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Exponential Stability and Hypoelliptic Regularization for the Kinetic Fokker–Planck Equation with Confining Potential

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Abstract
This paper is concerned with a modified entropy method to establish the large-time convergence towards the (unique) steady state, for kinetic Fokker–Planck equations with non-quadratic confinement potentials in whole space. We extend previous approaches by analyzing Lyapunov functionals with non-constant weight matrices in the dissipation functional (a generalized Fisher information). We establish exponential convergence in a weighted H^1-norm with rates that become sharp in the case of quadratic potentials. In the defective case for quadratic potentials, i.e. when the drift matrix has non-trivial Jordan blocks, the weighted L^2-distance between a Fokker–Planck-solution and the steady state has always a sharp decay estimate of the order $\mathcal{O}\big((1+t)e^{-t\nu/2}\big)$, with ν the friction parameter. The presented method also gives new hypoelliptic regularization results for kinetic Fokker–Planck equations (from a weighted L^2-space to a weighted H^1-space).

Keywords Kinetic theory · Fokker–Planck equation · Confinement potential · Degenerate evolution · Long time behavior · Convergence to equilibrium · Hypocoercivity · Hypoelliptic regularity · Lyapunov functional

Mathematics Subject Classification 35Q84 · 35B40 · 35Q82 · 82C40

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1 Introduction

This paper is devoted to the study of the long time behavior of the kinetic Fokker–Planck equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \nu \mathrm{div}_v(vf) + \sigma \Delta_v f, & x,\ v \in \mathbb{R}^n,\ t > 0 \\ f(t=0) = f_0 \in L^1(\mathbb{R}^{2n}) \end{cases} \tag{1}$$

describing the time evolution of the phase space probability density $f(t, x, v)$, e.g. in a plasma [30]. Applications range from plasma physics [13, 29] to stellar dynamics [17, 18]. Here $V = V(x)$ is a given smooth, bounded below confinement potential for the system, and $\nu > 0$, $\sigma > 0$ denote the friction and diffusion parameters, respectively. This equation is associated with the Langevin stochastic differential equation

$$\begin{cases} dx_t = v_t dt \\ dv_t = -\nu v_t dt - \nabla V(x_t)dt + \sqrt{2\sigma}\, dB_t, \end{cases}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$ with covariance $\langle B_t, B_{t'}\rangle = \delta_{t-t'}$.

Since the equation conserves mass, i.e.,

$$\int_{\mathbb{R}^{2n}} f(t, x, v)dxdv = \int_{\mathbb{R}^{2n}} f_0(x, v)dxdv, \quad t \geq 0,$$

we shall always assume (without restriction of generality) that $\displaystyle\int_{\mathbb{R}^{2n}} f_0(x, v)dxdv = 1$. The unique normalized steady state of (1) is given by

$$f_\infty(x, v) = c_V e^{-\frac{\nu}{\sigma}[V(x)+\frac{|v|^2}{2}]}, \quad x, v \in \mathbb{R}^n, \tag{2}$$

where c_V is a positive constant such that $\int_{\mathbb{R}^{2n}} f_\infty(x, v)dxdv = 1$. The following equation is also considered as the kinetic Fokker–Planck equation:

$$\partial_t h + v \cdot \nabla_x h - \nabla_x V \cdot \nabla_v h = \sigma \Delta_v h - \nu v \cdot \nabla_v h, \quad x,\ v \in \mathbb{R}^n,\ t > 0, \tag{3}$$

and to switch from (1) to (3) it suffices to set $h := f/f_\infty$.

It was shown in [22] that, if $V \in C^\infty(\mathbb{R}^n)$, (3) generates a C^∞ regularizing contraction semigroup in $L^2(\mathbb{R}^d, f_\infty) := \{g : \mathbb{R}^d \to \mathbb{R} : g \text{ is measurable and } \int_{\mathbb{R}^d} g^2 f_\infty dxdv < \infty\}$, $d = 2n$. For well-posedness with non-smooth potentials, we refer to [31, Theorems 6, 7].

The long time behavior and exponential convergence of the solution to the steady state has been studied and there are various results: in [19], algebraic decay was proved for potentials that are asymptotically quadratic (as $|x| \to \infty$) and for initial conditions that are bounded below and above by Gaussians. The authors used logarithmic Sobolev inequalities and entropy methods. In [24], exponential decay was obtained also for faster growing potentials and more general initial conditions. That proof is based on hypoellipticity techniques. By using hypoelliptic methods, Villani proved exponential convergence results in $H^1(\mathbb{R}^d, f_\infty) := \{g \in L^2(\mathbb{R}^d, f_\infty) : |\nabla g| \in L^2(\mathbb{R}^d, f_\infty)\}$ [31, Theorem 35] and in $L^2(\mathbb{R}^d, f_\infty)$ [31, Theorem 37]. The main conditions in Villani's theorems above, as well as in [9, 10, 14, 15, 20, 32], are the validity of the Poincaré inequality (5) and the criterion

$$\exists\, C \geq 0 \ : \quad \left\| \frac{\partial^2 V(x)}{\partial x^2} \right\| \leq C(1 + |\nabla V(x)|), \quad \forall x \in \mathbb{R}^n, \tag{4}$$

where $\left\| \frac{\partial^2 V(x)}{\partial x^2} \right\|$ denotes the Frobenius norm of $\frac{\partial^2 V(x)}{\partial x^2}$.

When $\frac{\partial^2 V}{\partial x^2}$ is bounded, Villani also proved that the solution converges to the steady state exponentially in the logarithmic entropy [31, Theorem 39]. This result was extended in [16] to potentials V satisfying a weighted log-Sobolev inequality and the condition that $V^{-2\eta} \frac{\partial^2 V}{\partial x^2}$ is bounded for some $\eta \geq 0$. Even though Villani's result allows for a general class of potentials, the growth condition (4) is not satisfied by potentials with singularities. This type of potentials, such as Lennard–Jones type interactions with confinement, are considered in [10] and their method relies on an explicit construction of a Lyapunov function and Gamma calculus. In [20], Dolbeault et al. developed a method to get exponential decay in L^2 for a large class of linear kinetic equations, including (1). Their method was also used to study the long time behavior of (1) when the potential V is zero or grows slowly as $|x| \to \infty$, see [11, 12]. Based on a probabilistic coupling method, Eberle et al. [21] obtained an exponential decay result in Wasserstein distance.

The associated semigroup of the kinetic Fokker–Planck equation has instantaneous regularizing properties which is called *hypoellipticity* [26]. This hypoelliptic regularization is obvious when the confining potential V is zero or quadratic as the fundamental solution can be explicitly computed (see [26, 28]). For potentials such that $\frac{\partial^2 V}{\partial x^2}$ is bounded, Hérau [23] obtained short time estimates for a $L^2(\mathbb{R}^d, f_\infty) \to H^1(\mathbb{R}^d, f_\infty)$ regularization by constructing a suitable Lyapunov functional. Based on interpolation inequalities and a system of differential inequalities, Villani [31, Appendix A.21] extended Hérau's result for potentials satisfying (4).

We provide a new method to establish exponential decay of the solution to the steady state in $H^1(\mathbb{R}^d, f_\infty)$ for a wide class of potentials: Our method extends [1, 3, 31] by allowing for more general Lyapunov functionals. Generalizing the previous approaches, the weight matrix in the dissipation functional (a generalized Fisher information) may now depend on x and v. This leads to a new criterion on the potential V. For this entropy method we need the time derivative of the dissipation functional, but we also provide its (x, v)–pointwise analog, in the spirit of the *Gamma calculus* [9]. We provide a formula to estimate easily the exponential decay rate depending on the parameters of the equation, the constants appearing in the Poincaré inequality (5) and the growth condition on the potential (see (6) below). As a test of the effectiveness of our method, we show that our estimate on the decay rate is sharp when the potential is a quadratic polynomial. Moreover, our method lets us obtain estimates on the hypoelliptic regularization for potentials that are more general than in [23].

The organization of this paper is as follows. In Sect. 2, we define the assumptions on the potential, state the main results, and present concrete examples of such potentials. In Sect. 3, we present the intuition and explain our method. Section 4 contains important lemmas about matrix inequalities which are important to construct suitable Lyapunov functionals. The final section presents the proof of the main results.

2 Main Results

We make the following assumptions.

Assumption 2.1 There exists a constant $C_{PI} > 0$ such that the Poincaré inequality

$$\int_{\mathbb{R}^{2n}} h^2 f_\infty dxdv - \left(\int_{\mathbb{R}^{2n}} hf_\infty dxdv\right)^2 \le \frac{1}{C_{PI}} \int_{\mathbb{R}^{2n}} \left(|\nabla_x h|^2 + |\nabla_v h|^2\right) f_\infty dxdv \tag{5}$$

holds for all $h \in H^1(\mathbb{R}^d, f_\infty)$.

Sufficient conditions on the potential appearing in f_∞ so that the Poincaré inequality holds, e.g. the Bakry–Emery criterion, are presented in [8, Chapter 4].

Assumption 2.2 There are constants $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ such that the following $\mathbb{R}^{m\times m}$ matrix, $m{:=}n(n+1)$,

$$\begin{pmatrix} \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V(x))}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V(x))}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V(x))}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V(x))}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V(x))}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V(x))}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) \end{pmatrix} \tag{6}$$

is positive semi-definite for all $x \in \mathbb{R}^n$, where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

Roughly speaking, Assumption 2.2 essentially means that the second order derivatives of V control the third order ones. It implies that $\frac{\partial^2 V(x)}{\partial x^2} + cI$ is positive semi-definite for all $x \in \mathbb{R}^n$, and hence the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$ are uniformly bounded from below. We note that, in contrast to the Bakry–Emery strategy [7], the eigenvalues here may take negative values.

Let $\alpha(x) \in \mathbb{R}$ denote the smallest eigenvalue of $\frac{\partial^2 V(x)}{\partial x^2}$ at $x \in \mathbb{R}^n$. Then the following condition implies Assumption 2.2. For its proof see Appendix 6.1.

Assumption 2.2' *There are constants* $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ *such that* $\frac{\partial^2 V(x)}{\partial x^2} + cI$ *is positive semi-definite and*[1]

$$-\sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c)I \le \frac{\partial^2(\partial_{x_i} V(x))}{\partial x^2} \le \sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c)I \tag{7}$$

for all $x \in \mathbb{R}^n$ *and* $i \in \{1, ..., n\}$.

We denote

$$\alpha_0{:=} \inf_{x\in\mathbb{R}^n} \alpha(x) \tag{8}$$

and assume in the sequel that $\alpha_0 > -\infty$. Hence Assumption 2.2 can only hold for some $c \ge -\alpha_0$.

In the following results, we require that $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2n}, f_\infty)$ which implies $f_0 \in L^1(\mathbb{R}^{2d})$ because of the Hölder inequality $\int_{\mathbb{R}^{2n}} f_0 dxdv \le \sqrt{(\int_{\mathbb{R}^{2n}} \frac{f_0^2}{f_\infty} dxdv)(\int_{\mathbb{R}^{2n}} f_\infty dxdv)}$ and

[1] For two matrices A and $B \in \mathbb{R}^{n\times n}$, $A \ge B$ means that $A - B$ is positive semi-definite.

$\int_{\mathbb{R}^{2n}} f_\infty dxdv = 1$. We now state our first result, i.e. exponential decay of a functional that is a linear combination of the weighted L^2-norm and a Fisher information-type functional:

Theorem 2.3 *Let V be a C^∞ potential in $\mathbb{R}^n$ satisfying Assumptions 2.1 and 2.2. Let C_{PI}, c, τ, and α_0 be the constants in (5), (6), and (8). Suppose the initial data f_0 satisfies $\frac{f_0}{f_\infty} \in H^1(\mathbb{R}^{2n}, f_\infty)$ and $\int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f_0}{f_\infty}\right) \frac{\partial^2 V}{\partial x^2} \nabla_v \left(\frac{f_0}{f_\infty}\right) f_\infty dxdv < \infty$. Then there are explicitly computable constants $C > 0$ and $\lambda > 0$ (independent of f_0) such that the solution $f(t)$ of (1) satisfies*

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv \\
&\quad + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv \\
&\le C e^{-2\lambda t} \left[\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f_0}{f_\infty}\right)\right|^2 f_\infty dxdv \right. \\
&\quad \left. + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f_0}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f_0}{f_\infty}\right) f_\infty dxdv\right]
\end{aligned} \tag{9}$$

for all $t \ge 0$. Moreover, we have:

(a) if $\alpha_0 > \frac{\nu^2}{4}$, $c \le -\frac{\nu^2}{4}$, then $2\lambda = \nu - \tau$;

(b) if $c = -\alpha_0 = -\frac{\nu^2}{4}$, then $2\lambda = \nu - \tau - \varepsilon$ for any $\varepsilon \in (0, \nu - \tau)$;

(c) if $c > -\frac{\nu^2}{4}$, $c + 2\alpha_0 > \frac{\nu^2}{4}$, then

$$2\lambda = \begin{cases} \nu - \tau - \dfrac{c + \frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} & \text{if } \nu - \tau \ge A_1^{-1} + \frac{c+\frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} \\ \dfrac{(\nu-\tau)\sqrt{c+\alpha_0} - \left(c + \frac{\nu^2}{4}\right)\left(\sqrt{1+s_1^2} - s_1\right)}{\sqrt{c+\alpha_0} + A_1 s_1 \left(c + \frac{\nu^2}{4}\right)} & \text{if } \nu - \tau < A_1^{-1} + \frac{c+\frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} \end{cases},$$

where $$A_1 := \frac{1 + \frac{\nu^2}{4} + c + \alpha_0 + \sqrt{\left(\frac{\nu^2}{4} + c + \alpha_0 - 1\right)^2 + \nu^2}}{2\sigma C_{PI}},$$

$$s_1 := \begin{cases} \frac{A_1^2\left(c+\frac{\nu^2}{4}\right)^2 - c - \alpha_0}{2A_1\left(c+\frac{\nu^2}{4}\right)\sqrt{c+\alpha_0}} & \text{if } (\nu-\tau)A_1 = 2 \\ \frac{1}{\nu-\tau}\left[\left|\frac{(\nu-\tau)A_1 - 1}{(\nu-\tau)A_1 - 2}\right| \sqrt{\frac{\left(c+\frac{\nu^2}{4}\right)^2}{c+\alpha_0} + 2(\nu-\tau)A_1^{-1} - (\nu-\tau)^2} - \frac{c+\frac{\nu^2}{4}}{((\nu-\tau)A_1 - 2)\sqrt{c+\alpha_0}}\right] & \text{if } (\nu-\tau)A_1 \ne 2 \end{cases};$$

(d) if $c > -\frac{\nu^2}{4}$, $c + 2\alpha_0 \le \frac{\nu^2}{4}$, then

$$2\lambda = \begin{cases} \nu - \tau - \sqrt{\nu^2 - 4\alpha_0} & \text{if } \nu - \tau \ge A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0} \\ \dfrac{\nu - \tau - \sqrt{\nu^2 - 4\alpha_0}\left(\sqrt{1+s_2^2} - s_2\right)}{1 + A_2 s_2 \sqrt{\nu^2 - 4\alpha_0}} & \text{if } \nu - \tau < A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0} \end{cases},$$

$$\text{where } A_2 := \frac{1+\frac{\nu^2}{2}-\alpha_0+\sqrt{\left(\frac{\nu^2}{2}-\alpha_0-1\right)^2+\nu^2}}{2\sigma C_{PI}},$$

$$s_2 := \begin{cases} \frac{A_2^2\left(\nu^2-4\alpha_0\right)-1}{2A_2\sqrt{\nu^2-4\alpha_0}} & \textit{if } (\nu-\tau)A_2 = 2 \\ \frac{1}{\nu-\tau}\left[\left|\frac{(\nu-\tau)A_2-1}{(\nu-\tau)A_2-2}\right|\sqrt{\nu^2-4\alpha_0+2(\nu-\tau)A_2^{-1}-(\nu-\tau)^2}-\frac{\sqrt{\nu^2-4\alpha_0}}{((\nu-\tau)A_2-2)}\right] & \textit{if } (\nu-\tau)A_2 \neq 2 \end{cases};$$

(e) *if $V(x)$ is a quadratic polynomial of x and $\frac{\partial^2 V}{\partial x^2}$ is positive definite, then Assumptions 2.1 and 2.2 are satisfied with $\tau = 0$, $c = -\alpha_0$ [this rules out the conditions in the case of (c)]. Moreover, the decay rates λ in (a) and (d) are sharp and, in the case of (d), $\nu \geq A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0}$ holds and so $2\lambda = \nu - \sqrt{\nu^2 - 4\alpha_0}$. In the case of (b), the decay rate $2\lambda = \nu - \varepsilon$ is sharp in the sense that (9) holds with the rate $2\lambda = \nu - \varepsilon$ for any small fixed $\varepsilon \in (0, \nu)$, but it does not hold with the rate $2\lambda = \nu$.*

Remark 2.4 1. It is possible to make weaker regularity hypothesis on the potential V, but we maintain the assumption that $V \in C^\infty$ to keep the presentation simple.

