LONG TIME BEHAVIOR OF FOKKER-PLANCK EQUATIONS

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I. General Fokker-Planck equations

Fokker-Planck equations describe the time evolution of a many-particle system undergoing diffusion

$$\begin{cases}\partial_t f = \mathrm{div}_\xi\left(D\nabla_\xi f + f(D+R)\nabla_\xi E(\xi)\right),\ \xi\in\mathbb{R}^d,\ t>0\\ f_{|t=0} = f_0 \in L^1(\mathbb{R}^d)\end{cases} \tag{1}$$

- $f(t,\xi)$ is the distribution function of the particles, i.e., $f(t,\xi)$ is the probability of having one particle at time t and position ξ.
- $D\in\mathbb{R}^{d\times d}-$ symmetric, positive semi-definite
  $R\in\mathbb{R}^{d\times d}-$ skew-symmetric, i.e., $R^T=-R$.
- $E:\mathbb{R}^d\to\mathbb{R}$ is a given smooth function s.t. $E(\xi)\to+\infty$ as $|\xi|\to+\infty$.
- $f_\infty(\xi) = ce^{-E(\xi)}$ is the unique (normalized) stationary solution or equilibrium, i.e.,
  $$\mathrm{div}_\xi(D\nabla_\xi f_\infty + (D+R)\nabla_\xi E f_\infty) = 0,\ \forall\xi\in\mathbb{R}^d.$$
- (1) is called non-degenerate if $\mathrm{rank}(D)=d$, degenerate if $\mathrm{rank}(D)<d$.

II. Important examples

$d=2n$, $\xi := \begin{pmatrix}x\\v\end{pmatrix}\in\mathbb{R}^{2n}$, $D=\begin{pmatrix}0&0\\0&I\end{pmatrix}\in\mathbb{R}^{2n\times2n}$, $R=\begin{pmatrix}0&-I\\I&0\end{pmatrix}\in\mathbb{R}^{2n\times2n}$

• If $E=V(x)+\frac{|v|^2}{2}$, then (1) is called the kinetic Fokker-Planck equation:

$$\begin{cases}\partial_t f + v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f = \mathrm{div}_v(vf)+\Delta_v f,\ t>0\\ f_{|t=0}=f_0\in L^1(\mathbb{R}^{2n})\end{cases} \tag{2}$$

The stationary solution $f_\infty(x,v) = ce^{-[V(x)+\frac{|v|^2}{2}]}$

• If $E=V(x)+\phi(x)+\frac{|v|^2}{2}$, $-\Delta_x\phi=\int_{\mathbb{R}^n} f dv$, then (1) is called the nonlinear Vlasov-Poisson-Fokker-Planck system:

$$\begin{cases}\partial_t f + v\cdot\nabla_x f - (\nabla_x V+\nabla_x\phi)\cdot\nabla_v f = \mathrm{div}_v(vf)+\Delta_v f,\ t>0\\ -\Delta_x\phi = \displaystyle\int_{\mathbb{R}^n} f dv,\quad f_{|t=0}=f_0\in L^1(\mathbb{R}^{2n}).\end{cases} \tag{3}$$

The stationary solution $f_\infty(x,v)=ce^{-[V(x)+\phi_\infty(x)+\frac{|v|^2}{2}]}$, where ϕ_∞ is a solution of

$$-\Delta_x\phi_\infty(x) = \frac{e^{-[V(x)+\phi_\infty(x)]}}{\int_{\mathbb{R}^n} e^{-[V(x')+\phi_\infty(x')]}dx'}.$$

⋆ D is positive semi-definite with $\mathrm{rank}(D)=n<2n$.

III. Goal

When $\mathrm{rank}(D)<d$, find sufficient (possible necessary) conditions on D, R and E so that

$$f(t,\cdot)\to f_\infty \quad \text{as} \quad t\to\infty$$

and obtain explicit and constructive estimates on rates of convergence.
In particular, solve it for

- the kinetic Fokker-Planck equation
- the nonlinear Vlasov-Poisson-Fokker-Planck system

IV. Bakry-Emery or Entropy method: rank$(D)=d$

• Theorem ([1]). Assume $D\in\mathbb{R}^{d\times d}$ is positive definite ($\mathrm{rank}(D)=d$),

$$\exists\lambda>0 \text{ such that } \frac{\partial^2E}{\partial\xi^2}(I+RD^{-1}) + \left(\frac{\partial^2E}{\partial\xi^2}(I+RD^{-1})\right)^T \geq 2\lambda D^{-1},\ \ \forall\xi\in\mathbb{R}^d.$$

Then

$$\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi \leq e^{-2\lambda t}\int_{\mathbb{R}^d}\left(\frac{f_0}{f_\infty}-1\right)^2 f_\infty d\xi.$$

Proof:
The key idea is to obtain a Grönwall inequality for the dissipation functional:

• $\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi = -2\int_{\mathbb{R}^d}\nabla_\xi^T\left(\frac{f(t)}{f_\infty}\right)D\nabla_\xi\left(\frac{f(t)}{f_\infty}\right)f_\infty d\xi =: -I[f(t)]\leq 0,$
$I[f(t)]$ is called the dissipation functional.

• Bakry-Emery condition implies:

$$\frac{d}{dt}I[f(t)] \leq -2\lambda I[f(t)] \quad\Longrightarrow\quad I[f(t)]\leq e^{-2\lambda t}I[f_0].$$

• Integrating over (t,∞) :

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi \leq -2\lambda\int_{\mathbb{R}^d}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty d\xi.\ \square$$

⋆ If D is semi-definite (i.e., $\mathrm{rank}(D)<d$), this method does not work!

