TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DISSERTATION

Forcing theory and combinatorics of the real line

ausgeführt zum Zwecke der Erlangung des akademischen Grades
eines Doktors der technischen Wissenschaften
unter der Leitung von

Dr. rer. nat Jakob Kellner
Institut fur Diskrete Mathematik und Geometrie
104

Assoc. Prof. Diego Mejía
Faculty of Science
Shizuoka University

eingereicht an der Technischen Universität Wien Fakultät für Mathematik und Geoinformation

von

Miguel Antonio Cardona-Montoya[1]

Mat.Nr. 11814233

Wien, October 28, 2021

[1]Recipient of a DOC Fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry, TU Wien

Dedicado a
mi madre Aracely, mi tía Melida y a mis hermanos.

CONTENTS

ACKNOWLEDGMENTS

I want to start by expressing my deepest gratitude to my advisors: Jakob Kellner, for giving me the awesome opportunity to come to Austria as his student in spite that he barely knew me. Without his help, it would have been very hard for me to come to Austria and become a Ph.D. student. His unconditional support and patience for my bad communication in English was fundamental to improving my communication skills. I hope that, during these years, my work has showed him I merited such an amazing opportunity; and Diego Mejía, it is difficult to find words that correctly express my gratitude to him. I would like to thank him for awakening my interests in Set Theory (my first mathematical love). His guidance and mathematical discussions during this time were crucial in the development of this thesis. His experience in the field of Set Theory taught me how to think like a researcher and immersed me in many more Set Theory topics. When I went to Japan, he offered a lot of his time to help me with any question and inquiry that I had (even those not related to mathematics). I did not think that I was going to like living in Japan so much, I carry beautiful moments in my heart. Even more, he introduced me to Jörg Brendle, Teruyuki Yorioka and other researchers in Set Theory, and supported me to attend conferences in Japan. I am grateful to them for the multiple discussions on the issues related to my Ph.D thesis and other topics in Set Theory.

I would like to thank both Jakob Kellner and Diego Mejía for offering me their help and support for my application to the DOC scholarship.

I am completely gratefully to the Austrian Academy of Sciences (OeAW) for awarding me the DOC fellowship scholarship, and the Austrian Science Fund (FWF) project P30666 (Kellner) for funding me when I arrived in Vienna.

Finally, I would like to thank all the support during the completion of my Ph.D. studies I have received from my family and friends. The support from my mother and my aunt Melida has been immense. This lifetime and if there were another, will not be enough to thank my mother for all her effort and sacrifice to get me and my brothers forward in our studies, she is a very strong woman. I also want to give special thanks: To my brothers Sebastian and Mateo for supporting me in my decisions; to my father for valuable advise and support; to my aunts Luz Dary and Marleny for having me in their prayers; to Laura, to whom I am happy to have met, for the words of encouragement during my doctoral studies; to Pablo for giving me a sincere friendship since we met; to Diana Montoya for all the help with some procedures and forms I had to fill when I arrived in Vienna; to Victor Torres for his valuable advice on life and for his friendship; and to all my Colombian friends in Vienna, in particular to Diego Jaramillo for his valuable friendship.

ABSTRACT

The purpose of this dissertation is to apply and develop new forcing techniques to obtain models where several cardinal characteristics are pairwise different as well as force many (even more, continuum many) different values of cardinal characteristics that are parametrized by reals. In particular, we look at cardinal characteristics associated with strong measure zero, Yorioka ideals, and localization and anti-localization cardinals.

This thesis consists of three main parts, each representing a significant advancement in its respective topic. The first part introduces the property "F-linked" of subsets of posets for a given free filter F on the natural numbers, and defines the properties "μ-F-linked" and "θ-F-Knaster" for posets in a natural way. We show that θ-F-Knaster posets preserve strong types of unbounded families and of maximal almost disjoint families.

This type of posets led to developing a general technique to construct θ-Fr-Knaster posets (where Fr is the Frechet ideal) via matrix iterations of $<\theta$-ultrafilter-linked posets (restricted to some level of the matrix). This technique allows to prove consistency results about Cichoń's diagram (without using large cardinals) and to prove the consistency of the fact that, for each Yorioka ideal, the four cardinal characteristics associated with it are pairwise different. Another application is to show that three strongly compact cardinals are enough to force that Cichoń's diagram can be separated into 10 different values.

The second part is dedicated to showing that certain types of tree forcings, including Sacks forcing, increase the covering of the strong measure zero ideal $\mathcal{SN}$. As a consequence, in Sacks model, such covering number is equal to the size of the continuum, which indicates that this covering number is consistently larger than any other classical cardinal characteristics of the continuum. Even more, Sacks forcing can be used to force that $\mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$, which is the first consistency result where more than two cardinal characteristics associated with $\mathcal{SN}$ are pairwise different. On the other hand, we prove a result providing bounds for $\mathrm{cof}(\mathcal{SN})$, which generalizes Yorioka's characterization of $\mathcal{SN}$ ([Yor02]). This is applied to prove that $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ is consistent with ZFC (via a matrix iteration forcing construction).

In the last part, we combine creature forcing approaches from [KS12] and [FGKS17] to show that, under CH, there is a proper ω^ω-bounding poset with $\aleph_2$-cc that forces continuum many pairwise different cardinal characteristics, parametrized by reals, for each one of the following six types: uniformity and covering numbers of Yorioka ideals as well as both kinds of localization and anti-localization cardinals, respectively. This answers several open questions from [KM21].

INTRODUCTION

Set theory, as a field of mathematics, had its beginnings with the work of Georg Cantor, who is considered the founder of set theory by many and one of the most original minds in the history of mathematics. He made the awesome discovery that the linear continuum, that is, the real line, is not countable, meaning that its points cannot be enumerated by the natural numbers. This opened the path to the investigation of the different sizes of infinity.

Cantor introduced the notion *cardinality*: two sets A and B have the same size if if they are bijectable, i.e., the elements of A can be put in a one-to-one correspondence with the elements of B. Thus, $\aleph_0 < \mathfrak{c}$ where $\aleph_0$ denotes the size of the set natural numbers and $\mathfrak{c}$, the *size of the continuum*, is the size of the set of real numbers. In 1878, he also conjectured the famous Continuum Hypothesis (CH), which asserts that every infinite set of real numbers is either countable, i.e., it has the same cardinality as $\aleph_0$, or has the same cardinality as $\mathfrak{c}$, which is the most famous problem of set theory. This hypothesis can also be restated as $\mathfrak{c} = \aleph_1$ where $\aleph_1$ is the cardinal following $\aleph_0$ in the well-ordering of cardinal numbers.

The rise of logic and the formalization of mathematics, considering that modern mathematics can be formalized in the formal system ZFC (Zermelo-Fraenkel set theory with the axiom of choice), led Kurt Gödel [Göd40] to prove in 1938 that CH cannot be refuted in ZFC (assuming the consistency of ZFC, that is), which also means that CH *is consistent with* ZFC. In formal terms, whenever φ is a mathematical statement (in the language of ZFC),"φ *is consistent with* ZFC" means that no contradiction arises when φ is added to the axioms of ZFC (assuming there is no contradiction in ZFC). In 1966, Paul Cohen [Coh66] was awarded the Fields medal for showing that CH cannot be proved in ZFC. For this, he creating the method of *forcing*. So the negation of CH is consistent with ZFC; and so we can conclude that CH cannot be proved nor refuted (in ZFC). Immediately following the introduction of forcing, Solovay proved that it is consistent with ZFC that there are as many cardinals as desired between $\aleph_1$ and $\mathfrak{c}$.

Both Gödel's and Cohen's methods have been refined and expanded ever since, and today they are at the core of set theoretic research. They are the main tools to prove that certain statements are consistent with (or even independent from) ZFC.

One recurring application has been independence proofs about the so-called *cardinal characteristics of the continuum* (see Section 1.3). These cardinals describe important properties of the combinatorial structure of the reals, with some properties reflecting ideas from measure theory, algebra, general topology and combinatorics. Typically, cardinal characteristics of the continuum lie between $\aleph_1$ and $\mathfrak{c}$.

For instance, whenever $\mathcal{I}$ is a (nontrivial) ideal of subsets of $\mathbb{R}$ which contains all singletons,[2] the *cardinal characteristics associated with* $\mathcal{I}$ are defined as follows:

add($\mathcal{I}$), the *additivity of* $\mathcal{I}$, is the least size of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$.

cov($\mathcal{I}$), the *covering of* $\mathcal{I}$, is the least size of a subfamily of $\mathcal{I}$ whose union covers the reals.

non($\mathcal{I}$), the *uniformity of* $\mathcal{I}$, is the least size of a set of reals which is not in $\mathcal{I}$.

cof($\mathcal{I}$), the *cofinality of* $\mathcal{I}$, is the least size of a subfamily $\mathcal{J} \subseteq \mathcal{I}$ such that each member of $\mathcal{I}$ is a subset of some member of $\mathcal{J}$.

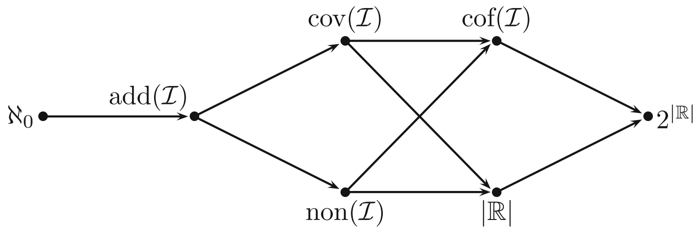

Figure 1: Diagram of the cardinal characteristics associated with $\mathcal{I}$. And arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

Figure 1 shows the "trivial" inequalities between the cardinal characteristics associated with $\mathcal{I}$. Often, connections between combinatorial properties of the real line correspond to relations between the corresponding cardinal characteristics. The best known example is *Cichoń's diagram* (Figure 2) which illustrates the cardinals associated with $\mathcal{N}$, the ideal of Lebesgue null sets, and with $\mathcal{M}$, the ideal of meager (or first category) subsets of $\mathbb{R}$, as well as the numbers $\mathfrak{b}$ and $\mathfrak{d}$ (the unbounding number and the dominating number, or equivalently, the additivity number and covering number of the σ-ideal generated by the compact sets of irrationals, see Theorem 1.3.6), and the cardinal numbers $\aleph_1$ and $\mathfrak{c} := 2^{\aleph_0}$. For example, the inequality $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M})$ means that, if κ is a cardinal and any union of at most κ-many null sets is null, then any union of at most κ-many meager sets is meager. ZFC also proves $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, so at most 10 different values can appear in this diagram.

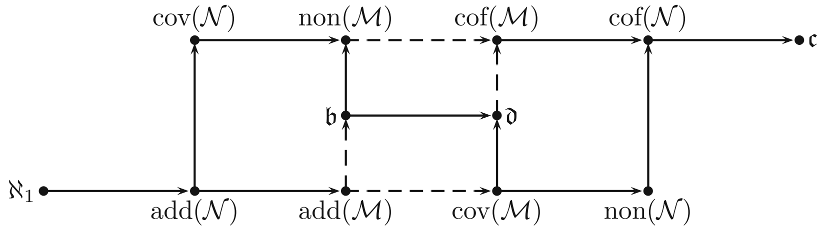

Figure 2: Cichoń's diagram. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$, and the dashed arrows indicate that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

[2] That is, $\mathcal{I}$ is a family of subsets of $\mathbb{R}$ which is closed under finite unions, any subset of a member of $\mathcal{I}$ is also in $\mathcal{I}$, it contains all the finite subsets of $\mathbb{R}$ and $\mathbb{R} \notin \mathcal{I}$.

An important theme in combinatorics of the reals is to understand the relationship between cardinal characteristics of the continuum. For instance, when $\mathfrak{x}$ and $\mathfrak{y}$ is a pair of such cardinals, the aim is to know whether, e.g., $\mathfrak{x} \leq \mathfrak{y}$ is provable in ZFC (which indicated a connection between the combinatorial concepts they represent), or whether both $\mathfrak{y} < \mathfrak{x}$ and $\mathfrak{x} < \mathfrak{y}$ are consistent with ZFC (which makes it less likely to find such a connection). In the case of Cichoń's diagram, it has been proved that the diagram is complete in the sense that no other inequalities (consistent with the diagram) can be proved in ZFC (see [BJ95] for a complete survey about this diagram and its completeness). Recently, Kellner, Goldstern and Shelah [GKS19], using four strongly compact cardinals, proved the consistency of Cichoń's diagram divided into 10 different values, situation known as *Cichoń's maximum*. Later on, it was shown in [GKMS19] that no large cardinals are needed for Cichoń's maximum.

Another well researched notion in combinatorics of the reals is the ideal of *strong measure zero set* $\mathcal{SN}$ (see Subsection 1.2.2), which has been receiving a lot of attention since it was discovered that the *Borel conjecture*, which states that every strong measure zero set is countable, cannot be proven nor refuted in ZFC: CH (the continuum hypothesis) implies that it is false and, on the other hand, R. Laver [Lav76] proved its consistency with ZFC using forcing. The cardinal characteristics associated with $\mathcal{SN}$ became interesting objects of research, in particular when related to the cardinals in Cichoń's diagram. Some of the earliest results are due to Miller [Mil81], who proved that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$, by Carlson [Car93], who proved that $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$, and by Cichoń (see [Ser89, Cor. 3.3]), who proved $\mathrm{cof}(\mathcal{SN}) \leq 2^{\mathfrak{d}}$. Later, Yorioka [Yor02] gave a very useful characterization of $\mathcal{SN}$ in terms of σ-ideals $\mathcal{I}_f$ parametrized by increasing functions $f \in \omega^\omega$, which are known as *Yorioka ideals* (see details in Definition 1.2.7). Concretely, $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \in \omega^\omega \text{ increasing}\}$ and $\mathcal{I}_f \subseteq \mathcal{N}$. This led to rediscover and greatly improve a result of Seredyński [Ser89] to get that no inequality between $\mathrm{cof}(\mathcal{SN})$ and $2^{\aleph_0}$ can be proven in ZFC.

Further research on Yorioka ideals has been continued by Kamo and Osuga [KO08], who proved that $\mathrm{add}(\mathcal{I}_f) \leq \mathfrak{b}$ and $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_f)$. Afterwards, in [CM19, Cor. 3.13, & 3.21], joint with Mejía, we proved that $\mathrm{cov}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{I}_f)$ and $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{N})$ (this was first mentioned by Osuga [Osu08] without proofs). Figure 3 summarizes relations between the cardinal characteristics associated with the Yorioka ideals and those in Cichoń's diagram (for more details, see [Osu06; KO08; Osu08; CM19]).

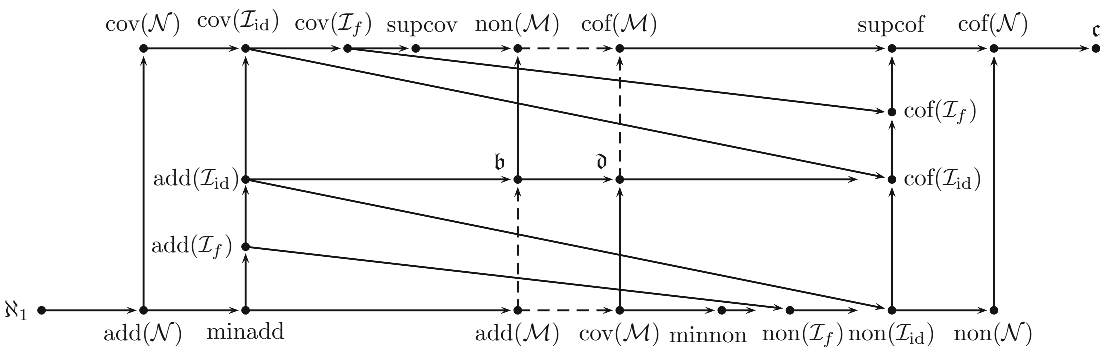

Figure 3: Cichoń's diagram with the cardinal characteristics associated with the Yorioka ideals. Here, minadd is the minimum over $\mathrm{add}(\mathcal{I}_f)$ for increasing $f : \omega \to \omega$; the cardinals supcov, minnon and supcof are defined likewise.

Both the covering and uniformity of Yorioka ideals are closely related to the definable cardinals $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ parametrized by functions b with domain ω, and $h \in \omega^\omega$ (see Definition 1.3.7). The first two cardinal characteristics are usually referred to as *localization cardinals*, while the later two as *anti-localization cardinals*. Figure 4 summarizes the provable inequalities among covering and uniformity of Yorioka ideals, localization, and anti-localization cardinals.

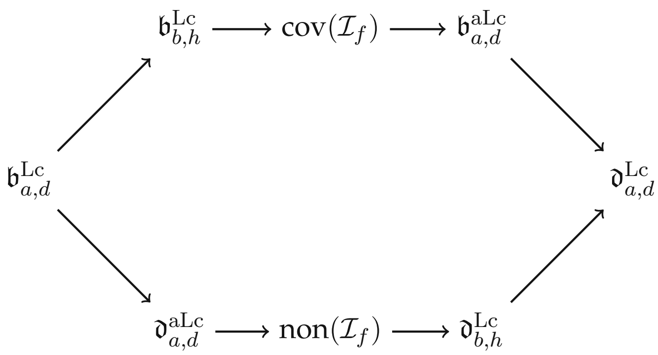

Figure 4: Diagram of inequalities for a block $\vec{\nu} = \langle d, h, g, b, f, a\rangle$ (see details in Chapter 4).

The localization and anti-localization cardinals have appeared in many contexts. The following are well-known characterizations (see e.g. [BJ95, Ch. 2]):

- (Bartoszyński [Bar84]) $\mathrm{add}(\mathcal{N}) = \mathfrak{b}^{\mathrm{Lc}}_{\omega,h}$ and $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}^{\mathrm{Lc}}_{\omega,h}$ when h diverges to infinity (here ω is interpreted as the constant sequence ω), and
- (Bartoszyński [Bar87], Miller [Mil82]) $\mathrm{non}(\mathcal{M}) = \mathfrak{b}^{\mathrm{aLc}}_{\omega,h}$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}^{\mathrm{aLc}}_{\omega,h}$ when $h \geq^* 1$ (for the definition of $\leq^*$ see Section 1.1).

In general, when $b(n)$ is infinite for infinitely many n, the localization and anti-localization cardinals coincide with other well-known cardinal characteristics (see [CM19, Sec. 3]), so the interesting case is when $0 < h(n) < |b(n)| < \omega$ for all (but finitely many) n.[3] Miller [Mil81] proved $\mathrm{non}(\mathcal{SN}) = \min\{\mathfrak{d}^{\mathrm{aLc}}_{b,h} : b \in \omega^\omega\}$ when $h \geq^* 1$, which can be dualized to $\mathrm{cof}(\mathcal{M}) = \sup(\{\mathfrak{d}\} \cup \{\mathfrak{b}^{\mathrm{aLc}}_{b,h} : b \in \omega^\omega\})$ (see [CM19, Thm. 3.23]).[4] On the other hand, due to Bartoszyński and Shelah [BS92] (see also [BJ95, Sec. 2.6]), $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$ whenever $\prod_{n<\omega} \frac{h(n)}{2^{\lfloor \log_2 b(n) \rfloor}} = 0$ and where $\mathcal{E}$ is the σ-ideal generated by the measure zero closed subsets of 2^ω, furthermore the localization cardinals (and some of its variations) play an important role for characterizing $\mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E})$.

Figure 5 illustrates the provable inequalities among localization (when h diverges to infinity), anti-localization cardinals (when $\frac{h(n)}{b(n)}$ converges to 0), and the cardinals in Cichoń's diagram (see [CM19] for a summary). In addition, as hinted in [GS93] and proved in [KM21, Lemma 2.3]: if $\sum_{n<\omega} \frac{h(n)}{b(n)}$ converges then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$; and if $\sum_{n<\omega} \frac{h(n)}{b(n)}$ diverges then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$.

The aim of this dissertation is to apply and develop new forcing techniques to obtain models where several cardinal characteristics are pairwise different, in particular those that we

[3] Other trivial instances also appear in this case as well: if h does not diverge to infinity then $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \mathfrak{c}$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ is finite (Goldstern and Shelah [GS93]); and if the quotient $\frac{h(n)}{b(n)}$ does not converge to 0 then $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ is finite and $\mathfrak{d}^{\mathrm{aLc}}_{b,h} = \mathfrak{c}$ (see [CM19, Sec. 3].

[4] However, it is consistent that $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$ (see [GJS93]), so we cannot dualize $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$ to prove $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{SN})\}$.

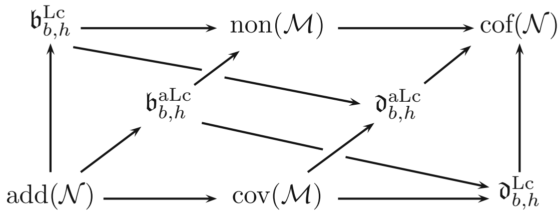

Figure 5: The ZFC provable inequalities among (non-trivial) localization, anti-localization cardinals and the cardinals in Cichoń's diagram.

discussed up to this point. In addition, we aim to force many (even more, continuum many) different values of cardinal characteristics that are parametrised by reals, in particular localization and anti-localization cardinals, and cardinal characteristics associated with $\mathcal{I}_f$. This naturally leads to the following main problems.

Main Problem A. *Produce models by forcing where the four cardinal characteristics associated with an ideal $\mathcal{I}$ are pairwise different. Particular emphasis will be given to $\mathcal{N}$, $\mathcal{M}$, $\mathcal{SN}$ and $\mathcal{I}_f$.*

Note that, for an ideal $\mathcal{I}$, there can be at most two cases for Main Problem A, namely

(A1)$_\mathcal{I}$ $\mathrm{add}(\mathcal{I}) < \mathrm{cov}(\mathcal{I}) < \mathrm{non}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$, and

(A2)$_\mathcal{I}$ $\mathrm{add}(\mathcal{I}) < \mathrm{non}(\mathcal{I}) < \mathrm{cov}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$.

Main Problem B. *Is it consistent with ZFC that $\mathfrak{c}$-many cardinal characteristics of the form* $\mathrm{cov}(\mathcal{I}_f)$ *are pairwise different? The same question is asked for cardinals of the type* $\mathrm{non}(\mathcal{I}_f)$, $\mathrm{cof}(\mathcal{I}_f)$ *and* $\mathrm{add}(\mathcal{I}_f)$*, and for localization and anti-localization cardinals.*

Some instances of both problems are known to be consistent, for example, concerning Main Problem A, Cichoń's maximum yields (A1)$_\mathcal{N}$ and (A2)$_\mathcal{M}$, but the former is known from [Mej13] and the latter is part of this thesis, which was solved before Cichoń's maximum without large cardinals [GKMS19]. Joint with Mejía, we solved some instances of (A1)$_{\mathcal{I}_f}$ in [CM19]; and (A2)$_\mathcal{N}$ is solved recently in [Bre19b]. See more details in the following sections, also concerning Main Problem B.

To attack these problems, we apply and improve the following forcing techniques: finite support (FS) iterations (like matrix iterations) [BS89; BF11; Mej13; GMS16; CM19; Mej19], creature forcing [Kel08; KS12; FGKS17; GK21] and countable support (CS) iterations. In the following sections, we address and describe the specific problems that are tackled and solved in this thesis.

The main results of this thesis were obtained in joint work with J. Brendle, L. Klausner and D. Mejía, and I. Rivera-Madrid and are contained in the papers [Car21; CMR21; BCM21; CKM21].

Ultrafilter-extendable matrix iterations

The main results of this section will be developed throughout Chapter 2, which corresponds to the publication [BCM21] (joint with J. Brendle and D. Mejía).

Miller [Mil81] proved that $\mathbb{E}$, the standard σ-centered poset that adds an eventually different real (see Subsection 1.5.5), does not add dominating reals. Later, Mejía [Mej19] introduced the notion of *Frechet-linkedness* (abbreviated Fr-*linkedness*) inspired by Miller's proof. He showed that $\mathbb{E}$ and random forcing are σ-Fr-linked, and that no σ-Fr-linked poset adds dominating reals. Moreover, it was proved that such posets preserve a certain type of mad (maximal almost disjoint) families (like those added by Hechler's poset $\mathbb{H}_\kappa$ for adding a mad family of size κ, see Subsection 1.5.3).

Frechet-linkedness is a notion of subsets of posets: given a poset $\mathbb{P}$ and $Q \subseteq \mathbb{P}$, *Q is Frechet-linked (in $\mathbb{P}$)* if, for any countable sequence $\langle p_n : n < \omega \rangle$ of members of Q, there is some $q \in \mathbb{P}$ forcing that p_n is in the generic filter for infinitely many n. Given a cardinal μ, a poset is *μ-Fr-linked* if it is the union of μ-many Fr-linked subsets. A Knaster-type notion can be defined in the natural way: a poset $\mathbb{P}$ is *θ-Fr-Knaster* if any subset of $\mathbb{P}$ of size θ contains a Fr-linked subset of the same size. It is clear that any μ-Fr-linked poset is μ^+-Fr-Knaster and, for regular θ, any θ-Fr-Knaster poset satisfies the θ-Knaster property (see Remark 2.1.3).

The notion of Fr-Knaster appears implicitly in several places. For example, using finitely additive measures along FS (finite support) iterations, Shelah [She00] constructed an $\aleph_1$-Fr-Knaster poset to force $\mathrm{cov}(\mathcal{N})$ with countable cofinality, while Kellner, Shelah and Tănasie [KST19] used the same technique to construct a θ-Fr-Knaster poset that forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} = \theta < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}. \qquad (\bigstar_1)$$

Using the analog of this technique for ultrafilters, Goldstern, Shelah and Mejía [GMS16] constructed a θ-Fr-Knaster poset that forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} = \theta < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}. \qquad (\bigstar_2)$$

These two results state the both possible ways to separate the cardinal invariants in the left side of Cichoń's diagram (Figure 2).

The main challenge in both results is to force $\mathfrak{b} = \theta$ while iterating restrictions of $\mathbb{E}$ to small models (for ($\bigstar_2$)), or similar restrictions of random forcing and of a variation of $\mathbb{E}$ (for ($\bigstar_1$)). In fact, from both arguments, it can be inferred that θ-Fr-Knaster posets preserve a strong type of unbounded families (see Theorem 2.2.2).

Since μ-Fr-linked posets preserve the mad family added by $\mathbb{H}_\theta$ for $\mu < \theta$, it is natural to ask:

Question C. *If θ is a regular uncountable cardinal, does any θ-Fr-Knaster poset preserve the mad family added by $\mathbb{H}_\theta$?*

Such a mad family falls into the category of what we call *θ-strong-$\mathbf{Md}$ a.d. family* (see Definition 2.2.3 and Lemma 2.2.5), which are also preserved by μ-Fr-linked posets for $\mu < \theta$ according to [Mej19]. Moreover, it was proved in [FFMM18; Mej19] that a large class of FS iterations preserve the mad family added by $\mathbb{H}_\theta$ for θ regular, which is used to prove that $\mathfrak{a} = \mathfrak{b}$ can be forced (where $\mathfrak{a}$ is the minimal size of an infinite mad family, see Definition 1.3.5(2)) in various models where Cichoń's diagram is divided into several values. In fact, this class is contained in the class of FS iterations of μ-Fr-linked posets with $\mu < \theta$, but since any such iteration yields a θ-Fr-Knaster poset (see [Mej19, Sect. 5]), the previous argument is nicely generalized with a positive answer to Question C. Even more, this will imply that it can be forced, in addition, that $\mathfrak{a} = \theta$ in both ($\bigstar_1$) and ($\bigstar_2$).

Theorem D (Theorem 2.2.6)**.** *If θ is a regular uncountable cardinal then any θ-Fr-Knaster poset preserves all the θ-strong-$\mathbf{Md}$ (a.d.) families from the ground model.*

Corollary. *In both ($\bigstar_1$) and ($\bigstar_2$) it can be forced, in addition, that $\mathfrak{a} = \mathfrak{b}$.*

Using four strongly compact cardinals, Goldstern, Kellner and Shelah [GKS19] applied Boolean ultrapowers (see [KTT18]) to the poset that forces ($\bigstar_2$) to prove the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c} \qquad (\bigstar_3)$$

With the same method, in [KST19] Boolean ultrapowers of the poset that forces ($\bigstar_1$) guarantee the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}. \qquad (\bigstar_4)$$

These results are examples of Cichoń's diagram divided into 10 different values (the maximum possible).

In this work, we are also interested to get strengthenings or variations of ($\bigstar_1$), ($\bigstar_2$), ($\bigstar_3$) and ($\bigstar_4$) with respect to ZFC alone or with weaker large cardinal assumptions. The following result strengthens ($\bigstar_2$).

Theorem E (Theorem 2.4.1)**.** *If $\theta_0 \leq \theta_1 \leq \theta_2 \leq \mu \leq \nu$ are uncountable regular cardinals and λ is a cardinal such that $\lambda^{<\theta_2} = \lambda$, then there is a ccc poset that forces (see Figure 6)*

$$\mathrm{add}(\mathcal{N}) = \theta_0 \leq \mathrm{cov}(\mathcal{N}) = \theta_1 \leq \mathfrak{b} = \mathfrak{a} = \theta_2 \leq \mathrm{non}(\mathcal{M}) = \mu \\ \leq \mathrm{cov}(\mathcal{M}) = \nu \leq \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda.$$

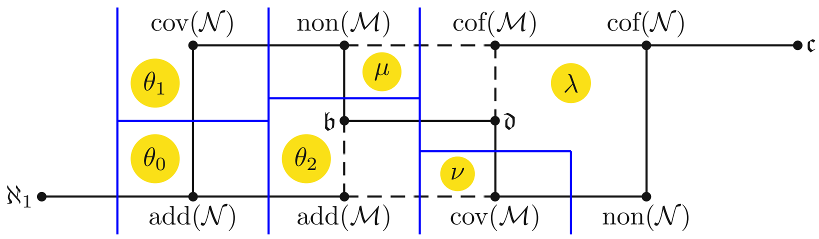

Figure 6: Seven values in Cichoń's diagram with the left side separated.

The method to prove Theorem E is a modification of the method in [GMS16] to prove ($\bigstar_2$), which is reviewed as follows. To force $\mathfrak{b} = \theta_2 < \mathrm{non}(\mathcal{M}) = \mu < \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$, the idea is to perform a FS iteration of Suslin ccc posets restricted to small models, this to guarantee that each cardinal invariant of the left gets its desired value. Though classical techniques from [JS90; Bre91] can be used, the main issue is to guarantee that $\mathfrak{b}$ does not get larger than desired. The reason is that restrictions of $\mathbb{E}$ are used (to increase $\mathrm{non}(\mathcal{M})$) along the iteration, and such restrictions may add dominating reals by a result of Pawlikowski [Paw92]. Hence, chains of ultrafilters on ω are used to guarantee that no dominating reals are added, even more, to guarantee that the iteration is θ_2-Fr-Knaster. To achieve this, the following is required.

(P1) $2^{\theta_2} \geq \lambda$, so that at most θ_2-sequences of ultrafilters are enough (by [EK65]).

(P2) $\theta^{\aleph_0} < \mu$ for any $\theta < \mu$.

(P3) The chains of ultrafilters and the iteration are constructed simultaneously by recursion.

Now, to prove Theorem E we need to additionally separate $\operatorname{cov}(\mathcal{M})$ and $\mathfrak{d}$, which lie on the right side of Cichoń's diagram, while separating all the left side. Mejia [Mej13] has shown that Blass's and Shelah's [BS89] method of matrix iterations works to separate several cardinals on the left and right side simultaneously, which we use to produce a method of matrix iterations with matrices of ultrafilters to extend the method from [GMS16]. Concretely, we introduce the concept of *$<\kappa$-uf-extendable matrix iteration* (see Definition 2.3.1) and prove the following result.

Theorem F (Theorem 2.3.3). *If κ is an uncountable regular cardinal then every $<\kappa$-uf-extendable matrix iteration is κ-*Fr*-Knaster.*

In order to define this type of matrix iterations, we required to generalize the notion of Fr-linked as follows. When F is a free filter on ω, $\mathbb{P}$ is a poset and $Q \subseteq \mathbb{P}$, we say that Q is *F-linked* if, for any sequence $\langle p_n : n < \omega \rangle$ of members of Q, there is some $q \in \mathbb{P}$ forcing that $\{n < \omega : p_n \in \dot{G}\}$ is F-positive (note that this is the same as Fr-linked when F is the Frechet filter). In the natural way, the notions *μ-F-linked* and *θ-F-Knaster* are defined for posets. We also say that $Q \subseteq \mathbb{P}$ is *uf-linked* if Q is F-linked for every free filter F (equivalently, for every non-principal ultrafilter); say that $\mathbb{P}$ is *μ-uf-linked* if it is the union of $\leq\mu$-many uf-linked subsets; and $\mathbb{P}$ is *θ-uf-Knaster* if every subset of $\mathbb{P}$ of size θ contains a uf-linked subset of the same size.[5]

A curious fact proved in [Mej19, Lemma 5.5] (see Lemma 2.1.2) is that, for ccc posets, the notions Fr-linked and uf-linked are equivalent, which means that the notions above are not generalizations in the context of ccc. However, the notion μ-uf-linked (for $\mu < \theta_2$) is implicitly used to construct the chains of ultrafilters in [GMS16], and it is also necessary to construct matrices of ultrafilters along an uf-extendable matrix iteration. For short, a $<\kappa$-uf extendable matrix iteration produces a FS iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \pi \rangle$ (at the top of the matrix) of κ-cc posets where each iterand $\dot{\mathbb{Q}}_\alpha$ is μ_α-uf-linked *with respect* to a complete subposet of $\mathbb{P}_\alpha$ (lying below in the matrix) for some $\mu_\alpha < \kappa$ (but not necessarily μ_α-uf-linked with respect to $\mathbb{P}_\alpha$).

The most surprising fact about our method is that it does not rely on conditions like (P1)-(P3), e.g., the matrix iteration can be defined before considering any matrix of ultrafilters, and no restriction on the amount of matrices of ultrafilters is required. For each quite uniform countable Δ-system $\langle p_n : n < \omega \rangle$ we can construct a matrix of ultrafilters along the matrix iteration and a condition q forcing that $\{n < \omega : p_n \in \dot{G}\}$ is infinite, which will be enough to guarantee that the construction is κ-Fr-Knaster.

The following constellation can also be proved by our method.

Theorem G (Theorem 2.4.2). *If $\theta_0 \leq \theta_1 \leq \mu \leq \nu$ are uncountable regular cardinals and λ is a cardinal such that $\lambda^{<\theta_1} = \lambda$, then there is a ccc poset that forces (see Figure 7)*

$$\operatorname{add}(\mathcal{N}) = \theta_0 \leq \mathfrak{b} = \mathfrak{a} = \theta_1 \leq \operatorname{cov}(\mathcal{N}) = \operatorname{non}(\mathcal{M}) = \mu$$
$$\leq \operatorname{cov}(\mathcal{M}) = \operatorname{non}(\mathcal{N}) = \nu \leq \mathfrak{d} = \mathfrak{c} = \lambda.$$

Theorem G shows (without using large cardinals) the consistency of $(\mathrm{A2})_{\mathcal{M}}$. The statement $(\mathrm{A1})_{\mathcal{N}}$ is also forced, though its consistency was already proved in [Mej13]. On the other hand, Brendle [Bre19b] introduced a new forcing method called *shattered iterations* to prove the consistency of $\aleph_1 < \operatorname{cov}(\mathcal{M}) = \operatorname{non}(\mathcal{N}) < \operatorname{non}(\mathcal{M}) = \operatorname{cov}(\mathcal{N})$. This led to get the consistency of $(\mathrm{A2})_{\mathcal{N}}$. The consistency of $(\mathrm{A1})_{\mathcal{M}}$ is still open.

Joint with Mejía [CM19], we produced a ccc poset, via a matrix iteration, that forces $\operatorname{add}(\mathcal{I}_f) < \operatorname{cov}(\mathcal{I}_f) < \operatorname{non}(\mathcal{I}_f) < \operatorname{cof}(\mathcal{I}_f)$ for any f above some fixed f^*, which solves $(\mathrm{A1})_{\mathcal{I}_f}$ but not for all f at the same time. Now, thanks to Theorem G, we can prove the consistency of $\operatorname{add}(\mathcal{I}_f) < \operatorname{cov}(\mathcal{I}_f) < \operatorname{non}(\mathcal{I}_f) < \operatorname{cof}(\mathcal{I}_f)$ for any f, i.e., $(\mathrm{A1})_{\mathcal{I}_f}$ for all f.

[5] In general, these notions are not equivalent to "μ-F-linked (resp. θ-F-Knaster) for every free filter F on ω".

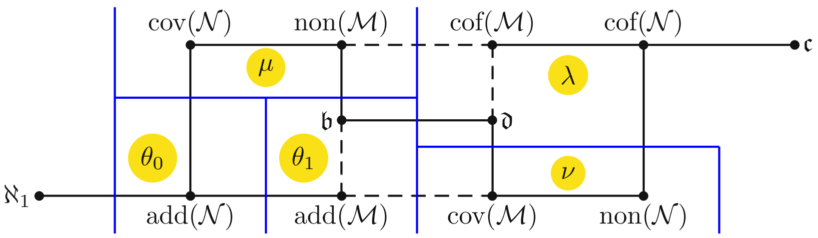

Figure 7: Separation of the cardinals associated with $\mathcal{M}$ and $\mathcal{N}$.

Theorem H (Corollary 2.4.5)**.** *There is a ccc poset that forces* $\mathrm{add}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ *for any increasing* $f \in \omega^\omega$*.*

The consistency of $(\mathrm{A2})_{\mathcal{I}_f}$ for some f is still known.

Finally, Boolean ultrapowers can be applied to the poset constructed for Theorem E to weaken the large cardinal hypothesis of ($\bigstar_3$).

Theorem I (Theorem 2.4.6)**.** *Assuming three strongly compact cardinals, there is a ccc poset that forces*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}.$$

Result ($\bigstar_3$) requires further hypotheses, for example, GCH is assumed in the ground model, the cardinals on the left side of Cichoń's diagram cannot be successors of cardinals of countable cofinality, and the value for $\mathfrak{b}$ should be a successor. These assumptions can be omitted for Theorem I except of GCH that can be weakened substantially.

Theorem E was a new result in the sense that no large cardinals are used to prove it, and it is another example of Cichoń's diagram divided into 7 values without using large cardinals. See more examples in [FFMM18; Mej19] for 7 values.

The consistency of ($\bigstar_3$) and ($\bigstar_4$) was proved in [GKMS19] without using large cardinals. Although Theorem E, G, H and I are covered by this new result, our methods are different, and in particular Theorem E may be useful as a starting point for further separation results for Cichoń's diagram. A more detailed discussion is provided in Section 2.5.

Cardinal characteristics associated with the strong measure zero ideal

The main results of this section will be developed throughout Chapter 3, which corresponds to the publications [CMR21] (joint with D. Mejía and I. Rivera-Madrid) and [Car21].

The consistency results about the cardinal characteristics of $\mathcal{SN}$ have been studied since it was discovered that the Borel conjecture is independent of ZFC. In this direction, Goldstern, Judah, and Shelah [GJS93] used a countable support (CS) iteration of proper forcings to prove the consistency of $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$, and with finite support (FS) iterations of ccc forcings Pawlikowski [Paw90] constructed a model where $\mathrm{cov}(\mathcal{SN}) < \mathrm{add}(\mathcal{M})$. Afterwards, Yorioka [Yor02] proved the consistency of $\mathrm{cof}(\mathcal{SN}) > \mathfrak{c}$, while $\mathrm{cof}(\mathcal{SN}) < \mathfrak{c}$ follows from CH.

We know that $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{N})$, however, due to the consistency results above, no other inequality between $\mathrm{add}(\mathcal{SN})$, $\mathrm{non}(\mathcal{SN})$ and another cardinal in Cichoń's diagram can be proved. On the other hand, unsolved problems about $\mathrm{cov}(\mathcal{SN})$ and $\mathrm{cof}(\mathcal{SN})$ still remain.

Question J. *(a) Is there a classical cardinal characteristics of the continuum (different from $\mathfrak{c}$ and $\mathrm{cof}(\mathcal{SN})$) that is an upper bound of $\mathrm{cov}(\mathcal{SN})$? In particular, is $\mathrm{cov}(\mathcal{SN}) \leq \mathrm{cof}(\mathcal{N})$?*

(b) Is there a classical cardinal characteristics of the continuum (different from the obvious ones, $\mathrm{cov}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{M})$) that is a lower bound of $\mathrm{cof}(\mathcal{SN})$? In particular, is $\mathrm{add}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{SN})$? Is $\mathrm{cof}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{SN})$?

In this work, we answer Question J(a) in the negative, that is, we show that $\mathrm{cov}(\mathcal{SN})$ is consistently larger than $\mathrm{cof}(\mathcal{N})$ and even larger than any classical cardinal invariant of the continuum (like the *almost disjointness number* $\mathfrak{a}$, the *independence number* $\mathfrak{i}$ and the *ultrafilter number* $\mathfrak{u}$, which are maximal among classical cardinal invariants of the continuum that could be below $\mathfrak{c}$, see Definition 1.3.5). The answer is a direct consequence of the following theorem.

Theorem K (Theorem 3.1.13)**.** *In Sacks model,* $\mathrm{cov}(\mathcal{SN}) = \aleph_2$.

As a consequence of Theorem K, we partially answer $(\mathrm{A2})_{\mathcal{SN}}$, more precisely, we get the first consistency result where more than two cardinal characteristics associated with $\mathcal{SN}$ are pairwise different. Concretely,

Theorem L (Theorem 3.1.14)**.** *Assume* CH *and that λ is an infinite cardinal such that $\lambda^{\aleph_1} = \lambda$. Then, there is a cofinality preserving poset that forces*

$$\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \aleph_1 < \mathrm{cov}(\mathcal{SN}) = \mathfrak{c} = \aleph_2 \text{ and } \mathrm{cof}(\mathcal{SN}) = \lambda$$

For this proof, we first add λ-many ω_1-Cohen reals and, afterwards, perform a CS iteration of Sacks forcing of length $\aleph_2$ to force $\mathrm{cov}(\mathcal{SN}) = \aleph_2$. This later is possible thanks to Theorem K. In terms of ideals, this implies that $\mathcal{SN} \subseteq s^0$ where $s^0 = \{X \subseteq 2^\omega : \forall p \in \mathbb{S} \exists q \leq p([q] \cap X = \emptyset)\}$ is the *Marczewski's ideal* (originally defined in [Mar35]) and $\mathbb{S}$ denotes *Sacks forcing*, so $\mathrm{cov}(s^0) \leq \mathrm{cov}(\mathcal{SN})$.[6] We also use Yorioka's characterization of $\mathrm{cof}(\mathcal{SN})$ ([Yor02], Theorem N in this work).

To continue this line of research, we get another result where more than two cardinal characteristics associated with $\mathcal{SN}$ are pairwise different. This partially solves $(\mathrm{A1})_{\mathcal{SN}}$. Concretely,

Theorem M (Theorem 3.2.18)**.** *Let $\kappa \leq \lambda$ be regular uncountable cardinals where $\kappa^{<\kappa} = \kappa$, $\lambda^{<\lambda} = \lambda$ and let λ_1, λ_2 be cardinals such that $\lambda \leq \lambda_1 = \lambda_1^{\aleph_0}$, and $\lambda < \lambda_2 = \lambda_2^{\lambda}$. Then there is a cofinality preserving poset that forces*

$$\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) = \kappa \leq \mathrm{non}(\mathcal{SN}) = \lambda < \mathrm{cof}(\mathcal{SN}) = \lambda_2 \text{ and } \mathfrak{c} = \lambda_1$$

Note that we did not specify which of λ_1 and λ_2 is larger, both variants are possible.

There are two key concepts in the proof of Theorem M. First is the dominating number of $\langle S^\lambda, \leq \rangle$, denoted by $\mathfrak{d}_S^\lambda$, where $\leq$ is interpreted pointwise and $\langle S, \leq_S \rangle$ is a directed preorder (see Example 1.3.10), and the second are Yorioka ideals.

Yorioka ideals play an important role in the following characterization of $\mathrm{cof}(\mathcal{SN})$, namely:

Theorem N ([Yor02, Thm. 3.9])**.** *If $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \lambda$ for all increasing $f \in \omega^\omega$, i.e,* minadd $=$ supcof, *then $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$ (the dominating number of λ^λ, see Definition 1.3.12) .*

To prove Theorem N, Yorioka constructed a dominating family $\langle f_\alpha : \alpha < \lambda \rangle$ in ω^ω along with a *matrix* $\langle A_\alpha^\beta : \alpha, \beta < \lambda \rangle$ of subsets of the Cantor space 2^ω satisfying the following properties:

[6] Also $\mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(s^0)$, but $\mathrm{non}(s^0) = \mathfrak{c}$ because $[2^\omega]^{<\mathfrak{c}} \subseteq s^0$.

- $\forall \alpha, \beta < \lambda \, (A_\alpha^\beta \subseteq 2^\omega$ is a dense G_δ set and $A_\alpha^\beta \in \mathcal{I}_{f_\alpha})$;
- $\forall \alpha, \beta, \beta' < \lambda \, (\beta \leq \beta' \to A_\alpha^\beta \subseteq A_\alpha^{\beta'})$;
- $\forall \alpha < \lambda \, \forall A \in \mathcal{I}_{f_\alpha} \, \exists \beta < \lambda \, (A \subseteq A_\alpha^\beta)$; and
- $\forall \alpha < \lambda \, \forall B \in \mathcal{I}_{f_\alpha} \, (\alpha > 0 \to \bigcap_{\gamma<\alpha} A_\gamma^0 \smallsetminus B \neq \varnothing)$.

This gives a Tukey equivalent between $\langle \mathcal{SN}, \subseteq \rangle$ and $\langle \lambda^\lambda, \leq \rangle$, which implies $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$ (for Tukey equivalence, see Definition 1.3.15 and Theorem 1.3.16).

In this work, we introduce the notion of λ-*dominating system* (see Definition 3.2.7), which lets us then refine Theorem N by providing bounds to $\mathrm{cof}(\mathcal{SN})$ with hypotheses weaker than $\mathrm{minadd} = \mathrm{supcof}$. We use this notion to study the relationship between $\mathfrak{d}_S^\lambda$ and the cofinality of $\mathcal{SN}$.

Theorem O (Theorem 3.2.9 and 3.2.13)**.** *If there is some λ-dominating system on a directed preorder $\langle S, \leq_S \rangle$ then*

1. *$\mathcal{SN}$ is Tukey below $\langle S^\lambda, \leq \rangle$.*
2. *If $\kappa \leq \lambda \leq \mathrm{minnon}$ and $\langle S, \leq_S \rangle = \langle \kappa \times \lambda, \leq \rangle$, then $\langle \lambda^\lambda, \leq \rangle$ is Tukey below $\mathcal{SN}$, in particular $\mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN})$ and $\mathrm{add}(\mathcal{SN}) \leq \lambda$.*

This theorem implies Theorem N because a λ-dominating system on λ can be obtained when $\lambda = \mathrm{minadd} = \mathrm{supcof}$, see details in Corollary 3.2.15. On the other hand, we can force in Theorem M that there is a λ-dominating system on $\langle \kappa \times \lambda, \leq \rangle$ while $\mathrm{add}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for all f.

To prove Theorem M we will perform a forcing matrix iteration to add a λ-dominating system on $\langle \kappa \times \lambda, \leq \rangle$. Concretely, we go through the following steps:

(P1) We will force $\mathfrak{c} = \lambda_1$ and $\mathfrak{d}_{\kappa\times\lambda}^\lambda = \mathfrak{d}_\lambda = \lambda_2$ by adding λ_2-many λ-Cohen reals and λ_1-many ω-Cohen reals (in that order).

(P2) In this generic extension, we construct a ccc forcing matrix iteration of height λ and length $\lambda\kappa$ (ordinal product). Along the matrix iteration, we construct a dominating family $\langle f_\gamma : \gamma < \lambda \rangle$ along with a λ-dominating system on $\langle \kappa \times \lambda, \leq \rangle$. Theorem O implies that the matrix iteration forces $\mathrm{cof}(\mathcal{SN}) = \lambda_2$. For the construction, we use restricted localization forcing.

(P3) The constructed matrix iteration forces $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \lambda$ and $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \kappa$, so $\kappa \leq \mathrm{add}(\mathcal{SN})$ and $\mathrm{non}(\mathcal{SN}) = \lambda$ because $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{N})$. On the other hand, since the matrix iteration results in a ccc FS finite support iteration of length with cofinality κ, $\mathrm{cov}(\mathcal{SN}) \leq \kappa$.

The main point of using a matix iteration is to produce the λ-dominating system in the generic extension. Although this could be achieved by a regular ccc FS iteration, many technical issues that would appear are avoided when constructing the iteration with a matrix structure instead.

A new result from the Creature World

The main results of this subsection will be developed throughout Chapter 4, which corresponds to the publication [CKM21] (joint with L. Klausner and D. Mejía).

Klausner and Mejía [KM21] used a CS (countable support) product of limsup creature-like forcings to prove that, consistently, uncountably many uniformity numbers of Yorioka ideals and uncountably many anti-localization numbers $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ are pairwise different. Our research is a continuation of this work and solves several instances from Main Problem B. To achieve this, we propose and prove the following result.

Theorem P. *Assume* CH. *Then there is an* $\aleph_2$*-cc* ω^ω*-bounding proper poset which forces that there are continuum many pairwise different cardinal characteristics of each one of the following six types:* $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$, $\mathrm{non}(\mathcal{I}_f)$ *and* $\mathrm{cof}(\mathcal{I}_f)$.

In the direction of this research topic, Goldstern and Shelah [GS93] used a CS product of *limsup tree* posets to force that $\aleph_1$-many cardinals of the type $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ are pairwise different, answering a question of Blass [Bla93]. Here "limsup" trees refers to trees like in Sacks' and Miller's perfect trees posets, where splitting can be delayed from any node. This result was improved by Kellner [Kel08] who showed that, consistently, continuum-many cardinals of the type $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ are pairwise different. Later, Kellner and Shelah [KS09] included $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ in this line of research: they introduced a CS product-like *liminf* forcing construction with *decisive creatures with halving* to force that there are $\aleph_1$-many pairwise different cardinals of each of the types $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, which was improved in [KS12] to continuum-many using the same forcing technique, but including creatures that determine the parameters b and h generically. Here "liminf" means that splitting becomes recurrent from some point, like trees in Laver's poset. We remark here that the construction of the suitable decisive creatures in these latter two references is very complex, which makes it rather difficult to understand the intuition behind the main forcing construction.

Concerning Yorioka ideals, Kamo and Osuga [KO14] discovered connections between these ideals and anti-localization cardinals, concretely, $\mathrm{cov}(\mathcal{I}_f)$ lies between two cardinals of the form $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ (and dually, $\mathrm{non}(\mathcal{I}_f)$ lies between two cardinals of the form $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$). This is their starting point to force, via a FS (finite support) iteration of ccc posets, infinitely many pairwise different cardinal characteristics of each of the types $\mathrm{cov}(\mathcal{I}_f)$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, and even continuum many under the existence of a weakly inaccessible cardinal.

These techniques inspired Brendle and the Mejía [BM14] to force, via FS iterations of ccc posets, infinitely many pairwise different cardinal characteristics of the type $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, even continuum many under the existence of a weakly inaccessible cardinal.[7] Much later, joint with Mejía [CM19], we combined the methods of [KO14] and [BM14] to force infinitely many pairwise different cardinal characteristics of each of the types $\mathrm{cov}(\mathcal{I}_f)$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, even continuum many under the existence of a weakly inaccessible cardinal. In this framework, it is not possible to force continuum-many different cardinals of these three types without using a weakly inaccessible: usually FS iterations of ccc posets are constructed with a length of uncountable cofinality, and this cofinality is forced between $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$; on the other hand, all these cardinal characteristics are below $\mathrm{non}(\mathcal{M})$ (see Figure 3 and 5), so if continuum-many different cardinals are forced, then the cofinality of the length of the iteration must be weakly inaccessible (and so it is forced $\mathrm{non}(\mathcal{M}) = \mathfrak{c}$ weakly inaccessible).

[7] The original motivation in [BM14] was to force infinitely many pairwise different cardinal characteristics associated with Rothberger gaps in F_σ ideals on ω, but these turned out to be connected with the localization cardinals of the form $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$.

Very recently, Klausner and Mejía [KM21] refined Kamo's and Osuga's connections between Yorioka ideals and anti-localization cardinals (see e.g. Lemma 4.2.1) to prove that, consistently, there are $\aleph_1$-many cardinal characteristics of each of the types $\mathrm{non}(\mathcal{I}_f)$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$. This leaves open the question on whether there could be continuum-many different cardinals of each of the forms $\mathrm{non}(\mathcal{I}_f)$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$, and also of the form $\mathrm{cov}(\mathcal{I}_f)$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ without using inaccessible cardinals (see [KM21, Questions C and E]). We answer all these questions in the positive, even more, as we claim in Theorem Q, we construct a single model where there are continuum-many different cardinals for each of the six types discussed.

We mainly work with a creature forcing construction as in [GK21] mixed with liminf creatures for parameters as in [KS12]. We review the evolution of these techniques for motivation. To force continuum many $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, Kellner and Shelah [KS12] proposed a mixed-limit creature forcing construction (i.e. one that mixes limsup and liminf creatures) that takes care of the continuum many parameters $\langle (b_\xi, h_\xi) : \xi < \mu \rangle$ while forcing $\mathfrak{c} = \mu$ and (pairwise different) values for $\mathfrak{b}^{\mathrm{aLc}}_{b_\xi,h_\xi} = \mathfrak{d}^{\mathrm{Lc}}_{b_\xi,h_\xi}$ (although liminf is only used in their applications). Two very important features of liminf creatures that come from [KS09] are halving and decisiveness: *halving*, along with the more common notion of *bigness* (in the context of creatures), is essential to prove properness and continuous reading, while, *decisiveness* is used to prove that the construction indeed increases $\mathfrak{b}^{\mathrm{aLc}}_{b_\xi,h_\xi}$ to the desired value. As mentioned earlier, the construction of decisive creatures is quite elaborated, making it hard to keep track of the intuition connecting the construction to what is to be forced.

A bit later, A. Fischer, Goldstern, Kellner and Shelah [FGKS17] considerably simplified the mixed-limit creature forcing framework by allowing *subatomic creatures* in the construction of liminf atomic creatures, ensuring decisiveness (implicitly, because the notion is not directly used) and a very intuitive construction of the creatures. This framework was used to force a constellation in Cichoń's diagram where five cardinal characteristics on the right side are pairwise different, namely $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$, $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$ and $\mathfrak{c}$ (see Theorem 4.9.1 and Figure 4.3).

Goldstern and Klausner [GK21] improved [FGKS17] to force the same constellation of Cichoń's diagram plus $\aleph_1$-many different localization cardinals of the form $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$. They managed to separate the components of the creatures and present the forcing construction as a CS product of one (large) liminf part to increase $\mathrm{non}(\mathcal{M})$, one large limsup part to increase the continuum (similar to a CS product of Sacks posets, original from [FGKS17]), and several single limsup posets to increase $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$ and the localization cardinals.[8]

In this work we combine the methods of [GK21] with a liminf creature forcing as in [KS12] for parameters, and construct a large creature forcing to prove Theorem P. This construction is not a CS product in the strict sense, but it looks like one.

The first component, called the *pr-part*, is a large liminf creature forcing, similar to the liminf part of the forcing in [GK21], that adds *blocks* of parameters.[9] A block is a finite sequence $\vec{\nu} = \langle d, h, g, b, f, a \rangle$ of increasing functions satisfying the requirements in Definition 4.2.4, whose important feature is that it satisfies the diagram of inequalities in Figure 4 (see Lemma 4.2.5).

With the notion of block we can restate Theorem P more precisely.

[8]In fact, the subatomic creatures corresponding to $\mathrm{non}(\mathcal{M})$ in [FGKS17; GK21] come from those defined in [KS09; KS12] to increase $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ (which we use in the current paper to add parameters); and the atomic creatures corresponding to $\mathrm{non}(\mathcal{N})$ actually increases some anti-localization cardinal $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ (that lies below $\mathrm{non}(\mathcal{N})$). We use an equivalent formulation of these atomic creatures in our framework.

[9]Although this can be done with limsup forcing, it may not (easily) ensure a property called "separated support" (from [KS09; KS12]), which we require to separate the continuum many cardinal characteristics.

Theorem Q. *Assume CH. Let $\mu = \mu^{\aleph_0}$ be an infinite cardinal, S^{pr} a set of size μ and, for $i \in S^{\mathrm{pr}}$, let $\kappa_i = \kappa_i^{\aleph_0}$ be a cardinal. Then there is a proper ω^ω-bounding $\aleph_2$-cc poset $\mathbb{Q}$ which forces, for all $i \in S^{\mathrm{pr}}$,*

(I) $\mathfrak{c} = \mu$,

(II) *blocks* $\vec{\nu}_i^{\mathrm{lc}} = \langle d_i^{\mathrm{lc}}, h_i^{\mathrm{lc}}, g_i^{\mathrm{lc}}, b_i^{\mathrm{lc}}, f_i^{\mathrm{lc}}, a_i^{\mathrm{lc}} \rangle$ *and* $\vec{\nu}_i^{\mathrm{al}} = \langle h_i^{\mathrm{al}}, h_i^{\mathrm{al}}, g_i^{\mathrm{al}}, b_i^{\mathrm{al}}, f_i^{\mathrm{al}}, a_i^{\mathrm{al}} \rangle$,

(III $\mathfrak{b}^{\mathrm{Lc}}_{b_i^{\mathrm{lc}}, h_i^{\mathrm{lc}}} = \mathrm{cov}(\mathcal{I}_{f_i^{\mathrm{lc}}}) = \mathfrak{b}^{\mathrm{aLc}}_{a_i^{\mathrm{lc}}, d_i^{\mathrm{lc}}} = \mathfrak{d}^{\mathrm{Lc}}_{a_i^{\mathrm{lc}}, d_i^{\mathrm{lc}}} = \kappa_i$ *(the upper part of Figure 4), and*

(IV) $\mathfrak{d}^{\mathrm{aLc}}_{a_i^{\mathrm{al}}, h_i^{\mathrm{al}}} = \mathrm{non}(\mathcal{I}_{f_i^{\mathrm{al}}}) = \mathfrak{d}^{\mathrm{Lc}}_{a_i^{\mathrm{al}}, h_i^{\mathrm{al}}} = \kappa_i$ *(the lower part of Figure 4).*

To prove this theorem, we use the pr-part to add the blocks in (II) (while forcing (I)), and we attach to the pr-part one (large) liminf creature forcing to increase $\mathfrak{b}^{\mathrm{Lc}}_{b_i^{\mathrm{lc}}, h_i^{\mathrm{lc}}}$ for all i, and several individual limsup creature forcings to increase $\mathfrak{d}^{\mathrm{aLc}}_{a_i^{\mathrm{al}}, h_i^{\mathrm{al}}}$. Although the pr-part can be isolated, the other components depend on the "possibilities" determined by the pr-part. The subatomic creatures used for these other components are the same ones used in [KM21] to manipulate $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ (which also works for $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ in the liminf context).[10]

To force $\mathfrak{d}^{\mathrm{Lc}}_{a_i^{\mathrm{lc}}, d_i^{\mathrm{lc}}} \leq \kappa_i$ and $\mathfrak{d}^{\mathrm{Lc}}_{a_i^{\mathrm{al}}, h_i^{\mathrm{al}}} \leq \kappa_i$, we use similar arguments as in the cited works on creature forcing: we force that the set of slaloms that can be continuously read only using a fixed set of indices of size κ_i is a witness of the cardinal characteristic. This is derived from a property we call (a_i, d_i)*-bounding over* $\{i\} \cup S_i^{\mathrm{lc}} \cup S_i^{\mathrm{al}}$, which is presented in Lemma 4.8.12.

Since our forcing construction is still quite complex, we need to motivate and explain more detailed features while building its different components. This is taken care of throughout Chapter 4.

[10] To prove Theorem Q we expected to add only one block $\vec{\nu}_i = \langle d_i, h_i, g_i, b_i, f_i, a_i \rangle$ for each $i \in S^{\mathrm{pr}}$, and force that all cardinals in Figure 4 corresponding to $\vec{\nu}_i$ equal κ_i, by using (besides the pr-part) one large liminf creature construction to increase $\mathfrak{b}^{\mathrm{Lc}}_{a_i, d_i}$ for all i (i.e. without limsup forcings). However, we could not find the right construction of the parameters that allows this.

1

PRELIMINARIES

Mathematicians who are not set theorists generally consider "null" as senior to "meagre", that is, as a more important case; set theorists inversely, as set-theoretically Cohen reals are much more manageable than random reals . . .

– Saharon Shelah

The main purpose of this chapter is to give an overview of some important definitions and results that will be used in this thesis.

1.1 Some notation

Most of our notation is quite standard and compatible with that of classical textbooks on Set Theory (see e.g. [Jec03; Kun11]).

Set $\omega^{\uparrow\omega} := \{d \in \omega^\omega : d(0) = 0 \text{ and } d \text{ is increasing}\}$. For any set A, id_A denotes the identity function on A. Denote $\mathrm{id} := \mathrm{id}_\omega$. If z is an ordered pair, z_0 and z_1 denote the first and second component of z respectively. For sets X and Y, Y^X denotes the set of functions from X into Y. Given a cardinal number κ, $[X]^{<\kappa}$ denotes the set of all subset of X of size $< \kappa$. Likewise, we define $[X]^{\leq\kappa}$ and $[X]^\kappa$.

Given a formula ϕ, $\forall^\infty n\,\phi$ means that all but finitely many natural numbers satisfy ϕ; $\exists^\infty n\,\phi$ means that infinitely many natural numbers satisfy ϕ.

For $x, y : \omega \to \mathrm{On}$, we write

- $x \leq^* y$ if $\forall^\infty n\,(x(n) \leq y(n))$, which is read x *is dominated by* y. Likewise, $x <^* y$ is defined.
- $x \neq^\infty y$ if $\forall^\infty n(x(n) \neq y(n))$, which is read x *is eventually different to* y.
- $x \ll y$ if $\forall k < \omega \forall^\infty n(x(n^k) \leq y(n))$.

When $b = \langle b(n) : n \in \omega\rangle$ is a sequence of non-empty sets and $h : \omega \to \omega$, denote

$$\prod b := \prod_{n<\omega} b(n),\ \mathcal{S}(b, h) := \prod_{n<\omega} [b(n)]^{\leq h(n)},\ \text{and } \mathrm{seq}_{<\omega}(b) := \bigcup_{n<\omega} \prod_{i<n} b(i).$$

For each $\sigma \in \mathrm{seq}_{<\omega}(b)$ define

$$[s] := [s]_b := \{x \in \prod b : s \subseteq x\}.$$

and for each $\sigma \in (\mathrm{seq}_{<\omega}(b))^\omega$ define

- $[\sigma]_\infty := [\sigma]_{b,\infty} = \{x \in \prod b : \exists^\infty n < \omega(\sigma(n) \subseteq x)\}$.
- $\mathrm{ht}_\sigma \in \omega^\omega$ by $\mathrm{ht}_\sigma(i) := |\sigma(i)|$ (see Figure 1.1).

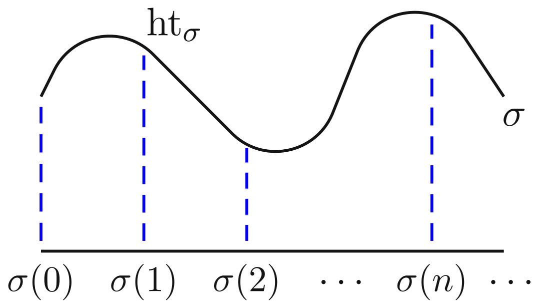

Figure 1.1: The function ht_σ represent the length of each value of σ, for σ in $(\mathrm{seq}_{<\omega}(b))^\omega$.

As a topological space, $\prod b$ is endowed with the product topology, where each $b(n)$ has the discrete topology. Standard cases are:

- The Cantor space 2^ω, when $b(n) = 2$ for all n, and
- The Baire space ω^ω, when $b(n) = \omega$ for all n.

The standard basis for the topology on $\prod b$ consists of all the sets of the form $[s]$ ($[s]$ is a clopen set, that is, open and closed) for $s \in \mathrm{seq}_{<\omega}(b)$. Therefore $\prod b$ is a zero dimensional space (recall that space is *zero dimensional* if it has a basis that consists of clopen sets). Even more, $\prod b$ is a perfect space when $\exists^\infty n(|b(n)| \geq 2)$, and $\prod b$ is a compact space for any $b \in \omega^\omega$.

Operations and relations between functions from ω into the ordinals are interpreted pointwise. For example, if b and c are such functions, $b \cdot c$ denotes the pairwise ordinal product of both functions, and $b < c$ indicates that $b(n) < c(n)$ for any $n < \omega$. Also, constant objects may be interpreted as constant functions with domain ω, for instance, the ω in $\mathcal{S}(\omega, h)$ is understood as the constant function ω.

Given a non-empty set A we say that $F \subseteq \mathcal{P}(A)$ is a *filter* on A if it fulfills the following:

- $A \in F$ and $\emptyset \notin F$,
- $X \cap Y \in F$ for any $X, Y \in F$, and
- for any $X, Y \in F$, if $Y \subseteq X$ and $Y \in F$ then $X \in F$.

Let F be a filter on ω and let $\mathrm{Fr} := \{x \subseteq \omega : |\omega \smallsetminus x| < \aleph_0\}$ be the *Frechet filter*. Say that F is *free* if $\mathrm{Fr} \subseteq F$. A set $x \subseteq \omega$ is *F-positive* if it intersects every member of F. Denote by F^+ the family of F-positive sets. Note that $x \in \mathrm{Fr}^+$ iff x is an infinite subset of ω.

Let X be a nonempty set. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal* if fulfills:

(i) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,

(i) if $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$, and

(iii) $[X]^{<\omega} \subseteq \mathcal{I}$ and $X \notin \mathcal{I}$.

An ideal $\mathcal{I}$ is called a *σ-ideal* if it is closed under countable unions.

We review the following notation about trees. Say that $T \subseteq \omega^{<\omega}$ is a *tree* if $\langle\,\rangle \in T$ and $\forall t \in T \forall s \subseteq t(s \in T)$. Denote by $\mathrm{Lv}_n(T) := T \cap \omega^n$ the *n-th level of* T and, for any $s \in T$, let $T^{[s]} := \{t \in T : s \subseteq t \text{ or } t \subseteq t\}$, which is also a tree. Denote by $[T] =: \{x \in \omega^\omega : \forall n < \omega(x{\restriction}n \in T)\}$ the set of infinite branches of T.

Let $T \subseteq \omega^{<\omega}$ be a tree. Say that $s \in T$ is a *splitting node of* T if $s^\frown\langle i\rangle, s^\frown\langle j\rangle \in T$ for some $i \neq j$. Denote by $\mathrm{spl}(T)$ the set of splitting nodes of T. For $n < \omega$, let $\mathrm{spl}_n(T)$ be the set of $s \in \mathrm{spl}(T)$ such that there are exactly n-many splitting nodes strictly below s. Given another tree $T' \subseteq \omega^\omega$, write $T' \subseteq_n T$ when $T' \subseteq T$ and there is some $m < \omega$ such that all the elements of $\mathrm{spl}_n(T)$ have length $< m$ and $T' \cap \omega^m = T \cap \omega^m$. Note that $T' \subseteq_{n+1} T$ implies $T' \subseteq_n T$, and that the relation $\subseteq_n$ is transitive.

1.2 Reals and ideals

1.2.1 Polish spaces and Lebesgue measure

Definition 1.2.1. Let $\langle X, \tau\rangle$ be a topological space. We say that $\langle X, \tau\rangle$ is a *Polish space* if it is completely metrizable and separable.

Classical examples of Polish spaces are the real line $\mathbb{R}$ with the usual topology, $\prod b$ when $|b(n)| \leq \aleph_0$ for all $n < \omega$, in particular the Cantor space 2^ω and the Baire space ω^ω. In practice, an uncountable Polish space is thought as space of reals. For instance, 2^ω is homeomorphic to the Cantor ternary set, and the Baire space ω^ω is homeomorphic to the space of the irrational numbers. From now on, we refer as *reals* to the members of any uncountable Polish space.

For a topological space X, we say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *σ-algebra* if

(i) $X \in \mathcal{A}$,

(ii) if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$, and

(iii) if $A_n \in \mathcal{A}$ for all $n \in \omega$, then $\bigcup_{n\in\omega} A_n \in \mathcal{A}$.

Let $\mathcal{B}(X)$ be the smallest σ-algebra containing all open subsets of X, which is called the *σ-algebra of the Borel subsets of X*.

Say that $A \subseteq X$ is *nowhere dense* (nwd) if $\mathrm{int}(\bar{A}) = 0$, and A is meager if it is the countable union of nwd sets. Let $\mathcal{M}(X)$ be the σ-ideal of meager subsets of X. When the space is clear from the context, we just write $\mathcal{M}$ to denote the ideal.

Given a σ-algebra $\mathcal{A}$ on $\mathcal{P}(X)$, say that $\mu : \mathcal{A} \to [0, +\infty]$ is a *measure* if it fulfills the following properties:

(a) $\mu(\varnothing) = 0$

(b) $\mu(\bigcup_{n<\omega} A_n) = \sum_{n<\omega} \mu(A_n)$ for any pairwise disjoint family $\{A_n : n < \omega\} \subseteq \mathcal{A}$

We say that $N \subseteq X$ is *μ-null* if there is a $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$. Denote by $\mathcal{N}(X, \mathcal{A}, \mu)$ the σ-ideal of μ-null subsets of X. When the space and the measure are clear from the context, we just write $\mathcal{N}$. We say that μ is a *probability measurable* if it is a measure such that $\mu(X) = 1$. A measure μ is *σ-finite* if $X = \bigcup_{n\in\omega} A_n$ where each A_n has finite measure. A measure μ is *continuous* if $\mu(\{x\}) = 0$ for all $x \in X$.

Let us conclude this subsection by introducing the Lebesgue measure on $\prod b$ when $b : \omega \to (\omega + 1) \smallsetminus \{0\}$. To do this first define, for $n \leq \omega$, the probability measure μ_n on the power set of n:

(i) when $n < \omega$, μ_n is the measure such that, for $k < n$, $\mu_n(\{k\}) = \frac{1}{n}$, and

(ii) when $n = \omega$, μ_ω is the measure such that, for $k < n$, $\mu_\omega(\{k\}) = \frac{1}{2^{k+1}}$.

Denote by $\mathbf{Lb}_b$ the product measure of $\langle \mu_{b(n)} : n < \omega \rangle$, so $\mathbf{Lb}_b$ is a continuous probability measure on the Borel σ-algabra of $\prod b$. To be more precise, $\mathbf{Lb}_b = \prod_{n<\omega} \mu_{b(n)} : \mathcal{B}(\prod b) \to [0, 1]$ is the unique measure on $\mathcal{B}(\prod b)$ such that, for any $s \in \text{seq}_{<\omega}(b)$, $\mathbf{Lb}_b([b]) = \prod_{i<|s|} \mathbf{Lb}_{b(i)}(\{s(i)\})$.

On the other hand, denote by $\mathbf{Lb}$ the Lebesgue measure on $\mathbb{R}$. Note that, for any open interval I, $\mathbf{Lb}(I)$ is the length of I.

1.2.2 The σ-ideal of strong measure zero sets

Definition 1.2.2. A set $X \subseteq \mathbb{R}$ has *strong measure zero* if, for every sequence $\langle \epsilon_n \rangle_{n\in\omega}$ of positive reals, there are open intervals $\langle I_n \rangle_{n\in\omega}$ such that $\mathbf{Lb}(I_n) \leq \varepsilon_n$ and $A \subseteq \bigcup_{n\in\omega} I_n$.

From the definition, it can be easily proved that each strong measure zero set has measure zero and they from a σ-ideal. Even more, no perfect set cannot have strong measure zero (see Corollary 1.2.9). This implies that the notions of measure zero and strong measure zero never coincide.

Definition 1.2.2 can be naturally generalized to any arbitrary metric space, simply using *balls* instead of intervals. For this, consider the following notation:

For a *metric space* $\langle X, d \rangle$, $x_0 \in X$ and $\varepsilon > 0$, the open ball around x_0 of radius ε is the subset

$$B_\varepsilon(x_0) := \{x \in X : d(x_0, x) < \varepsilon\}$$

of X. Therefore, the notion of strong measure zero can be generalized to metric space in the following way.

Definition 1.2.3. Let $A \subseteq X$. Say that A has *strong measure zero with respect to d* if for each sequence $\langle \varepsilon_n \rangle_{n<\omega}$ of positive reals there is some sequence $\langle x_n \rangle_{n<\omega}$ of elements of X such that $X \subseteq \bigcup_{n<\omega} B_{\varepsilon_n}(x_n)$. Denote by $\mathcal{SN}(\langle X, d \rangle)$ the family of all subsets of X with strong measure zero. We omit d when understood and simply write $\mathcal{SN}(X)$. We also write $\mathcal{SN}$ when the space X is understood.

The concept of strong measure zero, in general, may depend on the metric, but it is independent of the metric for locally compact Polish spaces.

Lemma 1.2.4. *Let X be a locally compact Polish space, and let d_0 and d_1 be two compatible metrics*[1]*. Then for every $A \subseteq X$, $A \in \mathcal{SN}(X, d_0)$ if only and if $A \in \mathcal{SN}(X, d_1)$.*

An analogous of Definition 1.2.2 applies to $\prod b$ when $b : \omega \to \omega \smallsetminus \{0\}$, however, one uses basic clopen sets instead of intervals. Working in $\prod b$ has the advantage that Definition 1.2.2 can be expressed in a "purely combinatorial" way, so looking ahead to Section 3.1 we use the following simple characterization of strong measure zero in $\prod b$ for combinatorial purposes, which is possible thanks to Lemma 1.2.4.

Lemma 1.2.5. *Let $b : \omega \to \omega \smallsetminus \{0\}$, let $A \subseteq \prod b$. Then $A \in \mathcal{SN}(\prod b)$ iff for every $f \in \omega^\omega$ there is some $\sigma \in (\text{seq}_{<\omega}(b))^\omega$ with $\text{ht}_\sigma = f$ such that $A \subseteq \bigcup_{i<\omega}[\sigma(i)]$.*

We conclude this subsection with the following practical characterization of $\mathcal{SN}(\prod b)$.

Lemma 1.2.6. *Let $b : \omega \to \omega \smallsetminus \{0\}$, $X \subseteq \prod b$ and let $D \subseteq \omega^\omega$ be a dominating family. Then $X \in \mathcal{SN}(\prod b)$ iff for every $f \in D$ there is some $\sigma \in (\text{seq}_{<\omega}(b))^\omega$ with $\text{ht}_\sigma = f$ such that $X \subseteq [\sigma]_\infty$.*

[1]This means that they generate the same topology of the Polish space.

1.2.3 Yorioka ideals

Yorioka ideals are defined as follows.

Definition 1.2.7 (Yorioka [Yor02]). For $f \in \omega^\omega$ define

$$\mathcal{I}_f := \{X \subseteq 2^\omega : \exists \sigma \in (2^{<\omega})^\omega (X \subseteq [\sigma]_\infty \text{ and } \mathrm{ht}_\sigma \gg f)\}.$$

Any family of the form $\mathcal{I}_f$ with f increasing is called a *Yorioka ideal*. When f is increasing $\mathcal{SN}(2^\omega) \subseteq \mathcal{I}_f \subseteq \mathcal{N}$.

Lemma 1.2.8 (Yorioka [Yor02]). *When f is increasing $\mathcal{I}_f$ is a σ-ideal . Even more, $\mathcal{SN}(2^\omega) = \bigcap\{\mathcal{I}_f : f \text{ increasing}\}$.*

The next lemma implies that no perfect set can have strong measure zero.

Corollary 1.2.9 ([Yor02, Lemma 3.7]). *Let A be a perfect subset of 2^ω. Then there some increasing $f \in \omega^\omega$ such that $A \notin \mathcal{I}_f$.*

1.3 Cardinal characteristics

In this section, we present some classical cardinal characteristic that are relevant for the main results of this thesis. For more details and the proofs of all the results of this section, see [BJ95; Bla10; CM19; KM21].

1.3.1 Some classical cardinal characteristics

Definition 1.3.1. Let $\mathcal{I}$ be an ideal on $\mathcal{P}(X)$ containing all the finite subsets of X. Define *the cardinal characteristics associated with* $\mathcal{I}$ by:

1. $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}$ *the additivity of* $\mathcal{I}$;
2. $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = X\}$ *the covering of* $\mathcal{I}$;
3. $\mathrm{non}(\mathcal{I}) := \min\{|A| : A \subseteq X \text{ and } A \notin \mathcal{I}\}$ *the uniformity of* $\mathcal{I}$;
4. $\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ is cofinal in } \langle \mathcal{I}, \subseteq \rangle\}$ *the cofinality of* $\mathcal{I}$.

For our applications, we are interested in the cardinal characteristics associated with $\mathcal{M}, \mathcal{N}$, $\mathcal{I}_f$, and $\mathcal{SN}$ for an uncountable Polish space. The following outcomes state that these cardinals of $\mathcal{M}$ and $\mathcal{N}$ do not depend on the chosen uncountable Polish space.

Theorem 1.3.2. *Let X, Y be perfect Polish spaces.*

(a) ([Kec95, Subsect. 8.F]) *$\mathcal{B}(X)/\mathcal{M}(X)$ and $\mathcal{B}(Y)/\mathcal{M}(Y)$ are isomorphic complete Boolean algebras.*

(b) ([Kec95, Subsect. 15.D]) *If $\Psi : \mathcal{B}(X)/\mathcal{M}(X) \to \mathcal{B}(Y)/\mathcal{M}(Y)$ is an isomorphism, then there exists a Borel isomorphism $f : Y \to X$ such that $\Psi([A]) = [f^{-1}[A]]$ for any $A \in \mathcal{B}(X)$.*

As an immediately consequence we get.

Corollary 1.3.3. *If X, Y are perfect Polish spaces then there exists a Borel isomorphism $f : Y \to X$ such that, for $A \subseteq X$, $A \in \mathcal{M}(X)$ iff $f^{-1}[A] \in \mathcal{M}(Y)$.*

Theorem 1.3.4 ([Kec95, Thm. 17.41]). *Let X be a Polish space and $\mu : \mathcal{B}(X) \to [0, 1]$ a continuous probability measure. Then, there exists a Borel isomorphism $f : X \to [0, 1]$ such that $\mu(f^{-1}[A]) = \mathbf{Lb}(A)$ for any $\in \mathcal{B}([0, 1])$.*

Let us notice that $\prod b$ when $b : \omega \to \omega \setminus \{0\}$ can be related to the interval $[0,1]$ through the one-to-one continuous function $F_b : \prod b \to [0,1]$ defined by

$$F_b(x) := \sum_{n<\omega} \frac{x(n)}{\prod_{i\leq n} b(i)}$$

Let us also notice that, when $b \not\leq^* 1$, instead of using strongly measure zero sets in $\prod b$, we may consider strongly measure zero sets in $[0,1]$, that is, this map preserves sets between $\mathcal{SN}(\prod b)$ and $\mathcal{SN}([0,1])$ via images and pre-images. Therefore, the value of the cardinal characteristics associated with $\mathcal{SN}$ do not depend on the space $\prod b$, neither on $[0,1]$ and $\mathbb{R}$.

