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Argumentation Theory and Alternative Semantics for Non-Classical Logics

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Declaration of Authorship

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Vienna, 1st October, 2021

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Abstract

We present argumentation frames defined on an intermediary level of abstraction between Dung's abstract argumentation frames and logical (fully instantiated) argumentation frames. On these semi-abstract frames, we define attack principles that refine the definition of the attack relations and allow us to recover new argumentative semantics for classical and non-classical logics. By interpreting the attack principles in sequent-based argumentation frames, we introduce additional elimination rules that can be used to make the dynamic derivations shorter, but with unchanged results. Through weighted attack principles, we introduced alternative semantic for the three main t-norm fuzzy logics: Gödel, Łukasiewicz and Product logic. Additional semantics for these logics are also recovered using bipolar argumentation frames and principles.

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CHAPTER 1

Introduction

Argumentation finds its motivation in one of the main human activities: reasoning. Arguing about something, defending our own opinions, persuading people or performing scientific reasoning. These are all daily life situations where argumentation plays a key role and many are the fields where it is an interesting line of research. In this thesis, we focus on an argumentative paradigm introduced by Dung [Dun95] that currently enjoys considerable popularity in Artificial Intelligence. In particular, we investigate some logical aspects of Dung's framework. From the argumentative point of view, we will see how certain logical principles can be relevant for the construction and analysis of argumentative models. From the logical point of view, we will construct new argumentative semantics characterising classical logic and some non-classical logics. While in the literature we can find many works aimed at using mathematical logic to solve argumentative issues, the use of concepts from argumentation theory to explore different logics from a perspective that is quite different from Tarski-style semantics is a new approach.

In his seminal paper Dung [Dun95] defines an argumentation frame as a pair consisting of a set of arguments and a binary relation over them intuitively understood as *attack*. In Dung's framework arguments are understood as abstract entities with no inner structure. Hence the attack relation is simply defined as a binary relation with no further requirements needed. One of the central issues analysed by Dung is, given an argumentation framework, to identify which sets of arguments can be considered acceptable in the light of the given attack relation. To this aim, he defines several kinds of *abstract argumentation semantics* using an extension-based approach and identifies sets of arguments that are "collectively acceptable" i.e. represent a reasonable viewpoint. Then, the justification status of each argument can be defined based on its extension membership. Different definitions of semantics refer to different levels of justification of the arguments.

The relationship between abstract argumentation and logic has been developed right from the start and Dung himself in [Dun95] establishes a formal equivalence between argumentation frameworks and logic programs. Since then, the investigation of this intrinsic connection between abstract argumentation theory and different logical aspects have been a fertile field of research. In particular, to allow for better modelling inconsistent information, arguments have been defined as entities made of three parts: the support (consisting of a set of formulas), the claim or conclusion (consisting of a single formula) and the method of inference between the support and the claim frequently understood as a deductive inference [BH01]. Besides, also the attack relation is instantiated using different kinds of *attack functions* that identify several ways of attacks between logical arguments. The theory that works with these logical arguments is referred as *logical* (or deductive) *argumentation theory*.

Once arguments are logically instantiated, even just by looking at their claims some pattern related to the definition of the attack relation emerges. In particular, we can observe that, for example, the arguments that in a given frame attack an argument with claim A should, at least *implicitly*, attack also an argument with claim $A \wedge B$, e.g. if A represents the proposition *"today it is rainy"*, B stands for *"today it is cold"* and X is *"today the sun is shining"*, the argument with claim X attacks the argument with claim A, therefore also the argument with claim $A \wedge B$ (*"today it is rainy and it is cold"*) is attacked. Principles like these are referred to as *attack principles*, they can be defined for all the connectives and some can also be derived from a more general logical principle: if an argument with claim X attacks an argument with claim F and F is logically entailed from F', then X attacks also the arguments with claim F'. The attack principles that derive from this more general one are referred to as *weak* attack principles since they are easier to justify. The attack principle about conjunction introduced above is a weak attack principle and it will be identified by $(\mathbf{A}.\wedge)$. The introduction of the attack principles is crucial for the development of the thesis and many of the bridges built between argumentation theory and logic are made considering several variations of these. Interestingly the attack principles are not related to any specific instantiation of the attack relation. For this reason, we introduce and work on an intermediary level of abstraction between Dung's abstract argumentation frames and fully instantiated frames. We refer to these new frames where only the claims of the arguments are instantiated as *semi-abstract argumentation frames*.

In this setting, only by focusing on the claims of the arguments, we introduce an argumentative entailment relation that equipped with a specific set of attack principles can be proved to be a sound and complete semantics for CL (classical logic). Some of the attack principles needed to establish such relation are hard to justify and for this reason, they are referred to as *strong* attack principles. For example, one of the strong and problematic attack principles needed to recover a complete semantics for CL is $(\mathbf{C}.\neg)$ that states that if an argument with claim X does not attack an argument with claim A, then it attacks the arguments with claim $\neg A$. In general, the strong principle for a connective is just the inverse of the weak principle of the same connective, therefore the strong principle for conjunction $(\mathbf{C}.\wedge)$ states that whenever an argument with claim X attacks another argument with claim $A \wedge B$, then either the argument with claim X attacks also the arguments with claim A, or it attacks the arguments with claim B. Motivated by the negative results for classical logic (too strong principles are required) we look for a way to sort-out these attack principles and isolate those that are easier to justify. We can do this through a modal interpretation of the attack relation by which we identify a complete argumentative semantics for LM: the logic that arises from dropping the rules $(\neg, r)$, $(\wedge, r)$ and $(\supset, l)$ from Gentzen's classical sequent calculus for classical logic LK [Gen35]. In this interpretation, "an argument (with claim) X attacks an argument (with claim) A" is interpreted as: there is a "viewpoint" (world w) according to which "$\neg A$ whenever X", i.e. in a standard Kripke interpretation, $\mathcal{M}, w \models \Box(X \wedge \neg A)$.

The attack principles that have been introduced on a semi-abstract level, can be interpreted in logical fully instantiated frames. In particular, by focusing on sequent-based logical argumentation frames where arguments are understood as Gentzen's sequents of the form $\Gamma \Rightarrow \psi$, the interpretation of the attack principles in this setting play a role in the quest of the assertions that may be concluded from a given argumentation framework. To this aim Arieli and Straßer [AS15], using sequent-based arguments, have introduced *dynamic derivations*: a machinery that uses Gentzen-style rules to generate new arguments from existing ones and *elimination rules*, defined using the attack functions, to exclude arguments that were previously derived. In this setting, we find an application of the attack principles. If we interpret them in sequent-based argumentation frames, then additional elimination rules can be defined making dynamic derivations shorter without altering the set of surviving arguments (see Chapter 5).

As pointed out in [DHM$^+$11], Dung's frameworks have some limitations: there can be multiple solutions without any criteria for choosing among them and it might be that the only solution is the empty set. For these reasons, Dung's frameworks have been extended in different ways and one possible solution to the problem cited above is the introduction of *weighted* attacks. We consider this kind of extension, but place it in logical frames and rather than applying fuzzy logic to argumentation theory, we investigate to which extent t-norm based fuzzy logics can be interpreted using weighted attacks as semantic units. In particular, we use specific *weighted attack principles* to recover argumentative semantics for the three main t-norm fuzzy logics: Gödel, Łukasiewicz and Product logic. The attack principles introduced are not simply a mathematical trick to recover a complete semantics, but some of them reflect an intuitive understanding of the connectives considered through the definition of the weights, e.g. an attack against a conjunction carries a weight that is at least as large as that against any of its conjuncts. However this is not the case for all the weighted attack principles introduced and, as already happened for the argumentative semantics of CL, some of them are hard to justify. To prove the completeness of the argumentative semantics with the fuzzy logics considered we use Hilbert's style proofs systems and interestingly the weighted attack principles related to the prelinearity axiom (that can be considered a hallmark of all deductive fuzzy logics) are rather reasonable principles.

Considering weighted argumentation frames we build and analyse an argumentative model of a field experiment aimed at studying how to strengthen intergroup cooperation.

From this argumentative analysis, it emerges the need for considering also a positive relation among arguments: the *support* relation. Besides, this argumentative model suggests a specific interpretation of the weights of the attack relations that are defined in terms of the strength of the arguments.

Another way to recover an argumentative semantics for t-norm based fuzzy logic is to define also the arguments with a t-norm standard consequence relation. In this case, we work with bipolar argumentation frames, i.e. frameworks where both the attack and support relation is defined and using *mixed* principles we identify sound and complete semantics for Gödel, Łukasiewicz and Product logic.

Outline

The dissertation is organised as follows: in Chapter 2 some key concepts of abstract argumentation are recalled. In Chapter 3 we focus on logical argumentation theory analysing two possible ways of defining an argument. Besides, we recall the several ways the attack relation can be instantiated and the method introduced in [AS15] to dynamically evaluate a given frame. In Chapter 4 we focus only on claims of the arguments, introduce semi-abstract argumentation frames (that place themselves on an intermediated level of abstraction between Dung's style argumentation frames and fully instantiated ones) and the attack principles. Then, through an argumentative entailment relation, we recover a semantics for classical logic and considering a modal interpretation of the attack relation we sort out the attack principles introduced and recover a semantics for LM. In Chapter 5 we interpret the attack principles in fully instantiated argumentation frames, this analysis helps to better understand the role of minimality and consistency sometimes imposed to the support part of the arguments. Moreover, we define additional elimination rules that can be seen as shortcuts in the process of dynamic derivations. In Chapter 6 we consider weighed argumentation frames and recover an argumentative semantics for Gödel, Łukasiewicz and Product logic. Then, we build and analyse a field experiment using bipolar weighted frames. Finally, in Chapter 7 we introduce t-norm based arguments and through bipolar argumentation frames, we construct alternative semantics for the main three t-norm based fuzzy logics.

Publications Related to this Dissertation

Some of the contributions described in this thesis have first appeared in other publications.

- Chapter 4: [CF18b] and [CF17].
- Chapter 6: [CF18a].

The results presented in Chapter 5 and Chapter 7 have not been published yet.

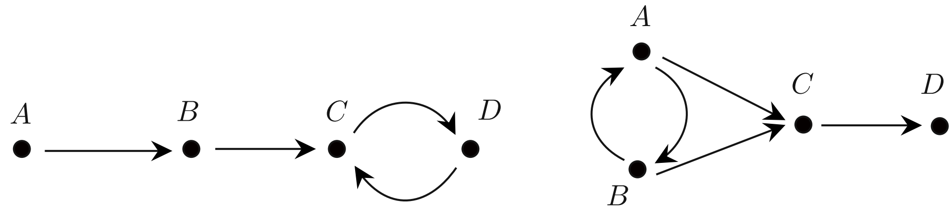
CHAPTER 2

Abstract Argumentation Theory

2.1 Abstract Argumentation Frameworks

Argumentation theory finds its motivations in real-life situations. It can be used as a tool to draw reasonable conclusions from a given set of arguments; it can help in the resolution of a disagreement or the analysis of debates. In the past three decades, argumentation theory has also been employed as a tool for drawing conclusions from databases possibly containing inconsistent sets of statements. In this approach, arguments are extracted from such databases and the central question is: *which set of arguments can be accepted?* In his seminal paper, Dung [Dun95] introduces abstract argumentation frameworks and addresses this question by defining different kinds of *extensions*. Even though understanding the structure of an argument is essential for a complete comprehension of argumentation, the arguments Dung considers are abstract entities, and he shows that we can already say a lot by working on this high level of abstraction. Also, the relations between arguments that Dung considers are very general. He refers only to an attack relation defined just as a binary relation among arguments. The argumentation frameworks introduced by Dung should not be seen as a comprehensive model of the process of argumentation, but rather as a very general, simple and nevertheless powerful framework that can be instantiated in various different ways. As indicated, Dung-style argumentation theory, "abstracting away" from concrete arguments and considering only the attack graph resulting from this abstraction, aims at AI and automatisation, particularly the algorithmic extractions of arguments from databases.

Abstract argumentation with a focus of its non-monotonic nature has been introduced and investigated by Pollock [Pol92], Simari and Loui [SL92] and, with a particular attention to the definition of an argument, also by Lin and Shoham [LS89] and Vreeswijk [Vre93]. In particular, abstract argumentation theory is relevant in the definition of non-monotonic reasoning because of the defeasible structure of the arguments. Arguments, in fact, differ from proofs because their claims cannot be considered definitively accepted, but they



(a) (b)

Figure 2.1: Examples of Argumentation Frameworks

can be challenged by other arguments. Thus, abstract argumentation theory mimics non-monotonic reasoning.

Definition 2.1.1 (Argumentation Frame (AF) ([Dun95]))**.** *An argumentation framework is a pair $\mathcal{A} = \langle Ar, R_{\rightarrow} \rangle$ where:*

1. *Ar is a set of arguments, and*
2. *$R_{\rightarrow}$ is a binary relation over Ar ($R_{\rightarrow} \subseteq Ar \times Ar$).*

Example 2.1.1. *In Figure 2.1 there are two graphic representations of two argumentation frameworks where $Ar = \{A, B, C, D\}$ is the same for both frames, $R^{(a)}_{\rightarrow} = \{(A, B), (B, C), (C, D), (D, C)\}$ and $R^{(b)}_{\rightarrow} = \{(A, B), (B, A), (A, C), (B, C), (C, D)\}$. For reference in examples below, we denote with $\mathcal{A}^{(a)}$ the argumentation frame corresponding to the directed graph (a) of Figure 2.1 and with $\mathcal{A}^{(b)}$ the other one.*

Whenever A and B are two arguments in Ar, $(A, B) \in R_{\rightarrow}$ means that the argument A attacks the argument B. In the sequel instead of writing $(A, B) \in R_{\rightarrow}$, we will use the more intuitive notation $A \longrightarrow B$ and also use $\longrightarrow$ to denote $R_{\rightarrow}$ itself.

Given an argumentation framework $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ and an argument $A \in Ar$, we denote with A^+ the set of arguments attacked by A, i.e. $A^+ = \{F | \ F \in Ar \text{ and } A \longrightarrow F\}$ with A^- the set of arguments that attack A, i.e. $A^- = \{F | \ F \in Ar \text{ and } F \longrightarrow A\}$. If $\mathsf{S} \subseteq Ar$ we say that $A \longrightarrow \mathsf{S}$ if there is $B \in \mathsf{S}$ s.t. $A \longrightarrow B$ and $\mathsf{S} \longrightarrow A$ if there is $B \in \mathsf{S}$ s.t. $B \longrightarrow A$.

Dung's argumentation frames can be used to model a wide range of application scenarios in Artificial Intelligence and related fields. In [Dun95] abstract argumentation theory is used to investigate the logical structure of the solution of n-person games and of the stable marriage problem. Still, he also shows that Reiter's default logic [Rei80] and Pollok's inductive defeasible logic [Pol87, Pol91] are different forms of argumentation. To enlarge the possible applications of abstract argumentation theory, frames with also a positive relation among arguments, called *support*, have been investigated [KP01, Ver02, ACLSL08, CLS13]. Argumentation frames with both attack and support relation are referred to as *bipolar*. In some specific situations, support and attack are dependent notions. For example, if in a given argumentation frame there are three arguments A, B

and C such that $A \longrightarrow B \longrightarrow C$, we have that the argument A defends C. Therefore, A *supports* C ($A \rightsquigarrow C$). However, whenever the arguments are instantiated with actual propositions, this way of reasoning does not always hold. Let us consider the following example [CLS05]:

Example 2.1.2. *There is a family that would like to go hiking. They prefer sunny weather to a cloudy one and a cloudy one to rainy weather. If it is rainy, they will not go hiking. Otherwise, they will go. However, clouds could be a sign of rain. They look at the sky early in the morning, and it is cloudy.*

The following exchange of informal arguments occurs:

A: Today it is a holiday, and we go hiking.

B: The weather will be cloudy, clouds are a sign of rain, we should cancel the hiking.

C: These clouds are early patches of mist; the day will be sunny without clouds, and the weather will be not cloudy.

D: Clouds will not grow. Thus, the weather will be cloudy, but not rainy.

In Example 2.1.2 we have that $D \longrightarrow C \longrightarrow B \longrightarrow A$ and in Dung's framework we have that D defends B and C defends A. However both C and D support the idea of going on a hike. Therefore the idea of considering chains of arguments and counter-arguments, counting the links between them and assess the odd ones as attacks and the even ones as supports is an oversimplification for the definition of support. For these reasons, the attack and support relations in bipolar argumentation frame are understood as independent relations.

Definition 2.1.2 (Bipolar Argumentation Frame (BAF))**.** *A bipolar argumentation framework is a triplet $\mathcal{A} = \langle Ar, \longrightarrow, \rightsquigarrow \rangle$ such that:*

1. *Ar is a set of arguments,*
2. *$\longrightarrow$ is the* attack *relation and it is a binary relation over Ar ($\longrightarrow \subseteq Ar \times Ar$),*
3. *$\rightsquigarrow$ is the* support[1]

   relation and it is a binary relation over Ar ($\rightsquigarrow \subseteq Ar \times Ar$).

We will refer again to bipolar argumentation frames in Section 4.5 and Chapter 7.

[1]In the literature, the *support* relation has also been indicated with the symbol "$\Longrightarrow$", however, to stress that it is an independent relation from *attack*, we prefer to use the symbol "$\rightsquigarrow$". Besides, we will later use the symbol "$\Rightarrow$" for a different purpose, namely as sequent arrow (see Section 3.2).

2.2 Argumentative Semantics

To work on the acceptability of arguments, or sets of arguments, Dung introduces argumentative *extensions*, i.e. sets of compatible arguments that provide counter-attacks (defences) to outside attacks. The acceptability of each argument is based on its membership in extensions, of which, as we shall see, there are various kinds. The first property a set of arguments needs to satisfy to be considered acceptable is that no arguments are attacking each other.

Definition 2.2.1 (Conflict-Free Set ([Dun95])). *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *and* $\mathsf{S} \subseteq Ar$*, we say that* S *is* conflict-free *if there are no* A *and* B *in* S *s.t.* $A \longrightarrow B$*.*

Definition 2.2.2 (Defense of an Argument ([BCM11])). *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$*,* $\mathsf{S} \subseteq Ar$ *and* $A \in Ar$*, we say that* S defends A[2] *if whenever there is* $B \in Ar$ *s.t.* $B \longrightarrow A$*, then* $\mathsf{S} \longrightarrow B$*, i.e. there is* $S \in \mathsf{S}$ *such that* $S \longrightarrow B$*.*

An *admissible extension* is a set of arguments with no inner conflict and able to defend all its elements.

Definition 2.2.3 (Admissible Extension ([Dun95])). *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *and* $\mathsf{S} \subseteq Ar$*, we say that* S *is* admissible *if* S *is conflict-free and defends all its elements, i.e. whenever there is* $B \in Ar$ *s.t.* $B \longrightarrow \mathsf{S}$*, then* $\mathsf{S} \longrightarrow B$*.*

Lemma 2.2.1 (Fundamental Lemma [Dun95]). *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$*,* S *an admissible set of arguments over* $\mathcal{A}$ *and* A *and* A' *arguments defended by* S*, then:*

1. $\mathsf{S}' = \mathsf{S} \cup \{A\}$ *is admissible, and*
2. S' *defends* A'*.*

Given an argumentation framework $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ and $\mathsf{S} \subseteq Ar$, the function $F \colon 2^{Ar} \longrightarrow 2^{Ar}$ such that $F(\mathsf{S}) = \{A \,|\, \mathsf{S} \text{ defends } A\}$ is the *characteristic function* of the frame. Therefore S is *admissible* if S is conflict-free and $\mathsf{S} \subseteq F(\mathsf{S})$. We denote with $\mathsf{Cf}(\mathcal{A})$ the set of all conflict-free sets of the frame $\mathcal{A}$ and with $\mathsf{Adm}(\mathcal{A})$ the set of all admissible extensions in $\mathcal{A}$. In particular we have that the empty set is always conflict-free and admissible. In Table 2.1 we list all the conflict-free sets and admissible extensions of the argumentation frames introduced in Example 2.1.1.

If a set S of arguments is conflict-free and includes every argument it defends, then it is a *complete* extension.

[2] The terminology used in [Dun95] is different. If S *defends* A, for Dung it means that the argument A is acceptable w.r.t. the set of arguments S. However, following [BCM11], we find it more intuitive to say that the set of arguments S defends the argument A.

	$\mathcal{A}^{(a)}$	$\mathcal{A}^{(b)}$
$\mathsf{Cf}(\mathcal{A})$	$\{A\},\{B\},\{C\},\{D\},\{A,C\},\{A,D\},\{B,D\},\emptyset$	$\{A\},\{B\},\{C\},\{D\},\{A,D\},\{B,D\},\emptyset$
$\mathsf{Adm}(\mathcal{A})$	$\{A\},\{D\},\{A,C\},\{A,D\},\emptyset$	$\{A\},\{B\},\{A,D\},\{B,D\},\emptyset$

Table 2.1: Conflict-Free sets and Admissible Extensions of Example 2.1

Definition 2.2.4 (Complete Extension ([Dun95]))**.** *Given an argumentation framework $\mathcal{A} = \langle Ar, \longrightarrow\rangle$ and $\mathsf{S} \subseteq Ar$, S is a* complete *extension if S is conflict-free and $\mathsf{S} = F(\mathsf{S})$. We denote with $\mathsf{Cmp}(\mathcal{A})$ the set of all complete extensions of the argumentation frame $\mathcal{A}$.*

Every complete extension is also an admissible extension, but not vice versa. To find the complete extensions of Example 2.1.1, we compute the characteristic function on every admissible set and verify whether it is a fixed point or not. Concerning the empty set, we have that the only arguments it defends are the unattacked arguments. In $\mathcal{A}^{(a)}$, $F(\emptyset) = \{A\}$ and in $\mathcal{A}^{(b)}$, $F(\emptyset) = \emptyset$, therefore $\emptyset$ is a complete extension only in $\mathcal{A}^{(b)}$ because it satisfies $\emptyset = F(\emptyset)$.

$\mathcal{A}^{(a)}$	$\mathcal{A}^{(b)}$
$F(\{A\}) = \{A\}$	$F(\{A\}) = \{A, D\}$
$F(\{D\}) = \{A, D\}$	$F(\{B\}) = \{B, D\}$
$F(\{A, C\}) = \{A, C\}$	$F(\{A, D\}) = \{A, D\}$
$F(\{A, D\}) = \{A, D\}$	$F(\{B, D\}) = \{B, D\}$
$F(\emptyset) = \{A\}$	$F(\emptyset) = \emptyset$

Table 2.2: Characteristic Function Applied on the Conflict-Free Sets of Example 2.1

Therefore, $\mathsf{Cmp}(\mathcal{A}^{(a)}) = \{\{A\}, \{A,C\}, \{A,D\}\}$ and $\mathsf{Cmp}(\mathcal{A}^{(b)}) = \{\{A,D\}, \{B,D\}, \emptyset\}$

If complete extensions represent reasonable positions one can take in the presence of conflicting information expressed through argumentation frames, *grounded* extensions are the least questionable ones based on complete extensions.

Definition 2.2.5 (Grounded Extension ([Dun95]))**.** *Given an argumentation framework $\mathcal{A} = \langle Ar, \longrightarrow\rangle$, $\mathsf{S} \subseteq Ar$ and $A \in Ar$, we say that S is a* grounded *extension if it is a minimal (w.r.t. set inclusion) complete extension of $\mathcal{A}$. We denote with $\mathsf{Grn}(\mathcal{A})$ the set of all grounded extensions of the argumentation frame $\mathcal{A}$.*

Therefore, $\mathsf{Grn}(\mathcal{A}^{(a)}) = \{\{A\}\}$ and $\mathsf{Grn}(\mathcal{A}^{(b)}) = \{\emptyset\}$.

While grounded semantics takes a skeptical, or least-commitment, viewpoint, one can consider the alternative view oriented at accepting as many arguments as reasonably possible. *Preferred* extensions express the idea of maximising accepted arguments [BCM11].

Definition 2.2.6 (Preferred Extension ([Dun95]))**.** *Given an argumentation framework $\mathcal{A} = \langle Ar, \longrightarrow\rangle$, $\mathsf{S} \subseteq Ar$, we say that S is a* preferred *extension if it is a maximal (w.r.t.*

set inclusion) admissible set. We denote with $\mathsf{Prf}(\mathcal{A})$ *the set of all preferred extensions of the argumentation frame* $\mathcal{A}$.

Therefore, $\mathsf{Prf}(\mathcal{A}^{(a)}) = \{\{A, C\}, \{A, D\}\}$ and $\mathsf{Prf}(\mathcal{A}^{(b)}) = \{\{A, D\}, \{B, D\}\}$.

The notion of *semi-stable* semantics [Ver96, Cam06] is quite similar to that of preferred semantics. The only difference is that instead of maximising S, one maximises $\mathsf{S} \cup \mathsf{S}^+$, called the *range* of a set of arguments [Ver96]. Thus, being a semi-stable extension is a stronger condition than being a preferred extension (see [CCD12], Theorem 3).

Definition 2.2.7 (Semi-Stable)**.** *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *and* $\mathsf{S} \subseteq Ar$, *we say that* S *is a* semi-stable *extension if* S *is a complete extension and there is no complete extension* S' *such that* $\mathsf{S} \cup \mathsf{S}^+ \subsetneq \mathsf{S}' \cup \mathsf{S}'^+$. *We denote with* $\mathsf{S\text{-}stb}(\mathcal{A})$ *the set of all semi-stable extensions of the argumentation frame* $\mathcal{A}$.

$\mathsf{Cmp}(\mathcal{A}^{(a)})$	Range
$\{A\}$	$\{A, B\}$
$\{A, C\}$	$\{A, B, C, D\}$
$\{A, D\}$	$\{A, B, C, D\}$

Table 2.3: Complete Extensions of Example 2.1 (a)

$\mathsf{Cmp}(\mathcal{A}^{(b)})$	Range
$\{A, D\}$	$\{A, B, C, D\}$
$\{B, D\}$	$\{A, B, C, D\}$
$\emptyset$	$\emptyset$

Table 2.4: Complete Extensions of Example 2.1 (b)

Therefore, $\mathsf{S\text{-}stb}(\mathcal{A}^{(a)}) = \{\{A, C\}, \{A, D\}\}$ and $\mathsf{S\text{-}stb}(\mathcal{A}^{(b)}) = \{\{A, D\}, \{B, D\}\}$.

Stable extensions are conflict-free sets of arguments that attack every argument that does not belong to the set. Not every argumentation frame has a stable extension.

Definition 2.2.8 (Stable Extension ([Dun95]))**.** *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *and* $\mathsf{S} \subseteq Ar$, *we say that* S *is a* stable *extension if* S *is conflict-free and* $\mathsf{S} \cup \mathsf{S}^+ = Ar$. *We denote with* $\mathsf{Stb}(\mathcal{A})$ *the set of all stable extensions of the argumentation frame* $\mathcal{A}$.

Therefore, $\mathsf{Stb}(\mathcal{A}^{(a)}) = \{\{A, C\}, \{A, D\}\}$ and $\mathsf{Stb}(\mathcal{A}^{(b)}) = \{\{A, D\}, \{B, D\}\}$. In the following tables we list for each conflict-free set of arguments of frameworks $\mathcal{A}^{(a)}$ and $\mathcal{A}^{(b)}$, its corresponding range.

$\mathsf{Prf}(\mathcal{A}^{(a)})$	$\bigcap \mathsf{Prf}(\mathcal{A}^{(a)})$	$\mathsf{Idl}(\mathcal{A}^{(a)})$
$\{A, C\}$ $\{A, D\}$	$\{A\}$	$\{A\}$

Table 2.7: Steps for Finding the Ideal Extension of Example 2.1 (a)

$\mathsf{Prf}(\mathcal{A}^{(b)})$	$\bigcap \mathsf{Prf}(\mathcal{A}^{(b)})$	$\mathsf{Idl}(\mathcal{A}^{(b)})$
$\{A, D\}$ $\{B, D\}$	$\{D\}$	$\emptyset$

Table 2.8: Steps for Finding the Ideal Extension of Example 2.1 (b)

$\mathsf{Cf}(\mathcal{A}^{(a)})$	Range
$\{A\}$	$\{A, B\}$
$\{B\}$	$\{B, C\}$
$\{C\}$	$\{C, D\}$
$\{D\}$	$\{C, D\}$
$\{A, C\}$	$\{A, B, C, D\}$
$\{A, D\}$	$\{A, B, C, D\}$
$\{B, D\}$	$\{B, C, D\}$
$\emptyset$	$\{A\}$

Table 2.5: Conflict-Free Sets and Range of Example 2.1 (a)

$\mathsf{Cf}(\mathcal{A}^{(b)})$	Range
$\{A\}$	$\{A, B, C\}$
$\{B\}$	$\{A, B, C\}$
$\{C\}$	$\{C, D\}$
$\{D\}$	$\{D\}$
$\{A, D\}$	$\{A, B, C, D\}$
$\{B, D\}$	$\{A, B, C, D\}$
$\emptyset$	$\emptyset$

Table 2.6: Conflict-Free Sets and Range of Example 2.1 (b)

Ideal extensions have been introduced in [ADP93]. We can better understand the informal meaning of ideal semantics in the context of judgment aggregation [CP11]. In particular, suppose to be in a meeting. Each preferred extension represents the point of view of a person present at the meeting. The general aim is to identify the biggest set of arguments accepted by all individuals, which is also defensible. At first, we consider the intersection of all the preferred extensions. However, this intersection might not be able to defend itself. Therefore, we rather move on to consider the biggest admissible set included in the intersection of all preferred extensions.

Definition 2.2.9 (Ideal Extension [ADP93])**.** *Given an argumentation framework* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *and* $\mathsf{S} \subseteq Ar$*, we say that* S *is an* ideal *extension if it is a maximal (w.r.t. set inclusion) admissible set that is contained within every preferred extension. We denote with* $\mathsf{Idl}(\mathcal{A})$ *the set of all ideal extensions of the argumentation frame* $\mathcal{A}$*.*

Fig.2.2 depicts the relation between the extensions introduced. An arrow from the extension A to extension B means that any A-extension is also a B-extension. As a direct consequence of Lemma 18 and Lemma 24 in [Dun95], every complete extension is also admissible. In Theorem 25 (1) in [Dun95], Dung shows that every preferred extension is also complete and still in [Dun95] in Theorem 25 (2) that every grounded extension is also complete. In Theorem 3 in [Cam06], Caminada shows that every semi-stable extension is also preferred and in Theorem 2 (still in [Cam06]) that every stable extension is also semi-stable. In Theorem 2.1 (ii) of [DMT07], it is shown that every ideal extension is also complete.

As we have seen, there are not only different kinds of extensions but, in general, there may be *several* different extension of the same kind for a given argumentation frame. This fact raises an important distinction regarding the overall conclusions that we might draw from a given frame. We may either (skeptically) accept an argument only if it appears in *every* extension of the chosen kind, or (creduously) be already satisfied if an argument in question appears in *at least one* extension of the relevant kind.

Definition 2.2.10 (Credulous and Skeptical Semantics)**.** *Let* $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ *be an argumentation frame and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Idl}, \mathsf{Prf}, \mathsf{S\text{-}Stb}, \mathsf{Stb}\}$. *We denote with:*

$\mathcal{A} \mathrel{|\!\!\sim}^{\forall}_{\mathsf{Sem}} \psi$ *if* $\psi \in \mathsf{E}$ *for every extension* $\mathsf{E} \in \mathsf{Sem}(\mathcal{A})$*. In this case we say that* ψ skeptically *follows from* $\mathcal{A}$*.*

$\mathcal{A} \mathrel{|\!\!\sim}^{\exists}_{\mathsf{Sem}} \psi$ *if there is an extension* $\mathsf{E} \in \mathsf{Sem}(\mathcal{A})$ *such that* $\psi \in \mathsf{E}$*. In this case we say that* ψ credulously *follows from* $\mathcal{A}$*.*

In the following chapters we will only occasionally refer to the various forms of extensions, i.e. to Dung-style semantics. However, the distinction between credulous and skepitcal (non-monotonic) inferences will reappear in Section 3.6 for concrete instantiations of abstract argumentation frames.

We will now move on to review the instance of argumentation that is most central to the whole thesis: logical argumentation.

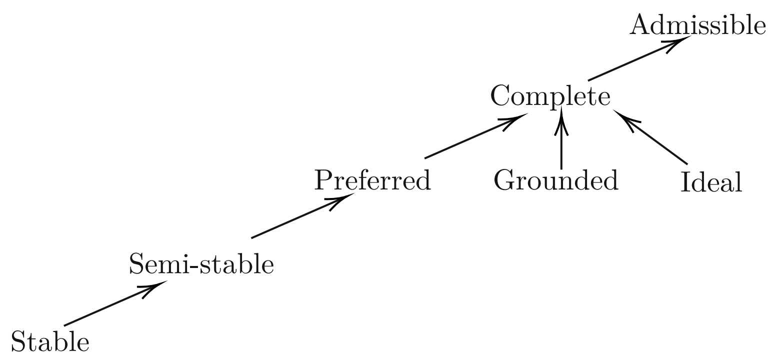


Figure 2.2: Relations Among Extensions

CHAPTER 3

Logical Argumentation

3.1 A Logic-Based Approach

Debates, resolution of disagreements and, more generally, a cognitive process description can be formalised by argumentation theory. In this cognitive process, there are two key aspects: *generating* and *evaluating* arguments [GH11]. The focus of Dung's work is in the characterisation of maximal sets of pairwise compatible arguments that can mutually defend themselves from outer attacks. These sets are referred as *extensions* and have been recalled in section 2.2. After Dung's seminal work [Dun95] there have been several works aimed at deductively formalise his approach [GS04, AB09, RS09, AB10, ABV11, BH01, GH11, GS04]. From the logical perspective of argumentation theory, not only the arguments are instantiated, but, correspondingly, also the attack relation. Using the different parts that constitute an argument, many attack relations can be defined showing the different ways through which an argument can challenge another argument. In the following section we will analyse two possibilities to define an argument. The first one has been largely used in the literature [BH01], [GH11],[AC02], but it has non-negligible complexity limitations for both generating and evaluating arguments. Therefore, we will consider an alternative definition, which uses weaker assumptions and gains a wider expressive and modelling power [AS15].

3.2 Definition of Arguments

In the definition of an argument we distinguish three parts: the *support* (or *reason*, or *premise*), the *claim* (or *conclusion*) and the *method of inference* used to draw the claim from the reason. Following Besnard-Hunter [BH01], we will focus on deductive inference and, in particular, on the deduction in classical logic.

We denote with $\mathcal{L}$ an arbitrary propositional language, with lower case letters (e.g. p, q) *atomic* formulas in $\mathcal{L}$, with uppercase greek letters (e.g. Γ, Δ) *finite sets* of formulas and with calligraphy uppercase letters (e.g. $\mathcal{S}$, $\mathcal{T}$) *arbitrary sets* of formulas in $\mathcal{L}$.

We assume that in the propositional language $\mathcal{L}$ there are at least $\neg$, $\wedge$, $\vee$, $\supset$ and $\perp$.

Definition 3.2.1 (Propositional Logic)**.** *A* propositional logic *for a language $\mathcal{L}$ is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ where $\vdash$ is a Tarskian consequence relation over $\mathcal{L}$, i.e. a binary operation between sets of formulas and formulas in $\mathfrak{L}$, s.t.*

Reflexivity: $\{\varphi\} \vdash \varphi$.

Monotonicity: If $\mathcal{S} \vdash \varphi$ and $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{S}' \vdash \varphi$.

Cut: If $\mathcal{S} \vdash \varphi$ for each $\varphi \in \mathcal{S}'$ and $\mathcal{S}' \vdash \psi$, then $\mathcal{S} \vdash \psi$.

Finitary: If $\mathcal{S} \vdash \varphi$, then there is a finite set $\Gamma \subseteq \mathcal{S}$ such that $\Gamma \vdash \varphi$

Two $\mathcal{L}$-formulas φ and ψ are *logical equivalent* ($\varphi \equiv \psi$) if $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

A set of possible arguments is determined by a *database* that can be understood as a large repository of information from which arguments can be constructed.

Definition 3.2.2 (Database)**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, a* database *$\Delta$ with respect to $\mathfrak{L}$ is a finite set of propositional formulas in $\mathcal{L}$.*

We say that Δ is inconsistent if for some $\alpha \in \mathcal{L}$ both $\Delta \vdash \alpha$ and $\Delta \vdash \neg\alpha$. If not, then Δ is *consistent*. In general, a database Δ is not expected to be consistent and this reflects the fact that we can have different arguments with contradictory claims.

Definition 3.2.3 (Deductive Argument ([AC02], [BH08], [GH11]))**.** *Let $\mathfrak{L}$ be a propositional logic and Δ a database, an* argument *based on $\mathfrak{L}$ and Δ is a pair $\langle \Gamma, \alpha \rangle$ s.t. $\Gamma \subseteq \Delta$ and:*

1. $\Gamma \vdash \alpha$
2. *Γ is consistent*
3. *Γ is a minimal subset of Δ satisfying (1).*

Γ and α are respectively referred to as the support *(or premise) and* claim *(or conclusion) of the argument $\langle \Gamma, \alpha \rangle$. $\mathsf{S}(\langle \Gamma, \alpha \rangle)$ denotes the support of the argument and $\mathsf{C}(\langle \Gamma, \alpha \rangle)$ its claim. For all $S \subseteq \Delta$, $Arg(S)$ denotes the set of all arguments that can be built from S.*

Note that α does not need to be in the database Δ.

In [GH11], the authors define *equivalent arguments*. Their notion of equivalence is semantic concerning the claim and syntactic for the support. This understanding of equivalence reflects the fact that having two ways to prove the same thing should give rise to two different arguments.

Definition 3.2.4 (Equivalent Arguments ([GH11])). *Let $\mathfrak{L}$ be a propositional logic, Δ a database and $Arg(\Delta)$ the set of all arguments built from Δ. If $\langle\Gamma,\alpha\rangle$ and $\langle\Sigma,\beta\rangle$ are two arguments in $Arg(\Delta)$, we say that they are* equivalent *($\langle\Gamma,\alpha\rangle \equiv \langle\Sigma,\beta\rangle$) if $\mathsf{S}(\langle\Gamma,\alpha\rangle) = \mathsf{S}(\langle\Sigma,\beta\rangle)$ and $\mathsf{C}(\langle\Gamma,\alpha\rangle) \equiv \mathsf{C}(\langle\Sigma,\beta\rangle)$ (the claims are logical equivalent).*

We present an example for generating deductive arguments from a given database as specified in Definition 3.2.3. For sake of concreteness, we assume that the underlying logic is classical logic (CL). Note that much weaker underlying logics yield the same arguments.

Example 3.2.1. *Let $\mathfrak{L} = \mathsf{CL}$ and $\Delta = \{\alpha, \beta, \gamma, \neg\delta, \alpha \supset \beta, \beta \supset \neg\delta, \beta \supset \neg\gamma, \beta \supset \neg\delta \wedge \neg\gamma, \neg\gamma \supset \neg\delta\}$, some possible arguments extracted from Δ are:*

$\langle\{\alpha\}, \alpha\rangle$

$\langle\{\alpha, \beta\}, \alpha \wedge \beta\rangle$

$\langle\{\beta, \beta \supset \neg\delta\}, \neg\delta\rangle$

$\langle\{\alpha, \alpha \supset \beta, \beta \supset \neg\delta\}, \neg\delta\rangle$

$\langle\{\alpha, \alpha \supset \beta, \beta \supset \neg\gamma, \neg\gamma \supset \neg\delta\}, \neg\delta\rangle$

While the following are not arguments:

$\langle\{\alpha, \beta\}, \alpha\rangle$ *(the support is not minimal)*

$\langle\{\beta, \gamma, \beta \supset \neg\gamma\}, \neg\gamma\rangle$ *(the support is inconsistent)*

$\langle\{\alpha, \alpha \supset \beta, \beta \supset \neg\gamma \wedge \neg\delta, \neg\gamma \supset \neg\delta\}, \neg\delta\rangle$ *(the support is not minimal)*

We define *logical independence* between two formulas, and then between two arguments, in the following way.

Definition 3.2.5 (Logical Independence). *Let $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ be a propositional logic, two formulas α and β in $\mathcal{L}$ are* logically independent *($\alpha \perp\!\!\!\perp_{\mathfrak{L}} \beta$) if $\alpha \nvdash \beta$, $\alpha \nvdash \neg\beta$, $\beta \nvdash \alpha$ and $\beta \nvdash \neg\alpha$.*

Definition 3.2.6 (Logical Independent Arguments). *Let Δ be a database and $\langle\Gamma_1, \alpha_1\rangle$ and $\langle\Gamma_2, \alpha_2\rangle$ deductive arguments in $Arg(\Delta)$ based on the underlying logic $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$. $\langle\Gamma_1, \alpha_1\rangle$ and $\langle\Gamma_2, \alpha_2\rangle$ are* logically independent *if $\alpha_1 \perp\!\!\!\perp_{\mathfrak{L}} \alpha_2$, $\Gamma_1 \nvdash \neg\alpha_2$, $\Gamma_2 \nvdash \neg\alpha_1$, $\Gamma_1 \nvdash \alpha_2$, $\Gamma_2 \nvdash \alpha_1$, $\alpha_2 \nvdash \neg\bigwedge\Gamma_1$ and $\alpha_1 \nvdash \neg\bigwedge\Gamma_2$.*

The Example 3.2.1 shows that many of the arguments generated from the same database are not logically independent. To make this more explicit, we introduce a partial order based on the *relative conservativity* of deductive arguments.

Definition 3.2.7 (Relative Conservativity ([BH08]))**.** *If Δ is a database and $\langle\Gamma_1, \alpha_1\rangle$ and $\langle\Gamma_2, \alpha_2\rangle$ are deductive arguments based on the underlying logic $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ such that $\Gamma_1, \Gamma_2 \subseteq \Delta$, we say that $\langle\Gamma_1, \alpha_1\rangle$ is* more (or at least as) conservative *than $\langle\Gamma_2, \alpha_2\rangle$ if $\Gamma_1 \subseteq \Gamma_2$ and $\alpha_2 \vdash \alpha_1$, i.e. $\langle\Gamma_1, \alpha_1\rangle$ is more conservative than $\langle\Gamma_2, \alpha_2\rangle$ if $\langle\Gamma_1, \alpha_1\rangle$ makes a logically stronger claim from a weaker claim.*

If a deductive argument $\langle\Gamma_1, \alpha_1\rangle$ is more conservative than the deductive argument $\langle\Gamma_2, \alpha_2\rangle$, by Definition 3.2.7 we have that $\Gamma_1 \subseteq \Gamma_2$. This relation among the support sets shows that the less conservative argument is the one that uses more assumptions. The other condition ($\alpha_2 \vdash \alpha_1$) tells us that the argument with more assumptions can have logically stronger claims. Thus, there is a trade-off between the assumptions that an argument uses (the support set) and the logical power of its claim.

With $Cn(\Delta)$ we denote $\{\varphi| \ \Delta \vdash \varphi\}$. Whenever a logic $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ admits tautologies, i.e. $Cn(\emptyset) \neq \emptyset$, then arguments with empty supports are built and those arguments are referred to as *tautological arguments*. For any $\alpha \in Cn(\emptyset)$, $\langle\emptyset, \alpha\rangle$ is a tautological argument. Tautological arguments are the most conservative arguments.

Recalling Example 3.2.1, we note that since, $\alpha \wedge \beta \vdash \alpha$ and $\{\alpha\} \subseteq \{\alpha, \beta\}$, the argument $\langle\{\alpha\}, \alpha\rangle$ is more conservative than $\langle\{\alpha, \beta\}, \alpha \wedge \beta\rangle$.

The relation of being more conservative is a pre-order[1] on the set of arguments as observed by Besnard and Hunter in the following Proposition.

Proposition 3.2.1 ([BH01])**.** *If Δ is a database based on $\mathfrak{L}$, the relation of* being more conservative *is a pre-order over $Arg(S)$ for any $S \subseteq \Delta$. Minimal arguments always exist, unless all formulas in Δ are inconsistent. If $Cn(\emptyset) \neq \emptyset$, maximal arguments always exist: they are $\langle\emptyset, \alpha\rangle$ for any $\alpha \in Cn(\emptyset)$.*

The general idea behind the definition of deductive arguments (Definition 3.2.3) is that arguments, through the support set and the method of inference, should justify their claims and make them acceptable ([Amg14]). An argument is convincing if the support set is consistent. Otherwise, e.g., considering CL as the underlying logic and Δ a database such that $\{\alpha, \gamma, \gamma \supset (\neg\alpha \wedge \beta)\} \subseteq \Delta$, from the *Ex Falso Quodlibet* rule (EFQ) we can build the argument $\langle\{\alpha, \gamma, \gamma \supset (\neg\alpha \wedge \beta)\}, \alpha \wedge \beta\rangle$. In the argument just introduced the support set contradicts itself, and this makes the assumptions that entail the contradiction questionable. Either α or $\gamma \supset (\neg\alpha \wedge \beta)$ should not be accepted as assumptions in the same support set. The minimality condition guarantees that, at least to some extend, that the support set of an argument does not contain unnecessary information. The absence of unneeded information in a support set links the support and the claim transparently, making the argument convincing.

Even if Definition 3.2.3 might seem a good starting point for the construction and the identification of strong arguments, we will now show some limitations of this approach.

[1] A pre-order is a relation over a set that satisfies reflexivity and transitivity.

As stated in [DM18], formalisations should be compatible with real-world modes of reasoning, and they should take into account that real-world computational abilities are resource-bounded. Deciding whether the support set of a given argument is consistent or not is in general and depending on the underlying logic, an NP-complete decision problem ([BH06]) and determining whether it is minimal is a Π_2^p-complete problem for CL and at least as hard for many other logics ([PWA03]). Since the two assumptions are, in general, computationally unfeasible, Definition 3.2.3 is subject to criticism. For these reasons, we consider alternative definitions of arguments that drop the minimality and consistency requirements.

The computational complexity issues for checking and generating deductive arguments are not the only reasons for dropping the consistency and minimality requirements in the definition of arguments.

Yet another problem for the minimality condition of Definition 3.2.3 is the syntax dependency. For example, if we consider a propositional logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and a database Δ such that $\{\alpha \wedge \beta, \alpha, \beta, \beta \supset \gamma\} \subseteq \Delta$, we have that by Definition 3.2.3, $\langle \{\alpha \wedge \beta, \beta \supset \gamma\}, \gamma \rangle$ is an argument, while $\langle \{\alpha, \beta, \beta \supset \gamma\}, \gamma \rangle$ is not. It is counterintuitive that given two tuples of formulas, $\langle \Gamma_1, \varphi \rangle$ and $\langle \Gamma_2, \varphi \rangle$, which share the same claim (φ), are built from the same database Δ, use the same underlying logic and have logically equivalent supports, one is considered an argument and the other one is not. We can avoid this situation by dropping the minimality condition in the definition of deductive arguments.

Besides, the inconsistency of a support set of an argument should be part of the process of argumentation itself and should not be precluded by definition in any realistic model of argumentation.

In a game-theoretic setting as the one considered in [GR08], where two players are involved in a public debate, they support opposed claims, and they aim to convince an audience, one might want to disregard the minimality condition of the support to persuade the audience. For example, suppose Player 1 wants to convince the audience that the claim β holds and the only argument that she has is $\langle \{\alpha, \alpha \supset \beta\}, \beta \rangle$. Since she already knows that Player 2 has a convincing argument against α, Player 1 might try to hide her weak point by adding irrelevant (and preferably not attackable) information to the support set, so that the audience might be disoriented by the amount of information that she is giving to them and would not perceive the attack on α so crucial.

The general setting of [AS20a] has been first introduced by Arieli in [Ari13] and later investigated by Arieli and Straßer in [AS14, AS15, AS19]. In this logical argumentation framework, arguments are understood as *sequents*, as introduced by Gentzen [Gen35]. Recently they have also investigated a more general proof-theoretic approach where arguments are *hypersequents* [BSA20].

We analyse now the *sequent*-based approach, and we will use the same notation of Arieli and Straßer.

Definition 3.2.8 ($\mathcal{L}$-Sequent)**.** *Given $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ a propositional logic, an* $\mathcal{L}$-sequent *is an ordered pair of finite multisets of $\mathcal{L}$-formulas, usually written as $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$. The left sequence represents a conjunction of formulas, the right, a disjunction. Arbitrary finite multisets of formulas will be denoted by upper case Greek letters. $\Rightarrow$ is a new symbol not present in the language of the logic $\mathfrak{L}$.*

The intuitive meaning of a sequent is that the conjunction of the φ_i's implies the disjunction of the ψ_j's. Thus, a sequent is equivalent in meaning to the formula

$$\bigwedge_{i=1}^{i=n} \varphi_i \supset \bigvee_{j=1}^{j=m} \psi_j .$$

A sequent calculus proof consists of a rooted tree in which the nodes are sequents. The root of the tree, written at the bottom, is called the *endsequent* and is the sequent proved by the proof. The leaves, at the top of the tree, are called *initial sequents* or *axioms*. Each sequent in a sequent calculus proof can be also inferred by one of the rules of inference. A rule of inference is denoted by a figure $\dfrac{S_1}{S}\ (R_1)$ or $\dfrac{S_1 \quad S_1}{S}\ (R_2)$ indicating that the sequent S may be inferred from S_1 or from the pair S_1 and S_2 . The conclusion, S , is called the *lower sequent* of the inference; each hypotheses is an *upper sequent* of the inference [Bus98]. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in the proof system $\mathfrak{C}$ if there is a sequent calculus proof where $\Gamma \Rightarrow \Delta$ is the *endsequent*.

Definition 3.2.9 (Sound and Complete Proof System)**.** *Let $\mathfrak{C}$ be a sequent-based proof system. $\mathfrak{C}$ is sound and complete w.r.t. the logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ if the following holds. The sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathfrak{C}$ iff $\Gamma \vdash \psi$ holds in the logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$.*

Figure 3.1 shows the proof system **LK** that is sound and complete with classical logic CL.

Definition 3.2.10 (Sequent-based Argument ([Ari13]))**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic, an* $\mathfrak{L}$-argument *is a pair $\langle \Gamma, \varphi \rangle_{\mathfrak{L}}$ where $\Gamma \Rightarrow \varphi$ is an $\mathcal{L}$-sequent such that $\Gamma \vdash \varphi$.*

If S is a set of $\mathcal{L}$-formulas, $Arg_{\mathfrak{L}}(S)$ is the set of all $\mathfrak{L}$-arguments $\langle \Gamma, \varphi \rangle_{\mathfrak{L}}$ based on S, i.e. the $\mathfrak{L}$-arguments whose support set is in S.

The set S of $\mathcal{L}$-formulas plays the role of the database of the Definition 3.2.3 of deductive arguments. The main difference between the two definitions is that in sequent-based arguments, the support sets do not need to be either consistent or minimal. Therefore if we consider the same underlying logic $\mathfrak{L}$ and the same database Δ all the deductive arguments based on $\mathfrak{L}$ and Δ are also sequent-based arguments defined on the same logic and database, i.e. $Arg(\Delta) \subsetneq Arg_{\mathfrak{L}}(\Delta)$. The converse does not hold. E.g., the last three pairs of $\mathcal{L}$-formulas introduced in the Example 3.2.1 are not deductive arguments (see Definition 3.2.3), but they are sequent-based arguments (see Definition 3.2.10).

Axiom:

$$\frac{}{\varphi \Rightarrow \varphi}\ (Ax)$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Weakening)$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (Cut)$$

Logical Rules:

$$\frac{}{\Gamma \Rightarrow \top, \Delta}\ (\top, r) \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot, l)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\neg, r) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\neg, l)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (\wedge, r) \qquad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge, l)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ (\vee, r) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee, l)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}\ (\supset, r) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}\ (\supset, l)$$

Figure 3.1: The proof system for **LK**

In the sequel we shall usually assume that the underlying logic has a sound and complete sequent calculus.

Seeing arguments as sequents allows using deeply studied sequent calculi for the generation of new arguments in an automated way.

3.3 Instantiating the Attack Relations

In Dung's framework, both the arguments and the attack relation are abstract entities. Yet, once the arguments have also been instantiated the conflicts among them can be of several kinds (see [CML00, PV01]). As already mentioned in the previous section, in logical argumentation, arguments are made of two parts: the support and the conclusion. Therefore, we will define instantiated attack relations in terms of these components, i.e. considering, for example, the conclusion of the attacking argument and the support of the attacked one, the conclusion of both the attacking argument and the attacked one, etc. ([Pol92, Ver99, Vre97]).

The notion of *defeater* captures an intuitive idea of a counterargument, understood as an argument whose claim refutes the support of another argument. [BH08, FKEG93].

Definition 3.3.1 (Defeat). *Let Δ be a database on the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ two deductive arguments in $Arg(\Delta)$. $\langle \Gamma_1, \psi_1 \rangle$* defeats *$\langle \Gamma_2, \psi_2 \rangle$ ($\langle \Gamma_1, \psi_1 \rangle \xrightarrow{[Def]} \langle \Gamma_2, \psi_2 \rangle$) if $\psi_1 \vdash \neg \bigwedge \Gamma_2$.*

Example 3.3.1. *If $\mathfrak{L} = \mathsf{CL}$ and $\Delta = \{\neg\alpha, \gamma \supset \alpha, \beta, \beta \supset \alpha\}$ we have that $\langle \{\beta, \beta \supset \alpha\}, \alpha \vee \gamma \rangle$ is a defeater for $\langle \{\neg\alpha, \gamma \supset \alpha\} \neg\gamma \rangle$.*

An argument is an *undercut* if it directly opposes the support of another argument. Undercuts are special cases of defeaters.

Definition 3.3.2 (Undercut). *Let Δ be a database on the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ two deductive arguments in $Arg(\Delta)$. $\langle \Gamma_1, \psi_1 \rangle$* undercuts *$\langle \Gamma_2, \psi_2 \rangle$ ($\langle \Gamma_1, \psi_1 \rangle \xrightarrow{[Ucut]} \langle \Gamma_2, \psi_2 \rangle$), if $\psi_1 \vdash \neg \bigwedge {\Gamma'}_2$ and $\neg \bigwedge {\Gamma'}_2 \vdash \psi_1$ for some ${\Gamma'}_2 \subseteq \Gamma_2$.*

Example 3.3.2. *If $\mathfrak{L} = \mathsf{CL}$ and $\Delta = \{\gamma, \gamma \supset \neg\beta, \beta, \beta \supset \alpha\}$ we have that $\langle \{\gamma, \gamma \supset \neg\beta\}, \neg\beta \rangle$ is an undercut for $\langle \{\beta, \beta \supset \alpha\}, \alpha \rangle$.*

A very intuitive way of understanding the conflict between arguments is to have two arguments with opposite conclusions. This type of attack is *rebuttal*.

Definition 3.3.3 (Rebuttal). *Let Δ be a database on the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ two deductive arguments in $Arg(\Delta)$. $\langle \Gamma_1, \psi_1 \rangle$ is a* rebuttal *of the argument $\langle \Gamma_2, \psi_2 \rangle$ ($\langle \Gamma_1, \psi_1 \rangle \xrightarrow{[Reb]} \langle \Gamma_2, \psi_2 \rangle$), if $\psi_1 \vdash \neg\psi_2$ and $\neg\psi_2 \vdash \psi_1$.*

Rebuttals are also specific cases of defeaters. Now that both arguments and attack relations are defined in a logical setting, we can introduce *logical argumentation frameworks*.

We review below several attack relations from the literature.

Definition 3.3.4 (Attack Relations ([GH11, BH18])). *Let Δ be a database on the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ two deductive arguments in $Arg(\Delta)$. We define the following attack relations by listing the conditions under which $\langle \Gamma_1, \psi_1 \rangle$ attacks $\langle \Gamma_2, \psi_2 \rangle$. On the left we list the symbol for each attack relation.*

[D-Def] *$\langle \Gamma_1, \psi_1 \rangle$ is a* direct defeater *of $\langle \Gamma_2, \psi_2 \rangle$ if there is $\gamma_2 \in \Gamma_2$ such that $\psi_1 \vdash \neg\gamma_2$.*

[C-Def] *$\langle \Gamma_1, \psi_1 \rangle$ is a* compact defeater *of $\langle \Gamma_2, \psi_2 \rangle$ if $\Gamma_1 \vdash \neg \bigwedge \Gamma_2$.*

[CD-Def] *$\langle \Gamma_1, \psi_1 \rangle$ is a* compact direct defeater *of $\langle \Gamma_2, \psi_2 \rangle$ if $\Gamma_1 \vdash \neg\gamma_2$ for some $\gamma_2 \in \Gamma_2$.*

[I-Def] *$\langle \Gamma_1, \psi_1 \rangle$ is a* indirect defeater *of $\langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg \bigwedge \Gamma'$ for some $\Gamma' \subseteq \Gamma$.*

[D-Ucut] *$\langle \Gamma_1, \psi_1 \rangle$ is a* direct undercut *of $\langle \Gamma_2, \psi_2 \rangle$ if there is $\gamma_2 \in \Gamma_2$ such that $\psi_1 \equiv \neg\gamma_2$.*

[Ca-Ucut] $\langle \Gamma_1, \psi_1 \rangle$ *is a* canonical undercut *of* $\langle \Gamma_2, \psi_2 \rangle$ *if* $\psi_1 \equiv \neg \bigwedge \Gamma_2$.

[D-Reb] $\langle \Gamma_1, \psi_1 \rangle$ *is a* defeating rebuttal *of* $\langle \Gamma_2, \psi_2 \rangle$ *if* $\psi_1 \vdash \neg\psi_2$.

[I-Reb] $\langle \Gamma_1, \psi_1 \rangle$ *is an* indirect rebuttal *of* $\langle \Gamma_2, \psi_2 \rangle$ *if there is* $\varphi \in \mathcal{L}$ *such that* $\psi_1 \vdash \varphi$ *and* $\psi_2 \vdash \neg\varphi$.

Note that the various attack relations, defined above, can be understood as sets of pairs, as usual. Hence their relative strength can be compared using set inclusion. In the following figure arrows represent set inclusions, i.e. [D-Ucut]$\longrightarrow$[D-Def] it means that [D-Ucut] $\subseteq$ [D-Def]. The dashed boxes include attack relations logically equivalent. [Def] is logically equivalent to [C-Def] (see [AS15], Proposition 4.4) and [D-Def] is logically equivalent to [CD-Def] (see [AS15], Proposition 4.6).

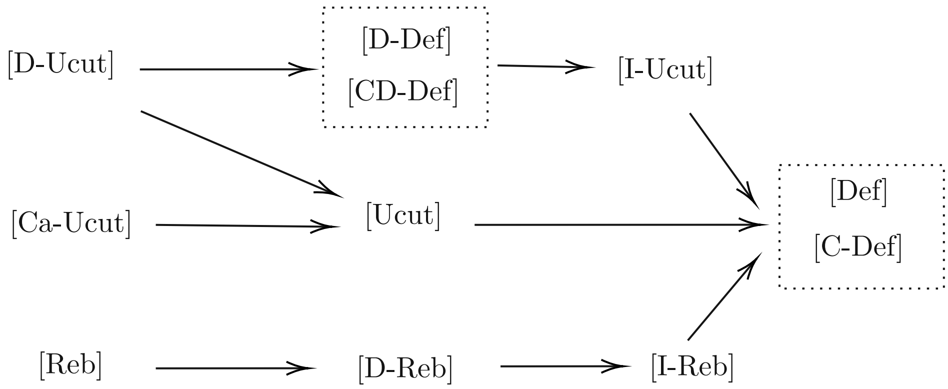


Figure 3.2: Relative Strength of Attack Relations

3.4 Logical and Sequent-based Argumentation Framework

Now that both arguments and attack relations have been defined in logical terms, we can introduce logical and sequent-based argumentation frameworks.

Definition 3.4.1 (Logical Argumentation Framework)**.** *Let* $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *be a propositional logic,* Δ *a database over* $\mathcal{L}$*-formulas and* $\mathscr{A}$ *a set of attack rules. A* logical argumentation framework *over* Δ *is a pair* $AF = \langle Arg(\Delta), Attack(\mathscr{A}) \rangle$ *where* $Arg(\Delta)$ *is the set of all arguments generated by* Δ *and* $Attack(\mathscr{A}) \subseteq Arg(\Delta) \times Arg(\Delta)$ *is an attack relation such that* $(A_1, A_2) \in Attack(\mathscr{A})$ *iff there is some* $\mathscr{R} \in \mathscr{A}$ *such that* A_1 $\mathscr{R}$*-attacks* A_2.

The only difference between logical argumentation framework and sequent-based argumentation framework is in the definition of the arguments. The definition of attack relations remains the same.

Definition 3.4.2 (Sequent-based Argumentation Framework)**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and $\mathscr{A}$ a set of attack rules defined over $\mathfrak{L}$. Let also S be a set of $\mathcal{L}$-formulas. The sequent-based argumentation framework for Δ, induced by $\mathfrak{L}$ and $\mathscr{A}$, is the pair $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$, where $Arg_{\mathfrak{L}}(\Delta)$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of S, and $Attack(\mathscr{A})$ is a relation on $Arg_{\mathfrak{L}}(S) \times Arg_{\mathfrak{L}}(S)$, defined by $(A_1, A_2) \in Attack(\mathscr{A})$ iff there is some $\mathscr{R} \in \mathscr{A}$ such that A_1 $\mathscr{R}$-attacks A_2 (that is, the pair (A_1, A_2) is an instance of the relation $\mathscr{R}$).*

Let see an example of a sequent-based argumentation framework.

Example 3.4.1. *Suppose to have a sequent-based argumentation framework $AF_{\mathfrak{L},\mathscr{A}}(S)$ where the underlying logic is* CL *and the attack rules* defeat, undercut *and* rebuttal *are in the set $\mathscr{A}$. Suppose that $\{\beta, \gamma, \neg\alpha, \neg\beta, \alpha \wedge \beta, \gamma \supset \alpha, \beta \supset \gamma, \beta \supset \neg\alpha, \neg\alpha \vee \neg\beta\} \subseteq S$. In Figure 3.3 there is the graphical representation of the argumentation frame.*

The arguments constructed from S are $\mathfrak{L}$-sequents and, by Definition 3.2.8, their support sets need to be neither consistent nor minimal subsets that entail the conclusion. Thus, the argument $\langle \{\gamma,\ \gamma \supset \alpha\}\ , \alpha\ \rangle_{\mathsf{CL}}$ [Def]-attacks $\langle \{\neg\alpha,\ \neg\beta\}\ , \beta \supset \neg\alpha\ \rangle_{\mathsf{CL}}$, an argument whose support is not a minimal subset that entails the conclusion, and it also [Def]-attacks $\langle \{\alpha \wedge \beta,\ \neg\alpha \vee \neg\beta\}\ , \beta \supset \neg\alpha\ \rangle_{\mathsf{CL}}$, an argument with an inconsistent support set.

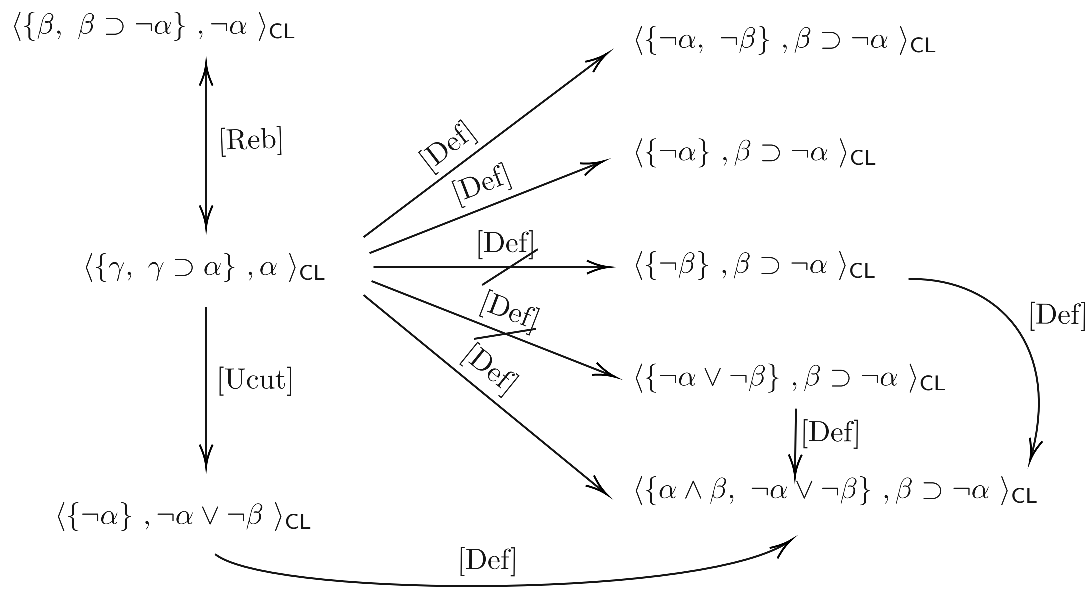


Figure 3.3: Example of a sequent-based argumentation framework

In Example 3.4.1, we have seen that minimality and consistency implicitly play a role in the definition of the attack relation. $\mathfrak{L}$-arguments whose support set is not a minimal subset that entails the conclusion are easier to attack. The inclusion of irrelevant information in the support set makes the argument more vulnerable, e.g. this is the case of the argument $\langle \{\neg\alpha,\ \neg\beta\}, \beta \supset \neg\alpha \rangle_{\mathsf{CL}}$ that it is [Def]-attacked by $\langle \{\gamma,\ \gamma \supset \alpha\}, \alpha \rangle_{\mathsf{CL}}$ while $\langle \{\neg\beta\}, \beta \supset \neg\alpha \rangle_{\mathsf{CL}}$, whose support set is minimal, is not attacked by $\langle \{\gamma,\ \gamma \supset \alpha\}, \alpha \rangle_{\mathsf{CL}}$.

The inconsistency of the support set also makes an $\mathfrak{L}$-argument more vulnerable. If we consider the *defeat* attack relation, $\mathfrak{L}$-arguments with inconsistent support are attacked by every argument of the frame.

3.5 Subset and Logical Minimality

Subset minimality and consistency of the support set discussed in Section 3.2 are not the only factors that make an argument *strong*, i.e. hard to attack. In [AS20b], the authors define a support ordering and show a way to focus on the arguments whose support is minimal with reference to a generic support ordering.

Definition 3.5.1 (Support Ordering ([AS20b]))**.** *Given an argumentation framework $AF_{\mathfrak{L},\mathscr{A}}(S)$, a* support ordering *for $AF_{\mathfrak{L},\mathscr{A}}(S)$ is a preorder $\preceq$ on the finite subsets of S ($\mathscr{P}_{\mathsf{fin}}(S)$).*

Definition 3.5.2 ($\min_{\preceq}(\mathscr{E})$ ([AS20b]))**.** *Given a framework $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$, a support ordering $\preceq$ for $AF_{\mathfrak{L},\mathscr{A}}(S)$, a set $\mathscr{E} \subseteq Arg_{\mathfrak{L}}(S)$, and an argument $\langle \Gamma_1, \psi_1\rangle \in Arg_{\mathfrak{L}}(S)$. We denote with $\min_{\preceq}(\mathscr{E})$ the set of arguments whose support is minimal among the arguments in $\mathscr{E}$, i.e.*

$$min_{\preceq}(\mathscr{E}) = \{\langle \Gamma_1, \psi_1\rangle \in \mathscr{E} |\ \textit{there is no}\ \langle \Gamma_2, \psi_2\rangle \in \mathscr{E}\ \textit{such that}\ \psi_1 = \psi_2\ \textit{and}\ \Gamma_2 \prec \Gamma_1\}$$

The subset relation $\subseteq$ is the most natural support ordering in our context, and it is the one used in the definition of deductive argument by Besnard and Hunter [BH01], see Definition 3.2.3. Another support ordering can be defined using the logical strength of the support sets.

Definition 3.5.3 (Logical Support Ordering ([AS20b]))**.**
Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ be a sequent-based argumentation framework and $\mathscr{P}_{\mathsf{fin}}(S)$ the set of finite subsets of S. If Γ_1 and Γ_2 are elements of $\mathscr{P}_{\mathsf{fin}}(S)$, $\Gamma_1 \preceq_{\vdash} \Gamma_2$ if $\Gamma_2 \vdash \bigwedge \Gamma_1$.

If $\langle \Gamma, \psi\rangle \in min_{\preceq_{\vdash}}(Arg_{\mathfrak{L}}(S))$, then Γ is $\vdash$-minimal. If $\langle \Gamma, \psi\rangle \in min_{\subseteq}(Arg_{\mathfrak{L}}(S))$, then Γ is $\subseteq$-minimal.

Instantiating Definition 3.5.1 with the logical support ordering (see Definition 3.5.3) we obtain *logical minimality.* For clarity we state the extended definition of logical minimality.

Definition 3.5.4 (Logical Minimality)**.** *If $\langle \Gamma, \psi\rangle$ is an $\mathfrak{L}$-argument in $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$, Γ is a* $\vdash$-minimal *support set if there is no $\Gamma' \subseteq S$ such that $\Gamma \vdash \bigwedge \Gamma'$, $\Gamma' \nvdash \bigwedge \Gamma$ and $\Gamma' \vdash \psi$.*

$\vdash$-minimal support sets are *logically equivalent* to their corresponding claims, i.e. if $\langle \Gamma, \psi \rangle$ is $\vdash$-minimal in $Arg_{\mathfrak{L}}(S)$, then $\Gamma \vdash \psi$ and $\psi \vdash \bigwedge \Gamma$.

Proposition 3.5.1. *Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$, be a sequent-based argumentation frame and $\langle \Gamma, \psi \rangle \in Arg_{\mathfrak{L}}(S)$. The set Γ is a $\vdash$-minimal support for ψ iff $\bigwedge \Gamma$ is logically equivalent to ψ.*

Proof. ($\Rightarrow$) For the left-to-right direction of the proof we argue indirectly. Suppose that Γ is not logically equivalent to ψ. Since $\langle \Gamma, \psi \rangle$ is an argument in $Arg_{\mathfrak{L}}(S)$, $\Gamma \vdash \psi$, and from our assumption, it follows $\psi \nvdash \bigwedge \Gamma$. Therefore we have found a $\Gamma' := \{\psi\}$ such that $\Gamma \vdash \bigwedge \Gamma'$, $\Gamma' \nvdash \bigwedge \Gamma$ and $\Gamma' \vdash \psi$, i.e. Γ is a non-$\vdash$-minimal support for ψ.

($\Leftarrow$) Suppose that Γ is a non-$\vdash$-minimal support for ψ. Therefore there is Γ' such that $\Gamma \vdash \bigwedge \Gamma'$, $\Gamma' \nvdash \bigwedge \Gamma$ $(*)$, $\Gamma' \vdash \psi$ and Γ' is a $\vdash$-minimal support for ψ. From the previous point of the proof $\bigwedge \Gamma' \equiv \psi$. Since $\Gamma' \vdash \psi$ and $\psi \vdash \bigwedge \Gamma$, it follows that $\Gamma' \vdash \bigwedge \Gamma$, but this is in contradiction with $(*)$. $\square$

As stated in the following proposition, except that in some trivial case, $\vdash$-minimality always covers $\subseteq$-minimality.

Proposition 3.5.2. *Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$, be a sequent-based argumentation frame. If $\langle \Gamma, \psi \rangle \in Arg_{\mathfrak{L}}(S)$, Γ is $\vdash$-minimal and it does not contain tautologies, then Γ is $\subseteq$-minimal.*

Proof. If a support set of a sequent-based argument $\langle \Gamma, \psi \rangle$ contains a tautology φ and it is $\vdash$-minimal it means that there is no Γ' such that $\Gamma \vdash \bigwedge \Gamma'$, $\Gamma' \nvdash \bigwedge \Gamma$ and $\Gamma' \vdash \psi$. However, if we define $\Gamma' := \Gamma \setminus \{\varphi\}$, we have that $\Gamma' \subsetneq \Gamma$ and $\Gamma' \vdash \psi$, i.e. $\langle \Gamma, \psi \rangle$ is not $\subseteq$-minimal.

If Γ is $\vdash$-minimal and it does not contain tautologies, we cannot find any proper subset of Γ that entails ψ. Otherwise, if there is $\Gamma' \subseteq S$ such that $\Gamma' \subsetneq \Gamma$ and $\Gamma' \vdash \psi$, then $\Gamma \vdash \bigwedge \Gamma'$, $\Gamma' \nvdash \bigwedge \Gamma$ and $\Gamma' \vdash \psi$, i.e. Γ is not $\vdash$-minimal. $\square$

In the logical partial ordering $\preceq_{\vdash}$, $\top$ is a bottom element and $\bot$ a top element. All tautologies are $\vdash$-minimal support sets and contradictions $\vdash$-maximal support sets. Considering a tautology as a support set does not imply any logical commitment, while in the case of contradictions, the logical commitment is maximal.

A sequent-based argument whose support is not $\vdash$-minimal or $\subseteq$-minimal is easier to attack. As we have seen in Section 3.2, $\subseteq$-minimality is syntax dependent while $\vdash$-minimality is not. Some concern about the computational complexity of checking whether a support set is $\vdash$-minimal, emerges. If checking whether a support set is $\subseteq$-minimal is a *local* condition, i.e. we only need to look at all the proper subsets of the support set, checking $\vdash$-minimality is a *global* condition. In the case of $\vdash$-minimality, given $\langle \Gamma, \psi \rangle \in AF_{\mathfrak{L},\mathscr{A}}(S)$ we need to look at all the formulas $\Gamma' \in S$, such that $\Gamma \vdash \bigwedge \Gamma'$,

$\Gamma' \nvdash \bigwedge\Gamma$ and $\Gamma' \vdash \psi$. However, already checking $\subseteq$-minimality and consistency of a given support set is computationally intractable [BH06, PWA03], but still feasible in practice in many situations. In the generating process of the arguments from a given set S of $\mathcal{L}$-formulas, $\subseteq$-minimality and consistency of the support sets can be checked while constructing the arguments. We can do the same with $\vdash$-minimal support sets. Once we find a $\vdash$-minimal support set for an $\mathcal{L}$-formula ψ, then all the arguments $\langle \Gamma^*, \psi \rangle$ with claim ψ and support Γ^* such that $\Gamma^* \vdash \bigwedge\Gamma$ and $\Gamma \nvdash \bigwedge\Gamma^*$, are not $\vdash$-minimal.

