho > 0$ there exists $\varepsilon > 0$ such that if $u \in \mathcal{U}$ and $\|y_u - \bar y\|_{C(\bar\Omega)} < \varepsilon$ then*

$$|[J''(\bar u + \theta(u - \bar u)) - J''(\bar u)]v^2| < \rho\|z_{\bar u,v}\|^2_{L^2(\Omega)} \quad \forall v \in L^2(\Omega) \text{ and } \forall\theta \in [0,1]. \tag{3.33}$$

*Proof.* First, let us denote $u_\theta$, $y_\theta$, and $\varphi_\theta$ as in the proof of Lemma II.3.7. From (3.21) we get

$$\begin{aligned}
|[J''(\bar u+\theta(u-\bar u))-J''(\bar u)]v^2| &\le \int_\Omega \Big|\Big[\frac{\partial^2 L}{\partial y^2}(x,y_\theta,u_\theta)-\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u)\Big]z^2_{u_\theta,v}\Big|\,\mathrm{d}x\\
&+\int_\Omega\Big|(\varphi_\theta-\bar\varphi)\frac{\partial^2 f}{\partial y^2}(x,y_\theta)z^2_{u_\theta,v}\Big|\,\mathrm{d}x+\int_\Omega\Big|\bar\varphi\Big[\frac{\partial^2 f}{\partial y^2}(x,y_\theta)-\frac{\partial^2 f}{\partial y^2}(x,\bar y)\Big]z^2_{u_\theta,v}\Big|\,\mathrm{d}x\\
&+\int_\Omega\Big|\Big[\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u)-\bar\varphi\frac{\partial^2 f}{\partial y^2}(x,\bar y)\Big](z^2_{u_\theta,v}-z^2_{\bar u,v})\Big|\,\mathrm{d}x = I_1+I_2+I_3+I_4.
\end{aligned}$$

Let us estimate the terms $I_i$. For $I_1$ we deduce from (3.19), (2.16), and (3.30) that for every $\rho>0$ there exists $\varepsilon>0$ such that $I_1\le\rho\|z_{\bar u,v}\|^2_{L^2(\Omega)}$ if $\|y_u-\bar y\|_{C(\bar\Omega)}<\varepsilon$. The same estimate can be deduced for $I_2$ using (2.6), (2.9), (2.16), and (3.32). The estimate for $I_3$ follows from (2.7), (2.9), (2.16), (3.30), and (3.31). Finally, we estimate $I_4$ by using (2.6), (2.9), (2.14), (2.16), (3.18), (3.30), and (3.31) to infer that

$$\begin{aligned}
I_4 &\le (C_{L,K_U}+M_UC_{f,K_U})\|z_{u_\theta,v}+z_{\bar u,v}\|_{L^2(\Omega)}\|z_{u_\theta,v}-z_{\bar u,v}\|_{L^2(\Omega)}\\
&\le \frac{5}{2}(C_{L,K_U}+M_UC_{f,K_U})C_{L^2(\Omega)}|\Omega|^{\frac12}\|z_{\bar u,v}\|_{L^2(\Omega)}\|y_\theta-\bar y\|_{C(\bar\Omega)}\|z_{\bar u,v}\|_{L^2(\Omega)}\\
&\le \rho\|z_{\bar u,v}\|^2_{L^2(\Omega)}\quad\text{if } \|y_u-\bar y\|_{C(\bar\Omega)}<\varepsilon.
\end{aligned}$$

Hence, (3.33) is a straightforward consequence of the above estimates. □

*Proof of Theorem II.3.5.* Let us take $u\in\mathcal{U}$ with $\|y_u-\bar y\|_{C(\bar\Omega)}<\alpha$. By performing a Taylor expansion and using that $J'(\bar u)(u-\bar u)\ge 0$ we obtain

$$\begin{aligned}
J(u) &= J(\bar u)+J'(\bar u)(u-\bar u)+\frac12 J''(u_\theta)(u-\bar u)^2\\
&\ge J(\bar u)+\frac12[J'(\bar u)(u-\bar u)+J''(\bar u)(u-\bar u)^2]+\frac12[J''(u_\theta)-J''(\bar u)](u-\bar u)^2\\
&\ge J(\bar u)+\frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}-\frac12|[J''(u_\theta)-J''(\bar u)](u-\bar u)^2|.
\end{aligned}$$

Lemma II.3.8 implies the existence of $\varepsilon\in(0,\alpha]$ such that $|[J''(u_\theta)-J''(\bar u)](u-\bar u)^2|<\frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}$ for every $u\in\mathcal{U}$ with $\|y_u-\bar y\|_{C(\bar\Omega)}<\varepsilon$. Inserting this estimate in the above expression and taking $\varepsilon$ still smaller if necessary, we can apply (2.15) to deduce

$$J(u)\ge J(\bar u)+\frac{\gamma}{4}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}\ge J(\bar u)+\frac{\gamma}{16}\|y_u-\bar y\|^2_{L^2(\Omega)}.$$

This inequality yields (3.28) with $\kappa=\frac{\gamma}{8}$. □

## II.4 Stability of the states

In this section, we consider the following perturbations of the control problem (P)

$$(\mathrm{P}_\varepsilon)\ \min_{u\in\mathcal{U}} J_\varepsilon(u) := \int_\Omega [L(x,y^\varepsilon_u(x),u(x))+\eta_\varepsilon(x)y^\varepsilon_u(x)]\,\mathrm{d}x,$$

where $y_u^\varepsilon$ is the solution of the equation

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x,y) = u + \xi_\varepsilon & \text{in } \Omega,\\ y = 0 \text{ on } \Gamma. \end{cases} \tag{4.34}$$

Here we assume that $\{\xi_\varepsilon\}_{\varepsilon>0}$ and $\{\eta_\varepsilon\}_{\varepsilon>0}$ are bounded families in $L^2(\Omega)$ satisfying that $(\xi_\varepsilon, \eta_\varepsilon) \to (0,0)$ in $L^2(\Omega)^2$ as $\varepsilon \to 0$. As a consequence of Theorem II.2.5 we get the existence and uniqueness of a solution $y_u^\varepsilon \in H_0^1(\Omega) \cap C(\bar\Omega)$ of (4.34). Moreover, using (2.8) with $r = 2$ and the boundedness of $\{\xi_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ we infer that the set $\{y_u^\varepsilon : u \in \mathcal{U} \text{ and } \varepsilon > 0\}$ is bounded in $H_0^1(\Omega)\cap C(\bar\Omega)$. Therefore, increasing the value of $K_U$, if necessary, we can assume that (2.9) and the inequality

$$\|y_u^\varepsilon\|_{H_0^1(\Omega)} + \|y_u^\varepsilon\|_{C(\bar\Omega)} \le K_U \quad \forall u \in \mathcal{U} \text{ and } \forall \varepsilon > 0 \tag{4.35}$$

hold. We will prove that the solutions of problems (P$_\varepsilon$) converge to the solutions of (P) in some sense to be made precise below. Conversely, we will also prove that any strict strong local minimizer of (P) can be approximated by strong local minimizers of problems (P$_\varepsilon$). Finally, the Lipschitz stability of the optimal states with respect to the perturbations is established. We start analyzing the difference between the solutions of (1.1) and (4.34).

**Theorem II.4.1.** *The following inequalities hold for every* $\varepsilon > 0$

$$\|y_u^\varepsilon - y_u\|_{H_0^1(\Omega)} + \|y_u^\varepsilon - y_u\|_{C(\bar\Omega)} \le C_2\|\xi_\varepsilon\|_{L^2(\Omega)} \ \forall u \in L^2(\Omega), \tag{4.36}$$

$$\|z_{u,v}^\varepsilon - z_{u,v}\|_{L^2(\Omega)} \le C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)} \ \forall (u,v) \in \mathcal{U}\times L^2(\Omega), \tag{4.37}$$

*where* $C_2$ *is the constant given in* (2.3) *for* $r = 2$, $C_{f,K_U}$ *is the constant* $C_{f,M}$ *of* (2.6) *with* $M = K_U$ *given in* (2.9) *or* (4.35), *and* $z_{u,v}^\varepsilon$ *denotes the solution of* (2.10) *with* $y_u^\varepsilon$ *replacing* $y_u$.

*Proof.* Subtracting the equations (4.34) and (1.1) and using the mean value theorem we obtain

$$\mathcal{A}(y_u^\varepsilon - y_u) + \frac{\partial f}{\partial y}(x, y_\theta)(y_u^\varepsilon - y_u) = \xi_\varepsilon.$$

Then, (2.3) implies (4.36). To prove (4.37) we subtract the equations satisfied by $z_{u,v}^\varepsilon$ and $z_{u,v}$ to obtain

$$\mathcal{A}(z_{u,v}^\varepsilon - z_{u,v}) + \frac{\partial f}{\partial y}(x, y_u^\varepsilon)(z_{u,v}^\varepsilon - z_{u,v}) = \Big[\frac{\partial f}{\partial y}(x, y_u) - \frac{\partial f}{\partial y}(x, y_u^\varepsilon)\Big] z_{u,v}.$$

Now, using (2.4) with $s = 2$, (2.6), (2.9), and (4.36) we obtain from the previous equation with the mean value theorem

$$\begin{aligned} \|z_{u,v}^\varepsilon - z_{u,v}\|_{L^2(\Omega)} &\le C_2\Big\|\Big[\frac{\partial f}{\partial y}(x, y_u) - \frac{\partial f}{\partial y}(x, y_u^\varepsilon)\Big] z_{u,v}\Big\|_{L^1(\Omega)} \le C_2 C_{f,K_U}\|(y_u^\varepsilon - y_u) z_{u,v}\|_{L^1(\Omega)}\\ &\le C_2 C_{f,K_U}\|y_u^\varepsilon - y_u\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)} \le C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)}. \end{aligned}$$

□

Now we analyze the convergence of problems (P$_\varepsilon$) to (P).

**Theorem II.4.2.** *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of problems* $(\mathrm{P}_\varepsilon)$. *Any control $\bar{u}$ that is a weak* limit in $L^\infty(\Omega)$ of a sequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ with $\varepsilon_k \to 0$ as $k \to \infty$ is a solution of* (P). *Moreover, the strong convergence $y^{\varepsilon_k}_{u_{\varepsilon_k}} \to y_{\bar{u}}$ in $H_0^1(\Omega) \cap C(\bar{\Omega})$ holds.*

*Proof.* The existence of the sequences $\{u_{\varepsilon_k}\}_{k=1}^\infty$ converging to $\bar{u}$ weakly* in $L^\infty(\Omega)$ is a consequence of the boundedness of $\mathcal{U}$ in $L^\infty(\Omega)$. From Theorem II.2.5 and (4.36) we infer

$$\begin{aligned}
&\|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{H_0^1(\Omega)} + \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{C(\bar{\Omega})} \\
&\le \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{u_{\varepsilon_k}}\|_{H_0^1(\Omega)} + \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{u_{\varepsilon_k}}\|_{C(\bar{\Omega})} + \|y_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{H_0^1(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{C(\bar{\Omega})} \\
&\le C_2\|\xi_\varepsilon\|_{L^2(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{H_0^1(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar{u}}\|_{C(\bar{\Omega})} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Using this fact, the convergence $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, (3.18), the optimality of $u_{\varepsilon_k}$ for $(\mathrm{P}_{\varepsilon,k})$, and again (4.36), we get

$$J(\bar{u}) = \lim_{k\to\infty} J_{\varepsilon_k}(u_{\varepsilon_k}) \le \lim_{k\to\infty} J_{\varepsilon_k}(u) = J(u) \quad \forall u \in \mathcal{U},$$

which proves that $\bar{u}$ is a solution of (P). □

Now, we establish a kind of converse result.

**Theorem II.4.3.** *Let $\bar{u}$ be a strict strong local minimizer of* (P). *Then, there exist $\varepsilon_0 > 0$ and a family of strong local minimizers $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of problems $(\mathrm{P}_\varepsilon)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$ and $y^\varepsilon_{u_\varepsilon} \to y_{\bar{u}}$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$ as $\varepsilon \to 0$.*

*Proof.* Since $\bar{u}$ is a strict strong local minimizer of (P), there exists $\rho > 0$ such that $\bar{u}$ is the unique solution of the problem

$$(\mathrm{P}_\rho) \ \min_{u\in\mathcal{U}_\rho} J(u),$$

where $\mathcal{U}_\rho = \{u \in \mathcal{U} : \|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} \le \rho\}$. Now, for every $\varepsilon > 0$ we define the problems

$$(\mathrm{P}_{\rho,\varepsilon}) \ \min_{u\in\mathcal{U}_\rho} J_\varepsilon(u).$$

Using Theorem II.2.5 we deduce that $\mathcal{U}_\rho$ is weakly* closed in $L^\infty(\Omega)$, hence the existence of a solution $u_\varepsilon$ of $(\mathrm{P}_{\rho,\varepsilon})$ can be proved as we indicated for (P). Moreover, arguing as in the proof of Theorem II.4.2, we deduce the existence of sequences $\{u_{\varepsilon_k}\}_{k=1}^\infty$ converging weakly* to a solution $u$ of $(\mathrm{P}_\rho)$ in $L^\infty(\Omega)$ and such that $y^{\varepsilon_k}_{u_{\varepsilon_k}} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$. Since $\bar{u}$ is the unique solution of $(\mathrm{P}_\rho)$, we conclude the convergence $u_\varepsilon \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$ and $y^\varepsilon_{u_\varepsilon} \to y_{\bar{u}}$ in $H_0^1(\Omega) \cap C(\bar{\Omega})$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon_0 > 0$ such that $\|y^\varepsilon_{u_\varepsilon} - y_{\bar{u}}\|_{C(\bar{\Omega})} < \rho$ for every $\varepsilon < \varepsilon_0$. This implies that $u_\varepsilon$ is a strong local minimizer of $(\mathrm{P}_\varepsilon)$ for every $\varepsilon < \varepsilon_0$, which completes the proof. □

Now we establish our main theorem of this section.

**Theorem II.4.4.** *Let $\bar{u}$ be a local minimizer of* (P) *satisfying Assumption II.3.4 and* $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ *a family of local solutions of problems* $(\mathrm{P}_\varepsilon)$ *such that* $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar{u}$ *in* $L^\infty(\Omega)$ *as* $\varepsilon \to 0$*. Then, there exist* $\hat{\varepsilon} \in (0, \varepsilon_0)$ *and a constant* $C > 0$ *such that*

$$\|y^\varepsilon_{u_\varepsilon} - \bar{y}\|_{L^2(\Omega)} \le C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \quad \forall \varepsilon < \hat{\varepsilon}, \tag{4.38}$$

*where* $\bar{y} = y_{\bar{u}}$*.*

Let us observe that Assumption II.3.4 implies that $\bar{u}$ satisfies (3.28). Hence, $\bar{u}$ is a strict strong local minimizer of (P) and, consequently, Theorem II.4.3 ensures the existence of a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of strong local minimizers of problems $(\mathrm{P}_\varepsilon)$ satisfying the conditions of the above theorem. Before proving this theorem we establish the following lemma.

**Lemma II.4.5.** *Let $\bar{u}$ satisfy the assumptions of Theorem II.4.4. Then, there exists $\varepsilon > 0$ such that*

$$J'(u)(u - \bar{u}) \ge \frac{\gamma}{2}\|z_{u,u-\bar{u}}\|^2_{L^2(\Omega)} \quad \forall u \in \mathcal{U} \text{ with } \|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon, \tag{4.39}$$

*where $\gamma$ is given in Assumption II.3.4.*

*Proof.* We denote by $H : \Omega \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ the Hamiltonian associated with the control problem (P):

$$H(x, y, \varphi, u) = L(x, y, u) + \varphi[u - f(x, y)].$$

For every $u \in \mathcal{U}$ and $v \in L^2(\Omega)$, we define $\psi_{u,v} \in H^1_0(\Omega) \cap C(\bar{\Omega})$ as the function satisfying

$$\mathcal{A}^*\psi_{u,v} + \frac{\partial f}{\partial y}(x, y_u)\psi_{u,v} = \frac{\partial^2 H}{\partial y^2}(x, y_u, \varphi_u, u) z_{u,v}.$$

We split the proof into two steps.

*Step I.-* Here we prove that for every $\rho > 0$ there exists $\varepsilon > 0$ such that for every $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$ we have

$$\Big|\int_\Omega (\varphi_u - \bar{\varphi} - \psi_{\bar{u},u-\bar{u}})(u - \bar{u})\,\mathrm{d}x\Big| \le \rho\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)}. \tag{4.40}$$

Setting $\pi = \varphi_u - \bar{\varphi} - \psi_{\bar{u},u-\bar{u}}$ and subtracting their respective equations it follows with the mean value theorem

$$\begin{aligned}
\mathcal{A}^*\pi + \frac{\partial f}{\partial y}(x,\bar y)\pi &= \frac{\partial H}{\partial y}(x,y_u,\varphi_u,u) - \frac{\partial H}{\partial y}(x,\bar y,\bar\varphi,\bar u)\\
&\quad - \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u) z_{\bar u,u-\bar u} - \frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar y,\bar\varphi,\bar u)(\varphi_u-\bar\varphi)\\
&= \frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar y) - \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u) z_{\bar u,u-\bar u}\\
&\quad + \Big[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar y,\bar\varphi,\bar u)\Big](\varphi_u-\bar\varphi)\\
&= \frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar y - z_{\bar u,u-\bar u})\\
&\quad + \Big[\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u)\Big] z_{\bar u,u-\bar u}\\
&\quad + \Big[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar y,\bar\varphi,\bar u)\Big](\varphi_u-\bar\varphi).
\end{aligned}$$

This implies

$$\begin{aligned}
\int_\Omega \pi(u-\bar u)\,\mathrm{d}x &= \int_\Omega \pi\Big(\mathcal{A} z_{\bar u,u-\bar u} + \frac{\partial f}{\partial y}(x,\bar y) z_{\bar u,u-\bar u}\Big)\,\mathrm{d}x\\
&= \int_\Omega \Big(\mathcal{A}^*\pi + \frac{\partial f}{\partial y}(x,\bar y)\pi\Big) z_{\bar u,u-\bar u}\,\mathrm{d}x\\
&= \int_\Omega \frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar y - z_{\bar u,u-\bar u}) z_{\bar u,u-\bar u}\,\mathrm{d}x\\
&\quad + \int_\Omega \Big[\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u)\Big] z^2_{\bar u,u-\bar u}\,\mathrm{d}x\\
&\quad + \int_\Omega \Big[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar y,\bar\varphi,\bar u)\Big](\varphi_u-\bar\varphi) z_{\bar u,u-\bar u}\,\mathrm{d}x = I_1 + I_2 + I_3.
\end{aligned}$$

We estimate every term $I_i$. For the first term we use (2.6), (2.9), (2.13) with $s = 2$, (2.15) with $X = L^2(\Omega)$, (3.18), and (3.31) as follows

$$\begin{aligned}
|I_1| &\le (C_{L,K_U} + M_U C_{f,K_U})\|y_u - \bar y - z_{\bar u,u-\bar u}\|_{L^2(\Omega)} \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le (C_{L,K_U} + M_U C_{f,K_U}) M_2 \|y_u - \bar y\|^2_{L^2(\Omega)} \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le 2(C_{L,K_U} + M_U C_{f,K_U}) M_2 \sqrt{|\Omega|}\varepsilon \|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}.
\end{aligned}$$

The second term is estimated with (2.7), (2.9), (3.19), (3.30), (3.31), (3.32), leading to $|I_2| \le \rho\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}$ for $\rho$ arbitrarily small if $\varepsilon$ is taken according to $\rho$. Finally, for the last term we use the same inequalities as for $I_2$, that estimate (3.32) holds true for $L^2(\Omega)$ instead of $C(\bar\Omega)$ and additionally (2.16) with $X = L^2(\Omega)$ to get

$$\begin{aligned}
|I_3| \le \rho\|\varphi_u - \bar\varphi\|_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)} &\le \rho C_2 (C_{L,K_U} + M_U C_{f,K_U})\|y_u - \bar y\|_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le 2\rho C_2 (C_{L,K_U} + M_U C_{f,K_U})\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)},
\end{aligned}$$

where again $\rho$ is arbitrarily small if $\varepsilon$ is chosen according to it. Thus, (4.40) follows from the proved estimates.

*Step II-* Now, we prove (4.39). First, we observe that for every $v \in L^2(\Omega)$

$$\int_\Omega \psi_{\bar u,v} v \,\mathrm{d}x = \int_\Omega \psi_{\bar u,v}\Big(\mathcal{A} z_{\bar u,v} + \frac{\partial f}{\partial y}(x,\bar y) z_{\bar u,v}\Big)\,\mathrm{d}x$$
$$= \int_\Omega \Big(\mathcal{A}^* \psi_{\bar u,v} + \frac{\partial f}{\partial y}(x,\bar y)\psi_{\bar u,v}\Big) z_{\bar u,v}\,\mathrm{d}x = \int_\Omega \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u) z_{\bar u,v}^2\,\mathrm{d}x = J''(\bar u)v^2,$$

where the last inequality follows from (3.21) and the definition of the Hamiltonian. Let $\varepsilon > 0$ be such that (4.40) holds with $\rho = \frac{\gamma}{2}$. Then, using Assumption II.3.4 and (4.40) we get for $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{C(\bar\Omega)} < \varepsilon$

$$J'(u)(u-\bar u) = \int_\Omega (\varphi_u + g)(u - \bar u)\,\mathrm{d}x$$
$$= \int_\Omega (\varphi_u - \bar\varphi - \psi_{\bar u,u-\bar u})(u-\bar u)\,\mathrm{d}x + \int_\Omega (\bar\varphi + g + \psi_{\bar u,u-\bar u})(u-\bar u)\,\mathrm{d}x$$
$$\geq -\frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)} + [J'(\bar u)(u-\bar u) + J''(\bar u)(u-\bar u)^2] \geq \frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}.$$

□

***Remark* II.4.6.** *Let us notice that if $\bar u$ is a local minimizer of* (P) *satisfying Assumption II.3.4, then there exists $\varepsilon > 0$ such that there is no stationary point $\hat u$ of* (P) *different from $\bar u$ such that $\|y_{\hat u} - \bar y\|_{C(\bar\Omega)} < \varepsilon$. We say that $\hat u$ is a stationary point of* (P) *if it satisfies the first-order optimality condition. In particular, if $\hat u$ is a stationary point then $J'(\hat u)(\bar u - \hat u) \geq 0$. This contradicts* (4.39) *if $\|y_{\hat u} - \bar y\|_{C(\bar\Omega)} < \varepsilon$.*

*Proof of Theorem II.4.4.* Using the local optimality of $u_\varepsilon$ we get

$$0 \geq J'_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar u) = J'(u_\varepsilon)(u_\varepsilon - \bar u) + \int_\Omega \Big[\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\Big] z_{u_\varepsilon,u_\varepsilon-\bar u}\,\mathrm{d}x$$
$$+ \int_\Omega \frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon)(z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u} - z_{u_\varepsilon,u_\varepsilon-\bar u})\,\mathrm{d}x + \int_\Omega \eta_\varepsilon z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u}\,\mathrm{d}x. \tag{4.41}$$

We estimate each one of these four terms. First, we observe that the convergence $u_\varepsilon \rightharpoonup \bar u$ in $L^2(\Omega)$ implies that $\|y_{u_\varepsilon} - \bar y\|_{C(\bar\Omega)} \to 0$; see Theorem II.2.5. Hence, from Lemma II.4.5 we deduce the existence of $\varepsilon_1 > 0$ such that

$$J'(u_\varepsilon)(u_\varepsilon - \bar u) \geq \frac{\gamma}{2}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|^2_{L^2(\Omega)} \quad \forall \varepsilon < \varepsilon_1. \tag{4.42}$$

For the second term, we use Schwarz's inequality, the mean value theorem, (2.9) and (4.35), (3.18), and (4.36)

$$\int_\Omega \Big|\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\Big| |z_{u_\varepsilon,u_\varepsilon-\bar u}|\,\mathrm{d}x \leq C_{L,K_U}\|y^\varepsilon_{u_\varepsilon} - y_{u_\varepsilon}\|_{L^2(\Omega)}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}$$
$$\leq C_{L,K_U}\sqrt{|\Omega|}C_2\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}. \tag{4.43}$$

Now we estimate the third term with (3.18) and (4.35), Schwarz's inequality, and (4.37)

$$\int_\Omega \left|\frac{\partial L}{\partial y}(x,y^\varepsilon_{u_\varepsilon},u_\varepsilon)\right||z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u} - z_{u_\varepsilon,u_\varepsilon-\bar u}|\,\mathrm{d}x \le \int_\Omega \psi_{K_U}|z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u} - z_{u_\varepsilon,u_\varepsilon-\bar u}|\,\mathrm{d}x$$
$$\le \|\psi_{K_U}\|_{L^2(\Omega)}C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}. \tag{4.44}$$

For the last term we use again (4.37) and the fact that $\{\xi_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega)$

$$\int_\Omega |\eta_\varepsilon z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u}|\,\mathrm{d}x \le \|\eta_\varepsilon\|_{L^2(\Omega)}\Big(\|z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u} - z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)} + \|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}\Big)$$
$$\le \Big(C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}+1\Big)\|\eta_\varepsilon\|_{L^2(\Omega)}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)} \le C\|\eta_\varepsilon\|_{L^2(\Omega)}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}. \tag{4.45}$$

Inserting the estimates (4.42)–(4.45) in (4.41) we obtain for some constant $C' > 0$ and every $\varepsilon < \varepsilon_1$

$$\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)} \le C'\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big).$$

Finally, using (2.15) and (4.36) we deduce the existence of $\varepsilon_2 \in (0,\varepsilon_1]$ such that for every $\varepsilon < \varepsilon_2$ we have

$$\|y^\varepsilon_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le \|y^\varepsilon_{u_\varepsilon} - y_{u_\varepsilon}\|_{L^2(\Omega)} + \|y_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le C_2\sqrt{|\Omega|}\|\xi_\varepsilon\|_{L^2(\Omega)} + 2\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}$$
$$\le C_2\sqrt{|\Omega|}\|\xi_\varepsilon\|_{L^2(\Omega)} + 2C'\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big),$$

which proves (4.38). □

## II.5 Stability of the controls

In the previous section, we established Lipschitz stability for the optimal states with respect to state perturbations in the objective functional and to the force in the state equation. In order to obtain stability of the optimal controls an additional assumption is usually required. The reader is referred to [21] for the following assumption

$$\exists C > 0 \text{ such that } |\{x \in \Omega : |(\varphi + g)(x)| \le \varepsilon\}| \le C\varepsilon \quad \forall \varepsilon > 0. \tag{5.46}$$

Using this assumption and sufficient second-order optimality conditions they proved Lipschitz stability of the controls in the $L^1(\Omega)$ norm. However, the assumption (5.46) implies that $\bar u$ is bang-bang. As far as we know, there is no proof for stability of the optimal controls when they are not bang-bang. Assumption II.3.4 that we have considered in the previous sections is applicable for the case of optimal controls that are not bang-bang. Nevertheless, it leads only to Lipschitz stability of the optimal states. Here, we modify Assumption II.3.4 as follows

**Assumption II.5.1.** *There exist numbers $\alpha > 0$ and $\gamma > 0$ such that for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{C(\bar\Omega)} < \alpha$ the following inequality is fulfilled*

$$J'(\bar u)(u-\bar u) + J''(\bar u)(u-\bar u)^2 \ge \gamma\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\|u-\bar u\|_{L^1(\Omega)}. \tag{5.47}$$

Under this assumption, we will prove Lipschitz stability of the optimal controls. It has been proved in [9] that the sufficient second order conditions plus the structural assumption (5.46) imply the existence of positive numbers $\gamma$ and $\alpha$ such that

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma\|u-\bar{u}\|^2_{L^1(\Omega)} \ \forall u \in \mathcal{U} \text{ with } \|u-\bar{u}\|_{L^1(\Omega)} < \alpha. \tag{5.48}$$

But we have the next equivalence:

**Proposition II.5.2.** *The statement* (5.48) *is equivalent to the existence of positive numbers* $\gamma'$ *and* $\alpha'$ *such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma'\|u-\bar{u}\|^2_{L^1(\Omega)} \ \forall u \in \mathcal{U} \textit{ with } \|y_u-\bar{y}\|_{C(\bar{\Omega})} < \alpha'. \tag{5.49}$$

*Proof.* Let us assume that (5.48) holds, but (5.49) is false. Then, for every integer $k \geq 1$ there exists an element $u_k \in \mathcal{U}$ such that

$$J'(\bar{u})(u_k-\bar{u}) + J''(\bar{u})(u_k-\bar{u})^2 < \frac{1}{k}\|u_k-\bar{u}\|^2_{L^1(\Omega)} \ \text{ and } \ \|y_{u_k}-\bar{y}\|_{C(\bar{\Omega})} < \frac{1}{k}. \tag{5.50}$$

Since $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ is bounded in $L^\infty(\Omega)$, we can extract a subsequence, denoted in the same way, such that $u_k \stackrel{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$. On one side, (5.50) implies that $y_{u_k} \to \bar{y}$ in $C(\bar{\Omega})$. On the other side, from Theorem II.2.5 the convergence $y_{u_k} \to y_u$ in $C(\bar{\Omega})$ follows. Then, $y_u = \bar{y}$ and, consequently, $u = \bar{u}$ holds. But (5.48) implies that $\bar{u}$ is bang-bang and, hence, the weak convergence $u_k \stackrel{*}{\rightharpoonup} \bar{u}$ yields the strong convergence $u_k \to \bar{u}$ in $L^1(\Omega)$; see [9, Proposition 12 and Lemma 6]. Then, (5.50) contradicts (5.48).

Let us prove the converse implication. First, we observe that given $u \in \mathcal{U}$ we get with the mean value theorem

$$\mathcal{A}(y_u-\bar{y}) + \frac{\partial f}{\partial y}(x,\bar{y}+\theta(y_u-\bar{y}))(y_u-\bar{y}) = u-\bar{u}.$$

Now, using (2.3) with $r=2$ we get

$$\|y_u-\bar{y}\|_{C(\bar{\Omega})} \leq C_2\|u-\bar{u}\|_{L^2(\Omega)} \leq C_2\sqrt{u_b-u_a}\|u-\bar{u}\|^{\frac{1}{2}}_{L^1(\Omega)}.$$

Then, taking $\alpha = \frac{\alpha'^2}{C_2^2(u_b-u_a)}$, we obtain that (5.49) implies (5.48) with $\gamma = \gamma'$. □

From (2.4) we infer that (5.49) implies (5.47). Hence, the combination of sufficient second-order conditions plus (5.46) is a stronger assumption than (5.47).

**Theorem II.5.3.** *Let* $\bar{u}$ *be a local minimizer of* (P) *satisfying Assumption II.5.1 and* $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ *a family of local solutions of problems* $(\mathrm{P}_\varepsilon)$ *such that* $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar{u}$ *in* $L^\infty(\Omega)$ *as* $\varepsilon \to 0$*. Then, there exist* $\hat{\varepsilon} \in (0,\varepsilon_0)$ *and a constant* $C > 0$ *such that*

$$\|u_\varepsilon-\bar{u}\|_{L^1(\Omega)} \leq C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \quad \forall \varepsilon < \hat{\varepsilon}, \tag{5.51}$$

*where* $\bar{y} = y_{\bar{u}}$.

The proof of this theorem follows the steps of the one of Theorem II.4.4 with Lemma II.4.5 replaced by the following:

**Lemma II.5.4.** *Let $\bar{u}$ satisfy the assumptions of Theorem II.5.3. Then, there exists $\varepsilon > 0$ such that*

$$J'(u)(u-\bar{u}) \geq \frac{\gamma}{2}\|z_{u,u-\bar{u}}\|_{L^2(\Omega)}\|u-\bar{u}\|_{L^1(\Omega)} \ \forall u \in \mathcal{U} \text{ with } \|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon, \tag{5.52}$$

*where $\gamma$ is given in Assumption II.5.1.*

*Proof.* We use (4.40) with $\rho = \frac{\gamma}{2C_2}$, Assumption II.5.1, and (2.4) to deduce for $\varepsilon > 0$ small enough

$$\begin{aligned} J'(u)(u-\bar{u}) &= \int_\Omega (\varphi_u + g)(u-\bar{u})\,\mathrm{d}x \\ &= \int_\Omega (\varphi_u - \bar{\varphi} - \psi_{\bar{u},u-\bar{u}})(u-\bar{u})\,\mathrm{d}x + \int_\Omega (\bar{\varphi} + g + \psi_{\bar{u},u-\bar{u}})(u-\bar{u})\,\mathrm{d}x \\ &\geq -\frac{\gamma}{2C_2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)} + [J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2] \\ &\geq -\frac{\gamma}{2}\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|u-\bar{u}\|_{L^1(\Omega)} + \gamma\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|u-\bar{u}\|_{L^1(\Omega)}, \end{aligned}$$

which proves (5.52). □

*Proof of Theorem II.5.3.* We follow the proof of Theorem II.4.4 replacing the estimate (4.42) by (5.52) to deduce with (4.41) and (4.43)–(4.45) the inequality

$$0 \geq J'_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar{u}) \geq \frac{\gamma}{2}\|z_{u_\varepsilon,u_\varepsilon-\bar{u}}\|_{L^2(\Omega)}\|u_\varepsilon - \bar{u}\|_{L^1(\Omega)} - C_1\|z_{u_\varepsilon,u_\varepsilon-\bar{u}}\|_{L^2(\Omega)}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big).$$

Then, dividing this inequality by $\|z_{u_\varepsilon,u_\varepsilon-\bar{u}}\|_{L^2(\Omega)}$ we get

$$\|u_\varepsilon - \bar{u}\|_{L^1(\Omega)} \leq \frac{2C_1}{\gamma}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big),$$

which proves (5.51) with $C = \frac{2C_1}{\gamma}$. □

## II.6 Some final state stability results

In this section we see how Assumption II.5.1 allows us to prove Lipschitz stability for the optimal states for more general perturbations of (P). Here, we consider simultaneous perturbations on the control and state variables of (P):

$$(\mathrm{P}_\varepsilon) \ \min_{u\in\mathcal{U}} J_\epsilon(u) := \int_\Omega L_\varepsilon(x, y_u^\varepsilon(x), u(x))\,\mathrm{d}x,$$

where $y_u^\varepsilon$ is the solution of (4.34) and for every $\epsilon > 0$

$$L_\varepsilon(x, y, u) = L_0(x, y) + \eta_\varepsilon y + g_\varepsilon u + \frac{\varepsilon}{2}u^2.$$

As in Section II.4, we assume that $\{\xi_\varepsilon\}_{\varepsilon>0}$ and $\{\eta_\varepsilon\}_{\varepsilon>0}$ are bounded families in $L^2(\Omega)$ satisfying that $(\xi_\varepsilon, \eta_\varepsilon) \to (0, 0)$ in $L^2(\Omega)^2$ as $\varepsilon \to 0$. Moreover, we suppose that $\|g_\varepsilon - g\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$. Under these assumptions, it is immediate to check that $(\mathrm{P}_\varepsilon)$ is an approximation of (P) in the sense of Theorems II.4.2 and II.4.3. Moreover, we have the following Lipschitz stability property for the optimal states:

**Theorem II.6.1.** *Let $\bar u$ be a local minimizer of* (P) *satisfying Assumption II.5.1 and $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ a family of local solutions of problems $(\mathrm{P}_\varepsilon)$ such that $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$. Then, there exist $\hat\varepsilon \in (0, \varepsilon_0)$ and a constant $C > 0$ such that*

$$\|y^\varepsilon_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)} + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \varepsilon\Big) \quad \forall \varepsilon < \hat\varepsilon, \tag{6.53}$$

*where $\bar y = y_{\bar u}$.*

*Proof.* Similarly to (4.41) we have

$$\begin{aligned} 0 \ge J'_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar u) &= J'(u_\varepsilon)(u_\varepsilon - \bar u) + \int_\Omega (\varepsilon u_\varepsilon + g_\varepsilon - g)(u_\varepsilon - \bar u)\,\mathrm{d}x \\ &+ \int_\Omega \Big[\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\Big] z_{u_\varepsilon, u_\varepsilon - \bar u}\,\mathrm{d}x \\ &+ \int_\Omega \frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon)(z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar u} - z_{u_\varepsilon, u_\varepsilon - \bar u})\,\mathrm{d}x + \int_\Omega \eta_\varepsilon z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar u}\,\mathrm{d}x. \end{aligned}$$

Then, using (5.52) and (4.43)–(4.45) we obtain with (2.4)

$$\begin{aligned} 0 &\ge \frac{\gamma}{2}\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\|u_\varepsilon - \bar u\|_{L^1(\Omega)} - \Big(\varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)} + \|g_\varepsilon - g\|_{L^\infty(\Omega)}\Big)\|u_\varepsilon - \bar u\|_{L^1(\Omega)} \\ &- C_1\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \ge \frac{\gamma}{2}\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\|u_\varepsilon - \bar u\|_{L^1(\Omega)} \\ &- C'\Big(\varepsilon + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big)\|u_\varepsilon - \bar u\|_{L^1(\Omega)}, \end{aligned}$$

where $C' = \max\{1, |u_a|, |u_b|, C_1 C_2\}$. Dividing the above expression by $\|u_\varepsilon - \bar u\|_{L^1(\Omega)}$ and using (2.15) we infer

$$\|y_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le \frac{4C'}{\gamma}\Big(\varepsilon + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big).$$

Now, the rest follows as in the proof of Theorem II.4.4. ☐

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# III. Finite Element Error Analysis and Solution Stability of Affine OCPs

## Outline



## Bibliographic Information

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## Author's contribution


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## Abstract

We consider affine optimal control problems subject to semilinear elliptic PDEs. The results are two-fold; first, we continue the investigation of the solution stability of control problems with respect to perturbations appearing jointly in the objective functional and the PDE. We prove that a certain modification of a coercivity-type property, that appears in the context of optimal control problems where the optimal control is of bang-bang structure, is sufficient for solution stability estimates for the optimal controls and states. The second result is the achievement of error estimates for the numerical approximation generated by a finite element and a variational discretization scheme. The error estimates for the optimal controls and states are obtained for the first time under new assumptions of different strengths on the joint growth of the first and second variation of the objective functional. These assumptions appeared recently in the context of solution stability. Additionally, under one of these assumptions, which assumes a Hölder-type joint growth of the first and second variation, we can provide an improvement of error estimates for a finite element scheme for both, the optimal controls and the states .


## III.1 Introduction

For the consideration of optimal control problems where the control appears at most linearly in the objective functional, it is common to add a Tikhonov regularization term. This is done due to the fact that by including such a term, the optimal control problem becomes in some sense coercive and the analysis of the control problem is then substantially easier. On the other hand, adding such a term represents a distortion of the original problem. Thus if possible, one wants to execute a thorough analysis of the properties of the optimal control problem without this distortion. To perform the study of solution stability and error estimates for finite element approximation schemes of affine optimal control problems, we need certain assumptions on the joint grwoth of the first and second variation. These growth-type assumptions are weaker than the coercivity-type growth of the second variation of the objective functional with respect to the $L^2$-distance of the controls guaranteed for Tikhonov regularized problems. In this paper, we consider several coercivity-type conditions on the joint growth of the first and second variation of the objective functional. In regard to solution stability of optimal controls and states under perturbations appearing in the objective functional and the state equation simultaneously, the result of this paper is a continuation of the investigation in [4, 11, 12]. Here, we consider a growth condition that is similar to the one in [11], but weaker and that still allows for a Lipschitz-type estimate for the optimal controls with respect to the perturbations. Further we address a claim made [4] on the structure of optimal controls satisfying an assumption introduced in [4]. The afterward, the main part of this paper deals with error estimates for the numerical approximation for problems where the control appears at most in an affine way in the PDE and the objective functional and builds upon the papers [7, 8, 10, 13]. In comparison to the results therein, we obtain error estimates for the optimal controls under the assumptions introduced in [4, 11], which are weaker than the ones assumed in [7, 8, 10, 13]. For instance, we do not need to demand that the so-called structural assumption on the adjoint state is satisfied together with the second variation being strict positive on a certain cone, see [6, 12] for a comparison. Instead, we work with the unified conditions established in [4, 11, 12]. To be precise, one of the growth assumptions we assume for the optimal control problem is the following assumption: Given a reference optimal control $\bar{u}$ and a number $\gamma \in (0, 1]$, there exist positive constants $\alpha$ and $c$ such that

$$J(u) - J(\bar{u}) \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{1.1}$$

for all controls $u$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$ and $(u - \bar{u}) \in D^{\tau}_{\bar{u}}$.
Here, $D^{\tau}_{\bar{u}}$ denotes a cone, that extends the cone of critical directions commonly used PDE-constrained optimal control and will be specified later on. See also [3], where it was first introduced.
Conditions of type (1.1) arise naturally in the characterization of strict bang-bang optimal controls, appearing as a consequence of sufficient second-order optimality conditions and the structural assumption on the adjoint state, see [10]. A slightly stronger assumption that implies (1.1) was first considered in [18] for affine ODE optimal control problems and [11] for PDE optimal control problems. Recently (1.1) appeared in [16, 17] in the context of eigenvalue optimization problems. There it was shown that for a certain type of eigenvalue optimization problem, condition (1.1) is implied by a growth of the second order shape derivatives. To relate (1.1) with the classical assumptions used in affine PDE-constrained optimal control problems we refer to Theorem III.7.4 in Section III.4. To apply condition (1.1) in the context of solution stability, we need that the controls corresponding to the perturbed problems are minimizers. This is not the case under the slightly stronger condition in [11], where it is sufficient that the controls corresponding to the perturbed problem satisfy a necessary first-order optimality condition.
Finally, we do not consider a sparsity-promoting term appearing in the objective functional, but the proofs in this paper can be easily adapted to include such a term. Also one can consider a semilinear elliptic non-monotone and non-coercive state equation as in [9] without any changes of the results in the section on solution stability. The sections on error estimates can be adapted to the case of a non-monotone and non-coercive state equation using the results in [5]. To the author's best knowledge, the assumptions considered in this paper are the weakest so far that still allow error estimates for the numerical approximation for problems where the control appears at most in an affine way in the objective functional.