We will be interested in the cardinal characteristics defined below:

Definition 1.3.5. (1) A family of functions $F \subseteq \omega^\omega$ is called *bounded* if it is eventually dominated by a single function, i.e there is a $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in F$. Otherwise F is *unbounded*. The *(un)bounding number* $\mathfrak{b}$, is defined as the least size of an unbounded family. On the other hand, a family D is *dominating* if every $g \in \omega^\omega$ is eventually dominated by a member of D. The *dominating number* $\mathfrak{d}$ is defined as the least size of a dominating family.

(2) A family $A \subseteq [\omega]^{\aleph_0}$ is said to be *almost disjoint*, abbreviated a.d., if the intersection of any two different members of A is finite. An almost disjoint family is called a *maximal almost disjoint family*, abbreviated *mad family*, if it is maximal under inclusion among a.d. families. The *almost disjointness number* $\mathfrak{a}$ is defined as the least size of an infinite mad family.

(3) A family $F \subseteq [\omega]^{\aleph_0}$ is called a *filter base* if it is closed under intersections. The *ultrafilter number* $\mathfrak{u}$ is defined as the least size of a filter base that generates a non-principal ultrafilter on ω.

(4) A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is an *independent family* (over ω) if for every pair $\mathcal{A}, \mathcal{B}$ of disjoint finite subsets of $\mathcal{F}$ the set $\bigcap \mathcal{A} \cap (\omega \setminus \cup \mathcal{B})$ is infinite. The *independence number* $\mathfrak{i}$ is defined as the minimum size of a maximal independent family of subsets of ω

Denote by $\mathcal{K}$ the σ-ideal generated by the compact subsets of the irrationals. Below we give the relationship between these cardinals. Proofs can be found in the cited references.

Theorem 1.3.6. *(i)* [BHH04] $\mathrm{cof}(\mathcal{M}) \leq \mathfrak{i}$.

(ii) [Bla10] $\mathfrak{b} \leq \mathfrak{a}, \mathfrak{u}$.

(iii) [Bla10] $\mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K}) = \mathfrak{b}$, *and* $\mathrm{cov}(\mathcal{K}) = \mathrm{cof}(\mathcal{K}) = \mathfrak{d}$.

1.3.2 Localization and anti-localization cardinals

Let x and φ be functions with domain ω. Denote:

(i) $x \in^* \varphi$ iff $\forall^\infty n\, (x(n) \in \varphi(n))$, which is read φ *localises* x (see Figure 1.2);

(ii) $x \in^\infty \varphi$ iff $\exists^\infty n\, (x(n) \notin \varphi(n))$. The expression $x \notin^\infty \varphi$ is read φ *anti-localises* x (see Figure 1.2).

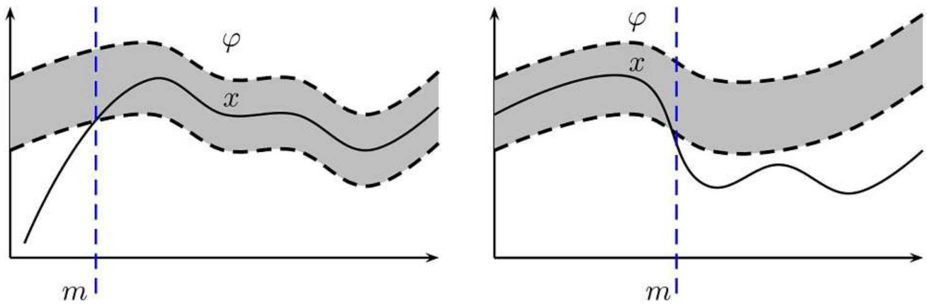

Figure 1.2: On the left side we have that $x(n) \in \varphi(n)$ for any $n \geq m$. On the right side we have that $x(n) \notin \varphi(n)$ for any $n \geq m$.

Definition 1.3.7. Let $b = \langle b(n) : n < \omega \rangle$ be a sequence of non-empty sets and let $h \in \omega^\omega$. Define the cardinals numbers $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ (called *localization cardinals*) and $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ (called *anti-localization cardinals*) as follows:

$$\mathfrak{d}^{\mathrm{Lc}}_{b,h} := \min\left\{|D| : D \subseteq \mathcal{S}(b,h),\ \forall x \in \prod b\, \exists \varphi \in D\, (x \in^* \varphi)\right\},$$

$$\mathfrak{b}^{\mathrm{Lc}}_{b,h} := \min\left\{|F| : F \subseteq \prod b,\ \neg\exists \varphi \in \mathcal{S}(b,h)\, \forall x \in F\, (x \in^* \varphi)\right\},$$

$$\mathfrak{b}^{\mathrm{aLc}}_{b,h} := \min\left\{|F| : F \subseteq \mathcal{S}(b,h),\ \forall x \in \prod b\, \exists \varphi \in F\, (x \in^\infty \varphi)\right\},$$

$$\mathfrak{d}^{\mathrm{aLc}}_{b,h} := \min\left\{|D| : D \subseteq \prod b,\ \forall \varphi \in \mathcal{S}(b,h)\, \exists x \in D\, (x \notin^\infty \varphi)\right\}.$$

1.3.3 Relational systems and Tukey-order

Many of the classical cardinal characteristics can be expressed by relational systems, and inequalities between these cardinals are induced by the *Tukey-Galois order* between the corresponding relational systems. These notions where defined by Vojtas [Voj93].

Definition 1.3.8. A *relational system* is a triple $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ where X and Y are non-empty sets and $\sqsubset$ is a relation.[2] For $x \in X$ and $y \in Y$, $x \sqsubset y$ is often read y *$\sqsubset$-dominates* x.

(1) A family $F \subseteq X$ is $\mathbf{R}$*-bounded* if there is a member of Y that $\sqsubset$-dominates every member of F, otherwise we say that the set is $\mathbf{R}$*-unbounded*.

(2) Dually, $D \subseteq Y$ is $\mathbf{R}$*-dominating* if every member of X is $\sqsubset$-dominated by some member of D.

These notions allow to define the following cardinal characteristics:

$$\mathfrak{b}(\mathbf{R}) := \min\{|F| : F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\}$$
$$\mathfrak{d}(\mathbf{R}) := \min\{|D| : D \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\}.$$

The relational system $\mathbf{R}^\perp := \langle Y, X, \not\sqsupset \rangle$ where $y \not\sqsupset x$ iff $\neg(x \sqsubset y)$, which is referred to as the *dual* of $\mathbf{R}$. It is clear that any $F \subseteq X$ is $\mathbf{R}$-undounded iff it is $\mathbf{R}^\perp$-domintaing, and $D \subseteq Y$ is $\mathbf{R}$-dominating iff it is $\mathbf{R}^\perp$-unbounded, so $\mathfrak{b}(\mathbf{R}^\perp) = \mathfrak{d}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}^\perp) = \mathfrak{b}(\mathbf{R})$. In addition, $(\mathbf{R}^\perp)^\perp = \mathbf{R}$.

[2]Although the relation $\sqsubset$ is only relevant when restricted to $X \times Y$, there is no need to demand it to be contained in $X \times Y$. See Example 1.3.10(2).

Remark 1.3.9. The cardinal invariants $\mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R})$ may not always exit. More concretely, $\mathfrak{b}(\mathbf{R})$ does not exist iff $\mathfrak{d}(\mathbf{R}) = 1$. Dually, $\mathfrak{d}(\mathbf{R})$ does not exist iff $\mathfrak{b}(\mathbf{R}) = 1$.

Many classical cardinal characteristics can be expressed through relational systems.

Example 1.3.10. (1) A *preorder* is pair $\langle S, \leq \rangle$ where $S \neq \varnothing$ and $\leq$ is a relation on S that satisfies reflexivity and transitivity. A *directed preorder* is a preorder $\langle S, \leq \rangle$ that satisfies $\forall x, y \in S \, \exists z \in S \, (x \leq z$ and $y \leq z)$. As a relational system, $S = \langle S, S \leq \rangle$.

(2) For any ideal $\mathcal{I}$ on X:

(i) $\mathbf{C}_{\mathcal{I}} := \langle X, \mathcal{I}, \in \rangle$, so $\mathfrak{b}(\mathbf{C}_{\mathcal{I}}) = \mathrm{non}(\mathcal{I})$ and $\mathfrak{d}(\mathbf{C}_{\mathcal{I}}) = \mathrm{cov}(\mathcal{I})$.

(ii) $\mathcal{I} := \langle \mathcal{I}, \subseteq \rangle$ is directed, $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$ and $\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$.

Example 1.3.11. Let $b = \langle b(n) : n < \omega \rangle$ be a sequence of non-empty sets and let $h \in \omega^\omega$.

(1) Denote $\mathbf{Lc}(b, h) := \langle \prod b, \mathcal{S}(b, h), \in^* \rangle$, which is a relational system. Note that $\mathfrak{b}^{\mathrm{Lc}}_{b,h} = \mathfrak{b}(\mathbf{Lc}(b, h))$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \mathfrak{d}(\mathbf{Lc}(b, h))$.

(2) Denote $\mathrm{aLc}(b, h) := \langle \mathcal{S}(b, h), \prod b, \not\ni^\infty \rangle$, which is a relational system. Note that $\mathfrak{b}^{\mathrm{aLc}}_{b,h} = \mathfrak{b}(\mathrm{aLc}(b, h))$ and $\mathfrak{d}^{\mathrm{aLc}}_{b,h} = \mathfrak{d}(\mathrm{aLc}(b, h))$.

Definition 1.3.12. Let κ and λ be non-zero cardinals, and let $\langle S, \leq_S \rangle$ be a directed preorder

(1) Consider the relational system $\mathbf{D}^\lambda_S := \langle S^\lambda, S^\lambda, \leq \rangle$ where $x \leq y$ iff $\forall \alpha < \lambda (x(\alpha) \leq_S y(\alpha))$. Define $\mathfrak{b}^\lambda_S := \mathfrak{b}(\mathbf{D}^\lambda_S)$ and $\mathfrak{d}^\lambda_S := \mathfrak{d}(\mathbf{D}^\lambda_S)$.

(2) Denote $\mathfrak{b}^\lambda_{\kappa \times \lambda} := \mathfrak{b}(\mathbf{D}^\lambda_{\kappa \times \lambda})$ and $\mathfrak{d}^\lambda_{\kappa \times \lambda} := \mathfrak{d}(\mathbf{D}^\lambda_{\kappa \times \lambda})$ where $\lambda \times \kappa$ is ordered by $(\alpha, \beta) \leq (\alpha', \beta')$ iff $\alpha \leq \alpha'$ and $\beta \leq \beta'$.

(3) Assume that λ is infinite. Consider the relational system $\mathbf{D}^\lambda_S(\leq^*) := \langle S^\lambda, S^\lambda, \leq^* \rangle$ where $x \leq^* y$ iff $\exists \beta < \lambda \forall \alpha \in [\beta, \lambda)(x(\alpha) \leq y(\alpha))$. Set $\mathfrak{b}^\lambda_S(\leq^*) := \mathfrak{b}(\mathbf{D}^\lambda_S(\leq^*))$ and $\mathfrak{d}^\lambda_S(\leq^*) := \mathfrak{d}(\mathbf{D}^\lambda_S(\leq^*))$.

(4) When κ is infinite, define $\mathfrak{b}_\kappa := \mathfrak{b}^\kappa_\kappa(\leq^*)$ and $\mathfrak{d}_\kappa := \mathfrak{d}^\kappa_\kappa(\leq^*)$ (a particular case of $\mathbf{D}^\lambda_S(\leq^*)$ with $S = \lambda = \kappa$). These are the well known *unbounding number of* κ^κ and *dominating number of* κ^κ respectively. The classical unbounded and dominating numbers are $\mathfrak{b} := \mathfrak{b}_\omega$ and $\mathfrak{d} := \mathfrak{d}_\omega$, respectively.

We define the product of relational systems, which will be used to characterize $\mathfrak{d}^\lambda_{\kappa \times \lambda}$ in Lemma 3.2.5.

Definition 1.3.13. Let $\mathbf{R} := \langle X, Y, \sqsubset \rangle$ and $\mathbf{R}' := \langle X', Y', \sqsubset' \rangle$ be two relational systems. Set $\mathbf{R} \otimes \mathbf{R}' := \langle X \times X', Y \times Y', \sqsubset_\otimes \rangle$, where $(x, x') \sqsubset_\otimes (y, y')$ iff $x \sqsubset x'$ and $y \sqsubset' y'$.

Fact 1.3.14 ([Bla10, Thm 4.11])**.** $\mathfrak{b}(\mathbf{R} \otimes \mathbf{R}') = \min\{\mathfrak{b}(\mathbf{R}), \mathfrak{b}(\mathbf{R}')\}$ *and* $\max\{\mathfrak{d}(\mathbf{R}), \mathfrak{d}(\mathbf{R}')\} \leq \mathfrak{d}(\mathbf{R} \otimes \mathbf{R}') \leq \mathfrak{d}(\mathbf{R}) \cdot \mathfrak{d}(\mathbf{R}')$

The Tukey order is a useful notion to determine relations between cardinal characteristics.

Definition 1.3.15 ([Bla10, Def. 4.8])**.** Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ and $\mathbf{R}' := \langle X', Y', \sqsubset' \rangle$ be relational systems. We say that $\mathbf{R}$ *is Tukey below* $\mathbf{R}'$, denoted by $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$, if there are maps $\Psi_1 : X \to X'$ and $\Psi_2 : Y' \to Y$ such that, for any $x \in X$ and $y' \in Y'$, if $\Psi_1(x) \sqsubset' y'$ then $x \sqsubset \Psi_2(y')$. Here, we say that the pair (Ψ_1, Ψ_2) *witnesses* $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$. Say that $\mathbf{R}$ *and* $\mathbf{R}'$ *are Tukey equivalent*, denoted by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$, if $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

Theorem 1.3.16 ([Bla10, Thm. 4.9]). *Assume* $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ *and that this is witnessed by* (Ψ_1, Ψ_2).

(1) If $D \subseteq Y'$ *is* $\mathbf{R}'$*-dominating then* $\Psi_2[D]$ *is* $\mathbf{R}$*-dominating.*

(2) $(\mathbf{R}')^{\perp} \preceq_{\mathrm{T}} \mathbf{R}^{\perp}$ *is witnessed by* (Ψ_2, Ψ_1).

(3) If $C \subseteq X$ *is* $\mathbf{R}$*-unbounded then* $\Psi_1[C]$ *is* $\mathbf{R}'$*-unbounded.*

In particular, $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ *and* $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$.

1.4 Overview of forcing

This section aims to give a review of some definitions and results known in the folklore of forcing theory. For basic knowledge about forcing see e.g. [Jec03; Kun11].

In this thesis, by *forcing notion* we mean a preorder $\langle \mathbb{P}, \leq \rangle$. Elements of $\mathbb{P}$ are called *conditions*. If $p, q \in \mathbb{P}$ we interpret $q \leq p$ as *q is stronger than p*.

The conditions $p, q \in \mathbb{P}$ are *compatible* in $\mathbb{P}$, denoted by $p \parallel q$, if there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. Otherwise, they are *incompatible* in $\mathbb{P}$, denoted by $p \perp q$.

The $\mathbb{P}$-name $\dot{G}$ usually denotes the canonical name of the $\mathbb{P}$-generic set.

Definition 1.4.1. Let M be a transitive model of ZFC (or of a finite fragment of it). Given two posets $\mathbb{P} \in M$ and $\mathbb{Q}$ (not necessarily in M), say that $\mathbb{P}$ *is a complete subposet of* $\mathbb{Q}$ *with respect to* M, denoted by $\mathbb{P} \lessdot_M \mathbb{Q}$, if $\mathbb{P}$ is a subposet of $\mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ that belongs to M is also a maximal antichain in $\mathbb{Q}$. Also define $\mathbb{P} \lessdot \mathbb{Q}$ by $\mathbb{P} \lessdot_V \mathbb{Q}$ where V is the universe.

In this case, if N is another transitive model of ZFC such that $N \supseteq M$ and $\mathbb{Q} \in N$, then $\mathbb{P} \lessdot_M \mathbb{Q}$ implies that, whenever G is $\mathbb{Q}$-generic over N, $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G \cap \mathbb{P}] \subseteq N[G]$ (see Figure 1.3). When $\mathbb{P} \in M$ it is clear that $\mathbb{P} \lessdot_M \mathbb{P}$.

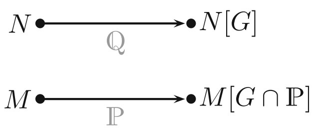

Figure 1.3: Generic extensions of pairs of posets when $\mathbb{P} \lessdot_M \mathbb{Q}$.

Definition 1.4.2. Let $\mathbb{P}$ be a forcing notion and let ν be an infinite cardinal.

(1) For $n < \omega$, $B \subseteq \mathbb{P}$ is *n-linked* if, for every $F \subseteq B$ of size $\leq n$, $\exists p \in \mathbb{P} \forall q \in F (p \leq q)$.

(2) $C \subseteq \mathbb{P}$ is *centered* if it is n-linked for every $n < \omega$.

(3) $\mathbb{P}$ is *ν-linked* if $\mathbb{P} = \bigcup_{\alpha<\nu} P_\alpha$ where each P_α is 2-linked. When $\nu = \omega$, we say that $\mathbb{P}$ is *σ-linked*.

(4) $\mathbb{P}$ is *ν-centered* if $\mathbb{P} = \bigcup_{\alpha<\nu} P_\alpha$ where each P_α is centered. When $\kappa = \omega$, we say that $\mathbb{P}$ is *σ-centered*.

(5) $\mathbb{P}$ has the *ν-chain condition* (ν-cc) if every antichain in $\mathbb{P}$ has size $< \nu$. When $\nu = \aleph_1$, this property is known as *the countable chain contidion* (ccc).

(6) $\mathbb{P}$ has the *ν-Knaster* if for every subset $A \subseteq \mathbb{P}$ of size ν there exists a linked $Q \in [A]^{\nu}$. When $\nu = \aleph_1$, this property is known as the *Knaster property*.

(7) $\mathbb{P}$ has *precaliber* ν if for every subset $A \subseteq \mathbb{P}$ of size ν there exists $Q \in [A]^\nu$ such that Q is centered.

The next notion of Suslin ccc forcing is due to Judah and Shelah [IS88]. The main reason for working with this type of order is to have nice definability, so many of its features are absolute when these are relativized to models of ZFC.

Definition 1.4.3. A poset $\mathbb{S}$ is called a *Suslin ccc forcing notion* if it is ccc and there is a Polish space X such that

(i) $\mathbb{S} \subseteq X$,

(ii) $\leq_\mathbb{S} \subseteq X \times X$ is Σ^1_1 and

(iii) $\perp_\mathbb{S} \subseteq X \times X$ is Σ^1_1.

Given a model M of a large enough fragment of ZFC, if the parameters of the poset $\mathbb{S}$ are in M, then $\mathbb{S}$ is coded in M and we denote its interpretation by $\mathbb{S}^M$, as usual for analytic sets. For instance, the statements "$p \in \mathbb{S}$", "$p \leq_\mathbb{S} q$" and "$p \perp_\mathbb{S} q$" are absolute for transitive models of ZFC. Note that $\mathbb{S}$ itself is a Σ^1_1-set because $x \in \mathbb{S}$ iff $x \leq x$. In the following Section 1.5 we introduce some examples of Suslin ccc forcing notions

1.5 Some forcing notions

All the forcing notions we introduce in this section are Suslin ccc forcings that we use throughout this work. For any sets I, J and any infinite cardinal κ denote by $\mathrm{Fn}_{<\kappa}(I, J)$ the poset of partial functions from I to J with domain of size $<\kappa$, ordered by $\supseteq$. When $\kappa = \omega$, this is well-known as $\mathrm{Fn}(I, J)$, that is, the poset of finite partial functions from I to J.

1.5.1 Cohen forcing

Cohen forcing is $\mathbb{C} := \mathrm{Fn}(\omega, 2)$, but it can be any atomless countable poset by Lemma 1.5.1. It is known that $\mathbb{C}$ is ccc, even more, it is σ-centered. For a nonempty set X, denote by $\mathbb{C}_X$ the poset that adds a family of Cohen reals indexed by the set X. Concretely, $\mathbb{C}_X = \mathrm{Fn}(X \times \omega, 2)$.

Lemma 1.5.1. *Any atomless countable forcing notion is forcing equivalent to* $\mathrm{Fn}(\omega, 2)$.

Lemma 1.5.2. *Let δ be a limit ordinal and let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a FS iteration where each $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be non-trivial. Then,* $\mathrm{Fn}(\delta, 2) \lessdot \mathbb{P}_\delta$.

As a consequence we get,

Corollary 1.5.3. *If $\mathbb{P}_\delta$ is as in Lemma 1.5.2 with δ is limit and $\alpha < \delta$, then $\mathbb{P}_\delta$ forces that there is some Cohen real over $V[\dot{G} \cap \mathbb{P}_\alpha]$.*

1.5.2 Random forcing

Random forcing is the poset $\mathbb{B} := \{T \subseteq 2^{<\omega} : T$ is a tree and $\mathbf{Lb}_2([T]) > 0\}$, and it is ordered by $\subseteq$.

For $(s, m) \in 2^{<\omega} \times \omega$ set

$$\mathbb{B}(s, m) := \{T \in \mathbb{B} : [T] \subseteq [s] \text{ and } 2^{|s|} \cdot \mathbf{Lb}_2([T]) \geq 1 - 2^{-10-m}\}.$$

Note that, for fixed $m < \omega$, $\bigcup_{s \in 2^{<\omega}} \mathbb{B}(s, m)$ is dense in $\mathbb{B}$. It is clear that that $\mathbb{B}$ is σ-linked.

Definition 1.5.4. A forcing notion $\mathbb{P}$ is called *ω^ω-bounding* iff for any $\mathbb{P}$-name $\dot{f}$ for a member of ω^ω and $p \in \mathbb{P}$ there is a $q \leq p$ in $\mathbb{P}$ and a $g \in \omega^\omega \cap V$ such that $q \Vdash \dot{f} \leq g$.

It is known that $\mathbb{B}$ is ω^ω-bounding.

Lemma 1.5.5. *(i) $\mathbb{B}$ does not add unbounded reals. So $\mathbb{B}$ does not add Cohen reals.*

(ii) No σ-centered poset adds random reals. So $\mathbb{C}$ does not add random reals.

Proof. It follows from the fact that Cohen reals are unbounded over V, and that any σ-centered poset is **Cn**-good (see Example 1.7.9(3)). ☐

1.5.3 Hechler forcing adding a mad family

For a set Z, let $\mathbb{H}_Z$ *be the Hechler forcing for adding an a.d. family (indexed by Z)* (see [Hec72; BF11]). Define $\mathbb{H}_Z := \{p : F_p \times n_p \to 2 : F_p \in [Z]^{<\aleph_0} \text{ and } n_p < \omega\}$ (demand $n_p = 0$ iff $F_p = \emptyset$), ordered by $q \leq p$ iff $p \subseteq q$ and $|q^{-1}[\{1\}] \cap (F_p \times \{i\})| \leq 1$ for every $i \in [n_p, n_q)$.

This poset has precaliber $\aleph_1$, and the a.d. family it adds is maximal when Z is uncountable. The following observations are due to Brendle and Fischer [BF11].

Remark 1.5.6. (1) For any $Z \subseteq Z'$, $\mathbb{H}_Z \lessdot \mathbb{H}_{Z'}$; and both $\mathbb{H}_Z$ and $\mathbb{H}_{Z'}$ are isomorphic whenever $|Z| = |Z'|$. The forcing $\mathbb{H}_Z$ is forcing equivalent to $\mathbb{C}$ when Z is countable and non-empty.

(2) Let $\mathbb{H}_{\omega_1}$ be a FS iteration of the quotients $\mathbb{H}_{\alpha+1}/\mathbb{H}_\alpha$ for $\alpha < \omega_1$. Since these quotients are countable, $\mathbb{H}_{\omega_1}$ is equivalent to the FS support iteration of length ω_1 of $\mathbb{C}$, which is $\mathbb{C}_{\omega_1}$.

1.5.4 Localization forcing

Localization forcing is the poset $\mathbb{LOC} := \{\varphi \in \mathcal{S}(\omega, \mathrm{id}) : \exists m < \omega \forall i < \omega(|\varphi(i)| \leq m)\}$ ordered by $\varphi' \leq \varphi$ iff $\varphi(i) \subseteq \varphi'(i)$ for every $i < \omega$. Recall that this poset is σ-linked and that it adds an slalom φ^* in $\mathcal{S}(\omega, \mathrm{id}_\omega)$ which *localizes the ground model reals in* ω^ω, that is, $x \in^* \varphi^*$ for any $x \in \omega^\omega \cap V$. $\mathbb{LOC}$ also adds a domating real, that is, $f \leq f_{\varphi^*}$ for all $f \in \omega^\omega \cap V$ where $f_{\varphi^*}(i) := \sup(\varphi^*(i))$.

1.5.5 Eventually different real forcing

The following poset is a generalization of the standard ccc poset that adds an eventually different real (see e.g. [KO14; CM19]).

Fix $b : \omega \to \omega + 1 \setminus \{0\}$ and $h \in \omega^\omega$ such that $\lim_{i \to +\infty} \frac{h(i)}{b(i)} = 0$ (when $b(i) = \omega$, interpret $\frac{h(i)}{b(i)}$ as 0). Define the (b, h)*-ED (eventually different real) forcing* $\mathbb{E}_b^h$ as the poset whose conditions are of the form $p = (s, \varphi)$ such that, for some $m := m_p < \omega$,

(i) $s \in \mathrm{seq}_{<\omega}(b)$, $\varphi \in \mathcal{S}(b, m \cdot h)$, and

(ii) $m \cdot h(i) < b(i)$ for every $i \geq |s|$,

ordered by $(t, \psi) \leq (s, \varphi)$ iff $s \subseteq t$, $\forall i < \omega(\varphi(i) \subseteq \psi(i))$ and $t(i) \notin \varphi(i)$ for all $i \in |t| \setminus |s|$.

Put $\mathbb{E}_b^h(s, m) := \{(t, \varphi) \in \mathbb{E}_b^h : t = s \text{ and } m_{(t,\varphi)} \leq m\}$ for $s \in \mathrm{seq}_{<\omega}(b)$ and $m < \omega$. $\mathbb{E}_b^h$ adds an real $e \in \prod b$ such that, $e \not\in^\infty \varphi$ for any $\varphi \in \mathcal{S}(b, h)$ in the ground model. Denote $\mathbb{E}_b := \mathbb{E}_b^1$, $\mathbb{E} := \mathbb{E}_\omega$, $\mathbb{E}_b(s, m) := \mathbb{E}_b^1(s, m)$, and $\mathbb{E}(s, m) := \mathbb{E}_\omega(s, m)$. In particular, when $h \geq^* 1$, $\mathbb{E}_b^h$ adds an $r \in \prod b$ which is *eventually different* from the ground model reals, that is, $r \neq^\infty x$ for all $x \in V \cap \prod b$.

Lemma 1.5.7 ([CM19, Corollary 2.23])**.** *Let $b : \omega \to \omega + 1 \setminus \{0\}$ and $h \in \omega^\omega$ such that $\frac{h(i)}{b(i)}$ goes to 0. Then, $\mathbb{E}_b^h$ is σ-linked. Even more, if $b \geq^* \omega$, then $\mathbb{E}_b^h$ is σ-centered.*

1.5.6 Hechler forcing

Hechler forcing is the poset $\mathbb{D} = \{(s, f) : s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subseteq f\}$ ordered by $(t, g) \leq (s, f)$ if $s \subseteq t$ and $f \leq g$. Recall that $\mathbb{D}$ is σ-centered and it adds a real d in ω^ω which *is dominating over the ground model reals in* ω^ω, which means that $f \leq^* d$ for any $\forall f \in V \cap \omega^\omega$. Even more, $\mathbb{D}$ adds Cohen reals.

1.5.7 Tree forcings

Definition 1.5.8. Let $b : \omega \to \omega \smallsetminus \{0\}$. We say that a poset $\mathbb{T}$ is a *b-tree forcing notion* if it satisfies the following properties

(T1) $\mathbb{T}$ is a non-empty set of trees contained in $\mathrm{seq}_{<\omega}(b)$.

(T2) If $T \in \mathbb{T}$ and $s \in T$, then there is some splitting note $t \in T$ extending s.

(T3) For $T, T' \in \mathbb{T}$, $T' \leq T$ implies $T' \subseteq T$.

(T4) If $T \in \mathbb{T}$ and $s \in T$ then $T^{[s]} \in \mathbb{T}$ and $T^{[s]} \leq T$.

(T5) If $T \in \mathbb{T}$, $n < \omega$ and $\{S_t : t \in \mathrm{Lv}_n(T)\} \subseteq \mathbb{T}$ such that $S_t \leq T^{[t]}$ for all $t \in \mathrm{Lv}_n(T)$, then $S := \bigcup_{t \in \mathrm{Lv}_n(T)} S_t \in \mathbb{T}$, $S \leq T$ and $\{S_t : t \in \mathrm{Lv}_n(T)\}$ is a maximal antichain below S.

(T6) If $\langle T_n : n < \omega \rangle$ is a decreasing sequence in $\mathbb{T}$ and $T_{n+1} \subseteq_n T_n$ for al $n < \omega$, then $T := \bigcap_{n<\omega} T_n \in \mathbb{T}$ and $T \leq T_n$ for all $n < \omega$.

When $\mathbb{T}$ is a b-tree forcing for some b we say that $\mathbb{T}$ is a *bounded-tree forcing notion*. Note that (T1) and (T2) imply $b \not\leq^* 1$. Denote by $\mathbb{T}_b$ the poset of all conditions satisfying (T1) and (T2), ordered by $\subseteq$. It is clear that this is a b-tree forcing notion.

Example 1.5.9. 1. Recall *Sacks forcing* $\mathbb{S} := \mathbb{T}_2$ (where 2 represents the constant function with value 2). It is clearly a 2-tree forcing notion.

2. Let $\mathbb{PT}_b$ be the poset of conditions $T \in \mathbb{T}_b$ such that, whenever $s \in \mathrm{spl}(T)$, $s^\frown\langle i \rangle \in T$ for every $i \in b(|s|)$. Judah, Goldstern and Shelah [GJS93] defined this poset and showed that, under CH, there is a CS iteration of such type of posets forcing $\mathrm{add}(\mathcal{SN}) = \aleph_2$ (even more, it forces that $\mathcal{SN} = [\mathbb{R}]^{\leq\aleph_1}$). In particular, these tree forcings are used to prove the consistency of $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$.

1.6 Coherent systems of FS iterations

This section is dedicated to introduce the general notion of a coherent system of FS iterations due to Fischer, Friedman, Mejía and Montoya [FFMM18], which generalizes and improves the matrix of iteration introduced by Blass and Shelah [BS89]. In our applications, we use a particular case called *simple matrix iterations*. In this type of matrix iterations only restricted generic reals are added, and preservation properties behave very nicely.

Definition 1.6.1 ([FFMM18, Def. 3.2]). A *coherent systems of FS iterations* $\mathbf{m}$ consists of

(I) a partially ordered set $I^{\mathbf{m}}$ and an ordinal $\pi^{\mathbf{m}}$;

(II) a system of posets $\langle \mathbb{P}^{\mathbf{m}}_{i,\xi} : i \in I^{\mathbf{m}}, \xi \leq \pi^{\mathbf{m}} \rangle$ such that

  (i) $\mathbb{P}^{\mathbf{m}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{m}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{m}}$, and

  (ii) $\mathbb{P}^{\mathbf{m}}_{i,\eta}$ is the direct limit of $\langle \mathbb{P}^{\mathbf{m}}_{i,\xi} : \xi < \eta \rangle$ for each limit $\eta \leq \pi^{\mathbf{m}}$;

(III) a sequence $\langle \dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi} : i \in I^{\mathbf{m}}, \xi < \pi^{\mathbf{m}} \rangle$ where each $\dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi}$ is a $\mathbb{P}^{\mathbf{m}}_{i,\xi}$-name for a poseet, $\mathbb{P}^{\mathbf{m}}_{i,\xi+1} = \mathbb{P}^{\mathbf{m}}_{i,\xi} * \dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi}$ and $\mathbb{P}^{\mathbf{m}}_{j,\xi}$ forces that $\dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi} \lessdot \dot{\mathbb{Q}}^{\mathbf{m}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{m}}$ and $\mathbb{P}^{\mathbf{m}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{m}}_{j,\xi}$

According to this notation, $\mathbb{P}^{\mathbf{m}}_{i,0}$ is the trivial poset and $\mathbb{P}^{\mathbf{m}}_{i,1} = \dot{\mathbb{Q}}^{\mathbf{m}}_{i,0}$. On the other hand, by Lemma 1.7.13, $\mathbb{P}^{\mathbf{m}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{m}}_{j,\xi}$ for all $i \leq j$ in $I^{\mathbf{m}}$ and $\xi \leq \pi^{\mathbf{m}}$.

For $j \in I^{\mathbf{m}}$ and $\eta \leq \pi^{\mathbf{m}}$ we denote by $V_{j,\eta}$ the $\mathbb{P}^{\mathbf{m}}_{j,\eta}$-generic extension. To be precise, when G is $\mathbb{P}^{\mathbf{m}}_{j,\eta}$-generic over V we denote $V_{i,\eta} := V[G]$ and $V_{i,\xi} := V[G \cap \mathbb{P}^{\mathbf{m}}_{i,\xi}]$ for all $i \leq j$ in $I^{\mathbf{m}}$ and $\xi \leq \eta$. Clearly, $V_{i,\xi} \subseteq V_{j,\eta}$ for all $i \leq j$ in $I^{\mathbf{m}}$ and $\xi \leq \eta \leq \pi^{\mathbf{m}}$.

We say the coherent system $\mathbf{m}$ has the ccc if, additionality, $\mathbb{P}^{\mathbf{m}}_{i,\xi}$ forces that $\dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi}$ has the ccc for each $i \in I^{\mathbf{m}}$ and $\xi < \pi^{\mathbf{m}}$. This implies that $\mathbb{P}^{\mathbf{m}}_{i,\xi}$ has the ccc for all $i \in I^{\mathbf{m}}$ and $\xi \leq \pi^{\mathbf{m}}$.

Remark 1.6.2. When $I^{\mathbf{m}}$ is a well-ordered set, we say that $\mathbf{m}$ is a 2D-coherent system of FS iterations. The idea of such a construction is to obtain a matrix $\langle V_{i,\xi} : i \in I,\ \xi \leq \pi \rangle$ of generic extensions as illustrated in Figure 1.4

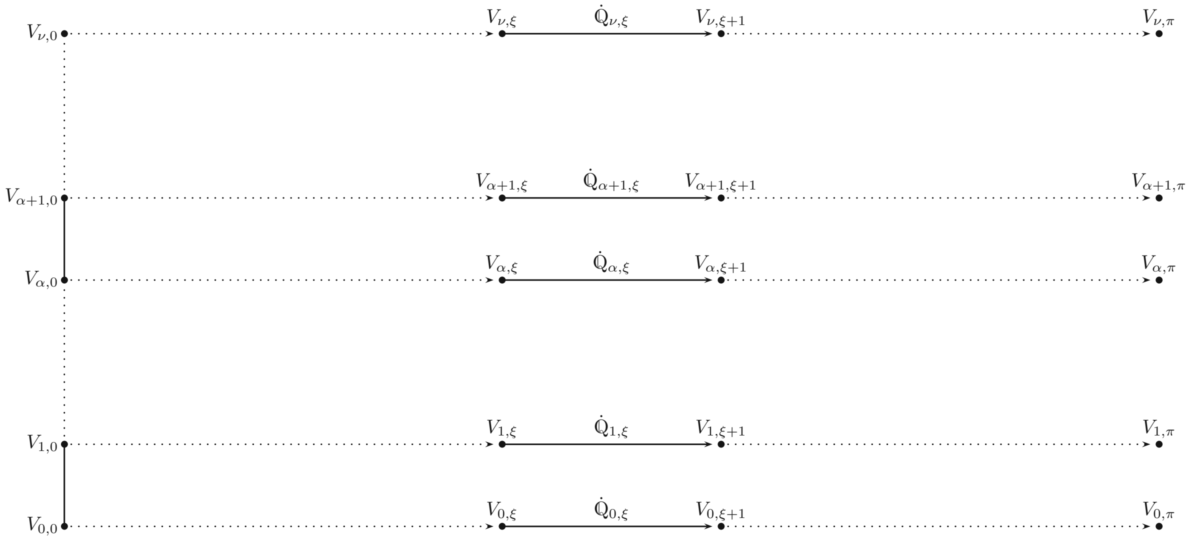

Figure 1.4: Matrix iteration with $I^{\mathbf{m}} = \nu + 1$ where ν is an ordinal.

For our applications, the following type of matrix iteration is the one we are going to deal with throughout the whole text.

Definition 1.6.3 (Simple matrix iteration)**.** A *simple matrix iteration* $\mathbf{m}$ is a matrix iteration, composed additionally of a function $\Delta^{\mathbf{m}} : \pi^{\mathbf{m}} \to I^{\mathbf{m}}$, that satisfies: for each $\xi < \pi^{\mathbf{m}}$, there is a $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),\xi}$-name $\dot{\mathbb{Q}}^{\mathbf{m}}_{\xi}$ of a poset such that, for each $i \in I^{\mathbf{m}}$,

$$\dot{\mathbb{Q}}^{\mathbf{m}}_{i,\xi} = \begin{cases} \dot{\mathbb{Q}}^{\mathbf{m}}_{\xi} & \text{if } i \geq \Delta^{\mathbf{m}}(\xi), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

The upper index $\mathbf{m}$ is omitted when there is no risk of ambiguity. If $\xi \leq \pi$, denote by $\mathbf{m}{\restriction}\xi$ (*horizontal restriction*) the matrix iteration with $I^{\mathbf{m}{\restriction}\xi} = I$ and $\pi^{\mathbf{m}{\restriction}\xi} = \xi$ where the FS iterations are the same as in (II) and (III) but restricted to ξ. On the other hand, for any $J \subseteq I$, denote by $\mathbf{m}|J$ (*vertical restriction*) the matrix iteration with $I^{\mathbf{m}|J} = J$ and $\pi^{\mathbf{m}|J} = \pi$ where the FS iterations for $i \in J$ are exactly as in (II) and (III).

Although $I^{\mathbf{m}}$ is an ordinal in all our applications, it is more practical to use it as a well order in general because it eases the notation when dealing with $\mathbf{m}|J$ in the case that J is a set of ordinals but not an ordinal (as in the last part of the proof of Lemma 2.3.6).

A simple matrix iteration is easily constructed by recursion on $\xi \leq \pi$. When $\mathbf{m}{\restriction}\xi$ is already constructed, $\Delta(\xi)$ and $\dot{\mathbb{Q}}_\xi$ are freely defined, which allows to extend the matrix to $\mathbf{m}{\restriction}(\xi+1)$. Limit steps are uniquely determined by taking direct limits. When $\dot{\mathbb{Q}}_\xi$ adds a real, it will be generic over $V_{\Delta(\xi),\xi}$ but not necessarily over $V_{i,\xi}$ for larger i, which is the reason we say that a *restricted generic real* is added at step ξ. For instance, when $\dot{\mathbb{Q}}_\xi = \mathbb{D}^{V_{\Delta(\xi),\xi}}$, the generic real added at ξ is dominating over $V_{\Delta(\xi),\xi}$. Moreover, more restricted generic sets are allowed, for example, when $\dot{\mathbb{Q}}_\xi = \mathbb{D}^{N_\xi}$ where $N_\xi \in V_{\Delta(\xi),\xi}$ is a (small) transitive model of ZFC, the generic real added at step ξ is dominating over N_ξ but not necessarily over $V_{\Delta(\xi),\xi}$.

Most of the time we deal with simple matrix iterations where $I^{\mathbf{m}} = \nu+1$ for some ordinal ν, unless we are reasoning with restrictions of such matrix iteration. In this case, if the simple matrix iteration is composed by ccc posets and ν has uncountable cofinality, then $\mathbb{P}_{\nu,\xi}$ is the direct limit of the posets below it in the matrix. More generally:

Lemma 1.6.4 ([BF11], see also [Mej19, Cor. 2.6])**.** *Let θ be an uncountable regular cardinal and let ν be an ordinal. Assume that $\mathbf{m}$ is a simple matrix iteration such that*

(1) $I^{\mathbf{m}} = \nu+1$, $\mathrm{cf}(\nu) \geq \theta$,

(2) $\nu \notin \mathrm{ran}\Delta$, and

(3) for each $\xi < \pi^{\mathbf{m}}$, $\mathbb{P}_{\nu,\xi}$ forces that $\dot{\mathbb{Q}}_{\Delta(\xi),\xi}$ is θ-cc.

Then, for any $\xi \leq \pi$,

(a) $\mathbb{P}_{\nu,\xi}$ is the direct limit of $\langle \mathbb{P}_{\alpha,\xi} : \alpha < \nu\rangle$, and

(b) if $\beta < \theta$ and $\dot{f}$ is a $\mathbb{P}_{\nu,\xi}$-name of a function from β into $\bigcup_{\alpha<\nu} V_{\alpha,\xi}$ then $\dot{f}$ is forced to be equal to a $\mathbb{P}_{\alpha,\xi}$-name for some $\alpha < \nu$. In particular, the reals in $V_{\nu,\xi}$ are precisely the reals in $\bigcup_{\alpha<\nu} V_{\alpha,\xi}$.

1.7 Preservation properties

In this section, we discuss the effect of such coherent systems on cardinal characteristcs of the form $\mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R})$ for some relational system $\mathbf{R}$. In the context of FS iterations, these cardinals are dealt with by using a strong type of unbounded and dominating families.

First we define a very special type of dominating (and unbounded) families for relational systems. They play an important role when forcing values to cardinal characteristics.

Definition 1.7.1. Let $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ be a relational system and let θ be a cardinal.

(1) For a set M,

   (i) An object $y \in Y$ is $\mathbf{R}$-*dominating over* M if $x \sqsubset y$ for all $x \in X \cap M$.

   (ii) An object $x \in X$ is $\mathbf{R}$-*unbounded over* M if it $\mathbf{R}^\perp$-dominating over M, that is, $x \not\sqsubset y$ for all $y \in Y \cap M$.

(2) A family $D \subseteq Y$ is *strongly* θ-$\mathbf{R}$-*dominating* if $|D| \geq \theta$ and, for any $x \in X$, $|\{y \in D : x \not\sqsubset y\}| < \theta$.

(3) A family $F \subseteq X$ is *strongly* θ-$\mathbf{R}$-*unbounded* if it strongly θ-$\mathbf{R}^\perp$-dominating, that is, $|F| \geq \theta$ and, for any $y \in Y$, $|\{x \in F : x \sqsubset y\}| < \theta$.

Remark 1.7.2. Any strongly θ-$\mathbf{R}$-dominating family is $\mathbf{R}$-dominating. Likewise for unbounded families.

The next result shows the effect of this special type of dominating (and unbounded) families on cardinal characteristics.

Lemma 1.7.3. *Let* $\mathbf{R}$ *be a relational system and let* θ *be regular.*

(i) If $D \subseteq Y$ *is a strongly* θ*-*$\mathbf{R}$*-dominating family then it is* $|D|$*-*$\mathbf{R}$*-dominating and* $\mathfrak{d}(\mathbf{R}) \leq \theta \leq |D| \leq \mathfrak{b}(\mathbf{R})$*.*

(ii) If $F \subseteq X$ *is a strongly* θ*-*$\mathbf{R}$*-unbounded family then it is* $|F|$*-*$\mathbf{R}$*-unbounded and* $\mathfrak{b}(\mathbf{R}) \leq \theta \leq |F| \leq \mathfrak{d}(\mathbf{R})$*.*

In the following we give the definition of a special type of dominating family, which is extracted from the property CBO introduced in [GKS19, Def. 1.8].

Definition 1.7.4. Let $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ be a relational system and let $\langle S, \leq_S\rangle$ be a direted preorder. Say that a subset D of Y is a *strongly* S*-*$\mathbf{R}$*-dominating family* if $D := \{y_i : i \in S\}$ and, for any $x \in X$, there is some $i_x \in S$ such that $x \sqsubset y_i$ for all $i \geq i_x$ in S.

Remark 1.7.5. The existence of a strongly S-$\mathbf{R}$-dominating family is equivalent to $\mathbf{R} \preceq_{\mathrm{T}} S$. So, if such a family exists then $\mathfrak{b}(S) \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(S)$.

The following two definitions are the central concepts for preservation of strongly unbounded families of the ground model.

Definition 1.7.6. Say that $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ is a *Polish relational system (Prs)* if the following is satisfied:

(i) X is a perfect Polish space,

(ii) Y is a non-empty analytic subspace of some Polish space Z and

(iii) $\sqsubset \cap (X \times Z) = \bigcup_{n<\omega} \sqsubset_n$ where $\langle \sqsubset_n\rangle_{n<\omega}$ is some increasing sequence of closed subsets of $X \times Z$ such that, for any $n < \omega$ and for any $y \in Y$, $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense.

By (iii), $\langle X, \mathcal{M}(X), \in\rangle \preceq_{\mathrm{T}} \mathbf{R}$. Therefore, $\mathfrak{b}(\mathbf{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathbf{R})$.

Definition 1.7.7 (Judah and Shelah [JS90]). Let $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ be a Prs and let θ be a cardinal. A poset $\mathbb{P}$ is θ-$\mathbf{R}$-*good* if, for any $\mathbb{P}$-name $\dot{h}$ for a member of Y, there is a non-empty $H \subseteq Y$ (in the ground model) of size $<\theta$ such that, for any $x \in X$, if x is $\mathbf{R}$-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.
Say that $\mathbb{P}$ is $\mathbf{R}$-*good* if it is $\aleph_1$-$\mathbf{R}$-good.

The definition above describes a property used to preserve strongly $\mathbf{R}$-unbounded families, which is respected by FS iterations. Concretely, when θ is uncountable regular,

(1) any θ-$\mathbf{R}$-good poset preserves all the strongly θ-$\mathbf{R}$-unbounded families from the ground model and

(2) FS iterations of θ-cc θ-$\mathbf{R}$-good posets produce θ-$\mathbf{R}$-good posets.

Hence, by Lemma 1.7.3, posets that are θ-$\mathbf{R}$-good work to preserve $\mathfrak{b}(\mathbf{R})$ small and $\mathfrak{d}(\mathbf{R})$ large. Clearly, θ-$\mathbf{R}$-good implies θ'-$\mathbf{R}$-good whenever $\theta \leq \theta'$, and any poset completely embedded into a θ-$\mathbf{R}$-good poset is also θ-$\mathbf{R}$-good. Also note the trivial fact that any poset is $\mathfrak{d}(\mathbf{R})^+$-good.
As a first general example, every small poset is always good.

Lemma 1.7.8 ([Mej13, Lemma 4])**.** *If* $\mathbf{R}$ *is a Prs and* θ *is an uncountable regular cardinal then any poset of size* $<\theta$ *is* θ-$\mathbf{R}$*-good. In particular, Cohen forcing is* $\mathbf{R}$*-good.*

Now, we present the instances of Prs and their corresponding good posets that we use in our applications.

Example 1.7.9. (1) *Preserving non-meager sets:* Consider the Polish relational system $\mathbf{Ed} := \langle\omega^\omega, \omega^\omega, \neq^\infty\rangle$. By [BJ95, Thm. 2.4.1 & Thm. 2.4.7], $\mathfrak{b}(\mathbf{Ed}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Ed}) = \mathrm{cov}(\mathcal{M})$.

(2) *Preserving unbounded families:* Let $\mathbf{D}$ be the relational system $\mathbf{D} := \langle\omega^\omega, \omega^\omega, \leq^*\rangle$, which is Polish. Clearly $\mathfrak{b}(\mathbf{D}) = \mathfrak{b}$ and $\mathfrak{d}(\mathbf{D}) = \mathfrak{d}$. Any μ-Fr-linked poset is μ^+-$\mathbf{D}$-good (see Theorem 2.2.1).

(3) *Preserving null-covering families:* Define $\Omega_n := \{a \in [2^{<\omega}]^{<\aleph_0} : \mathbf{Lb}_2(\bigcup_{s\in a}[s]) \leq 2^{-n}\}$ (endowed with the discrete topology) and put $\Omega := \prod_{n<\omega}\Omega_n$ with the product topology, which is a perfect Polish space. For every $x \in \Omega$ denote $N^*_x := \bigcap_{n<\omega}\bigcup_{s\in x(n)}[s]$, which is clearly a Borel null set in 2^ω.

Define the Prs $\mathbf{Cn} := \langle\Omega, 2^\omega, \sqsubset\rangle$ where $x \sqsubset z$ iff $z \notin N^*_x$. Recall that any null set in 2^ω is a subset of N^*_x for some $x \in \Omega$, so $\mathbf{Cn} \cong_{\mathrm{T}} \langle\mathcal{N}(2^\omega), 2^\omega, \not\ni\rangle$. Hence, $\mathfrak{b}(\mathbf{Cn}) = \mathrm{cov}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Cn}) = \mathrm{non}(\mathcal{N})$.

Any μ-centered poset is μ^+-$\mathbf{Cn}$-good (see e.g. [Bre91]). In particular, σ-centered posets are $\mathbf{Cn}$-good.

(4) *Preserving union of null sets is not null:* For each $k < \omega$ let $\mathrm{id}^k : \omega \to \omega$ such that $\mathrm{id}^k(i) = i^k$ for all $i < \omega$ and $\mathcal{H} := \{\mathrm{id}^{k+1} : k < \omega\}$. Let $\mathbf{Lc}^* := \langle\omega^\omega, \mathcal{S}(\omega, \mathcal{H}), \in^*\rangle$ be the Polish relational system where

$$\mathcal{S}(\omega, \mathcal{H}) := \{\varphi : \omega \to [\omega]^{<\aleph_0} : \exists h \in \mathcal{H}\forall i < \omega(|\varphi(i)| \leq h(i))\}.$$

As consequence of [BJ95, Thm. 2.3.9], $\mathfrak{b}(\mathbf{Lc}^*) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}^*) = \mathrm{cof}(\mathcal{N})$.

Any μ-centered poset is μ^+-$\mathbf{Lc}^*$-good (see [Bre91; JS90]) so, in particular, σ-centered posets are $\mathbf{Lc}^*$-good. Besides, Kamburelis [Kam89] showed that any Boolean algebra with a strictly positive finitely additive measure is $\mathbf{Lc}^*$-good (in particular, subalgebras of random forcing).

(5) *Preserving large continuum:* Consider the Polish relational system $\mathbf{Id} := \langle\omega^\omega, \omega^\omega, =\rangle$. It is clear that $\mathfrak{b}(\mathbf{Id}) = 2$ and $\mathfrak{d}(\mathbf{Id}) = \mathfrak{c}$. Though this is a quite trivial Prs, we are interested in the following simple facts:

(5.1) $x \in \omega^\omega$ is $\mathbf{Id}$-unbounded over M iff $x \notin M$.

(5.2) If $\theta \geq 2$ then $F \subseteq \omega^\omega$ is strongly θ-$\mathbf{Id}$-unbounded iff $|F| \geq \theta$.

(5.3) Any θ-cc poset is θ-$\mathbf{Id}$-good.[3]

Concretely, we use (5.2) as a simple resource to justify why the continuum is increased after Boolean ultrapowers of a ccc poset (Theorem 2.4.6).

The following results indicate that strongly unbounded families can be added with Cohen reals, and the effect on $\mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R})$ by a FS iteration of good posets.

[3]The converse is true when $\theta \leq \mathfrak{c}$. On the other hand, any poset is $\mathfrak{c}^+$-$\mathbf{Id}$-good.

Lemma 1.7.10. *Let μ be a cardinal with uncountable cofinality, $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ a Prs and let $\langle \mathbb{P}_\alpha \rangle_{\alpha<\mu}$ be a $\lessdot$-increasing sequence of $\mathrm{cf}(\mu)$-cc posets such that $\mathbb{P}_\mu = \mathrm{limdir}_{\alpha<\mu}\mathbb{P}_\alpha$. If $\mathbb{P}_{\alpha+1}$ adds a Cohen real $\dot{c}_\alpha \in X$ over $V^{\mathbb{P}_\alpha}$ for any $\alpha < \mu$, then $\mathbb{P}_\mu$ forces that $\{\dot{c}_\alpha : \alpha < \mu\}$ is a strongly μ-$\mathbf{R}$-unbounded family of size μ.*

Theorem 1.7.11. *Let θ be an uncountable regular cardinal, $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ a Prs, $\pi \geq \theta$ an ordinal, and let $\mathbb{P}_\pi = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \pi\rangle$ be a FS iteration such that, for each $\alpha < \pi$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name of a non-trivial θ-$\mathbf{R}$-good θ-cc poset. Then, $\mathbb{P}_\pi$ forces that $\mathfrak{b}(\mathbf{R}) \leq \theta$ and $|\pi| \leq \mathfrak{d}(\mathbf{R})$.*

Proof. See e.g. [CM19, Thm. 4.15] and [GMS16, Thm. 3.6]. □

For the remainder of this section, fix transitive models $M \subseteq N$ of ZFC and a Prs $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ coded in M. The next results are related to preservation of $\mathbf{R}$-unbounded reals along simple matrix iterations.

Lemma 1.7.12 ([BF11, Lemma 11], see also [Mej15, Lemma 5.13])**.** *Assume that $\mathbb{P} \in M$ is a poset. Then, in N, $\mathbb{P}$ forces that every $c \in X^N$ that is $\mathbf{R}$-unbounded over M is $\mathbf{R}$-unbounded over $M^{\mathbb{P}}$.*

Lemma 1.7.13 ([BF11])**.** *Assume that $\mathbb{P}_{0,\pi} = \langle \mathbb{P}_{0,\alpha}, \dot{\mathbb{Q}}_{0,\alpha} : \alpha < \pi\rangle \in M$ and $\mathbb{P}_{1,\pi} = \langle \mathbb{P}_{1,\alpha}, \dot{\mathbb{Q}}_{1,\alpha} : \alpha < \pi\rangle \in N$ are FS iterations such that, for any $\alpha < \pi$, if $\mathbb{P}_{0,\alpha} \lessdot_M \mathbb{P}_{1,\alpha}$ then $\mathbb{P}_{1,\alpha}$ forces that $\dot{\mathbb{Q}}_{0,\alpha} \lessdot_{M^{\mathbb{P}_{0,\alpha}}} \dot{\mathbb{Q}}_{1,\alpha}$. Then $\mathbb{P}_{0,\alpha} \lessdot_M \mathbb{P}_{1,\alpha}$ for any $\alpha \leq \pi$.*

In addition, if π is limit, $c \in X^N$ and, for any $\alpha < \pi$, $\mathbb{P}_{1,\alpha}$ forces (in N) that c is $\mathbf{R}$-unbounded over $M^{\mathbb{P}_{0,\alpha}}$, then $\mathbb{P}_{1,\pi}$ forces that c is $\mathbf{R}$-unbounded over $M^{\mathbb{P}_{0,\pi}}$.

Theorem 1.7.14 ([BF11], see also [Mej13, Thm. 10 & Cor. 1])**.** *Let $\mathbf{m}$ be a simple matrix iteration, and let $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ be a Polish relational system coded in V. Assume that, for any $\alpha \in I$, there is some $\xi_\alpha < \pi$ such that $\mathbb{P}_{\alpha+1,\xi_\alpha}$ adds a real $\dot{c}_\alpha \in X$ that is $\mathbf{R}$-unbounded over V_{α,ξ_α}. (Here, $\alpha + 1$ denotes the immediate successor of α in I.) Then, for any $\alpha \in I$, $\mathbb{P}_{\alpha+1,\pi}$ forces that $\dot{c}_\alpha$ is $\mathbf{R}$-unbounded over $V_{\alpha,\pi}$.*

In addition, if $\mathbf{m}$ satisfies the hypothesis of Lemma 1.6.4 with ν a cardinal of uncountable cofinality and $\theta = \mathrm{cf}(\nu)$, and $f : \mathrm{cf}(\nu) \to \nu$ is increasing and cofinal, then $\mathbb{P}_{\nu,\pi}$ forces that $\{\dot{c}_{f(\zeta)} : \zeta < \mathrm{cf}(\nu)\}$ is a strongly $\mathrm{cf}(\nu)$-$\mathbf{R}$-unbounded family.

2

CONSTELLATIONS OF CICHOŃ'S DIAGRAM VIA FILTER-LINKEDNESS

This chapter is based on the paper [BCM21]:

Filter-linkedness and its effect on preservation of cardinal characteristics

joint work with Jörg Brendle and Diego Mejía, published in the Annals of Pure and Applied Logic, volumen 172, number 1, 2021.

Mejía [Mej19] introduced a new notion of posets, called *μ-Frechet-linked* (denoted μ-Fr-linked) to prove that μ-Fr-linked posets are μ^+-$\mathbf{D}$-good, which turns out to be useful to preserve strongly μ^+-$\mathbf{D}$-unbounded families of ω^ω from the ground model. He also proves that those posets behave well to preserve certain type of mad families (like those added by $\mathbb{H}_\theta$ for $\theta > \mu$).

In this chapter, we generalize the notion of Fr-linked and show its use to preserve certain type of unbounded families and mad families from the ground model. We introduce this notion in Section 2.1 as well as its corresponding notions of linkedness and Knaster for posets. We also present examples related to these notions. In Section 2.2, we show Theorem D and we show how μ-F-linkedness and the θ-F-Knaster property behave in FS iterations and FS products. We define the notion of $<\kappa$-uf-extendable matrix iterations and prove Theorem F in Section 2.3. We show applications of Theorem F in Section 2.4, concretely, we prove Theorem E, G, H, and I. In Section 2.5, we discuss some open questions and recent updates related to this research.

2.1 Filter-linkedness

In this section, we introduce the notion of Filter-linkedness, and others related notions and examples. The next definition plays a central role in this chapter.

Definition 2.1.1. Let $\mathbb{P}$ be a poset, F a free filter on ω and let μ be an infinite cardinal.

(1) If $\bar{p} = \langle p_n : n < \omega \rangle$ is a sequence in $\mathbb{P}$, denote by $\dot{W}_{\mathbb{P}}(\bar{p})$ the $\mathbb{P}$-name of $\{n < \omega : p_n \in \dot{G}\}$. When the forcing is understood from the context, we just write $\dot{W}(\bar{p})$.

(2) A set $Q \subseteq \mathbb{P}$ is *F-linked* if, for any sequence $\bar{p} = \langle p_n : n < \omega \rangle$ in Q, there exists a $q \in \mathbb{P}$ that forces $\dot{W}(\bar{p}) \in F^+$.

(3) A set $Q \subseteq \mathbb{P}$ is *ultrafilter-linked*, abbreviated *uf-linked*, if Q is D-linked for any non-principal ultrafilter D on ω.

(4) The poset $\mathbb{P}$ is *μ-F-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} P_\alpha$ for some sequence $\langle P_\alpha : \alpha < \mu \rangle$ of F-linked subsets of $\mathbb{P}$.

When each P_α is uf-linked, we say that $\mathbb{P}$ is *μ-uf-linked*.

When $\mu = \aleph_0$, we write *σ-F-linked* and *σ-uf-linked*.

(5) When κ is an uncountable cardinal, say that $\mathbb{P}$ is *$<\kappa$-F-linked* if it is μ-F-linked for some infinite cardinal $\mu < \kappa$. Likewise, define *$<\kappa$-uf-linked*.

(6) The poset $\mathbb{P}$ is *μ-F-Knaster* if any subset of $\mathbb{P}$ of size μ contains an F-linked set of size μ.

Say that $\mathbb{P}$ is *μ-uf-Knaster* if any subset of $\mathbb{P}$ of size μ contains a uf-linked set of size μ.[1]

When $F \subseteq F'$ are free filters, it is clear that any F'-linked set is F-linked. In particular, a set is uf-linked iff it is F-linked for every free filter F. Though Fr-linked is the weakest, and uf-linked is the strongest among these properties, they are equivalent for some posets.

Lemma 2.1.2 ([Mej19, Lemma 5.5])**.** *Let $\mathbb{P}$ be a poset.*

(a) If F is a free filter on ω generated by $<\mathfrak{p}$-many sets, then any subset of $\mathbb{P}$ is F-linked iff it is Fr*-linked.*

(b) If $\mathbb{P}$ has $\mathfrak{p}$-cc then any subset of $\mathbb{P}$ is uf-linked iff it is Fr*-linked.*

Proof. It is enough to show that, if $\mathbb{P}$ is a poset and F is a free filter on ω such that either F is generated by $<\mathfrak{p}$-many sets or $\mathbb{P}$ is $\mathfrak{p}$-cc, then any Fr-linked subset of $\mathbb{P}$ is F-linked. Towards a contradiction, assume that $Q \subseteq \mathbb{P}$ is Fr-linked but not F-linked, so there are a countable sequence $\langle p_n : n < \omega \rangle$ in Q, a maximal antichain $A \subseteq \mathbb{P}$ and a sequence $\langle a_r : r \in A \rangle$ in F such that each $r \in A$ is incompatible with p_n for every $n \in a_r$. In any of the two cases of the hypothesis, it can be concluded that there is some pseudo-intersection $a \in [\omega]^{\aleph_0}$ of $\langle a_r : r \in A \rangle$. Hence each $r \in A$ forces $p_n \in \dot{G}$ for only finitely many $n \in a$, which means that $\mathbb{P}$ forces the same. However, since Q is Fr-linked, there is some $q \in \mathbb{P}$ that forces $\exists^\infty n \in a (p_n \in \dot{G})$, a contradiction. □

Remark 2.1.3. Let $\mathbb{P}$ be a poset and μ an infinite cardinal.

(1) Any Fr-linked subset of $\mathbb{P}$ cannot contain infinite antichains of $\mathbb{P}$, that is, it is *finite-cc*.[2]

(2) Any μ-Fr-linked poset is μ-finite-cc (i.e., the union of $\leq\mu$-many finite-cc sets). The property "finite-cc" is absolute for transitive models of ZFC. [3]

(3) Any μ-Fr-linked poset is μ^+-Fr-Knaster.

(4) By (1), if θ is an infinite cardinal then any θ-Fr-Knaster poset is θ-finite-cc-Knaster (that is, any subset of the poset of size θ contains a finite-cc set of size θ). Also, any θ-finite-cc-Knaster poset has the θ-Knaster property because, by Erdős–Dushnik–Miller [DM41], every finite-cc set of size θ contains a linked set of the same size.

[1] In general, this notion is stronger than "$\mathbb{P}$ is μ-D-Knaster for every non-principal ultrafilter D on ω". Likewise for the notion of μ-uf-linked.

[2] Say that $Q \subseteq \mathbb{P}$ is *finite-cc* if every antichain of $\mathbb{P}$ contained in Q is finite.

[3] Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Consider the tree $T_Q \subseteq Q^{<\omega}$ defined by $t \in T_Q$ iff $\{t(k) : k < |t|\}$ is an antichain of $\mathbb{P}$. Note that Q is finite-cc iff T_Q does not have an infinite branch, which is an absolute property for transitive models of ZFC.

(5) It is clear that any singleton is uf-linked. Hence, any poset of size $\leq\mu$ is μ-uf-linked. In particular, Cohen forcing is σ-uf-linked.

(6) Assume that $\mathbb{P}$ is a $\mathfrak{p}$-cc poset. In view of Lemma 2.1.2, $\mathbb{P}$ is μ-Fr-linked iff it is μ-uf-linked. In the same way, $\mathbb{P}$ is θ-Fr-Knaster iff it is θ-uf-Knaster. Note that, for $\theta \leq \mathfrak{p}$, θ-Fr-Knaster implies θ-Knaster (and hence $\mathfrak{p}$-cc) by (4).

(7) c

Now we look at instances of σ-uf-linked posets. The following result indicates that random forcing is σ-uf-linked. To this end, we define the following: For a Boolean algebra $\mathbb{B}$, say that $\mu : \mathbb{B} \to [0,1]$ is a *strictly positive finitely additive (s.p.f.a.) measure* if it fulfills:

- $\mu(\mathbf{1}_{\mathbb{B}}) = 1$,
- $\mu(a \vee a') = \mu(a) + \mu(a')$ for all $a, a' \in \mathbb{B}$ such that $a \wedge a' = 0_{\mathbb{B}}$, and
- $\mu(a) = 0$ iff $a = \mathbf{0}_{\mathbb{B}}$.

Note that any Boolean algebra with a s.p.f.a. measure is ccc.

Lemma 2.1.4 ([Mej19, Lemma 3.29]). *Any complete Boolean algebra that admits a strictly-positive σ-additive measure is σ-uf-linked. In particular, any random algebra is σ-uf-linked.*

Proof. By Lemma 2.1.2 it suffices to prove that any such algebra is σ-Fr-linked.

Let $\mathbb{B}$ be a complete Boolean algebra that admits a strictly positive σ-additive measure μ. For $m < \omega$, defefine

$$B_m = \left\{ a \in \mathbb{B} : \mu(a) \geq \frac{1}{m+1} \right\}.$$

It is clear that $\mathbb{B} = \bigcup_{m<\omega} B_m$. To finish the proof, it is enough to show that B_m is Fr-linked. Assume the contrary, so by Remark 2.1.3(7) there is a sequence $\langle a_n : n < \omega \rangle$ in B_m and a maximal antichain $\langle a'_n : n < \omega \rangle$ in $\mathbb{B}$ (where each $\mu(a'_n) > 0$) such that for each $n < \omega$, $a'_n \wedge a_k = \mathbf{0}_{\mathbb{B}}$ for all but finitely many $k < \omega$. Next, construct an increasing function $g : \omega \to \omega$ such that $a'_n \wedge a_k = \mathbf{0}_{\mathbb{B}}$ for all $k \geq g(n)$. Find $n^* < \omega$ such that the measure of $a^* := \bigvee_{n<n^*} a_n$ is strictly larger than $1 - \frac{1}{m+1}$, which is possible because $\mathbf{1}_{\mathbb{B}} = \bigvee_{n<\omega} A_n$. Hence $\mu(a^* \wedge a) > 0$ for any $a \in B_m$, but this contradicts that $\mu(a^* \wedge a_k) = 0$ for all $k \geq g(n^* - 1)$, which finishes the proof. $\square$

We finish this section by proving any poset of the form $\mathbb{E}^h_b$ (see Subsection 1.5.5) is σ-uf-linked. This actually follows the idea of Miller's proof that $\mathbb{E}$ is $\mathbf{D}$-good (see [Mil81], in fact, his proof indicates that $\mathbb{E}$ is σ-uf-linked). To see this, we use ultrafilter limits.

Definition 2.1.5. Let D be an ultrafilter on $\mathcal{P}(\omega)$, X a topological space. If $\bar{x} = \langle x_n : n < \omega \rangle$ is a sequence on X and $x \in X$, we say that $\bar{x}$ *D-converges to x* if, for every open neighborhood U of x, $\{n < \omega : x_n \in U\} \in D$. Here, we also say that *x is a D-limit of $\bar{x}$*.

Note that there is at most one D-limit for Hausdorff spaces. In this case, we denote by $\lim^D_n x_n$ the ultrafilter limit of $\bar{x}$. Existence can always be guaranteed from compactness.

Lemma 2.1.6. *If X is a compact Hausdorff space and D is an ultrafilter on ω, then any countable sequence in X has a unique ultrafilter limit.*

Proof. Towards a contradiction, assume that there is a sequence $\langle x_n : n < \omega \rangle$ on X without D-limit. So, for any $x \in X$, there is some open neighborhood U_x of x such that $a_x := \{n < \omega : x_n \notin U_x\} \in D$. By compactness, there is some finite $F \subseteq X$ such that $\bigcup_{x\in F} U_x = X$. On the other hand, $\bigcap_{x\in F} a_x \in D$, but $\bigcap_{x\in F} a_x = \{n < \omega : x_n \notin X\} = \emptyset$, a contradiction. $\square$

Note that, for any $h' \in \omega^\omega$, $\mathcal{S}(b,h')$ is a compact subspace of $\mathcal{P}(\omega)^\omega$ (with the product topology where $\mathcal{P}(\omega)$ is the Cantor space) so, for any $m<\omega$, $\mathcal{S}(b, m\cdot h)$ is a compact space.

Example 2.1.7. Recall from Subsection 1.2.1 that $\mathbf{Lb}_2$ denotes the Lebesgue measure on 2^ω.

(1) Given a real $\delta \in (0,1)$ set

$$\mathbb{B}_{\geq\delta} := \{T \in \mathbb{B} : \mathbf{Lb}_2([T]) \geq \delta\}$$

This set is a compact subspace of $2^{2^{<\omega}}$ (with the Cantor-space topology). In particular, for each $(s,m) \in 2^{<\omega}\times\omega$, $\mathbb{B}(s,m)$ is a compact subspace (see Subsection 1.5.2). Therefore, every sequence in $\mathbb{B}_{\geq\delta}$ has its D-limit in $\mathbb{B}_{\geq\delta}$ for every ultrafilter D on $\mathcal{P}(\omega)$. Even more, if $\bar{p} = \langle p_n : n<\omega\rangle$ is a sequence in $\mathbb{B}_{\geq\delta}$, then $q = \lim^D_n p_n$ satisfies that, for any $t \in 2^{<\omega}$, $t \in q$ iff $\{n<\omega : t \in p_n\} \in D$.

(2) Fix b, h as in Subsection 1.5.5. Let D be an ultrafilter on ω, $s \in \mathrm{seq}_{<\omega}(b)$ and let $\bar{p} = \langle p_n : n<\omega\rangle$, where $p_n = (s, \varphi_n)$, be a sequence in $\mathbb{E}^h_b(s,m)$. Then the sequence $\langle \varphi_n : n<\omega\rangle$ has its D-limit φ in $\mathcal{S}(b, m\cdot h)$. Define *the D-limit of $\bar{p}$* by $\lim^D_n p_n := (s,\varphi)$. Note that, for any $k<\omega$, $k \in \varphi(i)$ iff $\{n<\omega : k \in \varphi_n(i)\} \in D$.

Lemma 2.1.8. *Let D be a non-principal ultrafilter on $\mathcal{P}(\omega)$ and b, h as in Subsection 1.5.5. If G is $\mathbb{E}^h_b$-generic over V then, in $V[G]$, D can be extended to an ultrafilter D^* on $\mathcal{P}(\omega)\cap V[G]$ such that, for any $(s,m) \in \mathrm{seq}_{<\omega}(b)\times\omega$ and any sequence $\bar{p} \in \mathbb{E}^h_b(s,m)\cap V$ that has its D-limit in G, $\dot{W}(\bar{p})[G] \in D^*$.*
In particular, $\mathbb{E}^h_b$ is σ-uf-linked.

This lemma is a direct consequence of the following claim in V.

Claim 2.1.9. *Assume $N<\omega$, $\{(s_k,m_k) : k<N\} \subseteq \mathrm{seq}_{<\omega}(b)\times\omega$, $\{\bar{p}^k : k<N\}$ such that each $\bar{p}^k = \langle p_{k,n} : n<\omega\rangle$ is a sequence in $\mathbb{E}^h_b(s_k,m_k)$, q_k is the D-limit of $\bar{p}^k$ for each $k<N$, and $q \in \mathbb{E}^h_b$ is stronger than every q_k. If $a \in D$ then q forces that $a \cap \bigcap_{k<N}\dot{W}(\bar{p}^k) \neq \emptyset$.*

Proof. We can express the forcing conditions as $p_{k,n} = (s_k, \varphi_{k,n})$, $q_k = (s_k,\varphi_k)$ where each φ_k is the D-limit of $\langle \varphi_{k,n} : n<\omega\rangle$ in $\mathcal{S}(b, m_k\cdot h)$. Assume that $q' = (t,\psi) \leq q$ in $\mathbb{E}^h_b$. Wlog, by making q' stronger, we can assume that $m^*\cdot h(i) < b(i)$ for any $i \geq |t|$ where $m^* := m_{q'} + \sum_{k<N} m_k$. Note that $U_k := \{\varphi \in \mathcal{S}(b, m_k\cdot h) : \forall i \in |t|\smallsetminus|s_k| (t(i) \notin \varphi(i))\}$ is an open neighborhood of φ_k in $\mathcal{S}(b, m_k\cdot h)$, so $\{n<\omega : \forall i \in |t|\smallsetminus|s_k|(t(i)\notin\varphi_{k,n}(i))\} \in D$. Hence $a \cap \bigcap_{k<N}\{n<\omega : \forall i \in |t|\smallsetminus|s_k|(t(i)\notin\varphi_{k,n}(i))\}$ is non-empty. Choose an n in that set and put $r := (t,\psi')$ where $\psi'(i) := \psi(i)\cup\bigcup_{k<N}\varphi_{k,n}(i)$. This is a condition in $\mathbb{E}^h_b$ because $|\psi'(i)| \leq m^*\cdot h(i)$ for every $i<\omega$, and $m^*\cdot h(i) < b(i)$ for $i \geq |t|$. Moreover, r is stronger than q' and $p_{n,k}$ for any $k<N$, so it forces $n \in a\cap\bigcap_{k<N}\dot{W}(\bar{p}^k)$. □

By Lemma 2.1.8, it is clear that if D is a non-principal ultrafilter on ω and $\bar{p}$ is a countable sequence in $\mathbb{E}^h_b(s,m)$ then its D-limit forces that $\dot{W}(\bar{p})$ is infinite. However, the existence of an ultrafilter-limit for a forcing notion does not suffice to guarantee a version of Lemma 2.1.8. The following remark provides a concrete counter-example.

Remark 2.1.10. Let $0<k<\omega$, $\delta := 1-2^{-k}$ and let $\langle I_n : n<\omega\rangle$ be an interval partition of $[k,\omega)$ such that $\sum_{n<\omega} 2^{-|I_n|} < 1$. For each $n<\omega$ define

$$p_n := \{t \in 2^{<\omega} : \text{if } |t|\geq k \text{ and } t(i)=0 \text{ for all } i<k, \text{ then } t(i)=0 \text{ for all } i \in I_n\cap|t|.\}$$

It can be shown that $p_n \in \mathbb{B}_{\geq\delta}$ and that $q := 2^{<\omega}$ is the D-limit of $\bar{p} := \langle p_n : n<\omega\rangle$. As $\mathbf{Lb}_2([p_n]) = 1-2^{-k}+2^{-k-|I_n|}$, $\mathbf{Lb}_2(\bigcup_{n<\omega}[p_n]) \leq 1-2^{-k}+2^{-k}\sum_{n<\omega}2^{-|I_n|} < 1$, so $[q]\smallsetminus\bigcup_{n<\omega}[p_n]$ has positive measure. Hence, there is an $r \in \mathbb{B}$ such that $[r]\cap\bigcup_{n<\omega}[p_n] = \emptyset$, so r forces that $\dot{W}(\bar{p}) = \emptyset$.

2.2 Preservation of strongly unbounded families and of mad families

Linkedness and Knaster notions associated with filters actually work to preserve strongly $\mathbf{D}$-unbounded families and certain type of mad families (see Definition 2.2.3). In [Mej19] it was proved that μ-Fr-linked posets satisfy stronger properties than these type of preservation, for instance,

Theorem 2.2.1 ([Mej19, Thm. 3.30])**.** *Any μ-Fr-linked poset is μ^+-$\mathbf{D}$-good. In particular, it preserves all the strongly κ-$\mathbf{D}$-unbounded families from the ground model for any regular $\kappa \geq \mu^+$.*

Proof. Let $\mathbb{P}$ be a μ-Fr-linked poset witnessed by $\langle P_i : i \in I\rangle$ with $|I| \leq \mu$. Fix a $\mathbb{P}$-name $\dot{h}$ for a real in ω^ω. For each $i \in I$ define $h_i : \omega \to \omega + 1$ by

$$h_i(n) := \min\{l \leq \omega : \forall p \in P_i (p \nVdash l \leq \dot{h}(n))\}.$$

We first show that $h_i(n) < \omega$ for all $n < \omega$. If not, for each $l < \omega$ choose a $p_l \in P_i$ such that $p_l \Vdash l \leq \dot{h}(n)$. Since $\bar{p} = \langle p_l : l < \omega\rangle$ is a sequence in P_i, there is a condition $q \in \mathbb{P}$ such that $q \Vdash$"$\dot{W}(\bar{p})$ is infinite". Therefore, q forces that $l \leq \dot{h}(n)$ for infinitely many $l < \omega$, which is a contradiction.

Put $H := \{h_i : i \in I\}$. Suppose that $x \in \omega^\omega$ is unbounded over H and $p \in \mathbb{P}$. There exists an $i \in I$ and a $p_0 \leq p$ in P_i so, for any $n < \omega$ such that $h_i(n) < x(n)$, there is a $p_1 \leq p_0$ such that $p_1 \Vdash \dot{h}(n) < h_i(n) < x(n)$. As there are infinitely many such n, we can conclude that $\Vdash x \not\leq^* \dot{h}$. □

The preservation of strongly unbounded families via Frechet-Knaster posets actually generalizes [GMS16, Main Lemma 4.6].

Theorem 2.2.2. *If κ is an uncountable regular cardinal then any κ-Fr-Knaster poset preserves all the strongly κ-$\mathbf{D}$-unbounded families from the ground model.*

Proof. Let $\mathbb{P}$ be a κ-Fr-Knaster poset and let $F \subseteq \omega^\omega$ be a strongly κ-$\mathbf{D}$-unbounded family in the ground model. Towards a contradiction, assume that there is a $\mathbb{P}$-name $\dot{h}$ of a real in ω^ω and a $p \in \mathbb{P}$ such that $p \Vdash |\{x \in F : x \leq^* \dot{h}\}| \geq \kappa$. Find $F' \subseteq F$ of size κ, a family of conditions $\{p_x : x \in F'\} \subseteq \mathbb{P}$ and a natural number m such that, for each $x \in F'$, $p_x \leq p$ and $p_x \Vdash \forall n \geq m (x(n) \leq \dot{h}(n))$. As $\mathbb{P}$ is κ-Fr-Knaster, there is some $F'' \subseteq F'$ of size κ such that $\{p_x : x \in F''\}$ is Fr-linked.