In general, support sets that are not $\vdash$-minimal are more vulnerable than the corresponding argument with the support $\vdash$-minimal. They are easier to attack either because in the support sets, they include more formulas than those needed, or because the logical commitment is stronger than the logical commitment of the claim. For these and all the above reasons, we introduce the following definition.

Definition 3.5.5 (Strong Argument)**.** *Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$, be a sequent-based argumentation frame. An argument $\langle \Gamma, \psi \rangle \in Arg_{\mathfrak{L}}(S)$ is* strong *if Γ is both $\vdash$-minimal and consistent.*

3.6 Sequent-based Extensions

Dung-style extensions can be defined also in sequent-based argumentation frames.

Definition 3.6.1 (Sequent-based Extensions[AS20b])**.**
Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$ be a sequent-based argumentation frame and $\mathscr{E} \subseteq Arg_{\mathfrak{L}}(S)$.

- *For any $\mathscr{R} \in \mathscr{A}$, we say that $\mathscr{E}$ $\mathscr{R}$-*attacks* an argument $\langle \Gamma, \psi \rangle$, if there is an argument $\langle \Gamma', \psi' \rangle \in \mathscr{E}$ that $\mathscr{R}$-attacks $\langle \Gamma, \psi \rangle$, i.e. $(\langle \Gamma', \psi' \rangle, \langle \Gamma, \psi \rangle) \in Attack(\mathscr{A})$. The set of arguments that $\mathscr{E}$ attacks is denoted by $\mathscr{E}^+$.*
- *We say that $\mathscr{E}$* defends *and argument $\langle \Gamma, \psi \rangle$, if for some $\mathscr{R} \in \mathscr{A}$, $\mathscr{E}$ $\mathscr{R}$-attacks every argument that $\mathscr{R}$-attacks $\langle \Gamma, \psi \rangle$.*
- *The set $\mathscr{E}$ is* conflict-free *with respect to $AF_{\mathfrak{L},\mathscr{A}}(S)$, if for any $\mathscr{R} \in \mathscr{A}$, $\mathscr{E}$ does not $\mathscr{R}$-attack any of its elements, i.e. $\mathscr{E} \cap \mathscr{E}^+ = \emptyset$.*
- *An* admissible extension *of $AF_{\mathfrak{L},\mathscr{A}}(S)$ is a subset of $Arg_{\mathfrak{L}}(S)$ that is conflict-free and defends all its elements.*
- *A* preferred extension *of $AF_{\mathfrak{L},\mathscr{A}}(S)$ is a subset of $Arg_{\mathfrak{L}}(S)$ that is maximal (with respect to $\subseteq$) among the admissible extensions of $AF_{\mathfrak{L},\mathscr{A}}(S)$.*
- *A* stable extension *of $AF_{\mathfrak{L},\mathscr{A}}(S)$ is a conflict-free set $\mathscr{E} \subseteq Arg_{\mathfrak{L}}(S)$ such that $\mathscr{E} \cup \mathscr{E}^+ = Arg_{\mathfrak{L}}(S)$.*
- *A* complete extension *of $AF_{\mathfrak{L},\mathscr{A}}(S)$ is an admissible extension of $AF_{\mathfrak{L},\mathscr{A}}(S)$ that contains all the arguments that it defends.*

- *A* grounded extension *of* $AF_{\mathfrak{L},\mathscr{A}}(S)$ *is a subset of* $Arg_{\mathfrak{L}}(S)$ *that is minimal (with respect to* $\subseteq$*) among the complete extensions of* $AF_{\mathfrak{L},\mathscr{A}}(S)$*.*

We denote with $\mathsf{Adm}(AF_{\mathfrak{L},\mathscr{A}}(S))$ *(respectively , by* $\mathsf{Cmp}(AF_{\mathfrak{L},\mathscr{A}}(S))$*,* $\mathsf{Grd}(AF_{\mathfrak{L},\mathscr{A}}(S))$*,* $\mathsf{Stb}(AF_{\mathfrak{L},\mathscr{A}}(S))$ *and* $\mathsf{Prf}(AF_{\mathfrak{L},\mathscr{A}}(S))$*) the set of all the admissible (respectively, the complete, grounded, stable and preferred) extensions of* $AF_{\mathfrak{L},\mathscr{A}}(S)$*.*

Credulous and skeptical semantics are defined as follows.

Definition 3.6.2 (Credulous and Skeptical Semantics in Sequent-based Argumentation Frames [AS20b])**.** *Let* $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ *be a sequent-based argumentation frame and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}\}$*. We denote with:*

- $AF_{\mathfrak{L},\mathscr{A}}(S) \mid\!\sim^{\forall}_{\mathsf{Sem}} \psi$ *if* $\psi \in \mathsf{E}$ *for every extension* $\mathsf{E} \in \mathsf{Sem}(AF_{\mathfrak{L},\mathscr{A}}(S))$*. In this case we say that* ψ skeptically *follows from* $AF_{\mathfrak{L},\mathscr{A}}(S)$*.*
- $AF_{\mathfrak{L},\mathscr{A}}(S) \mid\!\sim^{\exists}_{\mathsf{Sem}} \psi$ *if there is an extension* $\mathsf{E} \in \mathsf{Sem}(AF_{\mathfrak{L},\mathscr{A}}(S))$ *such that* $\psi \in \mathsf{E}$*. In this case we say that* ψ credulously *follows from* $AF_{\mathfrak{L},\mathscr{A}}(S)$*.*

3.7 Postulates on the Attack Relations

Gorogiannis and Hunter in [GH11] introduce postulates about the attack relations. These postulates are useful for classifying the attack relations, and they also use them to characterise some of the attack relations, i.e. an arbitrary attack relation satisfies a specific list of postulates if and only if the attack relation $\mathscr{R}$ is a particular member of $\mathscr{A} = \{[Def], [D\text{-}Def], [Ucut], [Ca\text{-}Ucut], [D\text{-}Ucut], [Reb], [D\text{-}Reb]\}$. Let $\langle\Gamma_1, \psi_1\rangle$, $\langle\Gamma_2, \psi_2\rangle$ and $\langle\Gamma_3, \psi_3\rangle$ be three deductive arguments (Definition 3.2.3).

(D0) If $\langle\Gamma_1, \psi_1\rangle \equiv \langle\Gamma'_1, \psi'_1\rangle$ and $\langle\Gamma_2, \psi_2\rangle \equiv \langle\Gamma'_2, \psi'_2\rangle$ (see Definition 3.2.4), then $\langle\Gamma_1, \psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2, \psi_2\rangle$ iff $\langle\Gamma'_1, \psi'_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma'_2, \psi'_2\rangle$.

(D1) If $\langle\Gamma_1, \psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2, \psi_2\rangle$, then $\psi_1 \cup \Gamma_2 \vdash \perp$.

(D2) If $\langle\Gamma_1, \psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2, \psi_2\rangle$ and $\psi_3 \equiv \psi_1$, then $\langle\Gamma_3, \psi_3\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2, \psi_2\rangle$.

Postulate (D0) is a syntax-independence requirement. The syntax of the components of two arguments should not play a role in the definition of the attack relations.

Postulate (D1) affirms that if an argument attacks another argument, then the claim of the attacking argument is inconsistent with the support of the attacked one.

Postulate (D2) states that arguments with equivalent claims should attack the same argument. In order to characterise the several attack relations, some additional postulates are needed.

(D1′) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, there is $\phi \in \mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \neg\phi$.

(D1″) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, then $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \neg\mathsf{C}(\langle\Gamma_2,\psi_2\rangle)$.

(D2′) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, and $\mathsf{C}(\langle\Gamma_3,\psi_3\rangle) \vdash \mathsf{C}(\langle\Gamma_1,\psi_1\rangle)$, then $\langle\Gamma_3,\psi_3\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$.

(D1′) and (D1″) are variations of (D1) and they require stronger conditions that still imply the inconsistency of the attacking argument's claim and the attacked argument's support. (D2′) states that any argument with a logically stronger claim then $\langle\Gamma_1,\psi_1\rangle$ should attack anything that $\langle\Gamma_1,\psi_1\rangle$ attacks.

Postulate (D5) and its variations work in tandem with (D1) and its variations, e.g in (D1″) we require that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \neg\mathsf{C}(\langle\Gamma_2,\psi_2\rangle)$ while in (D5″) that $\neg\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \mathsf{C}(\langle\Gamma_2,\psi_2\rangle)$.

(D5) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, then $\neg\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \bigwedge\mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$.

(D5′) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, there is $\phi \in \mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$ such that $\neg\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \phi$.

(D5″) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, then $\neg\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \mathsf{C}(\langle\Gamma_2,\psi_2\rangle)$.

(D5‴) If $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$, then there is $X \subseteq \mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$ such that $\neg\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \equiv \bigwedge X$.

All the postulates introduced so far put constraints on the attack relations already defined in the frame. However, to characterise the attack relations, a set of postulates that guarantee the existence of some attacks are needed. These postulates are the following:

(D6) If $\{\mathsf{C}(\langle\Gamma_1,\psi_1\rangle)\} \cup \mathsf{S}(\langle\Gamma_2,\psi_2\rangle) \vdash \bot$, then there exists $\langle\Gamma_3,\psi_3\rangle$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \mathsf{C}(\langle\Gamma_3,\psi_3\rangle)$ and $\langle\Gamma_3,\psi_3\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$.

(D6′) If there is $\phi \in \mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \neg\phi$, then there exists $\langle\Gamma_3,\psi_3\rangle$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \mathsf{C}(\langle\Gamma_3,\psi_3\rangle)$ and $\langle\Gamma_3,\psi_3\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$.

(D6″) If $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \neg\mathsf{C}(\langle\Gamma_2,\psi_2\rangle)$, then there exists $\langle\Gamma_3,\psi_3\rangle$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \vdash \mathsf{C}(\langle\Gamma_3,\psi_3\rangle)$ and $\langle\Gamma_3,\psi_3\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$..

(D6‴) If there is $X \subseteq \mathsf{S}(\langle\Gamma_2,\psi_2\rangle)$ such that $\mathsf{C}(\langle\Gamma_1,\psi_1\rangle) \equiv \neg\bigwedge X$, then $\langle\Gamma_1,\psi_1\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_2,\psi_2\rangle$.

It is possible to characterise all the attack relations that belong to the set $\mathscr{A}$ defined at the beginning of this section using the postulates just introduced. In [GH11] we can find the proof of the following Proposition, additional principles and their interpretation.

Proposition 3.7.1 ([GH11]). *Let $AF = \langle Arg(\Delta), Attack(\mathscr{A})\rangle$ be a logical argumentation framework constructed over the database Δ and the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash\rangle$. Let $\mathscr{A} = \{[Def], [D\text{-}Def], [Ucut], [Ca\text{-}Ucut], [D\text{-}Ucut], [Reb], [D\text{-}Reb]\}$. The following characterisations hold:*

1. $\mathscr{R} = [Def]$ *iff* (D1), (D2′) *and* (D6) *hold.*
2. $\mathscr{R} = [D\text{-}Def]$ *iff* (D1′), (D2′) *and* (D6′) *hold.*
3. $\mathscr{R} = [D\text{-}Reb]$ *iff* (D1″), (D2′) *and* (D6″) *hold.*
4. $\mathscr{R} = [Ca\text{-}Ucut]$ *iff* (D1), (D2), (D5) *and* (D6) *hold.*
5. $\mathscr{R} = [D\text{-}Ucut]$ *iff* (D1′), (D2), (D5′) *and* (D6′) *hold.*
6. $\mathscr{R} = [Reb]$ *iff* (D1″), (D2), (D5″) *and* (D6″) *hold.*
7. $\mathscr{R} = [Ucut]$ *iff* (D5‴) *and* (D6‴) *hold.*

3.8 Dynamic Derivations

In a sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ (see Definition 3.4.2) the set $\mathscr{A}$ may include many attack relations. Attacks allow for the elimination or the discharging of sequent-based arguments from $Arg_{\mathfrak{L}}(S)$. We denote with $\Gamma \not\Rightarrow \psi$ the elimination of the sequent $\Gamma \Rightarrow \psi$. Attack relations can be seen as *sequent elimination rules* that exclude sequent-based arguments from a given frame if the premises of the rule are satisfied (see [AS19]). Sequent elimination rules have a form similar to the inference rules, except that the conclusion is the elimination of one of its premises. In general, the premise of a sequent elimination rule has three parts: on the left side, there is the *attacking argument* (that is an $\mathcal{L}$-sequent), on the right the *attacked argument* (still an $\mathcal{L}$-sequent) and between these two the *attacking condition*. As conclusion, there is the attacked argument, i.e. the argument that has been eliminated through the use of the attack.

$$\begin{array}{ccc} \textit{Attacking Argument} & \textit{Attacking Condition} & \textit{Attacked Argument} \end{array}$$

$$\frac{\Gamma_1 \;\Rightarrow \psi_1 \qquad\qquad \psi_1 \;\Rightarrow \neg\bigwedge\Gamma_2 \qquad\qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}\;[\text{Def}]$$

$$\textit{Eliminated Argument}$$

The sequent elimination rules relative to *undercut* and *rebuttal* are the following:

$$\frac{\Gamma_1 \;\Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg\bigwedge\Gamma_2 \quad \neg\bigwedge\Gamma_2 \Rightarrow \psi_1 \qquad \Gamma_2, \Gamma'_2 \Rightarrow \psi_2}{\Gamma_2, \Gamma'_2 \not\Rightarrow \psi_2}\;[\text{Ucut}]$$

Defeat:	[Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Defeat:	[C-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Direct Defeat:	[D-Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Indirect Defeat:	[I-Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Compact Direct Defeat:	[CD-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Compact Indirect Defeat:	[CI-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Undercut:	[Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Direct Undercut:	[D-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \gamma_2 \quad \neg \gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$
Compact Direct Undercut:	[CD-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \neg \gamma_2 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$
Canonical Undercut:	[Ca-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Rebuttal:	[Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \neg \psi_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Rebuttal 1:	[C-Reb-1]	$\dfrac{\Gamma_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Rebuttal 2:	[C-Reb-2]	$\dfrac{\Gamma_1 \Rightarrow \psi_2 \quad \Gamma_2 \Rightarrow \neg \psi_2}{\Gamma_2 \not\Rightarrow \neg \psi_2}$
Defeat Rebuttal:	[D-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Reductio Defeating Rebuttal:	[RD-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_2 \Rightarrow \neg \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Indirect Rebuttal:	[I-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \varphi \quad \psi_2 \Rightarrow \neg \varphi \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$

Table 3.1: Sequent Elimination Rules

$$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg \psi_2 \quad \neg \psi_2 \Rightarrow \psi_1 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \text{[Reb]}$$

One of the overall aims of argumentation theory is to identify which sets of arguments can be considered acceptable in the light of the given attack relation. In [AS14], the authors address this problem using *dynamic derivations*. Through a sequence of inference and sequent elimination rules, possibly new arguments are derived from the original framework and evaluated. At the end of the derivation, only some arguments are considered acceptable.

Definition 3.8.1 (Argumentation Setting ([AS14])). *An* Argumentation Setting *is a triple* $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ *where:*

$\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *is a propositional logic (see Definition 3.2.1)*

$\mathfrak{C}$ *is a sound and complete sequent calculus for* $\mathfrak{L}$ *(see Definition 3.2.9)*

$\mathfrak{A}$ *is a set of sequent elimination rules (see Table 3.1).*

For the following definition we consider two general forms of inference and sequent elimination rules:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta} \quad (1) \qquad \text{and} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma \not\Rightarrow \Delta} \quad (2)$$

Definition 3.8.2 (Inference and Elimination Rules ([AS19])). *Let* $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ *be an argumentation setting,* S *a set of* $\mathcal{L}$*-formulas and* θ *an* $\mathcal{L}$*-substitution, i.e. a function representing replacements, in* $\mathcal{L}$*-formulas, of atomic formulas by* $\mathcal{L}$*-formulas.*

An inference rule $\mathcal{R} \in \mathfrak{C}$ *of the form of (1) is* applicable *(for* $\mathfrak{S}$*, with respect to* θ*), if for every* $1 \leq i \leq n$ $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ *is* $\mathfrak{C}$*-provable.*

An elimination rule $\mathfrak{A}$ *of the form of (2) is* $Arg_{\mathfrak{L}}(S)$-applicable *(for* $\mathfrak{S}$*, with respect to* θ*), if* $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ *and* $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ *are in* $Arg_{\mathfrak{L}}(S)$ *and for each* $1 < i < n$*,* $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ *is* $\mathfrak{C}$*-provable.*

Given a sequent-based argumentation framework $AF_{\mathfrak{S}}(S)$ where $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$, dynamic proofs are sequences of tuples constructed by applications of inference rules from $\mathfrak{C}$ or sequent elimination rules from $\mathfrak{A}$. Each step of the poof is characterised by an index number (to exactly identify each step of the proof), a sequent or an eliminated sequent, an inference rule in $\mathfrak{C}$ or a sequent elimination rule in $\mathfrak{A}$ (depending on weather an argument has been introduced or eliminated) and finally, the sequents over which the rule have been applied.

Definition 3.8.3 (Derivation Step ([AS14])). *A* derivation step *is a quadruple* $\langle i, s, \mathsf{J}, \mathsf{A} \rangle$ *where:*

i *is a natural number and it is the index of the tuple.*

s *is either a sequent* $(\Gamma \Rightarrow \psi)$ *or an eliminated sequent* $(\Gamma \not\Rightarrow \psi)$

J *is a string. It is a rule in* $\mathfrak{C} \cup \mathfrak{A}$ *and a tuple of sequents over which the rule has been applied to recover* s *(*J *stands for* justification *of* s*).*

A *is the empty set or in case a rule in* $\mathfrak{A}$ *has been applied, the attacking sequent-based argument.*

Definition 3.8.4 (Simple Dynamic Derivation ([AS19]))**.** *Let* $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ *be an argumentation setting and* S *a set of formulas in* $\mathcal{L}$ *and* $AF_{\mathfrak{S}}(S)$ *a sequent-based argumentation framework. A* simple (dynamic) derivation *w.r.t.* $\mathfrak{S}$ *and* S *is a finite sequence* $\mathcal{D} = \langle T_1, \ldots, T_m \rangle$ *of derivation steps where each* $T_i \in \mathcal{D}$ *is one of the following forms:*

$T_i = \langle i, \theta(\Gamma) \Rightarrow \theta(\Delta), \mathsf{J}, \emptyset \rangle$ *if there is an inference rule* $\mathcal{R} \in \mathfrak{C}$ *of the form of (1) above that is applicable for some* $\mathcal{L}$*-substitution* θ *and for each* $1 \leq k \leq n$ *there is a proof tuple* $\langle i_k, s_k, \mathsf{J}_k, \emptyset \rangle$ *in which* $i_k < i$ *and* s_k *is the sequent* $\theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$*. In this case* $\mathsf{J} =$ "$\mathcal{R}; i_1, \ldots, i_n$"*.* T_i *is an* ***introduction tuple****.*

$T_i = \langle i, \theta(\Gamma_n) \not\Rightarrow \theta(\Delta_n), \mathsf{J}, \theta(\Gamma_1) \Rightarrow \theta(\Delta_1) \rangle$*, if there is an elimination rule* $\mathcal{R} \in \mathfrak{A}$ *of the form of (2) that is* $Arg_{\mathfrak{L}}(S)$*-applicable for some* $\mathcal{L}$*-substitution* θ *and for every* $1 \leq k \leq n$ *there is a proof tuple* $\langle i_k, s_k, \mathsf{J}_k, \emptyset \rangle$ *in which* $i_k < i$ *and* $s_k = \theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$*. In this case* $\mathsf{J} =$ "$\mathcal{R}; i_1, \ldots, i_n$"*.* T_i *is an* ***eliminating tuple****.*

In a simple dynamic derivation, an argument $\Gamma_2 \Rightarrow \psi_2$ is attacked by another argument $\Gamma_1 \Rightarrow \psi_1$ if $\Gamma_2 \Rightarrow \psi_2$ is in the conclusion of a sequent elimination rule and $\Gamma_1 \Rightarrow \psi_1$ is the first on the left premise of the rule. Once arguments are derived using a simple dynamic derivation $\mathcal{D}$, they can be evaluated with the following definition.

Definition 3.8.5 (Evaluation of a Derivation ([AS19]))**.** *Given a simple dynamic derivation* $\mathcal{D}$*, we evaluate the following three sets:*

$\mathsf{Elim}(\mathcal{D})$*: the sequents that at least once in* $\mathcal{D}$ *are attacked by an attacker which is not already attacked.*

$\mathsf{Attack}(\mathcal{D})$*: the sequents that attack a sequent in* $\mathsf{Elim}(\mathcal{D})$*.*

$\mathsf{Accept}(\mathcal{D})$*: the derived sequents in* $\mathcal{D}$ *that are not in* $\mathsf{Elim}(\mathcal{D})$*.*

Using the functions Elim and Attack, we can define a condition for the coherence of a derivation. This condition assures that eliminating tuples represent *firm* attacks, i.e. there is no sequent in the derivation that eliminates another sequent, and later on, is eliminated itself.

Definition 3.8.6 (Coherence in Simple Dynamic Derivations [AS19])**.** *A simple dynamic derivation* $\mathcal{D}$ *is* coherent*, if* $\mathsf{Elim}(\mathcal{D}) \cap \mathsf{Attack}(\mathcal{D}) = \emptyset$*.*

Given a simple dynamic derivation $\mathcal{D}$, we denote by $\mathcal{D}' = \mathcal{D} \oplus \langle T_1, \ldots, T_n \rangle$ the simple dynamic derivation whose prefix is $\mathcal{D}$ and whose suffix is $\langle T_1, \ldots, T_n \rangle$. $\mathcal{D}'$ is the *extension* of $\mathcal{D}$ by $\langle T_1, \ldots, T_n \rangle$.

Dynamic derivations are simple dynamic derivation extended in a restricted manner, and they illustrate the non-monotonicity of the reasoning process in argumentation theory. In these derivations arguments can be challenged by a counter-argument and a specific sequent, at a particular stage of the proof may be considered not derived even if in an earlier stage of the proof it was considered derived. Derived sequents may be eliminated in light of new proof tuples, but also the other way around is possible. An eliminated sequent may be restored if its attacking tuple is counter-attacked by a new eliminating tuple.

Definition 3.8.7 (Dynamic Derivation ([AS19]))**.** *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be an argumentation setting and S a set of formulas in $\mathcal{L}$. A* dynamic derivation *(for $\mathfrak{S}$, based on S) is a simple derivation $\mathcal{D}$ of one of the following forms:*

1. *$\mathcal{D} = \langle T \rangle$ where $T = \langle 1, s, \mathsf{J}, \emptyset \rangle$ is a proof tuple.*
2. *$\mathcal{D}$ is an extension of a dynamic derivation by a sequence $\langle T_1, \ldots, T_n \rangle$ of introducing tuples of the form $\langle i, s, \mathsf{J}, \emptyset \rangle$, whose derived sequents are not in $\mathsf{Elim}(\mathcal{D})$.*
3. *$\mathcal{D}$ is an extension of a dynamic derivation by a sequence $\langle T_1, \ldots, T_n \rangle$ of eliminating tuples of the form $\langle i, \bar{s}, \mathsf{J}, r \rangle$, such that:*
   a) $\mathcal{D}$ is coherent: $\mathsf{Elim} \cap \mathsf{Attack} = \emptyset$
   b) the new attacking sequents (the r's) are not $\mathfrak{A}$-attacked by any sequent in $\mathsf{Accept}(\mathcal{D}) \cap Arg_{\mathfrak{L}}(S)$.

Condition (a) of Definition 3.8.7 assures that neither of the attacking sequents of the additional elimination tuples is in $\mathsf{Elim}(\mathcal{D})$ and condition (b) guarantees that an accepted S-based sequent does not attack them.

An argument can be *safely* concluded by the dynamic proof only when it is *finally derived*, i.e. there is no extension of the derivation in which it can be eliminated.

Definition 3.8.8 (Finally Derivation ([AS19]))**.** *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be an argumentation setting and S a set of formulas in $\mathcal{L}$. A sequent s is* finally derived *(or* safely derived*) in a dynamic derivation $\mathcal{D}$ (for $\mathfrak{S}$, based on S), if $s \in \mathsf{Accept}(\mathcal{D})$ and $\mathcal{D}$ cannot be extended to a dynamic derivation $\mathcal{D}'$ (for $\mathfrak{S}$, based on S) such that $s \in \mathsf{Elim}(\mathcal{D}')$.*

Let see now an example of a dynamic derivation

Example 3.8.1. *Let consider an argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ where $\mathfrak{S} = \langle \mathsf{CL}, LK, D\text{-}Reb \rangle$ and $\mathcal{S} = \{p, \neg p, p \wedge q\}$*

1.	$p \Rightarrow p$	Axiom
2.	$\neg p \Rightarrow \neg p$	Axiom
3.	$p \wedge q \Rightarrow p \wedge q$	Axiom
4.	$p \Rightarrow \neg\neg p$	
5.	$\neg p \not\Rightarrow \neg p$	D-Reb, 1, 4, 2
6.	$p \not\Rightarrow p$	D-Reb, 2, 2, 1
7.	$\neg p \Rightarrow \neg p, \neg q$	
8.	$\neg p \Rightarrow \neg p \vee \neg q$	$[\Rightarrow, \vee]$, 7
9.	$\neg p \Rightarrow \neg(p \wedge q)$	
10.	$p \wedge q \not\Rightarrow p \wedge q$	D-Reb, 2, 9, 3

Table 3.2: Example of a Dynamic Derivation

In Table 3.2, there is an example of a dynamic derivation. In Table 3.3, we define the three sets $\mathsf{Elim}(\mathcal{D})$*,* $\mathsf{Attack}(\mathcal{D})$ *and* $\mathsf{Accept}(\mathcal{D})$ *at each corresponding point of the derivation.*

Step of the derivation	$\mathsf{Elim}(\mathcal{D})$	$\mathsf{Attack}(\mathcal{D})$	$\mathsf{Accept}(\mathcal{D})$
4.	$\emptyset$	$\emptyset$	$p \Rightarrow p,\ \neg p \Rightarrow \neg p,\ p \wedge q \Rightarrow p \wedge q$
5.	$\neg p \Rightarrow \neg p$	$p \Rightarrow p$	$p \Rightarrow p,\ p \wedge q \Rightarrow p \wedge q$
6.	$p \Rightarrow p$	$\neg p \Rightarrow \neg p$	$\neg p \Rightarrow \neg p,\ p \wedge q \Rightarrow p \wedge q$
10.	$p \Rightarrow p,\ p \wedge q \Rightarrow p \wedge q$	$\neg p \Rightarrow \neg p$	$\neg p \Rightarrow \neg p$

Table 3.3: Evaluation of the Derivation

CHAPTER 4

Attack Principles on Semi-Abstract AFs

4.1 Attack Principles Emerge

Abstract argumentation theory is intrinsically related to logic. As we have seen in the previous chapter, the use of logic represents a way to solve some core question of argumentation, such as finding the most acceptable arguments of a given frame; but there is more. Instantiating arguments with sequents is easier to recognise some hidden patterns between arguments that share some, or even all, atomic proposition. For example, it is reasonable to think that an argument with claim A, should be related to an argument with claim $A \wedge B$. One way to extract some of these relations from a given logic-based argumentation frame is to introduce *attack principles* which, intuitively, can be seen as rules that refine the existence of attack relations whenever the arguments involved share some atomic propositions. To rigorously introduce these attack principles, we first need to introduce semi-abstract argumentation frames. These argumentative frames are deductive frames where the conclusion is the only part instantiated. Thus, we will work on an intermediary level of abstraction between Dung style abstract argumentation and logical argumentation. Interestingly even instantiating only one part of the arguments (the claim) many logical aspects emerge.

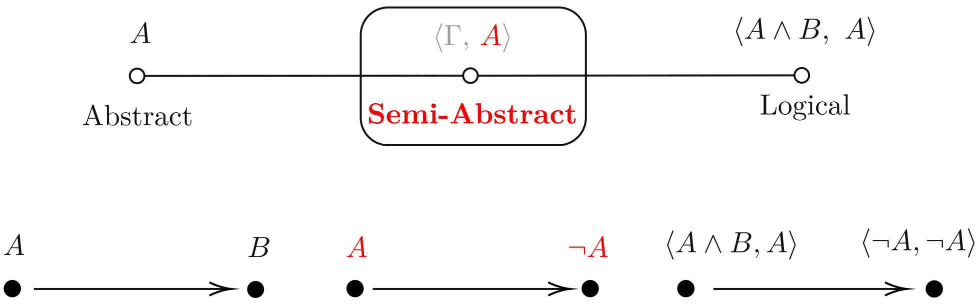


Figure 4.1: Levels of Abstraction of Argumentation Frames

Some attack principles can be related to the postulates introduced in [GH11] (see Section 3.7). Based on the logical structure of the arguments, postulates and attack principles aim at refining the definition of the attack relation. The postulate (D2′) is precisely our general attack principle **(B.gen)** introduced for the out-going attacks present in Section 4.5. However, as already pointed out, attack principles and postulates on different levels of abstraction and direct connections can be found only in specific cases.

Let PV be an infinite set of propositional variables and define the set of propositional formulas $\mathcal{PL}$ over PV by

$$\mathcal{F} ::= \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \supset \mathcal{F} \mid \neg\mathcal{F} \mid \mathcal{PV}$$

where $\mathcal{F}$ and $\mathcal{PV}$ are used as meta-variables for formulas and propositional variables, respectively.

Definition 4.1.1 (Semi-Abstract Argumentation Frame (SAF))**.** *A semi-abstract argumentation frame (SAF) is a directed graph of formulas. More precisely, in an SAF $\mathcal{S} = \langle Ar, R_{\rightarrow}\rangle$, Ar is a set of formulas of some underlying logic $\mathcal{L}$, representing claims of arguments and $R_{\rightarrow} \subseteq Ar \times Ar$ is a relation on Ar. We write $F \longrightarrow G$ if $(F, G) \in R_{\rightarrow}$. The latter represents the existence of an attack of an argument with claim F on an argument with claim G.*

An SAF is like an ordinary abstract argumentation frame as introduced by Dung [Dun95], except for attaching a unique formula to each node. We say that a formula F *attacks* a formula G and write $F \longrightarrow G$ if there is an edge from a node labeled by F to one labeled by G.[1] We abbreviate "not $F \longrightarrow G$" by $F \not\longrightarrow G$.

Note that formulas occur at most once in a given SAF. Of course, a fully instantiated argumentation frame may contain several arguments that feature the same formula as claim. We may consider the transition from an instantiated frame to a corresponding SAF as a process of abstraction. This process includes the mapping of all arguments that share the same claim into a single node.

[1] The same formula may occur as claim of different arguments. Thus we (implictly) refer to *occurrences* of formulas, rather than to formulas themselves when talking about attacks in a given SAF.

Given a SAF $\mathcal{S} = \langle Ar, R_{\rightarrow} \rangle$, we say that a formula A of $\mathcal{PL}$ belongs to Ar ($A \in Ar$) if there is a vertex of $\mathcal{S}$ labeled by A.

Example 4.1.1. *Consider the following statements:*
– *"The overall prosperity in society increases." (P)*
– *"Warlike conflicts about energy resources arise." (W)*
– *"The level of CO_2 emissions is getting dangerously high." (C)*
– *"Awareness about the need of environmental protection increases." (E)*
Consider an argumentation frame containing arguments, where the claims consist in some of these statements or in some simple logical compounds thereof. Using the indicated abbreviations and identifying vertices with theirs labels, a concrete corresponding SAF $\mathcal{S}_E = \langle Ar, R_{\rightarrow} \rangle$ is given by $Ar = \{P, E, W, P \supset C, E \vee C, P \wedge C\}$ and $R_{\rightarrow} = \{E \longrightarrow P \supset C,\ W \longrightarrow E \vee C,\ W \longrightarrow P \wedge C,\ E \vee C \longrightarrow P\}$.

Note that the various statements that are put forward as claims in the above example may well be thought as supported by additional statements that remain implicit here. Even without any access to such additional statements, one can identify certain logical connections between these claims that bear on the existence of further implicit arguments and attacks.

Example 4.1.2. *Suppose to have an argument with claim X that attacks an argument whose claim is "the majority of the population of some country strongly supports its government." Without analysing X and even without knowing X, one can reasonably assume that X implicitly also attacks another argument whose claim is the conjunction of "the majority of the population strongly supports its government"* and *"believes that the economic situation is improving." Note that this observation does not assert any particular connection between "support for the government" and "economic performance." It rather expresses the simple rationality principle that one cannot attack an argument with claim A without implicitly attacking the arguments whose claims are conjunctions like $A \wedge B$.*

It seems reasonable to expect that an argument that attacks another argument with claim G also attacks some arguments whose claims logically entail G. In our notation, this amounts to the following *general attack principle*:

(A.gen) If $F \longrightarrow G$ and $G' \models G$ then $F \longrightarrow G'$.

Applied naively, principle **(A.gen)** is problematic for at least two reasons. (1) We have not specified which logic the consequence relation $\models$ refers to. Classical logic may be a canonical choice, but we should not dismiss weaker logics that are potentially more adequate in the context of defeasible reasoning, too quickly. (2) Even for classical propositional logic deciding logical consequence is computationally intractable, in general. Arguably, a realistic argumentation model might insist on constraining **(A.gen)** to arguments G that *immediately* follow from G' in some appropriate sense. Thus, we will focus on principles that follow already from *transparent and straightforward instances* of **(A.gen)**:

(A.$\wedge$) If $F \longrightarrow A$ or $F \longrightarrow B$ then $F \longrightarrow A \wedge B$.

(A.$\vee$) If $F \longrightarrow A \vee B$ then $F \longrightarrow A$ and $F \longrightarrow B$.

(A.$\supset$)* If $F \longrightarrow A \supset B$ then $F \longrightarrow B$.

These specific instances of **(A.gen)** involve only very basic consequence claims, that are already valid in *minimal logic* [TS00], i.e. in the positive fragment of intuitionistic logic (and in fact in even weaker logics).

Proposition 4.1.1. *If "$\models$" refers to minimal logic, then the general attack principle **(A.gen)** entails the specific attack principles* **(A.$\wedge$)**, **(A.$\vee$)**, **(A.$\supset$)***.

Proof. Immediate from the fact that $A \wedge B \models A$, $A \wedge B \models B$, $A \models A \vee B$, $B \models A \vee B$, $B \models A \supset B$ in minimal logic. $\square$

Principle **(A.$\supset$)*** might be considered intuitively less obvious than **(A.$\wedge$)** and **(A.$\vee$)**. Indeed, the above justification of **(A.$\supset$)*** depends on the fact that $B \models A \supset B$ according to minimal logic and thus involves a logical principle that may be disputed, e.g., from the point of view of "relevant entailment" (see [DR02]). Therefore we prefer to replace **(A.$\supset$)*** by the following attack principle:

(A.$\supset$) If $F \longrightarrow B$ and $F \not\longrightarrow A$ then $F \longrightarrow A \supset B$.

As we will see in Sections 4.3 and 4.4, **(A.$\supset$)** relates to a basic inference principle about implicative premises in logical consequence claims, that holds in a very wide range of logics.

We have not yet specified any principle involving negation. Negation is often defined by $\neg F =_{df} F \supset \bot$, where $\bot$ is an atomic formula that signifies an elementary contradiction. But minimal logic treats $\bot$ just like an arbitrary propositional variable and thus does not give rise to any specific attack principle for negation. However the following principle seems intuitively plausible: If an argument attacks (an argument with claim) A then it does not simultaneously also attack the negation of A. In symbols:

(A.$\neg$) If $F \longrightarrow A$ then $F \not\longrightarrow \neg A$.

We will show in Section 4.4 that the (weak) attack principles mentioned so far give raise to a logic that arises from dropping some logical rules from Gentzen's classical sequent calculus LK. But we are also interested in the question, which (stronger) attack principles have to be imposed on semi-abstract argumentation frames in order to recover ordinary classical logic. To this aim we introduce the following additional attack principles that are inverse to **(A.$\wedge$)**, **(A.$\vee$)**, **(A.$\supset$)**, and **(A.$\neg$)**, respectively.

(C.$\wedge$) If $F \longrightarrow A \wedge B$ then $F \longrightarrow A$ or $F \longrightarrow B$.

(C.$\vee$) If $F \longrightarrow A$ and $F \longrightarrow B$ then $F \longrightarrow A \vee B$.

(C.$\supset$) If $F \longrightarrow A \supset B$ then $F \longrightarrow B$ and $F \not\longrightarrow A$.

(C.$\neg$) If $F \not\longrightarrow A$ then $F \longrightarrow \neg A$.

Conditions like **(A.$\wedge$)** seem to entail that the corresponding SAFs are infinite. However, we may *relativize* the attack principles to sets of formulas Γ (usually the set of claims of arguments of some finite argumentation frame). E.g.,

(A.$\wedge$) For every $A, B, F \in \Gamma$: $F \longrightarrow A$ or $F \longrightarrow B$ implies $F \longrightarrow A \wedge B$, if $A \wedge B \in \Gamma$.

In the following we will tacitly assume that attack principles are relativized to some (finite) set of formulas that will always be clear from the context.

4.2 Argumentative Consequence Relation

Viewing an argument attacking a certain claim F as a kind of counter-model to F suggests the following definition of consequence, that, in contrast to usual definitions of logical consequence, neither refers to truth values nor to interpretations in the usual (Tarskian) sense.

Definition 4.2.1 (Argumentative Consequence Relation). *F is an* argumentative consequence *of (the claims of) arguments $A_1, \ldots, A_n$ with respect to a SAF $\mathcal{S}$ ($A_1, \ldots, A_n \models^{\mathcal{S}}_{arg} F$) if all arguments in $\mathcal{S}$ that attack F also attack A_i for some $i \in \{1, \ldots, n\}$.*[2] *For a set of SAFs $\mathscr{S}$ $A_1, \ldots, A_n \models^{\mathscr{S}}_{arg} F$ if $A_1, \ldots, A_n \models^{\mathcal{S}}_{arg} F$ for all $\mathcal{S} \in \mathscr{S}$.*

To render this notion of consequence plausible from a logical perspective, the underlying SAFs should be rich enough to contain (potential) arguments that feature also the subformulas of occurring formulas as claims. Moreover, we want these SAFs to satisfy at least some of our logical attack principles.

Definition 4.2.2 (Syntactic Closure). *A SAF $\mathcal{S}$ is* syntactic closed *with respect to the set of formulas Δ if each subformula of a formula from Δ occurs as a claim of some argument in $\mathcal{S}$.*

Definition 4.2.3 (Argumentative $\mathcal{P}$-Consequence Relation). *Let $\mathcal{P}$ be a set of attack principles, then F is an* argumentative $\mathcal{P}$-consequence *of $A_1, \ldots, A_n$ ($A_1, \ldots, A_n \models^{\mathcal{P}}_{arg} F$) if $A_1, \ldots, A_n \models^{\mathcal{S}}_{arg} F$ for every SAF $\mathcal{S}$ that is syntactically closed with respect to $\{A_1, \ldots, A_n, F\}$ and moreover satisfies all (appropriately relativized) attack principles in $\mathcal{P}$.*

[2] If we identify arguments with counter-models, if S satisfies all the attack principles introduced in Section 4.1 and contains all relevant counter-models, then argumentative consequence coincides with an ordinary logical consequence. Every counter-model of the conclusion must invalidate some premise.

We do *not* suggest that argumentation frames should always be syntactically closed. We rather view syntactic closure as an operation that augments a given SAF by "potential claims of arguments", which are *implicit* according to logical attack principles. Argumentative consequence thus refers to a "logical completion" of interpreted argumentation frames, rather than directly to arbitrarily given collections of arguments.

Example 4.2.1. *Continuing Example 4.1.1, we observe the SAF $\mathcal{S}_E$ is almost, but not yet fully, syntactically closed with respect to the statements that appear as claims of arguments: we have to add one more vertex with label C (for "Awareness about the need of environmental protection increases") to obtain syntactic closure.*

More interestingly, we may check which additional (implicit) attacks are induced by which of our attack principles: Since there is an argument with claim $E \vee C$ that attacks an argument with claim P, principle **(A.∧)** *stipulates that there is also an attack from $E \vee C$ to an argument with the stronger claim $P \wedge C$. In other words* **(A.∧)** *induces the addition of the edge $E \vee C \longrightarrow P \wedge C$. In brief:*

Since $E \vee C \longrightarrow P$, by **(A.∧)** *we have $E \vee C \longrightarrow P \wedge C$.*

Note that this corresponds to the plausible assumption that an argument that attacks the claim that "the overall prosperity in society increases" also attacks the statement "the overall prosperity in society increases and (moreover) the level of CO_2 emissions is getting dangerously high."

Similarly, the principle **(A.∨)** *stipulates that an argument with claim C (added for syntactic closure, as explained above) should be attacked by an argument with claim W, since such an argument already attacks the weaker claim $E \vee C$, i.e.:*

Since $W \longrightarrow E \vee C$, by **(A.∨)** *we have $W \longrightarrow E$ and $W \longrightarrow C$.*

Moreover,

Since $E \longrightarrow P \supset C$ and $E \not\longrightarrow P$, by **(A.⊃)** *we have $E \longrightarrow C$.*

The stronger principle **(C.⊃)** *would call for $E \longrightarrow C$ even without $E \not\longrightarrow P$. In addition,*

Since $W \longrightarrow P \wedge C$, by **(C.∧)** *we have either $W \longrightarrow P$ or $W \longrightarrow C$.*

Both seem reasonable with respect to the intended interpretation of $\mathcal{S}_E$. However, $W \longrightarrow C$ is already present anyway if **(A.∨)** *is imposed, as explained above. Likewise, the strong disjunction principle* **(C.∨)** *is already satisfied.*

In the next section, we will investigate which collection $\mathcal{P}$ of attack principles allows recovering classical logical consequence as argumentative $\mathcal{P}$-consequence. Gentzen's classical sequent calculus LK turns out to be a perfect tool for this task. Hence, we generalize the consequence relation to (disjunctive) sets of premises, as usual in proof theory. Moreover, we adopt the convention to identify finite lists and sets of formulas, and write, e.g., Γ, F for $\Gamma \cup \{F\}$.

Definition 4.2.4 (Argumentative Consequence)**.** *Let Γ and Δ be finite sets of formulas and let $\mathcal{S}$ be a SAF. Δ is an* argumentative consequence *of Γ with respect to a SAF $\mathcal{S}$ ($\Gamma \models^{\mathcal{S}}_{arg} \Delta$) if all arguments in $\mathcal{S}$ that attack every $F \in \Delta$ attack at least some $G \in \Gamma$.*

The generalization to sets of SAFs and sets of attack principles is just as indicated above.

With respect to an SAF $\mathcal{S} = \langle Ar, R_{\rightarrow} \rangle$ we define:

- $\text{atts}_{\mathcal{S}}(F) =_{df} \{\langle \Gamma; A\rangle \mid \langle \Gamma; A\rangle \in Ar \textit{ and } A \longrightarrow F\}$, i.e. $\text{atts}_{\mathcal{S}}(F)$ consists of all the arguments of the SAF $\mathcal{S}$ that attack an argument with claim F.
- $\overline{\text{atts}}_{\mathcal{S}}(\Gamma) =_{df} \bigcup_{F \in \Gamma} \text{atts}_{\mathcal{S}}(F)$, i.e. $\overline{\text{atts}}_{\mathcal{S}}(\Gamma)$ consists of all the arguments of the SAF $\mathcal{S}$ that attack at least one argument whose claim is in Γ.
- $\underline{\text{atts}}_{\mathcal{S}}(\Gamma) =_{df} \bigcap_{F \in \Gamma} \text{atts}_{\mathcal{S}}(F)$, i.e. $\underline{\text{atts}}_{\mathcal{S}}(\Gamma)$ consists of all the arguments of the SAF $\mathcal{S}$ that attack every element of Γ.

The following simple facts will be useful below:

(a) $\Gamma \models^{\mathcal{S}}_{\text{arg}} \Delta$ iff $\underline{\text{atts}}_{\mathcal{S}}(\Delta) \subseteq \overline{\text{atts}}_{\mathcal{S}}(\Gamma)$, i.e. every argument (whose claim is) in $\mathcal{S}$ that attacks every argument (whose claim is) in Δ, attacks also at least one argument (whose claim is) in Γ, that is exactly the definition of $\models^{\mathcal{S}}_{\text{arg}}$.

(b) $\overline{\text{atts}}_{\mathcal{S}}(\Gamma, F) = \overline{\text{atts}}_{\mathcal{S}}(\Gamma) \cup \text{atts}_{\mathcal{S}}(F)$.

(c) $\underline{\text{atts}}_{\mathcal{S}}(\Gamma, F) = \underline{\text{atts}}_{\mathcal{S}}(\Gamma) \cap \text{atts}_{\mathcal{S}}(F)$.

We will drop the index $\mathcal{S}$ if no ambiguity arises.

4.3 Classical Logic Through an Argumentative Entailment Relation

In this version of Gentzen's classical sequent calculus LK [Gen35], sequents are pairs of sets of formulas, written as $\Gamma \Rightarrow \Delta$. Initial sequents (*axioms*) are of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$. The logical rules (for introducing logical connectives) are those listed in Figure 4.2.

Note that we do not need to use structural rules: weakening and exchange are redundant because of the more general form of axioms compared to Gentzen's $\varphi \Rightarrow \varphi$; contraction is

Axiom:

$$\frac{}{\varphi, \Gamma \Rightarrow \Delta, \varphi}\ (Ax)$$

Logical Rules:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\neg, r) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\neg, l)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (\wedge, r) \qquad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge, l)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ (\vee, r) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee, l)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}\ (\supset, r) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}\ (\supset, l)$$

Figure 4.2: A Proof System for **LK**

eliminated because we treat sequents as pairs of *sets* of formulas. Moreover the calculus is cut-free complete with respect to the classical consequence relation $\models_{cl}$ (generalized to disjunctions of premises, as usual)and it is also invertible. We rely on the following well known facts (see, e.g., [TS00]).

Proposition 4.3.1. *$\Gamma \Rightarrow \Delta$ is derivable in* LK *iff* $\Gamma \models_{cl} \Delta$.

Proposition 4.3.2. (e.g., [TS00], Proposition 3.5.4) *The rules of* LK *are invertible; i.e. if a sequent $\Gamma \Rightarrow \Delta$ is derivable and $\Gamma \Rightarrow \Delta$ is an instance of a lower sequent of an* LK*-rule then the corresponding instance(s) of the upper sequent(s) is (are) derivable, too.*

Let CAP consist of the attack principles **(A.$\wedge$)**, **(A.$\vee$)**, **(A.$\supset$)**, **(A.$\neg$)**, **(C.$\wedge$)**, **(C.$\vee$)**, **(C.$\supset$)**, and **(C.$\neg$)**. We first show that LK is sound with respect to argumentative CAP-consequence.

Theorem 4.3.1 (Soundness)**.** *If $\Gamma \Rightarrow \Delta$ is derivable in* LK *then* $\Gamma \models^{\mathsf{CAP}}_{arg} \Delta$.

Proof. Clearly, $A, \Gamma \models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A$. It remains to check that the inference rules of LK preserve CAP-consequence. We only present two cases, the others can be found in Appendix A.2

$(\neg, r)$ We have to show that $A, \Gamma \models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$ implies $\Gamma \models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, \neg A$. The premise states that $\overline{\mathrm{atts}}_{\mathcal{S}}(\Gamma, A) \supseteq \underline{\mathrm{atts}}_{\mathcal{S}}(\Delta)$ for every SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$ and satisfies the CAP-principles. The conclusion states that

$\overline{\text{atts}}_{\mathcal{S}'}(\Gamma) \supseteq \underline{\text{atts}}_{\mathcal{S}'}(\Delta, \neg A)$, where $\mathcal{S}'$ now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$. Since every SAF $\mathcal{S}'$ of the second kind reduces to one of the first kind (without $\neg A$) we may argue over any such SAF and drop the reference. We obtain:

$$\begin{aligned}
& \overline{\text{atts}}(\Gamma, A) \supseteq \underline{\text{atts}}(\Delta) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \supseteq \underline{\text{atts}}(\Delta) \\
\Rightarrow\ & (\overline{\text{atts}}(\Gamma) \cup \text{atts}(A)) \cap \text{atts}(\neg A) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & (\overline{\text{atts}}(\Gamma) \cap \text{atts}(\neg A)) \cup (\text{atts}(A) \cap \text{atts}(\neg A)) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cap \text{atts}(\neg A) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \quad [\text{using } \mathbf{(A.\neg)}] \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, \neg A).
\end{aligned}$$

Crucially, $\mathbf{(A.\neg)}$ amounts to $\text{atts}(A) \cap \text{atts}(\neg A) = \emptyset$.

$(\supset, l)$ We show that, if $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ and $B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$, then $A \supset B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$. Like above, the premises amount to $\overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A)$ and $\overline{\text{atts}}(B, \Gamma) \supseteq \underline{\text{atts}}(\Delta)$, respectively; whereas the conclusion is $\overline{\text{atts}}(A \supset B, \Gamma) \supseteq \underline{\text{atts}}(\Delta)$. We use $(\cdot)^c$ to denote the complement with respect to the set of arguments in question.

$$\begin{aligned}
& \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \subseteq \overline{\text{atts}}(\Gamma) \text{ and } \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup \text{atts}(B) \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \cap (\overline{\text{atts}}(\Gamma))^c = \emptyset \quad \text{and} \quad \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma) \cup \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap \text{atts}(A) = \emptyset \quad \text{and} \quad \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(A) \cup (\text{atts}(B))^c) = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap ((\text{atts}(A))^c \cap \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \cap \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup (\text{atts}(B) \setminus \text{atts}(A)) \quad [\text{using } \mathbf{(A.\supset)}] \\
\Rightarrow\ & \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup \text{atts}(A \supset B)
\end{aligned}$$

Note that $\mathbf{(A.\supset)}$ amounts to $\text{atts}(B) \setminus \text{atts}(A) \subseteq \text{atts}(A \supset B)$. □

Here below we summarise which attack principle has been used in every sub-case of the proof.

Rule	$(\wedge, r)$	$(\wedge, l)$	$(\vee, r)$	$(\vee, l)$	$(\supset, r)$	$(\supset, l)$	$(\neg, r)$	$(\neg, l)$
Attack Principle	$\mathbf{(C.\wedge)}$	$\mathbf{(A.\wedge)}$	$\mathbf{(A.\vee)}$	$\mathbf{(C.\vee)}$	$\mathbf{(C.\supset)}$	$\mathbf{(A.\supset)}$	$\mathbf{(A.\neg)}$	$\mathbf{(C.\neg)}$

Table 4.1: Attack principles used in each point of the proof of Theorem 4.3.1

To show the completeness of LK with respect to argumentative CAP-consequence, we rely on the invertibility of the logical rules (Proposition 4.3.2).

Theorem 4.3.2 (Completeness)**.** *If* $\Gamma \models^{\mathsf{CAP}}_{arg} \Delta$ *then* $\Gamma \Rightarrow \Delta$ *is derivable in* LK.

Proof. We have to check the inverse directions of the implications in the proof of Theorem 4.3.1. Again, we just present two cases, since the others are similar and can be found in Appendix A.2.

$(\neg, r)$ We show that $A, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, \neg A$. To this aim assume that $\overline{\text{atts}}_{\mathcal{S}}(A, \Gamma) \not\supseteq \underline{\text{atts}}_{\mathcal{S}}(\Delta)$ for some CAP-complying SAF $\mathcal{S} = (Ar^{\mathcal{S}}, R^{\mathcal{S}}_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$. In other words there is an $F \in \underline{\text{atts}}_{\mathcal{S}}(\Delta)$, such that $F \notin \overline{\text{atts}}_{\mathcal{S}}(A, \Gamma)$. The latter implies $F \not\longrightarrow A$ in $\mathcal{S}$. Now let $\mathcal{S}'$ be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$, where the attack relation restricted to those (claims of) arguments that already occur in $\mathcal{S}$ coincides with $R^{\mathcal{S}}_{\rightarrow}$. Since $F \not\longrightarrow A$ also in $\mathcal{S}'$, we obtain $F \longrightarrow \neg A$ from **(C.¬)**. Since $F \in \underline{\text{atts}}_{\mathcal{S}'}(\Delta)$ we conclude that $F \in \underline{\text{atts}}_{\mathcal{S}'}(\Delta, \neg A)$. On the other hand, $F \notin \overline{\text{atts}}_{\mathcal{S}'}(\Gamma)$, since otherwise we already had $F \in \overline{\text{atts}}_{\mathcal{S}}(A, \Gamma)$. Thus we have shown that $\overline{\text{atts}}_{\mathcal{S}'}(\Gamma) \not\supseteq \underline{\text{atts}}_{\mathcal{S}'}(\Delta, \neg A)$, which entails $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, \neg A$.

$(\supset, l)$ We again proceed indirectly and show that (1) $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ implies $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$, and (2) $B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$.

For (1) assume that $\overline{\text{atts}}_{\mathcal{S}}(\Gamma) \not\supseteq \underline{\text{atts}}_{\mathcal{S}}(\Delta, A)$ for some CAP-complying SAF $\mathcal{S} = (Ar^{\mathcal{S}}, R^{\mathcal{S}}_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$. Thus there is an $F \in \underline{\text{atts}}_{\mathcal{S}}(\Delta, A)$, such that $F \notin \overline{\text{atts}}_{\mathcal{S}}(\Gamma)$. In particular $F \in \underline{\text{atts}}_{\mathcal{S}}(\Delta)$. Let $\mathcal{S}'$ be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \supset B\}$, where the attack relation restricted to those (claims of) arguments that already occur in $\mathcal{S}$ coincides with $R^{\mathcal{S}}_{\rightarrow}$. Then **(C.⊃)** implies $F \not\longrightarrow A \supset B$ in $\mathcal{S}'$. Therefore $\overline{\text{atts}}_{\mathcal{S}'}(A \supset B, \Gamma) \not\supseteq \underline{\text{atts}}_{\mathcal{S}'}(\Delta)$, which in turn entails $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$.

For (2) assume that $\overline{\text{atts}}_{\mathcal{S}}(\Gamma, B) \not\supseteq \underline{\text{atts}}_{\mathcal{S}}(\Delta)$ for some CAP-complying SAF $\mathcal{S} = (Ar^{\mathcal{S}}, R^{\mathcal{S}}_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{B\}$. Thus there is an $F \in \underline{\text{atts}}_{\mathcal{S}}(\Delta)$, such that $F \notin \overline{\text{atts}}_{\mathcal{S}}(\Gamma, B)$. Let $\mathcal{S}'$ be an SAF like in case (1). Then $F \not\longrightarrow B$ also in $\mathcal{S}'$. Therefore **(C.⊃)** implies $F \not\longrightarrow A \supset B$ in $\mathcal{S}'$, which entails $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$, like in case (1). □

Here below we summarise which attack principle has been used in every sub-case of the proof.

Rule	$(\wedge, r)$	$(\wedge, l)$	$(\vee, r)$	$(\vee, l)$	$(\supset, r)$	$(\supset, l)$	$(\neg, r)$	$(\neg, l)$
Attack Principle	**(A.∧)**	**(C.∧)**	**(C.∨)**	**(A.∨)**	**(A.⊃)**	**(C.⊃)**	**(C.¬)**	**(A.¬)**

Table 4.2: Attack Principles Used in the Proof of Theorem 4.3.2

4.4 Modal Interpretation of the Attack Relation

We have seen that the collection CAP of attack principles leads to a characterisation of classical logical consequence, that replaces model theoretic (Tarskian) semantics by a reference to specific structural properties of syntactically closed semi-abstract argumentation frames. Remember that classical logic is the top element in the lattice of all possible consistent logics over the language $\mathcal{PL}$. Therefore it is hardly surprising that some of the principles in CAP might be considered too demanding to be adequate for models of logical argumentation. Consider for example **(C.¬)**: it says that any argument that does not attack a claim A can be understood as an argument that attacks $\neg A$. In other words every argument has to attack either A or $\neg A$. This is hardly plausible and should be contrasted with the inverse principle **(A.¬)**, which just stipulates that no argument attacks A and $\neg A$, simultaneously. We have justified **(A.∨)** and **(A.∧)** in Section 4.1 as immediate instances of the general principle that, if an argument attacks a claim A, then it (implicitly) also attacks claims *from which A logically follows*. But the plausibility of the inverse principles **(C.∨)** and **(C.∧)** remains in question. Intuitively, it seems justifiable to stipulate that an argument that attacks both, A and B, attacks also $A \vee B$ **(C.∨)**. However the requirement that any argument attacking a conjunction must attack also at least one of the conjuncts intuitively seems too strong: think of the instance $A \wedge \neg A$, against which an agent presumably may have a reasonable (general) argument, without knowing an argument that attacks either A or $\neg A$.

Rather than to simply appeal to pre-theoretic intuitions, as just outlined, we want to present a simple formal interpretation of attacks involving logically compound claims, that supports some, but not all of the attack principles in CAP. The interpretation is intended to provide a formal mechanism that allows one to separate attack principles that are intuitively too demanding to be plausible from weaker, more plausible attack principles. While deliberately simple, the suggested interpretation is not completely ad hoc, but rather based on the following idea. Any claim made by an argument will be compatible with certain possible states of affairs and incompatible with others. This gives raise to a relation between state of affairs, i.e. a relation between possible worlds in the sense of Kripke semantics for modal logics. Formally, we employ standard modal logic and refer to *Kripke interpretations* $\langle W, R, V\rangle$, where W is a non-empty set of *states*, $R \subseteq W \times W$ the *accessibility relation*, and V a *valuation* $V : W \times PV \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ that assigns a truth value to each propositional variable in each state. The language $\mathcal{PL}$ is enriched by a modal operator $\Box$ and its dual $\Diamond = \neg\Box\neg$. The valuation V is extended from propositional variables to classical formulas (elements of $\mathcal{PL}$) as usual. For $\Box$ we have

$$V(w, \Box F) = \mathbf{t} \text{ iff } \forall v\text{: } wRv \text{ implies } V(v, F) = \mathbf{t}.$$

A Kripke interpretation $\langle W, R, V\rangle$ is a *model* of formula F if $V(w, F) = \mathbf{t}$ for all $w \in W$. F is a (global) $\mathcal{K}$-consequence of $G_1, \ldots, G_n$ with respect to a class $\mathcal{K}$ of Kripke interpretations if every model of $G_1, \ldots, G_n$ is also a model of F.

We view the states W as possible states of affairs and interpret wRv as "v is a possible alternative from the viewpoint of w". We translate $F \longrightarrow G$ into a modal formula $\iota(F \longrightarrow G)$ which expresses that in all possible alternative states F holds, but G is false. The intuition behind this translation can be paraphrased as follows: We look at all possible states of affairs that arise when we accept the attacking argument. In each

Attack Principle	Premise	Conclusion
(A.$\wedge$)	$X \longrightarrow A$ or $X \longrightarrow B$	$X \longrightarrow A \wedge B$
(C.$\wedge$)	$X \longrightarrow A \wedge B$	$X \longrightarrow A$ or $X \longrightarrow B$
(A.$\vee$)	$X \longrightarrow A \vee B$	$X \longrightarrow A$ and $X \longrightarrow B$
(C.$\vee$)	$X \longrightarrow A$ and $X \longrightarrow B$	$X \longrightarrow A \vee B$
(A.$\supset$)	$X \longrightarrow B$ and $X \not\longrightarrow A$	$X \longrightarrow A \supset B$
(C.$\supset$)	$X \longrightarrow A \supset B$	$X \longrightarrow B$ and $X \not\longrightarrow A$
(A.$\neg$)	$X \longrightarrow A$	$X \not\longrightarrow A$
(C.$\neg$)	$X \not\longrightarrow A$	$X \longrightarrow \neg A$

Table 4.3: Premises and Conclusions of the Attack Principles in CAP

of these states F holds (because this is what the attacking argument claims), but G is false (because the argument claiming that G is true is attacked). More formally, this interpretation is defined as follows.

Definition 4.4.1 (Modal Interpretation of the Attack Relation)**.** *For all $F, G \in \mathcal{PL}$:*
– $\iota(F \longrightarrow G) =_{df} \Box(F \wedge \neg G)$;
– $\iota(F \not\longrightarrow G) =_{df} \Diamond(\neg F \vee G)$ *(or, equivalently,* $\neg\Box(F \wedge \neg G)$*).*

Recall that attack principles are implications between (disjunctions or conjunctions) of assertions of the form $F \longrightarrow G$ or $F \not\longrightarrow G$. In particular, for each attack principle we distinguish two parts: the *premise* and the *conclusion*. In Table 4.3 we summarise the *premise* and the *conclusion* of the attack principles in CAP (see Section 4.3).

We call an attack principle $\mathcal{K}$*-justified* if the implication translates into a valid $\mathcal{K}$-consequence claim via ι.

Definition 4.4.2 ($\mathcal{K}$-justified Attack Principles)**.** *An attack principle AP is $\mathcal{K}$-justified if the interpretation of the conclusion of AP via ι is a $\mathcal{K}$-consequence of the interpretation of the premise of AP via ι.*

We have not yet imposed any restriction on Kripke interpretations. We aim to separate principles that are $\mathcal{K}$-justified, according to some underlying logic $\mathcal{K}$, from others that should be rejected as not $\mathcal{K}'$-justified, according to a logic $\mathcal{K}'$. To render this separation meaningful, the logic $\mathcal{K}$, employed to check the logical validity of the translated principles should be weak, while the logic $\mathcal{K}'$, that rejects translated principles as not logically valid, should be strong. In other words, we want to confirm that certain intuitively acceptable attack principles translate into a consequence claim that is valid in almost any normal modal logic, while, in contrast, intuitively less plausible attack principles translate into consequence claims that do not hold even in a very strong normal modal logic, and hence are also invalid in most other logics. To this aim we choose the logic $\mathcal{D}$ for justification, i.e., the only constraint imposed on the accessibility relation is seriality. Informally this means that for each argument under consideration, there should exist at least one possible state of affairs that is compatible with the claim of the argument. On the other hand, we chose the logic $\mathcal{S}5$ for the rejection of principles. Note that principles that are not $\mathcal{S}5$-justified, a fortiori, are not $\mathcal{K}$-justified for weaker logics like $\mathcal{D}$, $\mathcal{T}$, $\mathcal{S}4$, etc, as well.

Theorem 4.4.1. *The attack principles* **(A.∧)**, **(A.∨)**, **(C.∨)**, **(C.⊃)**, *and* **(A.¬)** *are all $\mathcal{D}$-justified.*

Proof. To show that **(A.∧)** is $\mathcal{D}$-justified we have to check that $\Box(F \wedge \neg A) \models_{\mathcal{D}} \Box(F \wedge \neg(A \wedge B))$ as well as $\Box(F \wedge \neg B) \models_{\mathcal{D}} \Box(F \wedge \neg(A \wedge B))$, which is obvious. The cases for **(A.∨)**, **(C.∨)** and **(C.⊃)** are similar.

(A.¬) is $\mathcal{D}$-justified because $\Box(F \wedge \neg A) \models_{\mathcal{D}} \Diamond(\neg F \vee \neg A)$, where the seriality of $\mathcal{D}$-models is used. □

Theorem 4.4.2. *The attack principles* **(C.∧)**, **(C.¬)**, *and* **(A.⊃)** *are not $\mathcal{S}5$-justified.*

Proof. In Figure 4.3 there is a counter-model for $\Box(F \wedge \neg(A \wedge B)) \models_{\mathcal{S}5} \Box(F \wedge \neg A)$, and $\Box(F \wedge \neg(A \wedge B)) \models_{\mathcal{S}5} \Box(F \wedge \neg B)$.

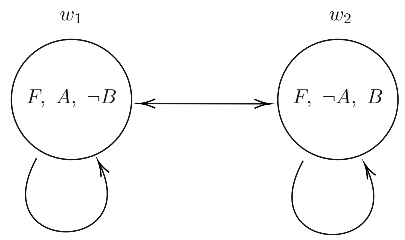


Figure 4.3: Counter Model for the Modal Interpretation of **(C.∧)**

In Figure 4.4 there is a counter-model for $\Diamond(\neg F \vee \neg A) \models_{\mathcal{S}5} \Box(F \wedge \neg A)$.

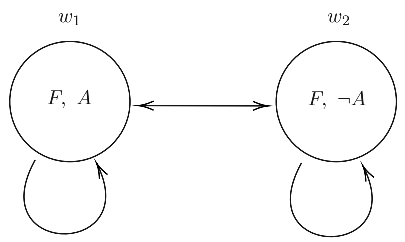


Figure 4.4: Counter Model for the Modal Interpretation of **(C.¬)**

In Figure 4.5 there is a counter-model for $\Diamond(\neg F \vee A),\ \Box(F \wedge \neg B) \models_{\mathcal{S}5} \Box(F \wedge (A \wedge \neg B))$.

Note that the above models feature an accessibility relation that is reflexive, symmetric and transitive, and hence are adequate for the logic $\mathcal{S}5$ as required. □

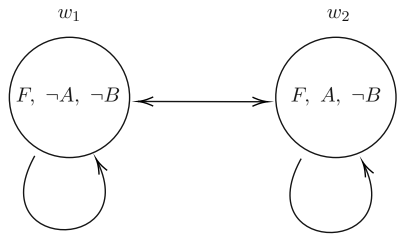


Figure 4.5: Counter Model for the Modal Interpretation of **(A.**$\supset$**)**

To sum up, we have seen that our (admittedly rather unsophisticated and coarse) modal interpretation of the attack relation supports a formal justification of the collection of attack principles $\mathsf{MAP} = \{$**(A.**$\wedge$**)**, **(A.**$\vee$**)**, **(C.**$\vee$**)**, **(C.**$\supset$**)**, **(A.**$\neg$**)** $\}$.

Moreover, the modal interpretation allows us to reject the attack principles **(C.**$\wedge$**)** and **(C.**$\neg$**)**, thus suggesting that a corresponding "logic of argumentation" should be weaker than classical logic.

Of course, the interpretation of the attack relation using modalities is not unique. We briefly discuss three alternatives.

Definition 4.4.3 (Alternative Modal Interpretations of the Attack Relation)**.** *For all $F, G \in \mathcal{PL}$:*

- $\iota_1(F \longrightarrow G) =_{df} \Diamond(F \wedge \neg G)$; – $\iota_1(F \not\longrightarrow G) =_{df} \Box(\neg F \vee G)$.
- $\iota_2(F \longrightarrow G) =_{df} \Diamond(\neg F \vee \neg G)$; – $\iota_2(F \not\longrightarrow G) =_{df} \Box(F \wedge G)$.
- $\iota_3(F \longrightarrow G) =_{df} \Box(\neg F \vee \neg G)$; – $\iota_3(F \not\longrightarrow G) =_{df} \Diamond(F \wedge G)$.

The interpretation ι_1 is similar to ι, but less demanding because (1) the claim of the attacking argument is asserted to hold in just one of the alternatives and (2) the negation of the claim of the attacked argument is asserted to hold only there. Interpretation ι_2 suggests that $F \longrightarrow G$ means that there is at least one possible state in which $F \supset \neg G$ (equivalently: $\neg F \vee \neg G$) holds; whereas according to ι_3 all possible states must be of this form.

These alternative interpretations of the attack relation (abstracted to the claims of the argument) are arguably more problematic than the interpretation ι suggested in Definition 4.4.1. This is also witnessed by the attack principles that are justified or rejected by the respective interpretations: The set of attack principles justified by ι_1 is $\mathsf{MAP}_1 = \{$**(A.**$\wedge$**)**, **(A.**$\vee$**)**, **(A.**$\supset$**)**, **(C.**$\wedge$**)** $\}$; ι_2 justifies $\mathsf{MAP}_2 = \{$ **(A.**$\wedge$**)**, **(A.**$\vee$**)**, **(A.**$\supset$**)**, **(C.**$\wedge$**)**, **(C.**$\neg$**)** $\}$; whereas ι_3 is extremely demanding and rejects all of our attack principles except **(A.**$\wedge$**)** and **(A.**$\vee$**)**.

We have seen that the attack principles in MAP are more plausible than the collection CAP that induces an entailment relation sound and complete with respect to the classical consequence relation. Thus the question arises, whether argumentative consequence relative to MAP can be characterized in a similar manner. We provide a positive answer by showing that $\models^{\mathsf{MAP}}_{\mathrm{arg}}$ matches the sequent calculus LM, that arises from dropping the rules $(\neg, l)$, $(\wedge, r)$, and $(\supset, l)$ from LK.

Theorem 4.4.3. *$\Gamma \Rightarrow \Delta$ is derivable in* LM *iff* $\Gamma \models^{\mathsf{MAP}}_{arg} \Delta$.

Proof. For the left-to-right direction (soundness of LM with respect to $\models^{\mathsf{MAP}}_{\mathrm{arg}}$) it suffices to observe that only attack principles in MAP are used to show the soundness of those LM-rules in the proof of Theorem 4.3.1.

For the other direction (completeness of LM with respect to $\models^{\mathsf{MAP}}_{\mathrm{arg}}$) we can no longer rely on the invertibility of rules, as we did for LK. We rather show that for every sequent $\Gamma \Rightarrow \Delta$ that is not derivable in LM there is an MAP *counter model*; i.e. an SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta$ and satisfies all principles in MAP, such that $\Gamma \not\models^{\mathcal{S}}_{\mathrm{arg}} \Delta$.

A sequent $\Gamma_0 \Rightarrow \Delta_0$ is called LM-*irreducible* if it is neither an axiom nor an instance of a lower sequent of an LM-rule. Note that every sequent that is not derivable in LM, results from applying LM-rules backwards until one hits an LM-irreducible sequent $\Gamma_0 \Rightarrow \Delta_0$. It follows from the soundness of LM with respect to $\models^{\mathsf{MAP}}_{\mathrm{arg}}$ that $\Gamma \Rightarrow \Delta$ has an MAP-counter-model if $\Gamma_0 \Rightarrow \Delta_0$ has one, too. It thus remains to show that every LM-irreducible sequent has an MAP-counter-model.

First observe that every LM-irreducible sequent $\Gamma_0 \Rightarrow \Delta_0$ is of the following form:
– $\Gamma_0 \cap \Delta_0 = \emptyset$;
– every $F \in \Gamma_0$ is atomic, a negation or an implication;
– every $F \in \Delta_0$ is atomic or a conjunction.
We construct an SAF $\mathcal{S} = (Ar, R_{\rightarrow})$ where the set of claims of arguments in Ar consists of all subformulas occurring in $\Gamma_0 \cup \Delta_0$. Additionally there is special argument in Ar with a claim x that is a new atomic formula (i.e. x does not occur as a subformula in $\Gamma_0 \cup \Delta_0$). The attack relation $R_{\rightarrow}$ is obtained by setting $x \rightarrow F$ for all $F \in \Delta_0$. Moreover, because of $(\mathbf{C.}\vee)$, we have to add an attack from x to a disjunction $G \vee H$, if x attacks G as well as H (and if $G \vee H$ occurs as a subformula of some formula in $\Gamma_0 \cup \Delta_0$). It is easy to check that S satisfies all attack principles in MAP and that $\Gamma_0 \not\models^{\mathcal{S}}_{\mathrm{arg}} \Delta_0$. (Remember in particular, that no disjunctions can occur in an irreducible sequent.) I.e., S is an MAP counter-model of $\Gamma_0 \Rightarrow \Delta_0$. □

We have identified a 'logic of argumentation' with a consequence relation arising from certain plausible principles about syntactically closed collections of (claims of) arguments and managed to characterize this logic in terms of a variant of Gentzen's classical sequent calculus LK, where some of the logical rules have been discarded. We finally ask whether LM appears already in a different context, pointing to a different type of semantics. A positive answer is provided by *nondeterministic matrix semantics* (see [AL05, AZ11]).

Definition 4.4.4 (Classical Nmatrix)**.** *A classical Nmatrix $\mathcal{N}$ consists in a function $\tilde{\neg} : \{\mathbf{t}, \mathbf{f}\} \rightarrow 2^{\{\mathbf{t},\mathbf{f}\}} \backslash \emptyset$ and a function $\tilde{\circ} : \{\mathbf{t}, \mathbf{f}\}^2 \rightarrow 2^{\{\mathbf{t},\mathbf{f}\}} \backslash \emptyset$ for each $\circ \in \{\wedge, \vee, \supset\}$.*

A corresponding dynamic valuation is a function $\tilde{v}_{\mathcal{N}} : \mathcal{PL} \to \{\mathbf{t}, \mathbf{f}\}$ sucht that $\tilde{v}_{\mathcal{N}}(\neg A) \in \tilde{\neg}(\tilde{v}_{\mathcal{N}}(A))$ and $\tilde{v}_{\mathcal{N}}(A \circ B) \in \tilde{\circ}(\tilde{v}_{\mathcal{N}}(A), \tilde{v}_{\mathcal{N}}(B))$ for $\circ \in \{\wedge, \vee, \supset\}$. $\tilde{v}_{\mathcal{N}}$ is a *model* of A if $\tilde{v}_{\mathcal{N}}(A) = \mathbf{t}$; it is a model of Γ if it is a model of every $A \in \Gamma$.

Δ is a *dynamical consequence of* Γ *with respect to* $\mathcal{N}$*, written* $\Gamma \models^{\mathcal{N}}_{dyn} \Delta$ *if every model of* Γ *is a model of some* $A \in \Delta$*.*

Consider the following classical Nmatrix $\mathcal{M}_{\mathsf{MAP}}$:

		$\tilde{\wedge}$	$\tilde{\vee}$	$\tilde{\supset}$
$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{t},\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$

	$\tilde{\neg}$
$\mathbf{t}$	$\{\mathbf{t},\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$

The following is just an instance of Theorem 62 of [AZ11].