2. Since $\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I \geq I$, (9) implies that the solution converges exponentially to the steady state in $H^1(\mathbb{R}^{2n}, f_\infty)$. If the eigenvalues of $\frac{\partial^2 V}{\partial x^2}$ are uniformly bounded, then (9) is equivalent to the exponential decay of the solution to the steady state in $H^1(\mathbb{R}^{2n}, f_\infty)$. Due to the Poincaré inequality (5), the L^2-term on the right hand side of (9) could be omitted.
3. If V satisfies Assumption 2.2 with some constants $c \in \mathbb{R}$ and $\tau \in [0, \nu)$, then V also satisfies Assumption 2.2 with any $\tilde{c} \geq c$ and $\tilde{\tau} \in [\tau, \nu)$. Therefore, these constants are not unique. But the exponential decay rate λ obtained in Theorem 2.3 depends on the choice of c and τ. To obtain a better rate, one has to optimize $\lambda = \lambda(c, \tau)$ with respect to all c and τ satisfying Assumption 2.2.
4. In Theorem 2.3 (b), the constant C in (9) depends on ε, and $C = C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.
5. The highest exponential rate is $\frac{\nu}{2}$ which can be attained by the quadratic potentials V with $\frac{\partial^2 V}{\partial x^2} \geq \frac{\nu^2}{4} I$.

When V is a quadratic polynomial as in Theorem 2.3 (e), we prove the following sharp estimates.

Proposition 2.5 *Let V be a quadratic polynomial and $\frac{\partial^2 V}{\partial x^2}$ be positive definite. Let $\alpha_0 > 0$ be the smallest eigenvalue of $\frac{\partial^2 V}{\partial x^2}$, then*[2]

$$\sup_{1 \neq \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{\|f(t)/f_\infty - 1\|_{L^2(\mathbb{R}^d, f_\infty)}}{\|f_0/f_\infty - 1\|_{L^2(\mathbb{R}^d, f_\infty)}} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \textit{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \textit{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu-\sqrt{\nu^2-4\alpha_0}}{2}t}, & \textit{if } \alpha_0 < \frac{\nu^2}{4} \end{cases} \quad \textit{as } t \to \infty. \tag{10}$$

We shall use this proposition to prove the sharpness of the decay rates in Theorem 2.3 (e). When V is a quadratic polynomial and $-\alpha_0 = -\frac{\nu^2}{4} =: c$, Theorem 2.3 (e) shows that the decay in (9) can be $e^{-(\nu-\varepsilon)t}$ for any small fixed $\varepsilon \in (0, \nu)$, but it can not be $e^{-\nu t}$. In this case, it is natural to expect a decay between $e^{-\nu t}$ and $e^{-(\nu-\varepsilon)t}$: Proposition 2.5 shows that this is indeed the case for the square of the L^2-norm, with the decay $(1+t)^2 e^{-\nu t}$. But an analogous

[2] For functions $\varphi = \varphi(t)$ and $\phi = \phi(t)$, $\varphi \asymp \phi$ as $t \to \infty$ means $\varphi = \mathcal{O}(\phi)$ and $\phi = \mathcal{O}(\varphi)$ as $t \to \infty$.

extension of this result for the functional on the left hand side of (9) (i.e., to replace the term $Ce^{-(\nu-\varepsilon)t}$ with $C(1+t)^2e^{-\nu t}$) has not been obtained so far.

Remark 2.6 Under assumptions of Proposition 2.5, we can construct special solutions $f_s(t)$ (see [3, Section 6]) which satisfy

$$\frac{||f_s(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu-\sqrt{\nu^2-4\alpha_0}}{2}t}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases} \quad \text{as } t \to \infty.$$

Our next result is about the estimates on the hypoelliptic regularization.

Theorem 2.7 *Assume V is a C^∞ potential on $\mathbb{R}^n$ and there are constants $c \in \mathbb{R}$ and $\tau \geq 0$ such that the matrix (6) is positive semi-definite for all $x \in \mathbb{R}^n$. Suppose the initial data f_0 satisfies $\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv < \infty$. Then, for any $t_0 > 0$, there are explicitly computable constants $C_1 = C_1(t_0) > 0$ and $C_2 = C_2(t_0) > 0$ (independent of f_0) such that the inequalities*

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv \leq \frac{C_1}{t^3} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv \quad (11)$$

and

$$\int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv$$
$$\leq \frac{C_2}{t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv \quad (12)$$

hold for all $t \in (0, t_0]$.

In Theorem 2.3 we assumed that the initial data f_0/f_∞ is in $H^1(\mathbb{R}^d, f_\infty)$. If we use the estimates in Theorem 2.7, this condition can be relaxed:

Corollary 2.8 *Let V be a C^∞ potential in $\mathbb{R}^n$ satisfying Assumptions 2.1 and 2.2. Suppose the initial data f_0 satisfies $\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv < \infty$. Then, for any $t_0 > 0$, there is an explicitly computable constant $C = C(t_0) > 0$ (independent of f_0) such that*

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dx dv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv$$
$$+ \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv$$
$$\leq Ce^{-2\lambda t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv \quad (13)$$

holds for all $t \geq t_0$ with λ defined in Theorem 2.3.

Remark 2.9 1. In contrast to Theorem 2.3, Theorem 2.7 holds even if the Poincaré inequality (5) is not satisfied by f_∞. Also, τ can be larger than ν.

2. The exponents of t in (11) and (12) are optimal when V is a quadratic polynomial (see [32, Appendix A]).

To illustrate our result, we present concrete examples of potentials V satisfying our Assumptions 2.1 and 2.2:

Example 2.10 (Polynomial confining potentials) (a) As mentioned in Theorem 2.3, if $V(x) = \frac{x^T M^{-1} x}{2} + p\cdot x + q,\ x \in \mathbb{R}^n$ with a positive definite covariance matrix $M^{-1} \in \mathbb{R}^{n\times n}$, a constant vector $p \in \mathbb{R}^n$ and a constant $q \in \mathbb{R}$, the convergence rate is

$$\lambda = \begin{cases} \frac{\nu}{2}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \qquad\qquad\qquad\qquad \text{(case (a))} \\ \frac{\nu-\varepsilon}{2}, & \text{if } \alpha_0 = \frac{\nu^2}{4},\ \text{for any } \varepsilon \in (0,\nu) \quad \text{(case (b))} \\ \frac{\nu-\sqrt{\nu^2-4\alpha_0}}{2}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \qquad\qquad\qquad\qquad \text{(case (d))} \end{cases},$$

and it is sharp for $\alpha_0 \neq \frac{\nu^2}{4}$, where α_0 is the smallest eigenvalue of M^{-1} (see Theorem 2.3 (e)).

(b) More generally, we consider potentials of the form

$$V(x) = r|x|^{2k} + V_0(x)$$

where $r > 0$, $k \in \mathbb{N}$ and $V_0\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $j < 2k$. Since we have already considered quadratic potentials, we assume $k \geq 2$. V satisfies the Poincaré inequality (5); this can be proven, for example, by showing that V satisfies one of the sufficient conditions given in [6, Corollary 1.6]. Concerning Assumption 2.2' we have

$$r\frac{\partial^2 |x|^{2k}}{\partial x^2} = 2kr|x|^{2k-2}I + 2k(2k-2)r|x|^{2k-4}\begin{pmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_1x_2 & x_2^2 & \dots & x_2x_n \\ \dots & \dots & \dots & \dots \\ x_1x_n & x_2x_n & \dots & x_n^2 \end{pmatrix} \geq 2kr|x|^{2k-2}I.$$

Since V_0 has degree $j < 2k$, there is a constant $A > 0$ such that

$$-A\big(1+|x|^{2k-3}\big)I \leq \frac{\partial^2 V_0(x)}{\partial x^2} \leq A\big(1+|x|^{2k-3}\big)I.$$

Therefore, we can estimate

$$\frac{\partial^2 V(x)}{\partial x^2} \geq \Big(2kr|x|^{2k-2} - A|x|^{2k-3} - A\Big)I. \tag{14}$$

We also observe that there exists a positive constant B such that

$$-B\big(1+|x|^{2k-3}\big)I \leq \frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2} \leq B\big(1+|x|^{2k-3}\big)I$$

for all $i \in \{1,...,n\}$. (14) shows that the smallest eigenvalue of $\frac{\partial^2 V(x)}{\partial x^2}$ satisfies $\alpha(x) \geq 2kr|x|^{2k-2} - A|x|^{2k-3} - A$. Since $2kr|x|^{2k-2} - A|x|^{2k-3} - A$ grows faster than $B(1+|x|^{2k-3})$ as $|x| \to \infty$, there are constants c and $\tau \in [0,\nu)$ such that (7) is satisfied. Thus, Theorem 2.3 applies to this type of potentials. In particular, it applies to double-well potentials of the form $V(x) = r_1|x|^4 - r_2|x|^2,\ \ r_1, r_2 > 0$.

Remark 2.11 1. Our decay and regularization results above extend those of [23], where a stronger assumption, i.e. $\partial^2_{x_ix_j}V \in \bigcap_{p=1}^{\infty} W^{p,\infty}(\mathbb{R}^n)$ for all $i, j \in \{1,...,n\}$, was made. By contrast, we did not require the boundedness of the second and higher derivatives of V.

2. Most of the previous works on the exponential convergence $f(t) \to f_\infty$ as $t \to \infty$ (e.g. [9, 10, 14, 15, 20, 31, 32]) used the growth condition (4) to get some weighted Poincaré type inequalities (see [31, Lemma A.24]), which are crucial in these works—and additional to the Poincaré inequality (5). Our technique is rather different, based on construction of appropriate state dependent matrices and state dependent matrix inequalities so that the (modified) dissipation functional (see (20) below) decays exponentially.
3. Most of the previous methods for proving the exponential convergence do not give an accurate decay rate, λ is typically much too small there (see [31, Section 7.2], [20, Section 1.4]). For example, in [31, Section 7.2], the exponential decay rate $\lambda = \frac{1}{40}$ was obtained for $V(x) = \frac{|x|^2}{2}$ and $\nu = \sigma = 1$. Since our decay rates are sharp for quadratic potentials, in this setting, the true rate $\lambda = \frac{1}{2}$ is given by Theorem 2.3 (a) and (e).

3 Modified Entropy Methods for Degenerate Fokker–Planck Equations

We first consider the following degenerate and non-symmetric Fokker–Planck equation [1, 2]:

$$\begin{cases} \partial_t f = \operatorname{div}(D\nabla f + (D+R)\nabla E f), \ \xi \in \mathbb{R}^d, \ t > 0, \\ f(t=0) = f_0 \in L^1_+(\mathbb{R}^d), \ \int_{\mathbb{R}^d} f_0 \, d\xi = 1 \end{cases} \tag{15}$$

where $D \in \mathbb{R}^{d\times d}$ is a constant, symmetric, positive semi-definite ($\operatorname{rank}(D) < d$) matrix, $R \in \mathbb{R}^{d\times d}$ is a constant skew-symmetric matrix. $E : \mathbb{R}^d \to \mathbb{R}$ is a function which only depends on the state variable ξ. We assume that E is confining (i.e. $E(\xi) \to \infty$ for $|\xi| \to \infty$) and smooth enough so that (15) has a unique and smooth solution. If E grows fast enough, (15) has a normalized steady state $f_\infty = c_E e^{-E}$, $c_E > 0$. The weak maximum principle for degenerate parabolic equations [25] can be applied to (15) and we can prove that $f(t,\xi) \geq 0$ for all $t > 0$, $\xi \in \mathbb{R}^d$. The divergence structure implies that the initial mass is conserved and $f(t,\cdot)$ describes the evolution of a probability density

$$\int_{\mathbb{R}^d} f(t,\xi)d\xi = \int_{\mathbb{R}^d} f_0(\xi)d\xi = 1, \quad \forall t \geq 0.$$

We are interested in the large-time behavior of the solution, in particular, when $\operatorname{rank}(D)$ is less than the dimension d. When D is positive definite ($\operatorname{rank}(D) = d$), the large time behavior and exponential convergence have been studied comprehensively (see [2, 4, 7]). One of the well-known conditions which provides the exponential decay of the solution to the steady state is called *the Bakry–Emery condition* (see (16) below) leading to:

Theorem 3.1 [2, Theorem 2.6] *Assume* $\int_{\mathbb{R}^d} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty d\xi < \infty$ *and*

$$\exists \lambda > 0 \ \textit{such that} \ \frac{\partial^2 E}{\partial \xi^2}(I + RD^{-1}) + \left(\frac{\partial^2 E}{\partial \xi^2}(I + RD^{-1})\right)^T \geq \lambda D^{-1}, \quad \forall \xi \in \mathbb{R}^d. \tag{16}$$

Then

$$\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \leq e^{-2\lambda t} \int_{\mathbb{R}^d} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty d\xi.$$

To prove the theorem above, one considers the time derivative of the L^2-norm and we see that it decreases

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi=-2\int_{\mathbb{R}^d}\nabla^T\left(\frac{f}{f_\infty}\right)D\nabla\left(\frac{f}{f_\infty}\right)f_\infty d\xi=:-I(f(t)|f_\infty)\leq 0. \tag{17}$$

$I(f(t)|f_\infty)$ is called the dissipation functional and since D is positive definite it vanishes if and only if $f=f_\infty$. It can be proven that, under the Bakry–Emery condition,

$$\frac{d}{dt}I(f(t)|f_\infty)\leq -2\lambda I(f(t)|f_\infty). \tag{18}$$

Integrating this inequality from (t,∞) and using the convergences $I(f(t)|f_\infty)\to 0$ and $\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi\to 0$ as $t\to\infty$, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi\leq -2\lambda\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi \tag{19}$$

and, by Grönwall's lemma, we get the desired result.

When D is only positive semi-definite, i.e. rank$(D)<d$, one observes that $I(f(t)|f_\infty)$ may vanish for certain probability densities $f\neq f_\infty$. Hence the inequalities (18) and (19) will not hold in general. Since the above problems stem from the singularity of D, one can modify the dissipation function and define a modified dissipation functional (see also [1, 3])

$$S(f):=2\int_{\mathbb{R}^d}\nabla_\xi^T\left(\frac{f}{f_\infty}\right)P(\xi)\nabla_\xi\left(\frac{f}{f_\infty}\right)f_\infty d\xi \tag{20}$$

where $P:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is a symmetric positive definite matrix which will be chosen later. Extending the approach of [1, 3], we allow the matrix P here to depend on $\xi\in\mathbb{R}^d$. Our goal is to derive a differential inequality similar to (18) (like the dissipation functional satisfied for non-degenerate equations), i.e.

$$\frac{d}{dt}S(f(t))\leq -2\lambda S(f(t)), \tag{21}$$

for some $\lambda>0$ and a "good" choice of the matrix P. If this holds true, we would obtain

$$S(f(t))\leq S(f_0)e^{-2\lambda t}.$$

If we can choose such $P=P(\xi)\geq\eta I$ for some $\eta>0$ and all $\xi\in\mathbb{R}^d$, under the validity of the Poincaré inequality (5) for $f_\infty(\xi)=c_E e^{-E(\xi)}$ (where $\begin{pmatrix}x\\v\end{pmatrix}$ in (5) is replaced with ξ) we have

$$\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi\leq\frac{1}{C_{PI}}\int_{\mathbb{R}^d}\left|\nabla_\xi\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty d\xi\leq\frac{1}{2C_{PI}\eta}S(f(t)),$$

which implies the exponential decay of the L^2-norm

$$\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi\leq\frac{1}{2C_{PI}\eta}S(f_0)e^{-2\lambda t}.$$

More generally, since the quadratic entropy is also a decreasing function of time t, instead of proving (21), we can consider the functional

$$\begin{aligned}\Phi(f(t)) &:= \gamma \int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi + S(f(t)) \\ &= \gamma \int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi + 2 \int_{\mathbb{R}^d} \nabla^T \left(\frac{f}{f_\infty}\right) P(\xi) \nabla \left(\frac{f}{f_\infty}\right) f_\infty d\xi \end{aligned} \tag{22}$$

and choose a suitable parameter $\gamma \geq 0$ and a matrix P such that

$$\frac{d\Phi(f(t))}{dt} \leq -2\lambda \Phi(f(t)) \leq 0 \tag{23}$$

for some $\lambda > 0$. This idea and method were successfully applied in [3] to (15) when the potential E is quadratic.