Let us list the novelties in the paper. In respect to solution stability, we prove that condition (1.1) is sufficient for solution stability for affine distributed control problems in Theorem III.5.2. Further, in Proposition III.4.4 we answer a question raised in [4] on the structure of optimal controls satisfying one of the assumption introduced in [4]. Under conditions similar to the one introduced in [4] in the context of solution stability and conditions (1.1), we derive error estimates for a finite element scheme in Theorem III.6.5. For this, we provide some lemmas that allow us to circumvent the use of the structural assumption on the adjoined state. Then in the proof of the main theorem, Theorem III.6.5, we argue similarly as in the first steps of the proof of [7, Theorem 7], but in contrast to the proof therein, we subsequently use the approximation property of the linearized state to conclude the proof. This new approach allows us to obtain error estimates for bang-bang optimal controls similar as in [7, Theorem 9], but under weaker assumptions, further we improve the error estimates for the optimal controls for $\gamma \in (0, 1)$, from $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \leq ch^{\gamma^2}$ to $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \leq ch^{\gamma}$ (and similarly for the states). Finally using the assumptions in [11, 4], we prove error estimates for a variational discretization scheme in Theorem III.6.8 and discuss a relationship of solution stability and finite element error estimates afterward. We believe this approach to be feasible also for the achievement of error estimates for the numerical approximation for a 2-dimensional Neumann boundary control problem, but postpone this analysis to future work.

The paper is structured as follows: In the remainder of this section, we state the main assumptions that hold throughout the paper and state some additional remarks on the notation. In Section III.2, we collect results on the involved PDEs, and in Section III.3 the optimal control problem is discussed. In Section III.4, we investigate the sufficient conditions for local optimality and provide the result that one of the assumptions introduced in [4] implies the bang-bang structure of optimal control. Section III.5 is concerned with solution stability. In Section III.6 we define the discretization schemes and prove error estimates.

Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded domain with Lipschitz boundary. Given functions $u_a, u_b \in L^\infty(\Omega)$ such that $u_a < u_b$ a.e in $\Omega$, define the set of feasible controls by

$$\mathcal{U} := \{u \in L^\infty(\Omega) |\ u_a \le u \le u_b \ \text{ for a.a. } x \in \Omega\} \tag{1.2}$$

and consider the optimal control problem

$$\text{(P)} \quad \min_{u \in \mathcal{U}} \left\{ J(u) := \int_\Omega L(x, y(x), u(x))\, \mathrm{d}x \right\}, \tag{1.3}$$

subject to

$$\begin{cases} \mathcal{A}y + f(\cdot, y) = u & \text{in } \ \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.4}$$

Denote by $y_u$, the unique solution of the state equation that corresponds to the control $u$. The objective integrand $L$ appearing in (1.3) satisfies additional smoothness conditions, given below in Assumption I.7.4.

### III.1.1 Main assumptions and notation

The following assumptions, close to those in [4, 6, 7, 10], are standing in all of the paper.

**Assumption III.1.1.** *The following statements are fulfilled.*

*(i) The operator* $A : H_0^1(\Omega) \to H^{-1}(\Omega)$, *is given by*

$$Ay := -\sum_{i,j=1}^n \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

*where* $a_{i,j} \in L^\infty(\Omega)$. *In Section III.6 we additionally assume that* $a_{i,j} \in C^{0,1}(\bar{\Omega})$. *Further, the* $a_{i,j}$ *satisfy the uniform ellipticity condition*

$$\exists \lambda_A > 0 : \ \lambda_A |\xi|^2 \le \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \quad \textit{for all } \xi \in \mathbb{R}^n \ \textit{ and a.a. } x \in \Omega.$$

(ii) *We assume that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class $C^2$ with respect to the second variable satisfying:*

$$\begin{cases} f(\cdot, 0) \in L^\infty(\Omega) \text{ and } \frac{\partial f}{\partial y}(x,y) \geq 0 \ \forall y \in \mathbb{R}, \\ \forall M > 0 \ \exists C_{f,M} > 0 \text{ s. t. } \left|\frac{\partial f}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \leq C_{f,M} \ \forall |y| \leq M, \\ \forall \rho > 0 \text{ and } \forall M > 0 \ \exists \ \varepsilon > 0 \text{ such that} \\ \left|\frac{\partial^2 f}{\partial y^2}(x,y_2) - \frac{\partial^2 f}{\partial y^2}(x,y_1)\right| < \rho \ \forall |y_1|, |y_2| \leq M \text{ with } |y_2 - y_1| \leq \varepsilon, \end{cases}$$

*for almost every $x \in \Omega$.*

**Assumption III.1.2.** *The function $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Carathéodory and of class $C^2$ with respect to the second variable. In addition, we assume that*

$$\begin{cases} L(x,y,u) = L_a(x,y) + L_b(x,y)u \quad \text{with} \quad L_a(\cdot,0), L_b(\cdot,0) \in L^1(\Omega), \\ \forall M > 0 \ \exists C_{L,M} > 0 \text{ such that} \\ \left|\frac{\partial L}{\partial y}(x,y,u)\right| + \left|\frac{\partial^2 L}{\partial y^2}(x,y,u)\right| \leq C_{L,M} \ \forall |y|, |u| \leq M, \\ \forall \rho > 0 \text{ and } M > 0 \ \exists \varepsilon > 0 \text{ such that} \\ \left|\frac{\partial^2 L}{\partial y^2}(x,y_2,u) - \frac{\partial^2 L}{\partial y^2}(x,y_1,u)\right| < \rho \ |y_1|, |y_2| \leq M \text{ with } |y_2 - y_1| \leq \varepsilon, \end{cases}$$

*for almost every $x \in \Omega$.*

## III.2 Auxiliary results for the state equation

We collect properties of solutions to linear and semilinear elliptic PDEs. The results in this section are standard by now, we refer to [4, 7]. Let $a_0 \in L^\infty(\Omega)$ be a nonnegative function. We consider the properties of solutions to the linear equation

$$Az + a_0 z = v \text{ in } \Omega, \ z = 0 \text{ on } \Gamma. \tag{2.5}$$

**Theorem III.2.1.** *[4, Lemma 2.2] Let $v \in L^r(\Omega)$ with $r > n/2$. Then the linear equation (2.5) has a unique solution $z_v \in H_0^1(\Omega) \cap C(\bar{\Omega})$. Further there exists a positive constant $C_r$ independent of $a_0$ and $v$ such that*

$$\|z_v\|_{H_0^1(\Omega)} + \|z_v\|_{C(\bar{\Omega})} \leq C_r \|v\|_{L^r(\Omega)}. \tag{2.6}$$

**Lemma III.2.2.** *[4, Lemma 2.3] Assume that $s \in [1, \frac{n}{n-2})$, $s'$ is its conjugate, and let $a_0 \in L^\infty(\Omega)$ be a nonnegative function. Then, there exists a constant $C_{s'}$ independent of $a_0$ such that*

$$\|y_h\|_{L^s(\Omega)} \leq C_{s'} \|h\|_{L^1(\Omega)}, \quad \forall h \in H^{-1}(\Omega) \cap L^1(\Omega), \tag{2.7}$$

*where $y_h$ satisfies the equation (2.5), and $C_{s'}$ is given by (2.6) with $r = s'$.*

For the semilinear state equation, we cite the following regularity result.

**Theorem III.2.3.** *[7, Theorem 1] For every $u \in L^r(\Omega)$ with $r > n/2$ there exists a unique $y_u \in Y := H_0^1(\Omega) \cap C(\bar{\Omega})$ solution of (1.2). Moreover, there exists a constant $T_r > 0$ independent of $u$ such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq T_r(\|u\|_{L^r(\Omega)} + \|f(\cdot, 0)\|_{L^\infty(\Omega)}).$$

*If $u_k \rightharpoonup u$ weakly in $L^r(\Omega)$, then we have the strong convergence*

$$\|y_{u_k} - y_u\|_{H_0^1(\Omega)} + \|y_{u_k} - y_u\|_{C(\bar{\Omega})} \to 0.$$

*Further if $u \in L^\infty(\Omega)$ and $\{a_{i,j}\} \in C^{0,1}(\bar{\Omega})$ we have $y_u \in W^{2,r}(\Omega)$ for all $r < \infty$ and*

$$\|y_u\|_{W^{2,r}(\Omega)} \leq M_0 r\Big(\|u\|_{L^\infty(\Omega)} + \|f(\cdot, 0)\|_{L^\infty(\Omega)}\Big)$$

*for a positive constant $M_0$ independent of $u$ and $r$.*

For each $r > n/2$, we define the map $G_r : L^r(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$ by $G_r(u) = y_u$.

**Theorem III.2.4.** *[4, Theorem 2.6] Let Assumption I.3.1 hold. For every $r > \frac{n}{2}$ the map $G_r$ is of class $C^2$, and the first and second derivatives at $u \in L^r(\Omega)$ in the directions $v, v_1, v_2 \in L^r(\Omega)$, denoted by $z_{u,v} = G_r'(u)v$ and $z_{u,v_1,v_2} = G_r''(u)(v_1, v_2)$, are the solutions of the equations*

$$Az + \frac{\partial f}{\partial y}(x, y_u)z = v \text{ in } \Omega, \ z = 0 \text{ on } \Gamma, \tag{2.8}$$

$$Az + \frac{\partial f}{\partial y}(x, y_u)z = -\frac{\partial^2 f}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} \text{ in } \Omega, \ z = 0 \text{ on } \Gamma, \tag{2.9}$$

*respectively.*

**Lemma III.2.5.** *[4, Lemma 2.7] The following statements are fulfilled.*

*(i) Suppose that $r > \frac{n}{2}$ and $s \in [1, \frac{n}{n-2})$. Then, there exist constants $K_r$ depending on $r$ and $M_s$ depending on $s$ such that for every $u, \bar{u} \in \mathcal{U}$*

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(\Omega)} \leq M_s\|y_u - y_{\bar{u}}\|^2_{L^2(\Omega)}. \tag{2.10}$$

*(ii) Taking $C_X = K_2\sqrt{|\Omega|}$ if $X = C(\bar{\Omega})$ and $C_X = M_2$ if $X = L^2(\Omega)$, the following inequality holds*

$$\|z_{u,v} - z_{\bar{u},v}\|_X \leq C_X\|y_u - y_{\bar{u}}\|_X\|z_{\bar{u},v}\|_X \ \forall u, \bar{u} \in \mathcal{U} \text{ and } \forall v \in L^2(\Omega). \tag{2.11}$$

*(iii) Let be $X$ as in (ii). There exists $\varepsilon > 0$ such that for all $\bar{u}, u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} \leq \varepsilon$ the following inequalities are satisfied*

$$1/2\|y_u - y_{\bar{u}}\|_X \leq \|z_{\bar{u},u-\bar{u}}\|_X \leq 3/2\|y_u - y_{\bar{u}}\|_X, \tag{2.12}$$

$$1/2\|z_{\bar{u},v}\|_X \leq \|z_{u,v}\|_X \leq 3/2\|z_{\bar{u},v}\|_X \quad \forall v \in L^2(\Omega). \tag{2.13}$$

## III.3 The optimal control problem

The optimal control problem (1.2)-(1.3) is well posed under Assumptions III.1.1 and III.1.2. By the direct method of calculus of variations we obtain the existence of at least one global minimizer, see [20, Theorem 5.7]. In this section, we discuss the some aspects of the optimal control problem.

**Definition III.3.1.** *We say that $\bar u \in \mathcal{U}$ is an $L^r(\Omega)$-weak local minimum of problem* (1.3)-(1.2)*, if there exists some positive $\varepsilon$ such that*

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar u\|_{L^r(\Omega)} \le \varepsilon.$$

*We say that $\bar u \in \mathcal{U}$ is a strong local minimum of* (P) *if there exists $\varepsilon > 0$ such that*

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le \varepsilon.$$

*We say that $\bar u \in \mathcal{U}$ is a strict weak (strong) local minimum if the above inequalities are strict for $u \neq \bar u$.*

Relations between these notions of optimality are obtained in [6, Lemma 2.8].

**Theorem III.3.2.** *For every $r > \frac{n}{2}$, the functional $J : L^r(\Omega) \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^r(\Omega)$ we have*

$$J'(u)v = \int_\Omega \Big[\frac{\partial L}{\partial y}(x, y_u, u)\Big] z_{u,v} + \Big[\frac{\partial L}{\partial u}(x, y_u, u)\Big] v \,\mathrm{d}x = \int_\Omega \Big[p_u + \frac{\partial L}{du}(x, y_u, u)\Big] v \,\mathrm{d}x,$$

$$J''(u)(v_1, v_2) = \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) - p_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\Big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x + \int_\Omega \Big[\frac{\partial^2 L}{\partial u \partial y}(x, y_u, u)\Big] (z_{u,v_1} v_2 + z_{u,v_2} v_1) \,\mathrm{d}x.$$

*Here, $p_u \in H_0^1(\Omega) \cap C(\bar\Omega)$ is the unique solution of the adjoint equation*

$$\begin{cases} Ap + \dfrac{\partial f}{\partial y}(x, y_u)p = \dfrac{\partial L}{\partial y}(x, y_u, u) \text{ in } \Omega, \\ p = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.14}$$

We introduce the Hamiltonian $\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x, y, p, u) \mapsto H(x, y, p, u) \in \mathbb{R}$ in the usual way:

$$H(x, y, p, u) := L(x, y, u) + p(u - f(x, y)). \tag{3.15}$$

The local form of the Pontryagin type necessary optimality conditions for problem (1.3)-(1.2) stated in the theorem below, is well known (see e.g. [2, 6, 20]).

**Theorem III.3.3.** *If $\bar u$ is a weak or strong local minimizer for problem* (1.3)-(1.2)*, then there exist unique elements $\bar y, \bar p \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that*

$$\begin{cases} A\bar y + f(x, \bar y) = \bar u \text{ in } \Omega, \\ \bar y = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.16}$$

$$\begin{cases} A\bar p = \dfrac{\partial H}{\partial y}(x, \bar y, \bar p, \bar u) \text{ in } \Omega, \\ \bar p = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.17}$$

$$\int_\Omega \frac{\partial H}{\partial u}(x, \bar y, \bar p, \bar u)(u - \bar u) \,\mathrm{d}x \ge 0 \quad \forall u \in \mathcal{U}. \tag{3.18}$$

## III.4 Sufficient assumption for local optimality

In this section, let us recall three assumptions of different strength that are sufficient for strict local optimality and that appeared recently in the context of affine optimal control problems in [4, 11]. We provide some supportive lemmas concerning these assumptions that are helpful in the later section on finite element error estimates. Let us begin with the strongest of these three assumptions.

**Assumption III.4.1.** *Let $\bar{u} \in \mathcal{U}$, $\gamma \in (n/(2+n), 1]$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.19}$$

*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

Assumption III.4.1($\beta = 1$) was considered in the context of elliptic PDE-constrained optimization in [11]. If $\bar{u}$ satisfies the first-order optimality condition (3.18), Assumption III.4.1($\beta = 1$) implies Assumption III.4.1($\beta = 1/2$). If the second variation of the objective funcional is nonnegative, the cases $\beta \in \{1/2, 1\}$ are equivalent. Indeed, the second variation can be negative for box-constrained optimal control problems, see for instance [12, Example 2]. Let $\bar{u}$ satisfy the first-order optimality condition, then in regard to solution stability the formulation of (4.19) with $\beta = 1$ allows for a stronger and more general result that is also easier established. By this we mean that solution stability holds for the controls corresponding to a perturbed problem if they satisfy the corresponding first-order necessary optimality condition, see [11]. In this paper, we prove for the first time, that Assumption III.4.1($\beta = 1/2$), is also sufficient for a solution stability. Here, it is necessary that the control corresponding to the perturbed problem is a global minimizer of said problem. For the achievement of finite element error estimates, this is not a constraint, here, we consider minimizers of the discretized problems, which can be interpreted as a perturbed version of the continuous problem. Let us consider two assumptions on the optimal control problem that were first introduced in [4] for $\beta = 1$. Due to (2.7), they present a weakening of Assumption III.4.1.

**Assumption III.4.2.** *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ with*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|u - \bar{u}\|_{L^1(\Omega)} \tag{4.20}$$

*for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})}$.*

**Assumption III.4.3.** *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants $c$ and $\alpha$ with*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}^2 \tag{4.21}$$

*for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} < \alpha$.*

Assumption III.4.3 is the weakest of the three Assumptions III.4.1, III.4.2 and III.4.3. Assumption III.4.3 is applicable for optimal controls that may not be of bang-bang structure. In [4] it was conjectured that Assumption III.4.2 may also hold if the optimal control is not bang-bang. If $\frac{\partial L_b}{\partial y} = 0$, we can answer this negatively in the following proposition.

**Proposition III.4.4.** *Let* $\frac{\partial L_b}{\partial y} = 0$ *in the objective integrand and let* $\bar{u} \in \mathcal{U}$ *satisfy Assumption III.4.2. Then* $\bar{u}$ *is bang-bang.*

*Proof.* Assume that $\bar{u}$ is not bang-bang and let it satisfy Assumption III.4.2. Since $\bar{u}$ is not bang-bang, there exists a set of positive measure $E \subset \Omega$, such that $\bar{H}_u = 0$ on $E$. Let $v_E$ denote a control with $v_E = \bar{u}$ on $E^c$. Then the first variation in direction $v_E - \bar{u}$ is zero and by (4.20) we find

$$\beta J''(\bar{u})(v_E - \bar{u}, v_E - \bar{u}) \geq c\|z_{\bar{u},v_E-\bar{u}}\|_{L^2(\Omega)}\|v_E - \bar{u}\|_{L^1(\Omega)}. \tag{4.22}$$

By the affine structure of the optimal control problem, we can infer the existence of a positive constant $C$ independent of the control $v_E$ such that

$$\beta J''(\bar{u})(v_E - \bar{u}, v_E - \bar{u}) \leq \beta C\|z_{\bar{u},v_E-\bar{u}}\|^2_{L^2(\Omega)}. \tag{4.23}$$

Thus, using (4.22) and (4.23), we conclude for all controls $v_E$ with $v_E = \bar{u}$ on $E^c$

$$\|\bar{u} - v_E\|_{L^1(\Omega)} \leq \beta C/c\|z_{\bar{u},v_E-\bar{u}}\|_{L^2(\Omega)}. \tag{4.24}$$

Since $\bar{u}$ is not bang-bang, we can select an $\varepsilon > 0$ and a set of positive measure $E$ such that $\bar{u}(x) \in [u_a(x)+\varepsilon, u_b(x)-\varepsilon]$ for a.e. $x \in E$. Now consider a sequence $\{v^k_\varepsilon\}_{k=1}^\infty$ with $v^k_\varepsilon \in \{-\varepsilon, \varepsilon\}$ a.e. on $\Omega$ and $v^k_\varepsilon \rightharpoonup^* 0$ in $L^\infty(\Omega)$. Finally, define a sequence $\{\bar{v}^k_\varepsilon\}_{k=1}^\infty$ by $\bar{v}^k_\varepsilon := \bar{u}$ on $E^c$ and $\bar{v}^k_\varepsilon := \bar{u} + v^k_\varepsilon$ on $E$. It is clear that $\bar{v}^k_\varepsilon \rightharpoonup^* \bar{u}$ in $L^\infty(\Omega)$ and $\|\bar{v}^k_\varepsilon - \bar{u}\|_{L^1(\Omega)} = \varepsilon|E|$ for all $k \in \mathbb{N}$. On the other hand, by Theorem III.2.3, $\bar{v}^k_E \rightharpoonup^* \bar{u}$ in $L^\infty(\Omega)$ implies that $\|z_{\bar{u},\bar{v}^k_E-\bar{u}}\|_{L^2(\Omega)} \to 0$ as $k \to \infty$. This contradicts (4.24). □

As a consequence we obtain that the notion of strong and weak local minimizer is equivalent if Assumption III.4.1 or Assumption III.4.2 is satisfied.

**Lemma III.4.5.** *Let* $\bar{u} \in \mathcal{U}$ *and* $\gamma \in (0, 1)$ *be given. It is equivalent:*

1. *There exist positive constants* $c$ *and* $\alpha$ *such that*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \tag{4.25}$$

   *for all* $u \in \mathcal{U}$ *with* $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.

2. *There exist positive constants* $c$ *and* $\alpha$ *such that* (4.25) *holds for all* $u \in \mathcal{U}$ *with* $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} < \alpha$.

*Further, if the objective integrand satisfies* $\frac{\partial L_b}{\partial y} = 0$ *it is equivalent*

1. *There exist positive constants* $c$ *and* $\alpha$ *such that*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|_{L^1(\Omega)}\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)} \tag{4.26}$$

   *for all* $u \in \mathcal{U}$ *with* $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.

2. *There exist positive constants* $c$ *and* $\alpha$ *such that* (4.26) *holds for all* $u \in \mathcal{U}$ *with* $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} < \alpha$.

*Proof.* Let us remark that the statement of Lemma III.4.5 for Assumption III.4.1 with $\gamma = 1$ was proven in [4, Proposition 5.2]. The proof relies on the fact that (4.25) implies the bang-bang structure of $\bar{u}$. But if $\gamma \in (0,1)$, (4.25) still implies the bang-bang structure and the arguments in [4, Proposition 5.2] hold true for $\gamma \in (0,1)$. By Proposition III.4.4, the Assumption III.4.2 implies the control $\bar{u}$ to be bang-bang, thus the results can be obtained by similar arguments as in [4, Proposition 5.2]. □

The next lemma is crucial for the estimations later on. It is well known for objective functionals with varying generality and was proven in several publications for case $\gamma = 1$, see . The proof for $\gamma \in (n/(2+n), 1)$ follows by exactly the same arguments.

**Lemma III.4.6.** *Given $\gamma \in (n/(2+n), 1]$ and $\bar{u}, u \in \mathcal{U}$. Define $u_\theta := \bar{u} + \theta(u - \bar{u})$ for some $\theta \in [0,1]$. For all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| J''(\bar{u})(u-\bar{u})^2 - J''(u_\theta)(u-\bar{u})^2 \right| \leq \epsilon \|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

*for all $\|u - \bar{u}\|_{L^1(\Omega)} < \delta$.*

Further, we have the following result.

**Proposition III.4.7.** *Let $\bar{u} \in \mathcal{U}$ and $\gamma \in (n/(2+n), 1]$ be given. It is equivalent:*

1. *The control $\bar{u}$ satisfies Assumption III.4.1($\beta = 1$) and the first-order necessary optimality condition.*

2. *There exist positive constants $c$ and $\alpha$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u} + \theta(u - \bar{u}))(u-\bar{u})^2 \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

   *for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$ and $\theta \in [0,1]$.*

3. *There exist positive constants $\mu, \delta$, such that*

$$J'(u)(u-\bar{u}) \geq \mu\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.27}$$

   *for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \delta$.*

*Proof.* The direction from 1 to 3 was proven in [11, Lemma 12]. The direction from 3 to 1 follows by using Taylor's theorem and Lemma III.4.6: Define $u_\theta : \bar{u} + \theta(u - \bar{u})$ for some $\theta \in [0,1]$. By Talyor's theorem, there exists $\theta$ such that

$$J'(u)(u-\bar{u}) - J'(\bar{u})(u-\bar{u}) = J''(u_\theta)(u-\bar{u})^2.$$

By Lemma III.4.6 we obtain:

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &= J'(u)(u-\bar{u}) + J'(\bar{u})(u-\bar{u}) - J'(u)(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \\ &\geq \mu\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} - \left| J''(\bar{u})(u-\bar{u})^2 - J''(u_\theta)(u-\bar{u})^2 \right|. \end{aligned}$$

Select $\alpha < \min\{\delta, \beta\}$, such that $\mu > \varepsilon$. Then defining $c := \mu - \varepsilon$ the claim follows for controls $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$. The case 2 to 1 is trivial and the argument from 1 to 2 follows again by using Taylor's theorem and Lemma III.4.6. □

The reformulation of Assumption III.4.1($\beta = 1$) to (4.27) is useful to provide short proofs for the error estimates for the variational discretization later on. It appeared first in [14] in the context of ODE optimal control.

**Proposition III.4.8.** *Let $\bar{u} \in \mathcal{U}$ be given. It is equivalent:*

1. *Assumption III.4.1($\beta = 1/2$) holds.*
2. *There exist positive constants $c$ and $\alpha$ such that*
$$J'(\bar{u})(u - \bar{u}) + \frac{1}{2} J''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2 \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$
*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)}$ and $\theta \in [0, 1]$.*
3. *There exists a positive constants $c$ and $\alpha$ such that*
$$J(u) - J(\bar{u}) \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}, \tag{4.28}$$
*for all $u \in \mathcal{U}$ with $\|u - \bar{u}\| < \alpha$.*

*Proof.* The proof follows by the same arguments as in Theorem III.4.7. □

For assumptions III.4.3 and III.4.2, we can formulate an analog to Theorem III.4.7. For a proof, we refer to [4, Lemma 4.5, Lemma 5.4].

**Proposition III.4.9.** *[4, Lemma 4.5, Lemma 5.4] We have the following equivalence.*

1. *Assumption III.4.3($\beta = 1$) holds for $\bar{u} \in \mathcal{U}$.*
2. *There exist positive constants $c$ and $\alpha$ such that*
$$J'(u)(u - \bar{u}) \geq c\|z_{\bar{u}, u-\bar{u}}\|_{L^2(\Omega)}^2$$
*for all $u \in \mathcal{U}$ with $\|y_{\bar{u}} - y_u\|_{C(\bar{\Omega})} < \alpha$.*

*Further, it is equivalent*

1. *Assumption III.4.2($\beta = 1$) holds for $\bar{u} \in \mathcal{U}$.*
2. *There exist positive constants $c$ and $\alpha$ such that*
$$J'(u)(u - \bar{u}) \geq c\|z_{\bar{u}, u-\bar{u}}\|_{L^2(\Omega)}\|\bar{u} - u\|_{L^1(\Omega)} \tag{4.29}$$
*for all $u \in \mathcal{U}$ with $\|\bar{u} - u\|_{L^1(\Omega)} < \alpha$.*

## III.5 Solution stability

We investigate stability under perturbations appearing in the objective functional and the PDE simultaneously. This was also considered for instance in [4, 11, 19] under Assumptions III.4.1($\beta = 1$), III.4.3($\beta = 1$) and III.4.2($\beta = 1$). In this section we provide the first solution stability result under Assumption III.4.1($\beta = 1/2$), III.4.3($\beta = 1/2$) and III.4.2($\beta = 1/2$). For this, we need to additionally assume that the second derivatives of $L$ and $f$ are Lipschitz with respect to the $y$ variable. Let us fix a positive constant $M$ and define the set of feasible perturbations by

$$\Sigma := \Big\{ \zeta := (\xi, \eta, \rho) \in L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) | \ \|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)} \le M \Big\}.$$

The perturbed problem is given by

$$\min_{u \in \mathcal{U}} \Big\{ J_\zeta(u) := \int_\Omega L(x, y_u, u) + \rho u + \eta y_u \, \mathrm{d}x \Big\} \tag{5.30}$$

subject to (1.2) and

$$\begin{cases} Ay + f(\cdot, y) & = u + \xi \quad \text{in } \ \Omega, \\ y & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.31}$$

The existence of a globally optimal solution to (5.30)-(5.31) is guaranteed by the assumptions on the optimal control problem and the direct method in the calculus of variations. We define

$$\bar{C} := \max_{u \in \mathcal{U}} \{ \|y_u\|_{L\infty(\Omega)}, \|p_u\|_{L^\infty(\Omega)} \}. \tag{5.32}$$

We need the next technical lemma, for a proof we refer to [4, Theorem 4.1].

**Lemma III.5.1.** *Given $\xi \in L^2(\Omega)$, $u \in \mathcal{U}$ and $v \in L^2(\Omega)$, it holds*

$$\|y_u^\xi - y_u\|_{L^2(\Omega)} \le C_2 \|\xi\|_{L^2(\Omega)},$$
$$\|z_{u,v}^\xi - z_{u,v}\|_{L^s(\Omega)} \le \bar{C} C_2^2 \|\xi\|_{L^2(\Omega)} \|v\|_{L^1(\Omega)}.$$

**Theorem III.5.2.** *Let $\bar{u}$ satisfy Assumption III.4.1($\beta = 1/2$) for some $\gamma \in (n/(n+2), 1]$. There exist positive constants $c$ and $\alpha$ such that*

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} \le c(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)})^\gamma,$$

*for any minimizer $(\bar{y}^\zeta, \bar{p}^\zeta, \bar{u}^\zeta)$ of (5.30)-(5.31) with $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

*Proof.* Denote by $y_u^\xi$ the solution to 5.31 corresponding to the perturbation $\xi$ and data $u$ and define

$$J_\xi(u) := \int_\Omega L(x, y_u^\xi, u) \, \mathrm{d}x.$$

Let $\bar{u}^\zeta$ be a minimizer of the perturbed problem with perturbation $\zeta = (\xi, \eta, \rho)$ and denote by $\bar{y}^\zeta$ the corresponding state and $\bar{p}^\zeta$ the corresponding adjoint state. We notice that $y_{\bar{u}^\zeta}^\zeta = y_{\bar{u}^\zeta}^\xi$ if $\xi$ is as in $\zeta$. In this case, it holds

$$J_\xi(\bar{u}^\zeta) + \int_\Omega \rho \bar{u}^\zeta + \eta y_{\bar{u}^\zeta}^\xi \, \mathrm{d}x \le J_\xi(\bar{u}) + \int_\Omega \rho \bar{u} + \eta y_{\bar{u}}^\xi \, \mathrm{d}x.$$

Therefore,

$$J_\xi(\bar u^\zeta) - J_\xi(\bar u) \le \int_\Omega \rho(\bar u - \bar u^\zeta) + \eta(y^\xi_{\bar u} - y^\xi_{\bar u^\zeta})\,\mathrm{d}x \le (\|\rho\|_{L^\infty(\Omega)} + C_2\|\eta\|_{L^2(\Omega)})\|\bar u - \bar u^\zeta\|_{L^1(\Omega)}. \tag{5.33}$$

We estimate the term on the left side. For this let us define $u_\theta := \bar u + \theta(\bar u^\zeta - \bar u)$ for some $\theta \in [0,1]$ and denote by $y_{u_\theta}$ and $p_{u\theta}$ the corresponding state and adjoint state.

$$\begin{aligned}
&J_\xi(\bar u^\zeta) - J_\xi(\bar u) = J'_\xi(\bar u)(\bar u^\zeta - \bar u) + \frac{1}{2}J''_\xi(u_\theta)(\bar u^\zeta - \bar u)^2\\
&= \Big[J'_\xi(\bar u)(\bar u^\zeta - \bar u) - J'(\bar u)(\bar u^\zeta - \bar u)\Big] + \Big[\frac{1}{2}(J''_\xi(u_\theta)(\bar u^\zeta - \bar u)^2 - J''(u_\theta)(\bar u^\zeta - \bar u)^2)\Big]\\
&+ \Big[J'(\bar u)(\bar u^\zeta - \bar u) + \frac{1}{2}J''(u_\theta)(\bar u^\zeta - \bar u)^2\Big] = I_1 + I_2 + I_3.
\end{aligned}$$

We estimate the terms $I_1$, $I_2$, and $I_3$. By Theorem III.4.8 there exist positive constants $c$ and $\alpha$ such that

$$I_3 \ge c\|u - \bar u\|^{1+1/\gamma}_{L^1(\Omega)} \quad \text{for all } u \in \mathcal{U} \text{ with } \|u - \bar u\|_{L^1(\Omega)} < \alpha.$$

By assumption, this is satisfied by $\bar u^\zeta$. We continue with the term $I_1$:

$$\begin{aligned}
|I_1| &\le \Big|J'_\xi(\bar u)(\bar u^\zeta - \bar u) - J'(\bar u)(\bar u^\zeta - \bar u)\Big| \le \Big|\int_\Omega \frac{\partial L}{\partial y}(x, y^\xi_{\bar u}, \bar u) z^\xi_{\bar u, \bar u^\zeta - \bar u} - \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u) z_{\bar u, \bar u^\zeta - \bar u}\,\mathrm{d}x\Big|\\
&+ \Big|\int_\Omega \Big[\frac{\partial L}{\partial u}(x, y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial u}(x, y_{\bar u}, \bar u)\Big](\bar u^\zeta - \bar u)\,\mathrm{d}x\Big| = J_1 + J_2.
\end{aligned}$$

The term $J_1$ is estimated using Assumption I.7.4, the mean value theorem and Lemma **??**

$$\begin{aligned}
|J_1| &\le \Big|\int_\Omega \Big[\frac{\partial L}{\partial y}(x, y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u)\Big] z^\xi_{\bar u, \bar u^\zeta - \bar u}\,\mathrm{d}x\Big| + \Big|\int_\Omega \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u)\Big[z^\xi_{\bar u, \bar u^\zeta - \bar u} - z_{\bar u, \bar u^\zeta - \bar u}\Big]\,\mathrm{d}x\Big|\\
&\le C_{L,M}C_2\|\xi\|_{L^2(\Omega)}\|z^\xi_{\bar u, u^\zeta - \bar u}\|_{L^2(\Omega)} + \|\psi_M\|_{L^2(\Omega)}\|z^\xi_{\bar u, \bar u^\zeta - \bar u} - z_{\bar u, \bar u^\zeta - \bar u}\|_{L^2(\Omega)}\\
&\le C_2(C_{L,M} + C_2C_{f,M}\|\psi_M\|_{L^2(\Omega)})\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.
\end{aligned}$$

The term $J_2$ is estimated by again using Assumption III.1.2, the mean value theorem and Lemma III.5.1

$$|J_2| \le \Big\|\frac{\partial L}{\partial u}(y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial u}(y_{\bar u}, \bar u)\Big\|_{L^\infty(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)} \le C_2C_{L,M}\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.$$

To estimate $I_2$ we write

$$
\begin{aligned}
I_2 = &\int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y^\xi_{u_\theta}, u_\theta) - \frac{\partial^2 L}{\partial y^2}(x, y_{u_\theta}, u_\theta)\Big](z^\xi_{u_\theta,\bar u^\zeta-\bar u})^2 \,\mathrm{d}x \\
&+ \int_\Omega \Big[p_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y_{u_\theta}) - p^\xi_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y^\xi_{u_\theta})\Big](z^\xi_{u_\theta,\bar u^\zeta-\bar u})^2 \,\mathrm{d}x \\
&+ \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_{u_\theta}, u_\theta) + p_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y_{u_\theta})\Big]\Big[(z^\xi_{u_\theta,\bar u^\zeta-\bar u})^2 - z^2_{u_\theta,\bar u^\zeta-\bar u}\Big] \,\mathrm{d}x \\
&+ 2\int_\Omega \Big[\frac{\partial L}{\partial u}(x, y^\xi_{u_\theta}, u_\theta) - \frac{\partial L}{\partial u}(x, y_{u_\theta}, u_\theta)\Big] z^\xi_{u_\theta,\bar u^\zeta-\bar u}(\bar u^\zeta - \bar u)\,\mathrm{d}x \\
&+ 2\int_\Omega \frac{\partial L}{\partial u}(x, y_{u_\theta}, u_\theta)\Big[z^\xi_{u_\theta,\bar u^\zeta-\bar u} - z_{u_\theta,\bar u^\zeta-\bar u}\Big](\bar u^\zeta - \bar u)\,\mathrm{d}x = \sum_{i=1}^5 K_i.
\end{aligned}
$$

For the first term, we find by Assumption III.1.2, Theorem III.2.1, Lemma III.2.2 and Lemma III.5.1

$$|K_1| \le \mathrm{Lip}_{L,M} C_2 C_\infty \|u_b\|_{L^\infty(\Omega)} \|\xi\|_{L^2(\Omega)} \|\bar u^\zeta - \bar u\|^2_{L^1(\Omega)}.$$

The estimate for the second and third terms follows by Assumption III.1.2 and III.1.1, Theorem III.2.1, Lemma III.2.2, Lemma III.5.1 and (5.32)

$$
\begin{aligned}
|K_2| &\le C_2\Big(C_{f,M}C_\infty + \mathrm{Lip}_{f,M}\bar C\Big)\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|^2_{L^1(\Omega)}, \\
|K_3| &\le 2\Big(C_{L,M} + \bar C C_{f,M}\Big)\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.
\end{aligned}
$$

For the fourth and fifth terms using the same arguments, we find

$$
\begin{aligned}
|K_4| &\le C_{L,M} C^2_\infty \bar C \|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}, \\
|K_5| &\le 2\bar C C_2^2 \|u_b\|_{L^\infty(\Omega)} \|L_b(\cdot, y_{u_\theta}(\cdot))\|_{L^\infty(\Omega)} \|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.
\end{aligned}
$$

Summarizing, we conclude the existence of a positive constant c such that

$$|I_1| \le c\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)},$$

and

$$|I_2| \le \sum_{i=1}^5 |K_i| \le c\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.$$

Further by (5.33), it holds

$$I_3 - |I_1| - |I_2| \le (\|\rho\|_{L^\infty(\Omega)} + \|\eta\|_{L^2(\Omega)})\|\bar u - \bar u^\zeta\|_{L^1(\Omega)}.$$

Thus by the estimates on terms $I_1$, $I_2$ and $I_3$ we conclude the existence of a positive constant $c$ with

$$\|\bar u^\zeta - \bar u\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \le c(\|\rho\|_{L^\infty(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)})\|\bar u - \bar u^\zeta\|_{L^1(\Omega)},$$

for all $\bar u^\zeta \in \mathcal{U}$ satisfying $\|\bar u^\zeta - \bar u\|_{L^1(\Omega)} < \alpha$. □

Similar argumentation as done in the proof of Theorem III.5.2 yields estimations under Assumption III.4.2($\beta = 1/2$) and III.4.3($\beta = 1/2$).

**Proposition III.5.3.** *Let $\frac{\partial^2 L}{\partial u \partial y} = 0$ and let $\bar{u}$ satisfy Assumption III.4.2($\beta = 1/2$). There exist positive constants $c$ and $\alpha$ such that*

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} \leq c(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}),$$

*for any minimizer $(\bar{y}^\zeta, \bar{p}^\zeta, \bar{u}^\zeta)$ of* (5.30)-(5.31) *(for $\rho = 0$) with $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

**Proposition III.5.4.** *Let $\frac{\partial^2 L}{\partial u \partial y} = 0$ and let $\bar{u}$ satisfy Assumption III.4.3($\beta = 1/2$). There exist positive constants $c$ and $\alpha$ such that*

$$\|y_{\bar{u}^\zeta} - y_{\bar{u}}\|_{L^2(\Omega)} \leq c(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}),$$

*for any minimizer $(\bar{y}^\zeta, \bar{p}^\zeta, \bar{u}^\zeta)$ of* (5.30)-(5.31) *(for $\rho = 0$) with $\|y_{\bar{u}^\zeta} - y_{\bar{u}}\|_{C(\bar{\Omega})} < \alpha$.*

## III.6 Discrete model and error estimates

We come to the main part of this manuscript. The goal is to prove error estimates for the numerical approximation under Assumption III.4.1 for $\gamma \in (n/(n+2), 1]$ and Assumptions III.4.2 and III.4.3. In a remark at the end of the next subsection, we mention assumptions that allow us to admit $\gamma \in (0, 1]$.

### III.6.1 The finite element scheme

The finite element scheme we consider, is close to the one in [7], we also refer to [1] for an overview of the finite element method. In this section, we assume $\Omega$ to be convex. Let $\{\tau_h\}_{h>0}$ be a quasi-uniform family of triangulations of $\bar{\Omega}$. Denote $\bar{\Omega}_h = \cup_{T \in \tau_h} T$ and assume that every boundary node of $\Omega_h$ is a point of $\Gamma$. Further, suppose that there exists a constant $C_\Gamma > 0$ independent of $h$ such that the distance $d_\Gamma$ satisfies $d_\Gamma(x) < C_\Gamma h^2$ for every $x \in \Gamma_h = \partial\Omega_h$. Under this assumptions, we can infer the existence of a constant $C_\Omega > 0$ independent of $h$ such that

$$|\Omega \setminus \Omega_h| \leq C_\Omega h^2, \tag{6.34}$$

where $|\cdot|$ denotes the Lebesgue measure. We define the finite-dimensional space

$$Y_h = \{z_h \in C(\bar{\Omega}) : z_{h|T} \in P_1(T) \ \forall T \in \tau_h \ \text{ and } z_h \equiv 0 \text{ on } \Omega \setminus \Omega_h\},$$

where $P_i(T)$ denotes the polynomials in $T$ of degree at most $i$.
For $u \in L^2(\Omega)$, the associated discrete state is the unique element $y_h(u) \in Y_h$ that solves

$$a(y_h, z_h) + \int_{\Omega_h} f(x, y_h) z_h \,\mathrm{d}x \,\mathrm{d}t = \int_{\Omega_h} u z_h \,\mathrm{d}x \quad \forall z_h \in Y_h, \tag{6.35}$$

where

$$a(y, z) = \sum_{i,j=1}^n \int_\Omega a_{ij} \partial_{x_i} y \partial_{x_j} z \,\mathrm{d}x \quad \forall y, z \in H^1(\Omega).$$

The proof of the existence and uniqueness of a solution for (6.35) is standard.