Corollary 4.4.1. $\Gamma \Rightarrow \Delta$ *is derivable in* LM *iff* $\Gamma \models^{\mathcal{M}_{\mathsf{MAP}}}_{dyn} \Delta$*.*

Combing this observation with Theorem 4.4.3, we have thus connected argumentative consequence with respect to the attack principles in MAP with logical consequence defined with respect to a specific nondeterministic valuation.

Thanks to the modularity of Theorem 4.3.1, it has been possible to characterize a fragment of LK using non-deterministic matrices. In principle, for every fragment LK' of LK, we could recover, as we have done for LM, a corresponding set of attack principles AP'. Then, we could verify if, for every sequent $\Gamma \Rightarrow \Delta$ that is not derivable in LK', there is an AP' counter model and construct a non-deterministic semantics.

4.5 Outgoing Attack Principles

In addition to the general attack principle **(A.gen)** introduced in [CF17] we can consider also an *outgoing* version of it where the attacked argument is fixed and what varies are the attacking ones:

(B.gen) If $F \longrightarrow G$ and $F' \models F$ then $F' \longrightarrow G$.

As previously done for **(A.gen)**, in the case of **(B.gen)**, we can recover several outgoing attack principles by instantiating the general one. Also, in this case, the specific instances of **(B.gen)** involve only very basic consequence claims, that are already valid in minimal logic.

(B.$\wedge$) If $A \longrightarrow X$ or $B \longrightarrow X$, then $A \wedge B \longrightarrow X$.

(B.$\vee$) If $A \vee B \longrightarrow X$, then $A \longrightarrow X$ and $B \longrightarrow X$.

(B.$\supset$)* If $A \supset B \longrightarrow X$ then $B \longrightarrow X$.

The attack principles **(B.$\supset$)*** follows from the **(B.gen)** only if $B \models A \supset B$ holds in the propositional logic considered and we prefer to replace the attack principle by the following one.

(B.$\supset$) If $B \longrightarrow X$ and $A \not\longrightarrow X$, then $A \supset B \longrightarrow X$.

Regarding the definition of the attack principles for negation, we need to rely again on the same intuition that led us to the definition of **(A.$\neg$)**. If an argument is attacking another argument, then its negation does not attack that same argument.

(B.$\neg$) If $A \longrightarrow X$ then $\neg A \not\longrightarrow X$.

The *strong outgoing attack principles* are inverse of the once derived from **(B.gen)**.

(D.$\wedge$) If $A \wedge B \longrightarrow X$ then $A \longrightarrow X$ or $B \longrightarrow X$.

(D.$\vee$) If $A \longrightarrow X$ and $B \longrightarrow X$ then $A \vee B \longrightarrow X$.

(D.$\supset$) If $A \supset B \longrightarrow X$, then $B \longrightarrow X$ and $A \not\longrightarrow X$.

(D.$\neg$) If $A \not\longrightarrow X$, then $\neg A \longrightarrow X$.

We introduce now the definition of argumentative entailment relations based on the outgoing attack principles.

Definition 4.5.1 (Argumentative $\mathcal{OP}$-consequence Relation)**.** *Let $\mathcal{OP}$ be a set of outgoing attack principles, then Δ is an* argumentative $\mathcal{OP}$-consequence *of Γ ($\Gamma \models^{\mathcal{OP}}_{o\text{-}arg} \Delta$) if for every SAF $\mathcal{S}$ that is syntactically closed with respect to $\{\Gamma, \Delta\}$ and it satisfies all (appropriately relativized) attack principles in $\mathcal{OP}$, whenever every element of Δ attacks an argument, then there is at least one element of Γ that attacks that same argument.*

If $\mathsf{CoAP} = \{$**(B.$\wedge$)**, **(B.$\vee$)**, **(B.$\supset$)**, **(B.$\neg$)**, **(D.$\wedge$)**, **(D.$\vee$)**, **(D.$\supset$)**, **(D.$\neg$)**$\}$ we can recover another argumentative semantics for classical logic. This time using the outgoing attack principles and the outgoing version of the argumentative entailment relation. The idea behind the proof is the same as the in-going version. However, the result is interesting because it gives us a complete image of the possible kinds of attack principles that can be defined only considering the claims of the arguments. Moreover, thanks to the proof's modularity, the outgoing attack principles CoAP can be sorted out using a modal interpretation of the attack relation.

Theorem 4.5.1 (Soundness)**.** *If $\Gamma \Rightarrow \Delta$ is derivable in LK then $\Gamma \models^{\mathsf{CoAP}}_{o\text{-}arg} \Delta$.*

Proof. Clearly, $A, \Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta, A$. It remains to check that the inference rules of LK preserve CoAP-consequence. We only present two cases.

$(\neg, r)$ We have to show that $A, \Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$ implies $\Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta, \neg A$. Suppose there is an argument X s.t. for all $\delta \in \Delta$, $\delta \longrightarrow X$ and $\neg A \longrightarrow X$. By **(B.$\neg$)** we have $A \not\longrightarrow X$, therefore in order to have the assumption hold there must be some $\gamma \in \Gamma$ s.t. $\gamma \longrightarrow X$ and the claim holds.

$(\supset, l)$ We have to show that if $\Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta, A$ and $B, \Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$ implies $A \supset B, \Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$. Suppose there is an argument X s.t. for all $\delta \in \Delta$, $\delta \longrightarrow X$. Therefore from the second assumption we have either $B \longrightarrow X$ or $\gamma \longrightarrow X$ for some $\gamma \in \Gamma$. If $B \longrightarrow X$ and $A \not\longrightarrow X$, by **(B.$\supset$)**we have $A \supset B \longrightarrow X$ and the assertion holds. If $B \longrightarrow X$ and $A \longrightarrow X$ by the first assumption we have there is a $\gamma \in \Gamma$ such that $\gamma \longrightarrow X$ and the assertion holds. If $B \not\longrightarrow X$, from the second assumption we have that there must be some $\gamma \in \Gamma$ such that $\gamma \longrightarrow X$ and the assertion holds.

The other cases are analogous. We list the attack principles needed to prove that the corresponding inference rule of LK preserves CoAP-consequence: **(D.$\neg$)** for $(\neg, l)$, **(D.$\wedge$)** for $(\wedge, r)$, **(B.$\wedge$)**for $(\wedge, l)$, **(B.$\vee$)**for $(\vee, r)$, **(D.$\vee$)** for $(\vee, l)$, and **(D.$\supset$)** for $(\supset, r)$. $\square$

Theorem 4.5.2 (Completeness)**.** *If* $\Gamma \models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is derivable in* LK*.*

Proof. We have to check the inverse directions of the implications in the proof of Theorem 4.5.1. Again, we just present two cases, since the others are similar.

$(\neg, r)$ We show that $A, \Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$ implies $\Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta, \neg A$. To this aim assume there is a CoAP-complying SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$ and that contains an argument X such that $\delta \longrightarrow X$ for any $\delta \in \Delta$, $A \not\longrightarrow X$ and $\gamma \not\longrightarrow X$ for any $\gamma \in \Gamma$. Since $A \not\longrightarrow X$, by **(D.$\neg$)** we have $\neg A \longrightarrow X$ from which it follows the claim.

$(\supset, l)$ We again proceed indirectly and show that (1) $\Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta, A$ implies $A \supset B, \Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$, and (2) $B, \Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$ implies $A \supset B, \Gamma \not\models^{\mathsf{CoAP}}_{\text{o-arg}} \Delta$.

For (1) assume that for some CoAP-complying SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$ there is an argument X such that $\delta \longrightarrow X$ for any $\delta \in \Delta$, $A \longrightarrow X$ and $\gamma \not\longrightarrow X$ for any $\gamma \in \Gamma$. Since $A \longrightarrow X$, by **(D.$\supset$)** we have $A \supset B \not\longrightarrow X$. Therefore the claim holds.

For (2) assume that for some CoAP-complying SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{B\}$ there is an argument X such that $\delta \longrightarrow X$ for any $\delta \in \Delta$, $\gamma \not\longrightarrow X$ and $B \longrightarrow X$. Since $B \longrightarrow X$, by **(D.$\supset$)** we have $A \supset B \not\longrightarrow X$ and the claim holds.

The other cases are analogous and we list here the attack principles used for the other cases of the proof: **(B.¬)**for $(\neg, l)$, **(D.∧)** for $(\wedge, r)$, **(B.∧)**for $(\wedge, l)$, **(B.∨)**for $(\vee, r)$, **(D.∨)** for $(\vee, l)$, and **(B.⊃)**for $(\supset, r)$. $\square$

If we consider the following set of outgoing attack principles, we can obtain another characterisation of LM.

$$\mathsf{MoAP} = \{(\mathbf{B.}\wedge), (\mathbf{D.}\vee), (\mathbf{B.}\vee), (\mathbf{D.}\supset), (\mathbf{B.}\neg)\}$$

The proof of the adequateness theorem is similar to the proof of Theorem 4.4.3. Even though the result is not surprising, it is still useful to have a complete picture of the role and "value" of all kinds of attack principles, some of which are intrinsically related.

Theorem 4.5.3. $\Gamma \Rightarrow \Delta$ *is derivable in* LM *iff* $\Gamma \models^{\mathsf{MoAP}}_{o\text{-}arg} \Delta$.

Proof. For the left-to-right direction (soundness of LM with respect to $\models^{\mathsf{MoAP}}_{\text{o-arg}}$) it suffices to observe that only attack principles in MoAP are used to show the soundness of those LM-rules in the proof of Theorem 4.5.1.

For the other direction (completeness of LM with respect to $\models^{\mathsf{MoAP}}_{\text{o-arg}}$) we can no longer rely on the invertibility of rules, as we did for LK. We rather show that for every sequent $\Gamma \Rightarrow \Delta$ that is not derivable in LM there is an MoAP *counter model*; i.e. an SAF $\mathcal{S}$ that is syntactically closed with respect to $\Gamma \cup \Delta$ and satisfies all principles in MoAP, such that $\Gamma \not\models^{\mathcal{S}}_{\text{o-arg}} \Delta$.

A sequent $\Gamma_0 \Rightarrow \Delta_0$ is called LM-*irreducible* if it is neither an axiom nor an instance of a lower sequent of an LM-rule. Note that every sequent that is not derivable in LM, results from applying LM-rules backwards until one hits an LM-irreducible sequent $\Gamma_0 \Rightarrow \Delta_0$. It follows from the soundness of LM with respect to $\models^{\mathsf{MoAP}}_{\text{o-arg}}$ that $\Gamma \Rightarrow \Delta$ has an MoAP-counter-model if $\Gamma_0 \Rightarrow \Delta_0$ has one, too. It thus remains to show that every LM-irreducible sequent has an MoAP-counter-model.

First observe that every LM-irreducible sequent $\Gamma_0 \Rightarrow \Delta_0$ is of the following form:
– $\Gamma_0 \cap \Delta_0 = \emptyset$;
– every $\gamma \in \Gamma_0$ is atomic, a negation or an implication;
– every $\delta \in \Delta_0$ is atomic or a conjunction.
We construct an SAF $\mathcal{S} = (Ar, \longrightarrow)$ where the set of claims of arguments in Ar consists of all subformulas occurring in $\Gamma_0 \cup \Delta_0$. Additionally there is special argument in Ar with a claim X that is a new atomic formula (i.e. X does not occur as a subformula in $\Gamma_0 \cup \Delta_0$). The attack relation is obtained by setting $\delta \to X$ for all $\delta \in \Delta_0$. Moreover if $A \longrightarrow X$ and $B \longrightarrow X$ and $A \vee B$ occurs as a subformula of some formula in $\Gamma_0 \cup \Delta_0$, because of **(D.∨)**, we have to add the attack $A \vee B \longrightarrow X$.

Therefore $\mathcal{S}$ satisfies all attack principles in MoAP and that $\Gamma_0 \not\models^{\mathcal{S}}_{\text{o-arg}} \Delta_0$ since no disjunctions can occur in an irreducible sequent.) Conclusively, S is an MoAP counter-model of $\Gamma_0 \Rightarrow \Delta_0$. $\square$

Bipolar abstract argumentation frames have been introduced in Section 2.1, Definition 2.1.2. Using the same level of abstraction of semi-abstract argumentation frames (Definition 4.1.1), we introduce *bipolar semi-abstract argumentation frames.*

Definition 4.5.2 (Bipolar Semi-Abstract Argumentation Frame (BiSAF))**.** *A* bipolar semi-abstract argumentation frame (BiSAF) *is a directed graph* $\mathcal{B} = \langle Ar, R_{\rightarrow}, R_{\rightsquigarrow} \rangle$*, where each vertex* $a \in Ar$ *is labeled by a formula of* $\mathcal{PL}$*, called the* claim *of the argument* a*, and the edges are either attack relations* ($R_{\rightarrow}$) *or support relations* ($R_{\rightsquigarrow}$) *between arguments.*

As done for the attack relation, we introduce *general support principles* from which specific *support principles* follow.

($\mathbf{A^S}$.gen) If $X \rightsquigarrow G'$ and $G' \models G$ then $X \rightsquigarrow G$.

($\mathbf{B^S}$.gen) If $G' \rightsquigarrow X$ and $G' \models G$ then $G \rightsquigarrow X$.

From $A \wedge B \models A$, $A \models A \vee B$ and the principles **($\mathbf{A^S}$.gen)** and **($\mathbf{B^S}$.gen)** we can derive the following support principles:

($\mathbf{A^S}.\wedge$) If $X \rightsquigarrow A \wedge B$, then $X \rightsquigarrow A$ and $X \rightsquigarrow B$.

($\mathbf{A^S}.\vee$) If $X \rightsquigarrow A$ or $X \rightsquigarrow B$, then $X \rightsquigarrow A \vee B$.

($\mathbf{B^S}.\wedge$) If $A \wedge B \rightsquigarrow X$, then $A \rightsquigarrow X$ and $B \rightsquigarrow X$.

($\mathbf{B^S}.\vee$) If $A \rightsquigarrow X$ or $B \rightsquigarrow X$, then $A \vee B \rightsquigarrow X$

We can define the support principles for implication and negation, but they do not directly follow from the general principles as for the attack relation.

In Chapter 7 we will see how *mixed* attack and support principles can be used to recover argumentative semantics for the three basic fuzzy logic: Łukasiewicz, Gödel and Product Logic.

4.6 Connection Between Attack Principles and Extensions

As we have seen in the previous sections, by understanding the arguments as complex entities and instantiating even just one part of them with propositional formulas, we identify implicit attack relations through the attack principles. In the following propositions, we investigate the relationship between the satisfaction of a specific attack principle by a given SAF and the different kinds of extensions introduced by Dung.

This connection is relevant because potentially new attacks tell us how we have to update the extensions of a given frame once an attack principle is enforced on that frame.

Suppose to have a SAF $\mathcal{S}$ whose extensions are $\mathsf{Sem}(\mathcal{S})$. Suppose we impose that $\mathcal{S}$ must satisfy the attack principles of conjunction and disjunction. We refer to this new SAF as $\mathcal{S}'$. In that case, the extensions of $\mathcal{S}'$ contain all the original extensions of $\mathcal{S}$, i.e. $\mathsf{Sem}(\mathcal{S}) \subseteq \mathsf{Sem}(\mathcal{S}')$.

Enforcing the attack principles of implication and negation tells us also how we cannot expand the original extension sets or which argument cannot belong to the original ones.

Proposition 4.6.1 (APs and Conflict-free Sets)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow\rangle$ be a semi-abstract argumentation frame, $\mathsf{Cf}(\mathcal{S}) := \{\mathsf{C} \subseteq Ar \mid \mathsf{C}$ is conflict free$\}$, and $\mathsf{C} \in \mathsf{Cf}(\mathcal{S})$, we have:*

(1) (a) Whenever **(C.$\wedge$)** *and* **(D.$\wedge$)** *hold in $\mathcal{S}$, if $A \in \mathsf{C}$ and $B \in \mathsf{C}$, then $\mathsf{C} \cup \{A \wedge B\} \in \mathsf{Cf}(\mathcal{S})$.*

(b) Whenever **(A.$\wedge$)** *and* **(B.$\wedge$)** *hold in $\mathcal{S}$, if $A \wedge B \in \mathsf{C}$, then $\mathsf{C} \cup \{A\} \in \mathsf{Cf}(\mathcal{S})$ and $\mathsf{C} \cup \{B\} \in \mathsf{Cf}(\mathcal{S})$.*

(2) Whenever **(A.$\vee$)** *and* **(B.$\vee$)** *hold in $\mathcal{S}$, if $A \in \mathsf{C}$ or $B \in \mathsf{C}$, then $\mathsf{C} \cup \{A \vee B\} \in \mathsf{Cf}(\mathcal{S})$.*

(3) (a) Whenever **(C.$\supset$)** *and* **(D.$\supset$)** *hold in $\mathcal{S}$, if $B \in \mathsf{C}$ and $A \notin \mathsf{C}$, then $\mathsf{C} \cup \{A \supset B\} \in \mathsf{Cf}(\mathcal{S})$.*

(b) Whenever **(A.$\supset$)** *and* **(B.$\supset$)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{C}$, then $\mathsf{C} \cup \{B\} \in \mathsf{Cf}(\mathcal{S})$ or $\mathsf{C} \cup \{A\} \notin \mathsf{Cf}(\mathcal{S})$.*

(4) Whenever **(C.$\neg$)** *holds in $\mathcal{S}$, if $A \in \mathsf{C}$, then $\neg A \notin \mathsf{C}$.*

Proof. (1) (a) Suppose $A \in \mathsf{C}$, $B \in \mathsf{C}$ and $\mathsf{C} \cup \{A \wedge B\} \notin \mathsf{Cf}(\mathcal{S})$, it means that it is not conflict-free and there is $C_i \in \mathsf{C}$ s.t. either $C_i \longrightarrow A \wedge B$ or $A \wedge B \longrightarrow C_i$. If $C_i \longrightarrow A \wedge B$, by **(C.$\wedge$)** we have $C_i \longrightarrow A$ or $C_i \longrightarrow B$ with both A and B in C, but this is against the assumption that C is conflict free. If $A \wedge B \longrightarrow C_i$ we obtain the same result, but with the use of **(D.$\wedge$)**.

(b) Suppose $A \wedge B \in \mathsf{C}$ and either $\mathsf{C} \cup \{A\} \notin \mathsf{Cf}(\mathcal{S})$, or $\mathsf{C} \cup \{B\} \notin \mathsf{Cf}(\mathcal{S})$. In the first case, it means there is $C_i \in \mathsf{C}$ s.t. either $C_i \longrightarrow A$ or $A \longrightarrow C_i$. If $C_i \longrightarrow A$, by **(A.$\wedge$)** we have $C_i \longrightarrow A \wedge B$ with $A \wedge B \in \mathsf{C}$, and this is against the hypothesis that C is conflict free. If $A \longrightarrow C_i$, we have the same result but with the use of **(B.$\wedge$)**. The same reasoning can be done also for the second case supposing $\mathsf{C} \cup \{B\} \notin \mathsf{Cf}(\mathcal{S})$.

(2) Suppose $\mathsf{C} \cup \{A \vee B\} \notin \mathsf{Cf}(\mathcal{S})$, it means either $\mathsf{C} \longrightarrow A \vee B$ or $A \vee B \longrightarrow \mathsf{C}$. If $\mathsf{C} \longrightarrow A \vee B$, by **(A.$\vee$)** we have $\mathsf{C} \longrightarrow A$ and $\mathsf{C} \longrightarrow B$ with either A or B in C, therefore we have reached an absurdum. If $A \vee B \longrightarrow \mathsf{C}$ we have the same result, but with the use of **(B.$\vee$)**.

(3) (a) Suppose $\mathsf{C} \cup \{A \supset B\} \notin \mathsf{Cf}(\mathcal{S})$, it means either $\mathsf{C} \longrightarrow A \supset B$ or $A \supset B \longrightarrow \mathsf{C}$. In the first case, by **(C.$\supset$)** we have $\mathsf{C} \longrightarrow B$, but this is against our hypothesis. In the second case we obtain the same result, but with the use of **(D.$\supset$)**.

Valid Attack Principles in $\mathcal{S}$	Whenever:	Then we have:
(C.$\wedge$**) (D.**$\wedge$**)**	$A \in \mathsf{C}$ and $B \in \mathsf{C}$	$\mathsf{C} \cup \{A \wedge B\} \in \mathsf{Cf}(\mathcal{S})$
(A.$\wedge$**) (B.**$\wedge$**)**	$A \wedge B \in \mathsf{C}$	$\mathsf{C} \cup \{A\} \in \mathsf{Cf}(\mathcal{S})$ and $\mathsf{C} \cup \{B\} \in \mathsf{Cf}(\mathcal{S})$
(A.$\vee$**) (B.**$\vee$**)**	$A \in \mathsf{C}$ or $B \in \mathsf{C}$	$\mathsf{C} \cup \{A \vee B\} \in \mathsf{Cf}(\mathcal{S})$
(C.$\supset$**) (D.**$\supset$**)**	$B \in \mathsf{C}$ and $A \notin \mathsf{C}$	$\mathsf{C} \cup \{A \supset B\} \in \mathsf{Cf}(\mathcal{S})$
(A.$\supset$**) (B.**$\supset$**)**	$A \supset B \in \mathsf{C}$	$\mathsf{C} \cup \{B\} \in \mathsf{Cf}(\mathcal{S})$ or $\mathsf{C} \cup \{A\} \notin \mathsf{Cf}(\mathcal{S})$
(C.$\neg$**)**	$A \in \mathsf{C}$	$\neg A \notin \mathsf{C}$

Table 4.4: Summary of Proposition 4.6.1

(b) Suppose $\mathsf{C} \cup \{B\} \notin \mathsf{Cf}(\mathcal{S})$ and $\mathsf{C} \cup \{A\} \in \mathsf{Cf}(\mathcal{S})$. We have either $\mathsf{C} \longrightarrow B$ or $B \longrightarrow \mathsf{C}$. Since $\mathsf{C} \cup \{A\}$ is conflict free, $\mathsf{C} \not\longrightarrow A$ and if $\mathsf{C} \longrightarrow B$ by **(A.**$\supset$**)** $\mathsf{C} \longrightarrow A \supset B$ with $A \supset B \in \mathsf{C}$ and this it cannot be. If $B \longrightarrow \mathsf{C}$ we have the same result, but with the use of **(B.**$\supset$**)**.

(4) Suppose $\neg A \in \mathsf{C}$, therefore if both A and $\neg A$ are in C $A \not\longrightarrow \neg A$, but by **(C.**$\neg$**)** $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$ and this it cannot be. □

The statement of Proposition 4.6.1 is summarised in Table 4.4.

Proposition 4.6.1 makes explicit the connection between attack principles and conflict-free sets of arguments. In particular, it tells us that if we enforce a given SAF to satisfy certain attack principles, conflict-free sets of arguments are closed under conjunction only if the strong attack principles for conjunction hold in the frame. As expected, conflict-free sets of arguments are closed under disjunction only if the weak attack principles for disjunction hold in the frame. For the case of implication, the strong attack principles are needed. Surprisingly, to have a very intuitive property hold in a conflict-free set, i.e., a conflict-free set should not contain arguments with contradictory claims, the strong principle for negation is needed. This last fact tells us indirectly that arguments with contradictory claims not necessarily attack each other. If this seems unreasonable in SAFs where the focus is only on the arguments' claims, it is not the case when you consider fully instantiated arguments. As seen in Section 3.3, arguments can attack each other in different ways and arguments with contradictory claims not necessarily satisfy the attacking conditions of a specific attack relation.

Proposition 4.6.2 (APs and Admissible Extensions)**.** *Let* $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ *be a semi-abstract argumentation frame,* $\mathsf{Adm}(\mathcal{S}) := \{\mathsf{D} \subseteq Ar \mid \mathsf{D}$ *is an admissible set*$\}$ *and* $\mathsf{D} \in \mathsf{Adm}(\mathcal{S})$*, we have:*

(1) (a) Whenever **(C.**$\wedge$**)** *and* **(D.**$\wedge$**)** *hold in* $\mathcal{S}$*, if* $A \in \mathsf{D}$ *and* $B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A \wedge B\} \in \mathsf{Adm}(\mathcal{S})$*.*

(b) Whenever **(A.**$\wedge$**)** *and* **(B.**$\wedge$**)** *hold in* $\mathcal{S}$*, if* $A \wedge B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A\} \in \mathsf{Adm}(\mathcal{S})$ *and* $\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$*.*

Valid Attack Principles in $\mathcal{S}$	Whenever:	Then we have:
(C.$\wedge$) **(D.$\wedge$)**	$A \in \mathsf{D}$ and $B \in \mathsf{D}$	$\mathsf{D} \cup \{A \wedge B\} \in \mathsf{Adm}(\mathcal{S})$
(A.$\wedge$) **(B.$\wedge$)**	$A \wedge B \in \mathsf{D}$	$\mathsf{D} \cup \{A\} \in \mathsf{Adm}(\mathcal{S})$ and $\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$
(A.$\vee$) **(B.$\vee$)**	$A \in \mathsf{D}$ or $B \in \mathsf{D}$	$\mathsf{D} \cup \{A \vee B\} \in \mathsf{Adm}(\mathcal{S})$
(C.$\supset$) **(D.$\supset$)**	$B \in \mathsf{D}$	$\mathsf{D} \cup \{A \supset B\} \in \mathsf{Adm}(\mathcal{S})$
(A.$\supset$) **(A.$\supset$)*** **(B.$\supset$)**	$A \supset B \in \mathsf{D}$	$\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$ or $\mathsf{D} \cup \{A\} \notin \mathsf{Adm}(\mathcal{S})$
(C.$\neg$)	$A \in \mathsf{D}$	$\neg A \notin \mathsf{D}$

Table 4.5: Summary of Proposition 4.6.2

(2) Whenever **(A.$\vee$)** *and* **(B.$\vee$)** *hold in $\mathcal{S}$, if $A \in \mathsf{D}$ or $B \in \mathsf{D}$, then $\mathsf{D} \cup \{A \vee B\} \in \mathsf{Adm}(\mathcal{S})$.*

(3) (a) Whenever **(C.$\supset$)** *and* **(D.$\supset$)** *hold in $\mathcal{S}$, If $B \in \mathsf{D}$, then $\mathsf{D} \cup \{A \supset B\} \in \mathsf{Adm}(\mathcal{S})$.*

(b) Whenever **(A.$\supset$)**, **(B.$\supset$)** *and* **(A.$\supset$)*** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{D}$, then either $\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$ or $\mathsf{D} \cup \{A\} \notin \mathsf{Adm}(\mathcal{S})$.*

(4) Whenever **(C.$\neg$)** *holds in $\mathcal{S}$, if $A \in \mathsf{D}$, then $\neg A \notin \mathsf{D}$.*

In Table 4.5, we summarise the statement of Proposition 4.6.2.

If we compare the attack principles needed in Proposition 4.6.1 with Proposition 4.6.2, we observe that the only difference is in the sub-case 3-*b*. In particular, in a given SAF, to have an admissible extension closed under the consequent of an implication already belonging to the set, the SAF needs to satisfies a *stronger*[3] version of **(A.$\supset$)**. The attack principle to which we refer is the following:

(A.$\supset$)* If $F \longrightarrow B$ then $F \longrightarrow A \supset B$.

In Proposition 4.6.3 and Proposition 4.6.4 we investigate the connection between the attack principles and complete extensions. The extended proofs are in Appendix A.2. Regarding Proposition 4.6.3, we only analyse the case of conjunction. We can prove similar results also for the other connectives. In Table 4.6 we summarise the results of the extended version of Proposition 4.6.4.

Proposition 4.6.3. *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and C a complete extension over $\mathcal{S}$ satisfying* **(A.$\wedge$)** *and* **(D.$\wedge$)**

(a) If $A \in \mathsf{C}$, $B \in \mathsf{C}$ and $\mathsf{C} \cup \{A \wedge B\} \in \mathsf{Cmp}(\mathcal{S})$, then $\mathsf{C} \cup \{A \wedge B\} = \mathsf{C}$, i.e. $A \wedge B \in \mathsf{C}$.

(b) If $A \wedge B \in \mathsf{C}$, $\mathsf{C} \cup \{A\} \in \mathsf{Cmp}(\mathcal{S})$ and $\mathsf{C} \cup \{B\} \in \mathsf{Cmp}(\mathcal{S})$, then $\mathsf{C} \cup \{A\} = \mathsf{C} \cup \{B\} = \mathsf{C}$, i.e. both A and B are in C.

[3]We say that this attack principle **(A.$\supset$)*** is *stronger* then **(A.$\supset$)** because by weakening the assumptions, it entails the same attack relation of **(A.$\supset$)**.

Proof. (a) In order to show that $\mathsf{C} \cup \{A \wedge B\} = \mathsf{C}$ we have to verify that C defends $A \wedge B$. Therefore, if $X \longrightarrow A \wedge B$, $\mathsf{C} \cup \{A \wedge B\} \longrightarrow X$, i.e. either $A \wedge B \longrightarrow X$ or $\mathsf{C} \longrightarrow X$. In the first case, by **(D.∧)** we have $A \longrightarrow X$ or $B \longrightarrow X$ with both A and B in C, which means that C defends $A \wedge B$ and by definition of complete extension, $A \wedge B \in \mathsf{C}$. The same can be affirmed if $A \wedge B \not\longrightarrow X$ and $\mathsf{C} \longrightarrow X$.

(b) To show that $\mathsf{C} \cup \{A\} = \mathsf{C} \cup \{B\} = \mathsf{C}$ we have to verify that C defends both A and B. In fact, if $X \longrightarrow A$, by **(A.∧)** we have $X \longrightarrow A \wedge B$ that by hypothesis is in C. Therefore, $\mathsf{C} \longrightarrow X$, i.e. C defends A (and similarly B). □

Proposition 4.6.4 (APs and Complete Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and C a complete extension over $\mathcal{S}$ ($\mathsf{C} \in \mathsf{Cmp}(\mathcal{S})$), we have:*

(1) Whenever **(A.∧)**, **(B.∧)**, **(D.∧)** *and* **(C.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{C}$ and $B \in \mathsf{C}$ iff $A \wedge B \in \mathsf{C}$.*

(2) If **(A.∨)** *and* **(B.∨)** *hold in $\mathcal{S}$, $A \in \mathsf{C}$ or $B \in \mathsf{C}$, then $A \vee B \in \mathsf{C}$*

(3) (a) Whenever **(C.⊃)** *and* **(D.⊃)** *hold in $\mathcal{S}$, if $B \in \mathsf{C}$,then $A \supset B \in \mathsf{C}$.*

(b) Whenever **(A.⊃)** *and* **(B.⊃)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{C}$, then $B \in \mathsf{C}$ or $A \notin \mathsf{C}$.*

(4) Whenever **(C.¬)** *holds in $\mathcal{S}$, if $A \in \mathsf{C}$, then $\neg A \notin \mathsf{C}$.*

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.∧) (D.∧)	$A \in \mathsf{C}$ and $B \in \mathsf{C}$	$A \wedge B \in \mathsf{C}$
(A.∧) (B.∧)	$A \wedge B \in \mathsf{C}$	$A \in \mathsf{C}$ and $B \in \mathsf{C}$
(A.∨) (B.∨)	$A \in \mathsf{C}$ or $B \in \mathsf{C}$	$A \vee B \in \mathsf{C}$
(C.⊃) (D.⊃)	$B \in \mathsf{C}$	$A \supset B \in \mathsf{C}$
(A.⊃) (B.⊃)	$A \supset B \in \mathsf{C}$	$B \in \mathsf{C}$ or $A \notin \mathsf{C}$
(C.¬)	$A \in \mathsf{C}$	$\neg A \notin \mathsf{C}$

Table 4.6: Summary of Proposition 4.6.4

In Proposition 4.6.5 and Proposition 4.6.6 we investigate the connection between the attack principles and preferred extensions. The extended proofs are in Appendix A.2. In Table 4.7 we summarise the results of the extended version of Proposition 4.6.6.

Proposition 4.6.5. *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame, P a preferred extension of $\mathcal{S}$ and $A \in Ar$. If $\mathsf{P} \cup \{A\}$ is conflict free and defends all its elements, then $A \in \mathsf{P}$.*

Proof. Since P is a maximal admissible set, if $\mathsf{P} \cup \{A\}$ is an admissible set of arguments, $A \in \mathsf{P}$. □

Proposition 4.6.6 (APs and Preferred Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and P a preferred extension over $\mathcal{S}$ ($\mathsf{P} \in \mathsf{Prf}(\mathcal{S})$), we have:*

(1) Whenever **(A.∧)**, **(B.∧)**, **(C.∧)** *and* **(D.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{P}$ and $B \in \mathsf{P}$ iff $A \wedge B \in \mathsf{P}$.*

(2) Whenever **(A.∨)** *and* **(B.∨)** *hold in $\mathcal{S}$, $A \in \mathsf{P}$ or $B \in \mathsf{P}$, then $A \vee B \in \mathsf{P}$*

(3) (a) Whenever **(C.⊃)** *and* **(D.⊃)** *holds in $\mathcal{S}$, if $B \in \mathsf{P}$, then $A \supset B \in \mathsf{P}$;*

(b) Whenever **(A.⊃)** *and* **(B.⊃)** *holds in $\mathcal{S}$, if $A \supset B \in \mathsf{P}$, then $B \in \mathsf{P}$ or $A \notin \mathsf{P}$*

(4) Whenever **(C.¬)** *holds in $\mathcal{S}$, if $A \in \mathsf{P}$, then $\neg A \notin \mathsf{P}$.*

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.∧) (D.∧)	$A \in \mathsf{P}$ and $B \in \mathsf{P}$	$A \wedge B \in \mathsf{P}$
(A.∧) (B.∧)	$A \wedge B \in \mathsf{P}$	$A \in \mathsf{P}$ and $B \in \mathsf{P}$
(A.∨) (B.∨)	$A \in \mathsf{P}$ or $B \in \mathsf{P}$	$A \vee B \in \mathsf{P}$
(C.⊃) (D.⊃)	$B \in \mathsf{P}$	$A \supset B \in \mathsf{P}$
(A.⊃) (B.⊃)	$A \supset B \in \mathsf{P}$	$B \in \mathsf{P}$ or $A \notin \mathsf{P}$
(C.¬)	$A \in \mathsf{P}$	$\neg A \notin \mathsf{P}$

Table 4.7: Summary of Proposition 4.6.6

Proposition 4.6.7 (APs and Semi-Stable Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and S a semi-stable extension over $\mathcal{S}$ ($\mathsf{S} \in \mathsf{S\text{-}Stb}(\mathcal{S})$), we have:*

(1) Whenever **(C.∧)** *and* **(A.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ and $B \in \mathsf{S}$ iff $A \wedge B \in \mathsf{S}$.*

(2) Whenever **(A.∨)** *hold in $\mathcal{S}$, if $A \in \mathsf{S}$ or $B \in \mathsf{S}$, then $A \vee B \in \mathsf{S}$*

(3) (a) Whenever **(C.⊃)** *holds in $\mathcal{S}$, if $B \in \mathsf{S}$,then $A \supset B \in \mathsf{S}$.*

(b) Whenever **(B.⊃)** *and* **(A.⊃)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{S}$, then $B \in \mathsf{S}$ or $A \notin \mathsf{C}$.*

(4) Whenever **(C.¬)** *holds in $\mathcal{S}$, if $A \in \mathsf{S}$, then $\neg A \notin \mathsf{S}$.*

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.$\wedge$)	$A \in \mathsf{S}$ and $B \in \mathsf{S}$	$A \wedge B \in \mathsf{S}$
(A.$\wedge$)	$A \wedge B \in \mathsf{S}$	$A \in \mathsf{S}$ and $B \in \mathsf{S}$
(A.$\vee$)	$A \in \mathsf{S}$ or $B \in \mathsf{S}$	$A \vee B \in \mathsf{S}$
(C.$\supset$)	$B \in \mathsf{S}$	$A \supset B \in \mathsf{S}$
(A.$\supset$) (B.$\supset$)	$A \supset B \in \mathsf{S}$	$B \in \mathsf{S}$ or $A \notin \mathsf{S}$
(C.$\neg$)	$A \in \mathsf{S}$	$\neg A \notin \mathsf{S}$

Table 4.8: Summary of Proposition 4.6.7

Proposition 4.6.8 (APs and Stable Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and $\mathsf{S} \in \mathsf{Stb}(\mathcal{S})$, we have:*

(1) Whenever **(A.$\wedge$)** *and* **(C.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ and $B \in \mathsf{S}$ iff $A \wedge B \in S$.*

(2) Whenever **(A.$\vee$)**, **(C.$\vee$)** *and* **(B.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ or $B \in \mathsf{S}$ iff $A \vee B \in \mathsf{S}$.*

(3) (a) Whenever **(C.$\supset$)** *holds in $\mathcal{S}$, if $B \in \mathsf{S}$, then $A \supset B \in S$.*

(b) Whenever **(A.$\supset$)** *holds in $\mathcal{S}$, if $A \supset B \in \mathsf{S}$, then $B \in \mathsf{S}$ or $A \notin \mathsf{S}$*

(4) Whenever **(C.$\neg$)** *and* **(A.$\neg$)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ iff $\neg A \notin \mathsf{S}$.*

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.$\wedge$)	$A \in \mathsf{S}$ and $B \in \mathsf{S}$	$A \wedge B \in \mathsf{S}$
(A.$\wedge$)	$A \wedge B \in \mathsf{S}$	$A \in \mathsf{S}$ and $B \in \mathsf{S}$
(A.$\vee$)	$A \in \mathsf{S}$ or $B \in \mathsf{S}$	$A \vee B \in \mathsf{S}$
(B.$\wedge$)(C.$\vee$)	$A \vee B \in \mathsf{S}$	$A \in \mathsf{S}$ or $B \in \mathsf{S}$
(C.$\supset$)	$B \in \mathsf{S}$	$A \supset B \in \mathsf{S}$
(A.$\supset$)	$A \supset B \in \mathsf{S}$	$B \in \mathsf{S}$ or $A \notin \mathsf{S}$
(C.$\neg$)	$A \in \mathsf{S}$	$\neg A \notin \mathsf{S}$

Table 4.9: Summary of Proposition 4.6.8

Proposition 4.6.9 (APs and Grounded Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and G a grounded extension over $\mathcal{S}$ ($\mathsf{G} \in \mathsf{Grn}(\mathcal{S})$), we have:*

(1) Whenever **(A.$\wedge$)** *and* **(C.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{G}$ and $B \in \mathsf{G}$ iff $A \wedge B \in \mathsf{G}$.*

(2) Whenever **(A.$\vee$)** *holds in $\mathcal{S}$, if $A \in \mathsf{G}$ or $B \in \mathsf{G}$, then $A \vee B \in \mathsf{G}$*

(3) *(a)* *Whenever* **(C.$\supset$)** *holds in* $\mathcal{S}$*, if* $B \in \mathsf{G}$ *,then* $A \supset B \in \mathsf{G}$.

(b) *Whenever* **(A.$\supset$)** *and* **(B.$\supset$)** *hold in* $\mathcal{S}$*, if* $A \supset B \in \mathsf{G}$*, then* $B \in \mathsf{G}$ *or* $A \notin \mathsf{G}$.

(4) *Whenever* **(C.$\neg$)** *holds in* $\mathcal{S}$*, if* $A \in \mathsf{G}$*, then* $\neg A \notin \mathsf{G}$.

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.$\wedge$)	$A \in \mathsf{S}$ and $B \in \mathsf{S}$	$A \wedge B \in \mathsf{S}$
(A.$\wedge$)	$A \wedge B \in \mathsf{S}$	$A \in \mathsf{S}$ and $B \in \mathsf{S}$
(A.$\vee$)	$A \in \mathsf{S}$ or $B \in \mathsf{S}$	$A \vee B \in \mathsf{S}$
(C.$\supset$)	$B \in \mathsf{S}$	$A \supset B \in \mathsf{S}$
(A.$\supset$) (B.$\supset$)	$A \supset B \in \mathsf{S}$	$B \in \mathsf{S}$ or $A \notin \mathsf{S}$
(C.$\neg$)	$A \in \mathsf{S}$	$\neg A \notin \mathsf{S}$

Table 4.10: Summary of Proposition 4.6.9

Proposition 4.6.10 (APs and Ideal Extensions)**.** *Let* $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ *be a semi-abstract argumentation frame and* I *a semi-stable extension over* $\mathcal{S}$ *(*$\mathsf{I} \in \mathsf{Idl}(\mathcal{S})$*), we have:*

(1) *Whenever* **(A.$\wedge$)**, **(B.$\wedge$)**, **(C.$\wedge$)** *and* **(D.$\wedge$)** *hold in* $\mathcal{S}$, $A \in \mathsf{I}$ *and* $B \in \mathsf{I}$ *iff* $A \wedge B \in \mathsf{I}$.

(2) *Whenever* **(A.$\vee$)** *and* **(B.$\vee$)** *hold in* $\mathcal{S}$*, if* $A \in \mathsf{I}$*, then* $B \in \mathsf{I}$ *iff* $A \vee B \in \mathsf{I}$

(3) *(a)* *Whenever* **(C.$\supset$)** *and* **(D.$\supset$)** *hold in* $\mathcal{S}$*, if* $B \in \mathsf{I}$*, then* $A \supset B \in \mathsf{I}$*;*

(b) *Whenever* **(A.$\supset$)** *and* **(B.$\supset$)** *hold in* $\mathcal{S}$*, if* $A \supset B \in \mathsf{I}$*, then* $B \in \mathsf{I}$ *or* $A \notin \mathsf{I}$

(4) *Whenever* **(C.$\neg$)** *holds in* $\mathcal{S}$*, if* $A \in \mathsf{I}$*, then* $\neg A \notin \mathsf{I}$.

Valid Attack Principles in $\mathcal{S}$	**Whenever:**	**Then we have:**
(C.$\wedge$) (D.$\wedge$)	$A \in \mathsf{I}$ and $B \in \mathsf{I}$	$A \wedge B \in \mathsf{I}$
(A.$\wedge$) (B.$\wedge$)	$A \wedge B \in \mathsf{I}$	$A \in \mathsf{I}$ and $B \in \mathsf{I}$
(A.$\vee$) (B.$\vee$)	$A \in \mathsf{I}$ or $B \in \mathsf{I}$	$A \vee B \in \mathsf{I}$
(C.$\supset$) (D.$\supset$)	$B \in \mathsf{I}$	$A \supset B \in \mathsf{I}$
(A.$\supset$) (B.$\supset$)	$A \supset B \in \mathsf{I}$	$B \in \mathsf{I}$ or $A \notin \mathsf{I}$
(C.$\neg$)	$A \in \mathsf{I}$	$\neg A \notin \mathsf{I}$

Table 4.11: Summary of Proposition 4.6.10

Following the Definition 3.2.7, we say that an argument with claim G in a SAF $\mathcal{S}$ is *more conservative* than an argument with claim G' in $\mathcal{S}$, if $G' \models G$. As already explained in Section 4.1, the consequence relation $\models$ does not refer to any specific logic.

Through the following definition and proposition, we show that all the extensions mentioned above are closed under some more conservative arguments if the SAFs over which the extensions are defined, satisfy only some of the attack principles justified by the modal interpretation introduced in Definition 4.4.1.

Definition 4.6.1 (Closure Under a Connective)**.** *Let* $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ *be a semi-abstract argumentation frame and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{S\text{-}Stb}, \mathsf{Stb}, \mathsf{Grn}, \mathsf{Idl}\}$*. We say that*

> $\mathsf{Sem}(\mathcal{S})$ *is* closed under conjuncts *if whenever an argument with claim* $A \wedge B$ *is in* $\mathsf{E} \in \mathsf{Sem}(\mathcal{S})$*, then both* $\mathsf{E} \cup \{A\} \in \mathsf{Sem}(\mathcal{S})$ *and* $\mathsf{E} \cup \{B\} \in \mathsf{Sem}(\mathcal{S})$*.*
>
> $\mathsf{Sem}(\mathcal{S})$ *is* closed under disjunction *if whenever an argument with claim* A *or an argument with claim* B *is in* $\mathsf{E} \in \mathsf{Sem}(\mathcal{S})$*, then* $\mathsf{E} \cup \{A \vee B\} \in \mathsf{Sem}(\mathcal{S})$*.*
>
> $\mathsf{Sem}(\mathcal{S})$ *is* closed under implication *if whenever an argument with claim* B *is in* $\mathsf{E} \in \mathsf{Sem}(\mathcal{S})$*, then* $\mathsf{E} \cup \{A \supset B\} \in \mathsf{Sem}(\mathcal{S})$*.*

We recall the two sets of attack principles MAP and MoAP introduced, respectively, in Section 4.4 and Section 4.5 are defined as $\mathsf{MAP} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg)\}$ and $\mathsf{MoAP} = \{(\mathbf{B.}\wedge), (\mathbf{D.}\vee), (\mathbf{B.}\vee), (\mathbf{D.}\supset), (\mathbf{B.}\neg)\}$.

We define EAP, an additional set of attack principles, in the following way:

$$\mathsf{EAP} = \{(\mathbf{A.}\wedge), (\mathbf{B.}\wedge), (\mathbf{A.}\vee), (\mathbf{B.}\vee), (\mathbf{C.}\supset), (\mathbf{D.}\supset)\}.$$

Let us notice that $\mathsf{EAP} \subset \mathsf{MAP} \cup \mathsf{MoAP}$.

Proposition 4.6.11. *Let* $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ *be a semi-abstract argumentation frame and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{S\text{-}Stb}, \mathsf{Stb}, \mathsf{Grn}, \mathsf{Idl}\}$*. If* $\mathcal{S}$ *satisfies the attack principles in* EAP*, then* $\mathsf{Sem}(\mathcal{S})$ *is closed under conjuncts, disjunction and implication.*

Proof. From Proposition 4.6.2, $\mathsf{Adm}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies $(\mathbf{A.}\wedge)$, $(\mathbf{B.}\wedge)$, $(\mathbf{A.}\vee)$, $(\mathbf{B.}\vee)$, $(\mathbf{C.}\supset)$ and $(\mathbf{D.}\supset)$.

From Proposition 4.6.4, $\mathsf{Cmp}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies $(\mathbf{A.}\wedge)$, $(\mathbf{B.}\wedge)$, $(\mathbf{A.}\vee)$, $(\mathbf{B.}\vee)$, $(\mathbf{C.}\supset)$ and $(\mathbf{D.}\supset)$.

From Proposition 4.6.6, $\mathsf{Prf}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies $(\mathbf{A.}\wedge)$, $(\mathbf{B.}\wedge)$, $(\mathbf{A.}\vee)$, $(\mathbf{B.}\vee)$, $(\mathbf{C.}\supset)$ and $(\mathbf{D.}\supset)$.

From Proposition 4.6.7, $\mathsf{S\text{-}Stb}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies $(\mathbf{A.}\wedge)$, $(\mathbf{A.}\vee)$ and $(\mathbf{C.}\supset)$.

From Proposition 4.6.8, $\mathsf{Stb}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies $(\mathbf{A.}\wedge)$, $(\mathbf{B.}\wedge)$, $(\mathbf{A.}\vee)$, $(\mathbf{B.}\vee)$ and $(\mathbf{C.}\supset)$.

From Proposition 4.6.9, $\mathsf{Grn}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies **(A.$\wedge$)**, **(A.$\vee$)** and **(C.$\supset$)**.

From Proposition 4.6.10, $\mathsf{Idl}(\mathcal{S})$ is closed under conjuncts, disjunction and implication if $\mathcal{S}$ satisfies **(A.$\wedge$)**, **(B.$\wedge$)**, **(A.$\vee$)**, **(B.$\vee$)**, **(C.$\supset$)** and **(D.$\supset$)**. □

We extend the definition of credulous and skeptical acceptability to SAFs in the obvious way.

Definition 4.6.2 (Credulous and Skeptical Acceptability on SAFs)**.** *Let $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ be an SAF and* $\mathsf{Sem} \in \{\mathsf{Cf}, \mathsf{Adm}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{S\text{-}Stb}, \mathsf{Stb}, \mathsf{Grn}, \mathsf{Idl}\}$

$\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Sem}} \psi$ *if* $\psi \in \mathsf{E}$ *for every extension* $\mathsf{E} \in \mathsf{Sem}(\mathcal{A})$*. In this case we say that* ψ skeptically *follows from* $\mathcal{A}$*.*

$\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Sem}} \psi$ *if there is an extension* $\mathsf{E} \in \mathsf{Sem}(\mathcal{A})$ *such that* $\psi \in \mathsf{E}$*. In this case we say that* ψ credulously *follows from* $\mathcal{A}$*.*

Considering the following sets of attack principles, we investigate how these affect credulous/skeptical acceptability and the entailment relations induced by these extensions.

$$\mathsf{AP}_0 = \{(\mathbf{A.}\wedge), (\mathbf{B.}\wedge), (\mathbf{C.}\wedge), (\mathbf{D.}\wedge), (\mathbf{A.}\vee), (\mathbf{B.}\vee), (\mathbf{A.}\supset), (\mathbf{A.}\supset)^*, (\mathbf{B.}\supset), (\mathbf{C.}\supset), (\mathbf{D.}\supset), (\mathbf{C.}\neg)\}$$

$$\mathsf{AP}_1 = \mathsf{AP}_0 \setminus \{(\mathbf{A.}\supset)^*\}$$

$$\mathsf{AP}_2 = \mathsf{AP}_1 \setminus \{(\mathbf{B.}\wedge), (\mathbf{D.}\wedge), (\mathbf{B.}\vee), (\mathbf{D.}\supset)\}$$

$$\mathsf{AP}_3 = \mathsf{AP}_2 \cup \{(\mathbf{B.}\wedge), (\mathbf{A.}\neg)\} \setminus \{(\mathbf{B.}\supset)\}$$

Proposition 4.6.12. *If $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ is an SAF such that it satisfies* AP_1*, then the following hold:*

(1) (a) *If* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Cf}} A$ *and* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Cf}} B$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \wedge B$*.*

(b) *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \wedge B$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A$ *and* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} B$*.*

(2) If $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A$ *or* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} B$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \vee B$*.*

(3) (a) *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} B$ *and* $\mathcal{A} \not\mid\!\sim^{\exists}_{\mathsf{Cf}} A$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \supset B$*.*

If $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Cf}} B$ *and* $\mathcal{A} \not\mid\!\sim^{\forall}_{\mathsf{Cf}} A$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \supset B$*.*

(b) *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A \supset B$*, then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} B$ *or* $\mathcal{A} \not\mid\!\sim^{\forall}_{\mathsf{Cf}} A$*.*

(4) (a) *If* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Cf}} A$*, then* $\mathcal{A} \not\mid\!\sim^{\exists}_{\mathsf{Cf}} \neg A$*.*

(b) *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Cf}} A$*, then* $\mathcal{A} \not\mid\!\sim^{\forall}_{\mathsf{Cf}} \neg A$*.*

Proof.

(1) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} A$ and $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} B$, it means that for all $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$, $A \in \mathsf{C_i}$ and $B \in \mathsf{C_i}$. Therefore, by Proposition 4.6.1, $\mathsf{C_i} \cup A \wedge B \in \mathsf{Cf}(\mathcal{A})$ and $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A \wedge B$ holds.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A \wedge B$, it means that $A \wedge B \in \mathsf{C_i}$ for all $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$. Therefore, by Proposition 4.6.1, $\mathsf{C_i} \cup \{A\} \in \mathsf{C_i}(\mathcal{A})$ and $\mathsf{C_i} \cup \{B\} \in \mathsf{C_i}(\mathcal{A})$, i.e. $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} A$ and $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} B$.

(2) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A$ or $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} B$, it means that there is $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$ such that either $A \in \mathsf{C_i}$, or $B \in \mathsf{C_i}$. Therefore by Proposition 4.6.1 $\mathsf{C_i} \cup \{A \vee B\} \in \mathsf{Cf}(\mathcal{A})$, i.e. $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A \vee B$.

(3) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} B$ and $\mathcal{A} \not\mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A$, it means that there $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$ such that $B \in \mathsf{C_i}$ and $A \notin \mathsf{C_i}$. Therefore, by Proposition 4.6.1, $\mathsf{C_i} \cup \{A \supset B\} \in \mathsf{Cf}(\mathcal{A})$, i.e. $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A \supset B$. The case of $\mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}}$ is similar.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A \supset B$, it means there is $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$ such that $A \supset B \in \mathsf{C_i}$. Then, by Proposition 4.6.1 we have either $\mathsf{C_i} \cup B \in \mathsf{Cf}(\mathcal{A})$ or $\mathsf{C_i} \cup A \notin \mathsf{Cf}(\mathcal{A})$. Therefore, either $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} B$ or $\mathcal{A} \not\mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} A$ holds.

(4) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} A$, it means that $A \in \mathsf{C_i}$ for all $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$. Therefore by Proposition 4.6.1 we have $\neg A \notin \mathsf{C_i}$, i.e. $\mathcal{A} \not\mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} \neg A$.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Cf}} A$, it means there is $\mathsf{C_i} \in \mathsf{Cf}(\mathcal{A})$ such that $A \in \mathsf{C_i}$. Therefore, by Proposition 4.6.1 $\neg A \notin \mathsf{C_i}$, i.e. $\mathcal{A} \not\mathrel{|\!\sim}^{\forall}_{\mathsf{Cf}} \neg A$. □

A proposition similar to the previous one holds also for admissible extensions. In this case the set of attack principles the argumentation frames need to satisfy is AP_0 and in the case (3)-(a) the hypothesis about the antecedent of the implication is redundant.

Proposition 4.6.13. *If $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ is an SAF such that it satisfies AP_1 and $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{S\text{-}Sbl}, \mathsf{Grn}\}$ then the following hold:*

(1) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A$ and $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} B$, then $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A \wedge B$.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A \wedge B$, then $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A$ and $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} B$.

If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A \wedge B$, then $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A$ and $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} B$.

(2) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A$ or $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} B$, then $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A \vee B$.

If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A$ or $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} B$, then $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A \vee B$.

(3) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} B$, then $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A \supset B$.

If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} B$, then $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A \supset B$.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A \supset B$, then $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} B$ or $\mathcal{A} \not\mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A$.

(4) (a) If $\mathcal{A} \mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} A$, then $\mathcal{A} \not\mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} \neg A$.

(b) If $\mathcal{A} \mathrel{|\!\sim}^{\exists}_{\mathsf{Sem}} A$, then $\mathcal{A} \not\mathrel{|\!\sim}^{\forall}_{\mathsf{Sem}} \neg A$.

For semi-stable and grounded semantics, Proposition 4.6.13 holds also for a smaller set of attack principles. The set of attack principles the argumentation frames need to satisfy is AP_2. The case of stable extensions is slightly different.

Proposition 4.6.14. *If $\mathcal{A} = \langle Ar, \longrightarrow \rangle$ is an SAF such that it satisfies AP_3, then the following hold:*

(1) $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A$ *and* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} B$ *iff* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A \wedge B$.

(2) *(a)* *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A$ *or* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} B$, *then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A \vee B$.
If $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A$ *or* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} B$, *then* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A \vee B$.
(b) *If* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A \vee B$, *then* $(\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A$ *or* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} B)$ *or* $(\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A$ *and* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} B)$
If $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A \vee B$, *then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A$ *or* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} B$.

(3) *(a)* *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} B$, *then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A \supset B$.
If $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} B$, *then* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A \supset B$.
(b) *If* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A \supset B$, *then* $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} B$ *or* $\mathcal{A} \not\mid\!\sim^{\forall}_{\mathsf{Stb}} A$.

(4) *(a)* $\mathcal{A} \mid\!\sim^{\forall}_{\mathsf{Stb}} A$ *iff* $\mathcal{A} \not\mid\!\sim^{\exists}_{\mathsf{Stb}} \neg A$.
(b) $\mathcal{A} \mid\!\sim^{\exists}_{\mathsf{Stb}} A$ *iff* $\mathcal{A} \not\mid\!\sim^{\forall}_{\mathsf{Stb}} \neg A$.

In [Rap18] the author has investigated which attack principles must hold in order to enforce consistency on the extensions and the following proposition has been proved.

Proposition 4.6.15. *Every conflict-free set in a syntactically closed SAF $\mathcal{S} = \langle Ar, \longrightarrow \rangle$, satisfying every attack principle in* CAP*, is consistent.*

As pointed out in [Rap18], the proof of Proposition 4.6.15 relies on the *interchangeability* of $\vdash$ and $\models^{\mathsf{CAP}}_{\mathrm{arg}}$ and on the strong assumptions of the *existence* of some attack given the *nonexistence* of some other attack relations imposed by **(C.¬)**.

In the next chapter, we will investigate the attack principles' role once they are instantiated in sequent-based argumentation frames.

CHAPTER 5

Attack Principles in Logical Argumentation

5.1 Interpretation of the Attack Principles in the Sequent Based Setting

The attack principles introduced in Section 4.1, impose constraints to the attack relations starting from an analysis of the arguments' claims. In defining the attack principles, the support part and the inference method between the support and the claim are always left abstract and not mentioned. However, by instantiating the support, the inference method and the attack relation, some interesting considerations emerge. We now analyse the attack principles in a sequent-based scenario and investigate under which conditions either on the support sets or the claims they hold. To interpret the attack relation in this fully instantiated frameworks, we need to remember that $X \longrightarrow A$ means that there exists an argument with claim X that attacks an argument with claim A. Therefore, the attack principle **(A.$\wedge$)** will be interpreted in the following way. If there exists an argument with claim X such that it attacks (using a specific attack relation) an argument with claim A, then there are arguments with claim $A \wedge B$ that are also attacked by the argument with claim X. The argument with claim X does not necessarily attack all the arguments with claim $A \wedge B$.

We analyse the interpretation of all the attack principles here in detail whenever the attack relation is *defeat rebuttal*, and the underlying logic is classical logic. While no formal proof system for CL is needed to check the adequateness of the attack principles, we will still occasionally refer to the sequent calculus **LK** presented in Figure 3.1. This will also serve as a reminder that the respective conditions can, in principle, be algorithmically checked by exhaustive proof search. We remind that $\langle \Gamma_X ; X \rangle_{\mathsf{CL}} \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A ; A \rangle_{\mathsf{CL}}$ if

$X \vdash \neg A$ holds. Since we are assuming to work only with CL-arguments, we will omit the CL sign in the sequel to enhance readability.

(A.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$, then $X \vdash \neg A$ or $X \vdash \neg B$. Thus, by Weakening, $(\vee, r)$, $(\vee, l)$ and $(\vee, r)$ we have that $X \vdash \neg(A \wedge B)$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$. Therefore, whenever the attack function is defeating rebuttal, the $(A.\wedge)$ attack principle holds.

(C.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$, then $X \vdash \neg(A \wedge B)$ and $X \vdash \neg A \vee \neg B$. Thus, we have that $X \vdash \neg A$ or $X \vdash \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$, only if $A \equiv B$. It is easy to see that the conditions needed to have $(C.\wedge)$ hold are very demanding and this is the reason why we will discard this principle.

(A.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $X \vdash \neg(A \vee B)$. Thus, $X \vdash \neg A \wedge \neg B$ holds and by $(\wedge, r)$, $X \vdash \neg A$ and $X \vdash \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$. The attack principle hold without any restriction on the attacked arguments.

(C.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$, then $X \vdash \neg A$ and $X \vdash \neg B$. Thus, by $(\wedge, r)$, $X \vdash \neg A \wedge \neg B$ and $X \vdash \neg(A \vee B)$ holds, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}; A \vee B\rangle$.

(A.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_1; X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$, then $X \vdash \neg B$ and $X \nvdash \neg A$. Therefore, $X \vdash \neg B \wedge A$, from which it follows $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, only if $X \vdash A$. We can deduce $X \vdash A$ from the assumptions only in the case in which $A \equiv \neg B$. Again, as in the case of **(C.$\wedge$)**, the conditions under which the attack principle holds are very strict.

(C.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $X \vdash \neg(A \supset B)$. Thus, $X \vdash \neg(\neg A \vee B)$ and $X \vdash A \wedge \neg B$. Therefore, $X \nvdash \neg A$ ($\langle \Gamma_X; X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$) and $X \vdash \neg B$ ($\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B; B\rangle$) only if X is consistent.

(A.$\neg$) If $\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$, then $X \vdash \neg A$. Thus, $\langle \Gamma_X; X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_{\neg A}; \neg A\rangle$, i.e. $X \nvdash A$, under the condition that X is consistent.

(C.$\neg$) Suppose that $\langle \Gamma_X; X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A; A\rangle$, i.e. $X \nvdash \neg A$. From this assumption we cannot deduce that $X \vdash A$
($\langle \Gamma_X; X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{\neg A}; \neg A\rangle$). Therefore the principle does not hold.

In the tables below, we summarise all the conditions needed to have the sequent-based arguments satisfy the attack principles according to the different attack relations. Whenever there is the symbol "–", it means that no condition is required. In Appendix B.1, there is a detailed analysis of the attack principles' interpretation considering other attack relations.

Attack Function	**(A.$\wedge$)**	**(C.$\wedge$)**
[Def]	$\Gamma_A \subseteq \Gamma_{A\wedge B}$	$\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$
[Ucut]	$\Gamma'_A \subseteq \Gamma_{A\vee B}$ or $\Gamma'_B \subseteq \Gamma_{A\vee B}$	$\Gamma'_{A\vee B} \subseteq \Gamma_A$ or $\Gamma'_{A\vee B} \subseteq \Gamma_B$
[Reb]	$A \equiv B$	$A \equiv B$
[C-Reb-1]	–	$A \equiv B$
[D-Reb]	–	$A \equiv B$
[I-Reb]	–	$A \equiv B$

Table 5.1: Conditions for the APs Concerning Conjunction

Attack Function	**(A.$\vee$)**	**(C.$\vee$)**
[Def]	$\Gamma_{A\vee B} \subseteq \Gamma_A \cap \Gamma_B$	$\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$
[Ucut]	$\Gamma'_{A\vee B} \subseteq \Gamma_A \cap \Gamma_B$	$\Gamma'_A \subseteq \Gamma_{A\vee B}$ or $\Gamma'_B \subseteq \Gamma_{A\vee B}$
[Reb]	$A \equiv B$	–
[C-Reb-1]	–	–
[D-Reb]	–	–
[I-Reb]	–	–

Table 5.2: Conditions for the APs Concerning Disjunction

Attack Function	**(A.$\supset$)**	**(C.$\supset$)**
[Def]	$\Gamma_B \subseteq \Gamma_{A\supset B}$	$\Gamma_{A\supset B} \subseteq \Gamma_B$ and $\Gamma_{A\supset B} \cap \Gamma_A = \emptyset$
[Ucut]	$\Gamma'_B \subseteq \Gamma_{A\supset B}$	$\Gamma'_{A\supset B} \subseteq \Gamma_B$ and $\Gamma'_{A\supset B} \not\subseteq \Gamma_A$
[Reb]	$A \equiv \neg B$	$A \equiv \neg B$
[C-Reb-1]	$A \equiv \neg B$	Γ_X consistent
[D-Reb]	$A \equiv \neg B$	X consistent
[I-Reb]	$\neg A \equiv B$	$\varphi \not\equiv \top$

Table 5.3: Conditions for the APs Concerning Implication

5.2 The Role of Minimality and Consistency Through APs

Sequent-based arguments (see Definition 3.2.10) are defined with the only requirement that the support infers the conclusion. Even if in the definition of a sequent-based argument,

Attack Function	**(A.$\neg$)**	**(C.$\neg$)**
[Def]	It does not hold	It does not hold
[Ucut]	$\Gamma'_A \neq \Gamma'_B$	It does not hold
[Reb]	–	$X \equiv A$
[C-Reb-1]	Γ_X is consistent	it does not hold
[D-Reb]	X is consistent	it does not hold
[I-Reb]	$\varphi \not\equiv \top$	It does not hold

Table 5.4: Conditions for the APs Concerning Negation

there is no additional condition on the support set, its minimality and consistency still play a role in determining which arguments can be attacked and in which manner. As pointed out in Section 3.2 and showed in Example 3.4.1, the arguments whose support is both minimal and consistent are in general harder to attack. We now investigate how this general observation is related to the attack principles.

To avoid misunderstandings, let us point out that we are not imposing on the supports of sequent-based arguments to be either minimal or consistent. Instead, we investigate how these two properties, which a sequent-based argument might, or might not have, interact with the definition of the attack relations.

An argument whose support is either inconsistent or not $\vdash$-minimal, as showed in Section 3.5, is vulnerable to more attacks compared to arguments with minimally consistent support.. For this reason, following Definition 3.5.5, an argument is *strong* if its support is both consistent and $\vdash$-minimal.

All the attack principles introduced in Chapter 4 have a schematic definition. For any of their instantiations, we can identify two parts: the *premise* and the *conclusion*.

Let us suppose that $AF_{\mathsf{CL},\mathscr{A}}(S) = \langle Arg_{\mathsf{CL}}(S), Attack(\mathscr{A})\rangle$ is a sequent-based argumentation framework such that $[Def] \in \mathscr{A}$ (see Definition 3.3.1) and $\langle \Gamma_X; X\rangle$, $\langle \Gamma_A; A\rangle$ and $\langle \Gamma_{A\wedge B}; A \wedge B\rangle$ are arguments in $Arg_{\mathsf{CL}}(S)$. In particular, we have that

$$\Gamma_X = \{A, A \supset X\}$$

$$\Gamma_A = \{\neg X, \neg X \supset A\}$$

$$\Gamma_{A\wedge B} = \{\neg X, \neg X \supset A, B\}$$

Since $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$ $(X \Rightarrow \neg \bigwedge \Gamma_A)$ and $\Gamma_A \subseteq \Gamma_{A\wedge B}$, the attack principle **(A.$\wedge$)** holds and $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$.

The *premise* of this instantiation of **(A.$\wedge$)** is "$\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$", the *conclusion* is "$\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$".

Attack Principle	Premise	Conclusion
(A.$\wedge$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$ or $\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\wedge B}; A \wedge B\rangle$
(C.$\wedge$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\wedge B}; A \wedge B\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$ or $\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$
(A.$\vee$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\vee B}; A \vee B\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$
(C.$\vee$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\vee B}; A \vee B\rangle$
(A.$\supset$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$ and $\langle\Gamma_X; X\rangle \not\xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\supset B}; A \supset B\rangle$
(C.$\supset$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{A\supset B}; A \supset B\rangle$	$\langle\Gamma_X; X\rangle \not\xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$
(A.$\neg$)	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$	$\langle\Gamma_X; X\rangle \not\xrightarrow{\mathscr{R}} \langle\Gamma_{\neg A}; \neg A\rangle$
(C.$\neg$)	$\langle\Gamma_X; X\rangle \not\xrightarrow{\mathscr{R}} \langle\Gamma_A; A\rangle$	$\langle\Gamma_X; X\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_{\neg A}; \neg A\rangle$

Table 5.5: Premises and Conclusions of Attack Principles Instantiated in Sequent-Based Argumentation Frames

In Table 5.5, we summarise the *premise* and the *conclusion* of the attack principles in CAP (see Section 4.3) instantiated in sequent-based argumentation frames. In Table 5.5, the $\mathscr{R}$ stands for a generic attack relation.

In the previous example, the attacking condition of the conclusion *logically follows* from the attacking condition of the premise. In the premise we have that $\langle\Gamma_X; X\rangle \xrightarrow{[Def]} \langle\Gamma_A; A\rangle$ because $X \vdash \neg \bigwedge \Gamma_A$ (the attacking condition of the premise). Since $\neg \bigwedge \Gamma_A \vdash \neg \bigwedge \Gamma_{A\wedge B}$, then $X \vdash \neg \bigwedge \Gamma_{A\wedge B}$ (the attacking condition of the conclusion). In the following definition, we formalise this concept of attacking conditions that logically follow from other attacking conditions.

Definition 5.2.1. *Let $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ be a sequent-based argumentation frame and AP an attack principle. Let us consider any instantiation of AP on the arguments $Arg_{\mathfrak{L}}(S)$. The attacking conditions of the conclusion of AP* logically follow *from the attacking conditions of the support of AP if from the latter we can derive, in the logic $\mathfrak{L}$, the former ones.*

In any of the attack principles concerning the three binary connectives $\wedge$, $\vee$ and $\supset$ we have that either in the premise or in the conclusion of the attack principle, an argument with claim $A \circ B$ with $\circ \in \{\wedge, \vee, \supset\}$ is attacked. A and B stand for any $\mathcal{L}$-formula of the underlying logic. We say that an attack principle is *instantiated on logically independent* *arguments* if the arguments present in the attack principle whose claims are either A or B are logically independent arguments as defined in Definition 3.2.6.

For example, if we consider the attack principle $(\mathbf{C.}\vee)$, we have that it is instantiated on logically independent arguments if the arguments attacked in the instantiation of the premise are logically independent.

Definition 5.2.2 (Direct Preservation of Strength)**.** *An attack principle AP* directly preserves strength *if for any instantiation of AP on logically independent arguments in a sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S)$ whenever the attacked arguments in the premise are strong, then there is a corresponding instantiation of the conclusion where the arguments attacked are also strong and the attacking conditions of the conclusion logically follow from the attacking conditions of the premise.*

From Definition 5.2.2 it follows that an attack principle AP *does not preserve strength* if there is an instantiation of AP on logically independent arguments such that whenever the attacked arguments in the premise are strong, then the arguments attacked in any corresponding instantiation of the conclusion whose attacking conditions logically follow from the attacking conditions of the premise are weak.

Proposition 5.2.1. *Let $\mathscr{A}$ be a set of attack relations.*

If $[Def] \in \mathscr{A}$, the attack principles that directly preserve strength in any sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ are the following:

$$\mathsf{AP}_{[Def]} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg), (\mathbf{C.}\neg)\}.$$

If $[Ucut] \in \mathscr{A}$, the attack principles that preserve strength in any sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S)$ are the following:

$$\mathsf{AP}_{[Ucut]} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg), (\mathbf{C.}\neg)\}.$$

If $[Reb] \in \mathscr{A}$, the attack principles that preserve strength in any sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S)$ are the following:

$$\mathsf{AP}_{[Reb]} = \{(\mathbf{A.}\vee), (\mathbf{A.}\neg), (\mathbf{C.}\neg)\}.$$

If $[C\text{-}Reb\text{-}1] \in \mathscr{A}$, or $[D\text{-}Reb] \in \mathscr{A}$, or $[I\text{-}Reb] \in \mathscr{A}$, the attack principles that preserve strength in any sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S)$ are the following:

$$\mathsf{AP}_{[C\text{-}Reb\text{-}1]} = \mathsf{AP}_{[D\text{-}Reb]} = \mathsf{AP}_{[I\text{-}Reb]} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg), (\mathbf{C.}\neg)\}.$$

Proof. We will analyse here only the case in which the *defeat* attack relation is in $\mathscr{A}$. The other cases are similar.

(A.$\wedge$)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_A, A\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_A, A\rangle$ is strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$. Thus, $X \vdash \neg \bigwedge \Gamma_A$.

Let us consider now the argument $\langle \Gamma_A \cup \{B\}; A \wedge B\rangle$. Since $\Gamma_A \subseteq \Gamma_A \cup \{B\}$, then $\neg \bigwedge \Gamma_A \vdash \neg \bigwedge(\Gamma_A \cup \{B\})$ and $X \vdash \neg \bigwedge(\Gamma_A \cup \{B\})$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A \cup \{B\}; A \wedge B\rangle$ and the attacking condition of this latter attack logically follows from the attacking condition of the hypothesis. If we show that $\langle \Gamma_A \cup \{B\}; A \wedge B\rangle$ is strong, then **(A.$\wedge$)** directly preserves strength.

From the hypothesis, we have that Γ_A is consistent. Since we are working with logically independent arguments, $\Gamma_A \nvdash \neg B$ and $B \nvdash \neg \bigwedge \Gamma_A$. If there is some Γ'_A such that $\Gamma_A \vdash \bigwedge \Gamma'_A$ and $B \vdash \neg \bigwedge \Gamma'_A$, it follows that $B \vdash \neg \bigwedge \Gamma_A$, against our hypothesis. Thus, $\Gamma_A \cup \{B\}$ is consistent.

Since $\Gamma_A \equiv A$ and $B \equiv B$, we have that $\Gamma_A \wedge B \vdash A \wedge B$ and $A \wedge B \vdash \Gamma_A \wedge B$, i.e. $\Gamma_A \wedge B \equiv A \wedge B$. From Proposition 3.5.1 it follows that $\Gamma_A \wedge B$ is a $\vdash$-minimal support set for $A \wedge B$.

(A.$\vee$)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_{A\vee B}, A \vee B\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_{A\vee B}, A \vee B\rangle$ is strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, i.e. $X \vdash \neg \bigwedge \Gamma_{A\vee B}$.

Let us consider the arguments $\langle \Gamma_{A\vee B} \cup \{\neg B \vee A\}; A\rangle$ and $\langle \Gamma_{A\vee B} \cup \{\neg A \vee B\}; B\rangle$. Since $\Gamma_{A\vee B} \subseteq \Gamma_A \cap \Gamma_B$, $X \vdash \neg \bigwedge(\Gamma_{A\vee B} \cup \{\neg B \vee A\})$ and $X \vdash \neg \bigwedge(\Gamma_{A\vee B} \cup \{\neg A \vee B\})$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B} \cup \{\neg B \vee A\}; A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B} \cup \{\neg A \vee B\}; B\rangle$ and the attacking conditions of these latest attacks logically follow from the attacking condition of the hypothesis.

If we show that both $\langle \Gamma_{A\vee B} \cup \{\neg B \vee A\}; A\rangle$ and $\langle \Gamma_{A\vee B} \cup \{\neg A \vee B\}; B\rangle$ are strong, then **(A.$\vee$)** directly preserves strength. We will analyse only the first case.

Since $\Gamma_{A\vee B}$ is consistent, in order to have $\Gamma_{A\vee B} \cup \{\neg B \vee A\}$ inconsistent, either $\Gamma_{A\vee B} \vdash B \wedge \neg A$, or $\neg B \vee A \vdash \neg \bigwedge \Gamma_{A\vee B}$.