Note that there is a $j \geq m$ such that the set $\{x(j) : x \in F''\}$ is infinite. (otherwise F'' would be bounded, which contradicts that F is strongly κ-$\mathbf{D}$-unbounded). Choose $\{x_n : n < \omega\} \subseteq F''$ such that $x_n(j) \neq x_{n'}(j)$ whenever $n \neq n'$. For each $n < \omega$, put $p_n := p_{x_n}$. As $\bar{p} = \langle p_n : n < \omega\rangle$ is a sequence in a Fr-linked set, there is a condition $q \in \mathbb{P}$ such that $q \Vdash$"$\dot{W}(\bar{p})$ is infinite". Therefore, q forces that $\exists^\infty n < \omega (x_n(j) \leq \dot{h}(j))$, which is a contradiction. □

We now turn to preservation of mad families. The relational system defined below is inspired by [BF11].

Definition 2.2.3. Fix $A \subseteq [\omega]^{\aleph_0}$.

(1) Let $P \subseteq \left[[\omega]^{\aleph_0}\right]^{<\aleph_0}$. For $x \subseteq \omega$ and $h : \omega \times P \to \omega$, define $x \sqsubset^* h$ by

$$\forall^\infty n < \omega \forall F \in P([n, h(n,F)) \smallsetminus \bigcup F \nsubseteq x).$$

(2) Define the relational system $\mathbf{Md}(A) := \langle [\omega]^{\aleph_0}, \omega^{\omega \times [A]^{<\aleph_0}}, \sqsubset^* \rangle$.

(3) If κ is an infinite cardinal, say that A is a *κ-strong-***Md** *family* if A is strongly κ-$\mathbf{Md}(A)$-unbounded. When $\kappa = \aleph_1$ we just say *strong-***Md** *family*.

Denote $\mathcal{I}(A) := \{w \subseteq \omega : \exists F \in [A]^{<\aleph_0}(w \subseteq^* \bigcup F)\}$. For $y \in [\omega]^{\aleph_0} \smallsetminus \mathcal{I}(A)$ we can define a function $h_y : \omega \times [A]^{<\aleph_0} \to \omega$ such that, for every $n < \omega$ and $F \subseteq A$ finite, $y \cap [n, h_y(n, F)) \smallsetminus \bigcup F \neq \emptyset$. Hence, if $x \in [\omega]^{\aleph_0}$ and $x \not\sqsubset^* h_y$ then $x \cap y$ is infinite. This actually proves the following result.

Lemma 2.2.4 ([BF11, Lemma 3]). *Let M be a transitive model of* ZFC *with $A \in M$. If $a^* \in [\omega]^{\aleph_0}$ is* $\mathbf{Md}(A)$*-unbounded over M then $|a^* \cap y| = \aleph_0$ for any $y \in [\omega]^{\aleph_0} \cap M \smallsetminus \mathcal{I}(A)$.*

Lemma 2.2.5. *Let Z be a set, $z^* \in Z$ and let $\dot{A} := \langle \dot{a}_z : z \in Z \rangle$ be the a.d. family added by $\mathbb{H}_Z$.*

(a) [BF11, Lemma 4] $\mathbb{H}_Z$ forces that $\dot{a}_{z^*}$ is $\mathbf{Md}(\dot{A}{\restriction}(Z \smallsetminus \{z^*\}))$-unbounded over $V^{\mathbb{H}_{Z \smallsetminus \{z^*\}}}$.

*(b) If Z is uncountable then $\mathbb{H}_Z$ forces that $\dot{A}$ is a strong-***Md** *a.d. family.*

Proof. We show (b). Let $\dot{h}$ be a $\mathbb{H}_Z$-name of a function in $\omega^{\omega \times [\dot{A}]^{<\aleph_0}}$. Note that the set

$$\{C \in [Z]^{\aleph_0} : \dot{h}{\restriction}(\omega \times [\dot{A}{\restriction}C]^{<\aleph_0}) \text{ is an } \mathbb{H}_C\text{-name}\}$$

is a club in $[Z]^{\aleph_0}$ (here, $\dot{A}{\restriction}C := \{\dot{a}_z : z \in C\}$), so choose some C in this club set. Hence, by (a), for any $z^* \in Z \smallsetminus C$, $\mathbb{H}_Z$ forces that $\dot{a}_{z^*} \not\sqsubset^* \dot{h}{\restriction}(\omega \times [\dot{A}{\restriction}C]^{<\aleph_0})$, which implies that $\dot{a}_{z^*} \not\sqsubset^* \dot{h}$. □

Theorem 2.2.6. *If κ is an uncountable regular cardinal then any κ-*Fr*-Knaster poset preserves all the κ-strong-***Md** *families from the ground model.*

Proof. Let $\mathbb{P}$ be a κ-Fr-Knaster poset and let A be a κ-strong-**Md** family. Assume, towards a contradiction, that there is some $p \in \mathbb{P}$ and some $\mathbb{P}$-name $\dot{h}$ of a function in $\omega^{\omega \times [A]^{<\aleph_0}}$ such that $p \Vdash |\{a \in A : a \sqsubset^* \dot{h}\}| \geq \kappa$. As in the proof of Theorem 2.2.2, find an $A' \subseteq A$ of size κ, $\{p_a : a \in A'\} \subseteq \mathbb{P}$ and an $m < \omega$ such that, for each $a \in A'$, $p_a \leq p$ and $p_a \Vdash \forall n \geq m \forall F \in [A]^{<\aleph_0}([n, \dot{h}(n, F)) \smallsetminus \bigcup F \not\subseteq a)$. We can also find an $A'' \subseteq A'$ of size κ such that $\{p_a : a \in A''\}$ is Fr-linked.

Claim 2.2.7. *The set of $k < \omega$ that satisfies $\exists F \in [A]^{<\aleph_0} \forall l \geq k \exists a \in A''([k, l) \smallsetminus \bigcup F \subseteq a)$ is infinite.*

Proof. Assume the contrary, that is, there is some $k_0 < \omega$ such that, for every $k \geq k_0$ and $F \in [A]^{<\aleph_0}$ there is a $g(k, F) < \omega$ such that $[k, g(k, F)) \smallsetminus \bigcup F \not\subseteq a$ for all $a \in A''$. This defines a function $g \in \omega^{\omega \times [A]^{<\aleph_0}}$ that $\sqsubset^*$-dominates all the members of A'', but this contradicts that A is strongly κ-$\mathbf{Md}(A)$-unbounded. This ends the proof of Claim 2.2.7. □

We continue the proof of Theorem 2.2.6. Choose a $k \geq m$ and one $F \in [A]^{<\aleph_0}$ as in Claim 2.2.7. Hence, for each $l \geq k$ there is some $a_l \in A''$ such that $[k, l) \smallsetminus \bigcup F \subseteq a_l$. Put $p_l := p_{a_l}$ and $\bar{p} := \langle p_l : l \geq k \rangle$, so there is a $q \in \mathbb{P}$ forcing that $\dot{W}(\bar{p})$ is infinite. Let G be $\mathbb{P}$-generic over V with $q \in G$ and work in $V[G]$. Denote $h := \dot{h}[G]$ and $W := \dot{W}(\bar{p})[G]$. Note that $[k, h(k, F)) \smallsetminus \bigcup F \not\subseteq a_l$ for any $l \in W$. On the other hand, $[k, l) \smallsetminus \bigcup F \subseteq a_l$ for any $l \geq k$, in particular, if $l \in W$ is chosen above $h(k, F)$ then $[k, h(k, F)) \smallsetminus \bigcup F \subseteq a_l$, a contradiction. This ends the proof of Theorem 2.2.6. □

We conclude this section by presenting some results about FS iterations and FS products of filter-linked and filter-Knaster posets. With the exception of the proof of Theorem 2.2.9, this part was taken care of, with a more general notation, in [Mej19, Sect. 5].

Theorem 2.2.8. *Let θ be an uncountable regular cardinal.*

(a) *Any FS iteration of θ-Fr-Knaster posets is θ-Fr-Knaster.*

(b) *Any FS iteration of θ-uf-Knaster posets is θ-uf-Knaster.*

(c) *If μ is an infinite cardinal, then any FS iteration of length $<(2^\mu)^+$ of μ-Fr-linked posets is μ-Fr-linked.*

Proof. See [Mej19, Rem. 5.11]. □

Theorem 2.2.9. *Let $\mathbb{Q}_0$ and $\mathbb{Q}_1$ be posets. If D_0 is a non-principal ultrafilter on ω and $Q_0 \subseteq \mathbb{Q}_0$ and $Q_1 \subseteq \mathbb{Q}_1$ are D_0-linked subsets, then $Q_0 \times Q_1$ is D_0-linked in $\mathbb{Q}_0 \times \mathbb{Q}_1$. In particular,*

(a) *The product of two μ-D_0-linked posets is μ-D_0-linked.*

(b) *If θ is regular, then the product of two θ-D_0-Knaster posets is θ-D_0-Knaster.*

Similar statements hold for "uf-linked" and "uf-Knaster".

To prove this theorem, we need the following result, which is a weaker version of [She00, Claim 1.6].

Lemma 2.2.10. *Let $M \subseteq N$ be transitive models of ZFC. In M, assume that $\mathbb{P}$ is a poset, D_0 is an ultrafilter on ω and, in N, assume that D is an ultrafilter that extends D_0. If G is $\mathbb{P}$-generic over N and $D'_0 \in M[G]$ is an ultrafilter on $\mathcal{P}(\omega) \cap M[G]$ that extends D_0 then, in $N[G]$, $D \cup D'_0$ can be extended to an ultrafilter on $\mathcal{P}(\omega) \cap N[G]$.*

Proof. Let $\dot{D}'_0 \in M$ be a $\mathbb{P}$-name of D'_0. Assume that $a \in D$, $\dot{b} \in M$ is a $\mathbb{P}$-name of a member of $\dot{D}'_0$, and $p \in \mathbb{P}$. Put $b'_0 := \{n < \omega : p \Vdash n \notin \dot{b}\}$. It is clear that $b'_0 \in M$ and that $p \Vdash b'_0 \cap \dot{b} = \emptyset$. Hence, $p \Vdash \omega \smallsetminus b'_0 \in \dot{D}'_0$, which implies that $\omega \smallsetminus b'_0 \in D_0$. Since $D_0 \subseteq D$ and $a \in D$, $a \smallsetminus b'_0 \in D$, so there is an $n \in a \smallsetminus b'_0$. Thus, in M, there is a $q \leq p$ that forces $n \in \dot{b}$, so q forces, in N, that $n \in a \cap \dot{b}$.[4] □

Proof of Theorem 2.2.9. Let $\bar{q} = \langle (q_{0,n}, q_{1,n}) : n < \omega \rangle$ be a sequence in $Q_0 \times Q_1$. Since both Q_0 and Q_1 are D_0-linked, for each $e \in \{0,1\}$ there is some $r_e \in \mathbb{Q}_e$ forcing $\dot{W}_{\mathbb{Q}_e}(\bar{q}_e) \in D_0^+$ where $\bar{q}_e = \langle q_{e,n} : n < \omega \rangle$. Now assume that G_0 is $\mathbb{Q}_0$-generic over V and G_1 is $\mathbb{Q}_1$-generic over $V[G_0]$ such that $(r_0, r_1) \in G_0 \times G_1$. Let $M := V$ and $N := V[G_0]$. In N, there is an ultrafilter $D \supseteq D_0 \cup \{W_{\mathbb{Q}_0}(\bar{p})\}$ and, in $M[G_1]$, there is an ultrafilter $D'_0 \supseteq D_0 \cup \{W_{\mathbb{Q}_1}(\bar{q})\}$. Thus, in $N[G_1]$, $D \cup D'_0$ has the finite intersection property, so $W_{\mathbb{Q}_0 \times \mathbb{Q}_1}(\bar{q}) = W_{\mathbb{Q}_0}(\bar{q}_0) \cap W_{\mathbb{Q}_1}(\bar{q}_1) \in D_0^+$. □

Theorem 2.2.11. *If κ is an uncountable regular cardinal, F is a free filter on ω, and $\mathbb{P}$ is a FS product of posets such that any finite subproduct is κ-F-Knaster, then $\mathbb{P}$ is κ-F-Knaster.*[5] *In particular, when F is an ultrafilter, any FS product of κ-F-Knaster posets is κ-F-Knaster (likewise for "uf-Knaster").*

Proof. Let λ be a cardinal and assume that $\mathbb{P}$ is the FS product of $\langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle$ as in the hypothesis. If $\langle p_\zeta : \zeta < \kappa \rangle \subseteq \mathbb{P}$ then, by the Δ-system Lemma, there is some $K \subseteq \kappa$ of size κ such that $\langle \mathrm{dom} p_\zeta : \zeta \in K \rangle$ forms a Δ-system with root R^*. Since $\prod_{\alpha \in R^*} \mathbb{Q}_\alpha$ is κ-F-Knaster, we can find a $K' \subseteq K$ of size κ such that $\{p_\zeta \restriction R^* : \zeta \in K'\}$ is F-linked.

Assume that $\langle \zeta_n : n < \omega \rangle \subseteq K'$. Hence, there is some $q \in \prod_{\alpha \in R^*} \mathbb{Q}_\alpha$ that forces $\{n < \omega : p_{\zeta_n} \restriction R^* \in \dot{G}\} \in F^+$. As a matter of fact, q forces that $\{n < \omega : p_{\zeta_n} \in \dot{G}\} \in F^+$. To see this, assume that $a \in F$ and $r \leq q$ in $\mathbb{P}$. Note that $\forall^\infty n < \omega(\mathrm{dom} r \cap \mathrm{dom} p_{\zeta_n} = R^*)$. On the

[4] Recall that $\mathbb{P} \subseteq M$ since $\mathbb{P} \in M$ and M is transitive.

[5] In the terminology of [Mej19, Sect. 5], the notion "F-linked" is *FS-productive*.

other hand, we can find some $s \le r{\restriction}R^*$ in $\prod_{\alpha\in R^*}\mathbb{Q}_\alpha$ and an $n \in a$ such that $s \le p_{\zeta_n}{\restriction}R^*$ and $\mathrm{dom} r \cap \mathrm{dom} p_{\zeta_n} = R^*$. Thus

$$r' := s \cup r{\restriction}(\mathrm{dom} r \smallsetminus R^*) \cup p_{\zeta_n}{\restriction}(\mathrm{dom} p_{\zeta_n} \smallsetminus R^*)$$

is a condition in $\mathbb{P}$ stronger than both r and p_{ζ_n}.

The latter statement is a consequence of Theorem 2.2.9. □

Theorem 2.2.12. *Let μ be an infinite cardinal, $\langle \mathbb{Q}_i : i \in I\rangle$ a sequence of μ-*Fr*-linked posets witnessed by $\langle Q_i(\zeta) : \zeta < \mu\rangle$ for each $i \in I$, and let $\mathbb{P}$ be the FS product of $\langle \mathbb{Q}_i : i \in I\rangle$. If*

(i) $|I| \le 2^\mu$ *and*

(ii) $\prod_{i\in u} Q_i(s(i))$ *is* Fr*-linked in* $\prod_{i\in u}\mathbb{Q}_i$ *for any finite $u \subseteq I$ and $s : u \to \mu$,*

*then $\mathbb{P}$ is μ-*Fr*-linked.*[6]

Proof. By a result of Engelking and Karłowicz [EK65], there is a set $H \subseteq \mu^I$ of size $\le \mu$ such that any finite partial function from I to μ is extended by some function in H.

For each $h \in H$ and $n < \omega$ define

$$Q_{h,n} := \{p \in \mathbb{P} : |\mathrm{dom} p| \le n \text{ and } \forall i \in \mathrm{dom} p (p(i) \in Q_i(h(i)))\}.$$

It is clear that these sets cover $\mathbb{P}$, so it remains to show that each $Q_{h,n}$ is Fr-linked. Let $\bar{p} = \langle p_k : k < \omega\rangle$ be a sequence in $Q_{h,n}$. By the Δ-system lemma, we can find $w \subseteq \omega$ infinite such that $\langle \mathrm{dom} p_k : k \in w\rangle$ form a Δ-system with root R^*. Hence, by (ii), there is some $q \in \prod_{i\in R^*}\mathbb{Q}_i$ forcing that $\{k \in w : p_k{\restriction}R^* \in \dot{G}\}$ is infinite. Similar to the last part of the proof of Theorem 2.2.11, it can be shown that q forces $w \cap \dot{W}_{\mathbb{P}}(\bar{p})$ is infinite. □

Before proceeding, a quick short remark on this lemma is in order: The reason the latter proof cannot guarantee the analog result for "F-linked" for other filters F in general is that, when finding the Δ-system, it cannot be guaranteed that $w \in F$. However, this can be done when F is a Ramsey ultrafilter, so Theorem 2.2.12 is valid for Ramsey ultrafilters in the place of Fr (even more, (ii) is redundant by Theorem 2.2.9).

2.3 Ultrafilter-extendable matrix iterations

This section is dedicated to prove Theorem F.

Definition 2.3.1. Let κ be an uncountable cardinal. A *$<\kappa$-ultrafilter-extendable matrix iteration* (abbreviated *$<\kappa$-uf-extendable*) is a simple matrix iteration $\mathbf{m}$ such that, for each $\xi < \pi^{\mathbf{m}}$, $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),\xi}$ forces that $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$ is a $<\kappa$-uf-linked poset.

As in Definition 1.6.3, we omit the upper index $\mathbf{m}$ when understood.

When $I^{\mathbf{m}} = \nu + 1$ for some ordinal ν, the FS iteration $\mathbb{P}_{\nu,\pi} = \langle \mathbb{P}_{\nu,\xi}, \dot{\mathbb{Q}}_{\nu,\xi} : \xi < \pi\rangle$ is not a FS iteration of $<\kappa$-uf-linked posets in general.

Definition 2.3.2. Let κ be uncountable regular. Given a $<\kappa$-uf-extendable matrix iteration $\mathbf{m}$, we define $\theta^{\mathbf{m}}_\xi$ and $\langle \dot{Q}^{\mathbf{m}}_\xi(\zeta) : \zeta < \theta^{\mathbf{m}}_\xi\rangle$ for $\xi < \pi^{\mathbf{m}}$ as follows. By Remark 2.1.3 (items (2)–(4)) it can be proved by induction on $\xi \le \pi$ that $\mathbb{P}_{\alpha,\xi}$ has the κ-Knaster property for every $\alpha \in I^{\mathbf{m}}$. Therefore, for each $\xi < \pi^{\mathbf{m}}$, we can find a cardinal $\theta^{\mathbf{m}}_\xi < \kappa$ (in the ground model) and a sequence $\langle \dot{Q}^{\mathbf{m}}_\xi(\zeta) : \zeta < \theta^{\mathbf{m}}_\xi\rangle$ of $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),\xi}$-names such that $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),\xi}$ forces that $\langle \dot{Q}^{\mathbf{m}}_\xi(\zeta) : \zeta < \theta^{\mathbf{m}}_\xi\rangle$ witnesses that $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$ is $<\kappa$-uf-linked. Again, upper indexes are omitted when understood.

[6] In the terminology of [Mej15, Sect. 5], if the notion "Fr-linked" is productive, then it is strongly productive.

Theorem 2.3.3. *Let κ be an uncountable regular cardinal and $\mathbf{m}$ a $<\kappa$-uf-extendable matrix iteration. Then $\mathbb{P}_{\alpha,\pi}$ is κ-uf-Knaster for any $\alpha \in I^{\mathbf{m}}$. In particular, it preserves any strongly κ-$\mathbf{D}$-unbounded family and any κ-strong-$\mathbf{Md}$ family from the ground model.*

Throughout this section, wlog we may assume that $I^{\mathbf{m}} = \gamma^{\mathbf{m}}$ is an ordinal (and again, we may omit the upper index).

The following is a version of the preceding result where the preserved strongly unbounded family is constructed within the matrix.

Theorem 2.3.4. *Let $\kappa \leq \mu$ be uncountable regular cardinals and let $\mathbf{m}$ be a $<\kappa$-uf-extendable matrix iteration. Assume that*

(i) $\gamma^{\mathbf{m}} > \mu$ *and* $\pi^{\mathbf{m}} \geq \mu$,

(ii) *for each* $\alpha < \mu$, $\Delta^{\mathbf{m}}(\alpha) = \alpha + 1$ *and* $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha} = \mathbb{C}$, *and*

(iii) $\dot{c}_\alpha$ *is the* $\mathbb{P}_{\alpha+1,\alpha+1}$*-name of the Cohen real added by* $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha}$.

Then, for any $\nu \in [\mu, \gamma^{\mathbf{m}})$, $\mathbb{P}_{\nu,\pi}$ *forces that* $\{\dot{c}_\alpha : \alpha < \mu\}$ *forms a* μ*-*$\mathbf{D}$*-strongly unbounded family.*

For the proof of both results, we need to work with special conditions of the matrix and with Δ-systems.

Definition 2.3.5. Let κ be a regular uncountable cardinal and let $\mathbf{m}$ be a $<\kappa$-uf-extendable matrix iteration. Let $\beta < \gamma$ and $\eta \leq \pi$.

(1) Define $\mathbb{P}^{+}_{\beta,\eta} = \mathbb{P}^{+\mathbf{m}}_{\beta,\eta}$ as the set of conditions $p \in \mathbb{P}_{\beta,\eta}$ such that, for each $\xi \in \operatorname{dom} p$ with $\Delta(\xi) \leq \beta$, $p(\xi)$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name.

Define $\mathbb{P}^{*}_{\beta,\eta} = \mathbb{P}^{*\mathbf{m}}_{\beta,\eta}$ as the set of conditions $p \in \mathbb{P}^{+}_{\beta,\eta}$ such that, for each $\xi \in \operatorname{dom} p$ with $\Delta(\xi) \leq \beta$, there is a $\zeta = \zeta_{p(\xi)} < \theta_\xi$ such that $\mathbb{P}_{\Delta(\xi),\xi}$ forces that $p(\xi) \in \dot{Q}_\xi(\zeta)$.

Note that $\mathbb{P}^{+}_{\beta,\eta}$ is a dense subset of $\mathbb{P}_{\beta,\eta}$, and $\mathbb{P}^{*}_{\beta,\eta}$ is a dense subset of $\mathbb{P}^{+}_{\beta,\eta}$.

(2) For each $p \in \mathbb{P}^{+}_{\beta,\eta}$, $\alpha \leq \beta$ and $\xi \leq \eta$, $p{\restriction}(\alpha,\xi)$ is the condition in $\mathbb{P}^{+}_{\alpha,\xi}$ defined by

(i) $\operatorname{dom}(p{\restriction}(\alpha,\xi)) = \operatorname{dom} p \cap \xi$, and

(ii) for each $\xi' \in \operatorname{dom}(p{\restriction}(\alpha,\xi))$,

$$p{\restriction}(\alpha,\xi)(\xi') = \begin{cases} p(\xi') & \text{if } \Delta(\xi') \leq \alpha, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Note that $p{\restriction}(\alpha,\xi) \in \mathbb{P}^{*}_{\alpha,\xi}$ whenever $p \in \mathbb{P}^{*}_{\beta,\eta}$

(3) A *uniform Δ-system in* $\mathbb{P}^{*}_{\beta,\eta}$ is a sequence $\bar{p} = \langle p_i : i \in J\rangle$ of conditions in $\mathbb{P}^{*}_{\beta,\eta}$ such that

(i) $\langle \operatorname{dom} p_i : i \in J\rangle$ forms a Δ-system with root R^*, and

(ii) for each $\xi \in R^*$ there is a $\zeta^*_\xi < \theta_\xi$ such that $\mathbb{P}_{\Delta(\xi),\xi}$ forces that $p_i(\xi) \in \dot{Q}_\xi(\zeta^*_\xi)$ for all $i \in J$.

Note that $\mathbb{P}^{+}_{\beta,\eta}$ and $p{\restriction}(\beta,\eta)$ can be defined for simple matrix iterations.

The core of our main result is the following lemma.

Lemma 2.3.6. *Let $\mathbf{m}$ be a $<\kappa$-uf-extendable matrix iteration with sequences of names as in Definition 2.3.2 (without assuming that κ is regular). If $\nu \in I^{\mathbf{m}}$ and $\bar{p} = \langle p_n : n < \omega\rangle$ is a uniform Δ-system in $\mathbb{P}^{*}_{\nu,\pi}$ then there is a $q \in \mathbb{P}_{\nu,\pi}$ forcing that $\dot{W}_{\mathbb{P}_{\nu,\pi}}(\bar{p})$ is infinite. Moreover, if D is a non-principal ultrafilter on ω in the ground model then there is some $q \in \mathbb{P}^{+}_{\nu,\pi}$ that forces $\dot{W}_{\mathbb{P}_{\nu,\pi}}(\bar{p}) \in D^{+}$.*

Proof of Theorem 2.3.3. Let $\langle p_\zeta : \zeta < \kappa \rangle$ be a sequence of conditions in $\mathbb{P}_{\alpha,\pi}$. For each $\zeta < \kappa$ find a $p'_\zeta \in \mathbb{P}^*_{\alpha,\pi}$ stronger than p_ζ. By the Δ-system lemma and some easy combinatorial arguments, we can find a $K \subseteq \kappa$ of size κ such that $\{p'_\zeta : \zeta \in K\}$ forms a uniform Δ-system in $\mathbb{P}^*_{\alpha,\pi}$. Therefore, by Lemma 2.3.6, $\{p'_\zeta : \zeta \in K\}$ is uf-linked. Hence, $\{p_\zeta : \zeta \in K\}$ is uf-linked. □

Proof of Theorem 2.3.4. First note that $\mathbb{P}_{\nu,\mu}$ results from a FS iteration of length μ of Cohen forcing and that, for each $\alpha < \mu$, $\dot{c}_\alpha$ is forced to be a Cohen real over $V_{\nu,\alpha}$. Even more, we can assume that $\theta_\alpha = \aleph_0$, $\dot{Q}_\alpha(n)$ is a singleton (in the ground model, not just a name), and $\omega^{<\omega} = \bigcup_{n<\omega} \dot{Q}_\alpha(n)$. Hence, $\mathbb{P}^*_{\nu,\mu} = \mathbb{P}^*_{\mu,\mu} = \mathbb{C}_\mu$.

Towards a contradiction, assume that there is a $\mathbb{P}_{\nu,\pi}$-name $\dot{h}$ of a real in ω^ω and a $p \in \mathbb{P}_{\nu,\pi}$ such that $p \Vdash_{\nu,\pi} |\{\alpha < \mu : \dot{c}_\alpha \leq^* \dot{h}\}| \geq \mu$. Find $K \subseteq \mu$ of size μ, a family of conditions $\{p_\alpha : \alpha \in K\} \subseteq \mathbb{P}^*_{\nu,\pi}$ and a natural number m such that, for each $\alpha \in K$, $\alpha \in \mathrm{dom} p_\alpha$, $p_\alpha \leq p$ and $p_\alpha \Vdash \forall n \geq m(\dot{c}_\alpha(n) \leq \dot{h}(n))$. Wlog, also assume that $|p(\alpha)| \geq m$ for all $\alpha \in K$. By the Δ-system lemma and some easy combinatorial arguments, we can find $K' \subseteq K$ of size μ such that $\{p_\alpha : \alpha \in K'\}$ forms a uniform Δ-system in $\mathbb{P}^*_{\nu,\pi}$ and there is some $t \in \omega^{<\omega}$ of length $m' \geq m$ such that, for all $\alpha \in K'$, $p_\alpha(\alpha) = t$. Choose $\{\alpha_n : n < \omega\} \subseteq K'$ (one-to-one enumeration). Define p'_n identical to p_{α_n} with the sole difference that $p'_n(\alpha) := p_{\alpha_n}(\alpha) \cup \{(m', n)\}$. Note that $\bar{p}' = \langle p'_n : n < \omega \rangle$ forms a countable uniform Δ-system. Therefore, by Lemma 2.3.6, there is a condition $q \in \mathbb{P}_{\nu,\pi}$ such that $q \Vdash$"$\dot{W}(\bar{p}')$ is infinite", so q forces that $\exists^\infty n < \omega(\dot{c}_{\alpha_n}(m') = n \leq \dot{h}(m'))$, which is a contradiction. □

We now focus on the proof of Lemma 2.3.6. We start with some preliminary results before developing the proof.

Lemma 2.3.7. *Let* $\mathbf{m}$ *be a simple matrix iteration,* $\alpha \leq \beta < \gamma$ *and* $\xi \leq \eta \leq \pi$. *Then:*

(a) For any $p \in \mathbb{P}^+_{\beta,\eta}$, *if* $q \leq p{\restriction}(\alpha,\xi)$ *in* $\mathbb{P}^+_{\alpha,\xi}$, *then there is some* $p' \leq p$ *in* $\mathbb{P}^+_{\beta,\eta}$ *such that* $q = p'{\restriction}(\alpha,\xi)$.

(b) If β *is limit and* $\beta \notin \mathrm{ran}\Delta^{\mathbf{m}}$ *then* $\mathbb{P}^+_{\beta,\xi} = \mathrm{limdir}_{\alpha<\beta}\mathbb{P}^+_{\alpha,\xi}$.

Even more, similar statements hold for $\mathbb{P}^*_{\beta,\eta}$ *when* $\mathbf{m}$ *is a* $<\kappa$*-uf-extendable matrix iteration.*

Proof. To see (a), define p' such that $\mathrm{dom} p' = \mathrm{dom} p \cup \mathrm{dom} q$ and $p'(\xi)$ is determined by the following cases: when $\xi \in \mathrm{dom} p \smallsetminus \mathrm{dom} q$, $p'(\xi) := p(\xi)$; when $\xi \in \mathrm{dom} q$, put $p'(\xi) := p(\xi)$ if $\alpha < \Delta(\xi)$, otherwise $p'(\xi) := q(\xi)$.

Now we show (b) by induction on ξ. The case $\xi = 0$ and the limit step are immediate. For the successor step, assume that $\mathbb{P}^+_{\beta,\xi} = \mathrm{limdir}_{\alpha<\beta}\mathbb{P}^+_{\alpha,\xi}$. If $\beta < \Delta(\xi)$ then $\mathbb{P}^+_{\alpha,\xi+1} = \mathbb{P}^+_{\alpha,\xi} * \mathbb{1}$ for any $\alpha \leq \beta$, so the conclusion follows; if $\Delta(\xi) \leq \beta$ then $\Delta(\xi) < \beta$ (because $\Delta(\xi) \neq \beta$) and, whenever $p \in \mathbb{P}^+_{\beta,\xi+1}$, by induction hypothesis $p{\restriction}(\beta,\xi) \in \mathbb{P}^+_{\alpha,\xi}$ for some $\alpha \in [\Delta(\xi), \beta)$. On the other hand, $p(\xi)$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a condition in $\dot{\mathbb{Q}}_\xi$, so $p \in \mathbb{P}^+_{\alpha,\xi+1}$. □

Lemma 2.3.8. *Let* $\mathbb{P}_\pi = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \pi \rangle$ *be a FS iteration with* π *limit. Assume:*

(i) $\bar{p} = \langle p_n : n < \omega \rangle$ *is a sequence of conditions in* $\mathbb{P}_\pi$.

(ii) $\langle \dot{D}_\xi : \xi < \pi \rangle$ *is a sequence such that each* $\dot{D}_\xi$ *is a* $\mathbb{P}_\xi$*-name of a non-principal ultrafilter on* ω *that contains* $\dot{D}_{\xi_0}$ *for any* $\xi_0 < \xi$.

(iii) $q \in \mathbb{P}_\pi$.

(iv) For any $\xi < \pi$, $q{\restriction}\xi$ *forces that* $\dot{W}_{\mathbb{P}_\xi}(\bar{p}{\restriction}\xi) \in \dot{D}_\xi$.

Then q *forces that* $\bigcup_{\xi<\pi} \dot{D}_\xi \cup \{\dot{W}_{\mathbb{P}_\pi}(\bar{p})\}$ *can be extended to an ultrafilter.*

Proof. Let $r \leq q$ in $\mathbb{P}_\pi$ and $\dot{b}$ a $\mathbb{P}_\pi$-name of a member of $\bigcup_{\xi<\pi} \dot{D}_\xi$. Wlog (by strengthening r if necessary), we may assume that there is a $\xi < \pi$ such that $r, q \in \mathbb{P}_\xi$ and $\dot{b}$ is (forced to be equal to) a $\mathbb{P}_\xi$-name of a member of $\dot{D}_\xi$. By (iv), there are some $r' \leq r$ in $\mathbb{P}_\xi$ and an $n < \omega$ such that $r' \leq p_n{\restriction}\xi$ and $r' \Vdash_\xi n \in \dot{b}$. Hence, $q' := r' \cup p_n{\restriction}[\xi, \pi)$ forces in $\mathbb{P}_\pi$ that $n \in \dot{b} \cap \dot{W}_{\mathbb{P}_\pi}(\bar{p})$. □

The last ingredient need for the proof of Lemma 2.3.6 is the next result.

Lemma 2.3.9. *Let* $\mathbf{s}$ *be a simple matrix iteration with* $I^{\mathbf{s}} = \{0, 1\}$*. Assume:*

(i) $\pi = \pi^{\mathbf{s}}$ *is limit.*

(ii) $\bar{p} = \langle p_n : n < \omega \rangle$ *is a sequence of conditions in* $\mathbb{P}^+_{1,\pi}$.

(iii) $\langle \dot{D}_{i,\xi} : i < 2, \xi < \pi \rangle$ *is a sequence such that each* $\dot{D}_{i,\xi}$ *is a* $\mathbb{P}_{i,\xi}$*-name of a non-principal ultrafilter on* ω *that contains* $\dot{D}_{i_0,\xi_0}$ *for any* $i_0 \leq i$ *and* $\xi_0 \leq \xi$.

(iv) $\dot{D}_{0,\pi}$ *is a* $\mathbb{P}_{0,\pi}$*-name of an ultrafilter containing* $\bigcup_{\xi<\pi} \dot{D}_{0,\xi}$.

(v) $q \in \mathbb{P}^+_{1,\pi}$.

(vi) For any $\xi < \pi$, $q{\restriction}(1, \xi)$ *forces that* $\dot{W}_{\mathbb{P}_{1,\xi}}(\bar{p}{\restriction}(1, \xi)) \in \dot{D}_{1,\xi}$.

(vii) $q{\restriction}(0, \pi)$ *forces that* $\dot{W}_{\mathbb{P}_{0,\pi}}(\bar{p}{\restriction}(0, \pi)) \in \dot{D}_{0,\pi}$.

Then, q *forces that* $\dot{D}_{0,\pi} \cup \bigcup_{\xi<\pi} \dot{D}_{1,\xi} \cup \{\dot{W}_{\mathbb{P}_{1,\pi}}(\bar{p})\}$ *can be extended to an ultrafilter. Even more,* $\mathbb{1}_{\mathbb{P}_{1,\pi}}$ *forces that* $\dot{D}_{0,\pi} \cup \bigcup_{\xi<\pi} \dot{D}_{1,\xi}$ *can be extended to an ultrafilter.*

Proof. We show that, for any $\mathbb{P}_{1,\pi}$-names $\dot{a}$ and $\dot{b}$ of members of $\bigcup_{\xi<\pi} \dot{D}_{1,\xi}$ and $\dot{D}_{0,\pi}$, respectively, q forces that $\dot{a} \cap \dot{b} \cap \dot{W}_{\mathbb{P}_{1,\pi}}(\bar{p}) \neq \emptyset$. Let $r \leq q$ in $\mathbb{P}^+_{1,\pi}$. Wlog (by strengthening r if necessary) we may assume that $\dot{b}$ is a $\mathbb{P}_{0,\pi}$-name and that there is a $\xi < \pi$ such that $\dot{a}$ is a $\mathbb{P}_{1,\xi}$-name, $r, q \in \mathbb{P}^+_{1,\xi}$ and r forces that $\dot{a} \in \dot{D}_{1,\xi}$. Consider the $\mathbb{P}_{0,\xi}$-name

$$\dot{b}'_0 := \{n < \omega : p_n{\restriction}(0, \xi) \in \dot{G}_{0,\xi} \text{ and } p_n{\restriction}(0, \pi) \Vdash_{\mathbb{P}_{0,\pi}/\mathbb{P}_{0,\xi}} n \notin \dot{b}\}.$$

It is clear that $\Vdash_{0,\pi} \dot{b} \cap \dot{W}_{\mathbb{P}_{0,\pi}}(\bar{p}{\restriction}(0, \pi)) \cap \dot{b}'_0 = \emptyset$ so, by (vii), $r{\restriction}(0, \xi)$ forces in $\mathbb{P}_{0,\xi}$ that

$$\dot{b}_0 := \dot{W}_{\mathbb{P}_{0,\xi}}(\bar{p}{\restriction}(0, \xi)) \smallsetminus \dot{b}'_0 = \{n < \omega : p_n{\restriction}(0, \xi) \in \dot{G}_{0,\xi} \text{ and } p_n{\restriction}(0, \pi) \nVdash_{\mathbb{P}_{0,\pi}/\mathbb{P}_{0,\xi}} n \notin \dot{b}\} \in \dot{D}_{0,\xi}.$$

Hence $r \Vdash_{1,\xi} \dot{b}_0 \in \dot{D}_{1,\xi}$, so by (vi) r forces that

$$\dot{a} \cap \dot{b}_0 \cap \dot{W}_{\mathbb{P}_{1,\xi}}(\bar{p}{\restriction}(1, \xi)) \in \dot{D}_{1,\xi}.$$

Find $n < \omega$ and $r' \in \mathbb{P}^+_{1,\xi}$ stronger than both r and $p_n{\restriction}(1, \xi)$ such that $r' \Vdash_{1,\xi} n \in \dot{a} \cap \dot{b}_0$. This implies that $r'{\restriction}(0, \xi) \Vdash_{0,\xi} n \in \dot{b}_0$, so there is a condition $s \leq p_n{\restriction}(0, \pi)$ in $\mathbb{P}^+_{0,\pi}$ such that $s{\restriction}(0, \xi) \leq r'{\restriction}(0, \xi)$ and $s \Vdash_{0,\pi} n \in \dot{b}$. Now, if we put $p' := r' \cup p_n{\restriction}[\xi, \pi)$, then $s \leq p'{\restriction}(0, \pi)$, which implies by Lemma 2.3.7(a) that $s = p''{\restriction}(0, \pi)$ for some $p'' \leq p'$ in $\mathbb{P}^+_{1,\pi}$. Hence, since p'' is stronger than both s and p', p'' forces that $n \in \dot{a} \cap \dot{b}$ and $p_n \in \dot{G}_{1,\pi}$.

The "even more" statement follows by the particular case when q and every p_n are the trivial condition. □

We conclude this section with the proof of Lemma 2.3.6.

Proof of Lemma 2.3.6. Recall that (in this section) $I^{\mathbf{m}}$ is an ordinal. Fix a uniform Δ-system $\bar{p} = \langle p_n : n < \omega \rangle$ with root R^* of $\mathbb{P}^*_{\nu,\pi}$ as in Definition 2.3.5(3) and an ultrafilter D on ω (in the ground model). By recursion on $\xi \leq \pi$ we construct $\mathbf{D}_\xi := \langle \dot{D}_{\alpha,\xi} : \alpha \leq \nu \rangle$ and $\langle q_{\alpha,\xi} : \alpha \leq \nu \rangle$ such that, for any $\alpha \leq \nu$,

(a) $\dot{D}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name of a non-principal ultrafilter on ω,

(b) $\mathbb{P}_{\alpha,\xi}$ forces that $D \subseteq \dot{D}_{\alpha_0,\xi_0} \subseteq \dot{D}_{\alpha,\xi}$ for any $\alpha_0 \leq \alpha$ and $\xi_0 \leq \xi$,

(c) $q_{\alpha,\xi} \in \mathbb{P}^+_{\alpha,\xi}$ with domain $R^* \cap \xi$,

(d) $q_{\alpha,\xi}{\restriction}(\alpha_0, \xi_0) = q_{\alpha_0,\xi_0}$ for any $\alpha_0 \leq \alpha$ and $\xi_0 \leq \xi$, and

(e) $q_{\alpha,\xi} \Vdash \dot{W}_{\mathbb{P}_{\alpha,\xi}}(\bar{p}{\restriction}(\alpha, \xi)) \in \dot{D}_{\alpha,\xi}$

After the construction, $q := q_{\nu,\pi}$ is the condition we are looking for.

Step $\xi = 0$. As $\mathbb{P}_{\alpha,0}$ is the trivial poset for any $\alpha \leq \nu$, $D_{\alpha,0} := D$ and $q_{\alpha,0} := \mathbb{1}$ work.

Successor step. Assume we have succeeded in our construction up to step ξ. For $\alpha < \Delta(\xi)$ it is clear that $\mathbb{P}_{\alpha,\xi+1} \simeq \mathbb{P}_{\alpha,\xi}$, so $\dot{D}_{\alpha,\xi+1}$ must be $\dot{D}_{\alpha,\xi}$. To define $q_{\alpha,\xi+1}$ (for all $\alpha \leq \nu$) and $\dot{D}_{\Delta(\xi),\xi+1}$ we consider two cases. If $\xi \notin R^*$ put $q_{\alpha,\xi+1} = q_{\alpha,\xi}$ and $\dot{D}_{\Delta(\xi),\xi+1}$ can be any $\mathbb{P}_{\Delta(\xi),\xi+1}$-name of an ultrafilter that contains $\dot{D}_{\Delta(\xi),\xi}$ (so it also contains $\dot{D}_{\alpha,\xi+1}$ for any $\alpha < \Delta(\xi)$); if $\xi \in R^*$, since $\dot{\mathbb{Q}}_{\Delta(\xi),\xi} = \dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a $<\kappa$-uf-linked forcing witnessed by $\langle \dot{Q}_\xi(\zeta) : \zeta < \theta_\xi \rangle$, and $\bar{p}(\xi) := \langle p_n(\xi) : n < \omega \rangle$ can be seen as a $\mathbb{P}_{\Delta(\xi),\xi}$-name of a sequence in $\dot{Q}_\xi(\zeta^*_\xi)$, there is a $\mathbb{P}_{\Delta(\xi),\xi}$-name $q(\xi)$ of a member of $\dot{\mathbb{Q}}_\xi$ such that $\mathbb{P}_{\Delta(\xi),\xi}$ forces that

$$q(\xi) \Vdash \text{“}\dot{W}_{\dot{\mathbb{Q}}_\xi}(\bar{p}(\xi)) \text{ intersects any member of } \dot{D}_{\Delta(\xi),\xi}\text{''}.$$

Put $q_{\alpha,\xi+1} := q_{\alpha,\xi} \cup \{(\xi, q(\xi))\}$ when $\Delta(\xi) \leq \alpha \leq \nu$, otherwise $q_{\alpha,\xi+1} := q_{\alpha,\xi} \cup \{(\xi, \mathbb{1})\}$, and choose $\dot{D}_{\Delta(\xi),\xi+1}$ as a $\mathbb{P}_{\Delta(\xi),\xi+1}$-name of an ultrafilter that contains $\dot{D}_{\Delta(\xi),\xi}$ and such that $q_{\Delta(\xi),\xi+1}$ forces that $\dot{W}_{\dot{\mathbb{Q}}_\xi}(\bar{p}(\xi)) \in \dot{D}_{\Delta(\xi),\xi+1}$.

No matter the case, for any $\alpha < \Delta(\xi)$, $\dot{D}_{\Delta(\xi),\xi+1}$ is forced to contain $\dot{D}_{\alpha,\xi+1}$ and

$$q_{\alpha,\xi+1} \Vdash_{\alpha,\xi+1} \dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1)) = \dot{W}_{\mathbb{P}_{\alpha,\xi}}(\bar{p}{\restriction}(\alpha, \xi)),$$

so this condition forces that $\dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1)) \in \dot{D}_{\alpha,\xi+1}$.

Now, by induction on $\alpha \in [\Delta(\xi), \nu]$, we define $\dot{D}_{\alpha,\xi+1}$ as required. We have already dealt with the case $\alpha = \Delta(\xi)$. For the successor step, assume we have defined $\dot{D}_{\alpha,\xi+1}$ accordingly. By Lemma 2.2.10, we can choose a $\mathbb{P}_{\alpha+1,\xi+1}$-name $\dot{D}_{\alpha+1,\xi+1}$ of an ultrafilter that contains $\dot{D}_{\alpha,\xi+1} \cup \dot{D}_{\alpha+1,\xi}$. For the limit step, let α be limit and assume we have already defined $\langle \dot{D}_{\alpha_0,\xi+1} : \alpha_0 < \alpha \rangle$. By Lemma 2.2.10, for any $\alpha_0 < \alpha$, $\mathbb{P}_{\alpha,\xi+1}$ forces that $\dot{D}_{\alpha_0,\xi+1} \cup \dot{D}_{\alpha,\xi}$ has the finite intersection property, hence $\dot{D}_{\alpha,\xi} \cup \bigcup_{\alpha_0<\alpha} \dot{D}_{\alpha_0,\xi+1}$ also has this property, i.e., it can be extended to an ultrafilter. Let $\dot{D}_{\alpha,\xi+1}$ be a $\mathbb{P}_{\alpha,\xi+1}$-name of such an ultrafilter.

It remains to show that item (e) holds for $(\alpha, \xi+1)$ when $\Delta(\xi) \leq \alpha \leq \nu$. If $\xi \in R^*$ then $q_{\alpha,\xi+1}$ forces $\dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1)) = \dot{W}_{\mathbb{P}_{\alpha,\xi}}(\bar{p}{\restriction}(\alpha, \xi)) \cap \dot{W}_{\dot{\mathbb{Q}}_\xi}(\bar{p}(\xi))$; else, if $\xi \notin R^*$ then $q_{\alpha,\xi+1}$ forces that

$$\dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1)) \subseteq \dot{W}_{\mathbb{P}_{\alpha,\xi}}(\bar{p}{\restriction}(\alpha, \xi)) \text{ and } |\dot{W}_{\mathbb{P}_{\alpha,\xi}}(\bar{p}{\restriction}(\alpha, \xi)) \smallsetminus \dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1))| \leq 1$$

(because $\langle \operatorname{dom} p_n : n < \omega \rangle$ forms a Δ-system and ξ is not in its root). Hence, in any case it is clear that $q_{\alpha,\xi+1}$ forces $\dot{W}_{\mathbb{P}_{\alpha,\xi+1}}(\bar{p}{\restriction}(\alpha, \xi+1)) \in \dot{D}_{\alpha,\xi+1}$.

Limit step. Let $\eta \leq \pi$ be a limit ordinal and assume we have succeeded in our construction for $\xi < \eta$. For each $\alpha \leq \nu$ put $q_{\alpha,\eta} := \bigcup_{\xi<\eta} q_{\alpha,\xi}$, which clearly satisfies (c) and (d). By recursion on $\alpha \leq \nu$ we define $\dot{D}_{\alpha,\eta}$ satisfying (a), (b) and (e). When $\alpha = 0$, by Lemma 2.3.8 applied to the FS iteration $\mathbb{P}^+_{0,\eta} = \langle \mathbb{P}^+_{0,\xi}, \dot{\mathbb{Q}}_{0,\xi} : \xi < \eta \rangle$, $q_{0,\eta}$ forces that $\dot{W}_{\mathbb{P}_{0,\eta}}(\bar{p}{\restriction}(0,\eta))$ intersects any member of $\bigcup_{\xi<\eta} \dot{D}_{0,\xi}$, so we can find a $\mathbb{P}_{0,\eta}$-name of an ultrafilter $\dot{D}_{0,\eta}$ that contains this union and such that $q_{0,\eta}$ forces $\dot{W}_{\mathbb{P}_{0,\eta}}(\bar{p}{\restriction}(0,\eta)) \in \dot{D}_{0,\eta}$.

For the successor step, assume we have found $\dot{D}_{\alpha,\eta}$. By Lemma 2.3.9 applied to $(\mathbf{m}|\{\alpha, \alpha+1\}){\restriction}\eta$, $q_{\alpha+1,\eta}$ forces that $\dot{D}_{\alpha,\eta} \cup \bigcup_{\xi<\eta} \dot{D}_{\alpha+1,\xi} \cup \{\dot{W}_{\mathbb{P}_{\alpha+1,\eta}}(\bar{p}{\restriction}(\alpha+1,\eta))\}$ has the finite intersection property, so we can find a $\mathbb{P}_{\alpha+1,\eta}$-name $\dot{D}_{\alpha+1,\eta}$ that satisfies (a), (b) and (e).

For the limit step, let $\alpha \leq \nu$ limit and assume we have defined $\dot{D}_{\alpha_0,\eta}$ for all $\alpha_0 < \alpha$. By Lemma 2.3.9 applied to $(\mathbf{m}|\{\alpha_0, \alpha\}){\restriction}\eta$, $q_{\alpha,\eta}$ forces that $\dot{D}_{\alpha_0,\eta} \cup \bigcup_{\xi<\eta} \dot{D}_{\alpha,\xi} \cup \{\dot{W}_{\mathbb{P}_{\alpha,\eta}}(\bar{p}{\restriction}(\alpha,\eta))\}$ has the finite intersection property. Hence, $q_{\alpha,\eta}$ forces that $\bigcup_{\alpha_0<\alpha} \dot{D}_{\alpha_0,\eta} \cup \bigcup_{\xi<\eta} \dot{D}_{\alpha,\xi} \cup \{\dot{W}_{\mathbb{P}_{\alpha,\eta}}(\bar{p}{\restriction}(\alpha,\eta))\}$ has the same property, so it can be extended to an ultrafilter $\dot{D}_{\alpha,\eta}$. □

2.4 Applications

In this section, we show applications of Theorem 2.3.3 to get new constellations of Cichoń's diagram, concretely, we prove Theorem E, G, I and H.

Theorem 2.4.1. *Let $\theta_0 \leq \theta_1 \leq \theta_2 \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta_2}$. Then there is a ccc poset that forces*

$$\mathrm{add}(\mathcal{N}) = \theta_0 \leq \mathrm{cov}(\mathcal{N}) = \theta_1 \leq \mathfrak{b} = \mathfrak{a} = \theta_2 \leq \mathrm{non}(\mathcal{M}) = \mu \leq \mathrm{cov}(\mathcal{M}) = \nu \leq \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda.$$

Proof. Denote $\mathbb{S}_0 = \mathbb{LOC}$, $\mathbb{S}_1 = \mathbb{B}$ and $\mathbb{S}_2 = \mathbb{D}$. Fix a bijection $g = (g_0, g_1, g_2) : \lambda \to 3 \times \lambda \times \lambda$ and a function $t : \nu\mu \to \nu$ such that $t(\nu\delta + \alpha) = \alpha$ for each $\delta < \mu$ and $\alpha < \nu$. For each $\rho < \nu\mu$ denote $\eta_\rho := \nu + \lambda\rho$, and put $R_i := \{\eta_\rho + 1 + \varepsilon : \varepsilon < \lambda,\ \rho < \nu\mu,\ g_0(\varepsilon) = i\}$ for each $i < 3$. Set $R := R_0 \cup R_1 \cup R_2$.

The poset we want is $\mathbb{H}_{\theta_2} * \mathbb{C}_\lambda * \mathbb{P}$ where $\mathbb{P}$ is constructed in $V_{0,0} := V^{\mathbb{H}_{\theta_2} * \mathbb{C}_\lambda}$ from a $<\theta_2$-uf-extendable matrix iteration $\mathbf{m}$, with $I^{\mathbf{m}} = \nu + 1$ and $\pi^{\mathbf{m}} = \nu + \lambda\nu\mu$, such that

(I) for any $\alpha < \nu$, $\Delta^{\mathbf{m}}(\alpha) = \alpha + 1$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_{\Delta(\alpha),\alpha} = \omega^{<\omega}$,

and the matrix iteration at each interval of the form $[\eta_\rho, \eta_{\rho+1})$ for $\rho < \nu\mu$ is defined as follows. Assume that $\mathbf{m}{\restriction}\eta_\rho$ has been constructed and that, for any $i < 3$ and $\xi \in R_i \cap \eta_\rho$, a $\mathbb{P}_{\Delta(\xi),\xi}$-name $\dot{N}_\xi$ of a transitive model of ZFC of size $<\theta_i$ has already been defined.

Choose

(0) for $i \in \{0, 2\}$, an enumeration $\{\dot{x}^\rho_{i,\zeta} : \zeta < \lambda\}$ of all the nice $\mathbb{P}_{\nu,\eta_\rho}$-names for all the members of ω^ω; for $i = 1$, $\{\dot{x}^\rho_{i,\zeta} : \zeta < \lambda\}$ enumerates all the (nice) $\mathbb{P}_{\nu,\eta_\rho}$-names for all the members of Ω (from $\mathbf{Cn}$, see Example 1.7.9(3));

(1) for $i < 3$, an enumeration $[\eta_\rho \cap R_i]^{<\theta_i} = \{A^\rho_{i,\zeta} : \zeta < \lambda\}$.

For $\xi \in [\eta_\rho, \eta_{\rho+1})$,

(II) if $\xi = \eta_\rho$, put $\Delta^{\mathbf{m}}(\xi) = t(\rho) + 1$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \mathbb{E}^{V_{\Delta(\xi),\xi}}$;

(III) if $\xi = \eta_\rho + 1 + \varepsilon$ for some $\varepsilon < \lambda$, then there is some $\alpha < \nu$ such that $\dot{x}^\rho_{g_0(\varepsilon),g_1(\varepsilon)}$ is a $\mathbb{P}_{\alpha,\eta_\rho}$-name, so we can choose

(III-1) a successor ordinal $\Delta^{\mathbf{m}}(\xi)$ such that $\sup_{\gamma \in A_{g_0(\varepsilon),g_2(\varepsilon)}} \Delta(\gamma) < \Delta(\xi)$ and $\alpha < \Delta(\xi) < \nu$, and

(III-2) a $\mathbb{P}_{\Delta(\xi),\xi}$-name $\dot{N}_\xi$ of a transitive model of ZFC of size $<\theta_{g_0(\varepsilon)}$ such that $\mathbb{P}_{\Delta(\xi),\xi}$ forces that $\bigcup_{\gamma\in A^\rho_{g_0(\varepsilon),g_2(\varepsilon)}} \dot{N}_\gamma \subseteq \dot{N}_\xi$ and $\dot{x}^\rho_{g_0(\varepsilon),g_1(\varepsilon)} \in \dot{N}_\xi$.

Put $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \mathbb{S}^{\dot{N}_\xi}_{g_0(\varepsilon)}$.

According to Definition 2.3.1, the above settles the construction of $\mathbf{m}$ as a $<\theta_2$-uf-extendable matrix iteration. Set $\mathbb{P} := \mathbb{P}_{\nu,\pi}$, which is ccc.

We need to show that $\mathbb{P}$ forces the statement of the theorem. Since this poset has size λ, it forces $\mathfrak{c} \leq \lambda$. On the other hand, by Theorem 2.3.3, $\mathbb{P}$ is a θ_2-uf-Knaster poset, so it preserves the mad family previously added by $\mathbb{H}_{\theta_2}$ and forces $\mathfrak{a} \leq \theta_2$. Even more, for any regular cardinal $\kappa \in [\theta_2, \lambda]$, $\mathbb{P}$ preserves the strongly κ-$\mathbf{D}$-unbounded family of size κ previously added by $\mathbb{C}_\kappa$. In particular, $\mathbb{P}$ forces $\mathfrak{b} \leq \theta_2$ and $\lambda \leq \mathfrak{d}$.

Observe that $\mathbb{P}$ can be obtained by the FS iteration $\langle \mathbb{P}_{\nu,\xi}, \dot{\mathbb{Q}}_{\nu,\xi} : \xi < \pi \rangle$ and that all its iterands are θ_0-$\mathbf{Lc}^*$-good and θ_1-$\mathbf{Cn}$-good. Therefore, by Theorem 1.7.11, $\mathbb{P}$ forces $\mathrm{add}(\mathcal{N}) \leq \theta_0$, $\mathrm{cov}(\mathcal{N}) \leq \theta_1$ and $\lambda \leq \mathrm{non}(\mathcal{N})$, in fact, $\mathbb{P}$ adds

(SU1) a strongly κ-$\mathbf{Lc}^*$-unbounded family of size κ for each regular $\kappa \in [\theta_0, \lambda]$, and

(SU2) a strongly κ-$\mathbf{Cn}$-unbounded family of size κ for each regular $\kappa \in [\theta_1, \lambda]$.

On the other hand, $\mathbb{P}$ adds μ-cofinally many Cohen reals that form a strongly μ-$\mathbf{Ed}$-unbounded family of size μ, hence $\mathbb{P}$ forces $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathbf{Ed}) \leq \mu$.

To see that $\mathbb{P}$ forces $\theta_0 \leq \mathrm{add}(\mathcal{N})$, $\theta_1 \leq \mathrm{cov}(\mathcal{N})$ and $\theta_2 \leq \mathfrak{b}$, we show that $\mathbb{P}$ adds the corresponding strongly dominating families. In the ground model, order R by $\eta \trianglelefteq \eta'$ iff $\eta \leq \eta'$, $\Delta(\eta) \leq \Delta(\eta')$ and $\Vdash_{\mathbb{H}_{\theta_2} * \mathbb{C}_\lambda * \mathbb{P}_{\Delta(\eta'),\eta'}} \dot{N}_\eta \subseteq \dot{N}_{\eta'}$, which is a partial order, even more, $\langle R_i, \trianglelefteq \rangle$ is $<\theta_i$-directed for any $i < 3$. To see this, if $A \subseteq R_i$ has size $<\theta_i$ then we can find some $\rho < \nu\mu$ such that $A \subseteq \eta_\rho$, so choose some $\zeta < \lambda$ such that $A := A^\rho_{i,\zeta}$. Put $\xi := \eta_\rho + 1 + \varepsilon$ where $\varepsilon = g^{-1}(i, \gamma, \zeta)$ for some γ chosen arbitrarily. Note that ξ is an upper bound of the set A with respect to $\trianglelefteq$.

In $V_{0,0}$, for $\xi \in R_0$ let $\dot{\varphi}_\xi$ be the $\mathbb{P}_{\Delta(\xi),\xi+1}$-name of the $\mathbf{Lc}^*$-dominating slalom over $\dot{N}_\xi$ added by $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \mathbb{LOC}^{\dot{N}_\xi}$; for $\xi \in R_1$ let $\dot{r}_\xi$ be the $\mathbb{P}_{\Delta(\xi),\xi+1}$-name of the random real over $\dot{N}_\xi$ added by $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \mathbb{B}^{\dot{N}_\xi}$; and for $\xi \in R_2$, let $\dot{d}_\xi$ be the $\mathbb{P}_{\Delta(\xi),\xi+1}$-name of the dominating real over $\dot{N}_\xi$ added by $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \mathbb{D}^{\dot{N}_\xi}$. Define $\dot{S} := \{\dot{\varphi}_\xi : \xi \in R_0\}$, $\dot{C} := \{\dot{r}_\xi : \xi \in R_1\}$, and $\dot{D} := \{\dot{d}_\xi : \xi \in R_2\}$.

We claim that $\mathbb{P}$ forces that $\dot{S}$ is a strongly θ_0-$\mathbf{Lc}^*$-dominating family, $\dot{C}$ is a strongly θ_1-$\mathbf{Cn}$-dominating family, and $\dot{D}$ is a strongly θ_2-$\mathbf{D}$-dominating family. We just show this fact for $\dot{S}$ (the others can be proved similarly). Let $\dot{x}$ be a $\mathbb{P}$-name for a real in ω^ω. We can find a $\rho < \nu\mu$ such that $\dot{x}$ is a $\mathbb{P}_{\nu,\eta_\rho}$-name, so there is some $\zeta < \nu$ such that $\dot{x} = \dot{x}^\rho_{0,\zeta}$. Put $\xi = \eta_\rho + 1 + \varepsilon$ where $\varepsilon := g^{-1}(0, \zeta, 0)$, so $\mathbb{P}_{\Delta(\xi),\xi}$ forces that $\dot{x} \in \dot{N}_\xi$. Fix any $\beta \trianglerighteq \xi$ in R_0. Then $\xi \leq \beta$, $\Delta(\xi) \leq \Delta(\beta)$ and $\Vdash_{\mathbb{P}_{\Delta(\beta),\beta}} \dot{N}_\xi \subseteq \dot{N}_\beta$, so $\Vdash_{\mathbb{P}_{\Delta(\beta),\beta}} \dot{x} \in \dot{N}_\beta$. Therefore, $\dot{\varphi}_\beta$ is forced to localize $\dot{x}$.

For each $\rho < \nu\mu$ denote by $\dot{e}_\rho$ the $\mathbb{P}_{\Delta(\eta_\rho),\eta_\rho+1}$-name of the eventually different real over $V_{t(\rho)+1,\eta_\rho}$ added by $\dot{\mathbb{Q}}_{t(\rho)+1,\eta_\rho}$. To show that $\mathrm{non}(\mathcal{M}) \geq \mu$ and $\mathrm{cov}(\mathcal{M}) \leq \nu$, it is enough to prove that $\mathbb{P}$ forces that $\dot{E} := \{\dot{e}_\rho : \rho < \nu\mu\}$ is a strongly μ-$\mathbf{Ed}$-dominating family. Consider the partial order on $\nu\mu$ defined by $\rho \trianglelefteq' \varrho$ iff $\rho \leq \varrho$ and $t(\rho) \leq t(\varrho)$, which is actually $<\mu$-directed. To see this, let $A \subseteq \nu\mu$ of size of $<\mu$. Since A is bounded with respect to $\leq$ (because $\mathrm{cf}(\nu\mu) = \mu$), in has an upper bound $\rho \in \nu\mu$. Define $\alpha := \sup_{\eta\in A}\{t(\eta) + 1\}$, which is $<\nu$ because ν is a regular cardinal. By the definition of t, there is some $\delta \in [\rho, \nu\mu)$ such that $\alpha = t(\delta)$, hence δ is an upper bound of A with respect to $\trianglelefteq'$.

Let $x \in V_{\nu,\pi} \cap \omega^\omega$. We can find $\alpha < \nu$ and $\rho < \nu\mu$ such that $x \in V_{\alpha,\eta_\rho}$. By the definition of t, there is some $\delta \in [\rho, \nu\mu)$ such that $t(\delta) = \alpha$, so $x \in V_{t(\delta),\eta_\delta}$. For any $\varrho \trianglerighteq' \delta$, $\delta \leq \varrho$ and $t(\delta) \leq t(\varrho)$, so $x \in V_{t(\varrho)+1,\eta_\varrho}$, which implies $x \neq^* e_\varrho$.

To finish the proof we conclude that, by Theorem 1.7.14, $\mathbb{P}$ forces $\text{cov}(\mathcal{M}) = \mathfrak{d}(\mathbf{Ed}) \geq \nu$. In fact, if c_α denotes the Cohen real added by $\mathbb{Q}_{\alpha+1,\alpha}$ for any $\alpha < \nu$, it is clearly $\mathbf{Ed}$-unbounded over $V_{\alpha,\alpha} = V_{\alpha,\alpha+1}$, so $\{c_\alpha : \alpha < \nu\}$ is a strongly ν-$\mathbf{Ed}$-unbounded family. □

Theorem 2.4.2. *Let $\theta_0 \leq \theta_1 \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta_1}$. Then there is a ccc poset that forces $\text{MA}_{<\theta_0}$ and*

$$\text{add}(\mathcal{N}) = \theta_0 \leq \mathfrak{b} = \mathfrak{a} = \theta_1 \leq \text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu \leq \text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \nu \leq \mathfrak{d} = \text{cof}(\mathcal{M}) = \mathfrak{c} = \lambda.$$

Proof. Fix a bijection $g = (g_0, g_1, g_2) : \lambda \to 2 \times \nu \times \lambda$ and a function $t : \nu\mu \to \nu$ such that $t(\nu\delta + \alpha) = \alpha$ for each $\delta < \mu$ and $\alpha < \nu$. Denote $\eta_\rho := \nu + \lambda\rho$ for each $\rho < \nu\mu$.

The desired poset is $\mathbb{H}_{\theta_1} * \mathbb{C}_\lambda * \mathbb{P}$ where $\mathbb{P}$ is constructed in $V_{0,0} = V^{\mathbb{H}_{\theta_1} * \mathbb{C}_\lambda}$ from a $<\theta_1$-uf-extendable matrix iteration $\mathbf{m}$.

Work in $V_{0,0}$. Put $I^{\mathbf{m}} := \nu + 1$, $\pi^{\mathbf{m}} := \nu + \lambda\nu\mu$,

(I) for any $\alpha < \nu$, $\Delta^{\mathbf{m}}(\alpha) = \alpha + 1$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_{\Delta(\alpha),\alpha} = \omega^{<\omega}$,

and define the matrix iteration in the intervals of the form $[\eta_\rho, \eta_{\rho+1})$ as follows. Assume that $\mathbf{m}{\restriction}\eta_\rho$ has been defined. For $\alpha < \nu$ choose

(0) an enumeration $\{\dot{\mathbb{Q}}^\rho_{0,\alpha,\zeta} : \zeta < \lambda\}$ of all the nice $\mathbb{P}_{\alpha,\eta_\rho}$-names for all the posets which underlining set is a subset of θ_0 of size $<\theta_0$ and $\Vdash_{\mathbb{P}_{\nu,\lambda\rho}}$"$\dot{\mathbb{Q}}^\rho_{0,\zeta}$ is ccc"; and

(1) an enumeration $\{\dot{\mathbb{Q}}^\rho_{1,\alpha,\zeta} : \zeta < \lambda\}$ of all the nice $\mathbb{P}_{\alpha,\eta_\rho}$-names for all the σ-centered subposets of Hechler forcing of size $<\theta_1$.

For $\xi \in [\eta_\rho, \eta_{\rho+1})$,

(II) if $\xi = \eta_\rho$ put $\Delta(\xi) = t(\rho) + 1$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_{\lambda\rho} = \mathbb{B}^{V_{\Delta(\xi),\xi}}$;

(III) if $\xi = \eta_\rho + 1 + \varepsilon$ for some $\rho < \nu\mu$ and $\varepsilon < \lambda$, put $\Delta(\xi) = g_1(\varepsilon) + 1$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi = \dot{\mathbb{Q}}^\rho_{g(\varepsilon)}$.

This settles the construction, which is clearly a $<\theta_1$-uf-extendable matrix iteration. □

Remark 2.4.3. It is possible to additionally force $\text{MA}_{<\theta_0}$ in Theorem 2.4.1 by slightly modifying the construction of the matrix iteration. On the other hand, the matrix of Theorem 2.4.2 could be modified to force the existence of a strongly-θ_0-$\mathbf{Lc}^*$-dominating family and a strongly θ_1-$\mathbf{D}$-dominating family.

For the reader convenience, before we prove Corollary 2.4.5 we summarize some results about cardinal characteristic of $\mathcal{I}_f$ mentioned in the introduction.

Theorem 2.4.4. *Let $f \in \omega^\omega$ be a strictly increasing function. Then*

(a) (Yorioka [Yor02]) $\text{cov}(\mathcal{N}) \leq \text{cov}(\mathcal{I}_f) \leq \text{cov}(\mathcal{SN})$ *and* $\text{non}(\mathcal{SN}) \leq \text{non}(\mathcal{I}_f) \leq \text{non}(\mathcal{N})$.

(b) (Kamo, see e.g. [CM19, Cor. 3.13]) $\text{add}(\mathcal{N}) \leq \text{add}(\mathcal{I}_f)$ *and* $\text{cof}(\mathcal{I}_f) \leq \text{cof}(\mathcal{N})$.

(c) (Kamo and Osuga [KO08]) $\text{add}(\mathcal{I}_f) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \text{cof}(\mathcal{I}_f)$.

(d) (Osuga [Osu08], see also [CM19, Cor. 3.21]) $\text{cov}(\mathcal{I}_f) \leq \text{non}(\mathcal{M})$ *and* $\text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{I}_f)$.

Corollary 2.4.5. *Let $\theta \leq \mu \leq \nu$ be uncountable regular cardinals and let $\lambda \geq \nu$ be a cardinal such that $\lambda^{<\theta} = \lambda$. Then, as in Figure 2.1, there is a ccc poset that forces $\text{add}(\mathcal{I}_f) = \theta$, $\text{cov}(\mathcal{I}_f) = \mu$, $\text{non}(\mathcal{I}_f) = \nu$, and $\text{cof}(\mathcal{I}_f) = \lambda$ for all increasing $f \in \omega^\omega$ (in the extension).*

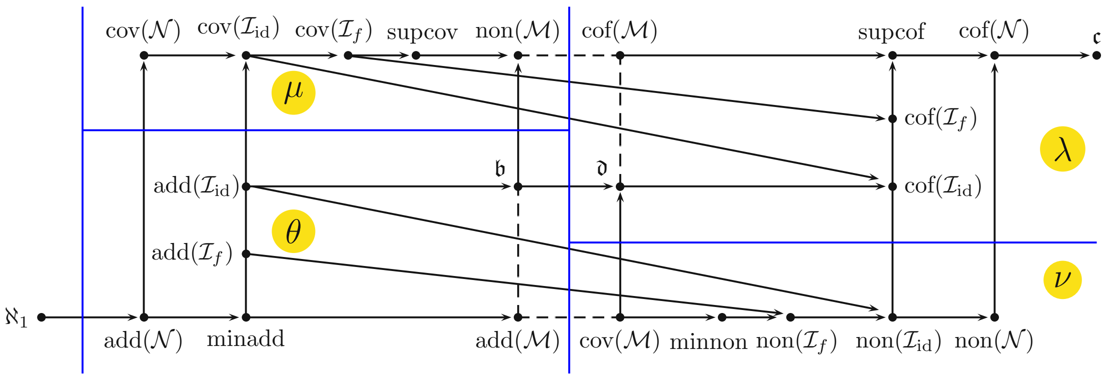

Figure 2.1: Separation of the cardinals associated with $\mathcal{I}_f$ for any f.

Proof. By application of Theorem 2.4.2 to $\theta := \theta_0 = \theta_1$, there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \mathfrak{b} = \theta$, $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mu$, $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \nu$ and $\mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \mathfrak{c} = \lambda$. This poset is as required by Theorem 2.4.4. □

We finally show that Cichoń's diagram can consistently be separated into 10 values, assuming the consistency of three strongly compact cardinals. Though in [GKS19] the same result is proved modulo four strongly compact cardinals and GCH, we avoid using GCH by tracking the exact necessary hypothesis about the cardinals.

Theorem 2.4.6. *Assume:*

(I) $\kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9$ *are cardinal numbers,*

(II) for $i \in [1, 9] \smallsetminus \{6\}$, λ_i *is regular,*

(III) $\lambda_6^{<\lambda_3} = \lambda_6$, *and*

(IV) for $j \in \{7, 8, 9\}$, κ_j *is strongly compact and* $\lambda_j^{\kappa_j} = \lambda_j$.

Then there is a ccc poset that forces

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 \leq \mathrm{non}(\mathcal{M}) = \lambda_4 \leq \mathrm{cov}(\mathcal{M}) = \lambda_5 \leq \mathfrak{d} = \lambda_6 \leq \mathrm{non}(\mathcal{N}) = \lambda_7 \leq \mathrm{cof}(\mathcal{N}) = \lambda_8 \leq \mathfrak{c} = \lambda_9.$$

see Figure 2.2.

This result is justified by application of Boolean ultrapowers to the poset constructed in the proof of Theorem 2.4.1 in the same way as in [KTT18; GKS19; KST19]. We review this technique as follows. Let κ be a strongly compact cardinal and $\lambda > \kappa$ regular such that[7] $\lambda^\kappa = \lambda$. Consider the Boolean completion $\mathbb{B}_{\kappa,\lambda}$ of the poset $\mathrm{Fn}_{<\kappa}(\lambda, \kappa)$.

Lemma 2.4.7 ([KTT18; GKS19])**.** *There is a κ-complete ultrafilter U on $\mathbb{B}_{\kappa,\lambda}$ such that its corresponding elementary embedding $j : V \to M$ satisfies:*

(a) M is closed under sequences of length $<\kappa$.

(b) j has critical point κ, $\mathrm{cf}(j(\kappa)) = \lambda$ *and* $\lambda \leq j(\kappa) < \lambda^+$.

[7] Without assuming GCH.

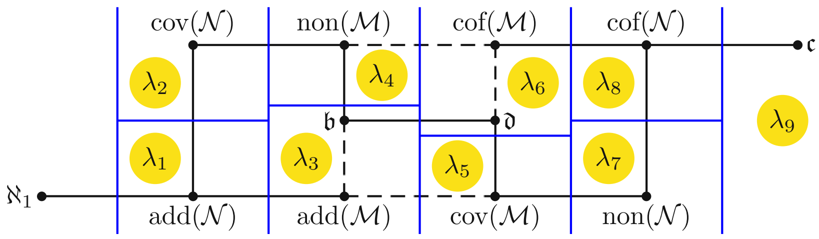

Figure 2.2: Cichoń's maximum

(c) If $|A| < \kappa$ then $j[A] = j(A)$.

(d) If $\theta \geq \kappa$ and either $\theta \leq \lambda$ or $\theta^\kappa = \theta$, then $\max\{\lambda, \theta\} \leq j(\theta) < \max\{\lambda, \theta\}^+$.

(e) If $\theta > \kappa$ and I is a $<\theta$-directed partial order then $j[I]$ is cofinal in $j(I)$.

(f) If $\mathrm{cf}(\alpha) \neq \kappa$ then $j[\alpha]$ is cofinal in $j(\alpha)$.

As a consequence,

Lemma 2.4.8 ([KTT18; GKS19], see also [GKMS20b, Thm. 1.13])**.** *Additionally to the above, assume that $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ is an analytic relational system (i.e., X, Y and $\sqsubset$ are analytic in some Polish space), θ is an uncountable regular cardinal and $\mathbb{P}$ is a ccc poset. Then:*

(a) $j(\mathbb{P})$ is ccc (in V, not just in M).

(b) If $\mathbb{P}$ adds a strongly θ-$\mathbf{R}$-unbounded family of size θ, then $j(\mathbb{P})$ adds a strongly $\mathrm{cf}(j(\theta))$-$\mathbf{R}$-unbounded family of size $\mathrm{cf}(j(\theta))$.

(c) If $\mathbb{P}$ adds a strongly θ-$\mathbf{R}$-dominating family with witnessing directed set L in the ground model such that $|L| = \lambda'$, then

(i) whenever $\theta < \kappa$, $j(\mathbb{P})$ adds a strongly θ-$\mathbf{R}$ dominating family with witnessing directed set of size $|j(\lambda')|$;

(ii) whenever $\kappa < \theta$, $j(\mathbb{P})$ adds a strongly θ-$\mathbf{R}$ dominating family with witnessing directed set of size λ'.

In both cases, the witnessing directed set can be obtained in the ground model.

Proof. We include the proof for completeness. Property (a) follows from Lemma 2.4.7(a). To see property (b), let $\{\dot{c}(\alpha) : \alpha < \theta\}$ be a strongly θ-$\mathbf{R}$-unbounded family added by $\mathbb{P}$. Since $\mathbb{P}$ is ccc, $\exists \alpha < \theta \forall \beta \in [\alpha, \theta)(\Vdash_{\mathbb{P}} \dot{c}(\beta) \not\sqsubset \dot{z})$ for any $\mathbb{P}$-name $\dot{z}$ of a real in Y, thus

$$M \models \exists \alpha < j(\theta) \forall \beta \in [\alpha, j(\theta))(\Vdash_{j(\mathbb{P})} j(\dot{c})(\beta) \not\sqsubset \dot{z}')$$

for any $j(\mathbb{P})$-name $\dot{z}'$ of a real in Y (note that every nice $j(\mathbb{P})$-name of a real is in M). Since $\mathbf{R}$ is analytic, the same statement holds in V. Therefore, if $f : \mathrm{cf}(j(\theta)) \to j(\theta)$ is an increasing cofinal function, then $\mathbb{P}$ forces that $\{j(\dot{c})(f(\xi)) : \xi < \mathrm{cf}(j(\theta))\}$ is a strongly $\mathrm{cf}(j(\theta))$-$\mathbf{R}$-unbounded family.

We finally show (c). Assume that $p \in \mathbb{P}$ forces that $\{\dot{a}(l) : l \in L\}$ is a strongly θ-$\mathbf{R}$-dominating family. Hence

$$M \models \text{“}j(p) \Vdash_{j(\mathbb{P})} \{j(\dot{a})(l) : l \in j(L)\} \text{ is a strongly } j(\theta)\text{-}\mathbf{R}\text{-dominating family”}.$$

If $\theta < \kappa$ then $j(\theta) = \theta$ and $j(L)$ is $<\theta$-directed (in M, but also in V) of size $|j(\lambda')|$; else, if $\kappa < \theta$, by Lemma 2.4.7(e) we have that $j[L]$ is cofinal in $j(L)$, so $j(p)$ forces (in V) that $\{j(\dot{a})(j(l)) : l \in L\}$ is a strongly θ-$\mathbf{R}$-dominating family. $\square$

Proof of Theorem 2.4.6. Denote $\mathbf{R}_0 := \mathbf{Id}$, $\mathbf{R}_1 := \mathbf{Lc}^*$, $\mathbf{R}_2 := \mathbf{Cn}$, $\mathbf{R}_3 := \mathbf{D}$, and $\mathbf{R}_4 := \mathbf{Ed}$. Let $\mathbb{P}_6$ be the poset constructed in Theorem 2.4.1 applied to $\theta_i = \lambda_{i+1}$ for $i < 3$, $\mu = \lambda_4$, $\nu = \lambda_5$ and $\lambda = \lambda_6$. Also let $\lambda_0 := \aleph_1$. Recall that $\mathbb{P}_6$ adds

- ($U_6 1$) a strongly κ-$\mathbf{R}_i$-unbounded family of size κ for $i < 4$ and each regular $\kappa \in [\lambda_i, \lambda_6]$;
- ($U_6 2$) a strongly λ_i-$\mathbf{R}_4$-unbounded family of size λ_i for $i \in \{4, 5\}$;
- ($D_6 1$) a strongly λ_4-$\mathbf{R}_4$-dominating family with witnessing directed set of size λ_5 in the ground model; and
- ($D_6 2$) a strongly λ_i-$\mathbf{R}_i$-dominating family with witnessing directed set of size λ_6 in the ground model, for $1 \leq i < 4$.

Let $j_7 : V \to M_7$ be the elementary embedding obtained from $\mathbb{B}_{\kappa_7,\lambda_7}$ as in the previous discussion, and let $\mathbb{P}_7 := j_7(\mathbb{P}_6)$. By Lemma 2.4.8, $\mathbb{P}_7$ is ccc and it adds

- ($U_7 1$) a strongly κ-$\mathbf{R}_i$-unbounded family of size κ for $i < 4$ and each regular $\kappa \in [\lambda_i, \lambda_6] \smallsetminus \{\kappa_7\}$;
- ($U_7 2$) a strongly λ_7-$\mathbf{R}_i$-unbounded family of size λ_7 for $i < 3$
- ($U_7 3$) a strongly λ_i-$\mathbf{R}_4$-unbounded family of size λ_i for $i \in \{4, 5\}$;
- ($D_7 1$) a strongly λ_4-$\mathbf{R}_4$-dominating family with witnessing directed set of size λ_5;
- ($D_7 2$) a strongly λ_3-$\mathbf{R}_3$-dominating family with witnessing directed set of size λ_6; and
- ($D_7 3$) a strongly λ_i-$\mathbf{R}_i$-dominating family with witnessing directed set of size λ_7 for $1 \leq i < 3$.