**Lemma III.6.1.** *[7, Lemma 3]. There exists a constant $c > 0$, which depends on the data of the problem but is independent of the discretization parameter $h$, such that for every $u \in \mathcal{U}$*

$$\|y_h(u) - y_u\|_{L^2(\Omega)} \le ch^2, \tag{6.36}$$

$$\|y_h(u) - y_u\|_{L^\infty(\Omega)} \le ch^2 |\log h|^2. \tag{6.37}$$

The set of feasible controls for the discrete problem is given by

$$U_h := \{u_h \in L^\infty(\Omega_h) : u_{h|T} \in P_0(T) \ \forall T \in \tau_h\}.$$

By $\Pi_h$ we denote the linear projection onto $U_h$ in the $L^2(\Omega_h)$ given by

$$(\Pi_h u)_{|T} = \frac{1}{|T|} \int_T u \, \mathrm{d}x, \quad \forall T \in \tau_h.$$

By $u_h \rightharpoonup u$ weak* in $L^\infty(\Omega)$ we mean, as in [7], the following

$$\int_{\Omega_h} u_h v \, \mathrm{d}x \to \int_\Omega uv \, \mathrm{d}x \ \ \forall \ v \in L^1(\Omega).$$

**Lemma III.6.2.** *[7, Lemma 4] Given $1 < p < \infty$ there exists a positive constant $K_p$ that depends on $p$ and $\Omega$ but is independent of $h$ such that*

$$\|u - \Pi_h u\|_{W^{-1,p}(\Omega_h)} \le K_p h \|u\|_{L^p(\Omega)} \ \ \forall \ u \in L^p(\Omega).$$

We define

$$J_h(u) := \int_{\Omega_h} L(x, y_h(u), u) \, \mathrm{d}x.$$

Let us define $\mathcal{U}_h := U_h \cap \mathcal{U}$. Then the discrete problem is given by

$$\min_{u_h \in \mathcal{U}_h} J_h(u_h). \tag{6.38}$$

The set $\mathcal{U}_h$ is compact and nonempty and the existence of a global solution of (6.38) follows from standard arguments. For $u \in L^2(\Omega)$, the discrete adjoint state $p_h(u) \in Y_h$ is the unique solution of

$$a(z_h, p_h) + \int_{\Omega_h} \frac{\partial f}{\partial y}(x, y_h(u)) p_h z_h \, \mathrm{d}x = \int_{\Omega_h} \frac{\partial L}{\partial y}(x, y_h(u), u) z_h \, \mathrm{d}x \ \ \forall z_h \in Y_h. \tag{6.39}$$

One can calculate that

$$J_h'(u)(v) = \int_{\Omega_h} (L_b(x, y_h(u)) + p_h(u)) v \, \mathrm{d}x.$$

A local solution of (6.38) satisfies the variational inequality

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) \ge 0 \ \ \forall u_h \in \mathcal{U}_h.$$

### III.6.2 Discretization with piecewise constant controls

The main goal of this section is to prove that Assumptions III.4.1, III.4.2 and III.4.3, allow finite element error estimates. Before stating the main theorems, let us consider two preliminary lemmas.

**Lemma III.6.3.** *Consider a bang-bang control $\bar{u} \in \mathcal{U}$ satisfying the first order optimality condition and define $\bar{\sigma} := p_{\bar{u}} + L_b(x, y_{\bar{u}})$. Assume that $\bar{\sigma}$ is Lipschitz on $\Omega$, then*

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) \leq \mathrm{Lip}_{\bar{\sigma}} h \|\bar{u} - \Pi_h \bar{u}\|_{L^1(\Omega)}.$$

*Proof.* For the convenience of the reader, we present a proof which follows the arguments in [7, Lemma 7]. Let $T$ be an element such that $\bar{\sigma}$ changes its sign. Since $\bar{\sigma}$ is Lipschitz, there exists a point $x_0 \in T$ with $\bar{\sigma}(x_0) = 0$. For $x \in T$ we obtain

$$|\bar{\sigma}(x)| = |\bar{\sigma}(x) - \bar{\sigma}(x_0)| \leq Lip_{\bar{\sigma}} |x - x_0| \leq Lip_{\bar{\sigma}} h. \tag{6.40}$$

Let us denote by $S$ the union of elements $T$ such that $\bar{\sigma}$ changes the sign. Then on the set $S$, we have the estimate $\|\bar{\sigma}\|_{L^\infty(S)} \leq Lip_{\bar{\sigma}} h$. If $\bar{\sigma}$ does not change the sign on an element $T$, the bang-bang structure implies $\Pi_h \bar{u} = \bar{u}$ on $\Omega \setminus S$. As a consequence we obtain the estimate

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) = \int_{\Omega_h} \bar{\sigma}(\Pi_h \bar{u} - \bar{u}) \;\; \mathrm{d}x \leq \|\bar{\sigma}\|_{L^\infty(S)} \|\Pi_h \bar{u} - \bar{u}\|_{L^1(S)} \leq Lip_{\bar{\sigma}} h \|\Pi_h \bar{u} - \bar{u}\|_{L^1(S)}.$$

□

The next lemma is crucial to obtain error estimates for the numerical approximation without assuming the structural assumption.

**Lemma III.6.4.** *We assume that $\bar{\sigma} = p_{\bar{u}} + L_b(x, y_{\bar{u}})$, corresponding to the reference solution $\bar{u}$, is Lipschitz continuous on $\Omega$. Let $\bar{u} \in \mathcal{U}$ satisfy Assumption III.4.1($\beta = 1/2$), here we allow $\gamma \in (0, 1]$. There exists a positive constant $c$ independent of $h$, such that for $h$ sufficiently small*

$$\|\bar{u} - \Pi_h \bar{u}\|_{L^1(\Omega_h)} \leq c h^\gamma. \tag{6.41}$$

The assumption that $p_{\bar{u}} + L_b(x, y_{\bar{u}})$ is Lipschitz, is not a big constraint. By the assumptions on the control problem in this section, it is already guaranteed that the adjoint state is Lipschitz continuous.

*Proof.* We begin with (6.41). If $\bar{u} \in \mathcal{U}$ satisfies Assumption III.4.1, then there exist positive constants $c$ and $\alpha$ such that for $h$ with $\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega_h)} < \alpha$

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) + 1/2 J''(\bar{u})(\Pi_h \bar{u} - \bar{u})^2 \geq c \|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega_h)}^{1+1/\gamma}.$$

By Lemma III.6.3, we obtain

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) = \int_\Omega (p_{\bar{u}} + L_b(x, y_{\bar{u}}))(\Pi_h \bar{u} - \bar{u}) \, \mathrm{d}x \leq \mathrm{Lip}_{\bar{\sigma}} h \|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega)}.$$

The right-hand side is further estimated employing the Peter-Paul inequality for some $\varepsilon < c$, where $c$ is the constant appearing in Assumption III.4.1,

$$\mathrm{Lip}_{\bar{\sigma}} h \|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega)} \leq \frac{(\mathrm{Lip}_{\bar{\sigma}} h)^2}{2\varepsilon} + \frac{\varepsilon \|\Pi_h \bar{u} - \bar{u}\|^2_{L^1(\Omega)}}{2}. \tag{6.42}$$

Given $n < p$, using [7, Lemma 1, Lemma 4], the second variation is estimated by

$$\begin{aligned} J''(\bar{u})(\Pi_h \bar{u} - \bar{u})^2 &\leq \left\| \frac{\partial^2 L}{\partial^2 y}(\cdot, y_{\bar{u}}) - p_{\bar{u}} \frac{\partial^2 f}{\partial^2 y}(\cdot, y_{\bar{u}}) \right\|_{L^\infty(\Omega)} \|z_{\bar{u}, \Pi_h \bar{u} - \bar{u}}\|^2_{L^2(\Omega)} \\ &\leq C_p(\bar{C} C_{f,M} + C_{L,M}) \|\Pi_h \bar{u} - \bar{u}\|^2_{W^{-1,p}} \leq C_p K_p^2 (\bar{C} C_{f,M} + C_{L,M}) h^2. \end{aligned}$$

The claim follows by absorbing the second term of (6.42). □

Now we are ready to state the main theorem of this section.

**Theorem III.6.5.** *Let $\bar{u}$ be a local solution of* (P). *Consider the constant $\alpha$ corresponding to the Assumptions III.4.1, III.4.2 or III.4.3. Consider a sequence of discrete optimal controls $\bar{u}_h \in \mathcal{U}_h$ of* (6.38) *that satisfy $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} < \alpha$. We recall that $\bar{y}$ is the solution of* (1.4) *and $y(\bar{u}_h)$ denotes the solution of* (6.35) *for $\bar{u}_h$.*

1. *Let $L_b = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption III.4.3($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq c\sqrt{h}. \tag{6.43}$$

2. *Let $L_b = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption III.4.2($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq c \min\left\{\sqrt{h}, \sqrt{h^2 + h\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega_h)}}\right\}. \tag{6.44}$$

3. *Let $\frac{\partial L_b}{\partial y} = 0$ in the objective functional and let $\bar{u}$ satisfy Assumption III.4.1($\beta = 1/2$). Then, there exists a positive constant $c$ independent of $h$ such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^\gamma. \tag{6.45}$$

   *If $\frac{\partial L_b}{\partial y} \neq 0$ we obtain the estimate*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^{\frac{(1+\min\{1/r,\gamma\})\gamma}{\gamma+1}}. \tag{6.46}$$

*Proof.* Let us consider a discrete control $\bar{u}_h$ that satisfies the assumptions of the theorem. We first prove the existence a positive constant $c$ such that

$$\begin{aligned} J(\bar{u}_h) - J(\bar{u}) &\leq \left|J(\bar{u}_h) - J_h(\bar{u}_h)\right| + J_h(\bar{u}_h) - J_h(\Pi_h \bar{u}) + \left|J_h(\Pi_h \bar{u}) - J(\Pi_h \bar{u})\right| \\ &\quad + \left|J(\Pi_h \bar{u}) - J(\bar{u})\right| = |I_1| + 0 + |I_3| + |I_4| \leq ch^{1+\gamma}. \end{aligned}$$

To estimate the first term, we use the estimates in Lemma III.6.1 and (6.34), to obtain

$$
\begin{aligned}
|I_1| &= \left| \int_{\Omega\setminus\Omega_h} L(x, y_{\bar u_h}, \bar u_h)\,\mathrm{d}x + \int_{\Omega_h} L(x, y(\bar u_h), \bar u_h) - L(x, y_{\bar u_h}, \bar u_h)\,\mathrm{d}x \right| \\
&\le h^2 \Big( \Big\| \frac{\partial L_a}{\partial y}(x, y_\theta) \Big\|_{L^2(\Omega)} + \Big\| \frac{\partial L_b}{\partial y}(x, y_\vartheta)\bar u_h \Big\|_{L^2(\Omega_h)} + C_\Omega \| L(x, y_{\bar u_h}, \bar u_h)\|_{L^\infty(\Omega)} \Big).
\end{aligned}
$$

For the second term, $I_2$, we have $I_2 \le 0$ since $\bar u_h$ is a minimizer of the discrete problem. The third term, $I_3$, can be estimated by similar arguments as used for the first term,

$$
\begin{aligned}
|I_3| &= \left| \int_{\Omega} L(x, y_{\Pi_h \bar u}, \Pi_h \bar u)\,\mathrm{d}x - \int_{\Omega_h} L(x, y(\Pi_h \bar u), \Pi_h \bar u)\,\mathrm{d}x \right| \\
&= \left| \int_{\Omega\setminus\Omega_h} L(x, y_{\Pi_h \bar u}, \Pi_h \bar u)\,\mathrm{d}x + \int_{\Omega_h} L(x, y_{\Pi_h \bar u}, \Pi_h \bar u) - L(x, y(\Pi_h \bar u), \Pi_h \bar u)\,\mathrm{d}x \right| \\
&\le h^2 \Big( \Big\| \frac{\partial L_a}{\partial y}(x, y_\theta) \Big\|_{L^2(\Omega)} + \Big\| \frac{\partial L_b}{\partial y}(x, y_\vartheta)\Pi_h \bar u \Big\|_{L^2(\Omega_h)} + C_\Omega \| L(x, y_\vartheta, \Pi_h \bar u)\|_{L^\infty(\Omega)} \Big).
\end{aligned}
$$

Finally, to estimate the last term, let us denote by $L_{a,y}$ and $L_{b,y}$ the derivatives of $L_a$ and $L_b$ by $y$. In the subsequent estimates, we use Lemma III.6.4 and the fact that by Lemma III.6.2,

$$
\|y_{\Pi_h \bar u} - \bar y\|_{L^2(\Omega)} \le c \|\Pi_h \bar u - \bar u\|_{W^{-1,2}} \le K_2 h \|\bar u\|_{L^2(\Omega)}.
$$

Now we prepare for the estimation. By Taylor's theorem

$$
\begin{aligned}
J(\Pi_h(\bar u)) - J(\bar u) &= \int_\Omega L(x, y_{\Pi_h(\bar u)}, \Pi_h(\bar u)) - L(x, y_{\bar u}, \bar u)\,\mathrm{d}x = \int_\Omega L_{a,y}(x, y_\theta)(y_{\Pi_h \bar u} - y_{\bar u})\,\mathrm{d}x \\
&+ \int_\Omega L_{b,y}(x, y_\theta)(y_{\Pi_h \bar u} - y_{\bar u})\Pi_h \bar u\,\mathrm{d}x + \int_\Omega L_b(x, y_{\bar u})(\Pi_h \bar u - \bar u)\,\mathrm{d}x \\
&= \int_\Omega L_{a,y}(x, y_{\bar u})(y_{\Pi_h \bar u} - y_{\bar u})\,\mathrm{d}x + \int_\Omega L_{b,y}(x, y_{\bar u})(y_{\Pi_h \bar u} - y_{\bar u})\bar u\,\mathrm{d}x \\
&+ \int_\Omega (L_{a,y}(x, y_\theta) - L_{a,y}(x, y_{\bar u}))(y_{\Pi_h \bar u} - y_{\bar u})\,\mathrm{d}x + \int_\Omega (L_{b,y}(x, y_\theta) - L_{b,y}(x, y_{\bar u}))(y_{\Pi_h \bar u} - y_{\bar u})\Pi_h \bar u\,\mathrm{d}x \\
&+ \int_\Omega L_{b,y}(x, y_{\bar u})(y_{\Pi_h \bar u} - y_{\bar u})(\Pi_h \bar u - \bar u)\,\mathrm{d}x + \int_\Omega L_b(x, y_{\bar u})(\Pi_h \bar u - \bar u)\,\mathrm{d}x.
\end{aligned}
$$

Thus,

$$\begin{aligned}J(\Pi_h(\bar u)) - J(\bar u) &= \int_\Omega (L_{a,y}(x,y_{\bar u}) + L_{b,y}(x,y_{\bar u})\bar u) z_{\bar u,\Pi_h\bar u-\bar u}\,\mathrm{d}x + \int_\Omega L_b(x,y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x\\ &+ \int_\Omega (L_{a,y}(x,y_{\bar u}) + L_{b,y}(x,y_{\bar u})\Pi_h\bar u)(y_{\Pi_h\bar u} - y_{\bar u} - z_{\bar u,\Pi_h\bar u-\bar u})\,\mathrm{d}x\\ &+ \int_\Omega (L_{a,y}(x,y_\theta) - L_{a,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\,\mathrm{d}x\\ &+ \int_\Omega (L_{b,y}(x,y_\vartheta) - L_{b,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\Pi_h\bar u\,\mathrm{d}x\\ &+ \int_\Omega L_{b,y}(x,y_{\bar u})(y_{\Pi_h\bar u}-y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x = \sum_{i=1}^6 K_i.\end{aligned}$$

We provide the estimates for (6.45) under Assumption III.4.1($\beta = 1/2$). Integration by parts, Lemma III.6.3 and Lemma III.6.4 guarantee the existence of a positive constant $c$ such that

$$|K_1 + K_2| = \int_\Omega (p_{\bar u} + L_b(x,y_{\bar u}))(\Pi_h\bar u - \bar u)\,\mathrm{d}x \le ch^{1+\gamma}.$$

The term $K_3$ can be estimated using (2.10) and (6.41)

$$|K_3| \le \|\psi_M\|_{L^2(\Omega)}\|y_{\Pi_h\bar u} - y_{\bar u}\|^2_{L^2(\Omega)} \le ch^2.$$

For the terms $K_4$ and $K_5$, we use (6.41) and the local Lipschitz property of $L_{a,y}$ and $L_{b,y}$ to infer the existence of a constant $c$ such that

$$|K_4|, |K_5| \le c\|y_{\Pi_h\bar u} - y_{\bar u}\|^2_{L^2(\Omega)} \le ch^2.$$

Finally, for some number $r > n/2$ and a positive constant $c$, we estimate $|K_6| \le c\|\Pi_h\bar u - \bar u\|^{1+1/r}_{L^1(\Omega)} \le ch^{1+1/r}$. If the term $K_6$ is absent, we obtain estimate (6.45), if not (6.46) holds. To continue the proof of (6.45), we conclude from the estimates of the terms $I_i$, by Theorem III.4.8 and Theorem 4.28, the existence of a positive constants $c$,$k$ and $\alpha$ such that

$$kh^{1+\gamma} \ge J(\bar u_h) - J(\bar u) \ge c\|\bar u_h - \bar u\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \text{ for all } \bar u_h \text{ with } \|\bar u_h - \bar u\|_{L^1(\Omega)} < \alpha. \tag{6.47}$$

But this is equivalent to

$$(c/k)^{\frac{\gamma}{\gamma+1}} h^\gamma \ge \|\bar u_h - \bar u\|_{L^1(\Omega)} \text{ for all } \bar u_h \text{ with } \|\bar u_h - \bar u\|_{L^1(\Omega)} < \alpha.$$

Using the estimate of the distance of the controls and (6.36), we estimate the distance of the states and the adjoint states, which completes the proof. The proof of the claims (6.43) and (6.44) follows by similar arguments. □

**Remark III.6.6.** *Assumption III.4.1($\beta = 1/2$) with $\gamma \in (n/(n+2), 1]$, is used to guarantee the existence of positive constants $c$ and $\alpha$ such that*

$$J(\bar{u}_h) - J(\bar{u}) \geq c\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}, \text{ for all } \bar{u}_h \text{ with } \|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} < \alpha. \tag{6.48}$$

*If $\frac{\partial^2 L}{\partial y \partial u} = 0$ holds for the objective integrand, then the growth (6.48) can be obtained by considering Assumption III.4.1($\beta = 1/2,\ \gamma \in (0, 1]$) together with Assumption III.4.3($\beta = 1/2$). To see this, let $c$ and $\alpha$ be positive constants for that Assumption III.4.1 and III.4.3 are both satisfied simultaneously. Then, applying Taylor's theorem, the assumptions, and an estimate on the distance of the second variations yields*

$$\begin{aligned} J(\bar{u}_h) &= J(\bar{u}) + J'(\bar{u})(\bar{u}_h - \bar{u}) + 1/2J''(\bar{u}_\theta)(\bar{u}_h - \bar{u})^2 \\ &\geq J(\bar{u}) + 1/2J'(\bar{u})(\bar{u}_h - \bar{u}) + 1/4J''(\bar{u})(\bar{u}_h - \bar{u})^2 \\ &+ 1/2J'(\bar{u})(\bar{u}_h - \bar{u}) + 1/4J''(\bar{u})(\bar{u}_h - \bar{u})^2 \\ &- 1/2\Big|J''(\bar{u}_\theta)(\bar{u}_h - \bar{u})^2 - J''(\bar{u})(\bar{u}_h - \bar{u})^2\Big| \\ &\geq c/2\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + c/4\|z_{\bar{u},\bar{u}_h - \bar{u}}\|_{L^2(\Omega)} \geq c/2\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}. \end{aligned}$$

*Thus, the constraint $\gamma \in (n/(2+n), 1]$ can be weakened to $\gamma \in (0, 1]$ for the cost of making both the assumptions III.4.1 and III.4.3 at the same time.*

### III.6.3 Variational discretization

We prove that Assumptions III.4.1, III.4.2 and III.4.3 with $\beta = 1$ are sufficient for approximation error estimates for a variational discretization. We refer to the [15] for the idea and introduction of variational discretization. Although we consider weaker conditions than the ones in [7], the estimates given in Theorem III.6.8 below agree with the estimates in [7, Remark 7] for the variational discretization.

**Theorem III.6.7.** *[7, Theorem 9]. Let $\bar{u}_h$ denote a solution to (6.38). We denote by $y_{\bar{u}_h}$ and $p_{\bar{u}_h}$ the solution to the continuous state equation and to the corresponding adjoint equation with respect to $\bar{u}_h$. By $p(\bar{u}_h)$ we denote the discrete adjoint equation corresponding to $\bar{u}_h$ and $p^h_{\bar{u}_h}$ denotes the solution to the following equation*

$$\begin{cases} \mathcal{A}^* p + \frac{\partial f}{\partial y}(\cdot, \bar{y}_h)p &= \frac{\partial L}{\partial y}(\cdot, y(\bar{u}_h)) \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma. \end{cases}$$

*Then the following estimates hold*

$$\|p_{\bar{u}_h} - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} \leq ch^2 \tag{6.49}$$

$$\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} \leq ch^2|\log h|^2. \tag{6.50}$$

We come to the error estimates for the variational discretization.

**Theorem III.6.8.** *Let $\{\bar{u}_h\}_h$ be a sequence of solutions to the first-order optimality condition of the discrete problems such that $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} < \alpha$. Here, $\alpha$ is the constant appearing in Proposition III.4.7 or Proposition III.4.9 depending on the selected growth assumption.*

1. *Let Assumption III.4.3($\beta = 1$) be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^\infty(\Omega)} \le ch. \tag{6.51}$$

2. *Let Assumption III.4.2($\beta = 1$) be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant $c$ independent of $h$ such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^\infty(\Omega)} \le c(h|\log h|)^2. \tag{6.52}$$

3. *Let Assumption III.4.1($\beta = 1$) be satisfied by $\bar{u} \in \mathcal{U}$ for some $\gamma \in (n/(2+n), 1]$. There exists a positive constant $c$ independent of $h$ such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^\infty(\Omega)} \le c(h|\log h|)^{2\gamma}. \tag{6.53}$$

*Proof.* We consider (6.53). Since $\bar{u}_h$ satisfies the first-order necessary optimality condition of the discrete problem, it holds

$$0 \ge J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}) = J'(\bar{u}_h)(\bar{u}_h - \bar{u}) + J_h'(\bar{u}_h)(\bar{u}_h - u) - J'(\bar{u}_h)(\bar{u}_h - \bar{u}),$$

and by Proposition III.4.7, (4.27)

$$c\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \le J'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}).$$

We use that $\bar{u}_h = \bar{u}$ on $\Omega \setminus \Omega_h$ by definition and write

$$J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J'(\bar{u}_h)(\bar{u}_h - \bar{u}) = \int_{\Omega_h} (p(\bar{u}_h) + L_b(x, y(\bar{u}_h)))(\bar{u}_h - \bar{u})\,\mathrm{d}x$$
$$- \int_{\Omega_h} (p_{\bar{u}_h} + L_b(x, y_{\bar{u}_h}))(\bar{u}_h - \bar{u})\,\mathrm{d}x = I.$$

To estimate the term $I$, we follow similar reasoning as in [7], using (6.37), (6.49), (6.50) and also using the local Lipschitz property of $L_b$ with respect to $y$, to infer

$$\begin{aligned}
I &\le C_{L,M}(\|\bar{p}(\bar{u}_h) - p_{\bar{u}_h}\|_{L^\infty(\Omega)} + \|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^\infty(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&\le C_{L,M}(\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} + \|p^h_{\bar{u}_h} - p_{\bar{u}_h}\|_{L^\infty(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&+ C_{L,M}\|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^\infty(\Omega)}\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&\le CC_{L,M}(h^2 + 2h^2|\log h|^2)\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)},
\end{aligned}$$

for some positive constant $C$. Altogether, we obtain

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \le CC_{L,M}(h^2 + 2h^2|\log h|^2)^\gamma. \tag{6.54}$$

Applying the estimates (6.36), (6.49) and (6.50) the claim (6.53) holds for the controls. For the states we use (6.37) to find

$$\|y(\bar{u}_h) - y_{\bar{u}}\|_{L^2(\Omega)} \leq \|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^2(\Omega)} + \|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \leq ch^2 + \|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}$$

and the estimate follows from (6.54). For Assumption III.4.2, by (4.29) of Proposition III.4.9

$$c\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^1(\Omega)} \leq J'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}).$$

Estimating as before, we obtain the existence of a positive constant $c$ that satisfies

$$\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \leq c(h^2 + h^2|\log h|^2).$$

By again (6.36), (6.49) and (6.50) the claim (6.52) holds. Finally, consider Assumption III.4.3. To estimate the term $I$, we use (6.49)-(6.50) to find

$$\begin{aligned} I &\leq \|p(\bar{u}_h) - p_{\bar{u}_h}\|_{L^2(\Omega)}\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \\ &\leq (\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^2(\Omega)} + \|p^h_{\bar{u}_h} - p_{\bar{u}_h}\|_{L^2(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \leq 2ch^2\|u_a - u_b\|_{L^\infty(\Omega)}, \end{aligned}$$

for some positive constant $c$. Taking the root, this leads to the estimate $\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \leq ch$, and by (6.36), (6.49) and the claim (6.51) holds. □

For a numerical example supporting the theoretical error estimates achieved in this paper, especially for the case $\gamma < 1$, we refer to [7].

### III.6.4 Solution stability and variational discretization

In this subsection, we aim to provide a link between the property of solution stability and the obtainment of finite element error estimates. In this sense, Theorem III.6.10 below shows that a property related to solution stability implies estimates for a finite element variational discretization scheme. The intuition is that once solution stability is achieved under certain growth conditions for an affine optimal control problem, we expect error estimates for a variational discretization scheme under the same conditions and strength.

Let us now define a property, which we will call strong solution stability.

**Definition III.6.9** (Strong solution stablility)**.** *We call the optimal control problem* (1.2)-(1.4) *strong solution stable at $\bar{u}$ for $V \subset \mathcal{U}$, with positive parameters $\kappa, \gamma$ and $\alpha$ if*

$$\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)} + \|\bar{y} - \bar{y}^\zeta\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}^\zeta\|_{L^\infty(\Omega)} \leq \kappa\Big(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)}\Big)^\gamma \qquad (6.55)$$

*for all triples $(\bar{u}^\zeta, \bar{y}^\zeta, \bar{p}^\zeta)$ related to the perturbed problem* (5.30)-(5.31) *that satisfy $\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)} < \alpha$ and*

$$J_\zeta'(\bar{u}^\zeta)(v - \bar{u}^\zeta) \geq 0 \quad \textit{for all } v \in V.$$

Assumption III.4.1 implies strong solution stability. This can be observed by investigating the proof of the strong metric subregularity of the optimality map in [12].

**Theorem III.6.10.** *Let the optimal control problem be strong solution stable at $\bar{u}$ for $\{\bar{u}\}$ with positive constants $\gamma, \kappa$ and $\alpha$. Let $\{\bar{u}_h\}_h$ be a sequence of solutions to the discrete problems* (6.38) *with $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} < \alpha$. Then*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^\infty(\Omega)} \leq \kappa(h|\log h|)^{2\gamma}. \tag{6.56}$$

*Proof.* Given a minimizer $\bar{u}_h$ of the discrete problem (6.38), let $\zeta = (0, 0, \rho)$, with $\rho := p(\bar{u}_h) - p_{\bar{u}_h}$. Then we define the perturbed optimal control problem

$$\min_{u \in \mathcal{U}} \left\{ J_\zeta(u) := \int_\Omega L(x, y(x), u(x))\, \mathrm{d}x + \int_\Omega (p(\bar{u}_h) - p_{\bar{u}_h}) u\, \mathrm{d}x \right\},$$

subject to

$$\begin{cases} Ay + f(\cdot, y) = u & \text{in } \ \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

It is easy to see that

$$J'_\zeta(\bar{u}_h)(\bar{u} - \bar{u}_h) = \int_\Omega p(\bar{u}_h)(\bar{u} - \bar{u}_h)\, \mathrm{d}x \geq 0. \tag{6.57}$$

But that is all we need of $\bar{u}_h$ to apply the strong solution stability at $\bar{u}$. That is, we obtain

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} \leq \kappa \|(p(\bar{u}_h) - p_{\bar{u}_h})\|^\gamma_{L^\infty(\Omega)}.$$

By Theorem III.2.1, Lemma III.2.2, Lemma III.6.1 and Theorem III.6.7 the claim follows. □

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## III.7 Appendix

For the reader's convenience, we collect some facts that relate the assumptions made in this paper to the usual assumptions in the literature. Also we relate the cases $\beta \in \{1/2, 1\}$ of Assumption III.4.1.

**Proposition III.7.1.** *Let Assumption III.4.1($\beta = 1/2$) be satisfied with constants $c$ and $\alpha$. Let there exist $\mu$ such that $c > \mu$ and*

$$J''(\bar{u}) \geq -\mu \|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \quad \text{for all } u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^1(\Omega)} < \alpha. \tag{7.58}$$

*Then Assumption III.4.1($\beta = 1$) is satisfied with constant $c := \gamma - \mu/2$.*

Let us define cones appearing in affine PDE-constrained optimal control.

**Definition III.7.2.** *We consider the set*

$$\Big\{v \in L^2(\Omega)\Big| v \geq 0 \text{ a.e. on } [\bar{u} = u_a] \text{ and } v \leq 0 \text{ a.e. on } [\bar{u} = u_b]\Big\}. \tag{7.59}$$

*Given $\tau > 0$, we define the sets*

$$\begin{aligned} D^\tau_{\bar{u}} &:= \Big\{v \in L^2(\Omega)\Big| v \text{ satisfies (7.59) and } v(x) = 0 \text{ if } \Big|\frac{\partial \bar{H}}{\partial u}(x)\Big| > \tau\Big\},\\ G^\tau_{\bar{u}} &:= \Big\{v \in L^2(\Omega)\Big| v \text{ satisfies (7.59) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^1(\Omega)}\Big\},\\ C^\tau_{\bar{u}} &:= D^\tau_{\bar{u}} \cap G^\tau_{\bar{u}}. \end{aligned}$$

*Here, $\bar{H}$ denotes the Hamiltonian* (3.15) *corresponding to the reference control $\bar{u}$.*

Assumption III.4.1 can be considered as acting only on the cone $D^\tau_{\bar{u}}$. For $\beta = 1$, a proof for elliptic problems was done in [11, Proposition 6.2] and for parabolic problems in [4, Corollary 14]. Thus, to ask for Assumption III.4.1 to be satisfied on cones considered in the context of second-order sufficient optimality conditions is not relaxation.

**Proposition III.7.3.** *[11, Proposition 6.2]. Assumption III.4.1, $\beta \in \{1/2, 1\}$, is equivalent to the assumption:*
*Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants $c$ and $\alpha$, such that*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \tag{7.60}$$

*for all $u \in \mathcal{U}$ with $(u - \bar{u}) \in D^\tau_{\bar{u}}$ and $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.*

Further, we have the following theorem that relates Assumption III.4.1 to assumptions made in [7, Theorem 9].

**Theorem III.7.4.** *Let $\frac{\partial L_b}{\partial y} = 0$ and let $\bar{u}$ satisfy the following conditions: There exist positive constants $c, k$ and $\alpha$ with $k < c$ such that*

$$J'(\bar{u})(u - \bar{u}) \geq c\|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \text{ for all } u \in \mathcal{U}, \tag{7.61}$$

*and*

$$J''(\bar{u})(u - \bar{u}) \geq -k\|u - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)} \text{ for all } (u - \bar{u}) \in C^\tau_{\bar{u}} \text{ with } \|u - \bar{u}\|_{L^1(\Omega)} < \alpha. \tag{7.62}$$

*Then Assumption III.4.1, $\beta \in \{1/2, 1\}$, holds for some appropriate constants.*

*Proof.* By Proposition III.7.3, it is sufficient to prove the statement for the Assumption III.4.1 on the cone $D^\tau_{\bar{u}}$. Thus, we only need to consider the case $(u - \bar{u}) \notin G^\tau_{\bar{u}}$. But by definition of $(u - \bar{u}) \notin G^\tau_{\bar{u}}$, $J'(\bar{u})(u - \bar{u}) > \tau\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)}$. We estimate for some constant $d$ independent of $u$

$$\Big|J''(\bar{u})(u - \bar{u})^2\Big| \leq d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)}\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)}.$$

By the assumption of the theorem, it also holds

$$J'(\bar{u})(u-\bar{u}) \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}.$$

Thus combining the estimates we obtain for $\|u-\bar{u}\|_{L^1(\Omega)}$ sufficiently small

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \frac{1}{2}J'(\bar{u})(u-\bar{u}) + (\frac{1}{2}\tau - d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)})\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)} \\
&\geq c/2\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + (\frac{1}{2}\tau - d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)})\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)} \geq c/2\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}.
\end{aligned}$$

□

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# IV. On the Solution Stability of Parabolic Optimal Control Problems

## Outline



## Bibliographic Information

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## Abstract

The paper investigates stability properties of solutions of optimal control problems constrained by semilinear parabolic partial differential equations. Hölder or Lipschitz dependence of the optimal solution on perturbations is obtained for problems in which the equation and the objective functional are affine with respect to the control. The perturbations may appear in both the equation and in the objective functional and may nonlinearly depend on the state and control variables. The main results are based on an extension of recently introduced assumptions on the joint growth of the first and second variation of the objective functional. The stability of the optimal solution is obtained as a consequence of a more general result obtained in the paper – the metric subregularity of the mapping associated with the system of first-order necessary optimality conditions. This property also enables error estimates for approximation methods. A Lipschitz estimate for the dependence of the optimal control on the Tikhonov regularization parameter is obtained as a by-product.

## IV.1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For a finite $T > 0$, denote by $Q := \Omega \times (0,T)$ the space-time cylinder and by $\Sigma := \partial\Omega \times (0,T)$ its lateral boundary. In the present paper, we investigate the following optimal control problem:

$$\text{(P)} \quad \min_{u\in\mathcal{U}} \left\{ J(u) := \int_Q L(x,t,y(x,t),u(x,t))\,\mathrm{d}x\,\mathrm{d}t \right\}, \tag{1.1}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(\cdot,y) = u & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

Here $y : Q \to \mathbb{R}$ is the state, $u : Q \to \mathbb{R}$ is the control and $\mathcal{A}$ is an elliptic operator. For functions $u_a, u_b \in L^\infty(Q)$ such that $u_a < u_b$ a.e in $Q$, the set of feasible controls is given by

$$\mathcal{U} := \{u \in L^\infty(Q) |\; u_a \leq u \leq u_b \;\text{ for a.a. } (x,t) \in Q\}. \tag{1.3}$$

Denote by $y_u$ the unique solution to the semilinear parabolic equation (1.2) that corresponds to the control $u \in L^r(Q)$, where $r$ is a fixed number satisfying the inequality $r > 1 + \frac{n}{2}$. The objective integrand in (1.1) is defined as

$$L(x,t,y,u) := L_0(x,t,y) + (my + g)u, \tag{1.4}$$

where $m$ is a number, $g$ is a function in $L^\infty(Q)$ and $L_0$ satisfies appropriate smoothness condition (see Assumption IV.1.2 in Subsection IV.1.1).

The goal of the present paper is to obtain stability results for the optimal solution of problem (1.1)–(1.3). The meaning of "stability" we focus on, is as follows. Given a reference optimal control $\bar{u}$ and the corresponding solution $\bar{y}$, the goal is to estimate the distance (call it $\Delta$) from the optimal solutions $(u, y_u)$ of a disturbed version of problem (1.1)–(1.3) to the pair $(\bar{u}, \bar{y})$, in terms of the size of the perturbations (call it $\delta$). The perturbations may enter either in the objective integrand or in the state equation, and the meaning of "distance" and "size" in the previous sentence will be clarified in the sequel in terms of appropriate norms. If an estimation $\Delta \leq \text{const.}\delta^\theta$ holds with $\theta \in (0,1)$, we talk about *Hölder stability*, while in the case $\theta = 1$ we have *Lipschitz stability*.

A powerful technique for establishing stability properties of the solutions of optimization problems is based on regularity properties of the system of first-order necessary optimality conditions (see e.g. [18]). In the case of problem (1.1)–(1.3), these are represented by a *differential variational inequality* (see e.g. [25, 15]), consisting of two parabolic equations (the primal equation (1.1) and the corresponding adjoint equation) and one variational inequality representing the condition for minimization of the Hamiltonian associated with the problem. The Lipschitz or Hölder stability of the solution of problem (1.1)–(1.3) is then a consequence of the property of *metric subregularity* (see [18, 16]) of the mapping defining this differential variational inequality. An advantage of this approach is that it unifies in a compact way the study of the stability of optimal solutions under a variety of perturbations (linear or nonlinear). Therefore, the main result in the present paper focuses on conditions for metric subregularity of the mapping associated with the first-order optimality conditions for problem (1.1)–(1.3). These conditions are related to appropriate second-order sufficient optimality conditions, which are revisited and extended in the paper. Several results for stability of the solutions are obtained as a consequence.

The commonly used second-order sufficient optimality conditions for ODE or PDE optimal control problems involve a *coercivity condition*, requiring strong positive definiteness of the objective functional as a function of the control in a Hilbert space. We stress that problem (1.1)–(1.3) is affine with respect to the control variable and such a coercivity condition is not fulfilled. The theory of sufficient optimality conditions and the regularity theory for affine optimal control of ODE systems have been developed in the past decade, see [24] and the bibliography therein. Sufficient conditions for weak or strong local optimality for optimal control problems with constraints given by elliptic or parabolic equations are developed in [3, 12, 10, 2, 17, 5, 8]. A detailed discussion thereof is provided in Section IV.2.1. In contrast with the elliptic setting, there are only a few stability results for semilinear parabolic optimal control problems. Results in this regard for a tracking type objective functional were obtained for instance in [10, 9] where stability with respect to perturbations in the objective functional was studied, and in [11], where stability with respect to perturbations in the initial data was investigated. We mention that for a linear state equation and a tracking type objective functional, Lipschitz estimates were obtained in [30] under an additional assumption on the structure of the optimal control. A more comprehensive discussion about the sufficiency theory and stability can be found in Section IV.2.

The main novelty in the present paper is the study of the subregularity property of the optimality mapping associated with problem (1.1)-(1.3). In contrast with the case of coercive problems, our assumptions in the affine case jointly involve the first and second-order variations of the objective functional with respect to the control. These assumptions are weaker than the ones in the existing literature in the context of sufficient optimality conditions, however, they are strong enough to imply metric subregularity of the optimality mapping. The subregularity result is used to obtain new Hölder- and Lipschitz estimates for the solution of the considered optimal control problem. An error estimate for the Tikhonov regularization is obtained as a consequence.

The obtained subregularity result provides a base for convergence and error analysis for discretization methods applied to problem (1.1)-(1.3). The point is, that numerical solutions of the discretized versions of the problem typically satisfy approximately first-order optimality conditions for the discretized problem and after appropriate embedding in the continuous setting (1.1)-(1.3), satisfy the optimality conditions for the latter problem with a residual depending on the approximation and the discretization error. Then the subregularity property of the optimality mapping associated with (1.1)-(1.3) provides an error estimate. Notice that the (Lipschitz) stability of the solution alone is not enough for such a conclusion, and this is an important motivation for studying subregularity of the optimality mapping rather than only stability of the solutions. However, we do not go into this subject, postponing it to a later paper based on the present one.

The paper is organized as follows. The analysis of the optimal control problem (1.1)-(1.3) begins in Section IV.2. We recall the state of the art regarding second-order sufficient conditions for weak and strong (local) optimality, as well as known sufficient conditions for stability of optimal controls and states under perturbations. In Section IV.3 we formulate and discuss the assumptions on which our further analysis on sufficiency and stability is based. The strong subregularity of the optimality mapping is proved in Section IV.4. In Section IV.5, we obtain stability results for the optimal control problem under non-linear perturbations, postponing some technicalities to Appendix IV.7. Finally, we support the theoretical results with some examples.

### IV.1.1 Preliminaries

We begin with some basic notations and definitions. Given a non-empty, bounded and Lebesgue measurable set $X \subset \mathbb{R}^n$, we denote by $L^p(X)$, $1 \le p \le \infty$, the Banach spaces of all measurable functions $f : X \to \mathbb{R}$ for which the usual norm $\|f\|_{L^p(X)}$ is finite. For a bounded Lipschitz domain $X \subset \mathbb{R}^n$ (that is, a set with Lipschitz boundary), the Sobolev space $H_0^1(X)$ consists of functions that vanish on the boundary (in the trace sense) and that have weak first-order derivatives in $L^2(X)$. The space $H_0^1(X)$ is equipped with its usual norm denoted by $\|\cdot\|_{H_0^1(X)}$. By $H^{-1}(X)$ we denote the topological dual of $H_0^1(X)$, equipped with the standard norm $\|\cdot\|_{H^{-1}(X)}$. Given a real Banach space $Z$, the space $L^p(0,T;\, Z)$ consist of all strongly measurable functions $y : [0,T] \to Z$ that satisfy

$$\|y\|_{L^p(0,T;\, Z)} := \Big(\int_0^T \|y(t)\|_Z^p \,\mathrm{d}t\Big)^{\frac{1}{p}} < \infty \qquad \text{if } 1 \le p < \infty,$$

or, for $p = \infty$,

$$\|y\|_{L^\infty(0,T;\, Z)} := \inf\{M \in \mathbb{R} \mid \|y(t)\|_Z \le M \text{ for a.e } t \in (0,T)\} < \infty.$$

The Hilbert space $W(0,T)$ consists of all of functions in $L^2(0,T;\, H_0^1(\Omega))$ that have a distributional derivative in $L^2(0,T;\, H^{-1}(\Omega))$, i.e.

$$W(0,T) := \left\{ y \in L^2(0,T; H_0^1(\Omega)) \Big| \; \frac{\partial y}{\partial t} \in L^2(0,T; H^{-1}(\Omega)) \right\},$$

which is endowed with the norm

$$\|y\|_{W(0,T)} := \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial y/\partial t\|_{L^2(0,T;H^{-1}(\Omega))}.$$

The Banach space $C([0,T];\, L^2(\Omega))$ consists of all continuous functions $y : [0,T] \to L^2(\Omega)$ and is equipped with the norm $\max_{t\in[0,T]} \|y(t)\|_{L^2(\Omega)}$. It is well known that $W(0,T)$ is continuously embedded in $C([0,T];\, L^2(\Omega))$ and compactly embedded in $L^2(Q)$. The duality pairing between a Banach space $X$ and its dual is denoted by $\langle \cdot, \cdot \rangle_X$. For proofs and further details regarding spaces involving time, see [31, 14, 20, 28].