We shall apply this method to the kinetic Fokker–Planck equation with non-quadratic $V(x)$. First, we denote $\xi := \begin{pmatrix} x \\ v \end{pmatrix} \in \mathbb{R}^{2n}$, $E(\xi) := \frac{\nu}{\sigma}[V(x) + \frac{|v|^2}{2}]$, $f_\infty = e^{-E}$. Then the kinetic Fokker–Planck equation (1) can be written in the form of (15),

$$\partial_t f = \mathrm{div}_\xi (D \nabla_\xi f + (D + R) \nabla_\xi E f) \tag{24}$$

with

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \sigma I \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \quad \text{and} \quad R = \frac{\sigma}{\nu} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{25}$$

The rank of the diffusion matrix D is $n < d = 2n$. Thus, (1) is both non-symmetric and degenerate and the arguments above apply to the equation.

We will develop a modified entropy method. We will choose ξ-dependent matrix P in the modified dissipation functional (20) so that (23) holds and $\lambda > 0$ is as large as possible.

We also mention that when the potential E is quadratic in (15), the question about the long time behavior can be reduced to an ODE problem:

Theorem 3.2 *Let $0 \neq D \in \mathbb{R}^{d \times d}$ be positive semi-definite, $R \in \mathbb{R}^{d \times d}$ be skew-symmetric and $\mathbb{R}^d \ni \xi \to E(\xi) = \frac{\xi^T K^{-1} \xi}{2}$ for some positive definite matrix K. Assume $(D + R)K^{-1}$ is positive stable and there is no non-trivial subspace of* Ker$D$ *which is invariant under $K^{-1}(D - R)$. If f is the solution of* (15) *and $\xi(t) \in \mathbb{R}^d$ is the solution of the ODE $\dot{\xi}(t) = -K^{-\frac{1}{2}}(D + R)K^{-\frac{1}{2}}\xi$ with initial datum $\xi(0) = \xi_0$, then*

$$\sup_{1 \neq \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} = \sup_{0 \neq \xi_0 \in \mathbb{R}^d} \frac{||\xi(t)||_2}{||\xi_0||_2}, \quad t \geq 0. \tag{26}$$

Proof See [5, Theorem 3.4]. □

One consequence of Theorem 3.2 is that the decay estimate of the ODE-solution carries over to the corresponding Fokker–Planck equation.

4 The Choice of the Matrix P

For future reference (in the proof of Theorem 2.7) we shall now also allow the matrix P to be time dependent. Hence we shall next consider the generalized functional

$$S(t, f) := 2\int_{\mathbb{R}^d} \nabla_\xi^T \left(\frac{f}{f_\infty}\right) P(t,\xi)\nabla_\xi \left(\frac{f}{f_\infty}\right) f_\infty d\xi.$$

The following lemmas will play a crucial role in our arguments.

Lemma 4.1 *Let* $P : [0,\infty)\times\mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ *be smooth and* f *be the solution of* (1), *then*

$$\begin{aligned}\frac{d}{dt}S(t, f(t)) = &-4\sigma\int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i}u)^T P\partial_{v_i}u\right\} f_\infty dxdv\\ &- 2\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T\right\} uf_\infty dxdv\\ &- 2\int_{\mathbb{R}^{2n}} u^T \{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v - \partial_t]P\}uf_\infty dxdv,\end{aligned} \tag{27}$$

where $u:=\nabla_{x,v}\left(\frac{f}{f_\infty}\right)$, $Q = Q(x):=\begin{pmatrix} 0 & I\\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I\end{pmatrix}$, *and* $[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v - \partial_t]$ *denotes a scalar differential operator that is applied to each element of the matrix* $P = P(t, x, v)$.

Proof We denote $u_1:=\nabla_x\left(\frac{f}{f_\infty}\right)$, $u_2:=\nabla_v\left(\frac{f}{f_\infty}\right)$, then u_1 and u_2 satisfy

$$\partial_t u_1 = \sigma\Delta_v u_1 - \nu\sum_{i=1}^n v_i\partial_{v_i}u_1 + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u_1 - \sum_{i=1}^n v_i\partial_{x_i}u_1 + \frac{\partial^2 V}{\partial x^2}u_2,$$

$$\partial_t u_2 = \sigma\Delta_v u_2 - \nu\sum_{i=1}^n v_i\partial_{v_i}u_2 + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u_2 - \sum_{i=1}^n v_i\partial_{x_i}u_2 - u_1 - \nu u_2.$$

These equations can be written with respect to $u = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$:

$$\partial_t u = \sigma\Delta_v u - \nu\sum_{i=1}^n v_i\partial_{v_i}u + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u - \sum_{i=1}^n v_i\partial_{x_i}u - Q^T u.$$

It allows us to compute the time derivative of the modified dissipation functional

$$\begin{aligned}\frac{d}{dt}S(t, f(t)) &= 4\int_{\mathbb{R}^{2n}} u^T P\partial_t uf_\infty dxdv + 2\int_{\mathbb{R}^{2n}} u^T\partial_t Puf_\infty dxdv\\ &= 4\sigma\int_{\mathbb{R}^{2n}} u^T P\Delta_v uf_\infty dxdv - 4\nu\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{v_i}uv_i f_\infty dxdv\\ &\quad + 4\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{v_i}u\partial_{x_i}Vf_\infty dxdv - 4\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{x_i}uv_i f_\infty dxdv\\ &\quad - 2\int_{\mathbb{R}^{2n}} u^T\{QP + PQ^T\}uf_\infty dxdv + 2\int_{\mathbb{R}^{2n}} u^T\partial_t Puf_\infty dxdv.\end{aligned} \tag{28}$$

First, we consider the term in the second line of (28) and use $\partial_{v_i} f_\infty = -\frac{\nu}{\sigma} v_i f_\infty$:

$$
\begin{aligned}
&4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial^2_{v_i v_i} u f_\infty dx dv - 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u v_i f_\infty dx dv \\
&= -4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} \partial_{v_i} u^T P \partial_{v_i} u f_\infty dx dv - 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv. \qquad (29)
\end{aligned}
$$

By integrating by parts the last term of (29) we obtain

$$
\begin{aligned}
&- 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv \\
&= 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv + 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial^2_{v_i v_i} P) u f_\infty dx dv \\
&- 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) u v_i f_\infty dx dv
\end{aligned}
$$

and we find

$$
\begin{aligned}
&- 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv = 2\sigma \int_{\mathbb{R}^{2n}} u^T (\Delta_v P) u f_\infty dx dv \\
&- 2\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (v_i \partial_{v_i} P) u f_\infty dx dv.
\end{aligned}
$$

If we use this equality in (29), we get

$$
\begin{aligned}
&4\sigma \int_{\mathbb{R}^{2n}} u^T P \Delta_v u f_\infty dx dv - 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P v_i \partial_{v_i} u f_\infty dx dv \\
&= -4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} (\partial_{v_i} u)^T P \partial_{v_i} u f_\infty dx dv - 2 \int_{\mathbb{R}^{2n}} u^T \{[\nu v \cdot \nabla_v - \sigma \Delta_v] P\} u f_\infty dx dv. \qquad (30)
\end{aligned}
$$

Next, we integrate by parts in the terms in the third line of (28):

$$
\begin{aligned}
&4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u \partial_{x_i} V f_\infty dx dv \\
&= -4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u \partial_{x_i} V f_\infty dx dv \\
&\quad - 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) u \partial_{x_i} V f_\infty dx dv + \frac{4\nu}{\sigma} \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P u \partial_{x_i} V v_i f_\infty dx dv, \qquad (31) \\
&\quad - 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{x_i} u v_i f_\infty dx dv
\end{aligned}
$$

$$= 4\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T P\partial_{x_i} u v_i f_\infty dxdv$$
$$+ 4\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T(\partial_{x_i} P)u v_i f_\infty dxdv - \frac{4\nu}{\sigma}\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T P u\partial_{x_i} V v_i f_\infty dxdv. \tag{32}$$

(31) and (32) show that the third line of (28) equals

$$- 2\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T(\partial_{v_i} P)u\partial_{x_i} V f_\infty dxdv + 2\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T(\partial_{x_i} P)u v_i f_\infty dxdv$$
$$= -2\int_{\mathbb{R}^{2n}} u^T\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x]P\}u f_\infty dxdv. \tag{33}$$

Combining (28), (30), and (33) we obtain the statement (27). □

Remark 4.2 We give now a (formal) generalization of the above result (27) to Markovian evolution equations using the *Gamma calculus*, see, e.g., [8–10]:

First, let L be the generator of some Markovian evolution on $\mathbb{R}^d$ with corresponding invariant measure $f_\infty d\xi$. Let $P = P(\xi)$ be a smooth matrix function (but it does not have to be symmetric or positive definite). We define the first order bilinear form

$$\Gamma^P(g,h) := \nabla_\xi g^T P\nabla_\xi h$$

and

$$\Gamma_2^P(g,h) := \frac{1}{2}\big(L\Gamma^P(g,h) - \Gamma^P(Lg,h) - \Gamma^P(g,Lh)\big).$$

For a solution $h(t)$ of $\partial_t h = Lh$, these definitions give

$$\frac{d}{dt}\Gamma^P(h,h) = \Gamma^P(Lh,h) + \Gamma^P(h,Lh) = -2\Gamma_2^P(h,h) + L\Gamma^P(h,h), \qquad \forall\xi\in\mathbb{R}^d. \tag{34}$$

We use Γ^P to define the modified dissipation functional

$$S(f) := 2\int_{\mathbb{R}^d}\Gamma^P(h,h)\, f_\infty d\xi \quad \text{with } h = \frac{f}{f_\infty}.$$

We obtain by integrating (34):

$$\frac{d}{dt}S(f(t)) = -4\int_{\mathbb{R}^d}\Gamma_2^P(h,h)\, f_\infty d\xi, \tag{35}$$

where we used that $\int_{\mathbb{R}^d} L\Gamma^P(h,h)\, f_\infty d\xi = 0$.

In particular, let L be the generator of the kinetic Fokker–Planck equation (3), and we recall that $\xi := \begin{pmatrix} x \\ v \end{pmatrix}$. Then, a straightforward (but lengthy) computation shows that

$$2\Gamma_2^P(h,h) = 2\sigma\sum_{i=1}^{n}(\partial_{v_i} u)^T P\partial_{v_i} u + u^T\left(QP + PQ^T\right)u + u^T(LP)u$$
$$+ 2\sigma\sum_{i=1}^{n}(\partial_{v_i} u)^T(\partial_{v_i} P)u + 4\sigma\sum_{i=1}^{n} u^T(\partial_{v_i} P)\partial_{v_i} u.$$

One can check (by integrating by parts the term $4\sigma \int_{\mathbb{R}^d} \sum_{i=1}^n u^T(\partial_{v_i} P)\partial_{v_i} u f_\infty d\xi$ in the right hand side of (35)) that (35) coincides with (27). Hence, (35) reproduces (27). But in contrast to (27), the preceding statement (34) is local in ξ and therefore stronger.

The key question for using the modified entropy dissipation functional $S(f)$ is how to choose the matrix P. To determine P we shall need the following algebraic result:

Lemma 4.3 *For any fixed matrix $\mathcal{Q} \in \mathbb{R}^{d\times d}$, let $\mu := \min\{Re(\beta) : \beta$ is an eigenvalue of $\mathcal{Q}\}$. Let $\{\beta_m : 1 \le m \le m_0\}$ be all the eigenvalues of $\mathcal{Q}$ with $\mu = Re(\beta)$, only counting their geometric multiplicity.*

(a) If β_m is non-defective for all $m \in \{1, ..., m_0\}$, then there exists a symmetric, positive definite matrix $P \in \mathbb{R}^{d\times d}$ with

$$\mathcal{Q}P + P\mathcal{Q}^T \ge 2\mu P.$$

(b) If β_m is defective for at least one $m \in \{1, ..., m_0\}$, then for any $\varepsilon > 0$ there exists a symmetric, positive definite matrix $P(\varepsilon) \in \mathbb{R}^{d\times d}$ with

$$\mathcal{Q}P(\varepsilon) + P(\varepsilon)\mathcal{Q}^T \ge 2(\mu - \varepsilon)P(\varepsilon).$$

Proof See [3, Lemma 4.3]. □

We consider the matrix function

$$Q(x) := \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}, \quad x \in \mathbb{R}^n, \tag{36}$$

which appears in (27). We want to construct a symmetric positive definite matrix $P(x)$ such that $Q(x)P(x) + P(x)Q^T(x)$ is positive definite and

$$Q(x)P(x) + P(x)Q^T(x) \ge 2\mu P(x)$$

for some $\mu > 0$ and for all $x \in \mathbb{R}^n$. We recall

$$\alpha(x) := \min_{i\in\{1,..,n\}} \left\{ \alpha_i(x) \ : \ \alpha_i(x) \text{ is an eigenvalue of } \frac{\partial^2 V(x)}{\partial x^2} \right\},$$

$$\alpha_0 := \inf_{x\in\mathbb{R}^n} \alpha(x),$$
$$\mu := \inf_{x\in\mathbb{R}^n,\, i\in\{1,..,n\}} \{\mathrm{Re}(\beta_i(x)) : \beta_i(x) \text{ is an eigenvalue of } Q(x)\}.$$

Lemma 4.4 *(1) The matrix $Q(x)$ is positive stable at any fixed $x \in \mathbb{R}^n$, if and only if $\frac{\partial^2 V(x)}{\partial x^2}$ is positive definite.*

(2) Let $\frac{\partial^2 V(x)}{\partial x^2}$ be positive definite for some $x \in \mathbb{R}^n$. Then:

(a) If $\alpha_0 > \frac{\nu^2}{4}$, then $\mu = \frac{\nu}{2}$ and there exists a symmetric positive definite matrix $P(x)$ such that

$$Q(x)P(x) + P(x)Q^T(x) = 2\mu P(x).$$

(b) If $0 < \alpha_0 < \frac{\nu^2}{4}$, then $\mu = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}$ and there exists a symmetric positive definite matrix $P(x)$ such that

$$Q(x)P(x) + P(x)Q^T(x) \ge 2\mu P(x).$$

(c) *If* $\alpha_0 = \frac{\nu^2}{4}$, *then* $\mu = \frac{\nu}{2}$ *and, for any* $\varepsilon \in (0, \nu)$, *there exists a symmetric positive definite matrix* $P(x, \varepsilon)$ *such that*

$$Q(x)P(x,\varepsilon) + P(x,\varepsilon)Q^T(x) \geq (2\mu - \varepsilon)P(x,\varepsilon).$$

Proof Part 1) Let x be any point of $\mathbb{R}^n$, we compute the eigenvalues $\beta(x)$ of $Q(x)$. If $\beta(x) \neq 0$ we have the condition

$$\begin{aligned}
\det(Q(x) - \beta(x)I) &= \begin{vmatrix} -\beta(x)I & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & (\nu - \beta(x))I \end{vmatrix} \\
&= \frac{1}{(\beta(x))^n} \begin{vmatrix} -\beta(x)I & 0 \\ -\frac{\partial^2 V(x)}{\partial x^2} & -\frac{\partial^2 V(x)}{\partial x^2} + \beta(x)(\nu - \beta(x))I \end{vmatrix} \\
&= (-1)^n \det\left(-\frac{\partial^2 V(x)}{\partial x^2} + \beta(x)(\nu - \beta(x))I\right) \\
&= 0.
\end{aligned}$$

Let $\alpha_i(x) \in \mathbb{R},\ i \in \{1, ..., n\}$ denote the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$, then the above eigenvalue condition reads

$$\prod_{i=1}^{n}(\beta^2(x) - \nu\beta(x) + \alpha_i(x)) = 0.$$

Hence the non-zero eigenvalues of $Q(x)$ are

$$\beta_i^{\pm}(x) = \begin{cases} \frac{\nu \pm \sqrt{\nu^2 - 4\alpha_i(x)}}{2}, & \text{if} \quad \nu^2 \geq 4\alpha_i(x) \\ \frac{\nu \pm \mathbf{i}\sqrt{4\alpha_i(x) - \nu^2}}{2}, & \text{if} \quad \nu^2 < 4\alpha_i(x) \end{cases}, \quad i \in \{1, ..., n\}, \tag{37}$$

where $\mathbf{i} = \sqrt{-1}$. Moreover, $\beta(x) = 0$ can be an eigenvalue of $Q(x)$ iff one of the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$ is zero. This shows that $Q(x)$ is positive stable (i.e., the eigenvalues $\beta_i(x)$ have positive real part) iff $\frac{\partial^2 V(x)}{\partial x^2} > 0$.

