TECHNISCHE UNIVERSITÄT WIEN

DIPLOMARBEIT

Overlapping Generations under Hyperbolic Discounting

zur Erlangung des akademischen Grades einer

Diplom-Ingenieurin

im Rahmen des Studiums

Statistik-Wirtschaftsmathematik

ausgeführt am Institut für Stochastik und Wirtschaftsmathematik der Fakultät für Mathematik und Geoinformation der Technischen Universität Wien

von

Katharina Ruhmer
Matrikelnummer 01227422

unter der Leitung von

Ass. Prof. Dipl.-Math. Dr. Timo Trimborn

Wien, 19.12.2018

(Unterschrift Verfasserin) (Unterschrift Betreuer)

Abstract

Hyperbolic discounting has been argued to inevitably lead to time-inconsistent behaviour. In this thesis a time-consistent method of hyperbolic discounting is introduced to the Blanchard-Yaari-Buiter model of overlapping generations in continuous time. Results are compared by using an argument that ensures equivalent present values of a constant utility stream for both discounting methods. Analysis at an aggregate level in a closed economy shows that for high life expectancies the economy is balanced at a lower interest rate under hyperbolic discounting than under conventional exponential discounting and vice versa. This is due to the fact that hyperbolically discounting individuals exhibit higher discount rates in the short run but more savings in the long run.

Contents

1 Introduction

The proper choice of a discounting method has always been crucial to economic analysis, as it is used to model intertemporal choices - decisions involving tradeoffs between costs and benefits occurring at various times. These decisions may include any aspect of life, such as one's health, wealth or happiness and are ultimately co-responsible for the economic prosperity of nations (Frederick et al., 2002). Thus, it is all the more important to apply the appropriate discount rate before making policy decisions about spendings on research, development, health, education and so forth (Berns et al., 2007).

So far, conventional economic analysis has been based on exponential discounting, which implies a constant discount rate. Empirical evidence from the field of economics and psychology, however, suggests that the individual time preference rate declines hyperbolically over time (Frederick et al., 2002; DellaVigna, 2009), as humans are present biased and exhibit higher impatience concerning short-run tradeoffs than tradeoffs in the distant future. For instance, in Thaler's (1981) early study, individuals were asked which amount of money in one month/one year/ten years would make them indifferent to receiving $15 now. His results imply an average annual discount rate that declines from 345% to 120% to eventually 19% as the delay gets longer.

In fact, countries such as France and the United Kingdom already apply declining discount rates in the evaluation of public projects (Treasury, 2003; Lebegue, 2005). A group of leading economists also recently concluded that this method of modelling time preference is compelling and should be taken into consideration by the Office of Management and Budget (OMB) in the United States (Arrow et al., 2014).

However, an important concern regarding hyperbolic discounting, is that it may lead to suboptimal and inconsistent decision making, which would violate the basic premise of rational behaviour in most economic models. In behavioural economics hyperbolic discounting has indeed often been used to explain irrational behaviour especially concerning self-control, such as procrastination, addiction or under-savings. This led to the belief that hyperbolic discounting inevitably entails a time inconsistency problem and is therefore incompatible with rational behaviours. Yet a non-constant subjective rate of time preference does not necessarily imply non-rational behaviour. It can be shown that intertemporal decision making is time-consistent if and only if the individual's discount function is multiplicatively separable in planning time and calendar time (Burness, 1976; Drouhin, 2009).

Strulik (2017) proposed the use of time-consistent hyperbolic discounting and applied a corresponding multiplicatively separable discount function to three canonical environmental problems. His results show that hyperbolic discounting outperforms conventional exponential discounting in various respects, despite the initially high present bias.

The aim of this thesis is now to extend the application of time-consistent hyperbolic discounting to one of the workhorse models of modern macroeconomics by applying Strulik's method to the Blanchard-Yaari-Buiter-model of overlapping generations (OLG) in continuous time (Blanchard, 1985; Yaari, 1965; Buiter, 1988). In this economy agents enjoy a life of perpetual youth, as they face a constant probability of death throughout their lives. As a consequence, individuals maximize their expected lifetime utility by choosing an optimal consumption path.

Whereas conventional exponential discounting leads to individual consumption that increases exponentially over time in this economy, hyperbolic discounting results in a significantly different behaviour. Due to the high impatience concerning the near future, optimal consumption starts off at a relatively high level early in life. This initially high level, however, decreases with age until a certain point where agents start to live off their savings and exhibit a growing consumption rate.

In order to achieve a fair comparison between the original and hyperbolically discounting model, an equivalent present value argument by Myerson et al. (2001) is applied, which disentangles the effect of different discounting methods from pure impatience and provides a parameter restriction for the exponential discount rate. As a consequence, it can be analytically shown that the initial propensity to consume out of wealth and therefore also initial consumption is indeed higher once hyperbolic discounting is implemented. This result is also being reflected at an aggregate macroeconomic level, as an analysis of a closed economy further shows. Since the property of convenient aggregation is no longer given under hyperbolic discounting, the focus lies solely on the steady state, which is numerically computed with the aid of a bisection method. A benchmark run with conventional parameters yields a higher steady state capital stock and thereby a lower steady state interest rate in the hyperbolic model. However, further analysis ultimately shows that this result is strongly dependent on the choice of certain parameter values.

The remainder of this thesis is organized as follows: In the next Section a basic version of the Blanchard-Yaari-Buiter model with an unspecified discount factor is introduced, which serves as framework. Section 3 and 4 cover general analytical results for the original and hyperbolically discounting model. The parameters used for subsequent numerical calculations are introduced in Section 5, together with the equivalent present value requirement. Section 6 addresses both discounting methods in a closed economy and their comparison on the basis of the steady state interest rate. Further, the sensitivity to parameter deviations of these results are analysed. Section 7 ultimately concludes.

2 The Model

2.1 An Economy of Perpetual Youth

The basis for the Blanchard-Yaari-Buiter overlapping generations model is built on Yaari's insights on lifetime uncertainty. His model assumes that agents face finite lives until they die at a random time $T \leq \bar{T}$. This randomness results in a stochastic decision problem for the consumer, for which the conventional way of maximizing lifetime utility, as in the well-known Ramsey-Cass-Koopmans- model, is unreasonable (Ramsey, 1928; Cass, 1965; Koopmans, 1963). Instead, the *expected* lifetime utility becomes the objective function (see Yaari, 1965).
Blanchard simplified assumptions by supposing the maximal attainable age to be infinite, i.e. $\bar{T} \to \infty$, and the probability density function of the individual's time of death $\phi(T)$ to be exponential, i.e. with parameter $\mu > 0$:

$$\phi(T) := \begin{cases} \mu e^{-\mu T} & \text{for } T \geq 0 \\ 0 & \text{for } T < 0. \end{cases} \tag{1}$$

This yields, first of all, that the probability of a consumer being still alive at time τ equals $e^{-\mu\tau}$ and second, one of Blanchard's central assumptions, that the instantaneous probability of death ("hazard rate") at any time is μ and therefore constant throughout life and independent of the consumer's age. The expected remaining lifetime is also constant and equals $1/\mu$. Thus, an agent who has already lived for a hundred years is no more likely to die in the next year than an agent, who was born yesterday. Hence he enjoys a life of *perpetual youth* (Heijdra, 2009).
A proper way of interpreting this rather unrealistic circumstance is to not think of agents in this economy, but of families or households. Then the hazard rate μ describes the probability that the family ends, which can either happen by death of family members that have no children or by them having no bequest motive (Blanchard, 1985).