The following assumptions, close to those in [2, 10, 11, 8, 5, 6, 12, 13], are standing in all the paper, together with the inequality

$$r > \max\left\{2, 1 + \frac{n}{2}\right\} \tag{1.5}$$

for the real number $r$ that appears in some assumptions and many statements below (we also remind that $n \in \{1, 2, 3\}$). Although for $n = 1$ it is admissible to have $r = 2$ (instead of $r > 2$), we keep the above restriction in order to treat all the cases in a unified way.

**Assumption IV.1.1.** *The operator* $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$, *is given by*

$$\mathcal{A}y = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{i,j}(x)\partial_{x_i}y),$$

*where* $a_{i,j} \in L^\infty(\Omega)$ *satisfy the uniform ellipticity condition*

$$\exists \lambda_{\mathcal{A}} > 0 : \; \lambda_{\mathcal{A}}|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \quad \text{for all } \xi \in \mathbb{R}^n \text{ and a.a. } x \in \Omega.$$

The functions $f$, $L_0 : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ of the variables $(x, t, y)$, and the "initial" function $y_0$ have the following properties.

**Assumption IV.1.2.** *For every* $y \in \mathbb{R}$, *the functions* $f(\cdot, \cdot, y) \in L^r(Q)$, $L_0(\cdot, \cdot, y) \in L^1(Q)$, *and* $y_0 \in L^\infty(\Omega)$. *For a.e.* $(x, t) \in Q$ *the first and the second derivatives of* $f$ *and* $L_0$ *with respect to* $y$ *exist and are locally bounded and locally Lipschitz continuous, uniformly with respect to* $(x, t) \in Q$. *Moreover,* $\frac{\partial f}{\partial y}(x, t, y) \ge 0$ *for a.e.* $(x, t) \in Q$ *and for all* $y \in \mathbb{R}$.

**Remark IV.1.3.** *The last condition in Assumption IV.1.2 can be relaxed in the following way:*

$$\exists C_f \in \mathbb{R} : \; \frac{\partial f}{\partial y}(x, t, y) \ge C_f \quad \text{a.a. } (x, t) \in Q \text{ and } \forall y \in \mathbb{R},$$

*see [8, 5]. However, this leads to complications in the proofs.*

### IV.1.2 Facts regarding the linear and the semilinear equation

Let $0 \le \alpha \in L^\infty(Q)$ and $u \in L^2(Q)$. We consider solutions of the following linear variational equality for $h \in W(0, T)$ with $h(\cdot, 0) = 0$:

$$\int_0^T \langle \frac{\partial h}{\partial t} + \mathcal{A}h, \psi \rangle_{H_0^1(\Omega)} \, dt = \int_0^T \langle u - \alpha h, \psi \rangle_{L^2(\Omega)} \, dt \tag{1.6}$$

for all $\psi \in L^2(0, T, H_0^1(\Omega))$, that is, for weak solutions of the equation (1.2) with $f(x, t, h) := \alpha(x, t)h$ and zero initial datum.

**Theorem IV.1.4.** *Let $0 \le \alpha \in L^\infty(Q)$ be given.*

1. *For each $u \in L^2(Q)$ the linear parabolic equation (1.6) has a unique weak solution $h_u \in W(0,T)$. Moreover, there exists a constant $C_2 > 0$ independent of $u$ and $\alpha$ such that*

$$\|h_u\|_{L^2(0,T,H_0^1(\Omega))} \le C_2 \|u\|_{L^2(Q)}. \tag{1.7}$$

2. *If, additionally, $u \in L^r(Q)$ (we remind (1.5)) then the weak solution $h_u$ of (1.6) belongs to $W(0,T) \cap C(\bar{Q})$. Moreover, there exists a constant $C_r > 0$ independent of $u$ and $\alpha$ such that*

$$\|h_u\|_{L^2(0,T,H_0^1(\Omega))} + \|h_u\|_{C(\bar{Q})} \le C_r \|u\|_{L^r(Q)}. \tag{1.8}$$

Besides the independence of the constants $C_2$, and $C_r$ on $\alpha$ all claims of the theorem are well known, see [29, Theorem 3.13, Theorem 5.5]. A proof of a similar independence statement can be found in [2] for a linear elliptic PDE of non-monotone type. We further remark that item 2 of Theorem IV.1.4 is true in dimension $n = 1$ even for $r = 2$, see [21, Section III.7].

*Proof.* For the convenience of the reader, we prove that the estimates are independent of $\alpha$. This is done along the lines of the proof of [2, Lemma 2.2]. By $h_{0,u}$ we denote a solution of (1.6) for $\alpha = 0$. It is well known that in this case there exist positive constants $C_r, C_2$ such that

$$\|h_{0,u}\|_{C(\bar{Q})} \le C_r \|u\|_{L^r(Q)}, \quad \|h_{0,u}\|_{L^2(Q)} \le C_2 \|u\|_{L^2(Q)}.$$

To apply this, we decompose $u$ in positive and negative parts, $u = u^+ - u^-$, $u^+, u^- \ge 0$. By the weak maximum principle [14, Theorem 11.9], it follows that $h_{\alpha,u^+}, h_{\alpha,u^-} \ge 0$. Again by the weak maximum principle, the equation

$$\frac{\partial}{\partial t}(h_{\alpha,u^+} - h_{0,u^+}) + \mathcal{A}(h_{\alpha,u^+} - h_{0,u^+}) + \alpha(h_{\alpha,u^+} - h_{0,u^+}) = -\alpha h_{0,u^+}$$

implies $0 \le h_{\alpha,u^+} \le h_{0,u^+}$, thus $\|h_{\alpha,u^+}\|_{C(\bar{Q})} \le \|h_{0,u^+}\|_{C(\bar{Q})}$. By the same reasoning, it follows that $0 \le h_{\alpha,u^-} \le h_{0,u^-}$ and $\|h_{\alpha,u^-}\|_{C(\bar{Q})} \le \|h_{0,u^-}\|_{C(\bar{Q})}$. Hence,

$$\begin{aligned}\|h_{\alpha,u}\|_{C(\bar{Q})} &\le \|h_{\alpha,u^+}\|_{C(\bar{Q})} + \|h_{\alpha,u^-}\|_{C(\bar{Q})} \le \|h_{0,u^+}\|_{C(\bar{Q})} + \|h_{0,u^-}\|_{C(\bar{Q})} \\ &\le C_r(\|u^+\|_{L^r(Q)} + \|u^-\|_{L^r(Q)}) \le 2C_r\|u\|_{L^r(Q)}.\end{aligned}$$

The estimate for $L^2(0,T,H_0^1(\Omega))$ can be obtained by similar arguments as in [2]. □

The next lemma is motivated by an analogous result for linear elliptic equations [2, Lemma 2.3], although, according to the nature of the parabolic setting, the interval of feasible numbers $s$, is smaller.

**Lemma IV.1.5.** *Let $u \in L^r(Q)$ and $0 \le \alpha \in L^\infty(Q)$. Let $h_u$ be the unique solution of (1.6) and let $p_u$ be a solution of the problem*

$$\begin{cases} -\frac{\partial p}{\partial t} + \mathcal{A}^* p + \alpha p = u \quad in \quad Q, \\ p = 0 \ \ on \ \Sigma, \ \ p(\cdot, T) = 0 \quad on \ \Omega. \end{cases} \tag{1.9}$$

*Then, for any $s_n \in [1, \frac{n+2}{n})$ there exists a constant $C_{s_n'} > 0$ independent of $u$ and $\alpha$ such that*

$$\max\{\|h_u\|_{L^{s_n}(Q)}, \|p_u\|_{L^{s_n}(Q)}\} \le C_{s_n'} \|u\|_{L^1(Q)}. \tag{1.10}$$

*Here $s_n'$ denotes the Hölder conjugate of $s_n$.*

*Proof.* First we observe that by Theorem IV.1.4, $h_u \in C(\bar{Q}) \cap W(0,T)$ and as a consequence, $|h_u|^{s_n - 1}\text{sign}(h_u) \in L^{s_n'}(Q)$. Moreover, $s_n < \frac{n+2}{n}$ implies that $s_n' > 1 + \frac{n}{2}$. By change of variables, see for instance [29, Lemma 3.17], a solution of equation (1.9) transforms into a solution of (1.6). Thus according to Theorem IV.1.4, the solution $q$ of

$$\begin{cases} -\frac{\partial q}{\partial t} + \mathcal{A}^* q + \alpha q = |h_u|^{s_n - 1}\text{sign}(h_u) \ \ \text{in} \ \ Q, \\ q = 0 \text{ on } \Sigma, \ \ q(\cdot, T) = 0 \ \ \text{on } \Omega. \end{cases}$$

belongs to $W(0,T) \cap C(\bar{Q})$ and satisfies

$$\|q\|_{C(\bar{Q})} \le C_{s_n'} \||h_u|^{s_n - 1}\text{sign}(h_u)\|_{L^{s_n'}(Q)} = C_{s_n'} \|h_u\|^{s_n - 1}_{L^{s_n}(Q)},$$

where $C_{s_n'}$ is independent of $\alpha$ and $v$. Using these facts we derive the equalities

$$\begin{aligned} \|h_u\|^{s_n}_{L^{s_n}(Q)} &= \int_Q |h_u|^{s_n} \, \mathrm{d}x = \Big\langle -\frac{\partial q}{\partial t} + \mathcal{A}^* q + \alpha q, h_u \Big\rangle = \Big\langle \frac{\partial h_u}{\partial t} + \mathcal{A} h_u + \alpha h_u, q \Big\rangle \\ &= \int_Q u q \, \mathrm{d}x \le \|u\|_{L^1(Q)} \|q\|_{C(\bar{Q})} \le C_{s_n'} \|u\|_{L^1(Q)} \|h_u\|^{s_n - 1}_{L^{s_n}(Q)}. \end{aligned}$$

This proves (1.10) for $h_u$. To obtain (1.10) for $p_u$, one tests (1.9) with a weak solution of

$$\begin{cases} \frac{\partial h}{\partial t} + \mathcal{A} h + \alpha h = |q_u|^{s_n - 1}\text{sign}(q_u) \ \ \text{in} \ \ Q, \\ h = 0 \text{ on } \Sigma, \ \ h(\cdot, 0) = 0 \ \ \text{on } \Omega, \end{cases}$$

and argues in an analogous way. □

Below we remind several results for the semilinear equation (1.2), which will be used further. The first part of the proof of the next theorem can be found in [4, Theorem 2.1], and the second in [5, Theorem 2.1].

**Theorem IV.1.6.** *For every $u \in L^q(0,T; L^p(\Omega))$ with $\frac{1}{q} + \frac{n}{2p} < 1$ and $q, p \ge 2$ there exists a unique solution $y_u \in L^\infty(Q) \cap W(0,T)$ of (1.2). Moreover, the following estimates hold*

$$\|y_u\|_{L^\infty(Q)} \le \eta(\|u\|_{L^q(0,T;L^p(\Omega))} + \|f(\cdot,\cdot,0)\|_{L^q(0,T;L^p(\Omega))} + \|y_0\|_{L^\infty(\Omega)}), \tag{1.11}$$

$$\|y_u\|_{C([0,T];L^2(\Omega))} + \|y_u\|_{L^2(0,T;H_0^1(\Omega))} \le K(\|u\|_{L^2(Q)} + \|f(\cdot,\cdot,0)\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}), \tag{1.12}$$

*for a monotone non-decreasing function $\eta : [0,\infty) \to [0,\infty)$ and some constant $K$ both independent of $u$. Finally, if $u_k \rightharpoonup u$ weakly in $L^q(0,T;L^p(Q))$, then*

$$\|y_{u_k} - y_u\|_{L^\infty(Q)} + \|y_{u_k} - y_u\|_{L^2(0,T;H_0^1(\Omega))} \to 0. \tag{1.13}$$

The differentiability of the control-to-state operator under the assumptions IV.1.1 and IV.1.2 is well known, see among others [8, Theorem 2.4].

**Theorem IV.1.7.** *The control-to-state operator $\mathcal{G} : L^r(Q) \to W(0,T) \cap L^\infty(Q)$, defined as $\mathcal{G}(v) := y_v$, is of class $C^2$ and for every $u, v, w \in L^r(Q)$, it holds that $z_{u,v} := \mathcal{G}'(u)v$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = v & \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \quad z(\cdot,0) = 0 & \text{on } \Omega \end{cases} \tag{1.14}$$

*and $\omega_{u,(v,w)} := \mathcal{G}''(u)(v,w)$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = -f_{yy}(x,t,y_u)z_{u,v}z_{u,w} & \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \quad z(\cdot,0) = 0 & \text{on } \Omega. \end{cases} \tag{1.15}$$

In the case $v = w$, we will just write $\omega_{u,v}$ instead of $\omega_{u,(v,v)}$.

**Remark IV.1.8.** *By the boundedness of $\mathcal{U}$ in $L^\infty(Q)$ and by Theorem IV.1.6, there exists a constant $M_{\mathcal{U}} > 0$ such that*

$$\max\{\|u\|_{L^\infty(Q)}, \|y_u\|_{L^\infty(Q)}\} \le M_{\mathcal{U}} \quad \forall u \in \mathcal{U}. \tag{1.16}$$

### IV.1.3 Estimates associated with differentiability

We employ results of the last subsection to derive estimates for the state equation (1.2) and its linearisation (1.14). These estimates constitute a key ingredient to deriving stability results in the later sections. The next lemma extends [2, Lemma 2.7] from elliptic equations to parabolic ones.

**Lemma IV.1.9.** *The following statements are fulfilled.*

*(i) There exists a positive constant $M_2$ such that for every $u, \bar{u} \in \mathcal{U}$ and $v \in L^r(Q)$*

$$\|z_{u,v} - z_{\bar{u},v}\|_{L^2(Q)} \le M_2 \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \|z_{\bar{u},v}\|_{L^2(Q)}. \tag{1.17}$$

*(ii) Let $X = L^\infty(Q)$ or $X = L^2(Q)$. Then there exists $\varepsilon > 0$ such that for every $u, \bar{u} \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ the following inequalities are satisfied*

$$\|y_u - y_{\bar{u}}\|_X \le 2\|z_{\bar{u},u-\bar{u}}\|_X \le 3\|y_u - y_{\bar{u}}\|_X, \tag{1.18}$$

$$\|z_{\bar{u},v}\|_X \le 2\|z_{u,v}\|_X \le 3\|z_{\bar{u},v}\|_X. \tag{1.19}$$

The proof is a consequence of Lemma IV.7.1 given in Appendix A.

## IV.2 The control problem

The optimal control problem (1.1)-(1.3) is well posed under assumptions IV.1.1 and IV.1.2. Using the direct method of calculus of variations one can easily prove that there exists at least one global minimizer, see [29, Theorem 5.7]. On the other hand, the semilinear state equation makes the optimal control problem nonconvex, therefore we allow global minimizers as well as local ones. In the literature, weak and strong local minimizers are considered.

**Definition IV.2.1.** *We say that $\bar{u} \in \mathcal{U}$ is an $L^r(Q)$-weak local minimum of problem* (1.1)-(1.3)*, if there exists some $\varepsilon > 0$ such that*

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^r(Q)} \leq \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ a strong local minimum of* (P) *if there exists $\varepsilon > 0$ such that*

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq \varepsilon.$$

*We say that $\bar{u} \in \mathcal{U}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.*

Relations between these types of optimality are obtained in [5, Lemma 2.8].

As a consequence of Theorem IV.1.7 and the chain rule, we obtain the differentiability of the objective functional with respect to the control.

**Theorem IV.2.2.** *The functional $J : L^r(Q) \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^r(Q)$ we have*

$$J'(u)v = \int_Q \Big(\frac{dL_0}{dy}(x,t,y_u) + mu\Big) z_{u,v} + (my_u + g)v \, \mathrm{d}x \, \mathrm{d}t \tag{2.20}$$

$$= \int_Q (p_u + my_u + g)v \, \mathrm{d}x \, \mathrm{d}t, \tag{2.21}$$

$$J''(u)(v_1, v_2) = \int_Q \Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) - p_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big] z_{u,v_1} z_{u,v_2} \, \mathrm{d}x \, \mathrm{d}t \tag{2.22}$$

$$+ \int_Q m(z_{u,v_1} v_2 + z_{u,v_2} v_1) \, \mathrm{d}x \, \mathrm{d}t, \tag{2.23}$$

*Here, $p_u \in W(0,T) \cap C(\bar{Q})$ is the unique solution of the adjoint equation*

$$\begin{cases} -\dfrac{dp}{dt} + \mathcal{A}^* p + \dfrac{\partial f}{\partial y}(x,t,y_u)p = \dfrac{\partial L}{\partial y}(x,t,y_u,u) \text{ in } Q, \\ p = 0 \text{ on } \Sigma, \ p(\cdot,T) = 0 \text{ on } \Omega. \end{cases} \tag{2.24}$$

We introduce the Hamiltonian $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x,t,y,p,u) \mapsto H(x,t,y,p,u) \in \mathbb{R}$ in the usual way:

$$H(x,t,y,p,u) := L(x,t,y,u) + p(u - f(x,t,y)).$$

The local form of the Pontryagin type necessary optimality conditions for problem (1.1)-(1.3) in the next theorem is well known (see e.g. [8, 5, 29]).

**Theorem IV.2.3.** *If $\bar{u}$ is a weak local minimizer for problem* (1.1)-(1.3)*, then there exist unique elements $\bar{y}, \bar{p} \in W(0,T) \cap L^\infty(Q)$ such that*

$$\begin{cases} \frac{d\bar{y}}{dt} + \mathcal{A}\bar{y} + f(x,t,\bar{y}) = \bar{u} \ \text{in } Q, \\ \bar{y} = 0 \ \text{on } \Sigma, \ \bar{y}(\cdot,0) = y_0 \ \text{on } \Omega. \end{cases} \tag{2.25}$$

$$\begin{cases} \dfrac{d\bar{p}}{dt} + \mathcal{A}^*\bar{p} = \dfrac{\partial H}{\partial y}(x,t,\bar{y},\bar{p},\bar{u}) \ \text{in } Q, \\ \bar{p} = 0 \ \text{on } \Sigma, \ \bar{p}(\cdot,T) = 0 \ \text{on } \Omega. \end{cases} \tag{2.26}$$

$$\int_Q \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall u \in \mathcal{U}. \tag{2.27}$$

### IV.2.1 Sufficient conditions for optimality and stability

In this subsection we discuss the state of the art in the theory of sufficient second-order optimality conditions in PDE optimal control, as well as related stability results for the optimal solution. For this purpose, we recall the definitions of several cones that are useful in the study of sufficient conditions. Given a triplet $(\bar{y}, \bar{p}, \bar{u})$ satisfying the optimality system in Theorem IV.2.3, and abbreviating $\frac{\partial \bar{H}}{\partial u}(x,t) := \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})$, we have from (2.27) that almost everywhere in $Q$

$$\bar{u} = u_a \ \text{ if } \ \frac{\partial \bar{H}}{\partial u} > 0 \quad \text{and} \quad \bar{u} = u_b \ \text{ if } \ \frac{\partial \bar{H}}{\partial u} < 0.$$

This motivates to consider the following set

$$\Big\{ v \in L^2(Q) \Big| v \geq 0 \text{ a.e. on } [\bar{u} = u_a] \text{ and } v \leq 0 \text{ a.e. on } [\bar{u} = u_b] \Big\}. \tag{2.28}$$

Sufficient second-order conditions for (local) optimality based on (2.28) are given in [8, 5, 10]. Following the usual approach in mathematical programming, one can define the critical cone at $\bar{u}$ as follows:

$$C_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.28) and } v(x,t) = 0 \text{ if } \Big|\frac{\partial \bar{H}}{\partial u}(x,t)\Big| > 0 \Big\}.$$

Obviously, this cone is trivial if $\frac{\partial \bar{H}}{\partial u}(x,t) \neq 0$ for a.e. $(x,t)$ (which implies bang-bang structure of $\bar{u}$) thus no additional information can be gained based on $C_{\bar{u}}$. To address this issue, it was proposed in [19, 22] to consider larger cones on which second-order conditions can be posed. Namely, for $\tau > 0$ one defines

$$D^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.28) and } v(x,t) = 0 \text{ if } \Big|\frac{\partial \bar{H}}{\partial u}(x,t)\Big| > \tau \Big\}, \tag{2.29}$$

$$G^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.28) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^1(Q)} \Big\}, \tag{2.30}$$

$$E^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.28) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^2(Q)} \Big\}, \tag{2.31}$$

$$C^\tau_{\bar{u}} := D^\tau_{\bar{u}} \cap G^\tau_{\bar{u}}. \tag{2.32}$$

The cones $D^\tau_{\bar{u}}$, $E^\tau_{\bar{u}}$ and $G^\tau_{\bar{u}}$ were introduced in [3, 10] as extensions of the usual critical cone. It was proven in [3, 9, 10] that the condition:

$$\exists \delta > 0, \tau > 0 \quad \text{such that} \quad J''(\bar{u})v^2 \geq \delta \|z_{\bar{u},v}\|^2_{L^2(Q)} \quad \forall v \in G \tag{2.33}$$

is sufficient for weak (in the case $G = D^\tau_{\bar{u}}$) or strong (in the case $G = E^\tau_{\bar{u}}$) local optimality in the elliptic and parabolic setting. Most recently, the cone $C^\tau_{\bar{u}}$ was defined in [5] and also used in [6]. It was proved in [5], that (2.33) with $G = C^\tau_{\bar{u}}$ is sufficient for strong local optimality.
Under (2.33) it is possible to obtain some stability results. In [10] and [9] the authors obtain Lipschitz stability in the $(L^2 - L^\infty)$-sense for the states[1], under perturbations appearing in a tracking type objective functional and under the assumption that the perturbations are Lipschitz. Further, they obtain Hölder stability for the states under a Tikhonov type perturbation. Hölder stability under (2.33) with exponent 1/2 was proved in [11] with respect to perturbations in the initial condition.

To improve the stability results an additional assumption is needed. This role is usually played by the structural assumption on the adjoint state or more general on the derivative of the Hamiltonian with respect to the control. In the case of an elliptic state equation, [26] uses the structural assumption

$$\exists \kappa > 0 \text{ such that } \quad \left| \left\{ x \in \Omega : \left| \frac{\partial \bar{H}}{\partial u} \right| \leq \varepsilon \right\} \right| \leq \kappa \varepsilon \quad \forall \varepsilon > 0. \tag{2.34}$$

In the parabolic case this assumption (with $\Omega$ replaced with $Q$) is used in [11]. We recall that the assumption (2.34) implies that $\bar{u}$ is of bang-bang type. Further, (2.34) implies the existence of a positive constant $\tilde{\kappa}$ such that the following growth property holds:

$$J'(\bar{u})(u - \bar{u}) \geq \tilde{\kappa} \|u - \bar{u}\|^2_{L^1(\Omega)} \; \forall u \in \mathcal{U}. \tag{2.35}$$

For a proof see [1], [23] or [27]. If the control constraints satisfy $u_a < u_b$ almost everywhere on $\Omega$, both conditions, (2.34) and (2.35) are equivalent, see [17, Proposition 6.4]. In [26], using (2.34) and (2.33) with $G = D^\tau_{\bar{u}}$, the authors prove $L^1 - L^2$-Lipschitz stability of the controls for an elliptic semilinear optimal control problem under perturbations appearing simultaneously in the objective functional and the state equation. Assuming (2.34), condition (2.33) may also be weakened to the case of negative curvature,

$$\exists \delta < \tilde{\kappa}, \; \exists \tau > 0 \quad \text{such that} \quad J''(\bar{u})v^2 \geq -\delta \|v\|^2_{L^1(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}. \tag{2.36}$$

This was done in [12], [13] where it was proved that (2.34) together with (2.36) implies, for the semilinear elliptic case, weak local optimality. Lipschitz stability results were also obtained in [17] in the elliptic case. Finally, for a semilinear parabolic equation with perturbed initial data, [11, Theorem 4.6] obtains, under (2.33) and (2.34), $L^2 - L^2$ and $L^1 - L^2$-Hölder stability (see Footnote 1), with exponent 2/3, for the optimal states and controls respectively. Additionally, $L^1 - L^\infty$ Lipschitz dependence on perturbations is obtained.

[1] For $p, r \in [1, \infty]$, we speak of stability in the $L^p - L^r$-sense for the optimal states $\bar{y}$ with respect to perturbations (may appear in the equation or the objective) $\xi$, if there exists a positive constant $\kappa$ such that $\|y^\xi - \bar{y}\|_{L^p(Q)} \leq \kappa \|\xi\|_{L^r(Q)}$, for all $\xi$ that are sufficiently small. Here, $y^\xi$ denotes the state corresponding to the perturbation $\xi$. We use this expression analogously for the optimal controls.

## IV.3 A unified sufficiency condition

In this section, we introduce an assumption that unifies the first and second-order conditions presented in the previous section.

**Assumption IV.3.1.** *Let* $\bar{u} \in \mathcal{U}$*. For a number* $k \in \{0,1,2\}$*, at least one of the following conditions is fulfilled:*
*(*$A_k$*): There exist constants* $\alpha_k, \gamma_k > 0$ *such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u-\bar{u}\|^{2-k}_{L^1(Q)} \tag{3.37}$$

*for all* $u \in \mathcal{U}$ *with* $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_k$*.*
*(*$B_k$*): There exist constants* $\tilde{\alpha}_k, \tilde{\gamma}_k > 0$ *such that* (3.37) *holds for all* $u \in \mathcal{U}$ *such that* $\|u-\bar{u}\|_{L^1(Q)} < \tilde{\alpha}_k$*.*

In the context of optimal control of PDEs the conditions $(A_0)$ and $(B_0)$ were first introduced in [17] and for $k = 1,2$ in [2]. Condition$(B_0)$ originates from optimal control theory of ODEs where it was first introduced in [24] to deal with nonlinear affine optimal control problems. The cases $k = 1,2$ are extensions, adapted to the nature of the PDE setting, while the case $k = 0$ can be hard to verify if a structural assumption like (2.34) is not imposed. The conditions corresponding to $k = 1,2$ are applicable for the case of optimal controls that need not be bang-bang, especially the case $k = 2$ seems natural for obtaining state stability. Condition $(A_k)$ implies strong (local) optimality, while Condition $(B_k)$ leads to weak (local) optimality. As seen below, in some cases the two conditions are equivalent.

For an optimal control problem subject to a semilinear elliptic equation the claim of the next proposition with $k = 0$ was proven in [2, Proposition 5.2].

**Proposition IV.3.2.** *For any* $k \in \{0,1,2\}$*, condition (*$A_k$*) implies (*$B_k$*). If* $\bar{u}$ *is bang-bang (that is,* $\bar{u}(x,t) \in \{u_a(x,t), u_b(x,t)\}$ *for a.e.* $(x,t) \in Q$*) then conditions (*$A_k$*) and (*$B_k$*) are equivalent.*

The proof is given in Appendix A.

**Remark IV.3.3.** *We compare the items in Assumption IV.3.1 to the ones using* (2.34) *and* (2.36) *or* (2.33)*.*

1. *Condition (*$A_0$*) is implied by the structural assumption* (2.34) *and also allows for negative curvature, similar to* (2.36)*. For details see [17, Theorem 6.3].*

2. *Let* $g = 0$*. Condition (*$A_1$*) is implied by the structural assumption* (2.34) *together with* (2.33)*, that is, by the conditions assumed in [11]. by* (2.35) *and by using* $v$ *and* $w$ *as defined in Lemma IV.3.8 and arguing as in Corollary IV.3.9, For the convenience of the reader, this is proven in Proposition IV.3.11.*

3. *Let* $m, g = 0$*. Condition (*$A_2$*) is implied by* (2.33) *together with the first order necessary optimality condition. This is a consequence of Corollary IV.3.10.*

### IV.3.1 Sufficiency for optimality of the unified condition

In this subsection we show that conditions ($A_k$) and ($B_k$) are sufficient either for strict weak or strict strong local optimality, correspondingly.

**Theorem IV.3.4.** *The following holds.*

1. *Let $m = 0$ in* (1.4). *Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (2.25)–(2.27) *and condition* ($A_k$) *with some $k \in \{0, 1, 2\}$. Then, there exist $\varepsilon_k, \kappa_k > 0$ such that:*

$$J(\bar{u}) + \frac{\kappa_k}{2}\|y_u - y_{\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)} \le J(u) \tag{3.38}$$

   *for all $u \in \mathcal{U}$ such that $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon_k$.*

2. *Let $m \in \mathbb{R}$ and let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (2.25)–(2.27) *and condition* ($B_k$) *with some $k \in \{0, 1, 2\}$. Then, there exist $\varepsilon_k, \kappa_k > 0$ such that* (3.38) *holds for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon_k$.*

Before presenting a proof of Theorem IV.3.4, we establish some technical results. The following lemma was proved for various types of objective functionals, see e.g. [10, Lemma 6],[9, Lemma 3.11]. Nevertheless, our objective functional is more general, therefore we present in Appendix A an adapted proof.

**Lemma IV.3.5.** *Let $\bar{u} \in \mathcal{U}$. The following holds.*

1. *Let $m = 0$. For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \le \rho\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \tag{3.39}$$

   *for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ and $\theta \in [0, 1]$.*

2. *Let $m \in \mathbb{R}$. For every $\rho > 0$ there exists $\varepsilon > 0$ such that* (3.39) *holds for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon$ and $\theta \in [0, 1]$.*

For the assumptions with $k \in \{0, 1\}$, we need the subsequent corollary, which is also given in Appendix A.

**Lemma IV.3.6.** *Let $\bar{u} \in \mathcal{U}$ and let $m = 0$. Then*

1. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \le \rho\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}\|u - \bar{u}\|_{L^1(Q)} \tag{3.40}$$

   *for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ and for all $\theta \in [0, 1]$.*

2. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \le \rho\|u - \bar{u}\|^2_{L^1(Q)} \tag{3.41}$$

   *for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$ and for all $\theta \in [0, 1]$.*

*The same assertions hold true for any $m \in \mathbb{R}$ with the inequality $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ replaced with $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon$.*

The next lemma claims that Assumption IV.3.1 implies a growth similar to (3.38) of the first derivative of the objective functional in a neighborhood of $\bar{u}$.

**Lemma IV.3.7.** *Let $\bar{u} \in \mathcal{U}$. The following claims are fulfilled.*

1. *Let $m = 0$ and $\bar{u}$ satisfy condition $(A_k)$, for some $k \in \{0, 1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that*
$$J'(u)(u - \bar{u}) \geq \bar{\gamma}_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} \tag{3.42}$$
*for every $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \bar{\alpha}_k$.*

2. *Let $m \in \mathbb{R}$ and let $\bar{u}$ satisfy condition $(B_k)$ for some $k \in \{0, 1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that (3.42) holds for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \bar{\alpha}_k$.*

*Proof.* Since $J$ is of class $C^2$ we can use the mean value theorem to infer the existence of a measurable function $\theta : Q \to [0, 1]$ such that
$$J'(u)(u - \bar{u}) - J'(\bar{u})(u - \bar{u}) = J''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2.$$
Select $k \in \{0, 1, 2\}$ such that condition $(A_k)$ is satisfied, we infer the existence of positive constants $\gamma_k$ and $\alpha_k$ such that
$$\begin{aligned} J'(u)(u - \bar{u}) &= J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 + [J'(u)(u - \bar{u}) - J'(\bar{u})(u - \bar{u}) - J''(\bar{u})(u - \bar{u})^2] \\ &\geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} - |[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2|, \end{aligned}$$
for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_k$. Using Lemma IV.3.5, we obtain that
$$J'(u)(u - \bar{u}) \geq (\gamma_k - \rho_k) \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)}$$
for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \bar{\alpha}_k$ and $\bar{\alpha}_k := \min\{\alpha_k, \varepsilon_k\}$, where $\varepsilon_k > 0$ is chosen such that $\bar{\gamma}_k := \gamma_k - \rho_k > 0$. This proves the first claim of the lemma. Using the last statement of Lemma IV.3.6 concerning the general case $m \in \mathbb{R}$ and the estimate
$$\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq C_r (2M_{\mathcal{U}})^{\frac{r-1}{r}} \|u - \bar{u}\|^{\frac{1}{r}}_{L^1(Q)}$$
we obtain the second claim. $\square$

Finally, we conclude this subsection with the proof of Theorem IV.3.4.

*Proof of Theorem IV.3.4.* Using the Taylor expansion and the first-order optimality condition satisfied by $\bar{u}$ we have
$$\begin{aligned} J(u) &= J(\bar{u}) + J'(\bar{u})(u - \bar{u}) + \frac{1}{2} J''(u_\theta)(u - \bar{u})^2 \\ &\geq J(\bar{u}) + \frac{1}{2} J'(\bar{u})(u - \bar{u}) + \frac{1}{2} J''(u_\theta)(u - \bar{u})^2 \end{aligned}$$

where $u_\theta := \bar{u} + \theta(u - \bar{u})$ for a measurable function $\theta : Q \to [0, 1]$. We select $k \in \{0, 1, 2\}$ such that the corresponding condition in Assumption IV.3.1 is satisfied. Then we continue the last inequality, using that, according to the condition, there exist positive $\alpha_k, \gamma_k$ such that (3.38) holds:

$$\begin{aligned} J(u) &\geq J(\bar{u}) + \frac{1}{2}[J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2] + \frac{1}{2}[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2] \\ &\geq J(\bar{u}) + \frac{\gamma_k}{2}\|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)} - \frac{1}{2}\big|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2\big| \end{aligned}$$

for all $u \in \mathcal{U}$ with either $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_k$ or $\|u - \bar{u}\|_{L^1(Q)} < \alpha_k$, depending on the chosen condition $(A_k)$ or $(B_k)$. Let $m = 0$, by Lemma IV.3.5 or Lemma IV.3.6 (depending on the condition) there exist $\varepsilon > 0$ and $\bar{\gamma}_k < \gamma_k$ such that

$$|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2| \leq \bar{\gamma}_k\|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)}$$

for every $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$. We may choose $\bar{\alpha}_k > 0$ and $\bar{\gamma}_k > 0$ according to Lemma IV.3.7 and depending on the chosen condition therein. Inserting this estimate in the above expression and applying (1.18) gives

$$J(u) \geq J(\bar{u}) + \frac{1}{2}(\gamma_k - \bar{\gamma}_k)\|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)} \geq J(\bar{u}) + \frac{3(\gamma_k - \bar{\gamma}_k)}{4}\|y_u - y_{\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)},$$

for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \min\{\varepsilon, \bar{\alpha}_k\}$ and condition $(A_k)$ follows. For condition $(B_k)$, we use that

$$\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}}\|u - \bar{u}\|^{\frac{1}{r}}_{L^1(Q)}$$

to apply Lemma IV.3.5 or Lemma IV.3.6 depending on $k \in \{0, 1, 2\}$. Finally, for $m \in \mathbb{R}$ and under $(B_k)$ the claim follows by the above arguments applying Lemma IV.3.5 or Lemma IV.3.6 depending on $k \in \{0, 1, 2\}$.

□

□

### IV.3.2 Some equivalence results for the assumptions on cones

In this subsection, we show that some of the items in Assumption IV.3.1 can be formulated equivalently on the cones $D^\tau_{\bar{u}}$ or $C^\tau_{\bar{u}}$ respectively. This applies to $(B_k)$ or to $(A_k)$ depending on whether the objective functional explicitly depends on the control or not. The results in this subsection are important to compare the conditions introduced in Assumption IV.3.1 with other conditions in the literature. We need the next lemma, the proof of which uses a result from [7].

**Lemma IV.3.8.** *Let $\bar{u} \in \mathcal{U}$ satisfy the first order optimality condition* (2.25)-(2.27) *and let $u \in \mathcal{U}$ be given. For any positive number $\tau$, we define*

$$v := \begin{cases} 0 & \text{on} \quad [\,|\frac{\partial \bar{H}}{\partial u}| > \tau\,], \\ u - \bar{u} & \text{else,} \end{cases}$$

*and $w := u - \bar u - v$. Let $\varepsilon > 0$ be given. Then there exists a positive constant $C$ such that*

$$\max\{\|z_{\bar u,w}\|_{L^\infty(Q)},\ \|z_{\bar u,v}\|_{L^\infty(Q)}\} < C\max\{\varepsilon,\varepsilon^{\frac{1}{r}}\} \tag{3.43}$$

*for all $u \in \mathcal{U}$ with $\|u-\bar u\|_{L^1(Q)} < \varepsilon$. If additionally, $\varepsilon$ is such that (1.18) holds. and the control does not appear explicitly in (1.1) (that is, $m = g = 0$ in (1.4)), then (3.43) holds for all $u \in \mathcal{U}$ such that $u - \bar u \in G^\tau_{\bar u}$ and $\|z_{\bar u,u-\bar u}\|_{L^\infty(Q)} < \varepsilon$.*

*Proof.* We define $\tilde u, \hat u \in \mathcal{U}$ by

$$\tilde u := \begin{cases} \bar u & \text{on} \quad [\,|\frac{\partial \bar H}{\partial u}| > \tau\,], \\ u & \text{else.} \end{cases} \qquad \hat u := \begin{cases} u & \text{on} \quad [\,|\frac{\partial \bar H}{\partial u}| > \tau\,], \\ \bar u & \text{else.} \end{cases}$$

Observe that $v = \tilde u - \bar u$, $w = \hat u - \bar u$ and $u - \bar u = v + w$. It is trivial by construction that

$$\{\|v\|_{L^1(Q)},\ \|w\|_{L^1(Q)}\} \le \|u-\bar u\|_{L^1(Q)}.$$

On the other hand, by (1.18), $\|z_{\bar u,u-\bar u}\|_{L^\infty(Q)} < \varepsilon$ implies $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < 2\varepsilon$. If $m, g = 0$, we can argue as in [7] using $u - \bar u \in G^\tau_{\bar u}$ and the definition of $w$, to estimate

$$\tau\|w\|_{L^1(Q)} \le J'(\bar u)(u-\bar u) \le \tau\|z_{\bar u,u-\bar u}\|_{L^1(Q)}. \tag{3.44}$$

Thus by Theorem IV.1.4, (1.16), and with $M := C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}}$,

$$\|z_{\bar u,w}\|_{L^\infty(Q)} \le \begin{cases} M\|z_{\bar u,u-\bar u}\|^{\frac{1}{r}}_{L^\infty(Q)} & \text{if} \quad m, g = 0,\ u - \bar u \in G^\tau_{\bar u}, \\ M\|u-\bar u\|^{\frac{1}{r}}_{L^1(Q)} & \text{else.} \end{cases}$$

For $z_{\bar u,v}$ we estimate with $C := 2(M+1)$

$$\|z_{\bar u,v}\|_{L^\infty(Q)} \le \|z_{\bar u,v+w}\|_{L^\infty(Q)} + \|-z_{\bar u,w}\|_{L^\infty(Q)} \le C\max\{\varepsilon,\varepsilon^{\frac{1}{r}}\}.$$

In the second case, the estimate holds trivially. □

Now we continue with the equivalence properties.

