Viscous–inviscid interaction of a just detached planar liquid film near the Taylor–Culick speed: waves, blow-up, reversed-flow breakdown

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Publications

- Sch., B. & P. (JFM **850**, 2018 & **926**, 2021)
- Sch., B. & P. (JFM) & Sch. (QJMAM): submitted soon — talk gives a taste

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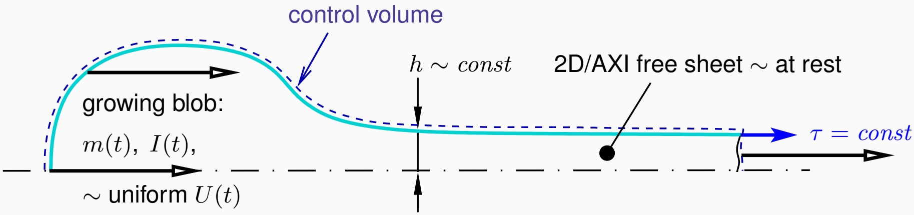
Outset: Taylor–Culick (TC) retraction speed U_{TC}: “infinitely strong Marangoni effect”



$$\left.\begin{array}{r} \dot m \sim \rho h U \\ I \sim mU, \quad \dot I \sim 2\tau \end{array}\right\} \Rightarrow \left\{\begin{array}{l} U \sim U_{\mathrm{TC}} = \sqrt{2\tau/(\rho h)} \\ T = \tau/(\rho h U^2): \quad T = T_{\mathrm{TC}} = 1/2 \end{array}\right.$$

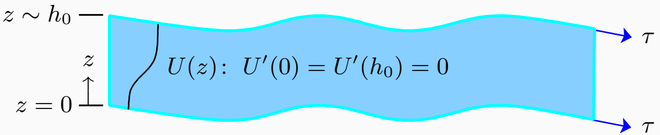
Taylor (1959), Culick (1960), Keller (AXI, 1983)

Motivation: modes on detached layer — we found a further interpretation

Momentum-flux-based reciprocal Weber number

$$T = \frac{\tau}{J} \to 1/2-, \quad J = \rho \int_0^{h_0} U^2(z)\,\mathrm{d}z,$$

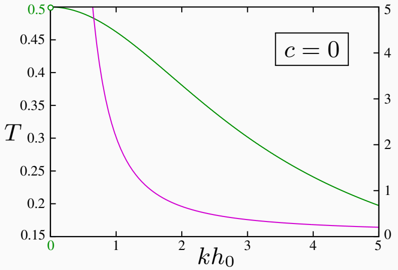
in high-Re & long-wave limit of steady planar sinusoidal perturbations



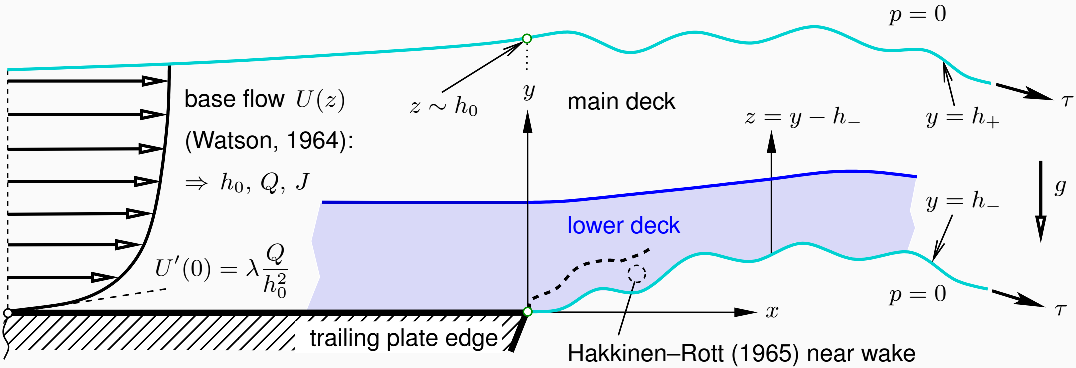
Linear inviscid waves, $U = const$: anomalous dispersion

$$(c-U)^2 = \frac{\tau k}{\rho} \times \begin{cases} \coth(kh_0/2) \ \ldots \ \text{sinuous: } kh_0 \to 0 \ \checkmark \\ \tanh(kh_0/2) \ \ldots \ \text{varicose} \end{cases}$$