Another simplifying assumption made by Blanchard is one of constant population growth. At each instant in time τ a large cohort of agents that hold no financial assets is assumed to be born according to a birth rate. This birth rate is additionally assumed to be equal to the instantaneous probability of death μ. Provided that the size of these cohorts is sufficiently large and every individual faces the same probability of death, by the law of large numbers, probabilities and frequencies coincide and the number of deaths matches exactly the number of births. This implies that the population remains constant and can therefore be normalized to 1. (Heijdra, 2009)

2.2 The Setup

Let the consumption and real financial assets of an individual born at time v as of time τ be denoted by $c(v,\tau)$ and $a(v,\tau)$, respectively. Real wage income $w(\tau)$ is assumed to be equally distributed over individual households, i.e. independent of the time of birth. The real interest rate r is assumed, unlike Blanchard, to be independent of time, which simplifies further analysis.
The objective of an agent born at time v in this economy is to plan lifetime consumption such that it maximizes expected lifetime utility, discounted to the present at planning time $t \geq v$. The discount factor is thereby given by the yet unspecified function $D(v,t,\tau)$ and instantaneous utility assumed to be the logarithm of consumption. This leads to the following expression of expected lifetime utility:

$$\mathbb{E}_t \left[\int_t^\infty \ln(c(v,\tau)) D(v,t,\tau) d\tau \right]. \tag{2}$$

Given that the time of death is the only uncertainty and follows an exponential distribution with parameter μ according to (1), maximizing this expected value is equivalent to solving the following problem (Blanchard, 1985; Heijdra, 2009; Blanchard and Fischer, 1989):

$$\max_{c(v,\tau)} \int_t^\infty \ln(c(v,\tau)) e^{\mu(t-\tau)} D(v,t,\tau) d\tau. \tag{3}$$

Furthermore, the agent simultaneously faces the budget constraint

$$\dot{a}(v,\tau) = (r+\mu)a(v,\tau) + w(\tau) - c(v,\tau), \tag{4}$$

where we use the notation $\dot{a}(v,\tau) = \partial a(v,\tau)/\partial \tau$ for the derivative with respect to time. At every instant in time agents are not only assumed to receive an interest r on their financial assets but also a rate of return μ from their insurance company. However, as soon as they die, their entire estate accrues to this insurance.
To prevent agents running a Ponzi game against the life insurance, that is, going infinitely into debt by accumulating wealth forever at a rate higher than the effective interest rate $r+\mu$, the following solvency condition has to be fulfilled:

$$\lim_{\tau \to \infty} a(v,\tau) e^{-R(t,\tau)} = 0, \tag{5}$$

where we define the used discount rate as

$$R(t,\tau) \coloneqq \int_t^\tau r + \mu \; ds. \tag{6}$$

To sum up, the Blanchard-Yaari overlapping generations model with a yet unspecified discount function reads as follows:

$$\max_{c(v,\tau)} \int_t^\infty \ln(c(v,\tau)) e^{\mu(t-\tau)} D(v,t,\tau) d\tau \tag{7}$$

$$\text{s.t.} \;\; \dot{a}(v,\tau) = (r+\mu)a(v,\tau) + w(\tau) - c(v,\tau) \;\; \text{and} \tag{8}$$

$$\lim_{\tau\to\infty} a(v,\tau)e^{-R(t,\tau)} = 0, \quad \text{with} \;\; R(t,\tau) \coloneqq \int_t^\tau r + \mu \; ds. \tag{9}$$

Using this as framework, we now first focus on the basic results of the original model featuring conventional exponential discounting before implementing hyperbolic discounting and investigating its outcome.

3 Exponential Discounting

Exponential discounting, originally proposed by Samuelson (1937) in his discounted utility model, is due to its simplicity presumably the most commonly applied discounting method in economics. It implies that the rate at which individuals discount future utility is constant throughout time. Let this rate be denoted by $\bar{\rho}$. The corresponding discount function is given, as the name already suggests, by an exponential function dependent on calendar time τ and decision time t

$$D(v,t,\tau) = e^{\bar{\rho}(t-\tau)}, \tag{10}$$

where the discount rate is formally defined as

$$\rho(v,t,\tau) = -(\partial D/\partial \tau)/D = \bar{\rho}. \tag{11}$$

This method of modelling time preference entails several advantages in the overlapping generations model. It allows, despite the distinction of agents by their age and therefore dealing with heterogeneous individual households, for a convenient analysis at an aggregate macroeconomic level.

3.1 Individual Households

By using this discounting method, an individual household, or agent, faces according to (7)-(9) the following decision problem at time t:

$$\max_{c(v,\tau)} \int_t^{\infty} \ln(c(v,\tau)) e^{(\mu+\bar{\rho})(t-\tau)} d\tau \tag{12}$$

$$\text{s.t.} \;\; \dot{a}(v,\tau) = (r+\mu)a(v,\tau) + w(\tau) - c(v,\tau) \;\; \text{and} \tag{13}$$

$$\lim_{\tau\to\infty} a(v,\tau)e^{-R(t,\tau)} = 0, \;\; \text{with} \;\; R(t,\tau) := \int_t^{\tau} r + \mu \; ds. \tag{14}$$

To derive a dynamic system characterizing the agents optimal behaviour and later in order to find an explicit solution for individual consumption, we first establish a new budget constraint by rewriting (13), multiplying both sides with $e^{-R(t,\tau)}$ and integrating with respect to τ.

$$\dot{a}(v,\tau) - (r+\mu)a(v,\tau) = w(\tau) - c(v,\tau)$$

$$\dot{a}(v,\tau)e^{-R(t,\tau)} - (r+\mu)a(v,\tau)e^{-R(t,\tau)} = \left(w(\tau) - c(v,\tau)\right)e^{-R(t,\tau)}$$

$$\frac{\partial}{\partial\tau}\left(e^{-R(t,\tau)}a(v,\tau)\right) = \left(w(\tau) - c(v,\tau)\right)e^{-R(t,\tau)}$$

$$\left[e^{-R(t,\tau)}a(v,\tau)\right]_{\tau=t}^{\infty} = \int_t^\infty \left(w(\tau) - c(v,\tau)\right)e^{-R(t,\tau)}d\tau$$

Now, by using the solvency condition (9) and defining

$$h(t) := \int_t^\infty w(\tau)e^{-R(t,\tau)}d\tau, \tag{15}$$

as the present value of lifetime wage income discounted by the annuity factor $R(t,\tau)$, we obtain the intertemporal budget constraint

$$\int_t^\infty c(v,\tau)e^{-R(t,\tau)}d\tau = a(v,t) + h(t). \tag{16}$$

It states, that the present value of the household's lifetime consumption plan equals the sum of current financial and human wealth.

This modified budget constraint can now be used to solve the optimization problem with the Lagrange method. The Lagrange function with $\lambda(t)$ as time dependent multiplier is given by

$$\mathcal{L} = \int_t^\infty \ln(c(v,\tau))e^{(\mu+\bar{\rho})(t-\tau)}d\tau + \lambda(t)\left[a(v,t) + h(t) - \int_t^\infty c(v,\tau)e^{-R(t,\tau)}d\tau\right]$$

The first order conditions are the intertemporal budget constraint (16) and the following Euler equation:

$$\int_t^\infty \frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)}d\tau = \lambda(t)\int_t^\infty e^{-R(t,\tau)}d\tau$$

$$\frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)} = \lambda(t)e^{-R^A(t,\tau)}. \tag{17}$$

Differentiation with respect to τ on both sides of this equation yields

$$-\frac{\dot{c}(v,\tau)}{c(v,\tau)^2}e^{(\mu+\bar{\rho})(t-\tau)} - (\mu+\bar{\rho})\frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)} = -(r+\mu)\lambda(t)e^{R^A(t,\tau)}$$

$$\left(-\frac{\dot{c}(v,\tau)}{c(v,\tau)} - (\mu+\bar{\rho})\right)\frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)} = -(r+\mu)\frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)}.$$

This consequently results in the well-known *Keynes-Ramsey rule*, an optimality condition for the growth rate of individual consumption (Ramsey, 1928):

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \bar{\rho}\ . \tag{18}$$

Thus, the rate of change of optimal consumption over time is determined by the difference of the interest and discount rate. If the former rate exceeds the latter, the interest, that a household would get on its assets later in life is relatively high compared to the pure impatience it exhibits. This means that the individual optimally forgoes consumption in early years in order to save up for consumption and thereby utility later, which leads to a upward sloping consumption path. If the reverse inequality holds, the opportunity costs for utility at an early age are higher than the interest on savings in the future, hence, the individual optimally chooses a downward sloping consumption profile by rather consuming in the present.

In order to solve for the consumption level $c(v,t)$ at planning time t, we first set $\tau = t$ in the Euler equation (17) and attain for the Lagrange multiplier

$$\frac{1}{c(v,t)} = \lambda(t), \tag{19}$$

which we insert back into the first order condition (17). Integrating with respect to τ and using the intertemporal budget constraint (16) yields

$$\frac{1}{c(v,\tau)}e^{(\mu+\bar{\rho})(t-\tau)} = \frac{1}{c(v,t)}e^{-R^A(t,\tau)}$$

$$\int_t^\infty c(v,t)e^{(\mu+\bar{\rho})(t-\tau)}d\tau = \int_t^\infty c(v,\tau)e^{-R^A(t,\tau)}d\tau$$

$$c(v,t)\int_t^\infty e^{(\mu+\bar{\rho})(t-\tau)}d\tau = a(v,t) + h(t)$$

$$c(v,t) = (\mu+\bar{\rho})(a(v,t)+h(t)). \tag{20}$$

Hence, the level of optimal individual consumption is a proportion of total individual wealth. The marginal propensity to consume is constant and equal to the effective rate of time preference $\mu + \bar{\rho}$.