This process is repeated a couple of times with κ_8 and κ_9. Let $j_8 : V \to M_8$ be the elementary embedding obtained from $\mathbb{B}_{\kappa_8,\lambda_8}$ and set $\mathbb{P}_8 := j_8(\mathbb{P}_7)$. This poset is ccc and it adds

- ($U_8 1$) a strongly κ-$\mathbf{R}_i$-unbounded family of size κ for $i < 4$ and each regular $\kappa \in [\lambda_i, \lambda_6] \smallsetminus \{\kappa_7, \kappa_8\}$;
- ($U_8 2$) a strongly λ_7-$\mathbf{R}_i$-unbounded family of size λ_7 for $i < 3$,
- ($U_8 3$) a strongly λ_8-$\mathbf{R}_i$-unbounded family of size λ_8 for $i < 2$
- ($U_8 4$) a strongly λ_i-$\mathbf{R}_4$-unbounded family of size λ_i for $i \in \{4, 5\}$;
- ($D_8 1$) a strongly λ_4-$\mathbf{R}_4$-dominating family with witnessing directed set of size λ_5;
- ($D_8 2$) a strongly λ_3-$\mathbf{R}_3$-dominating family with witnessing directed set of size λ_6;
- ($D_8 3$) a strongly λ_2-$\mathbf{R}_2$-dominating family with witnessing directed set of size λ_7; and
- ($D_8 4$) a strongly λ_1-$\mathbf{R}_1$-dominating family with witnessing directed set of size λ_8.

Let $j_9 : V \to M_9$ be the elementary embedding obtained from $\mathbb{B}_{\kappa_9,\lambda_9}$ and set $\mathbb{P}_9 := j_9(\mathbb{P}_8)$. This set is ccc and it satisfies the previous ($U_8 1$)–($U_8 4$) and ($D_8 1$)–($D_8 4$), with the exception that ($U_8 1$) does not hold for $\kappa = \kappa_9$. In addition, $\mathbb{P}_9$ adds a strongly λ_9-$\mathbf{R}_0$-unbounded family of size λ_9, so it forces $\lambda_9 \leq \mathfrak{c}$ (see Example 1.7.9(5)). On the other hand, $|\mathbb{P}_9| = |j_9(j_8(j_7(\lambda_6)))| = \lambda_9$, so $\mathbb{P}_9$ forces $\mathfrak{c} \leq \lambda_9$. By the properties listed above, $\mathbb{P}_9$ is the desired poset. □

2.5 Discussion and problems

This last section includes a short discussion related to the main results of this chapter. In Theorem 2.4.1 (Theorem E) we separated one additional value in the right side of Cichoń's diagram with respect to the constellation proved in [GMS16] (see ($\bigstar_2$) in the introduction). We ask if we could do the same to the constellation from [KST19], concretely,

Question R. *Can it be forced, without using large cardinals, that*

$$\aleph_1 < \text{add}(\mathcal{N}) < \mathfrak{b} < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < \text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}?$$

If this is possible, the large cardinal hypothesis from the main result in [KST19] can be reduced to three strongly compact cardinals.

The matrix iteration technique of this text seem not to be enough to deal with this problem since, to give desired values to $\text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{M})$ without increasing $\mathfrak{b}$ too much, we need to deal with restrictions of random forcing and $\mathbb{E}$ simultaneously, so they cannot be included in the same way in the matrix construction (a bit more in detail, only one could be the restriction to $V_{\Delta(\xi),\xi}$, but the other must be other type of restriction). On the other hand, similar to [KST19], dealing with ultrafilters may not be enough, so the matrix construction may include finitely additive measures instead.

The reader may have noticed that we did not force a value of $\mathfrak{a}$ in Theorem 2.4.6 after using Boolean ultrapowers. The reason is that the Boolean ultrapowers from $\mathbb{B}_{\kappa,\lambda}$ applied to a ccc poset $\mathbb{P}$ destroys all the mad families of size $\geq \kappa$ added by $\mathbb{P}$ in the same way as the ultrapower from a measurable cardinal destroys them (see [She04; Bre02]). This leads us to ask whether a value of $\mathfrak{a}$ can be forced in Theorem 2.4.6, or even in the consistency results from [GKS19; KST19].

By a slight modification, the poset constructed in Theorem 2.4.1 can force $\text{MA}_{<\lambda_1}$ (with $\lambda_1 = \theta_0$), and it is not hard to see that $\mathbb{P}_8$ from the proof of Theorem 2.4.6 also forces this. Though we can guarantee that $\mathbb{P}_9$ forces $\text{MA}_{<\kappa_9}$, it is unclear whether it forces $\text{MA}_{<\lambda_1}$.

As mentioned in the introduction, [GKMS19] showed that Cichoń's diagram can consistently be separated into 10 values without assuming large cardinals, concretely, for the instances ($\bigstar_3$) and ($\bigstar_4$) (see the introduction, in particular Question R is solved in the positive). The dynamic of the proof is similar to the original [GKS19]: start with a ccc poset $\mathbb{P}^0$ that separates the left hand side of Cichoń's diagram, e.g. ($\bigstar_1$) and ($\bigstar_2$), but instead of taking Boolean ultrapowers, intersect $\mathbb{P}^0$ with σ-closed elementary submodels of H_χ (for some large enough regular χ) so that the resulting poset forces Cichoń's diagram separated into 10 values.

This new method still relies on a forcing that separates the left side of the diagram. Although the poset from [GMS16] does this job, the new method is incompatible with conditions (P1)–(P3) (see the section "Ultrafilter-extendable matrix iterations" in the Introduction), so a modification of this forcing as in [GKS19] is necessary to get a poset compatible with the new method. The same happens with the Boolean ultrapower method. On the other hand, the poset we presented in Theorem 2.4.1 is already compatible with the new method, and less difficult to construct in comparison with the forcing from [GMS16; GKS19].

In relation with the previous discussion about MA, [GKMS20a] shows how to separate other classical cardinal characteristics of the continuum (without using large cardinals), in addition to those in Cichoń's diagram. In particular, $\mathfrak{m}$ (the smallest cardinal where MA fails) can be forced to be any chosen regular value between $\aleph_1$ and (the intended) $\mathrm{add}(\mathcal{N})$. With respect to the Boolean ultrapower method, in [GKMS20b] the forcing from Theorem 2.4.6 is modified to force, in addition, that $\mathfrak{m}$ can be any previously chosen regular value between $\aleph_1$ and κ_9.

Though we constructed a model of ZFC where $(\mathrm{A1})_{\mathcal{I}}$ for $\mathcal{I} \in \{\mathcal{N}, \mathcal{I}_f\}$ and $(\mathrm{A2})_{\mathcal{M}}$ hold, we still do not know how to construct a model for the following statements.

Question S. *Are each one of the following statements consistent with ZFC?:*

(a) $(\mathrm{A1})_{\mathcal{M}}$.

(b) $(\mathrm{A2})_{\mathcal{I}_f}$ *for any increasing* $f : \omega \to \omega$.

As mentioned in the Introduction, Brendle [Bre19b] constructed a ccc poset forcing Figure 2.3. However, tools to deal with $\mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$ in this situation are still unknown. In this model $\mathrm{non}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f)$, but the values of $\mathrm{add}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f)$ are unclear.

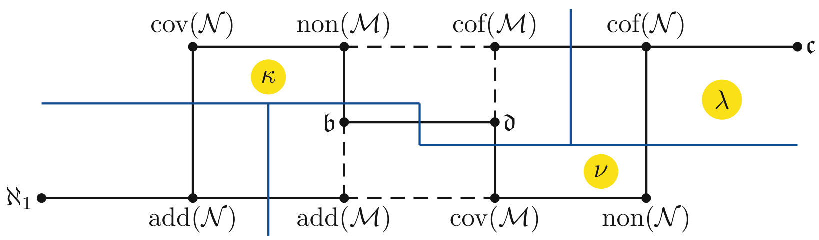

Figure 2.3: The constellation of Cichoń's diagram forced in [Bre19b].

3

MODELS WHERE MORE THAN TWO CARDINAL INVARIANTS ASSOCIATED WITH $\mathcal{SN}$ ARE PAIRWISE DIFFERENT

The main results of this chapter are based on the papers "On cardinal characteristics associated with the strong measure zero ideal" [Car21] and "The covering number of the strong measure zero ideal can be above almost everything else" [CMR21]. The latter is joint with Ismael Rivera-Madrid and Diego Mejía, and was accepted for publication in the Archive for Mathematical Logic while the first paper was accepted for publication in Fundamenta Mathematicae.

In [GJS93], it was shown that a CS iteration of forcings $\mathbb{PT}_b$ (Example 1.5.9) increases the additivity of $\mathcal{SN}$, and these iterands are particular cases of the tree forcing notions of Definition 1.5.8. In this chapter, we prove that any CS iterations of tree forcing notions, including Sacks forcing, increases the covering of $\mathcal{SN}$ (Theorem 3.1.13). Since countable support iteration is the universal method for constructing models with $\mathfrak{c} = \aleph_2$, our result indicates that we force $\mathrm{cov}(\mathcal{SN}) = \aleph_2$. We use this result to prove the consistency of $\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ (Theorem L).

On the other hand, based on an important characterization of the cofinality of $\mathcal{SN}$ by Yorioka [Yor02] (Theorem N), we introduce the notion of *λ-dominating system* (Definition 3.2.7), which is used to generalize Yorioka's characterization. This result provides bounds for the cofinality of $\mathcal{SN}$ (see Theorem O, Theorem 3.2.9 and 3.2.13), and Yorioka's characterization turns out to be a consequence (Corollary 3.2.15). This is used to prove the consistency of $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ (Theorem M).

This chapter is split into three sections: Section 3.1 is devoted to prove Theorem L. In Subsection 3.1.2 we present preservation results related to the dominating number of κ^κ for κ regular, this to ensure that the cofinality of $\mathcal{SN}$ can be manipulated as desired via Theorem N. We prove Theorem L, moreover, we show that tree forcings, when iterated, increase the covering of $\mathcal{SN}$ (Theorem 3.1.13 in Subsection 3.1.3).

Section 3.2 is dedicated to showing Theorem O and the consistency of $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ (Theorem M). In Subsection 3.2.1 we obtain several results about the dominating numbers $\mathfrak{d}(S)$ and $\mathfrak{d}^\lambda_S$. The notions of *$\mathcal{I}_f$-directed system* and *λ-dominating system* are introduced in Subsection 3.2.2, as well as the proof of Theorem O. In Subsection 3.2.3

we prove Theorem M.

The last Section 3.3 is devoted to discussion and open problems.

3.1 The first model

The objective of this chapter is to prove Theorem L.

3.1.1 More characterizations of the dominating number

The following characterization of the dominating number is due to Blass [Bla10], which will be useful to prove the main results of Section 3.1.

Definition 3.1.1. Denote by $\mathbb{I}$ the set of interval partitions of ω.

- For any $I, J \in \mathbb{I}$, write

$$I \sqsubseteq J \text{ iff } \forall^\infty n \exists m (I_m \subseteq J_n), \text{ and } I \not\trianglerighteq J \text{ iff } \forall^\infty n \forall m (I_n \not\supseteq J_m).$$

- Define the relational systems $\mathbf{D}_1 := \langle \mathbb{I}, \mathbb{I}, \sqsubseteq \rangle$ and $\mathbf{D}_2 := \langle \mathbb{I}, \mathbb{I}, \not\trianglerighteq \rangle$

For each $I \in \mathbb{I}$ we define $f_I : \omega \to \omega$ and $I^{*2} \in \mathbb{I}$ such that $f(n) := \min I_n$ and $I^{*2}_n := I_{2n} \cup I_{2n+1}$. For each increasing $f \in \omega^\omega$ define the increasing function $f^* : \omega \to \omega$ such that $f^*(0) = 0$ and $f^*(n+1) = f(f^*(n)+1)$, and define $I^f \in \mathbb{I}$ such that $I^f_n := [f^*(n), f^*(n+1))$.

Lemma 3.1.2. $\mathbf{D} \cong_{\mathrm{T}} \mathbf{D}_1 \cong_{\mathrm{T}} \mathbf{D}_2$. *Even more, if $D \subseteq \omega^\omega$ is a dominating family of increasing functions, then $\{I^f : f \in D\}$ is $\mathbf{D}_1$-dominating.*

Proof. To see $\mathbf{D}_1 \preceq_{\mathrm{T}} \mathbf{D}$ note that, for any $I \in \mathbb{I}$ and $f \in \omega^\omega$, if $f \in \omega^\omega$ is increasing then $f_I \leq^* f$ implies $I \sqsubseteq I^f$. Indeed, for n large enough, put $m := f^*(n)$, so $f^*(n) = m \leq f_I(m) < f_I(m+1) \leq f(m+1) = f^*(n+1)$, that is, $I_m \subseteq I^f_n$.

For $\mathbf{D}_2 \preceq_{\mathrm{T}} \mathbf{D}_1$ note that, for any $I, J \in \mathbb{I}$, $I \sqsubseteq J$ implies $I \not\trianglerighteq J^{*2}$. Finally $\mathbf{D} \preceq_{\mathrm{T}} \mathbf{D}_2$ because, for any increasing $f \in \omega^\omega$ and $I \in \mathbb{I}$, $I^f \not\trianglerighteq I$ implies $f \leq^* f_I$. To show this, notice that $I^f \not\trianglerighteq I$ is equivalent to say that $(f_I(n), f_I(n+1)) \cap \operatorname{ran} f^* \neq \emptyset$ for all but finitely many n. Split into cases: if $f = \mathrm{id}_\omega$, then $f^* = \mathrm{id}_\omega$, so $(f_I(n), f_I(n+1)) \neq \emptyset$ for n large enough. Hence, while $f(n+1) - f(n) = 1$, eventually $f_I(n+1) - f_I(n) \geq 2$, which guarantees $f \leq^* f_I$.

For the second case, assume $f(m_0) > m_0$ for some $m_0 < \omega$.[1] This implies that $f(n) > n$ for every $n \geq m_0$. To guarantee $f \leq^* f_I$, it is enough to show that $|I_n \cap \operatorname{ran} f| \geq 2$ for infinitely many n (recall that $I_n \cap \operatorname{ran} f \neq \emptyset$ for large enough n). If $n \in \omega$ is large enough, then there is some $m < \omega$ such that $f_I(n) < f^*(m) < f_I(n+1)$. On the other hand, since $(f_I(n+1), f_I(n+2)) \cap \operatorname{ran} f^* \neq \emptyset$, $f^*(m+1), f(f^*(m)) \in I_n \cup I_{n+1}$. This clearly implies that either I_n or I_{n+1} intersects $\operatorname{ran} f$ in 2 or more points. □

3.1.2 Preserving the dominating number $\mathfrak{d}_\kappa$

In this subsection, we show a method to preserve $\mathfrak{d}_\kappa$ large for κ regular. This is a natural generalization of preservation methods by Judah and Shelah [JS90] and Brendle [Bre91]. Our presentation is closer to [CM19, Sect. 4].

Lemma 3.1.3. *Let κ and λ be infinite cardinals. If $\lambda > \kappa^{<\kappa}$ then $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ forces $\mathfrak{d}_\kappa \geq \lambda$.*

[1] Since f is increasing, $f \geq \mathrm{id}_\omega$.

Proof. Let $\gamma < \lambda$ and let $\{\dot{y}_\alpha : \alpha < \gamma\}$ be a set of $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$-names of functions in κ^κ. Since this poset is $(\kappa^{<\kappa})^+$-cc, there is some $S \in [\lambda]^{<\lambda}$ such that each $\dot{y}_\alpha$ is a $\mathrm{Fn}_{<\kappa}(S \times \kappa, \kappa)$-name. A genericity argument guarantees that $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$ adds an unbounded function in κ^κ over the ground model, so $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ forces that the κ-Cohen real at $\xi \in \lambda \smallsetminus S$ is not dominated by any $\dot{y}_\alpha$. □

Definition 3.1.4. Let κ be an infinite cardinal. Say that a poset is *κ^κ-good* if, for any $\mathbb{P}$-name of a function in κ^κ, there is some $h \in \kappa^\kappa$ (in the ground model) such that, for any $x \in \kappa^\kappa$, if $x \not\leq^* h$ then $\Vdash x \not\leq^* \dot{y}$.

Lemma 3.1.5. *Any κ^κ-good poset forces that $\mathfrak{d}_\kappa \geq |\mathfrak{d}_\kappa^V|$.*

Proof. Assume that $\mathbb{P}$ is a κ^κ-good poset and that $\lambda = \mathfrak{d}_\kappa^V$. Let $\gamma < \lambda$ and assume that $\{\dot{y}_\alpha : \alpha < \gamma\}$ is a set of $\mathbb{P}$-names of functions in κ^κ. For each $\alpha < \gamma$ there is some $h_\alpha \in \kappa^\kappa$ satisfying goodness for $\dot{y}_\alpha$. Since $\gamma < \lambda$, there is some $x \in \kappa^\kappa$ such that $x \not\leq^* h_\alpha$ for any $\alpha < \gamma$. Therefore, by goodness, $\mathbb{P}$ forces that $x \not\leq^* \dot{y}_\alpha$. □

The following couple of lemmas illustrate simple examples of κ^κ-good posets.

Lemma 3.1.6 (cf. [Mon17, Lemma 1.46])**.** *If κ is regular then any poset of size $\leq \kappa$ is κ^κ-good.*

Proof. Let $\mathbb{P}$ be a poset of size $\leq \kappa$ and assume that $\dot{y}$ is a $\mathbb{P}$-name of a function in κ^κ. For each $p \in \mathbb{P}$ and $\xi < \kappa$ it is clear that there is some $h_p(\xi) < \kappa$ such that $p \not\Vdash \dot{y}(\xi) \neq h_p(\xi)$. Since $|\mathbb{P}| \leq \kappa < \mathfrak{b}_\kappa$, there is some $h \in \kappa^\kappa$ such that $h_p \leq^* h$ for any $p \in \mathbb{P}$.

It is not hard to see that $x \not\leq^* h$ implies $\Vdash x \not\leq^* \dot{y}$. Fix $p \in \mathbb{P}$ and $\eta < \kappa$. Since $x \not\leq^* h_p$, there is some $\xi > \eta$ such that $h_p(\xi) < x(\xi)$. On the other hand, there is some $q \leq p$ forcing $\dot{y}(\xi) = h_p(\xi)$, so $q \Vdash \dot{y}(\xi) < x(\xi)$. □

Lemma 3.1.7. *If κ is regular then any κ-cc poset is κ^κ-good.*

Proof. Let $\mathbb{P}$ be a κ-cc poset and let $\dot{y}$ be a $\mathbb{P}$-name of a function in κ^κ.

Claim 3.1.8. *If $\dot{\alpha}$ is a $\mathbb{P}$-name of a member of κ then there is some $\beta \in \kappa$ such that $\Vdash \dot{\alpha} < \beta$.*

Proof. Assume the contrary, that is, for any $\beta < \kappa$ there is some $p_\beta \in \mathbb{P}$ such that $p_\beta \Vdash \beta \leq \dot{\alpha}$. Since $\mathbb{P}$ is κ-cc and κ is regular, there is some $q \in \mathbb{P}$ forcing $|\{\beta < \kappa : p_\beta \in \dot{G}\}| = \kappa$, which implies that $q \Vdash \kappa \leq \dot{\alpha}$, a contradiction. □

For each $\xi < \kappa$, apply the claim to find some $h(\xi) \in \kappa$ such that $\Vdash \dot{y}(\xi) < h(\xi)$. It is clear that $\Vdash \dot{y} < h$, therefore, $x \not\leq^* h$ implies $\Vdash x \not\leq \dot{y}$. □

Montoya [Mon17, Sect. 1.2.2] defines a canonical forcing $\mathbb{E}_\kappa$ that adds a function in κ^κ eventually different from the ground model functions in κ^κ, and she proves that $\mathbb{E}_\kappa$ is κ^κ-good whenever $\mathbb{E}_\kappa$ forces that κ is measurable.

We finish this section with the following iteration result.

Lemma 3.1.9. *Assume that δ is a limit ordinal and that $\langle \mathbb{P}_\xi : \xi < \delta \rangle$ is a $\lessdot$-increasing sequence of κ^κ-good posets. Let $\mathbb{P} := \mathrm{limdir}_{\xi<\delta}\mathbb{P}_\xi$. If $\mathrm{cf}(\delta) > \kappa$ and $\mathbb{P}$ is $\mathrm{cf}(\delta)$-cc then $\mathbb{P}$ is κ^κ-good.*

Proof. If $\dot{y}$ is a $\mathbb{P}$-name of a function in κ^κ, then there is some $\alpha < \delta$ such that $\dot{y}$ is a $\mathbb{P}_\alpha$-name, this because $\mathbb{P}$ is $\mathrm{cf}(\delta)$-cc and $\mathrm{cf}(\delta) > \kappa$. Let $h \in \kappa^\kappa$ be a function obtained from the goodness of $\mathbb{P}_\alpha$ applied to $\dot{y}$. It is clear that $x \not\leq^* h$ implies $\Vdash_{\mathbb{P}} x \not\leq \dot{y}$. □

3.1.3 A model where $\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) < \mathrm{add}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$

In this subsection, we show that tree forcings increase $\mathrm{cov}(\mathcal{SN})$. Also, we prove Theorem L. First we start with the following result about b-tree forcings.

Lemma 3.1.10. *Any b-tree forcing notion is proper and strongly ω^ω-bounding.*[2]

Proof. The standard argument works (see e.g. [GS93]). □

We now present the key lemma that will allow us to prove Theorem 3.1.13.

Lemma 3.1.11. *Let $\mathbb{T}$ be a b-tree forcing notion and let $D \subseteq \omega^\omega$ be a dominating family of increasing functions. If $\sigma^f \in (\mathrm{seq}_{<\omega}(b))^\omega$ with $\mathrm{ht}_{\sigma^f} = f^*$ for each $f \in D$ then, for any $T \in \mathbb{T}$, there is some $S \leq T$ in $\mathbb{T}$ and some $f \in D$ such that $[\sigma^f]_\infty \cap [S] = \emptyset$. In particular, $\mathbb{T}$ forces that $\tau \notin \bigcap_{f\in D}[\sigma^f]_\infty$ where τ denotes generic real in $\prod b$ added by $\mathbb{T}$.*

Proof. Fix $T \in \mathbb{T}$. Define $f_0 : \omega \to \omega$ such that, for any $t \in \mathrm{Lv}_n(T)$, there is a splitting node of length $< f_0(n)$ extending t. By recursion, define $g(0) = 0$ and $g(n+1) = f_0(g(n))$, which clearly yields an increasing function. Set $I := (I^g)^{*2}$, that is, $I_n = [g(2n), g(2(n+1)))$ for each $n < \omega$. Since D is dominating, by Lemma 3.1.2 there is some $f \in D$ such that $I \sqsubseteq I^f$. For $n < \omega$, choose some k_n (if exists) such that $I_{k_n} \subseteq I^f_n$. Note that there are only finitely many $n < \omega$ for which k_n does not exist.

Now we define T_n by recursion on $n < \omega$ such that $T_n^{[t]} = T^{[t]}$ for any $t \in \mathrm{Lv}_{f^*(n)}(T_n)$. Put $T_0 = T$. For the successor step, if k_n does not exist then we set $T_{n+1} := T_n$; else, when k_n exists, for each $t \in \mathrm{Lv}_{g(2k_n)}(T_n)$ choose some $t' \in \mathrm{Lv}_{g(2k_n+1)}(T)$ extending t (recall that $f^*(n) \leq g(2k_n)$) such that t' is incompatible with σ^f_{n+1}. This is possible because there is a splitting node of length $< g(2k_n+1)$ extending t and $|\sigma^f_{n+1}| = f^*(n+1) \geq g(2(k_n+1))$. Put $T_{n+1} := \bigcup_{t\in\mathrm{Lv}_{f^*(n)}(T_n)} T^{[t']}$. Note that for each $t \in \mathrm{Lv}_{f^*(n)}(T_n)$, T_{n+1} contains a splitting node of length $< f^*(n+1)$ extending t'. This indicates that $\langle T_n : n < \omega\rangle$ satisfies the conditions of Definition 1.5.8(T6), so $S := \bigcap_{n<\omega} T_n \in \mathbb{T}$ and $S \leq T$. Even more, any branch of S is incompatible with $\sigma^f(k)$ for all but finitely many $k < \omega$, so $[\sigma^f]_\infty \cap [S] = \emptyset$. □

Corollary 3.1.12. $\mathcal{SN} \subseteq s^0$.

Proof. Apply Lemma 3.1.11 to $\mathbb{T} = \mathbb{S}$. □

The following theorem will be useful for the upcoming Theorem L.

Theorem 3.1.13. *Assume CH. Then, any CS (countable support) iteration of length ω_2 of bounded-tree forcing notions forces $\mathrm{cov}(\mathcal{SN}) = \aleph_2$.*

Proof. Assume that $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2\rangle$ results from such iteration and fix any dominating family D of increasing functions in the ground model (by CH, $|D| = \aleph_1$). Let $D^* := \{f^* : f \in D\}$, which is also a dominating family. Assume that $\{\dot X_\xi : \xi < \omega_1\}$ is a family of $\mathbb{P}$-names of members of $\mathcal{SN}(2^\omega)$. For each $\xi < \omega_1$ and $f \in D$, there is a $\mathbb{P}$-name $\dot\sigma^f_\xi$ for a function in $(2^{<\omega})^\omega$ such that $\mathbb{P}$ forces $\mathrm{ht}_{\dot\sigma^f_\xi} = f^*$ and $X_\xi \subseteq [\dot\sigma^f_\xi]_\infty$. Since $\mathbb{P}_{\omega_2}$ has $\aleph_2$-cc, there is some $\alpha < \aleph_1$ such that $\dot\sigma^f_\xi$ is a $\mathbb{P}_\alpha$-name for each $f \in D$ and $\xi < \omega_1$. Let $\dot{\mathbb{T}}$ be a $\mathbb{P}_\alpha$-name of a bounded-tree forcing notion such that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{T}}$.

Fix a $\mathbb{P}_\alpha$-generic set G over V. Work in $V[G]$. Let $b : \omega \to \omega$ be a function such that $\mathbb{T} := \dot{\mathbb{T}}[G]$ is a b-tree forcing notion. Thanks to the maps F_2 and F_b (see Subsection 1.3.1), since

[2]A poset $\mathbb{P}$ is *strongly ω^ω-bounding* if for any $p \in \mathbb{P}$ and any $\mathbb{P}$-name $\dot x$ of a function from ω into the ground model, there are a function f from ω into the finite sets and some $q \leq p$ that forces $\dot x(n) \in f(n)$ for any $n < \omega$.

D is still a dominating function in $V[G]$ (because CS iterations of ω^ω forcing notions are ω^ω-bounding, see [She17; Gol93]) and $\bigcap_{f\in D}[\sigma^f_\xi]_\infty \in \mathcal{SN}(2^\omega)$ for each $\xi < \omega_1$, we can find some $\rho^f_\xi \in (\mathrm{seq}_{<\omega}(b))^\omega$ with $\mathrm{ht}_{\rho^f_\xi} = f^*$ for each $f \in D$ and $\xi < \omega_1$ such that $F_b^{-1}F_2[\bigcap_{f\in D}[\sigma^f_\xi]_\infty] \subseteq [\rho^f_\xi]_\infty$. By Lemma 3.1.11, $\mathbb{T}$ forces $\tau_\alpha \notin \bigcap_{f\in D}[\rho^f_\xi]_\infty$ for each $\xi < \omega_1$ (here, $\tau_\alpha \in \prod b$ is the generic real added by $\mathbb{T}$), so $F_2^{-1}(F_b(\tau_\alpha)) \notin \bigcap_{f\in D}[\sigma^f_\xi]_\infty$ (since τ_α is a generic real, it can be shown by a density argument that $F_b(\tau_\alpha)$ has a unique pre-image under F_2).

Therefore, $\mathbb{P}_{\omega_2}$ forces that $F_2^{-1}(F_b(\tau_\alpha)) \notin \bigcup_{\xi<\omega_1} X_\xi$. □

Now we are ready to prove Theorem L.

Theorem 3.1.14. *Assume CH and that λ is an infinite cardinal such that $\lambda^{\aleph_1} = \lambda$. Then, there is a proper ω^ω-bounding poset with $\aleph_2$-cc forcing* $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \aleph_1$, $\mathrm{cov}(\mathcal{SN}) = \aleph_2$ *and* $\mathrm{cof}(\mathcal{SN}) = \lambda$. *In particular, it is consistent with ZFC that* $\mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

Proof. We show that $\mathrm{Fn}_{<\omega_1}(\lambda \times \omega_1, \omega_1)$ followed by the CS iteration of $\mathbb{S}$ of length $\aleph_2$ is the desired poset. By CH, $\mathrm{Fn}_{<\omega_1}(\lambda \times \omega_1, \omega_1)$ has $\aleph_2$-cc, and it is clear that it is $<\omega_1$-closed, so it is proper and preserves cofinalities (and it is obviously ω^ω-bounding since it does not add new reals). Even more, in the $\mathrm{Fn}_{<\omega_1}(\lambda \times \omega_1, \omega_1)$-forcing extension, CH still holds, $2^{\aleph_1} = \lambda$ and, by Lemma 3.1.3, $\mathfrak{d}_{\omega_1} = \lambda$.

Now work in the $\mathrm{Fn}_{<\omega_1}(\lambda \times \omega_1, \omega_1)$-extension. Let $\mathbb{Q} = \langle \mathbb{P}_\alpha, \mathbb{S} : \alpha < \omega_2 \rangle$ be the CS iteration of Sacks forcing of length ω_2. It is known that $\mathbb{Q}$ forces $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \aleph_1$ and, by Theorem 3.1.13, it forces $\mathrm{cov}(\mathcal{SN}) = \mathfrak{c} = \aleph_2$. In addition, since $\mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$, by Theorem N, $\mathbb{Q}$ forces that $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\omega_1}$.

It remains to show that $\mathbb{Q}$ forces $\mathfrak{d}_{\omega_1} = \lambda$. Since $\mathbb{Q}$ has $\aleph_2$-cc and size $\aleph_2$, it forces $2^{\aleph_1} = \lambda$. On the other hand, for each $\alpha < \omega_2$, $|\mathbb{P}_\alpha| = \aleph_1$, so $\mathbb{P}_\alpha$ is $\omega_1^{\omega_1}$-good by Lemma 3.1.6. Hence, by Lemma 3.1.9, $\mathbb{Q}$ is $\omega_1^{\omega_1}$-good and, by Lemma 3.1.5, $\mathbb{Q}$ forces $\lambda \leq \mathfrak{d}_{\omega_1}$. □

Remark 3.1.15. In the proof above it can be shown in addition that the first ω_2-many ω_1-Cohen reals form an unbounded family of $\omega_1^{\omega_1}$ even after the iteration of Sacks forcing. Hence, the final model satisfies $\mathfrak{b}_{\omega_1} = \aleph_2$.

Remark 3.1.16. Judah, Miller and Shelah [JMS92] have proved that, in Sacks model, $\mathrm{add}(s^0) = \aleph_1$ and $\mathrm{cov}(s^0) = \mathfrak{c}$. So Corollary 3.1.12 also implies that $\mathrm{cov}(\mathcal{SN}) = \mathfrak{c}$ in this model.

3.2 The second model

The aim of this chapter is to prove Theorem M.

3.2.1 On dominating numbers

In this subsection, we present some results about the cardinal characteristics associated with $\mathbf{D}^\lambda_S$ and $\mathbf{D}^\lambda_S(\leq^*)$ for an arbitrary directed preorder S.

Lemma 3.2.1. *Let λ be a non-zero cardinal. If S has no maximum then $\aleph_0 \leq \mathrm{cf}(\mathfrak{b}^\lambda_S) = \mathfrak{b}^\lambda_S \leq \mathrm{cf}(\mathfrak{d}^\lambda_S) \leq \mathfrak{d}^\lambda_S \leq |S|^\lambda$. In particular (for $\lambda = 1$), $\mathfrak{b}(S)$ is regular and $\mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S)) \leq \mathfrak{d}(S)$.*

Lemma 3.2.2. *(i) $\mathfrak{b}(S) = \mathfrak{b}^\lambda_S \leq \mathfrak{d}(S) \leq \mathfrak{d}^\lambda_S \leq \mathfrak{d}(S)^\lambda \leq |S|^\lambda$. Even more, $S \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$.*

(ii) If $\lambda < \mathfrak{b}(S)$ then $\mathbf{D}^\lambda_S \cong_{\mathrm{T}} S$.

(iii) If $\lambda \leq \lambda'$, then $\mathbf{D}^\lambda_S \preceq_{\mathrm{T}} \mathbf{D}^{\lambda'}_S$. In particular, $\mathfrak{d}^\lambda_S \leq \mathfrak{d}^{\lambda'}_S$.

(iv) If λ is infinite then $\mathbf{D}^\lambda_S(\leq^*) \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$, so $\mathfrak{d}^\lambda_S(\leq^*) \leq \mathfrak{d}^\lambda_S$ and $\mathfrak{b}(S) \leq \mathfrak{b}^\lambda_S(\leq^*)$.

Proof. (i) Clearly $\mathfrak{d}^\lambda_S \leq \mathfrak{d}(S)^\lambda \leq |S|^\lambda$ and $\mathfrak{b}(S) \leq \mathfrak{d}(S)$. It remains to prove that $\mathfrak{b}(S) \leq \mathfrak{b}^\lambda_S$ and $S \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$ (which implies $\mathfrak{b}^\lambda_S \leq \mathfrak{b}(S)$ and $\mathfrak{d}(S) \leq \mathfrak{d}^\lambda_S$). To see $\mathfrak{b}(S) \leq \mathfrak{b}^\lambda_S$, let $B \subseteq S^\lambda$ with $|B| < \mathfrak{b}(S)$. For $\alpha < \lambda$, since $|\{f(\alpha) : f \in B\}| \leq |B| < \mathfrak{b}(S)$, choose $g(\alpha) \in S$ such that $f(\alpha) \leq_S g(\alpha)$ for all $f \in B$. Then $g \in S^\lambda$ and it bounds B.

We now prove that $S \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$. Define $\Psi_1 : S \to S^\lambda$ and $\Psi_2 : S^\lambda \to S$ as follows. For $f \in S^\lambda$ put $\Psi_2(f) := f(0)$. On the other hand, for $i \in S$ define $f_i \in S^\lambda$ by $f_i(\alpha) := i$ for each $\alpha < \lambda$, so put $\Psi_1(i) := f_i$. It is clear that $\Psi_1(i) \leq f$ implies $i \leq_S \Psi_2(f)$.

(ii) By (i), it is enough to show that $\mathbf{D}^\lambda_S \preceq_{\mathrm{T}} S$. For $f \in S^\lambda$, since $f[\lambda] \leq \lambda < \mathfrak{b}(S)$, choose $\Psi'_1(f)$ in S such that $f(\alpha) \leq_S \Psi'_1(f)$ for each $\alpha < \lambda$. This defines $\Psi'_1 : S^\lambda \to S$.

Finally, put $\Psi_2(i) := f_i$ for $i \in S$, where f_i is as in (i). It remains to check that, if $\Psi'_1(f) \leq_S i$ then $f \leq f_i$ for $f \in S^\lambda$ and $i \in S$. This is clear because, for $\alpha < \lambda$, $f(\alpha) \leq_S \Psi'_1(f) \leq_S i = f_i(\alpha)$.

(iii) Define $\Psi''_2 : S^{\lambda'} \to S^\lambda$ by $\Psi''_2(f) := f{\restriction}\lambda$.

To define $\Psi_1 : S^\lambda \to S^{\lambda'}$, for $g \in S^\lambda$ set $g^* \in S^{\lambda'}$ such that, for any $\alpha < \lambda'$, $g^*(\alpha) := g(\alpha)$ if $\alpha < \lambda$, and $g^*(\alpha) = 0$ otherwise. Put $\Psi_1(g) := g^*$. It is clear that $\Psi_1(g) \leq f$ implies $g \leq \Psi''_2(f)$.

(iv) Obvious. □

Lemma 3.2.3. *If λ is an infinite cardinal, then $\mathfrak{d}^\lambda_S(\leq^*) > \lambda$.*

Proof. Let $F := \{f_\xi : \xi < \lambda\} \subseteq S^\lambda$ and let K be a bijection from λ onto $\lambda \times \lambda$. Define $f \in S^\lambda$ as follows: for any $\alpha < \lambda$ we choose $f(\alpha) > f_{K(\alpha)_0}(\alpha)$ (it exists because S has no maximum). For each $\xi, \beta < \lambda$ set $\alpha_{\xi,\beta} := K^{-1}(\xi, \beta)$, so $K(\alpha_{\xi,\beta})_0 = \xi$ and $f(\alpha_{\xi,\beta}) > f_\xi(\alpha_{\xi,\beta})$. Then $|\{\alpha < \lambda : f(\alpha) > f_\xi(\alpha)\}| = \lambda$. □

In the next theorem we give a characterization of $\mathfrak{d}^\lambda_S$.

Theorem 3.2.4. *If λ is an infinite cardinal, then*

$$\mathfrak{d}^\lambda_S = \mathfrak{d}^\lambda_S(\leq^*) \cdot \sup_{\alpha<\lambda}\{\mathfrak{d}^{|\alpha|}_S\}.$$

Proof. Clearly $\mathfrak{d}^\lambda_S(\leq^*) \cdot \sup_{\alpha<\lambda}\{\mathfrak{d}^{|\alpha|}_S\} \leq \mathfrak{d}^\lambda_S$ because $\mathfrak{d}^{|\alpha|}_S \leq \mathfrak{d}^\lambda_S$ and $\mathfrak{d}^\lambda_S(\leq^*) \leq \mathfrak{d}^\lambda_S$ by Lemma 3.2.2 (iii) and (iv), respectively.

For $\alpha < \lambda$, choose $D_\alpha \subseteq S^\alpha$ $\leq$-dominating with $|D_\alpha| = \mathfrak{d}^{|\alpha|}_S$, and choose a $\leq^*$-dominating family $D \subseteq S^\lambda$ of size $\mathfrak{d}^\lambda_S(\leq^*)$. For $g \in D_\alpha$ with $\alpha < \lambda$ and $h \in D$ define the function $f_{g,h} \in S^\lambda$ by

$$f_{g,h}(\beta) := \begin{cases} g(\beta) & \text{if } \beta < \alpha, \\ h(\beta) & \text{if } \beta \geq \alpha. \end{cases}$$

Since $|\{f_{g,h} : h \in D \wedge \exists \alpha < \lambda (g \in D_\alpha)\}| \leq \mathfrak{d}^\lambda_S(\leq^*) \cdot \sup_{\alpha<\lambda}\{\mathfrak{d}^{|\alpha|}_S\} \cdot \lambda = \mathfrak{d}^\lambda_S(\leq^*) \cdot \sup_{\alpha<\lambda}\{\mathfrak{d}^{|\alpha|}_S\}$ by Lemma 3.2.3, it sufficies to prove that this family is $\leq$-dominating. To this end let $f \in S^\lambda$. Find $h \in D$ and $\alpha < \lambda$ such that $f(\beta) \leq h(\beta)$ for all $\beta \geq \alpha$. Then, for $\beta < \alpha$ choose $g_\alpha(\beta) \in S$ above $f(\beta)$ and $h(\beta)$, so there is some $g \in D_\alpha$ such that $g_\alpha \leq g$. Therefore, $f_{g,h}$ dominates f everywhere. □

It is known that $\mathfrak{d}^\lambda_\lambda = \mathfrak{d}_\lambda$ when λ is regular, even more, this follows from Theorem 3.2.4 because $\mathfrak{d}^\kappa_\lambda = \lambda$ when $\kappa < \lambda$ (by Lemma 3.2.2 (ii)). However, $\mathfrak{d}^\lambda_\lambda = \mathfrak{d}^\lambda_{\mathrm{cf}(\lambda)}$ in general. More details about $\mathfrak{d}^\lambda_\kappa$ can be found in [Bre19a].

Lemma 3.2.5. $\mathbf{D}^\lambda_{\kappa\times\lambda} \cong_{\mathrm{T}} \mathbf{D}^\lambda_\kappa \otimes \mathbf{D}^\lambda_\lambda$. *In particular,* $\mathfrak{d}^\lambda_{\kappa\times\lambda} = \max\{\mathfrak{d}^\lambda_\kappa, \mathfrak{d}^\lambda_\lambda\}$.

Proof. To define $\Psi_1 : (\kappa\times\lambda)^\lambda \to \kappa^\lambda \times \lambda^\lambda$: for $F \in (\kappa\times\lambda)^\lambda$ define $f_F \in \kappa^\lambda$ and $g_F \in \lambda^\lambda$ by setting $f_F(\alpha) := F(\alpha)_0$ and $g_F(\alpha) := F(\alpha)_1$. Put $\Psi_1(F) := (f_F, g_F)$. Now, define $\Psi_2 : \kappa^\lambda \times \lambda^\lambda \to (\kappa\times\lambda)^\lambda$ as follows: for $f \in \kappa^\lambda$ and $g \in \lambda^\lambda$ set $F_{f,g} \in (\kappa\times\lambda)^\lambda$ by $F_{f,g}(\beta) = (f(\beta), g(\beta))$. Put $\Psi_2(f,g) := F_{f,g}$. It is clear that if $(f_F, g_F) \leq_\otimes (f,g)$ then $F \leq_{\kappa\times\lambda} F_{f,g}$. Also, $F_{f,g} \leq_{\kappa\times\lambda} F$ implies $(f,g) \leq_\otimes (f_F, g_F)$. $\square$

3.2.2 A connection between $\mathcal{SN}$ and $\mathfrak{d}^\lambda_S$

In this subsection, we introduce the notion of λ-dominating system on a direted preorder S; we discuss the relationship between $\mathbf{D}^\lambda_S$ and $\mathcal{SN}$, and we prove Theorem O.

Definition 3.2.6. Let S be a directed preorder. Given an increasing function $f \in \omega^\omega$, we say that a family $A^f = \langle A^f_i : i \in S\rangle$ of subsets of 2^ω is an $\mathcal{I}_f$*-directed system on* S if it satisfies:

(I) $\forall i \in S(A^f_i \subseteq 2^\omega$ is dense G_δ and $A^f_i \in \mathcal{I}_f)$;

(II) $\forall i,j \in S(i \leq_S j \to A^f_i \subseteq A^f_j)$ and

(III) $\langle A^f_i : i \in S\rangle$ is cofinal in $\mathcal{I}_f$.

Assume from now on that S is a directed partial order with a minimun i_0.

Definition 3.2.7. If λ is a cardinal and there is some dominating family $\{f_\alpha : \alpha < \lambda\}$ on ω^ω such that $A^{f_\alpha} = \langle A^{f_\alpha}_i : i \in S\rangle$ is an $\mathcal{I}_{f_\alpha}$-directed system and

$$\forall\alpha < \lambda\Big(\bigcap_{\beta<\alpha} A^{f_\beta}_{i_0} \notin \mathcal{I}_{f_\alpha}\Big),$$

then we say that $\langle A^{f_\alpha} : \alpha < \lambda\rangle$ is a λ*-dominating system on* S. For each $\alpha < \lambda$ and $i \in S$ we denote $A^\alpha := A^{f_\alpha}$ and $A^\alpha_i := A^{f_\alpha}_i$.

The matrix mentioned in the introduction is a λ-dominating system on λ, which was constructed by Yorioka [Yor02] under $\lambda = \mathrm{minadd} = \mathrm{supcof}$. The following lemma is inspired by Yorioka's proof of Theorem N, reproved in Corollary 3.2.15.

Lemma 3.2.8. *Let λ be a uncountable cardinal. Assume* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda$ *and that, for any increasing function $f \in \omega^\omega$, there is some $\mathcal{I}_f$-directed system on S. Then there is some λ-dominating system on S.*

Proof. Let $\langle h_\alpha : \alpha < \lambda\rangle$ be a dominating family and fix an $\mathcal{I}_f$-directed system $A^f = \langle A^f_i : i \in S\rangle$ for all increasing f. By recursion on $\alpha < \lambda$ we construct a dominating family $\langle f_\alpha : \alpha < \lambda\rangle$ that guarantees that $\langle A^{f_\alpha} : \alpha < \lambda\rangle$ is a λ-dominating system.

Assume that $\langle f_\beta : \beta < \alpha\rangle$ has been constructed. We can get a transitive model M for ZFC such that $|M| < \lambda = \mathrm{cov}(\mathcal{M})$ and $A^\beta_{i_0}$ is coded in M for any $\beta < \alpha$, i.e, $A^\beta_{i_0} = [\sigma^\beta]_\infty$ for some $\sigma^\beta \in M$ that witnesses $A^\beta_{i_0} \in \mathcal{I}_{f_\beta}$.

Recall that $M < \mathrm{cov}(\mathcal{M})$ implies that there is a Cohen real over M, hence it adds a perfect set P of Cohen reals over M (see [BJ95, Lemma 3.3.2]). Since each $A^\beta_{i_0}$ $(\beta < \alpha)$ is a dense G_δ

set coded in M, $P \subseteq \bigcap_{\beta<\alpha} A^{\beta}_{i_0}$. On the other hand, there is some $g \in \omega^\omega$ such that $P \notin \mathcal{I}_g$ by Corollary 1.2.9.

Choose $f_\alpha \in \omega^\omega$ increasing such that $h_\alpha \leq f_\alpha$ and $g \leq f_\alpha$. Then $\mathcal{I}_{f_\alpha} \subseteq \mathcal{I}_g$ and $P \notin \mathcal{I}_{f_\alpha}$. But $P \subseteq \bigcap_{\beta<\alpha} A^\beta$, hence $\bigcap_{\beta<\alpha} A^\beta \notin \mathcal{I}_{f_\alpha}$.

Clearly, $\langle f_\alpha : \alpha < \lambda \rangle$ is a dominating family and $\langle A^{f_\alpha} : \alpha < \lambda \rangle$ is a λ-dominating system on S. □

We now prove Theorem O(1).

Theorem 3.2.9. *Assume that there is a λ-dominating system on S. Then $\mathcal{SN} \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$.*

Proof. Fix a λ-dominating system $\langle A^{f_\alpha} : \alpha < \lambda \rangle$. For $X \in \mathcal{SN}$ define $\Psi_1(X) := G_X \in S^\lambda$ such that $X \subseteq \bigcap_{\alpha<\lambda} A^\alpha_{G_X(\alpha)}$, which can be choosen Definition 3.2.6 (III). Let $F \in S^\lambda$. Note that $\bigcap_{\alpha<\lambda} A^\alpha_{F(\alpha)} \in \mathcal{SN}$ because $\bigcap_{\alpha<\lambda} A^\alpha_{F(\alpha)} \subseteq A^\alpha_{F(\alpha)}$ and $A^\alpha_{F(\alpha)} \in \mathcal{I}_{f_\alpha}$. Define $\Psi_2(F) := \bigcap_{\alpha<\lambda} A^\alpha_{F(\alpha)}$.

We show that (Ψ_1, Ψ_2) witnesses $\mathcal{SN} \preceq_{\mathrm{T}} \mathbf{D}^\lambda_S$. Assume that $\Psi_1(X) \leq F$. Then $G_X(\alpha) \leq F(\alpha)$ for all $\alpha < \lambda$, so by Definition 3.2.6(II), $X \subseteq \bigcap_{\alpha<\lambda} A^\alpha_{G_X(\alpha)} \subseteq \bigcap_{\alpha<\lambda} A^\alpha_{F(\alpha)}$. □

As a consequence we get:

Corollary 3.2.10. *If there is an λ-dominating system on S then $\mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}^\lambda_S$ and $\mathfrak{b}(S) = \mathfrak{b}^\lambda_S \leq \mathrm{add}(\mathcal{SN})$.*

The rest of section is dedicated to prove Theorem O (2), which will be used in Subsection 3.2.3. To do this, we need the following lemma, which is inspired in the proof of [Yor02, Thm. 3.8].

Lemma 3.2.11. *Let $0 < \kappa < \lambda$ be cardinals with λ infinite. Assume $\mathrm{minnon} \geq \lambda$ and that there is a λ-dominating system $\langle A^{f_\alpha} : \alpha < \lambda \rangle$ on $\kappa \times \lambda$, $A^\alpha_{i,j} := A^{f_\alpha}_{i,j}$. Then, for any $f \in \lambda^\lambda$, there are $G \in (\kappa \times \lambda)^\lambda$ and $\{x^\alpha_\beta : \alpha < \lambda, \beta < \kappa\} \subseteq 2^\omega$ such that*

(i) $\forall \alpha < \lambda(\{x^{\alpha'}_\beta : \alpha' \leq \alpha, \beta < \kappa\} \subseteq A^\alpha_{G(\alpha)})$,

(ii) $\forall \alpha < \lambda \forall \beta < \kappa(x^\alpha_\beta \in \bigcap_{\alpha'<\alpha} A^{\alpha'}_{G(\alpha')} \smallsetminus A^\alpha_{\beta,f(\alpha)})$*, and*

(iii) $\forall \alpha < \lambda(f(\alpha) \leq G(\alpha)_1)$.

Proof. We will recursively construct $G(\alpha) \in \kappa \times \lambda$ and $x^\alpha_\beta \in 2^\omega$. Assume that we already have $G(\alpha')$ and $x^{\alpha'}_\beta$ for any $\alpha' < \alpha$ and $\beta < \kappa$. Set $B_\beta := A^\alpha_{\beta,f(\alpha)} \cup \{x^{\alpha'}_{\beta'} : \alpha' < \alpha, \beta' < \kappa\}$. Since $\{x^{\alpha'}_{\beta'} : \alpha' < \alpha, \beta' < \kappa\}$ has size $<\lambda$, $B_\beta \in \mathcal{I}_{f_\alpha}$ because $\kappa < \lambda \leq \mathrm{non}(\mathcal{I}_{f_\alpha})$. By Definition 3.2.7 $\bigcap_{\alpha'<\alpha} A^{\alpha'}_{0,0} \notin \mathcal{I}_{f_\alpha}$, so there is some $x^\alpha_\beta \in \bigcap_{\alpha'<\alpha} A^{\alpha'}_{0,0} \smallsetminus B_\beta$. Note that $\{x^{\alpha'}_\beta : \alpha' \leq \alpha, \beta < \kappa\} \in \mathcal{I}_{f_\alpha}$. Then there must be a $G(\alpha) \in \kappa \times \lambda$ such that $\{x^{\alpha'}_\beta : \alpha' \leq \alpha, \beta < \kappa\} \subseteq A^\alpha_{G(\alpha)}$ and $f(\alpha) \leq G(\alpha)_1$. This construction satisfies the required conditions. □

Lemma 3.2.12. *With the same assumptions as in Lemma 3.2.11 and with $G \in (\kappa \times \lambda)^\lambda$ fulfilling its conclusion, if $\alpha < \lambda$ and $\delta \leq f(\alpha)$ then $\bigcap_{\gamma<\lambda} A^\gamma_{G(\gamma)} \nsubseteq A^\alpha_{\beta,\delta}$ for all $\beta < \kappa$.*

Proof. By Lemma 3.2.11 (i) and (ii), $\{x^\alpha_\beta : \alpha < \lambda, \beta < \kappa\} \subseteq \bigcap_{\gamma<\lambda} A^\gamma_{G(\gamma)}$ and $x^\alpha_\beta \notin A^\alpha_{\beta,f(\alpha)}$. Hence $x^\alpha_\beta \notin A^\alpha_{\beta,\delta}$ because $\delta \leq f(\alpha)$. □

Theorem 3.2.13. *Assume $0 < \kappa \leq \lambda \leq \mathrm{minnon}$ and that there is some λ-dominating system on $\kappa \times \lambda$. Then $\mathbf{D}^\lambda_\lambda \preceq_{\mathrm{T}} \mathcal{SN}$.*

Proof. Fix a λ-dominating system $\langle A^{f_\alpha} : \alpha < \lambda \rangle$. When $\kappa = \lambda$, $\langle B^\beta_{0,\alpha} : \alpha, \beta < \lambda \rangle$ with $B^\beta_{0,\alpha} = A^\beta_{\alpha,\alpha}$ forms a λ-dominating system on $1 \times \lambda$. So the proof reduces to the case $\kappa < \lambda$. For $B \in \mathcal{SN}$, choose some $F_B \in (\kappa \times \lambda)^\lambda$ such that $B \subseteq \bigcap_{\alpha<\lambda} A^\alpha_{F_B(\alpha)}$. Define $\Psi_2(B) := f_B \in \lambda^\lambda$ by $f_B(\alpha) := F_B(\alpha)_1$ for every $\alpha < \lambda$.

Now for $f \in \lambda^\lambda$, by Lemma 3.2.11 and 3.2.12 we can find some $G_f \in (\kappa \times \lambda)^\lambda$ such that, for each $g \in \lambda^\lambda$ and for each $\alpha < \lambda$, if $g(\alpha) \leq f(\alpha)$ then

$$\bigcap_{\alpha<\lambda} A^\alpha_{G_f(\alpha)} \not\subseteq A^\alpha_{\beta,g(\alpha)} \text{ for all } \beta < \kappa.$$

Define $\Psi_1(f) := \bigcap_{\alpha<\lambda} A^\gamma_{G_f(\alpha)}$.

We show that (Ψ_1, Ψ_2) witnesses $\mathbf{D}^\lambda_\lambda \preceq_{\mathrm{T}} \mathcal{SN}$, i.e, if $f \not\leq f_B$ then $\Psi_1(f) \not\subseteq B$. Since $f \not\leq f_B$ we can choose $\alpha < \lambda$ such that $f(\alpha) > f_B(\alpha)$, so $\bigcap_{\alpha<\lambda} A^\alpha_{G_f(\alpha)} \not\subseteq A^\alpha_{F_B(\alpha)}$. Thus $\Psi_1(f) = \bigcap_{\alpha<\lambda} A^\alpha_{G_f(\alpha)} \not\subseteq B$ because $B \subseteq \bigcap_{\alpha<\lambda} A^\alpha_{F_B(\alpha)}$. □

As a consequence, we get:

Corollary 3.2.14. *With the same assumptions as in Theorem 3.2.13,* $\mathrm{cof}(\mathcal{SN}) \geq \mathfrak{d}^\lambda_\lambda$ *and* $\mathrm{add}(\mathcal{SN}) \leq \mathfrak{b}^\lambda_\lambda = \mathrm{cof}(\lambda)$.

Before ending this section we show that Theorem N is a consequence of the previous results.

Corollary 3.2.15 ([Yor02, Thm. 2.6])**.** *Let λ be an infinite cardinal. If* $\mathrm{minadd} = \mathrm{supcof} = \lambda$ *then* $\mathcal{SN} \cong_{\mathrm{T}} \mathbf{D}(\lambda^\lambda)$.

Proof. Since $\mathrm{minadd} = \mathrm{supcof} = \lambda$, for each $f \in \omega^\omega$ we can find an $\mathcal{I}_f$-directed system on λ. By Lemma 3.2.8, there is a λ-dominating system $\langle A^{f_\alpha} : \alpha < \lambda \rangle$ on λ. Since $\langle B^\beta_{0,\alpha} : \alpha, \beta < \lambda \rangle$ with $B^\beta_{0,\alpha} = A^\beta_\alpha$ forms a λ-dominating system on $1 \times \lambda$, $\mathbf{D}^\lambda_\lambda \preceq_{\mathrm{T}} \mathcal{SN}$ by Theorem 3.2.13

On the other hand, since $\langle A^{f_\alpha} : \alpha < \lambda \rangle$ is a λ-dominating system on λ, by Theorem 3.2.9 $\mathcal{SN} \preceq_{\mathrm{T}} \mathbf{D}^\lambda_\lambda$. □

3.2.3 A model where $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$

We begin this subsection by showing that a cofinal family in $\mathcal{I}_f$ is produced by a localizing family and a dominating family.

Lemma 3.2.16 ([CM19, Thm. 3.12])**.** *Let $f \in \omega^\omega$ be an increasing function. Then there is some definable function $\Psi^f : \omega^{\uparrow\omega} \times \mathcal{S}(\omega, \mathrm{id}_\omega) \to \mathcal{I}_f$ such that, if*

(1) $S \subseteq \mathcal{S}(\omega, \mathrm{id}_\omega)$ is a localizing family, i.e, for any $x \in \omega^\omega$ there is some $\varphi \in S$ such that $x \in^* \varphi$, and

(2) $D \subseteq \omega^{\uparrow\omega}$ is a dominating family,

then $\{\Psi^f(d, \varphi) : d \in D \text{ and } \varphi \in S\}$ is cofinal in $\mathcal{I}_f$.

The same proof actually yields:

Lemma 3.2.17. *Let M be a transitive model of ZFC with $f \in \omega^\omega \cap M$ increasing. If $d \in \omega^{\uparrow\omega}$ is dominating over M and $\varphi \in \mathcal{S}(\omega, \mathrm{id}_\omega)$ is localizing over $M \cap \omega^\omega$, then $A \subseteq \Psi^f(d, \varphi)$ for all $A \in \mathcal{I}_f$ coded in M.*[3]

[3] I.e, $A = [\sigma]_\infty$ for some $\sigma \in M$ witnessing $A \in \mathcal{I}_f$.

Now, we are ready to prove Theorem M.

Theorem 3.2.18. *Let $\kappa \leq \lambda$ be regular uncountable cardinals where $\kappa^{<\kappa} = \kappa$ and $\lambda^{<\lambda} = \lambda$, and let λ_1, λ_2 be cardinals such that $\lambda \leq \lambda_1 = \lambda_1^{\aleph_0}$, and $\lambda < \lambda_2 = \lambda_2^{\lambda}$. Then there is a cofinality preserving poset that forces*

(I) $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \kappa$ *and* $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \lambda$.

(II) $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) = \kappa \leq \mathrm{non}(\mathcal{SN}) = \lambda \leq \mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda = \mathfrak{d}^{\lambda}_{\kappa\times\lambda} = \lambda_2$

(III) $\mathfrak{c} = \lambda_1$.

Proof. Step 1. We start with $\mathbb{P}_0 := \mathrm{Fn}_{<\lambda}(\lambda_2 \times \lambda, \lambda)$. $\mathbb{P}_0$ is λ^+-cc and $<\lambda$-closed, so it preserves cofinalities, and $\mathbb{P}_0$ forces $\mathfrak{d}_\lambda = 2^\lambda = \lambda_2$.

Step 2. In $V^{\mathbb{P}_0}$, let $\mathbb{P}_1 := \mathrm{Fn}_{<\kappa}(\lambda_2 \times \lambda, \kappa)$. When $\kappa < \lambda$, $\mathbb{P}_1$ forces $\mathfrak{d}^\lambda_\kappa = 2^\lambda = \lambda_2$ and $\mathfrak{d}_\lambda = \lambda_2$, the latter preserved from $V^{\mathbb{P}_0}$ because $\mathbb{P}$ is λ-c.c (see Subsection 3.1.2); and if $\lambda = \kappa$, the same is forced by step 1.

Step 3. In $V^{\mathbb{P}_0 * \mathbb{P}_1}$, let $\mathbb{P}_2 := \mathrm{Fn}_{<\omega}(\lambda_1 \times \omega, \omega)$, which forces $\mathfrak{c} = \lambda_1$ and $2^\lambda = \max\{\lambda_1, \lambda_2\}$. In particular, $\mathfrak{d}^\lambda_{\kappa\times\lambda} = \mathfrak{d}_\lambda = \lambda_2$ by Lemma 3.2.5, which is preserved from $V^{\mathbb{P}_0*\mathbb{P}_1}$ because $\mathbb{P}_2$ is ccc.

Step 4. We work in $V_{0,0} := V^{\mathbb{P}_0*\mathbb{P}_1*\mathbb{P}_2}$. We define the simple matrix iteration of height $\gamma := \lambda$ and length $\pi := \lambda\kappa$ where, at each interval of the form $[\lambda\rho, \lambda(\rho+1))$ for each $\rho < \kappa$, is defined as follows: For each $\xi \in [\lambda\rho, \lambda(\rho+1))$, put $\Delta(\xi) = \varepsilon + 1$ and $\dot{\mathbb{Q}}_\xi := \mathbb{LOC}^{V_{\Delta(\xi),\xi}}$ when $\xi = \lambda\rho + \varepsilon$ for some (unique) $\rho < \kappa$ and $\varepsilon < \lambda$.

Set $\mathbb{P} := \mathbb{P}_{\lambda,\lambda\kappa}$ and $V_{\alpha,\xi} := V_{0,0}^{\mathbb{P}_{\alpha,\xi}}$. We first prove that $\mathbb{P}$ forces $\kappa \leq \mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N}) \leq \lambda$. For each $0 < \xi < \lambda\kappa$ denote by $\varphi^\xi \in V_{\Delta(\xi),\xi+1} \cap \mathcal{S}(\omega, \mathrm{id})$ the generic slalom over $V_{\Delta(\xi),\xi}$ added by $\dot{\mathbb{Q}}_{\Delta(\xi),\xi} = \dot{\mathbb{Q}}_{\lambda,\xi} = \mathbb{LOC}^{V_{\Delta(\xi),\xi}}$. Hence $V_{\lambda,\lambda\kappa} \models \kappa \leq \mathrm{add}(\mathcal{N})$ is a consequence of the following, which is a similar to one argument of the proof of Theorem 2.4.1:

$$\text{In } V_{\lambda,\lambda\kappa}\text{, each family of reals of size } {<}\kappa \text{ is localizated by some } \varphi^\xi. \tag{$\clubsuit$}$$

Since $\{\varphi^\xi : 0 < \xi < \lambda\kappa\}$ is a family of slaloms of size $\leq \lambda$, by ($\clubsuit$) and Lemma 1.6.4 any member of $V_{\lambda,\lambda\kappa} \cap \omega^\omega$ is localizated by some φ^ξ. Hence $V_{\lambda,\lambda\kappa} \models \mathrm{cof}(\mathcal{N}) \leq \lambda$.

On the other hand $\mathbb{P}$, as a finite support iteration of lenght $\lambda\kappa$, adds κ-cofinally many Cohen reals that form a strongly κ-**Ed**-unbounded family of size κ, hence $\mathbb{P}$ forces $\mathrm{non}(\mathcal{M}) \leq \kappa$, and by Theorem 1.7.14 (since Cohen reals are added by $\mathbb{LOC}$), $\mathbb{P}$ forces $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\mathbf{Ed}) \geq \lambda$. Therefore, $\mathbb{P}$ forces $\kappa = \mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \lambda$. In addition, $\mathbb{P}$ forces:

$\underline{\kappa \leq \mathrm{add}(\mathcal{SN})}$: because $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$ ([Car93]);

$\underline{\mathrm{cov}(\mathcal{SN}) \leq \kappa}$: because the lenght of the FS iteration on the top of the matrix has cofinality κ and it is well known that such cofinality becomes an upper bound of $\mathrm{cov}(\mathcal{SN})$ (see e.g. [BJ95, Lemma 8.2.6]);

$\underline{\mathrm{non}(\mathcal{SN}) = \lambda}$: because $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{N})$;

$\underline{\mathrm{cof}(\mathcal{SN}) = \lambda_2}$: Let $f \in V_{\lambda,\lambda\kappa} \cap \omega^\omega$ be an increasing function. Then, by Lemma 1.6.4 there are some $\varepsilon_f < \lambda$ and $\rho_f < \kappa$ such that $f \in V_{\varepsilon_f,\xi_f}$ with $\xi_f = \lambda\rho_f + \varepsilon_f > 0$.

For $\rho < \kappa$ and $\varepsilon < \lambda$ define $\xi_f(\rho,\xi) := \lambda(\rho_f + \rho) + \varepsilon_f + \varepsilon$. Let $\dot{\varphi}^f_{\rho,\varepsilon}$ be the $\mathbb{P}_{\Delta(\xi_f(\rho,\xi)),\xi_f(\rho,\xi)+1}$-name of the generic slalom over $V_{\Delta(\xi_f(\rho,\xi)),\,\xi_f(\rho,\xi)}$ and let $\dot{d}^f_{\rho,\varepsilon}$ be the $\mathbb{P}_{\Delta(\xi_f(\rho,\xi)),\,\xi_f(\rho,\xi)+1}$-name of some increasing dominating real over $V_{\Delta(\xi_f(\rho,\xi)),\xi_f(\rho,\xi)}$ added by $\dot{\mathbb{Q}}_{\xi_f(\rho,\xi)}$ (see Figure 3.1). Set $A^f_{\rho,\varepsilon} := \Psi^f(\dot{d}^f_{\rho,\varepsilon}, \dot{\varphi}^f_{\rho,\varepsilon})$.

$$\langle A^f_{\rho,\varepsilon} : \rho < \kappa \text{ and } \varepsilon < \lambda\rangle \text{ is an } \mathcal{I}_f\text{-directed system on } \kappa \times \lambda. \tag{$\spadesuit$}$$

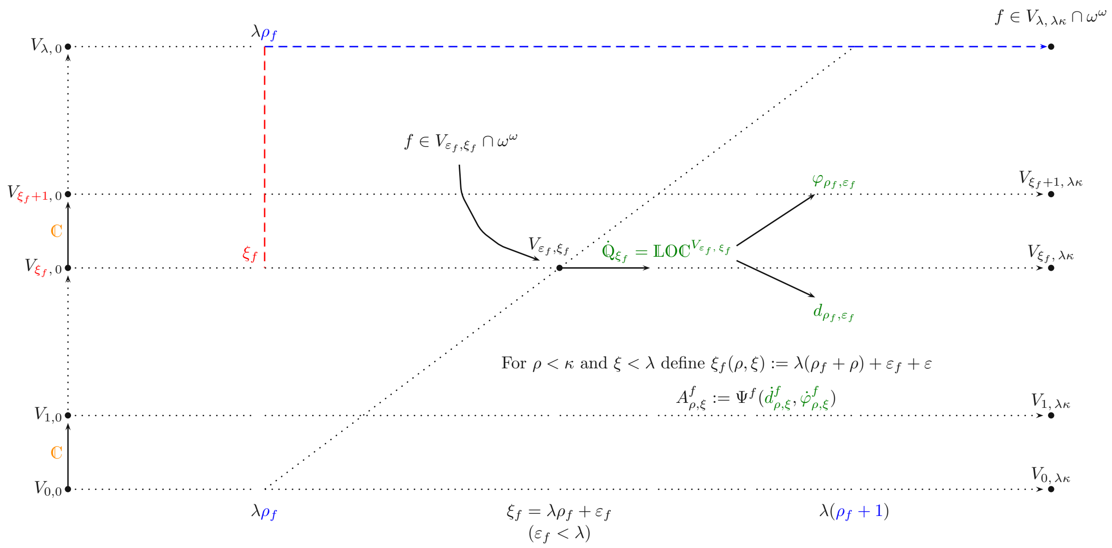

Figure 3.1: Matrix iteration

Proof of (♠). It is clear that (I) and (III) of Definition 3.2.6 follow by Lemma 3.2.17. To see (II) of Definition 3.2.6, note that $\dot{\varphi}^f_{\rho,\varepsilon}$ is localizing over $V_{\Delta(\xi_f(\rho,\xi)),\xi_f(\rho,\xi)}$ and $\dot{d}^f_{\rho,\varepsilon}$ is dominating over $V_{\Delta(\xi_f(\rho,\xi)),\xi_f(\rho,\xi)}$, so $A \subseteq A^f_{\rho,\varepsilon}$ for any $A \in \mathcal{I}_f$ coded in $V_{\Delta(\xi_f(\rho,\xi)),\xi_f(\rho,\xi)}$ by Lemma 3.2.17. In particular, $A^f_{\rho',\varepsilon'} \subseteq A^f_{\rho,\varepsilon}$ if $(\rho',\varepsilon') \le (\rho,\varepsilon)$. This shows (♠). □

We can choose a λ-dominating system $\langle A^{f_\gamma} : \gamma < \lambda \rangle$ on $\kappa \times \lambda$ by Lemma 3.2.8 because $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda$. Therefore, in $V_{\lambda,\lambda\kappa}$, $\mathrm{cof}(\mathcal{SN}) \le \mathfrak{d}^{\lambda}_{\kappa\times\lambda} = \lambda_2$ by Theorem 3.2.9 (recall that the values of $\mathfrak{d}^{\lambda}_{\kappa\times\lambda}$ and $\mathfrak{d}_\lambda$ do not change after λ-cc forcing) and since $\mathrm{minnon} = \lambda$, $\mathrm{cof}(\mathcal{SN}) \ge \mathfrak{d}_\lambda = \lambda_2$ by Theorem 3.2.13. □

3.3 Discussions and problems

Our Lemma 3.1.11 can also be proved for Silver-like type of posets, or more generally, for limsup creature type forcing notions obtained by finitary creating pairs as in Chapter 4. Therefore, these type of posets can be included as iterands in Theorem 3.1.13. Moreover, it can be concluded that $\mathcal{SN}$ is contained in the Marczewski-type ideal corresponding to Silver forcing.

Bartoszyński and Shelah [BS02, Thm. 3.3] proved that $\mathrm{non}(\mathcal{SN})$ can be increased by CS products of Silver-like posets. In fact, the same argument applies to CS products of posets of the form $\mathbb{PT}_b$ with b diverging to infinity. Concretely, assuming CH, if $\kappa^{\aleph_0} = \kappa$, I is a set of size κ and $\{b_i : i \in I\} \subseteq \omega^\omega$ is a family of functions diverging to infinity, then the CS product of $\mathbb{PT}_{b_i}$ with $i \in I$ forces $\mathfrak{d} = \aleph_1$ (because it is ω^ω-bounding) and $\mathrm{non}(\mathcal{SN}) = \mathfrak{c} = \kappa$.

A very natural question that comes from our main result is whether a version of Theorem 3.1.13 for CS products can be proved. By methods like in [GS93; KM21] it can be shown that any CS product of bounded-tree forcing notions remains proper and strongly ω^ω-bounding. However, it is not obvious how the proof of Lemma 3.1.11 can be translated to show that such a CS product increases $\mathrm{cov}(\mathcal{SN})$. This would generalize the consistency result of Theorem 3.1.13 in the sense that $\mathrm{cov}(\mathcal{SN})$ could be forced larger than $\aleph_2$.

By well known methods and results from [Yor02], the following open problem is the only one remaining to settle that the diagram of inequalities in Figure 3.2 is complete.

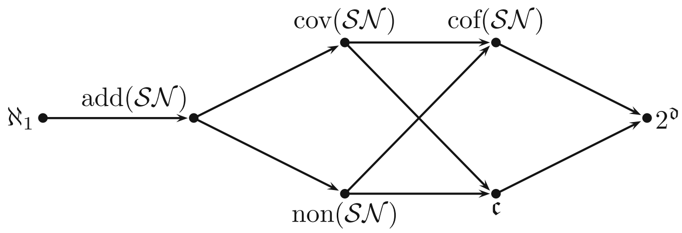

Figure 3.2: The arrows mean that $\leq$ is provable in ZFC.

Question T. *Is it consistent with ZFC that* $\mathrm{add}(\mathcal{SN}) < \min\{\mathrm{cov}(\mathcal{SN}), \mathrm{non}(\mathcal{SN})\}$*?*

In this chapter we provide the first two examples where 3 cardinals characteristics associated with $\mathcal{SN}$ can be pairwise different. To go one step further, we propose the following problem.

Question U. *Is it consistent with ZFC that the four cardinal invariants associated with* $\mathcal{SN}$ *are pairwise different?*

Any idea to solve Question T in the positive could be used to prove the consistency of Question U. In Theorem 2.4.2, we constructed a ccc poset forcing

$$\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}).$$

In the same model, $\mathrm{cov}(\mathcal{SN}) = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{SN}) = \mathrm{non}(\mathcal{N})$ because this model is obtained by a FS iteration of length with cofinality μ (where μ is the desired value for $\mathrm{non}(\mathcal{M})$), and it is well known that such cofinality becomes an upper bound of $\mathrm{cov}(\mathcal{SN})$ (see e.g. [BJ95, Lemma 8.2.6]). However, tools to deal with $\mathrm{add}(\mathcal{SN})$ and $\mathrm{cof}(\mathcal{SN})$ in this situation are still unknown.

4

FORCING WITH CREATURES

This chapter is based on the paper [CKM21]:

> "Continuum many different things: localisation, anti-localisation and Yorioka ideals"

joint with Lukas Klausner and Diego Mejía.

4.1 Structure of the chapter

This chapter is devoted to showing Theorem Q, which states that under CH there is an $\aleph_2$-cc ω^ω-bounding proper poset which forces that there are continuum many pairwise different cardinal characteristics of each one of the following six types: $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$, $\mathrm{non}(\mathcal{I}_f)$ and $\mathrm{cof}(\mathcal{I}_f)$.

We show some connections between the cardinal characteristics associated with Yorioka ideals, localization cardinals, and anti-localization cardinals, and then we present the notion of *block* in Section 4.2. In Section 4.3, we build the frame where the forcing lives, define the parameters of the forcing and present the rules they must obey to prove Theorem Q. Inspired in [FGKS17], in Section 4.4 we define *subatomic creatures*, *atomic creatures* and *compound creatures*, which are the building blocks of a condition in the forcing we want to construct. In Section 4.5 we present our forcing construction in detail and prove its basic properties. This construction follows the presentation in [GK21] and it is divided into three components: the *pr-part*, the *lc-part* and the *al-part*. As in [KS12], the atomic creatures of the lc-part and al-part depend on the possibilities given on the pr-part. The necessary bigness properties of the subatomic and compound creatures are developed in Section 4.6. These will be essential to prove main features of the forcing like *pure decision*, *continuous reading of names* and *rapid reading*, and for the proof of Theorem Q as well. In Section 4.7, we present continuous and rapid reading for our forcing. We prove that continuous reading follows by pure decision, as well as properness and ω^ω-bounding (in fact, our forcing satisfies a variation of Baumgartner's strong axiom A). Rapid reading follows by continuous reading and the bigness results from Section 4.6. The presentation of this section is very close to [GK21]. We prove Theorem Q in Section 4.8, and we present further discussion and open problems in Section 4.9.

4.2 Blocks of parameters and something else

The main purpose in this section is to introduce the notion of a *block of parameters* and to prove the inequalities in Figure 4 (Lemma 4.2.5). They are consequence of connections Yorioka ideals, localization and anti-localization cardinals that we present in this section.

The Tukey order between the relational systems in Example 1.3.11 and $\mathbf{Cv}(\mathcal{I}_f)$ are key to our definition of blocks.

Lemma 4.2.1 ([KM21, Lemma 2.4])**.** *Let $a, d \in \omega^\omega$ and let $\langle I_n : n < \omega\rangle$ be an interval partition of ω such that $|I_n| = d(n)$. Define $g_{a,d} \in \omega^\omega$ by $g_{a,d}(k) := \lfloor \log_2 a(n) \rfloor$ for each $k \in I_n$. If $a \geq^* 2$, $d \geq^* 1$, f is an increasing function and $g_{a,d} \gg f$, then $\mathbf{Cv}(\mathcal{I}_f) \preceq_{\mathrm{T}} \mathrm{aLc}(a,d)^\perp$. In particular, $\mathrm{cov}(\mathcal{I}_f) \leq \mathfrak{b}^{\mathrm{aLc}}_{a,d}$ and $\mathfrak{d}^{\mathrm{aLc}}_{a,d} \leq \mathrm{non}(\mathcal{I}_f)$.*

The following is a variation of [KM21, Lemma 2.5] where we use a relational system for localization instead of anti-localization.

Lemma 4.2.2. *Let $b, g, h \in \omega^\omega$ and let $\langle J_n : n < \omega\rangle$ be an interval partition of ω such that $|J_n| = g(n)$. Define $f_{b,g,h} \in \omega^\omega$ by $f_{b,g,h}(k) := \sum_{\ell \leq n} h(\ell)\lceil \log_2 b(\ell)\rceil$ for each $k \in J_n$. If $b \geq^* 2$, $g \geq^* 1$, $h \geq^* 1$, $f \in \omega^\omega$ is an increasing function and there is some $1 \leq m \leq \omega$ such that $f_{b,g,h}(k) \leq f(k^m)$ for all but finitely many $k < \omega$, and $g(n)h(n) < b(n)$ for all but finitely many $n < \omega$, then $\mathbf{Lc}(b,h)^\perp \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I}_f)$. In particular, $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{cov}(\mathcal{I}_f)$ and $\mathrm{non}(\mathcal{I}_f) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$.*

Proof. It is enough to construct two functions $F\colon \mathcal{S}(b,h) \to 2^\omega$ and $G\colon \mathcal{I}_f \to \prod b$ such that, for all $S \in \mathcal{S}(b,h)$ and $X \in \mathcal{I}_f$, $F(S) \in X$ implies $G(X) \notin^* S$.

For each $n < \omega$, fix a one-to-one function $\iota_n\colon b(n) \to 2^{\lceil \log_2 b(n)\rceil}$.

Let $S \in \mathcal{S}(b,h)$ and choose some $S' \in \mathcal{S}(b,h)$ such that $S(n) \subseteq S'(n) \neq \varnothing$ whenever $h(n) \neq 0$. For each n, let $S'(n) = \{m_{n,j} : j < h(n)\}$, and define $F(S)$ as the concatenation of $\{\iota_n(m_{i,j}) : i < \omega,\ j < h(n)\}$, where the indices (i,j) are ordered lexicographically.

To define G, let $X \in \mathcal{I}_f$, so choose $\sigma_X \in (2^{<\omega})^\omega$ such that $\mathrm{ht}_{\sigma_X} \gg f$ and $X \subseteq [\sigma_X]_\infty$. The hypothesis of the lemma implies that $\mathrm{ht}_{\sigma_X} \geq^* f_{b,g,h}$. Consider the interval partition $\langle I_{n,\ell} : n < \omega,\ \ell < h(\ell)\rangle$ of ω, ordered lexicographically, such that $|I_{n,\ell}| = \lceil \log_2 b(n)\rceil$. Then $I_{n,\ell} \subseteq \mathrm{ht}_{\sigma_X}(k)$ for all $k \in J_n$ and all but finitely many n (because $f_{b,g,h}(k) = \sum_{\ell \leq n} h(l)\lceil \log_2 b(\ell)\rceil \leq \mathrm{ht}_{\sigma_X}(k)$), so we can define

$$H(n) := \{{\iota_n}^{-1}(\sigma_X(k){\restriction}I_{n,\ell}) : \ell < h(n),\ k \in J_n,\ I_{n,\ell} \subseteq \mathrm{ht}_{\sigma_X}(k),\ \sigma_X(k){\restriction}I_{n,\ell} \in \mathrm{ran}\iota_n\}.$$

Clearly $H(n) \subseteq b(n)$ and, eventually, $|H(n)| \leq g(n)h(n) < b(n)$, so we can choose some $G(X)(n) \in a(n) \smallsetminus H(n)$. For the other finitely many n, choose any $G(X)(n) \in a(n)$.