In the first case, from $\Gamma_{A\vee B} \vdash B \wedge \neg A$ it follows $\Gamma_{A\vee B} \vdash B$ and $\Gamma_{A\vee B} \vdash \neg A$. Since $\Gamma_{A\vee B}$ is a $\vdash$-minimal support set for $A \vee B$, from Proposition 3.5.1 it follows that $\Gamma_{A\vee B} \equiv A \vee B$. Therefore, $A \vee B \vdash \Gamma_{A\vee B}$ and from $\Gamma_{A\vee B} \vdash B$ it follows that $A \vee B \vdash B$, but this cannot happen since $A \perp\!\!\!\perp_{\mathfrak{L}} B$.

In the second case, from $\neg B \vee A \vdash \neg \bigwedge \Gamma_{A\vee B}$ it follows both $\neg B \vdash \neg \bigwedge \Gamma_{A\vee B}$ and $A \vdash \neg \bigwedge \Gamma_{A\vee B}$. Therefore, $\Gamma_{A\vee B} \vdash B$ and $\Gamma_{A\vee B} \vdash A$, but for the same reason of the first point, this brings to a contradiction because $\Gamma_{A\vee B}$ is a $\vdash$-minimal support set for $A \vee B$ and $A \perp\!\!\!\perp_{\mathfrak{L}} B$.

Since $\Gamma_{A\vee B} \equiv A \vee B$ and $(A \vee B) \wedge (A \vee \neg B) \equiv A$, $\Gamma_{A\vee B} \cup \{A \vee \neg B\}$ is a $\vdash$-minimal support set for A.

(C.$\vee$)-[Def] Suppose that $\langle \Gamma_X, X\rangle$, $\langle \Gamma_A, A\rangle$ and $\langle \Gamma_B, B\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_A, A\rangle$ and $\langle \Gamma_B, B\rangle$ are strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$, i.e. $X \vdash \neg \bigwedge \Gamma_A$ and $X \vdash \neg \bigwedge \Gamma_B$.

Let us consider the argument $\langle (\bigwedge \Gamma_A) \vee (\bigwedge \Gamma_B); A \vee B\rangle$. Since $X \vdash \neg \bigwedge \Gamma_A$ and $X \vdash \neg \bigwedge \Gamma_B$, then $X \vdash (\neg \bigwedge \Gamma_A) \wedge (\neg \bigwedge \Gamma_B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle (\bigwedge \Gamma_A) \vee (\bigwedge \Gamma_B); A \vee B\rangle$ and the attacking condition of this latest attack logically follows from the attacking conditions of the hypothesis.

If we show that $\langle (\bigwedge \Gamma_A) \vee (\bigwedge \Gamma_B); A \vee B\rangle$ is a strong argument, then **(C.$\vee$)** directly preserves strength.

Since both Γ_A and Γ_B are consistent, then $(\bigwedge \Gamma_A) \vee (\bigwedge \Gamma_B)$ is consistent.

From the hypothesis we have that Γ_A is a $\vdash$-minimal support set for A and Γ_B is a $\vdash$-minimal support set for B. Thus, by Proposition 3.5.1, $\bigwedge \Gamma_A \equiv A$ and $\bigwedge \Gamma_B \equiv B$. From $\Gamma_A \vdash A$ and $A \vdash A \vee B$, it follows $\Gamma_A \vdash A \vee B$. In a similar way we obtain that $\Gamma_B \vdash A \vee B$. Thus, $\bigwedge \Gamma_A \vee \bigwedge \Gamma_B \vdash A \vee B$. Since $A \vdash \bigwedge \Gamma_A$ and $B \vdash \bigwedge \Gamma_B$, we have $A \vdash \bigwedge \Gamma_A \vee \bigwedge \Gamma_B$ and $B \vdash \bigwedge \Gamma_A \vee \bigwedge \Gamma_B$. Thus, the following holds: $A \vee B \vdash \bigwedge \Gamma_A \vee \bigwedge \Gamma_B$. In conclusion $A \vee B \equiv \bigwedge \Gamma_A \vee \bigwedge \Gamma_B$ and by Proposition 3.5.1 we have that $\bigwedge \Gamma_A \vee \bigwedge \Gamma_B$ is a $\vdash$-minimal support set for $A \vee B$.

(C.$\supset$)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_{A \supset B}, A \supset B\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_{A \supset B}, A \supset B\rangle$ is strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A \supset B}, A \supset B\rangle$, i.e. $X \vdash \neg \bigwedge \Gamma_{A \supset B}$.

Let us consider now the argument $\langle \Gamma_{A \supset B} \cup \{A \vee B\}, B\rangle$. Since $\Gamma_{A \supset B} \subseteq \Gamma_{A \supset B} \cup \{A \vee B\}$, then $X \vdash \neg \bigwedge(\Gamma_{A \supset B} \cup \{A \vee B\})$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A \supset B} \cup \{A \vee B\}, B\rangle$ and the attacking condition of this latest attack logically follows from the attacking condition of the hypothesis.

If we show that the argument $\langle \Gamma_{A \supset B} \cup \{A \vee B\}, B\rangle$ is strong, then **(C.$\supset$)** direclty preserves strength.

From the hypothesis we have that $\Gamma_{A \supset B}$ is consistent. Thus, $\Gamma_{A \supset B} \cup \{A \vee B\}$ is inconsistent if either $\Gamma_{A \supset B} \vdash \neg(A \vee B)$, or $A \vee B \vdash \neg \bigwedge \Gamma_{A \supset B}$.

In the first case, from $\Gamma_{A \supset B} \vdash \neg A \wedge \neg B$, it follows that $\Gamma_{A \supset B} \vdash \neg A$. Therefore, $\Gamma_{A \supset B} \vdash \neg A$, $\neg A \vdash A \supset B$ and $\Gamma_{A \supset B} \vdash A \supset B$. From the hypothesis we have $\Gamma_{A \supset B} \equiv A \supset B$, but $\neg A \not\equiv A \supset B$ and $\Gamma_{A \supset B} \nvdash \neg A \wedge \neg B$

In the second case, from $A \vee B \vdash \neg \bigwedge \Gamma_{A \supset B}$, it follows $\Gamma_{A \supset B} \vdash \neg A$ and $\Gamma_{A \supset B} \vdash \neg B$. Thus, we have $\Gamma_{A \supset B} \vdash \neg A \wedge \neg B$. From the hypothesis we have that $\bigwedge \Gamma_{A \supset B} \equiv A \supset B$, but $\neg A \wedge \neg B \not\equiv A \supset B$ and $A \vee B \nvdash \neg \bigwedge \Gamma_{A \supset B}$.

To show that $\Gamma_{A \supset B} \cup \{A \vee B\}$ is a $\vdash$-minimal support set for B, using the result of Proposition 3.5.1, it is sufficient to show that $\Gamma_{A \supset B} \cup \{A \vee B\}$ is logically equivalent to B. From the hypothesis we have that $\Gamma_{A \supset B} \equiv A \supset B$. Thus, $\Gamma_{A \supset B} \wedge (A \vee B) \equiv (A \supset B) \wedge (A \vee B)$ holds, and $(A \supset B) \wedge (A \vee B) \vdash B$. In addition, we have that $B \vdash A \vee B$ and $B \vdash A \supset B$, from which it follows $B \vdash (A \vee B) \wedge (A \supset B)$. Therefore, $(A \supset B) \wedge (A \vee B) \equiv B$ holds and $\Gamma_{A \supset B} \cup \{A \vee B\}$ is a $\vdash$-minimal support set for B.

(A.¬)-[Def] Since in the conclusion of the attack principle, there is no argument attacked, **(A.¬)** directly preserves strength vacuously.

(C.¬)-[Def] Since in the premise of the attack principle, there is no argument attacked, **(C.¬)** directly preserves strength vacuously. □

Proposition 5.2.2. *Let $\mathscr{A}$ be a set of attack relations.*

If $[Def] \in \mathscr{A}$*, the attack principles that* do not *directly preserve strength in any sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ *are the following:*

$$\overline{\mathsf{AP}}_{[Def]} = \{(\mathbf{C.}\wedge), (\mathbf{A.}\supset)\}.$$

If $[Ucut] \in \mathscr{A}$*, the attack principles that* do not *directly preserve strength in any sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S)$ *are the following:*

$$\overline{\mathsf{AP}}_{[Ucut]} = \{(\mathbf{C.}\wedge), (\mathbf{C.}\vee), (\mathbf{A.}\supset)\}.$$

If $[Reb] \in \mathscr{A}$*, the attack principles that* do not *directly preserve strength in any sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S)$ *are the following:*

$$\overline{\mathsf{AP}}_{[Reb]} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\supset), (\mathbf{A.}\vee), (\mathbf{A.}\supset), (\mathbf{C.}\supset)\}.$$

If $[C\text{-}Reb\text{-}1] \in \mathscr{A}$*, or* $[D\text{-}Reb] \in \mathscr{A}$*, or* $[I\text{-}Reb] \in \mathscr{A}$*, the attack principles that* do not *directly preserve strength in any sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S)$ *are the following:*

$$\overline{\mathsf{AP}}_{[C\text{-}Reb\text{-}1]} = \overline{\mathsf{AP}}_{[D\text{-}Reb]} = \overline{\mathsf{AP}}_{[I\text{-}Reb]} = \{(\mathbf{C.}\wedge), (\mathbf{A.}\supset)\}.$$

Proof. We will analyse here only the case in which the *defeat* attack relation is in $\mathscr{A}$. The other cases are similar.

(C.∧)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_{A\wedge B}, A\wedge B\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_{A\wedge B}, A\wedge B\rangle$ is strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}, A\wedge B\rangle$. Thus, $X \vdash \neg \bigwedge \Gamma_{A\wedge B}$.

Suppose there is an instantiation of the conclusion such that the arguments attacked are strong and that the attacking conditions of the conclusion logically follow from the attacking condition of the premise, i.e. either $\neg \bigwedge \Gamma_{A\wedge B} \vdash \neg \bigwedge \Gamma_A$, or $\neg \bigwedge \Gamma_{A\wedge B} \vdash \neg \bigwedge \Gamma_B$ for some strong argument $\langle \Gamma_A, A\rangle$ and $\langle \Gamma_B, B\rangle$ in $Arg_{\mathfrak{L}}(S)$. Therefore, either $\Gamma_A \vdash \bigwedge \Gamma_{A\wedge B}$, or $\Gamma_B \vdash \bigwedge \Gamma_{A\wedge B}$. If $\Gamma_A \vdash \bigwedge \Gamma_{A\wedge B}$ (the case in which $\Gamma_B \vdash \bigwedge \Gamma_{A\wedge B}$ is similar), from $A \vdash \bigwedge \Gamma_A$, it follows that $A \vdash A \wedge B$, but this cannot happen because $A \perp\!\!\!\perp_{\mathfrak{L}} B$.

(A.$\supset$)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_B, B\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that $\langle \Gamma_B, B\rangle$ is strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$, i.e. $X \vdash \neg \bigwedge \Gamma_B$.

Suppose there is an instantiation of the conclusion such that the argument attacked is strong and that the attacking condition of the conclusion logically follows from the attacking condition of the premise, i.e. $\neg \bigwedge \Gamma_B \vdash \neg \bigwedge \Gamma_{A\supset B}$ for some strong argument $\langle \Gamma_{A\supset B}, A \supset B\rangle$ in $Arg_{\mathfrak{L}}(S)$. From $\neg \bigwedge \Gamma_B \vdash \neg \bigwedge \Gamma_{A\supset B}$ it follows $\Gamma_{A\supset B} \vdash \bigwedge \Gamma_B$. Since we are under the assumption that $\langle \Gamma_{A\supset B}, A \supset B\rangle$ is strong, from Proposition 3.5.1 we have that $\bigwedge \Gamma_{A\supset B} \equiv A \supset B$. Thus, $A \supset B \vdash \bigwedge \Gamma_{A\supset B}$, $\Gamma_{A\supset B} \vdash \Gamma_B$ and $A \supset B \vdash B$, but this cannot happen because $A \perp\!\!\!\perp_{\mathfrak{L}} B$. □

The case of the attack principles concerning the negation is slightly different from the others. For these two cases, direct preservation of strength is satisfied vacuously. Thus, we use *direct justification* to sort out the weak from the strong attack principle.

Definition 5.2.3 (Direct Justification)**.** *The attack principle* **(A.¬)** *is* directly justified *if for any instantiation of* **(A.¬)** *on a sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S)$*, whenever the attacking and the attacked argument in the premise are strong, there is a corresponding instantiation of the conclusion where the argument explicitly not attacked is also strong. Besides, the not-attacking condition of the conclusion logically follows from the attacking condition of the premise.*

The attack principle **(C.¬)** *is* directly justified *if for any instantiation of* **(C.¬)** *on a sequent-based argumentation frame* $AF_{\mathfrak{L},\mathscr{A}}(S)$*, whenever the not-attacking and the not-attacked argument in the premise are strong, there is a corresponding instantiation of the conclusion where the argument attacked is also strong. Besides, the attacking condition of the conclusion logically follows from the not-attacking condition of the premise.*

Therefore an attack principle AP is *not directly justified* if there is an instantiation of AP on a sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S)$ such that whenever both the (not-)attacking and the (not-)attacked arguments in the premise are strong, for any instantiation of the conclusion, either the (not-)attacking conditions of the conclusion do not logically follow from the (not-)attacking conditions of the premise, or the arguments attacked or explicitly not attacked are weak.

Proposition 5.2.3. *Let* $\mathscr{A}$ *be a set of attack relations and* $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A})\rangle$ *a sequent-based argumentation frame. If* $[Def] \in \mathscr{A}$*, or* $[Ucut] \in \mathscr{A}$*, or* $[Reb] \in \mathscr{A}$*, or* $[C\text{-}Reb\text{-}1] \in \mathscr{A}$*, or* $[D\text{-}Reb] \in \mathscr{A}$*, or* $[I\text{-}Reb] \in \mathscr{A}$ *the attack principle* **(A.¬)** *is* direclty justified *and* **(C.¬)** *is* not-direclty justified*, i.e.:*

$$\mathsf{AP}_{[Def]} = \mathsf{AP}_{[Ucut]} = \mathsf{AP}_{[Ref]} = \mathsf{AP}_{[C\text{-}Reb\text{-}1]} = \mathsf{AP}_{[D\text{-}Reb]} = \mathsf{AP}_{[I\text{-}Ref]} = \{(\mathbf{A.\neg})\}$$

and

$$\overline{\mathsf{AP}}_{[Def]} = \overline{\mathsf{AP}}_{[Ucut]} = \overline{\mathsf{AP}}_{[Ref]} = \overline{\mathsf{AP}}_{[C\text{-}Reb\text{-}1]} = \overline{\mathsf{AP}}_{[D\text{-}Reb]} = \overline{\mathsf{AP}}_{[I\text{-}Ref]} = \{(\mathbf{C.\neg})\}.$$

Proof. We analyse here only the case in which the *defeat* attack relation is in $\mathscr{A}$. The other cases are similar.

(A.¬)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_A, A\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that they are both strong and that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$, i.e. $X \vdash \neg \bigwedge \Gamma_A$. Since $\langle \Gamma_A, A\rangle$ is strong, $\bigwedge \Gamma_A \equiv A$ and $X \vdash \neg A$.

Since $X \vdash \neg A$ and X is consistent, $X \not\vdash A$. Therefore for any strong argument $\langle \Gamma_{\neg A}, \neg A\rangle$, $\langle \Gamma_X, X\rangle \overset{[Def]}{\not\longrightarrow} \langle \Gamma_{\neg A}, \neg A\rangle$ and $\mathbf{(A.\neg)} \in \mathsf{AP}_{[Def]}$.

(C.¬)-[Def] Suppose that $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_A, A\rangle$ are arguments in $Arg_{\mathfrak{L}}(S)$, that they are both strong and that $\langle \Gamma_X, X\rangle \overset{[Def]}{\not\longrightarrow} \langle \Gamma_A, A\rangle$, i.e. $X \not\vdash \neg \bigwedge \Gamma_A$. However from $X \not\vdash \neg \bigwedge \Gamma_A$ it does not follow that $X \vdash \neg \bigwedge \Gamma_{\neg A}$ for any argument $\langle \Gamma_{\neg A}, \neg A\rangle$ and $\mathbf{(C.\neg)} \in \overline{\mathsf{AP}}_{[Def]}$. □

Attack Relation	(A.∧)	(C.∧)	(A.∨)	(C.∨)	(A.⊃)	(C.⊃)	(A.¬)	(C.¬)
[Def]	✓		✓	✓		✓	✓	
[Ucut]	✓		✓			✓	✓	
[Reb]				✓			✓	
[C-Reb-1]	✓		✓	✓		✓	✓	
[D-Reb]	✓		✓	✓		✓	✓	
[I-Reb]	✓		✓	✓		✓	✓	

Table 5.6: Attack Principles Justified by Definition 5.2.2 or Definition 5.2.3

Among the attack principles justified by Definition 5.2.2 or Definition 5.2.3, some of them are the same justified by the modal interpretation of the attack relation introduced in Section 4.4. Since the attack principles that we have analysed are defined only in terms of the conclusions of the arguments, it is not surprising that the attack functions for which we can find the correspondence on the attack principles justified, are exactly those defined in terms of the conclusions of arguments and *defeat*, i.e. *compact rebuttal 1* ([C-Reb-1]), *direct rebuttal* ([D-Reb]) and *indirect rebuttal* ([I-Reb]) (see Definition 3.3.4). This is not the case for *rebuttal* ([Reb]), but it is also not surprising given its very demanding definition (see Definition 3.3.3).

In the following section we take advantage of the correspondence between the attack principles that preserve strength through Definition 5.2.2 and the modal interpretation of the attack relation, to recover an alternative semantics for LM.

5.3 Alternative Semantics for LM

The attack principles justified through Definition 5.2.2 or Definition 5.2.3 by the attack relations [Def], [C-Reb-1], [D-Reb] and [I-Reb] are **(A.∧)**, **(A.∨)**, **(C.∨)**, **(A.⊃)** and

(A.¬) (see Table 5.6) and are the same attack principles justified by the modal interpretation of the attack relation introduced in Section 4.4. Following Definition 4.2.3, we define an argumentative consequence relation instantiating the attack relation in the following way:

Definition 5.3.1 (Argumentative $Attack(\mathscr{A})$-consequence Relation). *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and Γ and Δ finite sets of propositional $\mathcal{L}$-formulas representing claims of arguments. Δ is an* argumentative $Attack(\mathscr{A})$-consequence *of Γ ($\Gamma \models_{arg}^{Attack(\mathscr{A})} \Delta$) if for every sequent-based argumentation frame $AF_{\mathfrak{L},\mathscr{A}}(S) = \langle Arg_{\mathfrak{L}}(S), Attack(\mathscr{A}) \rangle$ that contains arguments whose claims are in $\{\Gamma, \Delta\}$, whenever an argument, for some $\mathscr{R} \in \mathscr{A}$, $\mathscr{R}$-attacks every argument with its claim in Δ, then it also $\mathscr{R}$-attacks at least one argument whose claim is in Γ*

We can give the same kind of definition even for any other attack relations, but in the specific case of [C-Reb-1], [D-Reb] and [I-Reb], the following result holds.

Theorem 5.3.1. *Let $Attack(\mathscr{A})_{\mathsf{MAP}} = \{[Def], [C\text{-}Reb\text{-}1], [D\text{-}Reb], [I\text{-}Reb]\}$ and Γ and Δ sets of claims of arguments. $\Gamma \models_{arg}^{\mathsf{MAP}} \Delta$ iff $\Gamma \models_{arg}^{Attack(\mathscr{A})_{\mathsf{MAP}}} \Delta$.*

Proof. For the left-to-right direction we have, by assumption that $\Gamma \models_{\mathrm{arg}}^{\mathsf{MAP}} \Delta$, i.e. whenever an argument, suppose with claim X, attacks every argument whose conclusion is in Δ, then it also attacks at least one argument whose claim is in Γ. By definition of $\models_{\mathrm{arg}}^{\mathsf{MAP}}$ we have that the only attack principles allowed to be used to find which element of Γ is attacked by the argument with claim X are **(A.∧)**, **(A.∨)**, **(C.∨)**, **(C.⊃)** and **(A.¬)**. Since all the attack relations in $Attack(\mathscr{A})_{\mathsf{MAP}}$ satisfy these attack principles we have that whenever an argument, suppose with claim X' attacks all the arguments whose conclusion is in Δ, then by applying the attack principles used to show that $\Gamma \models_{\mathrm{arg}}^{\mathsf{MAP}} \Delta$, we have that the argument with claim X' also attacks at least one argument whose conclusion is in Γ.

For the other direction, we observe that the attack principles justified by $Attack(\mathscr{A})_{\mathsf{MAP}}$ are precisely those in MAP. Thus, the proof of this direction of the theorem is specular to the other one. ☐

Thanks to Theorem 4.4.3 and by indicating with LM the sequent calculus that arises from dropping the rules $(\neg, l)$, $(\wedge, r)$, and $(\supset, l)$ from LK, we have:

Theorem 5.3.2. *Given Γ and Δ sets of claims of arguments, $\Gamma \models_{arg}^{Attack(\mathscr{A})_{\mathsf{MAP}}} \Delta$ iff $\Gamma \vdash \Delta$ is derivable in LM.*

The interest of Theorem 5.3.2 is twofold, on one side we have a further justification of only some of the attack principles, particularly those that are more naively acceptable. On the other side, we see, not surprisingly, that a possible "logic of arguments" should be weaker than classical logic

5.4 Additional Attack Principles

Attack relations are defined using different parts of the arguments. Some of them are defined in terms of the conclusion of the attacking argument and the support of the attacked one (e.g. [Def]), others only deal with the claims (e.g. [Reb], [I-Reb]), etc. In Table 5.7, we summarise the four different possible ways of defining them. We recall that S stands for the support, and C stands for the claim.

		Attacked Argument	
		S	C
Attacking Argument	S	[C-Def]	[C-Reb-1]
	C	[Def] [Ucut]	[Reb] [D-Reb] [I-Reb]

Table 5.7: Classification of attack relations

We can also do similar reasoning for the attack principles. The ones introduced in Section 4.1 are defined only in terms of claims of arguments. We now introduce attack principles that are defined in terms of the attacking argument's claims and the support of the attacked one. The attack principles are defined over the generic arguments $\langle \Gamma_X; X\rangle$, $\langle \Gamma_A; A\rangle$, $\langle \Gamma_B; B\rangle$, $\langle \Gamma_{A\vee B}; A\vee B\rangle$, $\langle \Gamma_{A\wedge B}; A\wedge B\rangle$, $\langle \Gamma_{A\supset B}; A\supset B\rangle$ and $\langle \Gamma_{\neg A}; \neg A\rangle$. We say that A attacks Γ_X ($A \longrightarrow \Gamma_X$) if there is an argument with claim A that attacks an argument with support Γ_X. To analyse situations more general as possible, we assume that the arguments attacked in the premise of the instantiation of the attack principles are not attacked by tautological arguments (see Section 3.2). Tautological arguments, following Definition 3.2.7, are the most conservative arguments, i.e. the *least informative* ones. If a tautological argument attacks another argument, then this last one is also attacked by all the other arguments of the frame.

($\mathbf{B^{CS}}.\wedge$) If $A \longrightarrow \Gamma_X$ or $B \longrightarrow \Gamma_X$, then $A\wedge B \longrightarrow \Gamma_X$.

($\mathbf{D^{CS}}.\wedge$) If $A\wedge B \longrightarrow \Gamma_X$, then $A \longrightarrow \Gamma_X$ or $B \longrightarrow \Gamma_X$.

($\mathbf{B^{CS}}.\vee$) If $A\vee B \longrightarrow \Gamma_X$, then $A \longrightarrow \Gamma_X$ and $B \longrightarrow \Gamma_X$.

($\mathbf{D^{CS}}.\vee$) If $A \longrightarrow \Gamma_X$ and $B \longrightarrow \Gamma_X$, then $A\vee B \longrightarrow \Gamma_X$.

($\mathbf{B^{CS}}.\supset$) If $A \not\longrightarrow \Gamma_X$ and $B \longrightarrow \Gamma_X$, then $A\supset B \longrightarrow \Gamma_X$.

($\mathbf{D^{CS}}.\supset$) If $A\supset B \longrightarrow \Gamma_X$, then $A \not\longrightarrow \Gamma_X$ and $B \longrightarrow \Gamma_X$.

($\mathbf{B^{CS}}.\neg$) If $A \longrightarrow \Gamma_X$, then $\neg A \not\longrightarrow \Gamma_X$.

($\mathbf{D^{CS}}.\neg$) If $A \not\longrightarrow \Gamma_X$, then $\neg A \longrightarrow \Gamma_X$.

We now analyse these attack principles interpreting the attack relation with *defeat* (see Definition 3.3.1).

($\mathbf{B^{CS}}.\wedge$) If $\langle\Gamma_A;A\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$, then $A \vdash \neg\bigwedge\Gamma_1$. Therefore $A \wedge B \vdash \neg\bigwedge\Gamma_1$, i.e. $\langle\Gamma_{A\wedge B};A\wedge B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$.

($\mathbf{D^{CS}}.\wedge$) If $\langle\Gamma_{A\wedge B};A\wedge B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$, then $A\wedge B \vdash \neg\bigwedge\Gamma_1$. Howerver, we cannot deduce either $A \vdash \neg\bigwedge\Gamma_1$ or $B \vdash \neg\bigwedge\Gamma_1$, i.e. $\langle\Gamma_A;A\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$ and $\langle\Gamma_B;B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$ respectively. Therefore, the conclusion of the attack principle does not follow from the premise.

($\mathbf{B^{CS}}.\vee$) If $\langle\Gamma_{A\vee B};A\vee B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$, then $A\vee B \vdash \neg\bigwedge\Gamma_1$. Therefore, $A \vdash \neg\bigwedge\Gamma_1$ and $B \vdash \neg\bigwedge\Gamma_1$, i.e. $\langle\Gamma_A;A\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$ and $\langle\Gamma_B;B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$.

($\mathbf{D^{CS}}.\vee$) If $\langle\Gamma_A;A\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$ and $\langle\Gamma_B;B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$, then $A \vdash \neg\bigwedge\Gamma_1$ and $B \vdash \neg\bigwedge\Gamma_1$. Therefore, $A\vee B \vdash \neg\bigwedge\Gamma_1$, i.e. $\langle\Gamma_{A\vee B};A\vee B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$.

($\mathbf{B^{CS}}.\supset$) If $\langle\Gamma_{A\supset B};A\supset B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$, then $A\supset B \vdash \neg\bigwedge\Gamma_X$, from which it follows $\neg A\vee B \vdash \neg\bigwedge\Gamma_X$. Therefore, $\neg A \vdash \neg\bigwedge\Gamma_X$ and $B \vdash \neg\bigwedge\Gamma_X$. Since $\neg A \vdash \neg\bigwedge\Gamma_X$, we can deduce that $A \nvdash \neg\bigwedge\Gamma_X$, otherwise we would have a *tautological* argument (with claim $\neg A\vee A$) attacking $\langle\Gamma_X;X\rangle$. In conclusion we have $\langle\Gamma_A;A\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!/\;\;\; \langle\Gamma_X;X\rangle$ and $\langle\Gamma_B;B\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$.

($\mathbf{D^{CS}}.\supset$) If $\langle\Gamma_A;A\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!/\;\;\; \langle\Gamma_X;X\rangle$, then $A \nvdash \neg\bigwedge\Gamma_1$. However we cannot deduce $\neg A \vdash \neg\bigwedge\Gamma_1$ that is needed to show $\neg A\vee B \vdash \neg\Gamma_X$. Therefore, the conclusion of the attack principle does not follow from the premise.

($\mathbf{B^{CS}}.\neg$) If $\langle\Gamma_A;A\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!/\;\;\; \langle\Gamma_X;X\rangle$, then $A \vdash \neg\bigwedge\Gamma_X$. Suppose that $\neg A \vdash \neg\bigwedge\Gamma_X$. Thus, $A\vee\neg A \vdash \neg\bigwedge\Gamma_X$, i.e. $\langle\Gamma_X;X\rangle$ is attacked by a tautological argument, but this goes against our assumptions and $\neg A \nvdash \neg\bigwedge\Gamma_X$, i.e. $\langle\Gamma_{\neg A};\neg A\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!/\;\;\; \langle\Gamma_X;X\rangle$.

($\mathbf{D^{CS}}.\neg$) If $\langle\Gamma_A;A\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!/\;\;\; \langle\Gamma_X;X\rangle$, then $A \nvdash \neg\bigwedge\Gamma_1$. However we cannot deduce that $\neg A \vdash \neg\bigwedge\Gamma_1$, and the attacking condition of $\langle\Gamma_{\neg A};\neg A\rangle \xrightarrow{[Def]} \langle\Gamma_X;X\rangle$ does not directly follow from the premise.

Under the only assumption that tautological arguments do not attack the arguments attacked in the instantiation of the premise of the attack principles, whenever the attack relation is instantiated with defeat, the following CS-attack principles hold:

$$\mathsf{AP}^{CS}_{[Def]} = \{(\mathbf{B^{CS}}.\wedge), (\mathbf{B^{CS}}.\vee), (\mathbf{D^{CS}}.\vee), (\mathbf{D^{CS}}.\supset), (\mathbf{B^{CS}}.\neg)\}.$$

As done for both the in-going and the out-going attack principles (see Definition 4.2.3 and 4.5.1, respectively), we can define an argumentative entailment relation in the case of CS-attack principles.

Definition 5.4.1 (Argumentative $\mathcal{CSP}$-[Def]-consequence Relation)**.** *Let* $\mathcal{CSP}$ *be a set of* CS*-attack principles, and* Γ *and* Δ *finite sets of propositional formulas representing claims of arguments.* Δ *is an* argumentative $\mathcal{CSP}$-$[Def]$-consequence *of* Γ *(*$\Gamma \models_{arg}^{\mathcal{CSP}\text{-}[Def]} \Delta$*) if for every sequent-based argumentation frame* $AF_{\mathsf{CL},[Def]}(S)$ *that contains arguments whose claims are in* $\{\Gamma, \Delta\}$ *and it satisfies all (appropriately relativized) attack principles in* $\mathcal{CSP}$*, whenever every argument whose claim is in* Δ *attacks an argument in* $AF_{\mathsf{CL},[Def]}(S)$*, then there is at least one argument with claim in* Γ *that attacks that same argument.*

Theorem 5.4.1. *Let* Γ *and* Δ *be sets of claims of arguments.* $\Gamma \models_{arg}^{\mathcal{CSP}\text{-}[Def]} \Delta$ *iff* $\Gamma \Rightarrow \Delta$ *is derivable in* LM*.*

We can construct the proof in the same way of the proof of Theorem 4.5.3.

In Appendix B.2 we introduce additional attack principles defined in terms of the support of the attacking argument and the attacked one's conclusion. We investigate under which condition the SC-attack principles are directly justified whenever the attack relation is *Compact-Rebuttal-1*. Even in this case, just by assuming that the involved arguments are not *contradictory*, we can recover an additional argumentative semantics for LM.

We can find an appropriate attack relation for every type of attack principles that justifies the attack principles needed to recover an argumentative semantic sound and complete for LM.

Whenever both the attack principles and the attack relation are defined using the same parts of the arguments involved, we can recover additional argumentative semantics for LM. E.g., in the case of the CS-attack principles, we consider the *defeat* attack relation because they are both defined in terms of the claim of the attacking argument and the support of the attacked one. For the CC-attack principles (those introduced in Section 4.1), an attack relation that allows recovering an argumentative semantics for LM is *defeating rebuttal* (see Definition 3.3.4).

The case in which we consider only the support part of the arguments, both for attack principles and attack relations, differs from the others. Intuitively, the *core* of an argument is its claim while the support has the role of justifying the claim, and it defines the overall strength of the argument. Besides, the same set of formulas can be a support set for two different claims. Thus, even if we can define attack principles only using the support parts of the involved arguments, it is not surprising that their interpretation in a sequent-based setting with the attack relation instantiated as well with attack relations defined only in terms of support, e.g. *compact direct defeat* (see Definition 3.3.4), lead to very poor conclusions. For example, even in the case of the intuitively justifiable attack principle **(A.**$\neg$**)**, the conclusion of the principle does not follow from its premise.

5.5 Attack Principles as Additional Elimination Rules

In Section 3.8.7, we have presented dynamic derivations. Some of the key elements of dynamic derivations are the *sequent elimination rules*.

In Section 5.1, we have investigated under which conditions the conclusion of the (CC-)attack principles logically follows from the premise. Thus, once through the application of an elimination rule, a certain argument is attacked; implicitly, all the arguments satisfying the conditions of tables 5.1-5.4 are also attacked. For example, suppose to use, in a dynamic derivation, the following sequent elimination rule:

$$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \tag{5.1}$$

Thus, $\langle \Gamma_1; \psi_1 \rangle$ implicitly attacks also the argument $\langle \Gamma_3; \psi_2 \wedge \psi_3 \rangle$ with $\Gamma_2 \subseteq \Gamma_3$. Thanks to the condition of Table 5.1, to conclude that the argument $\langle \Gamma_1; \psi_1 \rangle$ attacks $\langle \Gamma_3; \psi_2 \wedge \psi_3 \rangle$ we do not need to check the attacking condition, but only the inclusion condition of the support sets. In this specific case, the following additional elimination rule holds.

$$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \qquad \Gamma_2 \subseteq \Gamma_3 \qquad \Gamma_3 \Rightarrow \psi_3}{\Gamma_3 \not\Rightarrow \psi_2 \wedge \psi_3} \tag{5.2}$$

For each attack principle and attack relation, we can define additional sequent elimination rules. The other additional elimination rules referring to $(\mathbf{A.}\wedge)$ are the following:

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg \bigwedge \Gamma_A \qquad \Gamma_A \subseteq \Gamma_{A\wedge B} \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{Def -}(\mathbf{A.}\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \equiv \neg \bigwedge \Gamma'_A \qquad \Gamma'_A \subseteq \Gamma_{A\wedge B} \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{Ucut -}(\mathbf{A.}\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \equiv \neg A \qquad A \equiv B \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{Reb -}(\mathbf{A.}\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad \Gamma_1 \Rightarrow \neg A \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{C-Reb-1 -}(\mathbf{A.}\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg A \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{D -Reb-}(\mathbf{A.}\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \varphi \qquad A \Rightarrow \neg\varphi \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_4 \not\Rightarrow A \ \wedge B} \quad [\text{I-Reb -}(\mathbf{A.}\wedge)]$$

Additional elimination rules that refer to the *defeat* attack relation are presented in Appendix B.3.

We denote with $\mathsf{Co}(\mathscr{R}, AP)$ the conditions the arguments need to satisfy to apply the additional elimination rule [$\mathscr{R}$-AP], e.g. $\mathsf{Co}(Def, (\mathbf{A.}\wedge)) = (X \Rightarrow \neg \bigwedge \Gamma_A, \Gamma_A \subseteq \Gamma_{A\wedge B})$. The function π_i is the usual projection function, thus $\pi_1(\mathsf{Co}(Def, (\mathbf{A.}\wedge))) = X \Rightarrow \neg \bigwedge \Gamma_A$ and $\pi_2(\mathsf{Co}(Def, (\mathbf{A.}\wedge))) = \Gamma_A \subseteq \Gamma_{A\wedge B}$. We will use this notation in the following section.

5.6 Attack Principles in Dynamic Derivations

The definition of additional elimination rules can be relevant for reducing the computational complexity of the dynamic proofs. In the following example, we see the comparison of two dynamic derivations. On the left, the derivation follows Arieli-Straßer's definition (see Definition 3.8.7), while the one on the right also uses an additional elimination rule. We observe that the final result is identical except for the length of the derivation that is shorter in the one that uses the additional elimination rule.

1.	$p \Rightarrow p$	Axiom		1.	$p \Rightarrow p$	Axiom	
2.	$\neg p \Rightarrow \neg p$	Axiom		2.	$\neg p \Rightarrow \neg p$	Axiom	
3.	$p \wedge q \Rightarrow p \wedge q$	Axiom		3.	$p \wedge q \Rightarrow p \wedge q$	Axiom	
4.	$p \Rightarrow \neg\neg p$			4.	$p \Rightarrow \neg\neg p$		
5.	$\neg p \not\Rightarrow \neg p$	Def; 1, 4, 2	$p \Rightarrow p$	5.	$\neg p \not\Rightarrow \neg p$	Def; 1, 4, 2	$p \Rightarrow p$
6.	$p \not\Rightarrow p$	Def; 2, 2, 1	$\neg p \Rightarrow \neg p$	6.	$p \not\Rightarrow p$	Def; 2, 2, 1	$\neg p \Rightarrow \neg p$
7.	$\boldsymbol{\neg p \Rightarrow \neg p, \neg q}$			7.	$\mathsf{S}(\boldsymbol{p \Rightarrow p}) \subseteq \mathsf{S}(\boldsymbol{p \wedge q \Rightarrow p \wedge q})$	$\pi_2(\mathsf{Co}(Def, (\mathbf{A.}\wedge))); 1, 3$	
8.	$\boldsymbol{\neg p \Rightarrow \neg p \vee \neg q}$	$[\Rightarrow, \vee]$, 7		8.	$p \wedge q \not\Rightarrow p \wedge q$	Def-$(\mathbf{A.}\wedge)$; 2, 2, 7, 3	$\neg p \Rightarrow \neg p$
9.	$\boldsymbol{\neg p \Rightarrow \neg(p \wedge q)}$						
10.	$p \wedge q \not\Rightarrow p \wedge q$	Def; 2, 9, 3	$\neg p \Rightarrow \neg p$				

Table 5.8: Dynamic Derivation with Standard (Left Side) and Additional (Right Side) Elimination Rules

The use of the additional elimination rules does not change the result of the derivation. We will show now how derivations that use additional elimination rules are semantically equivalent to the dynamic derivations recalled in Definition 3.8.7. We define an *extended dynamic derivation (EDD)* as a standard dynamic derivation (DD) that can be extended in more ways than those introduced in Definition 3.8.7. An EDD can be extended introducing *condition tuples* that explicits some relations on the support sets or the claims of arguments already introduced. E.g., the tuple with index 7 on the right-side derivation of Table 5.8 is a condition tuple. An additional way to extend an EDD is through the *additional elimination rules*. These new kinds of tuples are standard elimination rules. The only difference is in the justification of the attack. E.g., the tuple with index 8 on the right-side derivation of Table 5.8 is an elimination rule that is justified by an additional elimination rule, in this case Def-$(\mathbf{A.}\wedge)$.

We recall that given an extended dynamic derivation $\mathfrak{D}$, the set $\mathsf{Elim}(\mathfrak{D})$ consists of the sequents that at least once in the derivation are attacked by an attacker which is not already attacked, $\mathsf{Attack}(\mathfrak{D})$ is the set of sequents that attack a sequent in $\mathsf{Elim}(\mathfrak{D})$, and $\mathsf{Accept}(\mathfrak{D})$ is the set of derived sequents in $\mathfrak{D}$ that are not in $\mathsf{Elim}(\mathfrak{D})$. Given a derivation $\mathfrak{D}$, the function $\mathsf{Evaluate}(\mathfrak{D})$ returns the three sets $\mathsf{Attack}(\mathfrak{D})$, $\mathsf{Elim}(\mathfrak{D})$ and $\mathsf{Accept}(\mathfrak{D})$.

Definition 5.6.1 (Extended Dynamic Derivation). *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be an argumentation setting and S a set of formulas in $\mathcal{L}$. An* extended dynamic derivation, *EDD for short, (for $\mathfrak{S}$, based on S) is a simple derivation $\mathcal{D}$ of one of the following forms:*

1. *$\mathcal{D} = \langle T \rangle$ where $T = \langle 1, s, \mathsf{J}, \emptyset \rangle$ is a proof tuple.*

2. *$\mathcal{D}$ is an extension of an extended dynamic derivation by a sequence $\langle T_1, \dots, T_n\rangle$ of introducing tuples of the form $\langle i, s, \mathsf{J}, \emptyset\rangle$, whose derived sequents are not in* $\mathsf{Elim}(\mathcal{D})$.

3. *$\mathcal{D}$ is an extension of an extended dynamic derivation by a sequence $\langle T_1, \dots, T_n\rangle$ of* condition tuples *of the form $\langle i, \mathcal{C}, \mathsf{J}, \emptyset\rangle$, where $\mathcal{C}$ is a relation on the support sets or on the conclusions of arguments and J is a string of the form $\mathsf{J} =$ "$\pi_k(\mathsf{Co}(\mathscr{R}, AP)); i_1, \dots, i_n$".*

4. *$\mathcal{D}$ is an extension of an extended dynamic derivation by a sequence $\langle T_1, \dots, T_n\rangle$ of eliminating tuples of the form $\langle i, \bar{s}, \mathsf{J}, r\rangle$, such that:*

   a) *$\mathcal{D}$ is coherent:* $\mathsf{Elim}(\mathcal{D}) \cap \mathsf{Attack}(\mathcal{D}) = \emptyset$

   b) *the new attacking sequents (the r's) are not $\mathfrak{A}$-attacked by any sequent in* $\mathsf{Accept}(\mathcal{D}) \cap Arg_{\mathfrak{L}}(S)$.

   c) *The justification J is a string of one of the following forms:*

      i. $\mathsf{J} =$ "$\mathscr{R}; i_1, \dots, i_n$" *where $\mathscr{R}$ is an elimination rule.*

      ii. $\mathsf{J} =$ "$\mathscr{R}$-$AP; i_1, \dots, i_n$" *where $\mathscr{R}$-AP is the* additional elimination rule *derived from the elimination rule $\mathscr{R}$ and the attack principle AP.*

Definition 5.6.2 (Equivalent Derivations)**.** *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two derivations, either dynamic derivation or extended dynamic derivation. $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent if* $\mathsf{Evaluate}(\mathfrak{D}) = \mathsf{Evaluate}(\mathfrak{D}')$ *and their attacks' order is the same.*

Given an extended dynamic derivation $\mathfrak{D}$, we say that $\mathsf{Type}(\mathfrak{D}, i) = j$ if the i-th tuple of $\mathfrak{D}$ is defined using the point j of Definition 5.6.1.

Now we are ready to prove the equivalence of dynamic and extended dynamic derivations.

Theorem 5.6.1. *Given $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$ an argumentation setting we have that for any extended dynamic derivation $\mathfrak{D}$ there is an equivalent dynamic derivation $\mathfrak{D}'$ and vice versa.*

Proof. For the left-to-right direction we have to show that given an EDD $\mathfrak{D}$, we can build a DD $\mathfrak{D}'$ that insists on the same argumentation setting and such that $\mathsf{Evaluate}(\mathfrak{D}) = \mathsf{Evaluate}(\mathfrak{D}')$. To construct $\mathfrak{D}' = \langle T'_1, \dots, T'_m\rangle$ we analyse $\mathfrak{D} = \langle T_1, \dots, T_n\rangle$ step by step, starting from the beginning.

If $\mathsf{Type}(\mathfrak{D}, i) \in \{1, 2, 4\text{-}(i)\}$, i.e. T_i is defined using points 1, 2 or 4-(i) of Definition 5.6.1, it follows that no additional elimination rules has been used and that $(\mathfrak{D}, i)$ is not a condition tuple. Thus, we set $T'_i := T_i$.

If $\mathsf{Type}(\mathfrak{D}, i) = 3$, then T_i is a condition tuple and we do not add any new tuple to $\mathfrak{D}'$.

If $\mathsf{Type}(\mathfrak{D}, i) = 4\text{-}(ii)$, then an additional elimination rule have been used and we replace it in $\mathfrak{D}'$ with an appropriate set of tuples of type 1, 2, or 4-(i) that preserve the attack.

It remains to show that any use of additional elimination rule can be replaced with a set of proof, introducing or eliminating tuples that preserve the attack relation. We will show this only for the case of [Def-**(A.**$\wedge$**)**]. The other cases are similar and are presented in Appendix B.3.

If [Def-$(A.\wedge)$] is used in $\mathfrak{D}$, then the following rule has been applied to some previous tuples.

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg \bigwedge \Gamma_A \qquad \Gamma_A \subseteq \Gamma_{A\wedge B} \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B} \quad [\text{Def -}(\mathbf{A.}\wedge)]$$

Therefore, the attacking argument, that in the general rule is denoted by $\Gamma_X \Rightarrow X$, the first attacking condition ($X \Rightarrow \neg \bigwedge \Gamma_A$) and the attacked argument ($\Gamma_{A\wedge B} \Rightarrow A \wedge B$) have all been introduced or derived in the previous points of the derivation using tuples of type 1 or 2.

Since $\Gamma_A \subseteq \Gamma_{A\wedge B}$, the proof tuple $T_i = \langle i, \Gamma_{A\wedge B} \Rightarrow \bigwedge \Gamma_A, \mathsf{J}, \emptyset\rangle$ can be added to $\mathfrak{D}'$. In addition, also the following tuples can be added to $\mathfrak{D}'$.

$$T_{i+1} = \langle i+1, \neg \bigwedge \Gamma_A \Rightarrow \neg \bigwedge \Gamma_{A\wedge B}, \mathsf{J}, \emptyset\rangle$$

$$T_{i+2} = \langle i+2, X \Rightarrow \neg \bigwedge \Gamma_{A\wedge B}, \mathsf{J}, \emptyset\rangle$$

At this point we can expand $\mathfrak{D}'$ by adding an elimination tuple of type 4-(i) that eliminates the same argument eliminated by the tuple i of the EDD $\mathfrak{D}$, i.e.:

$$T_{i+3} = \langle i+3, \Gamma_{A\wedge B} \not\Rightarrow A \ \wedge B, [Def]; i_X, i+2, i_{A\wedge B}, \Gamma_X \Rightarrow X\rangle$$

The indexes i_X and $i_{A\wedge B}$ stands for the tuples' indexes where the argument with claim X and the argument with claim $A \wedge B$ have been introduced. Both i_X and $i_{A\wedge B}$ are less than i.

In $\mathfrak{D}'$ instead of [Def-$(A.\wedge)$], we have introduced a set of tuples that define the same attack, but only using tuples of type 1, 2, or 4-(i). Thus, starting from an EDD we have recovered an equivalent DD.

The other direction of the theorem is straightforward. We only need to observe that every DD $\mathfrak{D}'$ is also an EDD. □

Given a derivation $\mathfrak{D}$ (DD or EDD), we define $\mathsf{Lenght}(\mathfrak{D})$ as the number of tuples present in the derivation.

Proposition 5.6.1. *Let $\mathfrak{D}$ be a dynamic derivation and $\mathfrak{D}'$ an extended dynamic derivation equivalent to $\mathfrak{D}$. Then, we have that $\mathsf{Lenght}(\mathfrak{D}) \geq \mathsf{Lenght}(\mathfrak{D}')$.*

Proof. We recall that two equivalent derivation preserves the attacks and their order. We will only analyse the case in which in the EDD $\mathfrak{D}'$ the additional elimination rule [Def-($A.\wedge$)] has been used. The others are similar and can be found in Appendix B.3.

In the proof of Theorem 5.6.1 it has been shown that if in a EDD [Def-($A.\wedge$)] has been applied, the same attack can be defined in an equivalent DD introducing 3 additional tuples. Thus, if $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent derivations and in $\mathfrak{D}'$ no additional elimination rule has been used, then $\mathsf{Lenght}(\mathfrak{D}) = \mathsf{Lenght}(\mathfrak{D}')$, otherwise $\mathsf{Lenght}(\mathfrak{D}) \geq \mathsf{Lenght}(\mathfrak{D}')$.

□

CHAPTER 6

Attack Principles With Weighted Attacks

6.1 Weighted Semi-Abstract Argumentation Frames (WSAFs)

One of the key purposes of argumentation is to handle inconsistency. In particular, as we have seen in Section 2.2, if we allow the arguments to attack one-another as in Dung's style argumentation frames, the general purpose cited above is addressed defining extensions and, through these, sceptical and credulous semantics. However, a common situation is that in a given frame some of the argumentative extensions, even if well defined, are the empty set. In contrast, it could happen that a given frame has many argumentative extensions without any criteria to distinguish between them. A possible way to overcome these difficulties is to consider the strength of the arguments and see how it affects the strength of the frame's attack relations.

In [BC03], the authors introduce *Value-based Argumentation Frameworks* (VAFs), where each frame's argument is associated with a value, and they define a preference relation over the set of possible values. In that approach, the value function is a primitive notion, while the attack relation weights are not. Furthermore, in [CLS05], Bipolar Argumentation Frameworks (BAFs) (see Definition 2.1.2) are associated with a gradual interaction-based valuation process. In this work, each argument's value only depends on the value of the arguments that directly interact with it in the given frame. In [MT08], the strength of the arguments is considered, and it is evaluated using the *minimax theorem* [Neu28] of Game Theory. In Section 6.8, we will model a real discussion using argumentation theory. We will define the weights of the attack relations in terms of strength of the arguments and the type of attack function considered for the attack relations.

In the literature there are many generalisations of Dung's argumentation frames that take argument strength as a primitive notion. However, considering the strength of the arguments as a primitive notion and the weights on the attacks as a derived one is a choice that should be dependent on the context. In the following example, introduced in [DHM+11], we see how it is more reasonable to assume the attacks' weight as a primitive notion in some specific case.

Example 6.1.1. *Let us consider the following arguments:*

A: *The house is in a good location, it is large enough for our family and it is affordable: we should buy it.*

B: *The house suffers from subsidence, which would be prohibitively expensive to fix: we should not buy it.*

These two arguments are mutually attacking and taken on their own are both strong *arguments, in the sense that the conclusions follow from the premises clearly and solidly. However, if we would have to choose between buying or not the house, we probably would opt for not buying it. We would not buy the house because the strength of the attack from B to A, is greater than the one from A to B i.e. $w(B, A) \geq w(A, B)$.*

In [DHM+11], a *weighted argumentation framework* is defined as follows.

Definition 6.1.1 (Weighted Argumentation Frame (WAF))**.** *A* weighted argumentation framework *is a triplet $\mathcal{A} = \langle Ar, R_{\rightarrow}, w\rangle$ where:*

1. *$\langle Ar, R_{\rightarrow}\rangle$ is a Dung-style abstract argumentation framework, and*

2. *$w : R_{\rightarrow} \longrightarrow [0, 1]$ is a function, and it is an assignment of weights in $[0, 1]$ to the attacks, i.e. to the ordered pairs of elements in Ar.*

In the definition of WAF proposed in [DHM+11], the weights on the attacks are not normalised to the unit interval and are not allowed to have value 0. However, if we normalise to [0,1] and also allow attacks of weight 0, (unweighted) AFs are just special cases of WAFs, where each attack is either of weight 1 or 0. The latter amounts to 'no attack at all', i.e., where we previously wrote $F \not\longrightarrow A$, we now write $F \xrightarrow{0} A$.

Before introducing weighted semi-abstract argumentation frames, it is relevant to recall that arguments in SAFs are still understood as complex entities as defined in Definition 3.2.3 or 3.2.10. Still, the only part that is instantiated is the claim. Thus, if in a given SAF $F \longrightarrow G$, it means that there is an argument with claim F that attacks an argument with claim G. In SAFs we identify the arguments that share the same claim and each formula, depicting the claim of an argument, occurs at most once. I.e., instantiating a

SAF entails that each node of the directed graph representing a semi-abstract argument is split into several arguments (those that share the same claim).

Suppose that $AF = \langle Ar, Attack(\mathscr{A})\rangle$ is an argumentation frame where the arguments are fully instantiated, e.g. see Definition 3.4.1 or Definition 3.4.2. Suppose that $\langle\Gamma_A; A\rangle$, $\langle\Gamma'_A; A\rangle$ and $\langle\Gamma_B; B\rangle$ are arguments in AF such that $\langle\Gamma_A; A\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$ and $\langle\Gamma'_A; A\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_B; B\rangle$ for some $\mathscr{R} \in \mathscr{A}$. In addition, suppose that there is no other argument in AF with claim A that $\mathscr{R}$-attacks $\langle\Gamma_B; B\rangle$ for any $\mathscr{R} \in \mathscr{A}$. Assume now that the following weights are assigned to these attacks: $w(\langle\Gamma_A; A\rangle, \langle\Gamma_B; B\rangle) = w_1$ and $w(\langle\Gamma'_A; A\rangle, \langle\Gamma_B; B\rangle) = w_2$. In the weighted SAF (WSAF) corresponding to AF we set $w(A, B) = w^*$ where $w^* = max\{w_1, w_2\}$. If in AF there are infinite arguments with claim A that attacks $\langle\Gamma_B; B\rangle$, we set $w(A, B) = w^*$ where $w^* = sup\{w_i \mid w(\langle\Gamma_i; A\rangle, \langle\Gamma_B; B\rangle) = w_i\}$.

In the abstraction process from a WAF to a WSAF, arguments with the same claim get summarised into a single semi-abstract argument. Concerning corresponding attacks (between arguments with identical claims) the abstraction step takes only the strongest attacks into account. Since in WSAFs, the support sets and the inference method between the support and the claim are left abstract, many are the WAFs that correspond to the same WSAF.

Suppose that A is a finite WAF and for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ $\langle\Gamma^i_A; A\rangle$ and $\langle\Gamma^j_B; B\rangle$ are arguments of A. If for any i, j $w(\langle\Gamma^i_A; A\rangle, \langle\Gamma^j_B; B\rangle) = w_{i,j}$, then in the corresponding WSAF $w(A, B) = w$ where $w = \max\{w_{i,j} \mid w(\langle\Gamma^i_A; A\rangle, \langle\Gamma^j_B; B\rangle) = w_{i,j}\}$. Again, if A is infinite, instead of the maximum we take the supremum.

Definition 6.1.2 (Weighted Semi-abstract Argumentation Frame (WSAF))**.** *A* weighted semi-abstract argumentation frame *is a triplet $A = \langle Ar, R_{\rightarrow}, w\rangle$, where:*

1. *$\langle Ar, R_{\rightarrow}\rangle$ is a SAF.*

2. *$w : R_{\rightarrow} \longrightarrow [0, 1]$ is a function, and it is an assignment of weights in $[0, 1]$ to the attacks, i.e. to the ordered pairs of elements in Ar.*

We write $F \xrightarrow{w} G$ if the weight w is assigned to $F \longrightarrow G$.

In WSAFs, weights are considered a primitive notion as in [DHM+11]. As pointed out in [DHM+11], this is one of the possible approaches to the definition of weights.

6.2 Weighted Attack Principles

The attack principles for SAFs can be generalised to WASFs in various ways, at least some of which are very straightforward, as indicated by the following example.

Example 6.2.1. *We revisit Example 4.1.2, where we considered an argument with claim X that attacks an argument with claim A, expressing that most of the population of some* *country strongly supports its government. Let us now assume we have some information regarding the actual strength of this attack. Note that we are not, at least not in any direct way, attaching a degree of truth or belief to the claim A itself. Instead, we only consider the given attack on the statement. Different meanings can be associated with the 'strength of the attack' as emphasized, e.g., in [DHM+11]. For example, it could simply reflect our (the modelers) degree of belief in the attack's validity. In a more sophisticated scenario, we can imagine a set of experts who are asked to judge whether the alleged attack of an argument with claim X on the argument claiming A is convincing or not. We could then stipulate the attack's weight to be equal the proportion of experts who find the attack compelling. Of course, many alternative interpretations of 'weight' are conceivable. But in any case, it should be clear that the argument with claim X should not attack any argument whose claim is formed by conjunctively attaching a further claim B to A attacked by (the argument with claim) X with a higher weight than A itself. Like in the unweighted case, this expresses a simple rationality principle that only considers the logical form of the attacked claim. Neither the content of the involved argument nor the nature of the attack and the particular interpretation of 'strength' or 'weight' matters when stipulating that any attack on an argument with claim A (implicitly) attacks any argument whose claim is of the form $A \wedge B$ with at least the same weight.*

Since on the semi-abstract level the weights on the attacks are understood as the supremum of the weights of the corresponding fully instantiated frames, in accordance to the above example we obtain the following generalization of principle **(A.$\wedge$)**:

(A^w.$\wedge$) If $F \xrightarrow{x} A$ and $F \xrightarrow{y} B$, then $F \xrightarrow{z} A \wedge B$, where $z \geq \max\{x, y\}$.
In words: for every argument with claim F that attacks with weight x_i an argument with claim A, and with weight y_i an argument with claim B, then it attacks an argument with claim $A \wedge B$ with a weight z_i at least as large as those against its conjuncts. Since $z_i \geq \max\{x_i, y_i\}$ holds for every x_i, y_i and z_i, it holds also for the corresponding supremum. Therefore, $z \geq \max\{x, y\}$.

Since we also consider attacks of weight 0 (equivalent to 'no attack edge' in SAFs), we may assume that the graph formed by $R_{\rightarrow}$ in a WSAF is complete without loss of generality. But this means that the above formulation of the attack principle for conjunction can be reformulated as follows:

(A^w.$\wedge$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \wedge B$, then $z \geq \max\{x, y\}$.

Likewise, the following principle refinement of **(A.$\vee$)** for (implicit) attacks on disjunctive claims should be intuitively uncontroversial.

(A^w.$\vee$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \vee B$, then $z \leq \min\{x, y\}$.
In words: if for every argument with claim F that attacks with weight x_i an

argument with claim A, with weight y_i an argument with claim B, and with weight z_i an argument witch claim $A \vee B$, then the weight of the attack against the disjunction entails attacks to both of its disjuncts of at least the same weight, i.e. for every x_i, y_i and z_i, $z_i \leq \min\{x_i, y_i\}$. Therefore, the constrain holds also for the corresponding supremum, i.e. $z \leq \min\{x, y\}$.

As shown in Section 4.4, **(C.$\vee$)**, the inverse of **(A.$\vee$)**, can be justified with a particular formal interpretation of the attack relation. It straightforwardly generalizes to the weighted scenario as follows.

(C^w.$\vee$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \vee B$, then $z \geq \min\{x, y\}$.
In words: if for every argument with claim F that attacks with weight x_i an argument with claim A, with weight y_i an argument with claim B, and with weight z_i an argument witch claim $A \vee B$, then the attack against the disjunction carries a weight that is at least as large as on of those against its disjuncts, i.e. $z_i \geq \min\{x_i, y_i\}$. Since the constraint holds for every x_i, y_i and z_i, then it holds also for the corresponding supremum, i.e. $z \geq \min\{x, y\}$.

Example 6.2.2. *Let us expand Examples 4.1.2 and 6.2.1 to disjunctive claims by considering the following statement "(A) The majority of the population strongly supports its government or (B) believes that the economic situation is improving". Assume that some argument with claim X attacks an argument with claim $(A \vee B)$ with some weight $w \in [0, 1]$ and w is the supremum of all such weights. Then* **(A^w.$\vee$)** *expresses the rationality principle that arguments claiming F cannot attack arguments with claim $A \vee B$ with a greater weight than that of a corresponding attack on arguments with claim A or B alone. Note that this makes sense independently of any concrete interpretation of weights of attacks, since $A \vee B$ logically follows from A as well as from B, even if we move from to classical logic to a many-valued one.*

The inverse principle **(C^w.$\vee$)** *is less obviously valid. However, if we adopt the interpretation of weights as reflecting degrees of belief in the validity of the proposed attacks, then the following principle seems reasonable: An agent who believes with degree x that arguments claiming F successfully attack arguments claiming A and believes with degree y that arguments claiming F also attack arguments claiming B successfully, should believe with a degree that is not lower than both, x and y, that those arguments, at least implicitly, also establish a valid attack on the claim that either $A \vee B$.*

From the above example, we see that if the attack principles are acceptable for individual arguments, then they transfer to the semi-abstract level. In the rest of this chapter all the weights on the attacks defined on WSAFs are understood as supremum of all the weights defined in any corresponding fully instantiated argumentation frames.

Since the attack principles suggest possibilities for 'closing off' given sets of coherent arguments with respect to simple logical consequence relations, also the following version of **(A.$\perp$)** should be obvious.

($\mathbf{A}^w.\bot$) $F \xrightarrow{1} \bot$, for every F.
Every argument fully attacks (at least implicitly) the clearly false claim $\bot$.

Note that $\bot$ is intended to stand for any *obviously* false statement. Therefore no incoherence should arise from stipulating that any argument implicitly rejects an argument with claim $\bot$ without qualification regarding the weight of the attack.

In [DHM+11], three different ways of imposing weights are analysed: weights can be interpreted as measures of votes in support of attacks, as a measure of inconsistency between arguments or, more generally, as rankings of different types of attack. Having in mind these possible interpretations of the weights helps justify the principles involving attacks on implicative claims. It is important to keep in mind that we only want to consider *material*, even truth-functional implication here, and hence do not investigate proper (intensional) conditionals or counter-factual statements. Thus, once more, we look for principles that only refer to the weights of attacks on the claim and its immediate subformulas, respectively. In light of the classical principles **($\mathbf{A}.\supset$)** and **($\mathbf{C}.\supset$)**, at least the following candidates are worth considering.

($\mathbf{A}^w.\supset$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{0} A \supset B$, then $x \geq y$.
If an implication is not attacked at all, then the implying formula is attacked with at least the same weight as the implied formula.

($\mathbf{C}^w.\supset$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z>0} A \supset B$[1], then $x < y$.
If an implication is attacked with some positive weight then the implying formula is attacked with a strictly smaller weight than the implied formula.

These two principles are equivalent to the following reformulations, respectively.

($\mathbf{A}^w.\supset$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $x < y$, then $F \xrightarrow{z>0} A \supset B$.
If the implied formula is attacked with a higher weight than the implying formula, then the implication is attacked with some positive weight.

($\mathbf{C}^w.\supset$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $x \geq y$, then $F \xrightarrow{0} A \supset B$.
If the implied formula is attacked with at least the same weight than the implying formula, then the implication is not attacked at all.

These reformulations make transparent that **($\mathbf{A}^w.\supset$)** and **($\mathbf{C}^w.\supset$)** jointly express the principle that an implication is attacked with some positive weight if and only if the implied formula is attacked with a higher weight than the implying formula. But no restriction is made on the amount of (positive) weight of the attack on the implication in relation to the weights of the attacks on its sub-formulas. The following principle bounds the weight of an attack on an implication by the corresponding attack's weight

[1] $F \xrightarrow{z>0} G$ abbreviates '$F \xrightarrow{z} G$, where $z > 0$'.

on the implied formula. This seems reasonable if we take into account that we aim at characterising strictly material implication, here.

($\mathbf{B}^w$.⊃) If $F \xrightarrow{y} B$ and $F \xrightarrow{z} A \supset B$, then $z \leq y$.
Since the claim $A \supset B$ is weaker than B, no implication is attacked to a higher degree than the implied formula.

Definition 6.2.1 ($\mathcal{P}_B$). *The set of* basic weighted attack principles $\mathcal{P}_B$ *consists of all principles mentioned in ths section; i.e.* $\mathcal{P}_B = \{(\mathbf{A}^w.\wedge), (\mathbf{A}^w.\vee), (\mathbf{C}^w.\vee), (\mathbf{A}^w.\bot), (\mathbf{A}^w.\supset), (\mathbf{C}^w.\supset), (\mathbf{B}^w.\supset)\}$.

It is probably not surprising that the basic weighted principles, even if imposed jointly, do not suffice to determine any specific truth functional semantics for the logical connectives. In other words: further principles will be needed to characterise particular fuzzy logics.

Example 6.2.3. *Let us consider a scenario similar to Example 4.1.1, referring to the recent economic growth of China and current debates on policies about pollution. The involved statements are the following:*
– *"Rapid economic growth occurs." (G)*
– *"Very high level of CO_2 emissions occur." (C)*
– *"Overall prosperity increases." (P)*
– *"Awareness about the need of environmental protection increases." (E)*
– *"Strict regulations concerning CO_2 emissions are put in place." (R)*
– *"Industry invests in 'green' production methods." (I)*
Again, in addition to such statements, also certain logical compounds of these statements might well be considered as claims of arguments. Let a concrete corresponding SAF $\mathcal{S}_E = (Ar, R_\rightarrow)$ *be given by* $Ar = \{I, C, P, G, C \wedge P, P \vee I, E \supset R\}$ *and* $R_\rightarrow = \{I \longrightarrow C, C \longrightarrow P \vee I, G \longrightarrow E \supset R\}$.

Imposing our attack principles for the unweighted frames results in additional (implicit) attacks that augment $R_\rightarrow$*; namely,* $I \longrightarrow C \wedge P$, $C \longrightarrow I$ *and* $G \longrightarrow R$ *using* **(A.∧)**, **(C.∨)** *and* **(C.⊃)**, *respectively.*

So far, we have not yet considered any weights attached to the indicated attacks. However, it is perfectly conceivable that not all of the mentioned attacks are equally plausible or equally agreed upon among a group of experts. Of course, to systematically derive certain weights of particular attacks, we would have to analyse the underlying arguments and not just the claims of these arguments. (Remember that only the later are recorded in SAFs.) But even without access to such information, it is plausible that, e.g., arguments claiming I *are only considered partly successful in attacking arguments claiming* C*. Similarly, also the two other attacks registered in the SAF* $\mathcal{S}_E$ *may receive weights less than* 1*. Concretely, suppose that we have the following weights on the attack relations:* $I \xrightarrow{0.2} C$, $C \xrightarrow{0.7} P \vee I$ *and* $G \xrightarrow{0.5} E \supset R$*. Then the logical principles discussed in this section entail that further attacks than those considered explicitly so far should be taken into account.*

For example, principle $(\mathbf{A}^w.\wedge)$*, applied to* $I \xrightarrow{0.2} C$ *yields that arguments claiming* I *will attack arguments claiming* $C \wedge P$ *with at least the same weight* (0.2) *with which they attack arguments claiming* C*. We record this by writing* $I \xrightarrow{z_1 \geq 0.2} C \wedge P$*. Similarly, we can apply* $(\mathbf{C}^w.\vee)$ *to* $C \xrightarrow{0.7} P \vee I$ *and* $(\mathbf{C}^w.\supset)$ *to* $G \xrightarrow{0.5} E \supset R$ *to obtain* $C \xrightarrow{z_2 \leq 0.7} I$ *and* $G \xrightarrow{z_3 > 0} R$*, respectively.*

Under all three interpretations of the weights mentioned in [DHM+11], at least some of our attack principles are straightforwardly justified. In particular, using the first interpretation it is easy to see that $(\mathbf{A}^w.\wedge)$, $(\mathbf{A}^w.\vee)$, $(\mathbf{A}^w.\bot)$, and $(\mathbf{A}^w.\supset)$ hold. For example, if x is the number of votes in support of the attack $F \longrightarrow A$ and y is the number of votes in support of $F \longrightarrow B$, then the number of votes in support of $F \longrightarrow A \wedge B$ can be neither below x nor below y, because agents that support either $F \longrightarrow A$ or $F \longrightarrow B$, if acting rationally, will also support the attack $F \longrightarrow A \wedge B$. Thus, $(\mathbf{A}^w.\wedge)$ is justified. If x is the number of votes in support of the attack $F \longrightarrow A$, y is the number of votes in support of $F \longrightarrow B$ and $x < y$ it means that there is at least one agent that supports the attack on B, but not the attack on A. Again, under the assumptions that agents are acting rationally, since $A \supset B$ is weaker than B, there is at least one agent that supports the attack $F \longrightarrow A \supset B$. $(\mathbf{A}^w.\vee)$, and $(\mathbf{A}^w.\bot)$ can be justified analogously.

We do not aim at an analysis of concrete arguments or at a new method for assigning weights to attacks between arguments. Rather we want to explore under which conditions *given* weighted argumentation frames can be used to extract a many-valued semantics for the involved claims. For this purpose we are now going to introduce a semantic notion that re-frames logical validity as immunity with respect to attacks that adhere to rationality principles like those discussed in this section, but also later, in Sections 6.4, 6.5, and 6.7.

6.3 Argumentative Immunity

Remember that we are not interested in concrete (weighted or unweighted) argumentation frameworks. Rather we want to relate fuzzy logics to the realm of *all* possible weighted argumentation frames that satisfy certain attack principles, like the ones discussed in the last section. Since we cannot expect any given WSAF to explicitly contain all arguments and all required attacks to make it possible to satisfy such principles, we define the following closure operation on WSAFs as the one introduced for SAFs in Definition 4.2.2.

Definition 6.3.1 (Syntactic Closure of WSAFs)**.** *A WSAF S is* syntactically closed *with respect to Δ if all formulas and subformulas of formulas in Δ occur as claims of some argument in S.*

We will suppress the explicit reference to Δ whenever the context makes clear what formulas are expected to be available as claims of arguments in the relevant WSAF. E.g., in speaking of an argumentation frame S that satisfies the principle $(\mathbf{A}^w.\wedge)$ it is implicitly understood that S is closed at least with respect to $\{A \wedge B\}$ and thus contain not only attacks (possibly of weight 0) on $A \wedge B$, but also attacks on A and on B.

Definition 6.3.2 ($\mathcal{P}$-argumentatively Immune Formula)**.** *Let $\mathcal{P}$ be a set of (weight related) attack principles, then we call a formula F* $\mathcal{P}$-argumentatively immune *(shortly:* $\mathcal{P}$-immune*) if in all syntactically closed WSAFs (with respect to $\{F\}$) that satisfy the principles in $\mathcal{P}$ F is attacked only with weight 0.*

Argumentative immunity is intended as a notion that provides a new view on *logical validity*, which is not based on Tarski-style semantics, but rather only refers to claims of arguments (that may or may not be interpreted in the usual way) and to the weights of explicit or implicit attacks between them. To illustrate its use consider the following example, that refers to the axiom of *(pre-)linearity* and is thus characteristic for all t-norm based fuzzy logics.

Proposition 6.3.1. *The formula $(A \supset B) \vee (B \supset A)$ is $\{(\mathbf{A}^w.\vee), (\mathbf{C}^w.\supset)\}$-immune.*

Proof. Let S be a WSAF that satisfies $(\mathbf{A}^w.\vee)$ and $(\mathbf{C}^w.\supset)$. We proceed indirectly and assume that S contains an argument X that attacks $(A \supset B) \vee (B \supset A)$ with some positive weight z. By $(\mathbf{A}^w.\vee)$, we obtain $X \xrightarrow{x>0} A \supset B$ and $X \xrightarrow{y>0} B \supset A$. Applying $(\mathbf{C}^w.\supset)$ to these attacks yields a contradiction: we obtain $X \xrightarrow{u} A$ and $X \xrightarrow{v} B$, where $u < v$ because of $X \xrightarrow{x>0} A \supset B$ and $v < u$ because of $X \xrightarrow{y>0} B \supset A$. □

More generally, our aim is to investigate with respect to which collections of attack principles some fundamental fuzzy logics are *argumentatively sound and complete*, respectively. By argumentative soundness we mean that all valid formulas are argumentatively immune; argumentative completeness is the converse: all argumentatively immune formulas are logically valid.

Regarding argumentative soundness, the following observation is crucial: argumentative immunity is preserved under applications of *modus ponens* whenever $(\mathbf{A}^w.\supset)$ is satisfied. More precisely the following holds.

Proposition 6.3.2. *If G and $G \supset F$ are argumentatively $\mathcal{P}$-immune, then also F is argumentatively $\mathcal{P}$-immune as long as $\mathcal{P}$ contains $(\mathbf{A}^w.\supset)$.*

Proof. Suppose that F is not argumentatively $\mathcal{P}$-immune. This means that there is a WSAF S that is syntactically closed (with respect to at least $\{G \supset F\}$), such that S satisfies all principles in $\mathcal{P}$ and contains an argument X attacking F with positive weight ($X \xrightarrow{z>0} F$). We make the following case distinction.

(1) $X \xrightarrow{x>0} G \supset F$: this means that $G \supset F$, too, is not $\mathcal{P}$-argumentatively immune.

(2) $X \xrightarrow{0} G \supset F$:, then according to principle $(\mathbf{A}^w.\supset)$ we have $X \xrightarrow{x} G$ and $X \xrightarrow{y} F$, where $x \geq y$. But by the assumption $X \xrightarrow{z>0} F$ we obtain that y and thus also x is greater than 0. In other words, in this case the first premise is not $\mathcal{P}$-argumentatively immune. □

6.4 Characterising Gödel Logic Through WSAFs

Propositional finite-valued Gödel logics were introduced (implicitly) by Gödel [Göd33] to show that intuitionistic logic does not have a characteristic finite matrix. Dummett [Dum59] later generalised these to an infinite set of truth-values, and showed that the set of its tautologies is axiomatised by intuitionistic logic extended by the prelinearity axiom $(A \supset B) \vee (B \supset A)$. Hence infinite-valued Gödel logic G is also called Gödel-Dummett logic or Dummett's LC. Gödel logics naturally turn up in a number of different areas of logic and computer science. For instance, Dunn and Meyer [DM71] pointed out their relation to relevance logics; Visser [Vis82] employed G in investigations of the provability logic of Heyting arithmetic. Most importantly in our context, G has been recognized as one of the most important formalizations of fuzzy logic [H́98].

We will first review the semantics and a Hilbert-style proof system for G and, then proceed in three steps.

1. We introduce two further attack principles $(\mathbf{G}^w.\supset)$ and $(\mathbf{C}^w.\wedge)$, that have not been considered in Section 6.2.
2. We show that all formulas that are derivable in the Hilbert-style system for Gödel logic are argumentatively immune with respect to $\mathcal{P}_B \cup \{(\mathbf{G}^w.\supset), (\mathbf{C}^w.\wedge)\}$.
3. Conversely, we show that formulas that all formulas that are argumentatively immune in this specific sense are also valid according to Gödel logic.

Recall the semantics of Gödel logic: every assignment I of truth values in $[0,1]$ to propositional variables is extended to non-atomic formulas as follows:

$$\|A \wedge B\|_I^{\mathsf{G}} = \min\{\|A\|_I^{\mathsf{G}}, \|B\|_I^{\mathsf{G}}\}, \qquad \|A \vee B\|_I^{\mathsf{G}} = \max\{\|A\|_I^{\mathsf{G}}, \|B\|_I^{\mathsf{G}}\},$$

$$\|A \supset B\|_I^{\mathsf{G}} = \begin{cases} 1 & \text{if } \|A\|_I^{\mathsf{G}} \leq \|B\|_I^{\mathsf{G}} \\ \|B\|_I^{\mathsf{G}} & \text{otherwise.} \end{cases}$$

$\neg A$ is defined as $A \supset \bot$, hence

$$\|\neg A\|_I^{\mathsf{G}} = \begin{cases} 1 & \text{if } \|A\|_I^{\mathsf{G}} = 0 \\ 0 & \text{otherwise} \end{cases}$$

For the atomic formula $\bot$ we have $\|\bot\|_I^{\mathsf{G}} = 0$. F is G-*valid* if $\|F\|_I^{\mathsf{G}} = 1$ for all assignments I.