Squire (1953), Taylor (1959), Drazin & Reid (1981)



Developed flow negotiating trailing edge

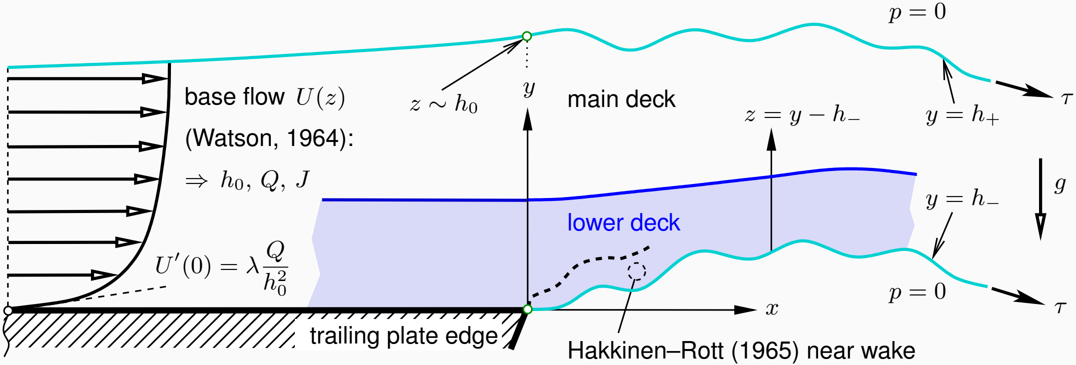


Least-degenerate limit: 2 control groups

$$T = \frac{\tau}{J} = O(1), \quad G = \frac{g h_0^3}{Q^2} \frac{1}{(\lambda^6 \, \epsilon^4)^{1/7}} = O(1)$$

$$Re = Q/\nu \to \infty, \quad \epsilon = (|T - 1|\lambda)^{1/2}/Re \to 0$$

Developed flow negotiating trailing edge



Lower-deck scalings & expansions: $X = \frac{1}{Re} \frac{\lambda^{5/7}}{\epsilon^{6/7}} \frac{x}{h_0} = O(1), \quad Z = \frac{\lambda^{4/7}}{\epsilon^{2/7}} \frac{z}{h_0} = O(1)$

$$\frac{\psi}{Q} \sim \frac{\epsilon^{4/7}}{\lambda^{1/7}} \Psi(X, Z), \quad \frac{p\, h_0^2}{\rho\, Q^2} \sim (\epsilon^4 \lambda^6)^{1/7}\, P(X), \quad \left[\frac{h_-}{h_0}, \frac{h_+}{h_0} - 1\right] \sim \frac{\epsilon^{2/7}}{\lambda^{4/7}} [H_-(X), H_+(X)]$$