Now assume $S \in \mathcal{S}(b,h)$, $X \in \mathcal{I}_f$ and $F(S) \in X$; we will show that $G(X)(n) \notin S(n)$ for infinitely many n. Since $F(S) \in X$, there are infinitely many n such that $\sigma_X(k_n) \subseteq F(S)$ for some $k_n \in J_n$. For such a sufficiently large n, $\iota_n(m_{n,j}) = \sigma_X(k_n){\restriction}I_{n,j}$ for all $j < h(n)$, so $m_{n,j} \in H(n)$, which implies $G(X)(n) \neq m_{n,j}$. Thus $G(X)(n) \notin S(n)$. □

We also consider the following easy fact.

Lemma 4.2.3. *Let $a, b, d, h \in \omega^\omega$. Then the following statements hold:*

(a) $\mathrm{aLc}(a,d)^\perp \preceq_{\mathrm{T}} \mathbf{Lc}(a,d)$*; in particular,* $\mathfrak{b}^{\mathrm{aLc}}_{a,d} \leq \mathfrak{d}^{\mathrm{Lc}}_{a,d}$ *and* $\mathfrak{b}^{\mathrm{Lc}}_{a,d} \leq \mathfrak{d}^{\mathrm{aLc}}_{a,d}$.

(b) If $b \leq^* a$ and $d \leq^* h$, then $\mathbf{Lc}(b,h) \preceq_{\mathrm{T}} \mathbf{Lc}(a,d)$; in particular, $\mathfrak{b}^{\mathrm{Lc}}_{a,d} \leq \mathfrak{b}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} \leq \mathfrak{d}^{\mathrm{Lc}}_{a,d}$.

Using the hypotheses of Lemma 4.2.1–4.2.3, we are finally ready to define the notion of a block of parameters. Properties (i)–(v) are chosen this way precisely to get the promised Figure 4 as a consequence of the following lemmata.

Definition 4.2.4. A sequence of increasing functions $\vec{\nu} = \langle d, h, g, b, f, a\rangle$ in ω^ω is a *block (of parameters)* if it fulfills, for all $n < \omega$,

(i) if $k \in I_n$, then $k^n \in J_n$, where $\langle I_n : n < \omega\rangle$ and $\langle J_n : n < \omega\rangle$ are interval partitions of ω such that $|I_n| = d(n)$ and $|J_n| = g(n)$;

(ii) $g(n)h(n) < b(n)$;

(iii) if $k \in J_n$, then $f(k) \geq \sum_{j\leq n} h(j)\lceil\log_2 b(j)\rceil$;

(iv) if $k \in I_n$, then $f(k^n) \leq \lfloor\log_2 a(n)\rfloor$; and

(v) $b(n) \leq a(n)$ and $d(n) \leq h(n)$.

Lemma 4.2.5. *If $\vec{\nu} = \langle d, h, g, b, f, a\rangle$ is a block, then $\mathfrak{b}^{\mathrm{Lc}}_{a,d} \leq \mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{cov}(\mathcal{I}_f) \leq \mathfrak{b}^{\mathrm{aLc}}_{a,d} \leq \mathfrak{d}^{\mathrm{Lc}}_{a,d}$ and $\mathfrak{b}^{\mathrm{Lc}}_{a,d} \leq \mathfrak{d}^{\mathrm{aLc}}_{a,d} \leq \mathrm{non}(\mathcal{I}_f) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h} \leq \mathfrak{d}^{\mathrm{Lc}}_{a,d}$ (cf. Figure 4).*

Proof. It is enough to show that $\mathbf{Lc}(a,d)^\perp \preceq_{\mathrm{T}} \mathrm{aLc}(a,d) \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I}_f)^\perp \preceq_{\mathrm{T}} \mathbf{Lc}(b,h) \preceq_{\mathrm{T}} \mathbf{Lc}(a,d)$. By Definition 4.2.4 (v) and Lemma 4.2.3, we have that $\mathbf{Lc}(a,d)^\perp \preceq_{\mathrm{T}} \mathrm{aLc}(a,d)$ and $\mathbf{Lc}(b,h) \preceq_{\mathrm{T}} \mathbf{Lc}(a,d)$ and, by Definition 4.2.4 (ii)–(iii) and Lemma 4.2.2, $\mathbf{Cv}(\mathcal{I}_f)^\perp \preceq_{\mathrm{T}} \mathbf{Lc}(b,h)$ follows. To show $\mathrm{aLc}(a,d) \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I}_f)^\perp$ by application of Lemma 4.2.1, it remains to prove that $g_{a,d} \gg f$. Fix $m < \omega$; for $n > m$, if $k \in I_n$, then $k^n \in J_n$, so $f(k^m) \leq f(k^n) \leq \lfloor\log_2 a(n)\rfloor = g_{a,d}(n)$. □

As mentioned in the introduction, we plan to use $\limsup$ forcing to increase the $\mathfrak{d}$ cardinals of (the relational systems determined by) a block, and $\liminf$ forcing for the $\mathfrak{b}$ cardinals (specifically, to increase $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$). For the latter, we consider a variation of the relational system $\mathbf{Lc}(b,h)$.

Definition 4.2.6. Let $\bar{I} := \langle I_n : n < \omega\rangle$ be a sequence of pairwise disjoint non-empty finite sets and let $D := \bigcup_{n<\omega} I_n$.

(1) For two functions x and φ with domain D, we write

$$x \in^*_{\bar{I}} \varphi \text{ iff } \forall^\infty n\, \exists \ell \in I_n (x(\ell) \in \varphi(\ell)).$$

(2) Let $b = \langle b(\ell) : \ell \in D\rangle$ be a sequence of non-empty sets and let $h\colon D \to \omega$. Let $\mathbf{Lc}_{\bar{I}}(b,h) := \langle \prod b, \mathcal{S}(b,h), \in^*_{\bar{I}}\rangle$ be a relational system, where we expand our notation $\prod b$ and $\mathcal{S}(b,h)$ to

$$\prod b := \prod_{\ell\in D} b(\ell), \qquad \mathcal{S}(b,h) := \prod_{\ell\in D} [b(\ell)]^{\leq h(\ell)}.$$

(3) We define $\mathfrak{b}^{\mathrm{Lc},\bar{I}}_{b,h} := \mathfrak{b}(\mathbf{Lc}_{\bar{I}}(b,h))$ and $\mathfrak{d}^{\mathrm{Lc},\bar{I}}_{b,h} := \mathfrak{d}(\mathbf{Lc}_{\bar{I}}(b,h))$.

Lemma 4.2.7. *With the notation from the previous definition, let $b^* := \langle b^*(n) : n < \omega\rangle$, $b^*(n) := \prod_{\ell\in I_n} b(\ell)$, and define $h^* \in \omega^\omega$ by $h^*(n) := b^*(n) - \prod_{\ell\in I_n}(b(\ell) - h(\ell))$. Then $\mathbf{Lc}(b^*, h^*) \preceq_{\mathrm{T}} \mathbf{Lc}_{\bar{I}}(b,h)$. In particular, $\mathfrak{b}^{\mathrm{Lc},\bar{I}}_{b,h} \leq \mathfrak{b}^{\mathrm{Lc}}_{b^*,h^*}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b^*,h^*} \leq \mathfrak{d}^{\mathrm{Lc},\bar{I}}_{b,h}$.*

Proof. Define $F\colon \prod b^* \to \prod b$ by $F(x) := \langle x(n)(\ell) : \ell \in I_n,\ n < \omega\rangle$, and $G\colon \mathcal{S}(b,h) \to \mathcal{S}(b^*,h^*)$ by $G(\varphi) := \varphi^*$ where $\varphi^*(n) = b^*(n) \smallsetminus \prod_{\ell\in I_n}(b(\ell) \smallsetminus \varphi(\ell))$ (which clearly has size $\leq h^*(n)$). For $x \in \prod b^*$ and $\varphi \in \mathcal{S}(b,h)$, $x(n) \in \varphi^*(n)$ iff $\exists \ell \in I_n\colon x(n)(\ell) \in \varphi(\ell)$ for all $n < \omega$. Therefore, $F(x) \in^*_{\bar{I}} \varphi$ iff $x \in^* \varphi^*$. □

In our forcing construction, we aim to increase cardinals of the form $\mathfrak{b}^{\mathrm{Lc},\bar{I}}_{b,h}$ using $\liminf$ forcing.

Observation 4.2.8. The slaloms we add in our forcing construction are not quite in $\mathcal{S}(b,h)$, but they are modulo finite modifications. For $b = \langle b(\ell) : \ell \in D\rangle$ and $h\colon D \to \omega$, define

$$\mathcal{S}^*(b,h) := \{\varphi : D \to [\omega]^{<\aleph_0} : \{\ell \in D : \varphi(\ell) \nsubseteq b(\ell) \text{ or } |\varphi(\ell)| > h(\ell)\} \text{ is finite}\}.$$

Our forcing will add slaloms in $\mathcal{S}^*(b,h)$. If we replace $\mathcal{S}(b,h)$ with $\mathcal{S}^*(b,h)$ in the relational systems $\mathbf{Lc}(b,h)$ and $\mathrm{aLc}(b,h)$, then we obtain Tukey equivalent relational systems (and the same localization and anti-localization cardinals); likewise for the relational system $\mathbf{Lc}_{\bar{I}}(b,h)$.

4.3 The Frame and Parameters of the Forcing

In this section we present the frame in which the forcing lives and the parameters we need for the forcing construction.

4.3.1 The frame

Our forcing construction has a support S^*; for each $\alpha \in S^*$, we add blocks or slaloms that increase cardinals of the form $\mathfrak{b}^{\mathrm{Lc},\bar{I}}_{b,h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{a,h}$. Concretely, we fix:

- a set of indices S^{pr} (where pr stands for "parameter");
- pairwise disjoint families $\langle S^{\mathrm{lc}}_i : i \in S^{\mathrm{pr}}\rangle$ and $\langle S^{\mathrm{al}}_i : i \in S^{\mathrm{pr}}\rangle$ (where lc stands for "localization", and al for "anti-localization") such that $S^{\mathrm{lc}}_i \cap S^{\mathrm{pr}} = S^{\mathrm{al}}_i \cap S^{\mathrm{pr}} = S^{\mathrm{lc}}_i \cap S^{\mathrm{al}}_j = \varnothing$ for $i, j \in S^{\mathrm{pr}}$; and
- $S^* := \bigcup_{\mathrm{t}\in\{\mathrm{pr},\mathrm{lc},\mathrm{al}\}} S^{\mathrm{t}}$, where $S^{\mathrm{t}} := \bigcup_{i\in S^{\mathrm{pr}}} S^{\mathrm{t}}_i$ for $\mathrm{t} \in \{\mathrm{lc},\mathrm{al}\}$.

For each $i \in S^{\mathrm{pr}}$, we intend to force several blocks $\bar{\nu}^{\mathrm{lc}}_i = \langle d^{\mathrm{lc}}_i, h^{\mathrm{lc}}_i, g^{\mathrm{lc}}_i, b^{\mathrm{lc}}_i, f^{\mathrm{lc}}_i, a^{\mathrm{lc}}_i\rangle$ and $\bar{\nu}^{\mathrm{al}}_i = \langle h^{\mathrm{al}}_i, h^{\mathrm{al}}_i, g^{\mathrm{al}}_i, b^{\mathrm{al}}_i, f^{\mathrm{al}}_i, a^{\mathrm{al}}_i\rangle$; in fact, we add an ω-sequence $y_i = \langle y_i(n) : n < \omega\rangle$ that defines both blocks. Definition 4.3.2 and Lemma 4.3.3 give details on how blocks are defined from reals.

For each $i \in S^{\mathrm{lc}}$ and each $\alpha \in S^{\mathrm{lc}}_i$, we add a slalom $\varphi_\alpha \in \mathcal{S}^*(b^{\mathrm{lc}}_i, h^{\mathrm{lc}}_i)$ that localizes all reals in $\prod b^{\mathrm{lc}}_i$ "not depending on α" (so $\mathfrak{b}^{\mathrm{Lc}}_{b^{\mathrm{lc}}_i,h^{\mathrm{lc}}_i}$ is increased); and for each $\alpha \in S^{\mathrm{al}}_i$ we add a slalom $\varphi_\alpha \in \mathcal{S}^*(a^{\mathrm{al}}_i, h^{\mathrm{al}}_i)$ that anti-localizes all reals in $\prod a^{\mathrm{al}}_i$ "not depending on α" (so $\mathfrak{d}^{\mathrm{aLc}}_{a^{\mathrm{al}}_i,h^{\mathrm{al}}_i}$ is increased). Details are presented in Lemma 4.8.2 and Lemma 4.8.4. Here, "not depending on α" means that continuous reading (which is discussed in detail in Section 4.7) takes place without using the index α.

For the moment, we are not going to add additional assumptions, but when it comes to actually prove Theorem Q, we will assume CH, $|S^*| = \mu = \mu^\alpha$ and $|S^{\mathrm{lc}}_i| = |S^{\mathrm{al}}_i| = \kappa_i = \kappa_i^{\aleph_0}$ for all $i \in S^{\mathrm{pr}}$. With these conditions, the collection of slaloms $\{\varphi_\alpha : \alpha \in S^{\mathrm{lc}}_i\}$ is used to force $\kappa_i \leq \mathfrak{b}^{\mathrm{Lc}}_{b^{\mathrm{lc}}_i,h^{\mathrm{lc}}_i}$ and the collection of slaloms $\{\varphi_\alpha : \alpha \in S^{\mathrm{al}}_i\}$ is used to force $\kappa_i \leq \mathfrak{b}^{\mathrm{Lc}}_{a^{\mathrm{al}}_i,h^{\mathrm{al}}_i}$.

In any case, we are adding an ω-sequence for each $\alpha \in S^*$. As in [GK21], we are going to use different levels for each type $\mathrm{t} \in \{\mathrm{pr},\mathrm{lc},\mathrm{al}\}$ of generic we add to make the construction more intuitive and less complex. For S^{al}, we use a $\limsup$ creature forcing construction, while for $\mathrm{t} \in \{\mathrm{pr},\mathrm{lc}\}$ we use $\liminf$ creature forcing. For this reason, the levels corresponding to pr and lc are divided into sublevels.

Definition 4.3.1. The *levels* (or *heights)* of the forcing construction are defined as follows.

(1) **pr-levels:** A *pr-level* is a natural number n, i.e. the set of pr-levels is ω. For each $n < \omega$, we will determine some $0 < \iota^{\mathrm{pr}}_n < \omega$. Each pr-level n is divided into **pr-sublevels** $\mathrm{ht}^{\mathrm{pr}}_n := \{(n,u) : u < \iota^{\mathrm{pr}}_n\}$. We refer to the set of pr-sublevels by $\mathrm{ht}^{\mathrm{pr}}$, i.e. $\mathrm{ht}^{\mathrm{pr}} := \bigcup_{n<\omega} \mathrm{ht}^{\mathrm{pr}}_n = \{(n,u) : n < \omega,\ u < \iota^{\mathrm{pr}}_n\}$. We often identify $(n,0)$ with n.

(2) **lc-levels:** An *lc-level* is $L^{\mathrm{lc}}_n := (n, \iota^{\mathrm{pr}}_n)$ for some $n < \omega$. Each lc-level is divided into **lc-sublevels** $\mathrm{ht}^{\mathrm{lc}}_n := I^*_n = \{(L^{\mathrm{lc}}_n, v) : v < \iota^*_n\}$ for some natural number ι^*_n. We refer to the set of lc-sublevels by $\mathrm{ht}^{\mathrm{lc}} := \bigcup_{n<\omega} I^*_n = \{(L^{\mathrm{lc}}_n, u) : n < \omega,\ v < \iota^*_n\}$. We often identify L^{lc}_n with $(L^{\mathrm{lc}}_n, 0)$.

(3) **al-levels:** An *al-level* is a pair $L^{\mathrm{al}}_n := (L^{\mathrm{lc}}_n, \iota^*_n)$ for $n < \omega$. We refer to the set of al-levels by $\mathrm{ht}^{\mathrm{al}} := \{L^{\mathrm{al}}_n : n < \omega\}$.

Define the *set of heights* by $\mathrm{ht} := \mathrm{ht}^{\mathrm{pr}} \cup \mathrm{ht}^{\mathrm{lc}} \cup \mathrm{ht}^{\mathrm{al}}$, ordered lexicographically, i. e.:

$$\begin{aligned} \ldots &< (n,0) < (n,1) < (n,2) < \ldots < (n, \iota^{\mathrm{pr}}_n - 1) < (L^{\mathrm{lc}}_n, 0) < \ldots < (L^{\mathrm{lc}}_n, \iota^*_n - 1) < L^{\mathrm{al}}_n \\ &< (n+1, 0) < (n+1, 1) < \ldots < (n+1, \iota^{\mathrm{pr}}_{n+1} - 1) < (L^{\mathrm{lc}}_{n+1}, 0) < \ldots \end{aligned}$$

For $n < \omega$, let $\mathrm{ht}_n := \mathrm{ht}^{\mathrm{pr}}_n \cup I^*_n \cup \{L^{\mathrm{al}}_n\}$. Also let

$$\mathrm{Ht} := \{(n,0) : n < \omega\} \cup \{(L^{\mathrm{lc}}_n, 0) : n < \omega\} \cup \{L^{\mathrm{al}}_n\} \text{ and } \Omega := \mathrm{Ht} \smallsetminus \mathrm{ht}^{\mathrm{pr}}.$$

Hence, our forcing poset will "live" on the set

$$\mathrm{DOM} := (S^{\mathrm{pr}} \times \mathrm{ht}^{\mathrm{pr}}) \cup (S^{\mathrm{lc}} \times \mathrm{ht}^{\mathrm{lc}}) \cup (S^{\mathrm{al}} \times \mathrm{ht}^{\mathrm{al}})$$

and we will add a generic real $y_\alpha \colon \mathrm{ht}^{\mathrm{t}} \to \omega$ for each $\mathrm{t} \in \{\mathrm{pr}, \mathrm{lc}, \mathrm{al}\}$ and each $\alpha \in S^{\mathrm{t}}$. In the following subsection, we will explain more about the nature and purpose of these generic reals.

We also fix the sequence $\bar{I}^* := \langle I^*_n : n < \omega \rangle$, which will be relevant to deal with the $\liminf$ construction increasing localization cardinals.

The numbers ι^{pr} and ι^* will be determined in Subsection 4.3.3 along with other many parameters we need to set for the forcing construction.

4.3.2 Parameters to build blocks

Our aim in this subsection is to define parameters following certain requirements to build the generic blocks added by the forcing. Along with the many parameters determined in Subsection 4.3.3, we will fix a sequence of natural numbers $\langle T^*_L : L \in \mathrm{ht}^{\mathrm{pr}} \rangle$ and let $T^*_n := \prod_{L \in \mathrm{ht}^{\mathrm{pr}}_n} T^*_L$ for all $n < \omega$. For every $i \in S^{\mathrm{pr}}$, we add a generic real $y_i \in \prod_{L \in \mathrm{ht}^{\mathrm{pr}}} T^*_L$, which gives us the real $y^*_i \in \prod_{n<\omega} T^*_n$ defined by $y^*_i(n) := \langle y_i(L) : L \in \mathrm{ht}^{\mathrm{pr}}_n \rangle$.[1]

We aim to define blocks for every real $y \in \prod_{n<\omega} T^*_n$. For each $t \in T^*_n$, we will later define functions

$$\begin{aligned} d_t &\colon I^*_n \cup \{L^{\mathrm{al}}_n\} \to \omega, \\ h_t &\colon I^*_n \cup \{L^{\mathrm{al}}_n\} \to \omega, \\ g_t &\colon \{L^{\mathrm{lc}}_n, L^{\mathrm{al}}_n\} \to \omega, \\ b_t &\colon I^*_n \cup \{L^{\mathrm{al}}_n\} \to \omega, \\ a_t &\colon \{L^{\mathrm{lc}}_n, L^{\mathrm{al}}_n\} \to \omega. \end{aligned} \tag{4.3.1}$$

Given these functions, we will define blocks as follows:

Definition 4.3.2. For $y \in \prod_{n<\omega} T^*_n$ and $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$, we make the followin definitions:

[1]The reason why T^*_n is decomposed as $\langle T^*_L : L \in \mathrm{ht}^{\mathrm{pr}}_L \rangle$ is due to the use of sublevels $\mathrm{ht}^{\mathrm{pr}}_L$ in the $\liminf$ construction of the forcing.

(1) $d_y\colon \mathrm{ht} \setminus \mathrm{ht}^{\mathrm{pr}} \to \omega$ is defined as $d_y := \bigcup_{n<\omega} d_{y(n)}$; the functions $h_y, b_y\colon \mathrm{ht} \setminus \mathrm{ht}^{\mathrm{pr}} \to \omega$ are defined analogously.

(2) $g_y\colon \Omega \to \omega$ is defined as $g_y := \bigcup_{n<\omega} g_{y(n)}$, and $a_y\colon \Omega \to \omega$ is defined analogously.

(3) $d_y^{\mathrm{t}}\colon \omega \to \omega$ is defined by $d_y^{\mathrm{t}}(n) := d_y(L_n^{\mathrm{t}})$, and define $g_y^{\mathrm{t}}, a_y^{\mathrm{t}} \in \omega^\omega$ analogously; in the case of $\mathrm{t} = \mathrm{al}$, h^{al} and b^{al} are also defined analogously.

(4) $b_y^{\mathrm{lc}}, h_y^{\mathrm{lc}}\colon \omega \to \omega$ are defined by

$$b_y^{\mathrm{lc}}(n) := \prod_{\ell \in I_n^*} b_y(\ell) \text{ and } h_y^{\mathrm{lc}}(n) := b_y^{\mathrm{lc}}(n) - \prod_{\ell \in I_n^*} (b_y(\ell) - h_y(\ell)) \text{ (cf. Lemma 4.2.7).}$$

(5) $\langle J_{y,n}^{\mathrm{t}} : n < \omega \rangle$ is the interval partition of ω such that $|J_{y,n}^{\mathrm{t}}| = g_y^{\mathrm{t}}(n)$.

(6) $f_y^{\mathrm{t}} \in \omega^\omega$ is defined by

$$f_y^{\mathrm{t}}(k) := \sum_{j \le n} h_y^{\mathrm{t}}(j) \lceil \log_2 b_y^{\mathrm{t}}(j) \rceil + k - \min J_{y,n}^{\mathrm{t}} \text{ for } k \in J_{y,n}^{\mathrm{t}} \text{ (cf. Lemma 4.2.2).}$$

(7) $\vec{\nu}_y^{\mathrm{lc}} := \langle d_y^{\mathrm{lc}}, h_y^{\mathrm{lc}}, g_y^{\mathrm{lc}}, b_y^{\mathrm{lc}}, f_y^{\mathrm{lc}}, a_y^{\mathrm{lc}} \rangle$ and $\vec{\nu}_y^{\mathrm{al}} := \langle h_y^{\mathrm{al}}, h_y^{\mathrm{al}}, g_y^{\mathrm{al}}, b_y^{\mathrm{al}}, f_y^{\mathrm{lc}}, a_y^{\mathrm{al}} \rangle$.

(8) Moreover, define $b_y^- := b_y {\restriction} \mathrm{ht}^{\mathrm{lc}}$ and $h_y^- := h_y {\restriction} \mathrm{ht}^{\mathrm{lc}}$.

In order to ensure that $\vec{\nu}_y^{\mathrm{lc}}$ and $\vec{\nu}_y^{\mathrm{al}}$ are blocks for each $y \in \prod_{n<\omega} T_n^*$, we will define the functions in Equation 4.3.1 as well as new functions $b_t^*\colon \{L_n^{\mathrm{lc}}, L_n^{\mathrm{al}}\} \to \omega$ and $h_t^*\colon \{L_n^{\mathrm{lc}}, L_n^{\mathrm{al}}\} \to \omega$ satisfying the requirements below for each $n < \omega$ and $t \in T_n^*$; we will refer to the blocks added for $i \in S^{\mathrm{pr}}$ by $\vec{\nu}_i^{\mathrm{lc}} := \vec{\nu}_{y_i}^{\mathrm{lc}}$ and $\vec{\nu}_i^{\mathrm{al}} := \vec{\nu}_{y_i}^{\mathrm{al}}$.

(fp1) $d_t(L_n^{\mathrm{t}}) < h_t(L_n^{\mathrm{t}})$ for $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$;

(fp2) $g_t(L_n^{\mathrm{lc}}) \ge \left(\sum\{d_t(L) : L \in \Omega,\ L \le L_n^{\mathrm{lc}}\}\right)^{\max\{n,1\}}$;

(fp3) $g_t(L_n^{\mathrm{al}}) \ge \left(\sum\{h_t(L) : L \in \Omega,\ L \le L_n^{\mathrm{al}}\}\right)^{\max\{n,1\}}$;

(fp4) $1 - \frac{1}{g_t(L_n^{\mathrm{lc}})} < \prod_{\ell \in I_n^*} \left(1 - \frac{h_t(\ell)}{b_t(\ell)}\right)$;

(fp5) $h_t(\ell) < b_t(\ell)$ for each $\ell \in I_n^*$;

(fp6) $b_t^*(L_n^{\mathrm{lc}}) := \prod_{\ell \in I_n^*} b_t(\ell)$ and $h_t^*(L_n^{\mathrm{lc}}) := b_t^*(L_n^{\mathrm{lc}}) - \prod_{\ell \in I_n^*} (b_t(\ell) - h_t(\ell))$;

(fp7) $b_t^*(L_n^{\mathrm{al}}) := b_t(L_n^{\mathrm{al}})$ and $h_t^*(L_n^{\mathrm{al}}) := h_t(L_n^{\mathrm{al}})$;

(fp8) $g_t(L_n^{\mathrm{al}}) h_t^*(L_n^{\mathrm{al}}) < b_t^*(L_n^{\mathrm{al}})$;

(fp9) $\log_2 a_t(L_n^{\mathrm{t}}) \ge \sum\{h_t^*(L) \lceil \log_2 b_t^*(L) \rceil : L \in \Omega,\ L \le L_n^{\mathrm{t}}\} + g_t(L_n^{\mathrm{t}})$ for $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$;

(fp10) $2 < d_t(L_n^{\mathrm{lc}})$, $2^{a_t(L_n^{\mathrm{lc}})} < d_t(L_n^{\mathrm{al}})$ and $2^{a_t(L_n^{\mathrm{al}})} < d_{t'}(L_{n+1}^{\mathrm{lc}})$ for any $t' \in T_n^*$.

Lemma 4.3.3. *The requirements (fp1)–(fp10) imply that, for each $y \in \prod_{n<\omega} T_n^*$, the sequences $\vec{\nu}_y^{\mathrm{lc}}$ and $\vec{\nu}_y^{\mathrm{al}}$ are blocks. Moreover,* $\mathfrak{b}^{\mathrm{Lc},\bar{I}^*}_{b_y^-, h_y^-} \le \mathfrak{b}^{\mathrm{Lc}}_{b_y^{\mathrm{lc}}, h_y^{\mathrm{lc}}}$.

Proof. For $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$, let $\langle I^{\mathrm{t}}_{y,n} : n < \omega\rangle$ be the interval partition of ω such that $|I^{\mathrm{lc}}_{y,n}| = d^{\mathrm{lc}}_y(n)$ and $|I^{\mathrm{al}}_{y,n}| = h^{\mathrm{lc}}_y(n)$. We check that the requirements (i)–(v) of Definition 4.2.4 hold and that all functions are increasing. Property (i) follows by (fp2) and (fp3). Property (ii) follows by (fp4) and (fp8), noting that (fp4) implies the inequality in (fp8) when L^{al}_n is replaced by L^{lc}_n. Property (iii) is obvious by the definition of f^{t}_y, cf. Definition 4.3.2 (5)–(6). To check property (iv), note that $k \in I^{\mathrm{t}}_{y,n}$ implies $k^n \in J^{\mathrm{t}}_{y,n}$ by (i), so

$$\begin{aligned} f^{\mathrm{t}}_y(k^n) &= \sum_{j \le n} h^{\mathrm{t}}_y(j)\lceil \log_2 b^{\mathrm{t}}_y(j)\rceil + k^n - \min J^{\mathrm{t}}_{y,n} \\ &\le \sum_{j\le n} h^{\mathrm{t}}_y(j)\lceil \log_2 b^{\mathrm{t}}_y(j)\rceil + g^{\mathrm{t}}_y(n) \le \lfloor \log_2 a^{\mathrm{t}}_y(n)\rfloor, \end{aligned}$$

where the last inequality follows from (fp9). To check property (v), it is clear from (fp9) that $a^{\mathrm{t}}_y(n) \ge b^{\mathrm{t}}_y(n)$; by (fp1) it is obvious that $d^{\mathrm{al}}_y(n) < h^{\mathrm{al}}_y(n)$, and $d^{\mathrm{lc}}_y(n) < h^{\mathrm{lc}}_y(n)$ follows by (using (pr5))

$$\prod_{\ell \in I^*_n}\left(1 - \frac{h_y(\ell)}{b_y(\ell)}\right) \le 1 - \frac{h_y(L^{\mathrm{lc}}_n)}{b_y(L^{\mathrm{lc}}_n)} < 1 - \frac{d_y(L^{\mathrm{lc}}_n)}{b^{\mathrm{lc}}_n(n)} = 1 - \frac{d^{\mathrm{lc}}_y(n)}{b^{\mathrm{lc}}_n(n)}.$$

With the exception of f^{t}_y, it is clear that all functions are increasing (also using (fp10)). To check that f^{t}_y is increasing, as well, it suffices to check that $f^{\mathrm{t}}_y(k) < f^{\mathrm{t}}_y(k+1)$ when $k = \max J^{\mathrm{t}}_{y,n}$. Indeed,

$$f^{\mathrm{t}}_y(k) = \sum_{j\le n} h^{\mathrm{t}}_y(j)\lceil \log_2 b^{\mathrm{t}}_y(j)\rceil + g^{\mathrm{t}}_y(n) - 1 < \sum_{j \le n+1} h^{\mathrm{t}}_y(j)\lceil \log_2 b^{\mathrm{t}}_y(j)\rceil = f^{\mathrm{t}}_t(k+1).$$

(For the inequality, we use $2^{g^{\mathrm{t}}_y(n)} < b^{\mathrm{t}}_y(n+1)$, which follows from (fp10).)

Finally, from (fp6) and Lemma 4.2.7 we obtain $\mathbf{Lc}(b^{\mathrm{lc}}_y, h^{\mathrm{lc}}_y) \preceq_{\mathrm{T}} \mathbf{Lc}_{\bar{I}^*}(b^-_y, h^-_y)$, so $\mathfrak{b}^{\mathrm{Lc},\bar{I}^*}_{b^-_y,h^-_y} \le \mathfrak{b}^{\mathrm{Lc}}_{b^{\mathrm{lc}}_y,h^{\mathrm{lc}}_y}$ follows. □

Recall that, for each $\mathrm{t} \in \{\mathrm{pr}, \mathrm{lc}, \mathrm{al}\}$, we add a generic real $y_\alpha\colon \mathrm{ht}^{\mathrm{t}} \to \omega$. Fix $i \in S^{\mathrm{pr}}$; we have already explained the nature of y_i. For $\alpha \in S^{\mathrm{lc}}_i$, $y_\alpha \in \mathcal{S}^*(b^-_{y^*_i}, h^-_{y^*_i})$ will be a generic slalom such that $x \in^*_{\bar{I}^*} y_\alpha$ for any real $x \in \prod b^-_{y^*_i}$ in the generic extension that does not depend on α, so $\mathfrak{b}^{\mathrm{Lc},\bar{I}^*}_{b^-_{y^*_i},h^-_{y^*_i}}$ is increased, and so is $\mathfrak{b}^{\mathrm{Lc}}_{b^{\mathrm{lc}}_{y^*_i},h^{\mathrm{lc}}_{y^*_i}}$ by the previous lemma. In more detail, as in Lemma 4.2.7 and its proof, setting $\varphi_\alpha(n) := b^{\mathrm{lc}}_{y^*_i}(n) \smallsetminus \prod_{\ell \in I^*_n}(b_{y^*_i}(\ell) \smallsetminus y_\alpha(\ell))$, we get that $\varphi_\alpha \in \mathcal{S}^*(b^{\mathrm{lc}}_{y^*_i}, h^{\mathrm{lc}}_{y^*_i})$ localizes all reals in $\prod b^{\mathrm{lc}}_{y^*_i}$ not depending on α.

For $\alpha \in S^{\mathrm{al}}_i$, we will have that $y_\alpha(L) \in [a_{y^*_i}(L)]^{\le h_{y^*_i}(L)}$ for all but finitely many $L \in \mathrm{ht}^{\mathrm{al}}$. Setting $\varphi_\alpha(n) := y_\alpha(L^{\mathrm{al}}_n)$, we will get that $\varphi_\alpha \in \mathcal{S}^*(a^{\mathrm{al}}_{y^*_i}, h^{\mathrm{al}}_{y^*_i})$ and that $x \in^\infty \varphi_\alpha$ for any $x \in \prod a^{\mathrm{al}}_{y^*_i}$ not depending on α.

(A formal description of these generic reals follows in Definition 4.5.23 and Lemma 4.5.25.)

4.3.3 More parameters for our forcing construction

In addition to the functions in Equation 4.3.1 and the numbers ι^{pr}_n, ι^*_n and T^*_L (for all $n < \omega$ and $L \in \mathrm{ht}^{\mathrm{pr}}$), our forcing construction requires several further parameters to ensure properties of the forcing (like properness or continuous reading of names) and to force the desired values in Theorem Q.

In this subsection, we list all the parameters and the conditions they must fulfill; the reader may not clearly recognize the intuition and motivation at this point of the paper, however. Although we will briefly state the reason for each condition on the parameters, we suggest it would be more helpful for the reader to go through this subsection quite quickly at first, concentrate on the content that follows, and come back here any time the conditions on these parameters are actually used. This will likely facilitate easier comprehension of our construction's ideas.

Notation 4.3.4. We introduce the notation of all the parameters we will use for the forcing construction. They all are integers larger than 2.

(I) $n^P_{<L}$ and n^B_L for $L \in \mathrm{ht}^{\mathrm{pr}} \cup \Omega$; P stands for "possibilities" and B for "bigness". ($n^P_{<L}$ will be an upper bound for the number of possibilities below L of a modest condition in the forcing, cf. Lemma 4.5.16; and n^B_L will be a lower bound for the bigness satisfied by a creature at level L, cf. Corollary 4.6.4, Lemma 4.6.2 and Lemma 4.6.5.)

(II) ι^{pr}_n and ι^*_n for $n < \omega$.

(III) T^*_L for $L \in \mathrm{ht}^{\mathrm{pr}}$. We also set $T^*_n := \prod_{L \in \mathrm{ht}^{\mathrm{pr}}} T^*_L$ with the lexicographic order $<_{\mathrm{lex}}$.

(IV) The functions in Equation 4.3.1.

(V) $n^S_{t,\ell}$ for each $t \in T^*_n$, $\ell \in I^*_n \cup \{L^{\mathrm{al}}_n\}$ and $n < \omega$. (This will be an upper bound for the number of possibilities of a subatomic creature at (t, ℓ), cf. Fact 4.4.6.)

(VI) n^S_L for all $L \in \mathrm{Ht}$. (This will be an upper bound for the number of possibilities of a creature at level L, cf. Fact 4.4.23.)

We further use the following terminology: Whenever $W = \langle W, \leq_W \rangle$ is a well-ordered set, we write (for $a \in W$)

(1) a^+ for the (immediate) successor of a in W, if it exists;

(2) a^- for the (immediate) predecessor of a in W, if it exists;

(3) $a{\downarrow} := \{x \in W : x <_W a\}$;

(4) $\mathrm{ot}(W)$ for the order type of W.

(Note that $\mathrm{ot}(\mathrm{ht}) = \omega$.)

In addition, we use:

(5) $t^{\mathrm{mx}}_n := \max_{<_{\mathrm{lex}}} T^*_n$, $t^{\mathrm{mn}}_n := \min_{<_{\mathrm{lex}}} T^*_n$,

(6) $\ell^{\mathrm{mx}}_n := \max I^*_n$.[2]

We also consider $T^*_n \times I^*_n$ with the lexicographic order, i.e. $(t, \ell) < (t', \ell')$ iff either $t <_{\mathrm{lex}} t'$ or $t = t'$ and $\ell < \ell'$.

We present two lists below. The first one indicates the order in which the parameters are defined and the second gives the conditions (pr1)–(pr14) on the parameters.

1. When we have $n^P_{<(n,0)}$, we define ι^{pr}_n as in (pr1).

2. For $L \in \mathrm{ht}^{\mathrm{pr}}_n$, given $n^P_{<L}$, we define larger $n^B_L < T^*_L < n^P_{<L^+}$ (in this order), see (pr2)–(pr4).

3. When we get to level $(L^{\mathrm{lc}}_n, 0)$, we define $n^S_{(n,0)}$ as in (pr5) and choose $n^P_{<L^{\mathrm{lc}}_n}$ as in (pr14) (which is stronger than (pr4) when L is the predecessor of L^{lc}_n in ht).

[2]There is no need to explicitly define $\ell^{\mathrm{mn}}_n := \min I^*_n$, since it clearly is $(L^{\mathrm{al}}_n, 0) = L^{\mathrm{al}}_n$.

4. Afterwards, we define ι_n^* as in (pr6) and define $n^B_{L_n^{\mathrm{lc}}}$ as in (pr2).

5. For each $t \in T_n^*$, we define the parameters $d_t(L_n^{\mathrm{al}}) < g_t(L_n^{\mathrm{al}}) < h_t(L_n^{\mathrm{al}}) < b_t(L_n^{\mathrm{al}}) < n^S_{t,L_n^{\mathrm{al}}}$. For $(t,\ell) > (t_n^{\mathrm{mn}}, L_n^{\mathrm{al}})$ in $T_n^* \times I_n^*$, we define $d_t(\ell) < h_t(\ell) < b_t(\ell) < n^S_{t,\ell}$, and get larger parameters for $(t',\ell') > (t,\ell)$ in $T_n^* \times I_n^*$. For $(t, \ell_n^{\mathrm{mx}})$, we additionally define $a_t(L_n^{\mathrm{al}})$. See (fp1)–(fp10) and (pr7)–(pr10).

6. Define $n^S_{L_n^{\mathrm{lc}}}$ as in (pr11).

7. At L_n^{al}, choose $n^P_{<L_n^{\mathrm{al}}}$ as in (pr14) and define $n^B_{L_n^{\mathrm{al}}} < d_t(L_n^{\mathrm{al}}) < h_t(L_n^{\mathrm{al}}) < g_t(L_n^{\mathrm{al}}) < b_t(L_n^{\mathrm{al}}) < a_t(L_n^s) < n^S_{t,L_n^s}$ for $t \in T_n^*$, while getting larger parameters for larger $t' \in T_n^*$. See (fp1)–(fp10), (pr2), (pr7)–(pr8) and (pr12).

8. Define $n^S_{L_n^{\mathrm{al}}}$ as in (pr13).

9. Define $n^P_{<(n+1,0)}$ as in (pr14), and repeat this process.

Conditions on the parameters

The following are the precise requirements of the parameters.

(pr1) $\iota_n^{\mathrm{pr}} \geq 3^{(n+1)\left(2^{(n+1)\cdot n^P_{<(n,0)}}+1\right)}$, which allows us to get $\mu^n(I_n^*) \geq 2^{(n+1)\cdot n^P_{<(n,0)}} + 1$ (Definition 4.4.8). (For modesty in the pr-part and non-trivial conditions, see Lemma 4.4.17 and 4.5.9.)

(pr2) $n^B_L > (n^P_{<L})^{n^P_{<L}}$. (For bigness on $\liminf$ compounds creatures and rapid reading, see Lemma 4.6.5 and 4.7.7.)

(pr3) For $L \in \mathrm{ht}_n^{\mathrm{pr}}$, $T_L^* \geq (n_L^B)^{n_L^B \cdot 2^{(n+1)\cdot n^P_{<(n,0)}}}$, which will imply $\|T_L^*\|_L \geq 2^{(n+1)\cdot n^P_{<(n,0)}}$ (Definition 4.4.4). (For non-trivial conditions in the pr-part, see Lemma 4.4.17 and 4.5.9.)

(pr4) For $L \in \mathrm{ht}_n^{\mathrm{pr}}$, $n^P_{<L^+} > n^P_{<L} \cdot T_L^*$. (For bigness on pr-compounds, see Lemma 4.6.5.)

(pr5) $n^S_{(n,0)} \geq T_n^*$. (Upper bound of the number of possibilities of a pr-compound at n, see Fact 4.4.23.)

(pr6) $\iota_n^* \geq 3^{(n\cdot T_n^*+1)\left(2^{(n+1)\cdot n^P_{<L_n^{\mathrm{lc}}}}+1\right)}$, which will imply $\mu^{n\cdot T_n^*}(\iota_n^*) \geq 2^{(n+1)\cdot n^P_{<L_n^{\mathrm{lc}}}} + 1$ (Definition 4.4.8). (For modesty in the $\liminf$ part and non-trivial conditions, cf. Lemma 4.4.16 and 4.5.10.)

(pr7) For $\ell \in I_n^*$, $h_t(\ell) \geq d_t(\ell)^{d_t(\ell)\cdot 2^{(n+1)\cdot n^P_{<L_n^{\mathrm{lc}}}}}$, which will imply $\|\mathrm{POSS}_\ell^t\|_\ell^t \geq 2^{(n+1)\cdot n^P_{<L_n^{\mathrm{lc}}}}$ (Definition 4.4.5). Likewise when $\ell = L_n^{\mathrm{al}}$, but here replace $n^P_{<L_n^{\mathrm{lc}}}$ by $n^P_{<L_n^{\mathrm{al}}}$. (For non-trivial conditions, see Lemma 4.4.16 and 4.5.10.)

(pr8) For $t \in T_n^*$ and $\ell \in I_n^* \cup \{L_n^{\mathrm{al}}\}$, $n^S_{t,\ell} \geq \left|[b_t(\ell)]^{\leq h_t(\ell)}\right|$ when $\ell \in \mathrm{ht}^{\mathrm{lc}}$, and $n^S_{t,\ell} \geq \left|[a_t(\ell)]^{\leq h_t(\ell)}\right|$ when $\ell \in \mathrm{ht}^{\mathrm{al}}$. This will imply $n^S_{t,\ell} \geq |\mathrm{POSS}_\ell^t|$ in Fact 4.4.6. (Also used for the (a,d)-bounding type property in Lemma 4.8.12.)

(pr9) For $t \in T^*_n$ and $\ell \in I^*_n$, if (t, ℓ) is not the maximum of $T^*_n \times \mathrm{ht}_{\mathrm{lc},n}$ then $b_t(\ell)^{m^S_{t,\ell}} \leq d_{t'}(\ell')$, where (t', ℓ') is the successor of (t, ℓ) in $T^*_n \times I^*_n$, and

$$m^S_{t,\ell} := \begin{cases} \prod_{\substack{\ell' \in I^*_n \\ \ell' \leq \ell}} n^S_{t,\ell'} \times \prod_{\substack{\ell' \in I^*_n \\ \ell' > \ell}} n^S_{t^-,\ell} & \text{if } t >_{\mathrm{lex}} t^{\mathrm{mn}}_n, \\ \prod_{\substack{\ell' \in I^*_n \\ \ell' \leq \ell}} n^S_{t,\ell'} & \text{if } t = t^{\mathrm{mn}}_n. \end{cases}$$

Recall that t^- denotes the predecessor of t in T^*_n. (For bigness on lc-compound creatures, see Lemma 4.6.3 and 4.8.4.)

(pr10) For $t <_{\mathrm{lex}} t'$ in T^*_n, $n^B_{L^{\mathrm{lc}}_n} < d_t(L^{\mathrm{lc}}_n)$ and $n^P_{<L^{\mathrm{lc}}_n} \cdot a_t(L^{\mathrm{lc}}_n)^{m^S_{t,\ell^{\mathrm{mx}}_n}} < d_{t'}(L^{\mathrm{lc}}_n)$. (For bigness on lc-compounds and the (a, d)-bounding type property, see Corollary 4.6.4 and Lemma 4.8.12.)

(pr11) $n^S_{L^{\mathrm{lc}}_n} \geq m^S_{t^{\mathrm{mx}}_n,\ell^{\mathrm{mx}}_n}$. (Upper bound of the number of possibilities of a lc-compound at L^{lc}_n, see Fact 4.4.23.)

(pr12) For $t <_{\mathrm{lex}} t'$ in T^*_n, $n^B_{L^{\mathrm{al}}_n} < d_t(L^{\mathrm{al}}_n)$ and $n^P_{<L^{\mathrm{al}}_n} \cdot (n^S_{t,L^{\mathrm{al}}_n} + 1) < d_{t'}(L^{\mathrm{al}}_n)$. (For the (a, d)-bounding type property, see Lemma 4.8.12.)

(pr13) $n^S_{L^{\mathrm{al}}_n} \geq n^S_{t^{\mathrm{mx}}_n,L^{\mathrm{al}}_n}$. (Upper bound of the number of possibilities of a subatom at L^{al}_n, see Fact 4.4.23.)

(pr14) For $L \in \mathrm{Ht}$, $n^P_{<L} > \prod\{n^S_{L'} : L' < L,\ L' \in \mathrm{Ht}\}$. (To ensure that $|\mathrm{poss}(p, {<}L)| < n^P_{<L}$ for any modest condition p and for rapid reading, see Lemma 4.5.16 and Theorem 4.7.8.)

Theorem 4.3.5. *There is a set of parameters as in Notation 4.3.4 satisfying all the requirements (fp1)–(fp10) and (pr1)–(pr14).*

Proof. Based on the properties (pr1)–(pr14), it is clear how to construct the parameters from steps 1–9. However, steps 5 and 7 need further explanation, also to guarantee properties (fp1)–(fp10).

Assume $t \in T^*_n$ and that we have taken care of step 5 for all $(t_0, \ell_0) < (t, L^{\mathrm{lc}}_n)$ in $T^*_n \times I^*_n$. Choose $d_t(L^{\mathrm{lc}}_n) > n^B_{L^{\mathrm{lc}}_n}$ such that, when $t > t^{\mathrm{mn}}_n$, $d_t(L^{\mathrm{lc}}_n) > n^P_{<L^{\mathrm{lc}}_n} \cdot a_{t^-}(L^{\mathrm{lc}}_n)^{m^S_{t^-,\ell^{\mathrm{mx}}_n}}$ as in (pr10). Define $h_t(L^{\mathrm{lc}}_n)$ as in (pr7) (so (fp1) holds for $\mathrm{t} = \mathrm{lc}$) and define $g_t(L^{\mathrm{lc}}_n)$ as in (fp2). Since $1 - \frac{1}{g_t(L^{\mathrm{lc}}_n)} < 1$, we can find some $b_t(L^{\mathrm{lc}}_n) > h_t(L^{\mathrm{lc}}_n)$ such that $1 - \frac{1}{g_t(L^{\mathrm{lc}}_n)} < 1 - \frac{h_t(L^{\mathrm{lc}}_n)}{b_t(L^{\mathrm{lc}}_n)}$. Define $n^S_{t,L^{\mathrm{lc}}_n}$ as in (pr8).

Now assume that $\ell > L^{\mathrm{lc}}_n$ in I^*_n and that we have defined $d_t(\ell_0)$, $h_t(\ell_0)$, $b_t(\ell_0)$ and n^S_{t,ℓ_0} for all $\ell_0 < \ell$ in I^*_n, such that $1 - \frac{1}{g_t(L^{\mathrm{lc}}_n)} < \prod_{\ell_0 \in I^*_n \cap \ell\downarrow} \left(1 - \frac{h_t(\ell)}{b_t(\ell)}\right)$. So m^S_{t,ℓ^-} can be defined as in (pr9) and we can pick $d_t(\ell) \geq b_t(\ell^-)^{m^S_{t,\ell^-}}$. Then define $h_t(\ell)$ as in (pr7) and pick $b_t(\ell) > h_t(\ell)$ such that $1 - \frac{1}{g_t(L^{\mathrm{lc}}_n)} < \left(1 - \frac{h_t(\ell)}{b_t(\ell)}\right) \prod_{\ell_0 \in I^*_n \cap \ell\downarrow} \left(1 - \frac{h_t(\ell)}{b_t(\ell)}\right)$.

When we get to $\ell = \ell^{\mathrm{mx}}_n$ and $n^S_{t,\ell^{\mathrm{mx}}_n}$ has been defined, we can define $a_t(L^{\mathrm{lc}}_n)$ as in (fp9). Note that (fp4) is guaranteed.

Step 7 is easier to explain: Choose $n^B_{L^{\mathrm{al}}_n}$ as in (pr2), $d_t(L^{\mathrm{al}}_n) > n^B_{L^{\mathrm{al}}_n}$ and, whenever $t > t^{\mathrm{mn}}_n$, $d_t(L^{\mathrm{al}}_n) > n^P_{<L^{\mathrm{al}}_n} \cdot (n^S_{t^-,L^{\mathrm{al}}_n} + 1)$ as in (pr12). Define $h_t(L^{\mathrm{al}}_n)$ as in (pr7), $g_t(L^{\mathrm{al}}_n)$ as in (fp3), $b_t(L^{\mathrm{al}}_n)$ as in (fp8), $a_t(L^{\mathrm{al}}_n)$ as in (fp9), and $n^S_{t,L^{\mathrm{al}}_n}$ as in (pr8).

Note that (pr2) implies (fp10). □

4.4 Creatures upon creatures

Now that we have the frame and the parameters, we are ready to introduce the basic components of the forcing, which we call *subatomic creatures*. Following the same idea as in [FGKS17; GK21], subatomic creatures are used to build *atomic creatures* and *compound creatures*.

4.4.1 Subatomic and atomic creatures

The notion of subatomic creature was introduced in [FGKS17, Definition 2.1.1]. We only need a weaker version thereof.

Definition 4.4.1. Let POSS is a finite non-empty set. A *subatomic family* living on POSS is a tuple $\langle \mathrm{K}, \|\cdot\|_{\mathrm{K}}\rangle$ such that

(K1) $\mathrm{K} \subseteq \mathcal{P}(\mathrm{POSS}) \smallsetminus \{\varnothing\}$ is nonempty, and

(K2) $\|\cdot\| = \|\cdot\|_{\mathrm{K}}$ is a function from K into $[0, +\infty)$ called *norm*,

and satisfying, for any $\mathbf{c} \in \mathrm{K}$,

(K3) if $\mathbf{c} \subseteq \mathbf{d}$ then $\|\mathbf{c}\| \leq \|\mathbf{d}\|$, and

(K4) if $|\mathbf{c}| = 1$ then $\|\mathbf{c}\| = 0$.

The elements of K are called *subatomic creatures*, or *subatoms* for short.

We call $\mathfrak{c} \in \mathrm{K}$ a *trivial subatom* if $|\mathbf{c}| = 1$.

We just write K for the subatomic family $\langle \mathrm{K}, \|\cdot\|\rangle$ when the norm is clear from the context.

Our forcing construction will use the following two types of subatomic families. The first one will be used to add the blocks of parameters, while the second will be used to add the generic slaloms.

Example 4.4.2. Fix the following subatomic families.

(1) Let $T \geq 1$ and $m \geq 2$ be natural numbers. Define the subatomic familly $\langle \mathbf{P}^m_T, \|\cdot\|^m_T\rangle$ living on T by:

  (i) $\mathbf{P}^m_T := \mathcal{P}(T) \smallsetminus \{\varnothing\}$;

  (ii) For $\mathbf{c} \in \mathbf{P}^m_T$, $\|\mathbf{c}\|^m_T := \frac{1}{m}\log_m(|\mathbf{c}|)$.

  Note that $\|\mathbf{c}\|^m_T \geq z$ iff $|\mathbf{c}| \geq m^{mz}$

(2) Given $c \geq 1$, $l \geq 1$ and $m \geq 2$ in ω, define the subatomic family $\langle \mathbf{S}^m_{c,l}, \|\cdot\|^m_{c,l}\rangle$ living on $\mathrm{POSS}^m_{c,l} := [c]^{\leq l}$ by

  (i) $\mathbf{S}^m_{c,l} := \mathcal{P}(\mathrm{POSS}^m_{c,l}) \smallsetminus \{\varnothing\}$;

  (ii) for $\mathbf{c} \in \mathbf{S}^m_{c,l}$,

$$\|\mathbf{c}\|^m_{c,l} := \frac{1}{m}\log_m(\|\mathbf{c}\|^{\mathrm{COV}}_c + 1)$$

  where $\|\mathbf{c}\|^{\mathrm{COV}}_c := \max\{k \leq c \colon \forall x \in [c]^{\leq k}\, \exists y \in \mathbf{c} \colon x \subseteq y\}$.

  Note that $\|\mathbf{c}\|^{\mathrm{COV}}_c \geq k$ iff every $x \subseteq c$ of size $\leq k$ is contained in some member of $\mathbf{c}$. Hence $\|\mathbf{c}\|^m_{c,l} \geq z$ iff $\|\mathbf{c}\|^{\mathrm{COV}}_c \geq m^{mz} - 1$, i. e. every $x \subseteq c$ of size $\leq\lfloor m^{mz} - 1\rfloor$ is contained in some member of $\mathbf{c}$.

Observation 4.4.3. For $\mathbf{c} \in \mathbf{S}^m_{c,l}$, note that $\|\mathbf{c}\|^{\mathrm{COV}}_c \geq 1$ iff $\bigcup \mathbf{c} = c$.

In our forcing construction we use creatures as in Example 4.4.2 at each level $\ell \in \mathrm{ht}$. The generic set of our forcing construction will choose one trivial subatom at each $(\alpha, \ell) \in \mathrm{DOM}$, which determines the value $y_\alpha(\ell)$ of the generic real at α. To determine $y_i(L) \in T^*_L$ for $i \in S^{\mathrm{pr}}$ and $L \in \mathrm{ht}^{\mathrm{pr}}$, we use:

Definition 4.4.4. Fix $L \in \mathrm{ht}^{\mathrm{pr}}$. Define the subatomic family $\langle \mathrm{K}_L, \|\cdot\|_L\rangle := \langle \mathbf{P}^{n^B_L}_{T^*_L}, \|\cdot\|^{n^B_L}_{T^*_L}\rangle$ living on $\mathrm{POSS}_L := T^*_L$ (see Example 4.4.2(1)), i. e.

(i) $\mathrm{K}_L := \mathcal{P}(\mathrm{POSS}_L) \smallsetminus \{\varnothing\}$;

(ii) For $\mathbf{c} \in \mathrm{K}_L$, $\|\mathbf{c}\|_L := \frac{1}{n^B_L} \log_{n^B_L}(|\mathbf{c}|)$.

The subatoms in K_L are also called *(sub)atomic creatures at L*

For the generic slaloms, we use the following subatomic families.

Definition 4.4.5. Fix $n < \omega$ and $t \in T^*_n$. Using Example 4.4.2(2):

(1) For each $\ell \in I^*_n$ define the subatomic family $\langle \mathrm{K}^t_\ell, \|\cdot\|^t_\ell\rangle := \langle \mathbf{S}^{d_t(\ell)}_{b_t(\ell),h_t(\ell)}, \|\cdot\|^{d_t(\ell)}_{b_t(\ell),h_t(\ell)}\rangle$ living on $\mathrm{POSS}^t_\ell := [b_t(\ell)]^{\leq h_t(\ell)}$, i. e.

   (i) $\mathrm{K}^t_\ell := \mathcal{P}(\mathrm{POSS}^t_\ell) \smallsetminus \{\varnothing\}$;

   (ii) for $\mathbf{c} \in \mathrm{K}^t_\ell$,

$$\|\mathbf{c}\|^t_\ell := \frac{1}{d_t(\ell)} \log_{d_t(\ell)}(\|\mathbf{c}\|^{\mathrm{COV}}_{\ell,t} + 1)$$

   where $\|\mathbf{c}\|^{\mathrm{COV}}_{\ell,t} = \|\mathbf{c}\|^{\mathrm{COV}}_{b_t(\ell)} := \max\{m \leq b_t(\ell) : \forall x \in [b_t(\ell)]^{\leq m}\, \exists y \in \mathbf{c}\colon x \subseteq y\}$.

(2) For $\ell = L^{\mathrm{al}}_n$ define the subatomic family $\langle \mathrm{K}^t_\ell, \|\cdot\|^t_\ell\rangle := \langle \mathbf{S}^{d_t(\ell)}_{a_t(\ell),h_t(\ell)}, \|\cdot\|^{d_t(\ell)}_{a_t(\ell),h_t(\ell)}\rangle$ living on $\mathrm{POSS}^t_\ell := [a_t(\ell)]^{\leq h_t(\ell)}$. Denote $\|\mathbf{c}\|^{\mathrm{COV}}_{\ell,t} := \|\mathbf{c}\|^{\mathrm{COV}}_{a_t(\ell)}$ for $\mathbf{c} \in \mathbf{K}^t_\ell$.

The subatoms in K^t_ℓ are also called *subatomic creatures at (t, ℓ) (or just at ℓ).*

Our parameters bound the size of a subatom at (t, ℓ).

Fact 4.4.6. *If $n < \omega$, $\ell \in \mathrm{ht} \smallsetminus \mathrm{ht}^{\mathrm{pr}}$ and $t \in T^*_n$ then $|\mathrm{POSS}^t_\ell| \leq n^S_{t,\ell}$.*

Proof. Immediate by (pr8). □

Recall that, for $i \in S^{\mathrm{pr}}$ and $\alpha \in S^{\mathrm{lc}}_i$, the generic y_α will be in $\mathcal{S}^*(b^-_{y^*_i}, h^-_{y^*_i})$, so for all but finitely many $n < \omega$ and for all $\ell \in I^*_n$, $y_\alpha(\ell) \subseteq b^-_{y^*_i}(\ell) = b_{y^*_i(n)}(\ell)$ will have size $\leq h^-_{y^*_i}(\ell) = h_{y^*_i(n)}(\ell)$. However, the value of $y^*_i(n) \in T^*_n$ is determined by the generic added at i. This indicates that the information a forcing contains at (α, ℓ) must also consider the possibilities of the value of $y^*_i(n)$, i. e. a subatom in K^t_ℓ for each possible $t \in T^*_n$. A similar situation happens when $(\alpha, \ell) \in S^{\mathrm{al}} \times \mathrm{ht}^{\mathrm{al}}$. This motivates the following notion of atomic creature.

Definition 4.4.7. Let $n < \omega$ and $\ell \in I^*_n \cup \{L^{\mathrm{al}}_n\}$.

(1) We say that $\mathbf{x}$ is an *atomic creature at ℓ* if $\mathbf{x} = \langle \mathbf{x}(t) : t \in P_{\mathbf{x}}\rangle$ where $P_{\mathbf{x}} \subseteq T^*_n$ is non-empty, and $\mathbf{x}(t)$ is a K^t_ℓ-subatom for any $t \in P_{\mathbf{x}}$.
   Here $P_{\mathbf{x}}$ is called the *set of pr-indices of* $\mathbf{x}$

(2) For such an atomic creature, define the norm $\|\mathbf{x}\|_{\min} := \min\{\|\mathbf{x}(t)\|^t_\ell : t \in P_{\mathbf{x}}\}$.

(3) Say that an atomic creature $\mathbf{x}$ at ℓ is *trivial* if $x(t)$ is a trivial subatom for each $t \in P_{\mathbf{x}}$.

(4) We consider the following partial order of atomic creatures at ℓ: $\mathbf{y} \leq \mathbf{x}$ iff $P_{\mathbf{y}} \subseteq P_{\mathbf{x}}$ and $\mathbf{y}(t) \subseteq \mathbf{x}(t)$ for all $t \in P_{\mathbf{y}}$.

4.4.2 Compound creatures

We present a method from [FGKS17] (also used in [GK21]) to build compounds of (sub)atomic creatures. This is essential for constructing the $\liminf$ parts of our forcing. We define another type of atomic creature, which uses the following "measure".

Definition 4.4.8 ([FGKS17, Definition 2.2.1])**.** For each $m \in \omega$ we define the "measure" μ^m on the finite sets by

$$\mu^m(A) := \frac{\log_3(|A|)}{m+1}.$$

Definition 4.4.9 ([FGKS17, Definition 2.2.3])**.** Fix a non-empty finite set J. Suppose that $\overline{\mathrm{K}} := \langle \mathrm{K}_j : j \in J \rangle$ is a sequence of subatomic families.

(1) We say that $\mathbf{x}$ *is a simple atomic creature along* J if $\mathbf{x} = \langle \mathbf{x}(j) : j \in J \rangle$ where each $\mathbf{x}(j)$ is a K_j-subatom.

(2) For $m < \omega$, *the* m*-norm* of a simple atomic creature $\mathbf{x}$ along J is

$$\|\mathbf{x}\|^m_{\overline{\mathrm{K}}} := \max\big\{ \min(\{\|\mathbf{x}(j)\|_{\mathrm{K}_j} : j \in A\} \cup \{\mu^m(A)\}) \mid A \subseteq J \big\},$$

i. e. it is the maximal r for which there is a set $A \subseteq J$ such that $\mu^m(A) \geq r$ and $\|\mathbf{x}(j)\|_{\mathrm{K}_j} \geq r$ for all $j \in A$. We say that such an A *witnesses the* m*-norm of* $\mathbf{x}$.

(3) For simple atomic creatures $\mathbf{x}$ and $\mathbf{y}$ along J, we write $\mathbf{y} \leq_{\overline{\mathrm{K}}} \mathbf{x}$ if $\mathbf{y}(j) \subseteq \mathbf{x}(j)$ for all $j \in J$. This determines a partial order of simple atomic creatures along J. When there is no confusion we just write $\mathbf{y} \leq \mathbf{x}$.

(4) We say that a simple atomic creature $\mathbf{x}$ along J is *trivial* if $\mathbf{x}(j)$ is a trivial subatom for each $j \in J$.

We use these simple atomic creatures to construct compound creatures. We first start to those corresponding to the forcing at S^{pr}. This is a variation of [FGKS17, Definition 2.5.1].

Definition 4.4.10. Say that $\mathbf{c}$ is a *pr-compound creature at level* $n < \omega$ (illustrated on the left side of Figure 4.1) if it is composed by

(1) a countable set $\mathrm{dom}\mathbf{c} \subseteq S^{\mathrm{pr}}$ (that could be empty) and a finite $\mathrm{supp}\mathbf{c} \subseteq \mathrm{dom}\mathbf{c}$;

(2) a rectangle of subatomic creatures $\langle \mathbf{c}(i, L) : i \in \mathrm{dom}\mathbf{c} \text{ and } L \in \mathrm{ht}^{\mathrm{pr}}_n \rangle$ such that each $\mathbf{c}(i) := \langle \mathbf{c}(i, L) : L \in \mathrm{ht}^{\mathrm{pr}}_n \rangle$ with $\mathbf{c}(i, L) \in \mathrm{K}_L$ is a simple atomic creature along $\mathrm{ht}^{\mathrm{pr}}_n$;

(3) a real number $d_{\mathbf{c}} \geq 0$ that we often call *halving parameter*.

We also demand that $\mathrm{supp}\mathbf{c} = \varnothing$ implies $d_{\mathbf{c}} = 0$.

In addition, $\mathbf{c}$ satisfies

(4) **Modesty:** for any $L \in \mathrm{ht}^{\mathrm{pr}}_n$ there is <u>at most one</u> $i \in \mathrm{supp}\mathbf{c}$ such that $\mathbf{c}(i, L)$ has size >1, i. e. it is a non-trivial subatom;

(5) if $i \in \mathrm{dom}\mathbf{c} \smallsetminus \mathrm{supp}\mathbf{c}$ then $|\mathbf{c}(i, L)| = 1$, i. e. $\mathbf{c}(i, L)$ is a trivial subatom;

(6) for $i \in \mathrm{supp}\mathbf{c}$, the *stacked norm* of $\mathbf{c}(i)$, denoted by $\|\mathbf{c}(i)\|^{\mathrm{pr}}_{\mathrm{stk}}$, is the n-norm from Definition 4.4.9(2), i. e. $\|\mathbf{c}(i)\|^{\mathrm{pr}}_{\mathrm{stk}}$ is the maximum real r for which there is a set $A \subseteq \mathrm{ht}^{\mathrm{pr}}_n$ with $\mu^n(A) = \frac{\log_3 |A|}{n+1} \geq r$ such that $\|\mathbf{c}(i, L)\|_L \geq r$ for all $L \in A$.

Let $\mathbf{c}$, $\mathbf{c}'$ be pr-compounds at level n. Define $\mathbf{c}' \leq \mathbf{c}$ iff the following holds:

(i) $\operatorname{dom}\mathbf{c} \subseteq \operatorname{dom}\mathbf{c}'$ and $\operatorname{supp}\mathbf{c}' \cap \operatorname{dom}\mathbf{c} = \operatorname{supp}\mathbf{c}$,

(ii) $\mathbf{c}'(i, L) \subseteq \mathbf{c}(i, L)$ for all $i \in \operatorname{dom}\mathbf{c}$ and $L \in \mathrm{ht}^{\mathrm{pr}}_n$, and

(iii) $d_{\mathbf{c}'} \geq d_{\mathbf{c}}$.

This determines a partial order of the pr-compound creatures at n.

By allowing empty domains, we can define the *trivial pr-compound at* n as $\mathbf{1}_{\mathrm{pr}}(n)$ with empty domain (the halving parameter must be 0 by (3)). Note that $\mathbf{c} \leq \mathbf{1}_{\mathrm{pr}}(n)$ for any pr-compound creature $\mathbf{c}$ at n.

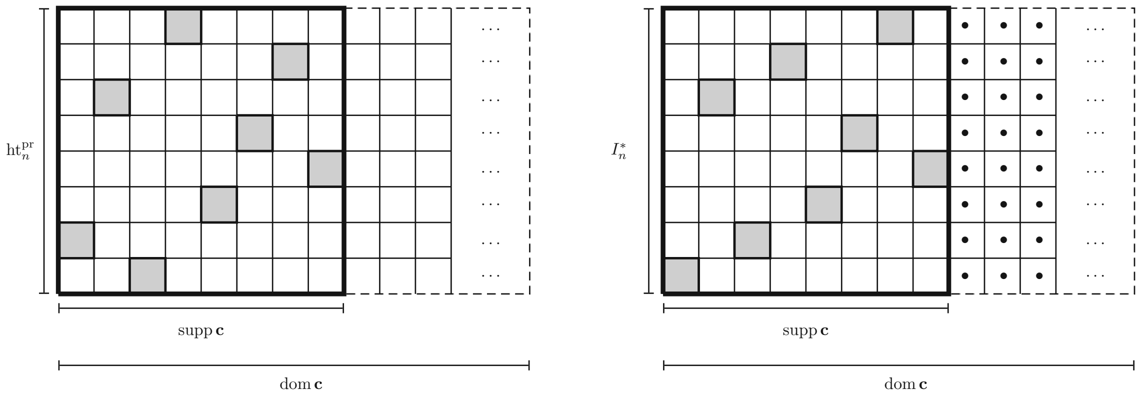

Figure 4.1: The structure of compound creatures. On the left side we have a pr-compound creature, the cells ■ denote subatomic creatures that may be non-trivial, while □ denotes a trivial subatomic creature. On the right side we have an lc-compound, the cells ■ denote atomic creatures (of the form $\mathbf{x} = \langle \mathbf{x}(t) : t \in P_{\mathbf{x}} \rangle$ as in Definition 4.4.7) that may be non-trivial, □ denotes a trivial atomic creature, and ⊡ denotes a trivial subatomic creature.

Definition 4.4.11. The *norm* of a pr-compound $\mathbf{c}$ at level n, when $\sup\mathbf{c} \neq \varnothing$, is defined by

$$\|\mathbf{c}\|^{\mathrm{pr}}_n := \frac{\log_2(\max\{1, \min\{\|\mathbf{c}(i)\|^{\mathrm{pr}}_{\mathrm{stk}} : i \in \operatorname{supp}\mathbf{c}\} - d_{\mathbf{c}}\})}{n^P_{<(n,0)}}.$$

In the case $\operatorname{supp}\mathbf{c} = \varnothing$ we stipulate[3]

$$\|\mathbf{c}\|^{\mathrm{pr}}_n = \begin{cases} 0 & \text{if } \operatorname{dom}\mathbf{c} \neq \varnothing, \\ n & \text{if } \operatorname{dom}\mathbf{c} = \varnothing. \end{cases}$$

In particular, $\|\mathbf{1}_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n = n$.

Since we require atomic creatures as in Definition 4.4.7 to add the generic slaloms, the compound creatures corresponding to the $\liminf$ part of the forcing adding slaloms has a more complex structure.

Definition 4.4.12. For $n < \omega$, say that $\mathbf{c}$ is a *lc-compound creature at level* L^{lc}_n (illustrated on the right side of Figure 4.1) if it is composed by

[3] The requirement $\|\mathbf{c}^{\mathrm{pr}}_n| = n$ when $\operatorname{dom}\mathbf{c} = \varnothing$ is to allow the empty set as the trivial condition in the forcing.

(1) a countable set $\text{dom}\mathbf{c} \subseteq S^{\text{lc}}$ (that could be empty) and a finite set $\text{supp}\mathbf{c} \subseteq \text{dom}\mathbf{c}$;

(2) a sequence of *pr-indices* $\bar{P}_{\mathbf{c}} := \langle P_{\mathbf{c},\alpha} : \alpha \in \text{supp}\mathbf{c} \rangle$ where each $P_{\mathbf{c},\alpha} \subseteq T_n^*$ is non-empty;

(3) a sequence $\langle \mathbf{c}(\alpha, \ell) : \alpha \in \text{dom}\mathbf{c},\ \ell \in I_n^* \rangle$ where:

   (i) for $\alpha \in \text{supp}\mathbf{c}$, $\mathbf{c}(\alpha, \ell) = \langle \mathbf{c}(\alpha, t, \ell) : t \in P_{\mathbf{c},\alpha} \rangle$ is an atomic creature at ℓ,

   (ii) for $\alpha \in \text{dom}\mathbf{c} \smallsetminus \text{supp}\mathbf{c}$, $\mathbf{c}(\alpha, \ell)$ is a trivial subatom in $\text{K}_\ell^{t_n^{\text{mx}}}$;

(4) and a non-negative real number $d_{\mathbf{c}}$ called *halving parameter*.

   We also demand that $\text{supp}\mathbf{c} = \varnothing$ implies $d_{\mathbf{c}} = 0$.

In addition, $\mathbf{c}$ fulfills:

(5) **Strong modesty**: for any $\ell \in I_n^*$ there is at most one pair $(\alpha, t) \in \text{supp}\mathbf{c} \times T_n^*$ such that $t \in P_{\mathbf{c},\alpha}$ and $\mathbf{c}(\alpha, t, \ell)$ is non-trivial.

(6) For $\alpha \in \text{supp}\mathbf{c}$ and $t_0 \in P_{\mathbf{c},\alpha}$, $\mathbf{c}(\alpha, t_0) = \langle \mathbf{c}(\alpha, t_0, \ell) : \ell \in I_n^* \rangle$ is considered as an atom along I_n^*, and its norm, denoted by $\|\mathbf{c}(\alpha, t_0)\|_{\text{stk}}^{t_0}$, is the $n \cdot T_n^*$-norm from Definition 4.4.9, i. e. $\|\mathbf{c}(\alpha, t_0)\|_{\text{stk}}^{t_0}$ is the maximal r for which there is a set $A \subseteq I_n^*$ with $\frac{\log_3 |A|}{n \cdot T_n^* + 1} \geq r$ such that $\|\mathbf{c}(\alpha, t_0, \ell)\|_\ell^{t_0} \geq r$ for all $\ell \in A$.

Given two lc-compound creatures $\mathbf{c}$ and $\mathbf{c}'$ at L_n^{lc}, write $\mathbf{c}' \leq \mathbf{c}$ iff

(i) $\text{dom}\mathbf{c} \subseteq \text{dom}\mathbf{c}'$ and $\text{supp}\mathbf{c}' \cap \text{dom}\mathbf{c} = \text{supp}\mathbf{c}$;

(ii) for all $\alpha \in \text{supp}\mathbf{c}$, $P_{\mathbf{c}',\alpha} \subseteq P_{\mathbf{c},\alpha}$;

(iii) $\mathbf{c}'(\alpha, t, \ell) \subseteq \mathbf{c}(\alpha, t, \ell)$ for every $\alpha \in \text{supp}\mathbf{c}$, $\ell \in I_n^*$ and $t \in P_{\mathbf{c}',\alpha}$;

(iv) $\mathbf{c}'(\alpha, \ell) = \mathbf{c}(\alpha, \ell)$ for all $\alpha \in \text{dom}\mathbf{c} \smallsetminus \text{supp}\mathbf{c}$; and

(v) $d_{\mathbf{c}'} \geq d_{\mathbf{c}}$.

This determines a partial order of the lc-compound creatures at L_n^{lc}. Items (ii) and (iii) mean that $\mathbf{c}'(\alpha, \ell) \leq \mathbf{c}(\alpha, \ell)$ as atomic creatures at ℓ (Definition 4.4.7) for any $\alpha \in \text{supp}\mathbf{c}'$ and $\ell \in I_n^*$.

Denote by $\mathbf{1}_{\text{lc}}(L_n^{\text{lc}})$ the lc-compound creature with empty domain (then $d_{\mathbf{1}_{\text{lc}}(L_n^{\text{lc}})} = 0$ by (4)). Clearly $\mathbf{c} \leq \mathbf{1}_{\text{lc}}(L_n^{\text{lc}})$ for any lc-compound creature $\mathbf{c}$ at L_n^{lc}.

Definition 4.4.13. Let $\mathbf{c}$ be a lc-compound creature at level L_n^{lc}. Define

$$\|\mathbf{c}\|_n^{\text{lc}} := \frac{\log_2(\max\{1, \min\{\|\mathbf{c}(\alpha, t_0)\|_{\text{stk}}^{t_0} : \alpha \in \text{supp}\mathbf{c},\ t_0 \in P_{\mathbf{c},\alpha}\} - d_{\mathbf{c}}\})}{n^P_{<L_n^{\text{lc}}}}.$$

when $\text{supp}\mathbf{c} \neq \varnothing$. In the case $\text{supp}\mathbf{c} = \varnothing$ we stipulate

$$\|\mathbf{c}\|_n^{\text{lc}} = \begin{cases} 0 & \text{if } \text{dom}\mathbf{c} \neq \varnothing, \\ n & \text{if } \text{dom}\mathbf{c} = \varnothing. \end{cases}$$

It is clear that $\|\mathbf{1}_{\text{lc}}(L_n^{\text{lc}})\|_n^{\text{lc}} = n$.

We list below some useful facts to calculate norms of compound creatures.

Lemma 4.4.14.

(a) *If $x, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and $m \geq 2$ in ω then*

$$\log_m(\max\{1, x-\varepsilon\}) \geq \log_m(\max\{1, x\}) - \varepsilon \log_m(2).$$

(b) *Let $n < \omega$ and let $\mathbf{c}$ be an lc-compound creature at L_n^{lc}. For each $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c},\alpha}$ assume that $A_{\alpha,t} \subseteq I_n^*$ witnesses $\|\mathbf{c}(\alpha,t)\|_{\mathrm{stk}}^t$. Let $\mathbf{d} \leq \mathbf{c}$ be an lc-compound creature at L_n^{lc} with the same domain, support, halving parameter and pr-indices, such that, for any $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c}_\alpha}$, there is some $B_{\alpha,t} \subseteq A_{\alpha,t}$ such that*

  (i) *$|A_{\alpha,t} \smallsetminus B_{\alpha,t}| \leq n^P_{<L_n^{\mathrm{lc}}}$ and*

  (ii) *for any $\ell \in B_{\alpha,t}$, $\|\mathbf{d}(\alpha,t,\ell)\|_\ell^t \geq \|\mathbf{c}(\alpha,t,\ell)\|_\ell^t - n^P_{<L_n^{\mathrm{lc}}}$.*

Then $\|\mathbf{d}\|_n^{\mathrm{lc}} \geq \|\mathbf{c}\|_n^{\mathrm{lc}} - 1$.

(c) *The version of (b) for pr-compound creatures also holds.*

Proof. We only show (b). For any $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c},\alpha}$, $\|\mathbf{c}(\alpha,t)\|_{\mathrm{stk}}^{\mathrm{lc}} - \mu^{n\cdot T_n^*}(B_{\alpha,t}) \leq \mu^{n\cdot T_n^*}(A_{\alpha,t}) - \mu^{n\cdot T_n^*}(B_{\alpha,t}) \leq \frac{n^P_{<L_n^{\mathrm{lc}}}}{n\cdot T_n^*+1} \leq n^P_{<L_n^{\mathrm{lc}}}$ by (i), so $\mu^{n\cdot T_n^*}(B_{\alpha,t}) \geq \|\mathbf{c}(\alpha,t)\|_{\mathrm{stk}}^{\mathrm{lc}} - n^P_{L_n^{\mathrm{lc}}}$. By (ii), $\|\mathbf{d}(\alpha,t,\ell)\|_\ell^t \geq \|\mathbf{c}(\alpha,t,\ell)\|_\ell^t - n^P_{<L_n^{\mathrm{lc}}} \geq \|\mathbf{c}(\alpha,t)\|_{\mathrm{stk}}^{\mathrm{lc}} - n^P_{L_n^{\mathrm{lc}}}$ for any $\ell \in B_{\alpha,t}$, so $\|\mathbf{d}(\alpha,t)\|_{\mathrm{stk}}^{\mathrm{lc}} \geq \|\mathbf{c}(\alpha,t)\|_{\mathrm{stk}}^{\mathrm{lc}} - n^P_{L_n^{\mathrm{lc}}}$. Therefore

$$\begin{aligned}
\|\mathbf{d}\|_n^{\mathrm{lc}} &\geq \frac{\log_2(\max\{1, \min\{\|\mathbf{c}(\alpha,t_0)\|_{\mathrm{stk}}^{t_0} : \alpha \in \operatorname{supp}\mathbf{c},\ t_0 \in P_{\mathbf{c},\alpha}\} - n^P_{L_n^{\mathrm{lc}}} - d_{\mathbf{c}}\})}{n^P_{<L_n^{\mathrm{lc}}}} \\
&\geq \|\mathbf{c}\|_n^{\mathrm{lc}} - \frac{n^P_{<L_n^{\mathrm{lc}}}}{n^P_{<L_n^{\mathrm{lc}}}} = \|\mathbf{c}\|_n^{\mathrm{lc}} - 1,
\end{aligned}$$

where the last inequality follows from (a). □

To make sense of the forcing we need to be able to construct, at least, subatoms, atoms and compound creatures with large norm (and arbitrary domain with some support). At this point we start using all the rules we presented in Subsection 4.3.3.