Combining obtained equations eventually yields that the optimal behaviour of an individual household is determined by the dynamic system:

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \bar{\rho} \tag{21}$$

$$\dot{a}(v,\tau) = (r+\mu)a(v,\tau) + w(\tau) - c(v,\tau) \tag{22}$$

$$c(v,t) = (\mu+\bar{\rho})(a(v,t)+h(t)). \tag{23}$$

The Solution

Now, by simply solving the Keynes-Ramsey rule (18) as separable differential equation, individual consumption can be derived explicitly:

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \bar{\rho}$$

$$\int_v^t \frac{\dot{c}(v,\tau)}{c(v,\tau)}d\tau = \int_v^t (r-\bar{\rho})d\tau$$

$$[\ln(c(v,\tau))]_{\tau=v}^t = (r-\bar{\rho})(t-v)$$

$$\ln(c(v,t)) - \ln(c(v,v)) = (r-\bar{\rho})(t-v)$$

$$\ln(c(v,t)) = (r-\bar{\rho})(t-v) + \ln(c(v,v))$$

$$c(v,t) = c(v,v)e^{(r-\bar{\rho})(t-v)} \tag{24}$$

The initial condition $c(v,v)$ is obtained by setting $t = v$ in the relationship of consumption with total wealth, given by equation (23) and using the fact, that newborns are assumed to hold no assets, i.e. $a(v,v) = 0$. Due to simplicity, we will additionally assume that wage income is not only independent of the generation index v but also of calendar time, thus constant, i.e. $w(v,\tau) = w$. All together yields that initial consumption is a proportion of discounted wage income.

$$
\begin{aligned}
c(v,v) &= (\mu+\bar{\rho})(a(v,v)+h(v)) = (\mu+\bar{\rho})h(v) \\
&= (\mu+\bar{\rho})\int_v^\infty w e^{-R(v,\tau)}d\tau = -(\mu+\bar{\rho})\left[\frac{w}{r+\mu}e^{-(r+\mu)(\tau-v)}\right]_{\tau=v}^{\infty} \\
&= (\mu+\bar{\rho})\frac{w}{r+\mu}. \qquad (25)
\end{aligned}
$$

By plugging this result in (24), we eventually derive a solution for optimal individual consumption with exponential discounting:

$$
c(v,t) = (\mu+\bar{\rho})\frac{w}{r+\mu}e^{(r-\bar{\rho})(t-v)}. \qquad (26)
$$

3.2 Aggregation

The assumption of large cohorts being born at every instant in time not only implies that the size of the total population in the economy can be normalized to unity, but also allows for easy determination of the size of any particular cohort over time. For instance, the amount of surviving members of a cohort at time t, which was born at time $v \leq t$, is of the size $\mu e^{-\mu(t-v)}$, whereas a fraction of $\mu\left(1-e^{-\mu(t-v)}\right)$ of this cohort will have died in the same time interval $[v,t]$ (Heijdra, 2009). Knowing the size of each cohort, it is now possible to analyse the dynamics at an aggregate level.

Thus, aggregate consumption at time t, denoted by $C(t)$, consists of consumption levels of all living agents that were born before time t.

$$
C(t) = \mu\int_{-\infty}^{t} c(v,t)e^{-\mu(t-v)}dv. \qquad (27)
$$

Using the optimal consumption rule (23) and the fact that it features a propensity to consume out of total wealth that is independent from the generation index v, we derive a similar statement for aggregate variables.

$$\begin{aligned} C(t) &= \mu \int_{-\infty}^{t} (\mu + \bar{\rho})(a(v,t) + h(t))e^{-\mu(t-v)}dv \\ &= (\mu + \bar{\rho}) \left(\mu \int_{-\infty}^{t} a(v,t)e^{-\mu(t-v)} + \mu \int_{-\infty}^{t} h(t)e^{-\mu(t-v)} \right) \\ &= (\mu + \bar{\rho}) \left(A(t) + H(t) \right), \end{aligned} \tag{28}$$

where aggregate financial and human wealth are defined analogously to aggregate consumption.

The next step is to derive the dynamic behaviour of $C(t)$ and aggregate non-human wealth $A(t)$, for which we will use the result above. To get the differential form of aggregate human wealth we impose the terminal boundary condition $\lim_{\tau\to\infty} w(\tau)e^{-R^A(t,\tau)} = 0$ and apply the Leibniz rule:

$$\begin{aligned} \dot{H}(t) &= \frac{\partial}{\partial t} \int_{t}^{\infty} w(\tau)e^{-R^A(t,\tau)}d\tau = \\ &= \int_{t}^{\infty} \frac{\partial}{\partial t} w(\tau)e^{-R^A(t,\tau)}d\tau + \lim_{\tau\to\infty} w(\tau)e^{-R^A(t,\tau)} - w(t)e^{-R^A(t,t)} \\ &= (r + \mu)H(t) - w(t). \end{aligned} \tag{29}$$

Using the Leibniz rule together with the individual budget constraint (13) and the condition $a(t,t) = 0$, yields an identity for aggregate asset accumulation.

$$\begin{aligned} \dot{A}(t) &= \frac{\partial}{\partial t} \mu \int_{-\infty}^{t} a(v,t)e^{-\mu(t-v)}dv \\ &= \mu \int_{-\infty}^{t} \frac{\partial}{\partial t} \left(a(v,t)e^{-\mu(t-v)} \right) dv + \mu a(t,t) - \lim_{v\to-\infty} \mu a(v,t)e^{-\mu(t-v)} \\ &= \mu \int_{-\infty}^{t} \dot{a}(v,t)e^{-\mu(t-v)}dv - \mu A(t) \end{aligned}$$

$$= (r + \mu)A(t) + w(t) - C(t) - \mu A(t)$$

$$= rA(t) + w(t) - C(t). \tag{30}$$

Whereas individual financial wealth accumulates at the rate $r + \mu$, its aggregate equivalent only increases at the rate r. That is because the amount $\mu A(t)$ is a transfer through insurance companies from those who pass away to those that remain alive, which does not accrue to aggregate wealth.

Now, we use the obtained differential identities of aggregate wealth, given by (29) and (30), and the condition (28) to get the following characterization of aggregate consumption:

$$\begin{aligned}\dot{C}(t) &= (\mu + \bar{\rho})\left(\dot{A}(t) + \dot{H}(t)\right)\\ &= (\mu + \bar{\rho})\left(rA(t) + w(t) - C(t) + (r + \mu)H(t) - w(t)\right)\\ &= r(\mu + \bar{\rho})(A(t) + H(t)) - (\mu + \bar{\rho})C(t) + \mu(\mu + \bar{\rho})H(t)\\ &= (r - \mu - \bar{\rho})C(t) + \mu C(t) - \mu(\mu + \bar{\rho})A(t)\\ &= (r - \bar{\rho})C(t) - \mu(\mu + \bar{\rho})A(t). \end{aligned} \tag{31}$$

Ultimately, collecting equations and dropping the time index yields a dynamic system of aggregate consumption and wealth:

$$\dot{C} = (r - \bar{\rho})C - \mu(\mu + \bar{\rho})A \tag{32}$$

$$\dot{A} = rA + w - C \tag{33}$$

If now $\mu = 0$, such that agents face infinite lives, the system simplifies to the dynamics of a Ramsey growth model with constant population (Ramsey, 1928; Cass, 1965; Koopmans, 1963).

4 Hyperbolic Discounting

For hyperbolic discounting we adapt the discount factor proposed by Strulik (2017). Therefore we consider the following multiplicatively separable function of the age at planning time $t-v$ and calendar time $\tau - v$:

$$D(v,t,\tau) = \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)}\right)^{\beta}, \quad \beta > 1. \tag{34}$$

The parameters α and β are being used for calibration, since higher values imply a higher discount rate at any time. The condition for β follows from the assumption of a finite present value of an infinite stream of utility, that is needed in order to make inferences (Strulik, 2017). If such a constant utility exists, then

$$\int_t^{\infty} (\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)})^{\beta} d\tau = \frac{(1+\alpha(t-v))}{\alpha(\beta-1)}(1+\alpha(\tau-v))^{1-\beta}|_t^{\infty}$$

needs to be finite, which in turn requires $\beta > 1$. The corresponding discount rate is obtained as

$$\rho(v,\tau) = -\left(\partial D/\partial \tau\right)/D = \frac{\alpha\beta}{1+\alpha(\tau-v)}, \tag{35}$$

and declines hyperbolically by approaches zero in calendar time . Furthermore, it is not dependent on the age at decision time, that is $t-v$. This feature is crucial, as it originates from multiplicative separability, which ensures time-consistent decisions (Burness, 1976; Drouhin, 2009).