Gödel logic can be axiomatized in various ways. Below, we will refer to the Hilbert-style system consisting in the following axioms:

$$\begin{aligned}
[\supset\text{-}1]: &\quad F \supset (G \supset F)\\
[\supset\text{-}2]: &\quad (F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))\\
[\wedge\text{-}1]: &\quad (F \wedge G) \supset F\\
[\wedge\text{-}2]: &\quad (F \wedge G) \supset G\\
[\wedge\text{-}3]: &\quad F \supset (G \supset (F \wedge G))\\
[\vee\text{-}1]: &\quad F \supset (F \vee G)\\
[\vee\text{-}2]: &\quad G \supset (F \vee G)\\
[\vee\text{-}3]: &\quad (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))\\
[\bot]: &\quad \bot \supset F\\
[Lin]: &\quad (F \supset G) \vee (G \supset F)
\end{aligned}$$

The only inference rule is *modus ponens*: from G and $F \supset G$ infer F. Note that the only axiom that is not already valid in intuitionistic logic is *Lin*. The following fact has been established by Dummett [Dum59].

Theorem 6.4.1. *The above Hilbert-style system is sound and complete for Gödel logic. In other words: a formula F is derivable in the system iff F is G-valid.*

To obtain a characterization of Gödel logic in terms of argumentative immunitiy we have to consider the following additional principles for weighted attacks.

($\mathbf{G}^w.\supset$) If $F \xrightarrow{x} A$ and $F \xrightarrow{y} B$, where $x < y$, then $F \xrightarrow{y} A \supset B$.
If the implying formula is attacked with a smaller weight than the implied formula, then the implication is attacked with the same weight as the implied formula.

($\mathbf{C}^w.\wedge$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, $F \xrightarrow{z} A \wedge B$, then $z \leq \max\{x, y\}$.
An attack against a conjunction entails an attack to at least one of its conjuncts with an equal or higher weight.

Definition 6.4.1. $\mathcal{P}_{\mathsf{G}} = \mathcal{P}_B \cup \{(\mathbf{G}^w.\supset), (\mathbf{C}^w.\wedge)\}$.

Remember that we do *not* suggest that all these principles regarding implicit attacks on logically compound claims should actually be respected in any given weight argumentation frame. We only claim that the principles collected in $\mathcal{P}_{\mathsf{G}}$ are fairly easy to understand and thus help to assess which conditions are necessary and sufficient for establishing a systematic relation between fuzzy logics and semi-abstract argumentation frames. While some of the principles, e.g. ($\mathbf{A}^w.\wedge$), ($\mathbf{A}^w.\vee$), indeed correspond to very natural assumptions on reasoning about conjunctive and disjunctive claims, other principles, e.g. ($\mathbf{C}^w.\wedge$) and ($\mathbf{G}^w.\supset$) are certainly much less natural and may well be too demanding to be imposed on given collections of (explicit and implicit) arguments.

Note that in the presence of $(\mathbf{C}^w.\supset)$, $(\mathbf{G}^w.\supset)$ amounts to a strengthening of $(\mathbf{B}^w.\supset)$. In other words, $(\mathbf{B}^w.\supset)$ is redundant in $\mathcal{P}_\mathsf{G}$. However, it is still interesting to see in which cases it suffices to refer to $(\mathbf{B}^w.\supset)$ instead of to the stronger principle $(\mathbf{G}^w.\supset)$.

Theorem 6.4.2 (Argumentative Soundness of G)**.** *Every* G*-valid formula is* $\mathcal{P}_\mathsf{G}$*-argumentatively immune.*

Proof. By Theorem 6.4.1 and Proposition 6.3.2, it remains to check that the axioms for Gödel logic are $\mathcal{P}_\mathsf{G}$-immune. In the following, we implicitly assume that all arguments occur in a WSAF that is syntactically closed with respect to the axiom in question. In each case we argue indirectly, deriving a contraction from the assumption that there is an argument X that attacks the axiom in question with some positive weight.

[$\supset$-1]**:** Assume that $X \xrightarrow{z>0} F \supset (G \supset F)$., then by $(\mathbf{C}^w.\supset)$ we obtain $f < y$, where f is given by $X \xrightarrow{f} F$ and y is given by $X \xrightarrow{y} G \supset F$. On the other hand, applying $(\mathbf{B}^w.\supset)$ to the latter statement yields $y \leq f$, which is a contradiction.

[$\supset$-2]**:** Assume that $X \xrightarrow{z>0} (F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))$. Then by $(\mathbf{C}^w.\supset)$, we obtain $x < y$, where $X \xrightarrow{x} F \supset (G \supset H)$ and $X \xrightarrow{y} (F \supset G) \supset (F \supset H)$. Since $y > 0$ we can apply $(\mathbf{C}^w.\supset)$ to obtain $v < w$, where $X \xrightarrow{v} F \supset G$ and $X \xrightarrow{w} F \supset H$. Since $w > 0$, $(\mathbf{C}^w.\supset)$ implies $f < h$, where $X \xrightarrow{f} F$ and $X \xrightarrow{h} H$. We can also apply $(\mathbf{G}^w.\supset)$ to obtain $y = w$. Applying $(\mathbf{G}^w.\supset)$ again, justified by $f < h$, yields $w = h$. For reference below, we assign the following labels to some of the facts established so far: (1) $x < y$, (2) $f < h$, and (3) $y = h$.

We show that each of the following cases leads to a contradiction.

$g < h$**:** By $(\mathbf{G}^w.\supset)$ this implies $u = h$. By (2) we obtain $f < u$ and thus can apply $(\mathbf{G}^w.\supset)$ to obtain $x = u$. Jointly, this yields $x = h$ and hence, by (3), also $x = y$, which contradicts (1).

$g \geq h$ **and** $f < g$**:** By $(\mathbf{G}^w.\supset)$ $f < g$ yields $g = v$. Because of (2), $(\mathbf{G}^w.\supset)$ also leads to $w = h$. Therefore $g \geq h$ implies $w \leq v$. By $(\mathbf{C}^w.\supset)$, the latter entails $y = 0$, which contradicts (1).

$g \geq h$ **and** $f \geq g$**:** By transitivity we have $f \geq h$, which contradicts (2).

[$\wedge$-1]**:** Assume that $X \xrightarrow{z>0} (F \wedge G) \supset F$. , then by $(\mathbf{C}^w.\supset)$ we obtain $x < f$, where $X \xrightarrow{x} F \wedge G$ and $X \xrightarrow{f} F$. On the other hand, by $(\mathbf{C}^w.\wedge)$, we obtain $x \geq \max\{f, g\}$, where g is given by $X \xrightarrow{g} G$. This in particular implies $x \geq f$. Thus we have a contradiction since X cannot attack $(F \wedge G) \supset F$ with a weight that is both, smaller as well as greater or equal to f.

[$\wedge$-2]**:** Analogous to [$\wedge$-1].

[$\wedge$-3]**:** Assume that $X \xrightarrow{z>0} F \supset (G \supset (F \wedge G))$. By applying **(C^w.$\supset$)** twice, we first obtain $f < y$ and, then $g < x$, where $X \xrightarrow{y} G \supset (F \wedge G)$, $X \xrightarrow{x} F \wedge G$, $X \xrightarrow{f} F$ and $X \xrightarrow{g} G$. On the other hand, by **(A^w.$\wedge$)**, we obtain $x \leq f$ or $x \leq g$. The latter case clearly contradicts $g < x$. To obtain a contradiction also in the first case, we appy **(G^w.$\supset$)** to $X \xrightarrow{y} G \supset (F \wedge G)$ and $X \xrightarrow{x} F \wedge G$ to obtain $x = y$, and consequently $f < y = x \leq f$.

[$\vee$-1]**:** Assume that $X \xrightarrow{z>0} F \supset (F \vee G)$. , then by **(C^w.$\supset$)** we obtain $f < y$, where $X \xrightarrow{f} F$ and $X \xrightarrow{y} F \vee G$. On the other hand, by **(A^w.$\vee$)**, we have $y \leq f$, which is in contradiction with the previous assertion.

[$\vee$-2]**:** analogous to [$\vee$-1]

[$\vee$-3]**:** Assume that $X \xrightarrow{z>0} (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))$. We first name the weights of attacks by X on subformulas: $X \xrightarrow{x} (H \supset F) \supset ((G \vee H) \supset F)$, $X \xrightarrow{y} G \supset F$, $X \xrightarrow{u} H \supset F$, $X \xrightarrow{v} (G \vee H) \supset F$, $X \xrightarrow{w} G \vee H$, $X \xrightarrow{f} F$, $X \xrightarrow{g} F$, and finally $X \xrightarrow{h} H$.

By successively applying **(C^w.$\supset$)**, we obtain $y < x$, $u < v$, and $w < f$.

Next, **(G^w.$\supset$)** yields $v = f$, because $w < f$, and $x = v$, because $u < v$. Below, we will refer to $x = v = f$ as $(*)$.

Finally, we show that each of the following cases leads to a contradiction.

$g < f$**:** By **(G^w.$\supset$)** this implies $y = f$. This contradicts $y < x$ combined with $(*)$.

$h < f$**:** By **(G^w.$\supset$)** this implies $u = f$. This contradicts $u < v$ combined with $(*)$.

$g \geq f$ **and** $h \geq f$**:** This means that $f \leq \min\{g, h\}$. By applying **(A^w.$\vee$)** to $X \xrightarrow{w} G \vee H$ we obtain that $g \leq w$ or $h \leq w$. But above we have shown $w < f$, and thus obtain a contradiction in both cases.

[$\bot$]**:** Assume that $X \xrightarrow{z>0} \bot \supset F$. By **(C^w.$\supset$)** we obtain that $x < f$, where $X \xrightarrow{x} \bot$ and $X \xrightarrow{f} F$. This directly contradicts principle **(A^w.$\bot$)**, which requires that $x = 1$.

[***Lin***]**:** By Proposition 6.3.1. □

We remarking in passing that, to guarantee the argumentative immunity of [$\supset$-2] and [$\vee$-3], one cannot trade **(G^w.$\supset$)** for any principle already contained in $\mathcal{P}_B$. Likewise one can show that the 'strong' principle **(C^w.$\wedge$)** is indeed needed to render [$\wedge$-1] argumentatively immune. All are other axioms are already $\mathcal{P}_B$-argumentatively immune.

Before showing the converse of Theorem 6.4.2—namely, argumentative completeness of G—let us observe that classical logic is not argumentatively sound with respect to $\mathcal{P}_\mathsf{G}$. $F \vee \neg F$ (i.e. $F \vee (F \supset \bot)$) is not $\mathcal{P}_\mathsf{G}$-immune. Consider a WSAF that just contains four arguments with claims $\bot$, F, $\neg F(= F \supset \bot)$ and $F \vee \neg F$, respectively, and where the weights of attacks between these arguments are as specified in the following matrix:

$\xrightarrow{w}$	$\bot$	F	$F \supset \bot$	$F \vee \neg F$
$\bot$	1	0	1	0
F	1	0.5	1	0.5
$F \supset \bot$	1	1	0	0
$F \vee \neg F$	1	0	1	0

It is straightforward to check that all principles of $\mathcal{P}_\mathsf{G}$ are satisfied in this WSAF. Since $F \vee \neg F$ is attacked with weight 0.5 by F it is not argumentatively valid.

Theorem 6.4.3 (Argumentative Completeness of G)**.** *Every $\mathcal{P}_\mathsf{G}$-argumentatively immune formula is G-valid.*

Proof. We proceed indirectly. Suppose that F is not G-valid. This means that there is an assignment I such that $\|F\|_I^\mathsf{G} < 1$. Taking I as a starting point, we construct a WSAF S_I that is syntactically closed with respect to $\{F\}$ and satisfies the attack principles in $\mathcal{P}_\mathsf{G}$ such that $X \xrightarrow{z>0} F$ for some (claim of an) argument X in S_I.

We define S_I by assigning the weight $1 - \|G\|_I^\mathsf{G}$ to each edge (H, G) of the attack relation of S_I. In other words, we stipulate that every (claim of an) argument is attacked by every other argument and by itself with a weight that is inverse to its degree of truth in I.

It remains to check that all attack principles in $\mathcal{P}_\mathsf{G}$ are satisfied in S_I.

($\mathbf{A}^w.\wedge$): Let $A \wedge B$ be an argument in S_I and let $\|A \wedge B\|_I^\mathsf{G} = u$, $\|A\|_I^\mathsf{G} = v$, and $\|B\|_I^\mathsf{G} = w$. Then, by definition of S_I, we have $F \xrightarrow{1-u} A \wedge B$, $F \xrightarrow{1-v} A$, and $F \xrightarrow{1-w} B$ for every argument F in S_I. Moreover, since $\|A \wedge B\|_I^\mathsf{G} = \min\{\|A\|_I^\mathsf{G}, \|B\|_I^\mathsf{G}\}$, i.e. $u = \min\{v, w\}$ we obtain $1 - u = \max\{1 - v, 1 - w\}$, as required by **($\mathbf{A}^w.\wedge$)**.

($\mathbf{A}^w.\vee$),($\mathbf{C}^w.\vee$): Let $A \vee B$ be an argument in S_I and let $\|A \vee B\|_I^\mathsf{G} = u$, $\|A\|_I^\mathsf{G} = v$, and $\|B\|_I^\mathsf{G} = w$. Then, by definition of S_I, we have $F \xrightarrow{1-u} A \vee B$, $F \xrightarrow{1-v} A$, and $F \xrightarrow{1-w} B$ for every argument F in S_I. Moreover, since $\|A \vee B\|_I^\mathsf{G} = \max\{\|A\|_I^\mathsf{G}, \|B\|_I^\mathsf{G}\}$, i.e. $u = \max\{v, w\}$ we obtain $1 - u = \min\{1 - v, 1 - w\}$. Consequently **($\mathbf{A}^w.\vee$)** and **($\mathbf{C}^w.\vee$)** are satisfied.

($\mathbf{C}^w.\supset$), ($\mathbf{A}^w.\supset$), ($\mathbf{B}^w.\supset$),($\mathbf{G}^w.\supset$): Let $A \supset B$ be an argument in S_I and let $\|A \supset B\|_I^\mathsf{G} = u$, $\|A\|_I^\mathsf{G} = v$, and $\|B\|_I^\mathsf{G} = w$. Then, by definition of S_I, we have $F \xrightarrow{1-u} A \supset B$, $F \xrightarrow{1-v} A$, and $F \xrightarrow{1-w} B$ for every argument F in S_I. By the definition of the truth function for implication in G, we obtain

$$u = \begin{cases} 1 & \text{if } v \leq w \\ w & \text{otherwise.} \end{cases}$$

and consequently

$$1 - u = \begin{cases} 0 & \text{if } 1 - w \leq 1 - v \\ 1 - w & \text{otherwise.} \end{cases}.$$

But this means that indeed all attack principles in $\mathcal{P}_{\mathsf{G}}$ regarding implication are satisfied.

($\mathbf{A}^w.\bot$): By definition every argument in S_I attacks $\bot$ with weight $1 = 1 - \|\bot\|_I^{\mathsf{G}}$. □

Remark 6.4.1. *It is well known that the truth of formulas in Gödel logic actually does not depend on the absolute values of the degrees of truth (other than* 0 *and* 1*) assigned to atomic propositions, but only on the relative order of these values. This fact has repercussions for the argumentation based interpretation of Gödel logics discussed above. It means that argumentative immunity with respect to the principles* $\mathcal{P}_{\mathsf{G}}$ *only concerns the relative order of weights. This in turn implies that we may focus on weighted argumentation frames, where the weights attached to attacks between arguments reflect* rankings *of attacks, which is one of three possible ways of assigning meaning to weights in [DHM+11].*

6.5 Characterizing Łukasiewicz and Product Logic Through WSAFs

Gödel logic G is one of three fundamental t-norm based fuzzy logics [H98́]. The other two are Łukasiewicz logic Ł and Product logic P.

Definition 6.5.1 (Triangular-norm)**.** *A* t-norm *(or* triangular-norm*) is a binary function* $*$ *on* $[0, 1]$ *such that*

1. $*$ *is commutative and associative, i.e. for all* $x, y, z \in [0, 1]$

$$x * y = y * x$$
$$(x * y) * z = x * (y * z)$$

2. $*$ *is non-decreasing in both arguments*

$$x_1 \leq x_2 \quad \textit{implies} \quad x_1 * y \leq x_2 * y$$
$$y_1 \leq y_2 \quad \textit{implies} \quad x * y_1 \leq x * y_2$$

3. $1 * x = x$ *and* $0 * x = 0$ *for all* $x \in [0, 1]$.

A function $f : [0, 1]^2 \to [0, 1]$ is *continuous* if for all convergent sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have

$$f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} f(x_n, y_n)$$

A t-norm $*$ is a *continuous t-norm* if t is a continuous function on $[0, 1]^2$. For an extensive study of t-norms and continuous t-norms see [KMP13].

Example 6.5.1. *The following three are prominent examples of continuous t-norms:*

- *The* Gödel t-norm *(*minimum*):* $x *_{\mathsf{G}} y = \min\{x, y\}$
- *The* Łukasiewicz t-norm*:* $x *_{\mathsf{Ł}} y = \max\{0, x + y - 1\}$
- *The* Product t-norm*:* $x *_{\mathsf{P}} y = x \cdot y$

If we consider continuous t-norms as truth functions for conjunction, we can define the corresponding truth function for implication in a unique way. This truth function for implication is called *residuum*.

Proposition 6.5.1 (Residuum)**.** *Let* $*$ *be a continuous t-norm. Then, for any* $x, y, z \in [0; 1]$*, the operation*

$$x \supset_* y = \max\{\ z \mid x * z \leq y\ \}$$

is the unique operation satisfying the condition

$$(x * z) \leq y \ \textit{iff}\ z \leq (x \supset_* y)$$

the operation $\supset_*$ *is called the* residuum *of the t-norm* $*$*.*

Example 6.5.2. *In the following table there are the risidua of the three main continuous t-norm*

	T-norm	Residuum
G	$x *_{\mathsf{G}} y = \min\{x, y\}$	$x \supset_{*_{\mathsf{G}}} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$
Ł	$x *_{\mathsf{Ł}} y = \max\{0, x + y - 1\}$	$x \supset_{*_{\mathsf{Ł}}} y = \min\{1, 1 - x + y\}$
P	$x *_{\mathsf{P}} y = x \cdot y$	$x \supset_{*_{\mathsf{P}}} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$

Table 6.1: Continuous t-norms with the corresponding residua

Since for any continuous t-norm $*$ the function min and max can be defined in terms of $*$ and $\supset_*$[2], and they extend the bivalent truth-tables for classical conjunction and disjunction on $\{0, 1\}$, respectively, we can consider them the truth-functions of *weak conjunction* ($\wedge$) and *weak disjunction* ($\vee$) while the one interpreted by the t-norm is referred as *strong conjunction* (&). In the case of Gödel logic the two conjunctions coincide, but they differ for all other t-norms. Negation can be defined as $\neg_* x := x \supset_* 0$, the fuzzy interpretation of *reductio ad absurdum*.

[2] $\min\{x, y\} = x * (x \supset_* y)$ and $\max\{x, y\} = \min\{(x \supset_* y) \supset_* y, (y \supset_* x) \supset_* x\}$

Example 6.5.3. *The (residual) negation of the three main continuous t-norm is defined as follows:*

$$\neg_{*_{\mathsf{G}}} x = \neg_{*_{\mathsf{P}}} x = \begin{cases} 1 & \textit{if } x = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

$$\neg_{*_{\mathsf{Ł}}} x = 1 - x.$$

The functions $*$, $\supset_*$, min and max equip the interval $[0,1]$ with an algebraic structure that can be used for a standard definition of algebraic semantics for fuzzy logic. These algebras are called *t-algebras* and are denote them by $[0,1]_*$.

Definition 6.5.2 (T-Algebra [CFHN15])**.** *Let $*$ be a continuous t-norm, we define the* t-algebra *of $*$ as the algebra*

$$[0,1]_* = \langle [0,1], *, \supset_*, \min, \max, 0, 1 \rangle$$

where $\supset_*$ is the residuum of $*$, min *and* max *are the minimum ad maximum in the usual order of reals.*

For any set of K of continuous t-norms we denote the corresponding set of t-algebras by $\mathbb{K}$, and vice versa.

Now we can define the syntax and standard semantics of logics based on continuous t-norms as follows:

Definition 6.5.3 (Syntax and Standard Semantics of Logics of Continuous T-norms [CFHN15])**.** *The language $\mathcal{L}$ of the propositional fuzzy logic L_K of continuous t-norms consists of the propositional variables p, q, r, ..., the binary propositional connectives & (strong conjunctions), $\supset$ (implication), $\wedge$ (weak conjunction), $\vee$ (weak disjunction), the unary propositional connective $\neg$ (negation) and the propositional constants $\bar{1}$ (truth) and $\bar{0}$ (falsity).*

The set of propositional variables of $\mathcal{L}$ will be denoted by Var and the set of all formulas of $\mathcal{L}$ by $Fm_{\mathcal{L}}$. Capital letters denotes formulas and upper case greek letters denotes sets of formulas.

A $[0,1]$-evaluation of propositional variables is a mapping $e_* \colon Var \to [0,1]$. The evaluation of propositional variables extends uniquely the $*$-evaluation to all formulas by the following recursive definition.

For any proposition variable p and any formula A, B:

$$\begin{array}{ll} e_*(p) = e(p) & e_*(A \& B) = e_*(A) * e_*(B) \\ e_*(\bar{0}) = 0 & e_*(A \supset B) = e_*(A) \supset_* e_*(B) \\ e_*(\bar{1}) = 1 & e_*(A \wedge B) = \min\{e_*(A), e_*(B)\} \\ e_*(\neg A) = \neg_*(e_*(A)) & e_*(A \vee B) = \max\{e_*(A), e_*(B)\} \end{array}$$

The *semantic consequence relation* of L_K is defined as follows.

Definition 6.5.4 (Semantic Consequence Relation of L_K [CFHN15])**.** *A $*$-evaluation e_* is a $*$-model of a set Γ of formulas if $e(A) = 1$ for all $A \in \Gamma$. A formula A is a semantic consequence of Γ in L_K ($\Gamma \models_K A$) if for each $* \in K$, all $*$-models e_* of Γ are $*$-models of A.*

A formula A is L_K-valid, $\models_K A$, iff $e_*(A) = 1$ for all $*$-evaluations e_*.

In general it is not possible to axiomatize the consequence relation $\models_K$ by rules with finitely many premises, but its finitary version[3] $\models_K^{fin}$ is finitely axiomatizable for any set $\mathbb{K}$ of t-algebras, with modus ponens as the only deduction rule.

Definition 6.5.5. *The logic of a set K of continuous t-norms (or equivalently the logic of $\mathbb{K}$) is identified with the finitary consequence relation $\models_K^{fin}$ and denoted by L_K .*

The logics of $*_{\mathsf{G}}$, $*_{\mathsf{Ł}}$ and $*_{\mathsf{P}}$ are, respectively, called Łukasiewicz *(Ł),* product *(P) and* Gödel logic *(G). The logic of all continuous t-norms is called* basic logic *(BL).*

A Hilbert-style axiomatisation for Łukasiewicz Logic is the following, where *modus ponens* is the only inference rule.

$$
\begin{array}{rl}
[Tr]: & (F \supset G) \supset ((G \supset H) \supset (F \supset H)) \\
[We]: & F \supset (G \supset F) \\
[Ex]: & (F \supset (G \supset H)) \supset (G \supset (F \supset H)) \\
[\wedge\text{-}1]: & (F \wedge G) \supset F \\
[\wedge\text{-}2]: & (F \wedge G) \supset G \\
[\wedge\text{-}3]: & (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G))) \\
[\vee\text{-}1]: & F \supset (F \vee G) \\
[\vee\text{-}2]: & G \supset (F \vee G) \\
[\vee\text{-}3]: & (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F)) \\
[Lin]: & (F \supset G) \vee (G \supset F) \\
[\bot]: & \bot \supset F \\
[Waj]: & ((F \supset G) \supset G) \supset ((G \supset F) \supset F)
\end{array}
$$

In this section we explore attack principles with respect to which Ł and P are argumentatively sound and complete. The discussion of possible interpretations of these principles is deferred to Section 6.6.

Attack principles that characterise strong conjunction for Ł and P are obtained by stipulating that the weight of an attack on a conjunction is determined by the respective co-t-norm:

(Ł^w.&) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \,\&\, B$, then $z = \min\{1, x + y\}$.

[3] $\Gamma \models_K^{fin} A$ iff there is a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma \models_K A$.

$(\mathbf{P}^w.\&)$ If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \& B$, then $z = x + y - xy$.

Correspondingly we obtain the following attack principles for implications:

$(\mathbf{Ł}^w.\supset)$ If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \supset B$, then $z = \max\{0, y - x\}$.

$(\mathbf{P}^w.\supset)$ If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, where $x < y$, and $F \xrightarrow{z} A \supset B$, then $z = \frac{y-x}{1-x}$.

The condition $x < y$ in $(\mathbf{P}^w.\supset)$ indicates that we assume the basic attack principles $(\mathbf{A}^w.\supset)$ and $(\mathbf{C}^w.\supset)$, which cover the case where $x \geq y$, are still present.

Definition 6.5.6.
$\mathcal{P}_{Ł} = \mathcal{P}_B \cup \{(\mathbf{Ł}^w.\&), (\mathbf{Ł}^w.\supset)\}$,
$\mathcal{P}_{\mathsf{P}} = \mathcal{P}_B \cup \{(\mathbf{P}^w.\&), (\mathbf{P}^w.\supset)\}$.

Given corresponding Hilbert-style proof systems, it is straightforward to show in analogy to Theorem 6.4.2 that Ł and P are argumentatively sound relative to $\mathcal{P}_{Ł}$ and $\mathcal{P}_{\mathsf{P}}$, respectively. Likewise, argumentative completeness can be checked in prefect analogy to the proof of Theorem 6.4.3. Since the proofs are routine, lengthy, but not very informative, we just state the corresponding results.

Theorem 6.5.1 (Argumentative Soundness and Completeness of Ł)**.** *Every formula is $\mathcal{P}_{Ł}$-argumentatively immune formula if and only if it is* Ł*-valid.*

Theorem 6.5.2 (Argumentative Soundness and Completeness of P)**.** *Every formula is $\mathcal{P}_{\mathsf{P}}$-argumentatively immune formula if and only if it is* P*-valid.*

6.6 Justifying Attack Principles for Ł and P

While the basic attack principles $\mathcal{P}_B$, but also the additional principles introduced in Section 6.4, are easy to grasp also independently of any specific knowledge about Gödel logic or fuzzy logics in general, this is hardly the case for $(\mathbf{Ł}^w.\&)$ and $(\mathbf{Ł}^w.\supset)$ or for $(\mathbf{P}^w.\&)$ and $(\mathbf{P}^w.\supset)$. Indeed, considering only what we have presented in Section 6.5, one may suspect that Theorems 6.5.1 and 6.5.2 amount to purely formal and in fact rather straightforward technical observations. It is therefore highly desirable to explore to which extend these results can be employed to establish connections between fuzzy logics, that shed new light on the informal meaning on argument (attack) strength on the one hand and degrees of truth or acceptability on the other hand.

Revisiting $(\mathbf{Ł}^w.\&)$ under the just mentioned perspective, we suggest to attach the following informal reading to it:

$(\mathbf{Ł}^w.\&)$ If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \& B$, then $z = \min(1, x + y)$.
A strong conjunction is attacked with the weight that results from summing up the weights of attacks on its conjuncts; but the sum is capped at the maximal weight.

Summing up weights of attack is certainly very reasonable if the underlying arguments are *independent*. At this point it is important to recall that formulas only denote claims of arguments, but—except in a degenerated case—are not already full arguments themselves. We stipulated that $F \xrightarrow{x} A$ means that x is the supremum of the all the weights that we obtain if we take into account all arguments with claim F that attack an argument with claim A in some specific way. This now provides a basis for a modelling scenario that is able to explain the difference of the meaning of weak conjunction ($\wedge$) and strong conjunction ($\&$): $A \wedge A$ is logically equivalent to A and consequently attacks on $A \wedge A$ are treated as indistinguishable from attacks on A; however determining the overall weight against the claim $A \& A$ calls for exhibiting two *independent* attacks on A, unless we find that already A alone is attacked with maximal weight. More generally, $F \xrightarrow{x} A \& B$ means that x is the (truncated) sum of weights of independent attacks with claim F on A and B, respectively. The consideration of strong conjunction in the sense of Łukasiewicz logic thus seems to be justified only with respect to argumentation frames that are rich enough to contain (also) independent arguments against corresponding claims.

Example 6.6.1. *Recall Examples 2 and 3, where we considered two arguments with the following respective claims: (A) "The majority of the population strongly supports its government" and (B) "The majority of the population believes that the economy is growing". Considering the further claim (X) "Many people are worried about their future". There clearly is some tension between X and A and, likewise, between X and B. Assume that this tension is witnessed by attacking arguments involving these claims. Suppose that we have no direct access to these arguments, but that we are informed that the following weights arise for an SAF, i.e. when we abstract away from the underlying arguments: $X \xrightarrow{0.7} A$ and $X \xrightarrow{0.9} B$. On the basis of just this information, it is difficult to decide which weight one should assign to implicit attacks of arguments claiming X to arguments that claim the conjunction of A and B. But under the following two assumptions it seems reasonable to follow principle* ($\mathbf{Ł}^w\mathbf{.}\&$) *and correspondingly assign the maximal weight to the implicit attack on the conjunctive claim: (1) The conjunction is understood in the strong sense, meaning that the degree of truth of the conjunction is, in general, strictly smaller than the degree of truth of each conjunct. (2) The (unkown) arguments that are represented in the abstraction as $X \xrightarrow{0.7} A$ and $X \xrightarrow{0.9} B$, respectively, are independent and therefore mutually reinforce each other. In other words,* ($\mathbf{Ł}^w\mathbf{.}\&$) *yields $X \xrightarrow{1} A \& B$, since we assume that we have independent arguments against A and B, respectively, where the sum of the weights of these arguments is at least as high as the maximal value for individual weights.*

The case for Product logic P seems to be more subtle than the one for Ł. To assist the reader, we restate the corresponding attack principle for strong conjunction:

($\mathbf{P}^w\mathbf{.}\&$) If $F \xrightarrow{x} A$, $F \xrightarrow{y} B$, and $F \xrightarrow{z} A \& B$, then $z = x + y - xy$.

The crucial expression $x + y - xy$ is not only the co-t-norm of the product t-norm, but is also known as *probabilistic sum*, which hints at a suitable interpretation. To this aim, we suggest to identify the weight of an attack on claim A by an argument with claim F with the conditional probability $p(\overline{A}|F)$, i.e. with the probability that A does not hold, given that F holds ($\overline{A}$ denotes the event that is complementary to that corresponding to proposition A). Argueably, this amounts to an intuitively sound interpretation of argument strength, or more appropriately: attack strength. [4] Similarly to the case for Łukasiewicz logic, let us assume that A and B correspond to two independent events. We then get $p(\overline{A \wedge B}|F) = x + y - xy$ if $x = p(\overline{A}|F)$ and $y = p(\overline{B}|F)$, where F corresponds to any non-empty event.

Note that the above scenario does not directly support the interpretation of arbitrarily nested logically compound statements, since A, B, F refer to classical events (and moreover the event F has to be non-empty). The scenario, however, suggests the use of a two-tiered language: (1) at the inner level, formulas are built up from atomic formulas using the classical connectives $\wedge$, $\vee$, $\neg$, intended to denote events; (2) at the outer level, one may combine classical formulas using connectives from Product logic. The intended meaning of formulas combined by strong (product) conjunction is then given via $(\mathbf{P}^w\mathbf{.}\,\&\,)$, interpreted as suggested. One might want to explore generalizations of this setting using fuzzy events [Yag82] and more general combinations of inner and outer language levels along the line of [HGE95] or [GHE03].

We have only addressed the interpretation of (strong) conjunction, so far. The corresponding principles for implication are uniquely determined, if we stipulate that the truth function for implication is the residuum of the truth function for strong implication. In our context we can enforce residuation by the following attack principle.

$$(\mathbf{R}^w\mathbf{.}{\supset}/\&) \quad F \xrightarrow{x} (A\,\&\,B) \supset C \text{ if and only if } F \xrightarrow{x} A \supset (B\,\&\,C)$$

In presence of $(\mathbf{A}^w\mathbf{.}{\supset})$ and $(\mathbf{C}^w\mathbf{.}{\supset})$, $(\mathbf{R}^w\mathbf{.}{\supset}/\&)$ ensures that $(\mathbf{Ł}^w\mathbf{.}\,\&\,)$ entails $(\mathbf{Ł}^w\mathbf{.}{\supset})$ and, likewise, that $(\mathbf{P}^w\mathbf{.}\,\&\,)$ entails $(\mathbf{P}^w\mathbf{.}{\supset})$.

Once implication is fixed, all other connectives—negation, weak (lattice) conjunction and disjunction, but also strong disjunction, the dual of strong conjunction—are uniquely defined as well. It should be obvious by now, how corresponding attack principles can be formulated.

In principle, we could use the same method to recover an alternative argumentative semantics for classical logic based on WSAFs. Starting from an axiomatization of the considered logic, we use the concept of immunity to recover the constraints on the weights that allow constructing a sound and complete semantics for that logic. We will apply the same procedure also in Chapter 7. In that case, we will use principles defined with either the attack or the support relation.

[4] We refer to [PF18] for a more detailed presentation and evaluation of this interpretation.

6.7 An Analysis of Prelinearity

Recall that by Proposition 6.3.1 of Section 6.3 the formula $(F \supset G) \vee (G \supset F)$ (prelinearity) is $\{(\mathbf{A}^w.\vee), (\mathbf{C}^w.\supset)\}$-immune. Given the centrality of prelinearity for t-norm based fuzzy logics, it may be useful to emphasize that only two rather reasonable principles on implicit attacks are needed to render this axiom argumentatively immune.

1. Corresponding to $(\mathbf{C}^w.\supset)$: An implication is attacked with some positive weight only if the implying formula is attacked with less weight than the implied formula.

2. Regarding $(\mathbf{A}^w.\vee)$: The proof of Proposition 6.3.1 shows that actually only a *weak form* of this principle is needed. Namely, if a claim F is not attacked at all, then neither is any (logically weaker) claim of the form $F \vee G$. Equivalently: any positive attack on a disjunction entails positive attacks on both disjuncts.

These observations are certainly encouraging from the perspective of fuzzy logic, since they seem to indicate that rather mild conditions on implicit attack already single out as possible 'logics of weighted argumentation' (in our current sense) those that satisfy an axiom that can be considered a hallmark of all deductive fuzzy logics. (See, e.g., [BC06] for a general characterization of fuzzy logics that focuses on prelinearity.) However, it is important to remember that prelinearity can also be expressed in a purely implicative form. In particular the standard proof systems for Hajek's BL, the logic of all continuous t-norms [Háj98], features the following version of the axiom:

$$\text{[PreLin:]} \quad ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H)$$

Likewise PreLin, rather than $(F \supset G) \vee (G \supset F)$, is among the axioms of MTL, the logic of all left-continuous t-norms [EG01]. Therefore it is important to take note of the following fact.

Proposition 6.7.1. PreLin *is not $\mathcal{P}_B$-argumentatively immune.*

Proof. Clearly, only the principles $(\mathbf{A}^w.\supset)$, $(\mathbf{B}^w.\supset)$, and $(\mathbf{C}^w.\supset)$ of $\mathcal{P}_B$ are relevant. Recall that $(\mathbf{A}^w.\supset)$ and $(\mathbf{C}^w.\supset)$ jointly express that an implication is attacked with a non-zero weight if and only if the implied formula is attacked with a higher weight than the implying formula. $(\mathbf{B}^w.\supset)$ bounds the weight of an attack on an implication by the weight of an attack on the implied formula. It is therefore straightforward to check that all three principles are satisfied, if, in a given WSAF, for an arbitrary (claim of an) argument X, the weights of corresponding attacks on the subformulas of *PreLin* are as follows:

$$
\begin{aligned}
&X \xrightarrow{0} F,\\
&X \xrightarrow{1} G,\\
&X \xrightarrow{1} H,\\
&X \xrightarrow{0.5} F \supset G,\\
&X \xrightarrow{0.5} (F \supset G) \supset H,\\
&X \xrightarrow{0} G \supset F,\\
&X \xrightarrow{0.5} (G \supset F) \supset H,\\
&X \xrightarrow{1} ((G \supset F) \supset H) \supset H,\\
&X \xrightarrow{1} ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H).
\end{aligned}
$$

Since *PreLin* is attacked with weight 1, it is not $\mathcal{P}_B$-argumentatively immune. □

Let us make two observations about the assignment of weights to attacks used in the above proof. (1) Although the weights on attacks to F, G, and H are in $\{0, 1\}$, some implications involving only these subformulas are attacked with the intermediary weight 0.5. (2) Although the respective weights of attacks on the immediate subformulas of $(F \supset G) \supset H$ and of $((G \supset F) \supset H) \supset H$ are identical (0.5 for the implying formula, and 1 for the implied formula), these formulas are attacked with different weights. This motivates the following definitions and further observation.

Definition 6.7.1. *A WSAP is* compatible with the unweighted case *if the weight of an attack on any formula whose all subformulas are attacked with weights in* $\{0, 1\}$ *is also either* 0 *or* 1*.*

Definition 6.7.2. *A WSAP has a* functional weight assignment *if for each logical connective, the weight of an attack on a compound formula only depends on the weights of attacks on its immediate subformulas.*

Proposition 6.7.1 can be strengthened as follows.

Proposition 6.7.2. *Let A be a WSAF with a functional weight assignments and compatible with the unweighted case.* PreLin *is not $\mathcal{P}_B$-argumentatively immune over A.*

Proof. It is straightforward to check that the following weight assignment is functional, compatible with the unweighted case, and still satisfies $(\mathbf{A}^w.\supset)$, $(\mathbf{B}^w.\supset)$, and $(\mathbf{C}^w.\supset)$.

$$\begin{aligned}
&X \xrightarrow{0.3} F,\\
&X \xrightarrow{0.6} G,\\
&X \xrightarrow{0.9} H,\\
&X \xrightarrow{0.4} F \supset G,\\
&X \xrightarrow{0.5} (F \supset G) \supset H,\\
&X \xrightarrow{0} G \supset F,\\
&X \xrightarrow{0.7} (G \supset F) \supset H,\\
&X \xrightarrow{0.8} ((G \supset F) \supset H) \supset H,\\
&X \xrightarrow{1} ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H).
\end{aligned}$$

Since *PreLin*is attacked with weight 1, it is not $\mathcal{P}_B$-argumentatively immune. □

The question arises which further principles guarantee the argumentative immunity of PreLin. Of course, since G, $\mathsf{Ł}$, and P are argumentatively sound, we know that each of $(\mathbf{G}^w.\supset)$, $(\mathbf{Ł}^w.\supset)$, and $(\mathbf{P}^w.\supset)$, separately, but in conjunction with $\mathcal{P}_B$, suffices to render PreLin argumentative immune with respect to corresponding, pairwise incompatible, set of attack principles. Motivated by the search for a *general*, not logic specific principle that suffices to justify PreLin, we suggest the following.

$(\mathbf{D}^w.\supset)$ If $F \xrightarrow{0} A$, $F \xrightarrow{x} B$ and $F \xrightarrow{y} A \supset B$, then $y \geq x$.
If the implying formula is not attacked at all, then the implication is attacked with at least the same weight as the implied formula.

Proposition 6.7.3. PreLin *is* $\mathcal{P}_B \cup \{(\mathbf{D}^w.\supset)\}$*-argumentatively immune.*

Proof. For some claim X, let the weights of corresponding attacks to subformulas of PreLin as follows:

$$\begin{aligned}
&X \xrightarrow{f} F,\\
&X \xrightarrow{g} G,\\
&X \xrightarrow{h} H,\\
&X \xrightarrow{w} F \supset G,\\
&X \xrightarrow{x} (F \supset G) \supset H,\\
&X \xrightarrow{v} G \supset F,\\
&X \xrightarrow{u} (G \supset F) \supset H,\\
&X \xrightarrow{y} ((G \supset F) \supset H) \supset H,\\
&X \xrightarrow{z} ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H).
\end{aligned}$$

Assume that $z > 0$; then $(\mathbf{C}^w.\supset)$ entails $x < y$, and, since $y > 0$, further also $u < h$. We now distinguish two cases.

$g \geq f$**:** By $(\mathbf{C}^w.\supset)$ we have $v = 0$, and therefore can apply $(\mathbf{D}^w.\supset)$ to obtain $u \geq h$, which contradicts $u < h$.

$g < f$**:** By $(\mathbf{C}^w.\supset)$ we have $w = 0$, and therefore, by $(\mathbf{D}^w.\supset)$, $h \leq x$. On the other hand, applying $(\mathbf{B}^w.\supset)$ to the right subformula of PreLin yields $y \leq h$. Since $x < y$, we obtain $x < h$, contradicting $h \leq x$.

These contradictions imply that PreLin cannot be attacked with positive weight. □

6.8 Instantiating Attack Weights Using Arguments Strength

In Section 6.1, we have considered the weights on the attacks as a primitive notion. However, as briefly discussed in Section 6.2, they can also be considered a derived one defined, for instance, in terms of the strength of the argument involved in each specific attack. The strength of an argument can be seen as its difficulty to be attacked. The harder is to attack an argument, the stronger it is.

We assume to operate in a fully instantiated frame where arguments have the usual structure (a support set, a claim and a method of inference between the support and the claim) and we can define the attack relations in the several ways introduced in Definition 3.3.4 or summarized, for the sequent-based frames, in Table 3.1. For example, suppose to assist to a public debate where Player 1 supports and wants to convince the audience that argument $\langle \Gamma_A; A\rangle$ is true. Player 2, on the contrary, wants to convince the audience that $\langle \Gamma_{\neg A}; \neg A\rangle$ is true. The two players, in turns, will try to attack the other player's last argument. For the winning condition, we follow Dung's criterion [Dun95]: whoever has the last word wins the debate. However, not all attacks should be allowed, or, at least, not all the attacks should be considered the same. Suppose that Player 1 during the discussion has used the argument $\langle \Gamma_{i+1}; \alpha_{i+1}\rangle$ to $\mathscr{R}$-attack, for some attack relation $\mathscr{R}$, Player 2's argument $\langle \Gamma_i; \alpha_i\rangle$. If Player 2 has no other argument against $\langle \Gamma_{i+1}; \alpha_{i+1}\rangle$ except that a contradictory one[5], Player 2's attack should not be considered valid for proceeding in the debate and Player 1 should be declared the winner. Contradictory arguments are the easiest to be attacked and therefore, the weakest ones.

We can generalize the idea expressed by this example with the definition of *relevant attacks*. The strength function S is a function with values in the unit interval [0, 1] and with domain all the arguments of the relative argumentation frame.

Definition 6.8.1 (Relevant Attacks)**.** *Let $AF = \langle Ar, Attack(\mathscr{A})\rangle$ be an argumentation frame with fully instantiated arguments, $\mathscr{A}$ a set of attack relations and $Attack(\mathscr{A})$ a relation over $Ar \times Ar$ defined as usual. Let $\langle \Gamma_1, \psi_1\rangle, \langle \Gamma_2, \psi_2\rangle \in Ar$ and S an argument strength function. If for some $\mathscr{R} \in \mathscr{A}$, $\langle \Gamma_1, \psi_1\rangle$ $\mathscr{R}$-attacks $\langle \Gamma_2, \psi_2\rangle$, then the attack*

[5] Contradictory arguments are arguments whose support set is inconsistent.

relation is relevant *only if the strength of the attacking argument is greater or equal to the strength of the attacked one, i.e.* $S(\langle \Gamma_1, \psi_1 \rangle) \geq S(\langle \Gamma_2, \psi_2 \rangle)$.

Note that in the above definition, the strength function S has not been instantiated. In principle, S can be defined in various ways and its definition might change according to the context considered. E.g., if we want to use argumentative tools to analyse political debates, the persuasion ability of the debaters should be used in the definition of the arguments' strength [CB11, CIGB$^+$12]. Therefore, S is not necessarily determined by the logic of the considered arguments. The attack principles only impose constraints on these functions.

Let us consider now the attack principles introduced in Section 3.3 interpreted in fully instantiated argumentation frames. If we consider only relevant attacks we have that the relevance of the attack (or attacks) in the conclusion of an attack principle directly follows from the relevance of the attack (or attacks) in the premise only if the strength of the arguments involved satisfy specific constraints. E.g. in the case of **(A.**$\wedge$**)**, if in a given argumentation frame, the argument $\langle \Gamma_X, X \rangle$ $\mathscr{R}$-attacks the argument $\langle \Gamma_A, A \rangle$, $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_A, A \rangle)$ and $\langle \Gamma_X, X \rangle$ $\mathscr{R}$-attacks $\langle \Gamma_{A\wedge B}, A \wedge B \rangle$, then $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle)$ directly follows from $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_A, A \rangle)$ only if $S(\langle \Gamma_A, A \rangle) \geq S(\Gamma_{A\wedge B}, A \wedge B)$. Thus, the general constraint relative to the attack principle **(A.**$\wedge$**)** on the strengths of the arguments is $S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle) \leq \min\{S(\langle \Gamma_A, A \rangle), S(\langle \Gamma_B, B \rangle)\}$. We now analyse all the other attack principles. Since the constraints on the strength of the argument are independent of the attack relation, we will omit the $\mathscr{R}$ sign in the following analysis.

(C.$\wedge$**)** Suppose that $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_{A\wedge B}, A \wedge B \rangle$ and $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle)$. The relevance of the attack $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_A, A \rangle$ or $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_B, B \rangle$ direclty follows from the premise only if $S(\langle \Gamma_A, A \rangle) \leq S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle)$ or of the attack $S(\langle \Gamma_B, B \rangle) \leq S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle)$, i.e. $S(\langle \Gamma_{A\wedge B}, A \wedge B \rangle) \geq \min\{S(\langle \Gamma_A, A \rangle), S(\langle \Gamma_B, B \rangle)\}$

(A.$\vee$**)** Suppose $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_{A\vee B}, A \vee B \rangle$ and $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_{A\vee B}, A \vee B \rangle)$. The relevance of the attacks $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_A, A \rangle$ and $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_B, B \rangle$ directly follows from the premise only if $S(\langle \Gamma_A, A \rangle) \leq S(\langle \Gamma_4, A \vee B \rangle)$ and $S(\langle \Gamma_B, B \rangle) \leq S(\langle \Gamma_4, A \vee B \rangle)$, i.e. $\max\{S(\langle \Gamma_A, A \rangle), S(\langle \Gamma_B, B \rangle)\} \leq S(\langle \Gamma_4, A \vee B \rangle)$.

(C.$\vee$**)** Suppose $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_A, A \rangle$, $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_B, B \rangle$, $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_A, A \rangle)$ and $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_B, B \rangle)$. The relevance of the attack $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_{A\vee B}, A \vee B \rangle$ directly follows from the premises only if $S(\langle \Gamma_{A\vee B}, A \vee B \rangle) \leq S(\langle \Gamma_A, A \rangle)$ or $S(\langle \Gamma_{A\vee B}, A \vee B \rangle) \leq S(\langle \Gamma_B, B \rangle)$, i.e. $S(\langle \Gamma_{A\vee B}, A \vee B \rangle) \leq \max\{S(\langle \Gamma_A, A \rangle), S(\langle \Gamma_B, B \rangle)\}$.

(A.$\supset$**)** Suppose $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_{A\supset B}, A \supset B \rangle$ and $S(\langle \Gamma_X, X \rangle) \geq S(\langle \Gamma_{A\supset B}, A \supset B \rangle)$. The relevance of the attack $\langle \Gamma_X, X \rangle \longrightarrow \langle \Gamma_B, B \rangle$ directly follows from the premise only if $S(\langle \Gamma_B, B \rangle) \leq s(\langle \Gamma_{A\supset B}, A \supset B \rangle)$.

(C.$\supset$) Suppose $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$ and $S(\langle \Gamma_X, X\rangle) \geq S(\langle \Gamma_B, B\rangle)$. The relevance of the attack $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\supset B}, A \supset B\rangle$ directly follows from the premise only if $S(\langle \Gamma_B, B\rangle) \geq S(\langle \Gamma_{A\supset B}, A \supset B\rangle)$.

The case of the attack principles for negation is different from the others because either in the premise or in the conclusion of the principle, there is no attack relation, only not-attacking ones. Thus, we cannot recover any constraint on the strength of the arguments involved.

In Section 6.1, there have mentioned some possible ways to interpret the weight attached to the attack relations. Using this general notion of *strength* function, we can introduce an additional way of interpreting the weights. Since we have not used any specific definition of *strength* of an argument, except for imposing its domain and range, the results deriving from this interpretation are still quite general. As done to study the attack principles, we introduce an intermediary level of abstraction between the possible ways of understanding the attack weights.

Definition 6.8.2 (Weights on Attacks)**.** *Let $AF = \langle Ar, Attack(\mathscr{A})\rangle$ be an argumentation frame with fully instantiated arguments and S an argument strength function. We define the weight on the attack relation between any two arguments in Ar as the positive part*[6] *of the difference between the strength of the attacking argument and the strength of the attacked one, i.e. $w : Ar \times Ar \longrightarrow [0,1]$ and for any $\langle \Gamma_1, \psi_1\rangle$ and $\langle \Gamma_2, \psi_2\rangle$ in Ar*

$$w(\langle \Gamma_1, \psi_1\rangle, \langle \Gamma_2, \psi_2\rangle) = \begin{cases} f^+(S(\langle \Gamma_1, \psi_1\rangle) - S(\langle \Gamma_2, \psi_2\rangle)) & \textit{if } \langle \Gamma_1, \psi_1\rangle \longrightarrow \langle \Gamma_2, \psi_2\rangle \\ 0 & \textit{otherwise} \end{cases}$$

Using this definition of weights, the constraints under which the relevance of the attack relation in the conclusion of an attack principle directly follow from the relevance of the attack relation in the premise can be reformulated as follows. For the case of **(A.$\wedge$)** if we denote with x the weight of the attack relation between the generic arguments $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_A, A\rangle$ ($x = w(\langle \Gamma_X, X\rangle, \langle \Gamma_A, A\rangle)$), with y the weight of the attack relation between $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_B, B\rangle$ ($y = w(\langle \Gamma_X, X\rangle, \langle \Gamma_B, B\rangle)$), and with z the weight of the attack between $\langle \Gamma_X, X\rangle$ and $\langle \Gamma_{A\wedge B}, A\wedge B\rangle$ ($z = w(\langle \Gamma_X, X\rangle, \langle \Gamma_{A\wedge B}, A\wedge B\rangle)$), then the constraint on the strengths of the arguments $S(\langle \Gamma_{A\wedge B}, A \wedge B\rangle) \leq \min\{S(\langle \Gamma_A, A\rangle), S(\langle \Gamma_B, B\rangle)\}$ can be reformulated as $z \geq \max\{x, y\}$. If we denote with z the weight of the attack on the generic argument $\langle \Gamma_{A*B}, A * B\rangle)$ with $* \in \{\wedge, \vee, \supset\}$, the constraints on the strength of the arguments deriving from the attack principles can be reformulated as shoed in Table 6.2.

[6]The positive part of a real-valued function $f(x)$ is defined as follows.

$$f^+(x) = \max\{f(x), 0\} = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Attack Principle	Constraints on the weights
(A.$\wedge$)	$z \geq \max\{x, y\}$
(C.$\wedge$)	$z \leq \max\{x, y\}$
(A.$\vee$)	$z \leq \min\{x, y\}$
(C.$\vee$)	$z \geq \min\{x, y\}$
(A.$\supset$)	$y \geq z$
(C.$\supset$)	$y \leq z$

Table 6.2: Reformulation of the constraints on the arguments' strength in terms of weights defined as in Definition 6.8.2

Interestingly the constraints that arise from this interpretation of the weights partially coincide with those introduced in Section 6.2. In particular, the constraints deriving from **(A.$\wedge$)** are the same of **(A^w.$\wedge$)**, those deriving from **(C.$\wedge$)** are **(C^w.$\wedge$)**, the constraints deriving from **(A.$\vee$)** and **(C.$\vee$)** are **(A^w.$\vee$)** and **(C^w.$\vee$)**, respectively. About implication we only have that the constraints deriving from **(A.$\supset$)** are those of **(B^w.$\supset$)**.

Having found similar constraints on the weights for both partially instantiated frames (see Section 6.2) and a more specific interpretation of the weights as in Definition 6.8.2, gives us reasons to consider the weight function introduced an appropriate interpretation of the weights.

If we go back to real-world situations, we have that some arguments are intrinsically stronger than others, and they can attack each other in different ways. In fact, in a public debate, one player can decide to attack the support of the opponent's argument. It could also be the case that she chooses to attack the opponent's claim without considering the support part. These different ways of attacks between arguments have been formalised in the literature through the several attack functions recalled in Table 3.1.

Having in mind these potentially very varied scenarios is reasonable to look back at the Definition 6.8.2 and make it dependent on the kind of attack function taken into consideration for the attack. Figure 3.3 shows the relations between the main attack relations considered in the literature. In particular, we can see that the *defeat* attack function is one of the *weakest* attack relations among those considered. The weakness of an attack relation is proportional to the ease to satisfy the corresponding attacking condition. Therefore we are looking for a variation of Definition 6.8.2 that takes into consideration not only the strength of the involved arguments but also the strength of the attack relation considered. For example, suppose that the arguments $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ both belong to the same argumentation frame $AF = \langle Ar, Attack(\mathscr{A}) \rangle$ and that $[Def]$, $[Reb] \in \mathscr{A}$. If $\langle \Gamma_1, \psi_1 \rangle \overset{[Reb]}{\longrightarrow} \langle \Gamma_2, \psi_2 \rangle$, then also $\langle \Gamma_1, \psi_1 \rangle \overset{[Def]}{\longrightarrow} \langle \Gamma_2, \psi_2 \rangle$ holds. However, even if the arguments involved are the same, the *rebuttal*-attack is intuitively stronger then the *defeat*-attack. To reflect this difference in the strength of the two attack relations, we introduce a weight function w with values in [0, 1] and with domain $Ar \times Ar \times \mathscr{A}$. Before introducing such *enriched weight function*, we need to define a *valuation function* over the attack relations.

Definition 6.8.3 (Valuation Attack Function)**.** *Let $AF = \langle Ar, Attack(\mathscr{A})\rangle$ be an argumentation framework such that the attack relations [D-Ucut], [D-Def], [CD-Def], [I-Ucut], [Ca-Ucut], [Ucut], [Reb], [D-Reb], [I-Reb], [Def] and [C-Def] are in $\mathscr{A}$. A* valuation attack function *is any function $v : \mathscr{A} \to [0,1]$ that satisfies the following inequalities.*

$$v([D\text{-}Ucut]) \geq v([D\text{-}Def]) \geq v([I\text{-}Ucut]) \geq v([Def]).$$

$$v([D\text{-}Ucut]) \geq v([CD\text{-}Def]) \geq v([I\text{-}Ucut]).$$

$$v([I\text{-}Ucut]) \geq v([C\text{-}Def]).$$

$$v([Ca\text{-}Ucut]) \geq v([Ucut]) \geq v([Def]).$$

$$v([Ucut]) \geq v([C\text{-}Def]).$$

$$v([D\text{-}Ucut]) \geq v([Ucut]).$$

$$v([Reb]) \geq v([D\text{-}Reb]) \geq v([I\text{-}Reb]) \geq v([Def]).$$

$$v([I\text{-}Reb]) \geq v([C\text{-}Def]).$$

The list of inequalities in Definition 6.8.3 guarantees that valuation attack functions (there could be many, not just one) need to preserve the strength of the attack relations summarised in Figure 3.3.

As done already with the *strength* function, also in this case, we have intentionally left the definition of the *valuation attack function* partially open to keep the results as general as possible. The enriched definition of *weight function* is the following:

Definition 6.8.4 (Enriched Weight Function)**.** *Let $AF = \langle Ar, Attack(\mathscr{A})\rangle$ be an argumentation framework, S a strength function defined over Ar and v a valuation attack function. We define the* enriched weight function *as a function with values in [0, 1] and domain $Ar \times Ar \times \mathscr{A}$ ($w : Ar \times Ar \times \mathscr{A} \longrightarrow [0,1]$) s.t. for any argument $\langle \Gamma_1, \psi_1\rangle$ and $\langle \Gamma_s, \psi_2\rangle$ in Ar and $\mathscr{R} \in \mathscr{A}$*

$$w(\langle \Gamma_1, \psi_1\rangle, \langle \Gamma_2, \psi_2\rangle, \mathscr{R}) = \begin{cases} f^+(S(\langle \Gamma_1, \psi_1\rangle) - S(\langle \Gamma_s, \psi_2\rangle))v(\mathscr{R}) & \textit{if } \langle \Gamma_1, \psi_1\rangle \xrightarrow{\mathscr{R}} \langle \Gamma_2, \psi_2\rangle \\ 0 & \textit{otherwise} \end{cases}$$

The constraints on the weights summarised in Table 6.2 deriving from Definition 6.8.2 and the attack principles are the same induced by Definition 6.8.4.

Proposition 6.8.1. *Let $AF = \langle Ar, Attack(\mathscr{A})\rangle$ be an argumentation framework, S a strength function defined over Ar, v a valuation attack function, w the derived enriched weight function and* AP *the following set of attack principles* AP $= \{(\mathbf{A.}\wedge), (\mathbf{C.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{A.}\supset), (\mathbf{C.}\supset)\}$.

For any $\langle\Gamma_1, \psi_1\rangle$ and $\langle\Gamma_2, \psi_2\rangle$ arguments in Ar such that $\langle\Gamma_1, \psi_1\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_2, \psi_2\rangle$ for some $\mathscr{R} \in \mathscr{A}$. We denote with $\mathscr{R}_{\psi_2}$ the strongest attack relation(s) in $\mathscr{A}$ s.t. $\langle\Gamma_1, \psi_1\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_2, \psi_2\rangle$, i.e.:

$$\mathscr{R}_{\psi_2} = \{\mathscr{R} \in \mathscr{A} | \langle\Gamma_1, \psi_1\rangle \xrightarrow{\mathscr{R}} \langle\Gamma_2, \psi_2\rangle \text{ and } v(\mathscr{R}) = \max\{v(\mathscr{R}_i)| \langle\Gamma_1, \psi_1\rangle \xrightarrow{\mathscr{R}_i} \langle\Gamma_2, \psi_2\rangle\}\}.$$

With w_0 we denote the weight of that attack function defined as in Definition 6.8.2, and with With w_e the enriched weight function defined as in Definition 6.8.4.

The constraints on the enriched weights deriving from imposing that the relevance of the attack(s) in the conclusion of any instantiation over AF of the attack principles in AP *directly follows from the relevance of the attacks(s) in the corresponding instantiation of the attack principle are the following:*

Attack Principle	Constraints on the weights
$(\mathbf{A.}\wedge)$	$z_e \geq \max\{x_e, y_e\}$
$(\mathbf{C.}\wedge)$	$z_e \leq \max\{x_e, y_e\}$
$(\mathbf{A.}\vee)$	$z_e \leq \min\{x_e, y_e\}$
$(\mathbf{C.}\vee)$	$z_e \geq \min\{x_e, y_e\}$
$(\mathbf{A.}\supset)$	$y_e \geq z_e$
$(\mathbf{C.}\supset)$	$y_e \leq z_e$

Proof. We will only show the case of $(\mathbf{A.}\wedge)$, the others are similar.

Suppose that $\langle\Gamma_X, \psi_X\rangle \xrightarrow{\mathscr{R}_A} \langle\Gamma_A, \psi_A\rangle$, $\langle\Gamma_X, \psi_X\rangle \xrightarrow{\mathscr{R}_B} \langle\Gamma_B, \psi_B\rangle$ and $\langle\Gamma_X, \psi_X\rangle \xrightarrow{\mathscr{R}_{A\wedge B}} \langle\Gamma_{A\wedge B}, \psi_{A\wedge B}\rangle$. Then, $v(\mathscr{R}_{A\wedge B}) \geq v(\mathscr{R}_A)$ and $v(\mathscr{R}_{A\wedge B}) \geq v(\mathscr{R}_B)$ from which it follows $v(\mathscr{R}_{A\wedge B}) \geq \max\{v(\mathscr{R}_A), v(\mathscr{R}_B)\}$. We denote with $x_0 = w_0(\langle\Gamma_X, \psi_X\rangle, \langle\Gamma_A, \psi_A\rangle)$, with $y_0 = w_0(\langle\Gamma_X, \psi_X\rangle, \langle\Gamma_B, \psi_B\rangle)$ and with $z_0 = w_0(\langle\Gamma_X, \psi_X\rangle, \langle\Gamma_{A\wedge B}, \psi_{A\wedge B}\rangle)$, then $z_0 \geq \max\{x_0, y_0\}$. Since $x_e = x_0 v(\mathscr{R}_A)$, $y_e = y_0 v(\mathscr{R}_B)$ and $z_e = z_0 v(\mathscr{R}_{A\wedge B})$, then $z_e \geq \max\{x_e, y_e\}$. □

6.9 Analysis of a Field Experiment Using WAFs

In the present section, we analyse using argumentative tools the transcript of a field experiment made in 2016 in Lebanon to study how to strengthen intergroup cooperation [CP19] using weighted bipolar argumentation frames.

Admittedly, this analysis does not directly relate to the attack principles. However, it illustrates the need for introducing weights and bipolar frames. Besides, we notice that people tend to use the same attack or support relation, and they implicitly assign weights to these relations. Thus, the study of weighed bi-polar argumentation frame is justified not only by a theoric question but also by a practical one. Starting from our argumentative analysis, as future work, we would like to use our findings to define an alternative *discussion quality index* and use the argumentation frames we constructed as an intermediate step for the definition of a causal model. Using causal models will then be interesting to test the causal relationships of the model using the statistical data collected from the authors of the experiment. The following analysis is only a starting point for a possible future interdisciplinary project.

In particular, the authors test how a cross-group expert appeal and participation in a cross-group discussion impact intergroup cooperation. The laboratory-in-the-field is set in Beirut and involves interactions between 180 Shia and 180 Sunni Muslim participants. All participants watch a prerecorded expert appeal about cooperation and then, in six-person groups, they are asked to discuss intersectarian cooperation. The discussions are lead by a moderator and lasted 28 minutes on average. The effect of the two interventions[7] on cooperation is tested in a series of games. The authors find that watching experts discussing increases unconditional cooperation, but there is no effect on conditional cooperation. Participation in a cross-sectarian group discussion about cooperation appears not to affect either unconditional or conditional cooperation. However, authors present suggestive observational evidence that group discussion's effectiveness is highly heterogeneous and varies with what they define as *discussion depth*.

We analyse the expert appeal transcript, the participant discussion with lowest discussion score (session 6 - table D) and the one with highest discussion score (session 11 - table D). Starting from some atomic formula whose meaning is described in Table 6.3, we construct the three texts' bipolar argumentation frames. The arguments constructed consist of two parts: the support and the conclusion. In addition to the usual attack relation, we also consider support. All the relations between the arguments are weighted. Both attack and support relations with the respective weight are deducted from the texts' interpretation.

6.9.1 Analysis of the Expert Appeal

The project's overall aim is to understand how to increase intergroup cooperation to make the country, in the specific example Lebanon, a better place to live. It is known that societies with multiple ethnic or religious groups have a lower level of public goods provision [ABE99]. The argument $\langle \Gamma, b \rangle$ denotes the main topic of the expert appeal. The support set Γ is intentionally left unspecified because it is the object of research. Considering some specific attack and support relations, we can deduce which atomic argument belongs to Γ and which is not. Looking at the frame of Figure 6.1, we observe that many are the arguments attacking $\langle \Gamma, b \rangle$; this means that the experts identify many possible causes that had led Lebanon to the current crisis. Besides, they identify also some relation of causality between these causes. More specifically, from the experts' appeal, it emerges that sectarianism is the primary cause that does not enable Lebanon

[7] We refer to the expert appeal and the discussion as *intervention*.

s	Sectarianism	**b**	Lebanon will become a better country
q	Quota system	**c**	Corruption
es	Current electoral system	**m**	Meritocracy
gc	Garbage crisis	**se**	School education
cm	Civic movement	**coo**	Cooperation
he	Home education	**rh**	Solution of the renting house situation
d	The country is devided	**qp**	Quota system for the parliament
p	Patriotism	**scp**	Separation between clerics and politics
eID	Elimination of the sect from the ID cards	**cu**	Curse another sect
clh	Clerics education	**cpa**	Clerics and political acts
pa	Personal acts	**uss**	Unified school system
esp	Eliminating sectarian parties	**ca**	Civic activities
cb	Common benefit	**pb**	Personal benefit
u	Unemployment	**dpl**	Dependence on political leaders
md	Media	**iop**	Impact of ordinary people

Table 6.3: Meaning of the Atomic Formulas

to become a better country. This concept can be transferred to the argumentative frame imposing $w_2 \geq w_1$. Moreover, they also assert that the current electoral system cannot be addressed as the only cause for sectarianism. Still, school education and home education also play a crucial role. For this reason, we assume $w_5 \geq w_4$.

If we look at how the specific attack or support relations are defined, we see that all the attack relations use *Defeating Rebuttal* ($\langle \Gamma_1, \psi_1 \rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg\psi_2$). The attacks labelled by w_6 and w_{11} can still be defined using Defeating Rebuttal. However, they can also be defined with some CS-attack relation, e.g., *Defeat* or any of its variants. All the support relations identified are defined using the positive version of Defeat.

Definition 6.9.1 ([S-Def] support relation)**.** *Let Δ be a set of atomic formulas, $\langle \Gamma_1; \psi_1 \rangle$ and $\langle \Gamma_2; \psi_2 \rangle$ two arguments in $Arg(\Delta)$. We say that $\langle \Gamma_1, \psi_1 \rangle$ [S-Def]-supports $\langle \Gamma_2, \psi_2 \rangle$ ($\langle \Gamma_1, \psi_1 \rangle \overset{[S\text{-}Def]}{\rightsquigarrow} \langle \Gamma_2, \psi_2 \rangle$) if $\mathsf{C}(\langle \Gamma_1, \psi_1 \rangle) \vdash \bigwedge \mathsf{S}(\langle \Gamma_2, \psi_2 \rangle)$.*

Recall that from the text's analysis, we have inferred that $w_2 \geq w_1$ and that $w_5 \geq w_4$. The attack relations labelled by w_2 and w_1 are defined by the same attack relation. The same also holds for the attack relations labelled by w_5 and w_4. Therefore, by Definition 6.8.4, $S(\langle s; \neg b \rangle_{RM}) \geq S(\langle c; \neg b \rangle_{JZ})$ and $S(\langle \neg es,\ se,\ he; \neg s \rangle_{NM}) \geq S(\langle \neg es; \neg s \rangle_{NM})$.

From the text of the expert appeal, we can also infer that the argument $\langle \Gamma_1; \psi_1 \rangle$ is stronger than $\langle \Gamma_2; \psi_2 \rangle$, i.e. $S(\langle \Gamma_1; \psi_1 \rangle) \geq S(\langle \Gamma_2; \psi_2 \rangle)$. From the specific definition of these two arguments, we deduce that the strength function S used within this particular context satisfies the following property.

Definition 6.9.2 (Strength Order Property)**.** *Let* $A = \langle Ar, Attack(\mathscr{A}), Support(\mathscr{S})$ *be a bipolar argumentation frame and* $\langle\Gamma_1, \psi_1\rangle, \langle\Gamma_2, \psi_2\rangle \in Ar$*. The argument* $\langle\Gamma_1, \psi_1\rangle$ *is* stronger *then* $\langle\Gamma_2, \psi_2\rangle$*, i.e.* $S(\langle\Gamma_1, \psi_1\rangle) \geq S(\langle\Gamma_2, \psi_2\rangle)$*, if* $\mathsf{S}(\langle\Gamma_1, \psi_1\rangle) \vdash \mathsf{S}(\langle\Gamma_2, \psi_2\rangle)$.

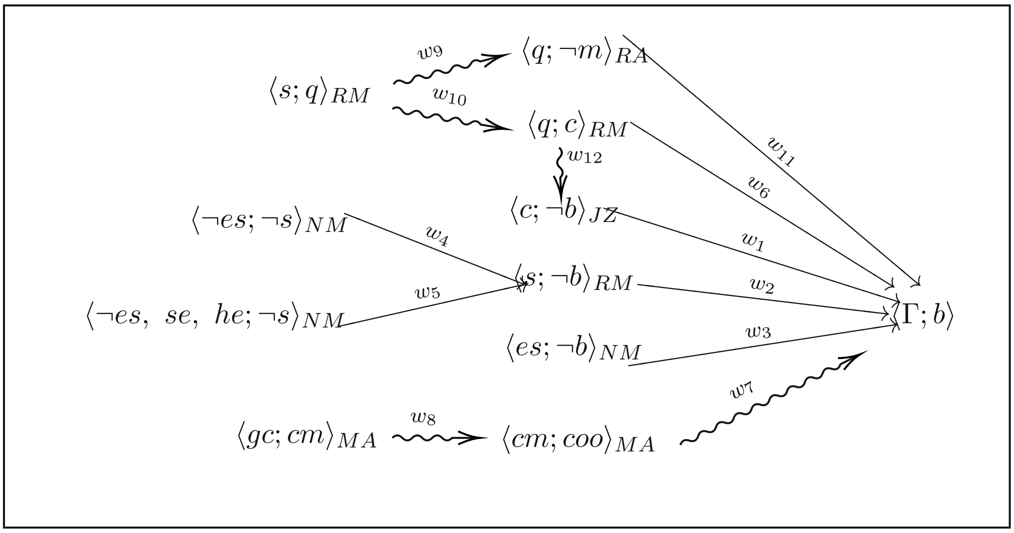


Figure 6.1: Argumentation Frame of the Expert Intervention

6.9.2 Analysis of the cross-group discussion [Session 11 - Table D]

After a first statistical analysis, the experiment authors found that participation in a group discussion does not affect cooperation. However, they observed that the effectiveness of group discussion is highly heterogeneous. For this reason, they defined an index that measures the depth of each discussion. Examining the effects of the depth discussion in each task, it emerges that higher quality of discussion appears to be strongly associated with more cooperation across sectarian lines.

Here we look for which argumentative factors make the discussion with highest discussion score (session 11 - table D) differs from the one with the lowest discussion score (session 6 - table D). From the frame of Figure 6.2, we see that the discussion participants identify cooperation as a critical factor to make Lebanon a better country. Many of the arguments they introduce are then directly related to the argument $\langle coo; b\rangle$. The support relation between $\langle coo; b\rangle$ and $\langle\Gamma; b\rangle$ can be defined using the following definition:

Definition 6.9.3 (Direct Support - [D-Sup])**.** *Let* $\langle\Gamma_1; \psi_1\rangle$ *and* $\langle\Gamma_2; \psi_2\rangle$ *two arguments in* Ar*, a set of fully instantiated arguments. We say that* $\langle\Gamma_1, \psi_1\rangle$ directly supports $\langle\Gamma_2; \psi_2\rangle$ *(*$\langle\Gamma_1, \psi_1\rangle \overset{[D\text{-}Sup]}{\rightsquigarrow} \langle\Gamma_2; \psi_2\rangle$*) if* $\psi_1 \vdash \psi_2$.

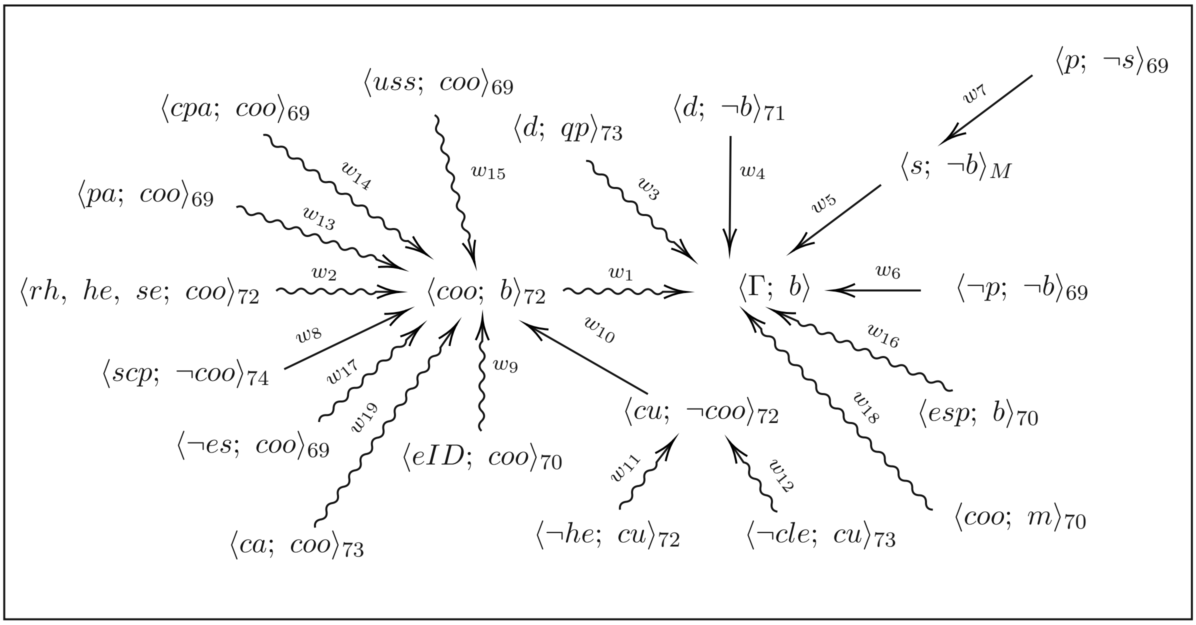


Figure 6.2: Argumentation Frame of the Discussion with Highest Discussion Rate

Like on the SAFs where we only looked at the claims, also in this specific case, the focus is on the claims of the arguments and not on the support sets.