We need the following fact for the construction of compound creatures.

Lemma 4.4.15 ([FGKS17, Lemma 2.2.2])**.** *Let $k \leq m+1$ and $\langle A_i : i \leq k\rangle$ a sequence of finite sets. Then there is a Squence $\langle B_i : i \leq k\rangle$ of pairwise disjoint sets such that $B_i \subseteq A_i$ and $\mu^m(B_i) \geq \mu^m(A_i) - 1$ for any $i \leq k$.*

Lemma 4.4.16. *Fix $B \subseteq S^{\mathrm{lc}}$ countable, $F \subseteq B$ of size $\leq n$, and a sequence $\bar{P} = \langle P_\alpha : \alpha \in F\rangle$ of non-empty subsets of T_n^*. Further assume it is not the case that F is empty and B is not. Then there is a lc-compound creature $\mathbf{c}$ at L_n^{lc} with norm $>n$ such that $\operatorname{dom}\mathbf{c} = B$, $\operatorname{supp}\mathbf{c} = F$ and $\bar{P}_{\mathbf{c}} = \bar{P}$.*

Proof. This is obvious when $F = \varnothing$ (because in this case $B = \varnothing$), so we assume that $F \neq \varnothing$. To find $\mathbf{c}$ we proceed as follows: First set $d_{\mathbf{c}} = 0$, $\operatorname{supp}\mathbf{c} = F$, $\operatorname{dom}\mathbf{c} = B$, and $P_{\mathbf{c},\alpha} = P_\alpha$ for each $\alpha \in F$. For $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c},\alpha}$, we could set $\mathbf{c}(\alpha,t,\ell) := \mathrm{POSS}_\ell^t$ for each $\ell \in I_n^*$, but this $\mathbf{c}$ would not satisfy strong modesty (Definition 4.4.12(5)). To get $\mathbf{c}$ satisfying this, we need to work a bit more.

Put $A_{\alpha,t} := I_n^*$ for each $\alpha \in F$ and $t \in P_\alpha$, and let $\mathcal{A} = \{A_{\alpha,t} : \alpha \in F,\ t \in P_\alpha\}$. Note that $|F| \leq n$ and $|P_\alpha| \leq T_n^*$, so $|\mathcal{A}| \leq n \cdot T_n^*$. By Lemma 4.4.15 choose a pairwise disjoint sequence $\langle B_{\alpha,t} : \alpha \in F,\ t \in P_\alpha\rangle$ such that $B_{\alpha,t} \subseteq A_{\alpha,t}$ and $\mu^{n\cdot T_n^*}(B_{\alpha,t}) \geq \mu^{n\cdot T_n^*}(A_{\alpha,t}) - 1$ for each $\alpha \in F$ and $t \in P_\alpha$. So by (pr6), $\mu^{n\cdot T_n^*}(B_{\alpha,t}) \geq 2^{(n+1)\cdot n^P_{<L_n^{\mathrm{lc}}}}$.

Finally define $\mathbf{c}$ as follows: for $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c},\alpha}$ we set $\mathbf{c}(\alpha,t,\ell) := \mathrm{POSS}^t_\ell$ for each $\ell \in B_{\alpha,t}$; $\mathbf{c}(\alpha,t,\ell)$ is an arbitrary singleton in K^t_ℓ for each $\ell \in I^*_n \setminus B_{\alpha,t}$; and $\mathbf{c}(\alpha,\ell)$ is a trivial subatom in $\mathrm{K}^{t^{\mathrm{mx}}_n}_\ell$ for $\alpha \in \operatorname{dom}\mathbf{c} \setminus \operatorname{supp}\mathbf{c}$ and $\ell \in \mathrm{ht}^{\mathrm{lc}}_n$. It is clear that $\mathbf{c}$ is a lc-compound creature at L^{lc}_n.

It remains to see that $\|\mathbf{c}\|^{\mathrm{lc}}_n > n$. To this end, note that $\|\mathbf{c}(\alpha,t,\ell)\|^t_\ell \geq 2^{(n+1)\cdot n^P_{<L^{\mathrm{lc}}_n}}$ for each $\ell \in B_{\alpha,t}$ by (pr7), so $\|\mathbf{c}(\alpha,t)\|^t_{\mathrm{stk}} \geq 2^{(n+1)\cdot n^P_{<L^{\mathrm{lc}}_n}}$ for $\alpha \in \operatorname{supp}\mathbf{c}$ and $t \in P_{\mathbf{c},\alpha}$. Therefore, $\|\mathbf{c}\|^{\mathrm{lc}}_n \geq n+1$. □

A similar proof using (pr1) and (pr3) gives us the existence of pr-compound creatures with large norm.

Lemma 4.4.17. *Fix $B \subseteq S^{\mathrm{al}}$ countable and $F \subseteq B$ of size $\leq n$, and assume it is not the case that F is empty and B is not. Then there is a pr-compound creature $\mathbf{c}$ at n with norm $>n$ such that* $\operatorname{dom}\mathbf{c} = B$ *and* $\operatorname{supp}\mathbf{c} = F$.

To understand the compatibility between conditions in the forcing we need to look at the compatibility between compound creatures.

Lemma 4.4.18. *Let $\mathbf{c}_0$ and $\mathbf{c}_1$ be two lc-compound creatures at level L^{lc}_n such that*

(i) $\forall\, \alpha \in \operatorname{dom}\mathbf{c}_0 \cap \operatorname{dom}\mathbf{c}_1\ \forall\, \ell \in I^*_n \colon \mathbf{c}_0(\alpha,\ell) = \mathbf{c}_1(\alpha,\ell)$;[4]

(ii) whenever $\operatorname{supp}\mathbf{c}_i \nsubseteq \operatorname{supp}\mathbf{c}_{1-j}$ *for all $j < 2$,* $|\operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1| \leq n$ *and $d_{\mathbf{c}_0} = d_{\mathbf{c}_1}$;*

(iii) whenever $\operatorname{supp}\mathbf{c}_i \subsetneq \operatorname{supp}\mathbf{c}_{1-j}$ *for some $j < 2$, $d_{\mathbf{c}_j} \leq d_{\mathbf{c}_{1-j}}$.*

Then there is some lc-compound creature $\mathbf{e}$ at L^{al}_n such that $\operatorname{dom}\mathbf{e} = \operatorname{dom}\mathbf{c}_0 \cup \operatorname{dom}\mathbf{c}_1$, $\operatorname{supp}\mathbf{e} = \operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1$, *$\mathbf{e} \leq \mathbf{c}_j$ for $j < 2$, and* $\|\mathbf{e}\|^{\mathrm{lc}}_n \geq \min\{\|\mathbf{c}_0\|^{\mathrm{lc}}_n, \|\mathbf{c}_1\|^{\mathrm{lc}}_n\} - \frac{1}{n^P_{L^{\mathrm{lc}}_n}}$.

Proof. To find $\mathbf{e}$ we proceed as follows: we first define $\operatorname{supp}\mathbf{e} := \operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1$, $\operatorname{dom}\mathbf{e} := \operatorname{dom}\mathbf{c}_0 \cup \operatorname{dom}\mathbf{c}_1$, $d_{\mathbf{e}} := \max\{d_{\mathbf{c}_0}, d_{\mathbf{c}_1}\}$, and $P_{\mathbf{e},\alpha} := P_{\mathbf{c}_j,\alpha}$ for $\alpha \in \operatorname{supp}\mathbf{c}_j$ and $j < 2$, which is fine because $\mathbf{c}_0$ and $\mathbf{c}_1$ are the same in their common domain. For the same reason we may define $\mathbf{e}(\alpha,\ell) = \mathbf{c}_j(\alpha,\ell)$ for $j < 2$, $\alpha \in \operatorname{dom}\mathbf{c}_j$ and $\ell \in I^*_n$, which is fine when one support contains the other (in which case $\|\mathbf{e}\|^{\mathrm{lc}}_n \in \{\|\mathbf{c}_0\|^{\mathrm{lc}}_n, \|\mathbf{c}_1\|^{\mathrm{lc}}_n\}$ by (ii)), but otherwise $\mathbf{e}$ may not satisfy strong modesty (Definition 4.4.12(5)). So we need to work more when no support is contained in the other, as in (iii). Hence we are assuming $|\operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1| \leq n$ and $d_{\mathbf{c}_0} = d_{\mathbf{c}_1}$.

Let $A_{\alpha,t} \subseteq I^*_n$ witness the norm $\|\mathbf{c}_j(\alpha,t)\|^t_{\mathrm{stk}}$ for $j < 2$, $\alpha \in \operatorname{supp}\mathbf{c}_j$ and $t \in P_{\mathbf{c}_j,\alpha}$ (which can be chosen independent on j in the common support). Set $\mathcal{A} := \{A_{\alpha,t} : \alpha \in \operatorname{supp}\mathbf{e},\ t \in P_{\mathbf{e},\alpha}\}$. It is clear that $|\mathcal{A}| \leq n \cdot T^*_n$.

By applying Lemma 4.4.15 to the family $\mathcal{A}$, we obtain a paiwise disjoint family $\mathcal{B} = \{B_{\alpha,t} : \alpha \in \operatorname{supp}\mathbf{e},\ t \in P_{\mathbf{e},\alpha}\}$ such that $B_{\alpha,t} \subseteq A_{\alpha,t}$ and $\mu^{n\cdot T^*_n}(B_{\alpha,t}) \geq \mu^{n\cdot T^*_n}(A_{\alpha,t}) - 1$ for $\alpha \in \operatorname{supp}\mathbf{e}$ and $t \in P_{\mathbf{e},\alpha}$.

Finally we use $\mathcal{B}$ to define $\mathbf{e}$ as follows:

- For $j < 2$, $\alpha \in \operatorname{supp}\mathbf{c}_j$ and $t \in P_{\mathbf{e},\alpha}$ we set $\mathbf{e}(\alpha,t,\ell) := \mathbf{c}_j(\alpha,t,\ell)$ for each $\ell \in B_{\alpha,t}$; $\mathbf{e}(\alpha,t,\ell)$ is a singleton contained in $\mathbf{c}_j(\alpha,t,\ell)$ for each $\ell \in I^*_n \setminus B_{\alpha,t}$.
- $\mathbf{e}(\alpha,\ell) := \mathbf{c}_j(\alpha,\ell)$ for $\alpha \in \operatorname{dom}\mathbf{c}_j \setminus \operatorname{supp}\mathbf{c}_j$ and $\ell \in I^*_n$.

[4]Considering the nature of $\mathbf{c}_j(\alpha,\ell)$ whenever α is in the support or not (a sequence of subatoms indexed with some subset of T^*_n, or a single trivial subatom, respectively), this condition (i) implies $\operatorname{supp}\mathbf{c}_{1-j} \cap \operatorname{dom}\mathbf{c}_j \subseteq \operatorname{supp}\mathbf{c}_j$ for $j < 2$.

It is clear that $\mathbf{e}$ is an lc-compound creature and $\mathbf{e} \leq \mathbf{c}_j$ for $j < 2$, so it remains to show that $\|\mathbf{e}\|_n^{\mathrm{lc}} \geq \min_{j<2}\{\|\mathbf{c}_j\|_n^{\mathrm{lc}}\} - 1$. It is clear that $\|\mathbf{e}(\alpha, t)\|_{\mathrm{stk}}^t \geq \|\mathbf{c}_j(\alpha, t)\|_{\mathrm{stk}}^t - 1$ for $j < 2$, $\alpha \in \operatorname{supp}\mathbf{c}_j$ and $t \in P_{\mathbf{e},\alpha}$. Hence

$$\begin{aligned}\|\mathbf{e}\|_n^{\mathrm{lc}} &\geq \frac{\log_2(\max\{1, \min\{\|\mathbf{c}_j(\alpha, t_0)\|_{\mathrm{stk}}^{t_0} : j < 2,\ \alpha \in \operatorname{supp}\mathbf{c}_j,\ t_0 \in P_{\mathbf{e},\alpha}\} - 1 - d_{\mathbf{e}}\})}{n^P_{<L_n^{\mathrm{lc}}}} \\ &\geq \min_{j<2}\{\|\mathbf{c}_j\|_n^{\mathrm{lc}}\} - \frac{1}{n^P_{<L_n^{\mathrm{lc}}}} \quad \text{(by Lemma 4.4.14(a)).} \qquad \square\end{aligned}$$

In a similar (and simpler) way, we can prove the corresponding fact for pr-compound creatures.

Lemma 4.4.19. *For $j < 2$ let $\mathbf{c}_j$ be a pr-compound creature at level n such that:*

(i) $\forall\, i \in \operatorname{dom}\mathbf{c}_0 \cap \operatorname{dom}\mathbf{c}_1\ \forall\, L \in \mathrm{ht}_n^{\mathrm{pr}} \colon \mathbf{c}_0(i, L) = \mathbf{c}_1(i, L)$*;*

(ii) whenever $\operatorname{supp}\mathbf{c}_i \nsubseteq \operatorname{supp}\mathbf{c}_{1-j}$ *for all* $j < 2$, $|\operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1| \leq n$ *and* $d_{\mathbf{c}_0} = d_{\mathbf{c}_1}$*;*

(iii) whenever $\operatorname{supp}\mathbf{c}_i \subsetneq \operatorname{supp}\mathbf{c}_{1-j}$ *for some* $j < 2$, $d_{\mathbf{c}_j} \leq d_{\mathbf{c}_{1-j}}$*;*

(iv) $\operatorname{supp}\mathbf{c}_{1-j} \cap \operatorname{dom}\mathbf{c}_j \subseteq \operatorname{supp}\mathbf{c}_j$ *for* $j < 2$.

Then there is some pr-compound creature $\mathbf{e}$ *at* n *such that* $\operatorname{dom}\mathbf{e} = \operatorname{dom}\mathbf{c}_0 \cup \operatorname{dom}\mathbf{c}_1$, $\operatorname{supp}\mathbf{c} = \operatorname{supp}\mathbf{c}_0 \cup \operatorname{supp}\mathbf{c}_1$, $\mathbf{e} \leq \mathbf{c}_j$ *for each* $j < 2$*, and* $\|\mathbf{e}\|_n^{\mathrm{pr}} \geq \min\{\|\mathbf{c}_0\|_n^{\mathrm{pr}}, \|\mathbf{c}_1\|_n^{\mathrm{pr}}\} - 1$.

Observation 4.4.20. In Lemma 4.4.18 and 4.4.19 we get a concrete construction of $\mathbf{e}$ when $\operatorname{supp}\mathbf{c}_j \subseteq \operatorname{supp}\mathbf{c}_{1-j}$ for some $j < 2$. We denote this $\mathbf{e}$ by $\mathbf{c}_0 \wedge \mathbf{c}_1$.

Notation 4.4.21. From now on, considering the norms of subatoms, atoms, and compound creatures, we are just going to write $\|\mathbf{c}\|$ when there is no place for confusion. This will be determined by the nature of $\mathbf{c}$. For example, if $\mathbf{c}$ is a lc-compound creature at level L_n^{lc} we abbreviate $\|\mathbf{c}\| = \|\mathbf{c}\|_n^{\mathrm{lc}}$, even $\|\mathbf{c}(\alpha, t_0)\| = \|\mathbf{c}(\alpha, t_0)\|_{\mathrm{stk}}^{t_0}$.

To conclude this section, we discuss about the set of possibilities of a compound creature.

Definition 4.4.22. Given a compound creature $\mathbf{c}$ we define its set of possibilities according to the following cases.

(1) If $\mathbf{c}$ is a compound at $n < \omega$ then $\operatorname{poss}\mathbf{c} := \prod\{\mathbf{c}(i, L) : i \in \operatorname{dom}\mathbf{c},\ L \in \mathrm{ht}_n^{\mathrm{pr}}\}$

(2) If $\mathbf{c}$ is a compound at L_n^{lc} and $\bar{t} = \langle t_\alpha : \alpha \in \operatorname{supp}\mathbf{c}\rangle \in \prod_{\alpha \in \operatorname{supp}\mathbf{c}} P_{\mathbf{c},\alpha}$ then define

$$\begin{aligned}\mathbf{c}(\bar{t}) &:= \langle \mathbf{c}(\alpha, t_\alpha, \ell) : \alpha \in \operatorname{supp}\mathbf{c},\ \ell \in I_n^*\rangle \cup \langle \mathbf{c}(\alpha, \ell) : \alpha \in \operatorname{dom}\mathbf{c} \smallsetminus \operatorname{supp}\mathbf{c},\ \ell \in I_n^*\rangle, \\ \operatorname{poss}(\mathbf{c}, \bar{t}) &:= \prod\{\mathbf{c}(\bar{t})(\alpha, \ell) : \alpha \in \operatorname{dom}\mathbf{c},\ \ell \in I_n^*\}.\end{aligned}$$

The upper bound of the number of possibilities is estimated by the parameters of the forcing.

Fact 4.4.23. *Let $n < \omega$.*

(a) If $\mathbf{c}$ *is a compound at* $n < \omega$ *then* $|\operatorname{poss}\mathbf{c}| \leq T_n^* \leq n^S_{(n,0)}$.

(b) If $\mathbf{c}$ *is a compound at* L_n^{lc} *and* $\bar{t} \in \prod_{\alpha \in \operatorname{supp}\mathbf{c}} P_{\mathbf{c},\alpha}$ *then* $|\operatorname{poss}(\mathbf{c}, \bar{t})| \leq n^S_{L_n^{\mathrm{lc}}}$.

(c) If $t \in T_n^*$ *then* $|\operatorname{POSS}^t_{L_n^{\mathrm{al}}}| \leq n^S_{L_n^{\mathrm{al}}}$.

Proof. For (a) and (b), if $\mathbf{c}$ has empty support then clearly the set of possibilities has only one element, so we assume that $\mathbf{c}$ has non-empty support.

The modesty of $\mathbf{c}$ is essential for this proof. For (a) we can find a function $f\colon \operatorname{ht}^{\mathrm{pr}}_n \to \operatorname{supp}\mathbf{c}$ such that $f(L)$ is the unique member of $\operatorname{supp}\mathbf{c}$ such that $\mathbf{c}(f(L), L)$ is non-trivial in case it exists. Hence, using (pr5),

$$|\operatorname{poss}(\mathbf{c})| = \left| \prod_{L \in \operatorname{ht}^{\mathrm{pr}}_n} \mathbf{c}(f(L), L) \right| \leq \prod_{L \in \operatorname{ht}^{\mathrm{pr}}_n} T^*_L = T^*_n \leq n^S_{(n,0)}.$$

For (b) we can find a function $f\colon I^*_n \to \operatorname{supp}\mathbf{c}$ such that $f(\ell)$ is the unique member of $\operatorname{supp}\mathbf{c}$ such that $\mathbf{c}(f(\ell), t_{f(\ell)}, \ell)$ is non-trivial in case it exists. Hence, using Fact 4.4.6, (pr9) and (pr11),

$$\begin{aligned} |\operatorname{poss}(\mathbf{c}, \bar{t})| &\leq \prod_{\ell \in I^*_n} |\mathbf{c}(f(\ell), t_{f(\ell),\ell})| \leq \prod_{\ell \in I^*_n} |\operatorname{POSS}^{t_{f(\ell)}}_\ell| \leq \prod_{\ell \in I^*_n} n^S_{t_{f(\ell)},\ell} \\ &\leq \prod_{\ell \in I^*_n} n^S_{t_n^{\mathrm{mx}},\ell} = m^S_{t_n^{\mathrm{mx}},\ell_n^{\mathrm{mx}}} \leq n^S_{L_n^{\mathrm{lc}}}. \end{aligned}$$

Property (c) is clear by Fact 4.4.6 and (pr13). □

4.5 The forcing construction

This section is devoted to defining our forcing and to prove some of its basic properties. This construction is based on [FGKS17; GK21].

4.5.1 The forcing

Our forcing is composed of three parts: the *pr-part* (also called *pr-forcing*), the *lc-part* and the *al-part*. As mentioned in many occasions, the pr-part is a $\liminf$ construction adding the blocks of parameters, the lc-part is also $\liminf$ and it increases $\mathfrak{b}^{\mathrm{Lc},\bar{I}^*}_{b_{y_i^*},h_{y_i^*}}$, and the al-part is a $\limsup$ construction that increases $\mathfrak{d}^{\mathrm{aLc}}_{a_{y_i^*},h_{y_i^*}}$. While the pr-forcing is independent, the lc-part and al-part are not independent, i. e. they depend a lot on the pr-part.

We start describing the pr-part. This is similar to the $\liminf$ part of the forcings from [FGKS17; GK21].

Definition 4.5.1. We define the *pr-forcing* $\mathbb{S}$ as follows:

Conditions. A condition $p \in \mathbb{S}$ consist of:

(1) a trunk length $\operatorname{trnklg}(p) \in \omega$,

(2) a sequence $\langle p(n) : n < \omega \rangle$ where each $p(n)$ is a pr-compound at level n,

(3) a countable set $\operatorname{supp} p \subseteq S^{\mathrm{pr}}$ (that could be empty),

and it satisfies

(4) $\operatorname{dom} p(n) = \operatorname{supp} p$ for all $n < \omega$,

(5) $\langle \operatorname{supp} p(n) : n < \omega \rangle$ is non-decreasing,

(6) $\operatorname{supp} p = \bigcup_{n<\omega} \operatorname{supp} p(n)$,

(7) $\operatorname{supp} p(n) = \varnothing$ for all $n < \operatorname{trnklg}(p)$,

(8) $\lim_{n\to\infty} \frac{|\mathrm{supp} p(n)|}{n} = 0$, and

(9) $\lim_{n\to\infty} \|p(n)\|_n^{\mathrm{pr}} = +\infty$.

Order. For $p, q \in \mathbb{S}$, $q \le p$ iff $\mathrm{trnklg}(q) \ge \mathrm{trnklg}(p)$ and $q(n) \le p(n)$ for all $n < \omega$.

It is clear that this is a partial order on $\mathbb{S}$ and that $\mathbf{1} = \mathbf{1}_{\mathrm{pr}} := \langle \mathbf{1}_{\mathrm{pr}}(n) : n < \omega \rangle$ with $\mathrm{trnklg}(\mathbf{1}) := 0$ is a maximum condition in $\mathbb{S}$.[5]

Before describing the other components of our forcing construction, we need to fix some terminology, particularly the definition of *possibilities* of the generic reals given by a condition in $\mathbb{S}$.

Notation 4.5.2. We fix the following terminology:

(1) For $\ell \in \mathrm{ht}$, denote by $n_*(\ell)$ the unique $n < \omega$ such that $\ell \in \mathrm{ht}_n$.

(2) For each $\alpha \in S^{\mathrm{t}}$ ($\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$) there is a unique $i \in S^{\mathrm{pr}}$ such that $\alpha \in S^{\mathrm{t}}_i$, which we denote by $i^*(\alpha)$. For any $\alpha \in S^{\mathrm{pr}}$ denote $i^*(\alpha) := \alpha$.

(3) A set $B \subseteq S^*$ is *closed* if, for any $\alpha \in B$, $i^*(\alpha) \in B$.

Definition 4.5.3. Fix a condition $p \in \mathbb{S}$ and $i \in \mathrm{supp} p$.

(1) For each height $\ell \in \mathrm{ht}$ we let

$$\mathrm{poss}(p(i), {<}\ell) := \prod_{\substack{L<\ell \\ L\in \mathrm{ht}^{\mathrm{pr}}}} p(i, L) \text{ and } \mathrm{poss}(p, {<}\ell) := \prod_{i \in \mathrm{supp} p} \mathrm{poss}(p(i), {<}\ell)$$

(2) For $A \subseteq \mathrm{ht}^{\mathrm{pr}}$ denote by $\mathrm{poss} p(i, A) := \prod_{L \in A} p(i, L)$ the *set of possibilities on A at i*.

(3) Let $(\alpha, \ell) \in \mathrm{DOM} \smallsetminus (S^{\mathrm{pr}} \times \mathrm{ht}^{\mathrm{pr}})$ such that $i^*(\alpha) \in \mathrm{supp} p$. Denote

$$\mathrm{pss}(p, \alpha, \ell) := \mathrm{poss} p(i^*(\alpha), \mathrm{ht}^{\mathrm{pr}}_{n_*(\ell)}) = \prod_{L \in \mathrm{ht}^{\mathrm{pr}}_{n_*(\ell)}} p(i^*(\alpha), L).$$

We now add the lc-part to the pr-forcing. For a condition, the atomic creatures at $(\alpha, \ell) \in S^{\mathrm{lc}} \times I^*_n$ must depend on the pr-part of the condition, concretely, on $\mathrm{pss}(p_{\mathrm{pr}}, \alpha, \ell)$.

Definition 4.5.4. We define the forcing $\mathbb{Q}_{\mathrm{lc}}$ as follows:
Conditions. A condition $p = (p_{\mathrm{pr}}, p_{\mathrm{lc}})$ in $\mathbb{Q}_{\mathrm{lc}}$ consist of:

(1) $p_{\mathrm{pr}} \in \mathbb{S}$;

(2) a trunk length $\mathrm{trnklg}(p) = \mathrm{trnklg}(p_{\mathrm{pr}}) \in \omega$;

(3) a countable set $\mathrm{supp} p_{\mathrm{lc}} \subseteq S^{\mathrm{lc}}$ and $\mathrm{supp} p := \mathrm{supp} p_{\mathrm{pr}} \cup \mathrm{supp} p_{\mathrm{lc}} \subseteq S^*$ (which is closed by (8));

(4) p_{lc} is a sequence $p_{\mathrm{lc}} := \langle p(L^{\mathrm{lc}}_n) : n < \omega \rangle$ such that

  (i) $p(L^{\mathrm{lc}}_n)$ is an lc-compound creature at level L^{lc}_n,
  (ii) $\mathrm{dom} p(L^{\mathrm{lc}}_n) = \mathrm{supp} p_{\mathrm{lc}}$,
  (iii) $\langle \mathrm{supp} p(L^{\mathrm{lc}}_n) : n < \omega \rangle$ is non-decreasing,

[5] We allow conditions like **1** with larger trunk length.

(iv) $\operatorname{supp} p_{\mathrm{lc}} = \bigcup_{n<\omega} \operatorname{supp} p(L_n^{\mathrm{lc}})$,

(v) $P_{p(L_n^{\mathrm{lc}}),\alpha} = \operatorname{pss}(p_{\mathrm{pr}}, \alpha, L_n^{\mathrm{lc}})$ for any $\alpha \in \operatorname{supp} p(L_n^{\mathrm{lc}})$;

and it satisfies:

(5) $\operatorname{supp} p(L_n^{\mathrm{lc}}) = \varnothing$ and $\operatorname{supp} p_{\mathrm{pr}}(n) = \varnothing$ for all $n < \operatorname{trnklg}(p)$;

(6) $\lim_{n\to\infty} \frac{|\operatorname{supp} p(L_n^{\mathrm{lc}})|}{n} = 0$;

(7) $\lim\limits_{n\to\infty} \|p(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}} = \infty$.

(8) If $\alpha \in \operatorname{supp} p(L_n^{\mathrm{lc}})$ then $i^*(\alpha) \in \operatorname{supp} p_{\mathrm{pr}}(n)$.

Order. For $p, q \in \mathbb{Q}_{\mathrm{lc}}$ we say that $q \leq p$ iff $\operatorname{trnklg}(q) \geq \operatorname{trnklg}(p)$, $q_{\mathrm{pr}} \leq p_{\mathrm{pr}}$ and, for all $n < \omega$:

(i) if either $n < \operatorname{trnklg}(p)$ or $n \geq \operatorname{trnklg}(q)$ then $q(L_n^{\mathrm{lc}}) \leq p(L_n^{\mathrm{lc}})$,

(ii) if $\operatorname{trnklg}(p) \leq n < \operatorname{trnklg}(q)$ then, for all $\alpha \in \operatorname{supp} p$ and $\ell \in I_n^*$: if $\alpha \in \operatorname{supp} p(n)$ then $q(\alpha, \ell) \subseteq p(\alpha, t, \ell)$ where t is the unique member of $\operatorname{pss}(q, \alpha, \ell)$; else, if $\alpha \notin \operatorname{supp} p(n)$ then $q(\alpha, \ell) = p(\alpha, \ell)$.

It is routine to show that this defines a partial order on $\mathbb{Q}_{\mathrm{lc}}$. Moreover, $\mathbf{1}$ is a maximum condition in $\mathbb{Q}_{\mathrm{lc}}$ (considering $\mathbf{1}_{\mathrm{lc}}(L_n^{\mathrm{lc}})$ as in Definition 4.4.12).

We denote, for $\alpha \in \operatorname{supp} p_{\mathrm{lc}}$,

$$\operatorname{trnklg}(p, \alpha) := \min\{n < \omega : \ \alpha \in \operatorname{supp} p(L_n^{\mathrm{lc}})\}.$$

Property (8) is only relevant to force the (a_i, d_i)-bounding property in Subsection 4.8.2.

We now define the forcing with the al-part (first without the lc-part for simplicity).

Definition 4.5.5. We define the forcing $\mathbb{Q}_{\mathrm{al}}$ as follows:

Conditions. A condition $p = (p_{\mathrm{pr}}, p_{\mathrm{al}}) \in \mathbb{Q}_{\mathrm{al}}$ consists of:

(1) $p_{\mathrm{pr}} \in \mathbb{S}$;

(2) a trunk length $\operatorname{trnklg}(p) = \operatorname{trnklg}(p_{\mathrm{pr}}) \in \omega$;

(3) a countable set $\operatorname{supp} p_{\mathrm{al}} \subseteq S^{\mathrm{al}}$ such that $\operatorname{supp} p := \operatorname{supp} p_{\mathrm{pr}} \cup \operatorname{supp} p_{\mathrm{al}}$ is closed;

(4) each $\alpha \in \operatorname{supp} p_{\mathrm{al}}$ has a $\limsup$ trunk length $\operatorname{trnklg}(p, \alpha) \geq \operatorname{trnklg}(p)$ in ω;

(5) p_{al} is a sequence $\langle p(\alpha, \ell) : \alpha \in \operatorname{supp} p_{\mathrm{al}} \text{ and } \ell \in \mathrm{ht}^{\mathrm{al}}\rangle$ such that, for $\alpha \in \operatorname{supp} p_{\mathrm{al}}$ and $\ell \in \mathrm{ht}^{\mathrm{al}}$:

(i) if $\ell \geq \operatorname{trnklg}(p, \alpha)$ then $p(\alpha, \ell) = \langle p(\alpha, t, \ell) : t \in \operatorname{pss}(p_{\mathrm{pr}}, \alpha, \ell)\rangle$ is an atomic creature at ℓ, and

(ii) if $\ell < \operatorname{trnklg}(p, \alpha)$ then $p(\alpha, \ell)$ is a trivial subatom in $\mathrm{K}_\ell^{t_n^{\mathrm{mx}}}$;

and it satisfies:

(6) For all $\alpha \in \operatorname{supp} p_{\mathrm{al}}$,

$$\limsup_{\ell \in \mathrm{ht}_{\mathrm{al}}} \|p(\alpha, \ell)\|_{\mathrm{al}} = \infty$$

where

$$\|p(\alpha, \ell)\|_{\mathrm{al}} := \begin{cases} \|p(\alpha, \ell)\|_{\min} & \text{(as in Definition 4.4.7) if } \ell \geq \operatorname{trnklg}(p, \alpha), \\ \|p(\alpha, \ell)\|_\ell^{t_{n*(\ell)}^{\mathrm{mx}}} & \text{if } \ell < \operatorname{trnklg}(p, \alpha). \end{cases}$$

Note that, in the second case, $\|p(\alpha, \ell)\|_\ell^{t_{n*(\ell)}^{\mathrm{mx}}} = 0$ by (5)(ii).

Order. For $p, q \in \mathbb{Q}_{\mathrm{al}}$ we say that $q \leq p$ iff $\mathrm{trnklg}(q) \geq \mathrm{trnklg}(p)$, $\mathrm{supp}q \supseteq \mathrm{supp}p$, $q_{\mathrm{pr}} \leq p_{\mathrm{pr}}$ and, for all $\alpha \in \mathrm{supp}p_{\mathrm{al}}$ and $\ell \in \mathrm{ht}^{\mathrm{al}}$:

(i) $\mathrm{trnklg}(q, \alpha) = \max\{\mathrm{trnklg}(p, \alpha), \mathrm{trnklg}(q)\}$;

(ii) if $\ell < \mathrm{trnklg}(p, \alpha)$ then $q(\alpha, \ell) = p(\alpha, \ell)$;

(iii) if $\mathrm{trnklg}(p, \alpha) \leq \ell < \mathrm{trnklg}(q, \alpha)$ then $q(\alpha, \ell) \subseteq p(\alpha, t, \ell)$ where t is the unique member of $\mathrm{pss}(q_{\mathrm{pr}}, \alpha, \ell)$;

(iv) if $\ell \geq \mathrm{trnklg}(q, \alpha)$ then $q(\alpha, \ell) \leq p(\alpha, \ell)$ as atomic creatures at ℓ (Definition 4.4.7).

It is routine to check that this is a partial order on $\mathbb{Q}_{\mathrm{al}}$. Even more, $\mathbf{1}$ is a maximum condition in $\mathbb{Q}_{\mathrm{al}}$ (by considering $\mathrm{supp}(\mathbf{1})_{\mathrm{al}} = \varnothing$).

We can finally put everything together and get our desired forcing.

Definition 4.5.6. Define the poset $\mathbb{Q}$ whose conditions are those $p = (p_{\mathrm{pr}}, p_{\mathrm{lc}}, p_{\mathrm{al}})$ satisfying $(p_{\mathrm{pr}}, p_{\mathrm{al}}) \in \mathbb{Q}_{\mathrm{al}}$ and $(p_{\mathrm{pr}}, p_{\mathrm{lc}}) \in \mathbb{Q}_{\mathrm{lc}}$. The order is defined by $q \leq p$ iff $(q_{\mathrm{pr}}, q_{\mathrm{al}}) \leq (p_{\mathrm{pr}}, p_{\mathrm{al}})$ and $(q_{\mathrm{pr}}, q_{\mathrm{lc}}) \leq (p_{\mathrm{pr}}, p_{\mathrm{lc}})$. This is a partial order with maximum condition $\mathbf{1}$. Figure 4.2 illustrates the structure of a condition $p \in \mathbb{Q}$.

For each $p \in \mathbb{Q}$ we define $\mathrm{supp}p := \mathrm{supp}p_{\mathrm{pr}} \cup \mathrm{supp}p_{\mathrm{lc}} \cup \mathrm{supp}p_{\mathrm{al}}$. This allows to define restrictions of $\mathbb{Q}$ for closed $B \subseteq S^*$:

(1) $\mathbb{Q}_B := \{p \in \mathbb{Q} : \mathrm{supp}p \subseteq B\}$, with the same order as $\mathbb{Q}$.

(2) For $p \in \mathbb{Q}$, $p|B$ denotes the condition in $\mathbb{Q}_B$ obtained from p, in the natural way, by restricting the support (as well as the supports and domains of the pr and lc-compounds) to B.[6]

It is not always the case that $\mathbb{Q}_B$ is a complete subposet of $\mathbb{Q}$ (the halving parameters are a problem for this), but in some cases this can be guaranteed. For example $\mathbb{Q}_{\mathrm{pr}} = \mathbb{Q}_{S^{\mathrm{pr}}}$, $\mathbb{Q}_{\mathrm{lc}} = \mathbb{Q}_{S^{\mathrm{pr}} \cup S^{\mathrm{lc}}}$ and $\mathbb{Q}_{\mathrm{al}} = \mathbb{Q}_{S^{\mathrm{pr}} \cup S^{\mathrm{al}}}$ are complete subposets of $\mathbb{Q}$ (see the general case in Corollary 4.5.19).

4.5.2 Basic properties of the forcing

We present some basic properties and features of our forcing construction.

By the construction, the $\liminf$ part of any condition $p \in \mathbb{Q}$ satisfies *modesty*, i. e. for any $\ell \in \mathrm{ht} \smallsetminus \mathrm{ht}^{\mathrm{al}}$ there is at most one $\alpha \in \mathrm{supp}p$ such that $p(\alpha, \ell)$ is not a trivial (sub)atom. Although it is not necessary to demand modesty at $\ell \in \mathrm{ht}^{\mathrm{al}}$, we are going to use this property most of the time. Just considering modest conditions at all levels gives us an equivalent forcing notion.

Definition 4.5.7. A condition $p \in \mathbb{Q}$ *is modest* if for any $\ell \in \mathrm{ht}$ there is at most one $\alpha \in \mathrm{supp}p$ such that $p(\alpha, \ell)$ is not a trivial (sub)atom.

Lemma 4.5.8. *Given a condition $p \in \mathbb{Q}$ there is some modest $q \leq p$ in $\mathbb{Q}$ with same support, trunk lengths and with identical pr and lc-parts. In particular, the set of modest conditions is dense in $\mathbb{Q}$.*

Proof. Let p be a condition in $\mathbb{Q}$. We only consider the case when $\mathrm{supp}p_{\mathrm{al}}$ is non-empty, so we can enumerate it by $\mathrm{supp}p_{\mathrm{al}} =: \{\alpha_m : m < \omega\}$ where each member of $\mathrm{supp}p_{\mathrm{al}}$ is enumerated infinitely many times. Define q with the same support, trunk lengths, and same pr and lc-parts as p and, by recursion on m, we define q_{al} up to some height ℓ_m. At step $m = 0$, according to

[6] So $\mathrm{trnklg}(p|B) = \mathrm{trnklg}(p)$ even when $B = \varnothing$, and the halving parameter at some level becomes 0 when the support of the compound at the same level is disjoint with B.

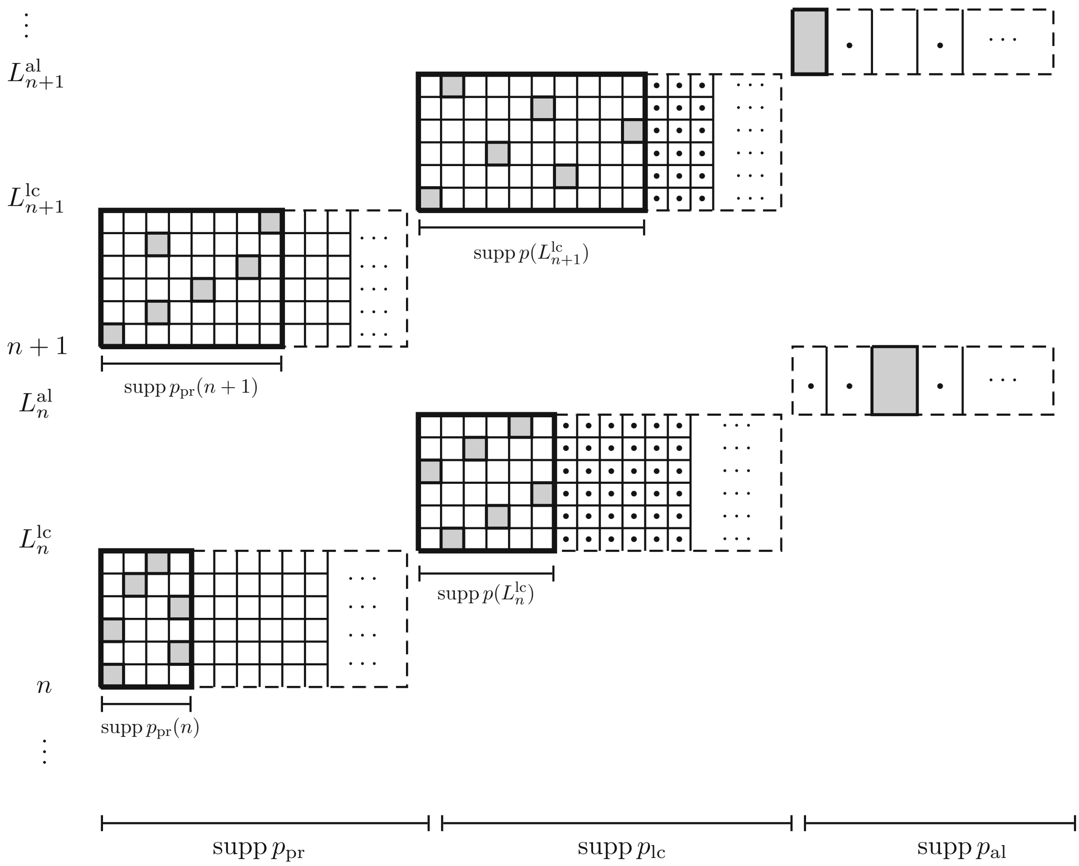

Figure 4.2: The structure of a condition $p \in \mathbb{Q}$. The lim inf compound creatures are as described in Figure 4.1; the al-part with support $\operatorname{supp} p_{\text{al}}$ is a lim sup construction and, like the lc-compound creatures, ■ is an atomic creature that may be non-trivial, □ is a trivial atomic creature, and ⊡ denotes a trivial subatomic creature (i. e. a singleton).

Definition 4.5.5(6), there is some $\ell_0 \in \text{ht}^{\text{al}}$ such that $\|p(\alpha_0, \ell_0)\|_{\text{al}} > 0$, so set $q(\alpha_0, \ell_0) := p(\alpha_0, \ell_0)$, and choose a trivial atom (or subatom) $q(\alpha_j, \ell') \leq p(\alpha_j, \ell')$ for any $j < \omega$ and $\ell' < \ell_0$ in ht^{al}, and also when $\alpha_j \neq \alpha_0$ and $\ell' = \ell_0$.

At step m, we can find an $\ell_m > \ell_{m-1}$ in ht^{al} such that $\|p(\alpha_m, \ell_m)\|_{\text{al}} > m$. So define $q(\alpha_m, \ell_m) := p(\alpha_m, \ell_m)$ and choose a trivial atom $q(\alpha_j, \ell') \leq p(\alpha_j, \ell')$ for any $j < \omega$ and $\ell_{m-1} < \ell' < \ell_m$ in ht^{al}, and when $\alpha_j \neq \alpha_m$ and $\ell' = \ell_m$.

Hence the resulting q is as required. □

In the following results we show that we can construct conditions with arbitrary closed support.

Lemma 4.5.9. *For any countable set of indices $B \subseteq S^{\text{pr}}$ there is some $p \in \mathbb{Q}_{\text{pr}}$ with $\operatorname{supp} p = B$.*

Proof. If $B = \varnothing$ we have $p = \mathbf{1}$, so assume $B \neq \varnothing$. Enumerate $B \cap S^{\text{pr}} := \{i_j : j < \omega\}$ (allowing repetitions). To define the desired p, set the trunk length $\operatorname{trnklg}(p) := 0$ and choose some non-decreasing sequence $\langle x_n : n < \omega \rangle$ of natural numbers such that $x_n \leq n$, $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} \frac{x_n}{n} = 0$. Set $\operatorname{supp} p(n) := \{i_j : j < x_n\}$. Then Definition 4.5.1(8) holds, that is, $\lim_{n\to\infty} \frac{|\operatorname{supp} p(n)|}{n} = 0$. Let $n_0 := \min\{n < \omega : x_n > 0\}$. For each $n_0 \leq n < \omega$, by Lemma 4.4.17 there is a pr-compound $p(n)$ at level n with norm $>n$, support $\operatorname{supp} p(n)$ and domain B. For

$n < n_0$, let $p(n)$ be any pr-compound with empty support and domain B. Then $p \in \$$ and $\mathrm{supp} p = \bigcup_{n<\omega} \mathrm{supp} p(n) = B$. □

Lemma 4.5.10. *Let B be a closed countable subset of S^* and $p_{\mathrm{pr}} \in \mathbb{Q}_{\mathrm{pr}}$ with support $B \cap S^{\mathrm{pr}}$. Then there is some condition p with support B whose pr-part is p_{pr}*

Proof. We assume that $B \cap S^{\mathrm{lc}} \neq \varnothing$ (otherwise there is no need to go through the "lc-part" below). Find the condition p as follows:

Trunk lengths. Define the trunk lengths $\mathrm{trnklg}(p) = \mathrm{trnklg}(p, \alpha) := \mathrm{trnklg}(p_{\mathrm{pr}})$ for all $\alpha \in B \cap S^{\mathrm{al}}$.

The pr-part. As required, we set the pr-part of p as the given p_{pr}.

The lc-part. It is important to ensure (8) of Definition 4.5.4, i. e. for any $\alpha \in \mathrm{supp} p(L^{\mathrm{lc}}_n)$, $i^*(\alpha) \in \mathrm{supp} p_{\mathrm{pr}}(n)$. First, define $B^{\mathrm{lc}}_n := \{\alpha \in B \cap S^{\mathrm{lc}} : i^*(\alpha) \in \mathrm{supp} p_{\mathrm{pr}}(n)\}$. Note that $B^{\mathrm{lc}}_n \subseteq B^{\mathrm{lc}}_{n+1}$ and $\bigcup_{n<\omega} B^{\mathrm{lc}}_n = B \cap S^{\mathrm{lc}}$.

Fix $n_0 := \min\{n < \omega : B^{\mathrm{lc}}_n \neq \varnothing\}$ (clearly $n_0 \geq \mathrm{trnklg}(p)$) and a bijection $f\colon \omega \smallsetminus n_0 \to (\omega \smallsetminus n_0) \times \omega$ such that $f(n) = (m, k)$ implies $m \leq n$. Enumerate $B^{\mathrm{lc}}_n := \{\beta_{n,i} : i < \omega\}$ (allowing repetitions) for $n \geq n_0$, and define $\gamma^{\mathrm{lc}}_n := \beta_{f(n)}$. So, if $f(n) = (m, k)$ then $m \leq n$ and $\gamma^{\mathrm{lc}}_n \in B^{\mathrm{lc}}_m \subseteq B^{\mathrm{lc}}_n$. Note that $B \cap S^{\mathrm{lc}} = \{\gamma^{\mathrm{lc}}_n : n \geq n_0\}$

As in the proof of Lemma 4.5.9, choose some non-decreasing Squence $\langle x_n : n < \omega\rangle$ of natural numbers such that $x_n \leq n$, $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} \frac{x_n}{n} = 0$. Set $\mathrm{supp} p(L^{\mathrm{lc}}_n) := \varnothing$ for $n < n_0$ and $\mathrm{supp} p(L^{\mathrm{lc}}_n) := \{\gamma^{\mathrm{lc}}_{n_0}, \dots \gamma^{\mathrm{lc}}_{x_n-1}\}$ for $n \geq n_0$. It is clear that Definition 4.5.4 (6) and (8) hold. Setting $n_1 := \min\{n < \omega : x_n > n_0\}$, by Lemma 4.4.16, for $n \geq n_1$ there is some lc-compound $p(L^{\mathrm{al}}_n)$ at L^{lc}_n with norm $>n$, support $\mathrm{supp} p(L^{\mathrm{lc}}_n)$, $\mathrm{dom} p(L^{\mathrm{lc}}_n) = B \cap S^{\mathrm{lc}}$ and $P_{p(L^{\mathrm{lc}}_n),\alpha} = \mathrm{pss}(p_{\mathrm{pr}}, \alpha, L^{\mathrm{lc}}_n)$ for all $\alpha \in \mathrm{supp} p(L^{\mathrm{lc}}_n)$. For $n < n_1$ let $p(L^{\mathrm{lc}}_n)$ be any lc-compound creature at L^{lc}_n with empty support and domain $B \cap S^{\mathrm{lc}}$. This determines the lc-part p_{lc} of p.

The al-part. For $n < \omega$ and $\alpha \in B \cap S^{\mathrm{al}}$, if $n \geq \mathrm{trnklg}(p)$ set $p(\alpha, t, L^{\mathrm{al}}_n) := \mathrm{POSS}^t_\ell$ for all $t \in \mathrm{pss}(p_{\mathrm{pr}}, \alpha, L^{\mathrm{al}}_n)$, otherwise let $p(\alpha, L^{\mathrm{al}}_n)$ be any trivial subatom in $\mathrm{K}^{t^{\mathrm{mx}}_n}_{L^{\mathrm{al}}_n}$. This determines the al-part p_{al} of p, and it is clear that $(p_{\mathrm{pr}}, p_{\mathrm{lc}}) \in \mathbb{Q}_{\mathrm{lc}}$.

Set $p := (p_{\mathrm{pr}}, p_{\mathrm{lc}}, p_{\mathrm{al}})$. To see that $p \in \mathbb{Q}$ it remains to check Definition 4.5.5(6). By (pr7), $\|\mathrm{POSS}^t_\ell\| \geq n+1$, so $\|p(\alpha, L^{\mathrm{al}}_n)\|_{\min} \geq n+1$ for any $\alpha \in B \cap S^{\mathrm{al}}$ and $n \geq \mathrm{trnklg}(p)$, that is, $\|p(\alpha, L^{\mathrm{al}}_n)\|_{\mathrm{al}} \geq n+1$. Therefore, $\limsup_{\ell \in \mathrm{ht}_{\mathrm{al}}} \|p(\alpha, \ell)\|_{\mathrm{al}} = \infty$. Hence the resulting p is indeed a condition in $\mathbb{Q}$ with $\mathrm{supp} p = B$. □

Corollary 4.5.11. *For any closed countable set of indices $B \subseteq S^*$, there is some $p \in \mathbb{Q}$ such that $\mathrm{supp} p = B$. In particular, given any $\alpha \in S^*$, there is some condition p such that $\mathrm{supp} p = \{i^*(\alpha), \alpha\}$.*

We know look at the set of possibilities of the generic reals determined by a condition in $\mathbb{Q}$.

Definition 4.5.12. Fix a condition $p \in \mathbb{Q}$. For any $n < \omega$ and any function η extending a member of $\mathrm{poss} p_{\mathrm{pr}}(n)$ we denote, according to Definition 4.4.22:

(1) for $\alpha \in \mathrm{supp} p$, $\hat{\eta}(\alpha, n) := \langle \eta(i^*(\alpha), L) : L \in \mathrm{ht}^{\mathrm{pr}}_n\rangle$, which is in T^*_n; in general, we write $\hat{\eta}(\alpha, \ell) := \hat{\eta}(\alpha, n_*(\ell))$ for any $\ell \in \mathrm{ht}$;

(2) $\hat{\eta}(L^{\mathrm{lc}}_n) := \langle \hat{\eta}(\alpha, n) : \alpha \in \mathrm{supp} p(L^{\mathrm{lc}}_n)\rangle$;

(3) $p(L^{\mathrm{lc}}_n, \eta) := p(L^{\mathrm{lc}}_n)(\hat{\eta}(L^{\mathrm{lc}}_n))$.

For $L \in \mathrm{Ht}$ define $\mathrm{poss}(p, {<}L)$ as the set of functions η with domain $\mathrm{DOM} \cap (\mathrm{supp} p \times L{\downarrow})$ such that, for any $n < L$ in ω,

(i) $\eta\restriction S^{\mathrm{pr}} \times \mathrm{ht}_n^{\mathrm{pr}} \in \mathrm{poss}p_{\mathrm{pr}}(n)$,

(ii) $\eta\restriction S^{\mathrm{lc}} \times I_n^* \in \mathrm{poss}p(L_n^{\mathrm{lc}}, \eta)$ when $L_n^{\mathrm{lc}} < L$,

(iii) when $L_n^{\mathrm{al}} < L$, for $\alpha \in \mathrm{supp}p_{\mathrm{al}}$: if $n \geq \mathrm{trnklg}(p,\alpha)$ then $\eta(\alpha, L_n^{\mathrm{al}}) \in p(\alpha, \hat{\eta}(\alpha, n), L_n^{\mathrm{al}})$, otherwise $\eta(\alpha, L_n^{\mathrm{al}})$ is the unique member of $p(\alpha, L_n^{\mathrm{al}})$.

When using possibilities $\eta \in \mathrm{poss}(p, {<}L)$, we allow the following abuse of notation:

- for $B \subseteq S^*$, $\eta|_B$ abbreviates $\eta\restriction B \times \mathrm{ht}$;
- for $A \subseteq \mathrm{Ht}$, $\eta\restriction A$ abbreviates $\eta\restriction S^* \times A$.

As we expect, any $\eta \in \mathrm{poss}(p, {<}L)$ denotes the set of possibilities of the generic on $\mathrm{DOM} \cap (\mathrm{supp}(p) \times L{\downarrow})$.

Fact 4.5.13. *Let $L \in \mathrm{Ht}$, $p \in \mathbb{Q}$ and $\eta \in \mathrm{poss}(p, {<}L)$. Then, for any $(\alpha, \ell) \in \mathrm{DOM}$ with $\alpha \in \mathrm{supp}p$ and $\ell < L$:*

(a) if $\alpha = i \in S^{\mathrm{pr}}$ then $\eta(i, \ell) \in T_\ell^*$;

(b) if $\alpha \in \mathrm{supp}p(L_{n_*(\ell)}^{\mathrm{lc}})$ then $\eta(\alpha, \ell) \subseteq b_{\hat{\eta}(\alpha, n_*(\ell))}(\ell)$ has size $\leq h_{\hat{\eta}(\alpha, n_*(\ell))}(\ell)$; else, if $\alpha \in S^{\mathrm{lc}} \setminus \mathrm{supp}p(L_{n_*(\ell)}^{\mathrm{lc}})$ then $\eta(\alpha, \ell) \subseteq b_{t^{\mathrm{mx}}_{n_*(\ell)}}(\ell)$ has size $\leq h_{t^{\mathrm{mx}}_{n_*(\ell)}}(\ell)$;

(c) if $\alpha \in S^{\mathrm{al}}$ and $\ell \geq \mathrm{trnklg}(p, \alpha)$ then $\eta(\alpha, \ell) \subseteq a_{\hat{\eta}(\alpha, n_*(\ell))}(\ell)$ has size $\leq h_{\hat{\eta}(\alpha, n_*(\ell))}(\ell)$; else, if $\ell < \mathrm{trnklg}(p, \alpha)$ then $\eta(\alpha, \ell) \subseteq a_{t^{\mathrm{mx}}_{n_*(\ell)}}(\ell)$ has size $\leq h_{t^{\mathrm{mx}}_{n_*(\ell)}}(\ell)$.

For any $\eta \in \mathrm{poss}(p, {<}L)$ we can define a condition $p \wedge \eta \leq p$ deciding the generic below L to be η. This condition is basically constructed by replacing creatures below L by the subatoms determined by η (and increasing trunk lengths if required).

Definition 4.5.14. Let $p \in \mathbb{Q}$. We define for every $L \in \mathrm{Ht}$ and $\eta \in \mathrm{poss}(p, {<}L)$ a condition $q = p \wedge \eta$ satisfying:

(i) $\mathrm{supp}q = \mathrm{supp}p$.

(ii) $\mathrm{trnklg}(q) := \max\{\mathrm{trnklg}(p), n_*(L)\}$, $\mathrm{trnklg}(q, \alpha) := \max\{\mathrm{trnklg}(p, \alpha), n_*(L)\}$ for $\alpha \in \mathrm{supp}p_{\mathrm{al}}$.

(iii) Any (pr and lc) compound creature at a level $\geq \mathrm{trnklg}(q)$ has the same support, domain, halving parameter and pr-indices as the compound in p at the same level.

(iv) For $i \in \mathrm{supp}p_{\mathrm{pr}}$ and $\ell \in \mathrm{ht}^{\mathrm{pr}}$:

- when $\ell < L$ set $q(i, \ell) := \{\eta(i, \ell)\}$;
- when $\ell \geq L$ set $q(i, \ell) := p(i, \ell)$.

(v) For $(\alpha, \ell) \in \mathrm{DOM}$ with $\alpha \in \mathrm{supp}q \setminus S^{\mathrm{pr}}$:

- when $\ell < \mathrm{trnklg}(q)$ set $q(\alpha, \ell) := \{\eta(\alpha, \ell)\}$;
- when $\mathrm{trnklg}(q) \leq \ell < \mathrm{trnklg}(q, \alpha)$ set $q(\alpha, \ell) := p(\alpha, \ell)$ (if there is such an ℓ then we must have $\mathrm{trnklg}(q, \alpha) = \mathrm{trnklg}(p, \alpha)$);
- when $\mathrm{trnklg}(q, \alpha) \leq \ell < L$ set $q(\alpha, \ell) = \langle q(\alpha, t, \ell) : t \in \{\hat{\eta}(\alpha, n_*(\ell))\}\rangle$ where $q(\alpha, \hat{\eta}(\alpha, n_*(\ell)), \ell) := \{\eta(\alpha, \ell)\}$ (note that $\mathrm{pss}(q, \alpha, \ell) = \{\hat{\eta}(\alpha, n_*(\ell))\}$);
- when $\ell \geq \max\{L, \mathrm{trnklg}(q, \alpha)\}$ and $t \in \mathrm{pss}(q_{\mathrm{pr}}, \alpha, \ell)$ set $q(\alpha, t, \ell) := p(\alpha, t, \ell)$.

From the definition above we get the following simple consequences.

Fact 4.5.15. *For $p \in \mathbb{Q}$ and $L \in \mathrm{Ht}$:*

(a) If $\eta \in \mathrm{poss}(p, {<}L)$ then $p \wedge \eta \in \mathbb{Q}$ and it is stronger than p.

(b) $\{p \wedge \eta \leq p \mid \eta \in \mathrm{poss}(p, {<}L)\}$ is an antichain.

(c) For any $q \leq p$ and $\eta' \in \mathrm{poss}(q, {<}L)$, there is a unique $\eta \in \mathrm{poss}(p, {<}L)$ such that $q \wedge \eta' \leq p \wedge \eta$.

(d) $\{p \wedge \eta \leq p \mid \eta \in \mathrm{poss}(p, {<}L)\}$ is a maximal antichain below p.

Proof. Property (a) is clear by Definition 4.5.14. For (b), note that two different $\eta, \eta' \in \mathrm{poss}(p, {<}L)$ determine incompatible $p \wedge \eta$ and $p \wedge \eta'$ (if η and η' differ at some (α, ℓ) then the trivial subatoms at this position are disjoint for both conditions).

For (c), it suffices to define $\eta := \eta'|_{\mathrm{supp} p}$. Uniqueness follows by (b).

To check (d), let $q \leq p$ in $\mathbb{Q}$, and choose some $\eta' \in \mathrm{poss}(q, {<}L)$. By (c), there is some $\eta \in \mathrm{poss}(p, {<}L)$ such that $q \wedge \eta' \leq p \wedge \eta$, so q is compatible with $p \wedge \eta$ by (a). □

For our calculations, we use the parameter $n^P_{<L}$ to bound the number of possibilities of a modest condition.

Lemma 4.5.16. *Assume that $p \in \mathbb{Q}$ is modest and $L \in \mathrm{Ht}$. Then $|\mathrm{poss}(p, {<}L)| < n^P_L$.*

Proof. Since p is modest, for any $\ell \in \mathrm{ht}$ we can pick the unique point $\alpha_\ell \in \mathrm{supp} p$ such that the creature $p(\alpha_\ell, \ell)$ is non-trivial in case it exists, otherwise let $\alpha_\ell \in \mathrm{supp} p$ be anything such that $(\alpha_\ell, \ell) \in \mathrm{DOM}$. In what follows, when we write $p(\alpha, \bar{t}, \ell)$ we ignore $\bar{t}$ when $\ell < \mathrm{trnklg}(p, \alpha)$ (i. e. it is interpreted as the trivial subatom $p(\alpha, \ell)$). In the case $L \in \mathrm{ht}^{\mathrm{pr}}$, using Fact 4.4.23, and (pr14),

$$\begin{aligned}
|\mathrm{poss}(p, {<}L)| &\leq \Big| \prod_{n \in \omega \cap L\downarrow} \bigcup_{\eta \in \mathrm{poss} p_{\mathrm{pr}}(n)} \{\eta\} \times \mathrm{poss} p(L^{\mathrm{lc}}_n, \hat{\eta}(L^{\mathrm{lc}}_n)) \times p(\alpha_{L^{\mathrm{al}}_n}, \hat{\eta}(\alpha, n), L^{\mathrm{al}}_n) \Big| \\
&\leq \prod_{n \in \omega \cap L\downarrow} \sum_{\eta \in \mathrm{poss} p_{\mathrm{pr}}(n)} n^S_{L^{\mathrm{lc}}_n} n^S_{L^{\mathrm{al}}_n} = \prod_{n \in \omega \cap L\downarrow} |\mathrm{poss} p_{\mathrm{pr}}(n)| n^S_{L^{\mathrm{lc}}_n} n^S_{L^{\mathrm{al}}_n} \\
&\leq \prod_{n \in \omega \cap L\downarrow} n^S_{(n,0)} n^S_{L^{\mathrm{lc}}_n} n^S_{L^{\mathrm{al}}_n} = \prod \{n^S_{L'} : L' < L,\ L' \in \mathrm{Ht}\} < n^P_{<L}.
\end{aligned}$$

In the case $L = L^{\mathrm{lc}}_n$, we get

$$|\mathrm{poss}(p, {<}L)| = |\mathrm{poss}(p, {<}n) \times \mathrm{poss}(p_{\mathrm{pr}}(n))| \leq \prod \{n^S_{L'} : L' < L,\ L' \in \mathrm{Ht}\} < n^P_{<L}.$$

The inequality $|\mathrm{poss}(p, {<}L^{\mathrm{al}}_n)| < n^P_{<L^{\mathrm{al}}_n}$ is checked similarly. □

We now look at several results about compatibility between conditions. We also get several relevant consequences such as cases when $\mathbb{Q}|B \lessdot \mathbb{Q}$ (complete embedabbility) and the $\mathfrak{c}^+$-cc on $\mathbb{Q}$.

Lemma 4.5.17. *Let $B \subseteq S^*$ be closed, $p \in \mathbb{Q}$. Assume that $r \in \mathbb{Q}_B$, $r \leq p|B$ and that, for any compound creature in p, one of the following conditions hold:*

(i) its support is contained in B, or

(ii) the compound in r at the same level has empty support, or

(iii) it has the same halving parameter as the compound in r at the same level.

Then there is some $q \leq p$ in $\mathbb{Q}$ with support $\operatorname{supp} p \cup \operatorname{supp} q$ such that $q|B \leq r$. Even more, if $\operatorname{trnklg}(r) = \operatorname{trnklg}(p)$ and (i) or (ii) always hold then we can even get $q|B = r$.

Proof. Let $n_0 \geq \operatorname{trnklg}(r)$ be minimal such that, for any $n \geq n_0$,

- if (i) and (ii) do not hold at level n then $|\operatorname{supp} r_{\mathrm{pr}}(n) \cup \operatorname{supp} p_{\mathrm{pr}}(n)| \leq n$,
- if (i) and (ii) do not hold at level L_n^{lc} then $|\operatorname{supp} r(L_n^{\mathrm{lc}}) \cup \operatorname{supp} p(L_n^{\mathrm{lc}})| \leq n$.

Note that, in case (i) or (ii) always hold, $n_0 = \operatorname{trnklg}(r)$. Choose some $\eta_0 \in \operatorname{poss}(r, {<}n_0)$ and set $r' := r \wedge \eta_0$, so $\operatorname{trnklg}(r') = n_0$. Since $r \leq p|B$, we can find some $\eta_1 \in \operatorname{poss}(p, {<}n_0)$ compatible with η_0 and set $p' := p \wedge \eta_1$. Note that the hypothesis of the lemma still holds when p and r are replaced by p' and r' respectively. Note that $r' = r$ when $n_0 = \operatorname{trnklg}(r)$.

We define q as follows.

Trunk lengths. $\operatorname{trnklg}(q) := n_0$, $\operatorname{trnklg}(q, \alpha) := \operatorname{trnklg}(r', \alpha)$ if $\alpha \in \operatorname{supp} r'_{\mathrm{al}}$, and $\operatorname{trnklg}(q, \alpha) := \operatorname{trnklg}(p', \alpha)$ if $\alpha \in \operatorname{supp} p'_{\mathrm{al}} \smallsetminus B$.

The pr-part. Fix $n < \omega$. Let $\mathbf{c}_n$ be the compound creature at n that results from $p'_{\mathrm{pr}}(n)$ by restricting its domain to $\operatorname{supp} p'_{\mathrm{pr}} \smallsetminus B$. Then, by Lemma 4.4.19, there is some pr-compound creature $q_{\mathrm{pr}}(n)$ at n such that

- $\operatorname{dom} q_{\mathrm{pr}}(n) = \operatorname{dom} \mathbf{c}_n \cup \operatorname{dom} r'_{\mathrm{pr}}(n) = \operatorname{supp} p_{\mathrm{pr}} \cup \operatorname{supp} r_{\mathrm{pr}}$,
- $\operatorname{supp} q_{\mathrm{pr}}(n) = \operatorname{supp} \mathbf{c}_n \cup \operatorname{supp} r'_{\mathrm{pr}}(n) = \operatorname{supp} p'_{\mathrm{pr}}(n) \cup \operatorname{supp} r'_{\mathrm{pr}}(n)$,
- $q_{\mathrm{pr}}(n) \leq \mathbf{c}_n$ and $q_{\mathrm{pr}}(n) \leq r'_{\mathrm{pr}}(n)$, and
- $\|q_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}} \geq \min\{\|\mathbf{c}_n\|_n^{\mathrm{pr}}, \|r'_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}}\} - 1$, even more

  $$\|q_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}} \geq \min\{\|p'_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}}, \|r'_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}}\} - 1$$

  (clear when $\mathbf{c}_n$ has non-empty support, otherwise (i) holds and $\|q_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}} = \|r'_{\mathrm{pr}}(n)\|_n^{\mathrm{pr}}$).

In fact $q_{\mathrm{pr}}(n) \leq p_{\mathrm{pr}}(n)$. This defines a condition $q_{\mathrm{pr}} \in \mathbb{Q}_{\mathrm{pr}}$ stronger than both p'_{pr} and $r'_{\mathrm{pr}}|B$. Moreover, if (i) or (ii) hold at level n then $q_{\mathrm{pr}}(n) = \mathbf{c}_n \wedge r'_{\mathrm{pr}}(n)$ (see Observation 4.4.20).

The lc-part. Fix $n < \omega$. Produce $\mathbf{c}_{n,0}$ from $r'(L_n^{\mathrm{lc}})$ by just reducing the pr-indices to $P_{\mathbf{c}_{n,0},\alpha} := \operatorname{pss}(q_{\mathrm{pr}}, \alpha, L_n^{\mathrm{lc}})$ for each $\alpha \in \operatorname{supp} r'(L_n^{\mathrm{lc}})$. Also define $\mathbf{c}_{n,1}$ from $p'(L_n^{\mathrm{lc}})$ by restricting its domain to $\operatorname{supp} p'_{\mathrm{lc}} \smallsetminus B$ and reducing the pr-indices to $P_{\mathbf{c}_{n,1},\alpha} := \operatorname{pss}(q_{\mathrm{pr}}, \alpha, L_n^{\mathrm{lc}})$ for each $\alpha \in \operatorname{supp} p'(L_n^{\mathrm{lc}}) \smallsetminus B$. Hence, by Lemma 4.4.18, there is an lc-compound creature $q_{\mathrm{lc}}(L_n^{\mathrm{lc}})$ at L_n^{lc} such that

- $\operatorname{dom} q_{\mathrm{lc}}(L_n^{\mathrm{lc}}) = \operatorname{dom} \mathbf{c}_{n,0} \cup \operatorname{dom} \mathbf{c}_{n,1} = \operatorname{supp} p_{\mathrm{lc}} \cup \operatorname{supp} r_{\mathrm{lc}}$,
- $\operatorname{supp} q_{\mathrm{lc}}(L_n^{\mathrm{lc}}) = \operatorname{supp} \mathbf{c}_{n,0} \cup \operatorname{supp} \mathbf{c}_{n,1} = \operatorname{supp} p'_{\mathrm{lc}}(L_n^{\mathrm{lc}}) \cup \operatorname{supp} r'_{\mathrm{lc}}(L_n^{\mathrm{lc}})$,
- $q(L_n^{\mathrm{lc}}) \leq \mathbf{c}_{n,0}$ and $q(L_n^{\mathrm{lc}}) \leq \mathbf{c}_{n,1}$, and
- $\|q_{\mathrm{lc}}(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}} \geq \min\{\|\mathbf{c}_{n,0}\|_n^{\mathrm{lc}}, \|\mathbf{c}_{n,1}\|_n^{\mathrm{lc}}\} - 1$, even more

  $$\|q_{\mathrm{lc}}(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}} \geq \min\{\|p'(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}}, \|r'(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}}\} - 1.$$

This clearly determines a condition $(q_{\mathrm{pr}}, q_{\mathrm{lc}}) \in \mathbb{Q}_{\mathrm{lc}}$ stronger than $(p'_{\mathrm{pr}}, p'_{\mathrm{lc}})$ and $(r'_{\mathrm{pr}}, r'_{\mathrm{lc}})$. Moreover, if (i) or (ii) hold at L_n^{lc}, we can set $q(L_n^{\mathrm{al}}) := \mathbf{c}_{n,0} \wedge \mathbf{c}_{n,1}$.

The al-part. Set $\operatorname{supp} q_{\mathrm{al}} = \operatorname{supp} p'_{\mathrm{al}} \cup \operatorname{supp} t'_{\mathrm{al}}$. Define $q(\alpha, \ell)$ for $(\alpha, \ell) \in \operatorname{supp} q_{\mathrm{al}} \times \mathrm{ht}^{\mathrm{al}}$ as follows: when $\ell \geq \operatorname{trnklg}(q, \alpha)$,

- if $\alpha \in \operatorname{supp} r'_{\mathrm{al}}$ then
$$q(\alpha, \ell) := \langle r'(\alpha, t, \ell) : t \in \operatorname{pss}(q_{\mathrm{pr}}, \alpha, \ell)\rangle,$$
- and if $\alpha \in \operatorname{supp} p'_{\mathrm{al}} \smallsetminus B$ then
$$q(\alpha, \ell) = \langle p'(\alpha, t, \ell) : t \in \operatorname{pss}(q_{\mathrm{pr}}, \alpha, \ell)\rangle;$$

when $\ell < \operatorname{trnklg}(q, \alpha)$ set $q(\alpha, \ell) := r'(\alpha, \ell)$ if $\alpha \in \operatorname{supp} r'_{\mathrm{al}}$, and $q(\alpha, \ell) := p'(\alpha, \ell)$ if $\alpha \in \operatorname{supp} p'_{\mathrm{al}} \smallsetminus B$. This determines a condition $(q_{\mathrm{pr}}, q_{\mathrm{al}}) \in \mathbb{Q}_{\mathrm{al}}$ stronger than $(p'_{\mathrm{pr}}, p'_{\mathrm{al}})$ and $(r'_{\mathrm{pr}}, r'_{\mathrm{al}})$.

It is clear that $q := (q_{\mathrm{pr}}, q_{\mathrm{lc}}, q_{\mathrm{al}}) \in \mathbb{Q}$, $q \leq p$ and $q|B \leq r$.

In the case when $\operatorname{trnklg}(r) = \operatorname{trnklg}(p)$ and (i) or (ii) always hold, we get from the construction that $n_0 = \operatorname{trnklg}(r) = \operatorname{trnklg}(p)$, $r' = r$, $p' = p$, and $q|B = r$. □

Observation 4.5.18. In the previous result, when $\operatorname{trnklg}(r) = \operatorname{trnklg}(p)$ and (i) or (ii) always hold, q has a concrete construction. We denote this q by $p \wedge r$, which becomes a very useful notation. Specially when $B = S^{\mathrm{pr}}$: if $p \in \mathbb{Q}$ and $r \leq p_{\mathrm{pr}}$ in $\mathbb{Q}_{\mathrm{pr}}$ with the same trunk length as p, we can define the condition $p \wedge r$ (at any level L_n^{lc} the lc-compound of r has empty domain, so (ii) is satisfied). In fact, $p \wedge r$ just results by shrinking the pr-indices of the atomic creatures at the lc-part and al-part of p to those given by the possibilities on r.

Corollary 4.5.19. *Let $B \subseteq S^*$ be closed. If for $\mathrm{t} \in \{\mathrm{pr}, \mathrm{lc}\}$, either $S^{\mathrm{t}} \subseteq B$ or $S^{\mathrm{t}} \cap B = \varnothing$, then $\mathbb{Q}_B$ is a complete subforcing of $\mathbb{Q}$. In particular, $\mathbb{Q}_{\mathrm{pr}}$, $\mathbb{Q}_{\mathrm{lc}}$ and $\mathbb{Q}_{\mathrm{al}}$ are complete subforcings of $\mathbb{Q}$.*

We also derive the following criteria for compatibility of conditions.

Corollary 4.5.20. *If $p, r \in \mathbb{Q}$ are identical on the intersection $A := \operatorname{supp} p \cap \operatorname{supp} r$ (that is $p|A = r|A$) and they have the same halving parameters at the levels where the supports of both compound creatures from both conditions are non-empty, then there is a condition q stronger than both p and r such that* $\operatorname{supp} q = \operatorname{supp} p \cup \operatorname{supp} r$.

Proof. Let $B := \operatorname{supp} r$. Clearly $r \in \mathbb{Q}_B$ and $r \leq p|B$ because $p|B = p|A = r|A$. Since the hypothesis of Lemma 4.5.17 holds, there is some $q \leq p$ in $\mathbb{Q}$ with $\operatorname{supp} q = \operatorname{supp} p \cup \operatorname{supp} r$ such that $q|B \leq r$. But $q \leq q|B$, so q is as required. □

As a consequence, we can show that the generic real at any $\alpha \in S^*$ is always defined, i. e. the set of conditions p with $\alpha \in \operatorname{supp} p$ is dense.

Corollary 4.5.21. *Given $p \in \mathbb{Q}$ and $\alpha \in S^*$, there is some $q \leq p$ such that* $\operatorname{supp} q = \operatorname{supp} p \cup \{i^*(\alpha), \alpha\}$.

Proof. If $\alpha \in \operatorname{supp} p$ then we can just set $q := p$, so assume that $\alpha \notin \operatorname{supp} p$. In case $i^*(\alpha) \notin \operatorname{supp} p$, by Corollary 4.5.11 find $r_0 \in \mathbb{Q}$ with support $\{i^*(\alpha), \alpha\}$, even more, according to the proofs of Lemma 4.4.16, 4.4.17, 4.5.9 and 4.5.10, we can find r_0 such that all its subatoms are the largest possible and with halving parameters equal to 0. So we can modify r_0 by setting the same halving parameters as p at levels where the compounds of both conditions have non-empty support, and the resulting r is still in $\mathbb{Q}$ with support $\{i^*(\alpha), \alpha\}$. Since p and r have disjoint support, we can find the desired q by Corollary 4.5.20.

Assume otherwise that $i^*(\alpha) \in \operatorname{supp} p$, so let $r_{\mathrm{pr}} := p|\{i^*(\alpha)\}$ and use Lemma 4.5.10 to find some $r \in \mathbb{Q}$ with support $\{i^*(\alpha), \alpha\}$ whose pr-part is r_{pr}. Again, the subatoms of r at (α, ℓ) for $\ell > \operatorname{trnklg}(r) = \operatorname{trnklg}(p)$ are the largest possible and halving parameters are 0 if $\alpha \in S^{\mathrm{lc}}$, so in this case we can modify r by changing the halving parameters as before, and still obtain a condition in $\mathbb{Q}$ with the same support. Since p and r are the same in $\operatorname{supp} p \cap \operatorname{supp} r = \{i^*(\alpha)\}$, we can find the desired q by Corollary 4.5.20. □

As a consequence of Corollary 4.5.20, we can calculate a chain condition for our forcing.

Lemma 4.5.22. *The poset $\mathbb{Q}$ has the $\mathfrak{c}^+$-cc. In particular, under* CH*, $\mathbb{Q}$ has the $\aleph_2$-cc.*

Proof. Let $A = \{p_\xi : \xi < \mathfrak{c}^+\}$ be a collection of conditions in $\mathbb{Q}$. Note that the set of conditions with all halving parameters in the rationals is dense, so we may assume that each p_ξ satisfies this property.

By the Δ-system lemma applied to A, we can find $C \subseteq \mathfrak{c}^+$ of size $\mathfrak{c}^+$ such that $\{\operatorname{supp} p_\xi : \xi \in C\}$ forms a Δ-system with root $D \subseteq S^*$. We can even shrink C in such a way that:

(i) for $n < \omega$, there is some $f_n^{\mathrm{pr}} : D \cap S^{\mathrm{pr}} \to 2$ such that f_n^{pr} coincides with the characteristic function of $\operatorname{supp} p_{\xi,\mathrm{pr}}(n) \cap D$ in $D \cap S^{\mathrm{pr}}$ for all $\xi \in C$ (this implies that $\{\operatorname{supp} p_{\xi,\mathrm{pr}}(n) : \xi \in C\}$ forms a Δ-system with root $D_n^{\mathrm{pr}} := (f_n^{\mathrm{pr}})^{-1}[\{1\}]$, and that $\operatorname{supp} p_{\xi,\mathrm{pr}}(n) \smallsetminus D_n^{\mathrm{pr}} \subseteq S^{\mathrm{pr}} \smallsetminus D$ for all $\xi \in C$);

(ii) for $n < \omega$, there is some $f_n^{\mathrm{lc}} : D \cap S^{\mathrm{lc}} \to 2$ such that f_n^{lc} coincides with the characteristic function of $\operatorname{supp} p_\xi(L_n^{\mathrm{lc}}) \cap D$ in $D \cap S^{\mathrm{lc}}$ for all $\xi \in C$ (this implies that $\{\operatorname{supp} p_\xi(L_n^{\mathrm{lc}}) : \xi \in C\}$ forms a Δ-system with root $D_n^{\mathrm{lc}} := (f_n^{\mathrm{lc}})^{-1}[\{1\}]$, and that $\operatorname{supp} p_\xi(L_n^{\mathrm{lc}}) \smallsetminus D_n^{\mathrm{lc}} \subseteq S^{\mathrm{lc}} \smallsetminus D$ for all $\xi \in C$);

(iii) the halving parameters at all levels are the same for all p_ξ with $\xi \in C$;

(iv) there is some $r \in \mathbb{Q}_D$ such that $p_\xi | D = r$ for all $\xi \in C$.

By Corollary 4.5.20, p_ξ and p_η are compatible for all $\xi, \eta \in C$. Therefore A is not an antichain. $\square$

We finally establish how the generic reals are defined from the generic set.