4.1 Individual Households

Once hyperbolic discounting is introduced, the maximization problem of the agent and its solution remain similar. An individual decides on consumption such that it solves

$$\max_{c(v,\tau)} \int_t^{\infty} \ln(c(v,\tau)) e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)}\right)^{\beta} d\tau \tag{36}$$

subject to the individuals budget constraint (8) and no-Ponzi game condition (9).

The steps that are being used in order to derive a dynamic system, characterizing this individual decision problem, are identical to the exponential case. First, the intertemporal budget constraint (16) is established, which is then used for solving the problem by the Lagrange method. The first order conditions are given by (16) and

$$\int_t^\infty \frac{1}{c(v,\tau)} e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)}\right)^\beta d\tau = \lambda(t) \int_t^\infty e^{-R(t,\tau)} d\tau$$

$$\frac{1}{c(v,\tau)} e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)}\right)^\beta = \lambda(t) e^{-R(t,\tau)}. \tag{37}$$

Differentiation with respect to τ yields the Keynes-Ramsey rule featuring the hyperbolic time preference rate

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \frac{\alpha\beta}{1+\alpha(\tau-v)} = r - \rho(v,\tau). \tag{38}$$

Whereas the rate of change of optimal consumption remains the same throughout time in the exponential case, hyperbolic discounting leads to a rate that changes over the years. As the hyperbolic discount rate $\rho(v,t)$ decreases and converges to 0 with time the individual consumption rate $\dot{c}/c$ increases and approaches the real interest rate r. In fact, if it holds that $r < \alpha\beta$ this implies a change of sign of the consumption slope during an individuals life. A hyperbolically discounting agent generally exhibits high impatience rates concerning the near future. Coupled with a relative low interest rate this leads to a downward sloping consumption profile early in his life, as savings get neglected in favour of instantaneous consumption. However, this impatience decreases and approaches 0 in the course of time, such that the interest rate eventually exceeds the time preference rate, which is accompanied by more valuable savings and an upward sloping consumption path.
Figure 6.2 in Section 6 illustrates both exponentially and hyperbolically discounting individual consumption profiles for different values of r.

In order to derive an identity for the consumption level similar to (23) we first set $\tau = t$ in the first order condition (37) to obtain the same result for the Lagrange multiplier, i.e. $1/c(v,t) = \lambda(t)$. This result plugged back in the first order condition (37), integration on both sides with respect to τ and the application of the intertemporal budget constraint (16) yields

$$\frac{1}{c(v,\tau)} e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)} \right)^{\beta} = \frac{1}{c(v,t)} e^{-R^A(t,\tau)}$$

$$\int_t^{\infty} c(v,t) e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)} \right)^{\beta} d\tau = \int_t^{\infty} c(v,\tau) e^{-R^A(t,\tau)} d\tau$$

$$c(v,t) \int_t^{\infty} e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)} \right)^{\beta} d\tau = a(v,t) + h(t)$$

$$c(v,t) = \gamma(v,t) \left(a(v,t) + h(t) \right). \tag{39}$$

In contrast to the exponential case, here the marginal propensity to consume out of total wealth, given by $\gamma(v,t)$, where

$$\gamma(v,t)^{-1} := \int_t^{\infty} e^{\mu(t-\tau)} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)} \right)^{\beta} d\tau, \tag{40}$$

is dependent on both the generation v and time t, which consequently complicates aggregation.

To summarize by collecting equations, the optimal behaviour of an individual household is characterized by the system:

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \rho(v,\tau) \tag{41}$$

$$\dot{a}(v,\tau) = (r+\mu) a(v,\tau) + w(\tau) - c(v,\tau) \tag{42}$$

$$c(v,t) = \gamma(v,t)(a(v,t) + h(t)). \tag{43}$$

The Solution

Again, we obtain a solution for the optimal consumption path by solving the given Keynes-Ramsey rule as separable differential equation.

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r - \rho(v,\tau)$$

$$\int_v^t \frac{\dot{c}(v,\tau)}{c(v,\tau)} d\tau = \int_v^t r - \frac{\alpha\beta}{1+\alpha(\tau - v)} d\tau$$

$$\big[\ln(c(v,\tau))\big]_{\tau=v}^{t} = r(t-v) - \big[\beta \ln(1+\alpha(\tau - v))\big]_{\tau=v}^{t}$$

$$\ln(c(v,t)) - \ln(c(v,v)) = r(t-v) - \beta \ln(1+\alpha(t-v))$$

$$\ln(c(v,t)) = r(t-v) + \ln\left(\frac{c(v,v)}{(1+\alpha(t-v))^{\beta}}\right)$$

$$c(v,t) = c(v,v)e^{r(t-v)}(1+\alpha(t-v))^{-\beta} \tag{44}$$

The initial condition is attained by using the identity of the consumption level (43) and setting $t = v$. For now, labour income is assumed to be constant, just as in the solution for exponential discounting, which results in the same expression for human wealth as in the original model, i.e. $h(v) = h = w/(r+\mu)$. The initial propensity $\bar{\gamma}$ remains the same for all newborns (substitute $y = \tau - v$):

$$\bar{\gamma}^{-1} := \gamma(v,v)^{-1} = \gamma(t,t)^{-1} = \int_0^\infty e^{-\mu y}(1+\alpha y)^{-\beta} dy. \tag{45}$$

Moreover, it can be written as generalized exponential integral defined as

$$E_s(x) := \int_1^\infty e^{-xt} t^{-s} dt \qquad (s \in \mathbb{R},\ x > 0), \tag{46}$$

which is closely related to the incomplete gamma function (e.g. Chiccoli et al., 1990):

$$\bar{\gamma}^{-1} = \frac{1}{\alpha}\int_1^\infty e^{-\frac{\mu}{\alpha}(x-1)} x^{-\beta} dx = \frac{e^{\frac{\mu}{\alpha}}}{\alpha} E_\beta\left(\frac{\mu}{\alpha}\right). \tag{47}$$

By using one of its properties, in particular, $e^x E_s(x) \leq \frac{1}{x+s-1}$ for $x > 0, s \geq 1$ this results in an estimation that allows for a direct comparison to the exponentially discounting case. It holds that

$$\bar{\gamma}^{-1} \leq \frac{1}{\alpha} \cdot \frac{1}{\beta + \frac{\mu}{\alpha} - 1} = \frac{1}{\mu + \alpha(\beta - 1)} \tag{48}$$

Provided that both discounting methods yield the same present value, which is

given by the condition $\bar{\rho} = \alpha(\beta - 1)$ (see Section 5.1), we observe that the initial propensity to consume for an hyperbolically discounting household will always be equal or bigger than for an exponentially discounting one, i.e. $\bar{\gamma} \geq \mu + \bar{\rho}$. This originates from the relatively high impatience that a hyperbolically discounting agent exhibits early in his life.
Consequently, also initial consumption is greater, as it is given by

$$\bar{c} \coloneqq c(v,v) = c(t,t) = \bar{\gamma} h = \bar{\gamma}\frac{w}{r+\mu}. \tag{49}$$

Ultimately, the optimal individual consumption path under hyperbolic discounting is given by the function

$$c(v,t) = \bar{\gamma}\frac{w}{r+\mu}e^{r(t-v)}(1+\alpha(t-v))^{-\beta}. \tag{50}$$

4.2 Aggregation

Results obtained about aggregate wealth in the exponentially discounting model are identically applicable to the hyperbolic case, as they are not effected by a change of discounting methods. Thus, particularly aggregate non-human wealth $A(t)$ and aggregate human wealth $H(t)$ accumulate according to equations (33) and (29) respectively. However, when trying to derive the dynamics of aggregate consumption analogously, difficulties arise. Due to the time dependent propensity to consume, it is virtually impossible to find a closed equation in terms of aggregate wealth similar to equation (32). In fact, by using the explicit solution for individual consumption, one observes that aggregate consumption equals a generalized exponential integral defined by (46).