**Corollary IV.3.9.** *For $k \in \{0,2\}$, condition ($B_k$) is equivalent to the following condition ($\bar B_k$): there exist positive constants $\alpha_k, \gamma_k$ and $\tau$ such that*

$$J'(\bar u)(u-\bar u) + J''(\bar u)(u-\bar u)^2 \ge \gamma_k\|z_{\bar u,u-\bar u}\|^k_{L^2(Q)}\|u-\bar u\|^{2-k}_{L^1(Q)}, \tag{3.45}$$

*for all $u \in \mathcal{U}$ for which $(u-\bar u) \in D^\tau_{\bar u}$ and $\|u-\bar u\|_{L^1(Q)} < \alpha_k$.*

*Proof.* Let $k \in \{0,2\}$. If ($B_k$) holds then ($\bar B_k$) is obviously also fulfilled. Now let ($\bar B_k$) hold. The numbers $\tilde\alpha_k$ and $\tilde\gamma_k$ will be chosen later so that assumption ($B_k$) will hold with these numbers. For now, we only require that $0 < \tilde\alpha_k < \alpha_k$. Choose an arbitrary $u \in \mathcal{U}$ with $\|u-\bar u\|_{L^1(Q)} < \tilde\alpha_k$. We only need to prove (3.37) in the case $u - \bar{u} \notin D_{\bar{u}}^{\tau}$. Take $v$ and $w$ as defined in Lemma IV.3.8. Clearly by definition $v \in D_{\bar{u}}^{\tau}$. As a direct consequence of (2.22)-(2.23) and Assumption IV.1.1 and IV.1.2 there exists a positive constant $M$ such that

$$|J''(\bar{u})(w)^2| \leq M\|z_{\bar{u},w}\|_{L^\infty(Q)}\|w\|_{L^1(Q)}, \tag{3.46}$$

$$|J''(\bar{u})(w,v)| \leq M\|z_{\bar{u},v}\|_{L^\infty(Q)}\|w\|_{L^1(Q)}. \tag{3.47}$$

Since $\tilde{\alpha}_k < \alpha_k$ and $v \in D_{\bar{u}}^{\tau}$ we may apply (3.45) with $v$ instead of $u - \bar{u}$. Using also (3.46) and (3.47), we estimate

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 = J'(\bar{u})(v+w) + J''(\bar{u})(v+w)^2 \\
&\geq J'(\bar{u})(v) + J'(\bar{u})(w) + J''(\bar{u})(v)^2 + J''(\bar{u})(w)^2 + 2J''(\bar{u})(w,v) \\
&\geq \gamma_k\|z_{\bar{u},v}\|_{L^2(Q)}^k\|v\|_{L^1(Q)}^{2-k} + \tau\|w\|_{L^1(Q)} - 3M(\|z_{\bar{u},w}\|_{L^\infty(Q)} + \|z_{\bar{u},v}\|_{L^\infty(Q)})\|w\|_{L^1(Q)} \\
&\geq \gamma_k\|z_{\bar{u},v}\|_{L^2(Q)}^k\|v\|_{L^1(Q)}^{2-k} + \frac{\tau}{2}\|w\|_{L^1(Q)}.
\end{aligned}$$

In the last inequality we use that by choosing $\tilde{\alpha}_k > 0$ sufficiently small we may ensure that

$$\tau - 3M(\|z_{\bar{u},w}\|_{L^\infty(Q)} + \|z_{\bar{u},v}\|_{L^\infty(Q)}) \geq \tau - 3MC\max\{\tilde{\alpha}, \tilde{\alpha}^{\frac{1}{r}}\} \geq \frac{\tau}{2}.$$

This holds because by Lemma IV.3.8, there exists a positive constant $C$ such that

$$\max\{\|z_{\bar{u},w}\|_{L^\infty(Q)},\ \|z_{\bar{u},v}\|_{L^\infty(Q)}\} \leq C\max\{\tilde{\alpha}_2, \tilde{\alpha}_2^{\frac{1}{r}}\}. \tag{3.48}$$

Further, we use that $\|u - \bar{u}\|_{L^1(Q)} < 2M_{\mathcal{U}}$ for all $u \in \mathcal{U}$, (1.8) and (1.10) in Lemma IV.1.5 for $s = 1$, to estimate

$$\|z_{\bar{u},w}\|_{L^2(Q)}^2 \leq \|z_{\bar{u},w}\|_{L^\infty(Q)}\|z_{\bar{u},w}\|_{L^1(Q)} \leq 2C_\infty C_r M_{\mathcal{U}}|Q|^{\frac{1}{r}}\|w\|_{L^1(Q)} \tag{3.49}$$

By this, we find

$$\|w\|_{L^1(Q)} \geq \begin{cases} \frac{1}{2M_{\mathcal{U}}|Q|}\|w\|_{L^1(Q)}^2, \\ \frac{1}{2C_\infty C_r M_{\mathcal{U}}|Q|^{\frac{1}{r}}}\|z_{\bar{u},w}\|_{L^2(Q)}^2. \end{cases} \tag{3.50}$$

Finally, we make the estimations for the different cases.
For $k = 0$:

$$\begin{aligned}
J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq \gamma_0\|v\|_{L^1(Q)}^2 + \frac{\tau}{2M_{\mathcal{U}}|Q|}\|w\|_{L^1(Q)}^2 \\
&\geq \min\left\{\gamma_0, \frac{\tau}{2M_{\mathcal{U}}|Q|}\right\}(\|v\|_{L^1(Q)}^2 + \|w\|_{L^1(Q)}^2) \\
&\geq \frac{1}{2}\min\left\{\gamma_0, \frac{\tau}{2M_{\mathcal{U}}|Q|}\right\}(\|u-\bar{u}\|_{L^1(Q)}^2.
\end{aligned}$$

For $k = 2$:

$$\begin{aligned}
J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq \gamma_2\|z_{\bar{u},v}\|_{L^2(Q)}^2 + \frac{\tau}{2}\|w\|_{L^1(Q)} \\
&\geq \min\left\{\gamma_2, \frac{\tau}{2C_\infty C_r M_{\mathcal{U}}|Q|^{\frac{1}{r}}}\right\}(\|z_{\bar{u},v}\|_{L^2(Q)}^2 + \|z_{\bar{u},w}\|_{L^2(Q)}^2) \\
&\geq \frac{1}{2}\min\left\{\gamma_2, \frac{\tau}{2C_\infty C_r M_{\mathcal{U}}|Q|^{\frac{1}{r}}}\right\}\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}^2.
\end{aligned}$$

This proves that (3.37) is satisfied with an appropriate number $\tilde{\gamma}_k$. □

If the control does not appear explicitly in the objective functional, we obtain a stronger result.

**Corollary IV.3.10.** *Let $m, g = 0$. Then condition $(A_2)$ is equivalent to the following condition $(\bar{A}_2)$: there exist positive constants $\alpha_2, \gamma_2, \tau$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_2 \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \tag{3.51}$$

*for all $u \in \mathcal{U}$ for which $(u-\bar{u}) \in C^\tau_{\bar{u}}$ and $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_2$.*

*Proof.* It is obvious that $(A_2)$ implies $(\bar{A}_2)$. For the reverse, if $u-\bar{u} \in C^\tau_{\bar{u}}$ the estimate holds trivially. We need to consider the cases $u-\bar{u} \notin G^\tau_{\bar{u}}$ and $u-\bar{u} \notin D^\tau_{\bar{u}}$ with $u-\bar{u} \in G^\tau_{\bar{u}}$. For the first, we argue as follows. Since $u-\bar{u} \notin G^\tau_{\bar{u}}$ it holds

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u}) > \frac{\tau}{2}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \geq \frac{\tau}{4C_r M_{\mathcal{U}} |Q|^{\frac{1}{r}}}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}.$$

For the second case $u-\bar{u} \in G^\tau_{\bar{u}}$ and $u-\bar{u} \notin D^\tau_{\bar{u}}$, let $\tilde{\alpha} > 0$ be smaller than $\alpha_2$, so that (3.51) and the prerequisite of Lemma IV.3.8 is satisfied. We define $w, v$ as in Lemma IV.3.8. By the choice of $\alpha_2$, Lemma IV.3.8 gives the existence of a positive constant $C$ such that $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty} < \alpha_2$ implies

$$\max\{\|z_{\bar{u},w}\|_{L^\infty(Q)},\ \|z_{\bar{u},v}\|_{L^\infty(Q)}\} < C \max\{\alpha_2, \alpha_2^{\frac{1}{r}}\}.$$

Now we can proceed by the same arguments as in Corollary IV.3.9

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 = J'(\bar{u})(v+w) + J''(\bar{u})(v+w)^2 \geq \gamma_2 \|z_{\bar{u},v}\|^2_{L^2(Q)} + \frac{\tau}{2}\|w\|_{L^1(Q)}.$$

Finally, we use (3.50) to obtain that

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq \gamma_2 \|z_{\bar{u},v}\|^2_{L^2(Q)} + \frac{\tau}{2C_\infty C_r M_{\mathcal{U}} |Q|^{\frac{1}{r}}}\|w\|_{L^1(Q)} \\ &\geq \min\left\{\gamma_2, \frac{\tau}{2C_\infty C_r M_{\mathcal{U}} |Q|^{\frac{1}{r}}}\right\}(\|z_{\bar{u},v}\|^2_{L^2(Q)} + \|z_{\bar{u},w}\|^2_{L^2(Q)}) \\ &\geq \min\left\{\gamma_2, \frac{\tau}{2C_\infty C_r M_{\mathcal{U}} |Q|^{\frac{1}{r}}}\right\}(\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}, \end{aligned}$$

for all $(u-\bar{u}) \in C^\tau_{\bar{u}}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \alpha_2$. □

Although we can not prove a similar equivalence property for the condition $(A_1)$ in Assumption IV.3.1, below we show that it is implied by the structural assumption (2.34) and a second-order sufficient condition.

**Proposition IV.3.11.** *Let $m, g = 0$. Then the structural assumption (2.34) and the second order sufficient condition (2.33) (for $G = C^\tau_{\bar{u}}$) imply condition $(A_1)$.*

*Proof.* Let $u$ be an arbitrary element of $\mathcal{U}$. We consider several cases.

**1.** If $u-\bar{u} \in C_{\bar{u}}^{\tau}$ we employ the structural assumption (2.34) that implies the existence of a positive constant $\gamma_1$ such that

$$J'(\bar{u})(u-\bar{u}) \geq \gamma_1 \|u-\bar{u}\|_{L^1(Q)}^2 \quad \text{for all } u \in \mathcal{U}. \tag{3.52}$$

Further since $u-\bar{u} \in C_{\bar{u}}^{\tau}$, by the second order sufficient optimality condition (2.33) there exists a positive constant $\gamma_2$ such that

$$J''(\bar{u})(u-\bar{u})^2 \geq \gamma_2 \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}^2. \tag{3.53}$$

Altogether, using the inequality $a^2+b^2 \geq 2ab$ we obtain that

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq 2\sqrt{\gamma_1\gamma_2}\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}\|u-\bar{u}\|_{L^1(Q)}, \tag{3.54}$$

which implies $(A_1)$ with $\gamma = 2\sqrt{\gamma_1\gamma_2}$ and any $\alpha_1 > 0$.

**2.** Now we consider the case where $u-\bar{u} \notin G_{\bar{u}}^{\tau}$. it holds that

$$J'(\bar{u})(u-\bar{u}) > \tau\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}. \tag{3.55}$$

On the other hand by the structural assumption (2.34) we have (3.52). Further for a positive constant $M$ such that for all $u \in \mathcal{U}$ we have

$$|J''(\bar{u})(u-\bar{u})^2| \leq M\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}. \tag{3.56}$$

Splitting the first variation into two parts and applying either (3.52) or (3.55) we conclude also using (3.56) and taking $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)}$ sufficiently small, such that by (1.18), $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)}$ is sufficiently small, that

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq \frac{1}{2}(\gamma_1\|u-\bar{u}\|_{L^1(Q)}^2 + \tau\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}) - M\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \\ &\geq \frac{1}{2}(\gamma_1\|u-\bar{u}\|_{L^1(Q)}^2 + \frac{\tau}{2}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}). \end{aligned}$$

Applying the estimate

$$\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}^2 \leq \|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \leq 2C_r M_{\mathcal{U}}|Q|^{\frac{1}{r}}\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}$$

and the inequality $a^2+b^2 \geq 2ab$, the claim follows.

**3.** Finally, we consider the case $u-\bar{u} \in G_{\bar{u}}^{\tau}$ and $u-\bar{u} \notin D_{\bar{u}}^{\tau}$. We select $v, w$ as defined in Lemma IV.3.8. By definition $v \in C_{\bar{u}}^{\tau}$. We proceed by splitting the first and second variation accordingly and applying (3.44),(3.52), (3.56) and taking $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)}$ sufficiently small to estimate

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &= J'(\bar{u})(v) + J'(\bar{u})(w) + J''(\bar{u})(v)^2 + J'(\bar{u})(w)^2 + 2J''(\bar{u})(v,w) \\ &\geq \gamma_1\|v\|_{L^1(Q)}^2 + \frac{\gamma_1}{2}\|w\|_{L^1(Q)}^2 + \frac{\tau}{2}\|w\|_{L^1(Q)} + \gamma_2\|z_{\bar{u},v}\|_{L^2(Q)}^2 \\ &\quad - M\max\{\|z_{\bar{u},v}\|_{L^\infty(Q)}, \|z_{\bar{u},w}\|_{L^\infty(Q)}\}\|z_{\bar{u},w}\|_{L^1(Q)} \\ &\geq 2\sqrt{\frac{\gamma_1^2}{2}}\|u-\bar{u}\|_{L^1(Q)}^2 + \gamma_2\|z_{\bar{u},v}\|_{L^2(Q)}^2 + \frac{\tau}{4}\|w\|_{L^1(Q)}. \end{aligned}$$

Then $(A_1)$ follows from the second estimation in (3.50) and the inequality $a^2+b^2 \geq 2ab$. ☐

## IV.4 Strong metric Hölder subregularity and auxiliary results

We study the strong metric Hölder subregularity property (SMHSr) of the optimality map. This is an extension of the strong metric subregularity property (see, [18, Section 3I] or [16, Section 4]) dealing with Lipschitz stability of set-valued mappings. The SMHSr property is especially relevant to the parabolic setting where Lipschitz stability may fail.

### IV.4.1 The optimality mapping

We begin by defining some mappings used to represent optimality in a more convenient way. This is done analogously to [17, Section 2.1]. Given the initial data $y_0$ in (1.2), we define the set

$$D(\mathcal{L}) := \Big\{ y \in W(0,T) \cap L^\infty(Q) \Big| \Big(\frac{d}{dt} + \mathcal{A}\Big) y \in L^r(Q), y(\cdot,0) = y_0 \Big\}. \tag{4.57}$$

To shorten notation, we define $\mathcal{L} : D(\mathcal{L}) \to L^r(Q)$ by $\mathcal{L} := \frac{d}{dt} + \mathcal{A}$. Additionally, we define the mapping $\mathcal{L}^* : D(\mathcal{L}^*) \to L^r(Q)$ by $\mathcal{L}^* := (-\frac{d}{dt} + \mathcal{A}^*)$, where

$$D(\mathcal{L}^*) := \Big\{ p \in W(0,T) \cap L^\infty(Q) \Big| \Big(-\frac{d}{dt} + \mathcal{A}^*\Big) p \in L^r(Q), p(\cdot,T) = 0 \Big\}.$$

With the mappings $\mathcal{L}$ and $\mathcal{L}^*$, we recast the semilinear state equation (1.2) and the linear adjoint equation (2.26) in a short way:

$$\mathcal{L}y = u - f(\cdot, y)$$

$$\mathcal{L}^* p = L_y(\cdot, y_u, u) - p f_y(\cdot, y_u) = \frac{\partial H}{\partial y}(\cdot, y_u, p, u).$$

The normal cone to the set $\mathcal{U}$ at $u \in L^1(Q)$ is defined in the usual way:

$$N_{\mathcal{U}}(u) := \begin{cases} \{\nu \in L^\infty(Q) \big| \int_Q \nu(v-u)\,dx\,dt \le 0 \quad \forall v \in \mathcal{U}\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

The first order necessary optimality condition for problem (1.1)-(1.3) in Theorem IV.2.3 can be recast as

$$\begin{cases} 0 &= \mathcal{L}y + f(\cdot,y) - u, \\ 0 &= \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ 0 &\in H_u(\cdot, y, p) + N_{\mathcal{U}}(u). \end{cases} \tag{4.58}$$

For (4.58) to make sense, a solution $(y,p,u)$ must satisfy $y \in D(\mathcal{L})$, $p \in D(\mathcal{L}^*)$ and $u \in \mathcal{U}$. For a local solution $\bar u \in \mathcal{U}$ of problem (1.1)-(1.3), by Theorem IV.2.3, the triple $(y_{\bar u}, p_{\bar u}, \bar u)$ is a solution of (4.58). We define the sets

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(Q) \times L^2(Q) \times L^\infty(Q), \tag{4.59}$$

and consider the set-valued mapping $\Phi : \mathcal{Y} \rightrightarrows \mathcal{Z}$ given by

$$\Phi \begin{pmatrix} y \\ p \\ u \end{pmatrix} := \begin{pmatrix} \mathcal{L}y + f(\cdot,y) - u \\ \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y, p, u) \\ \frac{\partial H}{\partial u}(\cdot, y, p, u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{4.60}$$

With the abbreviation $\psi := (y, p, u)$, the system (4.58) can be rewritten as the inclusion $0 \in \Phi(\psi)$. Our goal is to study the stability of the system (4.58), or equivalently, the stability of the solutions of the inclusion $0 \in \Phi(\psi)$ under perturbations. For elements $\xi, \eta \in L^r(Q)$ and $\rho \in L^\infty(Q)$ we consider the perturbed system

$$\begin{cases} \xi &= \mathcal{L}y + f(\cdot, y) - u, \\ \eta &= \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ \rho &\in \frac{\partial H}{\partial u}(\cdot, y, p) + N_{\mathcal{U}}(u), \end{cases} \tag{4.61}$$

which is equivalent to the inclusion $\zeta := (\xi, \eta, \rho) \in \Phi(\psi)$.

**Definition IV.4.1.** *The mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called the* optimality mapping *of the optimal control problem (1.1)-(1.3).*

**Theorem IV.4.2.** *For any perturbation $\zeta := (\xi, \eta, \rho) \in L^r(Q) \times L^r(Q) \times L^\infty(Q)$ there exists a triple $\psi := (y, p, u) \in \mathcal{Y}$ such that $\zeta \in \Phi(\psi)$.*

*Proof.* We consider the optimal control problem

$$\min_{u \in \mathcal{U}} \Big\{ \mathcal{J}(u) + \int_Q \eta y \,\mathrm{dxdt} - \int_Q \rho u \,\mathrm{dxdt} \Big\},$$

subject to

$$\begin{cases} \mathcal{L}y + f(x,t,y) = u + \xi \ \text{ in } \ Q, \\ y = 0 \ \text{ on } \ \Sigma, \ y(\cdot, 0) = y_0 \text{ in } \Omega. \end{cases}$$

Under assumptions IV.1.1 and IV.1.2, we have by standard arguments the existence of a global solution $\tilde{u}$. Then $\tilde{u}$ and the corresponding state $y_{\tilde{u}}$ and adjoint state $p_{\tilde{u}}$ satisfy (4.61). □

Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$, we denote by $B_{\mathcal{X}}(c, \alpha)$ the closed ball of center $c \in \mathcal{X}$ and radius $\alpha > 0$. The spaces $\mathcal{Y}$ and $\mathcal{Z}$, introduced in (4.59), are endowed with the metrics

$$\begin{aligned} d_{\mathcal{Y}}(\psi_1, \psi_2) &:= \|y_1 - y_2\|_{L^2(Q)} + \|p_1 - p_2\|_{L^2(Q)} + \|u_1 - u_2\|_{L^1(Q)}, \\ d_{\mathcal{Z}}(\zeta_1, \zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(Q)} + \|\eta_1 - \eta_2\|_{L^2(Q)} + \|\rho_1 - \rho_2\|_{L^\infty(Q)}, \end{aligned} \tag{4.62}$$

where $\psi_i = (y_i, p_i, u_i)$ and $\zeta_i = (\xi_i, \eta_i, \rho_i)$, $i \in \{1, 2\}$. From now on, we denote $\bar{\psi} := (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ to simplify notation.

The following extension of the previous theorem can be proved along the lines of [17, Theorem 4.12].

**Theorem IV.4.3.** *Let condition $(A_0)$ hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in B_Z(0; \delta)$ there exists $\psi \in B_Y(\bar{\psi}; \varepsilon)$ satisfying the inclusion $\zeta \in \Phi(\psi)$.*

### IV.4.2 Strong metric Hölder subregularity: main result

This subsection contains one of the main results in this paper: estimates of the difference between the solutions of the perturbed system (4.61) and a reference solution of the unperturbed one, (4.58), by the size of the perturbations. This will be done using the notion of *strong metric Hölder subregularity* introduced in the next paragraphs.

**Definition IV.4.4.** *Let $\bar\psi$ satisfy $0 \in \Phi(\bar\psi)$. We say that the optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is* strongly metrically Hölder subregular *(SMHSr) at $(\bar\psi, 0)$ with exponent $\theta > 0$ if there exist positive numbers $\alpha_1, \alpha_2$ and $\kappa$ such that*

$$d_{\mathcal{Y}}(\psi, \bar\psi) \le \kappa d_{\mathcal{Z}}(\zeta, 0)^\theta$$

*for all $\psi \in B_{\mathcal{Y}}(\bar\psi; \alpha_1)$ and $\zeta \in B_{\mathcal{Z}}(0; \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.*

Notice that applying the definition with $\zeta = 0$ we obtain that $\bar\psi$ is the unique solution of the inclusion $0 \in \Phi(\psi)$ in $B_{\mathcal{Y}}(\bar\psi; \alpha_1)$. In particular, $\bar u$ is a strict local minimizer for problem (1.1)-(1.3).

In the next assumption we introduce a restriction on the set of admissible perturbations, call it $\Gamma$, which is valid for the remaining part of this section.

**Assumption IV.4.5.** *For a fixed positive constant $C_{pe}$, the admissible perturbation $\zeta = (\xi, \eta, \rho) \in \Gamma \subset \mathcal{Z}$ satisfy the restriction*

$$\|\xi\|_{L^r(Q)}, \|\eta\|_{L^r(Q)} \le C_{pe}. \tag{4.63}$$

For any $u \in \mathcal{U}$ and $\zeta \in \Gamma$ we denote by $(y_u^\zeta, p_u^\zeta, u)$ a solution of the first two equations in (4.61). Using (1.11) in Theorem IV.1.6 we obtain the existence of a constant $K_y$ such that

$$\|y_u^\zeta\|_{L^\infty(Q)} \le K_y \quad \forall u \in \mathcal{U} \;\; \forall \zeta \in \Gamma. \tag{4.64}$$

Then for every $u \in \mathcal{U}$, every admissible disturbance $\zeta$, and the corresponding solution $y$ of the first equation in (4.61) it holds that $(y_u^\zeta(x,t), u(x,t)) \in R := [-K_y, K_y] \times [u_a, u_b]$.

**Remark IV.4.6.** *We apply the local properties in Assumption IV.1.2 to the interval $[-K_y, K_y]$, and denote further by $\bar C$ a positive constant that majorates the bounds and the Lipschitz constants of $f$ and $L_0$ and their first and second derivatives with respect to $y \in [-K_y, K_y]$.*

By increasing the constant $K_y$, if necessary, we may also estimate the adjoint state:

$$\|p_u^\zeta\|_{L^\infty(Q)} \le K_y(1 + \|\eta\|_{L^r(Q)}) \quad \forall u \in \mathcal{U} \;\; \forall \zeta \in \Gamma. \tag{4.65}$$

This follows from Theorem IV.1.4 with $\alpha = -\frac{\partial f}{\partial y}(x,t,y_u^\zeta)$ and with $\frac{\partial L}{\partial y}(x,t,y_u^\zeta,u)$ at the place of $u$.

We need some technical lemmas before stating our main result.

**Lemma IV.4.7.** *Let $u \in \mathcal{U}$ be given and $v, \eta \in L^r(Q)$, $\xi \in L^\infty(Q)$. Consider solutions $y_u, p_u, z_{u,v}$ and $y_u^\xi, p_u^\eta, z_{u,v}^\xi$ of the equations*

$$\begin{cases} \mathcal{L}y + f(\cdot, y) &= u, \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y_u, p, u) &= 0, \\ \mathcal{L}_0 z + f_y(\cdot, y_u)z &= v. \end{cases} \quad \begin{cases} \mathcal{L}y + f(\cdot, y) &= u + \xi, \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y_u^\xi, p, u) &= \eta, \\ \mathcal{L}_0 z + f_y(\cdot, y_u^\xi)z &= v. \end{cases} \tag{4.66}$$

*Here, $\mathcal{L}_0$ is defined as $\mathcal{L}$, but on the domain* (4.57) *with $y_0 = 0$. There exist positive constants $K_s, K_2$ and $R_2$, independent of $\zeta \in \Gamma$, such that the following inequalities hold*

$$\|y_u^\xi - y_u\|_{L^2(Q)} \le C_2 \|\xi\|_{L^2(Q)}, \tag{4.67}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^2(Q)} \le K_2 \|\xi\|_{L^r(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{4.68}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^s(Q)} \le K_s \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{4.69}$$

$$\|p_u^\eta - p_u\|_2 \le R_2(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}), \tag{4.70}$$

*where $C_2$ is the constant given in* (1.7) *and $s \in [1, \frac{n+2}{n})$.*

*Proof.* Subtracting the state equations in (4.66) and using the mean value theorem we obtain

$$\frac{d}{dt}(y_u^\xi - y_u) + \mathcal{A}(y_u^\xi - y_u) + \frac{\partial f}{\partial y}(x, t, y_\theta)(y_u^\xi - y_u) = \xi.$$

Then, (1.7) implies (4.67). To prove (4.68) we subtract the equations satisfied by $z_{u,v}^\xi$ and $z_{u,v}$ to obtain

$$\frac{d}{dt}(z_{u,v}^\xi - z_{u,v}) + \mathcal{A}(z_{u,v}^\xi - z_{u,v}) + \frac{\partial f}{\partial y}(x, t, y_u^\xi)(z_{u,v}^\xi - z_{u,v}) = \Big[\frac{\partial f}{\partial y}(x, t, y_u) - \frac{\partial f}{\partial y}(x, t, y_u^\xi)\Big] z_{u,v}.$$

Now, using (1.7), the mean value theorem and (4.63), (4.64) with regard to Remark IV.4.6 we obtain that

$$\begin{aligned}
&\|z_{u,v}^\xi - z_{u,v}\|_{L^2(Q)} \le C_2 \Big\| \Big[\frac{\partial f}{\partial y}(x, t, y_u) - \frac{\partial f}{\partial y}(x, t, y_u^\xi)\Big] z_{u,v} \Big\|_{L^2(Q)} \\
&\le C_2 \bar{C} \|(y_u^\xi - y_u) z_{u,v}\|_{L^2(Q)} \le C_2 \bar{C} \|y_u^\xi - y_u\|_{L^\infty(Q)} \|z_{u,v}\|_{L^2(Q)} \\
&\le C_2 C_r \bar{C} \|\xi\|_{L^r(Q)} \|z_{u,v}\|_{L^2(Q)}.
\end{aligned}$$

Defining $K_2 := C_2 C_r \bar{C}$, (4.68) follows. The proof for estimate (4.69) follows by the same argumentation but using (1.10) and defining the constant $K_s$ accordingly. Finally, we subtract the adjoint states and employ the mean value theorem to find

$$\begin{aligned}
&-\frac{d}{dt}(p_u^\eta - p_u) + \mathcal{A}^*(p_u^\eta - p_u) + \frac{\partial f}{\partial y}(x, t, y_u^\xi)(p_u^\eta - p_u) \\
&= \frac{\partial^2 L}{\partial y^2}(x, t, y_\theta)(y_u^\xi - y_u) + \frac{\partial^2 f}{\partial y^2}(x, t, y_\theta)(y_u^\xi - y_u) p_u + \eta.
\end{aligned}$$

The claim follows using (1.7), (1.16) and (4.64), (4.65) for Remark IV.4.6 to estimate

$$\|p_u^\eta - p_u\|_{L^2(Q)} \le (C_2^2 \bar{C} + M_{\mathcal{U}} C_2^2 \bar{C} + C_2)(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}).$$

□

**Lemma IV.4.8.** *Let $s \in [1, \frac{n+2}{n}) \cap [1,2]$. Let $u \in \mathcal{U}$ and let $y_u$, $p_u$ be the corresponding state and adjoint state. Further, let $y_u^\zeta$ and $p_u^\zeta$ be solutions to the perturbed state and adjoint equation in (4.61) for the control $u$. There exist positive constants $C, \tilde{C}$, independent of $\zeta \in \Gamma$, such that for $v \in \mathcal{U}$, the following estimates hold.*

1. *For $m = 0$ in (1.4):*

$$\begin{aligned} &\Big| \int_Q \Big( \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta) \Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t \Big| \\ &\quad \leq C(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}) \|z_{u,u-v}\|_{L^2(Q)} && (4.71) \\ &\quad \leq \tilde{C}(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}) \|v-u\|_{L^1(Q)}^{\frac{3s-2}{2s}}. && (4.72) \end{aligned}$$

2. *For a general $m \in \mathbb{R}$:*

$$\begin{aligned} &\Big| \int_Q \Big( \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta) \Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t \Big| \\ &\quad \leq \tilde{C}(\|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)}) \|v-u\|_{L^1(Q)}. && (4.73) \end{aligned}$$

*Proof.* We consider the first case, $m = 0$. We begin with integrating by parts

$$\begin{aligned} &\Big| \int_Q \Big( \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta) \Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t \Big| \\ &\leq \Big| \int_Q \Big[ \frac{\partial L_0}{\partial y}(x,t,y_u) z_{u,u-v} - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) z_{u,u-v}^\zeta \Big] \mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_Q z_{u,u-v}^\zeta \eta \,\mathrm{d}x\,\mathrm{d}t \Big| \\ &\leq \int_Q \Big| \frac{\partial L_0}{\partial y}(x,t,y_u) - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) \Big| \Big| z_{u,u-v} \Big| \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_Q \Big| \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) + \eta \Big| \Big| z_{u,u-v} - z_{u,u-v}^\zeta \Big| \,\mathrm{d}x\,\mathrm{d}t \\ &+ \Big| \int_Q \eta z_{u,u-v} \,\mathrm{d}x\,\mathrm{d}t \Big| = I_1 + I_2 + I_3. \end{aligned}$$

For the first term we use the Hölder inequality, the mean value theorem, (1.10), (1.16), Remark IV.4.6 and (4.67) to estimate

$$\begin{aligned} I_1 &\leq \int_Q \Big| \frac{\partial L_0}{\partial y}(x,t,y_u) - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) \Big| |z_{u,u-v}| \,\mathrm{d}x\,\mathrm{d}t \leq \bar{C} \|y_u^\zeta - y_u\|_{L^2(Q)} \|z_{u,u-v}\|_{L^2(Q)} \\ &\leq \bar{C} C_2 \|\xi\|_{L^2(Q)} \|z_{u,u-v}\|_{L^2(Q)} \leq \bar{C} C_2 C_{s'}^{1+\frac{2-s}{2}} (2M_{\mathcal{U}})^{\frac{(s'-1)(2-s)}{2s'}} \|\xi\|_{L^2(Q)} \|u-v\|_{L^1(Q)}^{1+\frac{s-2}{2s}}. \end{aligned}$$

Here we used that by Theorem IV.1.4 and Lemma 1.10 it holds

$$\|z_{u,u-v}\|_{L^2(Q)} \leq \|z_{u,u-v}\|_{L^\infty(Q)}^{\frac{2-s}{2}} \|z_{u,u-v}\|_{L^s(Q)}^{\frac{s}{2}} \leq C_{s'}^{1+\frac{2-s}{2}} (2M_{\mathcal{U}})^{\frac{(s'-1)(2-s)}{2s'}} \|u-v\|_{L^1(Q)}^{\frac{2-s}{2s'}+\frac{s}{2}},$$

and noticing that $\frac{2-s}{2s'}+\frac{s}{2}=1-\frac{2-s}{2s}$. The second term is estimated by using (1.16), Hölder's inequality, Remark IV.4.6 and (4.68):

$$I_2 \le \int_Q \Big|\frac{\partial L_0}{\partial y}(x,t,y_u^\zeta)+\eta\Big|\Big|z^\zeta_{u,u-v}-z_{u,u-v}\Big| \,\mathrm{d}x\,\mathrm{d}t \le 2K_s\bar{C}(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)})\|z_{u,u-v}\|_{L^2(Q)}$$
$$\le K(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)})\|u-v\|_{L^1(Q)}^{1+\frac{s-2}{2s}},$$

where $K := 2K_s\bar{C}C_{s'}^{1+\frac{2-s}{2}}(2M_{\mathcal{U}})^{\frac{(s'-1)(2-s)}{2s'}}$. For the last term, we estimate

$$I_3 \le \Big|\int_Q \eta z_{u,u-v}\,\mathrm{d}x\,\mathrm{d}t\Big| \le \|z_{u,u-v}\|_{L^2(Q)}\|\eta\|_{L^2(Q)}.$$

We prove the second case (4.73). By applying (1.8) and arguing as in the proof of (4.67) and (4.70) but for $r$, we infer the existence of a positive constant, denoted by $\tilde{C}$, such that:

$$\Big|\int_Q \Big(\frac{\partial H}{\partial u}(x,t,y_u,p_u)-\frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta)\Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t\Big| = \Big|\int_Q \Big[p_u-p_u^\zeta+m(y_u-y_u^\zeta)\Big](v-u)\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\le \|p_u-p_u^\zeta+m(y_u-y_u^\zeta)\|_{L^\infty(Q)}\|u-\bar{u}\|_{L^1(Q)}$$
$$\le \tilde{C}(\|\xi\|_{L^r(Q)}+\|\eta\|_{L^r(Q)})\|v-u\|_{L^1(Q)}.$$

□

The main result in the paper follows.

**Theorem IV.4.9.** *Let condition $(A_0)$ be fulfilled for the reference solution $\bar{\psi}=(y_{\bar{u}},p_{\bar{u}},\bar{u})$ of $0\in\Phi(\psi)$. Then the mapping $\Phi$ is strongly metrically Hölder subregular at $(\bar{\psi},0)$. More precisely, for every $\varepsilon\in(0,1/2]$ there exist positive constants $\alpha_n$ and $\kappa_n$ (with $\alpha_1$ and $\kappa_1$ independent of $\varepsilon$) such that for all $\psi\in\mathcal{Y}$ with $\|u-\bar{u}\|_{L^1(Q)}\le\alpha_n$ and $\zeta\in\Gamma$ satisfying $\zeta\in\Phi(\psi)$, the following inequalities are satisfied.*

1. *In the case $m=0$ in (1.4):*

$$\|u-\bar{u}\|_{L^1(Q)} \le \kappa_n\Big(\|\rho\|_{L^\infty(Q)}+\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}\Big)^{\theta_0}, \tag{4.74}$$

$$\|y_u^\zeta-y_{\bar{u}}\|_{L^2(Q)}+\|p_u^\zeta-p_{\bar{u}}\|_{L^2(Q)} \le \kappa_n\Big(\|\rho\|_{L^\infty(Q)}+\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}\Big)^{\theta}, \tag{4.75}$$

*where*

$$\theta_0=\theta=1 \qquad \textit{if } n=1, \tag{4.76}$$
$$\theta_0=\theta=1-\varepsilon \qquad \textit{if } n=2, \tag{4.77}$$
$$\theta_0=\frac{10}{11}-\varepsilon,\quad \theta=\frac{9}{11}-\varepsilon \qquad \textit{if } n=3. \tag{4.78}$$

2. *In the general case $m\in\mathbb{R}$:*

$$\|u-\bar{u}\|_{L^1(Q)} \le \kappa_n\Big(\|\rho\|_{L^\infty(Q)}+\|\xi\|_{L^r(Q)}+\|\eta\|_{L^r(Q)}\Big), \tag{4.79}$$

$$\|y_u^\zeta-y_{\bar{u}}\|_{L^2(Q)}+\|p_u^\zeta-p_{\bar{u}}\|_{L^2(Q)} \le \kappa_n\Big(\|\rho\|_{L^\infty(Q)}+\|\xi\|_{L^r(Q)}+\|\eta\|_{L^r(Q)}\Big)^{\theta_0}. \tag{4.80}$$

*Proof.* We begin with the proof for $m = 0$. We select $\alpha_1 < \tilde{\alpha}_0$ according to Lemma IV.3.7. Let $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$ and $\psi = (y_u^\zeta, p_u^\zeta, u)$ with $\|u - \bar{u}\|_{L^1(Q)} \leq \alpha_1$ such that $\zeta \in \Phi(\psi)$, i.e.

$$\begin{cases} \xi &= \mathcal{L}y_u^\zeta + f(\cdot,\cdot,y_u^\zeta) - u, \\ \eta &= \mathcal{L}^* p_u^\zeta - \frac{\partial H}{\partial y}(\cdot, y_u^\zeta, p_u^\zeta, u), \\ \rho &\in \frac{\partial H}{\partial u}(\cdot, y_u^\zeta, p_u^\zeta) + N_{\mathcal{U}}(u). \end{cases}$$

Let $y_u$ and $p_u$ denote the solutions to the unperturbed problem with respect to $u$, i.e.

$$0 = \mathcal{L}y_u + f(\cdot,\cdot,y_u) - u \text{ and } 0 = \mathcal{L}^* p_u - \frac{\partial H}{\partial y}(\cdot, y_u, p_u, u).$$

By Lemma IV.4.7, there exist positive constants $C_2, R_2$ independent of $\psi$ and $\zeta$ such that

$$\|y_u^\zeta - y_u\|_{L^2(Q)} + \|p_u^\zeta - p_u\|_{L^2(Q)} \leq (C_2 + R_2)\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big). \tag{4.81}$$

By the definition of the normal cone, $\rho \in \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta) + N_{\mathcal{U}}(u)$ is equivalent to

$$0 \geq \int_Q (\rho - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(w - u) \quad \forall w \in \mathcal{U}.$$

We conclude for $w = \bar{u}$,

$$\begin{aligned} 0 &\geq \int_Q \frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u)(u - \bar{u}) + \int_Q (\rho + \frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u) - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(\bar{u} - u) \\ &\geq J'(u)(u - \bar{u}) - \|\rho\|_{L^\infty(Q)}\|\bar{u} - u\|_{L^1(Q)} \\ &- \Big|\int_Q (\frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u) - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(\bar{u} - u)\,\mathrm{d}x\,\mathrm{d}t\Big|. \end{aligned} \tag{4.82}$$

By Lemma IV.4.8, we have an estimate on the third term. Since $\|u - \bar{u}\|_{L^1(Q)} < \tilde{\alpha}_0$, we estimate by Lemma IV.3.7 and Lemma IV.4.8

$$\|u - \bar{u}\|_{L^1(Q)}^2 \tilde{\gamma} \leq J'(u)(u - \bar{u}) \leq \tilde{C}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)\|u - \bar{u}\|_{L^1(Q)}^{1+\frac{s-2}{2s}} + \|\rho\|_{L^\infty(Q)}\|\bar{u} - u\|_{L^1(Q)}$$

and consequently for an adapted constant, denoted in the same way

$$\|\bar{u} - u\|_{L^1(Q)} \leq \tilde{C}\Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)^{\frac{2s}{s+2}}.$$

To estimate the states, we use the estimate for the controls. We notice that $(2 - s)/(2s') + s/2 = 1 + (s - 2)(2s)$ and obtain

$$\|y_{\bar{u}} - y_u\|_{L^2(Q)} \leq \|y_{\bar{u}} - y_u\|_{L^\infty(Q)}^{\frac{2-s}{2}}\|y_{\bar{u}} - y_u\|_{L^s(Q)}^{\frac{s}{2}} \leq C_r^{\frac{2-s}{2}}\|\bar{u} - u\|_{L^1(Q)}^{1+\frac{s-2}{2s}}. \tag{4.83}$$

Thus, for a constant again denoted by $\tilde{C}$ and with

$$(1 + \frac{s-2}{2s})\frac{2s}{s+2} = \frac{3s-2}{2+s},$$

$$\|y_{\bar u}-y_u\|_{L^2(Q)} \le \tilde C\Big(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}+\|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}}.$$

Next, we realize that by Lemma IV.4.7 and (IV.4.2)

$$\begin{aligned}\|y_{\bar u}-y_u^\zeta\|_{L^2(Q)} &\le \|y_{\bar u}-y_u\|_{L^2(Q)}+\|y_u-y_u^\zeta\|_{L^2(Q)}\\ &\le \max\{\tilde C, C_2\}\Big(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}+\|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}}.\end{aligned}$$

Using $\|p_{\bar u}-p_u\|_{L^2(Q)} \le C_2\|y_{\bar u}-y_u\|_{L^2(Q)}$ and (4.70), the same estimate holds for the adjoint state

$$\begin{aligned}\|p_{\bar u}-p_u^\zeta\|_{L^2(Q)} &\le \|p_{\bar u}-p_u\|_{L^2(Q)}+\|p_u-p_u^\zeta\|_{L^2(Q)}\\ &\le (C_2\tilde C+R_2)\Big(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}+\|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}},\end{aligned}$$

subsequently we define $\kappa := \max\{\tilde C, C_2\}$. Finally, we consider the case $m \ne 0$. Using estimate 4.73 in (4.82) and arguing from that as for the case $m = 0$, we infer the existence of a constant $\tilde C > 0$ such that

$$\|u-\bar u\|_{L^1(Q)} \le \tilde C\Big(\|\rho\|_{L^\infty(Q)}+\|\xi\|_{L^r(Q)}+\|\eta\|_{L^r(Q)}\Big).$$

This implies under (4.83) the estimate for the states and adjoint-states

$$\|y_{\bar u}-y_u^\zeta\|_{L^2(Q)}+\|p_{\bar u}-p_u^\zeta\|_{L^2(Q)} \le \max\{\tilde C, C_2\tilde C+R_2\}\Big(\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}+\|\rho\|_{L^\infty(Q)}\Big)^{1+\frac{s-2}{2s}}.$$

To determine $\theta$ and $\theta_0$ we notice that the functions

$$s \to \frac{s-2}{2s} \text{ and } s \to \frac{3s-2}{2+s}$$

are monotone. Inserting the value for $(n+2)/2$ for each case $n \in \{1,2,3\}$ completes the proof. □

To obtain results under Assumption IV.3.1 for $k \in \{1,2\}$, we need additional restrictions. We either don't allow perturbations $\rho$ (appearing in the inclusion in (4.61)) or they need to satisfy

$$\rho \in D(\mathcal{L}^*). \tag{4.84}$$

**Theorem IV.4.10.** *Let $m = 0$ and let some of the conditions $(A_1), (B_1)$ and $(A_2), (B_2)$ be fulfilled for the reference solution $\bar\psi = (y_{\bar u}, p_{\bar u}, \bar u)$ of $0 \in \Phi(\psi)$. Let, in addition, the set $\Gamma$ of feasible perturbations be restricted to such $\zeta \in \Gamma$ for which the component $\rho$ is either zero or satisfies (4.84). The numbers $\alpha_n$, $\kappa_n$ and $\varepsilon$ are as in Theorem IV.4.9. Then the following statements hold for $n \in \{1,2,3\}$:*

*1. Under Assumption IV.3.1, cases $(A_1)$ and $(B_1)$, the estimations*

$$\begin{aligned}&\|u-\bar u\|_{L^1(Q)} \le \kappa_n\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)}+\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}\Big),\\ &\|y_u^\zeta-y_{\bar u}\|_{L^2(Q)}+\|p_u^\zeta-p_{\bar u}\|_{L^2(Q)} \le \kappa_n\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)}+\|\xi\|_{L^2(Q)}+\|\eta\|_{L^2(Q)}\Big)^{\theta_0},\end{aligned}$$

*with $\theta_0$ as in Theorem IV.4.9, hold for all $u \in \mathcal{U}$ with $\|y_u-y_{\bar u}\|_{L^\infty(Q)} < \alpha_n$, in the case of $(A_1)$, or $\|u-\bar u\|_{L^1(Q)} < \alpha_n$ in the case $(B_1)$, and for all $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

2. *Under Assumption IV.3.1, cases* $(A_2)$ *and* $(B_2)$*, the estimation*

$$\|y_u^\zeta - y_{\bar u}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar u}\|_{L^2(Q)} \leq \kappa_n\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)$$

*hold for all* $u \in \mathcal{U}$ *with* $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \alpha_n$*, in the case of* $(A_2)$*, or* $\|u - \bar u\|_{L^1(Q)} < \alpha_n$ *in the cases* $(B_2)$*, and for all* $\zeta \in \Gamma$ *satisfying* $\zeta \in \Phi(\psi)$*.*

*Proof.* We first notice that if the perturbation $\rho$ satisfies (4.84), it holds

$$\begin{aligned}\int_Q \rho(u-\bar u)\,\mathrm{d}x\,\mathrm{d}t &= \int_Q ((\frac{d}{dt} + \mathcal{A})z_{\bar u, u-\bar u} + f_y(x,t,y_{\bar u})z_{\bar u,u-\bar u})\rho\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q ((-\frac{d}{dt} + \mathcal{A}^*)\rho + f_y(x,t,y_{\bar u})\rho)z_{\bar u,u-\bar u}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus

$$\Big|\int_Q \rho(u-\bar u)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \|z_{\bar u,u-\bar u}\|_{L^2(Q)}(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|f_y(x,t,y_{\bar u})\|_{L^\infty(Q)}\|\rho\|_{L^2(Q)}).$$

Under Assumption $(A_1)$, we can proceed as in the proof of Theorem IV.4.9 using Lemma IV.3.7 and (4.71) in Lemma IV.4.8, to infer the existence of positive constants $\alpha, \kappa$ such that

$$\|\bar u - u\|_{L^1(Q)} \leq \kappa\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),$$

and by standard estimates and using (1.18) the existence of a positive constant $C$ such that

$$\begin{aligned}&\|y_{\bar u} - y_u\|_{L^2(Q)} + \|p_{\bar u} - p_u\|_{L^2(Q)} \leq C\|y_{\bar u} - y_u\|_{L^2(Q)} \leq 2C\|z_{u,u-\bar u}\|_{L^2(Q)}\\ &\leq 2C\kappa^{\frac{2s}{s+2}}\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)^{\frac{2s}{s+2}},\end{aligned}$$

for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \alpha$ or $\|u - \bar u\|_{L^1(Q)} < \alpha$ depending on the assumption. From here on, one can proceed as in the proof of Theorem IV.4.9 and define the final constant $\kappa > 0$ and the exponent $\theta_0$ accordingly. Finally, by similar reasoning, under condition $(A_2)$ with Lemma IV.3.7 and Lemma IV.4.8, one obtains the existence of a positive constant $\kappa$ such that

$$\|y_{\bar u} - y_u\|_{L^2(Q)} + \|p_{\bar u} - p_u\|_{L^2(Q)} \leq \kappa\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),$$

for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \alpha$ or $\|u - \bar u\|_{L^1(Q)} < \alpha$. Again, proceeding as in Theorem IV.4.9 and increasing the constant $\kappa$ if needed proves the claim. □

**Remark IV.4.11.** *Theorems IV.4.9 and IV.4.10 concern perturbations which are functions of* $x$ *and* $t$ *only. On the other hand, [16, Theorem ] suggests that SMHSr implies a similar stability property under classes of perturbations that depend (in a non-linear way) on the state and control. This fact will be used and demonstrated in the next section.*

## IV.5 Stability of the optimal solution

In this section, we obtain stability results for the optimal solution under non-linear perturbations in the objective functional. Namely, we consider a disturbed problem

$$(\mathrm{P}_\zeta) \ \min_{u\in\mathcal{U}} J_\zeta(u) := \int_Q [L(x,t,y(x,t),u(x,t)) + \mu(x,t,y(x,t),u(x,t))]\, \mathrm{d}x\, \mathrm{d}t, \tag{5.85}$$

subject to

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y + f(x,t,y) = u + \xi \ \text{ in } \ Q, \\ y = 0 \ \text{ on } \ \Sigma, \ y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{5.86}$$

where $\zeta := (\xi,\mu)$ is a perturbation. The corresponding solution will be denoted by $y_u^\zeta$. In contrast with the previous section, the perturbation $\mu$ may be state and control dependent. For this reason, here we change the notation of the set of admissible perturbations to $\hat{\Gamma}$. However, Assumption IV.4.5 will still be valid for the set $\hat{\Gamma}$. The notations $C_{pe}$, $K_y$ and $R$ used below have the same meaning as in Subsection IV.4.2 (see Assumption IV.4.5 and the subsequent the paragraph).