Definition 4.5.23. Let G be a $\mathbb{Q}$-generic over the ground model V. Denote by $\dot{y}$ the $\mathbb{Q}$-name of

$$\{((\alpha,\ell),z) : (\alpha,\ell) \in \mathrm{DOM} \text{ and } \exists p \in G\colon \alpha \in \operatorname{supp} p,\ \operatorname{trnklg}(p) > \ell \text{ and } p(\alpha,\ell) = \{z\}\}.$$

For each $\mathrm{t} \in \{\mathrm{pr}, \mathrm{al}, \mathrm{lc}\}$ and each $\alpha \in S^{\mathrm{t}}$ let $\dot{y}_\alpha$ be a $\mathbb{Q}$-name of $\{(\ell,z) : ((\alpha,\ell),z) \in \dot{y}\}$.

As in Subsection 4.3.2 and in view of Lemma 4.5.25, we define the following $\mathbb{Q}$-names: For $i \in S^{\mathrm{pr}}$ and $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$, use Definition 4.3.2 to define:

(1) $\dot{y}_i^*$ is a $\mathbb{Q}$-name of the member of $\prod_{n<\omega} T_n^*$ defined by $\dot{y}_i^*(n) := \langle \dot{y}_i(L) : L \in \mathrm{ht}_n^{\mathrm{pr}} \rangle$;

(2) $\dot{d}_i$ is a $\mathbb{Q}$-name of $d_{\dot{y}_i^*}$; $\dot{h}_i$, $\dot{b}_i$, $\dot{g}_i$ and $\dot{a}_i$ are defined likewise.

(3) $\dot{b}_i^-$ is a $\mathbb{Q}$-name of $b_{\dot{y}_i^*}^-$, and $\dot{h}_i^-$ is defined similarly.

(4) $\dot{d}_i^{\mathrm{t}}$ is a $\mathbb{Q}$-name of $d_{\dot{y}_i^*}^{\mathrm{t}}$; g_i^{t}, h_i^{t}, b_i^{t}, f_i^{t} and a_i^{t} are defined similarly.

(5) $\dot{\vec{\nu}}_i^{\mathrm{t}}$ is a $\mathbb{Q}$-name of $\vec{\nu}_{\dot{y}_i^*}$.

(6) For $\alpha \in S_i^{\mathrm{lc}}$, $\dot{\varphi}_\alpha$ is a $\mathbb{Q}$-name of a function with domain ω such that $\dot{\varphi}_\alpha(n) := \dot{b}_i^{\mathrm{lc}}(n) \smallsetminus \prod_{i \in I_n^*}(\dot{b}_i(\ell) \smallsetminus \dot{y}_\alpha(\ell))$.

(7) For $\alpha \in S_i^{\mathrm{al}}$, $\dot{\varphi}_\alpha$ is a $\mathbb{Q}$-name of a function with domain ω such that $\dot{\varphi}_\alpha(n) := y_\alpha(L_n^{\mathrm{al}})$.

We remove the dots when these names are evaluated in any generic extension.

The following results show the type of reals added by $\mathbb{Q}$.

Fact 4.5.24. *If $p \in \mathbb{Q}$, $L \in \mathrm{Ht}$ and $\eta \in \mathrm{poss}(p, <L)$ then $p \wedge \eta$ forces that $\dot{y}$ is a function extending η.*

Lemma 4.5.25. *Assume that G is a $\mathbb{Q}$-generic over V and fix $i \in S^{\mathrm{pr}}$. Then $\mathbb{Q}$ forces that:*

(a) $\dot{y}$ is a function with domain DOM.

(b) $\dot{y}_i$ is a function in $\prod_{L \in \mathrm{ht}^{\mathrm{pr}}} T^*_L$, thus $\dot{y}^*_i$ is in $\prod_{n<\omega} T^*_n$ and both $\dot{\vec{\nu}}^{\mathrm{lc}}_i$ and $\dot{\vec{\nu}}^{\mathrm{al}}_i$ are blocks.

(c) For each $\alpha \in S^{\mathrm{lc}}_i$, $\dot{y}_\alpha$ is in $\mathcal{S}^*(\dot{b}^-_i, \dot{h}^-_i)$ (thus $\dot{\varphi}_\alpha$ is a slalom in $\mathcal{S}^*(\dot{b}^{\mathrm{al}}_i, \dot{h}^{\mathrm{al}}_i)$ by the proof of Lemma 4.2.7).

(d) For each $\alpha \in S^{\mathrm{al}}_i$, $\dot{\varphi}_\alpha$ is a slalom in $\mathcal{S}^*(\dot{a}^{\mathrm{al}}_i, \dot{h}^{\mathrm{al}}_i)$.

Proof. (a): Clearly, $\mathbb{Q}$ forces that $\dot{y}$ is a function with domain contained in DOM. To show equality it is enough to prove that, for any $(\alpha, \ell) \in \mathrm{DOM}$, the set $D_{\alpha,\ell} := \{p \in \mathbb{Q} : \alpha \in \mathrm{supp} p,\ \mathrm{trnklg}(p) > \ell\}$ is dense. If $p \in \mathbb{Q}$, by Corollary 4.5.21 there is some $q \leq p$ in $\mathbb{Q}$ such that $\alpha \in \mathrm{supp} q$. Choosing $n > \ell$ in ω and $\eta \in \mathrm{poss}(q, <n)$, we get $q \wedge \eta \in D_{\alpha,\ell}$ stronger than q.
(b): Let $p \in \mathbb{Q}$ and $L \in \mathrm{ht}^{\mathrm{pr}}$, and choose $q \in D_{i,L}$ stronger than p. So $\mathrm{poss}(q, <\mathrm{trnklg}(1))$ only contains one possibility η and, by Fact 4.5.13(a) and Fact 4.5.24, q forces $\dot{y}_i(L) = \eta(i, L) \in T^*_L$. The rest is clear by Lemma 4.3.3.
(c): It is enough to show that, for any $p \in \mathbb{Q}$, $\alpha \in \mathrm{supp} p_{\mathrm{lc}}$ and $\ell \in \mathrm{ht}^{\mathrm{lc}}$, p forces that $\dot{y}_\alpha(\ell) \subseteq b_{t^{\mathrm{mx}}_n}(\ell)$ has size $\leq h_{t^{\mathrm{mx}}_n}(\ell)$ where $n := n_*(\ell)$, and even more, whenever $\ell \geq \mathrm{trnklg}(p, \alpha)$, p forces that $\dot{y}_\alpha(\ell) \subseteq \dot{b}_i(\ell)$ has size $\leq \dot{h}_i(\ell)$. Pick some $n' > \ell$ and let $\eta \in \mathrm{poss}(p, <n')$. By Fact 4.5.13(b) and 4.5.24, $p \wedge \eta$ forces $\dot{y}_\alpha(\ell) = \eta(\alpha, \ell) \subseteq b_{t^{\mathrm{mx}}_n}(\ell)$ of size $\leq h_{t^{\mathrm{mx}}_n}(\ell)$ and, whenever $\ell \geq \mathrm{trnklg}(p, \alpha)$, $\dot{y}_\alpha(\ell) = \eta(\alpha, \ell) \subseteq b_{\hat{\eta}(\alpha,\ell)}(\ell)$ has size $\leq h_{\hat{\eta}(\alpha,\ell)}(\ell)$. But note that $p \wedge \eta$ forces $\hat{\eta}(\alpha, \ell) = y^*_i(n)$, so $p \wedge \eta$ forces the desired conclusion. By Fact 4.5.15(d) p forces the same.
(d): Similar to (c) we can show that, for any $p \in \mathbb{Q}$, $\alpha \in \mathrm{supp} p_{\mathrm{lc}}$ and $\ell \in \mathrm{ht}^{\mathrm{al}}$, p forces that $\dot{y}_\alpha(\ell) \subseteq a_{t^{\mathrm{mx}}_n}(\ell)$ has size $\leq h_{t^{\mathrm{mx}}_n}(\ell)$ where $n := n_*(\ell)$, and even more, whenever $\ell \geq \mathrm{trnklg}(p, \alpha)$, p forces that $\dot{y}_\alpha(\ell) \subseteq \dot{a}_i(\ell)$ has size $\leq \dot{h}_i(\ell)$. □

4.6 Bigness

The notion of bigness is a tool in forcing with creatures that allows to homogenize the decisions made by a condition or a creature, like deciding the name of an ordinal. This is not only essential for proving that $\mathbb{Q}$ is proper, but also for the proof of the main theorem.

Bigness is described for subatomic creatures as follows.

Definition 4.6.1. Let $\varepsilon > 0$ be a real number, $B \in \omega$ and let K be a subatomic family.

(1) A subatom $\mathbf{c} \in \mathrm{K}$ has (B, ε)-*bigness* if for each function $F\colon \mathbf{c} \to B$ there is a $\mathbf{d} \subseteq \mathbf{c}$ in $\mathbf{K}$ such that $F{\restriction}\mathbf{d}$ is constant and $\|\mathbf{d}\| \geq \|\mathbf{c}\| - \varepsilon$.

(2) We say that K has (B, ε)-*bigness* if each $\mathbf{c} \in \mathrm{K}$ has (B, ε)-bigness.

One of the reasons we built the parameters as in Subsection 4.3.3 is to be able to calculate the bigness of the subatomic creatures in our forcing.

Lemma 4.6.2. *Let $\ell \in \mathrm{ht}$.*

(a) If $\ell \in \mathrm{ht}^{\mathrm{pr}}$ then K_ℓ has $(n^B_\ell, \frac{1}{n^B_\ell})$-bigness.

(b) If $\ell \in \mathrm{ht}^{\mathrm{lc}} \cup \mathrm{ht}^{\mathrm{al}}$ and $t \in T^*_{n_*(\ell)}$ then K^t_ℓ has $(d_t(\ell), \frac{1}{d_t(\ell)})$-bigness.

Proof. To see (a), let $\mathbf{c} \in K_\ell$ and $F\colon \mathbf{c} \to n_\ell^B$. Find $\mathbf{d} \subseteq \mathbf{c}$ with $|\mathbf{d}| \geq \frac{|\mathbf{c}|}{n_\ell^B}$ such that $F{\restriction}\mathbf{d}$ is constant (in fact, $\mathbf{d}$ is the $F^{-1}[\{k\}]$ of largest size). Hence $\|\mathbf{d}\|_\ell = \frac{1}{n_\ell^B} \log_{n_\ell^B} |\mathbf{d}| \geq \frac{1}{n_\ell^B}(\log_{n_\ell^B} \frac{|\mathbf{c}|}{n_\ell^B}) = \frac{1}{n_\ell^B}(\log_{n_\ell^B} |\mathbf{c}| - \log_{n_\ell^B} n_\ell^B) = \|\mathbf{c}\|_n^{\mathrm{pr}} - \frac{1}{n_\ell^B}$.

For (b), see [KM21, Lemma 3.10]. □

We also describe and calculate the bigness of a compound creature. Bigness here homogenize functions with domain the set of possibilities of the compound, but note that possibilities on the lc-compounds also depend on the pr-indices. Inspired in the proof of the main result in [KS12] (where continuum many $\mathfrak{d}_{b,h}^{\mathrm{Lc}}$ and $\mathfrak{b}_{b,h}^{\mathrm{aLc}}$ are separated) we state our bigness results for compound creatures, which are essential to separate the many cardinal characteristics in the proof of the main theorem. We start with lc-compounds, here we use the lexicographic order of $T_n^* \times I_n^*$ presented in Subsection 4.3.3.

Lemma 4.6.3. *Let $n < \omega$, $\mathbf{c}$ a lc-compound creature at L_n^{lc}, $\bar{t} = \mathrm{seq}t_\alpha\alpha \in \mathrm{supp}\mathbf{c} \in \prod_{\alpha\in\mathrm{supp}\mathbf{c}} P_{\mathbf{c},\alpha}$, $t_0 \in T_n^*$, and let $\ell_0 \in I_n^*$. Assume $M \in \omega$, $f\colon \mathrm{poss}(\mathbf{c},\bar{t}) \to M$ and $M^{m^S_{t_0,\ell_0}} \leq d_{t'_0}(\ell'_0)$, the latter whenever (t'_0,ℓ'_0) is the successor of (t_0,ℓ_0) in $T_n^* \times I_n^*$ w.r.t. the lexicographic order. Then there is a lc-compound creature $\mathbf{c}' \leq \mathbf{c}$ with the same domain, halving parameter and same pr-indices such that, for any $\alpha \in \mathrm{supp}\mathbf{c}$, $\ell \in I_n^*$ and $t \in P_{\mathbf{c},\alpha}$:*

(i) if $t = t_\alpha$ and $(t_\alpha,\ell) > (t_0,\ell_0)$ then $\|\mathbf{c}'(\alpha,t,\ell)\|_\ell^t \geq \|\mathbf{c}(\alpha,t,\ell)\|_\ell^t - \frac{1}{d_t(\ell)}$;

(ii) otherwise $\mathbf{c}'(\alpha,t,\ell) = \mathbf{c}(\alpha,t,\ell)$;

(iii) $f{\restriction}\mathrm{poss}(\mathbf{c}',\bar{t})$ only depends on $C := \{(\alpha,\ell) : (t_\alpha,\ell) \leq (t_0,\ell_0)\}$, i. e. if $\eta,\eta' \in \mathrm{poss}(\mathbf{c}',\bar{t})$ coincide in C then $f(\eta) = f(\eta')$.

Proof. When (t_0,ℓ_0) is the maximum of $T_n^* \times I_n^*$, we can set $\mathbf{c}' := \mathbf{c}$. So assume that (t_0,ℓ_0) is not the maximum of $T_n^* \times I_n^*$. Note that $\{\eta|_{\mathrm{dom}\mathbf{c}\smallsetminus\mathrm{supp}\mathbf{c}} : \eta \in \mathrm{poss}(\mathbf{c},\bar{t})\}$ has only one element, which we denote by η_-. Define $F\colon \prod_{(\alpha,\ell)\notin C} \mathbf{c}(\alpha,t_\alpha,\ell) \to M^{\prod_{(\alpha,\ell)\in C}\mathbf{c}(\alpha,t_\alpha,\ell)}$ such that $F(\eta_0)(\eta_1) := f(\eta_0 \cup \eta_1 \cup \eta_-)$. Note that $\prod_{(\alpha,\ell)\in C} |\mathbf{c}(\alpha,t_\alpha,\ell)| \leq m^S_{t_0,\ell_0}$ by modesty, (pr9) and Fact 4.4.6, so $|\mathrm{ran}F| \leq M^{m^S_{t_0,\ell_0}} \leq d_{t'_0}(\ell'_0)$ by hypothesis. In this way, we just need to find $\mathbf{c}'$ satisfying (i) and (ii) and such that F is constant on $\prod_{(\alpha,\ell)\notin C} \mathbf{c}'(\alpha,t_\alpha,\ell)$.

Consider the increasing enumeration $\{(t'_k,\ell'_k) : k < m\}$ of all $(t,\ell) \in T_n^* \times I_n^*$ larger than (t_0,ℓ_0) such that, for some $\alpha \in \mathrm{supp}\mathbf{c}$, $t_\alpha = t$ and $\mathbf{c}(\alpha,t,\ell)$ is non-trivial. By strong modesty, such α is unique, so we denote by α_k the one corresponding to (t'_k,ℓ'_k). Moreover, the non-trivial creatures in $\{\mathbf{c}(\alpha,t_\alpha,\ell) : (\alpha,\ell) \notin C\}$ are precisely $\{\mathbf{c}(\alpha_k,t'_k,\ell'_k) : k < m\}$, hence F only depends on $\{(\alpha_k,\ell'_k) : k < m\}$. So we can define a function $F'\colon \prod_{k<m} \mathbf{c}(\alpha_k,t'_k,\ell'_k) \to \mathrm{ran}F$ that determines F.

Define $\langle \mathbf{c}'(\alpha_k,t'_k,\ell'_k) : k < m\rangle$ by decreasing induction on k, in such a way that:

(k-i) $\mathbf{c}'(\alpha_k,t'_k,\ell'_k) \subseteq \mathbf{c}(\alpha_k,t'_k,\ell'_k)$, $\|\mathbf{c}'(\alpha_k,t'_k,\ell'_k)\| \geq \|\mathbf{c}(\alpha,t'_k,\ell'_k)\| - \frac{1}{d_{t'_k}(\ell'_k)}$ and

(k-ii) $F'{\restriction}\prod_{k'<k} \mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'}) \times \prod_{k'\geq k} \mathbf{c}'(\alpha_{k'},t'_{k'},\ell'_{k'})$ only depends on $\prod_{k'<k} \mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'})$.

At step k, by induction hypothesis, $F'{\restriction}\prod_{k'\leq k} \mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'}) \times \prod_{k'>k} \mathbf{c}'(\alpha_{k'},t'_{k'},\ell'_{k'})$ only depends on $\prod_{k'\leq k} \mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'})$. So we can define $F_k\colon \mathbf{c}(\alpha_k,t'_k,\ell'_k) \to (\mathrm{ran}F)^{\prod_{k'<k}\mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'})}$ such that $F_k(i_k)(\langle i_{k'} : k' < k\rangle)$ is the value of F' calculated from $\langle i_{k'} : k' \leq k\rangle$.

We claim that $\left|(\mathrm{ran}F)^{\prod_{k'<k}\mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'})}\right| \leq d_{t'_k}(\ell'_k)$: the case $k = 0$ is $|\mathrm{ran}F| \leq d_{t'_0}(\ell'_0)$, which is already checked; if $k > 0$ then, by (pr9),

$$\left|(\mathrm{ran}F)^{\prod_{k'<k}\mathbf{c}(\alpha_{k'},t'_{k'},\ell'_{k'})}\right| \leq |\mathrm{ran}F|^{m^S_{t'_{k-1},\ell'_{k-1}}} \leq d_{t'_0}(\ell'_0)^{m^S_{t'_{k-1},\ell'_{k-1}}} \leq d_{t'_k}(\ell'_k).$$

So we can apply Lemma 4.6.2(b) to find $\mathbf{c}'(\alpha_k, t'_k, \ell'_k) \leq \mathbf{c}(\alpha_k, t'_k, \ell'_k)$ in $\mathbf{K}^{t'_k}_{\ell'_k}$ such that $F_k{\restriction}\mathbf{c}'(\alpha_k, t'_k, \ell'_k)$ is constant and $\|\mathbf{c}'(\alpha_k, t'_k, \ell'_k)\|^{t'_k}_{\ell'_k} \geq \|\mathbf{c}(\alpha_k, t'_k, \ell'_k)\|^{t'_k}_{\ell'_k} - \frac{1}{d_{t'_k}(\ell'_k)}$. It is clear that both (k-i) and (k-ii) hold.

Define the other components of $\mathbf{c}'$ identical to $\mathbf{c}$. Therefore, $\mathbf{c}'$ is as required. □

Corollary 4.6.4. *Let $n < \omega$, $\mathbf{c}$ be a* lc*-compound creature at L^{lc}_n, $\bar{t} = \langle t_\alpha : \alpha \in \operatorname{supp}\mathbf{c}\rangle \in \prod_{\alpha\in\operatorname{supp}\mathbf{c}} P_{\mathbf{c},\alpha}$ and let $f\colon \operatorname{poss}(\mathbf{c}, \bar{t}) \to n^B_{L^{\mathrm{lc}}_n}$. Then there is an* lc*-compound creature $\mathbf{c}' \leq \mathbf{c}$ with the same domain, halving parameter and same pr-indices, such that, for any $\alpha \in \operatorname{supp}\mathbf{c}$, $\ell \in I^*_n$ and $t \in P_{\mathbf{c},\alpha}$:*

(i) If $t = t_\alpha$ then $\|\mathbf{c}'(\alpha, t, \ell)\|^t_\ell \geq \|\mathbf{c}(\alpha, t, \ell)\|^t_\ell - \frac{1}{d_t(\ell)}$;

(ii) otherwise $\mathbf{c}'(\alpha, t, \ell) = \mathbf{c}(\alpha, t, \ell)$; and

(iii) $f{\restriction}\operatorname{poss}(\mathbf{c}', \bar{t})$ is constant.

Proof. Let $(t_0, \ell_0) := \min(T^*_n \times I^*_n)$. Since $n^B_{L^{\mathrm{lc}}_n} \leq d_{t_0}(\ell_0) < b_{t_0}(\ell_0)$ and $b_{t_0}(\ell_0)^{m^S_{t_0,\ell_0}} \leq d_{t'_0}(\ell'_0)$ by (pr9), where (t'_0, ℓ'_0) is the successor of (t_0, ℓ_0) in $T^*_n \times I^*_n$, we can obtain $\mathbf{c}^1 \leq \mathbf{c}$ as in Lemma 4.6.3, so $f{\restriction}\operatorname{poss}(\mathbf{c}^1, \bar{t})$ depends only on $\{(\alpha, \ell_0) : \alpha \in \operatorname{supp}\mathbf{c}^1,\ t_\alpha = t_0\}$, and note that $\mathbf{c}^1(\bar{t})$ and $\mathbf{c}(\bar{t})$ coincide in those coordinates. If this set is empty we are done, otherwise choose $\alpha_0 \in \operatorname{supp}\mathbf{c}$ such that $t_{\alpha_0} = t_0$ and $\mathbf{c}(\alpha_0, t_0, \ell_0)$ is non-trivial (if it exists, in which case it is unique by modesty). Hence $f{\restriction}\operatorname{poss}(\mathbf{c}^1, \bar{t})$ depends only on $\{(\alpha_0, \ell_0)\}$, i. e. it can be reconstructed from some function $f_0\colon \mathbf{c}(\alpha_0, t_0, \ell_0) \to n^B_{L^{\mathrm{lc}}_n}$. By (pr10) $n^B_{L^{\mathrm{lc}}_n} \leq d_{t_0}(\ell_0)$, so Lemma 4.6.2(b) implies that there is some $\mathbf{c}'(\alpha_0, t_0, \ell_0) \leq \mathbf{c}(\alpha_0, t_0, \ell_0)$ such that $\|\mathbf{c}'(\alpha_0, t_0, \ell_0)\|^{t_0}_{\ell_0} \geq \|\mathbf{c}(\alpha_0, t_0, \ell_0)\|^{t_0}_{\ell_0} - \frac{1}{d_{t_0}(\ell_0)}$ and $f_0{\restriction}\mathbf{c}'(\alpha_0, t_0, \ell_0)$ is constant. If the other components of $\mathbf{c}'$ are defined identical to those in $\mathbf{c}^1$, then $\mathbf{c}'$ is as required. □

A similar (and simpler) argument using Lemma 4.6.2(a), (pr2) and (pr4) allows us to calculate the bigness of pr-compound creatures.

Lemma 4.6.5. *Let $\mathbf{c}$ be a pr-compound at $(n, 0)$, $L_0 \in \mathrm{ht}^{\mathrm{pr}}_n$, $M \in \omega$ and let $f\colon \operatorname{poss}(\mathbf{c}) \to M$ such that $M^{\prod\{T^*_L : L\in\mathrm{ht}^{\mathrm{pr}}_n\cap L_0\downarrow\}} \leq n^B_{L_0}$. Then there is a pr-compound $\mathbf{c}' \leq \mathbf{c}$ with the same domain and halving parameter such that, for any $i \in \operatorname{supp}\mathbf{c}$ and $L \in \mathrm{ht}^{\mathrm{pr}}_n$:*

(i) if $L \geq L_0$ then $\|\mathbf{c}'(i, L)\| \geq \|\mathbf{c}(i, L)\| - \frac{1}{n^B_L}$;

(ii) otherwise $\mathbf{c}'(i, L) = \mathbf{c}(i, L)$;

(iii) for $\eta \in \operatorname{poss}(\mathbf{c}')$, $f(\eta)$ only depends on $\eta{\restriction}\operatorname{supp}\mathbf{c} \times L_0{\downarrow}$.

In particular, if $L_0 = (n, 0)$ and $M \leq n^B_{L_0}$ then $f{\restriction}\operatorname{poss}(\mathbf{c}')$ is constant.

Proof. We only check that, for $L' \geq L_0$ in $\mathrm{ht}^{\mathrm{pr}}_n$, $M^{\prod\{T^*_L : L\in\mathrm{ht}^{\mathrm{pr}}_n\cap L'\downarrow\}} \leq n^B_{L'}$ (the rest follows similarly to the proof of Lemma 4.6.3). We proceed by induction on L'. The case $L' = L_0$ holds from the assumption; if L'' is the successor of L' in $\mathrm{ht}^{\mathrm{pr}}_n$ then

$$M^{\prod\{T^*_L : L\in\mathrm{ht}^{\mathrm{pr}}_n\cap L''\downarrow\}} \leq (n^B_{L'})^{T^*_{L'}} < (n^P_{<L''})^{n^P_{<L''}} < n^B_{L''},$$

where the first "<" holds by (pr4), and the second by (pr2). □

4.7 Continuous and rapid reading

This section is dedicated to the continuous reading of names, whose proof gives us fundamental properties of $\mathbb{Q}$ like properness and ω^ω-bounding. We also define a notion of rapid reading of names.

Definition 4.7.1. Let $B \subseteq S^*$, $\dot{\tau}$ a $\mathbb{Q}$-name of a set in the ground model V, and let $p \in \mathbb{Q}$.

(1) For $L \in \mathrm{Ht}$, say that $\dot{\tau}$ is *L-decided by p*, if $p \wedge \eta$ decides the values of $\dot{\tau}$ for each $\eta \in \mathrm{poss}(p, {<}L)$; in other words, there is some map T in the ground model with domain $\mathrm{poss}(p, {<}L)$ such that $p \wedge \eta \Vdash \dot{\tau} = T(\eta)$ for all $\eta \in \mathrm{poss}(p, {<}L)$.

(2) In (1), say that $\dot{\tau}$ is L-decided by p *only using indices in B* if the value $T(\eta)$ as in (1) only depends on $\eta|_B$ for all $\eta \in \mathrm{poss}(p, {<}L)$, i. e. if $\eta' \in \mathrm{poss}(p, {<}L)$ coincides with η in $B \times L{\downarrow}$ then both $p \wedge \eta$ and $p \wedge \eta'$ decide the same value of $\dot{\tau}$.

(3) We say that p *essentially decides $\dot{\tau}$ (only using indices in B)*, if $\dot{\tau}$ is L-decided by p (only using indices in B) for some $L \in \mathrm{Ht}$.

(4) Assume that $p \Vdash \dot{r} \colon X \to V$ where $X \in V$. We say *p continuously reads $\dot{r}$ (only using indices in B)* if p essentially decides each $\dot{r}(z)$ (only using indices in B) for all $z \in X$.

(5) Assume that $p \Vdash \dot{r} \colon \mathrm{Ht} \to V$. We say that *$p$ rapidly reads $\dot{r}$ (only using indices in B)* if, for each $L \in \mathrm{Ht}$, $\dot{r} \restriction L{\downarrow}$ is L-decided (only using indices in B).

(6) In the previous notions we can alternatively define deciding and reading *without using indices in B* meaning *only using indices in $S^* \setminus B$*.

One of the main results of this section is:

Theorem 4.7.2. *The forcing $\mathbb{Q}$ is proper, ω^ω-bounding and it has continuous reading of names, i. e. if X be a countable set (in the ground model) and $\dot{r}$ is a $\mathbb{Q}$-name of a function from X into the ground model, then the set of conditions continuously reading $\dot{r}$ is dense.*

The extensive proof of this theorem is presented in Subsection 4.7.2. Before then we discuss some consequences and related properties, e. g. that for some $\dot{r}$ we actually get that the set of conditions rapidly reading $\dot{r}$ is dense (Corollary 4.7.9). Preservation of cardinals also follows under CH by Lemma 4.5.22.

Corollary 4.7.3. *Under* CH*, $\mathbb{Q}$ preserves all cofinalities and cardinalities.*

For $p, q \in \mathbb{Q}$, $q \leq^* p$ usually denotes that any condition in $\mathbb{Q}$ stronger than q is compatible with p. This is equivalent to the fact that q forces p inside the generic set. We write $p =^* q$ when $p \leq^* q$ and $q \leq^* p$.

Lemma 4.7.4. *Let $p, q \in \mathbb{Q}$, $\dot{\tau}$ a $\mathbb{Q}$-name of a set in the ground model V, $X \in V$ and let $\dot{r}$ be a $\mathbb{Q}$-name of a function from X into V. If $q \leq^* p$ and $\mathrm{supp} q \subseteq \mathrm{supp} p$ then:*

(a) If $L \in \mathrm{Ht}$ and $\dot{\tau}$ is L-decided by p then it is also L-decided by q.

(b) If p continuously reads $\dot{r}$ then q continuously reads $\dot{r}$.

(c) If $X = \mathrm{Ht}$ and p rapidly reads $\dot{r}$, then q rapidly reads $\dot{r}$.

The same results hold when adding "only (or without) using indices in B" for any $B \subseteq S^*$.

Proof. To see (a): Since $\dot{\tau}$ is L-decided by p, there is a map T as in Definition 4.7.1(1), i. e. $p \wedge \eta \Vdash \dot{\tau} = T(\eta)$ for any $\eta \in \mathrm{poss}(p, {<}L)$. Note that $\mathrm{poss}(q|_{\mathrm{supp} p}, {<}L) \subseteq \mathrm{poss}(p, {<}L)$ because $q \leq^* p$ and $\mathrm{supp} p \subseteq \mathrm{supp} q$. If $\eta \in \mathrm{poss}(q, {<}L)$ then $\eta' := \eta|_{\mathrm{supp} p} \in \mathrm{poss}(p, {<}L)$, so $q \wedge \eta \leq^* p \wedge \eta'$ and it follows that $q \wedge \eta \Vdash \dot{\tau} = T(\eta')$.

(b) and (c) are immediate consequences of (a). □

A note about the previous lemma: [FGKS17, Lemma 5.1.3] and [GK21, Lemma 6.4] claim that, whenever $q \leq^* p$ and $\dot{\tau}$ is essentially decided by p, then it is so by p, without assuming $\mathrm{supp} q \subseteq \mathrm{supp} p$. However, in our construction this is not true without further assuming $\mathrm{supp} q \subseteq \mathrm{supp} p$. For example, let $i_0 \in S^{\mathrm{pr}}$ and let $p_0 \in \mathbb{Q}$ be the condition with $\mathrm{supp} p = \{i_0\}$ and $\mathrm{trnklg}(p_0) = 0$, where the subatom at $L \in \mathrm{ht}^{\mathrm{pr}}$ is the full POSS_L and halving parameters are 0. It is clear that every $q \in \mathbb{Q}$ is compatible with p_0, i. e. $\mathbf{1} \leq^* p_0$. Hence $q \leq^* p_0$ for any $q \in \mathbb{Q}$, but if $i_0 \notin \mathrm{supp} q$ then we could easily construct some name $\dot{\tau}$ of a natural number that could be essentially read by p_0 (e. g. (2,0)-decided) but not by q.

Continuous reading allows to estimate the size of the continuum in any generic extension.

Lemma 4.7.5. *In V, let $B \subseteq S^*$ and set $\kappa := \max\{\aleph_0, |B|\}^{\aleph_0}$. Then, in any $\mathbb{Q}$-generic extension, there are at most κ many reals which are continuously read by some condition in the generic set and only using indices in B. To be more precise, define the following $\mathbb{Q}$-name of a subset of 2^ω:*

$$\dot{R}(B) := \{(\dot{r}, p) \mid \dot{r} \text{ is a (nice) } \mathbb{Q}\text{-name of a real in } 2^\omega, p \in \mathbb{Q} \text{ and } \dot{r} \text{ is continuously read by } p \text{ only using indices in } B\}.$$

Then $\Vdash |\dot{R}(B)| \leq \kappa$, i. e. if G is $\mathbb{Q}$-generic over V then, in $V[G]$, there are at most $|\kappa|$ many reals $x \in 2^\omega$ such that there are $p \in G$ and a $\mathbb{Q}$-name $\dot{r} \in V$ such that $x = \dot{r}[G]$ and $V \models$"p continuously reads $\dot{r}$ only using indices in B".[7]

Proof. Let P be the set of countable partial functions from $\mathrm{DOM} \cap (B \times \mathrm{ht})$ into $H_{\aleph_0}$ (the collection of hereditarily finite sets). Note that $|P| \leq \kappa^{\aleph_0} = \kappa$ (equality when $B \neq \varnothing$). And let E be the set of countable partial functions from P into 2. Hence $|E| \leq |P|^{\aleph_0} \leq \kappa$ and $|E^\omega| \leq \kappa$.

Let G be $\mathbb{Q}$-generic over V. In $V[G]$, if $x \in R := \dot{R}(B)[G]$ then we can choose some pair $(\dot{r}_x, p_x) \in \dot{R}(B)$ with $p_x \in G$, and define $\bar{f}^x := \langle f^x_n : n < \omega\rangle \in E^\omega \cap V$ such that, in V, f^x_n is the map that witnesses essential decision of $\dot{r}_x(n)$ by p_x only using indices in B, more precisely, f^x_n is a function with domain $D^x_n := \{\eta|_B : \eta \in \mathrm{poss}(p_x, {<}m_x(n))\}$ for some $m_x(n) \in \omega$ such that, for any $\eta \in \mathrm{poss}(p_x, {<}m_x(n))$, $p_x \wedge \eta \Vdash \dot{r}(n) = f^x_n(\eta^x_n|_B)$.

In $V[G]$, for any $n < \omega$ there is a unique $\eta^x_n \in \mathrm{poss}(p_x, {<}m_x(n))$ such that $p_x \wedge \eta^x_n \in G$, so $x(n) = \dot{r}_x[G](n) = f^x_n(\eta^x_n|_B)$. This implies that $x \mapsto \bar{f}^x$ is a one-to-one map from R into $E^\omega \cap V$, so $|R| \leq |\kappa|$. In more detail, assume that $x, y \in R$ and $\bar{f}^x = \bar{f}^y$. For $n < \omega$, $D^x_n = D^y_n$, so $\eta^x_n|_B$ and $\eta^y_n|_B$ must have the same domain. On the other hand $p_x \wedge \eta^x_n, p_y \wedge \eta^y_n \in G$, so η^x_n and η^y_n are compatible, hence $\eta^x_n|_B = \eta^y_n|_B$. Therefore $x(n) = f^x_n(\eta^x_n|_B) = f^y_n(\eta^y_n|_B) = y(n)$. □

As a direct consequence of Theorem 4.7.2:

Corollary 4.7.6. *Let $\kappa := \max\{|S^*|, \aleph_0\}^{\aleph_0}$. Then $\mathbb{Q}$ forces that $\mathfrak{c} \leq |\kappa|$.*

4.7.1 Rapid reading

We show that some $\mathbb{Q}$-names of reals allow rapid reading, meaning that (as a consequence of Theorem 4.7.2) the set of conditions rapidly reading it is dense. We start with the following general result that simplifies the proof of rapid reading, which also helps in the proof of continuous reading of names (so Theorem 4.7.2 is not used in its proof).

[7] Note that κ may be an ordinal in $V[G]$, but according to Corollary 4.7.3 it is still a cardinal if CH is assumed in the ground model.

Lemma 4.7.7. *Let $p \in \mathbb{Q}$ be a modest condition and $L_0 \in \omega$.*[8] *Assume that $\langle A_L : L \in \mathrm{Ht},\ L \leq L_0 \rangle$ is a sequence of sets such that*

(i) $|A_L| \leq n^P_{<L}$,

(ii) for $L < L'$ in Ht, $\pi_{L',L} \colon A_{L'} \to A_L$ such that, for any $L < L' < L''$ in Ht, $\pi_{L'',L} = \pi_{L',L} \circ \pi_{L'',L'}$.

If $h_{L_0} \colon \mathrm{poss}(p, {<}L_0) \to A_{L_0}$ then there is some $q \leq p$ in $\mathbb{Q}$ and, for each $L < L_0$ in Ht, there is some function $h_L \colon \mathrm{poss}(p, {<}L) \to A_L$ such that

(a) p and q have the same support, trunk lengths, halving parameters, and they are identical at levels $\geq L_0$;

(b) below L_0, the norm of any subatomic creature from q decreases at most 1 w.r.t. the subatomic creature in p at the same position;

(c) for $L < L_0$ in Ht, if $\eta \in \mathrm{poss}(p, {<}L_0)$ and $\eta(\alpha, \ell) \in q(\alpha, \hat{\eta}(\alpha, n_*(\ell)), \ell))$ for all $\alpha \in \mathrm{supp}\, p$ and $L \leq \ell < L_0$ in ht (when $\alpha \in S^{\mathrm{pr}}$ or $\ell < \mathrm{trnklg}(p, \alpha)$ we abuse of the notation and set $q(\alpha, \hat{\eta}(\alpha, n_*(\ell)), \ell)) := q(\alpha, \ell)$), then $h_L(\eta{\restriction}L{\downarrow}) = \pi_{L_0,L}(h_{L_0}(\eta))$.

Proof. Fix a decreasing enumeration $\{L_j : 1 \leq j < m\}$ of $\mathrm{Ht} \cap L_0{\downarrow}$. Set $q_0 := p$ and $h_0 := h_{\ell_0}$, and denote $A_j := A_{L_j}$ and $\pi_{j,k} := \pi_{L_j,L_k}$ for $j < k < m$. By induction on j we construct $h_j = h_{L_j}$ and $q_j \leq q_{j-1}$ such that

(i) q_{j-1} and q_j have the same support, trunk lengths, halving parameters, and they are identical except at L_j (and their sublevels); however, in the case $L_j = (n, 0)$ for some $n < \omega$, they may also differ at levels $\ell \in I^*_n \cup \{L^{\mathrm{al}}_n\}$ but only in that the set of pr-indices of the atomic creatures in q_j may be smaller than those in q_{j-1} (at the same coordinates);

(ii) at L_j the norm of each subatomic creature from q_j decreases at most 1 w.r.t. the subatomic creature in q_{j-1} at the same position;

(iii) if $\eta \in \mathrm{poss}(p, {<}L_{j-1})$ and $\eta(\alpha, \ell) \in q_j(\alpha, \hat{\eta}(\alpha, n_*(\ell)), \ell))$ for all $\alpha \in \mathrm{supp}\, p$ and $L_j \leq \ell < L_{j-1}$ (abusing of the notation when $\alpha \in S^{\mathrm{pr}}$ or $\ell < \mathrm{trnklg}(p, \alpha)$), then $h_j(\eta{\restriction}L_j{\downarrow}) = \pi_{j-1,j}(h_{L_{j-1}}(\eta))$.

By (i), notice that q_j is identical to p at levels ${<}L_j$ and ${\geq}L_0$.

So assume $1 \leq j < m$ and that we have the desired h_i and q_i for $i < j$. We proceed by cases on the height. When $L_j = L^{\mathrm{al}}_n$ for some $n < \omega$, first assume that there is some $\alpha_j \in \mathrm{supp}\, p$ such that $p(\alpha_j, L_j)$ is non-trivial (which is unique). Recall that $q_{j-1}(\alpha_j, L_j) = p(\alpha_j, L_j)$. Fix $t \in \mathrm{pss}(p_{\mathrm{pr}}, \alpha_j, L_j)$ and denote $B^t := \{\eta' \in \mathrm{poss}(p, {<}L_j) : \hat{\eta}'(\alpha_j, n) = t\}$.

Consider the function $F^t \colon p(\alpha_j, t, L_j) \to {A_j}^{B^t}$ such that $F^t(x) = F^t_x$ maps η to the value $\pi_{j-1,j}(h_{j-1}(\eta' {}^\frown x))$, where $\eta' {}^\frown x$ is the unique possibility $\eta \in \mathrm{poss}(p, {<}L_{j-1})$ extending η' with $\eta(\alpha, L_j) = x$ (all other values at L_j are determined by modesty).

Since both B^t and A_j have size at most $n^P_{<L_j}$ (Lemma 4.5.16), $|{A_j}^{B^t}| \leq (n^P_{<L_j})^{n^P_{<L_j}} < n^B_{L_j}$ by (pr2). Then, by bigness (Lemma 4.6.2(b)), there is some subatom $q_j(\alpha_j, t, L_j) \leq p(\alpha_j, t, L_j)$ such that $\|q_j(\alpha_j, t, L_j)\|^t_{L_j} \geq \|p(\alpha_j, t, L_j)\|^t_{L_j} - 1$ and $F^t{\restriction}q_j(\alpha_j, t, L_j)$ is constant with value $f_t \colon B^t \to A_j$.

For $\eta' \in \mathrm{poss}(p, {<}L_j)$ we define $h_j(\eta') := f_{\hat{\eta}'(\alpha_j, n)}(\eta')$. We can define q_j as in (i) such that $q_j(\alpha_j, L_j) = \langle q_j(\alpha_j, t, L_j) : t \in \mathrm{pss}(p_{\mathrm{pr}}, \alpha_j, L_j) \rangle$ and coinciding with q_{j-1} (and p) at (α', L_j) with $\alpha' \neq \alpha_j$. Hence q_j and h_j are as required. Concretely, if $\eta \in \mathrm{poss}(p, {<}L_{j-1})$ and $\eta(\alpha_j, L_j) \in q_j(\alpha_j, \hat{\eta}(\alpha_j, n), L_j)$ then $h_j(\eta{\restriction}L_j{\downarrow}) = \pi_{j-1,j}(h_{j-1}(\eta))$.

[8] Recall that we identify $n \in \omega$ with the level $(n, 0)$.

In the case when all creatures are trivial at L_j we can just set $q_j := q_{j-1}$ and $h_j(\eta) := \pi_{j-1,j}(h_{j-1}(\eta^+))$ for $\eta \in \mathrm{poss}(p, < L_j)$, where η^+ is the unique possibility in $\mathrm{poss}(p, <L_{j-1})$ extending η.

Now consider the case $L_j = L_n^{\mathrm{lc}}$. Enumerate $\mathrm{poss}(p, <L_j) = \{\eta'_k : k < m'\}$. By recursion, we define a decreasing sequence $\langle \mathbf{c}_k : k \leq m' \rangle$ of lc-compounds, all with the same support, halving parameter and pr-indices, where $\mathbf{c}_0 := p(L_n^{\mathrm{lc}})$. Assume we have defined $\mathbf{c}_k$. Define $f_k \colon \mathrm{poss}(\mathbf{c}_k, \bar{t}^k) \to A_j$ where $\bar{t}^k := \hat{\eta}'_k(L_n^{\mathrm{lc}})$ and $f_k(\nu)$ is the value of $\pi_{j-1,j} \circ h_{j-1}$ evaluated in the unique possibility in $\mathrm{poss}(p, <L_{j-1})$ obtained from η'_k and ν.

By Corollary 4.6.4 applied to f_k, there is some $\mathbf{c}_{k+1} \leq \mathbf{c}_k$ as required, such that each subatom in $\mathbf{c}_{k+1}$ decreases its norm by at most $\frac{1}{n^B_{L_n^{\mathrm{lc}}}}$ and f_k has constant value $h_j(\eta'_k)$ on $\mathrm{poss}(\mathbf{c}_{k+1}, \bar{t}^k)$. This defines h_j, and define q_j such that $q_j(L_n^{\mathrm{lc}}) := \mathbf{c}_{m'}$ and identical to q_{j-1} at other levels. It is routine to check that they are as required.

In the case $L_j = (n, 0)$ construct $(q_j)_{\mathrm{pr}}(n)$ as in the previous case (which is simpler because the parameter t is not required), but use bigness from Lemma 4.6.5. However, since $(q_j)_{\mathrm{pr}}(n)$ may be different from $(q_{j-1})_{\mathrm{pr}}(n) = p_{\mathrm{pr}}(n)$, we must also reduce the set of pr-indices of the atoms of q_{j-1} at levels $\ell \in I_n^* \cup \{L_n^{\mathrm{al}}\}$ (as indicated in (i)). So define $q_{j,\mathrm{pr}} \in \mathbb{Q}_{\mathrm{pr}}$ identical to $(q_{j-1})_{\mathrm{pr}}$ except at L_j where $q_{j,\mathrm{pr}}(n)$ is the constructed $(q_j)_{\mathrm{pr}}(n)$, and set $q_j := q_{j-1} \wedge q_{j,\mathrm{pr}}$. By Observation 4.5.18, q_j is as required.

We have defined h_j for all $j < m$. Set $q := q_{m-1}$, which is clearly as required. □

Theorem 4.7.8. *Let $\dot{r}$ be a $\mathbb{Q}$-name of a member of $\prod_{L \in \mathrm{Ht}} n_L^S$. Assume that $p \in \mathbb{Q}$ is modest and continuously read $\dot{r}$. Then there is a $q \leq p$ with the same support, trunk lengths and halving parameters as p, that rapidly reads $\dot{r}$.*

Proof. For each $L \in \mathrm{Ht}$, set

$$h(L) := \max\{L' \in \mathrm{Ht} : L' \leq L,\ p\ L\text{-decides } \dot{r}{\restriction}L'{\downarrow}\} \tag{$*_1$}$$

Note that $\langle h(L) : L \in \mathrm{Ht} \rangle$ is non-decreasing, and continuous reading implies that h is an unbounded function. For $L' \leq L$ let $\dot{x}_{L,L'}$ be a $\mathbb{Q}$-name of $\dot{r}{\restriction}\min\{h(L), L'\}{\downarrow}$. Note that $\dot{x}_{L,L'}$ is L-decided and that there are at most $|\prod_{L<L'} n_L^S|$-many possibilities for $\dot{x}_{L,L'}$.

Fix $L_0 \in \omega$. For all $L \leq L_0$ in Ht define

$$A_{L_0,L} := \prod\{n_{L'}^S : L' < \min\{h(L_0), L\},\ L' \in \mathrm{Ht}\}$$

and $\psi_{L_0,L_0} \colon \mathrm{poss}(p, <L_0) \to A_{L_0,L_0}$ such that $p \wedge \eta \Vdash \dot{x}_{L_0,L_0} = \psi_{L_0,L_0}(\eta)$. By (pr14) and Lemma 4.7.7 applied to these objects and to the projections $\pi_{L,L'} \colon A_{L_0,L} \to A_{L_0,L'}$ for $L' < L \leq L_0$, find $q_{L_0} \leq p$ and $\psi_{L_0,L} \colon \mathrm{poss}(p, <L) \to A_{L_0,L}$ for each $L < L_0$ in Ht that satisfies:

(i) p and q_{L_0} have the same support, trunk lengths, halving parameters, and they are identical at levels $\geq L_0$;

(ii) below L_0, any subatomic creature from q_{L_0} decreases at most 1 w.r.t. the subatomic creature in p at the same position;

(iii) for $L < L_0$ in Ht, if $\eta \in \mathrm{poss}(p, <L_0)$ and $\eta(\alpha, \ell) \in q_{L_0}(\alpha, \hat{\eta}(\alpha, n_*(\ell)), \ell)$ for all $\alpha \in \mathrm{supp}\, p$ and $L \leq \ell < L_0$ in ht (allowing abuse of notation), then $p \wedge \eta \Vdash \dot{x}_{L_0,L} = \psi_{L_0,L}(\eta{\restriction}L{\downarrow})$.

In (iii) we say that $\eta{\restriction}L{\downarrow}$ *decides* $\dot{x}_{L_0,L}$ $(= \psi_{L_0,L}(\eta{\restriction}L{\downarrow}))$ *modulo* $q_{L_0}{\restriction}[L, L_0)$.

Note that, given $L_0 \in \omega$, there are only finitely many possibilities for $q_{L_0}{\restriction}L_0{\downarrow}$ (restriction on ht) and $\psi_{L_0,L}$ for $L \leq L_0$ in Ht. Thus, by Konig's Lemma, there is some $q \leq p$ in $\mathbb{Q}$ and a sequence $\langle \psi_L^* : L \in \mathrm{Ht} \rangle$ such that

($*_2$) for all $L \in \mathrm{Ht}$, there is some $L_0 \geq L$ in ω such that q_{L_0} and q are identical below L and $\psi_{L_0,L'} = \psi^*_{L'}$ for all $L' \leq L$ in Ht.

Here q is constructed in the natural way, and it is clear that any subatom of q decreases its norm by at most 1 w.r.t. the subatom in p at the same position, hence $q \in \mathbb{Q}$. Note that q has the same support, trunk lengths, and halving parameters as p, and $q \leq p$.

To complete the proof it remains to show that q rapidly reads $\dot{r}$, that is, each $\eta \in \mathrm{poss}(q, {<}L)$ decides $\dot{r}{\restriction}L$ for all $L \in \mathrm{Ht}$. Fix $L \in \mathrm{Ht}$ and choose an $L' \geq L$ in Ht such that $h(L') \geq L$. According to ($*_1$), we get that $\dot{r}{\restriction}L$ is L'-decided. Choose $L_0 \geq L'$ as in ($*_2$) and recall (from (iii)) that $\dot{x}_{L_0,L}$ is decided by $\psi_{L_0,L}$ modulo $q_{L_0}{\restriction}[L, L_0)$. Notice that $L \leq h(L') \leq L' \leq L_0$, so $\min\{h(L_0), L\} = L$. Therefore $\Vdash \dot{x}_{L_0,L} = \dot{r}{\restriction}L{\downarrow}$ and, since $h(L') \geq L$, $\dot{x}_{L_0,L}$ is already L'-decided (by the original condition p). So we can decide $\dot{x}_{L_0,L}$ modulo $q_{L_0}{\restriction}[L, L')$ using $\psi_{L_0,L}$.

On the other hand, q and q_{L_0} coincide below L' and $\psi_{L',L} = \psi_{L_0,L} = \psi^*_{L'}$, so $q \wedge \eta$ forces $\dot{r}{\restriction}L = \dot{x}_{L_0,L} = \psi^*_L(\eta)$ for any $\eta \in \mathrm{poss}(q, {<}L)$. □

As a direct consequence of Lemma 4.5.8, 4.7.4 and Theorem 4.7.2 we obtain:

Corollary 4.7.9. *If $\dot{r}$ is as in Theorem 4.7.8 then the set of modest conditions rapidly reading $\dot{r}$ is dense in $\mathbb{Q}$.*

A consequence of rapid reading is that $\mathbb{Q}$ does not add random reals.

Lemma 4.7.10. *The forcing $\mathbb{Q}$ does not add random reals.*
In particular, under CH, *$\mathbb{Q}$ forces $\mathrm{cov}(\mathcal{N}) = \aleph_1$.*

Proof. Let $\dot{r}$ be a $\mathbb{Q}$-name for a real in 2^ω and $p \in \mathbb{Q}$. For $k < \omega$ set $n_k = \lfloor \log_2 n^S_{(k,0)} \rfloor$, and define $\dot{f} \colon \mathrm{Ht} \to \omega$ by

$$\dot{f}(L) = \begin{cases} \dot{r}{\restriction}n_k & \text{if } L = (k,0), \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 4.7.9 there is some modest $q \leq p$ that rapidly reads $\dot{f}$. Then $\dot{r}{\restriction}n_k$ is $(k,0)$-decided by q, that is, $\dot{r}{\restriction}n_k$ is determined by $\eta \in \mathrm{poss}(q, {<}k)$. We then denote by A^q_k the set of possible values of $\dot{r}{\restriction}n_k$ determined by $\mathrm{poss}(q, {<}k)$, so $\frac{|A^q_k|}{2^{n_k}} \leq \frac{n^P_{<k}}{2^{n_k}}$ by Lemma 4.5.16. The sequence $\langle A^q_k : k < \omega \rangle$ allows us to define $N := \{x \in 2^\omega : \forall k < \omega\colon x{\restriction}n_k \in A^q_k\}$ (in the ground model). It is clear from the definition of N that $\mathbf{Lb}(N) = \lim_{k\to\infty} \frac{|A^q_k|}{2^{n_k}} \leq \lim_{k\to\infty} \frac{n^P_{<k}}{2^{n_k}} = 0$ (where $\mathbf{Lb}$ denotes the Lebesgue measure) and $q \Vdash \dot{r} \in N$. □

4.7.2 The proof of continuous reading and more

In this subsection we prove Theorem 4.7.2 and present more features of the forcing, like fusion. This is the only place of the paper where the halving parameters are really used. This presentation is based on [GK21, Section 7].

Remember that the norm of an lc-compound $\mathbf{c}$ with non-empty support is defined by

$$\|\mathbf{c}\|^{\mathrm{lc}}_n := \frac{\log_2(\max\{1, \min\{\|\mathbf{c}(\alpha, t_0)\|^{t_0}_{\mathrm{stk}} : \alpha \in \mathrm{supp}\,\mathbf{c},\ t_0 \in P_{\mathbf{c},\alpha}\} - d_{\mathbf{c}}\})}{n^P_{<L^{\mathrm{lc}}_n}}.$$

Set $D_{\mathbf{c}} := \min\{\|\mathbf{c}(\alpha, t_0)\|^{t_0}_{\mathrm{stk}} : \alpha \in \mathrm{supp}\,\mathbf{c},\ t_0 \in P_{\mathbf{c},\alpha}\}$. If we change $d_{\mathbf{c}}$ to

$$d'_{\mathbf{c}} := d_{\mathbf{c}} + \frac{D_{\mathbf{c}} - d_{\mathbf{c}}}{2} = \frac{D_{\mathbf{c}} + d_{\mathbf{c}}}{2}$$

then the norm of the resulting lc-compound decreases by at most $1/n^P_{<L^{\mathrm{lc}}_n}$. We call this procedure *halving*, which also applies to pr-compound creatures.

Definition 4.7.11 ([FGKS17, Definition 5.2.2])**.** Given any pr or lc compound creature $\mathbf{c}$ we define $\mathrm{half}(\mathbf{c})$, *the half of* $\mathbf{c}$, to be the same creature as $\mathbf{c}$, except that we replace each halving parameter $d_{\mathbf{c}}$ by the $d'_{\mathbf{c}}$ described above. In case $\mathrm{supp}\mathbf{c} = 0$ we set $d'_{\mathbf{c}} = d_{\mathbf{c}} = 0$, i. e. $\mathrm{half}(\mathbf{c}) = \mathbf{c}$.

In this way we can also halve conditions.

Definition 4.7.12. Given a condition $q \in \mathbb{Q}$ and $m < \omega$, we define $r := \mathrm{half}(q, m)$ to be the same condition q, except that all compound creatures $q_{\mathrm{pr}}(n)$ and $q(L^{\mathrm{lc}}_n)$ for $n \geq m$ are halved, i. e. $r_{\mathrm{pr}}(n) = \mathrm{half}(q_{\mathrm{pr}}(n))$ and $r(L^{\mathrm{lc}}_n) = \mathrm{half}(q(L^{\mathrm{lc}}_n))$.

We also have a procedure to *unhalve* conditions, illustrated in the following result.

Lemma 4.7.13. *Let $M \in \mathbb{R}$, $p \in \mathbb{Q}$ and $N \in \omega$, $N \geq \mathrm{trnklg}(p)$ such that all compound creatures at levels $\geq N$ have norm $\geq M$. Assume that $r \leq \mathrm{half}(p, N)$ in $\mathbb{Q}$ such that $\mathrm{trnklg}(r) = N$ and all compound creatures at levels $\geq N$ have norm >0. Then there is a condition $q \leq p$ and $N^* > N$ such that*

(i) q is identical to r except of the halving parameters of the compound creatures at heights in $[(N, 0), (N^*, 0))$,

(ii) $\|q_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \geq M$ for $n \geq N^*$,

(iii) $\|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq M$ for $n \geq N^*$,

(iv) $\|q_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \geq M - 1/n^P_{<(n,0)}$ for all $N \leq n < N^*$,

(v) $\|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq M - 1/n^P_{<L^{\mathrm{lc}}_n}$ for all $N \leq n < N^*$,

In addition $q =^* r$, so by Lemma 4.7.4 r essentially decides some $\mathbb{Q}$-name $\dot{\tau}$ of a ground model object iff q does.

Proof. Choose $N^* > N$ such that $\|r_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \geq M$ and $\|r(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq M$ for $n \geq N^*$. We set q to be identical to r except that, for all $N \leq n < N^*$, we replace the halving parameters $d_{r_{\mathrm{pr}}(n)}$ and $d_{r(L^{\mathrm{lc}}_n)}$ by $d_{p_{\mathrm{pr}}(n)}$ and $d_{p(L^{\mathrm{lc}}_n)}$, respectively, that is, $d_{q_{\mathrm{pr}}(n)} := d_{p_{\mathrm{pr}}(n)}$ and $d_{q(L^{\mathrm{lc}}_n)} := d_{p(L^{\mathrm{lc}}_n)}$. It is clear that $r \leq q$ and $q \leq^* r$, so $q =^* r$.

It is straightforward to see that $q \leq p$ and (i)–(iii) hold. We only show (v) (since (iv) is similar). Assume $N \leq n < N^*$. If $\mathrm{supp} q(L^{\mathrm{lc}}_n) = \varnothing$ then $q(L^{\mathrm{lc}}_n) = r(L^{\mathrm{lc}}_n)$ and $p(L^{\mathrm{lc}}_n)$ has empty support, too. But $\|r(L^{\mathrm{lc}}_n)\| > 0$, so $\mathrm{supp} r_{\mathrm{lc}} = \varnothing$, which implies that $\mathrm{supp} p_{\mathrm{lc}} = \varnothing$ and $\|q(L^{\mathrm{lc}}_n)\| = \|r(L^{\mathrm{lc}}_n)\| = \|p(L^{\mathrm{lc}}_n)\| = n \geq M$.

So, assuming $\mathrm{supp} q(L^{\mathrm{lc}}_n) \neq \varnothing$, it remains show that

$$\begin{aligned}\|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n &= \frac{\log_2(\min\{\|q(\alpha, t_0)\|^{t_0}_{\mathrm{stk}} : (\alpha, t_0) \in Q_n\} - d_{q(L^{\mathrm{lc}}_n)})}{n^P_{<L^{\mathrm{lc}}_n}} \\ &\geq M - \frac{1}{n^P_{<L^{\mathrm{lc}}_n}}\end{aligned}$$

where $Q_n = \{(\alpha, t) \in \mathrm{supp} q(L^*_n) \times T^*_n : t \in \mathrm{pss}(q, \alpha, L^{\mathrm{lc}}_n)\}$ (since $\|r(L^{\mathrm{lc}}_n)\| > 0$, the term inside the logarithm is already >1).

Recall that $0 < \|r(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \leq \|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n$. Fix any $(\alpha, t_0) \in Q_n$. Since q and r only differ on the halving parameter, we have

$$0 < \log_2(\|q(\alpha, t_0)\|^{t_0}_{\mathrm{stk}} - d_{r(L^{\mathrm{lc}}_n)}),$$

which implies that

$$\begin{aligned}\|q(\alpha,t_0)\|^{t_0}_{\text{stk}} &> d_{r(L^{\text{lc}}_n)} + 1\\ &\geq d_{\text{half}(p(L^{\text{lc}}_n))} + 1\\ &= d_{p(L^{\text{lc}}_n)} + \frac{D_{p(L^{\text{lc}}_n)} - d_{p(L^{\text{lc}}_n)}}{2} + 1.\end{aligned}$$

Since $d_{q(L^{\text{lc}}_n)} = d_{p(L^{\text{lc}}_n)}$,

$$\|q(\alpha,t_0)\|^{t_0}_{\text{stk}} - d_{q(L^{\text{lc}}_n)} > \frac{D_{p(L^{\text{lc}}_n)} - d_{p(L^{\text{lc}}_n)}}{2} + 1$$

for any $(\alpha, t_0) \in Q_n$, hence

$$\frac{\log_2(\|q(\alpha,t_0)\|^{t_0}_{\text{stk}} - d_{q(L^{\text{lc}}_n)})}{n^P_{<(L^{\text{lc}}_n,0)}} \geq \|p(L^{\text{lc}}_n))\|^{\text{lc}}_n - \frac{1}{n^P_{<L^{\text{lc}}_n}}$$

Therefore,

$$\begin{aligned}\|q(L^{\text{lc}}_n)\|^{\text{lc}}_n &\geq \|p(L^{\text{lc}}_n)\|^{\text{lc}}_n - \frac{1}{n^P_{<L^{\text{lc}}_n}}\\ &\geq M - \frac{1}{n^P_{<L^{\text{lc}}_n}},\end{aligned}$$

which proves (iv). □

Before engaging in the proof of continuous reading, as in [GK21, Section 7] we present some orders that give us a variation of Baumgartner's strong axiom A for our forcing $\mathbb{Q}$, from which properness, ω^ω-bounding and continuous reading of names follow.

Definition 4.7.14. (1) Let $N < \omega$. Define the order $\leq_N$ on $\mathbb{Q}$ by $q \leq_N p$ iff the following properties hold:

(i) $\text{trnklg}(q) = \text{trnklg}(p) \leq N$,

(ii) $q \leq p$,

(iii) at each level $<N$ (excluding ht^{al}), the compound creature of both conditions have the same support and halving parameter (although $\text{supp} q$ could be larger than $\text{supp} p$),

(iv) $\text{trnklg}(q, \alpha) \geq N$ for $\alpha \in \text{supp} q_{\text{al}} \setminus \text{supp} p_{\text{al}}$,

(v) $q(\alpha, \ell) = p(\alpha, \ell)$ for any $(\alpha, \ell) \in \text{DOM}$ with $\alpha \in \text{supp} p$ and $\ell < (N, 0)$.

Properties (i) and (ii) imply that $\text{trnklg}(q, \alpha) = \text{trnklg}(p, \alpha)$ for all $\alpha \in \text{supp} p_{\text{al}}$ (by the definition of the order of $\mathbb{Q}$).

(2) Let $n < \omega$ and let $F \subseteq S^{\text{al}}$ be finite. Define the order $\leq_{n,F}$ on $\mathbb{Q}$ by $q \leq_{n,F} p$ if there is some $n' \geq n$ such that

(i) $q \leq_{n'} p$,

(ii) every compound creature at any level $\geq (n', 0)$ has norm $>n$,

(iii) for each $\alpha \in F \cap \text{supp} p$ there is some $\ell < (n', 0)$ in ht^{al} such that $\|p(\alpha, \ell)\| > n$.

(3) Say that a sequence $\bar{p} = \langle p_n : n < \omega \rangle$ of elements of $\mathbb{Q}$ is a *fusion sequence* if there is some sequence $\bar{F} = \langle F_n : n < \omega \rangle$ of finite subsets of S^{al} such that

(F1) $F_n \subseteq F_{n+1}$,

(F2) $p_{n+1} \leq_{n,F_n} p_n$,

(F3) $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \operatorname{supp}(p_n)_{\mathrm{al}}$.

Here we say that $\bar{F}$ *witnesses the fusion sequence* $\bar{p}$.

It is easy to check that $\leq_{n,F}$ is a preorder on $\mathbb{Q}$.[9] Also, whenever $n_0 \leq n$ and $F_0 \subseteq F$, $q \leq_{n,F} p$ implies $q \leq_{n_0,F_0} p$; and whenever $\operatorname{trnklg}(p) \leq n_0$, $q \leq_n p$ implies $q \leq_{n_0} p$. These properties of the preorders $\leq_{n,F}$, along with *fusion* (Lemma 4.7.15) and *pure decision* (Lemma 4.7.16) is what defines our variation of the strong axiom A.

We can always construct a *fusion condition* from a fusion sequence.

Lemma 4.7.15. *If $\langle p_n : n < \omega\rangle$ is a fusion sequence witnessed by $\bar{F} = \langle F_n : n < \omega\rangle$ then there is a condition $q \in \mathbb{Q}$ satisfying $\operatorname{supp} q = \bigcup_{n<\omega} \operatorname{supp} p_n$ and $q \leq_{n,F_n} p_n$ for all $n < \omega$.*

Proof. For each n let $h(n)$ be like n' in Definition 4.7.14 (2) for $p_{n+1} \leq_{n,F_n} p_n$. Choose an increasing sequence $\langle n_j : j < \omega\rangle$ of natural numbers with $n_0 = 0$ such that $\langle h(n_j) : j < \omega\rangle$ is strictly increasing. Set $\operatorname{trnklg}(q) := \operatorname{trnklg}(p_0)$, $\operatorname{supp} q := \bigcup_{n<\omega} \operatorname{supp} p_n$ and, for $\alpha \in \operatorname{supp} q_{\mathrm{al}}$, $\operatorname{trnklg}(q,\alpha) := \operatorname{trnklg}(p_{n_j},\alpha)$ for some j such that $\alpha \in \operatorname{supp} p_{n_j}$ (this value does not depend on the chosen j).

Fix $j < \omega$ and define q at heights $\ell \in \mathrm{ht}_m$ for $h(n_{j-1}) \leq m < h(n_j)$ as follows (with $h(n_{-1}) := 0$). Set $\operatorname{supp} q_{\mathrm{pr}}(m) := \operatorname{supp}(p_{n_j})_{\mathrm{pr}}(m)$, $\operatorname{supp} q(L^{\mathrm{lc}}_m) := \operatorname{supp} p_{n_j}(L^{\mathrm{lc}}_m)$, $d_{q_{\mathrm{pr}}(m)} := d_{(p_{n_j})_{\mathrm{pr}}(m)}$, $d_{q(L^{\mathrm{lc}}_m)} = d_{p_{n_j}(L^{\mathrm{lc}}_m)}$, and $q(\alpha,\ell) := p_{n_j}(\alpha,\ell)$ for all $\alpha \in \operatorname{supp} p_{n_j}$. Note that all these objects are the same when j is replaced by any $j' > j$. In the case $\alpha \in \operatorname{supp} q \smallsetminus \operatorname{supp} p_{n_j}$, set $q(\alpha,\ell) := p_{n_{j'}}(\alpha,\ell)$ when $\alpha \in \operatorname{supp} p_{n_{j'}}$ (this value does not depend on the chosen j').

Routine calculations give $q \in \mathbb{Q}$ and $q \leq_{n_j,F_{n_j}} p_{n_j}$ for all $j < \omega$. If $n \leq n_j$ then $p_{n_j} \leq_{n,F_n} p_n$, so $q \leq_{n,F_n} p_n$. □

The following lemma concludes our variation of strong axiom A.

Lemma 4.7.16 (Pure decision). *Assume that $\dot{\tau}$ is a $\mathbb{Q}$-name for a ground model object, $p \in \mathbb{Q}_{S_0}$, $N_0 \in \omega$ and $M_0 > 1$ is a real such that any compound creature of p at any level $\geq N_0$ has norm $\geq M_0 + 1$ (so $\operatorname{trnklg}(p) \leq N_0$). Then there is some $q \in \mathbb{Q}$ such that*

(i) $q \leq_{N_0} p$,

(ii) q essentially decides $\dot{\tau}$,

(iii) any compound creature of q at any level $\geq N_0$ has norm $\geq M_0$.

In particular, for any $n < \omega$ and finite $F \subseteq \operatorname{supp} p_{\mathrm{al}}$, there is some $q \leq_{n,F} p$ in $\mathbb{Q}$ essentially deciding $\dot{\tau}$.

Proof. This proof consist of three parts:

Part 1: Halving, the single step.

Suppose that we are given $p \in \mathbb{Q}$, $N \in \omega$ and $M \in \mathbb{R}$ such that $N \geq \operatorname{trnklg}(p)$ and any compound creature of p at any level $\geq N$ has norm $\geq M + 1$. We show how to construct a condition $r := r(p, N, M) \in \mathbb{Q}$ satifying:

(H1) $r \leq_N p$,

(H2) any compound creature of r at any level $\geq N$ has norm $\geq M$, and

[9] Although the order $\leq_N$ is transitive, reflexivity fails for conditions with trunk length larger than N.

(H3) if $\eta \in \text{poss}(r, <N)$ and if there is an $s \leq r \wedge \eta$ such that s essentially decides $\dot{\tau}$, $\text{trnklg}(s) = (N, 0)$ and the compound creatures of s at any level $\geq N$ have norm >0, then $r \wedge \eta$ already essentially decides $\dot{\tau}$.

To see this, first enumerate $\text{poss}(p, <N)$ as $\eta^1, \dots, \eta^m$. Set $q^0 := p$. By induction on m we construct conditions $\hat{q}^1, q^1, \dots, \hat{q}^m, q^m$ such that, for each $j < m$:

(1) $\hat{q}^{j+1}$ is derived from q^j by replacing the creature at any $(\alpha, \ell) \in \text{DOM} \cap (\text{supp}p \times (N, 0)\downarrow)$ by the trivial creature $\{\eta^{j+1}(\alpha, \ell)\}$.

(2) $q^{j+1} \leq \hat{q}^{j+1}$.

(3) $\text{trnklg}(\hat{q}^{j+1}) = \text{trnklg}(q^{j+1}) = N$.

(4) $\|q^{j+1}_{\text{pr}}(n)\|^{\text{pr}}_n \geq M + 1 - \frac{j+1}{n^P_{<(n,0)}}$ for $n \geq N$.

(5) $\|q^{j+1}(L^{\text{lc}}_n)\|^{\text{lc}}_n \geq M + 1 - \frac{j+1}{n^P_{<L^{\text{lc}}_n}}$ for $n \geq N$.

(6) One of the following two cases holds:

- q^{j+1} essentially decides $\dot{\tau}$
- $q^{j+1} = \text{half}(\hat{q}^{j+1}, N)$.

Note that $\hat{q}^1$ is just $p \wedge \eta^1$, but q^1 may have support larger than $\text{supp}p$, so we can not write $q^j \wedge \eta^{j+1}$ in (1). The definition of $\hat{q}^{j+1}$ is correct because $\text{trnklg}(q^j) = N$, so $q^j(\alpha, \ell)$ is trivial for (α, ℓ) in $\text{DOM} \times (\text{supp}q^j \times (N, 0)\downarrow)$ with $\alpha \notin \text{supp}p$. The condition q^{j+1} is easy to get: if there is some q^{j+1} essentially deciding $\dot{\tau}$ and satisfying (2)–(5) we just choose it, otherwise we set $q^{j+1} = \text{half}(\hat{q}^{j+1}, N)$ (which clearly satisfies (2)–(6)). We will always give priority to choosing an essentially deciding q^{j+1} over halving $\hat{q}^{j+1}$.

Set r as follows: In $\text{DOM} \cap (\text{supp}p \times (N, 0)\downarrow)$, r is identical to p; otherwise r is identical to q^m. In more detail:

- $\text{supp}r := \text{supp}q^m$;
- $\text{trnklg}(r) = \text{trnklg}(p)$, $\text{trnklg}(r, \alpha) := \text{trnklg}(p, \alpha)$ for any $\alpha \in \text{supp}p_{\text{al}}$, and $\text{trnklg}(r, \alpha) := \text{trnklg}(q_m, \alpha)$ for any $\alpha \in \text{supp}r_{\text{al}} \setminus \text{supp}p_{\text{al}}$ (so it is $\geq N$);
- for $(\alpha, \ell) \in \text{DOM}$ with $\ell < N$:
  - if $\alpha \in \text{supp}p$ then $r(\alpha, \ell) := p(\alpha, \ell)$,
  - if $\alpha \in \text{supp}r \setminus \text{supp}p$ then $r(\alpha, \ell) := q^m(\alpha, \ell)$;
- for $(\alpha, \ell) \in \text{DOM}$ with $\alpha \in \text{supp}r$ and $\ell \geq N$, $r(\alpha, \ell) := q^m(\alpha, \ell)$;
- the supports and halving parameters of compound creatures in r coincide with p at levels $<N$, and with q^m at levels $\geq N$.

It is clear by the construction that r satisfies (H1) and (H2). To see (H2) note that, for $n \geq N$, $\|r(L^{\text{lc}}_n)\| = \|q_m(L^{\text{lc}}_n)\| \geq M + 1 - \frac{m}{n^P_{<L^{\text{lc}}_n}} \geq M$, the last equality because $m \leq n^P_{<L^{\text{lc}}_n}$ by Lemma 4.5.16.

So it remains to prove (H3). Let $\eta \in \text{poss}(r, <N)$ and s as in (H3). Note that η extends some $\eta^j \in \text{poss}(p, <N)$, so $s \leq r \wedge \eta \leq q^j \leq \hat{q}^j$. It suffices to show that q^j was constructed using the "decision" case. Assume towards a contradiction that "halving" was used. Then s is

stronger than $\mathrm{half}(\hat{q}^j, N)$, so we can use Lemma 4.7.13 and unhalve s to get some $s' \le \hat{q}^j$, with $\|s'_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \ge M + 1 - \frac{j}{n^P_{<(n,0)}}$ and $\|s'(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \ge M + 1 - \frac{j}{n^P_{<L^{\mathrm{lc}}_n}}$ for $n \ge N$, such that $s' =^* s$. By Lemma 4.7.4, we could have used the "decision" case after all by setting $q^j := s'$, which is a contradiction.

Part 2: Iterating the single step.

Given p, N_0 and M_0 as in the lemma's statement, we inductively construct conditions p_j and natural numbers N_j for $j < \omega$, considering we start with N_0. Define $p_0 := r(p, N_0, M_0)$. Given p_j and N_j, define p_{j+1} and N_{j+1} as follows:

- Choose $N_{j+1} > N_j$ such that
  - $\|(p_j)_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \ge M_0 + j + 2$ for $n \ge N_{j+1}$,
  - $\|p_j(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \ge M_0 + j + 2$ for $n \ge N_{j+1}$,
  - for all $m, i \le j$ there is some $\ell \in \mathrm{ht}^{\mathrm{al}}$ with $(N_j, 0) < \ell < (N_{j+1}, 0)$ such that $\|p_j(\alpha_{m,i}, \ell)\| \ge M_0 + j + 1$, where $\mathrm{supp}_{\mathrm{al}}(p_j) := \{\alpha_{j,i} : i < \omega\}$.
- Set $p_{j+1} := r(p_j, N_{j+1}, M_0 + j + 1)$.

Thus $\langle p_j : j < \omega\rangle$ is a fusion Squence, which converges to a condition $q \in \mathbb{Q}$ constructed as in the proof of Lemma 4.7.15 (with $n_j = j$, $h(n_j) = N_j$ and $F_n = \{\alpha_{j,i} : j, i \le n\}$). By the construction, $\|q_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n \ge M_0 + j$ and $\|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \ge M_0 + j$ for $n \ge N_j$. On the other hand $\mathrm{supp}_{\mathrm{al}} q = \bigcup_{j<\omega} \mathrm{supp}_{\mathrm{al}} p_j = \{\alpha_{m,i} : m, i < \omega\}$ so, for any $m, i < \omega$, if $j := \max\{m, i\}$ then there is an $\ell \in \mathrm{ht}^{\mathrm{al}}$ with $(N_j, 0) < \ell < (N_{j+1}, 0)$ such that $\|q(\alpha_{m,i}, \ell)\| = \|p_j(\alpha_{m,i}, \ell)\| \ge M_0 + j + 1$. This guarantees $p_{j+1} \le_{N_j} p_j$ and $p_{j+1} \le_{j,F_j} p_j$ as well as properties (i) and (iii) of the lemma.

It remains to show essential decision for q. The following property will be crucial for this proof.

($\star_1$) If $j < \omega$, $\eta \in \mathrm{poss}(q, {<}N_j)$, $r \le q \wedge \eta$ essentially decides $\dot\tau$, $\mathrm{trnklg}(r) = N_j$, $\|r_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n > 0$ and $\|r(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n > 0$ for all $n \ge N_j$, then $q \wedge \eta$ already essentially decides $\dot\tau$.

Let j, η and r be as in ($\star_1$). Then $r \le q \wedge \eta \le p_j \wedge \eta'$ for some $\eta' \in \mathrm{poss}(p_j, {<}N_j)$ by Fact 4.5.15(c), hence by (H3) $p_j \wedge \eta'$ already essentially decides $\dot\tau$, and so does $q \wedge \eta$ by Lemma 4.7.4.

Part 3: Bigness, thinning out.

We use Lemma 4.7.7 to homogenize on whether $q \wedge \eta$ essentially decides $\dot\tau$ or not. For $L_0 \in \omega$ and $L \le L_0$ in Ht set $A_{L_0,L} := 2$ and $h_{L_0,L_0} \colon \mathrm{poss}(q, {<}L_0) \to 2$ such that $h_{L_0,L_0}(\eta) = 1$ if $q \wedge \eta$ essentially decides $\dot\tau$. Let B_{L_0,L_0} be the set of $\eta \in \mathrm{poss}(q, {<}L_0)$ such that $h_{L_0,L_0}(\eta) = 1$. By Lemma 4.7.7 applied to these objects and to the identity functions $\pi_{L,L'} \colon A_{L_0,L} \to A_{L_0,L'}$ for $L' < L \le L_0$ in Ht, find $q_{L_0} \le q$ and, for $L < L_0$ in Ht, a function $h_{L_0,L} \colon \mathrm{poss}(q, {<}L) \to 2$ that satisfies:

(i) q_{L_0} and q have the same support, trunk lengths, halving parameters, and they are identical at any level $\ge L_0$;

(ii) below L_0, any subatomic creature from q_{L_0} decreases at most 1 w.r.t. the subatomic creature in q at the same position;

(iii) for $L < L_0$ in Ht, if $\eta \in \mathrm{poss}(q, {<}L_0)$ and $\eta(\alpha, \ell) \in q_{L_0}(\alpha, \hat\eta(\alpha, n_*(\ell)), \ell)$ for all $\alpha \in \mathrm{supp} q$ and $L \le \ell < L_0$ (allowing abuse of notation), then $h_{L_0,L}(\eta{\restriction}L{\downarrow}) = h_{L_0,L_0}(\eta)$.

Set $B_{L_0,L} := \{\eta \in \mathrm{poss}(q, {<}L) : h_{L_0,L}(\eta) = 1\}$ for $L \le L_0$ in Ht. Hence (iii) says that, for η as in there, $\eta{\restriction}L{\downarrow} \in B_{L_0,L}$ iff $q \wedge \eta$ essentially decides $\dot\tau$.

Given $L \le L_0$ and $\eta \in \mathrm{poss}(q, {<}L)$, if $q \wedge \eta$ essentially decides $\dot\tau$ and $\eta' \in \mathrm{poss}(q, {<}L_0)$ extends η, then $q \wedge \eta'$ essentially decides $\dot\tau$, too. Therefore, by (iii):

($\star_2$) If $L \in \mathrm{Ht}$, $\eta \in \mathrm{poss}(q, {<}L)$ and $q \wedge \eta$ essentially decides $\dot\tau$ then $\eta \in B_{L_0,L}$ for any $L_0 \geq L$ in ω.

The converse holds (in some sense) as follows.

($\star_3$) For $j < \omega$ and $L_0 \in \omega$, if $L_0 \geq N_j$ and $\eta \in B_{L_0,N_j}$ then $q \wedge \eta$ essentially decides $\dot\tau$.

To see this, find a condition r identical to q_{L_0} at levels in $[N_j, L_0)$, and identical to $q \wedge \eta$ at the remaining levels. Any $\eta' \in \mathrm{poss}(r, {<}L_0) \subseteq \mathrm{poss}(q, {<}L_0)$ is in B_{L_0,L_0} by (iii), so $q \wedge \eta'$ essentially decides $\dot\tau$, and so does $r \wedge \eta'$. Hence r essentially decides $\dot\tau$. Also $\|r_{\mathrm{pr}}(n)\|^{\mathrm{pr}}_n > 0$ and $\|r(L^{\mathrm{lc}}_{n^*})\|^{\mathrm{lc}}_{n^*} > 0$ for all $n \geq N_j$ (because $M_0 - 1 > 0$), so by ($\star_1$) $q \wedge \eta$ essentially decides $\dot\tau$.