Well-posed marching problem: $X \geq 0$, jet-type P/A law

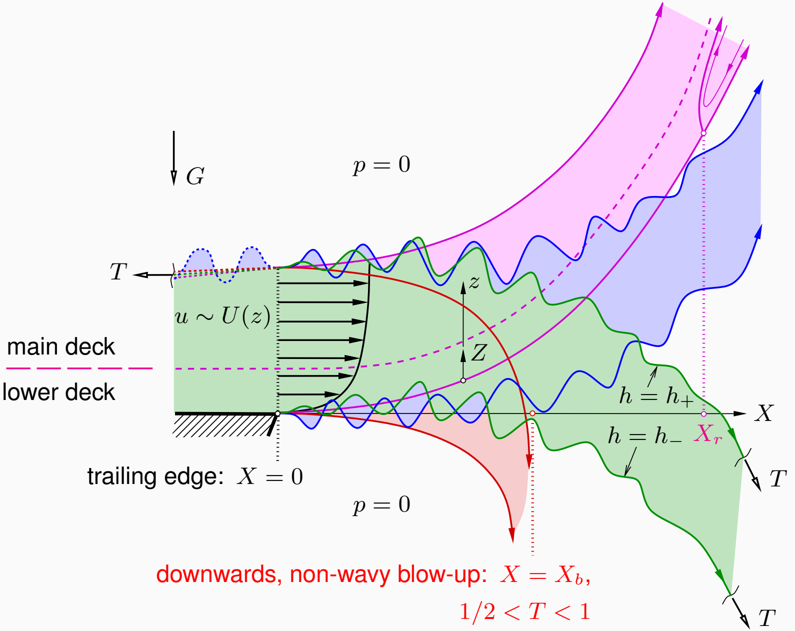
$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P'(X) + \Psi_{ZZZ}$$
$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$
$$\Psi_{ZZ}(X, \infty) = 1, \qquad A(X) = \lim_{z\to\infty}(\Psi_Z - Z)$$
$$P = C(G + SA''), \qquad C = T/(2T-1), \qquad S = \operatorname{sgn}(T-1)$$
$$\Psi_0(Z) := \Psi(0+, Z) = \Psi(0-, Z), \quad A'(0+) = A'(0-), \quad A''(0+) = -SG \quad \Leftarrow \quad P(0) = 0$$

Classification — streamline curvature vs. capillarity

$$P' = \sigma A''', \quad \sigma(T) = SC \begin{cases} > 0 & (0 < T < 1/2 \text{ or } T > 1) & \text{\ldots stabilising feedback: waves} \\ < 0 & (1/2 < T < 1) & \text{\ldots compressive/expansive} \\ = \mp\infty & (T = 1/2\pm) & \text{\ldots choking (cf. linear waves)} \\ = \pm 1 & (T = 1\pm) & \text{\ldots regular limits} \\ = 0 & (T = 1) & \text{\ldots choking (excluded)} \end{cases}$$

$T \to 1/2-$, $A'' \to G$: wavelenght $\to \infty$, interaction condensed, layer “falls down”

Four fundamental detached-jet manifestations



upwards, non-wavy
reversed-flow breakdown:
$1/2 < T < 1$

upwards, symmetric (varicose)
modes, phase shift indefinite:
$1 < T$

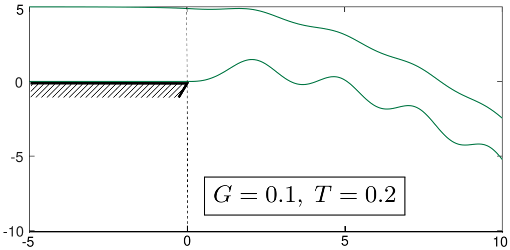
cf. inviscid slender jets
(Keller & Weitz 1957,
Benilov 2023)

downwards,
antisymmetric (sinuous) modes,
self-consistent for all X:
$0 < T < 1/2$

Classification — numerical results: $H-$, $H_+ + 5$, P vs. X

Sinuous / “flapping” modes:

supercritical



Super-exponential branching from

separatrix:

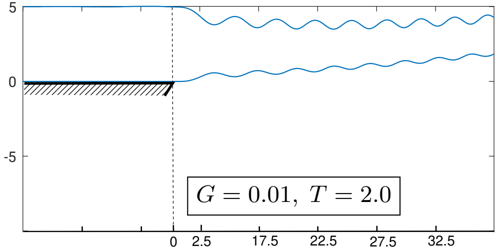
compressive flow reversal

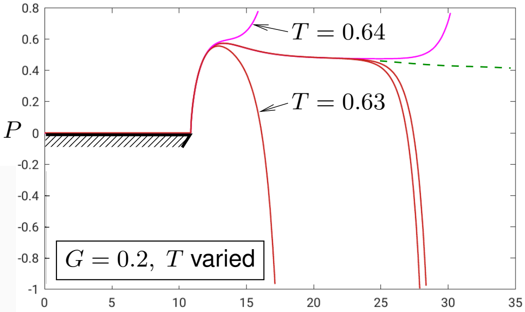
vs.

expansive blow-up

Varicose / “sausage-type” modes:

subcritical — fixing phase for $X < 0$?