The attack relations labelled by w_8, w_{10} and w_7 are defined by the *Defeat* attack relation while w_4, w_5 and w_6 by *Defeating Rebuttal*.

As done for the experts' appeal argumentative analysis, we deduce some relations between the weights from the text. In particular, we have that $w_6 \geq w_5$ and $w_{15} \geq w_{14}$. By Definition 6.8.4, it follows $S(\langle \neg p; \neg b\rangle) \geq S(\langle s; \neg b\rangle)$ and $S(\langle cpa; coo\rangle) \geq S(\langle pa; coo\rangle)$. Since $\neg p \vdash s$ and $cpa \vdash pa$ are implicitly valid entailments that state that "lack of patriotism brings to sectarianism" and that "clerics and political acts have an influence on private acts", and not *vice versa*, Definition 6.9.1 justifies the strength order.

6.9.3 Analysis of the Cross-Group Discussion [Session 6 - Table D]

The authors of the experiment indicate the discussion 6-D as the one with the lowest discussion score, i.e. a low-quality discussion. From an argumentative point of view, the only attack relation that explicitly uses [D-Reb] is the one labelled by w_6. The others can be defined by *Defeat* or any of its variations. Regarding the support relation, we have that $\langle \neg cb; pb\rangle$ supports $\langle s; q\rangle$ because sectarianism can follow from having the administrators of the country only focused on their own benefits, i.e. $pb \vdash s$. Moreover, from the text, we infer that $w_1 \leq w_2$ and the strength function considered in this case, satisfies the strength order property (see Definition 6.9.1). From the text, we can also infer that $w_5 \geq w_2$ because unemployment leads the people to be dependent on the political leaders and if the political leaders only care about their own benefits, then the condition of the country cannot improve. By contrast, if people are working, even if the political leaders only focus on their own benefits, the country can still be a good place to live because people do not depend on them, i.e. the entaliments $\neg u \vdash \neg dpl$ and $\neg dpl \wedge pb \nvdash \neg b$ holds. The strength function S from which it follows that $S(\langle u; dpl\rangle) \geq S(\langle \neg cb; pb\rangle)$ satisfies the following property.

Definition 6.9.4 (Strength Order Property - 2)**.** *Let* $A = \langle Ar, Attack(\mathscr{A}), Support(\mathscr{S})$ *be a bipolar argumentation frame and* $\langle \Gamma_1, \psi_1\rangle, \langle \Gamma_2, \psi_2\rangle, \langle \Gamma_3, \psi_3\rangle \in Ar$*. The argument* $\langle \Gamma_1, \psi_1\rangle$ *is* stronger *then* $\langle \Gamma_2, \psi_2\rangle$*, i.e.* $S(\langle \Gamma_1, \psi_1\rangle) \geq S(\langle \Gamma_2, \psi_2\rangle)$*, if* $\langle \Gamma_1, \psi_1\rangle$ *and* $\langle \Gamma_2, \psi_2\rangle$ $\mathscr{R}$*-attack* $\langle \Gamma_3, \psi_3\rangle$ *and, in addition,* $\neg \bigwedge \Gamma_1 \vdash \neg\psi_1$ *and* $\neg\psi_1 \wedge \psi_2 \nvdash \neg\psi_3$*.*

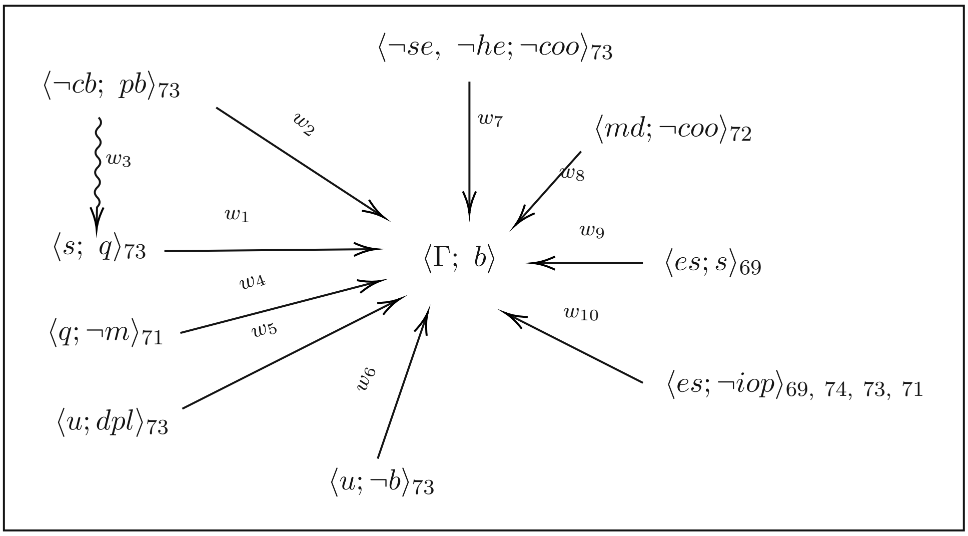


Figure 6.3: Argumentation Frame of the Discussion With Lowest Discussion Rate

6.9.4 Argumentative Comparison Between the Three Frames Considered

After analysing the three most significant interventions of the experiment, we observe that even though different people were involved, many of the attack or support relations used are of the same type. In particular, two kinds of strength relations are used. This homogeneity of language makes it easier to compare these interventions. Since discussion 11-D positively affects the participants on their will to cooperate, a key question we will address is the following: *Why is discussion 11-D more influent on the participants' level of cooperation than discussion 6-D?* And also: *What impact have the expert appeal on the two discussions?*

At first sight, we recognise a difference in the number of arguments used in the two discussions. In discussion 11-D, the participants introduce almost double the number of arguments raised in discussion 6-D. Thus, in discussion 6-D the question of how to make Lebanon a better country has been more deeply analysed.

Another difference is in the number of arguments that affect positively $\langle \Gamma, b \rangle$. In the discussion 11-D, the majority of the arguments introduced directly or indirectly support $\langle \Gamma, b \rangle$. Let us recall that support relation is a transitive relation while attack relation is not. By contrast, in discussion 6-D, the participants are focused only on which factors prevent Lebanon from becoming a better country. Therefore, supporting arguments might affect the participants in the discussion positively because they contribute instantiating Γ. For example, we can assume that all the supports sets of the arguments supporting $\langle \Gamma, b \rangle$ are in Γ. The same cannot be said of discussion 6-D where the attacking arguments of $\langle \Gamma, b \rangle$ only contribute to exploring Γ^c or give reasons that prevent b from happening.

An additional factor of the more effectiveness of discussion 11-6 could be the number of arguments that involve a personal involvement of ordinary citizens like, for example, the role of *home education*.

If we want to explore how the experts' appeal has influenced the discussions, we can look at which arguments or atomic components introduced by the experts have been re-used. In particular, the experts mainly focus on the problem of sectarianism (s), then also analyse the importance of the electoral system (es) and explore which factors can help to promote cooperation (coo). In discussion 11-D, the participants are mainly focused on how to encourage cooperation. Sectarianism has a role, but not predominant as in the experts' appeal; their attitude towards the problem is positive and looks at possible solutions. The participants of discussion 11-D re-use some atoms first introduced by the experts, but they also raise many new arguments. In discussion 6-D almost all the participants' arguments were already presented by the experts except for those arguments that refer to the problem of unemployment.

In the following table, we report some data that help analyse the experts' effect on the discussions. The use in both discussions of many of the experts' atomic concepts tells us that their appeal had an impact. However, the participants of discussion 6-D have introduced only three new concepts. This might be why this discussion did not affect the will to cooperate of the participants. The lack of introduction of new atomic concepts or arguments might reveal that the participants of that discussion were not truly committed, thereby invalidating the experiment results.

In their paper [CP19], the authors construct an index that measures the discussions' quality. In particular, they assign 1 point if a participant of the discussions affirm that a specific argument is relevant, -1 if she states that it is irrelevant, 0 if she does not express herself.

After having analysed two discussions and the expert appeal using argumentative tools, it might be interesting to test a new way of defining this index that takes into account our findings. We would distinguish between the value given to the cause of sectarianism (a "negative" approach to the problem) and finding concrete actions in support of cooperation. Moreover, the introduction of new atomic concepts and arguments seems relevant to

	Experts Appeal	Discussion 11-D	Discussion 6-D
Number of atomic concepts used	11	20	14
Number of atomic concepts already introduced in the expert appeal	-	7	11
Number of arguments used	10	19	9
Number of arguments already introduced in the expert appeal	-	1	3

Table 6.4: Data on the Influence of the Expert Appeals on Discussion 11-D and 6-D

measure the participants' involvement in the discussion. Another issue worth exploring is the potential personal impact of ordinary people to the cause of making Lebanon a better country. For example, the presence of concepts such as *home education*, or *civic activities* in the supports of arguments might be relevant to measure the effectiveness of a discussion. As counterfactual, we saw that in discussion 6-D, the personal impact on the issue emerges and all participants agree that change cannot come from ordinary people. Therefore it is not surprising that discussion 6-D did not affect cooperation.

Conclusively we see a lot of potential in using argumentation theory in the studying of very concrete issues like how to strengthen intergroup cooperation. We could do new kinds of experiments and analysis.

CHAPTER 7

Argumentative semantics of t-norm based fuzzy logic

7.1 T-norm Based Arguments

This chapter introduces alternative sound and complete semantics for the three main t-norm based fuzzy logics: *Łukasiewicz* (Ł), *Gödel* (G) and *product* logic (P). We still understand arguments as complex entities, but we consider the three t-norm consequence relations corresponding to the logics as the link between the support and the claim.

As a starting point, we focus on arguments defined using the logical consequence in Łukasiewicz logic. In particular, we investigate two possible ways of defining an argument using first the standard consequence relation, then the order-based one [BCH11].

After having analysed the properties of these two definitions of a t-norm based argument, we will consider fully instantiated bipolar argumentation frames and investigate which attack or support principles are needed to recover a sound and complete semantic for Ł.

For the syntax and standard semantics of logics based on continuous t-norms see Definition 6.5.3.

Definition 7.1.1 (Standard Consequence Relation – $\models_{Ł}$). *A formula F is a* standard consequence *of* Γ *in* Ł ($\Gamma \models_{Ł} F$) *if for all* $[0,1]_{Ł}$-evaluation e *s.t. for every* $\gamma \in \Gamma$ $e(\gamma) = 1$, $e(F) = 1$.

An alternative way of defining a semantic consequence relation is the following.

Definition 7.1.2 (Order-Based Consequence Relation – $\models^{\leq}_{Ł}$). *A formula* φ *is a* order-based consequence *of* Γ in Ł ($\Gamma \models^{\leq}_{Ł} F$) *if for all* $[0,1]_{Ł}$-evaluation e $\inf_{\gamma\in\Gamma} e(\gamma) \leq e(F)$.

From Definition 7.1.1 and Definition 7.1.2 we recover two definitions of t-norm based arguments.

Definition 7.1.3 (Argument based on $\models_{Ł}$)**.** *An argument based on $\models_{Ł}$ is a pair $\langle \Gamma_X, X\rangle$ with $\Gamma_X \cup \{X\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma_X \models_{Ł} X$.*

Definition 7.1.4 (Argument based on $\models^{\leq}_{Ł}$)**.** *An argument based on $\models^{\leq}_{Ł}$ is a pair $\langle \Gamma_X, X\rangle$ with $\Gamma_X \cup \{X\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma_X \models^{\leq}_{Ł} X$.*

To better understand the argumentative characteristics of these two possible ways of defining a t-norm based argument, we consider several attack relations and explore which attack principles are justified. We will analyse the attack principles defined only in terms of the claims of arguments and the following attack relations: *Defeat*, *Compact Rebuttal 1*, *Defeating Rebuttal* and *Indirect Rebuttal*.

Defeat:	$\langle \Gamma_A, A\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$	$A \models \neg \bigwedge \Gamma_B$
Compact Rebuttal 1:	$\langle \Gamma_A, A\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$	$\Gamma_A \models \neg B$
Defeat Rebuttal:	$\langle \Gamma_A, A\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$	$A \models \neg B$
Indirect Rebuttal:	$\langle \Gamma_A, A\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$	$B \models \phi$ and $B \models \neg\phi$

Table 7.1: Attack Relations Considered for the Analysis of the T-Norm Based Argument Definitions

We interpret the attack relations using both definitions of consequence relation. For example, if in a given framework the argument $\langle \Gamma_A, A\rangle$ attacks the argument $\langle \Gamma_B, B\rangle$ using *defeat*, using Definition 7.1.1, then $A \models_{Ł} \neg \bigwedge \Gamma_B$, i.e. for any evaluation e such that $e(A) = 1$, then $e(\neg \bigwedge \Gamma_B) = 1$. A similar instantiation can be done using Definition 7.1.2.

We say that an attack principle is justified if the attacking condition of the conclusion of the principle logically follows from the attacking condition of the premise. We now analyse the attack principles considering the *defeating rebuttal* attack relation and both definitions of Ł-based consequence relation. The other cases can be found in Appendix D.3.

Analysis of the attack principles considering $[D\text{-}Reb]$ and $\models_{Ł}$.

We assume to work in a fully instantiated argumentation frames. In such frames, the arguments mentioned in the attack principles belong to the set of arguments.

(A.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $X \models_{Ł} \neg A$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg A) = 1 - e(A) = 1$. Since for any evaluation e $e(\neg(A \wedge B)) = 1 - e(A \wedge B) =$

$1 - \min\{e(A), e(B)\} \geq 1 - e(A)$, if $1 - e(A) = 1$, then also $1 - e(A \wedge B) = 1$, i.e. $X \models_{Ł} \neg(A \wedge B)$ and the principle holds.

(C.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then for any evaluation e such that $e(X) = 1$, $e(\neg(A \wedge B)) = 1$ i.e. $1 - e(A \wedge B) = 1 - \min\{e(A), e(B)\}$=1. Therefore, either $e(A) = 0$ or $e(B) = 0$ i.e. either $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, or $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$ and the principle holds.

(A.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then $X \models_{Ł} \neg(A \vee B)$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg(A \vee B)) = 1 - e(A \vee B) = 1 - \max\{e(A), e(B)\} = 1$. Therefore, $\max\{e(A), e(B)\} = 0$, which implies both $e(A) = 0$ and $e(B) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$. The principle holds.

(C.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$, then whenever there is an evaluation e s.t $e(X) = 1$, $e(\neg A) = 1$ and $e(\neg B) = 1$, i.e. $e(A) = 0$ and $e(B) = 0$. Therefore, $e(\neg(A \vee B)) = 1 - \max\{e(A), e(B)\} = 1$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$ and the principle holds.

(A.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $X \models_{Ł} \neg B$ and $X \not\models_{Ł} \neg A$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\neg B) = 1 - e(B) = 1$, which implies $e(B) = 0$. Moreover, there exists at least one evaluation e^* s.t. $e^*(X) = 1$ and $e^*(A) > 0$. To have the principle hold we would need that for any evaluation e s.t. $e(X) = 1$, then $e(\neg(A \supset B)) = 1$, i.e. $1 - e(A \supset B) = 1$ from which it follows that $e(A \supset B) = 0$. Since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, $e(A \supset B) = 0$ only if $1 - e(A) + e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$, but this does not follow from the premises and the principle does not hold.

(C.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $X \models_{Ł} \neg(A \supset B)$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1 - \min\{1, 1 - e(A) + e(B)\} = 1$. Thus, $1 - e(A) + e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$. From $e(B) = 0$ and $e(A) = 1$ it follows that $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$ and the attack principle holds. From the hypothesis it follows also that $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{\neg A}, \neg A\rangle$.

(A.$\neg$) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $X \models_{Ł} \neg A$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, $e(\neg A) = 1 - e(A) = 1$ i.e. $e(A) = 0$. We want to show that there is some evaluation e^* s.t. $e^*(X) = 1$ and $e^*(\neg(\neg A)) < 1$, i.e. $e^*(A) < 1$. From the hypothesis we have $e(A) = 0$. Therefore, $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{\neg A}, \neg A\rangle$ and the principle holds.

(C.$\neg$) If $\langle \Gamma_X, X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_{\neg A}, \neg A\rangle$, then for some evaluation e^* s.t. $e^*(X) = 1$, $e^*(\neg\neg A) = e^*(A) < 1$. However, from this hypothesis it does not follow that for any evaluation e s.t. $e(X) = 1$ $e(\neg A) = 1 - e(A) = 1$, i.e. $e(A) = 0$ and the principle does not hold.

Analysis of the attack principles considering $[D\text{-}Reb]$ **and** $\models^{\leq}_{\text{Ł}}$**.**

(A.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg A$, i.e. for any evaluation e, $e(X) \leq e(\neg A) = 1 - e(A)$. Since for any evaluation e $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(A)$, $1 - e(A) \leq 1 - e(A \wedge B)$. Therefore, $e(X) \leq 1 - e(A \wedge B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$ and the principle holds.

(C.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg(A \wedge B)$ i.e. for any evaluation e $e(X) \leq e(\neg(A \wedge B)) = 1 - \min\{e(A), e(B)\}$. It could be then, that for some specific evaluation e_i, $e_i(A \wedge B) = e_i(A)$ and for some other evaluation $e_j(A \wedge B) = e_j(B)$. Therefore, we cannot deduce that for any evaluation e either $e(X) \leq e(\neg A)$ or $e(X) \leq e(\neg B)$. The principle does not hold.

(A.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg(A \vee B)$ i.e. for any evaluation e, $e(X) \leq e(\neg(A \vee B)) = 1 - e(A \vee B) = 1 - \max\{e(A), e(B)\}$. Since for any evaluation e, $e(A \vee B) \geq e(A)$ and $e(A \vee B) \geq e(B)$, it follows that $1 - e(A \vee B) \leq 1 - e(A)$ and $1 - e(A \vee B) \leq 1 - e(B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$. The principle holds.

(C.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg A$ and $X \models^{\leq}_{\text{Ł}} \neg B$, i.e. for any evaluation e, $e(X) \leq 1 - e(A)$ and $e(X) \leq 1 - e(B)$. Since $e(A \wedge B) = \max\{e(A), e(B)\} = e(A)$ or $e(A \wedge B) = \max\{e(A), e(B)\} = e(B)$, $e(X) \leq 1 - \max\{e(A), e(B)\}$ i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$ and the principle holds.

(A.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg B$ and $X \not\models^{\leq}_{\text{Ł}} \neg A$, i.e. for any evaluation e, $e(X) \leq 1 - e(B)$ and for some evaluation e^*, $e^*(X) > 1 - e^*(A)$. We would need to show that for every evaluation e, $e(X) \leq 1 - \min\{1, 1 - e(A) + e(B)\}$, but this does not follow from the premises and the principle does not hold.

(C.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then for any evaluation e, $X \models^{\leq}_{\text{Ł}} \neg(A \supset B)$, i.e. $e(X) \leq 1 - e(A \supset B) = 1 - \min\{1, 1 - e(A) + e(B)\}$. Since for any evaluation e, $\min\{1, 1 - e(A) + e(B)\} \geq e(B)$, it follows that $1 - \min\{1, 1 - e(A) + e(B)\} \leq 1 - e(B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_B, B\rangle$. However, the other part of the claim, $\langle \Gamma_X, X\rangle \not\xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, does not follow from the premises and the principle does not hold.

(A.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{[D\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $X \models^{\leq}_{Ł} \neg A$, i.e. for any evaluation e, $e(X) \leq 1 - e(A)$. To have the principle holds, we need to find some evaluation e^* s.t. $e^*(X) > e^*(A)$, but this does not follow from the premise and the principle does not hold.

(C.¬) It does not hold.

If the attack relation is instantiated with *direct rebuttal*, the attack principles holding whenever the entailment relation considered is the one of Definition 7.1.1 (standard consequence relation) are

$$\mathsf{AP}^{[D\text{-}Reb]}_{\models_Ł} = \{(\mathbf{A.}\wedge), (\mathbf{C.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg)\}.$$

If the entailment relation is defined using Definition 7.1.2 (order-based consequence relation), the attack principles justified are

$$\mathsf{AP}^{[D\text{-}Reb]}_{\models^{\leq}_Ł} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee)\}.$$

In the following tables, we summarise which attack principles are justified in relation to the attack relation and the definition of the entailment relation considered. For the case of *defeat*, we have indicated which additional conditions the arguments need to satisfy to have the attack principle hold. Interestingly, the attack principles satisfied by most of the attack relation considered whenever Definition 7.1.3 is used, are the same satisfied by the modal interpretation of the attack relation[1] introduced in Section 4.4. This is not the case whenever the order-based definition is considered. In this case, none of the attack relations considered satisfying either **(C.⊃)** or **(A.¬)**. The attack principle **(C.¬)** is problematic with both definitions. However, this is not surprising since it is a very demanding principle. Again, we have a confirmation that not all the attack principles are on the same level, and some are more acceptable, i.e. easier to justify, than others. In particular, those attack principles satisfied by the modal interpretation of the attack relation are justified in different scenarios: in logical argumentation, see Section 5.3 and now also in a t-norm based argumentation frame.

	(A.∧)		**(C.∧)**	
Attack Relation	$\models_Ł$	$\models^{\leq}_Ł$	$\models_Ł$	$\models^{\leq}_Ł$
[Def]	$\Gamma_A \subseteq \Gamma_{A\wedge B}$	$\Gamma_A \subseteq \Gamma_{A\wedge B}$	$\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$	$\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$
[C-Reb-1]	✓	✓	✗	✗
[D-Reb]	✓	✓	✓	✗
[I-Reb]	✓	✓	✗	✗

Table 7.2: Conditions Needed in Order to Satisfy **(A.∧)** and **(C.∧)**

[1] We recall that the attack principles satisfied by the modal interpretation of the attack relations are: $\mathsf{MAP} = \{(\mathbf{A.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\vee), (\mathbf{C.}\supset), (\mathbf{A.}\neg)\}$.

Attack Relation	(A.∨) $\models_Ł$	$\models_Ł^{\leq}$	(C.∨) $\models_Ł$	$\models_Ł^{\leq}$
[Def]	$\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$	$\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$	$\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$	$\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$
[C-Reb-1]	✓	✓	✓	✗
[D-Reb]	✓	✓	✓	✓
[I-Reb]	✓	✓	✓	✓

Table 7.3: Conditions Needed in Order to Satisfy **(A.∨)** and **(C.∨)**

Attack Relation	(A.⊃) $\models_Ł$	$\models_Ł^{\leq}$	(C.⊃) $\models_Ł$	$\models_Ł^{\leq}$
[Def]	$\Gamma_B \subseteq \Gamma_{A\supset B}$	$\Gamma_B \subseteq \Gamma_{A\supset B}$	$\Gamma_{A\supset B} \subseteq \Gamma_B$, $\Gamma_A \subset \Gamma_{A\supset B}$ and $\gamma_i^* \notin \Gamma_A$	✗
[C-Reb-1]	✗	✗	✓	✗*
[D-Reb]	✗	✗	✓	✗*
[I-Reb]	✗	✗	✗*	✗*

Table 7.4: Conditions Needed in Order to Satisfy **(A.⊃)** and **(C.⊃)**

Attack Relation	(A.¬) $\models_Ł$	$\models_Ł^{\leq}$	(C.¬) $\models_Ł$	$\models_Ł^{\leq}$
[Def]	✗	✗	✗	✗
[C-Reb-1]	✓	✗	✗	✗
[D-Reb]	✓	✗	✗	✗
[I-Reb]	✗	✗	✗	✗

Table 7.5: Conditions Needed in Order to Satisfy **(A.¬)** and **(C.¬)**

7.2 Argumentative Semantics for Ł

The argumentation frames where the attack relations are not graded are not adequate to characterize Ł. We can recover an argumentative complete semantics for Ł in two ways: either considering weighted attacks (as done in Chapter 6) or working on bipolar argumentation frames. By defining a support relation over the arguments, we can introduce new principles. In particular, we will introduce principles where both relations are involved.

To help this section's readability, we recall a Hilbert-style axiomatisation for Łukasiewicz

Logic to which we will refer in the sequel.

$$
\begin{aligned}
[Tr]: &\quad (F \supset G) \supset ((G \supset H) \supset (F \supset H)) \\
[We]: &\quad F \supset (G \supset F) \\
[Ex]: &\quad (F \supset (G \supset H)) \supset (G \supset (F \supset H)) \\
[\wedge\text{-}1]: &\quad (F \wedge G) \supset F \\
[\wedge\text{-}2]: &\quad (F \wedge G) \supset G \\
[\wedge\text{-}3]: &\quad (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G))) \\
[\vee\text{-}1]: &\quad F \supset (F \vee G) \\
[\vee\text{-}2]: &\quad G \supset (F \vee G) \\
[\vee\text{-}3]: &\quad (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F)) \\
[Lin]: &\quad (F \supset G) \vee (G \supset F) \\
[\bot]: &\quad \bot \supset F \\
[Waj]: &\quad ((F \supset G) \supset G) \supset ((G \supset F) \supset F)
\end{aligned}
$$

The only inference rule is *modus ponens*: from F and $F \supset G$ infer G.

Theorem 7.2.1. *The above Hilbert-style system is sound and complete for Łukasiewicz logic, i.e. a formula F is derivable in the system iff F is Ł-valid.*

Note that if we only consider validity, the two consequence relations coincide, i.e. $\models_{Ł} F$ iff $\models^{\leq}_{Ł} F$. However, *modus ponens* is not sound with the order-based consequence relation.

We can obtain an alternative semantics for Łukasiewicz logic using again the concept of argumentative immunity (see Section 6.3), but also Ł-based arguments and the two kinds of relations among arguments: the attack relation, as usual, and the support relation. Even though the introduction of the support relation is not necessary to recover a complete argumentative semantics for Ł, see Section 6.5, it is still rational to consider principles that also use the support relation. For example, the attack principle concerning implication that we will consider is the following.

($\mathbf{C}_{Bi}.\supset$) If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$.
In words: If an argument with claim X attacks an argument with claim $A \supset B$, then the argument with claim X also attacks an argument with claim B and supports an argument with claim A.

We will instantiate both the attack and support relations. Moreover, we will introduce additional attack principles involving both relations. Since we will work with CC-principles[2], we instantiate the attack relation with *direct rebuttal* because, in turn, it is defined using the claim of both the attacking argument and the attacked one.

[2] CC-principles are defined in terms of the claim of both the attacking/supporting and attacked/supported arguments

Definition 7.2.1 (Bi-ŁLAF)**.** *A* Bipolar Ł-based Logical Argumentation Frame *is a triplet* $AF = \langle Ar, Attack(\mathscr{A}), Support(\mathscr{S})\rangle$ *such that:*

Ar is a set of Ł-based sequents arguments, i.e. $\langle \Gamma, \psi\rangle \in Ar$ *only if* $\Gamma \models_{Ł} \psi$.

$\mathscr{A}$ is a set of attack rules and $\mathscr{S}$ a set of support rules.

$(\langle \Gamma_A, A\rangle, \langle \Gamma_B, B\rangle) \in Attack(\mathscr{A})$ *iff there is some* $\mathscr{R} \in \mathscr{A}$ *such that* $\langle \Gamma_A, A\rangle$ *$\mathscr{R}$-attacks* $\langle \Gamma_B, B\rangle$.

$(\langle \Gamma_A, A\rangle, \langle \Gamma_B, B\rangle) \in Support(\mathscr{S})$ *iff there is some* $\mathscr{R} \in \mathscr{S}$ *such that* $\langle \Gamma_A, A\rangle$ *$\mathscr{S}$-supports* $\langle \Gamma_B, B\rangle$.

Let us recall the definition of the [D-Sup] support relation.

Definition 7.2.2 (Direct Support - [D-Sup])**.** *Let* $\langle \Gamma_A; A\rangle$ *and* $\langle \Gamma_B; B\rangle$ *two arguments in* Ar*, a set of fully instantiated arguments. We say that* $\langle \Gamma_A, A\rangle$ directly supports $\langle \Gamma_B, B\rangle$ *(*$\langle \Gamma_A, A\rangle \overset{[D\text{-}Sup]}{\rightsquigarrow} \langle \Gamma_B, B\rangle$*) if* $A \vdash B$.

If we consider a Bi-ŁLAF argumentation frame such that $\mathscr{A} = \{[D\text{-}Reb]\}$ and $\mathscr{S} = \{[D\text{-}Sup]\}$, the following attack-support principles holds.

($\mathbf{A}_S.\vee$) If $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_{A\vee B}, A \vee B\rangle$, then $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$ or $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_B, B\rangle$.

If $X \models_{Ł} A \vee B$, then that whenever $e(X) = 1$ for any evaluation e, then $e(A \vee B) = \min\{e(A), e(B)\} = 1$ from which it follows $e(A) = 1$ or $e(B) = 1$, i.e. $X \models_{Ł} A$ or $X \models_{Ł} B$.

($\mathbf{C}_{Bi}.\supset$) If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$.

In fact, if $X \models_{Ł} \neg(A \supset B)$, then that whenever $e(X) = 1$ for any evaluation e, then $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1$ from which it follows $e(A \supset B) = 0$. Since $e(A \supset B) = \min\{1, 1-e(A)+e(B)\}$, if $e(A \supset B) = 0$, then that $1-e(A)+e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$. Therefore for any evaluation e such that $e(X) = 1$, $e(A) = 1$ and $e(B) = 0$, i.e. $X \models_{Ł} A$ and $X \models_{Ł} \neg B$.

($\mathbf{A}_S.\supset$) If $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_B, B\rangle$.

In fact, if $X \models_{Ł} A$ and $X \models_{Ł} A \supset B$, then that whenever $e(X) = 1$ for any evaluation e, then $e(A) = 1$ and $e(A \supset B) = 1$. Since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$ and $e(A) = 1$, $e(A \supset B) = \min\{1, e(B)\} = e(B)$ from which it follows $e(B) = 1$.

($\mathbf{A}_S$.$\neg$) If $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$, then $\langle \Gamma_X, X\rangle \not\rightsquigarrow \langle \Gamma_{\neg A}, \neg A\rangle$.

In fact, if $X \models_{Ł} A$ whenever $e(X) = 1$ for any evaluation e, then $e(A) = 1$, from which it follows $e(\neg A) = 0$ and $X \not\models_{Ł} \neg A$.

($\mathbf{C}_{Bi}$.$\neg$) If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_A, A\rangle$, then $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_{\neg A}, \neg A\rangle$

If $X \models_{Ł} \neg A$, then that whenever $e(X) = 1$ for any evaluation e, then $e(\neg A) = 1$, i.e. $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_{\neg A}, \neg A\rangle$.

Let us define a set of principles. Some of them are attack principles, the others are the ones just introduced.

Definition 7.2.3. $\mathsf{ŁP} = \{(\mathbf{A}.\wedge), (\mathbf{C}.\wedge), (\mathbf{A}.\vee), (\mathbf{C}.\vee), (\mathbf{C}_{Bi}.\supset), (\mathbf{A}_S.\supset), (\mathbf{A}_S.\vee), (\mathbf{C}_{Bi}.\neg)\}$.

Following the same steps used in Section 6.3, we first define *immune* formulas in Bi-ŁLAFs, then we prove the closure of ŁP-immune arguments over *modus ponens*, and finally the bipolar argumentative soundness of *Ł*.

Definition 7.2.4. *Let* ASP *be a set of attack-support principles. A formula F is* ASP-argumentatively immune *(shortly:* P-immune*) if in all syntactically closed Bi-ŁLAFs (with respect to F) that satisfy the principles in* ASP *F is not attacked.*

Proposition 7.2.1 (Closure of ŁP-immune arguments over Modus Ponens)**.** *If A and $A \supset B$ are argumentatively $\mathscr{P}_{\mathsf{ŁP}}$-immune, then also B is argumentatively $\mathscr{P}_{\mathsf{ŁP}}$-immune.*

Proof. Since both A and $A \supset B$ are $\mathscr{P}_{\mathsf{ŁP}}$-immune , then that for any X in a syntactically closed Bi-ŁLAF frame $X \not\models_{Ł} \neg A$ and $X \not\models_{Ł} \neg(A \supset B)$, i.e. there is an evaluation e_1 s.t. $e_1(X) = 1$ and $e_1(\neg A) < 1$ and an evaluation e_2 s.t. $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$. Our claim is that $X \not\models_{Ł} \neg B$ for any X in the frame, i.e. there is an evaluation e_3 s.t. $e_3(X) = 1$ and $e_3(\neg B) < 1$. Since A belongs to the frame, also $X \wedge A$ is the frame and from the hypothesis it must happen that $X \wedge A \not\models_{Ł} \neg A$ and $X \wedge A \not\models_{Ł} \neg(A \supset B)$, i.e. there is some evaluation e_4 s.t. $e_4(X \wedge A) = 1$ and $e_4(\neg(A \supset B)) < 1$. Since $e_4(X \wedge A) = \min\{e_4(A), e_4(A)\}$ and it must be 1, we have $e_4(A) = 1$. Moreover since $e_4(\neg(A \supset B)) = 1 - e_4(A \supset B) = 1 - \min\{1, 1 - e_4(A) + e_4(B)\}$, i.e. $e_4(\neg(A \supset B)) = 1 - e_4(B)$ and it must be greater than 1, it follows $e_4(B) > 0$ and $e_4(\neg B) < 1$. Therefore for any argument X in the frame we can find an evaluation that shows $X \not\models_{Ł} \neg B$. $\square$

Theorem 7.2.2 (Bipolar Argumentative Soundness of Ł)**.** *Every Ł-valid formula is argumentatively $\mathscr{P}_{\mathsf{ŁP}}$-immune.*

Proof. By Theorem 7.2.1 and Proposition 7.4.1, it remains to check that the axiom of the axiomatic system for Łukasiewicz logic recalled at the beginning of this chapter are $\mathscr{P}_{\mathsf{ŁP}}$-immune.

[Tr] If $X \longrightarrow (F \supset G) \supset ((G \supset H) \supset (F \supset H))$, then by $(\mathbf{C}_{Bi}.\supset)$ we have that $X \rightsquigarrow F \supset (G \supset H)$ and $X \longrightarrow ((F \supset G) \supset (F \supset H))$. Again by $(\mathbf{C}_{Bi}.\supset)$ we have $X \rightsquigarrow F \supset G$ and $X \longrightarrow F \supset H$ from which it follows, by $(\mathbf{C}_{Bi}.\supset)$, $X \longrightarrow H$ and $X \rightsquigarrow F$. Since $X \rightsquigarrow F$ and $X \rightsquigarrow F \supset (G \supset H)$, by $(\mathbf{A}_S.\supset)$, we have $X \rightsquigarrow G \supset H$ and from $X \rightsquigarrow F \supset G$ and $X \rightsquigarrow F$, by $(\mathbf{A}_S.\supset)$, it follows $X \rightsquigarrow G$. Still by $(\mathbf{A}_S.\supset)$, from $X \rightsquigarrow G \supset H$ and $X \rightsquigarrow G$ it follows $X \rightsquigarrow H$ which goes against $X \longrightarrow H$.

[We] If $X \longrightarrow F \supset (G \supset F)$, by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow F \supset G$ and $X \rightsquigarrow F$. From $X \longrightarrow G \supset G$ it follows by $(\mathbf{C}_{Bi}.\supset)$ $X \rightsquigarrow G$ and $X \longrightarrow F$ which is incompatible with $X \rightsquigarrow F$.

[Ex] If $X \longrightarrow (F \supset (G \supset H)) \supset (G \supset (F \supset H))$ by $(\mathbf{C}_{Bi}.\supset)$ it follows that $X \longrightarrow (G \supset (F \supset H))$ and $X \rightsquigarrow (F \supset (G \supset H))$. Again, by $(\mathbf{C}_{Bi}.\supset)$, we have that $X \longrightarrow F \supset H$ and $X \rightsquigarrow G$. Since $X \longrightarrow F \supset H$ it follows also $X \longrightarrow H$ and $X \rightsquigarrow F$. From $X \rightsquigarrow (F \supset (G \supset H))$ and $X \rightsquigarrow F$ by $(\mathbf{A}_S.\supset)$ it follows $X \rightsquigarrow G \supset H$ and by $X \rightsquigarrow G \supset H$ and $X \rightsquigarrow G$ it follows $X \rightsquigarrow H$ which is incompatible with $X \longrightarrow H$.

[$\wedge$-1] If $X \longrightarrow (F \wedge G) \supset F$ by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow F$, $X \rightsquigarrow F \wedge G$ and $X \not\longrightarrow F \wedge G$. Moreover by $(\mathbf{A}.\wedge)$ we have $X \not\longrightarrow F$ and $X \not\longrightarrow G$ which contradicts $X \longrightarrow F$.

[$\wedge$-2] See the previous case.

[$\wedge$-3] If $X \longrightarrow (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G)))$ by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow (H \supset G) \supset (H \supset (F \wedge G))$ and $X \rightsquigarrow H \supset F$ and again, by the same principle we have $X \longrightarrow H \supset (F \wedge G)$ and $X \rightsquigarrow H \supset G$. Since $X \longrightarrow H \supset (F \wedge G)$ it follows $X \longrightarrow F \wedge G$ and $H \rightsquigarrow H$. From $X \longrightarrow F \wedge G$ by $(\mathbf{C}.\wedge)$ it follows that $X \longrightarrow F$ or $X \longrightarrow G$. Since $X \rightsquigarrow H \supset F$ and $X \rightsquigarrow H$, by $(\mathbf{A}_S.\supset)$ we have $X \rightsquigarrow F$ and by $X \rightsquigarrow H$ and $X \rightsquigarrow H \supset G$ it follows $X \rightsquigarrow G$ and this is in contradiction with $X \longrightarrow F$ or $X \longrightarrow G$.

[$\vee$-1] If $X \longrightarrow F \supset (F \vee G)$ by $(\mathbf{C}_{Bi}.\supset)$ we have $X \longrightarrow F \vee G$ and $X \rightsquigarrow F$. From $X \longrightarrow F \vee G$ by $(\mathbf{A}.\vee)$ it follows $X \longrightarrow G$ and $X \longrightarrow F$ which is in contradiction with $X \rightsquigarrow F$.

[$\vee$-2] See the previous case.

[$\vee$-3] If $X \longrightarrow (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))$ by $(\mathbf{C}_{Bi}.\supset)$ we have $X \longrightarrow (H \supset F) \supset ((G \vee H) \supset F)$ and $X \rightsquigarrow G \supset F$. Again, by $(\mathbf{C}_{Bi}.\supset)$, we have also $X \longrightarrow (G \vee H) \supset F$ and $X \rightsquigarrow H \supset F$. From $X \longrightarrow (G \vee H) \supset F$ it follows also $X \longrightarrow F$ and $X \rightsquigarrow G \vee H$.

Since $X \rightsquigarrow G \vee H$, by $(\mathbf{A}_S.\vee)$ either (a) $X \rightsquigarrow G$ or (b) $X \rightsquigarrow H$. In the (a)-case, since $X \rightsquigarrow G$ and $X \rightsquigarrow G \supset F$, by $(\mathbf{A}_S.\supset)$ we have that $X \rightsquigarrow F$. However from $X \longrightarrow F$ and $(\mathbf{C}_{Bi}.\neg)$ it follows $X \rightsquigarrow \neg F$ and this it cannot happen.

In the (b)-case we reach the contradiction in the same way.

Axiom	Attack and Support Principles used in the proof
[Tr]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}_S.\supset)$
[We]	$(\mathbf{C}_{Bi}.\supset)$
[Ex]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}_S.\supset)$
[$\wedge$-1]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}.\wedge)$
[$\wedge$-2]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}.\wedge)$
[$\wedge$-3]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}_S.\supset)$, $(\mathbf{C}.\wedge)$
[$\vee$-1]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}.\vee)$
[$\vee$-2]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}.\vee)$
[$\vee$-3]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}_S.\supset)$, $(\mathbf{A}_S.\vee)$, $(\mathbf{C}_{Bi}.\neg)$
[Lin]	$(\mathbf{C}_{Bi}.\supset)$, $(\mathbf{A}.\vee)$
[$\perp$]	$(\mathbf{C}_{Bi}.\supset)$
[Waj]	$(\mathbf{C}_{Bi}.\supset)$

Table 7.6: Attack and Support Principles used in Theorem 7.2

[*Lin*] If $X \longrightarrow (F \supset G) \vee (G \supset F)$ from $(\mathbf{A}.\vee)$ it follows $X \longrightarrow F \supset G$ and $X \longrightarrow G \supset F$. From $X \longrightarrow F \supset G$ it follows by $(\mathbf{C}_{Bi}.\supset)$ $X \longrightarrow G$ and $X \rightsquigarrow F$ while from $X \longrightarrow G \supset F$ it follows $X \longrightarrow F$ and $X \rightsquigarrow G$ and we have reached a contradiction.

[$\perp$] If $X \longrightarrow \perp \supset F$ by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow F$ and $X \rightsquigarrow \perp$ which is absurd.

[*Waj*] If $X \longrightarrow ((F \supset G) \supset G) \supset ((G \supset F) \supset F)$ by $(\mathbf{C}_{Bi}.\supset)$ we have $X \longrightarrow (G \supset F) \supset F$ and $X \rightsquigarrow (F \supset G) \supset G$. From $X \longrightarrow (G \supset F) \supset F$ again by $(\mathbf{C}_{Bi}.\supset)$ we have $X \longrightarrow F$ and $X \rightsquigarrow G \supset F$. Therefore (1) $X \models_{Ł} \neg F$, (2) $X \models_{Ł} G \supset F$ and (3) $X \models_{Ł} (F \supset G) \supset G$. From (1) it follows that whenever $e(X) = 1$, $e(\neg F) = 1 - e(F) = 1$, i.e. $e(F) = 0$. From (2) it follows that whenever $e(X) = 1$, $e(G \supset F) = \min\{1, 1 - e(G) + e(F)\} = 1$ and given $e(F) = 0$ this is satisfied only if $e(G) = 0$. From (3) we have that whenever $e(X) = 1$ for some evaluation e, $e((F \supset G) \supset G) = 1$, i.e. $\min\{1, 1 - e(F \supset G) + e(G)\} = 1$. $e(F \supset G) = \min\{1, 1 - e(F) + e(G)\}$ and given $e(F) = 0$ and $e(G) = 0$, $e(F \supset G) = 1$. Therefore $\min\{1, 1 - e(F \supset G) + e(G)\} = 0$ while it should have been 1. □

In the table 7.6 we summarise which attack and support principle have been used in the corresponding section of the proof.

Theorem 7.2.3 (Bipolar Argumentative Completeness of Ł). *Every argumentatively $\mathscr{P}_{ŁP}$-immune formula is Ł-valid .*

Proof. We have to show that if F is not a Ł-valid valid formula, then it is not $\mathscr{P}_{ŁP}$-immune. If F is not a Ł-valid valid formula, then that there is an evaluation e such that $e(F) < 1$. Since $e(\neg F) = 1 - e(F)$, any formula F is attacked by its negation, i.e. $\neg F \longrightarrow F$. □

7.3 Argumentative Semantics for G

Concerning the attack principles needed to prove a completeness theorem with G-based argumentation frames, we only need to verify that the interpretation of **(C.**$\wedge$**)**, **(A.**$\vee$**)**, and **(C.**$\supset$**)** are satisfied. In Appendix D.3, there are all the control calculations and proofs of the following theorems.

We define G-based arguments as those considered for the Ł case.

Definition 7.3.1 (Standard Consequence Relation – $\models_{\mathsf{G}}$)**.** *A formula F is a* standard consequence *of* Γ *in* G *(*$\Gamma \models_{\mathsf{G}} F$*) if for any* $[0,1]_{\mathsf{G}}$*-evaluation e s.t. $e(\gamma) = 1$ for all $\gamma \in \Gamma$, $e(F) = 1$.*

Definition 7.3.2 (G-based Argument)**.** *An argument based on* G *is a pair $\langle \Gamma_X, X\rangle$ with $\Gamma \cup \{X\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma_X \models_{\mathsf{G}} X$.*

As axiomatic system we consider $[Tr]$, $[We]$, $[Ex]$, $[\wedge$-1$]$, $[\wedge$-2$]$, $[\wedge$-3$]$, $[\vee$-1$]$, $[\vee$-2$]$, $[\vee$-3$]$, $[Lin]$, $[\bot]$ plus $[Con]$.

$[Con]$ $(F \supset (F \supset G)) \supset (F \supset G)$.

This additional attack and support principle required is the following.

(C$_{Bi2}$**.**$\supset$**)** If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \rightsquigarrow \langle \Gamma_A, A\rangle$

Definition 7.3.3. $\mathsf{GP} = \{(\mathbf{C.}\wedge), (\mathbf{A.}\vee), (\mathbf{C.}\supset), (\mathbf{C}_{Bi2}\mathbf{.}\supset)\}$

Proposition 7.3.1 (Closure of GP-immune arguments over Modus Ponens)**.** *If A and $A \supset B$ are argumentatively $\mathscr{P}_{\mathsf{GP}}$-immune, then also B is argumentatively $\mathscr{P}_{\mathsf{GP}}$-immune.*

Theorem 7.3.1 (Adequateness Theorem for G)**.** *Any formula F is* G*-valid iff it is $\mathscr{P}_{\mathsf{GP}}$-immune.*

In Table 7.7 we summarise which attack principle has been used in the corresponding sub-case of the soundness theorem.

7.4 Argumentative Semantics for P

To recover a complete semantics for product logic, we procede in a similar way.

We define P-based arguments as those considered for the Ł and G case.

Definition 7.4.1 (Standard Consequence Relation – $\models_{\mathsf{P}}$)**.** *A formula F is a* standard consequence *of* Γ *in* P *(*$\Gamma \models_{\mathsf{P}} F$*) if for all* $[0,1]_{\mathsf{P}}$*-evaluation e s.t. $e(\gamma) = 1$ for any $\gamma \in \Gamma$, $e(F) = 1$.*

Axiom	Attack and Support Principles used in the proof
[Tr]	$(\mathbf{C}_{Bi2}.\supset)$
[We]	$(\mathbf{C}_{Bi2}.\supset)$
[Ex]	$(\mathbf{C}_{Bi2}.\supset)$
[$\wedge$-1]	$(\mathbf{C}_{Bi2}.\supset)$
[$\wedge$-2]	$(\mathbf{C}_{Bi2}.\supset)$
[$\wedge$-3]	$(\mathbf{C}_{Bi2}.\supset)$ $(\mathbf{C}.\wedge)$
[$\vee$-1]	$(\mathbf{C}_{Bi2}.\supset)$ $(\mathbf{A}.\vee)$
[$\vee$-2]	$(\mathbf{C}_{Bi2}.\supset)$
[$\vee$-3]	$(\mathbf{C}_{Bi2}.\supset)$
[Lin]	$(\mathbf{C}.\supset)$
[$\perp$]	$(\mathbf{C}.\supset)$
[Con]	$(\mathbf{C}_{Bi2}.\supset)$

Table 7.7: Attack and Support Principles used in the Completeness Theorem

Definition 7.4.2 (P-Based Argument)**.** *An argument based on* P *is a pair* $\langle \Gamma_X, X\rangle$ *with* $\Gamma_X \cup \{X\} \subseteq Fm_{\mathcal{L}}$ *such that* $\Gamma_X \models_{\mathsf{P}} X$*.*

As axiomatic system we consider

$$\begin{aligned}
[BL1]: &\quad (F \supset B) \supset ((G \supset H) \supset (F \supset H))\\
[BL4]: &\quad F\&(F \supset G) \supset G\&(G \supset F)\\
[BL5a]: &\quad (F\&G \supset H) \supset (F \supset (G \supset H))\\
[BL5b]: &\quad (F \supset (G \supset H)) \supset (F\&G \supset H)\\
[BL6]: &\quad ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H)\\
[BL7]: &\quad \perp \supset F\\
[P]: &\quad \neg F \vee ((F \supset F\&G) \supset G)
\end{aligned}$$

Together with the deduction rule of modus ponens: [MP] from A and $A \supset B$, infer B.

The numbering of the axioms follows the original presentation in [H9́8]. Later in [Chv12] it has been shown that two of the axioms introduced in [H9́8] are redundant. Namely $[BL2]$ and $[BL3]$.

$$\begin{aligned}
[BL2]: &\quad (F\&G) \supset F\\
[BL3]: &\quad (F\&G) \supset (G\&F)
\end{aligned}$$

The attack principles needed to prove a completeness theorem are: $(\mathbf{A}.\vee)$, $(\mathbf{A}.\neg)$, $(\mathbf{C}.\supset)$, $(\mathbf{A}.\supset)$ and an additional attack principle regarding strong conjunction.

(A.&) If $X \longrightarrow A$ or $X \longrightarrow B$, then $X \longrightarrow A\&B$.

(C.&) If $X \longrightarrow A\&B$, then $X \longrightarrow A$ or $X \longrightarrow B$.

Axiom	Attack and Support Principles used in the proof
[BL1]	$(\mathbf{A.}\supset)$ $(\mathbf{C.}\supset)$
[BL4]	$(\mathbf{A.}\supset)$ $(\mathbf{C.}\supset)$ $(\mathbf{A.}\&)$ $(\mathbf{C.}\&)$
[BL5a]	$(\mathbf{A.}\supset)$ $(\mathbf{C.}\supset)$ $(\mathbf{C.}\&)$
[BL5b]	$(\mathbf{A.}\supset)$ $(\mathbf{C.}\supset)$ $(\mathbf{A.}\&)$
[BL6]	$(\mathbf{A.}\supset)$ $(\mathbf{C.}\supset)$
[BL7]	$(\mathbf{A.}\vee)$ $(\mathbf{C.}\supset)$ $(\mathbf{A.}\neg)$

Table 7.8: Attack and Support Principles used in the Completeness Theorem

Definition 7.4.3. $\mathsf{PP} = \{(\mathbf{A.}\vee), (\mathbf{A.}\neg), (\mathbf{A.}\supset), (\mathbf{C.}\supset), (\mathbf{A.}\&), (\mathbf{C.}\&)\}$

Proposition 7.4.1 (Closure of PP-immune arguments over Modus Ponens)**.** *If A and $A \supset B$ are argumentatively $\mathscr{P}_{\mathsf{PP}}$-immune, then also B is argumentatively $\mathscr{P}_{\mathsf{PP}}$-immune.*

Theorem 7.4.1 (Adequateness Theorem for P)**.** *Any formula F is P-valid iff it is $\mathscr{P}_{\mathsf{PP}}$-immune.*

In Table 7.8 we summarise which attack principle has been used in the corresponding sub-case of the soundness theorem. The complete proofs can be found in Appendix D.3.

CHAPTER 8

Summary and Future Work

In this dissertation, we have analysed some logical aspects of argumentation theory through the attack and support principles. We have introduced an intermediate level of abstraction (the SAFs) between Dung's style argumentation frames and fully instantiated ones. On the SAFs we have defined attack and support principles. If fully instantiated argumentation frames are in general infinite, the corresponding semi-abstract one is potentially finite, inducing an advantage on the complexity level. Even at this intermediate level of abstraction, is it possible to recover sound and complete semantics for classical logics and some fragment of LK that refer to the logical form of the claims of attacking and attacked arguments (and analogously in case of support between arguments.)

We have also seen how the attack principles, once interpreted in fully instantiated frames, can be used to analyse the role of minimality and consistency of the support sets of complex arguments. Besides, we can define additional elimination rules that make the dynamic derivation process shorter. In future work, it remains to analyse the impact of the additional rules on the complexity of the derivations.

We have then introduced weighted semi-abstract argumentation frames (WSAFs) and defined over these argumentative frames weighted attack principles. Using these have recovered sound and complete semantics for the three main t-norm based fuzzy logics (Gödel, Łukasiewicz and Product logic). We have also seen how it is possible to define the weights on the attack relations in terms of the strength of the arguments. Using argumentative tools, we have analysed the transcripts of the discussions that occurred in a field experiment made in 2016 in Lebanon aimed at studying the factors that influence intergroup cooperation. As future work, it will be interesting to integrate the analysis made by the authors of the experiment with our theoretical findings.

Then, we have seen how using bipolar argumentation frames, where arguments are defined using t-norm entailment relations, is it possible to recover alternative argumentative semantics for Gödel, Łukasiewicz and Product logic.

In the future, it would be interesting to explore in more detail bipolar logical argumentation frames, to define in a systematic way principles that use both the attack and the support relation and then recover an argumentative semantics for classical logic, but also for weaker logics. It would also be interesting to define dynamic derivations that work with bipolar frames.

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APPENDIX A

Extended Proofs of Chapter 4

A.1 Extended Proofs of Section 4.3

Theorem A.1.1 (4.3.1) (Soundness)**.** *If* $\Gamma \Rightarrow \Delta$ *is derivable in* LK *then* $\Gamma \models^{\mathsf{CAP}}_{arg} \Delta$.

Proof. Clearly, $A, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$. It remains to check that the inference rules of LK preserve CAP-consequence. We only present two cases.

$(\neg, r)$ We have to show that $A, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, \neg A$. The premise states that $\overline{\text{atts}}_S(\Gamma, A) \supseteq \underline{\text{atts}}_S(\Delta)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$ and satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma) \supseteq \underline{\text{atts}}_{S'}(\Delta, \neg A)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$. Since every SAF S' of the second kind reduces to one of the first kind (without $\neg A$) we may argue over any such SAF and drop the reference. We obtain:

$$\begin{aligned}
& \overline{\text{atts}}(\Gamma, A) \supseteq \underline{\text{atts}}(\Delta) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \supseteq \underline{\text{atts}}(\Delta) \\
\Rightarrow\ & (\overline{\text{atts}}(\Gamma) \cup \text{atts}(A)) \cap \text{atts}(\neg A) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & (\overline{\text{atts}}(\Gamma) \cap \text{atts}(\neg A)) \cup (\text{atts}(A) \cap \text{atts}(\neg A)) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cap \text{atts}(\neg A) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \qquad [\text{using } (\mathbf{A.}\neg)] \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(\neg A) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, \neg A).
\end{aligned}$$

Crucially, $(\mathbf{A.}\neg)$ amounts to $\text{atts}(A) \cap \text{atts}(\neg A) = \emptyset$.

$(\neg, l)$ We have to show that $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ implies $\neg A, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$. The premise states that $\overline{\text{atts}}_S(\Gamma) \supseteq \underline{\text{atts}}_S(\Delta, A)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$ and satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma, \neg A) \supseteq \underline{\text{atts}}_{S'}(\Delta)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$. Since every SAF S' of the second kind reduces to one of the first kind (without $\neg A$) we may argue over any such SAF and drop the reference. We obtain:

$$
\begin{aligned}
& \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \supseteq (\underline{\text{atts}}(\Delta) \cap \text{atts}(A)) \cup (\text{atts}(A))^c \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \supseteq (\underline{\text{atts}}(\Delta) \cup (\text{atts}(A))^c) \cap (\text{atts}(A) \cup (\text{atts}(A))^c) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \supseteq \underline{\text{atts}}(\Delta) \cup (\text{atts}(A))^c \supseteq \underline{\text{atts}}(\Delta) \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \supseteq \underline{\text{atts}}(\Delta) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(\neg A) \supseteq \underline{\text{atts}}(\Delta) \qquad [\text{using } \textbf{(C.}\neg\textbf{)}] \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma, \neg A) \supseteq \underline{\text{atts}}(\Delta).
\end{aligned}
$$

Crucially, **(C.¬)** amounts to $(\text{atts}(A))^c = \text{atts}(\neg A)$.

$(\wedge, r)$ We have to show that from $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ and $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, B$ it follows $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \wedge B$. The premises state that $\overline{\text{atts}}_S(\Gamma) \supseteq \underline{\text{atts}}_S(\Delta, A)$ and $\overline{\text{atts}}_S(\Gamma) \supseteq \underline{\text{atts}}_S(\Delta, B)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$ and all S satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma) \supseteq \underline{\text{atts}}_{S'}(\Delta, A \wedge B)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{A \wedge B\}$. Since every SAF S' of the third kind reduces to one of the first or second kind we may argue over any such SAF and drop the reference. We obtain:

$$
\begin{array}{ccc}
\overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A) & & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, B) \\
\Updownarrow & & \Updownarrow \\
\overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A) & & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(B) \\
& \Downarrow &
\end{array}
$$

$$
\begin{aligned}
& \overline{\text{atts}}(\Gamma) \supseteq (\underline{\text{atts}}(\Delta) \cap \text{atts}(A)) \cup (\underline{\text{atts}}(\Delta) \cap \text{atts}(B)) \\
\Leftrightarrow & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap (\text{atts}(A) \cup \text{atts}(B)) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A \wedge B) \quad [\text{using } \textbf{(C.}\wedge\textbf{)}] \\
\Leftrightarrow & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A \wedge B)
\end{aligned}
$$

Crucially, **(C.∧)** amounts to $\text{atts}(A \wedge B) \subseteq \text{atts}(A) \cup \text{atts}(B)$.

$(\wedge, l)$ We have to show that $A, B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $A \wedge B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$. The premise states that $\overline{\text{atts}}_S(\Gamma, A, B) \supseteq \underline{\text{atts}}_S(\Delta)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$ and satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma, A \wedge B) \supseteq \underline{\text{atts}}_{S'}(\Delta)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{A \wedge B\}$. Since every SAF S' of the second kind reduces to one of the first kind we may argue over any such SAF and drop the reference. We obtain:

$$
\begin{aligned}
& \overline{\text{atts}}(\Gamma, A, B) \supseteq \underline{\text{atts}}(\Delta) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \cup \text{atts}(B) \supseteq \underline{\text{atts}}(\Delta) \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A \wedge B) \supseteq \overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \cup \text{atts}(B) \supseteq \underline{\text{atts}}(\Delta) && \text{[using } (\mathbf{A.}\wedge)\text{]} \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma, A \wedge B) \supseteq \underline{\text{atts}}(\Delta)
\end{aligned}
$$

Crucially, $(\mathbf{A.}\wedge)$ amounts to $\text{atts}(A) \cup \text{atts}(B) \subseteq \text{atts}(A \wedge B)$.

$(\vee, r)$ We have to show that $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A, B$ implies $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \vee B$. The premise states that $\overline{\text{atts}}_S(\Gamma) \supseteq \underline{\text{atts}}_S(\Delta, A, B)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$ and satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma) \supseteq \underline{\text{atts}}_{S'}(\Delta, A \vee B)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{A \vee B\}$. Since every SAF S' of the second kind reduces to one of the first kind we may argue over any such SAF and drop the reference. We obtain:

$$
\begin{aligned}
& \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A, B) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \cap \text{atts}(B) \\
\Rightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A \vee B) && \text{[using } (\mathbf{A.}\vee)\text{]} \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A \vee B)
\end{aligned}
$$

Crucially, $(\mathbf{A.}\vee)$ amounts to $\text{atts}(A \vee B) \subseteq \text{atts}(A \vee B)$.

$(\vee, l)$ We have to show that $A, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$ and $B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $A \vee B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$. The premises state that $\overline{\text{atts}}_S(\Gamma, A) \supseteq \underline{\text{atts}}_S(\Delta)$ and $\overline{\text{atts}}_S(\Gamma, B) \supseteq \underline{\text{atts}}_S(\Delta)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$ and satisfies the CAP-principles. The conclusion states that $\overline{\text{atts}}_{S'}(\Gamma, A \vee B) \supseteq \underline{\text{atts}}_{S'}(\Delta)$, where S' now ranges over the CAP-complying SAFs that are closed with respect to $\Gamma \cup \Delta \cup \{A \wedge B\}$. Since every SAF S' of the second kind reduces to one of the first kind we may argue over any such SAF and drop the reference. We obtain:

$$
\begin{array}{ccc}
\overline{\text{atts}}(\Gamma, A) \supseteq \underline{\text{atts}}(\Delta) & & \overline{\text{atts}}(\Gamma, B) \supseteq \underline{\text{atts}}(\Delta) \\
\Updownarrow & & \Updownarrow \\
\overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \supseteq \underline{\text{atts}}(\Delta) & & \overline{\text{atts}}(\Gamma) \cup \text{atts}(B) \supseteq \underline{\text{atts}}(\Delta) \\
& \Downarrow &
\end{array}
$$

$$
\begin{aligned}
& (\overline{\text{atts}}(\Gamma) \cup \text{atts}(A)) \cap (\overline{\text{atts}}(\Gamma) \cup \text{atts}(B)) \supseteq \underline{\text{atts}}(\Delta) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A \vee B) \supseteq \overline{\text{atts}}(\Gamma) \cup (\text{atts}(A) \cap \text{atts}(B)) \supseteq \underline{\text{atts}}(\Delta) \ \text{[using } (\mathbf{C.}\vee)\text{]} \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma, A \vee B) \supseteq \underline{\text{atts}}(\Delta)
\end{aligned}
$$

Crucially, $(\mathbf{C.}\vee)$ amounts to $\text{atts}(A) \cap \text{atts}(B) \subseteq \text{atts}(A \vee B)$.

$(\supset, r)$ We show that, if $A, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, B$, then $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \supset B$. The premise states that $\overline{\text{atts}}_S(\Gamma, A) \supseteq \underline{\text{atts}}_S(\Delta, B)$ for every SAF S that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$ and satisfies the CAP-principles and the conclusion that $\overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A \supset B)$ for the same set of SAFs. As before, $(\cdot)^c$ denotes the complement with respect to the set of arguments in question.

$$
\begin{aligned}
& \overline{\text{atts}}(\Gamma, A) \supseteq \underline{\text{atts}}(\Delta, B) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \cup \text{atts}(A) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(B) \\
\Leftrightarrow\ & (\overline{\text{atts}}(\Gamma) \cup \text{atts}(A))^c \cap \underline{\text{atts}}(\Delta) \cap \text{atts}(B) = \emptyset \\
\Leftrightarrow\ & (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(A))^c \cap \underline{\text{atts}}(\Delta) \cap \text{atts}(B) = \emptyset \\
\Rightarrow\ & (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(A))^c \cap \underline{\text{atts}}(\Delta) \cap \text{atts}(A \supset B) = \emptyset & [\text{using } \mathbf{(C.\supset)}\ ] \\
\Rightarrow\ & (\overline{\text{atts}}(\Gamma))^c \cap \underline{\text{atts}}(\Delta) \cap \text{atts}(A \supset B) = \emptyset & [\text{since atts}(A \supset B) \subseteq (\text{atts}(A))^c] \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta) \cap \text{atts}(A \supset B) \\
\Leftrightarrow\ & \overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A \supset B)
\end{aligned}
$$

Note that $\mathbf{(C.\supset)}$ amounts to $\text{atts}(A \supset B) \subseteq \text{atts}(B)$ and $\text{atts}(A \supset B) \cap \text{atts}(A) = \emptyset$.

$(\supset, l)$ We show that, if $\Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ and $B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$, then $A \supset B, \Gamma \models^{\mathsf{CAP}}_{\text{arg}} \Delta$. Like above, the premises amount to $\overline{\text{atts}}(\Gamma) \supseteq \underline{\text{atts}}(\Delta, A)$ and $\overline{\text{atts}}(B, \Gamma) \supseteq \underline{\text{atts}}(\Delta)$, respectively; whereas the conclusion is $\overline{\text{atts}}(A \supset B, \Gamma) \supseteq \underline{\text{atts}}(\Delta)$. We use $(\cdot)^c$ to denote the complement with respect to the set of arguments in question.

$$
\begin{aligned}
& \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \subseteq \overline{\text{atts}}(\Gamma) \text{ and } \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup \text{atts}(B) \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap \text{atts}(A) \cap (\overline{\text{atts}}(\Gamma))^c = \emptyset \quad \text{and} \quad \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma) \cup \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap \text{atts}(A) = \emptyset \quad \text{and} \quad \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap (\text{atts}(A) \cup (\text{atts}(B))^c) = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma))^c \cap ((\text{atts}(A))^c \cap \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \cap (\overline{\text{atts}}(\Gamma) \cup (\text{atts}(A))^c \cap \text{atts}(B))^c = \emptyset \\
\Leftrightarrow\ & \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup (\text{atts}(B) \setminus \text{atts}(A)) \quad [\text{using } \mathbf{(A.\supset)}] \\
\Rightarrow\ & \underline{\text{atts}}(\Delta) \subseteq \overline{\text{atts}}(\Gamma) \cup \text{atts}(A \supset B)
\end{aligned}
$$

Note that $\mathbf{(A.\supset)}$ amounts to $\text{atts}(B) \setminus \text{atts}(A) \subseteq \text{atts}(A \supset B)$. □

Theorem A.1.2 (4.3.2) (Completeness)**.** *If* $\Gamma \models^{\mathsf{CAP}}_{arg} \Delta$ *then* $\Gamma \Rightarrow \Delta$ *is derivable in* LK*.*

Proof. We have to check the inverse directions of the implications in the proof of Theorem 4.3.1.

$(\neg, r)$ We show that $A, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, \neg A$. To this aim assume that $\overline{\text{atts}}_S(A, \Gamma) \not\supseteq \underline{\text{atts}}_S(\Delta)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$. In other words there is an $F \in \underline{\text{atts}}_S(\Delta)$, such that $F \notin \overline{\text{atts}}_S(A, \Gamma)$. The latter implies $F \not\longrightarrow A$ in S. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. Since $F \not\longrightarrow A$ also in S', we obtain $F \longrightarrow \neg A$ from **(C.¬)**. Since $F \in \underline{\mathrm{atts}}_{S'}(\Delta)$ we conclude that $F \in \underline{\mathrm{atts}}_{S'}(\Delta, \neg A)$. On the other hand, $F \notin \overline{\mathrm{atts}}_{S'}(\Gamma)$, since otherwise we already had $F \in \overline{\mathrm{atts}}_S(A, \Gamma)$. Thus we have shown that $\overline{\mathrm{atts}}_{S'}(\Gamma) \not\supseteq \underline{\mathrm{atts}}_{S'}(\Delta, \neg A)$, which entails $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, \neg A$.

$(\neg, l)$ We proceed indirectly and we have to show that $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A$ implies $\neg A, \Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$. To this aim assume that $\overline{\mathrm{atts}}_S(\Gamma) \not\supseteq \underline{\mathrm{atts}}_S(\Delta, A)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$, i.e. there is an $F \in \underline{\mathrm{atts}}_S(\Delta, A)$, such that $F \notin \overline{\mathrm{atts}}_S(\Gamma)$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{\neg A\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. Since $F \in \underline{\mathrm{atts}}_S(\Delta, A)$, we have $F \longrightarrow A$ also in S' and from **(A.¬)** $F \not\longrightarrow \neg A$. Since $F \notin \overline{\mathrm{atts}}_{S'}(\Gamma)$, then $F \notin \overline{\mathrm{atts}}_{S'}(\Gamma, \neg A)$. Thus we have shown $\overline{\mathrm{atts}}_{S'}(\neg A, \Gamma) \not\supseteq \underline{\mathrm{atts}}_{S'}(\Delta)$, i.e. $\neg A, \Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$.

$(\wedge, r)$ We proceed indirectly and we have to show that (1) $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A \wedge B$ and (2) $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, B$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A \wedge B$. We only analyse the case (1) since the case (2) is similar. If we assume that $\overline{\mathrm{atts}}_S(\Gamma) \not\supseteq \underline{\mathrm{atts}}_S(\Delta, A)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$, i.e. there is an $F \in \underline{\mathrm{atts}}_S(\Delta, A)$, such that $F \notin \overline{\mathrm{atts}}_S(\Gamma)$. Since $F \longrightarrow A$, from **(A.∧)** it follows $F \longrightarrow A \wedge B$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \wedge B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. Therefore $F \in \underline{\mathrm{atts}}_S(\Delta, A \wedge B)$ and $\overline{\mathrm{atts}}_{S'}(\Gamma) \not\supseteq \underline{\mathrm{atts}}_{S'}(\Delta, A \wedge B)$, i.e. $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A \wedge B$.

$(\wedge, l)$ We proceed indirectly and we have to show that $\Gamma, A, B \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$ implies $\Gamma, A \wedge B \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$. If we assume that $\overline{\mathrm{atts}}_S(\Gamma, A, B) \not\supseteq \underline{\mathrm{atts}}_S(\Delta)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$, there is an $F \in \underline{\mathrm{atts}}_S(\Delta)$, such that $F \notin \overline{\mathrm{atts}}_S(\Gamma, A, B)$, i.e. $F \not\longrightarrow A$ and $F \not\longrightarrow B$. Therefore from **(C.∧)** we have $F \not\longrightarrow A \wedge B$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \wedge B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. We have then $F \notin \underline{\mathrm{atts}}_{S'}(\Gamma, A \wedge B)$ and $\overline{\mathrm{atts}}_{S'}(\Gamma, A \wedge B) \not\supseteq \underline{\mathrm{atts}}_{S'}(\Delta)$, i.e. $\Gamma, A \wedge B \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta$.

$(\vee, r)$ We proceed indirectly and we have to show that $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A, B$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\mathrm{arg}} \Delta, A, B$. If we assume that $\overline{\mathrm{atts}}_S(\Gamma) \not\supseteq \underline{\mathrm{atts}}_S(\Delta, A, B)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$, it means there is an $F \in \underline{\mathrm{atts}}_S(\Delta, A, B)$, such that $F \notin \overline{\mathrm{atts}}_S(\Gamma)$, i.e. $F \longrightarrow A$, $F \longrightarrow B$ and by **(C.∨)** $F \longrightarrow A \wedge B$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \vee B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$.

We have then $F \in \underline{\text{atts}}'_S(\Delta, A \vee B)$ from which it follows $\overline{\text{atts}}_{S'}(\Gamma) \not\supseteq \underline{\text{atts}}_{S'}(\Delta, A \vee B)$, i.e. $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \vee B$.

$(\vee, l)$ We proceed indirectly and we have to show that (1) $\Gamma, A \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $\Gamma, A \vee B \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ and (2) $\Gamma, B \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $\Gamma, A \vee B \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$. We only show the case (1) since the case (2) is completely similar. If we assume that $\overline{\text{atts}}_S(\Gamma, A) \not\supseteq \underline{\text{atts}}_S(\Delta)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$, it means there is an $F \in \underline{\text{atts}}_S(\Delta)$, such that $F \notin \overline{\text{atts}}_S(\Gamma, A)$, i.e. $F \not\longrightarrow A$, and by **(A.$\vee$)** $F \not\longrightarrow A \vee B$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \vee B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. We have then $F \notin \overline{\text{atts}}_{S'}(\Gamma, A \vee B)$ from which it follows $\overline{\text{atts}}_{S'}(\Gamma, A \vee B) \not\supseteq \underline{\text{atts}}_{S'}(\Delta)$, i.e. $\Gamma, A \vee B \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$.

$(\supset, r)$ We again proceed indirectly and show that $\Gamma, A \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, B$ implies $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \supset B$. If we assume that $\overline{\text{atts}}_S(\Gamma, A) \not\supseteq \underline{\text{atts}}_S(\Delta, B)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\} \cup \{B\}$, it means there is an $F \in \underline{\text{atts}}_S(\Delta, B)$, such that $F \notin \overline{\text{atts}}_S(\Gamma, A)$, i.e. $F \longrightarrow B$, $F \not\longrightarrow A$ and by **(A.$\supset$)** $F \longrightarrow A \supset B$. Now let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \supset B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. We have then $F \in \underline{\text{atts}}_{S'}(\Delta, A \supset B)$ from which it follows $\overline{\text{atts}}_{S'}(\Gamma) \not\supseteq \underline{\text{atts}}_{S'}(\Delta, A \supset B)$, i.e. $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, A \supset B$.

$(\supset, l)$ We again proceed indirectly and show that (1) $\Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta, A$ implies $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$, and (2) $B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$ implies $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$.

For (1) assume that $\overline{\text{atts}}_S(\Gamma) \not\supseteq \underline{\text{atts}}_S(\Delta, A)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A\}$. Thus there is an $F \in \underline{\text{atts}}_S(\Delta, A)$, such that $F \notin \overline{\text{atts}}_S(\Gamma)$. In particular $F \in \underline{\text{atts}}_S(\Delta)$. Let S' be an CAP-complying SAF that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{A \supset B\}$, where the attack relation restricted to those (claims of) arguments that already occur in S coincides with $R^S_{\rightarrow}$. Then **(C.$\supset$)** implies $F \not\longrightarrow A \supset B$ in S'. Therefore $\overline{\text{atts}}_{S'}(A \supset B, \Gamma) \not\supseteq \underline{\text{atts}}_{S'}(\Delta)$, which in turn entails $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$.