For Part 2) we shall construct matrices $P(x)$, which relies on the proof of Lemma 4.3 (Lemma 4.3 in [3]).

(a) Let $\alpha_0 > \frac{\nu^2}{4}$. In this case, because of (37) the matrix $Q(x)$ is positive stable and $\mu = \frac{\nu}{2} > 0$. We define the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} \end{pmatrix},$$

and for this choice, it is easy to check that

$$Q(x)P(x) + P(x)Q^T(x) = \nu P(x) = 2\mu P(x).$$

To make sure that $P(x)$ is positive definite, we compute the eigenvalues $\eta(x)$ of $P(x)$ at each $x \in \mathbb{R}^n$: For $\eta(x) \neq 2$ we have the condition

$$\det(P(x) - \eta(x)I) = \begin{vmatrix} (2-\eta(x))I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I \end{vmatrix}$$
$$= \frac{1}{(2-\eta(x))^n} \begin{vmatrix} (2-\eta(x))I & 0 \\ \nu I & (2-\eta(x))\left(2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I\right) - \nu^2 I \end{vmatrix}$$
$$= \det\left((2-\eta(x))\left(2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I\right) - \nu^2 I\right) = 0.$$

$\eta(x) = 2$ is not an eigenvalue of $P(x)$ and so the eigenvalues of $P(x)$ satisfy

$$\prod_{i=1}^{n} \left(\eta^2(x) - (2+2\alpha_i(x))\eta(x) + 4\alpha_i(x) - \nu^2\right) = 0.$$

We conclude that the eigenvalues are

$$\eta_i^{\pm}(x) = 1 + \alpha_i(x) \pm \sqrt{(\alpha_i(x)+1)^2 - (4\alpha_i(x) - \nu^2)},\ \ i \in \{1, ..., n\}.$$

Since we assumed $\alpha_i(x) \geq \alpha(x) \geq \alpha_0 > \frac{\nu^2}{4}$ for all $i \in \{1, ..., n\}$, the eigenvalues are positive and satisfy

$$\eta := \inf_{x\in\mathbb{R}^n,\, i\in\{1,...,n\}} \eta_i^{\pm}(x) = 1 + \alpha_0 - \sqrt{(\alpha_0+1)^2 - (4\alpha_0 - \nu^2)} > 0.$$

Thus, $P(x)$ is positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$.

$(b) - (c)$ Let $0 < \alpha_0 \leq \frac{\nu^2}{4}$. Then (37) shows $\mu = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}$. Let $\varepsilon > 0$ be a fixed small number. We define

$$\omega := \begin{cases} \alpha_0, & \text{if } \alpha_0 < \frac{\nu^2}{4} \\ \alpha_0 - \frac{\varepsilon^2}{4}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \end{cases}$$

and consider the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + (\nu^2 - 4\omega)I \end{pmatrix}.$$

We compute its eigenvalues $\eta(x)$ by a similar computation as above:

$$\eta_i^{\pm}(x) = 1 + \zeta_i(x) \pm \sqrt{(\zeta_i(x)+1)^2 - (4\zeta_i(x) - \nu^2)}, \tag{38}$$

where $\zeta_i(x) := \alpha_i(x) + \frac{\nu^2}{2} - 2\omega > \frac{\nu^2}{4}$. We also have

$$\eta := \inf_{x\in\mathbb{R}^n,\, i\in\{1,...,n\}} \eta_i^{\pm}(x) = 1 + \alpha_0 + \frac{\nu^2}{2} - 2\omega - \sqrt{\left(\alpha_0 + \frac{\nu^2}{2} - 2\omega - 1\right)^2 + \nu^2} > 0.$$

Thus, $P(x)$ is positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$. Then we compute

$$Q(x)P(x) + P(x)Q^T(x)$$
$$= \left(\nu - \sqrt{\nu^2 - 4\omega}\right)P(x) + \sqrt{\nu^2 - 4\omega}\begin{pmatrix} 2I & (\nu + \sqrt{\nu^2 - 4\omega})I \\ (\nu + \sqrt{\nu^2 - 4\omega})I & 2\frac{\partial^2 V}{\partial x^2} + \sqrt{\nu^2 - 4\omega}(\nu + \sqrt{\nu^2 - 4\omega})I \end{pmatrix}. \tag{39}$$

Since $\frac{\partial^2 V}{\partial x^2} \geq \omega I$, the second matrix in the last line of (39) is bounded below by

$$\begin{pmatrix} 2I & \left(\nu + \sqrt{\nu^2 - 4\omega}\right) I \\ \left(\nu + \sqrt{\nu^2 - 4\omega}\right) I & 2\omega + \sqrt{\nu^2 - 4\omega}\left(\nu + \sqrt{\nu^2 - 4\omega}\right) I \end{pmatrix}$$
$$= \begin{pmatrix} 2I & \left(\nu + \sqrt{\nu^2 - 4\omega}\right) I \\ \left(\nu + \sqrt{\nu^2 - 4\omega}\right) I & \frac{1}{2}\left(\nu + \sqrt{\nu^2 - 4\omega}\right)^2 I \end{pmatrix} \geq 0.$$

Consequently, we get

$$Q(x)P(x) + P(x)Q^T(x) \geq \left(\nu - \sqrt{\nu^2 - 4\omega}\right) P(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

□

Lemma 4.4 shows that, if $\frac{\partial^2 V(x)}{\partial x^2}$ is not positive definite at some $x \in \mathbb{R}^n$ (and hence $\alpha_0 \leq 0$), then $Q(x)$ is not positive stable. In this case, it is not possible to find a positive constant μ and a positive definite matrix $P(x)$ such that $Q(x)P(x) + P(x)Q^T(x) \geq \mu P(x)$. If α_0 is just finite and not necessarily positive, we have the following modified inequality.

Lemma 4.5 *Let $\alpha_0 > -\infty$. Then there exist $\gamma \geq 0$, $\delta \in [0, \nu)$, and a symmetric positive definite matrix function $P(x)$ such that*

$$Q(x)P(x) + P(x)Q^T(x) + \gamma D \geq (\nu - \delta)P(x), \quad \forall x \in \mathbb{R}^d, \tag{40}$$

where $D = \begin{pmatrix} 0 & 0 \\ 0 & \sigma I \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$ is the matrix defined in (24).

Proof Let $a \geq 0$ be any constant such that $a + \alpha_0 > \frac{\nu^2}{4}$. We consider the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + 2aI \end{pmatrix}.$$

In analogy to (38) we find its eigenvalues as

$$\eta_i^{\pm}(x) = 1 + \zeta_i(x) \pm \sqrt{(\zeta_i(x) + 1)^2 - (4\zeta_i(x) - \nu^2)},$$

where $\zeta_i(x) := \alpha_i(x) + a \geq a + \alpha_0 > \frac{\nu^2}{4}$, and $\alpha_i(x) \in \mathbb{R}$, $i \in \{1, ..., n\}$ denote the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$. We also have

$$\eta := \inf_{x \in \mathbb{R}^n, i \in \{1,\dots,n\}} \eta_i^{\pm}(x) = \frac{4\left(a + \alpha_0 - \frac{\nu^2}{4}\right)}{1 + a + \alpha_0 + \sqrt{(a + \alpha_0 - 1)^2 + \nu^2}} > 0. \tag{41}$$

Thus, $P(x)$ is uniformly positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$.

Next we compute

$$QP + PQ^T + \gamma D = \nu P + \begin{pmatrix} 0 & 2aI \\ 2aI & (2\nu a + \gamma\sigma)I \end{pmatrix}$$

$$= (\nu - \delta)P + \begin{pmatrix} 2\delta I & (\nu\delta + 2a)I \\ (\nu\delta + 2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma)I \end{pmatrix}, \tag{42}$$

where $\delta \in [0, \nu)$ will be chosen later. We compute the (real) eigenvalues θ of the symmetric matrix

$$\begin{pmatrix} 2\delta I & (\nu\delta + 2a)I \\ (\nu\delta + 2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma)I \end{pmatrix} \tag{43}$$

which appears in (42):

For $\theta(x) \neq 2\delta$ we have the condition

$$\begin{aligned} &\begin{vmatrix} (2\delta - \theta)I & (\nu\delta + 2a)I \\ (\nu\delta + 2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I \end{vmatrix} \\ &= \frac{1}{(2\delta - \theta)^n} \begin{vmatrix} (2\delta - \theta)I & 0 \\ (\nu\delta + 2a)I & (2\delta - \theta)\left(\delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I\right) - (\nu\delta + 2a)^2 I \end{vmatrix} \\ &= \left|(2\delta - \theta)\left(\delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I\right) - (\nu\delta + 2a)^2 I\right| \\ &= \prod_{i=1}^{n}\left(\theta^2 - \theta\left[2\delta(\alpha_i(x) + a) + 2\delta + 2\nu a + \gamma\sigma\right] + 4\delta^2(\alpha_i(x) + a - \nu^2/4) + 2\delta\gamma\sigma - 4a^2\right) = 0. \end{aligned}$$

Let us consider the following equations with $i \in \{1, ..., n\}$:

$$\theta^2 - \theta[2\delta(\alpha_i(x) + a) + 2\delta + 2\nu a + \gamma\sigma] + [4\delta^2(\alpha_i(x) + a - \nu^2/4) + 2\delta\gamma\sigma - 4a^2] = 0, \tag{44}$$

and we shall show that they have non-negative solutions for an appropriate choice of δ and γ. To this end we see first that

$$2\delta(\alpha_i(x) + a) + 2\delta + 2\nu a + \gamma\sigma \geq 2\delta(\alpha_0 + a) + 2\delta \geq \frac{\delta\nu^2}{2} + 2\delta \geq 0.$$

Next, we choose

$$\delta = \delta(a, \gamma) := \frac{1}{\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}}\left[\sqrt{\left(\frac{\gamma\sigma}{4\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}}\right)^2 + a^2} - \frac{\gamma\sigma}{4\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}}\right] \geq 0, \tag{45}$$

which satisfies

$$4\delta^2\left(a + \alpha_0 - \frac{\nu^2}{4}\right) + 2\delta\gamma\sigma - 4a^2 = 0. \tag{46}$$

Hence, the last term of (44) satisfies

$$4\delta^2\left(\alpha_i(x) + a - \frac{\nu^2}{4}\right) + 2\delta\gamma\sigma - 4a^2 \geq 4\delta^2\left(a + \alpha_0 - \frac{\nu^2}{4}\right) + 2\delta\gamma\sigma - 4a^2 = 0$$

for all $i \in \{1, ..., n\}$. Therefore, the quadratic equations (44) have non-negative coefficients and so their solutions, i.e. the eigenvalues of (43), are non-negative. Consequently, we get (40).

We note that δ from (45) satisfies, for any fixed $a > \frac{\nu}{4} - \alpha_0$, $\delta(a, \gamma) \to 0$ as $\gamma \to \infty$. Hence, choosing γ large enough, we have $\delta \in [0, \nu)$. □

Remark 4.6 If $\alpha_0 > 0$, we can take $\gamma = 0$ in Lemma 4.5. This follows by choosing in the proof of Lemma 4.5

$$a = \begin{cases} 0, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\varepsilon^2}{2}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ \frac{\nu^2-4\alpha_0}{2}, & \text{if } 0 < \alpha_0 < \frac{\nu^2}{4} \end{cases}, \quad \delta = \begin{cases} 0, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\varepsilon}{\sqrt{2}}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ \sqrt{\nu^2 - 4\alpha_0}, & \text{if } 0 < \alpha_0 < \frac{\nu^2}{4} \end{cases},$$

with any $\varepsilon \in (0, \nu)$. Therefore, Lemma 4.5 includes the second part of Lemma 4.4. However, if $\alpha_0 \le 0$, we have to choose $\gamma > 0$.

5 Proofs

5.1 Proof of Theorem 2.3

Proof We denote $u_1 := \nabla_x \left(\frac{f}{f_\infty}\right)$, $u_2 := \nabla_v \left(\frac{f}{f_\infty}\right)$, and $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

We consider the modified dissipation functional

$$S(f(t)) = 2 \int_{\mathbb{R}^{2n}} u^T(t) P u(t) f_\infty dx dv$$

for some symmetric positive definite matrix $P = P(x, v) \in \mathbb{R}^{2n \times 2n}$. By Lemma 4.1 (for a t-independent matrix P) we have

$$\begin{aligned} \frac{d}{dt} S(f(t)) = &-4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv \\ &- 2 \int_{\mathbb{R}^{2n}} u^T \left\{ QP + PQ^T \right\} u f_\infty dx dv \\ &- 2 \int_{\mathbb{R}^{2n}} u^T \left\{ [\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P \right\} u f_\infty dx dv, \end{aligned} \tag{47}$$

with $Q(x) = \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}$. Let $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ are the constants such that Assumption 2.2 is satisfied. Since (6) is positive semi-definite, $\frac{\partial^2 V(x)}{\partial x^2} + cI$ is also positive semi-definite and so $\frac{\partial^2 V(x)}{\partial x^2} \ge -cI$ for all $x \in \mathbb{R}^n$. We define the matrix P depending on the constant c.