$$\begin{aligned} C(t) &= \mu \int_{-\infty}^{t} \bar{c} e^{(r-\mu)(t-v)}(1+\alpha(t-v))^{-\beta} dv = \mu\bar{c}\int_{1}^{\infty} e^{\frac{(r-\mu)(y-1)}{\alpha}} y^{-\beta}\frac{1}{\alpha}dy \\ &= \frac{\mu}{\alpha}\bar{c}e^{\frac{(\mu-r)}{\alpha}}\int_{1}^{\infty} e^{-\frac{(\mu-r)}{\alpha}y}y^{-\beta}dy = \frac{\mu}{\alpha}\bar{c}e^{\frac{(\mu-r)}{\alpha}}E_{\beta}\left(\frac{(\mu-r)}{\alpha}\right). \end{aligned} \tag{51}$$

Nonetheless, applying the Leibniz rule yields an expression for aggregate consumption growth, which allows for a straight forward interpretation.

$$\dot{C}(t) = \mu c(t,t) - \lim_{v \to -\infty} c(v,t)e^{\mu(v-t)} + \mu \int_{-\infty}^{t} \dot{c}(v,t)e^{\mu(v-t)} - \mu c(v,t)e^{\mu(v-t)} dv$$

$$= \mu\bar{c} - \mu C(t) + \int_{-\infty}^{t} \dot{c}(v,t)e^{\mu(v-t)} dv \tag{52}$$

At every instant in time, aggregate consumption increases by the consumption of individuals being born $\mu\bar{c}$ and gets reduced by $\mu C(t)$, the proportion of aggregate consumption of those who pass away. What is left, is the third term that depicts the aggregated change in consumption of agents staying alive.

5 Calibration

In this Section benchmark specifications are established for subsequent numerical computations, where the aim is to compare the introduced hyperbolic discounting model to its original equivalent by means of the steady state in a closed economy. For this purpose we as well already calibrate parameters that are introduced in the next Section. Specifically, we assume in equation (55) the aggregate capital stock to depreciate at a rate of $\delta = 10\%$ and the capital's share of output to be $\varepsilon = 30\%$, which is in line with standard literature (Collins and Bosworth (1996) believe in a plausible range for ε of 0.3 to 0.4, where developing economies evidently show higher capital elasticities compared to industrial ones).
Regarding the instantaneous probability of death, we assume a benchmark value of $\mu = 0.0167$, which is equivalent to a constant life expectancy of 60 years. With this value, the probability of being alive for at least 100 years is around 19%.
To calibrate parameters used for hyperbolic discounting, we use estimations obtained by Weitzman's (2001) gamma discounting approach. He has shown that individual uncertainty or disagreement about a constant exponential discount rate yields in the aggregate a hyperbolically declining time preference rate. Eventually, by asking over two thousand at least Ph.D.-level economists what discount rate they think should be proposed to mitigate the possible effects of global climate change, he attains the discount rate

$$\rho(t) = \frac{m}{1 + \frac{\sigma^2}{m} \cdot t}$$

with values of $m = 0.04$ for the mean and $\sigma = 0.03$ for the standard deviation. These estimates imply $\alpha = \sigma^2/m = 0.0225$ and $\beta = m^2/\sigma^2 = 1.7778$ in our setting, which supports the requirement of $\beta > 1$.

Parameter	Description	Value
μ	Instantaneous probability of death (hazard rate)	0.0167
ε	Output elasticity of capital	0.3
δ	Depreciation rate of aggregate capital	0.1
α	Parameter used for hyperbolic discounting	0.0225
β	Parameter used for hyperbolic discounting	1.7778
$\bar{\rho}$	Exponential discount rate	0.0175

Table 5.1: Summary of parameter values used for numerical calculations

5.1 Equivalent Present Value

Further, in order to separate the effect of different discounting methods from pure impatience and therefore to achieve a fair comparison we apply an argument introduced by Myerson et al. (2001) (see also Strulik, 2015, 2017; Caliendo and Findley, 2014). We determine the value of the exponential discount rate $\bar{\rho}$ such that an infinite stream of for example utility provides the same present value for either discounting method. This leads us to the *Equivalent Present Value* requirement:

$$\int_t^{\infty} e^{\bar{\rho}(t-\tau)} d\tau = \int_t^{\infty} \left(\frac{1+\alpha(t-v)}{1+\alpha(\tau-v)} \right)^{\beta} d\tau. \quad (53)$$

Integrating both sides explicitly and setting the time of birth $v = t$ as the present, yields the requirement

$$\bar{\rho} = \alpha(\beta - 1). \quad (54)$$

Thus the condition of a finite present value of an infinite stream of utility, for which we obtained that $\beta > 1$ is equivalent to the condition that the associated equivalent constant discount rate is positive (Strulik, 2017).
By using Weitzman's calibration from above, we from now on set $\bar{\rho} = 0.0175$.

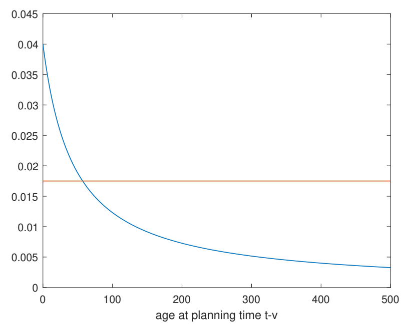


Figure 5.1: Equivalent Discount Rates for exponential discounting (red) and hyperbolic discounting (blue) using Weitzman's estimations.

As shown in Figure 5.1, $\rho(v, t)$ starts off at a higher value than $\bar{\rho}$, until it hyperbolically approaches zero in the long run. The time of intersection between the two equivalent discount rates is given by $t - v = \beta/\bar{\rho} - 1/\alpha \approx 57$ years.

6 The Closed Economy

In a closed economy, the interest rate r and labour income w are no longer exogenously given, but determined instead by the aggregate capital stock K (from now on we will drop the time index wherever it is not misleading). The technology that determines the output of this economy features two factors of production, capital K and labour force L. The latter is assumed to equal the size of the population, which is because of the large-cohort assumption, equal to one. Let this technology function be given by

$$F(K,1) = K^{\varepsilon} - \delta K, \tag{55}$$

a Cobb-Douglas production function with constant returns to scale, which implies that the capital's and labour's share of output is given by $\varepsilon \in (0,1)$ and $1-\varepsilon \in (0,1)$ respectively. The capital is assumed to depreciate at a rate equal to δ.
The interest rate r is determined as the net marginal product of capital and non-interest wage income w as the marginal product of labour. Thus,

$$r = r(K) = F_K(K,1) = \varepsilon K^{\varepsilon-1} - \delta \tag{56}$$

$$w = w(K) = F_L(K,1) = (1-\varepsilon)K^{\varepsilon}. \tag{57}$$

Equating aggregate non-human wealth A from the previous sections with the capital stock K and using the new identities for r and w in its accumulation equation (30), we get for both exponential and hyperbolic discounting the dynamics of capital in a closed economy:

$$\dot{K} = K^{\varepsilon} - \delta K - C. \tag{58}$$

Combined with the dynamics of aggregate consumption we will now analyse the steady state of this economy, particularly the interest rate at which it is balanced.

6.1 The Steady State

6.1.1 Exponential Discounting

By using equation (58) and the dynamics of aggregate consumption in (32), that we derived earlier, the original OLG model in a closed economy is given by

$$\dot{C} = (r - \bar{\rho})C - \mu(\mu + \bar{\rho})K \tag{59}$$

$$\dot{K} = K^{\varepsilon} - \delta K - C. \tag{60}$$

This two dimensional dynamical system allows for an analysis on the basis of the associated phase diagram given by Figure 6.1.
The isocline $\dot{C} = 0$, given by the expression $C = \mu(\mu + \bar{\rho})K/(r(K) - \bar{\rho})$, is upward sloping, convex and asymptotically reaching the point, where $r(K) = \bar{\rho}$, denoted by K^K, that is attained when agents face infinite horizons ($\mu = 0$). The $\dot{K} = 0$ isocline traces the net production function $C = F(K, 1) = K^{\varepsilon} - \delta K$. The golden rule capital stock K^G depicts the level that maximizes steady state consumption, i.e. $r(K) = 0$, and lies to the right of K^K, since the returns to capital are diminishing.
The two equilibria, the origin and (C^*, K^*) are attained by intersection of the two isoclines. The latter, non-trivial equilibrium has a saddle point structure, as indicated by the arrows. It can be shown that any trajectory other than the upward sloping saddle point path leads to inconsistencies, such as negative levels for consumption or capital. Thus, given an initial stock of capital, consumption is uniquely determined (Blanchard, 1985).