For (2) assume that $\overline{\text{atts}}_S(\Gamma, B) \not\supseteq \underline{\text{atts}}_S(\Delta)$ for some CAP-complying SAF $S = (Ar^S, R^S_{\rightarrow})$ that is syntactically closed with respect to $\Gamma \cup \Delta \cup \{B\}$. Thus there is an $F \in \underline{\text{atts}}_S(\Delta)$, such that $F \notin \overline{\text{atts}}_S(\Gamma, B)$. Let S' be an SAF like in case (1). Then $F \not\longrightarrow B$ also in S'. Therefore **(C.$\supset$)** implies $F \not\longrightarrow A \supset B$ in S', which entails $A \supset B, \Gamma \not\models^{\mathsf{CAP}}_{\text{arg}} \Delta$, like in case (1). $\square$

A.2 Extended Proofs of Section 4.6

Proposition A.2.1 (4.6.2) (APs and Admissible extensions). *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame,* $\mathsf{Adm}(\mathcal{S}) := \{\mathsf{D} \subseteq Ar \mid \mathsf{D}$ *is an admissible set*$\}$ *and* $\mathsf{D} \in \mathsf{Adm}(\mathcal{S})$*, we have:*

(1) *(a) Whenever* **(C.$\wedge$)** *and* **(D.$\wedge$)** *hold in* $\mathcal{S}$*, if* $A \in \mathsf{D}$ *and* $B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A \wedge B\} \in \mathsf{Adm}(\mathcal{S})$*.*

(b) Whenever **(A.$\wedge$)** *and* **(B.$\wedge$)** *hold in* $\mathcal{S}$*, if* $A \wedge B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A\} \in \mathsf{Adm}(\mathcal{S})$ *and* $\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$*.*

(2) Whenever **(A.$\vee$)** *and* **(B.$\vee$)** *hold in* $\mathcal{S}$*, if* $A \in \mathsf{D}$ *or* $B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A \vee B\} \in \mathsf{Adm}(\mathcal{S})$*.*

(3) *(a) Whenever* **(C.$\supset$)** *and* **(D.$\supset$)** *hold in* $\mathcal{S}$*, If* $B \in \mathsf{D}$*, then* $\mathsf{D} \cup \{A \supset B\} \in \mathsf{Adm}(\mathcal{S})$*.*

(b) Whenever **(A.$\supset$)**, **(B.$\supset$)** *and* **(A.$\supset$)*** *hold in* $\mathcal{S}$*, if* $A \supset B \in \mathsf{D}$*, then either* $\mathsf{D} \cup \{B\} \in \mathsf{Adm}(\mathcal{S})$ *or* $\mathsf{D} \cup \{A\} \notin \mathsf{Adm}(\mathcal{S})$*.*

(4) Whenever **(C.$\neg$)** *holds in* $\mathcal{S}$*, if* $A \in \mathsf{D}$*, then* $\neg A \notin \mathsf{D}$*.*

Proof. (1) (a) If $A \in \mathsf{D}$ and $B \in \mathsf{D}$, then $\mathsf{D} \not\longrightarrow A$, $A \not\longrightarrow \mathsf{D}$, $\mathsf{D} \not\longrightarrow B$ and $B \not\longrightarrow \mathsf{D}$. Therefore, by **(C.$\wedge$)** and **(D.$\wedge$)** we have $\mathsf{D} \not\longrightarrow A \wedge B$ and $A \wedge B \not\longrightarrow \mathsf{D}$, i.e. $\mathsf{D} \cup \{A \wedge B\}$ is conflict-free.

Moreover whenever an argument $X \in Ar \setminus \mathsf{D}$ attacks $A \wedge B$ ($X \longrightarrow A \wedge B$), by **(C.$\wedge$)** we have either $X \longrightarrow A$ or $X \longrightarrow B$. Therefore all the attackers of $A \wedge B$ are also either attackers of A or attackers of B. Since both A and B are in D and D defend itself, $\mathsf{D} \longrightarrow X$, i.e. D defends also $A \wedge B$. Conclusively $\mathsf{D} \cup \{A \wedge B\}$ is an admissible set.

(b) If $A \wedge B \in \mathsf{D}$, then $\mathsf{D} \not\longrightarrow A \wedge B$ and $A \wedge B \not\longrightarrow \mathsf{D}$. Therefore, by **(A.$\wedge$)** and **(B.$\wedge$)** we have $\mathsf{D} \not\longrightarrow A$, $\mathsf{D} \not\longrightarrow B$, $A \not\longrightarrow \mathsf{D}$ and $B \not\longrightarrow \mathsf{D}$, i.e. both $\mathsf{D} \cup \{A\}$ and $\mathsf{D} \cup \{B\}$ are conflict-free. It remains to show that they can defend themselves. Whenever there is $X \in Ar \setminus \mathsf{D}$ s.t. $X \longrightarrow A$, by **(A.$\wedge$)** we have $X \longrightarrow A \wedge B$ with $A \wedge B \in \mathsf{D}$, which implies $\mathsf{D} \longrightarrow X$. The same can be affirmed also for all the attackers of B. Conclusively both $\mathsf{D} \cup \{A\}$ and $\mathsf{D} \cup \{B\}$ defend themselves.

(2) (a) If $A \in \mathsf{D}$, then $\mathsf{D} \not\longrightarrow A$ and $A \not\longrightarrow \mathsf{D}$. Therefore, by **(A.$\vee$)** and **(B.$\vee$)** we have $\mathsf{D} \not\longrightarrow A \vee B$ and $A \vee B \not\longrightarrow \mathsf{D}$, i.e. $\mathsf{D} \cup \{A \vee B\} \in \mathsf{Cf}(\mathcal{S})$. It remains to show that $\mathsf{D} \cup \{A \vee B\}$ can defend itself knowing that D already does. If there is an argument $X \in Ar \setminus \mathsf{D}$ s.t. $X \longrightarrow A \vee B$, then, by **(A.$\vee$)**, we have $X \longrightarrow A$ and $X \longrightarrow B$. Since $A \in \mathsf{D}$ and $\mathsf{D} \longrightarrow X$, $\mathsf{D} \cup \{A \vee B\}$ can defend itself.

(3) (a) If $B \in \mathsf{D}$, then $\mathsf{D} \not\longrightarrow B$ and $B \not\longrightarrow \mathsf{D}$. Therefore, by **(C.**$\supset$**)** and **(D.**$\supset$**)** we have $\mathsf{D} \not\longrightarrow A \supset B$ and $A \supset B \not\longrightarrow \mathsf{D}$, i.e. $\mathsf{D} \cup \{A \supset B\} \in \mathsf{Cf}(\mathcal{S})$. It remains to show that $\mathsf{D} \cup \{A \supset B\}$ can defend itself knowing that D already does. Whenever there is an argument $X \in Ar \setminus \mathsf{D}$ s.t. $X \longrightarrow A \supset B$, by **(C.**$\supset$**)** we have $X \longrightarrow B$. Since $B \in \mathsf{D}$, $\mathsf{D} \longrightarrow X$, i.e. D defends $A \supset B$.

(b) If $A \supset B \in \mathsf{D}$, then $\mathsf{D} \not\longrightarrow A \supset B$ and $A \supset B \not\longrightarrow \mathsf{D}$. Therefore, by **(A.**$\supset$**)** we have either $\mathsf{D} \not\longrightarrow B$ or $\mathsf{D} \longrightarrow A$, and by **(B.**$\supset$**)** either $B \not\longrightarrow \mathsf{D}$ or $A \longrightarrow \mathsf{D}$. If $\mathsf{D} \longrightarrow A$ or $A \longrightarrow \mathsf{D}$, then $\mathsf{D} \cup \{A\} \notin \mathsf{Cf}(\mathcal{A})$ which implies $\mathsf{D} \cup \{A\} \notin \mathsf{Adm}(\mathcal{S})$. If $\mathsf{D} \not\longrightarrow B$ and $B \not\longrightarrow \mathsf{D}$, then $\mathsf{D} \cup \{B\} \in \mathsf{Cf}(\mathcal{A})$. It remains to show that $\mathsf{D} \cup \{B\}$ can defend itself knowing that D already does. Whenever there is an argument $X \in Ar \setminus \mathsf{D}$ s.t. $X \longrightarrow B$, by **(A.**$\supset$**)*** $X \longrightarrow A \supset B$. Since $A \supset B \in \mathsf{D}$, $\mathsf{D} \longrightarrow X$, i.e. D defends B.

(4) Suppose $\neg A \in \mathsf{D}$, therefore if both A and $\neg A$ are in D, then $A \not\longrightarrow \neg A$, but by **(C.**$\neg$**)** $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$ and this it cannot be. □

Proposition A.2.2 (4.6.4) (APs and Complete extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and C a complete extension over $\mathcal{S}$ ($\mathsf{C} \in \mathsf{Cmp}(\mathcal{S})$), we have:*

(1) Whenever **(A.**$\wedge$**)**, **(B.**$\wedge$**)**, **(D.**$\wedge$**)** *and* **(C.**$\wedge$**)** *hold in $\mathcal{S}$, $A \in \mathsf{C}$ and $B \in \mathsf{C}$ iff $A \wedge B \in \mathsf{C}$.*

(2) If **(A.**$\vee$**)** *and* **(B.**$\vee$**)** *hold in $\mathcal{S}$, $A \in \mathsf{C}$ or $B \in \mathsf{C}$, then $A \vee B \in \mathsf{C}$*

(3) (a) Whenever **(C.**$\supset$**)** *and* **(D.**$\supset$**)** *hold in $\mathcal{S}$, if $B \in \mathsf{C}$,then $A \supset B \in \mathsf{C}$.*

(b) Whenever **(A.**$\supset$**)** *and* **(B.**$\supset$**)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{C}$, then $B \in \mathsf{C}$ or $A \notin \mathsf{C}$.*

(4) Whenever **(C.**$\neg$**)** *holds in $\mathcal{S}$, if $A \in \mathsf{C}$, then $\neg A \notin \mathsf{C}$.*

Proof. (1) ($\Rightarrow$) We have to show that $\mathsf{C} \cup \{A \wedge B\}$ is (i) conflict-free, (ii) it defends all its elements and (iii) it contains every element it defends.

(i) If $\mathsf{C} \cup \{A \wedge B\}$ is not conflict-free, then either $\mathsf{C} \longrightarrow A \wedge B$ or $A \wedge B \longrightarrow \mathsf{C}$. In the first case by **(C.**$\wedge$**)** it follows either $\mathsf{C} \longrightarrow A$ or $\mathsf{C} \longrightarrow B$ and in the second case by **(D.**$\wedge$**)** either $A \longrightarrow \mathsf{C}$ or $B \longrightarrow \mathsf{C}$ which goes, in any case, against our assuption.

(ii) Since C defends all its elements, it remains to show that $A \wedge B$ can be defended by $\mathsf{C} \cup \{A \wedge B\}$. Suppose there is an argument $X \in Ar \setminus \mathsf{C}$ s.t. $X \longrightarrow A \wedge B$. By **(C.**$\wedge$**)** we have either $X \longrightarrow A$ or $X \longrightarrow B$. Since both A and B are in C, $\mathsf{C} \longrightarrow X$, i.e. C defends $A \wedge B$ from any attacker and being C a complete extension, $A \wedge B \in \mathsf{C}$.

(*iii*) Since C contains every element it defends, we have to show that every element that $A \wedge B$ defends is already in C. If $A \wedge B$ defends an argument F, then $A \wedge B$ attacks every attacker of F, i.e. for any $X \longrightarrow F$, $A \wedge B \longrightarrow X$. From **(D.$\wedge$)** it follows that either $A \longrightarrow X$ or $B \longrightarrow X$, i.e. either A defends F or B defends F. In any case, $A \wedge B$ does not defend anything not already defended by either A or B.

($\Leftarrow$) We have to show that $\mathsf{C} \cup \{A\}$ is (*i*) conflict-free, (*ii*) it defends all its elements and (*iii*) it contains every element it defends.

(*i*) If $\mathsf{C} \cup \{A\}$ is not conflict-free, then either $\mathsf{C} \longrightarrow A$ or $A \longrightarrow \mathsf{C}$. In the first case, by **(A.$\wedge$)**, it follows $\mathsf{C} \longrightarrow A \wedge B$. In the second case, by **(B.$\wedge$)**, it follows $A \wedge B \longrightarrow \mathsf{C}$ which goes, in both cases, against the assuption of having C conflict-free.

(*ii*) Since C defends all its elements, it remains to show that A can be defended by $\mathsf{C} \cup \{A\}$. Suppose there is an argument $X \in Ar \setminus \mathsf{C}$ s.t. $X \longrightarrow A$. By **(A.$\wedge$)**, we have $X \longrightarrow A \wedge B$. Since $A \wedge B \in \mathsf{C}$, $\mathsf{C} \longrightarrow X$, i.e. C defends A from any attacker and being C a complete extension, $A \in \mathsf{C}$.

(*iii*) Since C contains every element it defends, we have to show that every element that A defends is already in C. If A defends an argument F, then A attacks every attacker of F, i.e. for any $X \longrightarrow F$, $A \longrightarrow X$. By **(B.$\wedge$)**, we have $A \wedge B \longrightarrow X$, i.e. $A \wedge B$ defends F. In any case, A does not defend anything not already defended by $A \wedge B$.

(2) We have to show that $\mathsf{C} \cup \{A \vee B\}$ is (*i*) conflict-free, (*ii*) it defends all its elements and (*iii*) it contains every element it defends.

(*i*) If $\mathsf{C} \cup \{A \vee B\}$ is not conflict-free, then either $\mathsf{C} \longrightarrow A \vee B$ or $A \vee B \longrightarrow \mathsf{C}$. In the first case, by **(A.$\vee$)**, it follows that $A \longrightarrow \mathsf{C}$ and $B \longrightarrow \mathsf{C}$ while from $A \vee B \longrightarrow \mathsf{C}$ and **(B.$\vee$)**it follows $\mathsf{C} \longrightarrow A$ and $\mathsf{C} \longrightarrow B$. In both cases we are against the assumption that C is conflict-free.

(*ii*) Since C defends all its elements, it remains to show that $A \vee B$ can be defended by $\mathsf{C} \cup \{A \vee B\}$. Suppose there is an argument $X \in Ar \setminus \mathsf{C}$ s.t. $X \longrightarrow A \vee B$. By **(A.$\vee$)**, we have $X \longrightarrow A$ and $X \longrightarrow B$. Since either A or B is in C, $\mathsf{C} \longrightarrow X$, i.e. C defends $A \vee B$ from any attacker and being C a complete extension, $A \vee B \in \mathsf{C}$.

(*iii*) We have to show that every element that $A \vee B$ defends is already in C. If $A \vee B$ defends an argument F, then $A \vee B$ attacks every attacker of F, i.e. for any $X \longrightarrow F$, $A \vee B \longrightarrow X$. By **(B.$\vee$)**, we have $A \longrightarrow X$ and $B \longrightarrow X$, i.e. both A and B defend F. In any case, $A \vee B$ does not defend anything not already defended by both A and B.

(3) (a) We have to show that $\mathsf{C} \cup \{A \supset B\}$ is (*i*) conflict-free, (*ii*) it defends all its elements and (*iii*) it contains every element it defends.

(*i*) If $\mathsf{C} \cup \{A \supset B\}$ is not conflict-free, then either $\mathsf{C} \longrightarrow A \supset B$ or $A \supset B \longrightarrow \mathsf{C}$. Therefore, by **(C.$\supset$)** and **(D.$\supset$)**, it follows either $\mathsf{C} \longrightarrow B$ or $B \longrightarrow \mathsf{C}$ which is against the assumption that C is conflict-free.

(*ii*) Since C defends all its elements, we only need to show that $A \supset B$ can be defended by $\mathsf{C} \cup \{A \supset B\}$. Suppose there is an argument $X \in Ar \setminus \mathsf{C}$ s.t. $X \longrightarrow A \supset B$. By **(C.$\supset$)** we have $X \longrightarrow B$ with $B \in \mathsf{C}$. Therefore, $\mathsf{C} \longrightarrow X$, i.e. C defends $A \supset B$.

(*iii*) We have to show that every element that $A \supset B$ defends is already in C. If $A \supset B$ defends an argument F, then $A \supset B$ attacks every attacker of F, i.e. for any $X \longrightarrow F$, $A \supset B \longrightarrow X$. By **(D.$\supset$)**, we have $B \longrightarrow X$, with $B \in \mathsf{C}$. Therefore, $A \vee B$ does not defend anything not already defended by B.

(b) Suppose that $B \notin \mathsf{C}$ and $A \in \mathsf{C}$. The reasons why $B \notin \mathsf{C}$ could be several: (*i*) it might be that $B \cup \mathsf{C}$ is not conflict-free or, (*ii*) C might not defend B, or (*iii*)B might defend some argument not in C.

(*i*) If $B \cup \mathsf{C}$ is not conflict-free, then either $B \longrightarrow \mathsf{C}$ or $\mathsf{C} \longrightarrow B$. Since $A \in \mathsf{C}$, then $\mathsf{C} \not\longrightarrow A$ and $A \not\longrightarrow \mathsf{C}$. Therefore, by **(B.$\supset$)** and **(A.$\supset$)**, it follows $A \supset B \longrightarrow \mathsf{C}$ and $\mathsf{C} \longrightarrow A \supset B$ which is against the assuption of having C conflict-free.

(*ii*) If C does not defend B, then there is some argument X such that $X \longrightarrow B$, $X \not\longrightarrow A$ and $\mathsf{C} \not\longrightarrow X$. Therefore, by **(A.$\supset$)**, it follows $X \longrightarrow A \supset B$, but this would imply that C should defend X.

(*iii*) Suppose there is an argument F s.t. $B \longrightarrow X \not\longrightarrow F$ with $F \notin \mathsf{C}$. Since $A \in \mathsf{C}$, $A \not\longrightarrow X$ and by **(B.$\supset$)** we would have $A \supset B \longrightarrow X$, i.e. B cannot defend anything not already in C.

(4) Suppose $\neg A \in \mathsf{C}$. Thus, if both A and $\neg A$ are in C, then $A \not\longrightarrow \neg A$. However, by **(C.$\neg$)**, $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$ and this it cannot be. □

Proposition A.2.3 (4.6.6) (APs and Preferred extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and P a preferred extension over $\mathcal{S}$ ($\mathsf{P} \in \mathsf{Prf}(\mathcal{S})$), we have:*

(1) Whenever **(A.$\wedge$)**, **(B.$\wedge$)**, **(C.$\wedge$)** *and* **(D.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{P}$ and $B \in \mathsf{P}$ iff $A \wedge B \in \mathsf{P}$.*

(2) Whenever **(A.$\vee$)** *and* **(B.$\vee$)** *hold in $\mathcal{S}$, $A \in \mathsf{P}$ or $B \in \mathsf{P}$, then $A \vee B \in \mathsf{P}$*

(3) (a) Whenever **(C.$\supset$)** *and* **(D.$\supset$)** *holds in $\mathcal{S}$, if $B \in \mathsf{P}$, then $A \supset B \in \mathsf{P}$;*

(b) Whenever **(A.$\supset$)** *and* **(B.$\supset$)** *holds in $\mathcal{S}$, if $A \supset B \in \mathsf{P}$, then $B \in \mathsf{P}$ or $A \notin \mathsf{P}$*

(4) Whenever **(C.$\neg$)** *holds in $\mathcal{S}$, if $A \in \mathsf{P}$, then $\neg A \notin \mathsf{P}$.*

Proof. (1) ($\Rightarrow$) We have to show that $A \wedge B \in \mathsf{P}$, which is equivalent to showing that $\mathsf{P} \cup \{A \wedge B\}$ is (i) conflict-free, (ii) and it defends all its elements.

(i) If $\mathsf{P} \cup \{A \wedge B\}$ is not conflict-free, then either $A \wedge B \longrightarrow \mathsf{P}$, or $\mathsf{P} \longrightarrow A \wedge B$. Therefore, by **(D.**$\wedge$**)** and **(C.**$\wedge$**)**, $A \longrightarrow \mathsf{P}$, or $B \longrightarrow \mathsf{P}$, or $\mathsf{P} \longrightarrow A$, or $\mathsf{P} \longrightarrow B$. However, in any of the above cases we are in contradiction with the assumption of having both A and B in P.

(ii) If $\mathsf{P} \cup \{A \wedge B\}$ does not defend itself, then there is some argument X such that $X \longrightarrow A \wedge B$ and $\mathsf{P} \cup \{A \wedge B\} \not\longrightarrow X$. However, from $X \longrightarrow A \wedge B$ and **(C.**$\wedge$**)** it follows either $X \longrightarrow A$ or $X \longrightarrow B$. By having both A and B in P it follows that $\mathsf{P} \longrightarrow X$ and this is against our assumption.

($\Leftarrow$) We will only show that $A \in \mathsf{P}$. The proof to show that $B \in \mathsf{P}$ is similar. Therefore, we will prove that (i) $\mathsf{P} \cup \{A\}$ is conflict-free and that (ii) it defend all its elements.

(i) If $\mathsf{P} \cup \{A\}$ is not conflict-free, then either $A \longrightarrow \mathsf{P}$ or $\mathsf{P} \longrightarrow A$. Thus, by **(B.**$\wedge$**)**and **(A.**$\wedge$**)**, $A \wedge B \longrightarrow \mathsf{P}$ and $\mathsf{P} \longrightarrow A \wedge B$ which goes against the assumption of having $A \wedge B \in \mathsf{P}$.

(ii) If $\mathsf{P} \cup \{A\}$ does not defend itself, then there is some argument X such that $X \longrightarrow A$ and $\mathsf{P} \cup \{A\} \not\longrightarrow X$. However, from $X \longrightarrow A$ and **(A.**$\wedge$**)** it follows $X \longrightarrow A \wedge B$. Since $A \wedge B \in \mathsf{P}$, we have $\mathsf{P} \longrightarrow X$, but this is against our assumption.

(2) We have to show that $\mathsf{P} \cup \{A \vee B\}$ is (i) conflict-free and that (ii) it defends all its elements.

(i) If $\mathsf{P} \cup \{A \vee B\}$ is not conflict-free, then either $A \vee B \longrightarrow \mathsf{P}$ or $\mathsf{P} \longrightarrow A \vee B$ from which it follows, by **(B.**$\vee$**)**and **(A.**$\vee$**)**, $A \longrightarrow \mathsf{P}$ and $\mathsf{P} \longrightarrow B$, or $\mathsf{P} \longrightarrow A$ and $\mathsf{P} \longrightarrow B$. In any case we are against the hypothesis of having either A or B in P.

(ii) If $\mathsf{P} \cup \{A \vee B\}$ does not defend itself, then there is some argument X such that $X \longrightarrow A \vee B$ and $\mathsf{P} \cup \{A \vee B\} \not\longrightarrow X$. However, from $X \longrightarrow A \vee B$ and **(B.**$\vee$**)**it follows $X \longrightarrow A$ and $X \longrightarrow B$. Thus, $\mathsf{P} \longrightarrow X$, but this is against our assumptions.

(3) (a) We have to show that $\mathsf{P} \cup \{A \supset B\}$ is (i) conflict-free and that (ii) it defends all its elements.

(i) If $\mathsf{P} \cup \{A \supset B\}$ is not conflict-free, then either $A \vee B \longrightarrow \mathsf{P}$ or $\mathsf{P} \longrightarrow A \vee B$ from which it follows, by **(C.**$\supset$**)** and **(D.**$\supset$**)**, $B \longrightarrow \mathsf{P}$ or $\mathsf{P} \longrightarrow B$. In any case we are against the assumption of having B in P.

(ii) If $\mathsf{P} \cup \{A \supset B\}$ does not defend itself, then there is some argument X such that $X \longrightarrow A \supset B$ and $\mathsf{P} \cup \{A \supset B\} \not\longrightarrow X$. However, from $X \longrightarrow A \supset B$ and **(C.**$\supset$**)** it follows $X \longrightarrow B$. Thus, $\mathsf{P} \longrightarrow X$, but this is against our assumptions.

(b) Suppose that $B \notin \mathsf{P}$ and $A \in \mathsf{P}$. The reasons why $B \notin \mathsf{P}$ are multiple and it might be that (i) $\mathsf{P} \cup \{B\}$ is not conflict-free, or that (ii) $\mathsf{P} \cup \{B\}$ does not defend itself.

(i) If $\mathsf{P} \cup \{B\}$ is not conflict-free, then either $\mathsf{P} \longrightarrow B$, or $B \longrightarrow \mathsf{P}$. Since $A \in \mathsf{P}$, then $A \not\longrightarrow \mathsf{P}$ and $\mathsf{P} \not\longrightarrow A$. Therefore, by **(B.$\supset$)**and **(A.$\supset$)**, we have $A \supset B \longrightarrow \mathsf{P}$ or $\mathsf{P} \longrightarrow A \supset B$ which goes against the assumption of having $A \supset B \in \mathsf{P}$.

(ii) If $\mathsf{P} \cup \{B\}$ does not defend itself, then there is some argument X such that $X \longrightarrow B$ and $\mathsf{P} \cup \{B\} \not\longrightarrow X$. However, from $X \longrightarrow B$, $X \not\longrightarrow A$ (if $X \longrightarrow A$ P should defend A attacking X) and **(A.$\supset$)** we have $X \longrightarrow A \supset B$ with $A \supset B \in \mathsf{P}$. Therefore, P must defend $A \supset B$ and attack X, which is against our assumptions.

(4) Suppose that both A and $\neg A$ are in P. Therefore, $P = \{A, \neg A, P_1, \ldots, P_n\}$. Since P is conflict-free, it means that $A \not\longrightarrow \neg A$ and by **(C.$\neg$)** $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$ which it cannot be. $\square$

Proposition A.2.4 (4.6.7) (APs and Semi-Stable extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and S a semi-stable extension over $\mathcal{S}$ ($\mathsf{S} \in \mathsf{S\text{-}Stb}(\mathcal{S})$), we have:*

(1) Whenever **(C.$\wedge$)** *and* **(A.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ and $B \in \mathsf{S}$ iff $A \wedge B \in \mathsf{S}$.*

(2) Whenever **(A.$\vee$)** *hold in $\mathcal{S}$, if $A \in \mathsf{S}$ or $B \in \mathsf{S}$, then $A \vee B \in \mathsf{S}$*

(3) (a) Whenever **(C.$\supset$)** *holds in $\mathcal{S}$, if $B \in \mathsf{S}$,then $A \supset B \in \mathsf{S}$.*

(b) Whenever **(B.$\supset$)** *and* **(A.$\supset$)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{S}$, then $B \in \mathsf{S}$ or $A \notin \mathsf{C}$.*

(4) Whenever **(C.$\neg$)** *holds in $\mathcal{S}$, if $A \in \mathsf{S}$, then $\neg A \notin \mathsf{S}$.*

Proof. (1) ($\Rightarrow$) If $A \in \mathsf{S}$, $B \in \mathsf{S}$ and there is an argument X such that $X \longrightarrow A \wedge B$, then, by **(C.$\wedge$)**, we have that $X \longrightarrow A$ or $X \longrightarrow B$. Since S defends both A and B, S defends also $A \wedge B$ and being S a *special* complete extension, $A \wedge B \in \mathsf{S}$.

($\Leftarrow$) If $A \wedge B \in \mathsf{S}$ and there is an argument X such that $X \longrightarrow A$ (the case in which $X \longrightarrow B$ is similar), then, by **(A.$\wedge$)**, we have $X \longrightarrow A \wedge B$. Since S is a complete extension and it defends A (and B), $A \in \mathsf{S}$ (and $B \in \mathsf{S}$).

(2) If $A \in \mathsf{S}$ or $B \in \mathsf{S}$ and there is an argument X such that $X \longrightarrow A \vee B$, then, by **(A.$\vee$)**, we have $X \longrightarrow A$ and $X \longrightarrow B$. Therefore, S defends $A \vee B$ and $A \vee B \in \mathsf{S}$.

(3) (a) If $B \in \mathsf{S}$ and there is an argument X such that $X \longrightarrow A \supset B$, then, by **(C.$\supset$)**, $X \longrightarrow B$ (and $X \not\longrightarrow A$). Therefore, S defends $A \supset B$ and $A \supset B \in \mathsf{S}$.

(b) Suppose that $B \notin \mathsf{S}$, $A \in \mathsf{S}$ and that there is an argument X such that $X \longrightarrow B$. If $B \notin \mathsf{S}$ it might be for different reasons: (*i*) $\mathsf{S} \cup \{B\}$ is not conflict-free, (*ii*) $\mathsf{S} \cup \{B\}$ does not defend itself or, (*iii*) B defends something not in S.

(i) If $\mathsf{S} \cup \{B\}$ is not conflict-free, then either $B \longrightarrow \mathsf{S}$ or $\mathsf{S} \longrightarrow B$. In the first case, since $A \not\longrightarrow \mathsf{S}$, by **(B.⊃)**, we have $A \supset B \longrightarrow \mathsf{S}$ and this is against our hypothesis. In the second case, by **(A.⊃)**, we have $\mathsf{S} \longrightarrow A \supset B$ and again we are against our hypothesis.

(ii) If $\mathsf{S} \cup \{B\}$ does not defend itself, then $\mathsf{S} \cup \{B\}$ does not defend B, i.e. there is an argument X such that $X \longrightarrow B$, $X \not\longrightarrow A$ and $\mathsf{S} \cup \{B\} \not\longrightarrow X$. By **(A.⊃)** we have $X \longrightarrow A \supset B$ with $A \supset B \in \mathsf{S}$.

(iii) If B defends something not in S, then there is an argument F such that for any attacker X of F, $B \longrightarrow X \longrightarrow F$ and $A \not\longrightarrow X$. Therefore, by **(B.⊃)**, we have $A \supset B \longrightarrow X$ with $A \supset B \in \mathsf{S}$, i.e. S defends F and this is against our assumptions.

(4) Suppose both A and $\neg A$ are in S. Since S is conflict-free, it follows that $A \not\longrightarrow \neg A$ and by **(C.¬)** $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$ which it cannot be. □

Proposition A.2.5 (4.6.8) (APs and Stable extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and $\mathsf{S} \in \mathsf{Stb}(\mathcal{S})$, we have:*

(1) Whenever **(A.∧)** *and* **(C.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ and $B \in \mathsf{S}$ iff $A \wedge B \in S$.*

(2) Whenever **(A.∨)**, **(C.∨)** *and* **(B.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ or $B \in \mathsf{S}$ iff $A \vee B \in \mathsf{S}$.*

(3) (a) Whenever **(C.⊃)** *holds in $\mathcal{S}$, if $B \in \mathsf{S}$, then $A \supset B \in S$.*

(b) Whenever **(A.⊃)** *holds in $\mathcal{S}$, if $A \supset B \in \mathsf{S}$, then $B \in \mathsf{S}$ or $A \notin \mathsf{S}$*

(4) Whenever **(C.¬)** *and* **(A.¬)** *hold in $\mathcal{S}$, $A \in \mathsf{S}$ iff $\neg A \notin \mathsf{S}$.*

Proof. (1) (⇒) If $A \wedge B \notin \mathsf{S}$, then, there exists $S_i \in \mathsf{S}$ s.t. $S_i \longrightarrow A \wedge B$. Thus, by **(C.∧)**, we have $S_i \longrightarrow A$ or $S_i \longrightarrow B$, but it cannot be since S_i, A and B belong to the same stable extension S.

(⇐) If $A \notin \mathsf{S}$, then there exists $S_i \in \mathsf{S}$ s.t. $S_i \longrightarrow A$. Thus, by **(A.∧)**, $S_i \longrightarrow A \wedge B$, but it cannot be since S_i and $A \wedge B$ belong to the same stable extension S.

(2) (⇒) If $A \vee B \notin \mathsf{S}$, then there exists $S_i \in \mathsf{S}$ s.t. $S_i \longrightarrow A \vee B$ and from **(A.∨)** $S_i \longrightarrow A$ and $S_i \longrightarrow B$. However, this is against the assumption of having either S_i and A or S_i and B in the same stable extension S.

(⇐) If $A \notin \mathsf{S}$ and $B \notin \mathsf{S}$, then there exist S_i and S_j in S s.t. $S_i \longrightarrow A$ and $S_j \longrightarrow B$. Thus, by **(B.∧)**, we have $S_i \wedge S_j \longrightarrow A$ and $S_i \wedge S_j \longrightarrow B$ and by **(C.∨)** $S_i \wedge S_j \longrightarrow A \vee B$. However, this is against the assumption of having $A \vee B$ in S since also $S_i \wedge S_j$ is in S (it follows from point (1) of the proof).

(3) (a) If $A \supset B \notin \mathsf{S}$, then there is $S_i \in \mathsf{S}$ such that $S_i \longrightarrow A \supset B$. By **(C.$\supset$)**, $S_i \longrightarrow B$ and $S_i \not\longrightarrow A$, but this cannot happen since, by assumption, S_i and B belong to the same stable extension.

(b) If $B \notin \mathsf{S}$ and $A \in \mathsf{S}$, then there is $S_i \in \mathsf{S}$ s.t. $S_i \longrightarrow B$ and $S_i \not\longrightarrow A$. Therefore, by **(A.$\supset$)**, we have $S_i \longrightarrow A \supset B$ and this is against the assumption of having both S_i and $A \supset B$ in the same stable extension.

(4) ($\Rightarrow$) If both A and $\neg A$ belong to the same stable extension S, then, since stable extensions are conflict-free, $A \not\longrightarrow \neg A$. From **(C.$\neg$)** it follows that $A \longrightarrow \neg\neg A$, i.e. $A \longrightarrow A$, but this is against our assumptions.

($\Leftarrow$) If $\neg A \notin \mathsf{S}$, then that $\mathsf{S} \longrightarrow \neg A$. By **(A.$\neg$)**, we have that $\mathsf{S} \not\longrightarrow A$ which implies $A \in \mathsf{S}$. □

Proposition A.2.6 (4.6.9) (APs and Grounded Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and G a grounded extension over $\mathcal{S}$ ($\mathsf{G} \in \mathsf{Grn}(\mathcal{S})$), we have:*

(1) Whenever **(A.$\wedge$)** *and* **(C.$\wedge$)** *hold in $\mathcal{S}$, $A \in \mathsf{G}$ and $B \in \mathsf{G}$ iff $A \wedge B \in \mathsf{G}$.*

(2) Whenever **(A.$\vee$)** *holds in $\mathcal{S}$, if $A \in \mathsf{G}$ or $B \in \mathsf{G}$, then $A \vee B \in \mathsf{G}$*

(3) (a) Whenever **(C.$\supset$)** *holds in $\mathcal{S}$, if $B \in \mathsf{G}$,then $A \supset B \in \mathsf{G}$.*

(b) Whenever **(A.$\supset$)** *and* **(B.$\supset$)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{G}$, then $B \in \mathsf{G}$ or $A \notin \mathsf{G}$.*

(4) Whenever **(C.$\neg$)** *holds in $\mathcal{S}$, if $A \in \mathsf{G}$, then $\neg A \notin \mathsf{G}$.*

Proof. (1) ($\Rightarrow$) If $A \in \mathsf{G}$ and $B \in \mathsf{G}$ and there is an argument X such that $X \longrightarrow A \wedge B$, then, by **(C.$\wedge$)**, we have $X \longrightarrow A$ or $X \longrightarrow B$ with both A and B in G. Therefore, $\mathsf{G} \longrightarrow X$, i.e. G defends $A \wedge B$ from which it follows $A \wedge B \in \mathsf{G}$.

($\Leftarrow$) If $A \wedge B \in \mathsf{G}$ and there is an argument X such that $X \longrightarrow A$, then, by **(A.$\wedge$)**, we have $X \longrightarrow A \wedge B$, i.e. every attacker of A (or B) is also an attacker of $A \wedge B \in \mathsf{G}$. Thus, G defends both A and B, i.e. $A \in \mathsf{G}$ and $B \in \mathsf{G}$.

(2) If $A \in \mathsf{G}$ and there is an argument X such that $X \longrightarrow A \wedge B$, then, by **(A.$\vee$)**, we have $X \longrightarrow A$ and $X \longrightarrow B$. Therefore, every attacker of $A \vee B$ is also an attacker of both A and B. Thus, G defends $A \vee B$ and $A \vee B \in \mathsf{G}$.

(3) (a) If there is an argument X such that $X \longrightarrow A \supset B$, then, by **(C.$\supset$)**, it follows that $X \longrightarrow B$, i.e. every attacker of $A \supset B$ is also an attacker of $B \in \mathsf{G}$. Thus, G defends $A \supset B$, which implies that $A \supset B \in \mathsf{G}$.

(b) Suppose that $B \notin \mathsf{G}$ and that $A \in \mathsf{G}$. The reasons why $B \notin \mathsf{G}$ are several. It might be that (*i*) $\{B\} \cup \mathsf{G}$ is not conflict-free, or (*ii*) $\{B\} \cup \mathsf{G}$ does not defend B, or (*iii*) B defends something not in G.

(i) If $\{B\} \cup \mathsf{G}$ is not conflict-free, then, by **(A.⊃)** and **(B.⊃)**, we have either $\mathsf{G} \longrightarrow A \supset B$ or $A \supset B \longrightarrow \mathsf{G}$ and this is against our assumptions.

(ii) If $\{B\} \cup \mathsf{G}$ does not defend B, then there is some argument X such that $X \longrightarrow B$ and $X \not\longrightarrow A$. Therefore, by **(A.⊃)**, $X \longrightarrow A \supset B$ from which it follows that $\mathsf{G} \longrightarrow X$ and this is against our assumptions.

(iii) If B defends something not in G, then there is an argument F such that for any of its attackers X_i, $B \longrightarrow X_i \longrightarrow F$. Moreover, $A \not\longrightarrow X_i$ and by **(B.⊃)** we have $A \supset B \longrightarrow X_i$. Thus, G defends F, but this is against our assumptions.

(4) Suppose both A and $\neg A$ are in G. Since G is conflict-free, $A \not\longrightarrow \neg A$ and by **(C.¬)**, $A \longrightarrow \neg\neg A$. Thus, A attacks itself, but this it cannot be. □

Proposition A.2.7 (4.6.10) (APs and Ideal Extensions)**.** *Let $\mathcal{S} = \langle Ar, \longrightarrow \rangle$ be a semi-abstract argumentation frame and I a semi-stable extension over $\mathcal{S}$ ($\mathsf{I} \in \mathsf{Idl}(\mathcal{S})$), we have:*

(1) Whenever **(A.∧)**, **(B.∧)**, **(C.∧)** *and* **(D.∧)** *hold in $\mathcal{S}$, $A \in \mathsf{I}$ and $B \in \mathsf{I}$ iff $A \wedge B \in \mathsf{I}$.*

(2) Whenever **(A.∨)** *and* **(B.∨)** *hold in $\mathcal{S}$, if $A \in \mathsf{I}$, then $B \in \mathsf{I}$ iff $A \vee B \in \mathsf{I}$*

(3) (a) Whenever **(C.⊃)** *and* **(D.⊃)** *hold in $\mathcal{S}$, if $B \in \mathsf{I}$, then $A \supset B \in \mathsf{I}$;*

(b) Whenever **(A.⊃)** *and* **(B.⊃)** *hold in $\mathcal{S}$, if $A \supset B \in \mathsf{I}$, then $B \in \mathsf{I}$ or $A \notin \mathsf{I}$*

(4) Whenever **(C.¬)** *holds in $\mathcal{S}$, if $A \in \mathsf{I}$, then $\neg A \notin \mathsf{I}$.*

Proof. (1) (⇒) If $A \wedge B \notin \mathsf{I}$, then either (i) $\mathsf{I} \cup \{A \wedge B\}$ is not conflict-free, or (ii) $\mathsf{I} \cup \{A \wedge B\}$ does not defend $A \wedge B$, or (iii) there is a preferred extension P^* such that $A \wedge B \notin \mathsf{P}^*$.

(i) If $\mathsf{I} \cup \{A \wedge B\}$ is not conflict-free, then either $\mathsf{I} \longrightarrow A \wedge B$ or $A \wedge B \longrightarrow \mathsf{I}$. Thus, by **(C.∧)** and **(D.∧)**, it follows either $\mathsf{I} \longrightarrow A$, or $\mathsf{I} \longrightarrow B$, or $A \longrightarrow \mathsf{I}$, or $B \longrightarrow \mathsf{I}$. In any of these above cases we are against the assumption of having both A and B in I.

(ii) If $\mathsf{I} \cup \{A \wedge B\}$ does not defend $A \wedge B$, then there is an argument X such that $X \longrightarrow A \wedge B$ and $\mathsf{I} \cup \{A \wedge B\} \not\longrightarrow X$. However, from **(C.∧)**, it follows that $X \longrightarrow A$ or $X \longrightarrow B$ with both A and B in I, but it cannot be that $\mathsf{I} \not\longrightarrow X$.

(iii) Suppose that $\mathsf{I} \cup \{A \wedge B\}$ is an admissible set and that there is $\mathsf{P}^* \in \mathsf{Prf}(\mathcal{A})$ s.t. $A \wedge B \notin \mathsf{P}^*$. This can happen for two reasons: either $\mathsf{P}^* \cup \{A \wedge B\}$ is not conflict-free, or $\mathsf{P}^* \cup \{A \wedge B\}$ does not defend $A \wedge B$. In both cases, by the use of **(C.∧)** and **(D.∧)**, we reach a contradiction.

(⇐) If $A \notin \mathsf{I}$, then either

(i) $\mathsf{I} \cup \{A\}$ is not conflict-free, or

(ii) $\mathsf{I} \cup \{A\}$ does not defend A, or

(iii) $\mathsf{I} \cup \{A\}$ is an admissible set, but there is $\mathsf{P}^* \in \mathsf{Prf}(\mathcal{A})$ s.t. $A \notin \mathsf{P}^*$

In any of the above cases, by **(A.$\wedge$)** and **(B.$\wedge$)**, we reach a contradiction.

(2) If $A \vee B \notin \mathsf{I}$, then either

(i) $\mathsf{I} \cup \{A \vee B\}$ is not conflict-free, or

(ii) $\mathsf{I} \cup \{A \vee B\}$ does not defend $A \vee B$, or

(iii) $\mathsf{I} \cup \{A \vee B\}$ is an admissible set, but there is $\mathsf{P}^* \in \mathsf{Prf}(\mathcal{A})$ s.t. $A \vee B \notin \mathsf{P}^*$.

In any of the above cases, by **(A.$\vee$)** and **(B.$\vee$)**, we reach a contradiction.

(3) (a) If $B \in \mathsf{I}$ and $A \supset B \notin \mathsf{I}$, then either

(i) $\mathsf{I} \cup \{A \supset B\}$ is not conflict-free, or

(ii) $\mathsf{I} \cup \{A \supset B\}$ does not defend $A \supset B$, or

(iii) $\mathsf{I} \cup \{A \supset B\}$ is an admissible set, but there is $\mathsf{P}^* \in \mathsf{Prf}(\mathcal{A})$ s.t. $A \supset B \notin \mathsf{P}^*$.

In any of the above cases, by **(C.$\supset$)** and **(D.$\supset$)**, we reach a contradiction.

(b) Suppose that $A \supset B \in \mathsf{I}$, $B \notin \mathsf{I}$ and $A \in \mathsf{I}$. The reasons why $B \notin \mathsf{I}$ are several and it might be that:

(i) $\mathsf{I} \cup \{B\}$ is not conflict-free, or

(ii) $\mathsf{I} \cup \{B\}$ does not defend B, or

(iii) $\mathsf{I} \cup \{B\}$ is an admissible set, but there is $\mathsf{P}^* \in \mathsf{Prf}(\mathcal{A})$ s.t. $B \notin \mathsf{P}^*$.

In any of the above cases, by **(A.$\supset$)** and **(B.$\supset$)**, we reach a contradiction.

(4) If both A and $\neg A$ are in I, then $\mathsf{I} = \{A, \neg A, I_1, \ldots, I_n\}$. Since I is conflict-free $A \not\longrightarrow \neg A$ and by **(C.$\neg$)** $A \longrightarrow \neg\neg A$, i.e. A attacks itself, but this is against our assumptions. $\square$

APPENDIX B

Extended Proofs of Chapter 5

B.1 Attack Principles in Logical Argumentation Theory

Let us analyse the CC-attack principles in logical argumentation frameworks where the attack relation is instantiated with *defeat*.

(A.∧) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_A$ and if $\Gamma_A = \{\Gamma^A_X, \dots, \gamma^A_n\}$, then $X \Rightarrow \neg\Gamma^A_X \vee \cdots \vee \neg\gamma^A_n$. Therefore, if $\Gamma_A \subseteq \Gamma_{A\wedge B}$, then $X \Rightarrow \neg \bigwedge \Gamma_{A\wedge B}$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$.

(C.∧) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_{A\wedge B}$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_B; B\rangle$, i.e. $X \Rightarrow \neg \bigwedge \Gamma_A$ or $X \Rightarrow \neg \bigwedge \Gamma_B$ only in the case in which $\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$.

(A.∨) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_{A\vee B}$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_B; B\rangle$, i.e. $X \Rightarrow \neg \bigwedge \Gamma_A$ and $X \Rightarrow \neg \bigwedge \Gamma_B$ only in the case in which $\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$, i.e. $\Gamma_{A\vee B} \subseteq \Gamma_A \cap \Gamma_B$.

(C.∨) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_B; B\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_A$ and $X \Rightarrow \neg \bigwedge \Gamma_B$. Therefore $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, i.e. $X \Rightarrow \neg \bigwedge \Gamma_{A\vee B}$ only in the case in which $\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$.

(A.⊃) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[Def]} \langle \Gamma_A; A\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_B$ and $X \not\Rightarrow \neg \bigwedge \Gamma_A$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\supset B}; A \supset B\rangle$ (i.e. $X \Rightarrow \neg \bigwedge \Gamma_{A\supset B}$) only in the case in which $\Gamma_B \subseteq \Gamma_{A\supset B}$.

(C.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_{A\supset B}$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[Def]} \langle \Gamma_A; A\rangle$ (i.e. $X \Rightarrow \neg \bigwedge \Gamma_B$ and $X \not\Rightarrow \neg \bigwedge \Gamma_A$, respectively) only in the case in which $\Gamma_{A\supset B} \subseteq \Gamma_B$ and $\Gamma_{A\supset B} \cap \Gamma_A = \emptyset$.

(A.$\neg$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_A; A\rangle$, then $X \Rightarrow \neg \bigwedge \Gamma_A$. Nevertheless we cannot deduce that $X \Rightarrow \neg \bigwedge_{\neg A}$, i.e. $\langle \Gamma_X; X\rangle \not\xrightarrow{[Def]} \langle \Gamma_{\neg A}; \neg A\rangle$. In fact, even if $X \Rightarrow \neg \bigwedge \Gamma_A$ and $\Gamma_A \cap \Gamma_{\neg A} = \emptyset$, it could be that $X \Rightarrow \neg \gamma_i^{\neg A}$ for some i given $\Gamma_{\neg A} = \{\Gamma_X^{\neg A}, \dots, \gamma_m^{\neg A}\}$.

(C.$\neg$) If $\langle \Gamma_X; X\rangle \not\xrightarrow{[Def]} \langle \Gamma_A; A\rangle$, then $X \not\Rightarrow \neg \bigwedge \Gamma_A$, but we cannot deduce that $X \Rightarrow \neg \bigwedge \Gamma_{\neg A}$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Def]} \langle \Gamma_{\neg A}; \neg A\rangle$.

We now analyse the same attack principles, but with a different attack function: *undercut*.

(A.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$, then $X \equiv \neg \bigwedge \Gamma'_A$. Therefore, $X \equiv \neg \bigwedge \Gamma'_{A\wedge B}$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$ only in the case in which $\Gamma'_A \subseteq \Gamma_{A\wedge B}$.

(C.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$, then $X \equiv \neg \bigwedge \Gamma'_{A\wedge B}$. Therefore, $X \equiv \neg \bigwedge \Gamma'_A$ or $X \equiv \neg \bigwedge \Gamma'_B$, (i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_B; B\rangle$, respectively) only in the case in which $\Gamma'_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma'_{A\wedge B} \subseteq \Gamma_3$.

(A.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $X \equiv \neg \bigwedge \Gamma'_{A\vee B}$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_B; B\rangle$, i.e. $X \equiv \neg \bigwedge \Gamma'_A$ and $X \equiv \neg \bigwedge \Gamma'_3$ only in the case in which $\Gamma'_{A\vee B} \subseteq \Gamma_A$ and $\Gamma'_{A\vee B} \subseteq \Gamma_B$, i.e. $\Gamma'_{A\vee B} \subseteq \Gamma_A \cap \Gamma_B$.

(C.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_B; B\rangle$, then $X \equiv \neg \bigwedge \Gamma'_A$ and $X \equiv \neg \bigwedge \Gamma'_B$. Therefore, $\Gamma'_A \equiv \Gamma'_B$ and $X \equiv \neg \bigwedge \Gamma'_{A\vee B}$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\vee B}; A \vee B\rangle$ only in the case in which $\Gamma'_A \subseteq \Gamma_{A\vee B}$ or $\Gamma'_B \subseteq \Gamma_{A\vee B}$.

(A.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$, then $X \equiv \neg \bigwedge \Gamma'_B$ and $X \not\equiv \neg \bigwedge \Gamma'_A$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, i.e. $X \equiv \neg \bigwedge \Gamma'_{A\supset B}$ only in the case in which $\Gamma'_B \subseteq \Gamma_{A\supset B}$.

(C.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $X \equiv \neg \bigwedge \Gamma'_{A\supset B}$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$ (i.e. $X \equiv \neg \bigwedge \Gamma'_B$ and $X \not\equiv \neg \bigwedge \Gamma'_A$, respectively) only in the case in which $\Gamma'_{A\supset B} \subseteq \Gamma_B$ and $\Gamma'_{A\supset B} \not\subseteq \Gamma_A$

(A.$\neg$) If $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle \Gamma_A; A\rangle$, then $X \equiv \neg \bigwedge \Gamma'_A$. Therefore, $X \not\equiv \neg \bigwedge \Gamma'_{\neg A}$, i.e. $\langle \Gamma_X; X\rangle \not\xrightarrow{[Def]} \langle \Gamma_{\neg A}; \neg A\rangle$, only in the case in which $\Gamma'_A \neq \Gamma'_{\neg A}$ for any $\Gamma'_{\neg A} \subseteq \Gamma_{\neg A}$.

(C.¬) If $\langle \Gamma_X; X\rangle \overset{[Ucut]}{\not\longrightarrow} \langle \Gamma_A; A\rangle$, then $X \not\equiv \neg \bigwedge \Gamma'_A$ for any $\Gamma'_A \subseteq \Gamma_A$. However, we cannot deduce that $X \not\equiv \bigwedge \Gamma'_{\neg A}$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Ucut]} \Gamma_{\neg A} \Rightarrow \neg A$.

We continue the analysis of the CC-APs, but this time considering the *Rebuttal* attack relation.

(A.∧) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_A; A\rangle$, then $X \equiv \neg A$. Therefore, $X \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$ only in the case in which $A \equiv B$.

(C.∧) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$, then $X \equiv \neg(A \wedge B)$. Therefore, $X \equiv \neg A$ or $X \equiv \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_B; B\rangle$ respectively, only in the case in which $A \equiv B$.

(A.∨) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $X \equiv \neg(A \vee B)$, i.e. $X \equiv \neg A \wedge \neg B$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_B; B\rangle$ ($X \equiv \neg A$ and $X \equiv \neg B$) only in the case in which $A \equiv B$.

(C.∨) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_B; B\rangle$, then $X \equiv \neg A$ and $X \equiv \neg B$. Therefore, $A \equiv B$ and $X \equiv \neg A \wedge \neg B \equiv \neg(A \vee B)$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\vee B}; A \vee B\rangle$.

(A.⊃) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \overset{[Reb]}{\not\longrightarrow} \langle \Gamma_A; A\rangle$, then $X \equiv \neg B$ and $X \not\equiv \neg A$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, i.e. $X \equiv \neg(A \supset B) \equiv A \wedge \neg B$ only in the case in which $A \equiv \neg B$.

(C.⊃) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $X \equiv \neg(A \supset B) \equiv A \wedge \neg B$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \overset{[Reb]}{\not\longrightarrow} \langle \Gamma_A; A\rangle$, i.e. $X \equiv \neg B$ and $X \not\equiv \neg A$ respectively, only in the case in which $A \equiv \neg B$.

(A.¬) If $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_A; A\rangle$, then $X \equiv \neg A$. Therefore, $X \not\equiv \neg(\neg A)$. Therefore, $X \not\equiv A$ ($\langle \Gamma_X; X\rangle \overset{[Reb]}{\not\longrightarrow} \langle \Gamma_{\neg A}; \neg A\rangle$).

(C.¬) If $\langle \Gamma_X; X\rangle \overset{[Reb]}{\not\longrightarrow} \langle \Gamma_A; A\rangle$, then $X \not\equiv \neg A$. Therefore, $X \equiv \neg(\neg A) \equiv A$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[Reb]} \langle \Gamma_{\neg A}; \neg A\rangle$ only in the limit case where $X \equiv A$.

We continue with the analysis of the same attack principles, but considering the *Compact Rebuttal 1* attack function.

(A.∧) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg A$. Therefore, $\Gamma_X \Rightarrow \neg(A \wedge B)$ and $\Gamma_X \Rightarrow \neg A \vee \neg B$, that is the condition for having $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$

(C.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}; A\wedge B\rangle$, then $\Gamma_X \Rightarrow \neg(A\wedge B)$. Therefore, $\Gamma_X \Rightarrow \neg A \vee \neg B$ and $\Gamma_X \Rightarrow \neg A$ or $\Gamma_X \Rightarrow \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ respectively, only in the case in which $A \equiv B$.

(A.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}; A\vee B\rangle$, then $\Gamma_X \Rightarrow \neg(A\vee B)$. Therefore, $\Gamma_X \Rightarrow \neg A \wedge \neg B$, from which it follows $\Gamma_X \Rightarrow \neg A$ and $\Gamma_X \Rightarrow \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ respectively.

(C.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$, then $\Gamma_X \Rightarrow \neg A$ and $\Gamma_X \Rightarrow \neg B$. Therefore, by $[\Rightarrow \wedge]$, $\Gamma_X \Rightarrow \neg A \wedge \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}; A \vee B\rangle$.

(A.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg B$ and $\Gamma_X \not\Rightarrow \neg A$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, i.e. $\Gamma_X \Rightarrow \neg(A \supset B)$ that is equivalent to $\Gamma_X \Rightarrow A \wedge \neg B$, only in the case in which $A \equiv \neg B$.

(C.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $\Gamma_X \Rightarrow \neg(A \supset B)$ from which it follows $\Gamma_X \Rightarrow \neg(\neg A \vee B)$ and $\Gamma_X \Rightarrow A \wedge \neg B$. Therefore, $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, i.e. $\Gamma_X \Rightarrow \neg B$ and $\Gamma_X \not\Rightarrow \neg A$ respectively, only in the case in which Γ_X is consistent.

(A.$\neg$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg A$. Therefore $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{\neg A}; \neg A\rangle$, i.e. $\Gamma_X \not\Rightarrow \neg(\neg A)$, which is equivalent to $\Gamma_X \not\Rightarrow A$ only in the case in which Γ_X is consistent.

(C.$\neg$) If $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \not\Rightarrow \neg A$. However, we cannot deduce that $\Gamma_X \Rightarrow A$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{\neg A}; \neg A\rangle$.

We proceede the analysis with the *Indirect Rebuttal* attack relation.

(A.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A; A\rangle$, then $X \Rightarrow \varphi$ and $A \Rightarrow \neg\varphi$ for some φ. Therefore, by Weakening and $[\wedge \Rightarrow]$ we have $A \wedge B \Rightarrow \neg\varphi$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$.

(C.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$, then $X \Rightarrow \varphi$ and $A \wedge B \Rightarrow \neg\varphi$ for some φ. Therefore, we can deduce that $X \Rightarrow \psi$ and $A \Rightarrow \neg\psi$ for some ψ or $X \Rightarrow \xi$ and $B \Rightarrow \neg\xi$ for some ξ, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A; A\rangle$ or $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B; B\rangle$ respectively, only in the case in which $A \equiv B$.

(A.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $X \Rightarrow \varphi$ and $A \vee B \Rightarrow \neg\varphi$ for some φ. Therefore, by *Cut*, $A \Rightarrow A \wedge B$ and $B \Rightarrow A \wedge B$, we can deduce that $\langle \Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]}$

$\langle\Gamma_A; A\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_B; B\rangle$, i.e. $X \Rightarrow \psi$ and $A \Rightarrow \neg\psi$ for some ψ and $X \Rightarrow \xi$ and $B \Rightarrow \neg\xi$ for some ξ respectively.

(C.$\vee$) If $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_B; B\rangle$, then $X \Rightarrow \psi$ and $A \Rightarrow \neg\psi$ for some ψ and $X \Rightarrow \xi$ and $B \Rightarrow \neg\xi$ for some ξ. Therefore, we have that $X \Rightarrow \phi \wedge \xi$ and $A \vee B \Rightarrow \neg(\phi \wedge \xi)$, i.e. $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_{A\vee B}; A \vee B\rangle$.

(A.$\supset$) If $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_B; B\rangle$ and $\langle\Gamma_X; X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$, then $X \Rightarrow \varphi$ and $B \Rightarrow \neg\varphi$ for some φ and that there is no ξ such that $X \Rightarrow \xi$ and $A \Rightarrow \neg\xi$. We can deduce that $X \Rightarrow \varphi$ and $\neg A \vee B \Rightarrow \neg\varphi$ for some φ, i.e. $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_{A\supset B}; A \supset B\rangle$ only in the case in which $\neg A \equiv B$.

(C.$\supset$) If $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_{A\supset B}; A \supset B\rangle$, then $X \Rightarrow \varphi$ and $A \supset B \Rightarrow \neg\varphi$ for some φ. Therefore, $\neg A \vee B \Rightarrow \neg\varphi$, from which it follows that $B \Rightarrow \neg\varphi$ and $\neg A \Rightarrow \varphi$, i.e. $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_B; B\rangle$ and $\langle\Gamma_X; X\rangle \xrightarrow{[Ucut]} \langle\Gamma_{\neg A}; \neg A\rangle$. Our claim is that $\langle\Gamma_X; X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$. Let us suppose that $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$. Therefore, there exists ξ such that $X \Rightarrow \xi$ and $A \Rightarrow \neg\xi$, from which it follows $X \Rightarrow \varphi \wedge \xi$ and $\neg A \vee A \Rightarrow \neg(\varphi \wedge \xi)$, i.e. $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_{A\wedge\neg A}; A \wedge \neg A\rangle$, a *tautological* argument, but this is against our assumptions.

(A.$\neg$) If $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$, then $X \Rightarrow \varphi$ and $A \Rightarrow \neg\varphi$ for some φ. Moreover, it cannot be that $X \Rightarrow \xi$ and $\neg A \Rightarrow \neg\xi$ for any ξ because otherwise we would have $X \Rightarrow \varphi \wedge \xi$ and $\neg A \vee A \Rightarrow \neg(\varphi \wedge \xi)$, i.e. $\Gamma_X \Rightarrow X$ attacking a *tautological* argument. Therefore $\langle\Gamma_X; X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle\Gamma_{\neg A}; \neg A\rangle$

(C.$\neg$) If $\langle\Gamma_X; X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle\Gamma_A; A\rangle$, then there is no ξ such that $X \Rightarrow \xi$ and $\neg A \Rightarrow \neg\xi$. However, we cannot deduce that there exists a φ such that $X \Rightarrow \varphi$ and $\neg A \Rightarrow \varphi$, i.e. $\langle\Gamma_X; X\rangle \xrightarrow{[I\text{-}Reb]} \langle\Gamma_{\neg A}; \neg A\rangle$.

B.2 SC-Attack Principles in Logical Argumentation Theory

We introduce attack principles defined in terms of the support of the attacking argument and the conclusion of the attacked one. As for the case of Conclusion-Support introduced in Section 5.4 , the attack principles maintain some similarity with the first introduced attack principles (the conclusion-conclusion case). Let us consider the following arguments: $\langle\Gamma_X; X\rangle$, $\langle\Gamma_A; A\rangle$, $\langle\Gamma_B; B\rangle$, $\langle\Gamma_{A*B}; A * B\rangle$ (with $*$ one of the three classical connectives $\wedge$, $\vee$ and $\supset$) and $\langle\Gamma_{\neg A}; \neg A\rangle$. In the definition of the SC-attack principles we will refer to the arguments just introduced.

($\mathbf{A^{SC}}.\wedge$) If $\Gamma_X \longrightarrow A$ or $\Gamma_X \longrightarrow B$, then $\Gamma_X \longrightarrow A \wedge B$.

($\mathbf{C^{SC}}.\wedge$) If $\Gamma_X \longrightarrow A \wedge B$, then $\Gamma_X \longrightarrow A$ or $\Gamma_X \longrightarrow B$.

($\mathbf{A^{SC}}.\vee$) If $\Gamma_X \longrightarrow A \vee B$, then $\Gamma_X \longrightarrow A$ and $\Gamma_X \longrightarrow B$.

($\mathbf{C^{SC}}.\vee$) If $\Gamma_X \longrightarrow A$ and $\Gamma_X \longrightarrow B$, then $\Gamma_X \longrightarrow A \vee B$.

($\mathbf{A^{SC}}.\supset$) If $\Gamma_X \longrightarrow A \supset B$, then not $\Gamma_X \longrightarrow A$ and $\Gamma_X \longrightarrow B$.

($\mathbf{C^{SC}}.\supset$) If not $\Gamma_X \longrightarrow A$ and $\Gamma_X \longrightarrow B$, then $\Gamma_X \longrightarrow A \supset B$.

($\mathbf{A^{SC}}.\neg$) If $\Gamma_X \longrightarrow A$, then not $\Gamma_X \longrightarrow \neg A$.

($\mathbf{C^{SC}}.\neg$) If not $\Gamma_X \longrightarrow A$, then $\Gamma_X \longrightarrow \neg A$.

We proceede now with the interpretation of the Support-Conclusion attack principles just introduced in a sequent-based argumentation frame and where the attack function considered is *Compact Rebuttal 1*.

(A.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg A$. Therefore $\Gamma_X \Rightarrow \neg A \vee \neg B$, which is equivalent to $\Gamma_X \Rightarrow \neg(A \wedge B)$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$.

(C.$\wedge$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}; A \wedge B\rangle$ then $\Gamma_X \Rightarrow \neg A \vee B$. However, we cannot deduce either $\Gamma_X \Rightarrow \neg A$ or $\Gamma_X \Rightarrow \neg B$. Therefore, the principle does not hold.

(A.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}; A \vee B\rangle$, then $\Gamma_X \Rightarrow \neg(A \vee B)$. Therefore, $\Gamma_X \Rightarrow \neg A \wedge \neg B$, from which it follows $\Gamma_X \Rightarrow \neg A$ and $\Gamma_X \Rightarrow \neg B$, i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$.

(C.$\vee$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$ and $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$, then $\Gamma_X \Rightarrow \neg A$ and $\Gamma_X \Rightarrow \neg B$. Therefore, $\Gamma_X \Rightarrow \neg A \wedge \neg B$ and $\Gamma_X \Rightarrow \neg(A \vee B)$ i.e. $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}; A \vee B\rangle$.

(C.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}; A \supset B\rangle$, then $\Gamma_X \Rightarrow \neg(A \supset B)$. Therefore, $\Gamma_X \Rightarrow A$ and $\Gamma_X \Rightarrow \neg B$, from which it follows $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]}$ $\Gamma_2 \Rightarrow A$. Otherwise we would have both $\Gamma_X \Rightarrow A$ and $\Gamma_X \Rightarrow \neg A$, i.e. $\Gamma_X \Rightarrow X$ would be a *contradictory* argument.

(A.$\supset$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B; B\rangle$ and $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg B$ and $\Gamma_X \not\Rightarrow \neg A$. However, from $\Gamma_X \not\Rightarrow \neg A$ we cannot deduce $\Gamma_X \Rightarrow A$ and the principle does not hold.

(A.$\neg$) If $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \Rightarrow \neg A$. Therefore $\Gamma_X \not\Rightarrow A$, i.e. $\langle \Gamma_X; X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]}$ $\langle \Gamma_{\neg A}; \neg A\rangle$ otherwise, $\langle \Gamma_X; X\rangle$ would be a *contradictory* argument.

(C.$\neg$) If $\langle \Gamma_X; X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A; A\rangle$, then $\Gamma_X \not\Rightarrow \neg A$. However we cannot deduce that $\Gamma_X \Rightarrow A$, therefore neither $\langle \Gamma_X; X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{\neg A}; \neg A\rangle$.

B.3 Additional Elimination Rules for CC-APs in Sequent-Based Frames

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg \bigwedge \Gamma_A \qquad \Gamma_A \Rightarrow A \qquad \Gamma_A \subseteq \Gamma_{A\wedge B} \qquad \Gamma_{A\wedge B} \Rightarrow A \wedge B}{\Gamma_{A\wedge B} \not\Rightarrow A \wedge B}\ [\text{Def-}(A.\wedge)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg \bigwedge \Gamma_{A\vee B} \qquad \Gamma_{A\vee B} \Rightarrow A \vee B \qquad \Gamma_{A\vee B} \subseteq \Gamma_A \qquad \Gamma_A \Rightarrow A}{\Gamma_A \not\Rightarrow A}\ [\text{Def-}(A.\vee)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \underset{i=A,B}{\Rightarrow} \neg \bigwedge \Gamma_i \qquad \Gamma_A \Rightarrow A \qquad \Gamma_B \Rightarrow B \qquad \Gamma_A \subseteq \Gamma_{A\vee B} \quad \Gamma_{A\vee B} \Rightarrow A \wedge B}{\Gamma_{A\vee B} \not\Rightarrow A \wedge B}\ [\text{Def-}(C.\vee)]$$

$$\frac{\Gamma_X \Rightarrow X \qquad X \Rightarrow \neg \bigwedge \Gamma_{A\supset B} \qquad \Gamma_{A\supset B} \Rightarrow A \supset B \qquad \Gamma_{A\supset B} \subseteq \Gamma_B \qquad \Gamma_B \Rightarrow B}{\Gamma_B \not\Rightarrow B}\ [\text{Def-}(A.\supset)]$$

APPENDIX C

Transcripts of the Field Experiment of Chapter 6

C.1 Expert Video Transcript (English)

Josephine Zoughaib (moderator): We have now reached the end of 2016. We have a lot of big problems in the country. Lebanon has become one of the most corrupt countries. I imagine we have a lot of big problems, so I don't know Mr. Radwan if you have reached a conclusion that there is a problem bigger than the others, which have gotten us here. Can you give us a little bit...[of what you think]

Radwan Mortada (Shia): The main idea is that the biggest problem, which is bigger than all the rest, which if we solve we can reach a better country, in principle is the crisis of sectarianism and the sectarianism that exists in Lebanon and the parties' leaders who are accordingly dividing the country and transforming Lebanon to...

JZ: So the sectarianism is leading to the quota system? Everyone is taking their own share and that's why we got here?

RM: Everyone is taking their own share and in the same time when they are corrupt and from a certain sect, the other sects cannot reach them (hold them accountable) unless their own sect lifts the political cover (political immunity) over them.

JZ: What is the reason to why we got to this system, which has become a corrupt system and all countries acknowledge this issue, we have ranked the 3rd most corrupt country in the world?

Nabil Moukaddem (Sunni): I think the main problem we have is the sectarian distribution. Our main and first problem is our election system. I want to talk a little bit about our election system. I think if we make a new and modern electoral system and we get rid of sectarianism, we are able to solve a big part of the problem. I think, today, the question asked is, after a couple of months we will be having parliamentary elections if everything goes smoothly supposedly. We are still today prisoners of laws and legislations that are so far from...

JZ: The 60's Law [Election Law] is being discussed today.

NM: Yes true...that are so far from the spirit and progress. The question that we want to ask is: Until when are we going to keep digging out worn out and silly laws and until when is Lebanon going to keep perpetuating this sectarian system. I want to tell you something. The process of dismantling sectarianism, just so we are not very hopeful, doesn't come from changing only the electoral text. It is a long-term educational process. It starts from history books, starts from the way we raise our kids, a number of things, but an electoral law is one of the examples. Lebanon today is susceptible to explode at any second. [unclear] That is why if we do not have an established national unity, we are susceptible to things escalating at any second as a result to certain regional factors. These guys know it better than I do.

JZ: But the Lebanese society is finding the problem elsewhere. It is finding that if there is no cooperation currently between 2 very big sects, which are the Sunni and the Shia, I can't imagine that we can agree on an electoral law or a transparent judiciary system.

Mohammad Abdullah (Sunni): After we called it the Cedar Revolution in 2005, the big phenomenon that we witnessed in Martyr's Square, we saw that the hunkering down came back over a certain political period. Until there was a major crisis in Lebanon that unified all of Lebanon, that is the garbage crisis that generated another phenomenon: everyone, the Sunni, the Shiaa, the Christian, and other religions went down to the streets for a certain cause. But we know that regardless of the size of movement in Lebanon, we know that sectarianism leads to no results. When the garbage crisis happened and what we call the civic movement, which I participated in, sectarianism came in to break down this collective. You are going today to the streets because you saw that the garbage is at your doorstep, but when it became such that my leader or your leader or his leader are supposed to take action and responsibility, there was a smart political move, if you may, from the leaders of the sects to dismantle this gathering...

JZ: You are confirming that the citizen is unable to abandon sectarianism for the sake of his leader even over a very important issue such as the garbage crisis.

MA: The citizen did abandon it, and he protested in the streets. However, afterwards, there was work done to dismantle this movement because they united, then they became factions again and politics entered with its divisional, confessional, and sectarian methodology and led to a division and inability to arrive at a conclusion.

Radwan Aqil (Shia): Starting from what my colleagues have talked about, we don't live in a normal country. For example, if Lebanon were to play a soccer game with any foreign country, we wouldn't find all the Lebanese people. While we find the Iranian people all as one, in a soccer game for example, the same with Egypt, we live in a state of sectarianism, as soon we mention the name of the street, we know the political affiliation of the street and the area. Let's admit it, if today we want to go buy a house from a person from another sect, it hasn't been that easy to do it and its very well known. Today there is a percentage, but this doesn't mean...

JZ: It has to be 10% Sunni in a certain building...

RA: But this doesn't mean that the relationships in the country between people is really that bad. No! On the contrary, we still have communities in universities and institutions etc. But I, as a journalist, I say we do not live normally in a normal country. To go back to the Sunni/Shia conversations, which has been brought up a lot. I don't want to say it's a disagreement; it's a contrast that goes back 1400 years and unfortunately is still until today used to bring up a conflict. Let's also admit that in the last 10 years the marriage between the Sunni and Shia has decreased, before it hasn't been this bad. I want to talk a little bit about the Shia movement. In the civil war, the Shia movement was a container for the National Movement led by Kamal Jounblat. It wasn't lead by a Shia leader although there were a lot of Shia leaders. Lately there has been a lot of talk about the conflict, I just want to say, today, there is a Sunni in Tripoli who doesn't know Tyre, and there is a Shia in Tyre who doesn't know Tarii Ljdideh and doesn't know Tripoli.

JZ: They don't know or they're not letting them know?

RA: We all take responsibility as press, as political powers, as parties, and sects. And because they are scared. Imagine this happens in this small country. Lets go back to the problem, which is that once someone is born, his confession follows him from birth to the grave. And today, let us admit it, Wasta is killing us. Wherever it is: in getting a degree, a job, etc. The Sunni-Shia conflict also made us think about the region with the Gulf countries and the Arab countries and how it affects this region. Otherwise, we as Sunni and Shia in this country, we lived together. Shia, still until today, Jamal Abdel Naser pictures are put up in Shia houses. Our sectarian system, the quota system, and sectarianism are pushing for continuation of this conflict. Also in this country, there is no nationality. For example, why are the quotas calculated for the biggest sects: Maronites, Otrthodox, Shia, Sunna, and the Druze have no right to be represented. If Einstein were Druze he wouldn't be allowed to run the ministry of foreign affairs. Kamal Jounblat ran the Ministry of interior and he was amongst the best ministers. Prince Majeed Erslen ran the Ministry of Defense. Today everything is constrained and there is injustice even within the same sect.

RM: So we have the option: we either agree to rebuild the country on the right fundamentals and I agree that we cannot do this before we transcend sectarianism in our Electoral Law, because the big figures with the influence to change things, are benefiting from the current situation.

JZ: we are talking about benefits between the sects today.

RM: True because and there is another issue in the sectarian realm if we can say. When the youth want to apply for the military, or the judiciary school, or any job, they first need to go to the alleged leader of their sect so he can work it out for them. And they can't make him angry because he controls what they do for a living. He is the one who guarantees whether they are accepted or not.

JZ: Where is the Lebanese society going and if there is an opportunity to fix this sectarian system or the elections, how far can we go as you see it as Lebanese citizens, not as journalists? As a Lebanese citizen Radwan, where are we headed?

RM: When someone knows what his or her problem is, that is half of the treatment. At least we know that we have a problem, and we know that sectarianism is a problem, and we know that if we don't agree to sit together and actually sit together to discuss and plan a common future that is good for everyone, we won't be able to succeed. And it seems that in all the sects, all the Lebanese people, educated or not, know that this is the problem.

NM: I want to stress on one thing about the word "peoples". We are one unified peoples with unified goals and benefits, there are political and sectarian disagreements, yes, but it can be fixed even if it's very difficult at times. There is still in inter-religious marriage in Lebanon, this also a very important thing. Even the Lebanese people, in their nature, are peaceful not violent. Of course in the Civil war there were people from all sects that held weapons and fought, but the majority was expressing their opinion verbally, the percentage that held weapons and killed people depending on religion is small relative to the rest of the Lebanese people. Even the people who forced the migration of others, they are minority in their sects. If you are thorough in checking, you will see that in every sect the majority refuses these actions and practices and the evidence is that after the war the harmony, more or less, resurfaced again.

JZ: So we are confirming that the Lebanese people are not sectarian in nature.

NM: The Lebanese people are not sectarian. They are creative people who love life but the only problem is that they are constrained by sectarian laws after the false independence in 1943. The Lebanese people need to look after their benefits. As my colleagues were saying, if you want to put your child in schools or get a job, you need to go back to your sect's political reference and leader. I, as a Lebanese person today, they are saying they are forming a government, why do I care if it is all Muslims or all Christians if the members are qualified and are able to perform their tasks, why do I care if they are 30 Muslims and 30 Christians as long as they are providing me good service and look after me. If they are all Orthodox, why do I care?

RA: It is your right to be with a political leader and you have the right to support him/her, but regardless, this leader today can deprive you from [unclear word]. Be open to the other and marry whomever you want. Hopefully, that with the new generation we are able to reach real nationalism that Hussein Fadl-Allah and Gregoire Haddad used to talk about, and hopefully we think about what we are leaving our children and grandchildren in this country.

JZ: Thank you. I want to conclude now about how much we are connected to this country, which is why we are staying and I imagine that the Lebanese youth who is participating in the streets and in university elections is insisting to stay with good values. I think we started the conversation with Sunni-Shia conflict and ended with Gregoire Haddad and Sir Fadl-Allah, we arrived at the conclusion that we are sitting together on one table with one hope that hopefully hopefully hopefully in a new era or an era in the future because I have hope in the youth as you said Radwan, the coming generations that are raised on these values and that what we witness over the past 40 years isn't going to get us anywhere.