P/A or interaction law

$$P = \frac{T}{2T-1}\left[G + \operatorname{sgn}(T-1)A''\right]$$

entails least-degenerate distinguished limit
(self-similarity) near condensed interaction:

Let first

$$0 < X \ll \alpha^{-3}.$$

Then the 2-term inviscid-flow expansion

$$\Psi = \Psi_0 + A(X)\,\Psi_0' + \cdots\,, \quad [P,\,A] = O\big[X^{2/3},\,\alpha^7 X^{8/3}\big], \quad Z \gg X^{1/3}$$

collapses in Hakkinen–Rott near wake where

$$\Psi = O(Z^2), \quad Z = O(X^{1/3})$$

for

$$A\,\Psi_0' \sim \Psi_0 \quad \text{or} \quad Z \sim X^{1/3} \sim \alpha^{-1}.$$

Choking of a capillary wave & non-wavy breakdown: $\alpha = (4\,|T - 1/2|)^{1/7} \to 0$

P/A or interaction law

$$P = \frac{T}{2T-1}\left[G + \operatorname{sgn}(T-1)A''\right]$$

entails least-degenerate distinguished limit
(self-similarity) near condensed interaction:

$$\left[\hat{X}, \hat{Z}, \hat{\Psi}, \hat{A}, \hat{P}, \hat{G}\right] \sim \left[\alpha^3 X,\ \alpha Z,\ \alpha^2 \Psi,\ \alpha A, \alpha^2 P,\ \alpha^{-5} G\right] = O(1)$$

$$\hat{S}\hat{P} = \hat{G} - \hat{A}'', \quad \hat{S} = \operatorname{sgn}(T - 1/2)$$

$$\hat{\Psi}(0, \hat{Z}) = \hat{Z}^2/2, \quad \hat{A}'(0) = 0, \quad \hat{A}''(0) = \hat{G}$$

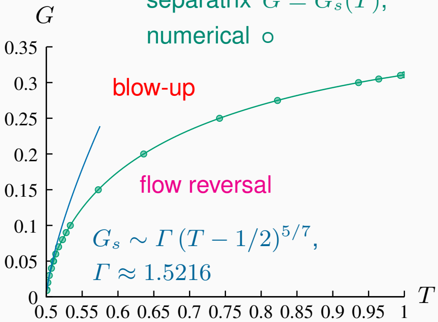
$\hat{S} = -1:$ (cnoidal) waves $(\hat{G} \to \infty)$

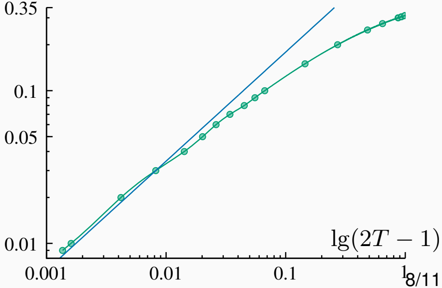
$$\hat{S} = +1: \quad 4^{5/7}\hat{G} \begin{cases} < \Gamma: & \text{flow reversal} \quad (\hat{X} \to \infty) \\ = \Gamma: & \text{Goldstein far wake} \quad (\hat{X} \to \infty) \\ > \Gamma: & \text{finite-}\hat{X}\text{ blow-up} \end{cases}$$

$T > 1/2:$

separatrix $G = G_s(T)$,

numerical ∘



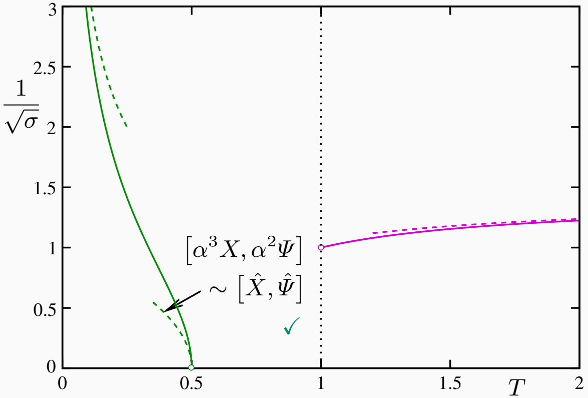


WKBJ analysis for $X \to \infty$: recovers linear long-wave limit

$$\frac{\Psi}{X^{2/3}} = \underbrace{G(\eta)}_{\text{Goldstein wake}} + X^{\mu} \exp\left(\frac{6\mathrm{i}K}{7} X^{7/6} + \cdots\right)\Big[\underbrace{f(\eta)}_{\text{eigenfunction}} + \cdots\Big] + c.c., \quad \eta = \frac{Z}{X^{1/3}}$$