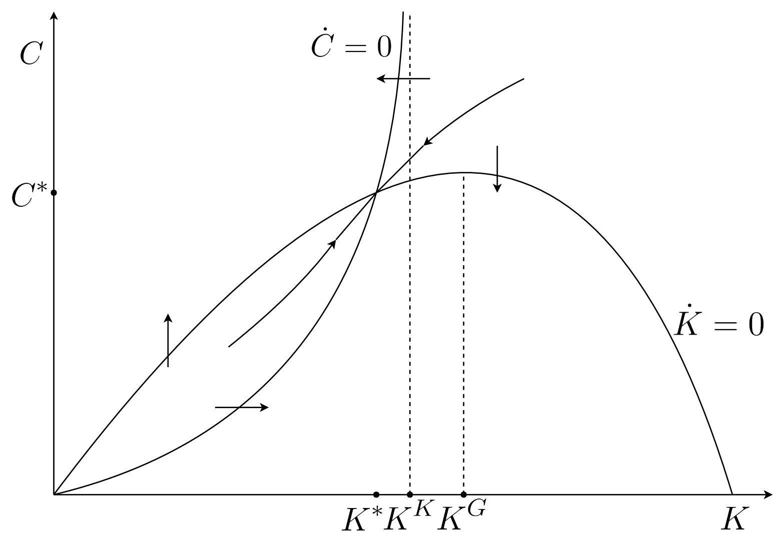


Figure 6.1: Phase Diagram

The steady state interest r^* lies between $\bar{\rho}$ and $\mu + \bar{\rho}$, that is $\bar{\rho} \leq r^* < \bar{\rho} + \mu$. The first inequality follows from the $\dot{C} = 0$ isocline. This result becomes clear when returning to the Keynes-Ramsey rule of individual consumption in (18). In order to generate positive aggregate capital, individual consumption must be increasing by agents initially saving, which corresponds to an interest rate greater than or equal to the pure rate of time preference.
The second statement can be shown by contradiction. Suppose $r^* \geq \mu + \bar{\rho}$ holds, such that $(r^* - \bar{\rho})C^* \geq \mu C^*$. Using the consumption isocline, this is equivalent

to $\mu(\mu+\bar{\rho})K^* \geq \mu C^*$ and by using the $\dot{K}=0$ locus this results in $(\mu+\bar{\rho})K^* \geq K^{*\varepsilon} - \delta K^*$. But $r^*K^* = \varepsilon K^{*\varepsilon} - \delta K^* < K^{*\varepsilon} - \delta K$, which contradicts the assumption.

An explicit analytical solution for the steady state interest rate r^* and thereby C^*, K^* can be obtained in different ways. One, for instance, is to use individual consumption explicitly to express aggregate consumption as a function of r which is then used in the $\dot{K}=0$ isocline. Another way is to express the capital stock in terms of r and solve the intersection condition:

$$C = K^\varepsilon - \delta K = \frac{\mu(\mu+\bar{\rho})}{r-\bar{\rho}} K = C$$

$$K^{\varepsilon-1} - \delta = \frac{\mu(\mu+\bar{\rho})}{r-\bar{\rho}}$$

$$\frac{r}{\varepsilon} + \frac{(1-\varepsilon)\delta}{\varepsilon} = \frac{\mu(\mu+\bar{\rho})}{r-\bar{\rho}}$$

$$r^2 - r\bar{\rho} + (1-\varepsilon)\delta r - (1-\varepsilon)\delta\bar{\rho} = \varepsilon\mu(\mu+\bar{\rho})$$

$$r^2 - r(\bar{\rho} - (1-\varepsilon)\delta) = (1-\varepsilon)\delta\bar{\rho} + \varepsilon\mu(\mu+\bar{\rho})$$

By using basic algebraic operations we attain a quadratic expression for r, of which the solution is given by:

$$r^* = \frac{\bar{\rho} - (1-\varepsilon)\delta}{2} \overset{+}{\underset{(-)}{}} \sqrt{\left(\frac{\bar{\rho} - (1-\varepsilon)\delta}{2}\right)^2 + ((1-\varepsilon)\delta\bar{\rho} + \varepsilon\mu(\mu+\bar{\rho})}. \qquad (61)$$

The second term under the root is positive, hence we obtain a both positive and negative solution for the steady state interest rate. However, we rule out the negative result and focus solely on a positive interest rate, as a negative value would imply capital depreciating at a rate larger than its marginal product.
Now, using the benchmark parameter values given by Table 5.1, we obtain as numerical value for the steady state interest rate

$$r^* = 1.94\%,$$

which is consistent with the observation that $r^* \in [\bar{\rho}, \bar{\rho} + \mu)$.

6.1.2 Hyperbolic Discounting

As mentioned earlier in Section 4.2, there is no closed form in terms of capital for the dynamics of aggregate consumption under hyperbolic discounting. However, focussing merely on the steady state allows for simplification. In an equilibrium, all aggregate variables are constant, including human wealth H and thus labour income. Hence, we can use the explicit expression for individual consumption in equation (50) to investigate characteristics of r^*. Just as in the exponential model, we assume the interest to be non-negative.

As it turns out, the steady state interest under hyperbolic discounting does not exceed the instantaneous probability of death, i.e. $r^* \leq \mu$. To see this, we return to equation (51) and express aggregate consumption as exponential integral:

$$C = \frac{\mu}{\alpha}\bar{c}e^{\frac{(\mu-r)}{\alpha}}E_\beta\left(\frac{(\mu-r)}{\alpha}\right). \tag{62}$$

A steady state interest exceeding the hazard rate, results in a negative second argument and thus a complex value for aggregate consumption (e.g. Navas-Palencia, 2018), which clearly is inconsistent.
However, interest rates smaller than or equal the hazard rate lead to a finite, time independent integral and hence to constant aggregate consumption that is in line with a steady state. In fact, since the exponential integral satisfies $\frac{1}{s+x} \leq e^x E_s(x) \leq \frac{1}{s+x-1}$ for $x > 0, s \geq 1$ (e.g. Chiccoli et al., 1990), aggregate consumption can be estimated the following way:

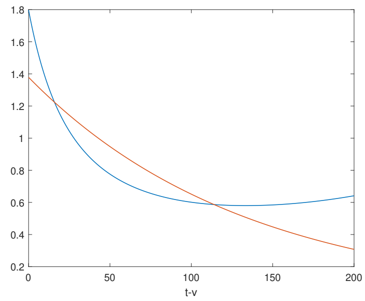
$$\frac{\mu\bar{c}}{\mu+\alpha\beta-r} \leq C \leq \frac{\mu\bar{c}}{\mu+\bar{\rho}-r} \tag{63}$$

Whereas r^* and also (K^*, C^*) can be easily calculated analytically in the exponentially discounting model, in the hyperbolically discounting economy this is not the case. However, we derive a solution numerically by calculating the intersection point between the two isoclines. With the assumption of a steady state, all variables remain constant, which allows us to use aggregate consumption explicitly in the $\dot{K} = 0$ isocline (see equation (58)). We obtain an equation in r, of which the root determines the steady state value. By using a standard bisection method (e.g. fzero in Matlab) we eventually attain for the given setting

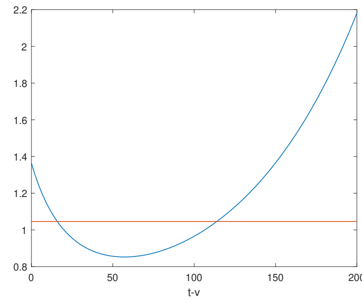
$$r^* = 1.54\%.$$

6.1.3 Comparison

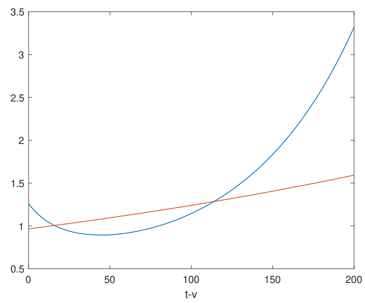
Whereas in the exponential model r^* lies between $\bar{\rho}$ and $\mu + \bar{\rho}$, the only restriction for the non-negative steady state interest rate in the hyperbolically discounting economy is from above by the hazard rate μ.
Figure 6.2 illustrates the impact of the interest rate on individual consumption in both models. As it can be seen in the Keynes-Ramsey rule, individual consumption in the original model increases or decreases exponentially dependent on whether the interest rate exceeds or falls short of the constant time preference rate. Hyperbolically discounting households, on the other hand, generally exhibit a different behaviour, as their consumption rate approaches the interest rate from below over the course of time. As long as the interest lies below a certain point ($r < \alpha\beta$) young individuals will always prefer utility in the near future over utility in the distant future such that their consumption slope exhibits a change of sign.