C.2 Transcripts of Group Discussions (English) [11-D]

Moderator: We are Session 11 Table D. Welcome. After we saw the 30 min video, there were a lot of topics discussed: How can there be cooperation between Sunni and Shia sects? They talked about this problem as a historical problem. Some talked about the judiciary system and the problems with this system. Someone also traced back the problems in Lebanon to the electoral law. The political parties and sectarian parties were also mentioned in details. Was the video clear to everyone? And first, do you agree with everything that was said in the video?

73: Not everything.

Moderator: You do not agree on everything.

Unidentified: Same.

Unidentified: Me too.

Moderator: Ok. What are the specific things said that are beneficial in your opinion? Or the ideas that you support in this discussion.

72: I support one of the ideas someone suggested about the necessity of cooperation between sects. In principle, they are mostly talking about Sunni-Shia. As he said it is necessary to have cooperation between Sunni and Shia sects, but some people suggested ways of cooperation that are rejected. It doesn't mean that if we have civil marriage then we've cooperated. Some religions forbid civil marriage from a religious point of view not legislative one. But we support what they said about the issue of renting houses and the issue of education and the necessity to raise our kids from a young age that the Sunni is their brother and the Shia is their brother and the Christian is their brother and that they are all the same. We shouldn't be raising them that certain areas have certain sectarian affiliations. We were also raised on this perspective. We'd be in the car with our parents and we get to a specific area...

Unidentified: "oh this is Tariq Jdide"

72: Tariq Jdide for example, we'd directly ask, "oh dad, whose is this area?" "This area is not with us, they are all against us." That is wrong. We should teach them that this is all Lebanon.

Unidentified: True.

Moderator: Sir, what did you object about?

73: I objected about the Parliamentary elections and about the idea that the delegates can be anyone. They are saying if the delegates are all Sunni or all Christians...

Moderator: But as long as they are good, if they are all Christian then where is the problem?

Unidentified: If they all serve the country, where is the problem?

73: No this is wrong.

Moderator: It's wrong. They need to be divided.

73: They need to be divided. Because as we all know, the country is divided.

Moderator: He talked about the sectarian politics and the quotas system. Sometimes the wrong or bad person fills a position just because they need to fill it with someone from a certain sect.

73: Yes but the country is divided, and this is known.

Moderator: So it is still better to divide it over all the sects.

73: Yes, of course.

Moderator: Who among the speakers was the most convincing in your opinion?

71: The one who was talking about the elections and how the candidates should be from all sects not limited to one sect. He even talked with respect to Druze, Muslims, and Christians. And regarding whether or not the country is divided, after it was divided in 1970's and 1980's, it honestly never went back to how it was and it has just been worsening. As the guy here said that every generation is teaching its children that these people are X and these follow X and those follow Y. So it was divided and it has only been in decline.

Unidentified: To the worse.

71: We want to work against this flow that the rising youth is following. And it is very unfortunate that the ministers and rulers support everyone who supports their sects and follow them. Although, we want leaders and rulers to work with all sects and solve the problem from the roots.

73: Why are they focused on Sunni-Shia only?

69: The study is just about Sunni-Shia.

71: Because it is the most intense.

Unidentified: Sunni-Shia is the more prevalent.

Moderator: Sir, after watching the video and in your opinion, is the lack of cooperation between Sunni and Shia the reason behind all these problems?

69: No, its reason is the lack of love for the country and citizenship as they say because that is fundamental. The Sunni is Sunni and the Shia is Shia, to each his/her own personal life, but they should talk about patriotism. Patriotism is fundamental to raise a good society that doesn't think about its sects but rather about Lebanon, the country, and the civic society. This is the basis. Three quarters of what they were saying was about sects. They didn't talk about the country and patriotism.

Moderator: So if we switch the rhetoric to talk about how this cooperation can serve the national interest, do you think this cooperation will affect the political, economical, and societal situation as a whole?

69: Of course. Everything changes. The citizen will be psychologically relived.

73: Why can't the country be for all sects and not necessarily the Sunni and Shia sects? We are a diverse country and we have many sects.

Moderator: So your opinion is that it is not just about limiting the cooperation between the Sunni and Shia?

73: Yes, cooperate with everyone. It doesn't have to become Sunni-Shia. In the end of the day, both Sunni and Shia are Muslims. The same Qur'an.

Unidentified: Same Qur'an.

72: But we need to pay attention that when she addressed the questions, she addressed them based on a Sunni-Shia rhetoric because the aim of the study is explore the Sunni-Shia rhetoric.

Moderator: Yes that is the aim of the study but his opinion is that the problem is not just a Sunni-Shia one.

72: Oh ok.

73: Yes as a whole. Lebanon as a whole, we want to live together.

71: We want to live together.

Moderator: In your opinion and from what you saw in the video, what are the barriers that stand in the way of the cooperation between sects and specifically between Sunni and Shia sects?

74: Just like she said that we raise our kids on how this area is ours and that area is not ours and this is Sunni and that is Shia. Please repeat the question, I forget. Moderator: So in your opinion, are these reasons that prevent the cooperation between these two sects?

74: Yes of course. And the leaders who give incitement speeches and that religion and politics should not mix in the government. Even as someone here said, they go on podiums during Friday prayers and they talk about politics. You are here to talk about religion, what does politics have anything to do with this?

Moderator: So clerics...

74: Should stick to religion.

Moderator: affect this cooperation negatively?

74: Yes, yes.

71: Yes.

73: There needs to be a separation between clerics and politics. It is necessary.

71: Yes.

74: If I go to the mosque to pray, I want to hear a religious talk that I can understand. I can watch the news to know about politics.

Moderator: Go ahead.

70: I want to say something. Why don't we unite like we united once for the garbage crisis and at some point they managed to break us apart?

Moderator: Why?

70: They started sending infiltrators to vandalize and beat people. In the end, we are all Muslims, whether Sunni or Shia, we cannot abandon each other. It is absolutely impossible for these two sects to abandon each other because there are a lot of Sunni married to Shia and vice versa.

Unidentified: Me.

70: What are the sects of the children of mixed parents then? Sunni or Shia? His mom is Shia and his dad is Sunni. We need to get rid of these things and distinctions between Sunni and Shia. I liked the idea of eliminating the sect from the ID cards.

73: Yes the sect.

70: So that the person taking your ID card at a checkpoint, he might be Shia and sees you are Sunni, so he might start showing off and act superior to you. The first step in order to fix the country, we need to get rid of the...

73: sect.

74: sect.

70: the sect from ID cards. When that happens, no one will know who is what.

74: True.

Moderator: The sect is also on the Personal Status Record.

70: The sect should be taken off the ID cards and the Personal Status Records. We need to get rid of the Sunni-Shia rhetoric.

74: You can tell from the names though.

70: I am from the south and I live in the middle Tariq Jdide. If I am sitting in a group of people and I say my last name they get surprised and they start winking at each other. Why would you do that? You've known me for a while now! I live with you and I have Sunni kids. Why are you discriminating? If I discriminate I wouldn't have married one of you and my husband wouldn't have married me. And I speak Beiruti more than they do. Why do they discriminate? There is discrimination from my sect and from the Sunni sect as well, I am not saying one or the other.

73: Both sects the same.

70: We need to get rid of this idea of labeling Sunni or Shia. Eliminate it on the ID cards as well. It starts with the ID cards and once your sect is erased, everything gets erased.

71: We wish.

70: It will unite all the Muslims and we won't have this Sunni-Shia issue.

Moderator: The madam gave a reason on why she thinks there is a lack of cooperation between Sunni and Shia in the society. Can you give us another example that proves the lack of cooperation between Sunni and Shia? From your daily life.

71: I'm going to tell you a story about something that happened with a friend of mine. Her daughter went to apply somewhere. Her last name is not very clear to what her sect is and whether she is Sunni or Shia. They accepted her and everything and told her to bring her ID card. After showing them her ID he asked her if she was from X city. His face turned yellow and told her that we will contact you later after he had confirmed with her and everything. These stories about the hatred between Sunni and Shia in general, of course there are a lot of people from both sects who love each other, have become serious and the problem is so complicated and it is very difficult to be resolved. From what I see there is a Sunni-Shia problem and it is just as prevalent among the youth as it is among the elders. It has become a very difficult problem, but we hope that it gets resolved somehow.

70: By the way, the new rising generation will eliminate the idea of Sunni-Shia because while our grandparents still hold grudges and hatred from the civil war, the new generation is going to be more aware. It will not care about the Sunni and Shia labels because the Lebanese economy is on decline and we are the reason behind that due to our backwards mentality. It shouldn't be about Sunni or Shia. We are all Muslims and we all have the Qur'an. Why don't we say that? Why do we discriminate between Sunni and Shia when half of our children are Sunni and the other half is Shia? Why? For example if someone from Tariq Jdide went to Barboor (Predominantly Shia) they start winking to let each other know that he's from Tariq Jdide. And it the same thing the other way around. Why?

74: They'd beat him.

70: Why? You are his friend! Why are you acting like this? The main reason is that the big important figures squeezed this idea into the regular people's heads. I hope that any mother and any father would take this idea of Sunni-Shia out of their heads and the head of any child.

73: This has to be done through clerics. Unidentified: Clerics and leaders.

73: No just clerics.

72: We also need to shed light that clerics and political leaders are not the only ones to blame. I will give an example from both sects. First, the Sunni religion prohibits cursing any faith and they say that our without Ali, prophet Omar would've perished (Ali being Shia and Omar being Sunni). In the same time, the big religious Shia references such as Sayyed Mohammad Hussein Fadlallah and Sayyed Ali Khamenei, they absolutely prohibit cursing any (unclear word).

70: Yes.

74: It still happens though.

72: Any ignorant Sunni or Shia guy, not an educated one, curses X or Y to their friends. By that, he is provoking the other side. As a Sunni or Shia guy, if I hear someone cursing Imam Ali or Omar, I will inevitably react.

73: True.

70: You will feel something.

74: Yes.

72: So the youth's ignorance is what got us here not the politicians at all.

73: I was just saying that.

72: Because we cannot just blame clerics and politicians.

73: On the contrary, this is the cleric's business.

74: No as he said, it is these ignorant people on the streets.

73: Clerics need to raise awareness among the youth.

Moderator: He said in the video that in the internal circles, the clerics' speeches are going to be sectarian.

70: Yes exactly, sectarian.

73: Why can't it be unifying? All the clerics need to unite...

Moderator: So you think there is a responsibility on clerics?

73: Of course.

70: Any cleric that goes on TV needs to promote love between Muslims and not use the Sunni and Shia labels but instead use Muslims.

73: Sayyed Mohammad Hussein used to talk about the Islamic unity.

Moderator: I want to hear your opinion on this, the absence of cooperation and its reasons, and how we can solve it.

69: The main reasons lies in the civic society. We were divided because of our clerics and because of our leaders. They constructed their own schools like Al Maqased or (unclear).

74: Al Masharee'.

69: And that was the beginning of the Sunni-Shia division in this country. On this principle, they started teaching religion in their own ways. Whether Sunni or Shia, they founded sectarianism. And sectarianism, as I told you, does not end except if with our love and commitment to the country. That is fundamental because if we keep thinking Sunni-Shia or Christian-Druze we will not be able to make it.

74: I swear; before they used to all marry each other.

69: They need to unify the schoolbooks and not allow schools to teach its own thing so that they can feed into sectarianism.

Moderator: Do you think that someone ordinary like you can contribute in enhancing this cooperation?

69: He can contribute through his family. He can contribute through his neighbor if he trusts them. He can make a change but very slightly. The main contribution needs to come form the leaders and clerics. They are the ones who need to make a change.

74: For example, I am Sunni and you are Sunni and you are Shia and we are really good friends. But if my Shia neighbor or Sunni neighbor interferes, my friend turns against me.

70: Yes.

Moderator: This happens?

74: Yes, I hear them. We would be sitting together and all is well but when someone from her sect joins, she just becomes aggressive.

70: That's our problem. We become sectarian.

74: I have been married twice: one was Sunni and one was Shia. I have Sunni and Shia kids.

73: Nice.

74: Both of them are with me and they do not know he is Sunni or he is Shia, they just know that they are Muslims.

70: I lived in an area where no one knew where I was from. And everyone loved me. When they knew where I was from they said, “We wish they are all like you.” I do not involve myself in politics. I tell them if either of you, Shia or Sunni, wants to talk politics you have to respect each other. They ask me where I’m from; I say I’m Muslim. It is not your business.

74: I have a Shia neighbor whom I didn’t know was Shia. We used to laugh and joke about things and even Sunni and Shia tease each other. In the end I learned that she was Shia and she never showed it.

Moderator: The video discussed that the judiciary is a problem and the electoral law is a problem. And someone considered the Communist Party and the Nationalist Party are actual political parties because they include people from different sects while other parties are sectarian parties because they are not represented by all the sects.

Unidentified: True.

Moderator: Which topic convinced you most? The one about the judiciary, the electoral law, or the talk about political and sectarian parties?

74: The political parties and sectarianism. Because these are the ones who ruined the country.

70: True, same. A whole sect follows certain parties.

71: Yes, political parties and sectarianism. If we resolve the sectarianism problem, everything will be resolved.

70: If they get rid of all parties, Lebanon will become a great country again. Moderator: Get rid of sectarian parties?

70: Yes sectarian ones.

71: Yes.

Moderator: So you agree that in Lebanon there are political parties represented by different sects and there are sectarian parties?

70: Yes of course.

Unidentified: They should make a shared party.

70: They should found parties to unite all the Muslims together. Why does the Lebanese Army indulge in sectarianism? Because it is part of their training and teachings not to be sectarian.

Moderator: I asked a question: if we achieve this cooperation, does it affect the economy and political and social situations? Give me examples.

74: Yes, they start to like each other.

71: Of course!

70: The Shia will start employing the Sunni and the Sunni will start employing the Shia.

74: True.

70: You'll stop needing a Wasta to find an opportunity or employment. And the same goes for the Sunni.

73: the regions mix.

70: Finding a job then just becomes about competency and degrees not about Wasta. Moderator: On the societal level, you think that will create some sort of familiarity between the citizens?

73: Of course. Someone from Tripoli wouldn't go to Dahyeh.

70: True.

71: That's what I wanted to say.

73: Or take someone from Dahyeh to Tripoli... last time there were fights and people beat each other.

71: That's too bad, Lebanon is for everyone not for certain sects.

73: The whole country would just freshen up. Dahyeh freshens up, Beirut freshens up, Tripoli freshens up...There'd be communication between the citizens. Moderator: How would the cooperation affect the political situation?

70: The politicians will start liking each other.

69: You witness how that helped the political atmosphere.

Moderator: how they recently were all in agreement?

73: Yes.

69: In agreement but it is all about personal interests. When the parliamentary system is changed and when the electoral law is changed, then you'll have a hope to build something for the future. But while they are sticking with these outdated laws, nothing is going to change.

Moderator: So you are focusing on the electoral as the basis?

69: Of course.

Moderator: Do you consider that it is part of solving the problem and it is not just about the text because there are a lot of reasons that can lead to the absence of cooperation?

69: But the majority of it is about the electoral law. Three quarters of our problems are because of the law. When the law enforces that only Sunni can occupy this position and only Shia can occupy that position, then it is dividing between Sunni and Shia.

72: I just want to say that legislatively, the president shouldn't be Christian. They made it this way based on customs.

71: It was imposed on us.

73: That is the Lebanese constitution.

72: Legally, the Prime Minister, or the Chairman of the House of Representatives, or the President of the Republic is not limited to one sect. Legally that is.

73: Our constitution says that.

72: No, legally it is not. That is just customs.

71: This law was set by the French colonialists.

Moderator: But what is the origin of these customs? Al Taif Agreement.

72: Yes.

Moderator: Al Taif Agreement is the constitution of this country.

72: But I am saying that the Lebanese law does not specify sectarian requirements for certain positions. If our politicians follow the Lebanese Law properly, we'd get rid of sectarianism in politics.

73: Yes you mean eliminating sectarian politics.

72: But everyone has their own law tailored to fit their needs. If the politicians follow the proper Lebanese law, it is a good law. But each of them wants to customize their own laws. They were the ones to specify that the Prime Minister is Shia and Chairman of the House of Representatives is Sunni...But legally, this law doesn't exist.

Moderator: Now I want you to think about what kind of societies your children will live in 20-30 years from today. Will they live in a society that actually has cooperation or is it going to be the same society we live in today? How do you imagine their society would look like?

70: Same society. Let me tell you something. The Sunni-Shia issue might disappear but in 7, 10, 12 years it will spark again.

74: True.

71: In my opinion, it is not going to disappear to start with.

70: No matter how much it disappears, a day will come when one of the leaders will need something and will need to exploit this country and its weakest point, which is the Sunni-Shia conflict, so he will incite the people and walk away.

72: I'm going to say it again: if the politicians start eliminating sectarianism from the top, and treating everyone equally where all of Lebanon is for us regardless of which area, and get rid of the Sunni-Shia rhetoric even on the ID cards, I think we can improve bit by bit and this issue will be over.

73: I have hope that things will change.

Moderator: Is there hope?

73: Yes.

Moderator: Is there something that we should do for our children so that such change can happen in the future?

73: Of course! We need to put in a big effort, and there should be civic activities mainly to distance clerics from politics.

74: Yes.

69: There is no change.

Moderator: You think it will stay like this?

69: As it is.

70: It might disappear but in 7, 10, 12 years but one day a leader will want something and will create a conflict, which is the easiest thing to do in Lebanon because we have birds-brains. If someone curses Omar go beat him and if someone curses the Hussein go beat him. That's how it works here.

73: That is wrong.

70: Just so that the leaders can exploit his position so that they can benefit on our children's sake.

74: We need a leader whose mom is Shia and dad is Sunni or otherwise so that they wouldn't dare marginalize either group.

73: We are still doing the same mistakes and our clerics are wrong. The speeches are wrong, Friday speeches are wrong, everything is wrong whether in this sect or that one.

70: Who told you though that Shia don't love the Sunni or the Sunni don't love the Shia. Moderator: there is love.

70: There is love.

74: But it is subtle.

71: We are talking in general. Of course there are some who love each other, but generally, the conflict is there.

74: I love my husband and my children.

70: No there is love. There is love.

Moderator: Ok we are end the discussion about the video now.

C.3 Transcripts of Group Discussions (English) [6-D]

Moderator: Welcome to Table D Session 6. After listening to the video and the topics that were discussed, I would like to start with those topics. First, do you agree with everything that was said in the video? Who would like to start?

73: What is being said is all about the situation we are living in. They are not adding anything more to what is really going on: sectarianism, quota system, job opportunities, ministers, House of Representatives...This is what is actually happening. Nothing is changing, whether it is in the Sunni sect or Shia sect or Christian sect or Druze sect, they all have a quota system. And the people are the ones paying. For example, X leader or minister has his own group and says, "No I'll serve my group so that they can back me up in the elections, why would I serve the other sect?"

Moderator: His benefit is only in his sect.

73: his benefit is more important than anything. More important than the country to be honest. His priority is in his personal benefits more than it is in the benefits of the country. If the country meant anything to him, he wouldn't have created this group around him so that it backs him with its votes. They'd say "I have nothing to do with the Mountain region or South region. I am from Beirut or the North or Biqaa. Every leader is controlling his sect and even if they approve of him or not, he says to them "who else are you going to vote for? To the Sunni or the Shia or the Christian?" That is what is happening.

69: The theory he is talking about is true. But also, you cannot say that the country is following this path. For example, you get a representative who you vote for but you don't see him. This is called the complete lists where you just vote for everyone on the list.

73: True.

69: There are a lot of representatives who come and do not serve anyone, not even themselves. He sits in an office for a short while then he's out.

74: Yes.

73: 100%.

73: This depends on the cluster and the person because I might want the people on this list to all make it. You don't look at the individuals you just like the title of the group so you vote for all of them. That's the wrong thing to do.

69: 100%.

71: Sometimes the names change. Sometimes someone from a sect other than your own serves you better than someone from your own sect.

72: True.

Moderator: And that was said in the video that sometimes better service comes from outside your sect. Is that happening?

71: Yes of course.

74: Yes.

72: What I liked about the video most is that we all united over the garbage crisis and we all protested: Sunni, Shia, and Christians. That was what united us. Other than that, unfortunately, everyone follows his or her own leader.

71: But generally, sectarianism was not as deeply rooted.

69: True true. We used to live as one, we didn't use to think X is Shia, Y is Sunni, Z is Christian. You might find a better friend from another sect than your own.

74: They also even said that if someone is Shia he couldn't go to Tripoli, why not?

71: yea why not?

74: That he fears that he'd be killed because they are not from the same sect as him. All of this also...

71: This recently started happening.

74: Yes.

69: But if you look back at the wars in 1975 and 1981 and the elimination war, there wasn't this Sunni-Shia rhetoric. Yes sure we passed through a Muslim-Christian period during the elimination war, but the Sunni-Shia thing only started after the 1996.

74: True.

71: Absolutely true.

69: That we only witness after 1996. If you want to look at us, the people present in this room, we lived half of our lives 1996, 1997, 1998 and we still live normally, then someone introduced this conflict. Of course nothing happens out of nothing, as they say "there is no smoke without a fire."

74: Of course.

69: They introduced this conflict for political and divisional reasons.

74: True.

71: It serves their purposes.

Moderator: And sectarianism furthered this.

69: Of course.

Moderator: It is a tool.

69: Exactly.

72: And since they assassinated the martyr Rafic Al Hariri, these things started happening.

74: That's true.

71: True.

72: They got what they wanted.

Moderator: By assassinating the martyr Rafic Al Hariri?

72: Yes.

74: Yes, this never used to show before.

71: Yes.

Moderator: I want to hear your voice now. Do you agree about most things said in the video? First I want to ask you, are we one people or multiple peoples? So are we just sects or do we truly belong to one Lebanese people.

74: Sects

Unidentified: Sects, and very much so.

71: No but our belonging is to Lebanon if you want the truth.

73: The truth is that we are 18 sects.

71: Even if 20 sects, but who do we belong to?

73: To our leaders.

Moderator: In their opinion, we are more sectarian than we are nationalistic.

71: True, but in the end our belonging is to our country.

73: They only agree with each other when they are abroad and they love Lebanon.

71: No here too.

72: Yes.

74: True.

73: You see a Lebanese abroad, you love him but when you're in Lebanon...

71: True.

73: you got into a hospital for example, you see someone and you start thinking, "they are from my sect so they will do me more favors than someone from another sect." Unidentified: In hospitals, they are all the same.

74: We are saying that even sometimes people in our own sect don't help us, so we have to go to another sect to get help.

71: True.

73: There is something we need to pay attention to is that, true we have sects but for example, if you are not affiliated with a specific leader or political party you find that you won't get as much help. That is the idea. Even religious figures do not cooperate to unite a certain sect.

74: True.

72: True.

71: True.

73: They've done this so that they can create this sort of an army. The Lebanese people are people who love life and fun in life. They made them poor.

Moderator: So you stress that the people in Lebanon are not sectarian but they love life.

73: 100%. But they are exploiting that you are poor.

74: True.

69: Take for example the events that happened on the 7th of May and other events; it was all about money. Who sent protesters to the streets other than those who have money? They paid them $500.

73: They are after the poor people. There are no jobs so they pushed you to follow political parties. The day I get a job I won't care about weapons. If I cannot support myself, I have to follow them so that I can survive. That's the point.

74: True.

69: From 2000 until the end of 2004, it was the best 4 years ever. After 2005...

73: Because people were busy with work. Now you sit at home and you bring it out on your family because there is nothing that is distracting you.

69: Yes, true. Before 2005 everyone was working and no one had time for anything. Now you find yourself without work, you go to your relative's place and you start fighting. Unemployment makes problems.

74: True.

71: And generally, one person cannot support the family.

74: Of course, one hand doesn't clap.

73: Even if you notice, in every house the dad is married to either a Sunni or Shia women. There is no mixing. There are slogans and things they say in schools that are wrong. The government wants this disorder.

69: You've seen the extremists that we've been hearing about. Where do these people come from? From the education.

73: That is wrong education and wrong mentality.

69: Someone is teaching them to be like that. They're not born like that.

74: Someone is paying for them to become like that.

73: Lebanon is small and has 18 sects not like the countries around us that have 2 or 3 sects.

70: They plant the doctrine.

73: She is right.

69: Even words like doctrine, funding...These are words that the media started using only recently.

70: Yes these are new.

72: The media also plays a big role.

69: The media war is more important even than the actual war and influences a lot.

72: Even if there is nothing happening they start exaggerating things.

Moderator: In your opinion these problems that we have in Lebanon are the result of the lack of cooperation between sects specifically, Sunni-Shia?

70: Yes of course. If they were united we wouldn't have gotten here.

Moderator: Is it possible that if this cooperation happens, it can improve the social, political, and economical situation? And how can this cooperation be translated in your opinion?

70: When the big shots agree with each other, the ones below them follow.

72: True.

71: True.

72: Unfortunately, we are talking about people above and people below.

73: The big shots are agreeing not for the people, but for their own benefits.

74: Of course.

73: They are just playing roles like in a play.

69: If we go back a couple of months, it was chaos. The people who were shooting each other are now fine but those who died, it is now over for them.

72: Yes.

74: Yes.

69: Their worth has become nothing.

71: they can make us love each other and they can make us hate each other.

69: Any leader of any sect goes on TV and starts threatening. The country goes into chaos.

73: But why? If you are working and have money you wouldn't follow a leader, you can support yourself. But they have made the people poor so they keep following them.

74: Where can you find jobs anyway? There are no jobs.

73: Even if you find the ministers and leaders fighting with each other, it is not for us, it is because of their benefits. I'm sorry but we keep saying Lebanon was united over the garbage crisis, no it wasn't united. There was just portions to be divided between the Druze and Sunni sect.

74: They united over the garbage crisis. Why don't we actually become all one hand in everything without caring about Druze sect or Sunni sect or Shia sect.

73: I mean we united to get rid of the garbage and smell that is surrounding us. But if you actually listen to them speak, because...before 2005 or even now as they make the new quotas, the Druze sect was the most benefited because there was someone from the Sunni sect who was also involved and made a huge deal about it (negatively). Because the people of X or not the people of Y are going to benefit, they agitate the citizens whenever they want. Just like in the civic movement. There were a lot of people who had a benefit in it. When they found that it was getting risky they sent their forces to beat people up.

74: True.

71: True.

73: Even there were officers who talked. The government is the base. For example, I as Shia, or Sunni or Druze and you as Shia you want to take 4 officers, Sunni 4 officers, Druze 2 officers and Christians 3 officers. This is our law. Even first class officers are Christians. They force it on you.

69: True that is what is requested.

Moderator: So sectarianism is rooted in the law even.

74: Yes.

73: Exactly, they are forcing it on you.

69: You are talking about the 60's law.

Moderator: Election Law then.

69: Yes, we are talking about a law that has been followed for a while. When they were done with all the benefits, they created sectarianism.

73: True true.

69: And if we manage to get rid of sectarianism, they'll create something else. 71: They'll distract us with something else, true.

72: They enforced this system. House of Representatives is for Shia, the head of government is for Sunni's, president for Christians.

73: True.

Moderator: is this something wrong in your opinion?

72: In my opinion, if there is someone Sunni and he is qualified, I do not mind if he becomes president.

74: But it is important that he provides us with what we want.

71: This is an old law; they didn't set it up recently. It is since we got our independence. Because they were all united back then, so they divided and it's been like that since then.

73: Yes, but they took away the authority from the president. In another words, he has no role, just a chair to be filled.

Moderator: So the parliament is for the Shia sect and the government is for the Sunni sect, which was since Al Taif in 1990 until today.

71: True.

69: Yes from 1975 until the 1990s we didn't have a country. After the 90's they set up Al Taif and we've been following it since.

71: Yes that's it. It's a wrong law and we haven't changed it.

Moderator: Ok. In your opinion, what should be done to reinforce this cooperation? We talked about a lot of problems and the reasons behind this absence of cooperation. Now what should we do so that we create this cooperation? And let's try to be realistic in our answers.

71: If the people all come to an agreement...

73: But the people cannot come to an agreement if the big leaders don't resolve these issues. What is supposed to happen is...

69: Are we talking about predictions? Or something on the ground?

Moderator: No not predictions. On the ground, yes.

69: It is impossible.

73: Everything can be fixed in Lebanon today, but how? For example, there wasn't going to be a president. A couple of days before, they pushed for it and we got one. Everything will work if you want it to happen.

71: True.

Moderator: Ok. People like you, do you think you are capable of making a changing that can lead to more cooperation? You as ordinary people, you don't think you play a role?

74: No no.

73: No we don't have an influence.

71: Of course not.

69: No. In dreams maybe.

73: When the civic movement happened, we supported them thinking that they were doing something for the better. Turned out people involved in it were politicized and political parties pushed them to talk in certain ways. It was revealed in the end. This one is affiliated with X and that one is affiliated with Y. They identified them in pictures and names. You need to eliminate or fix the law of political parties in Lebanon. When this is resolved and we'll stop having political parties, you become just a Lebanese citizen without affiliations.

69: This is very difficult.

73: No we can get rid of political parties.

69: It is impossible. Why did someone become a leader? Because of his group. Take away his support and he becomes just like anyone one of us.

73: My friend, yes but they'll become leaders because of their popularity not his political group. When you say "movement" or whatever, take out the word "parties" in the Lebanese law, you'll become just citizens...

69: Let me just give you this example. They told Wiaam Wahaab that they wanted to dismantle his United Party, so he made it the United Movement.

73: No just dismantle all parties in the law. You can still love your leader but not under a specific party.

69: It won't work.

73: Why not!

Moderator: In your opinion, if we want to think about our children and as youth, in our future. Do you think we're going to be living in a similar society as the one we are living in today or maybe it'll be better?

69: 100%. It will not change.

72: Of course. It'll be the same.

74: Depends on our optimism.

73: If you want Ziad El Rahbani plays (satirical plays that discussed the political and social problems in Lebanon) they are still the same reality. What our parents watched and laughed about, we can still relate to and laugh about as well.

71: No hopefully everything changes. Nothing stays the same.

74: We just need to be hopeful.

69: The lady here. You've witnessed so many governments and ministries and it is still the same.

71: We used to live a more luxurious life. Older people know how it was. A man used to get 300,000 LBP and live like a king, now if he gets a million it's not enough.

74: yes true.

73: The economical situation is like that globally.

71: Yes, I'm telling you, it just might change again.

74: because back then you didn't pay TVA. They drown you. No you pay 10% of your salary, on the food you buy, on everything. In the end up of the day you find that you've paid 30-40% of your salary. There are policies that me and you don't know about.

71: I'm telling you everything changed. And you never know, maybe it'll change again one day.

Moderator: I want to thank you for this effective discussion.

APPENDIX D

Extended Analysis of Chapter 7

D.1 Attack Principles and Ł-Based Arguments

[Def] and Ł-Based Arguments

(A.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$, then $X \models_{Ł} \neg \bigwedge \Gamma_A$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg \bigwedge \Gamma_A) = 1$. For any evaluation e we have $e(\neg \bigwedge \Gamma_A) = 1 - e(\bigwedge \Gamma_A) = 1 - \min_{\gamma \in \Gamma_A}(e(\gamma))$. Therefore considering the argument $\langle \Gamma_{A\wedge B}, A \wedge B\rangle$, if $\Gamma_A \subseteq \Gamma_{A\wedge B}$, we have $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) \leq 1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma))$ and if $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) = 1$ also $1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma)) = 1$ i.e. $X \models_{Ł} \neg \bigwedge \Gamma_{A\wedge B}$. The attack principle holds.

(C.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then $X \models_{Ł} \neg \bigwedge \Gamma_{A\wedge B}$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg \bigwedge \Gamma_{A\wedge B}) = 1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma)) = 1$. Therefore $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$ or $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$ only if either $\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$, but this is a quite demanding requirement to ask.

(A.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then $X \models_{Ł} \neg \bigwedge \Gamma_{A\vee B}$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg \bigwedge \Gamma_{A\vee B}) = 1 - \min_{\gamma\in\Gamma_{A\vee B}}(e(\gamma)) = 1$. Therefore $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$ only if $\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$.

(C.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$, then $X \models_{Ł} \neg \bigwedge \Gamma_A$ and $X \models_{Ł} \neg \bigwedge \Gamma_B$, i.e. for any evaluation e s.t. $e(X) = 1$ $e(\neg \bigwedge \Gamma_A) = 1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) = e(\neg \bigwedge \Gamma_B) = 1 - \min_{\gamma\in\Gamma_B}(e(\gamma)) = 1$. Therefore $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\vee B}, A \vee B\rangle$ only if either $\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$.

(A.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\neg \bigwedge \Gamma_{A\supset B}) = 1 - \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) = 1$. If $\Gamma_{A\supset B} \subseteq \Gamma_B$, then

$\min_{\gamma\in\Gamma_B}(e(\gamma)) \leq \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)$ and $1-\min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) \leq 1-\min_{\gamma\in\Gamma_B}(e(\gamma))$. Therefore $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$. To show that $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ follows from the hypothesis, we need to find a specific evaluation e^* s.t. $e^*(X) = 1$, but $1-\min_{\gamma\in\Gamma_A}(e^*(\gamma)) < 1$ while $1-\min_{\gamma\in\Gamma_{A\supset B}}(e^*(\gamma)) = 1$. Given an evaluation e s.t. $e(X) = 1$, if we indicate with γ_i^* the elements of $\Gamma_{A\supset B}$ s.t. $e(\gamma_i^*) = \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma))$ if $\Gamma_A \subset \Gamma_{A\supset B}$ and $\gamma_i^* \notin \Gamma_A$ for any i, we have $\min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) < \min_{\gamma\in\Gamma_A}(e(\gamma))$ from which it follows $1-\min_{\gamma\in\Gamma_A}(e(\gamma)) < 1-\min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma))$.

(C.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ and $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then for any evaluation e s.t. $e(X) = 1$, $e(\neg\bigwedge\Gamma_B) = 1-\min_{\gamma\in\Gamma_B}(e(\gamma)) = 1$ and there is an evaluation e^* s.t. $e^*(X) = 1$, but $e^*(\neg\bigwedge\Gamma_A) = 1-\min_{\gamma\in\Gamma_A}(e^*(\gamma)) < 1$. Therefore $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$ only if $\Gamma_B \subseteq \Gamma_{A\supset B}$.

(A.$\neg$) The principle does not hold

(C.$\neg$) The principle does not hold

[Def] and Order Ł-Based Arguments

(A.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg\bigwedge\Gamma_A$, i.e. for any evaluation e $e(X) \leq e(\neg\bigwedge\Gamma_A) = 1-\min_{\gamma\in\Gamma_A}(e(\gamma))$. If we consider the argument $\langle\Gamma_{A\wedge B}, A\wedge B\rangle$, if $\Gamma_A \subseteq \Gamma_{A\wedge B}$, we have $1-\min_{\gamma\in\Gamma_A}(e(\gamma)) \leq 1-\min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma))$. Therefore for any e we have $e(X) \leq e(\neg\bigwedge\Gamma_{A\wedge B})$, i.e. $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\wedge B}, A\wedge B\rangle$ and the principle holds.

(C.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\wedge B}, A\wedge B\rangle$, then it follows either $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ or $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ if either $\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$, respectively.

(A.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$, then it follows $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ only if $\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$.

(C.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$, then it follows $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$ only if either $\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$.

(A.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$, then for any evaluation e $e(X) \leq e(\neg\bigwedge\Gamma_{A\supset B})$, therefore if $\Gamma_{A\supset B} \subseteq \Gamma_B$ $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$. However from the hypothesis we cannot deduce, even by using very strong conditions, that $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and the principle does not hold.

(C.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ and $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then it follows $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$ only if $\Gamma_B \subseteq \Gamma_{A\supset B}$.

(A.¬) The principle does not hold

(C.¬) The principle does not hold

[C-Reb-1] and Ł-Based arguments

(A.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$, then $\Gamma_X \models_{Ł} \neg A$, i.e. for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg A) = 1 - e(A) = 1$. Since for any evaluation e $e(A) \geq (A \wedge B)$, if $e(A) = 0$ also $e(A \wedge B) = 0$ and $e(\neg(A \wedge B)) = 1$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$.

(C.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg(A \wedge B)) = 1 - e(A \wedge B) = 1$, i.e. $\min(e(A), e(B)) = 0$. However this is different from having either $e(A) = 0$ or $e(B) = 0$. If could be that for a certain e_i s.t. $e_i(\bigwedge \Gamma_X) = 1$ $e_i(A) = 0$ while for another e_j s.t. $e_j(\bigwedge \Gamma_X) = 1$ $e_j(B) = 0$. Therefore the principle does not hold.

(A.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A \vee B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, we have $e(\neg(A \vee B)) = 1$, i.e. $1 - e(A \vee B) = 1 - \max(e(A), e(B)) = 1$ from which it follows $\max(e(A), e(B)) = 0$. Since $e(A) \leq \max(e(A), e(B))$ and $e(B) \leq \max(e(A), e(B))$ for any evaluation e, if $\max(e(A), e(B)) = 0$ also $e(A) = 0$ and $e(B) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$.

(C.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$ we have $e(\neg A) = 1 - e(A) = 1$ and $e(\neg B) = 1 - e(B) = 1$, i.e. $e(A) = 0$ and $e(B) = 0$. Therefore $\max(e(A), e(B)) = 0$, $1 - \max(e(A), e(B)) = 1$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}, A \vee B\rangle$.

(A.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1$, i.e. $e(A \supset B) = 0$. Since $e(A \supset B) = \min(1, 1 - e(A) + e(B))$, $1 - e(A) + e(B) = 0$, i.e. $1 + e(B) = e(A)$ which implies $e(B) = 0$ and $e(A) = 1$. Therefore whenever an evaluation e is such that $e(\bigwedge \Gamma_X) = 1$, then $e(\neg B) = 1 - e(B) = 1$ and $e(\neg A) = 1 - e(A) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \overset{[C\text{-}Reb\text{-}1]}{\not\longrightarrow} \langle \Gamma_A, A\rangle$. The principle holds.

(C.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \overset{[C\text{-}Reb\text{-}1]}{\not\longrightarrow} \langle \Gamma_A, A\rangle$, then whenever an evaluation e is s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg B) = 1 - e(B) = 1$, i.e. $e(B) = 0$. Since $e(B) = 0$ $e(A \supset B) = \min(1, 1 - e(A) + e(B)) = \min(1, 1 - e(A)) = 1 - e(A)$. In order to have $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A \supset B\rangle$ we need to shoe that $e(A \supset B) \leq e(B)$, i.e. $1 - e(A) \leq e(B)$. In fact, if this last inequality holds we would have $1 = 1 - e(B) \leq 1 - 1 + e(A) = 1 - e(A \supset B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A \supset B\rangle$.

However we are under the hypothesis that $e(B) = 0$, therefore we would need $e(A) = 1$ for any e while from the hypothesis that $\langle \Gamma_X, X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ we can only have that for some e^* s.t. $e^*(\bigwedge \Gamma_X) = 1$, $e^*(A) > 0$. The principle does not hold.

(A.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg A) = 1 - e(A) = 1$ i.e. $e(A) = 0$. Therefore $e(\neg\neg A) = 1 - 1 + e(A) = 0$, i.e. $\langle \Gamma_X, X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, \neg A\rangle$. The principle holds.

(C.¬) If $\langle \Gamma_X, X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, \neg A\rangle$, then there is some evaluation e^* s.t. $e^*(\bigwedge \Gamma_X) = 1$ and $e^*(\neg\neg A) = e^*(A) < 1$. In order to have $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ we would need $e^*(\neg A) = 1 - e^*(A) = 1$ i.e. $e^*(A) = 0$, but from the hypothesis we can only infer $e^*(A) < 1$. Therefore, the principle does not hold

[C-Reb-1] and Order Ł-based arguments

(A.∧) The principle holds for the same reason **(A.∧)** holds using Definition 7.1.3.

(C.∧) The principle does not hold because from $e(\bigwedge \Gamma_X) \leq 1 - e(A \wedge B)$ we cannot deduce neither $e(\bigwedge \Gamma_X) \leq 1 - e(A)$ nor $e(\bigwedge \Gamma_X) \leq 1 - e(B)$ since we only know that $1 - e(A) \leq 1 - \min(e(A), e(B))$.

(A.∨) The principle holds for the same reason **(A.∨)** holds using Definition 7.1.3.

(C.∨) The principle does not hold.

(A.⊃) $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then for any evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(A \supset B)$. Since $e(A \supset B) = \min(1, 1 - e(A) + e(B))$, if $1 - e(A) + e(B) < 1$ $e(A \supset B) = 1 - e(A) + e(B)$, therefore $e(B) \leq e(A \supset B)$. If $1 - e(A) + e(B) \geq 1$ $e(A \supset B) = 1$ and also in this case $e(B) \leq e(A \supset B)$. In any case $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$. However from the hypothesis we cannot deduce that $\langle \Gamma_X, X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$. Therefore the principle does not hold, but it holds its shorter version **(A.⊃)**.

(C.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$, then for any evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(B)$ and in order to show that $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A \supset B\rangle$ we would need $1 - e(B) \leq 1 - e(A \supset B)$, but this it cannot be because for any evaluation e, $e(B) \leq e(A \supset B)$. The principle does not hold.

(A.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$, then for any evaluation e $e(\bigwedge \Gamma_X) \leq e(\neg A) = 1 - e(A)$. In order to show that $\langle \Gamma_X, X\rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, \neg A\rangle$ we would need $e^*(\bigwedge \Gamma_X) > 1 - e^*(\neg A) = e^*(A)$ for some evaluation e^*, but it does not follow from the hypothesis and the principle does not hold.

(C.¬) If $\langle \Gamma_X, X\rangle \overset{[C\text{-}Reb\text{-}1]}{\not\longrightarrow} \langle \Gamma_B, \neg A\rangle$, then there is some evaluation e^* s.t. $e^*(\bigwedge \Gamma_X) > e^*(A)$ and in order to have $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ we would need that for any evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(A)$, but this cannot be deduced from the hypothesis. Therefore, the principle does not hold.

[I-Reb] and Ł-Based Arguments

(A.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \models_{\text{Ł}} \neg\varphi$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$ $e(\varphi) = 1$ and whenever $e(A) = 1$, $e(\neg\varphi) = 1$. Since $e(A \wedge B) = \min\{e(A), e(B)\}$, whenever $e(A \wedge B) = 1$, both $e(A) = 1$ and $e(B) = 1$. From the hypothesis we have that from $e(A) = 1$ it follows $e(\neg\varphi) = 1$. Conclusively $A \wedge B \models_{\text{Ł}} \neg\varphi$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$ and the principle holds.

(C.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \wedge B \models_{\text{Ł}} \neg\varphi$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$ $e(\varphi) = 1$ and whenever $e(A \wedge B) = 1$ $e(\neg\varphi) = 1$. If $e(A \wedge B) = 1$, then $\min\{e(A), e(B)\} = 1$, i.e. both $e(A) = 1$ and $e(B) = 1$ and from this we cannot deduce that having just, for example, $e(A) = 1$ is enough to conclude $e(\neg\varphi) = 1$. Therefore the principle does not hold.

(A.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \vee B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \vee B \models_{\text{Ł}} \neg\varphi$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\varphi) = 1$ and whenever $e(A \vee B) = 1$, $e(\neg\varphi) = 1$. If $e(A \vee B) = 1$, then $\max\{e(A), e(B)\} = 1$. Conclusively whenever there is an evaluation e s.t. $e(A) = 1$, $e(A \vee B) = 1$ and $e(\neg\varphi) = 1$. The same holds with B. This implies that both $A \models_{\text{Ł}} \neg\varphi$ and $B \models_{\text{Ł}} \neg\varphi$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$ and the principle holds.

(C.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there are φ and φ' in $Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$, $X \models_{\text{Ł}} \varphi'$, $A \models_{\text{Ł}} \neg\varphi$ and $B \models_{\text{Ł}} \neg\varphi'$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\varphi) = 1$ and $e(\varphi') = 1$, which implies $e(\varphi \wedge \varphi') = \min\{e(\varphi), e(\varphi')\} = 1$. Whenever there is an evaluation e s.t. $e(A \vee B) = 1$, , then $\max\{e(A), e(B)\} = 1$. This implies that at least one between $e(A)$ and $e(B)$ is 1. From the hypothesis it follows either $e(\neg\varphi) = 1$ or $e(\neg\varphi') = 1$, i.e. $e(\varphi) = 0$ or $e(\varphi') = 0$. Conclusively we have that whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi \wedge \varphi') = 1$ and whenever $e(A \wedge B) = 1$, $e(\neg(\varphi \wedge \varphi')) = 1 - e(\varphi \wedge \varphi') = 1$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$.

(A.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \supset B \models_{\text{Ł}} \neg\varphi$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi) = 1$ and whenever $e(A \supset B) = 1$, $e(\neg\varphi) = 1$. Our first claim is that

$\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. Therefore we have to show that there is $\varphi' \in Fm_Ł$ s.t. $X \models_Ł \varphi'$ and $B \models_Ł \neg\varphi$. Whenever there is an evaluation e s.t. $e(B) = 1$, since $e(A \supset B) = \min\{1, 1-e(A)+e(B)\}$, also $e(A \supset B) = 1$ and from the hypothesis we have that $e(\neg\varphi) = 1$. Therefore $X \models_Ł \varphi$ and $B \models_Ł \neg\varphi$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. However from the hypothesis it does not follows that $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and the principle does not hold.

(C.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models_Ł \varphi$ and $B \models_Ł \neg\varphi$. If $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then for any formula $\varphi' \in Fm_Ł$ s.t. $X \models_Ł \varphi'$ $A \not\models_Ł \neg\varphi'$, i.e. there is at least an evaluation e^* s.t. $e^*(A) = 1$ and $e^*(\neg\varphi') = 1 - e^*(\varphi') < 1$, which implies $e^*(\varphi') > 0$. Therefore whenever $e(A \supset B) = 1$, then $\min\{1, 1 - e(A) + e(B)\} = 1$ and this happens if $e(B) \geq e(A)$. However for some evaluations e^*, $e^*(A) = 1$ and $e^*(\varphi) > 0$. At the same time, since we are under the assumption that $e^*(A \supset B) = 1$, $e^*(B) \geq e^*(A) = 1$. Therefore $e^*(B) = 1$ and from the first hypothesis $e^*(\varphi) = 0$, but this is a contradiction and the principle does not hold.

(A.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models_Ł \varphi$ and $A \models_Ł \neg\varphi$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi) = 1$ and whenever $e(A) = 1$, $e(\neg\varphi) = 1$. We should then show that there is an evaluation e^* s.t. $e^*(\neg A) = 1$ and $e^*(\neg\varphi) < 1$, but this does not follow from the hypothesis and the principle does not hold.

(C.¬) It does not hold.

[I-Reb] and Order Ł-Based Arguments

(A.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models_Ł^{\leq} \varphi$ and $A \models_Ł^{\leq} \neg\varphi$. Therefore for any evaluation e $e(X) \leq e(\varphi)$ and $e(A) \leq e(\neg\varphi)$. Since $e(A \wedge B) = \min\{e(A), e(B)\}$, $e(A \wedge B) \leq e(A)$ from which it follows $e(A \wedge B) \leq e(A) \leq e(\neg\varphi)$, i.e. $A \wedge B \models_Ł^{\leq} \neg\varphi$ and the principle holds.

(C.∧) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}, A\wedge B\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models_Ł^{\leq} \varphi$ and $A\wedge B \models_Ł^{\leq} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A\wedge B) \leq e(\neg\varphi)$. However, since $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(A)$ and $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(B)$ from the hypothesis we cannot conclude neither $A \models_Ł^{\leq} \neg\varphi$ or $B \models_Ł^{\leq} \neg\varphi$ and the principle does not hold.

(A.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models_Ł^{\leq} \varphi$ and $A \vee B \models_Ł^{\leq} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A \vee B) \leq e(\neg\varphi)$. Since $e(A \vee B) = \max\{e(A), e(B)\}$ we have both $e(A) \leq e(A \vee B) \leq e(\neg\varphi)$ and $e(B) \leq e(A \vee B) \leq e(\neg\varphi)$, i.e. $A \models_Ł^{\leq} \neg\varphi$ and $B \models_Ł^{\leq} \neg\varphi$ and the principle holds.

(C.∨) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there are φ and φ' in $Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi$, $X \models^{\leq}_{Ł} \varphi'$, $A \models^{\leq}_{Ł} \neg\varphi$ and $B \models^{\leq}_{Ł} \neg\varphi'$. Therefore for any evaluation e $e(X) \leq e(\varphi)$, $e(A) \leq e(\neg\varphi)$, $e(X) \leq e(\varphi')$ and $e(B) \leq e(\neg\varphi')$. Since $e(X) \leq e(\varphi)$ and $e(X) \leq e(\varphi')$, $e(x) \leq e(\varphi \wedge \varphi')$. In fact, whenever $e(\varphi \wedge \varphi') = \min\{e(\varphi), e(\varphi')\} = e(\varphi)$ from $e(X) \leq e(\varphi)$ we have $e(X) \leq e(\varphi \wedge \varphi')$ and the same holds if $e(\varphi \wedge \varphi') = e(\varphi')$. Since $e(A \vee B) = \max\{e(A), e(B)\}$, whenever $e(A \vee B) = e(A)$, $e(A \vee B) = e(A) \leq 1 - e(\varphi) \leq 1 - e(\varphi \wedge \varphi')$ and the same holds if $e(A \vee B) = e(B)$. Therefore in both cases we have that $e(A \wedge B) \leq 1 - e(\varphi \wedge \varphi')$ and the principle holds.

(A.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi$ and $A \supset B \models^{\leq}_{Ł} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A \supset B) \leq e(\neg\varphi)$. Since for any evaluation e $e(B) \leq 1 - e(A) + e(B)$ and $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, $e(B) \leq e(A \supset B)$ and from the hypothesis $e(B) \leq e(A \supset B) \leq e(\neg\varphi)$, i.e. $B \models^{\leq}_{Ł} \neg\varphi$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. However from the hypothesis we cannot deduce $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and the principle does not hold.

(C.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi$ and $B \models^{\leq}_{Ł} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(B) \leq e(\neg\varphi)$. If $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then $\varphi' \in Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi'$ and $A \not\models^{\leq}_{Ł} \neg\varphi$, i.e. for some evaluation e^* $e^*(X) \leq e^*(\varphi)$ and $e^*(A) > e^*(\neg\varphi)$. We would need to show that there is a formula $\varphi'' \in Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi''$ and $A \supset B \models^{\leq}_{Ł} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A \supset B) \leq e(\neg\varphi'')$, but this does not follows from the hypothesis and the principle does not hold.

(A.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_Ł$ s.t. $X \models^{\leq}_{Ł} \varphi$ and $A \models^{\leq}_{Ł} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A) \leq e(\neg\varphi)$. We would need to show that for some evaluation e^*, $e^*(\neg A) > e^*(\neg\varphi)$ and the principle does not hold.

(C.¬) The principle does not hold.

D.2 Argumentative Bipolar Semantics for Gödel Logic

We report below the analysis of of the principles **(C.∧)**, **(A.∨)**, **($\mathbf{C}_{Bi2}$.⊃)** and **(C.⊃)** with the *defeat-rebuttal* attack function.

(C.∧) If $X \models_{\mathsf{G}} \neg(A \wedge B)$, then whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg(A \wedge B)) = 1$. Therefore $\min\{e(A), e(B)\} = 0$ which implies either $e(A) = 0$ or $e(B) = 0$ from which it follows either $X \models_{\mathsf{G}} \neg A$, or $X \models_{\mathsf{G}} \neg B$.

(A.$\vee$) If $X \models_{\mathsf{G}} \neg(A \vee B)$, , then whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg(A \vee B)) = 1$. Therefore $\max\{e(A), e(B)\} = 0$, which implies $e(A) = 0$ and $e(B) = 0$, i.e. $X \models_{\mathsf{G}} \neg B$ and $X \models_{\mathsf{G}} \neg A$.

($\mathbf{C}_{Bi2}$.$\supset$) If $X \models_{\mathsf{G}} \neg(A \supset B)$, then whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg(A \supset B)) = 1$. Therefore $e(A \supset B) = 0$, from which it follows $e(B) = 0$ and $e(A) > 0$. From $e(B) = 0$ it follows $e(\neg B) = 1$ and from $e(A) > 0$ it follows $e(\neg A) = 0$ and $e(\neg\neg A) = 1$. Conclusively, $X \models_{\mathsf{G}} \neg B$ and $X \models_{\mathsf{G}} \neg\neg A$.

(C.$\supset$) If $X \models_{\mathsf{G}} \neg(A \supset B)$, then for any evaluation e such that $e(X) = 1$, then also $e(\neg(A \supset B)) = 1$ and this happens only if $e(A \supset B) = 0$. The only case in which the implication has value 0 is whenever $e(B) = 0$ and $e(A) > 0$. Therefore $e(\neg B) = 1$ and $e(\neg A) = 0$, i.e. $X \models_{\mathsf{G}} \neg B$ and $X \not\models_{\mathsf{G}} \neg A$.

Proposition D.2.1 (7.3.1) (Closure of [D-Reb]-GP-immune arguments over MP)**.** *If A and $A \supset B$ are argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune, then also B is argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune.*

Proof. We have to show that if (a) $X \not\models_{\mathsf{G}} \neg A$ and (b) $X \not\models_{\mathsf{G}} \neg(A \supset B)$, then $X \not\models_{\mathsf{G}} \neg B$. From (a) it follows that there is an evaluation e_1 such that $e_1(X) = 1$ and $e_1(\neg A) < 1$, from which it follows $e_1(\neg A) = 0$ and $e_1(A) > 0$. From (b) it follows there is an evaluation e_2 such that $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$, i.e. $e_2(\neg(A \supset B)) = 0$ and $e_2(A \supset B) > 0$. If we now consider an evaluation e^* such that $e^*(X) = 1$, $e^*(A) = 1$ and $e^*(A \supset B) = 1$. This evaluation e^* is compatible with both hypothesis (a) and (b). Since $e^*(A \supset B) = 1$, we have that $e^*(A) \leq e^*(B)$ and having $e^*(A) = 1$, it follows $e^*(B) = 1$. Therefore $e^*(\neg B) = 0$, i.e. $X \not\models_{\mathsf{G}} \neg B$. □

Theorem D.2.1 (7.3) (Adequateness Theorem for G)**.** *Any formula F is G-valid iff it is $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune.*

Proof. ($\Rightarrow$) Given Proposition D.3, it remain to show that every G-axiom is argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune.

[Tr] If there is an argument X such that $X \longrightarrow (F \supset G) \supset ((G \supset H) \supset (F \supset H))$, then, by **($\mathbf{C}_{Bi2}$.$\supset$)** it follows $X \longrightarrow (G \supset H) \supset (F \supset H)$ (1) and $X \rightsquigarrow \neg\neg(F \supset G)$ (2). From (1) and **($\mathbf{C}_{Bi2}$.$\supset$)** it follows $X \longrightarrow F \supset H$ (3) and $X \rightsquigarrow \neg\neg G \supset H$ (4). From (3) and **($\mathbf{C}_{Bi2}$.$\supset$)** it follows $X \longrightarrow H$ (7) and $X \rightsquigarrow \neg\neg F$ (8). From (8) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg\neg F) = 1$. Therefore $e(\neg F) = 0$ and $e(F) > 0$ (9). From (4) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg\neg(G \supset H)) = 1$, i.e. $e(\neg(G \supset H)) = 0$ and $e(G \supset H) > 0$. Since $e(H) = 0$, it follows from (7), $e(G) = 0$. From (2) it follows that $e(\neg\neg(F \supset G)) = 1$, $e(\neg(F \supset G)) = 0$ and $e(F \supset G) > 0$. Since $e(G) = 0$, it follows that $e(F) = 0$, but this is in contradiction with (9).

[We] If there is some argument X such that $X \longrightarrow F \supset (G \supset F)$, then, by $(\mathbf{C}_{Bi2}.\supset)$ we have that $X \longrightarrow G \supset F$ (1) and $X \rightsquigarrow \neg\neg F$ (2). From (1) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow F$ (3) and $X \rightsquigarrow \neg\neg G$. From (2) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(\neg F) = 1$, i.e. $e(F) = 0$. From (3) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(\neg\neg F) = 1$, i.e. $e(\neg F) = 0$ and $e(F) > 0$, but this is in contradiction with $e(F) = 0$.

[Ex] If there is some argument X such that $X \longrightarrow (F \supset (G \supset H)) \supset (G \supset (F \supset H))$, then, by $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow G \supset (F \supset H)$ (1) and $X \rightsquigarrow \neg\neg(F \supset (G \supset H))$ (2). From (1) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow F \supset H$ (3) and $X \rightsquigarrow \neg\neg G$ (4). From (3) and $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow H$ (5) and $X \rightsquigarrow \neg\neg F$ (6). From (5) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(H) = 0$, from (4) it follows that $e(F) > 0$ and from (4) it follows $e(G) > 0$. From (2) it follows $e(\neg\neg(F \supset (G \supset H))) = 1$. Therefore $e(F \supset (G \supset H)) > 0$. This last inequality holds if either (i) $e(G \supset H) > 0$ or (ii) $e(F) = e(G \supset H)$. From (i) and $e(H) = 0$, it follows $e(G) = 0$, but this is in contradiction with $e(G) > 0$. From (ii), $e(G) > 0$ and $e(H) = 0$ it follows $e(G \supset H) = 0$. Therefore $e(F) = 0$, but this is in contradiction with $e(F) = 0$.

[$\wedge$-1] If there is some argument X such that $X \longrightarrow (F \wedge G) \supset F$, then, by $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow F$ and $X \rightsquigarrow \neg\neg(F \wedge G)$. Therefore whenever there is an evaluation e such that $e(X) = 1$, then $e(F) = 0$ and $e(\neg\neg(F \wedge G)) = 1$ from which it follows $\min\{e(F), e(G)\} > 0$, in particular $e(F) > 0$ that is in contradiction with $e(F) = 0$.

[$\wedge$-2] This case is similar to the previous one.

[$\wedge$-3] If there is some argument X such that $X \longrightarrow (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G)))$, then, by $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow (H \supset G) \supset (H \supset (F \wedge G))$ (1) and $X \rightsquigarrow \neg\neg(H \supset F)$ (2). From (1) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow H \supset (F \wedge G)$ (3) and $X \rightsquigarrow \neg\neg(H \supset G)$ (4). From (3) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow (F \wedge G)$ (5) and $X \rightsquigarrow \neg\neg H$ (6). From (7) and $(\mathbf{C}.\wedge)$ it follows either (i) $X \longrightarrow F$, or (ii) $X \longrightarrow G$. In the case (i) we have that whenever there is an evaluation e such that $e(X) = 1$, then $e(F) = 0$. From (6) it follows that $e(H) > 0$ and from (2) we have that $e(\neg\neg(H \supset F)) = 1$. Therefore $e(H \supset F) > 0$. However from $e(H) > 0 = e(F)$ it follows $e(H \supset F) = 0$ and we have found a contradiction. In the case (ii) we can reach a contradiction in a very similar way.

[$\vee$-1] If there is some argument X such that $X \longrightarrow F \supset (F \vee G)$, then, by $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow F \vee G$ (1) and $X \rightsquigarrow \neg\neg F$ (2). From (1) and $(\mathbf{A}.\vee)$ it follows $X \longrightarrow F$ and $X \longrightarrow G$. Therefore whenever there is an evaluation e such that $e(X) = 1$, $e(F) = 0$ and $e(G) = 0$, but this is in contradiction with what follows from (2): $e(F) > 0$.

[$\vee$-2] This case is similar to the previous one.

[$\vee$-3] If there is some argument X such that $X \longrightarrow (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))$, then, by $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow (H \supset F) \supset ((G \vee H) \supset F)$ (1) and $X \rightsquigarrow \neg\neg(G \supset F)$ (2). From (1) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow (G \vee H) \supset F$ (3) and $X \rightsquigarrow \neg\neg(H \supset F)$ (4). From (3) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow F$ (5) and $X \rightsquigarrow \neg\neg(G \vee H)$ (6). From (5) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(F) = 0$. From (6) it follows $e(G \vee H) > 0$, i.e. $\max\{e(G), e(H)\} > 0$. Therefore either (i) $e(G) > 0$ or (ii) $e(H) > 0$. In the first case we have that $e(G) > e(F) = 0$ from which it follows $e(G \supset F) = e(F) = 0$, but this is in contradiction with what follows from (2): $e(G \supset F) > 0$. In the case (ii) we reach a contradiction form (4) as in the previous point.

[Lin] If there is some argument X such that $X \longrightarrow (F \supset G) \vee (G \supset F)$, by $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow F \supset G$ (1) and $X \longrightarrow G \supset F$ (2). From (1) and $(\mathbf{C}.\supset)$ it follows $X \longrightarrow G$ and $X \not\longrightarrow F$, but this is in contradiction with what follows from (2) and $(\mathbf{C}.\supset)$: $X \longrightarrow F$ and $X \not\longrightarrow G$.

[$\perp$] If there is some argument X such that $X \longrightarrow \perp \supset F$, then, by $(\mathbf{C}.\supset)$ we have $X \longrightarrow F$ and $X \not\longrightarrow \perp$. However, this it cannot be since every argument is supposed to attack *falsum*.

[Con] If there is some argument X such that $X \longrightarrow (F \supset (F \supset G)) \supset (F \supset G)$, by $(\mathbf{C}_{Bi2}.\supset)$ we have $X \longrightarrow F \supset G$ (1) and $X \rightsquigarrow \neg\neg(F \supset (F \supset G))$ (2). From (1) and $(\mathbf{C}_{Bi2}.\supset)$ it follows $X \longrightarrow G$ (3) and $X \rightsquigarrow \neg\neg F$ (4). From (3) and (4) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(G) = 0$ and $e(F) > 0$. Therefore, since $e(F) > e(G) = 0$, $e(F \supset G) = 0$ and $e(F \supset (F \supset G)) = 0$, but this is in contradiction with what follows from (2): $e(F \supset (F \supset G)) > 0$. □

Theorem D.2.2 (Bipolar Argumentative Completeness of G). *Every argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune formula is G-valid .*

Proof. We prove the theorem indirectly. Suppose there is a formula F that is not G-valid, we will show that F is not argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{GP}}$-immune. Since F that is not G-valid, , then there is an evaluation e such that $e(F) < 1$. We distinguish two possibilities: (a) $e(F) = 0$, and (b) $e(F) \in (0, 1)$.

(a) If $e(F) = 0$, then $e(\neg F) = 1$ and $\neg F \models_{\mathsf{G}} \neg F$, i.e. $\neg F \longrightarrow F$.

(b) If $e(F) \in (0, 1)$ then $e(\neg F) = 0$. Therefore $e(\neg F) \leq e(F)$ and $e(F \supset \neg F) = 0$, i.e. also the formula $F \supset \neg F$ is not G-valid and $\neg(F \supset \neg F) \longrightarrow (F \supset \neg F)$. □

D.3 Argumentative Bipolar Semantics for Product Logic

(A.$\neg$) If $X \models_{\mathsf{P}} \neg\neg A$, then for any evaluation e such that $e(X) = 1$, $e(\neg\neg A) = 1$. Therefore $e(\neg A) = 0$, i.e. $X \not\models_{\mathsf{P}} \neg A$

(A.$\vee$) If $X \models_{\mathsf{P}} \neg(A \vee B)$, , then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \vee B)) = 1$. Therefore $e(A \vee B) = \max\{e(A), e(B)\} = 0$, from which it follows $e(A) = 0$ and $e(B) = 0$, i.e. $X \models_{\mathsf{P}} \neg A$ and $X \models_{\mathsf{P}} \neg B$.

(A.&) If $X \models_{\mathsf{P}} \neg A$ or $X \models_{\mathsf{P}} \neg B$, then for any evaluation e such that $e(X) = 1$, then either $e(\neg A) = 1$, i.e. $e(A) = 0$, or $e(\neg B) = 1$, i.e. $e(B) = 0$. Therefore $e(A) \cdot e(B) = 0$, i.e. $X \models_{\mathsf{P}} \neg(A \& B)$.

(C.&) If $X \models_{\mathsf{P}} \neg(A \& B)$, then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \& B)) = 1$. Therefore $e(A \& B)$=0 which implies either $e(A) = 0$, or $e(B) = 0$.

(A.$\supset$) If $X \models_{\mathsf{P}} \neg B$ and $X \not\models_{\mathsf{P}} \neg A$, then, whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg B) = 1$ and $e(\neg A) < 1$. Therefore $e(B) = 0$ and $e(A) > 0$ which implies $e(A \supset B) = 0$ and $e(\neg(A \supset B)) = 1$.

(C.$\supset$) If $X \models_{\mathsf{P}} \neg(A \supset B)$, then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \supset B)) = 1$. Therefore $e(A \supset B) = 0$ and this happens only if $e(B) = 0$ and $e(A) > 0$. From this it follows $e(\neg B) = 1$ and $e(\neg A) = 0$, i.e. $X \models_{\mathsf{P}} \neg B$ and $X \not\models_{\mathsf{P}} \neg A$.

Proposition D.3.1 (Closure of [D-Reb]-P-immune arguments over Modus Ponens)**.** *If A and $A \supset B$ are argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune, then also B is argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune.*

Proof. We have to show that if, for any X in the frame, (a) $X \not\models_{\mathsf{P}} \neg A$ and (b) $X \not\models_{\mathsf{P}} \neg(A \supset B)$, then $X \not\models_{\mathsf{P}} \neg B$. From (a) it follows that there is an evaluation e_1 such that $e_1(X) = 1$ and $e_1(\neg A) < 1$, from which it follows $e_1(A) > 0$. From (b) it follows there is an evaluation e_2 such that $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$, i.e. $e_2(\neg(A \supset B)) = 0$ and $e_2(A \supset B) > 0$. In particular, we have that $e_2(A \supset B) > 0$ if either (i) $e_2(A) \leq e_2(B)$, from which it follows $e_2(A \supset B) = 0$, or (ii) $e_2(A) > e_2(B)$ and $e_2(B) > 0$, from which it follows $e_2(A \supset B) = \frac{e(B)}{e(A)}$. If (ii) happens we have already identified an evaluation such that $e(X) = 1$ and $e(\neg B) = 0$ from which it follows $X \not\models_{\mathsf{P}} \neg B$. Otherwise, since both X and A belong to the frame, also $X \wedge A$ it does and from the hypothesis we have $X \wedge A \not\models_{\mathsf{P}} \neg A$ and $X \wedge A \not\models_{\mathsf{P}} \neg(A \supset B)$, i.e. there is an evaluation e_3 such that $e_3(X \wedge A) = 1$ (from which it follows $e_3(X) = 1$ and $e_3(A) = 1$) and $e_3(\neg(A \supset B)) < 1$. Therefore $e_3(A \supset B) > 0$ and if $e_3(A \supset B) = 1$, $1 = e_3(A) \leq e_3(B)$ which implies $e_3(B) = 1$; if $e_3(A \supset B) \in (0; 1)$, then $e_3(A) > e_3(B) > 0$. Conclusively in both cases $e_3(\neg B) = 0$. □

Theorem D.3.1 (Adequateness Theorem for G)**.** *Any formula F is G-valid iff it is $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune.*

Proof. Given Proposition D.3.1, it remain to show that every P-axiom is argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune.

[BL1] Suppose there is an argument X such that $X \longrightarrow (F \supset B) \supset ((G \supset H) \supset (F \supset H))$. By **(C.$\supset$)** we have that $X \longrightarrow (G \supset H) \supset (F \supset H)$ (1) and $X \not\longrightarrow F \supset B$ (2). From (1) and **(C.$\supset$)** it follows $X \longrightarrow F \supset H$(3) and $X \not\longrightarrow G \supset H$ (4). From (3) and **(C.$\supset$)** it follows $X \longrightarrow H$ (5) and $X \not\longrightarrow F$ (6). From (2) and **(A.$\supset$)** it follows $X \not\longrightarrow G$ (7) or $X \longrightarrow F$ (8). From (4) and **(A.$\supset$)** it follows $X \not\longrightarrow H$ or $X \longrightarrow G$. Since (5) holds, then also $X \longrightarrow G$ and $X \longrightarrow F$, but this is in contradiction with (6).

[BL4] Suppose there is an argument X such that $X \longrightarrow F\&(F \supset G) \supset G\&(G \supset F)$. By **(C.$\supset$)** we have $X \longrightarrow G\&(G \supset F)$ (1) and $X \not\longrightarrow F\&(F \supset G)$ (2). By (1) and **(C.&)** it follows $X \longrightarrow G$ either $X \longrightarrow G$ (3) or $X \longrightarrow G \supset F$ (4). From (4) and **(C.$\supset$)** it follows $X \longrightarrow F$ (5) and $X \not\longrightarrow G$ (6). From (2) and **(A.&)** it follows $X \not\longrightarrow F$ (7) and $X \not\longrightarrow F \supset G$ (8). From (8) and **(A.$\supset$)** it follows either $X \longrightarrow F$ (9) or $X \not\longrightarrow G$ (10). Since (7) holds, then (9) does not hold and $X \not\longrightarrow G$ it does. Therefore (3) does not hold and (5) it does, but this is in contradiction with (7).

[BL5a] Suppose there is an argument X such that $X \longrightarrow (F\&G \supset H) \supset (F \supset (G \supset H))$. Therefore by **(C.$\supset$)** we have $X \longrightarrow F \supset (G \supset H)$ (1) and $X \not\longrightarrow F\&G \supset H$ (2). From (1) and **(C.$\supset$)** we have that $X \longrightarrow G \supset H$(3) and $X \not\longrightarrow F$ (4). From (3) and **(C.$\supset$)** it follows $X \longrightarrow H$ (5) and $X \not\longrightarrow G$ (6). From (2) and **(A.$\supset$)** it follows either $X \longrightarrow F\&G$ (7), or $X \longrightarrow G$ (8). Since (5) holds, (8) does not and from (7) and **(C.&)** we have wither $X \longrightarrow F$, that cannot be because of (4), or $X \longrightarrow G$ that is in contradiction with (6).

[BL5b] Suppose there is an argument X such that $X \longrightarrow (F \supset (G \supset H)) \supset (F\&G \supset H)$. By **(C.$\supset$)** we have $X \longrightarrow (F\&G) \supset H$ (1) and $X \not\longrightarrow F \supset (G \supset H)$ (2). From (1) and **(C.$\supset$)** it follows $X \longrightarrow H$ (3) and $X \not\longrightarrow F\&G$ (4). From (4) and **(A.&)** it follows $X \not\longrightarrow F$ (5) and $X \not\longrightarrow G$ (6). From (2) and **(A.$\supset$)** it follows $X \not\longrightarrow G \supset H$ (7) or $X \longrightarrow F$ (8) and from (7) and **(A.$\supset$)** we have $X \not\longrightarrow H$ (9) or $X \longrightarrow G$ (10). Since (5) holds, (8) does not. Since (3) holds, (9) does not. Therefore should hold both $X \not\longrightarrow G$ (6) and $X \longrightarrow G$ (10), and this cannot happen.

[BL6] Suppose there is an argument X such that $X \longrightarrow ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H)$. Therefore, by **(C.$\supset$)** we have $X \longrightarrow ((G \supset F) \supset H) \supset H$ (1) and $X \not\longrightarrow (F \supset G) \supset H$ (2). From (1) and **(C.$\supset$)** it follows $X \longrightarrow H$ (3) and $X \not\longrightarrow (G \supset F) \supset H$ (4). From (2) and **(A.$\supset$)** we have either $X \not\longrightarrow H$ (5) or $X \longrightarrow F \supset G$ (6). From (6) and **(C.$\supset$)** it follows $X \longrightarrow G$ (7) and $X \not\longrightarrow F$ (8). From (4) and **(A.$\supset$)** it follows either $X \not\longrightarrow H$ (9) or $X \longrightarrow G \supset F$ (10), from which it follows, by **(C.$\supset$)**, $X \longrightarrow F$ and $X \not\longrightarrow G$. Since (3) holds, (5) and (9) do not and (6) it does. However, both (7) and (8) are in contradiction with what follows from (10), and this it cannot happen.

[BL7] Suppose there is an argument X such that $X \longrightarrow \bot \supset F$. Therefore by **(C.$\supset$)** we have $X \longrightarrow F$ and $X \not\longrightarrow \bot$, but this it cannot be since every argument is supposed to attack $\bot$.

[P] Suppose there is an argument X such that $X \longrightarrow \neg F \vee ((F \supset F\&G) \supset G)$. Therefore by **(A.$\vee$)** we have $X \longrightarrow \neg F$ (1) and $X \longrightarrow (F \supset (F\&G)) \supset F$ (2). From (1) and **(A.$\neg$)**, it follows $X \not\longrightarrow F$ (3). From (2) and **(C.$\supset$)** it follows $X \longrightarrow F$ and $X \not\longrightarrow F \supset (F\&G)$. However $X \longrightarrow F$ is in contradiction with (3).

□

Theorem D.3.2 (Bipolar Argumentative Completeness of P)**.** *Every argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune formula is P-valid .*

Proof. We prove the theorem indirectly. Suppose there is a formula F that is not P-valid, we will show that F is not argumentatively $\mathscr{P}_{[D\text{-}Reb]\text{-}\mathsf{P}}$-immune. Since F that is not P-valid, , then there is an evaluation e such that $e(F) < 1$. We distinguish two possibilities: (a) $e(F) = 0$, and (b) $e(F) \in (0, 1)$.

(a) If $e(F) = 0$, then $e(\neg F) = 1$ and $\neg F \models_{\mathsf{P}} \neg F$, i.e. $\neg F \longrightarrow F$.

(b) If $e(F) \in (0, 1)$ then $e(\neg F) = 0$. Therefore $e(\neg F) < e(F)$ and $e(F \supset \neg F) = e(\neg F) = 0$, i.e. also the formula $F \supset \neg F$ is not P-valid and $\neg(F \supset \neg F) \longrightarrow (F \supset \neg F)$. □