Note that q essentially decides $\dot\tau$ iff, for some $L \in \mathrm{Ht}$, $q \wedge \eta$ essentially decides $\dot\tau$ for all $\eta \in \mathrm{poss}(q, {<}L)$. So, by ($\star_3$), it suffices to prove that, for all $\eta \in \mathrm{poss}(q, {<}N_0)$, there is some $L_0 \geq N_0$ in ω such that $\eta \in B_{L_0,N_0}$.

As in Theorem 4.7.8, since for fixed $L_0 \in \omega$ there are only finitely many possibilities for $q_{L_0}{\restriction}L_0{\downarrow}$ and $B_{L_0,L}$ for $L \leq L_0$ in Ht, by König's Lemma there is some $q^* \leq q$ and some sequence $\langle B^*_L : L \in \mathrm{Ht},\ L \geq N_0\rangle$ such that, for any $L \geq N_0$ in Ht, there is some $L_0 \geq L$ in ω such that q^* is identical to q below N_0 and identical to q_{L_0} at levels in $[N_0, L]$, and $B^*_{L'} = B_{L_0,L'}$ for all $N_0 \leq L' \leq L$ in Ht.

Fix $\eta \in \mathrm{poss}(q, {<}N_0)$. Find any $r \leq q^* \wedge \eta$ deciding $\dot\tau$. Without loss of generality, for some m, $\mathrm{trnklg}(r) = N_m$, $\|r(n)\|^{\mathrm{pr}}_n \geq 1$ and $\|r(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq 1$ for all $n \geq N_m$. Let η' be the unique possibility in $\mathrm{poss}(r, {<}N_m)$ restricted to $\mathrm{supp}(q)$, which ensures $\eta' \in \mathrm{poss}(q, {<}N_m)$ and $r \leq q \wedge \eta'$. So by ($\star_1$), $q \wedge \eta'$ already essentially decides $\dot\tau$.

Pick some $L_0 \geq N_m$ in ω such that q^* and q_{L_0} are identical below N_m and $B^*_L = B_{L_0,L}$ for all $N_0 \leq L \leq N_m$ in Ht. According to ($\star_2$), $\eta' \in B_{L_0,N_m} = B^*_{N_m}$, so $\eta = \eta'{\restriction}N_0{\downarrow} \in B^*_{N_0} = B_{L_0,N_0}$ because $\eta'(\alpha, \ell) \in q^*(\alpha, \hat\eta(\alpha, n_*(\ell)), \ell) = q_{L_0}(\alpha, \hat\eta(\alpha, n_*(\ell)), \ell)$ for all $\alpha \in \mathrm{supp} q$ and $\ell \in [N_0, N_m)$ in ht. □

We can finally prove Theorem 4.7.2 using pure decision. This theorem is reformulated as follows.

Theorem 4.7.17. *Let $\dot r$ be a $\mathbb{Q}$-name of a function from ω into the ground model.*

(a) For any $p \in \mathbb{Q}$ there is some fusion sequence $\langle p_n : n < \omega\rangle$ such that $p_0 \leq p$ and each p_n essentially decides $\dot r(n)$.

(b) The set of conditions continuously reading $\dot r$ is dense in $\mathbb{Q}$.

(c) $\mathbb{Q}$ is ω^ω-bounding,

(d) $\mathbb{Q}$ is proper.

Proof. (a): By induction on $n < \omega$, we want to find a sequence $\bar F = \langle F_n : n < \omega\rangle$ of finite subsets of S^{al} such that

(i) $F_n \subseteq F_{n+1}$,

(ii) $p_{n+1} \leq_{n,F_n} p_n$,

(iii) $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \mathrm{supp}(p_n)_{\mathrm{al}}$.

(iv) p_n essentially decides $\dot r(n)$.

Find any $p_0 \leq p$ in $\mathbb{Q}$ deciding $\dot{r}(0)$. Now assume that p_n have been constructed and set $\operatorname{supp}(p_m)_{\mathrm{al}} = \{\alpha_{m,j} : j < \omega\}$ for all $m \leq n$. Next, define $F_n := \{\alpha_{i,j} : i, j \leq n\}$. By application of Lemma 4.7.16, we can find $p_{n+1} \leq_{n,F_n} p_n$ essentially deciding $\dot{r}(n+1)$.

(b),(c): Given $p \in \mathbb{Q}$, by (a) choose a fusion sequence $\langle p_n : n < \omega \rangle$ such that $p_0 \leq p$ and each p_n essentially decides $\dot{r}(n)$. Next, by Lemma 4.7.15, there is a $q \in \mathbb{Q}_{S_0}$ satisfying $\operatorname{supp} q = \bigcup_{n<\omega} F_n$ and $q \leq_{n,F_n} p_n$ for all $n < \omega$. This implies that $q \leq p$ and q continuously read $\dot{r}$.

In addition, if $\dot{r}$ is a name of a function in ω^ω then, since $\operatorname{poss}(q, {<}L)$ is finite for all $L \in \mathrm{Ht}$, we can find some $f \in \omega^\omega$ (in the ground model) such that $q \Vdash \dot{r}(n) \leq f(n)$. This shows (c).

(d): Let χ be large enough regular cardinal, $p \in \mathbb{Q}$, and $N \preceq H_\chi$ countable that contains (as elements) p and all the parameters of the forcing $\mathbb{Q}$. Let $\langle A_n : n \in \omega \rangle$ enumerate all maximal antichains in $\mathbb{Q}$ that belong to N, and choose a $\mathbb{Q}$-name $\dot{\tau}_n \in N$ of the element of A_n chosen by the generic. As in (a), we can construct a fusion sequence $\langle p_n : n < \omega \rangle$ such that $p_n \in N$ and p_n essentially decides $\dot{\tau}_n$ (however, the whole sequence may not be in N). For each $n < \omega$, as in the argument for (c) we can find a finite set $E_n \in N$, $E_n \subseteq A_n$ such that $p_n \Vdash \dot{\tau}_n \in E_n$. Hence $q \Vdash \dot{\tau}_n \in E_n \cap \dot{G} \subseteq A_n \cap N \cap \dot{G}$ (where $\dot{G}$ is the $\mathbb{Q}$-name of the generic set). □

4.8 The proof of the main theorem

In this section we prove Theorem Q. There the assumptions are CH, $|S^{\mathrm{pr}}| = \mu = \mu^{\aleph_0}$ infinite, and $\kappa_i = \kappa_i^{\aleph_0} \leq \mu$ is an infinite cardinal for all $i \in S^{\mathrm{pr}}$. In accordance to our forcing construction:

Assumption 4.8.1. We consider the following assumptions,

(1) CH.

(2) $|S^{\mathrm{pr}}| = \mu = \mu^{\aleph_0}$ is infinite.

(3) For each $i \in S^{\mathrm{pr}}$, $\kappa_i := |S_i^{\mathrm{lc}}| = |S_i^{\mathrm{al}}|$ is infinite and $\kappa_i = \kappa_i^{\aleph_0} \leq \mu$.

Note that (2) and (3) imply that $|S^*| = \mu$.

We present several results that, together, prove Theorem Q. We do not need to assume all (1)–(3) of Assumption 4.8.1 in all of them, so we present each one of these results with the necessary hypothesis. This means that we will not use Assumption 4.8.1 all the time.

First of all, recall from Theorem 4.7.2 that $\mathbb{Q}$ is proper and ω^ω-bounding and, under CH, $\mathbb{Q}$ has the $\aleph_2$-cc by Lemma 4.5.22, so it preserves all cofinalities and cardinalities. On the other hand, under CH, $\mathbb{Q}$ forces $\operatorname{cov}(\mathcal{N}) = \aleph_1$ by Lemma 4.7.10.

According to Lemma 4.5.25(b), at each $i \in S^{\mathrm{pr}}$ we add a real generic $y_i^* \in \prod_{n<\omega} T_n^*$ and the sequences $\vec{\nu}_i^{\mathrm{lc}}$ and $\vec{\nu}_i^{\mathrm{al}}$ are the generic blocks added at $i \in S^{\mathrm{pr}}$. This proves (II) of Theorem Q.

In the rest of the section, we will prove the remaining parts of Theorem Q, i. e. under Assumption 4.8.1, $\mathbb{Q}$ forces, for all $i \in S^{\mathrm{pr}}$,

(I) $\mathfrak{c} = \mu$,

(III) $\mathfrak{b}^{\mathrm{Lc}}_{\dot{b}_i^{\mathrm{lc}}, \dot{h}_i^{\mathrm{lc}}} = \operatorname{cov}(\mathcal{I}_{\dot{f}_i^{\mathrm{lc}}}) = \mathfrak{b}^{\mathrm{aLc}}_{\dot{a}_i^{\mathrm{lc}}, \dot{d}_i^{\mathrm{lc}}} = \mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{lc}}, \dot{d}_i^{\mathrm{lc}}} = \kappa_i$, and

(IV) $\mathfrak{d}^{\mathrm{aLc}}_{\dot{a}_i^{\mathrm{al}}, \dot{h}_i^{\mathrm{al}}} = \operatorname{non}(\mathcal{I}_{\dot{f}_i^{\mathrm{al}}}) = \mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{al}}, \dot{h}_i^{\mathrm{al}}} = \kappa_i$.

In order to prove (III) and (IV), by Lemma 4.2.5 it suffices to show that $\mathbb{Q}$ forces, for all $i \in S^{\mathrm{pr}}$, $\kappa_i \leq \mathfrak{b}^{\mathrm{Lc}}_{\dot{b}_i^{\mathrm{lc}}, \dot{h}_i^{\mathrm{lc}}}, \mathfrak{d}^{\mathrm{aLc}}_{\dot{a}_i^{\mathrm{al}}, \dot{h}_i^{\mathrm{al}}}$, and $\mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{lc}}, \dot{d}_i^{\mathrm{lc}}}, \mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{al}}, \dot{h}_i^{\mathrm{al}}} \leq \kappa_i$. In Subsection 4.8.1 we prove that the κ_i are lower bounds by using the generic slaloms.

In Subsection 4.8.2, we force (I) and $\mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{lc}}, \dot{d}_i^{\mathrm{lc}}}, \mathfrak{d}^{\mathrm{Lc}}_{\dot{a}_i^{\mathrm{al}}, \dot{h}_i^{\mathrm{al}}} \leq \kappa_i$, basically by forcing that the reals in $\dot{R}(\{i\} \cup S_i^{\mathrm{lc}} \cup S_i^{\mathrm{al}})$ (see Lemma 4.7.5) give witnesses of both cardinals. A feature of the forcing called *separated support* is essential in the proofs.

4.8.1 Forcing the lower bounds

We start with the anti-localization cardinals. Recall from Corollary 4.5.19 that, whenever $\alpha \in S^{\mathrm{al}}$, $\mathbb{Q}_{S^*\smallsetminus\{\alpha\}}$ is a complete subforcing of $\mathbb{Q}$.

Lemma 4.8.2. *Let $i \in S^{\mathrm{pr}}$, $\alpha \in S^{\mathrm{al}}_i$ and let $\dot r$ be a $\mathbb{Q}_{S^*\smallsetminus\{\alpha\}}$-name for a real in $\prod_{n<\omega} \dot a^{\mathrm{al}}_i(n)$. Then $\Vdash \dot r \in^\infty \dot\varphi_\alpha$.*

Proof. Fix $p \in \mathbb{Q}$, wlog $\alpha \in \operatorname{supp} p$. It is enough to show that, for any $n_0 < \omega$, there are $n \geq n_0$ and some $q \leq p$ that forces $\dot r(n) \in \dot\varphi_\alpha(n)$.

Pick $n \geq n_0$ such that $\|p(\alpha, L^{\mathrm{al}}_n)\|_{\mathrm{al}} \geq 1$. Denote $\ell_n := L^{\mathrm{al}}_n$. So we choose $t_0 \in \operatorname{pss}(p,\alpha,\ell_n)$ such that $\|p(\alpha,t_0,\ell_n)\|^{t_0}_{\ell_n} \geq 1$. Pick $\eta \in \operatorname{poss}(p,{<}n+1)$ such that $\hat\eta(\alpha,n) = t_0$. Then, there are some k and $p' \leq p \wedge \eta$ that forces $\dot r(n) = k$. We define q being identical to p' except on (α,ℓ_n), where $q(\alpha,\ell_n) = \{w\}$ for some $w \in p(\alpha,t_0,\ell_n)$ that contains k (which exists by Observation 4.4.3 because $\|p(\alpha,t_0,\ell_n)\|^{t_0}_{\ell_n} \geq 1$). It is clear that $q \in \mathbb{Q}$, $q \leq p$ and $q \Vdash \dot r(n) \in w = \dot y_\alpha(L^{\mathrm{al}}_n) = \dot\varphi_\alpha(n)$. ☐

As an immediate consequence, we get:

Corollary 4.8.3. *Assume* CH. *Then $\mathbb{Q}$ forces $|S^{\mathrm{al}}_i| \leq \mathfrak{d}^{\mathrm{aLc}}_{\dot a^{\mathrm{al}}_i, \dot h^{\mathrm{al}}_i}$ for all $i \in S^{\mathrm{pr}}$.*

Proof. Let $\langle \dot r_\zeta : \zeta < \kappa\rangle$ be a sequence of $\mathbb{Q}$-names of members of $\prod \dot a^{\mathrm{al}}_i$ with $\kappa < |S^{\mathrm{al}}_i|$ a cardinal. If $\kappa \leq \aleph_0$ it is clear that $\mathbb{Q}$ forces that, for some slalom $\varphi \in \mathcal{S}(\dot a^{\mathrm{al}}_i, \dot h^{\mathrm{al}}_i)$, $\dot r_\zeta \in^\infty \varphi$ for all $\zeta < \kappa$ (because $\mathfrak{d}^{\mathrm{aLc}}_{a,h}$ is uncountable whenever $\lim_{n\to\infty} \frac{h(n)}{a(n)} = 0$). So assume that κ is uncountable.

For each $\zeta < \kappa$, we can assume wlog that $\dot r_\zeta$ is a nice-name, i. e. each $\dot r_\zeta(n)$ is determined by a maximal antichain $A_{\zeta,n} \subseteq \mathbb{Q}$. Then $B := \bigcup\{\operatorname{supp}(q) : q \in A_{\zeta,n},\ \zeta < \kappa,\ n < \omega\}$ has size $\leq \kappa < |S^{\mathrm{al}}_i|$ because $\mathbb{Q}$ is $\aleph_2$-cc, so we can pick some $\alpha \in S^{\mathrm{al}}_i \smallsetminus B$.

It is clear that $\dot r_\zeta$ is a $\mathbb{Q}_{S^*\smallsetminus\{\alpha\}}$-name for all $\zeta < \kappa$. Therefore, by Lemma 4.8.2, $\Vdash \dot r_\zeta \in^\infty \dot\varphi_\alpha$, which finishes the proof. ☐

For the previous results we could, alternatively, use continuous reading. Namely, in Lemma 4.8.2, if instead of assuming that $\dot r$ is a $\mathbb{Q}_{S^*\smallsetminus\{\alpha\}}$-name we assume that $p \in \mathbb{Q}$ continuously read $\dot r$ without using the index α, then we can show that there is some $q \leq p$ forcing $\dot r \in^\infty \dot\varphi_\alpha$. This is actually the approach we must use to increase the localization cardinals because we cannot say that $\mathbb{Q}_{S^*\smallsetminus\{\alpha\}}$ is a complete subforcing of $\mathbb{Q}$ whenever $\alpha \in S^{\mathrm{lc}}$.

Lemma 4.8.4. *Let $i \in S^{\mathrm{pr}}$, $\alpha_0 \in S^{\mathrm{lc}}_i$, $\dot r$ a $\mathbb{Q}$-name for a real in $\prod \dot b^-_i$, and let $p_0 \in \mathbb{Q}$. Assume that, for all $n < \omega$, $\dot r(n) := \langle \dot r(\ell) : \ell \in I^*_n\rangle$ is L^{al}_n-decided by p_0 without using the index α_0. Then there is some $q \leq p_0$ forcing $\dot r \in^*_{\bar I^*} \dot y_{\alpha_0}$.*

Proof. Fix a modest $p \leq p_0$ in $\mathbb{Q}$ with $\alpha_0 \in \operatorname{supp} p$. Pick $n_0 < \omega$ such that $\alpha_0 \in \operatorname{supp} p(L^{\mathrm{lc}}_n)$ and $\|p(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq 2$ for all $n \geq n_0$. We construct q identical to p except on the lim inf part above $L^{\mathrm{lc}}_{n_0}$, so we shall construct $q(L^{\mathrm{lc}}_n)$ by induction on $n \geq n_0$. Fix $n \geq n_0$. Assume that we have constructed q up to ${<}L^{\mathrm{lc}}_n$. Since $\dot r(n)$ is L^{al}_n-decided by p without using the index α_0, there is a function $F^n\colon \operatorname{poss}(p,{<}L^{\mathrm{al}}_n) \to b^*_{t^n_{\mathrm{mx}}}(L^{\mathrm{lc}}_n)$ (see (fp6)) that calculates $\dot r(n)$ and such that $F^n(\eta) = F^n(\eta')$ whenever $\eta|_{\operatorname{supp} p\smallsetminus\{\alpha_0\}} = \eta'|_{\operatorname{supp} p\smallsetminus\{\alpha_0\}}$. For each $\eta \in \operatorname{poss}(p,{<}L^{\mathrm{lc}}_n)$ denote $\bar t^\eta := \hat\eta(L^{\mathrm{lc}}_n) = \langle t^\eta_\alpha : \alpha \in \operatorname{supp} p(L^{\mathrm{lc}}_n)\rangle$ (so $t^\eta_\alpha := \hat\eta(\alpha,n)$) and $t^\eta_* := t^\eta_{\alpha_0}$. Also define $F^n_\eta\colon \operatorname{possp}(L^{\mathrm{lc}}_n,\eta) \to b^*_{t^\eta_*}(L^{\mathrm{lc}}_n)$ such that $F^n_\eta(\bar x)$ is F^n evaluated on the unique member of $\operatorname{poss}(p,{<}L^{\mathrm{al}}_n)$ constructed from η and $\bar x$.

Enumerate $\operatorname{poss}(p,{<}L^{\mathrm{lc}}_n)$ as $\eta_0,\ldots,\eta_{m-1}$. By induction on $k \leq m$ we construct compound lim inf creatures $\mathbf{d}_k \leq \mathbf{d}_{k-1} \leq p(L^{\mathrm{lc}}_n)$ and $\ell_k \in I^*_n$ when $k > 0$, such that the creatures have the same domain, halving parameters and same pr-indices, and

$$\|\mathbf{d}_{k+1}(\alpha,t,\ell)\|^t_\ell \geq \|\mathbf{d}_k(\alpha,t,\ell)\|^t_\ell - \frac{1}{d_t(\ell)},$$

for each valid (α, t, ℓ), except at $(\alpha_0, t_*^{\eta_k}, \ell_{k+1})$ where $\mathbf{d}_{k+1}(\alpha_0, t_*^{\eta_k}, \ell_{k+1})$ is a singleton. Set $\mathbf{d}_0 := p(L_n^{\mathrm{lc}})$. Assume we have constructed $\mathbf{d}_k$. Choose $\ell_{k+1} \in I_n^* \smallsetminus \{\ell_{k'} : 0 < k' \leq k\}$ such that $\|\mathbf{d}_k(\alpha_0, t_*^k, \ell_{k+1})\|_{\ell_{k+1}}^{t_*^k} \geq 1$ (this is possible because $\|\mathbf{d}_k\|_n^{\mathrm{lc}} > 1$). Denote $\bar{t}^k := \bar{t}^{\eta_k}$, $t_*^k := t_*^{\eta_k}$ and $F_k^n := F_{\eta_k}^n$. Let $f_k^n \colon \mathrm{poss}p(L_n^{\mathrm{lc}}, \eta_k) \to b_{t_*^k}(\ell_{k+1})$ where $f_k^n(\bar{x})$ is the ℓ_{k+1}-th coordinate of $F_k^n(\bar{x})$. According to Lemma 4.6.3 and (pr9), there is a compound creature $\mathbf{d}'_k \leq \mathbf{d}_k$ with same domain, halving parameter and same pr-indices, such that $f_k^n{\restriction}\mathrm{poss}(\mathbf{d}'_k, \bar{t}^k)$ only depends on

$$C_{k+1} := \{(\alpha, \ell) \in (\mathrm{supp}p(L_n^{\mathrm{lc}})) \times I_n^* \colon (t_\alpha^k, \ell) \leq (t_*^k, \ell_{k+1})\}.$$

But recall that $\dot{r}(n)$ does not depend on α_0, and that $\mathbf{d}_k(\alpha_0, \ell_{k+1})$ is not trivial, so $p(\alpha, \ell_{k+1})$ is trivial for all $\alpha \neq \alpha_0$ by strong modesty, thus the decision of $\dot{r}(n)$ does not depend on level ℓ_{k+1}. Therefore, f_k^n does not depend on (α_0, ℓ_{k+1}), so $f_k^n{\restriction}\mathrm{poss}(\mathbf{d}'_k, \bar{t}^k)$ only depends on $C_{k+1}^* = \{(\alpha, \ell) \in (\mathrm{supp}p(L_n^{\mathrm{lc}})) \times I_n^* : (t_\alpha^k, \ell) < (t_*^k, \ell_{k+1})\}$. Considering this, we can redefine $\mathbf{d}'_k(\alpha_0, t_*^k, \ell_{k+1}) := \mathbf{d}_k(\alpha_0, t_*^k, \ell_{k+1})$

Define $H_k := \{f_k^n(\bar{x}) : \bar{x} \in \mathrm{poss}(\mathbf{d}'_k, \bar{t}^k)\}$. Then $|H_k| \leq \prod_{(\alpha,\ell) \in C_{k+1}^*} |\mathbf{d}'_k(\alpha, t_\alpha^k, \ell)| < d_{t_*^k}(\ell_{k+1})$ by (pr9). Since $\|\mathbf{d}'_k(\alpha_0, t_*^k, \ell_{k+1})\|_{\ell_{k+1}}^{t_*^k} \geq 1$, we have $\|\mathbf{d}'_k(\alpha_0, t_*^k, \ell_{k+1})\|_{\ell_{k+1}, t_*^k}^{\mathrm{COV}} > d_{t_*^k}(\ell_{k+1})$, so there is some $W_k \in \mathbf{d}'_k(\alpha_0, t_\alpha^{\eta_{k+1}}, \ell_{k+1})$ such that $H_k \subseteq W_k$. Define $\mathbf{d}_{k+1}$ identical to $\mathbf{d}'_k$ except at $(\alpha_0, t_*^{\eta_{k+1}}, \ell_{k+1})$ where we set $\mathbf{d}_{k+1}(\alpha_0, t_*^{\eta_{k+1}}, \ell_{k+1}) := \{W_k\}$.

Define $q(L_n^{\mathrm{lc}}) := \mathbf{d}_m$. According to the construction each subatom of $q(L_n^{\mathrm{lc}})$ loses at most $\frac{n^P_{<L_n^{\mathrm{lc}}}}{n^B_{L_n^{\mathrm{lc}}}} \leq 1$ of the norm of the subatom of $p(L_n^{\mathrm{lc}})$ at the same coordinate, except at $(\alpha_0, t_*^k, \ell_{k+1})$ for $k < m$ where we get singletons. Hence, by Lemma 4.4.14(b), $\|q(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}} \geq \|p(L_n^{\mathrm{lc}})\|_n^{\mathrm{lc}} - 1$.

We now prove that $p \wedge \eta \Vdash \exists \ell \in I_n^* \colon \dot{r}(\ell) \in \dot{y}_{\alpha_0}(\ell)$ for all $\eta \in \mathrm{poss}(q, {<}L_n^{\mathrm{al}})$ (although q has not been fully defined, we can talk about $\mathrm{poss}(q, {<}L_n^{\mathrm{al}})$ because q is already defined ${<}L_n^{\mathrm{al}}$). Each $\eta \in \mathrm{poss}(q, {<}L_n^{\mathrm{al}})$ depends on some $\eta_k \in \mathrm{poss}(p, {<}L_n^{\mathrm{lc}})$ and $\bar{x} \in \mathrm{poss}q(L_n^{\mathrm{lc}})(\bar{t}^{\eta_k})$, and by the construction of W_k, we know that $p \wedge \eta \Vdash \dot{r}(\ell_{k+1}) = f_k^n(\bar{x}) \in H_k \subseteq W_k = \dot{y}_{\alpha_0}(\ell_{k+1})$.

This finishes the construction, and it is clear that q is in $\mathbb{Q}$ and that it is as required. □

Just as in Corollary 4.8.3, this implies:

Corollary 4.8.5. *Assume* CH. *Then, for all* $i \in S^{\mathrm{pr}}$, $\mathbb{Q}$ *forces* $|S_i^{\mathrm{lc}}| \leq \mathfrak{b}^{\mathrm{Lc}, \bar{I}^*}_{\dot{b}_i^-, \dot{h}_i^-} \leq \mathfrak{b}^{\mathrm{Lc}}_{\dot{b}_i^{\mathrm{lc}}, \dot{h}_i^{\mathrm{lc}}}$.

Proof. This proof is a variation of the proof of Corollary 4.8.3. Let $\kappa < |S_i^{\mathrm{lc}}|$ be a cardinal and let $\langle \dot{r}_\zeta : \zeta < \kappa \rangle$ be a sequence of name of members of $\prod \dot{b}_i^-$. If $\kappa \leq \aleph_0$ it is clear that $\mathbb{Q}$ forces that, for some slalom $\varphi \in \mathcal{S}(\dot{b}_i^-, \dot{h}_i^-)$, $\dot{r}_\zeta \in^*_{\bar{I}^*} \varphi$ for all $\zeta < \kappa$. So assume that κ is uncountable.

For each $\zeta < \kappa$ define a $\mathbb{Q}$-name $\dot{r}'_\zeta$ of a function with domain Ht such that $\mathbb{Q}$ forces

$$\dot{r}'_\zeta(L) = \begin{cases} \langle \dot{r}_\zeta(\ell) : \ell \in I_n^* \rangle & \text{if } L = L_n^{\mathrm{lc}}, \\ 0 & \text{otherwise,} \end{cases}$$

and, by Corollary 4.7.9, pick a maximal antichain $A_\zeta \subseteq \mathbb{Q}$ such that every condition in A_ζ rapidly reads $\dot{r}'_\zeta$. Then $B := \bigcup_{\zeta<\kappa} \bigcup_{p \in A_\zeta} \mathrm{supp}(p)$ has size $\leq \kappa < |S_i^{\mathrm{lc}}|$, so we can pick some $\alpha \in S_i^{\mathrm{lc}} \smallsetminus B$.

It is enough to show that, for any $\zeta < \kappa$ and $p_0 \in A_\zeta$, $p_0 \Vdash \dot{r}_\zeta \in^*_{\bar{I}^*} \dot{y}_\alpha$. If $p \leq p_0$ then, by Lemma 4.7.4 p rapidly reads $\dot{r}'_\zeta$ without using the index α, so by Lemma 4.8.4 there is some $q \leq p$ forcing $\dot{r}_\zeta \in^*_{\bar{I}^*} \dot{y}_\alpha$. □

4.8.2 Forcing the upper bounds

To calculate the upper bounds we need a property, denominated *separated support* in [KS09; KS12], that basically says that the set of conditions $p \in \mathbb{Q}_{\mathrm{pr}}$ satisfying that $\langle \hat{\eta}(i,n) : i \in \operatorname{supp} p(n)\rangle$ is a one-to-one sequence for all $n < \omega$ and $\eta \in \operatorname{poss}(p, {<}n+1)$ is dense. This is the main reason why we constructed the pr-part as a $\liminf$ forcing.

Motivated by [KS12, Lemma 2.8], we start proving a version of separated support for pr-compound creatures.

Lemma 4.8.6. *Let $\mathbf{c}$ be a pr-compound creature at level n and $i_0 \in \operatorname{supp}\mathbf{c}$. Then there is some pr-compound $\mathbf{d} \leq \mathbf{c}$ with the same domain and halving parameter such that $\|\mathbf{d}\|^{\mathrm{pr}}_n \geq \|\mathbf{c}\|^{\mathrm{pr}}_n - \frac{1}{n^B_{(n,0)}}$ and, for $i \neq i_0$ in $\operatorname{supp}\mathbf{c}$ and $L \in \mathrm{ht}^{\mathrm{pr}}_n$, if $\mathbf{d}(i,L)$ is non-trivial then $\mathbf{d}(i,L) \cap \mathbf{d}(i_0,L) = \varnothing$.*

Proof. Set $d_{\mathbf{d}} := d_{\mathbf{c}}$, $\operatorname{supp}\mathbf{d} := \operatorname{supp}\mathbf{c}$, $\operatorname{dom}\mathbf{d} := \operatorname{dom}\mathbf{c}$ and, for each $i \in \operatorname{dom}\mathbf{c}$ and $L \in \mathrm{ht}^{\mathrm{pr}}_n$,

$$\mathbf{d}(i,L) = \begin{cases} \mathbf{c}(i,L) \smallsetminus \mathbf{c}(i_0,L) & \text{if } i \neq i_0 \text{, and } \mathbf{c}(i,L) \text{ is non-trivial,} \\ \mathbf{c}(i_0,L) & \text{otherwise.} \end{cases}$$

In the first case $\mathbf{c}(i_0,L)$ is a singleton by modesty, so $\|\mathbf{d}(i,L)\| \geq \|\mathbf{c}(i,L)\| - \frac{1}{n^B_L}$. It is clear from the definition above that $\mathbf{d}$ works. □

Corollary 4.8.7. *Let $\mathbf{c}$ be a pr-compound creature at level n. Then there is some pr-compound $\mathbf{d} \leq \mathbf{c}$ with the same domain and halving parameter such that $\|\mathbf{d}\|^{\mathrm{pr}}_n \geq \|\mathbf{c}\|^{\mathrm{pr}}_n - \frac{|\operatorname{supp}\mathbf{c}|}{n^B_{(n,0)}}$ and, for any $i \in \operatorname{supp}\mathbf{c}$, and $L \in \mathrm{ht}^{\mathrm{pr}}_n$, if $\mathbf{d}(i,L)$ is non trivial then $\mathbf{d}(i,L) \cap \mathbf{d}(i',L) = \varnothing$ for all $i' \in \operatorname{supp}\mathbf{c} \smallsetminus \{i\}$.*

This motivates the following notion.

Definition 4.8.8. We fix the following terminology.

(1) A pr-compound $\mathbf{c}$ at level n has *separated support* if, for any $i \in \operatorname{supp}\mathbf{c}$, and $L \in \mathrm{ht}^{\mathrm{pr}}_n$, if $\mathbf{c}(i,L)$ is non trivial then $\mathbf{c}(i,L) \cap \mathbf{c}(i',L) = \varnothing$ for all $i' \in \operatorname{supp}\mathbf{c} \smallsetminus \{i\}$.

(2) A condition $p \in \mathbb{Q}$ has *separated support* if it satisfies, for any $n < \omega$:

  (i) if $\|p_{\mathrm{pr}}(n)\| = 0$ then $\operatorname{supp} p_{\mathrm{pr}}(n) = \varnothing$;

  (ii) $p_{\mathrm{pr}}(n)$ has separated support;

  (iii) if $\alpha \in \operatorname{supp} p_{\mathrm{al}}$ and $p(\alpha, L^{\mathrm{al}}_n)$ is non-trivial then $i^*(\alpha) \in \operatorname{supp} p_{\mathrm{pr}}(n)$.

Note that property (iii) for $\alpha \in \operatorname{supp} p_{\mathrm{lc}}$ already holds by property (8) of Definition 4.5.4 (lc-part of the forcing). This property of the lc-part has not been used so far.

We then can easily obtain conditions in $\mathbb{Q}_{\mathrm{pr}}$ with separated support.

Lemma 4.8.9. *Let $p \in \mathbb{S}$. Then there is some $q \leq p$ in $\mathbb{S}$ with the same support and halving parameters, satisfying separated support.*

Proof. Fix $n_0 \geq \operatorname{trnklg}(p)$ such that, for all $n \geq n_0$, $\|p(n)\| > 2$ and $|\operatorname{supp} p(n)| \leq n$ (fine by Definition 4.5.1 (8)–(9)). Extend the trunk length to n_0, in detail, choose $\eta \in \operatorname{poss}(p, {<}n_0)$ and set $p^* = p \wedge \eta$.

We define q by cases: for $n < n_0$, define $q(n) := p^*(n)$; for $n \geq n_0$, by Corollary 4.8.7 there exists some pr-compound creature $q(n)$ at level n with the same domain, support and halving parameter as $p^*(n)$ such that $q(n) \leq p^*(n)$ satisfies separated support and $\|q(n)\|^{\mathrm{pr}}_n \geq \|p^*(n)\|^{\mathrm{pr}}_n - \frac{|\operatorname{supp} p(n)|}{n^B_{(n,0)}} \geq \|p^*(n)\|^{\mathrm{pr}}_n - \frac{n}{n^B_{(n,0)}} > \|p^*(n)\|^{\mathrm{pr}}_n - 1$.

Finally, define $q := \langle q(n) : n < \omega\rangle$, which satisfies the requirements. □

Lemma 4.8.10. *If $p \in \mathbb{Q}$ then there is some modest $q \leq p$ with the same support and halving parameters, satisfying separated support.*

Proof. By Lemma 4.8.9, there is some $s \leq p_{\mathrm{pr}}$ in $\mathbb{S}$ with the same support and halving parameters, satisfying separated support. Let $n_0 := \mathrm{trnklg}(s)$ and let η' be the only possibility in $\mathrm{poss}(s, <n_0)$, choose some $\eta \in \mathrm{poss}(p, <n_0)$ extending η' and set $r := (p \wedge \eta) \wedge s$ (see Observation 4.5.18).

It is clear that conditions (i) and (ii) of Definition 4.8.8 hold, so we strengthen r to also get (iii). We only need to modify the al-part, so we define q' identical to r at the pr-part and lc-part, with the same trunk lengths, and define q' at any $\alpha \in \mathrm{supp} r_{\mathrm{al}}$ as follows: we can find some $n_\alpha \geq \mathrm{trnklg}(r, \alpha)$ such that $i^*(\alpha) \in \mathrm{supp} r_{\mathrm{pr}}(n_\alpha)$, so define $q'(\alpha, L_n^{\mathrm{al}}) := r(\alpha, L_n^{\mathrm{al}})$ for any $n \geq n_\alpha$, and pick any trivial $q'(\alpha, L_n^{\mathrm{al}}) \leq r(\alpha, l_n^{\mathrm{al}})$ for any $n < n_\alpha$.

Finally, by Lemma 4.5.8 find a modest $q \leq q'$ with same support and trunk lengths as q' and with the same pr-part and lc-part as q. This resulting q works. $\square$

One direct consequence of separated support is that we can calculate the size of the continuum in Theorem Q.

Lemma 4.8.11. *Under* CH, *if $|S^*| = |S^{\mathrm{pr}}| = \mu = \mu^{\aleph_0}$ is infinite then $\mathbb{Q}$ forces $\mathfrak{c} = \mu$.*[10]

Proof. By Corollary 4.7.6, $\mathbb{Q}$ forces $\mathfrak{c} \leq \mu$. On the other hand, Lemma 4.8.10 implies that, in any generic extension, $\langle y_i^* : i \in S^{\mathrm{pr}} \rangle$ is a one-to-one sequence of reals, so $\mathbb{Q}$ forces $\mu = |S^{\mathrm{pr}}| \leq \mathfrak{c}$. $\square$

We use separated support to prove that $\mathbb{Q}$ *is (a_i, d_i)-bounding over* $\{i\} \cup S_i^{\mathrm{lc}} \cup S_i^{\mathrm{al}}$, which means that $\mathbb{Q}$ forces that any real in $\prod \dot{a}_i$ is localized by some slalom in $\mathcal{S}(\dot{a}_i, \dot{d}_i) \cap \dot{R}(\{i\} \cup S_i^{\mathrm{lc}} \cup S_i^{\mathrm{al}})$ (see notation in Lemma 4.7.5).

Lemma 4.8.12. *Fix $i \in S^{\mathrm{pr}}$ and let $\dot{r}$ be a $\mathbb{Q}$-name of a function in $\prod_{L \in \Omega} \dot{a}_i(L)$. Then, for any $p \in \mathbb{Q}$, there are some $q \leq p$ in $\mathbb{Q}$ and some $\mathbb{Q}$-name $\dot{\varphi}$ of a slalom in $\mathcal{S}(\dot{a}_i, \dot{d}_i)$ such that*

(i) $q \Vdash \dot{r} \in^* \varphi$,

(ii) for $n \in \omega$, $\dot{\varphi}(L_n^{\mathrm{lc}})$ is L_n^{al}-decided by q only using $\{i\} \cup S_i^{\mathrm{lc}}$, and

(iii) $\dot{\varphi}(L_n^{\mathrm{al}})$ is $n+1$-decided by q only using $\{i\} \cup S_i^{\mathrm{al}}$.

Proof. Fix $i \in S^{\mathrm{pr}}$ and set $E_{\mathrm{al}} := \{i\} \cup S_i^{\mathrm{al}}$ and $E_{\mathrm{lc}} := \{i\} \cup S_i^{\mathrm{lc}}$. Assume that $\dot{r}$ is a $\mathbb{Q}$-name for an element of $\prod_{L \in \Omega} a_i(L)$ and let p be a condition of $\mathbb{Q}$. Wlog we can assume that $i \in \mathrm{supp} p_{\mathrm{pr}}(\mathrm{trnklg}(p))$, p rapidly reads $\dot{r}'$ (by Theorem 4.7.8) and that it is modest with separated support (by Lemma 4.8.10), where $\dot{r}'$ is a $\mathbb{Q}$-name of a function with domain Ht such that

$$\dot{r}'(L) := \begin{cases} \dot{r}(L) & \text{if } L \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

We set $q_{\mathrm{pr}} := p_{\mathrm{pr}}$, $\mathrm{supp} q := \mathrm{supp} p$ and define q to be identical to p below $N_0 := \mathrm{trnklg}(q) = \mathrm{trnklg}(p)$. For $L < (N_0, 0)$ in Ω let $\dot{\varphi}(L)$ be the canonical name of $\varnothing$ (so it is L-decided without using any index at all). So we construct $\dot{\varphi}(L)$ and q at level L by induction on $L \in \Omega$, $L \geq N_0$. Fix $L \geq N_0$ in Ω and assume we have defined q below L. Split into two cases:

Case 1: $L = L_n^{\mathrm{lc}}$. Let $\{\eta_k : k < m_n\}$ enumerate $\mathrm{poss}(q, <L_n^{\mathrm{lc}})$. For each $k < m_n$ set

- $S_{n,k} := \{\alpha \in \mathrm{supp} p(L_n^{\mathrm{lc}}) : \hat{\eta}_k(\alpha, n) <_{\mathrm{lex}} \hat{\eta}_k(i, n)\}$,

[10] Actually we do not need to assume $|S^{\mathrm{pr}}| = |S^*|$. By a method similar to the proof of Lemma 4.8.6 we can force $\mathfrak{c}$ above $|S^{\mathrm{pr}}|$ and $|S_i^{\mathrm{t}}|$ for any $i \in S^{\mathrm{pr}}$ and $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$, which implies $\mathfrak{c} \geq |S^*|$.

- $E_{n,k} := \{\alpha \in \operatorname{supp} p(L^{\mathrm{lc}}_n) : \hat\eta_k(\alpha, n) = \hat\eta_k(i, n)\}$.

According to separated support (and using Definition 4.5.4 (8)), $E_{n,k} = \operatorname{supp} p(L^{\mathrm{lc}}_n) \cap S^{\mathrm{lc}}_i =: E_n$. On the other hand, by rapid reading, we can define $F_k\colon p(L^{\mathrm{lc}}_n, \eta_k) \to a_{\hat\eta_k(i,n)}(L^{\mathrm{lc}}_n)$ such that $F_k(\bar x)$ is the value of $\dot r(L^{\mathrm{lc}}_n)$ decided by $p \wedge \eta'$, where η' is the unique possibility in $\operatorname{poss}(p, {<}L^{\mathrm{al}}_n)$ determined by η_k and $\bar x$.

By induction on $k < m_n$ we construct a decreasing sequence of lc-compound creatures $\langle \mathbf{d}_k : k < m_n\rangle$ stronger than $p(L^{\mathrm{lc}}_n)$ with the same domain, halving parameters, and pr-indices, and a $\mathbb{Q}$-name $\dot s_{n,k}$ of a subset of $\dot a_i(L^{\mathrm{lc}}_n)$ as follows.

At step k let $\mathbf{d} := \mathbf{d}_{k-1}$ where $\mathbf{d}_{-1} := p(L^{\mathrm{lc}}_n)$. By (pr10) and Lemma 4.6.3 applied to $(\hat\eta_k(i,n), \ell^{\mathrm{mx}}_n)$, there is a compound creature $\mathbf{d}_k \leq \mathbf{d}$ with same domain, halving parameter and same pr-indices, such that $F_k{\restriction}\operatorname{poss}(\mathbf{d}_k, \hat\eta_k(L^{\mathrm{lc}}_n))$ depends on $(S_{n,k} \cup E_n) \times I^*_n$. Denote by $F'_k\colon \prod_{(\alpha,\ell)\in(S_{n,k}\cup E_n)\times I^*_n} \mathbf{d}_k(\alpha, \hat\eta_k(\alpha,n), \ell) \to a_{\hat\eta_k(i,n)}(L^{\mathrm{lc}}_n)$ the corresponding function that gives the same values as F_k.

We now define a $\mathbb{Q}$-name $\dot s_{n,k}$ such that, for any $q' \in \mathbb{Q}$ with $\operatorname{trnklg}(q') > n$ and $\{i\} \cup E_n \subseteq \operatorname{supp} q'$: if $q'(i, L) = \{\eta_k(i, L)\}$ for all $L \in \mathrm{ht}^{\mathrm{pr}}_n$ and $q'(\alpha, \ell) = \{v(\alpha,\ell)\} \subseteq \mathfrak{d}_k(\alpha, \hat\eta_k(\alpha,\ell), \ell)$ for all $\alpha \in E_n$ and $\ell \in I^*_n$, then

$$q' \Vdash \text{“}\dot s_{n,k} = \left\{ F'_k(\bar u \cup \langle v(\alpha,\ell) : (\alpha,\ell) \in E_n \times I^*_n\rangle) \;\middle|\; \bar u \in \prod_{(\alpha,\ell)\in S_k \times I^*_n} \mathbf{d}_k(\alpha, \hat\eta_k(\alpha,n), \ell) \right\}\text{”};$$

otherwise q' forces $\dot s_{n,k} = \varnothing$. Clearly, $\mathbb{Q}$ forces $\dot s_{n,k} \subseteq \dot a_i(L^{\mathrm{lc}}_n)$ and that $|\dot s_{n,k}| \leq m^S_{\hat\eta_k(i,n)^-, \ell^{\mathrm{mx}}_n}$ if $\hat\eta_k(i,n)$ is not minimal in T^*_n, otherwise $|\dot s_{n,k}| \leq 1$. It is also clear that, for any $q_1 \in \mathbb{Q}$ containing $\{i\} \cup E_n$ in its support, $\dot s_{n,k}$ is L^{al}_n-decided by q_1 (and in particular by p) only using $\{i\} \cup E_n \subseteq E_{\mathrm{lc}}$.

Set $q(L^{\mathrm{lc}}_n) := \mathfrak{d}_{m_n - 1}$. Due to the applications of Lemma 4.6.3, all subatoms in $q(L^{\mathrm{lc}}_n)$ decreases at least $\frac{n^P_{<L^{\mathrm{lc}}_n}}{n^B_{L^{\mathrm{lc}}_n}} < 1$ on norm with respect of $p(L^{\mathrm{lc}}_n)$, so we can conclude that $\|q(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n \geq \|p(L^{\mathrm{lc}}_n)\|^{\mathrm{lc}}_n - 1$. Define $\dot\varphi(L^{\mathrm{lc}}_n)$ as a $\mathbb{Q}$-name of $\bigcup_{k<m_n} \dot s_{n,k}$. Note that $\mathbb{Q}$ forces that $|\dot\varphi(L^{\mathrm{lc}}_n)| \leq n^P_{<L^{\mathrm{lc}}_n} \cdot m^S_{\dot y^*_i(n)^-, \ell^{\mathrm{mx}}_n} < d_{\dot y^*_i(n)}(L^{\mathrm{lc}}_n) = \dot d_i(L^{\mathrm{lc}}_n)$ by (pr10) and Lemma 4.5.16. Moreover, $\dot\varphi(L^{\mathrm{lc}}_n)$ is L^{al}_n-decided by p only using $\{i\} \cup E_n$.

We prove that, for all $\eta \in \operatorname{poss}(q, {<}L^{\mathrm{al}}_n)$ (which we can use because q has been defined ${<}L^{\mathrm{al}}_n$), $p \wedge \eta \Vdash \dot r(L^{\mathrm{lc}}_n) \in \dot\varphi(L^{\mathrm{lc}}_n)$. Each $\eta \in \operatorname{poss}(q, {<}L^{\mathrm{al}}_n)$ depends on some $\eta_k \in \operatorname{poss}(p, {<}L^{\mathrm{lc}}_n)$ and $\bar x \in \operatorname{poss} q(L^{\mathrm{lc}}_n, \hat\eta_k(L^{\mathrm{lc}}_n))$, so $p \wedge \eta \Vdash \dot r(L^{\mathrm{lc}}_n) = F_k(\bar x) = F'_k(\bar x{\restriction}((S_{n,k} \cup E_n) \times I^*_n)$, which implies that $p \wedge \eta \Vdash \dot r(L^{\mathrm{lc}}_n) \in \dot s_{n,k}$. Hence $p \wedge \eta \Vdash \dot r(L^{\mathrm{lc}}_n) \in \dot\varphi(L^{\mathrm{lc}}_n)$.

Case 2: $L = L^{\mathrm{al}}_n$. First consider the case when all atomic creatures of p at level L^{al}_n are trivial, so $\dot r(L^{\mathrm{al}}_n)$ is L^{al}_n-decided by rapid reading. We define q identical to p at this level. Let $F\colon \operatorname{poss}(q, {<}L^{\mathrm{al}}_n) \to a_{t^{\mathrm{mx}}_n}(L^{\mathrm{al}}_n)$ such that $p \wedge \eta \Vdash \dot r(L^{\mathrm{al}}_n) = F(\eta)$. Notice that $|\operatorname{ran} F| \leq n^P_{<L^{\mathrm{al}}_n} < d_{t^{\mathrm{mn}}_n}(L^{\mathrm{al}}_n)$. So define $\dot\varphi(L^{\mathrm{al}}_n)$ as a $\mathbb{Q}$-name of $\dot a_i(L^{\mathrm{al}}_n) \cap \operatorname{ran} F$, which is forced to have size $<\dot d_i(L^{\mathrm{al}}_n)$. It is clear that $q \wedge \eta \Vdash \dot r(L^{\mathrm{al}}_n) = F(\eta) \in \dot\varphi(L^{\mathrm{al}}_n)$ for all $\eta \in \operatorname{poss}(q, {<}L^{\mathrm{al}}_n)$. It is clear that $\dot\varphi(L^{\mathrm{al}}_n)$ is L^{lc}_n-decided only using $\{i\}$.

Now, consider the case when p contains some non-trivial atomic creature at L^{al}_n, which means by modesty that there is a unique $\alpha_n \in \operatorname{supp} p$ such that $p(\alpha_n, L^{\mathrm{al}}_n)$ is non-trivial. For $\eta \in \operatorname{poss}(p, {<}L^{\mathrm{al}}_n)$ we can define $F_\eta\colon p(\alpha_n, \hat\eta(\alpha_n, n), L^{\mathrm{al}}_n) \to a_{\hat\eta(i,n)}(L^{\mathrm{al}}_n)$ such that $F_\eta(s)$ is the value of $\dot r(L^{\mathrm{al}}_n)$ forced by p restricted to the unique member of $\operatorname{poss}(p, {<}n+1)$ obtained from η and s (which is possible by rapid reading). Set $i_n := i^*(\alpha_n)$, which is in $\operatorname{supp}_{\mathrm{pr}}(n)$ by Definition 4.8.8 (2) (iii).

To find $q(L^{\mathrm{al}}_n)$ and $\dot\varphi(L^{\mathrm{al}}_n)$, we break into two subcases:

Subcase 2.1: $i = i_n$. So $\alpha_n \in S^{\mathrm{al}}_i$. We define q identical to p at level L^{al}_n. Also define $\dot\varphi(L^{\mathrm{al}}_n)$ as a $\mathbb{Q}$-name such that, for any $q' \in \mathbb{Q}$ with $\mathrm{trnklg}(q') > n$ and $i, \alpha_n \in \mathrm{supp} q'$: if $q'(i, \mathrm{ht}^{\mathrm{pr}}_n) = \{t\} \subseteq q(i, \mathrm{ht}^{\mathrm{pr}}_n)$ and $q'(\alpha_n, L^{\mathrm{al}}_n) = \{v\} \subseteq p(\alpha_n, t, L^{\mathrm{al}}_n)$ then

$$q' \Vdash \dot\varphi(L^{\mathrm{al}}_n) = \{F_\eta(v) : \eta \in \mathrm{poss}(q, {<}L^{\mathrm{al}}_n),\ \hat\eta(\alpha_n, n) = t\};$$

otherwise q' forces $\dot\varphi(L^{\mathrm{al}}_n) = \varnothing$. Note that $\dot\varphi(L^{\mathrm{al}}_n)$ is $n+1$-decided by p only using $\{i, \alpha_n\} \subseteq E_{\mathrm{al}}$.

It is easy to see that $p \wedge \eta' \Vdash$"$\dot r(L^{\mathrm{al}}_n) \in \dot\varphi(L^{\mathrm{al}}_n)$ and $|\dot\varphi(L^{\mathrm{al}})| \leq n^P_{<L^{\mathrm{al}}_n} < \dot d_i(L^{\mathrm{al}}_n)$" for all $\eta' \in \mathrm{poss}(q, {<}n+1)$.

Subcase 2.2: $i \neq i_n$. Fix $t \in \mathrm{pss}(p, \alpha_n, L^{\mathrm{al}}_n)$. Enumerate $\{\eta \in \mathrm{poss}(q, {<}L^{\mathrm{al}}_n) : \hat\eta(i_n, n) = t\} = \{\eta^t_k : k < m^t\}$. By induction we construct subatomic creatures $\mathbf{c}^t_{m^t} \leq \cdots \leq \mathbf{c}^t_1 \leq \mathbf{c}^t_0 := p(\alpha_n, t, L^{\mathrm{al}}_n)$, so given $\mathbf{c}^t_k$, define $\mathbf{c}^t_{k+1}$ as follows: When $\hat\eta(i,n) <_{\mathrm{lex}} t$, since $a_{\hat\eta(i,n)}(L^{\mathrm{al}}_n) < n^S_{\hat\eta(i,n), L^{\mathrm{al}}_n} < d_t(L^{\mathrm{al}}_n)$ by (pr8) and (pr12), we can apply Lemma 4.6.2(b) to $\mathbf{c}^t_k$ and F_η to get $\mathbf{c}^t_{k+1} \leq \mathbf{c}^t_k$ such that $F_{\eta^t_k}{\restriction}\mathbf{c}^t_{k+1}$ is constant with value $j^t_k \in a_{\hat\eta(i,n)}(L^{\mathrm{al}}_n)$ and $\|\mathbf{c}^t_{k+1}\| \geq \|\mathbf{c}^t_k\| - \frac{1}{d_t(L^{\mathrm{al}}_n)}$. Set $s^t_k := \{j^t_k\}$.

In the case $\hat\eta(i,n) >_{\mathrm{lex}} t$, set $\mathbf{c}^t_{k+1} = \mathbf{c}^t_k$ and $s^t_k := F_{\eta^t_k}[\mathbf{c}^t_k]$, so $|s^t_k| \leq n^S_{t, L^{\mathrm{al}}_n}$ by (pr8). Note that the case $\hat\eta(i,n) = t$ cannot happen because p satisfies separated support.

To define q at level L^{al}_n, set $q(\alpha_n, t, L^{\mathrm{al}}_n) := \mathbf{c}^t_{m^t}$ for all $t \in \mathrm{pss}(q, \alpha, L^{\mathrm{al}}_n)$, and $q(\alpha, L^{\mathrm{al}}_n) := p(\alpha, L^{\mathrm{al}}_n)$ for all $\alpha \neq \alpha_n$ in $\mathrm{supp} p_{\mathrm{al}}$. By construction, $\|q(\alpha_n, L^{\mathrm{al}}_n)\| \geq \|p(\alpha_n, L^{\mathrm{al}}_n)\| - \frac{n^P_{<L^{\mathrm{al}}_n}}{d_{t^{\mathrm{mn}}_n}(L^{\mathrm{al}}_n)} > \|p(\alpha_n, L^{\mathrm{al}}_n)\| - 1$. Define $\dot\varphi(L^{\mathrm{al}}_n)$ as a $\mathbb{Q}$-name such that, for all $q' \in \mathbb{Q}_i$ with $\mathrm{trnklg}(q') > n$ and $i \in \mathrm{supp} q'$: if $q'(i, \mathrm{ht}^{\mathrm{pr}}_n) = \{t_0\} \subseteq q(i, \mathrm{ht}^{\mathrm{pr}}_n)$ then

$$q' \Vdash \dot\varphi(L^{\mathrm{al}}_n) = \bigcup\{s^t_k : t \in \mathrm{pss}(p, \alpha_n, L^{\mathrm{al}}_n),\ k < m^t,\ \hat\eta^t_k(i,n) = t_0\};$$

otherwise q' forces $\dot\varphi(L^{\mathrm{al}}_n) = \varnothing$. In the first case, q' forces $|\dot\varphi(L^{\mathrm{al}}_n)| \leq n^P_{<L^{\mathrm{al}}_n} \cdot (n^S_{t_0^-, L^{\mathrm{al}}_n} + 1) < d_{t_0}(L^{\mathrm{al}}_n) = \dot d_i(L^{\mathrm{al}}_n)$ by (pr12) (in the case that $t_0 = t^{\mathrm{mn}}_n$, replace $n^S_{t_0^-, L^{\mathrm{al}}_n}$ by 0). Therefore, $\mathbb{Q}$ forces that $|\dot\varphi(L^{\mathrm{lc}}_n)| < \dot d_i(L^{\mathrm{al}}_n)$. Moreover, $\dot\varphi(L^{\mathrm{al}}_n)$ is L^{lc}_n-decided by p only using $\{i\}$.

We show that $p \wedge \eta' \Vdash \dot r(L^{\mathrm{al}}_n) \in \dot\varphi(L^{\mathrm{al}}_n)$ for all $\eta' \in \mathrm{poss}(q, {<}n+1)$. Each $\eta' \in \mathrm{poss}(q, {<}n+1)$ depends on some $\eta^t_k \in \mathrm{poss}(p, {<}L^{\mathrm{al}}_n)$ and $s \in q(\alpha_n, t, L^{\mathrm{al}}_n)$ where $t = \hat\eta'(\alpha_n, n) = \hat\eta^t_k(\alpha_n, n)$. Let $t_0 := \hat\eta_k(i,n)$. If $t_0 <_{\mathrm{lex}} t$ then $p \wedge \eta \Vdash \dot r(L^{\mathrm{al}}_n) = F_{\eta^t_k}(s) = j^t_k \in s^t_k \subseteq \dot\varphi(L^{\mathrm{al}}_n)$; else, if $t <_{\mathrm{lex}} t_0$, we then have that $p \wedge \eta \Vdash \dot r(L^{\mathrm{al}}_n) \in F_{\eta^t_k}[\mathbf{c}^t_k] = s^t_k \subseteq \dot\varphi(L^{\mathrm{al}}_n)$.

This finishes the construction of q and $\dot\varphi$. It is clear that they are as required. □

As an immediate consequence, we conclude:

Corollary 4.8.13. *Under* CH, *for* $\mathrm{t} \in \{\mathrm{lc}, \mathrm{al}\}$ *and any* $i \in S^{\mathrm{pr}}$, *if* $\kappa = \max\{|S^{\mathrm{t}}_i|, \aleph_0\}^{\aleph_0}$ *then* $\mathbb{Q}$ *forces* $\mathfrak{d}^{\mathrm{Lc}}_{a^{\mathrm{t}}_i, d^{\mathrm{t}}_i} \leq \kappa$. *In the case* $\mathrm{t} = \mathrm{al}$ *we also have* $\mathfrak{d}^{\mathrm{Lc}}_{a^{\mathrm{t}}_i, h^{\mathrm{t}}_i} \leq \kappa$.

Proof. Let $E_{\mathrm{t}} := \{i\} \cup S^{\mathrm{t}}_i$. By Lemma 4.7.5, $|\dot R(E_{\mathrm{t}})| \leq \kappa$. So it is enough to show that, for any $\mathbb{Q}$-name $\dot r$ of a real in $\prod a^{\mathrm{t}}_i$ and any $p \in \mathbb{Q}$, there is some (nice) $\mathbb{Q}$-name $\dot\varphi$ of a member of $\mathcal{S}(\dot a^{\mathrm{t}}_i, \dot d^{\mathrm{t}}_i)$ and some $q \leq p$ continuously reading $\dot\varphi$ only using E_{t} such that $q \Vdash \dot r \in^* \dot\varphi$.

Define a $\mathbb{Q}$-name $\dot r'$ of a member of $\prod_{L \in \Omega} a_i(L)$ by

$$\dot r'(L) := \begin{cases} \dot r(n) & \text{if } L = L^{\mathrm{t}}_n, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.8.12, there is a $q \leq p$ and a $\mathbb{Q}$-name $\dot\varphi'$ of a slalom in $\mathcal{S}(\dot a_i, \dot d_i)$ satisfying (i)–(iii) (where $\dot\varphi(L^{\mathrm{t}'}_n)$ with $\mathrm{t}' \in \{\mathrm{lc}, \mathrm{al}\} \smallsetminus \{\mathrm{t}\}$ can be taken as the canonical name of $\{0\}$). Let $\dot\varphi(n)$ be a (nice) name of $\dot\varphi'(L^{\mathrm{lc}}_n)$ for each $n < \omega$. It is clear that $q \Vdash$"$\dot\varphi \in \mathcal{S}(\dot a^{\mathrm{lc}}_i, \dot d^{\mathrm{al}}_i)$ and $\dot r \in^* \dot\varphi$" and that q continouosly read $\dot\varphi$ only using E_{t}.

The additional claim when $\mathrm{t} = \mathrm{al}$ follows by Lemma 4.2.3 (2) because $\Vdash d^{\mathrm{al}}_i \leq^* h^{\mathrm{al}}_i$. □

Using Assumption 4.8.1, the proof of Theorem Q is concluded by Corollary 4.8.3, 4.8.5 and 4.8.13.

4.9 Discussions

The forcing construction developed in Section 4.4–4.7 can be easily generalized as in the presentation in [FGKS17], in the sense that we can replace the subatomic families K_L and K^t_ℓ by any arbitrary subatomic family (even allowing different subatomic families at any (i, L) and (α, t, ℓ)). However, we decided to proceed with the current particular construction to ease the presentation, particularly in relation with the parameters of the forcing. In the generalized framework, to make sense of results like the existence of conditions with arbitrary support (Corollary 4.5.11), continuous reading of names (Theorem 4.7.2) and rapid reading (Theorem 4.7.8), we would need a more complicated setting of the parameters, which we think would obscure the presentation. Our forcing construction can be adapted to different situations, and we believe it would be simpler to modify the current construction to other contexts than presenting an utterly obscure abstract construction.

This forcing construction, as well as those in [GS93; KO14; KS09; KS12; FGKS17; GK21], satisfy enough rapid reading to guarantee that random reals are never added (see Lemma 4.7.10). It would be interesting to discover how to modify these constructions to allow a part adding random reals, while ensuring rapid reading outside the random-part and force the same results with larger values of $\mathrm{cov}(\mathcal{N})$.

Concerning Cichoń's diagram, our construction forces $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ and the other cardinals equal to the continuum because, as indicated in "Creature forcing" in the Introduction, the cardinal $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$ is below $\mathrm{non}(\mathcal{E})$ (under some conditions meet by our parameters), and $\mathrm{non}(\mathcal{E})$ is below $\mathrm{non}(\mathcal{M})$ and $\mathrm{non}(\mathcal{N})$.

Recall:

Theorem 4.9.1 ([FGKS17]). *Under* CH, *if $\lambda_1 \leq \lambda_3 \leq \lambda_4$ and $\lambda_2 \leq \lambda_3$ are infinite cardinals such that $\lambda_i^{\aleph_0} = \lambda_i$ for $i \in \{1, 2, 3, 4\}$, then there is some proper ω^ω-bounding poset with $\aleph_2$-cc forcing* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$, $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \lambda_1$, $\mathrm{non}(\mathcal{N}) = \lambda_2$, $\mathrm{cof}(\mathcal{N}) = \lambda_3$ *and* $\mathfrak{c} = \lambda_4$ *(see Figure 4.3).*

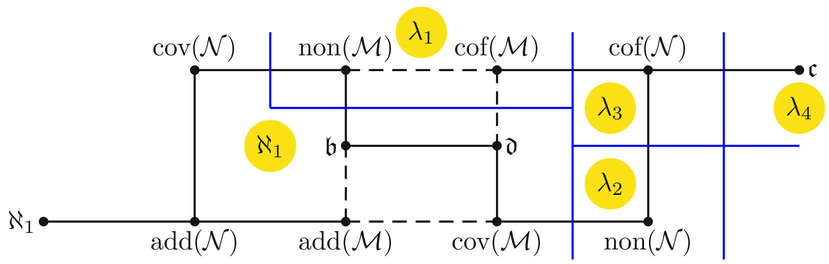

Figure 4.3: The constellation of Cichoń's diagram forced in [FGKS17; GK21].

The construction in [GK21] forces the same and, in addition, forces $\aleph_1$ many different localization cardinals $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$ (with parameters in the ground model). It looks like the methods of [GK21] to force the constellation in Figure 4.3 and the forcing construction in this paper can be combined to force the same constellation along with $\min\{\lambda_1, \lambda_2\}$-many different cardinals of each one of the six types discussed in this work. A bit of more work would be needed to force the constellation of Figure 4.3 along with $\min\{\lambda_1, \lambda_2\}$-many different cardinals of the type $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, λ_1-many of each type $\mathfrak{b}^{\mathrm{aLc}}_{b,h}$ and $\mathrm{cov}(\mathcal{I}_f)$, λ_2-many of each type $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ and $\mathrm{non}(\mathcal{I}_f)$, and

λ_3-many of type $\mathfrak{d}^{\mathrm{Lc}}_{b,h}$. This is quite optimal because (under certain conditions) $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E})$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{M})$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} \leq \mathrm{cof}(\mathcal{N})$ (as discussed in "Creature forcing" in the Introduction).

This work solves many open questions from [KM21], but the questions about additivities and cofinalities of Yorioka ideals remain open.

Question 4.9.2. *Is it consistent with* ZFC *that there are two different cardinals of the form* $\mathrm{add}(\mathcal{I}_f)$*? We ask the same for* $\mathrm{cof}(\mathcal{I}_f)$*.*

Forcing two different additivities or cofinalities would indicate a method to separate infinitely (and even continuum) many of them. Although the method of this paper could be used for the cofinalities, it does not work for the additivities because $\mathrm{add}(\mathcal{I}_f) \leq \mathfrak{b}$ ([KO08], see Figure 3). For the same reason, continuum many different additivities of Yorioka ideals implies that $\mathfrak{b} = \mathfrak{c}$ is weakly inaccessible.

BIBLIOGRAPHY

[Bar84] Tomek Bartoszyński. "Additivity of measure implies additivity of category". In: *Trans. Amer. Math. Soc.* 281.1 (1984), pp. 209–213. DOI: 10.2307/1999530.

[Bar87] Tomek Bartoszyński. "Combinatorial aspects of measure and category". In: *Fund. Math.* 127.3 (1987), pp. 225–239. DOI: 10.4064/fm-127-3-225-239.

[BCM21] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. "Filter-linkedness and its effect on preservation of cardinal characteristics". In: *Ann. Pure Appl. Logic* 172.1 (2021), pp. 102856, 30. DOI: 10.1016/j.apal.2020.102856.

[BF11] Jörg Brendle and Vera Fischer. "Mad families, splitting families and large continuum". In: *J. Symbolic Logic* 76.1 (2011), pp. 198–208. DOI: 10.2178/jsl/1294170995.

[BHH04] B. Balcar, F. Hernández-Hernández, and M. Hrušák. "Combinatorics of dense subsets of the rationals". In: *Fund. Math.* 183.1 (2004), pp. 59–80. DOI: 10.4064/fm183-1-4.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.

[Bla10] Andreas Blass. "Combinatorial cardinal characteristics of the continuum". In: *Handbook of set theory. Vols. 1, 2, 3*. Springer, Dordrecht, 2010, pp. 395–489. DOI: 10.1007/978-1-4020-5764-9_7.

[Bla93] Andreas Blass. "Simple cardinal characteristics of the continuum". In: *Set theory of the reals (Ramat Gan, 1991)*. Vol. 6. Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, 1993, pp. 63–90.

[BM14] Jörg Brendle and Diego Alejandro Mejía. "Rothberger gaps in fragmented ideals". In: *Fund. Math.* 227.1 (2014), pp. 35–68. DOI: 10.4064/fm227-1-4.

[Bre02] Jörg Brendle. "Mad families and iteration theory". In: *Logic and algebra*. Vol. 302. Contemp. Math. Amer. Math. Soc., Providence, RI, 2002, pp. 1–31. DOI: 10.1090/conm/302/05083.

[Bre19a] Joerg Brendle. *The higher Cichon diagram in the degenerate case*. 2019. arXiv: 1907.03111 [math.LO].

[Bre19b] Jörg Brendle. "Forcing and cardinal invariants". Tutorial for the Young Set Theory Workshop. 2019. URL: https://sites.google.com/view/estc2019/advanced-class-yst.

[Bre91] Jörg Brendle. "Larger cardinals in Cichoń's diagram". In: *J. Symbolic Logic* 56.3 (1991), pp. 795–810. DOI: 10.2307/2275049.

[BS02] Tomek Bartoszyński and Saharon Shelah. "Strongly meager and strong measure zero sets". In: *Arch. Math. Logic* 41.3 (2002), pp. 245–250. DOI: 10.1007/s001530000068.

[BS89] Andreas Blass and Saharon Shelah. "Ultrafilters with small generating sets". In: *Israel J. Math.* 65.3 (1989), pp. 259–271. DOI: 10.1007/BF02764864.

[BS92] Tomek Bartoszyński and Saharon Shelah. "Closed measure zero sets". In: *Ann. Pure Appl. Logic* 58.2 (1992), pp. 93–110. DOI: 10.1016/0168-0072(92)90001-G.

[Car21] Miguel A. Cardona. "On cardinal characteristics associated with the strong measure zero ideal". In: *Fundam. Math.* (2021). Accepted. arXiv: 2003.07066 [math.LO].

[Car93] Timothy J. Carlson. "Strong measure zero and strongly meager sets". In: *Proc. Amer. Math. Soc.* 118.2 (1993), pp. 577–586. DOI: 10.2307/2160341.

[CKM21] Miguel Antonio Cardona, Lukas Daniel Klausner, and Diego Alejandro Mejía. *Continuum Many Different Things: Localisation, Anti-Localisation and Yorioka Ideals*. 2021. arXiv: 2110.11614 [math.LO].

[CM19] Miguel Antonio Cardona and Diego Alejandro Mejía. "On Cardinal Characteristics of Yorioka Ideals". In: *Math. Log. Q.* 65.2 (2019), pp. 170–199. DOI: 10.1002/malq.201800034.

[CMR21] Miguel A. Cardona, Diego A. Mejía, and Ismael E. Rivera-Madrid. "The covering number of the strong measure zero ideal can be above almost everything else". In: *Arch. Math. Logic* (2021). DOI: 10.1007/s00153-021-00808-0. arXiv: 1902.01508 [math.LO].

[Coh66] Paul J. Cohen. *Set theory and the continuum hypothesis*. W. A. Benjamin, Inc., New York-Amsterdam, 1966, pp. vi+154.

[DM41] Ben Dushnik and E. W. Miller. "Partially ordered sets". In: *Amer. J. Math.* 63 (1941), pp. 600–610. DOI: 10.2307/2371374.

[EK65] Ryszard Engelking and Monika Karłowicz. "Some theorems of set theory and their topological consequences". In: *Fund. Math.* 57 (1965), pp. 275–285.

[FFMM18] Vera Fischer, Sy D. Friedman, Diego A. Mejía, and Diana C. Montoya. "Coherent systems of finite support iterations". In: *J. Symbolic Logic* 83.1 (2018), pp. 208–236. DOI: 10.1017/jsl.2017.20.

[FGKS17] Arthur Fischer, Martin Goldstern, Jakob Kellner, and Saharon Shelah. "Creature forcing and five cardinal characteristics in Cichoń's diagram". In: *Arch. Math. Logic* 56.7-8 (2017), pp. 1045–1103. DOI: 10.1007/s00153-017-0553-8.

[GJS93] Martin Goldstern, Haim Judah, and Saharon Shelah. "Strong measure zero sets without Cohen reals". In: *J. Symbolic Logic* 58.4 (1993), pp. 1323–1341. DOI: 10.2307/2275146.

[GK21] Martin Goldstern and Lukas Daniel Klausner.
"Cichoń's diagram and localisation cardinals".
In: *Archive for Mathematical Logic* 60.3 (2021), pp. 343–411.
DOI: 10.1007/s00153-020-00746-3.

[GKMS19] Martin Goldstern, Jakob Kellner, Diego Alejandro Mejía, and Saharon Shelah.
"Cichoń's maximum without large cardinals".
In: *J. Eur. Math. Soc. (JEMS)* to appear (2019). arXiv: 1906.06608 [math.LO].

[GKMS20a] Martin Goldstern, Jakob Kellner, Diego A. Mejia, and Saharon Shelah.
"Controlling cardinal characteristics without adding reals".
In: *J. Math. Log.* (2020). Online first. DOI: 10.1142/S0219061321500185.

[GKMS20b] Martin Goldstern, Jakob Kellner, Diego Alejandro Mejía, and Saharon Shelah.
"Controlling classical cardinal characteristics while collapsing cardinals".
In: (2020). Submitted. arXiv: 1904.02617 [math.LO].

[GKS19] Martin Goldstern, Jakob Kellner, and Saharon Shelah. "Cichoń's maximum".
In: *Ann. of Math.* 190.1 (2019), pp. 113–143.
DOI: 10.4007/annals.2019.190.1.2.

[GMS16] Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah.
"The left side of Cichoń's diagram".
In: *Proc. Amer. Math. Soc.* 144.9 (2016), pp. 4025–4042.
DOI: 10.1090/proc/13161.

[Göd40] Kurt Gödel. *The Consistency of the Continuum Hypothesis*.
Annals of Mathematics Studies, no. 3.
Princeton University Press, Princeton, N. J., 1940, p. 66.

[Gol93] Martin Goldstern. "Tools for your forcing construction".
In: *Set theory of the reals (Ramat Gan, 1991)*. Vol. 6. Israel Math. Conf. Proc.
Bar-Ilan Univ., Ramat Gan, 1993, pp. 305–360.

[GS93] Martin Goldstern and Saharon Shelah. "Many simple cardinal invariants".
In: *Arch. Math. Logic* 32.3 (1993), pp. 203–221. DOI: 10.1007/BF01375552.

[Hec72] Stephen H. Hechler. "Short complete nested sequences in $\beta N \setminus N$ and small maximal almost-disjoint families".
In: *General Topology and Appl.* 2 (1972), pp. 139–149.

[IS88] Jaime I. Ihoda and Saharon Shelah. "Souslin Forcing".
In: *Journal of Symbolic Logic* 53.4 (1988), pp. 1188–1207. DOI: 10.2307/2274613.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics.
The third millennium edition, revised and expanded.
Springer-Verlag, Berlin, 2003, pp. xiv+769.

[JMS92] Haim Judah, Arnold W. Miller, and Saharon Shelah.
"Sacks forcing, Laver forcing, and Martin's axiom".
In: *Arch. Math. Logic* 31.3 (1992), pp. 145–161. DOI: 10.1007/BF01269943.

[JS90] Haim Judah and Saharon Shelah. "The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing)".
In: *J. Symbolic Logic* 55.3 (1990), pp. 909–927. DOI: 10.2307/2274464.

[Kam89] Anastasis Kamburelis. "Iterations of Boolean algebras with measure".
In: *Arch. Math. Logic* 29.1 (1989), pp. 21–28. DOI: 10.1007/BF01630808.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*. Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. DOI: 10.1007/978-1-4612-4190-4.

[Kel08] Jakob Kellner. "Even More Simple Cardinal Invariants". In: *Arch. Math. Logic* 47.5 (2008), pp. 503–515. DOI: 10.1007/s00153-008-0094-2.

[KM21] Lukas Daniel Klausner and Diego Alejandro Mejía. "Many Different Uniformity Numbers of Yorioka Ideals". In: *Arch. Math. Logic* (2021). To appear. arXiv: 1805.11005 [math.LO].

[KO08] Shizuo Kamo and Noboru Osuga. "The cardinal coefficients of the ideal $\mathcal{I}_f$". In: *Arch. Math. Logic* 47.7-8 (2008), pp. 653–671. DOI: 10.1007/s00153-008-0091-5.

[KO14] Shizuo Kamo and Noboru Osuga. "Many Different Covering Numbers of Yorioka's Ideals". In: *Arch. Math. Logic* 53.1 (2014), pp. 43–56. DOI: 10.1007/s00153-013-0354-7.

[KS09] Jakob Kellner and Saharon Shelah. "Decisive creatures and large continuum". In: *J. Symbolic Logic* 74.1 (2009), pp. 73–104. DOI: 10.2178/jsl/1231082303.

[KS12] Jakob Kellner and Saharon Shelah. "Creature forcing and large continuum: the joy of halving". In: *Arch. Math. Logic* 51.1-2 (2012), pp. 49–70. DOI: 10.1007/s00153-011-0253-8.

[KST19] Jakob Kellner, Saharon Shelah, and Anda Tănasie. "Another ordering of the ten cardinal characteristics in Cichoń's Diagram". In: *Comment. Math. Univ. Carolin.* 60.1 (2019), pp. 61–95. DOI: 10.14712/1213-7243.2015.273.

[KTT18] Jakob Kellner, Anda Tănasie, and Fabio Tonti. "Compact Cardinals and Eight Values in Cichoń's Diagram". In: *J. Symbolic Logic* 83.2 (2018), pp. 790–803. DOI: 10.1017/jsl.2018.17.

[Kun11] Kenneth Kunen. *Set theory*. Vol. 34. Studies in Logic (London). College Publications, London, 2011, pp. viii+401.

[Lav76] Richard Laver. "On the consistency of Borel's conjecture". In: *Acta Math.* 137.3-4 (1976), pp. 151–169. DOI: 10.1007/BF02392416.

[Mar35] Edward Marczewski. "Sur une classe de fonctions de W. Sierpiński et la classe correspondante d'ensembles". In: *Fund. Math.* 24 (1935), pp. 17–34.

[Mej13] Diego Alejandro Mejía. "Matrix iterations and Cichon's diagram". In: *Arch. Math. Logic* 52.3-4 (2013), pp. 261–278. DOI: 10.1007/s00153-012-0315-6.

[Mej15] Diego A. Mejía. "Template iterations with non-definable ccc forcing notions". In: *Ann. Pure Appl. Logic* 166.11 (2015), pp. 1071–1109. DOI: 10.1016/j.apal.2015.06.001.

[Mej19] Diego A. Mejía. "Matrix iterations with vertical support restrictions". In: *Proceedings of the 14th and 15th Asian Logic Conferences*. Ed. by Byunghan Kim, Jörg Brendle, Gyesik Lee, Fenrong Liu, R Ramanujam, Shashi M Srivastava, Akito Tsuboi, and Liang Yu. World Sci. Publ., 2019, pp. 213–248.

[Mil81] Arnold W. Miller. “Some properties of measure and category”. In: *Trans. Amer. Math. Soc.* 266.1 (1981), pp. 93–114. DOI: 10.2307/1998389.

[Mil82] Arnold W. Miller. “A characterization of the least cardinal for which the Baire category theorem fails”. In: *Proc. Amer. Math. Soc.* 86.3 (1982), pp. 498–502. DOI: 10.2307/2044457.

[Mon17] Diana C. Montoya. *Some cardinal invariants of the generalized Baire spaces*. PhD thesis, University of Vienna, 2017.

[Osu06] Noboru Osuga. “The covering number and the uniformity of the ideal $\mathcal{I}_f$”. In: *MLQ* 52.4 (2006), pp. 351–358. ISSN: 0942-5616. DOI: 10.1002/malq.200610001.

[Osu08] Noboru Osuga. “The cardinal invariants of certain ideals related to the strong measure zero ideal”. In: *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku* 1619 (2008), pp. 83–90. URL: http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1619-11.pdf.

[Paw90] Janusz Pawlikowski. “Finite support iteration and strong measure zero sets”. In: *J. Symbolic Logic* 55.2 (1990), pp. 674–677. DOI: 10.2307/2274657.

[Paw92] Janusz Pawlikowski. “Adding dominating reals with ω^ω bounding posets”. In: *J. Symbolic Logic* 57.2 (1992), pp. 540–547. DOI: 10.2307/2275289.

[Ser89] Witold Seredyński. “Some operations related with translation”. In: *Colloq. Math.* 57.2 (1989), pp. 203–219. DOI: 10.4064/cm-57-2-203-219.

[She00] Saharon Shelah. “Covering of the null ideal may have countable cofinality”. In: *Fund. Math.* 166.1-2 (2000), pp. 109–136.

[She04] Saharon Shelah. “Two cardinal invariants of the continuum $(\mathfrak{d} < \mathfrak{a})$ and FS linearly ordered iterated forcing”. In: *Acta Math.* 192.2 (2004), pp. 187–223. DOI: 10.1007/BF02392740.

[She17] Saharon Shelah. *Proper and improper forcing*. Vol. 5. Cambridge University Press, 2017.

[Voj93] Peter Vojtáš. “Generalized Galois-Tukey-connections between explicit relations on classical objects of real analysis”. In: *Set theory of the reals (Ramat Gan, 1991)*. Vol. 6. Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, 1993, pp. 619–643.

[Yor02] Teruyuki Yorioka. “The cofinality of the strong measure zero ideal”. In: *J. Symbolic Logic* 67.4 (2002), pp. 1373–1384. DOI: 10.2178/jsl/119015\-0290.