Case (a) : Assume $c \le -\frac{\nu^2}{4}$, $\alpha_0 > \frac{\nu^2}{4}$. By Lemma 4.4 $(2a)$ and by its proof, the matrix $P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} \end{pmatrix}$ satisfies

$$Q(x)P(x) + P(x)Q^T(x) = \nu P(x) \quad \text{and} \quad P(x) \ge \eta I$$

for all $x \in \mathbb{R}^n$ and $\eta := 1 + \alpha_0 - \sqrt{(\alpha_0 + 1)^2 - (4\alpha_0 - \nu^2)} > 0$. For this choice of the matrix P,

$$[\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P(x) = \begin{pmatrix} 0 & 0 \\ 0 & -2\frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix}. \tag{48}$$

Then (47) can be written as

$$\begin{aligned}\frac{d}{dt}S(f(t)) &= -4\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u)^T P\partial_{v_i}u\right\}f_\infty dxdv \\ &\quad - 2\nu\int_{\mathbb{R}^{2n}}u^T Puf_\infty dxdv + 4\int_{\mathbb{R}^{2n}}u^T\begin{pmatrix}0 & 0\\ 0 & \frac{\partial^2(v\cdot\nabla_x V)}{\partial x^2}\end{pmatrix}uf_\infty dxdv \\ &= -4\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u)^T P\partial_{v_i}u\right\}f_\infty dxdv - \nu S(f(t)) \\ &\quad + 4\int_{\mathbb{R}^{2n}}u^T\begin{pmatrix}0 & 0\\ 0 & \frac{\partial^2(v\cdot\nabla_x V)}{\partial x^2}\end{pmatrix}uf_\infty dxdv. \end{aligned} \tag{49}$$

We shall now consider each term of this equation. First we compute

$$\begin{aligned} S(f(t)) &= 2\int_{\mathbb{R}^{2n}}\left\{2|u_1|^2 + 2\nu u_1\cdot u_2 + 2u_2^T\frac{\partial^2 V}{\partial x^2}u_2\right\}f_\infty dxdv \\ &= 4\int_{\mathbb{R}^{2n}}|u_1 + \frac{\nu}{2}u_2|^2 f_\infty dxdv + 4\int_{\mathbb{R}^{2n}}u_2^T\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4}I\right)u_2 f_\infty dxdv \\ &\geq 4\int_{\mathbb{R}^{2n}}u_2^T\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4}I\right)u_2 f_\infty dxdv. \end{aligned} \tag{50}$$

Then

$$\begin{aligned} &4\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u)^T P\partial_{v_i}u\right\}f_\infty dxdv \\ &= 4\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}\left(2|\partial_{v_i}u_1|^2 + 2\nu\partial_{v_i}u_1\cdot\partial_{v_i}u_2 + 2(\partial_{v_i}u_2)^T\frac{\partial^2 V}{\partial x^2}\partial_{v_i}u_2\right)\right\}f_\infty dxdv \\ &= 8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}|\partial_{v_i}u_1 + \frac{\nu}{2}\partial_{v_i}u_2|^2\right\}f_\infty dxdv \\ &\quad + 8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4}I\right)\partial_{v_i}u_2\right\}f_\infty dxdv \\ &\geq 8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4}I\right)\partial_{v_i}u_2\right\}f_\infty dxdv. \end{aligned} \tag{51}$$

Now we consider the last term in (49)

$$\begin{aligned} &4\int_{\mathbb{R}^{2n}}u^T\begin{pmatrix}0 & 0\\ 0 & \frac{\partial^2(v\cdot\nabla_x V)}{\partial x^2}\end{pmatrix}uf_\infty dxdv = 4\int_{\mathbb{R}^{2n}}u_2^T\frac{\partial^2(v\cdot\nabla_x V)}{\partial x^2}u_2 f_\infty dxdv \\ &= 4\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j=1}^{n}u_{2,i}v\cdot\nabla_x V_{ij}u_{2,j}\right\}f_\infty dxdv = 4\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j,k=1}^{n}u_{2,i}v_k V_{ijk}u_{2,j}\right\}f_\infty dxdv \\ &= -\frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j,k=1}^{n}u_{2,i}V_{ijk}u_{2,j}(\partial_{v_k}f_\infty)\right\}dxdv \end{aligned}$$

$$
\begin{aligned}
&= \frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j,k=1}^{n}\partial_{v_k}(u_{2,i}u_{2,j})V_{ijk}\right\}f_\infty dxdv\\
&= \frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j,k=1}^{n}(\partial_{v_k}u_{2,i})u_{2,j}V_{ijk}+u_{2,i}(\partial_{v_k}u_{2,j})V_{ijk}\right\}f_\infty dxdv\\
&= \frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i,j,k=1}^{n}(\partial_{v_k}u_{2,i})u_{2,j}V_{ijk}\right\}f_\infty dxdv\\
&= \frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{k=1}^{n}(\partial_{v_k}u_2)^T\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}u_2\right\}f_\infty dxdv,
\end{aligned}\tag{52}
$$

where we integrated by parts and used $\partial_{v_k}f_\infty=-\frac{\nu}{\sigma}v_kf_\infty$ and the notations $u_{2,i}:=\partial_{v_i}\left(\frac{f}{f_\infty}\right)$, $V_{ij}:=\partial^2_{x_ix_j}V$, $V_{ijk}:=\partial^3_{x_ix_jx_k}V$. By (49), (51), (52), and (50) we obtain

$$
\begin{aligned}
&\frac{d}{dt}S(f(t))+(\nu-\tau)S(f(t))\le-\tau S(f(t))\\
&\quad-8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)\partial_{v_i}u_2\right\}f_\infty dxdv\\
&\quad+\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv\\
&\le-4\tau\int_{\mathbb{R}^{2n}}u_2^T\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)u_2f_\infty dxdv\\
&\quad-8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)\partial_{v_i}u_2\right\}f_\infty dxdv\\
&\quad+\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv\\
&=-\frac{8\sigma}{\nu}\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}}\left\{\nu(\partial_{v_i}u_2)^T\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)\partial_{v_i}u_2-(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv\\
&\quad-\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\frac{\tau\nu}{2\sigma}u_2^T\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)u_2f_\infty dxdv.
\end{aligned}
$$

The right hand side of this inequality is a quadratic polynomial with respect to $\partial_{v_i}u_2,\ i\in\{1,...,n\}$, and u_2. The corresponding matrix of this quadratic polynomial is

$$
\begin{pmatrix}
\nu\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right) & 0 & ... & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2}\\
0 & \nu\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right) & ... & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2}\\
... & ... & ... & ... & ...\\
0 & 0 & ... & \nu\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2}\\
-\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} & ... & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2V}{\partial x^2}-\frac{\nu^2}{4}I\right)
\end{pmatrix}.\tag{53}
$$

The assumption $\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \geq \frac{\partial^2 V}{\partial x^2} + cI$ and the Assumption 2.2 imply that (53) is positive semi-definite.

Thus we have obtained

$$\frac{d}{dt} S(f(t)) + (\nu - \tau) S(f(t)) \leq 0$$

and by Grönwall's lemma

$$S(f(t)) \leq e^{-(\nu-\tau)t} S(f_0). \tag{54}$$

The estimate $P(x) \geq \eta I$ and the Poincaré inequality (5) imply

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1 \right)^2 f_\infty dx dv \leq \frac{1}{2C_{PI}\eta} S(f(t)) \leq \frac{1}{2C_{PI}\eta} e^{-(\nu-\tau)t} S(f_0). \tag{55}$$

The matrix inequalities (see Lemma 6.1 in Appendix 6.2)

$$\begin{aligned} \frac{1}{1+\alpha_0+\sqrt{(1-\alpha_0)^2+\nu^2}} P &\leq \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I \end{pmatrix} \\ &\leq \frac{1+\alpha_0+\sqrt{(1-\alpha_0)^2+\nu^2}}{4\alpha_0 - \nu^2} P \end{aligned} \tag{56}$$

show that $S(f(t))$ is equivalent to the functional

$$\int_{\mathbb{R}^{2n}} \left| \nabla_x \left(\frac{f(t)}{f_\infty} \right) \right|^2 f_\infty dx dv + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty} \right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I \right) \nabla_v \left(\frac{f(t)}{f_\infty} \right) f_\infty dx dv.$$

This equivalence, and (55) let us obtain (9).

Case (b) : Assume $c = -\alpha_0 = -\frac{\nu^2}{4}$.

Then by Lemma 4.4 $(2c)$, for any $\varepsilon \in (0, \nu - \tau)$, the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + \varepsilon^2 I \end{pmatrix}$$

satisfies

$$Q(x)P(x) + P(x)Q^T(x) \geq (\nu - \varepsilon) P(x) \quad \text{and} \quad P(x) \geq \eta I \tag{57}$$

for all $x \in \mathbb{R}^n$ and $\eta := 1 + \frac{\nu^2+2\varepsilon^2}{4} - \sqrt{(\frac{\nu^2+2\varepsilon^2}{4} - 1)^2 + \nu^2} > 0$. With this matrix we have

$$\begin{aligned} S(f(t)) &= 4 \int_{\mathbb{R}^{2n}} |u_1 + \frac{\nu}{2} u_2|^2 f_\infty dx dv + 4 \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + \frac{2\varepsilon^2 - \nu^2}{4} I \right) u_2 f_\infty dx dv \\ &\geq 4 \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + \frac{2\varepsilon^2 - \nu^2}{4} I \right) u_2 f_\infty dx dv, \end{aligned} \tag{58}$$

$$\begin{aligned} &4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv \\ &\quad = 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n |\partial_{v_i} u_1 + \frac{\nu}{2} \partial_{v_i} u_2|^2 \right\} f_\infty dx dv \\ &\qquad + 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + \frac{2\varepsilon^2 - \nu^2}{4} I \right) \partial_{v_i} u_2 \right\} f_\infty dx dv, \end{aligned} \tag{59}$$

and by using (48), $\partial_{v_i} f_\infty = -\frac{\nu}{\sigma} v_i f_\infty$:

$$
\begin{aligned}
&-2\int_{\mathbb{R}^{2n}} u^T\left\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v]P\right\} u f_\infty dx dv \\
&= \frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^{n}(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\} f_\infty dx dv. \qquad (60)
\end{aligned}
$$

(47), (57), (58), (59), (60), and similar estimates as for Case $a)$ show that

$$
\begin{aligned}
&\frac{d}{dt}S(f(t)) + (\nu-\tau-\varepsilon)S(f(t)) \\
&\le -\frac{8\sigma}{\nu}\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}}\left\{\nu(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)\partial_{v_i}u_2-(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\} f_\infty dx dv \\
&\quad -\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\frac{\tau\nu}{2\sigma}u_2^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)u_2 f_\infty dx dv.
\end{aligned}
$$

The right hand side of this inequality is a quadratic polynomial with respect to $\partial_{v_i}u_2,\ i\in\{1,...,n\}$, and u_2. The corresponding matrix of this quadratic polynomial is

$$
\begin{pmatrix}
\nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} \\
0 & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} \\
-\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)
\end{pmatrix}. \qquad (61)
$$

Because of $\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I > \frac{\partial^2 V}{\partial x^2}+cI$ and Assumption 2.2, (61) is positive definite and we get

$$
\frac{d}{dt}S(f(t)) + (\nu-\tau-\varepsilon)S(f(t)) \le 0
$$

and by Grönwall's lemma

$$
S(f(t)) \le e^{-(\nu-\tau-\varepsilon)t}S(f_0). \qquad (62)
$$

Similar to (55), we have

$$
\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dx dv \le \frac{1}{2C_{PI}\eta}S(f(t)) \le \frac{1}{2C_{PI}\eta}e^{-(\nu-\tau-\varepsilon)t}S(f_0). \qquad (63)
$$

The functional

$$
\int_{\mathbb{R}^{2n}}\left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv + \int_{\mathbb{R}^{2n}}\nabla_v^T\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv
$$

and $S(f(t))$ are equivalent because of (see Lemma 6.1 in Appendix 6.2)

$$\frac{1}{1+\frac{\nu^2+2\varepsilon^2}{4}+\sqrt{\left(1-\frac{\nu^2+2\varepsilon^2}{4}\right)^2+\nu^2}}P \leq \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I \end{pmatrix} \\ \leq \frac{1+\frac{\nu^2+2\varepsilon^2}{4}+\sqrt{\left(1-\frac{\nu^2+2\varepsilon^2}{4}\right)^2+\nu^2}}{2\varepsilon^2}P. \tag{64}$$

This equivalence, and (63) imply (9).

Case (c) **and** (d), **exponential decay**: Assume $c > -\frac{\nu^2}{4}$. For some $\gamma \geq 0$ to be chosen later, we consider the functional

$$\Phi(f(t)){:=}\gamma \int_{\mathbb{R}^{2n}} \left(\frac{f}{f_\infty}-1\right)^2 f_\infty dx dv + S(f(t)) \\ =\gamma \int_{\mathbb{R}^{2n}} \left(\frac{f}{f_\infty}-1\right)^2 f_\infty dx dv + 2\int_{\mathbb{R}^{2n}} u^T P u f_\infty dx dv. \tag{65}$$

Using (17) and (47) its time derivative reads

$$\frac{d\Phi(f(t))}{dt} = -4\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u\right\} f_\infty dx dv \\ -2\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T + \gamma D\right\} u f_\infty dx dv \\ -2\int_{\mathbb{R}^{2n}} u^T \{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v]P\} u f_\infty dx dv. \tag{66}$$

Let a, to be chosen later, be any number such that $a \geq c+\frac{\nu^2}{4} > 0$ and $a+\alpha_0 > \frac{\nu^2}{4}$. We consider the matrix

$$P(x){:=}\begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2}+2aI \end{pmatrix}. \tag{67}$$

Then, by Lemma 4.5 we have

$$Q(x)P(x)+P(x)Q^T(x)+\gamma D \geq (\nu-\delta)P(x), \quad \forall x\in\mathbb{R}^d, \tag{68}$$

with a constant δ defined in (45). If γ is large enough, (45) shows that $\delta \in (0, \nu-\tau)$.

The choice of the matrix P in (67), (66), and (68) lets us estimate

$$\frac{d\Phi(f(t))}{dt} \leq -4\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u\right\} f_\infty dx dv \\ -(\nu-\delta)S(f(t)) + 4\int_{\mathbb{R}^{2n}} u^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 (v\cdot\nabla_x V)}{\partial x^2} \end{pmatrix} u f_\infty dx dv. \tag{69}$$

Similar computations as for Case (a) as well as (58) (but with $\varepsilon^2 = 2a$) lead to

$$\begin{aligned}&\frac{d}{dt}\Phi(f(t)) + (\nu-\delta-\tau)S(f(t))\\&\quad\le -\frac{8\sigma}{\nu}\sum_{i=1}^n\int_{\mathbb{R}^{2n}}\left\{\nu(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right)\partial_{v_i}u_2-(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv\\&\qquad-\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\frac{\tau\nu}{2\sigma}u_2^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right)u_2 f_\infty dxdv.\end{aligned}$$

The two integrands of the right hand side are together a quadratic polynomial of $\partial_{v_i}u_2$, $i\in\{1,...,n\}$, and u_2, and its corresponding matrix is

$$\begin{pmatrix}\nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right) & 0 & \ldots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2}\\ 0 & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right) & \ldots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2}\\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} & \ldots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2}+\frac{4a-\nu^2}{4}I\right)\end{pmatrix}. \tag{70}$$

Because of $a-\frac{\nu^2}{4}\ge c$ and Assumption 2.2, the matrix (70) is positive semi-definite, thus, we have

$$\frac{d}{dt}\Phi(f(t)) + (\nu-\tau-\delta)S(f(t)) \le 0. \tag{71}$$

The estimate $P(x)\ge\eta I$ ($\eta>0$ defined in (41)) and the Poincaré inequality (5) imply

$$\int_{\mathbb{R}^{2n}}\left(\frac{f}{f_\infty}-1\right)^2 f_\infty dxdv \le \frac{1}{2\eta C_{PI}}S(f(t))$$

and so

$$\frac{1}{1+\frac{\gamma}{2\eta C_{PI}}}\Phi(f(t)) \le S(f(t)).$$

This estimate and (71) let us conclude

$$\frac{d}{dt}\Phi(f(t)) + 2\lambda\Phi(f(t)) \le 0 \tag{72}$$

for

$$2\lambda = \frac{\nu-\tau-\delta}{1+\frac{\gamma}{2\eta C_{PI}}} > 0. \tag{73}$$

By Grönwall's lemma we obtain

$$\Phi(f(t)) \le e^{-2\lambda t}\Phi(f_0). \tag{74}$$

One can check that (see Lemma 6.1 in Appendix 6.2)

$$\frac{1}{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}P \leq \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I \end{pmatrix} \leq \frac{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{4(a+\alpha_0)-\nu^2}P. \tag{75}$$

Hence, $S(f(t))$ is equivalent to the functional

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv.$$

Subsequently, $\Phi(f(t))$ and the functional on the left hand side of (9) are equivalent. This equivalence and (74) let us obtain (9).