Secularity conditions: wavenumber growth & amplitude decay

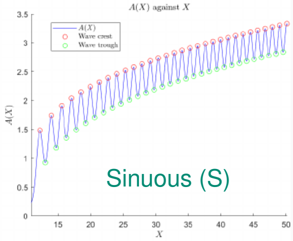
$$K = \frac{1}{\sqrt{I\,\sigma(T)}}, \quad I = \int_0^\infty \frac{\mathrm{d}\eta}{G'^2(\eta)} \approx 0.8525, \quad \mu = -\frac{5}{12} - \int_0^\infty \frac{Gf''/3 + f'''/2}{G'^2}\,\mathrm{d}\eta \approx -0.4704$$



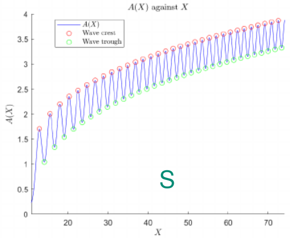
○ choking

WKBJ vs. numerical analysis: $A(X)$ for $X \gg 1$, $G = 0.2$, T increased

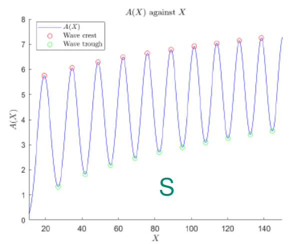
Numerical trends confirm: amplitude $\to \infty$, $k_0 \to 0$ as $T \to 1/2-$ (choking)



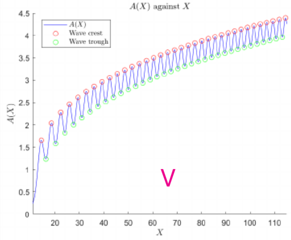
(a) $G = 0.2,\ T = 0.1$



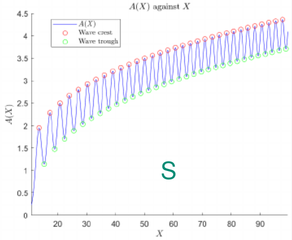
(b) $G = 0.2,\ T = 0.2$



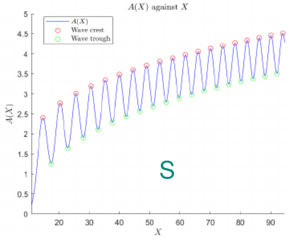
(e) $G = 0.2,\ \boxed{T = 0.49}$



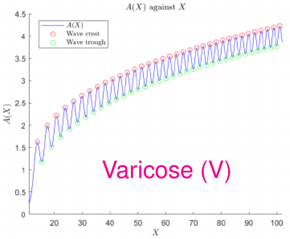
(f) $G = 0.2,\ \boxed{T = 1.1}$



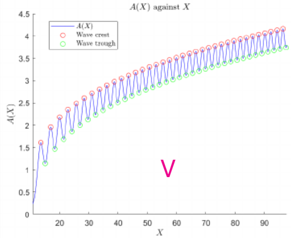
(c) $G = 0.2,\ T = 0.3$



(d) $G = 0.2,\ T = 0.4$



(g) $G = 0.2,\ T = 2$



(h) $G = 0.2,\ T = 5$

Central results

- Surprisingly rich self-consistent theory of developed film passing plate edge
- $T < 1/2,\ T > 1:$ nonlinear extension of stationary Squire–Taylor modes
- $T \sim T_{\mathrm{TC}} = 1/2,\ T \sim 1:$ choking
- $1/2 < T < 1:$ breakdowns by flow reversal or blow-up

To-dos

- Regularise: breakdowns, $T \sim 1$
- Unsteadiness & stability
- Axial-symmetry breaking in pipe exit problem
- Careful experiments!

Thanks for your attention!