In addition to Assumption IV.4.5 we require the following that holds through the remainder of the section.

**Assumption IV.5.1.** *For every $\zeta := (\xi,\mu) \in \hat{\Gamma}$, it holds that $\mu \in L^1(Q\times R)$. For a.e. $(x,t) \in Q$ the function $\mu(x,t,\cdot,\cdot)$ is of class $C^2$ and is convex with respect to the last argument, $u$. Moreover, the functions $\frac{\partial\mu}{\partial y}$ and $\frac{\partial^2\mu}{\partial y^2}$ are bounded on $Q\times R$, and the second one is continuous in $(y,u)\in R$, uniformly with respect to $(t,x)\in Q$.*

Due to the linearity of (5.86) and the convexity of the objective functional (5.85) with respect to $u$, the proof of the next theorem is standard.

**Theorem IV.5.2.** *For perturbations $\zeta \in \hat{\Gamma}$ satisfying Assumption IV.5.1, the perturbed problem* $(\mathrm{P}_\zeta)$ *has a global solution.*

In the next two theorems, we consider sequences of problems $\{(\mathrm{P}_{\zeta_k})\}$ with $\zeta_k \in \hat{\Gamma}$. The proofs repeat the arguments in [2, Theorem 4.2, Theorem 4.3].

**Theorem IV.5.3.** *Let a sequence $\{\zeta_k \in \hat{\Gamma}\}_k$ converge to zero in $L^2(Q)\times L^2(Q\times R)$ and let $u_k$ be a local solution of problem $(P_{\zeta_k})$, $k = 1, 2, \ldots$. Then any control $\bar{u}$ that is a weak* limit in $L^\infty(Q)$ of this sequence is a weak local minimizer in problem (P), and for the corresponding solutions, it holds that $y_{u_k} \to y_{\bar{u}}$ in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$.*

**Theorem IV.5.4.** *Let $\{\zeta_k\}_k$ be as in Theorem IV.5.3. Let $\bar{u}$ be a strict strong local minimizer of* (P)*. Then there exists a sequence of strong local minimizers $\{u_k\}$ of problems $(P_{\zeta_k})$ such that $u_k \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(Q)$ and $y_{u_k}$ converges strongly in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$.*

The next theorem is central in this section.

**Theorem IV.5.5.** *Let condition $(A_0)$ be fulfilled for the reference solution $\bar{\psi} = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ of $0 \in \Phi(\psi)$. Then there exist positive numbers $C$ and $\alpha$ for which the following is fulfilled. For all $\psi \in \mathcal{Y}$ with $\|u-\bar{u}\|_{L^1(Q)} \le \alpha$ and $\zeta \in \hat{\Gamma}$ satisfying $\zeta \in \Phi(\psi)$ it holds:*

1. *If* $m = 0$ *in* (1.4)*:*

$$\|u-\bar u\|_{L^1(Q)} \le C\Big[\|\xi\|_{L^2(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^2(Q))} + \Big\|\frac{d}{du}\mu\Big\|_{L^\infty(Q\times R)}\Big]^{\theta_0},$$

$$\|y_u^\zeta - y_{\bar u}\|_{L^2(Q)} \le C\Big[\|\xi\|_{L^2(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^2(Q))} + \Big\|\frac{d}{du}\mu\Big\|_{L^\infty(Q\times R)}\Big]^{\theta}.$$

2. *For* $m \in \mathbb{R}$*:*

$$\|u-\bar u\|_{L^1(Q)} \le C\Big[\|\xi\|_{L^r(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^r(Q))} + \Big\|\frac{d}{du}\mu\Big\|_{L^\infty(Q\times R)}\Big],$$

$$\|y_u^\zeta - y_{\bar u}\|_{L^2(Q)} \le C\Big[\|\xi\|_{L^r(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^r(Q))} + \Big\|\frac{d}{du}\mu\Big\|_{L^\infty(Q\times R)}\Big]^{\theta_0}.$$

*Here* $\theta_0$ *and* $\theta$ *are defined as in Theorem IV.4.9.*

*Proof.* The reference solution $(y_{\bar u}, \bar u)$ satisfies, together with the corresponding adjoint variable, the relations (4.58). Similarly, $(y_u^\zeta, u)$ satisfies, together with the corresponding $p_u^\zeta$ the perturbed optimality system (4.61) with the left-hand side given by the triple

$$\begin{pmatrix} \xi(\cdot) \\ \frac{d}{dy}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot)) \\ \frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot)). \end{pmatrix} \tag{5.87}$$

Since it is assumed that $\|u-\bar u\|_{L^1(Q)} \le \alpha$ we may apply Theorem IV.4.9 (here we choose the same $\alpha$ as in this theorem) to prove the inequalities in the theorem. □

The proof of theorems IV.5.6 and IV.5.7 follows in the same spirit but using Theorem IV.4.10 instead of Theorem IV.4.9. We make an additional assumption for the perturbation $\mu$ in the objective functional, namely, that $\rho := \frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot))$ satisfies (4.84), i.e.

$$\frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot)) \in D(\mathcal{L}^*). \tag{5.88}$$

For an explanation of the condition (5.88), we refer to the proof of Theorem IV.4.10.

**Theorem IV.5.6.** *Let* $m = 0$ *and let condition* $(A_1)$ *be fulfilled for the reference solution* $\bar\psi = (y_{\bar u}, p_{\bar u}, \bar u)$ *of* $0 \in \Phi(\psi)$*. Then there exist positive numbers* $\alpha$ *and* $C$ *for which the following is fulfilled. For all* $\psi \in \mathcal{Y}$ *with* $\|y_u - y_{\bar u}\|_{L^\infty(Q)} \le \alpha$ *and* $\zeta \in \hat\Gamma$ *satisfying* $\zeta \in \Phi(\psi)$ *and* (5.88) *the following estimates hold:*

$$\|u-\bar u\|_{L^1(Q)} \le C\Big(\|\mathcal{L}^*\frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^2(Q))}\Big)$$

*and*

$$\|y_u^\zeta - y_{\bar u}\|_{L^2(Q)} \le C\Big(\|\mathcal{L}^*\frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^2(Q))}\Big)^{\theta_0},$$

*where* $\theta_0$ *is defined in Theorem IV.4.9.*

**Theorem IV.5.7.** *Let $m = 0$ and let condition $(A_2)$ be fulfilled for the reference solution $\bar{\psi} = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ of $0 \in \Phi(\psi)$. Then there exist positive numbers $C$ and $\alpha$ for which the following is fulfilled. For all $\psi \in \mathcal{Y}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq \alpha$ and $\zeta \in \hat{\Gamma}$ satisfying $\zeta \in \Phi(\psi)$ and* (5.88) *the following estimate holds.*

$$\|y_u^\zeta - y_{\bar{u}}\|_{L^2(Q)} \leq C\Big(\|\mathcal{L}^* \frac{d}{du}(\mu(\cdot, y_u^\zeta(\cdot), u(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \Big\|\frac{d}{dy}\mu\Big\|_{L^\infty(R;L^2(Q))}\Big).$$

**Remark IV.5.8.** *The constraint that $u_\zeta$ needs to be close to the reference solution $\bar{u}$ in the theorems above is not a big restriction. This is clear, since Assumption IV.3.1 implies that $\bar{u}$ satisfies* (3.38)*. Hence, $\bar{u}$ is a strict strong local minimizer of* (P) *and, consequently, Theorem IV.5.4 ensures the existence of a family $\{u_{\zeta_k}\}$, $\zeta_k \in \hat{\Gamma}$, of strong local minimizers of problems* $(\mathrm{P}_\zeta)$ *satisfying the conditions of Theorem IV.4.9 or IV.4.10.*

## IV.6 Examples

Here, we present three examples that show particular applications in which different assumptions are involved.

**Example IV.6.1** (Tikhonov regularization). *We consider the optimal control problem*

$$(\mathrm{P}_\lambda)\ \min_{u\in\mathcal{U}} J_\lambda(u) := \int_Q L(x,t,y(x,t),u(x,t)) + \frac{\lambda}{2}\int_Q u(x,t)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

*subject to* (1.2) *and* (1.3)*. As before, $\bar{u}$ denotes a strict strong solution of problem (P)$\equiv$* $(\mathrm{P}_0)$*. We assume that $\bar{u}$ satisfies condition* $(A_0)$*. From Theorem IV.5.4 we know that for every sequence $\lambda_k > 0$ converging to zero there exists a sequence of strong local minimizer $\{u_{\lambda_k}\}_{k=1}^\infty$ such that $u_{\lambda_k} \to \bar{u}$ in $L^1(Q)$ for $k \to \infty$, thus for a sufficiently large $k_0$ we have that for all $k > k_0$ and a positive constant $C$*

$$\|y_{\bar{u}} - y_{u_{\lambda_k}}\|_{L^2(Q)} + \|p_{\bar{u}} - p_{u_{\lambda_k}}\|_{L^2(Q)} \leq C\Big(\lambda_k\Big)^\theta,$$

$$\|\bar{u} - u_{\lambda_k}\|_{L^1(Q)} \leq C\lambda_k,$$

*where $\theta$ is defined in Theorem IV.4.9.*

**Example IV.6.2** (Negative curvature). *We consider an optimal control problem, that has negative curvature. The parabolic equation has the form*

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y + \exp(y) & = u \quad \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) & = 0 \quad \text{on } \Omega. \end{cases} \tag{6.89}$$

*Let $0 \leq g \in L^2(Q)$ be a function satisfying the structural assumption, i.e. $g$ satisfies* (2.34) *in place of $\frac{\partial \bar{H}}{\partial u}$. We consider the optimal control problem*

$$\min_{u\in\mathcal{U}}\Big\{J(u) := \int_Q (y_u + gu)\,\mathrm{d}x\,\mathrm{d}t\Big\}$$

*subject to* (6.89) *and with control constraints*

$$\mathcal{U} := \{u \in L^\infty(Q) |\ 0 \le u_a \le u \le u_b \ \ \textit{for a.a.}\ (x,t) \in Q\}. \tag{6.90}$$

*By the weak maximum principle,* $y_{u_a} - y_u \le 0$ *for all* $u \in \mathcal{U}$ *and* $\bar{u} := u_a$ *constitutes an optimal solution. Further, by the weak maximum principle, the adjoint-state* $p_{\bar{u}}$ *and the linearized states* $z_{\bar{u},u-\bar{u}}$ *for all* $u \in \mathcal{U}$*, are non-negative. Moreover, we have*

$$J'(\bar{u})(u-\bar{u}) = \int_Q (p_{\bar{u}} + g)(u - \bar{u})\,\mathrm{d}x\,\mathrm{d}t \ge 0,$$

$$J''(\bar{u})(u-\bar{u})^2 = \int_Q w_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t = \int_Q -p_{\bar{u}} \exp(\bar{y}) z^2_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t < 0,$$

*for all* $u \in \mathcal{U} \setminus \bar{u}$*. Since* $g$ *satisfies the structural assumption, there exists a constant* $C > 0$ *such that*

$$\int_Q g(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t \ge C\|u - \bar{u}\|^2_{L^1(Q)} \quad \forall u \in \mathcal{U}.$$

*On the other hand, integrating by parts we obtain*

$$\int_Q p_{\bar{u}}(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t = \int_Q z_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t. \tag{6.91}$$

*If for* $u \in \mathcal{U}$*,* $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)}$ *is sufficiently small such that*

$$\frac{1}{2\|p_{\bar{u}} \exp(y_{\bar{u}})\|_{L^\infty(Q)}} > \|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)},$$

*we can absorb the term* $J''(\bar{u})(u-\bar{u})^2$ *by estimating*

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &= \int_Q z_{\bar{u},u-\bar{u}}(1 - p_{\bar{u}} \exp(y_{\bar{u}}) z_{\bar{u},u-\bar{u}})\,\mathrm{d}x\,\mathrm{d}t & (6.92)\\ &\ge \frac{1}{2}\int_Q z_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t \ge \frac{K}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}, & (6.93)\end{aligned}$$

*where the last inequality is a consequence of the boundedness of* $\mathcal{U} \subset L^\infty(Q)$ *that implies the existence of a positive constant* $K$ *such that*

$$\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \ge K\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}$$

*for all* $u \in \mathcal{U}$*. Altogether, we find*

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\ge C\|u-\bar{u}\|^2_{L^1(Q)} + \frac{K}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}\\ &\ge \sqrt{\frac{CK}{2}}\|u-\bar{u}\|_{L^1(Q)}\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \quad \forall u \in \mathcal{U}.\end{aligned}$$

*Thus, condition* $(A_1)$ *is fulfilled, and we can apply Theorem IV.4.10 to obtain a stability result.*

**Example IV.6.3** (State stability)**.** *We will discuss* $(A_2)$ *for an optimal control problem with tracking type objective functional where the control does not appear explicitly in the objective functional:*

$$\min_{u\in\mathcal{U}} \left\{ J(u) := \frac{1}{2}\int_Q (y_u - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t \right\}$$

*subject to* (1.3) *and equation* (6.89) *and a given function* $y_d \in L^r(Q)$. *As perturbations, we consider functions* $\zeta := (\xi, \eta, \rho) \in L^r(Q) \times L^r(Q) \times D(\mathcal{L}^*)$. *Denote by* $\bar{\psi} = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ *the reference solution of* $0 \in \Phi(\psi)$ *satisfying* $(A_2)$ *and consider the perturbed problem*

$$\min_{u\in\mathcal{U}} \left\{ J(u) := \frac{1}{2}\int_Q (y(x,t) - y_d(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q \eta y \,\mathrm{d}x\,\mathrm{d}t + \int_Q \rho u \,\mathrm{d}x\,\mathrm{d}t \right\},$$

*subject to* (1.3) *and*

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y + \exp(y) & = u + \xi \quad \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) & = 0 \quad \text{on } \Omega. \end{cases}$$

*Condition* $(A_2)$ *implies that* $\bar{u}$ *is a strong local minimizer of the unperturbed problem* $(\zeta = 0)$, *thus it holds*

$$\begin{aligned} J'(\bar{u})(u - \bar{u}) &= \int_Q (y_{\bar{u}}(x,t) - y_d(x,t)) z_{\bar{u},u-\bar{u}} \; dxdt \geq 0 \quad \forall u \in \mathcal{U}, \\ J''(\bar{u})(u - \bar{u}) &= \int_Q (y_{\bar{u}}(x,t) - y_d(x,t)) w_{\bar{u},u-\bar{u}} + z^2_{\bar{u},u-\bar{u}} \; dxdt \\ &= \int_Q (1 - p_{\bar{u}}\, exp(y_{\bar{u}})) z^2_{\bar{u},u-\bar{u}} \; dxdt \quad \forall u \in \mathcal{U}, \end{aligned}$$

*where* $p_{\bar{u}}$ *solves*

$$\begin{cases} -\frac{dp_{\bar{u}}}{dt} + \mathcal{A}^* p_{\bar{u}} + \exp(y_{\bar{u}}) p_{\bar{u}} = y_{\bar{u}} - y_d & \text{in } Q, \\ p_{\bar{u}} = 0 \text{ on } \Sigma, \; p_{\bar{u}}(\cdot, T) = 0 & \text{on } \Omega. \end{cases}$$

*If the optimal state tracks* $y_d$ *such that* $\|y_{\bar{u}} - y_d\|_{L^r(Q)} < \frac{1}{C_r \|\exp(y_{\bar{u}})\|_{L^\infty(Q)}}$ *we find that* $(A_2)$ *holds. From Theorem IV.5.6 we obtain the existence of positive constants* $\alpha$ *and* $\kappa$ *such that*

$$\|y_{\bar{u}} - y^\zeta_u\|_{L^2(Q)} + \|p_{\bar{u}} - p^\zeta_u\|_{L^2(Q)} \leq \kappa \Big( \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\mathcal{L}^*\rho\|_{L^2(Q)} \Big),$$

*for all* $(y^\zeta_u, p^\zeta_u, u) = \psi \in \Gamma$ *with* $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq \alpha$ *and* $\zeta \in \Gamma$ *satisfying* (4.84) *and* $\zeta \in \Phi(\psi)$.

## IV.7 Appendix

**Lemma IV.7.1.** *Suppose* $r > 1 + \frac{n}{2}$ *and* $s \in [1, \frac{n+2}{n}) \cap [1, 2]$. *The following statement is fulfilled for all* $u, \bar{u} \in \mathcal{U}$. *There exist positive constants* $K_r$, $M_s$ *and* $N_{r,s}$ *depending on* $s$ *and* $r$ *such that*

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)} \leq K_r \|y_u - y_{\bar{u}}\|^2_{L^{2r}(Q)}, \tag{7.94}$$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(Q)} \leq M_s \|y_u - y_{\bar{u}}\|^{2-s}_{L^\infty(Q)} \|y_u - y_{\bar{u}}\|^s_{L^s(Q)}, \tag{7.95}$$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \leq N_{r,s} \|y_u - y_{\bar{u}}\|^{2-\frac{s^2}{2}}_{L^\infty(Q)} \|y_u - y_{\bar{u}}\|^{\frac{s^2}{2}}_{L^s(Q)}. \tag{7.96}$$

*Proof.* Let us denote $\phi := y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}} \in W(0,T) \cap L^\infty(Q)$. From the equations satisfied by the three functions and by the mean value theorem $\phi$ satisfies

$$\frac{d\phi}{dt} + \mathcal{A}\phi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\phi = \Big[\frac{\partial f}{\partial y}(x,t,y_{\bar{u}}) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big](y_u - y_{\bar{u}}),$$

where $y_\theta(x,t) = y_{\bar{u}}(x,t) + \theta(x,t)(y_u(x,t) - y_{\bar{u}}(x,t))$ with $\theta : Q \longrightarrow [0,1]$ measurable. Applying again the mean value theorem we obtain

$$\frac{d\phi}{dt} + \mathcal{A}\phi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\phi = \theta\frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_u - y_{\bar{u}})^2$$

with $y_\vartheta(x,t) = y_{\bar{u}}(x,t) + \vartheta(x,t)(y_\theta(x,t) - y_{\bar{u}}(x,t))$ and $\vartheta : Q \longrightarrow [0,1]$ measurable. By Theorem IV.1.4 and Remark IV.4.6 we infer the existence of constants $C_r, \bar{C}$ independent of $u, \bar{u} \in \mathcal{U}$ and $\frac{\partial f}{\partial y}(x,t,y_{\bar{u}})$ such that

$$\|\phi\|_{L^\infty(Q)} \leq C_r\bar{C}\|(y_u - y_{\bar{u}})^2\|_{L^r(Q)} = C_r\bar{C}\|y_u - y_{\bar{u}}\|^2_{L^{2r}(Q)},$$

which proves (7.94) with $K_r := C_r\bar{C}$. To prove (7.95), we use Lemma IV.1.5, Remark IV.4.6 and (1.16) to obtain that

$$\|\phi\|_{L^s(Q)} \leq C_{s'}\bar{C}\|(y_u - y_{\bar{u}})^2\|_{L^1(Q)} \leq C_{s'}\bar{C}\|y_u - y_{\bar{u}}\|^{2-s}_{L^\infty(Q)}\|y_u - y_{\bar{u}}\|^s_{L^s(Q)}.$$

Taking $M_s := C_{s'}\bar{C}$, (7.95) follows. The inequality, (7.96), follows from (7.95) and (7.94) of Lemma IV.7.1 by estimating

$$\begin{aligned}\|\phi\|_{L^2(Q)} &\leq \|\phi\|^{\frac{2-s}{2}}_{L^\infty(Q)}\|\phi\|^{\frac{s}{2}}_{L^s(Q)} \leq K_r^{\frac{2-s}{2}}\|y_u - y_{\bar{u}}\|^{\frac{2(2-s)}{2}}_{L^{2r}(Q)}\Big[M_s^{\frac{s}{2}}\|y_u - y_{\bar{u}}\|^{\frac{(2-s)s}{2}}_{L^\infty(Q)}\|y_u - y_{\bar{u}}\|^{\frac{s^2}{2}}_{L^s(Q)}\Big]\\ &\leq K_r^{\frac{(2-s)}{2}}M_s^{\frac{s}{2}}|Q|^{\frac{2-s}{2r}}\|y_u - y_{\bar{u}}\|^{2-s+\frac{(2-s)s}{2}}_{L^\infty(Q)}\|y_u - y_{\bar{u}}\|^{\frac{s^2}{2}}_{L^s(Q)}.\end{aligned}$$

Defining $N_{r,s} := K_r^{\frac{(2-s)}{2}}M_s^{\frac{s}{2}}|Q|^{\frac{2-s}{2r}}$ and noticing that

$$2 - s + \frac{(2-s)s}{2} = 2 - \frac{s^2}{2},$$

proves the claim. □

*Proof of Lemma IV.1.9.* We prove (1.17) by applying Theorem 1 to $\psi := z_{\bar{u},v} - z_{u_\theta,v}$, that solves

$$\begin{aligned}\frac{d\psi}{dt} + \mathcal{A}\psi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\psi &= \Big[\frac{\partial f}{\partial y}(x,t,y_{u_\theta}) - \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\Big]z_{u_\theta,v}\\ &= \theta\frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_{\bar{u}} - y_{u_\theta})z_{u_\theta,v}.\end{aligned} \tag{7.97}$$

To prove (1.18), we use (7.96) with $s = \sqrt{2}$ to estimate

$$\|y_u - y_{\bar{u}}\|_{L^2(Q)} \leq \|\phi\|_{L^2(Q)} + \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \leq N_{r,\sqrt{2}}\|y_u - y_{\bar{u}}\|_{L^\infty(Q)}\|y_u - y_{\bar{u}}\|_{L^{\sqrt{2}}(Q)} + \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}.$$

Using fact that by the Hölder inequality $\|y_u - y_{\bar{u}}\|_{L^{\sqrt{2}}(Q)} \leq |Q|^{\frac{1}{\sqrt{2}}-\frac{1}{2}}\|y_u - y_{\bar{u}}\|_{L^2(Q)}$, the claim follows. For the other direction, we select again $s = \sqrt{2}$ in (7.96) and find

$$\begin{aligned}\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \leq \|\phi\|_{L^2(Q)} + \|y_u - y_{\bar{u}}\|_{L^2(Q)} &\leq N_{r,\sqrt{2}}\|y_u - y_{\bar{u}}\|_{L^\infty(Q)}\|y_u - y_{\bar{u}}\|_{L^{\sqrt{2}}(Q)} + \|y_u - y_{\bar{u}}\|_{L^2(Q)} \\ &\leq \left(N_{r,\sqrt{2}}|Q|^{\frac{1}{\sqrt{2}}-\frac{1}{2}}\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} + 1\right)\|y_u - y_{\bar{u}}\|_{L^2(Q)}.\end{aligned}$$

Finally, for (1.19) we use (1.17) and estimate

$$\begin{aligned}\|z_{\bar{u},v}\|_{L^2(Q)} &\leq \|z_{\bar{u},v} - z_{u,v}\|_{L^2(Q)} + \|z_{u,v}\|_{L^2(Q)} \\ &\leq K_2 \sqrt[2]{|Q|}\|y_u - y_{\bar{u}}\|_{L^\infty(Q)}\|z_{\bar{u},v}\|_{L^2(Q)} + \|z_{u,v}\|_{L^2(Q)}.\end{aligned}$$

Choosing $\varepsilon = [2K_2 \sqrt[2]{|Q|}]^{-1}$ proves the first part. The second inequality follows in a similar way. The estimates with respect to the $\|\cdot\|_{L^\infty(Q)}$ follow by similar reasoning, using (7.94). □

*Proof of Proposition IV.3.2.* Let us prove first the implication $(A_k)\Rightarrow(B_k)$ for any $k \in \{0, 1, 2\}$. Given $u \in \mathcal{U}$, by the mean value theorem

$$\frac{d(y_u - y_{\bar{u}})}{dt} + \mathcal{A}(y_u - y_{\bar{u}}) + \frac{\partial f}{\partial y}(x, y_{\bar{u}} + \theta(y_u - y_{\bar{u}}))(y_u - y_{\bar{u}}) = u - \bar{u}.$$

Using (1.8) in Theorem IV.1.4 we obtain that

$$\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq C_r\|u - \bar{u}\|_{L^r(Q)} \leq C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}}\|u - \bar{u}\|_{L^1(Q)}^{\frac{1}{r}}.$$

Then, by $\tilde{\alpha}_k := \frac{\alpha_k^r}{C_r^r(2M_{\mathcal{U}})^{r-1}}$, we obtain that $(A_k)$ implies $(B_k)$ with $\gamma_k = \tilde{\gamma}_k$.

To prove the converse implication, $(B_k)\Rightarrow(A_k)$, we assume that $(B_k)$ holds, but $(A_k)$ fails. Then for every integer $l \geq 1$ there exists an element $u_l \in \mathcal{U}$ such that

$$J'(\bar{u})(u_l - \bar{u}) + J''(\bar{u})(u_l - \bar{u})^2 < \frac{1}{l}\|u_l - \bar{u}\|_{L^1(Q)}^{2-k}\|z_{\bar{u},u_l-\bar{u}}\|_{L^2(Q)}^k \quad \text{and} \quad \|y_{u_l} - y_{\bar{u}}\|_{L^\infty(Q)} < \frac{1}{l}. \tag{7.98}$$

Since $\{u_l\}_{l=1}^\infty \subset \mathcal{U}$ is bounded in $L^\infty(Q)$, we can extract a subsequence, denoted in the same way, such that $u_l \overset{*}{\rightharpoonup} u$ in $L^\infty(Q)$. On one side, (7.98) implies that $y_{u_l} \to y_{\bar{u}}$ in $L^\infty(Q)$. On the other side, $u_l \overset{*}{\rightharpoonup} u$ in $L^\infty(Q)$ implies weak convergence in $L^r(Q)$. From (1.13), the convergence $y_{u_l} \to y_u$ in $L^\infty(Q)$ follows. Then, $y_u = y_{\bar{u}}$ and, consequently, $u = \bar{u}$ holds. But condition $(B_0)$ implies that $\bar{u}$ is bang-bang, and hence the weak convergence $u_l \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(Q)$ yields the strong convergence $u_l \to \bar{u}$ in $L^1(Q)$; see [17, Proposition 4.1 and Lemma 4.2]. Then, for $k = 0$, (7.98) contradicts $(B_0)$. The same argument holds for $(B_1)$ and $(B_2)$ under the additional condition that $\bar{u}$ is bang-bang and noticing that $\|z_{\bar{u},u_l-\bar{u}}\|_{L^\infty(Q)} \leq 3/2\|y_{u_l} - y_{\bar{u}}\|_{L^\infty(Q)}$ by Lemma IV.1.9.

□

A proof of the following lemma can be found in [8, Lemma 3.5].

**Lemma IV.7.2.** *Given $\bar{u} \in \mathcal{U}$ with associated state $y_{\bar{u}}$. Then, the following estimate holds*

$$\|y_{\bar{u}+\theta(u-\bar{u})} - y_{\bar{u}}\|_{L^\infty(Q)} \le B\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \quad \forall \theta \in [0,1] \quad \text{and} \quad \forall u \in \mathcal{U}, \tag{7.99}$$

*where $B := (2C_r\bar{C}\sqrt[r]{|Q|}M_{\mathcal{U}}+1)$, $C_r$ is the constant of Lemma 2 and $\bar{C}$ is the one from Remark IV.4.6.*

We prove an analogous statement for the adjoint state. For an elliptic state equation, a similar result is proved in [2, Lemma 3.7].

**Lemma IV.7.3.** *Given $\bar{u} \in \mathcal{U}$ with associated state $y_{\bar{u}}$ and adjoint-state $p_{\bar{u}}$, there exists a positive constant $\tilde{B}$ such that*

$$\|p_{\bar{u}+\theta(u-\bar{u})} - p_{\bar{u}}\|_{L^\infty(Q)} \le \tilde{B}(\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} + |m|\|u - \bar{u}\|_{L^1(Q)}^{\frac{1}{r}}), \tag{7.100}$$

*for all $\theta \in [0,1]$ and $u \in \mathcal{U}$.*

*Proof.* Let us prove (7.100). Given $u \in \mathcal{U}$ and $\theta \in [0,1]$, let us denote $u_\theta = \bar{u} + \theta(u - \bar{u})$, $y_\theta = y_{u_\theta}$, and $p_\theta = p_{u_\theta}$. Subtracting the equations satisfied by $p_\theta$ and $p_{\bar{u}}$ we get with the mean value theorem

$$\begin{aligned}
&-\frac{d}{dt}(p_\theta - p_{\bar{u}}) + \mathcal{A}^*(p_\theta - p_{\bar{u}}) + \frac{\partial f}{\partial y}(x,t,\bar{y})(p_\theta - p_{\bar{u}}) \\
&= \frac{\partial L}{\partial y}(x,t,y_\theta,u_\theta) - \frac{\partial L}{\partial y}(x,t,y_{\bar{u}},\bar{u}) + \Big[\frac{\partial f}{\partial y}(x,t,y_{\bar{u}}) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big]p_\theta \\
&= \Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_\vartheta) - p_\theta\frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)\Big](y_\theta - y_{\bar{u}}) + m(u_\theta - \bar{u}),
\end{aligned}$$

where $y_\vartheta = y_{\bar{u}} + \vartheta(y_\theta - y_{\bar{u}})$ for some measurable function $\vartheta : Q \longrightarrow [0,1]$. Now, we can apply again Theorem IV.1.4 and Remark IV.4.6 to conclude from the above equation

$$\begin{aligned}
\|p_\theta - p_{\bar{u}}\|_{L^\infty(Q)} &\le C_r(\bar{C} + M_{\mathcal{U}}\bar{C})\sqrt[r]{|Q|}\|y_\theta - y_{\bar{u}}\|_{L^\infty(Q)} + |m|\theta C_r\|u - \bar{u}\|_{L^r(Q)} \\
&\le \tilde{B}(\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} + |m|\|u - \bar{u}\|_{L^1(Q)})^{\frac{1}{r}},
\end{aligned}$$

where $\tilde{B} := C_r((\bar{C} + M_{\mathcal{U}}\bar{C})|Q|^{\frac{1}{r}}B + (2M_{\mathcal{U}})^{\frac{r-1}{r}})$, with $B$ being the constant from Lemma IV.7.2. $\square$

*Proof of Lemma IV.3.5.* The second variation of the objective functional is given by Theorem IV.2.2.

Let us denote $u_\theta$, $y_\theta$, and $p_\theta$ as in the proof of Lemma IV.7.3. From (2.23) we obtain

$$
\begin{aligned}
&|[J''(\bar u + \theta(u-\bar u)) - J''(\bar u)](u-\bar u)^2| \\
&\le \int_Q \Big| \Big[\frac{\partial^2 L_0}{\partial y^2}(x,t,y_\theta) - \frac{\partial^2 L_0}{\partial y^2}(x,t,y_{\bar u})\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|(p_{\bar u} - p_\theta)\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta) z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|p_{\bar u}\Big[\frac{\partial^2 f}{\partial y^2}(x,t,y_{\bar u}) - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|\Big[\frac{\partial^2 L_0}{\partial y^2}(x,t,y_{\bar u}) - p_{\bar u}\frac{\partial^2 f}{\partial y^2}(x,t,y_{\bar u})\Big](z^2_{u_\theta,u-\bar u} - z^2_{\bar u,u-\bar u})\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\Big|\int_Q (u-\bar u) m \Big[z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\Big] \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Let us estimate the terms $I_i$, $i \in \{1,..,5\}$. For $I_1$, we deduce from Remark IV.4.6, (7.99), (1.10) and (1.19) that for every $\rho_1 > 0$ there exists $\varepsilon_1 > 0$ such that

$$I_1 \le \rho_1 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon_1.$$

We consider $I_2$. Let $m = 0$, we use Remark IV.4.6, (1.10), (1.16), (1.19), and (7.100) to obtain for every $\rho_2 > 0$ the existence of a $\varepsilon_2 > 0$ such that

$$I_2 \le \rho_2 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon_2.$$

For the general case $m \in \mathbb{R}$, we use Remark IV.4.6, (1.10), (1.16), (1.19), and (7.100), to infer for any $\rho_2 > 0$ the existence of a $\varepsilon_2 > 0$ such that

$$
\begin{aligned}
I_2 &\le \bar C \tilde B (C_r (2M_{\mathcal U})^{\frac{r-1}{r}} + |m|)\|u-\bar u\|^{\frac{1}{r}}_{L^1(Q)} \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \\
&\le \rho_2 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|u-\bar u\|_{L^1(Q)} < \varepsilon_2.
\end{aligned}
$$

The estimate for $I_3$ follows from (1.10), (1.16), (1.19) and Remark IV.4.6. Thus for every $\rho_3 > 0$, there exists $\varepsilon_3 > 0$ with

$$I_3 \le \rho_3 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon_3.$$

For $I_4$ we infer by Remark IV.4.6, (7.96), (1.10), (1.16), (1.17), (1.19) and (7.99) that for every $\rho_4 > 0$ there exists $\varepsilon_4 > 0$ such that

$$
\begin{aligned}
I_4 &\le (\bar C + M_{\mathcal U}\bar C)\|z_{u_\theta,u-\bar u} + z_{\bar u,u-\bar u}\|_{L^2(Q)}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^2(Q)} \\
&\le \frac{5C_2}{2}(\bar C + M_{\mathcal U}\bar C)\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|y_\theta - y_{\bar u}\|_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)} \\
&\le \rho_4 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon_4.
\end{aligned}
$$

The term $I_5$ must only be considered in the general case $m \in \mathbb{R}$. We recall that in this case, we assume $\|u-\bar u\|_{L^1(Q)}$ to be sufficiently small. To estimate $I_5$ we use that $z_{\bar u,v}$ satisfies equation (1.14) and that $\psi := z_{\bar u,u-\bar u} - z_{u_\theta,u-\bar u}$ solves (7.97). Then, by Remark IV.4.6, applying (1.10) to (7.97), (1.16), (1.19), Lemma IV.1.5 and (7.99) we estimate

$$2\Big|\int_Q (u-\bar u)m\Big[z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\Big]\,\mathrm{d}x\,\mathrm{d}t\Big| \le 2|m|\|u-\bar u\|_{L^{s'}(Q)}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^s(Q)}$$
$$\le 2|m|(2M_{\mathcal{U}})^{\frac{s'-1}{s'}}\|u-\bar u\|_{L^1(Q)}^{\frac{1}{s'}}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^s(Q)}$$
$$\le 2|m|\bar C C_{s'}B(2M_{\mathcal{U}})^{\frac{s'-1}{s'}}\|u-\bar u\|_{L^1(Q)}^{\frac{1}{s'}}\|y_{u_\theta} - y_{\bar u}\|_{L^2(Q)}\|z_{\bar u_\theta,u-\bar u}\|_{L^2(Q)}.$$

We remark that to make the last step, we used that (7.99) holds also if the $\|\cdot\|_{L^\infty(Q)}$-norm is exchanged with the $\|\cdot\|_{L^2(Q)}$-norm. This can be seen in the proof of [2, Lemma 3.5]. Thus we infer that for every $\rho_5 > 0$ there exists a $\epsilon_5 > 0$ such that

$$I_5 \le \rho_5\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|u-\bar u\|_{L^1(Q)} < \varepsilon_5.$$

Now if $m = 0$ the validity of the estimates for $I_i$ for $i \in \{1,...,4\}$ holds under the condition that $\|y_u - y_{\bar u}\|_{L^\infty(Q)}$ is sufficiently small. For general $m \in \mathbb{R}$ the validity of the estimates holds under the condition that $\|u-\bar u\|_{L^1(Q)}$ is sufficiently small by the additional arguments given above for the terms $I_2$ and $I_5$ and for the other terms by the fact that by (1.8), $\|u-\bar u\|_{L^1(Q)} < \frac{\varepsilon^r}{C_r^r(2M_{\mathcal{U}})^{\frac{r-1}{2r}}}$, implies $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon$. Taking $\varepsilon := \min_{1\le i\le 5}\varepsilon_i$, completes the proof. □

*Proof of Lemma IV.3.6.* Let $s \in [1, \frac{n+2}{n}) \cap [1,2]$. We first consider the case $m = 0$. Using that $L_0$ and $f$ satisfy the assumption in Remark IV.4.6 and arguing as in the proof of Lemma IV.3.5, there exists $\varepsilon > 0$ and a positive constant $P$ such that

$$|[J''(\bar u + \theta(u-\bar u)) - J''(\bar u)](u-\bar u)^2| \le P\|y_u - y_{\bar u}\|_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)}$$

for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon$. To prove (3.40), we select $l_1, l_2 \ge 0$ with $l_1 + l_2 = 1$ and use the estimate

$$\|z_{\bar u,u-\bar u}\|_{L^2(Q)} \le \|z_{\bar u,u-\bar u}\|^{\frac{2-s}{2}}_{L^\infty(Q)}\|u-\bar u\|^{\frac{s}{2}}_{L^1(Q)}. \tag{7.101}$$

By (7.101), (1.8), (1.10), (1.16) and (1.18) we find

$$\|y_u - y_{\bar u}\|_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \le \|y_u - y_{\bar u}\|_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|z_{\bar u,u-\bar u}\|^{\frac{2-s}{2}}_{L^\infty(Q)}\|u-\bar u\|^{\frac{s}{2}}_{L^1(Q)}$$
$$\le C_{s'}\sup_{\mathcal{U}}\|u-\bar u\|^{\frac{s-1}{s'}}_{L^\infty(Q)}\|y_u - y_{\bar u}\|^{l_1+l_2}_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|u-\bar u\|^{\frac{2-s}{2s'}+\frac{s}{2}}_{L^1(Q)}$$
$$\le C_{s'}^2\tilde M_{\mathcal{U}}\|y_u - y_{\bar u}\|^{l_1}_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|u-\bar u\|^{\frac{l_2}{s'}+\frac{s}{2}}_{L^1(Q)}\|u-\bar u\|^{\frac{2-s}{2s'}}_{L^1(Q)},$$

with $\tilde M := M_{\mathcal{U}}^{\frac{s-1}{s'}(l_2+\frac{2-s}{2})}$. We select $l_2$ such that

$$\frac{l_2}{s'} + \frac{2-s}{2s'} + \frac{s}{2} = 1.$$

We have that $1/s' = 1 - 1/s$ is equivalent to $(1 + l_2)(1 - 1/s) + s/2(1 - 1 + 1/s) = 1$. Thus we find

$$l_2 = s'/2 - 1.$$

Defining $\varepsilon := \frac{1}{C_{s'}^2 \tilde{M}} \rho^{\frac{1}{l_1}}$ proves the first claim. For the proof of (3.41) we use (7.101), (1.8), (1.10), (1.16) and (1.18) to infer

$$\begin{aligned}
&\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \|z_{\bar{u},v}\|_{L^2(Q)}^2 \leq C_{s'} \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \|z_{\bar{u},v}\|_{L^\infty(Q)}^{(2-s)} \|u - \bar{u}\|_{L^1(Q)}^s \\
&\leq C_{s'}^2 M_{\mathcal{U}}^{\frac{s'-1}{s'}} \|y_u - y_{\bar{u}}\|_{L^\infty(Q)}^{l_1 + l_2} \|u - \bar{u}\|_{L^1(Q)}^{\frac{2-s}{s'}} \|u - \bar{u}\|_{L^1(Q)}^s \\
&\leq C_{s'}^3 \tilde{M} \|y_u - y_{\bar{u}}\|_{L^\infty(Q)}^{l_1} \|u - \bar{u}\|_{L^1(Q)}^{\frac{l_2}{s'}} \|u - \bar{u}\|_{L^1(Q)}^{\frac{2-s}{s'}} \|u - \bar{u}\|_{L^1(Q)}^s,
\end{aligned} \tag{7.102}$$

with $\tilde{M} := M_{\mathcal{U}}^{\frac{(s-1)(l_2+2-s)}{s'}}$. Select $l_2$ such that

$$\frac{l_2}{s'} + \frac{2-s}{s'} + s = 2.$$

By $1/s' = 1 - 1/s$, this is equivalent to $l_2 = (2 - s)/(s - 1)$. Setting $\varepsilon := \frac{1}{C_{s'}^3 \tilde{M}} \rho^{\frac{1}{l_1}}$ proves the case for $m = 0$. For $m \in \mathbb{R}$, we recall that the $L^1(Q)$-distance of the controls is assumed to be sufficiently small. This is used to estimate the terms where the difference of the controls appears explicitly. For the terms where the controls do not appear explicitly, we use the estimations for $m = 0$ above and apply the estimate (1.8) to $y_u - y_{\bar{u}}$ to conclude that the $L^\infty(Q)$-distance of the states is close if the $L^1(Q)$-distance of the controls is close. □

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# V. Solution Stability of Parabolic OCPs with Fixed State-Distribution of the Controls

## Outline



## Bibliographic Information

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The publication is a joint work of the three authors, each contributing substantially. The author of this thesis contributed crucially in all proofs throughout all the sections.