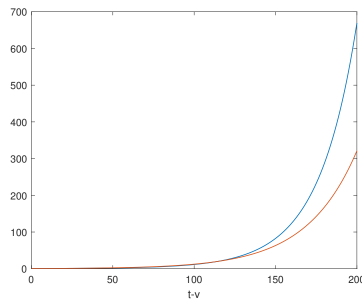
(a) $r = 0.01 < \mu$



(b) $r = 0.0175 = \bar{\rho}$



(c) $\bar{\rho} < r = 0.02 < \bar{\rho} + \mu$



(d) $r = 0.05 > \alpha\beta > \bar{\rho} + \mu$

Figure 6.2: Individual Consumption $c(v,t)$ for exponential discounting (red) and hyperbolic discounting (blue)

Regarding the benchmark computations for the steady state, the original model exhibits with $r_e^* = 1.94\%$ a clearly higher interest rate than the hyperbolic model ($r_h^* = 1.54\%$). Thus, the exponentially discounting economy is balanced at a lower capital stock. Also, the level of aggregate consumption is lower. This result, however, is strongly dependent on the choice of certain parameters as further analysis shows.

6.1.4 Sensitivity Analysis

We now investigate the sensitivity of the results to deviations from the benchmark parameter specifications introduced in Section 5. For this purpose we alter parameter values ceteris paribus and analyse how this effects the steady state of the economy in both the hyperbolic and exponential model. Thereby we are particularly interested in the steady state interest rates r_e^* and r_h^*. Table 6.1 depicts the key figures for each scenario of parameter alteration.

Scenario	r_e^*	r_h^*
–	0.0194	0.0154
μ^+	0.0204	0.0189
μ^-	0.0186	0.0111
ε^+	0.0203	0.0157
ε^-	0.0187	0.0150
δ^+	0.0201	0.0156
δ^-	0.0190	0.0152
α^+	0.0254	0.0164
α^-	0.0134	0.0133
β^+	0.0246	0.0166
β^-	0.0129	0.0134

Table 6.1: Sensitivity Analysis

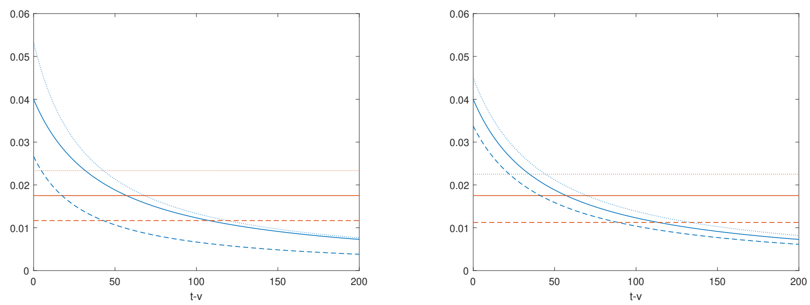
First, we increase the hazard rate by a third to a value of $\mu^+ = 0.0222$, which is equivalent to 45 years of expected remaining life at any given time. This leads in both economies to a relatively clear rise in the key figures. A decrease of μ by the same factor to a value of $\mu^- = 0.0111$ means an increase in life expectancy to 90 years and has the opposite effect on the steady state interest rates.

In the next scenario we increase the output elasticity of capital by a third to a value of 0.4 (ε^+), which causes both steady state interest rates to increase slightly. A decrease by the same factor ($\varepsilon^- = 0.2$) has the reverse minimal effect.

A similar variation of the depreciation rate of capital results in the same behaviour. An increase to $\delta^+ = 13.33\%$ leads to a rise of both key figures, whereas a decrease to $\delta^- = 6.66\%$ to a decline. Again, the effect is rather minimal.

Next, we investigate how deviations of discounting parameters affect the steady state outcome. Therefore we first set α to a higher value of 0.03. In order to maintain a fair comparison between discounting methods, we recalibrate the exponential

discount rate by the same factor according to the equivalent present value condition (54). This causes a higher value of $\bar{\rho} = 0.0233$. Also the hyperbolic discount rate is affected and increases significantly, especially concerning the near future, as it can be seen in Figure 6.3a. This means that in this scenario both economies exhibit higher impatience towards future utility, of which consequences can be clearly seen at an individual level. A higher α and thereby higher $\bar{\rho}$ generate a lower discount factor in both economies and consequently a higher propensity to consume out of wealth. The optimal consumption rate given by the Keynes-Ramsey rule in equation (18) and (38) respectively, however, declines to a lower level due to higher discount rates. Meaning that, young individuals would optimally cut back on savings in favour of instantaneous consumption which in turn entails a smaller consumption growth later in life due to the lack of accumulated wealth. As a result, the capital stock aggregates to a lower level, which involves considerably higher steady state interest rates as given in Table 6.1. In the exponential model this can also be properly seen at an aggregate level by investigating the explicit solution for r_e^* in equation (61) or the phase diagram in Figure 6.1. An increase in $\bar{\rho}$ shifts, due to the increased consumption, the $\dot{C} = 0$ isocline to the left, causing a lower steady state capital stock K^* and thus a higher equilibrium interest rate r_e^*.
A decrease to $\alpha^- = 0.015$, which means that we set $\bar{\rho} = 0.0117$, results in the opposite effect. Lower impatience causes young individuals to reduce their consumption in the short run in order to save up for future utility which in turn leads to more aggregate capital and therefore lower steady state interest rates. The impact in this case is as well significant. In fact, reducing α slightly further would eventually lead to a higher steady state interest rate in the hyperbolic model than in the exponential model.



(a) Variation of α (b) Variation of β

Figure 6.3: Time preference rates of exponential (red) and hyperbolic (blue) discounting for different values of α and β. The solid lines depict the initial benchmark specifications, dotted lines positive parameter changes ($\alpha^+ = 0.03$; $\beta^+ = 2$) and dashed lines negative deviations ($\alpha^- = 0.015$; $\beta^- = 1.5$).

The argumentation behind alterations of the second discounting parameter β remains the same as in case of α, as higher values also result in higher time preference rates and vice versa. The impact, however, is more significant. Already a slight increase to $\beta^+ = 2$ ($\bar{\rho} = \alpha = 0.0225$) yields slightly higher results than in the scenario of α^+. A small decrease to $\beta^- = 1.5$ and therefore $\bar{\rho} = 0.113$ even causes the original model to be actually balanced at a lower interest rate than its hyperbolic counterpart, which obviously differs from the benchmark outcome.
In general, a change in β affects the discount rates differently than a change in α, which can be seen in Figure 6.3. Due to the form of the equivalent present value requirement, a change in β causes a more significant adjustment of $\bar{\rho}$ than the same percentage change in α would cause. Also the hyperbolic discount rate is affected in a different way. Whereas a variation of α yields a higher discount rate especially concerning the near future, deviations of β affect the discount rate in a more consistent way.
To conclude, the results are clearly robust to changes in the hazard rate, the output elasticity and the depreciation rate of capital, but react rather sensitive to similar deviations in parameters concerning the time preference rate. Particularly decreasing them sufficiently enough easily leads to a different result compared to the benchmark computations.

6.1.5 A Change in Life Expectancy

As life expectancy generally appears to have a crucial impact on the economy we now take a closer look on an exogenous change in the hazard rate. A ceteris paribus decrease in life expectancy and hence an increase in μ has generally the same effect on the steady state in both models, as a sensitivity analysis has shown. It leads to a higher equilibrium interest rate and a lower level of aggregate capital stock and consumption. Since the higher μ, the lower expected life is, the smaller average individual savings and thus aggregate capital.
For the case of exponential discounting, this can either be seen in equation (61), where r_e^* explicitly is an increasing function of μ or by investigating the phase diagram given by Figure 6.1. An increase in μ shifts the $\dot{C} = 0$ isocline to the left, causing a smaller steady state capital stock and therefore a higher interest rate.
As for the hyperbolic model, there is no closed form for either the equilibrium interest rate nor the aggregate consumption isocline to simplify the analysis. Nevertheless, one can numerically observe the same effect, which can be seen in Table 6.2 and in Figure 6.4.