Case (c) **and** (d), **estimated decay rate**: Next, we shall estimate λ from (73) explicitly, and we shall choose the parameters a and γ such that λ is (rather) large. By (41) and (46), $\eta = \eta(a)$ and $\delta = \delta(a,\gamma)$ are functions of $a \in [c+\frac{\nu^2}{4}, \infty) \bigcap (\frac{\nu^2}{4}-\alpha_0, \infty)$ and $\gamma \in [0,\infty)$. Since $\delta > 0$, and η is monotonically increasing up to 2, we have the following uniform estimate and choice of the decay rate:

$$2\lambda := \sup_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty),\, \gamma\geq 0} \frac{\nu-\tau-\delta(a,\gamma)}{1+\frac{\gamma}{2\eta(a)C_{PI}}} \leq \sup_{\gamma\geq 0} \frac{\nu-\tau}{1+\frac{\gamma}{4C_{PI}}} \leq \nu-\tau.$$

Next, we shall estimate this supremum (in fact it is a maximum). First we introduce a new variable $s := \frac{\gamma\sigma}{4a\sqrt{a+\alpha_0-\frac{\nu^2}{4}}} \in [0,\infty)$, then

$$\delta(a,\gamma) = \frac{a}{\sqrt{a+\alpha_0-\frac{\nu^2}{4}}}(\sqrt{1+s^2}-s).$$

With the notations $A(a) := \frac{1+a+\alpha_0+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{2\sigma C_{PI}} > 0$ and $B(a) := \frac{a}{\sqrt{a+\alpha_0-\frac{\nu^2}{4}}} > 0$, we have

$$2\lambda = \max_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty),\, s\geq 0} \frac{\nu-\tau-B(a)(\sqrt{1+s^2}-s)}{1+A(a)B(a)s}.$$

Next, we shall fix the parameter a. To estimate λ as accurately as possible, we choose a as the argmin of $B(a)$ such that $\nu-\tau-B(a)(\sqrt{1+s^2}-s)$ is maximal with respect to a. The minimal value of $B(a)$ is

$$\min_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty)} B(a) = \begin{cases} B(a_1) = \frac{c+\frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} & \text{if } c+2\alpha_0 > \frac{\nu^2}{4} \\ B(a_2) = \sqrt{\nu^2-4\alpha_0} & \text{if } c+2\alpha_0 \leq \frac{\nu^2}{4} \end{cases},$$

and this minimum is attained at $a_1 := c+\frac{\nu^2}{4}$ if $c+2\alpha_0 > \frac{\nu^2}{4}$ (i.e. in Case (c)), and $a_2 := 2(\frac{\nu^2}{4}-\alpha_0)$ if $c+2\alpha_0 \leq \frac{\nu^2}{4}$ (i.e. in Case (d)).

If $c + 2\alpha_0 > \frac{\nu^2}{4}$, then $c > -\alpha_0$ and so a varies in

$$\Big[c + \frac{\nu^2}{4}, \infty\Big) \bigcap \Big(\frac{\nu^2}{4} - \alpha_0, \infty\Big) = \Big[c + \frac{\nu^2}{4}, \infty\Big) = \Big[a_1, \infty\Big).$$

Since $A(a)$ is increasing, both $A(a)$ and $B(a)$ attain their minimal values at a_1. Thus, a_1 is optimal, i.e.

$$\max_a \frac{\nu - \tau - B(a)(\sqrt{1+s^2} - s)}{1 + A(a)B(a)s} = \frac{\nu - \tau - B(a_1)(\sqrt{1+s^2} - s)}{1 + A(a_1)B(a_1)s}.$$

If $c+2\alpha_0 \le \frac{\nu^2}{4}$, $a_2 = 2(\frac{\nu^2}{4} - \alpha_0)$ may not be optimal as $A(a)$ does not attain its minimum at this point, i.e.

$$\max_a \frac{\nu - \tau - B(a)(\sqrt{1+s^2} - s)}{1 + A(a)B(a)s} \ge \frac{\nu - \tau - B(a_2)(\sqrt{1+s^2} - s)}{1 + A(a_2)B(a_1)s}.$$

But it is the optimal choice when $s = 0$ and so it gives a good approximation if s is small. From now on we assume that a is fixed as

$$a := \begin{cases} a_1 = c + \frac{\nu^2}{4} & \text{if } \ c + 2\alpha_0 > \frac{\nu^2}{4} \\ a_2 = 2(\frac{\nu^2}{4} - \alpha_0) & \text{if } \ c + 2\alpha_0 \le \frac{\nu^2}{4} \end{cases} . \tag{76}$$

Note that this choice is independent of s.

Let $\Lambda(a, s) := \dfrac{\nu - \tau - B(a)(\sqrt{1+s^2} - s)}{1 + A(a)B(a)s}$ and we seek its maximum with respect to $s \in [0, \infty)$. We compute

$$\begin{aligned} &\partial_s \Lambda(a, s) \\ &= \frac{B(a)}{(1 + A(a)B(a)s)^2 \sqrt{s^2+1}} \Big([1 - (\nu - \tau - B(a))A(a)]\sqrt{s^2+1} \\ &\quad - A(a)B(a)(\sqrt{s^2+1} - 1) - s\Big). \end{aligned} \tag{77}$$

If $1-(\nu-\tau-B(a))A(a) \le 0$, then $\partial_s \Lambda(a, s) \le 0$ which implies that $\Lambda(a, s)$ is a decreasing function of s and the maximum in $[0, \infty)$ is attained at $s = 0$.

If $1 - (\nu - \tau - B(a))A(a) > 0$, then $\partial_s \Lambda(a, 0) = B(a)[1 - (\nu - \tau - B(a))A(a)] > 0$ and $\Lambda(a, s)$ is increasing in a neighborhood of $s = 0$. We also see $\partial_s \Lambda(a, s)$ is negative if s is large enough (since $\nu - \tau > 0$). This means that $\Lambda(a, s)$ starts to grow at $s = 0$ and it decreases as $s \to \infty$. Therefore, there is a point in $(0, \infty)$ at which $\Lambda(a, s)$ takes its maximum. Setting $\partial_s \Lambda(a, s) = 0$ we obtain

$$[1 - (\nu - \tau)A(a)]\sqrt{s^2+1} - s + A(a)B(a) = 0.$$

It has only one solution in $(0, \infty)$ given by

$$s(a) = \begin{cases} \frac{A^2(a)B^2(a) - 1}{2A(a)B(a)} & \text{if } (\nu-\tau)A(a) = 2 \\ \frac{1}{\nu-\tau}\Big[\Big|\frac{(\nu-\tau)A(a)-1}{(\nu-\tau)A(a)-2}\Big|\sqrt{B^2(a) + 2(\nu-\tau)A^{-1}(a) - (\nu-\tau)^2} - \frac{B(a)}{(\nu-\tau)A(a)-2}\Big] & \text{if } (\nu-\tau)A(a) \ne 2 \end{cases} \tag{78}$$

and at this point $\Lambda(a, s)$ attains its maximum with respect to s.

Considering the computations above, we conclude that the decay rate can be estimated by:

$$2\lambda = \begin{cases} \nu - \tau - B(a) & \text{if } \nu - \tau \geq A^{-1}(a) + B(a) \\ \frac{\nu - \tau - B(a)(\sqrt{1+s^2(a)} - s(a))}{1 + A(a)B(a)s(a)} & \text{if } \nu - \tau < A^{-1}(a) + B(a) \end{cases}, \tag{79}$$

where two cases correspond to the two cases discussed after (77). Moreover, a and $s(a)$ are defined in (76) and (78), respectively. If we denote $A_1 := A(a_1)$, $A_2 := A(a_2)$, $s_1 := s(a_1)$ and $s_2 := s(a_2)$ and take into account that $B(a_1) = \frac{c + \frac{\nu^2}{4}}{\sqrt{c + \alpha_0}}$ and $B(a_2) = \sqrt{\nu^2 - 4\alpha_0}$, we obtain the explicit decay rates stated in the theorem.

Case (e) : Let $V(x)$ be a quadratic function of x and $\frac{\partial^2 V}{\partial x^2}$ be positive definite. Then, $\frac{\partial^2(\partial_{x_i} V)}{\partial x^2}$ are zero matrices for all $i \in \{1, ..., n\}$. Thus, V satisfies Assumption 2.2 with $\tau = 0$, $-c = \alpha_0 > 0$.

If $\alpha_0 < \frac{\nu^2}{4}$, then $c + 2\alpha_0 = \alpha_0 < \frac{\nu^2}{4}$ which falls into Case (d).

The constant in the Poincaré inequality (5) equals $C_{PI} = \frac{\nu}{\sigma} \min\{1, \alpha_0\}$ (see [4]). It lets us compute A_2^{-1} explicitly:

$$A_2^{-1} = \frac{2\nu \min\{1, \alpha_0\}}{1 + \frac{\nu^2}{2} - \alpha_0 + \sqrt{\left(\frac{\nu^2}{2} - \alpha_0 - 1\right)^2 + \nu^2}}.$$

In Appendix 6.3 we prove the following inequality:

$$\nu \geq A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0}. \tag{80}$$

Thus Case (d) implies

$$\lambda = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}. \tag{81}$$

If $\alpha_0 \geq \frac{\nu^2}{4}$, the decay rate is explicit by Case (a) and Case (b) :

$$\lambda = \begin{cases} \frac{\nu}{2} & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\nu - \varepsilon}{2} & \text{if } \alpha_0 = \frac{\nu^2}{4}, \text{ for any } \varepsilon \in (0, \nu) \end{cases}. \tag{82}$$

We now prove that the decay rates in (81) and (82) are sharp: From Corollary 2.8

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dx dv \\ \leq C e^{-2\lambda t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|^2 + 1\right) f_\infty dx dv, \quad \forall t \geq t_0$$

holds with the same λ given in (81) and (82). Since $\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right| + 1$ is constant, this estimate implies

$$\sup_{1 \neq \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} \leq \tilde{C} e^{-\lambda t}, \quad \forall t \geq t_0 \tag{83}$$

for some constant $\tilde{C} > 0$. On the one hand this means that the estimated decay rate λ can not be larger than the (true) decay rate of the propagator norm given on the left hand side of (83). On the other hand, Proposition 2.5 gives the sharp decay rates for this propagator norm. The decay rates in (81) and (82) coincide with the ones in Proposition 2.5 except in the case of $\alpha_0 = \frac{\nu^2}{4}$. Thus, the exponential decay rates in Case (a) and Case (d) are sharp. When $\alpha_0 = \frac{\nu^2}{4}$, Proposition 2.5 provides the sharp decay $(1+t)e^{-\frac{\nu}{2}t}$ for the propagator norm. Hence, (9) can hold with rates $\lambda = \frac{\nu-\varepsilon}{2}$ for any small fixed $\varepsilon \in (0, \nu)$, but it does not hold for $\varepsilon = 0$. □

5.2 Proof of Proposition 2.5

Proof of Proposition 2.5 Let V be a quadratic polynomial and $\frac{\partial^2 V}{\partial x^2} =: M^{-1} \in \mathbb{R}^{n\times n}$ be positive definite. Then there are $x_0 \in \mathbb{R}^n$ and $\mathcal{C} \in \mathbb{R}$ such that $V(x) = \frac{(x-x_0)^T M^{-1}(x-x_0)}{2} + \mathcal{C}, \forall x \in \mathbb{R}^n$. Since the change $x \to x + x_0$ does not affect the supremum in (10) and only the gradient of V appears in (1), without loss of generality we assume that $x_0 = 0$ and $\mathcal{C} = 0$.

Step 1, reformulation as an ODE-problem: To this end we use Theorem 3.2. We check the conditions of this theorem for the kinetic Fokker–Planck equation. With the notation $\xi = \begin{pmatrix} x \\ v \end{pmatrix}$, we write

$$E(\xi) = \frac{\nu}{\sigma}\left(V(x) + \frac{|v|^2}{2}\right) = \frac{\nu}{\sigma}\left(\frac{x^T M^{-1} x}{2} + \frac{|v|^2}{2}\right) = \frac{1}{2}\xi^T \begin{pmatrix} \frac{\nu}{\sigma} M^{-1} & 0 \\ 0 & \frac{\nu}{\sigma} I \end{pmatrix} \xi = \frac{\xi^T K^{-1} \xi}{2} \tag{84}$$

with $K^{-1} := \frac{\nu}{\sigma}\begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix}$.

From (25) we see that $\mathrm{Ker} D = \{(\psi, 0)^T : \psi \in \mathbb{R}^n\}$. Let $(\psi, 0)^T \in \mathrm{Ker} D$, then its image under $K^{-1}(D-R)$ is

$$K^{-1}(D-R)\begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & M^{-1} \\ -I & \nu I \end{pmatrix}\begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi \end{pmatrix}$$

and it is in $\mathrm{Ker} D$ iff $\psi = 0$. Therefore, there is no non-trivial $K^{-1}(D-R)$-invariant subspace of $\mathrm{Ker} D$. Next we compute the eigenvalues β of $K^{-1/2}(D+R)K^{-1/2} = \begin{pmatrix} 0 & -M^{-1/2} \\ M^{-1/2} & \nu I \end{pmatrix}$:

$$\begin{vmatrix} -\beta I & -M^{-1/2} \\ M^{-1/2} & (\nu-\beta) I \end{vmatrix} = \begin{vmatrix} -\beta I & 0 \\ M^{-1/2} & (\nu-\beta) I - \beta^{-1} M^{-1} \end{vmatrix}$$
$$= \det(\beta(\beta-\nu) I + M^{-1}) = \prod_{i=1}^{n} (\beta^2 - \nu\beta + \alpha_i) = 0,$$

where α_i, $i \in \{1, ..., n\}$ denote the eigenvalues of M^{-1}. By solving the latter equation, we find that the eigenvalues of $K^{-1/2}(D+R)K^{-1/2}$ are $\beta_i^- = \frac{\nu - \sqrt{\nu^2 - 4\alpha_i}}{2}$, $\beta_i^+ = \frac{\nu + \sqrt{\nu^2 - 4\alpha_i}}{2}$, $i \in \{1, ..., n\}$. If $\alpha_0 > 0$ is the smallest eigenvalue of M^{-1}, then

$$\mu := \min_i \{\mathrm{Re}(\beta_i) : \beta_i \text{ is an eigenvalue of } K^{-1/2}(D+R)K^{-1/2}\} = \begin{cases} \frac{\nu}{2} & \text{if } \alpha_0 \ge \frac{\nu^2}{4} \\ \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2} & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases}.$$

Hence μ is positive, so $K^{-1/2}(D+R)K^{-1/2}$ and $(D+R)K^{-1}$ are positive stable. Therefore, Theorem 3.2 applies to the kinetic Fokker–Planck equation.

Step 2, decay rates of the ODE-solution: We consider the ODE

$$\dot{\xi}(t) = -K^{-1/2}(D+R)K^{-1/2}\xi$$

with the initial data $\xi(0) = \xi_0$. Since $K^{-1/2}(D+R)K^{-1/2}$ is positive stable, the solution $\xi(t)$ is stable. To quantify the decay rate, we continue to analyze the eigenvalues of $K^{-1/2}(D+R)K^{-1/2}$. Let m_i be the multiplicity of $\alpha_i > 0$ as an eigenvalue of M^{-1} (now the α_i with $i \in \{1, ..., \tilde{n}\}$ are labeled without multiplicity). Since M^{-1} is symmetric, there are linearly independent eigenvectors $\psi_{ij} \in \mathbb{R}^n,\ \ j \in \{1, ..., m_i\}$ of M^{-1} corresponding to α_i. Then we can check that the vectors

$$\begin{pmatrix} -\frac{\alpha_i^{1/2}}{\beta_i^-}\psi_{ij} \\ \psi_{ij} \end{pmatrix} \in \mathbb{R}^{2n},\ \ j \in \{1, ..., m_i\} \tag{85}$$

are linearly independent eigenvectors of $K^{-1/2}(D+R)K^{-1/2}$ corresponding to β_i^-, $i \in \{1, ..., \tilde{n}\}$. Moreover, these vectors form a basis of the space of eigenvectors corresponding to β_i^-. Similarly, the vectors

$$\begin{pmatrix} -\frac{\alpha_i^{1/2}}{\beta_i^+}\psi_{ij} \\ \psi_{ij} \end{pmatrix} \in \mathbb{R}^{2n},\ \ j \in \{1, ..., m_i\}. \tag{86}$$

satisfy the same property for β_i^+.

If $\alpha_i \neq \frac{\nu^2}{4}$ for all $i \in \{1, ..., \tilde{n}\}$ (i.e., $\beta_i^- \neq \beta_i^+$), the algebraic multiplicities of β_i^- and β_i^+ are equal to m_i. Then β_i^- (resp. β_i^+) has m_i eigenvectors given by (85) (resp. (86)). Thus, the geometric multiplicities of β_i^- and β_i^+ also equal m_i. In particular, $K^{-1/2}(D+R)K^{-1/2}$ is diagonalizable.