V. IDEA: modified entropy method, rank$(D)<d$

- If $\mathrm{rank}(D)<d$, the dissipation functional $I[f(t)]$ is 'lacking information' on some partial derivatives. We define the modified dissipation functional
  $$I_P[f(t)] := 2\int_{\mathbb{R}^d}\nabla_\xi^T\left(\frac{f(t)}{f_\infty}\right)P\nabla_\xi\left(\frac{f(t)}{f_\infty}\right)f_\infty d\xi$$
  with a positive definite matrix $P=P(\xi)\in\mathbb{R}^{d\times d}$.
- We want to choose a positive definite matrix $P\in\mathbb{R}^{d\times d}$ and make appropriate assumptions on D, R, E such that I_P is a Lyapunov functional and
  $$\frac{d}{dt}I_P[f(t)]\leq -2\lambda I_P[f(t)],\quad \forall t>0 \ \Rightarrow\ I_P[f(t)]\leq e^{-2\lambda t}I_P[f_0],\quad \forall t>0.$$
- Lemma: *For any positive stable matrix Q, there is a positive definite matrix P such that*
  $$QP+PQ^T\geq 2\mu P,$$
  where $\mu>0$ is the smallest real part of the eigenvalues of Q.

VI. Results

Result 1: General Fokker-Planck equation (1) with $E(\xi)=\frac{\xi^TK\xi}{2}$

• Theorem ([2]). Let $D\in\mathbb{R}^{d\times d}$ be positive semi-definite, and $E(\xi)=\frac{\xi^TK\xi}{2}$ for some positive definite matrix K. Assume $Q:=(D-R)K$ is positive stable . Let $\lambda>0$ be the minimal real part of the eigenvalues of $(D-R)K$. Then the solution $f(t)$ of (1) satisfies

$$\left\|\frac{f(t)}{f_\infty}-1\right\|_{H^1(\mathbb{R}^d,f_\infty)}\leq Ce^{-\lambda t}\left\|\frac{f_0}{f_\infty}-1\right\|_{H^1(\mathbb{R}^d,f_\infty)},\quad C>0.$$

Proof: P is chosen such that $QP+PQ^T\geq 2\lambda P$.

Result 2: The kinetic Fokker-Planck equation (2)

• Theorem ([3]). Assume $\exists\alpha\in\mathbb{R}$ such that $\frac{\partial^2V(x)}{\partial x^2}+\alpha I$ is positive definite for all $x\in\mathbb{R}^n$, and the eigenvalues of $\frac{\partial^2V(x)}{\partial x^2}$ grow faster than the eigenvalues of $\frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2}$, $i\in\{1,\dots,n\}$, as $|x|\to\infty$. Then there are $\lambda>0$, $C>0$ such that the solution $f(t)$ of (2) satisfies

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}}\left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv\\ &\quad+\int_{\mathbb{R}^{2n}}\nabla_v^T\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2V}{\partial x^2}+\alpha I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right)f_\infty dxdv\\ &\leq Ce^{-2\lambda t}\left[\int_{\mathbb{R}^{2n}}\left(\frac{f_0}{f_\infty}-1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}}\left|\nabla_x\left(\frac{f_0}{f_\infty}\right)\right|^2 f_\infty dxdv\right.\\ &\quad\left.+\int_{\mathbb{R}^{2n}}\nabla_v^T\left(\frac{f_0}{f_\infty}\right)\left(\frac{\partial^2V}{\partial x^2}+\alpha I\right)\nabla_v\left(\frac{f_0}{f_\infty}\right)f_\infty dxdv\right].\end{aligned}$$

Proof: We choose $P=P(x)=\begin{pmatrix}2I & I\\ I & 2\frac{\partial^2V(x)}{\partial x^2}+2aI\end{pmatrix}$, $a>0$.

Result 3: The Vlasov-Poisson-Fokker-Planck system (3)

• Theorem ([4]). Assume $n=3$, $\beta\in\left(\frac{1}{2},\frac{2}{3}\right)$, there exists a constant $C>0$ such that $\left|\frac{\partial^2V(x)}{\partial x^2}\right|\leq C(1+|\nabla_xV(x)|)$, $\forall x\in\mathbb{R}^d$. Assume

$$\left\|\frac{f_0}{f_\infty}-1\right\|_{H_x^\beta(\mathbb{R}^6,f_\infty)}+\left\|\frac{f_0}{f_\infty}-1\right\|_{H_v^1(\mathbb{R}^6,f_\infty)}\leq\delta$$

for an explicitly computable constant $\delta>0$. Then there exist explicitly computable constants $\lambda>0$, $C_1>0$, and $C_2>0$ (independent of h_0, but depending on δ) such that the solutions f and ϕ of (3) satisfy

$$\left\|\frac{f(t)}{f_\infty}-1\right\|_{H_x^\beta(\mathbb{R}^6,f_\infty)}+\left\|\frac{f(t)}{f_\infty}-1\right\|_{H_v^1(\mathbb{R}^6,f_\infty)}\leq C_1e^{-\lambda t},$$

$$||\nabla_x\phi(t)-\nabla_x\phi_\infty||_{L^\infty(\mathbb{R}^3)}\leq C_2e^{-\lambda t}.$$

Proof: Suitable matrix P, weighted Poincaré inequalities, hypoelliptic regularity.

⋆ Polynomials $V(x)\approx|x|^k$ as $|x|\to\infty$, $k\in\mathbb{N}$, satisfy all assumptions.

VII. References

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