## Abstract

The paper presents results on strong metric subregularity of the optimality mapping associated with the system of first-order necessary optimality conditions for a problem of optimal control of a semilinear parabolic equation. The control has a predefined spatial distribution and only the magnitude at any time is a subject of choice. The obtained conditions for subregularity imply, in particular, sufficient optimality conditions that extend the known ones. The paper is complementary to a companion one by the same authors, in which a distributed control is considered.

## V.1 Introduction

Let $T \in \mathbb{R}$ and let $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Denote by $Q := \Omega \times (0,T)$ the space-time cylinder and by $\Sigma := \partial\Omega \times (0,T)$ its lateral boundary. In the present paper, we investigate the following optimal control problem:

$$\text{(P)} \quad \min_{u\in\mathcal{U}} \left\{ J(u) := \int_Q [L_0(x,t,y(x,t)) + \langle L_1(x,t,y(x,t)), u(t)\rangle] \,\mathrm{d}x\,\mathrm{d}t \right\}, \tag{1.1}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(x,t,y) = \langle g(x), u(t)\rangle & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

Here $y : Q \to \mathbb{R}$ is the state, $u : [0,T] \to \mathbb{R}^m$, is the control, $m \in \mathbb{N}$, $\langle \cdot,\cdot \rangle$ is the scalar product in $\mathbb{R}^m$, the functions $L_0, L_1, f, g$ are of corresponding dimensions, $\mathcal{A}$ is an elliptic operator. Moreover, $g := (g_1, ..., g_m)$ with $g_j \in L^\infty(\Omega)$ satisfies $\mathrm{supp}(g_j) \cap \mathrm{supp}(g_i) = \emptyset$ for all $i,j = 1,...,m$, $i \ne j$ and $\mathrm{meas}(\mathrm{supp}(g_i)) > 0$ for at least one $i$. The set of admissible controls is

$$\mathcal{U} := \{u \in L^\infty(0,T)^m |\ u_{a,j} \le u_j \le u_{b,j} \ \text{ for a.e. } t \in [0,T],\ 1 \le j \le m\}, \tag{1.3}$$

where $u_a, u_b \in L^\infty(0,T)^m$ and $u_{a,j}(t) < u_{b,j}(t)$ a.e. in $[0,T]$, $1 \le j \le m$.

In the stability analysis and approximation methods for optimization problems, in general, an important role is played by several regularity properties of the system of first-order necessary optimality conditions, see e.g. [15]. The *Strong Metric subRegularity* (SMsR) property, [15, 12], of the mapping associated with this system, the so-called *optimality mapping*, is especially relevant to the analysis of numerical methods. This property of the optimality mapping associated with problem (1.1)–(1.3) is the subject of investigation of the present paper.

Sufficient conditions for the SMsR property are usually formulated as strong positive definiteness (coercivity) of the second derivative of the objective functional with respect to feasible control variations (or on the so-called critical cone) with respect to the $L^2$-norm of the controls. Conditions of this type are also sufficient for optimality. In the paper, we present several sufficient conditions for SMsR of the optimality mapping of problem (1.1)–(1.3), combining in a *unified* way strong and weak coercivity requirements relative to the $L^1$-norm. Due to the affine structure of the problem with respect to the control, the conditions involve simultaneously the first and the second derivative of the objective functional. The importance of including the first derivative in the coercivity condition in $L^1$ is known from the existing works on ODE affine optimal control problems (see e.g. [16]). Moreover, the coercivity condition involves not only a quadratic function of the $L^1$-norm of the control variation; instead it involves a more general homogeneous function of second order jointly depending on the control and the corresponding state variation, therefore we call it "unified".

The sufficient conditions for SMsR are proved in the paper to imply sufficiency of the first-order optimality condition (the Pontryagin principle). Moreover, these conditions are then equivalently reformulated in terms of several "critical cones" that appear in the literature (see e.g. [4]), showing the generality of the former.

In the recently submitted companion paper [14], we consider a similar problem where the control, $u(x,t)$, depends on the spacial position $x$ and the time. In the present paper, following [5], the control function $u(t)$ depends only on the time, and each control component $u_j(t)$ has a fixed spacial distribution given by the function $g_j(x)$, $j = 1, \dots, m$. For the reader's convenience, here we repeat several auxiliary results from [14] in a slightly modified form. The main results—the strong subregularity theorems in Section V.5—are also similar to the ones in [14]. However, there are important differences: (i) the objective functional is more general (in the companion paper, it is essential that the function $L_1$ in the objective functional is affine in $y$ or even independent of $y$ in some of the results); (ii) the hierarchy of the sufficient conditions for optimality and subregularity introduced in Section V.4 is similar to that in the elliptic case. However, this hierarchy is not true for parabolic problems with controls depending on space and time; (iii) in contrast to the present paper, several of the results about SMsR in [14] have the weaker form of Hölder SMsR. We refer to the companion paper [14] for comprehensive discussions about the relationship between conditions for SMsR, second-order sufficient optimality conditions, and stability analysis of optimal control problems for elliptic and parabolic equations, which we do not repeat here.

The optimal control problem considered in this paper resembles the one in [6][1]. First-order Pontryagin-type necessary optimality conditions, as well as second-order sufficient optimality conditions for strong local minimum, are established in this paper. In the present paper, we build upon a-priori estimates for the linearized states established in [6] and study metric subregularity of the optimality mapping, hence also the stability of the solution.

The paper is organized as follows. Section V.2 presents notations, assumptions, and known facts about semilinear parabolic equations. Preliminary results about the optimal control problem (1.1)–(1.3) are given in Section V.3. The unified conditions for SMsR are introduced in Section V.4 and their sufficiency for optimality is discussed. Section V.5 presents the main results – two theorems claiming that the SMsR property of the optimality mapping holds under several sets of conditions. Some technical auxiliary results and proofs are given in Appendix.

## V.2 Notations, assumptions, and known facts

We begin with some notations and definitions. Given a non-empty, bounded and Lebesgue measurable set $\Omega \subset \mathbb{R}^n$, we denote by $L^p(\Omega)$, $1 \le p \le \infty$, the Banach spaces of all measurable functions $\Omega \to \mathbb{R}$ for which the usual norm $\| \cdot \|_{L^p(\Omega)}$ is finite. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ (that is, a set with Lipschitz boundary), the Sobolev space $H_0^1(\Omega)$ consists of all functions $\Omega \to \mathbb{R}$ that have weak first order derivatives in $L^2(\Omega)$ and vanish on the boundary of $\Omega$ (in the trace sense). The space

[1]We are thankful to Eduardo Casas, who brought to our attention the problem with control depending only on time.

$H_0^1(\Omega)$ is equipped with its usual norm denoted by $\|\cdot\|_{H_0^1(\Omega)}$. By $H^{-1}(\Omega)$ we denote the topological dual of $H_0^1(\Omega)$, equipped with the standard norm $\|\cdot\|_{H^{-1}(\Omega)}$. Given a real Banach space $Z$, the space $L^p(0,T;\,Z)$ consist of all strongly measurable functions $y:[0,T]\to Z$ that satisfy

$$\begin{aligned}\|y\|_{L^p(0,T;\,Z)} &:= \Big(\int_0^T \|y(t)\|_Z^p\,\mathrm{d}t\Big)^{1/p} < \infty \qquad \text{if } 1\le p<\infty,\\ \|y\|_{L^\infty(0,T;\,Z)} &:= \inf\{M\in\mathbb{R} \mid \|y(t)\|_Z \le M \text{ for a.e. } t\in(0,T)\} < \infty.\end{aligned}$$

The Hilbert space $W(0,T)$ consists of all functions in $L^2(0,T;\,H_0^1(\Omega))$ that have a distributional derivative in $L^2(0,T;\,H^{-1}(\Omega))$, that is

$$W(0,T) := \left\{ y \in L^2(0,T;H_0^1(\Omega)) \Big| \, \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) \right\},$$

and is endowed with the norm

$$\|y\|_{W(0,T)} := \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial y/\partial t\|_{L^2(0,T;H^{-1}(\Omega))}.$$

The Banach space $C([0,T];\,L^2(\Omega))$ consists of all continuous functions $y:[0,T]\to L^2(\Omega)$ and is equipped with the norm $\max_{t\in[0,T]}\|y(t)\|_{L^2(\Omega)}$. It is well known that $W(0,T)$ is continuously embedded in $C([0,T];\,L^2(\Omega))$ and compactly embedded in $L^2(Q)$. We use the notation $\langle\cdot,\cdot\rangle_X$ for the duality pairing between a Banach space $X$ and its dual.

The following assumptions, close to those in [2, 4, 5, 6, 8, 9, 10, 11] are standing in all the paper.

**Assumption V.2.1.** *The operator $\mathcal{A}: H_0^1(\Omega)\to H^{-1}(\Omega)$, is given by*

$$\mathcal{A} = -\sum_{i,j=1}^n \partial_{x_j}(a_{i,j}(x)\partial_{x_i}),$$

*where $a_{i,j}\in L^\infty(\Omega)$ satisfy the uniform ellipticity condition*

$$\exists\lambda_{\mathcal{A}}>0:\ \lambda_{\mathcal{A}}|\xi|^2 \le \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \ \ \forall\xi\in\mathbb{R}^n \ \text{ and a.a. } x\in\Omega.$$

*The matrix with components $a_{i,j}$ is denoted by $A$.*

**Assumption V.2.2.** *For every $y\in\mathbb{R}$, the functions $f(\cdot,\cdot,y)\in L^r(Q)$, $L_0(\cdot,\cdot,y)\in L^1(Q)$, $L_{1,j}(\cdot,\cdot,y)\in L^1(Q)$ $1\le j\le m$, and $y_0\in L^\infty(\Omega)$, where $r$ is a real number satisfying the inequality*

$$r > \max\Big\{2, 1+n/2\Big\}. \tag{2.4}$$

*We remark that the constraint $r>1+n/2$ on the number appearing through the paper is used to guarantee the boundedness of solutions to the PDEs. We use (2.4) to fix a number $r$ that is feasible for all dimensions $n\in\{1,2,3\}$. For a.e. $(x,t)\in Q$ the first and the second derivatives of $f$,$L_0$ and $L_{1,j}$ with respect to $y$ exist and are locally bounded and locally Lipschitz continuous, uniformly with respect to $(x,t)\in Q$. Moreover, $\frac{\partial f}{\partial y}(x,t,y)\ge 0$ for a.e. $(x,t)\in Q$ and for all $y\in\mathbb{R}$.*

Next, for the reader's convenience, we remind some facts about linear and semilinear parabolic equations.

By definition, the function $y$ is a weak solution of the semilinear parabolic initial-boundary value problem (1.2) if $y \in W(0,T)$ with $y(\cdot,0)=0$, and

$$\int_0^T \langle \frac{\partial y}{\partial t} + \mathcal{A}y, \psi \rangle_{H_0^1} dt = -\int_0^T \langle f(\cdot,y),\psi\rangle_{L^2(\Omega)}\, \mathrm{d}t + \int_0^T \langle h,\psi\rangle_{L^2(\Omega)}\, \mathrm{d}t \tag{2.5}$$

for all $\psi \in L^2(0,T,H_0^1(\Omega))$, where $h(x,t) := \langle g(x), u(t)\rangle$.

A proof of the next theorem can be found in [4, Theorem 2.1].

**Theorem V.2.3.** *For any $u \in L^2(0,T)^m$ the initial-boundary value problem (1.2) has a unique weak solution $y_u \in W(0,T)$. If $u \in L^r(0,T)^m$ (see (2.4)) then $y_u \in W(0,T) \cap L^\infty(Q)$. Moreover, there exists a positive constant $D_r$, independent of $u$, $g$, $f$ and $y_0$, such that*

$$\|y_u\|_{L^2(0,T;H_0^1(\Omega))} + \|y_u\|_{L^\infty(Q)} \le D_r\big(\|\langle u,g\rangle\|_{L^r(Q)} + \|f(\cdot,\cdot,0)\|_{L^r(Q)} + \|y_0\|_{L^\infty(\Omega)}\big). \tag{2.6}$$

*Finally, if $u_k \rightharpoonup u$ weakly in $L^r(Q)$, then*

$$\|y_{u_k} - y_u\|_{L^\infty(Q)} + \|y_{u_k} - y_u\|_{L^2(0,T;H_0^1(\Omega))} \to 0. \tag{2.7}$$

Below we remind some results concerning the linearized version of (1.2) and its adjoint equation. We consider weak solutions (in the same sense as above) of the following linear parabolic equation:

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + \alpha y = h & \text{in } \ Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{2.8}$$

**Lemma V.2.4.** *Let $0 \le \alpha \in L^\infty(Q)$ be given.*

1. *For each $h \in L^2(Q)$ equation (2.8) has a unique weak solution $y_h \in W(0,T)$. Moreover, there exists a constant $C_2 > 0$ independent of $h$ and $\alpha$ such that*

$$\|y_h\|_{L^2(0,T,H_0^1(\Omega))} \le C_2 \|h\|_{L^2(Q)}. \tag{2.9}$$

2. *If, additionally, $h \in L^r(Q)$ (we remind (2.4)) and $y_0 \in C(\bar{Q})$, then the weak solution $y_h$ of (2.8) belongs to $W(0,T) \cap L^\infty(Q)$. Moreover, there exists a constant $C_r > 0$ independent of $h$ and $\alpha$ such that*

$$\|y_h\|_{L^2(0,T,H_0^1(\Omega))} + \|y_h\|_{L^\infty(Q)} \le C_r \|h\|_{L^r(Q)}. \tag{2.10}$$

All claims of the lemma are well-known, see [21, Theorem 3.13, Theorem 5.5] for the first statements of the two items; for a proof of the independence of the constants $C_2$ and $C_r$ on $\alpha$ see [2] for a linear elliptic PDE of non-monotone type, and [14] for the parabolic setting.

The differentiability of the control-to-state operator under Assumptions V.2.1 and V.2.2 is well known, see [6, Theorem 2.4].

**Theorem V.2.5.** *Let $q > 1$. The control-to-state operator $\mathcal{G} : L^q(0,T)^m \to W(0,T) \cap L^\infty(Q)$, given by $\mathcal{G}(u) := y_u$, is of class $C^2$ and for every $u, v, w \in L^q(0,T)$, it holds that $z_{u,v} := \mathcal{G}'(u)v$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = \langle g, v\rangle \quad in \quad Q, \\ z = 0 \quad on \quad \Sigma, \;\; z(\cdot,0) = 0 \quad on \; \Omega, \end{cases} \tag{2.11}$$

*and $\omega_{u,(v,w)} := \mathcal{G}''(u)(v,w)$ is the solution of*

$$\begin{cases} \frac{d\omega}{dt} + \mathcal{A}\omega + f_y(x,t,y_u)\omega = -f_{yy}(x,t,y_u)z_{u,v}z_{u,w} \quad in \quad Q, \\ \omega = 0 \quad on \quad \Sigma, \;\; \omega(\cdot,0) = 0 \quad on \; \Omega. \end{cases} \tag{2.12}$$

**Lemma V.2.6.** *([6, Lemma 2.5]) Let $\alpha_0 \in L^\infty(Q)$, $u \in L^1(0,T)^m$, and let $z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ be the solution of*

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + \alpha_0 y & = \langle g(x), u(t)\rangle \quad in \;\; Q, \\ y = 0 \;\; on \; \Sigma, \quad y(\cdot,0) & = y_0 \qquad\qquad on \; \Omega. \end{cases} \tag{2.13}$$

*Then, the following inequality holds:*

$$\|z\|_{L^\infty(0,T;L^2(\Omega))} \leq 2\exp(\|\alpha_0\|_{L^\infty(Q)}T) \max_{1\leq j\leq m} \|g_j\|_{L^2(\Omega)}\|u\|_{L^1(0,T)^m}. \tag{2.14}$$

**Remark V.2.7.** *By the boundedness of $\mathcal{U}$ in $L^\infty(0,T)^m$ and Theorem V.2.3, there exists a constant $M_{\mathcal{U}} > 0$ such that*

$$\max\{\|u\|_{L^\infty(0,T)^m}, \|y_u\|_{L^\infty(Q))}\} \leq M_{\mathcal{U}} \quad \forall u \in \mathcal{U}. \tag{2.15}$$

The estimates in the next lemma constitute a key ingredient to deriving stability results in the later sections. It extends [2, Lemma 2.7] from elliptic equations to parabolic ones and was proved in [14].

**Lemma V.2.8.** *([14, Lemma 5]) The following statements are fulfilled.*

*(i) There exists a positive constant $M_2$ such that for every $u, \bar{u} \in \mathcal{U}$ and $v \in L^r(Q)$*

$$\|z_{u,v} - z_{\bar{u},v}\|_{L^2(Q)} \leq M_2\|y_u - y_{\bar{u}}\|_{C(\bar{Q})}\|z_{\bar{u},v}\|_{L^2(Q)}. \tag{2.16}$$

*(ii) Let $X = L^\infty(Q)$ or $X = L^2(Q)$. Then there exists $\varepsilon > 0$ such that for every $u, \bar{u} \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ the following inequalities are satisfied*

$$\|y_u - y_{\bar{u}}\|_X \leq 2\|z_{\bar{u},u-\bar{u}}\|_X \leq 3\|y_u - y_{\bar{u}}\|_X, \tag{2.17}$$

$$\|z_{\bar{u},v}\|_X \leq 2\|z_{u,v}\|_X \leq 3\|z_{\bar{u},v}\|_X. \tag{2.18}$$

## V.3 The optimal control problem

The optimal control problem (1.1)–(1.3) has a global solution due to the linearity with respect to the control, the convexity and closedness of the set of admissible control values, and Theorem V.2.3 (see e.g. [21, Theorem 5.7]). On the other hand, the semilinear state equation makes the optimal control problem possibly nonconvex, therefore it may have local minimizers. We recall the following definitions of local optimality.

**Definition V.3.1.** The control $\bar{u} \in \mathcal{U}$ is called *weak local minimizer* of problem (1.1)–(1.3), if there exists a number $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^1(0,T)^m} \leq \varepsilon;$$

$\bar{u} \in \mathcal{U}$ is called *strong local minimizer* of (P) if there exists $\varepsilon > 0$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar{u}}\|_{L^\infty(Q)} \leq \varepsilon.$$

Moreover, $\bar{u} \in \mathcal{U}$ is called *strict (weak or strong) local minimizer* if the above inequalities are strict for every admissible $u \neq \bar{u}$.

Due to the boundedness of the set of admissible control values, one can equivalently replace the inequality $\|u - \bar{u}\|_{L^1(0,T)^m} \leq \varepsilon$ in the definition of weak local optimality with $\|u - \bar{u}\|_{L^q(0,T)^m} \leq \varepsilon$, where $q$ is any (finite) number $\geq 1$ (see [4, Lemma 2.8]).

The analysis below involves first and second-order optimality conditions for problem (1.1)–(1.3). Further, we use the abbreviation

$$L(x,t,y,u) := L_0(x,t,y) + \langle L_1(x,t,y), u\rangle.$$

The next theorem provides a basis for obtaining such conditions. It is a consequence of Theorem IV.1.7 and the chain rule, and adapts [6, Theorem 2.7] to the more general objective functional considered in the present paper.

**Theorem V.3.2.** *Given, $q > 1$, the functional $J : L^q(0,T)^m \longrightarrow \mathbb{R}$ is of class $C^2$. Moreover, given $u, v, v_1, v_2 \in L^q(0,T)^m$ we have*

$$J'(u)v = \int_Q \Big(\frac{\partial L_0}{dy}(x,t,y_u) + \Big\langle \frac{\partial L_1}{dy}(x,t,y_u), u\Big\rangle\Big) z_{u,v} + \langle L_1(x,t,y_u), v\rangle \,\mathrm{d}x\,\mathrm{d}t \tag{3.19}$$

$$= \int_Q \langle p_u g + L_1(x,t,y_u), v\rangle \,\mathrm{d}x\,\mathrm{d}t, \tag{3.20}$$

$$J''(u)(v_1,v_2) = \int_Q \Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) - p_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x\,\mathrm{d}t \tag{3.21}$$

$$+ \int_Q \Big\langle \frac{\partial L_1}{dy}(x,t,y_u), v_2 z_{u,v_1} + v_1 z_{u,v_2}\Big\rangle \,\mathrm{d}x\,\mathrm{d}t, \tag{3.22}$$

*Here, $p_u \in W(0,T) \cap C(\bar{Q})$ is the unique solution of the adjoint equation*

$$\begin{cases} -\dfrac{dp}{dt} + \mathcal{A}^* p + \dfrac{\partial f}{\partial y}(x,t,y_u)p = \dfrac{\partial L_0}{dy}(x,t,y_u) + \Big\langle \dfrac{\partial L_1}{dy}(x,t,y_u), u\Big\rangle & \text{in } Q, \\ p = 0 \text{ on } \Sigma, \ p(\cdot,T) = 0 & \text{on } \Omega, \end{cases} \tag{3.23}$$

*where $\mathcal{A}^*$ is the adjoint operator to $\mathcal{A}$.*

We introduce the Hamiltonian $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \ni (x,t,y,p,u) \mapsto H(x,t,y,p,u) \in \mathbb{R}$ in the usual way:

$$H(x,t,y,p,u) := L(x,t,y,u) + p(\langle u, g\rangle - f(x,t,y)).$$

We denote the derivative at $\bar{u}$ in direction $v \in L^r(0,T)^m$ of $H$ by $\frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})(v) := \langle L_1(x,t,\bar{y}) + \bar{p}(x,t)g(x), v(t)\rangle$ and further abbreviate $\frac{\partial \bar{H}}{\partial u}(x,t) := \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})$. Notice that $\frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})$ is actually independent of the last argument. The Pontryagin type necessary optimality conditions for problem (1.1)-(1.3) appearing in the next theorem are well known (see e.g. [4, 6, 21]). For a problem with controls depending only on time, we refer to [6, Theorem 3.3].

**Theorem V.3.3.** *If $\bar{u}$ is a weak local minimizer for problem* (1.1)-(1.3)*, then there exist unique elements $\bar{y}, \bar{p} \in W(0,T) \cap L^\infty(Q)$ such that*

$$\begin{cases} \frac{d\bar{y}}{dt} + \mathcal{A}\bar{y} + f(x,t,\bar{y}) = \langle \bar{u}, g\rangle \ \textit{in } Q, \\ \bar{y} = 0 \ \textit{on } \Sigma, \ \bar{y}(\cdot,0) = y_0 \ \textit{on } \Omega. \end{cases} \tag{3.24}$$

$$\begin{cases} -\dfrac{d\bar{p}}{dt} + \mathcal{A}^*\bar{p} = \dfrac{\partial H}{\partial y}(x,t,\bar{y},\bar{p},\bar{u}) \ \textit{in } Q, \\ \bar{p} = 0 \ \textit{on } \Sigma, \ \bar{p}(\cdot,T) = 0 \ \textit{on } \Omega. \end{cases} \tag{3.25}$$

$$\int_\Omega \frac{\partial H}{\partial u_j}(x,t,\bar{y},\bar{p},\bar{u})\, \mathrm{d}x\, (u_j - \bar{u}_j(t)) \geq 0 \quad \forall j \in \{1,\dots,m\}, \forall u_j \in [u_{a,j}, u_{b,j}], \text{ and for a.e. } t \in [0,T]. \tag{3.26}$$

As a consequence of (3.25), for any triplet $(\bar{y},\bar{p},\bar{u})$, $j \in \{1,\dots,m\}$ and for a.e. $t \in [0,T]$ it holds that

$$\bar{u}_j(t) = \begin{cases} u_{a,j}(t) & \text{if } \int_\Omega \frac{\partial \bar{H}}{\partial u_j}(x,t)\,\mathrm{d}x > 0, \\ u_{b,j}(t) & \text{if } \int_\Omega \frac{\partial \bar{H}}{\partial u_j}(x,t)\,\mathrm{d}x < 0. \end{cases}$$

## V.4 Sufficient optimality conditions

In this section, we present a second-order sufficient optimality condition, which is a version of [14, Assumption 3] adapted to the case of controls depending only on time. Below, $\bar{u}$ is an admissible reference control and $\bar{y}$ is an element of $W(0,T) \cap L^\infty(Q)$ (presumably the solution of (1.2)).

**Assumption V.4.1.** *For a number $k \in \{0,1,2\}$, at least one of the following conditions is fulfilled:*

*$(A_k)$: There exist constants $\alpha_k, \gamma_k > 0$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u-\bar{u}\|^{2-k}_{L^1(0,T)^m} \tag{4.27}$$

*for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_k$.*

*$(B_k)$: There exist constants $\tilde{\alpha}_k, \tilde{\gamma}_k > 0$ such that* (4.27) *holds for all $u \in \mathcal{U}$ such that $\|u-\bar{u}\|_{L^1(0,T)^m} < \tilde{\alpha}_k$.*

Assumption V.4.1($B_0$) was first introduced in [17] in the ODE optimal control context, and was extended to parabolic PDEs in [2], where also ($A_0$) was introduced.

As it is proved in [14, Proposition 8], for any $k \in \{0, 1, 2\}$, Assumption ($A_k$) implies ($B_k$); if $\bar{u}$ is bang-bang (that is, $\bar{u}(t) \in \{u_a(t), u_b(t)\}$ for a.e. $t \in [0, T]$) then assumptions ($A_k$) and ($B_k$) are equivalent.

Next, we obtain growth estimations for the objective functional, which show, in particular, that assumptions V.4.1($A_k$) and ($B_k$) are sufficient either for strict weak or strict strong local optimality, correspondingly.

**Theorem V.4.2.** *The following statements hold.*

1. *Let the function $L_1$ in the objective functional be independent of $y$. Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (3.24)–(3.26) *and Assumption V.4.1($A_k$) with some $k \in \{0, 1, 2\}$. Then, there exist positive constants $\varepsilon_k$ and $c_k$ such that:*

$$J(\bar{u}) + c_k \|y_u - \bar{y}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(0,T)^m} \le J(u) \tag{4.28}$$

   *for all $u \in \mathcal{U}$ such that $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_k$.*

2. *Let the function $L_1$ in the objective functional be affine with respect to $y$. Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (3.24)–(3.26) *and Assumption V.4.1($B_k$) with some $k \in \{1, 2\}$. Then, there exist $\varepsilon_k, c_k > 0$ such that* (4.28) *holds for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(0,T)^m} < \varepsilon_k$.*

3. *Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (3.24)–(3.26) *and Assumption V.4.1($B_0$). Then, there exist $\varepsilon_0, c_0 > 0$ such that* (4.28) *holds for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(0,T)^m} < \varepsilon_0$.*

A proof of Theorem V.4.2 in the case of a less general objective functional can be found in [14]. It is a consequence of the next two lemmas, which will be used also in Section V.5. The first of them has been proved for various types of objective functionals, see e.g. [8, Lemma 6],[7, Lemma 3.11] or [14, Lemma 10]. Due to the more general objective functional in the present paper and for readers' convenience, we present a proof in the Appendix.

**Lemma V.4.3.** *Let $\bar{u} \in \mathcal{U}$. The following holds.*

1. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \le \rho \|u - \bar{u}\|^2_{L^1(0,T)^m} \tag{4.29}$$

   *holds for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \varepsilon$ and every $\theta \in [0, 1]$.*

2. *Let the function $L_1$ in the objective functional be affine with respect to $y$. For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \le \rho \|z_{\bar{u}, u-\bar{u}}\|^2_{L^2(Q)}. \tag{4.30}$$

   *holds for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \varepsilon$ and $\theta \in [0, 1]$.*

3. *Let the function $L_1$ in the objective functional be independent of $y$. For every $\rho > 0$ there exists $\varepsilon > 0$ such that (4.30) holds for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$ and $\theta \in [0, 1]$.*

The next lemma shows that Assumption V.4.1 implies a growth similar to (4.28) of the first derivative of the objective functional in a neighborhood of $\bar{u}$.

**Lemma V.4.4.** *The following claims are fulfilled.*

1. *Let the function $L_1$ in the objective functional be independent of $y$. Let $\bar{u}$ satisfy assumption $(A_k)$, for some $k \in \{0, 1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that*
$$J'(u)(u - \bar{u}) \geq \bar{\gamma}_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(0,T)^m} \tag{4.31}$$
*for every $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \bar{\alpha}_k$.*

2. *Let the function $L_1$ in the objective functional be affine with respect to $y$. Let $\bar{u}$ satisfy assumption $(B_k)$ for some $k \in \{1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that (4.31) holds for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \bar{\alpha}_k$.*

3. *Let $\bar{u}$ satisfy assumption $(B_0)$. Then, there exist $\bar{\alpha}_0, \bar{\gamma}_0 > 0$ such that (4.31) holds for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} < \bar{\alpha}_0$.*

### Reformulations of Assumption 3 using cones.

We recall that some of the items in Assumption V.4.1 can be formulated equivalently by restricting the admissible control variations $v = u - \bar{u}$ to appropriate cones. This applies to $(B_k)$ or to $(A_k)$ depending on whether the objective functional explicitly depends on the control or not.

Obviously any admissible control variation $v = u - \bar{u}$, $u \in \mathcal{U}$, satisfies the conditions
$$v \in L^2(0,T)^m, \quad v_j(t) \geq 0 \text{ whenever } \bar{u}_j(t) = u_{a,j}(t) \text{ and } v_j(t) \leq 0 \text{ whenever } \bar{u}_j(t) = u_{b,j}(t). \tag{4.32}$$
Then, for $\tau > 0$ define
$$D^\tau_{\bar{u}} := \Big\{ v \in L^2(0,T)^m \Big| v \text{ satisfies (4.32) and } v_j(x,t) = 0 \text{ if } \Big|\frac{\partial \bar{H}}{\partial u_j}(x,t)\Big| > \tau,\ 1 \leq j \leq m \Big\}, \tag{4.33}$$
$$G^\tau_{\bar{u}} := \Big\{ v \in L^2(0,T)^m \Big| v \text{ satisfies (4.32) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^1(Q)} \Big\}, \tag{4.34}$$
$$E^\tau_{\bar{u}} := \Big\{ v \in L^2(0,T)^m \Big| v \text{ satisfies (4.32) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^2(Q)} \Big\}, \tag{4.35}$$
$$C^\tau_{\bar{u}} := D^\tau_{\bar{u}} \cap G^\tau_{\bar{u}}. \tag{4.36}$$
The cones $D^\tau_{\bar{u}}$, $E^\tau_{\bar{u}}$ and $G^\tau_{\bar{u}}$ were introduced in [3, 8] as extensions of the usual critical cone. Most recently, the cone $C^\tau_{\bar{u}}$ was defined in [4] and also used in [5]. In the ODE control literature, a cone similar to $D^\tau_{\bar{u}}$ has been in use for a long time, see [18].

**Theorem V.4.5.** *1. For $k \in \{0, 2\}$, Assumption V.4.1$(B_k)$ is equivalent to the following condition $(\bar{B}_k)$: there exist constants $\alpha_k, \gamma_k, \tau > 0$ such that*
$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(0,T)^m}, \tag{4.37}$$
*for all $u \in \mathcal{U}$ for which $(u - \bar{u}) \in D^\tau_{\bar{u}}$ and $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha_k$.*

2. *Let the function $L_1$ in the objective functional be independent of $y$, then Assumption V.4.1($A_2$) is equivalent to the following condition ($\bar{A}_2$): there exist constants $\alpha_2, \gamma_2, \tau > 0$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_2 \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \tag{4.38}$$

*for all $u \in \mathcal{U}$ for which $(u-\bar{u}) \in C^{\tau}_{\bar{u}}$ and $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_2$.*

The proof goes along the lines of [14, Corollary 14,15].
By Theorem V.4.2, the conditions (4.37) and (4.38) constitute sufficient conditions for strict weak or strong local optimality.

Sufficient second-order conditions for (local) optimality based on (4.32)-(4.36) are given in [6, 4, 8]. For instance, it was proved in [3, 7, 8] that the condition:

$$\exists \delta > 0, \tau > 0 \quad \text{such that} \quad J''(\bar{u})v^2 \geq \delta \|z_{\bar{u},v}\|^2_{L^2(Q)} \quad \forall v \in G \tag{4.39}$$

is sufficient for weak (in the case $G = D^{\tau}_{\bar{u}}$) or strong (in the case $G = E^{\tau}_{\bar{u}}$) local optimality in the elliptic and parabolic setting. It was proven in [4], that (4.39) with $G = C^{\tau}_{\bar{u}}$ is sufficient for strong local optimality. To obtain and improve stability results, an additional assumption is usually imposed, called the structural assumption. Adapted to the problem considered in this paper, it reads

$$\exists \kappa > 0 \text{ such that} \quad \text{meas} \left\{ t \in [0,T] \, : \, \left| \int_\Omega \frac{\partial \bar{H}}{\partial u_j}(x,t)\, \mathrm{d}x \right| \leq \varepsilon \right\} \leq \kappa\varepsilon \quad \forall \varepsilon > 0,\ j = 1, ..., m. \tag{4.40}$$

It is known that the assumption (4.40) implies that $\bar{u}$ is of bang-bang type and the existence of a constant $\tilde{\kappa} > 0$ such that the following growth property holds:

$$J'(\bar{u})(u-\bar{u}) \geq \tilde{\kappa} \|u-\bar{u}\|^2_{L^1(0,T)^m} \quad \forall u \in \mathcal{U}. \tag{4.41}$$

For a proof see [1], [16] or [20]. For stability results under these and additional conditions, see [7, 8, 9, 19, 10, 11, 14].

**Remark V.4.6.** *We compare the items in Assumption V.4.1 to the ones using* (4.39) *and* (4.40)*.*

1. *Assumption V.4.1($A_0$) is implied by the structural assumption* (4.40) *and possible negative curvature as in [10, 11]. For details see [13, Theorem 6.3].*

2. *Assumption V.4.1($A_1$) is implied by the structural assumption* (4.40) *together with*

$$J''(\bar{u})(u-\bar{u})^2 \geq -\tilde{\delta} \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \|u-\bar{u}\|_{L^1(0,T)^m}$$

*for all $u \in \mathcal{U}$ and any $\tilde{\delta} > 0$ sufficiently small. This is clear by Lemma V.2.6 and* (4.41)*.*

3. *By item two in Theorem V.4.5, Assumption V.4.1($A_2$) is implied by* (4.39)*.*

## V.5 Strong metric subregularity and auxiliary results

In this section we study the strong metric subregularity property (SMSr) of the optimality mapping (see, [15, Section 3I] or [12, Section 4]), beginning with a precise definition of the latter.

### The optimality mapping

We begin by defining some mappings used to represent the optimality map in a convenient way. This is done by a sight modification of [13, Section 2.1] and [14, Section 4.1]. Given the initial data $y_0$ in (1.2), we define the set

$$D(\mathcal{L}) := \Big\{ y \in W(0,T) \cap L^\infty(Q) \Big| \Big(\frac{d}{dt} + \mathcal{A}\Big) y \in L^r(Q), y(\cdot,0) = y_0 \Big\}. \tag{5.42}$$

To shorten notation, we define $\mathcal{L} : D(\mathcal{L}) \to L^r(Q)$ by $\mathcal{L} := \frac{d}{dt} + \mathcal{A}$. Additionally, we define the mapping $\mathcal{L}^* : D(\mathcal{L}^*) \to L^r(Q)$ by $\mathcal{L}^* := (-\frac{d}{dt} + \mathcal{A}^*)$, where

$$D(\mathcal{L}^*) := \Big\{ p \in W(0,T) \cap L^\infty(Q) \Big| \Big( -\frac{d}{dt} + \mathcal{A}^* \Big) p \in L^r(Q), p(\cdot,T) = 0 \Big\}.$$

With the mappings $\mathcal{L}$ and $\mathcal{L}^*$, we recast the semilinear state equation (1.2) and the linear adjoint equation (3.25) in a short way:

$$\mathcal{L}y = \langle u, g \rangle - f(\cdot,\cdot,y),$$
$$\mathcal{L}^* p = \frac{\partial L}{\partial y}(\cdot,\cdot,y_u,u) - p\frac{\partial f}{\partial y}(\cdot,\cdot,y_u) = \frac{\partial H}{\partial y}(\cdot,\cdot,y_u,p,u).$$

The normal cone to the set $\mathcal{U}$ at $u \in L^1(0,T)^m$ is defined in the usual way:

$$N_{\mathcal{U}}(u) := \begin{cases} \{\nu \in L^\infty(0,T)^m \,|\, \int_0^T \nu(v-u)\,\mathrm{d}t \le 0 \quad \forall v \in \mathcal{U}\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

The first order necessary optimality condition for problem (1.1)-(1.3) in Theorem V.3.3 can be recast as

$$\begin{cases} 0 &= \mathcal{L}y + f(\cdot,\cdot,y) - \langle u, g \rangle \\ 0 &= \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot,\cdot,y,p,u), \\ 0 &\in \int_\Omega \frac{\partial H}{\partial u}(x,\cdot,y,p,u)\,\mathrm{d}x + N_{\mathcal{U}}(u). \end{cases} \tag{5.43}$$

For (5.43) to make sense, a solution $(y,p,u)$ must satisfy $y \in D(\mathcal{L})$, $p \in D(\mathcal{L}^*)$ and $u \in \mathcal{U}$. For a local solution $\bar{u} \in \mathcal{U}$ of problem (1.1)-(1.3), by Theorem V.3.3, the triple $(y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ is a solution of (5.43). We define the sets

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(Q) \times L^2(Q) \times L^\infty(0,T)^m, \tag{5.44}$$

and consider the set-valued mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ given by

$$\Phi \begin{pmatrix} y \\ p \\ u \end{pmatrix} := \begin{pmatrix} \mathcal{L}y + f(\cdot,\cdot,y) - \langle u, g \rangle \\ \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot,\cdot,y,p,u) \\ \int_\Omega \frac{\partial H}{\partial u}(x,\cdot,y,p,u)\,\mathrm{d}x + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{5.45}$$

With the abbreviation $\psi := (y,p,u)$, the system (5.43) can be rewritten as the inclusion $0 \in \Phi(\psi)$. Therefore, the mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called the *optimality mapping* of the optimal control problem (1.1)-(1.3). Our goal is to study the stability of the system (5.43), or equivalently, the stability of the solutions of the inclusion $0 \in \Phi(\psi)$ under perturbations. For elements $\xi, \eta \in L^r(Q)$ and $\rho \in L^\infty(0,T)^m$ we consider the perturbed system

$$\begin{cases} \xi &= -\mathcal{L}y + f(\cdot,\cdot,y) - \langle g, u\rangle, \\ \eta &= -\mathcal{L}^* p + \frac{\partial H}{\partial y}(\cdot,\cdot,y,p,u), \\ \rho &\in \int_\Omega \frac{\partial H}{\partial u}(x,\cdot,y,p)\,\mathrm{d}x + N_{\mathcal{U}}(u), \end{cases} \tag{5.46}$$

or equivalently, the inclusion $\zeta \in \Phi(\psi)$, where $\zeta := (\xi,\eta,\rho) \in \mathcal{Z}$.