If $\alpha_{i_0} = \frac{\nu^2}{4}$ for some $i_0 \in \{1, ..., \tilde{n}\}$, then the algebraic multiplicity of $\beta_{i_0}^- = \beta_{i_0}^+ = \frac{\nu}{2}$ equals $2m_{i_0}$. Since the vectors (85) and (86) coincide in this case, the geometric multiplicity of $\frac{\nu}{2}$ equals m_{i_0}. Thus, in this case, $\frac{\nu}{2}$ is a defective[3] eigenvalue of $K^{-1/2}(D+R)K^{-1/2}$ with the corresponding eigenvectors

$$\begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix} \in \mathbb{R}^{2n},\ \ j \in \{1, ..., m_{i_0}\}. \tag{87}$$

By solving the following linear system (with respect to ξ)

$$K^{-1/2}(D+R)K^{-1/2}\xi - \frac{\nu}{2}\xi = \begin{pmatrix} -\frac{\nu}{2}I & -M^{-1/2} \\ M^{-1/2} & \frac{\nu}{2}I \end{pmatrix}\xi = \begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix},\quad \xi \in \mathbb{R}^{2d},$$

we find that the solution $\xi = \begin{pmatrix} 0 \\ \frac{2}{\nu}\psi_{i_0 j} \end{pmatrix}$ is a generalized eigenvector of $\frac{\nu}{2}$ corresponding to the eigenvector $\begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix}$. Since $\psi_{i_0 j}$, $j \in \{1, ..., m_{i_0}\}$ are linearly independent, the vectors

$$\begin{pmatrix} 0 \\ \frac{2}{\nu}\psi_{i_0 j} \end{pmatrix},\ \ j \in \{1, ..., m_{i_0}\} \tag{88}$$

[3] An eigenvalue is *defective* if its geometric multiplicity is strictly less than its algebraic multiplicity.

form a set of linearly independent generalized eigenvectors of $\frac{\nu}{2}$. Since the vectors in (87) and (88) are linearly independent and their total number equals $2m_{i_0}$ (which is the algebraic multiplicity of $\frac{\nu}{2}$), we conclude that each eigenvector of $\frac{\nu}{2}$ has only one generalized eigenvector. Therefore, all Jordan blocks associated to $\frac{\nu}{2}$ have the same size 2×2. In particular, if $\alpha_0 = \frac{\nu^2}{4}$, then the eigenvalue $\mu = \frac{\nu}{2}$ is defective and the maximal size of the Jordan blocks associated to $\frac{\nu}{2}$ is 2.

Then, the classical stability theory for ODEs shows that

$$\sup_{1\neq \frac{f_0}{f_\infty}\in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}$$
$$= \sup_{0\neq \xi_0\in\mathbb{R}^d} \frac{||\xi(t)||_2}{||\xi_0||_2} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu-\sqrt{\nu^2-4\alpha_0}}{2}t}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases}$$

as $t\to\infty$. □

Remark 5.1 With the eigenvalues of $C:=(D+R)K^{-1}$ (see (24), (84)) obtained at the end of Step 1 in the above proof, the sharpness of the decay rate μ in the cases 1 and 3 of (10) would also follow from [3, Theorem 6.1].

5.3 Proof of Theorem 2.7 and Corollary 2.8

Proof of Theorem 2.7 **Step 1, an auxiliary inequality**: As we assume the matrix (6) is positive semi-definite, then the following submatrices of (6) are positive semi-definite:

$$Y_k := \begin{pmatrix} \nu\left(\frac{\partial^2 V}{\partial x^2} + cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_k} V)}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_k} V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2} + cI\right)\end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad k\in\{1,...,n\}.$$

Letting $\delta > 0$, we consider

$$X_\delta := \begin{pmatrix} I & \delta I \\ \delta I & \delta^2 I\end{pmatrix} \otimes \left(\frac{\partial^2 V}{\partial x^2} + cI\right) = \begin{pmatrix} \frac{\partial^2 V}{\partial x^2} + cI & \delta\frac{\partial^2 V}{\partial x^2} + \delta cI \\ \delta\frac{\partial^2 V}{\partial x^2} + \delta cI & \delta^2\frac{\partial^2 V}{\partial x^2} + \delta^2 cI\end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

X_δ is positive semi-definite as it is the Kronecker product [27, Corollary 4.2.13] of two positive semi-definite matrices. Hence, we have for all $k\in\{1,...,n\}$:

$$\begin{aligned} \mathrm{Tr}(X_\delta^{1/2} Y_k X_\delta^{1/2}) &= \mathrm{Tr}(X_\delta Y_k) \\ &= (\nu + \delta^2\frac{\tau\nu}{2\sigma})\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)^2\right] \\ &\quad -\delta\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)\frac{\partial^2(\partial_{x_k} V)}{\partial x^2}\right] \geq 0. \end{aligned}$$

This implies

$$\frac{2\sigma\nu + \delta^2\tau\nu}{2\sigma\delta}\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)^2\right] \geq \mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)\frac{\partial^2(\partial_{x_k} V)}{\partial x^2}\right] \tag{89}$$

and by minimizing the constant on the left hand side of (89) with respect to δ (i.e., by choosing $\delta = \sqrt{\frac{2\sigma}{\tau}}$), we obtain

$$\sqrt{\frac{2\tau\nu^2}{\sigma}}\mathrm{Tr}\left[\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)^2\right]\geq\mathrm{Tr}\left[\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V(x))}{\partial x^2}\right] \quad \text{for all} \quad x\in\mathbb{R}^n. \tag{90}$$

□

Step 2, growth estimate for the r.h.s. of (11), (12): We denote $u_1:=\nabla_x\left(\frac{f(t)}{f_\infty}\right)$, $u_2:=\nabla_v\left(\frac{f(t)}{f_\infty}\right)$, and $u:=\begin{pmatrix}u_1\\u_2\end{pmatrix}$. Since $\frac{f(t)}{f_\infty}-1$ satisfies

$$\begin{aligned}\partial_t\left(\frac{f(t)}{f_\infty}-1\right) &= -v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\\ &\quad+\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)-\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\end{aligned}$$

and by integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dxdv=-2\sigma\int_{\mathbb{R}^{2n}}|u_2|^2 f_\infty dxdv. \tag{91}$$

Next, we compute (with $||\cdot||$ denoting the Frobenius norm)

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\ &=2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\partial_t\left(\frac{f(t)}{f_\infty}-1\right)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\ &=2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)\right.\\ &\quad\left.+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\ &\quad+2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)-\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv.\end{aligned} \tag{92}$$

Integrating by parts with respect to v, we obtain

$$\begin{aligned}&2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)-\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\ &=-2\sigma\int_{\mathbb{R}^{2n}}|u_2|^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv.\end{aligned} \tag{93}$$

Next, we work on the term in the second line of (92):

$$
\begin{aligned}
&2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\
&=\int_{\mathbb{R}^{2n}}\left(-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)^2+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)^2\right)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\
&=\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left[v\cdot\nabla_x\left(\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty\right)-\nabla_x V\cdot\nabla_v\left(\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty\right)\right]dxdv\\
&=\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2 v\cdot\nabla_x\left(\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2\right)f_\infty dxdv\\
&=\frac{2\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)u_2\cdot\nabla_x\left(\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2\right)f_\infty dxdv\\
&=\frac{2\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\sum_{k=1}^n u_{2,k}\partial_{x_k}\left(\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2\right)f_\infty dxdv\\
&=\frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n u_{2,k}\sum_{i,j=1}^n(\partial^2_{x_ix_j}V+\delta_{ij}c)\partial^2_{x_ix_j}(\partial_{x_k}V)\right\}f_\infty dxdv, \qquad (94)
\end{aligned}
$$

where we integrated by parts twice, and used $-\frac{\nu}{\sigma}vf_\infty=\nabla_v f_\infty$ and the notations

$$
u_{2,k}:=\partial_{v_k}\left(\frac{f(t)}{f_\infty}\right)\quad\text{and}\quad \delta_{ij}:=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}.
$$

Using the identity

$$
\sum_{i,j=1}^n(\partial^2_{x_ix_j}V+\delta_{ij}c)\partial^2_{x_ix_j}(\partial_{x_k}V)=\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}\right],
$$

the estimate (90), and the discrete Hölder inequality, (94) can be estimated as

$$
\begin{aligned}
&\frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{i,j,k=1}^n u_{2,k}(\partial^2_{x_ix_j}V+\delta_{ij}c)\partial^2_{x_ix_j}(\partial_{x_k}V)\right\}f_\infty dxdv\\
&\quad=\frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n u_{2,k}\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}\right]\right\}f_\infty dxdv\\
&\quad\le 4\sqrt{2\sigma\tau}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n |u_{2,k}|\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right]\right\}f_\infty dxdv\\
&\quad\le 4\sqrt{2\sigma\tau n}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)|u_2|\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right]f_\infty dxdv\\
&\quad\le \sigma\int_{\mathbb{R}^{2n}}|u_2|^2\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right]f_\infty dxdv
\end{aligned}
$$

$$+ 8\tau n \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 \mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)^2\right] f_\infty dxdv. \tag{95}$$

Combining the equations from (92) to (95) and the identity

$$\left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 = \mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)^2\right],$$

we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 f_\infty dxdv \\
&\quad \le -\sigma \int_{\mathbb{R}^{2n}} |u_2|^2 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv + 8\tau n \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv.
\end{aligned} \tag{96}$$

(96) can be reformulated as

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-8\tau nt}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 f_\infty dxdv\right) \\
&\quad \le -\sigma e^{-8\tau nt} \int_{\mathbb{R}^{2n}} |u_2|^2 \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 f_\infty dxdv.
\end{aligned} \tag{97}$$

Step 3, t-dependent functional Ψ: In order to prove the short-time regularization of (11) and (12) we introduce now an auxiliary functional that depends explicitly on time. Our strategy is the generalization of the approach in [31, Theorem A.12], [23, Theorem 1.1], [3, Theorem 4.8].

For $t \in (0, t_0]$, we consider the following functional

$$\begin{aligned}
\Psi(t, f(t)) &:= \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 \left(\gamma_1 e^{-8\tau nt} \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 + \gamma_2\right) f_\infty dxdv \\
&\quad + \int_{\mathbb{R}^{2n}} u^T P u f_\infty dxdv,
\end{aligned} \tag{98}$$

with the t- and x-dependent matrix in $\mathbb{R}^{2n\times 2n}$,

$$P = P(t,x) := \begin{pmatrix} 2\varepsilon^3 t^3 I & \varepsilon^2 t^2 I \\ \varepsilon^2 t^2 I & 2\varepsilon t I + t(\frac{\partial^2 V}{\partial x^2} + cI) \end{pmatrix}. \tag{99}$$

ε, γ_1, and γ_2 are positive constants which we shall fix later. We note that, for all $t \in (0, t_0]$,

$$P(t,x) \ge \begin{pmatrix} \varepsilon^3 t^3 I & 0 \\ 0 & t(\frac{\partial^2 V}{\partial x^2} + cI) + \varepsilon t I \end{pmatrix} > \begin{pmatrix} \varepsilon^3 t^3 I & 0 \\ 0 & t(\frac{\partial^2 V}{\partial x^2} + cI) \end{pmatrix} \ge 0 \tag{100}$$

as $\frac{\partial^2 V}{\partial x^2} + cI$ is positive semi-definite. Thus, $\Psi(t, f(t))$ is non-negative and satisfies

$$\Psi(t, f(t)) \geq \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 \left(\gamma_1 e^{-8\tau n t} \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 + \gamma_2\right) f_\infty dx dv$$
$$+ \varepsilon^3 t^3 \int_{\mathbb{R}^{2n}} |u_1|^2 f_\infty dx dv$$
$$+ t \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + (c+\varepsilon) I\right) u_2 f_\infty dx dv. \tag{101}$$

Our goal is to show that $\Psi(t, f(t))$ decreases. To this end we estimate the time derivative of the second term in (98). First, (27) yields

$$\frac{d}{dt} \int_{\mathbb{R}^{2n}} u^T P u f_\infty dx dv$$
$$= -2\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u\right\} f_\infty dx dv - \int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T - \partial_t P\right\} u f_\infty dx dv$$
$$- \int_{\mathbb{R}^{2n}} u^T \{[\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P\} u f_\infty dx dv, \tag{102}$$

with $Q = \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}$. We consider each terms of (102). Because of (100), the first term can be estimated as

$$-2\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u\right\} f_\infty dx dv$$
$$\leq -2t\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + cI\right) \partial_{v_i} u_2\right\} f_\infty dx dv. \tag{103}$$

For the third term of (102) we have

$$[\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P = \begin{pmatrix} 0 & 0 \\ 0 & -t \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix}$$

and using $v f_\infty = -\frac{\sigma}{\nu} \nabla_v f_\infty$ yields

$$- \int_{\mathbb{R}^{2n}} u^T \{[\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P\} u f_\infty dx dv$$
$$= \frac{2t\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2\right\} f_\infty dx dv. \tag{104}$$

For the second term of (102) we compute

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T - \partial_t P\right\} u f_\infty dxdv \\
&= -\int_{\mathbb{R}^{2n}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 0 & (t-2\varepsilon^3 t^3)\left(\frac{\partial^2 V}{\partial x^2}+cI\right) \\ (t-2\varepsilon^3 t^3)\left(\frac{\partial^2 V}{\partial x^2}+cI\right) & (-1+2\nu t-2\varepsilon^2 t^2)\left(\frac{\partial^2 V}{\partial x^2}+cI\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} f_\infty dxdv \\
&- \int_{\mathbb{R}^{2n}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 2\varepsilon^2 t^2 (1-3\varepsilon) I & [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I \\ [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I & [2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon] I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} f_\infty dxdv.
\end{aligned} \tag{105}
$$

Using the estimates

$$
\begin{aligned}
&-(t-2\varepsilon^3 t^3)\int_{\mathbb{R}^{2n}} u_1^T \left(\frac{\partial^2 V}{\partial x^2}+cI\right) u_2 f_\infty dxdv \\
&\le \varepsilon^3 t^2 |1-2\varepsilon^3 t^2| \int_{\mathbb{R}^{2n}} |u_1|^2 f_\infty dxdv + \frac{|1-2\varepsilon^3 t^2|}{4\varepsilon^3} \int_{\mathbb{R}^{2n}} |u_2|^2 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv
\end{aligned}
$$

and

$$
\begin{aligned}
&-(-1+2\nu t-2\varepsilon^2 t^2)\int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2}+cI\right) u_2 f_\infty dxdv \\
&\le |1-2\nu t+2\varepsilon^2 t^2| \int_{\mathbb{R}^{2n}} |u_2|^2 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\| f_\infty dxdv,
\end{aligned}
$$

we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T - \partial_t P\right\} u f_\infty dxdv \\
&\le \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 + |1-2\nu t+2\varepsilon^2 t^2| \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\right] f_\infty dxdv \\
&- \int_{\mathbb{R}^{2n}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 2\varepsilon^2 t^2 (1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2|) I & [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I \\ [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I & [2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon] I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} f_\infty dxdv.
\end{aligned} \tag{106}
$$

We fix $\varepsilon = \varepsilon(t_0) > 0$ so that the element in the upper left corner of the matrix in (106) is positive for $t > 0$; more precisely we require

$$
1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2| > 0 \quad \text{for all} \quad t \in [0, t_0]. \tag{107}
$$

Then, the matrix in the last line of (106) can be estimated as

$$
\begin{aligned}
&\begin{pmatrix} 2\varepsilon^2 t^2 (1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2|) I & [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I \\ [2c\varepsilon^3 t^3 + \nu\varepsilon^2 t^2 + 2(1-\varepsilon)\varepsilon t] I & [2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon] I \end{pmatrix} \\
&\qquad \ge \begin{pmatrix} 0 & 0 \\ 0 & [2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon] I - \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2|)} I \end{pmatrix}.
\end{aligned}
$$

Using this matrix inequality, we obtain from (106):

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T - \partial_t P\right\} u f_\infty dxdv \\
&\le \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 + |1 - 2\nu t + 2\varepsilon^2 t^2| \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\| \right. \\
&\left. -2c\varepsilon^2 t^2 - 4\varepsilon\nu t + 2\varepsilon + \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1-3\varepsilon - \varepsilon|1-2\varepsilon^2 t^2|)}\right] f_\infty dxdv. \qquad (108)
\end{aligned}
$$