What should be noted is that for relatively high hazard rates, i.e. higher than 2.26%, which is equivalent to less than around 38 expected remaining years of

Life Expectancy in Years	μ	r_e^*	C_e^*	K_e^*	r_h^*	C_h^*	K_h^*
30	0.0333	0.0230	1.1080	3.5754	0.0250	1.1060	3.4924
38	0.0263	0.0213	1.1096	3.6463	0.0213	1.1096	3.6478
50	0.0200	0.0200	1.1107	3.7024	0.0175	1.1128	3.8135
60	0.0167	0.0194	1.1112	3.7285	0.0154	1.1145	3.9173
70	0.0143	0.0190	1.1116	3.7456	0.0136	1.1157	4.0019
90	0.0111	0.0186	1.1120	3.7660	0.0111	1.1174	4.1358

Table 6.2: Steady state values for different values of μ

life, the hyperbolic r_h^* exceeds the interest rate originating from the original OLG model, whereas for life expectancies higher than 38 years, the reverse inequality holds. This originates from the difference in individual consumption behaviour. If life expectancy is low, hyperbolically discounting households, accumulate less wealth than in the original setting, due to the high impatience early in their lives. This leads in the aggregate to a lower steady state capital stock and thus higher interest. Conversely, hyperbolically discounting individuals exhibit higher consumption rates later in life which is accompanied by more savings. If life expectancy is high, this would lead to higher steady state aggregate capital or a lower steady state interest rate. Figure 6.4 shows the steady state interest as function of the hazard rate and K^* as function of life expectancy for both discounting methods.

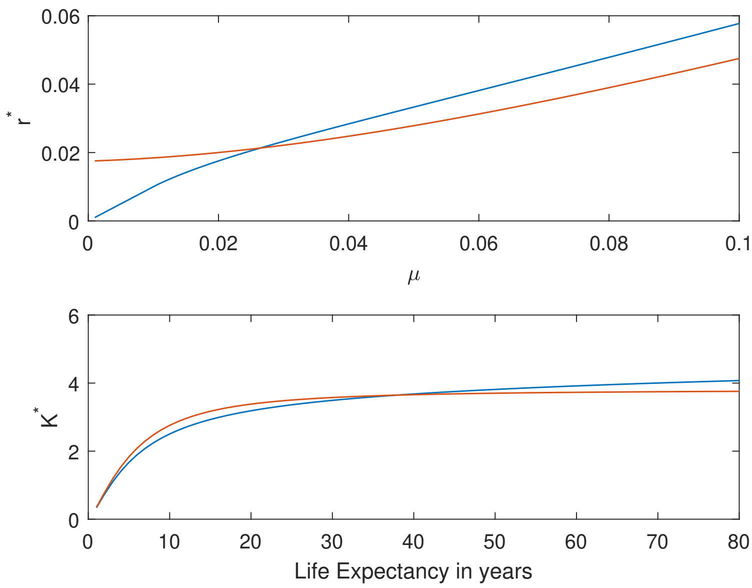


Figure 6.4: Steady state interest rate r^* and capital stock K^* Exponential discounting is represented by red and hyperbolic discounting by blue.

Although both discounting models exhibit the same reaction to a change in life expectancy, they differ fundamentally as soon as μ approaches 0.
The exponential model, as already mentioned, resembles the well known Ramsey growth model for $\mu = 0$. In this case the standard modified golden rule result $r^* = \bar{\rho}$ obtains, which means that individuals choose an entirely flat consumption profile in the steady state (Blanchard, 1985) .
The hyperbolically discounting model, however, is only defined for strictly positive μ. As the horizon converges to infinity, the steady state interest rate converges to 0, since it holds that $r^* \leq \mu$. This results in a negative individual consumption rate, which ultimately causes $c(v, t)$ to decrease infinitely, which clearly leads to an inconsistency.

7 Conclusion

In this thesis, a time-consistent method of hyperbolic discounting proposed by Strulik (2017) was introduced and adapted to the Blanchard-Yaari-Buiter model of overlapping generations in continuous time. Before investigating the steady state and the impact of the instantaneous probability of death in a closed economy, results were compared to the original model in a general setting. In order to establish a fair comparison, an equivalent present value argument introduced by Myerson et al. (2001) was applied, which ensures that both discounting methods provide the same present value of an infinite stream of utility.

The implementation leads at first to a clear hyperbolic behaviour in individual households, which is reflected in the optimal consumption path. The property of convenient aggregation, which characterizes the original overlapping generations model, however, gets lost due to the age dependent time preference rate. Therefore a simple bisection method was applied in order to numerically determine the steady state in the setting of a closed economy. A benchmark run using conventional parameter values and Weitzman's (2001) estimations for the hyperbolic discounting factor eventually yields a lower steady state aggregate capital stock and consumption level in the original model. A sensitivity analysis ultimately shows that this result is strongly dependent on certain parameter specifications, especially concerning the time preference rate.

Bibliography

Arrow, Kennet J., Cropper, Maureen L., Gollier, Christian, Groom, Ben, Heal, Geoffrey M., Newell, Richard G., Nordhaus, William D., Pindyck, Robert S., Pizer, William A., Portney, Paul R., Sterner, Thomas, Tol, Richard S. J., and Weitzman, Martin L. (2014). Should governments use a declining discount rate in project analysis. *Review of Environmental Economics and Policy*, 8(2): 145–163.

Berns, Gregory S., Laibson, David, and Loewenstein, George (2007). Intertemporal choice - toward an integrative framework. *Trends in Cognitive Science*, 11(11): 482–488.

Blanchard, Olivier J. (1985). Debt, deficits, and finite horizons. *The Journal of Political Economy*, 93(2): 223–247.

Blanchard, Olivier J. and Fischer, Stanley (1989). *Lectures on Macroeconomics*. The MIT Press, Cambridge Massachusetts.

Buiter, Willem H. (1988). Death, birth, productivity growth and debt neutrality. *The Economic Journal*, 98(391): 279–293.

Burness, H. Stuart (1976). A note on consistent naive intertemporal decision making and an application to the case of uncertain lifetime. *The Review of Economic Studies*, 43(3): 547–549.

Caliendo, Frank N. and Findley, T. Scott (2014). Discount functions and self-control problems. *Economics Letters*, 122(3): 416–419.

Cass, David (1965). Optimum growth in an aggregative model of capital accumulation. *The Review of Economic Studies*, 32(3): 233–240.

Chiccoli, C., Lorenzutta, S., and Maino, G. (1990). Recent results for generalized exponential integrals. *Computers & Mathematics with Applications*, 19(5): 21–29.

Collins, Susan M. and Bosworth, Barry P. (1996). Economic growth in east asia: Accumulation versus assimilation. *Brookings Papers on Economic Activity*, 27(2): 135–204.

DellaVigna, Stefano (2009). Psychology and economics: Evidence from the field. *Journal of Economic Literature*, 47(2): 315–372.

Drouhin, Nicolas (2009). Hyperbolic discounting may be time consistent. *Economics Bulletin*, 29(4): 2549–2555.

Frederick, Shane, Loewenstein, George, and O'Donoghue, Ted (2002). Time discounting and time preferance: A critical review. *Journal of Economic Literature*, 40(2): 351–401.

Heijdra, Ben J. (2009). *Foundations of Macroeconomics*, chapter 16, pages 553–615. Oxford University Press, New York, 2nd edition.

Koopmans, Tjalling C. (1963). On the concept of optimal economic growth. Cowles Foundation Discussion Papers 163, Cowles Foundation for Research in Economics, Yale University.

Lebegue, Daniel (2005). *Revision du taux d'actualisation des investissements publics.* Rapport du groupe d'experts.

Maddison, Angus (1987). Growth and slowdown in advanced capitalist economies: Techniques of quantitative assessment. *American Economic Association*, 25(2): 649–698.

Myerson, Joel, Green, Leonard, and Warusawitharana, Missaka (2001). Area under the curve as a measure of discounting. *Journal of the Experimental Analysis of Behaviour*, 76(2): 235–243.

Navas-Palencia, Guillermo (2018). Fast and accurate algorithm for the generalized exponential integral $e_v(x)$ for positive real order. *Numerical Algorithms*, 77(2): 603–630.

Ramsey, Frank P. (1928). A mathematical theory of saving. *The Economic Journal*, 38(152): 543–559.

Samuelson, Paul A. (1937). A note on measurement of utility. *The Review of Economic Studies*, 4(2): 155–161.

Strulik, Holger (2015). Hyperbolic discounting and endogenous growth. *Economics Letters*, 126: 131–134.

Strulik, Holger (2017). Hyperbolic discounting and the time-consistent solution of three canonical environmental problems. *Center for European, Governance and Economic Development Research Discussion Papers*, (319).

Thaler, Richard (1981). Some empirical evidence on dynamic inconsistency. *Economics Letters*, 8(3): 201–207.

Treasury, HM (2003). *The Green Book: Appraisal and Evaluation in Central Government.* HMSO, London.

Weitzman, Martin L. (2001). Gamma discounting. *The American Economic Review*, 91(1): 260–271.

Yaari, Menahem E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *The Review of Economic Studies*, 32(2): 137–150.