The next theorem is a consequence of the fact that (5.46) represents the Pontryagin maximum principle for an appropriately perturbed version of problem (1.1)-(1.3), for which a solution exists by the same argument as in the beginning of Section V.3.

**Theorem V.5.1.** *For any perturbation $\zeta := (\xi,\eta,\rho) \in L^r(Q) \times L^r(Q) \times L^\infty(0,T)^m$ there exists a triple $\psi := (y,p,u) \in \mathcal{Y}$ such that $\zeta \in \Phi(\psi)$.*

Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$, we denote by $B_{\mathcal{X}}(c,\alpha)$ the closed ball of radius $\alpha > 0$ centered at $c \in \mathcal{X}$. The spaces $\mathcal{Y}$ and $\mathcal{Z}$, introduced in (5.44), are endowed with the metrics

$$\begin{aligned} d_{\mathcal{Y}}(\psi_1,\psi_2) &:= \|y_1 - y_2\|_{L^2(Q)} + \|p_1 - p_2\|_{L^2(Q)} + \|u_1 - u_2\|_{L^1(0,T)^m}, \\ d_{\mathcal{Z}}(\zeta_1,\zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(Q)} + \|\eta_1 - \eta_2\|_{L^2(Q)} + \|\rho_1 - \rho_2\|_{L^\infty(0,T)^m}, \end{aligned} \tag{5.47}$$

where $\psi_i = (y_i,p_i,u_i)$ and $\zeta_i = (\xi_i,\eta_i,\rho_i)$, $i \in \{1,2\}$. Further on, we denote $\bar\psi := (y_{\bar u}, p_{\bar u}, \bar u)$.

The following extension of the previous theorem can be proved along the lines of [13, Theorem 4.12].

**Theorem V.5.2.** *Let Assumption V.4.1($A_0$) hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in B_{\mathcal{Z}}(0;\delta)$ there exists $\psi \in B_{\mathcal{Y}}(\bar\psi;\varepsilon)$ satisfying the inclusion $\zeta \in \Phi(\psi)$.*

### Strong metric subregularity: main result

This subsection contains one of the main results in this paper: estimates of the difference between the solutions of the perturbed system (5.46) and a reference solution of the unperturbed one, (5.43), by the size of the perturbations. This will be done using the notion of *strong metric subregularity* recalled in the next paragraphs.

**Definition V.5.3.** *Let $\bar\psi$ satisfy $0 \in \Phi(\bar\psi)$. We say that the optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is* strongly metrically subregularity *(SMsR) at $(\bar\psi, 0)$ if there exist positive numbers $\alpha_1, \alpha_2$ and $\kappa$ such that*

$$d_{\mathcal{Y}}(\psi,\bar\psi) \le \kappa d_{\mathcal{Z}}(\zeta, 0)$$

*for all $\psi \in B_{\mathcal{Y}}(\bar\psi;\alpha_1)$ and $\zeta \in B_{\mathcal{Z}}(0;\alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.*

Notice that applying the definition with $\zeta = 0$ we obtain that $\bar\psi$ is the unique solution of the inclusion $0 \in \Phi(\psi) \cap B_{\mathcal{Y}}(\bar\psi;\alpha_1)$. In particular, $\bar u$ is a strict local minimizer for problem (1.1)-(1.3).

In the next assumption we introduce a restriction on the set of admissible perturbations, call it $\Gamma$, which is valid for the remaining part of this section.

**Assumption V.5.4.** *For a fixed positive constant $C_{pe}$, the admissible perturbation $\zeta = (\xi, \eta, \rho) \in \Gamma \subset \mathcal{Z}$ satisfy the restriction*

$$\|\xi\|_{L^r(Q)} \le C_{pe}. \tag{5.48}$$

For any $u \in \mathcal{U}$ and $\zeta \in \Gamma$ we denote by $(y_u^\zeta, p_u^\zeta, u)$ a solution of the first two equations in (5.46). Using (2.6) in Theorem V.2.3 we obtain the existence of a constant $K_y$ such that

$$\|y_u^\zeta\|_{L^\infty(Q)} \le K_y \quad \forall u \in \mathcal{U} \;\; \forall \zeta \in \Gamma. \tag{5.49}$$

Then for every $u \in \mathcal{U}$, every admissible disturbance $\zeta$, and the corresponding solution $y$ of the first equation in (5.46) it holds that $(y_u^\zeta(x,t), u(t)) \in R := [-K_y, K_y] \times [u_a, u_b]^m$.

**Remark V.5.5.** *We apply the local properties in Assumption V.2.2 to the interval $[-K_y, K_y]$, and denote by $\bar{C}$ a constant that majorates the bounds and the Lipschitz constants of $f$, $L_0$ and $L_1$ and their first and second derivatives with respect to $y \in [-K_y, K_y]$.*

By increasing the constant $K_y$, if necessary, we may also estimate the adjoint state:

$$\|p_u^\zeta\|_{L^\infty(Q)} \le K_y(1 + \|\eta\|_{L^r(Q)}) \quad \forall u \in \mathcal{U} \;\; \forall \zeta \in \Gamma.$$

This follows from Theorem V.2.4 with $\alpha = -\frac{\partial f}{\partial y}(x,t,y_u^\zeta)$ and with $\frac{\partial L}{\partial y}(x,t,y_u^\zeta,u)$ at the place of $u$.

The main result of this paper follows.

**Theorem V.5.6.** *Let assumption V.4.1($B_0$) be fulfilled for the reference solution $\bar{\psi} = (\bar{y}, \bar{p}, \bar{u})$ of $0 \in \Phi(\psi)$. Then the mapping $\Phi$ is strongly metrically subregular at $(\bar{\psi}, 0)$. More precisely, there exist $\alpha_n, \kappa_n > 0$ such that for all $\psi \in \mathcal{Y}$ with $\|u - \bar{u}\|_{L^1(0,T)^m} \le \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$, the following inequality is satisfied:*

$$\|\bar{u} - u\|_{L^1(0,T)^m} + \|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \tag{5.50}$$

$$\le \kappa_n \Big( \max_{1 \le j \le m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big). \tag{5.51}$$

To prove Theorem V.5.6, we need some technical lemmas.

**Lemma V.5.7.** *([14, Lemma 18]) Let $u \in \mathcal{U}$ be given and $v \in L^r(0,T)^m$, $\xi, \eta \in L^r(Q)$. Consider solutions $y_u$, $p_u$, $z_{\bar{u},v}$ and $y_u^\xi$, $p_u^\eta$, $z_{\bar{u},v}^\xi$ of the equations*

$$\begin{cases} \mathcal{L}y + f(\cdot,\cdot,y) &= \langle g, u\rangle, \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot,\cdot,y_u,p,u) &= 0, \\ \mathcal{L}_0 z + f_y(\cdot,\cdot,y_u)z &= \langle g, v\rangle, \end{cases} \quad \begin{cases} \mathcal{L}y + f(\cdot,\cdot,y) &= \langle g, u\rangle + \xi, \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot,\cdot,y_u^\xi,p,u) &= \eta, \\ \mathcal{L}_0 z + f_y(\cdot,\cdot,y_u^\xi)z &= \langle g, v\rangle, \end{cases} \tag{5.52}$$

*Here, $\mathcal{L}_0$ is defined as $\mathcal{L}$, but on the domain (5.42) with $y_0 = 0$. Then for every $s \in [1, \frac{n+2}{n})$ there exist constants $K_s, K_2, R_2 > 0$, independent of $\zeta \in \Gamma$, such that the following inequalities hold*

$$\|y_u^\xi - y_u\|_{L^2(Q)} \le C_2 \|\xi\|_{L^2(Q)}, \tag{5.53}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^2(Q)} \le K_2 \|\xi\|_{L^r(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{5.54}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^s(Q)} \le K_s \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{5.55}$$

$$\|p_u^\eta - p_u\|_2 \le R_2(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}), \tag{5.56}$$

*where $C_2$ is the constant given in (2.9).*

**Lemma V.5.8.** *Let $u \in \mathcal{U}$ and $y_u$, $p_u$ be the corresponding state and adjoint state. Further, let $y_u^\zeta$ and $p_u^\zeta$ be solutions to the perturbed state and adjoint equation in* (5.46) *for the control $u$.*

1. *Let the function $L_1$ in the objective functional be independent of $y$. There exists a constant $C > 0$, independent of $\zeta \in \Gamma$, such that for all $v \in L^r(0,T)^m$, the following estimate holds:*
$$\Big| \int_Q \Big\langle \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), v \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \Big| \leq C(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|z_{u,v}\|_{L^2(Q)}. \quad (5.57)$$

2. *There exists a constant $\tilde{C} > 0$, independent of $\zeta \in \Gamma$, such that for all $v \in L^r(0,T)^m$, the following estimate holds:*
$$\Big| \int_Q \Big\langle \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), v \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \Big| \leq \tilde{C}(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|v\|_{L^1(0,T)^m}. \quad (5.58)$$

*Proof.* We begin with integrating by parts

$$\begin{aligned}
&\Big| \int_Q \Big\langle \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), v \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \Big| \leq \Big| \int_Q \Big[\frac{\partial L_0}{\partial y}(x,t,y_u) z_{u,v} - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) z_{u,v}^\zeta\Big] \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&+ \Big| \int_Q \Big\langle \frac{\partial L_1}{\partial y}(x,t,y_u), v\rangle z_{u,v} - \langle \frac{\partial L_1}{\partial y}(x,t,y_u^\zeta), v \Big\rangle z_{u,v}^\zeta \,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_Q \Big\langle L_1(x,t,y_u) - L_1(x,t,y^\zeta), v \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&+ \Big| \int_Q \eta z_{u,v}^\zeta \,\mathrm{d}x\,\mathrm{d}t \Big| = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the first term, we use the Hölder inequality and the mean value theorem to estimate

$$\begin{aligned}
I_1 &\leq \int_Q \Big| \frac{\partial L_0}{\partial y}(x,t,y_u) - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) \Big| \, |z_{u,v}| \,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big| \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) \Big| \, |z_{u,v} - z_{u,v}^\zeta| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Big\| \frac{\partial^2 L_0}{\partial y^2}(x,t,y_\theta) \Big\|_{L^\infty(Q)} \|y_u^\zeta - y_u\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)} + K_s \Big\| \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) \Big\|_{L^{s'}(Q)} \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)},
\end{aligned}$$

where $L^{s'}$ is the dual space to $L^s$. By the mean value theorem, Assumption V.2.2, (2.15), (5.53) and (5.68), we can infer the existence of a constant $B_1 > 0$ such that

$$I_1 \leq B_1 \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}. \quad (5.59)$$

The second term is estimated by using Assumption V.2.2, (2.15), Hölder's inequality, and (5.54):

$$\begin{aligned}
I_2 &\leq \Big| \int_Q \Big\langle \frac{\partial L_1}{\partial y}(x,t,y_u) - \frac{\partial L_1}{\partial y}(x,t,y_u^\zeta), v \Big\rangle z_{u,v} \,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_Q \Big\langle \frac{\partial L_1}{\partial y}(x,t,y_u^\zeta), v \Big\rangle \Big[z_{u,v} - z_{u,v}^\zeta\Big] \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\leq K_2 \max_{1 \leq i \leq m} \Big\| \frac{\partial^2 L_{1,i}}{\partial y^2}(x,t,y_{\theta_i}) v_i \Big\|_{L^\infty(Q)} \|y_u - y_u^\zeta\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)} \\
&+ K_s \Big\| \Big\langle \frac{\partial L_1}{\partial y}(x,t,y_u^\zeta), v \Big\rangle \Big\|_{L^{s'}(Q)} \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}
\end{aligned}$$

By the mean value theorem, Assumption V.2.2, (2.15), (5.53) and (5.68), we can infer the existence of a constant $B_2 > 0$ such that

$$I_2 \leq B_2 \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}. \tag{5.60}$$

Applying the mean value theorem $m$ times, we obtain for the third term

$$\begin{aligned} I_3 &\leq \Big| \int_Q \langle L_1(x,t,y_u) - L_1(x,t,y^\zeta), v\rangle \,\mathrm{d}x\,\mathrm{d}t \Big| \\ &\leq \max_{1\leq j\leq m} \Big\| \frac{\partial L_{1,j}}{\partial y}(x,t,y_{\theta_j}) \Big\|_{L^\infty(Q)} \|y_u - y_u^\zeta\|_{L^\infty(0,T,L^2(\Omega))} \|v\|_{L^1(0,T)^m} \end{aligned}$$

and infer by Assumption V.2.2, (2.15), (5.68) and (5.53), the existence of a constant $B_3 > 0$ with

$$I_3 \leq B_3 \|\xi\|_{L^2(Q)} \|v\|_{L^1(0,T)^m}.$$

For the last term, we estimate by Assumption V.2.2, (5.68), (2.15), (5.54) and (5.53)

$$I_4 \leq \|\eta\|_{L^2(Q)} (\|z_{u,v}\|_{L^2(Q)} + \|z_{u,v}^\zeta - z_{u,v}\|_{L^2(Q)}) \leq (1 + K_2 C_{pe}) \|\eta\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}$$

and define $B_4 := 1 + K_2 C_{pe}$. If the function $L_1$ in the objective functional is independent of $y$, the term $I_3$ does not appear and the first estimate (5.57) holds for $C := 4 \max_{1\leq i\leq 4} B_i$. For the other case, (5.58), we use that by Theorem V.2.5 and Lemma V.2.6 it holds

$$\|z_{u,v}\|_{L^2(Q)} \leq 2 \exp\Big( \Big\| \frac{\partial f}{\partial y}(\cdot,\cdot,y_u(\cdot)) \Big\|_{L^\infty(Q)} T \Big) \max_{1\leq j\leq m} \|g_j\|_{L^2(\Omega)} \|v\|_{L^1(0,T)^m}$$

and define $\tilde{C}$ in a similar way. □

*Proof of Theorem V.5.6.* We select $\alpha < \tilde{\alpha}_0$ according to Lemma V.4.4. Let $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$ and $\psi = (y_u^\zeta, p_u^\zeta, u)$ with $\|u - \bar{u}\|_{L^1(0,T)^m} \leq \alpha$ be such that $\zeta \in \Phi(\psi)$, i.e.

$$\begin{cases} \xi &= \mathcal{L} y_u^\zeta + f(\cdot,\cdot,y_u^\zeta) - u, \\ \eta &= \mathcal{L}^* p_u^\zeta - \frac{\partial H}{\partial y}(\cdot,\cdot,y_u^\zeta,p_u^\zeta,u), \\ \rho &\in \int_\Omega \frac{\partial H}{\partial u}(x,\cdot,y_u^\zeta,p_u^\zeta)\,\mathrm{d}x + N_{\mathcal{U}}(u). \end{cases}$$

Let $y_u$ and $p_u$ denote the solutions to the unperturbed problem with respect to $u$, i.e.

$$\langle u, g\rangle = \mathcal{L} y_u + f(\cdot,\cdot,y_u) \text{ and } 0 = \mathcal{L}^* p_u - \frac{\partial H}{\partial y}(\cdot,\cdot,y_u,p_u,u).$$

By Lemma V.5.7, there exists $C_2, R_2 > 0$ independent of $\psi$ and $\zeta$ such that

$$\|y_u^\zeta - y_u\|_{L^2(Q)} + \|p_u^\zeta - p_u\|_{L^2(Q)} \leq (C_2 + R_2)\Big( \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big). \tag{5.61}$$

By the definition of the normal cone, $\rho \in \int_\Omega \frac{\partial H}{\partial u}(x,\cdot,y_u^\zeta,p_u^\zeta)\,\mathrm{d}x + N_{\mathcal{U}}(u)$ is equivalent to

$$0 \geq \int_Q \Big\langle \rho - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), w - u \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in \mathcal{U}.$$

We conclude for $w = \bar{u}$,

$$
\begin{aligned}
0 &\geq \int_Q \Big\langle \frac{\partial H}{\partial u}(x,t,y_u,p_u), u - \bar{u} \Big\rangle \,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big\langle \rho + \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), \bar{u} - u \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\geq J'(u)(u-\bar{u}) - \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} \|\bar{u} - u\|_{L^1(0,T)^m} \\
&\quad - \Big| \int_Q \Big\langle \frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta), \bar{u} - u \Big\rangle \,\mathrm{d}x\,\mathrm{d}t \Big|.
\end{aligned}
$$

By Lemma V.5.8, we have an estimate on the third term. Since $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha$, we estimate by Lemma V.4.4 and Lemma V.5.8

$$
\begin{aligned}
\|u - \bar{u}\|^2_{L^1(0,T)^m} \bar{\gamma}_0 &\leq J'(u)(u-\bar{u}) \\
&\leq \tilde{C}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)\|u - \bar{u}\|_{L^1(0,T)^m} + \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} \|\bar{u} - u\|_{L^1(0,T)^m}
\end{aligned}
$$

and consequently for an adapted constant, denoted in the same way

$$
\|\bar{u} - u\|_{L^1(0,T)^m} \leq \tilde{C}\Big( \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big).
$$

To estimate the states, by Lemma V.2.6, we use the estimate for the controls and obtain

$$
\|y_{\bar{u}} - y_u\|_{L^2(Q)} \leq 2\exp(\|\frac{\partial f}{\partial y}(\cdot,\cdot,\bar{y}(\cdot))\|_{L^\infty(Q)}T) \max_{1\leq j\leq m} \|g_j\|_{L^2(\Omega)} \|\bar{u} - u\|_{L^1(0,T)^m}. \tag{5.62}
$$

Thus, for a constant again denoted by $\tilde{C}$

$$
\|y_{\bar{u}} - y_u\|_{L^2(Q)} \leq \tilde{C}\Big( \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big).
$$

Next, we realize that by Lemma V.5.7 and (V.5)

$$
\begin{aligned}
\|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} &\leq \|y_{\bar{u}} - y_u\|_{L^2(Q)} + \|y_u - y_u^\zeta\|_{L^2(Q)} \\
&\leq \max\{\tilde{C}, C_2\}\Big( \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big).
\end{aligned}
$$

Using $\|p_{\bar{u}} - p_u\|_{L^2(Q)} \leq C_2\|y_{\bar{u}} - y_u\|_{L^2(Q)}$ and (5.56), the same estimate holds for the adjoint state

$$
\begin{aligned}
\|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} &\leq \|p_{\bar{u}} - p_u\|_{L^2(Q)} + \|p_u - p_u^\zeta\|_{L^2(Q)} \\
&\leq (C_2\max\{\tilde{C}, C_2\} + R_2)\Big( \max_{1\leq j\leq m} \|\rho_j\|_{L^\infty(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),
\end{aligned}
$$

subsequently we define $\kappa := C_2 \max\{\tilde{C}, C_2\} + R_2$. □

To obtain results under Assumption V.4.1 for $k \in \{1,2\}$, we need additional restrictions. We either don't allow perturbations $\rho$ (appearing in the inclusion in (5.46)) or they need to satisfy

$$
\rho = \mu\sigma \tag{5.63}
$$

where $\mu = \int_\Omega g\,\mathrm{d}x \in \mathbb{R}^m$ and $\sigma \in W^{1,2}(0,T)$ with $\sigma(T) = 0$.

**Theorem V.5.9.** *Let some of the assumptions $(A_1),(B_1)$ and $(A_2),(B_2)$ be fulfilled for the reference solution $\bar\psi = (\bar y, \bar p, \bar u)$ of $0 \in \Phi(\psi)$. Further, for $(A_1),(A_2)$ let the function $L_1$ in the objective functional be independent of $y$. For $(B_1),(B_2)$ let $L_1$ be affine with respect to $y$. In addition, the set $\Gamma$ of feasible perturbations is restricted to such $\zeta \in \Gamma$ for which the component $\rho$ is either zero or satisfies* (5.63)*. The numbers $\alpha_n$, $\kappa_n$ and $\varepsilon$ are as in Theorem V.5.6. Then the following statements hold for $n \in \{1,2,3\}$:*

1. *Under Assumption V.4.1, cases $(A_1)$ and $(B_1)$, the estimation*

$$\|\bar u - u\|_{L^1(0,T)^m} + \|y_{\bar u} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar u} - p_u^\zeta\|_{L^2(Q)} \le \kappa_n \Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),$$

*hold for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \alpha_n$, in the case of $(A_1)$, or $\|u - \bar u\|_{L^1(0,T)^m} < \alpha_n$ in the case $(B_1)$, and for all $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

2. *Under Assumption V.4.1, cases $(A_2)$ and $(B_2)$, the estimation*

$$\|\bar y - y_u^\zeta\|_{L^2(Q)} + \|\bar p - p_u^\zeta\|_{L^2(Q)} \le \kappa_n \Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)$$

*hold for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \alpha_n$, in the case of $(A_2)$, or $\|u - \bar u\|_{L^1(0,T)^m} < \alpha_n$ in the case $(B_2)$, and for all $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.*

*Proof.* If the perturbation $\rho \in L^2(0,T,H^{-1}(\Omega))$ satisfies (5.63), it holds

$$\begin{aligned}\int_0^T \langle \rho, u - \bar u\rangle \,\mathrm{d}t &= \nu \int_0^T \langle \sigma, u - \bar u\rangle \,\mathrm{d}t = \int_0^T \int_\Omega \sigma \langle g, u - \bar u\rangle \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega (\mathcal{L} z_{\bar u, u-\bar u} + f_y(\cdot,t,y_{\bar u}) z_{\bar u,u-\bar u}) \sigma \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega (-\frac{d\sigma}{dt} + f_y(\cdot,t,y_{\bar u})\sigma) z_{\bar u,u-\bar u} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus we can estimate

$$\begin{aligned}\Big|\int_0^T \langle \rho, u - \bar u\rangle \,\mathrm{d}t\Big| &\le K(\|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|f_y(x,t,y_{\bar u})\|_{L^\infty(Q)}\|\sigma\|_{L^2(0,T)})\|z_{\bar u,u-\bar u}\|_{L^2(Q)} \\ &\le K(\|\frac{d\sigma}{dt}\|_{L^2(0,T)} + C_2\|f_y(x,t,y_{\bar u})\|_{L^\infty(Q)}\|\frac{d\sigma}{dt}\|_{L^2(0,T)})\|z_{\bar u,u-\bar u}\|_{L^2(Q)}.\end{aligned}$$

Under Assumptions $(A_1)$, $(B_1)$, we can proceed as in the proof of Theorem V.5.6 using Lemma V.4.4 and (5.57) in Lemma V.5.8, to infer the existence of constants $\alpha_1, \kappa_1 > 0$ such that

$$\|\bar u - u\|_{L^1(0,T)^m} \le \kappa_1 \Big( \|\frac{d\sigma}{dt}\|_{L^2(Q)^m} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \alpha_1$ or $\|u - \bar u\|_{L^1(0,T)^m} < \alpha_1$ depending on the assumption. By standard estimates, using (2.17), there exists a constant $E > 0$, such that

$$\begin{aligned}&\|y_{\bar u} - y_u\|_{L^2(Q)} + \|p_{\bar u} - p_u\|_{L^2(Q)} \le E\|y_{\bar u} - y_u\|_{L^2(Q)} \le 2E\|z_{u,u-\bar u}\|_{L^2(Q)} \\ &\le 2\kappa_1 E\Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),\end{aligned}$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_1$ or $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha_1$ depending on the assumption. From here on, we can proceed as in the proof of Theorem V.5.6 and redefine the constant $\kappa_1 > 0$ accordingly. Finally, by similar reasoning, under Assumption $(A_2)$, $(B_2)$ with Lemma V.4.4 and Lemma V.5.8, one obtains the existence of a constant $\kappa_2 > 0$ such that

$$\|y_{\bar{u}} - y_u\|_{L^2(Q)} + \|p_{\bar{u}} - p_u\|_{L^2(Q)} \le \kappa_2 \Big( \|\frac{d\sigma}{dt}\|_{L^2(0,T)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big),$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_2$ or $\|u - \bar{u}\|_{L^1(0,T)^m} < \alpha_2$. Again, proceeding as in Theorem V.5.6 and increasing the constant $\kappa_2$ if needed, proves the claim. □

**Remark V.5.10.** *Theorems V.5.6 and V.5.9 concern perturbations which are functions of $x$ and $t$ only. On the other hand, [12, Theorem ] suggests that SMSr implies a similar stability property under classes of perturbations that depend (in a non-linear way) on the state and control. We refer to [14, Section 5] for a detailed discussion on this. By straightforward adaptations, the results therein hold also for the problem considered in this paper.*

# Appendix

A proof of the following lemma can be found in [2, Lemma 3.5] or [6, Lemma 3.5].

**Lemma V.5.11.** *Let $X = L^\infty(Q)$ or $L^2(Q)$. Given $\bar{u} \in \mathcal{U}$ with associated state $\bar{y}$, there exists a constant $B_X > 0$ such that the following estimate holds*

$$\|y_{\bar{u}+\theta(u-\bar{u})} - \bar{y}\|_X \le B_X \|y_u - \bar{y}\|_X \quad \forall \theta \in [0,1] \text{ and } \forall u \in \mathcal{U}. \tag{5.64}$$

We prove the analogous statement for the adjoint state. For an elliptic state equation, a proof is given in [2, Lemma 3.7].

**Lemma V.5.12.** *Let $X = L^\infty(Q)$ or $L^2(Q)$. Given $\bar{u} \in \mathcal{U}$ with associated state $\bar{y}$ and adjoint-state $\bar{p}$, then there exists a constant $\tilde{B}_X > 0$ such that*

$$\|p_{\bar{u}+\theta(u-\bar{u})} - \bar{p}\|_X \le \tilde{B}_X (\|y_u - \bar{y}\|_X + \|u - \bar{u}\|_{L^1(0,T)^m}^{\frac{1}{r}}), \tag{5.65}$$

*for all $\theta \in [0,1]$ and $u \in \mathcal{U}$. If the function $L_1$ in the objective functional is independent of $y$, then there exists a constant $\tilde{B}_X > 0$ such that*

$$\|p_{\bar{u}+\theta(u-\bar{u})} - \bar{p}\|_X \le \tilde{B}_X \|y_u - \bar{y}\|_X, \tag{5.66}$$

*for all $\theta \in [0,1]$ and $u \in \mathcal{U}$.*

*Proof.* Let us prove (5.65). Given $u \in \mathcal{U}$ and $\theta \in [0,1]$, let us denote $u_\theta = \bar{u} + \theta(u - \bar{u})$, $y_\theta = y_{u_\theta}$, and

$p_\theta = p_{u_\theta}$. Subtracting the equations satisfied by $p_\theta$ and $\bar{p}$ we get with the mean value theorem

$$\begin{aligned}
&-\frac{d}{dt}(p_\theta - \bar{p}) + \mathcal{A}^*(p_\theta - \bar{p}) + \frac{\partial f}{\partial y}(x,t,\bar{y})(p_\theta - \bar{p}) = \frac{\partial L}{\partial y}(x,t,y_\theta,u_\theta) - \frac{\partial L}{\partial y}(x,t,\bar{y},\bar{u})\\
&+\Big[\frac{\partial f}{\partial y}(x,t,\bar{y}) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big]p_\theta\\
&= \frac{\partial L_0}{\partial y}(x,t,y_\theta) - \frac{\partial L_0}{\partial y}(x,t,\bar{y}) + \Big\langle \frac{\partial L_1}{\partial y}(x,t,y_\theta) - \frac{\partial L_1}{\partial y}(x,t,\bar{y}), u_\theta\Big\rangle\\
&+\Big\langle \frac{\partial L_1}{\partial y}(x,t,\bar{y}), u_\theta - \bar{u}\Big\rangle + \Big[\frac{\partial f}{\partial y}(x,t,\bar{y}) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big]p_\theta\\
&= \frac{\partial^2 L_0}{\partial y^2}(x,t,y_{\vartheta_1})(y_\theta - \bar{u}) + \sum_{1\le j\le m} \frac{\partial^2 L_{1,j}}{\partial^2 y}(x,t,y_{\vartheta_j})u_{j,\theta}(y_\theta - \bar{y})\\
&+\Big\langle \frac{\partial L_1}{\partial y}(x,t,\bar{y}), u_\theta - \bar{u}\Big\rangle + \frac{\partial^2 f}{\partial y^2}(x,t,y_{\vartheta_{m+1}})(\bar{y} - y_\theta)p_\theta,
\end{aligned}$$

where $y_{\vartheta_i} = \bar{y} + \vartheta_i(y_\theta - \bar{y})$ for some measurable functions $\vartheta_i : Q \longrightarrow [0,1]$, $i = 0, ..., m+1$. Now, we can apply Theorem V.2.4 and Remark V.5.5 to conclude from the above equation the existence of a constant $C_X > 0$ such that

$$\|p_\theta - \bar{p}\|_X \le C_X(\|y_\theta - \bar{y}\|_X + \|u - \bar{u}\|_{L^r(Q)}) \le C_X(B_X\|y_u - \bar{y}\|_X + |\Omega|(2M_{\mathcal{U}})^{\frac{r-1}{r}}\|u - \bar{u}\|^{\frac{1}{r}}_{L^1(0,T)^m}).$$

Defining $\tilde{B}_X := C_X(B_X + |\Omega|(2M_{\mathcal{U}})^{\frac{r-1}{r}})$, with $B_X$ being the constant from Lemma V.5.11, concludes the proof of the first claim. The second claim follows by the same argument and the fact that the right-hand side of the equation satisfied by $p_\theta - \bar{p}$ does not depend on $L_1$. □

Below we shall use the next lemma, the proof of which can be found for linear elliptic equations in [2, Lemma 2.3] and for parabolic equations in [14].

**Lemma V.5.13.** *Let $u \in L^r(Q)$ and $0 \le \alpha \in L^\infty(Q)$. Let $y_u$ be the unique solution of (2.5) and let $p_u$ be a solution of the problem*

$$\begin{cases} -\frac{\partial p}{\partial t} + \mathcal{A}^* p + \alpha p = u & \text{in } Q,\\ p = 0 \text{ on } \Sigma,\ p(\cdot,T) = 0 & \text{on } \Omega. \end{cases} \tag{5.67}$$

*Then, for any $s_n \in [1, \frac{n+2}{n})$ there exists a constant $C_{s_n'} > 0$ independent of $u$ and $\alpha$ such that*

$$\max\{\|y_u\|_{L^{s_n}(Q)}, \|p_u\|_{L^{s_n}(Q)}\} \le C_{s_n'}\|u\|_{L^1(Q)}. \tag{5.68}$$

*Here $s_n'$ denotes the Hölder conjugate of $s_n$.*

*Proof of Lemma V.4.3.* The second variation of the objective functional is given by Theorem V.3.2.

Let us denote $u_\theta$, $y_\theta$, and $\varphi_\theta$ as in the proof of Lemma V.5.12. From (3.22) we obtain that

$$\begin{aligned}
&|[J''(\bar u+\theta(u-\bar u)) - J''(\bar u)](u-\bar u)^2| \\
&\le \int_Q \Big|\Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_\theta,u_\theta) - \frac{\partial^2 L}{\partial y^2}(x,t,\bar y,\bar u)\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big|(\bar\varphi-\varphi_\theta)\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta) z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|\bar\varphi\Big[\frac{\partial^2 f}{\partial y^2}(x,t,\bar y) - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|\Big[\frac{\partial^2 L}{\partial y^2}(x,t,\bar y) - \bar\varphi\frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\Big](z^2_{u_\theta,u-\bar u} - z^2_{\bar u,u-\bar u})\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\int_Q \Big|\Big\langle \frac{\partial L_1}{\partial y}(x,t,y_\theta) - \frac{\partial L_1}{\partial y}(x,t,\bar y), z_{u_\theta,u-\bar u}(u-\bar u)\Big\rangle\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\int_Q \Big|\Big\langle \frac{\partial L_1}{\partial y}(x,t,\bar y), (z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u})(u-\bar u)\Big\rangle\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

The first term, $I_1$, can be estimated as

$$\begin{aligned}
I_1 &\le \int_Q \Big|\Big[\frac{\partial^2 L_0}{\partial y^2}(x,t,y_\theta) - \frac{\partial^2 L_0}{\partial y^2}(x,t,\bar y)\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_Q \Big|\Big[\Big\langle \frac{\partial^2 L_1}{\partial y^2}(x,t,y_\theta) - \frac{\partial^2 L_1}{\partial y^2}(x,t,\bar y), u_\theta\Big\rangle + \Big\langle \frac{\partial^2 L_1}{\partial y^2}(x,t,\bar y), u_\theta - \bar u\Big\rangle\Big] z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&= I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned}$$

For the first two terms, we deduce from Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (2.17) and (2.18), that for every $\rho_{1,i} > 0$ there exists $\varepsilon_{1,i} > 0$ such that

$$I_{1,i} \le \rho_{1,i} \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon_{1,i}, \quad i = 1,2.$$

For $I_{1,3}$ we estimate under Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (2.14), (2.17), (2.18), that for $\|y_u - \bar y\|_{C(\bar Q)}$ sufficiently small

$$\begin{aligned}
&\int_Q \Big|\Big\langle \frac{\partial^2 L_1}{\partial y^2}(x,t,\bar y), u_\theta - \bar u\Big\rangle z^2_{u_\theta,u-\bar u}\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&\le \|z_{u_\theta,u-\bar u}\|^2_{L^\infty(0,T,L^2(\Omega))} \|u_\theta - \bar u\|_{L^1(0,T)^m} \max_{j=1,\dots,m} \Big\|\frac{\partial^2 L_{1,j}}{\partial y^2}(x,t,\bar y)\Big\|_{L^\infty(Q)} \\
&\le \Big[\frac{9}{4}|\Omega|^{\frac12} \exp(\|\frac{\partial f}{\partial y}(\cdot,\cdot,y)\|_{L^\infty(Q)}) \|\bar y - y_u\|_{L^\infty(Q)} \|u-\bar u\|^2_{L^1(0,T)^m} \\
&\cdot \max_{j=1,\dots,m} \|g_j\|_{L^\infty(Q)} \max_{j=1,\dots,m} \Big\|\frac{\partial^2 L_{1,j}}{\partial y^2}(x,t,\bar y)\Big\|_{L^\infty(Q)}\Big].
\end{aligned}$$

We can therefore infer, that for every $\rho_{1,3} > 0$ there exists $\varepsilon_{1,3} > 0$ such that

$$I_{1,3} \le \rho_1 \|u-\bar u\|^2_{L^1(0,T)^m} \quad \text{if} \quad \|y_u - \bar y\|_{L^\infty(Q)} < \varepsilon_{1,3}.$$

For the term $I_2$, we first consider the general case. Using Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (5.65), (2.17) and (2.18), we find for any $\rho_2 > 0$ a $\varepsilon_2 > 0$ such that

$$I_2 \leq \frac{9}{4}\bar{C}\tilde{B}C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}})\|u-\bar{u}\|^{\frac{1}{r}}_{L^1(0,T)^m}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \leq \rho_2\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \quad \text{if} \quad \|u-\bar{u}\|_{L^1(0,T)^m} < \varepsilon_2.$$

In the case that $\frac{\partial L_1}{\partial y} \equiv 0$, we deduce from Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (5.65), (2.17) and (2.18), that for every $\rho_2 > 0$ there exists $\varepsilon_2 > 0$ such that

$$I_2 \leq \rho_j\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_2.$$

For the term $I_3$ we deduce from Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (5.65), (2.17) and (2.18), that for every $\rho_3 > 0$ there exists $\varepsilon_3 > 0$ such that

$$I_3 \leq \rho_3\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_3$$

For $I_4$ we define $\psi := z_{\bar{u},u-\bar{u}} - z_{u_\theta,u-\bar{u}}$. $\psi$ solves the equation

$$\frac{d\psi}{dt} + \mathcal{A}\psi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\psi = \Big[\frac{\partial f}{\partial y}(x,t,y_{u_\theta}) - \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\Big]z_{u_\theta,u-\bar{u}} = \frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_{\bar{u}} - y_{u_\theta})z_{u_\theta,u-\bar{u}},$$

where we used the mean value theorem to infer the existence of a function $\vartheta$ such that the above holds. We apply (2.9) to $\psi$ and estimate

$$\begin{aligned} I_4 &\leq \Big\|\frac{\partial^2 L}{\partial y^2}(x,t,\bar{y}) - \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x,t,\bar{y})\Big\|_{L^\infty(Q)}\|z_{u_\theta,u-\bar{u}} + z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}\|z_{u_\theta,u-\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \\ &\leq \frac{3M_2}{2}\|y_\theta - \bar{y}\|_{L^\infty(Q)}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}. \end{aligned}$$

Then by Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (2.17) and (2.18), for every $\rho_4 > 0$ there exists $\varepsilon_4 > 0$ such that

$$I_4 \leq \rho_4\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_4. \tag{5.69}$$

The term $I_5$, can be estimate similar as $I_{1,3}$, therefore under under Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (2.14), (2.17), (2.18), for every $\rho_5 > 0$ there exists $\varepsilon_5 > 0$ such that

$$I_5 \leq \rho_5\|u-\bar{u}\|^2_{L^1(0,T)^m} \quad \text{if} \quad \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_5.$$

To estimate $I_6$, we select $s$ as in Lemma V.5.13 and apply (5.68) to $\psi$ and estimate for $\|y_u - \bar{y}\|_{L^\infty(Q)}$ sufficiently small

$$\begin{aligned} I_6 &\leq 2M_{\mathcal{U}}^{\frac{s'-1}{s'}} \max_{j=1,..,m}\Big\|\frac{\partial L_{1,j}}{\partial y}(\cdot,\bar{y}(\cdot))\Big\|_{L^\infty(Q)}\|u-\bar{u}\|^{\frac{1}{s'}}_{L^1(0,T)^m}\|z_{u_\theta,u-\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(Q)} \\ &\leq \Big[2M_{\mathcal{U}}^{\frac{s'-1}{s'}}\|\frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)\|_{L^\infty(Q)} \max_{j=1,..,m}\Big\|\frac{\partial L_{1,j}}{\partial y}(\cdot,\bar{y}(\cdot))\Big\|_{L^\infty(Q)} \\ &\cdot \|u-\bar{u}\|^{\frac{1}{s'}}_{L^1(0,T)^m}\|y_{u_\theta} - \bar{y}\|_{L^2(Q)}\|z_{\bar{u}_\theta,u-\bar{u}}\|_{L^2(Q)}\Big]. \end{aligned}$$

Thus, depending on the chosen estimation, under Assumption V.2.2, Remark V.2.7, Remark V.5.5, (5.64), (2.17) and (2.18), for every $\rho_6 > 0$ there exists $\varepsilon_6 > 0$ such that

$$I_6 \leq \rho_6 \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \quad \text{if} \quad \|u - \bar{u}\|_{L^1(0,T)^m} < \varepsilon_6.$$

We remark that by (2.10),

$$\|u - \bar{u}\|_{L^1(0,T)^m} < \frac{\varepsilon^r}{(C_r |\Omega|^{\frac{1}{r}} \|g\|_{L^\infty(\Omega)^m} \|u_a - u_b\|^{\frac{r-1}{r}}_{L^\infty(0,T)^m})^r}$$

implies $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$.
If the function $L_1$ in the objective functional is independent of $y$, the problematic parts in the terms $I_1, I_2, I_5$ and $I_6$ are absent. Further, if the function $L_1$ is affine with respect to $y$, the problematic parts in the terms $I_1, I_2, I_5$ and $I_6$ are either absent or can be estimated under the condition that $\|u - \bar{u}\|_{L^1(0,T)^m}$ is sufficiently small. If this is not the case, we only obtain item 1 of the Lemma V.4.3. Depending on the terms in the objective functional, by taking $\rho_i$ so small that $I_i < \frac{\rho}{6}$ for every $i \in \{1, .., 6\}$ and setting $\varepsilon = \min_{1 \leq i \leq 6} \varepsilon_i$, we complete the proof. □

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