(102), (103), (104), and (108) show that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{2n}} u^T P u f_\infty dxdv \le -2t\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + cI\right) \partial_{v_i} u_2\right\} f_\infty dxdv \\
&+ \frac{2t\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2\right\} f_\infty dxdv \\
&+ \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 + |1 - 2\nu t + 2\varepsilon^2 t^2| \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\| \right. \\
&\left. -2c\varepsilon^2 t^2 - 4\varepsilon\nu t + 2\varepsilon + \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1-3\varepsilon - \varepsilon|1-2\varepsilon^2 t^2|)}\right] f_\infty dxdv.
\end{aligned}
$$

As the matrix (6) is positive semi-definite, we have

$$
\begin{aligned}
&-2t\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + cI\right) \partial_{v_i} u_2\right\} f_\infty dxdv \\
&+ \frac{2t\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2\right\} f_\infty dxdv \\
&\le \tau t \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + cI\right) u_2 f_\infty dxdv \le \tau t \int_{\mathbb{R}^{2n}} |u_2|^2 \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\| f_\infty dxdv.
\end{aligned}
$$

Subsequently,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{2n}} u^T P u f_\infty dxdv \\
&\le \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\|^2 + (|1 - 2\nu t + 2\varepsilon^2 t^2| + \tau t) \left\|\frac{\partial^2 V}{\partial x^2} + cI\right\| \right. \\
&\left. -2c\varepsilon^2 t^2 - 4\varepsilon\nu t + 2\varepsilon + \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1-3\varepsilon - \varepsilon|1-2\varepsilon^2 t^2|)}\right] f_\infty dxdv. \qquad (109)
\end{aligned}
$$

Step 4, decay of the functional Ψ:

We estimate the time derivative of (98): Combining (91), (97), and (109) yield

$$\begin{aligned}&\frac{d}{dt}\Psi(t, f(t))\\&\leq -\int_{\mathbb{R}^{2n}} |u_2|^2 \Bigg[\left(\sigma e^{-8\tau n t}\gamma_1 - \frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3}\right)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2\\&\quad -(|1-2\nu t+2\varepsilon^2 t^2|+\tau t)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\\&+2\sigma\gamma_2+2c\varepsilon^2 t^2+4\varepsilon\nu t-2\varepsilon-\frac{[2c\varepsilon^2 t^2+\nu\varepsilon t+2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2|)}\Bigg] f_\infty dx dv. \end{aligned} \tag{110}$$

We fix $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\begin{aligned}&\left(\sigma e^{-8\tau n t}\gamma_1 - \frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3}\right)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 - (|-1+2\nu t-2\varepsilon^2 t^2|+\tau t)\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\\&+2\sigma\gamma_2+2c\varepsilon^2 t^2+4\varepsilon\nu t-2\varepsilon-\frac{[2c\varepsilon^2 t^2+\nu\varepsilon t+2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2 t^2|)}\geq 0\end{aligned} \tag{111}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, t_0]$. We recall that we have fixed $\varepsilon = \varepsilon(t_0)$ so that (107) holds, which makes the above denominator positive. The existence of such $\gamma_1 > 0$ and $\gamma_2 > 0$ can be proven by the following arguments: We can consider the left hand side of (111) as a quadratic polynomial of $\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\| \in [0,\infty)$. As time t varies in a bounded interval $[0, t_0]$, the terms containing t are bounded. Therefore, we can choose large values for $\gamma_1 = \gamma_1(t_0)$ and $\gamma_2 = \gamma_2(t_0)$ so that this quadratic polynomial is non-negative for all $t \in [0, t_0]$.

Consequently, we obtain that

$$\frac{d}{dt}\Psi(t, f(t)) \leq 0.$$

Hence $\Psi(t, f(t))$ is decreasing and

$$\Psi(t, f(t)) \leq \Psi(0, f_0) \quad \text{for all} \quad t \in [0, t_0]. \tag{112}$$

(101) and (112) show that

$$\int_{\mathbb{R}^{2n}} |u_1|^2 f_\infty dx dv \leq \frac{1}{\varepsilon^3 t^3}\int_{\mathbb{R}^{2n}}\left(\frac{f_0}{f_\infty}-1\right)^2\left(\gamma_1\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dx dv, \tag{113}$$

$$\int_{\mathbb{R}^{2n}} |u_2|^2 f_\infty dx dv \leq \frac{1}{\varepsilon t}\int_{\mathbb{R}^{2n}}\left(\frac{f_0}{f_\infty}-1\right)^2\left(\gamma_1\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dx dv, \tag{114}$$

and

$$\int_{\mathbb{R}^{2n}} u_2^T\left(\frac{\partial^2 V}{\partial x^2}+cI\right)u_2 f_\infty dx dv \leq \frac{1}{t}\int_{\mathbb{R}^{2n}}\left(\frac{f_0}{f_\infty}-1\right)^2\left(\gamma_1\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dx dv. \tag{115}$$

It is clear that there is a positive constant C such that

$$\gamma_1 \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\|^2 + \gamma_2 \le C \left(\left\| \frac{\partial^2 V}{\partial x^2} \right\|^2 + 1 \right). \tag{116}$$

(113), a proper linear combination of (114) and (115), and (116) imply the claimed estimates (11), (12). □

Proof of Corollary 2.8 Theorems 2.3 and 2.7 show that, for $t \ge t_0 > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1 \right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left| \nabla_x \left(\frac{f(t)}{f_\infty} \right) \right|^2 f_\infty dxdv \\
&\quad + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty} \right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I \right) \nabla_v \left(\frac{f(t)}{f_\infty} \right) f_\infty dxdv \\
&\le C e^{-2\lambda(t-t_0)} \left[\int_{\mathbb{R}^{2n}} \left(\frac{f(t_0)}{f_\infty} - 1 \right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left| \nabla_x \left(\frac{f(t_0)}{f_\infty} \right) \right|^2 f_\infty dxdv \right. \\
&\quad \left. + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t_0)}{f_\infty} \right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I \right) \nabla_v \left(\frac{f(t_0)}{f_\infty} \right) f_\infty dxdv \right]
\end{aligned} \tag{117}$$

holds with the constant C and the rate λ given in Theorem 2.3. Using (11) and (12) at $t = t_0$, we get

$$\int_{\mathbb{R}^{2n}} \left| \nabla_x \left(\frac{f(t_0)}{f_\infty} \right) \right|^2 f_\infty dxdv \le \frac{C_1}{t_0^3} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1 \right)^2 \left(\left\| \frac{\partial^2 V}{\partial x^2} \right\|^2 + 1 \right) f_\infty dxdv \tag{118}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t_0)}{f_\infty} \right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I \right) \nabla_v \left(\frac{f(t_0)}{f_\infty} \right) f_\infty dxdv \\
&\quad \le \frac{C_2}{t_0} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1 \right)^2 \left(\left\| \frac{\partial^2 V}{\partial x^2} \right\|^2 + 1 \right) f_\infty dxdv.
\end{aligned} \tag{119}$$

Combining (117), (118), and (119), we obtain (13). □

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Declarations

Conflict of interest The authors have no conflict of interest to declare.

6 Appendix

6.1 Proof that Assumption 2.2′ Implies Assumption 2.2

Assume Assumption 2.2' is satisfied. Let $(u_1, u_2, ..., u_{n+1})^T$ be any vector in $\mathbb{R}^{n(n+1)}$, where u_i is a vector in $\mathbb{R}^n$ for all $i \in \{1, ..., n+1\}$. We compute the quadratic form of the matrix (6)

$$\begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_{n+1} \end{pmatrix}^T \begin{pmatrix} \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V(x))}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V(x))}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V(x))}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V(x))}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V(x))}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V(x))}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_{n+1} \end{pmatrix}$$

$$= \sum_{i=1}^{n}\left\{\nu u_i^T\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)u_i - u_i^T\frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2}u_{n+1}\right\} + \frac{\tau\nu}{2\sigma}u_{n+1}^T\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)u_{n+1}.$$

semi-definite, it is enough to show the quadratic form above is non-negative. Assumption 2.2' implies

$$\left|u_i^T\frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2}u_{n+1}\right| \le |u_i||u_{n+1}|\sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c) \le \nu(\alpha(x)+c)|u_i|^2 + \frac{\tau\nu}{2n\sigma}(\alpha(x)+c)|u_{n+1}|^2.$$

therefore, we get the desired result

$$\sum_{i=1}^{n}\left\{\nu u_i^T\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)u_i - u_i^T\frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2}u_{n+1} + \frac{\tau\nu}{2n\sigma}u_{n+1}^T\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)u_{n+1}\right\}$$
$$\ge \sum_{i=1}^{n}\left\{\nu u_i^T\left(\frac{\partial^2 V(x)}{\partial x^2}-\alpha(x)I\right)u_i + \frac{\tau\nu}{2n\sigma}u_{n+1}^T\left(\frac{\partial^2 V(x)}{\partial x^2}-\alpha(x)I\right)u_{n+1}\right\} \ge 0. \quad \square$$

6.2 Matrix Inequalities for Section 5.1

Lemma 6.1 *Let $\alpha_0 > -\infty$ be the constant defined by (8), $a \in \mathbb{R}$ be some constant such that $a+\alpha_0 > \frac{\nu^2}{4}$, and $P := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V}{\partial x^2}+2aI \end{pmatrix}$. Then*

$$c_1 P \le \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I \end{pmatrix} \le c_2 P \tag{120}$$

holds with $c_1 := \frac{1}{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}} > 0, \quad c_2 := \frac{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{4(a+\alpha_0)-\nu^2} > 0.$

Proof We consider, for some $k \in \mathbb{R}$ to be chosen later as $\frac{1}{2c_{1,2}}$,

$$A{:=}P - 2k \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I \end{pmatrix} = \begin{pmatrix} 2(1-k)I & \nu I \\ \nu I & 2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I \end{pmatrix}.$$

We check the (real) eigenvalues η of the symmetric matrix A (depending on k). It is easy to check that $\eta = 2(1-k)$ is not an eigenvalue of A. If $\eta \neq 2(1-k)$, then we have the condition

$$\begin{aligned}
\det(A-\eta I) &= \begin{vmatrix} 2(1-k)I - \eta I & \nu I \\ \nu I & 2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I \end{vmatrix} \\
&= \frac{1}{(2(1-k)-\eta)^n} \begin{vmatrix} 2(1-k)I - \eta I & 0 \\ \nu I & (2(1-k)-\eta)\left[2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I\right] - \nu^2 I \end{vmatrix} \\
&= \det\left((2(1-k)-\eta)\left[2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I\right] - \nu^2 I\right) = 0.
\end{aligned}$$

If α_i, $i \in \{1, ..., n\}$ are the eigenvalues of $\frac{\partial^2 V}{\partial x^2}$, then the eigenvalues η of A satisfy

$$\begin{aligned}
&\prod_{i=1}^{n} \big(\eta^2 - 2\eta[(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1] \\
&+4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a+\alpha_0-1) - \nu^2\big) = 0. \qquad (121)
\end{aligned}$$

Right inequality of (120): From (121), we see that A is positive semi-definite (i.e., all $\eta \geq 0$) if the following three conditions hold:

$$1-k \geq 0, \qquad \text{(due to the first minor of} A) \qquad (122)$$

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \geq 0, \quad \forall i \in \{1, ..., n\}, \qquad (123)$$

$$4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a+\alpha_0-1) - \nu^2 \geq 0, \quad \forall i \in \{1, ..., n\}. \qquad (124)$$

We set

$$k{:=}\frac{1}{2c_2} > 0.$$

Then, (122) holds:

$$1-k = \frac{\sqrt{(a+\alpha_0-1)^2 + \nu^2} - (a+\alpha_0-1)}{2} > 0. \qquad (125)$$

Using $\alpha_i \geq \alpha_0$ for all $i \in \{1, ..., n\}$ we see that (123) also holds:

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \geq 2(1-k) + a + \alpha_0 - 1 = \sqrt{(a+\alpha_0-1)^2 + \nu^2} > 0.$$

To verify (124) we estimate using $\alpha_i \geq \alpha_0$ for all $i \in \{1, ..., n\}$ and (125)

$$\begin{aligned}
&4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a+\alpha_0-1) - \nu^2 \\
&\geq 4(1-k)^2 + 4(1-k)(a+\alpha_0-1) - \nu^2 = 0.
\end{aligned}$$

Therefore, for k defined in (125), A is positive semi-definite. Hence, the inequality on the right hand side of (120) holds.

Left inequality of (120): Similarly, A is negative semi-definite if the following three conditions hold:

$$1-k \leq 0, \tag{126}$$

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \leq 0, \quad \forall i \in \{1, ..., n\}, \tag{127}$$

$$4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a + \alpha_0 - 1) - \nu^2 \geq 0, \quad \forall i \in \{1, ..., n\}. \tag{128}$$

Setting

$$k := \frac{1}{2c_1} > 0$$

we find

$$1 - k = \frac{-\sqrt{(a+\alpha_0-1)^2 + \nu^2} - (a + \alpha_0 - 1)}{2} < 0 \tag{129}$$

and

$$\begin{aligned}&(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \leq 2(1-k) + a + \alpha_0 - 1\\&= -\sqrt{(a+\alpha_0-1)^2 + \nu^2} < 0.\end{aligned}$$

Finally, we check using $\alpha_i \geq \alpha_0$ for all $i \in \{1, ..., n\}$ and (129)

$$\begin{aligned}&4(1-k)^2(\alpha_i - \alpha_0 + 1)\\&+4(1-k)(a+\alpha_0-1) - \nu^2 \geq 4(1-k)^2 + 4(1-k)(a+\alpha_0-1) - \nu^2 = 0.\end{aligned}$$

Therefore, for k defined in (129), A is negative semi-definite. Hence, the inequality on the left hand side of (120) holds. □

Remark 6.2 Lemma 6.1 proves the following matrix inequalities from Sect. 5.1:

(a) If $a = 0$ and $\alpha_0 > \frac{\nu^2}{4}$, then (120) is the matrix inequality (56).
(b) If $a = \frac{\varepsilon^2}{2}$ and $\alpha_0 = \frac{\nu^2}{4}$, then (120) is the matrix inequality (64).
(c) (120) coincides with the matrix inequality (75).

6.3 Proof of Inequality (80)

We recall the assumption $\alpha_0 < \frac{\nu^2}{4}$. We first rewrite

$$\begin{aligned}A_2^{-1} &= \frac{2\nu \min\{1, \alpha_0\}}{1 + \frac{\nu^2}{2} - \alpha_0 + \sqrt{(\frac{\nu^2}{2} - \alpha_0 - 1)^2 + \nu^2}}\\&= \frac{4 \min\{1, \alpha_0\}}{\nu + 2(1-\alpha_0)\nu^{-1} + \sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}}},\\ \nu - \sqrt{\nu^2 - 4\alpha_0} &= \frac{4\alpha_0}{\nu + \sqrt{\nu^2 - 4\alpha_0}}.\end{aligned}$$

Then (80) is equivalent to

$$\frac{\alpha_0}{\nu + \sqrt{\nu^2 - 4\alpha_0}} \geq \frac{\min\{1, \alpha_0\}}{\nu + 2(1-\alpha_0)\nu^{-1} + \sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}}}. \tag{130}$$

If $\min\{1, \alpha_0\} = \alpha_0$, then (130) is true because of

$$\nu + 2(1-\alpha_0)\nu^{-1} + \sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}} > \nu + \sqrt{\nu^2 - 4\alpha_0}.$$

If $\min\{1, \alpha_0\} = 1$, then (130) is equivalent to

$$\alpha_0\nu - 2\alpha_0(\alpha_0 - 1)\nu^{-1} + \alpha_0\sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}} \geq \nu + \sqrt{\nu^2 - 4\alpha_0},$$

or equivalently

$$(\alpha_0 - 1)(\nu^2 - 2\alpha_0)\nu^{-1} + \alpha_0\sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}} \geq \sqrt{\nu^2 - 4\alpha_0}.$$

The last inequality holds since

$$(\alpha_0 - 1)(\nu^2 - 2\alpha_0)\nu^{-1} \geq (\alpha_0 - 1)(\nu^2 - 4\alpha_0)\nu^{-1} \geq 0$$

and

$$\alpha_0\sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0+1)^2\nu^{-2}} > \sqrt{\nu^2 - 4\alpha_0}.$$

These two cases show that inequality (80) holds. □

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