Wavenumber-explicit analysis of Maxwell's equations in piecewise smooth media

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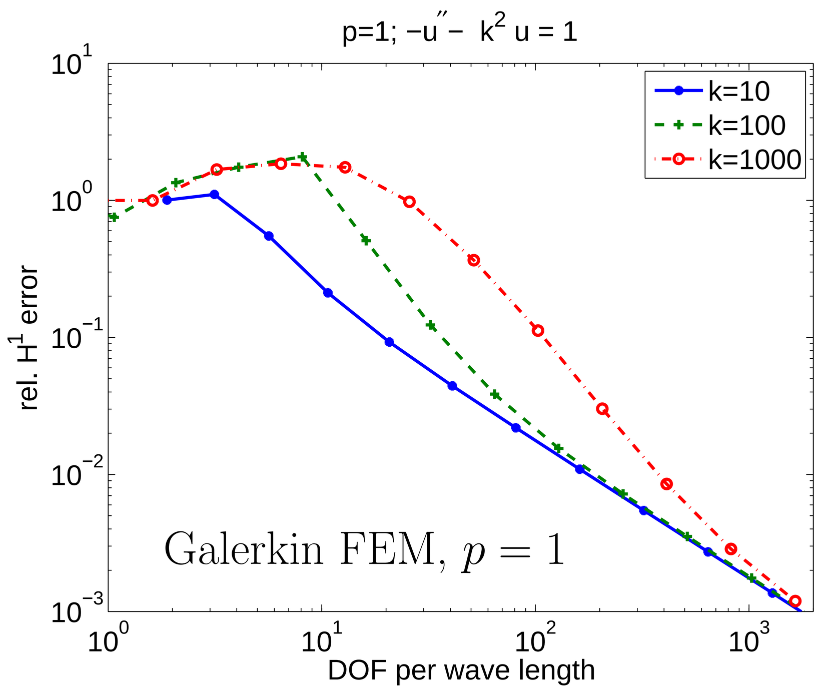
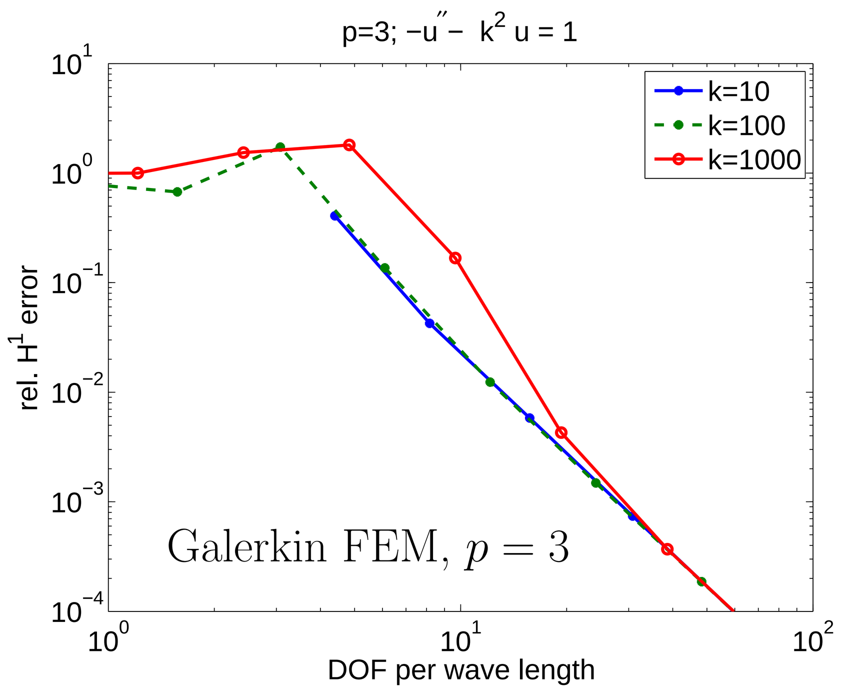
Motivation

Lowest order finite element discretizations for time-harmonic wave problems suffer from the pollution effect. That is, as the wavenumber $|k|$ increases, the gap between FEM error and best approximation widens. E.g., for the 1D Helmholtz equation we observe





- Higher polynomial degrees are better!
- Analogous result for Maxwell's equations with constant scalar coefficients: We need $|hk/p|$ sufficiently small and $p \gtrsim \log|k|$ to suppress pollution, [2].
- **Question:** Extension to Maxwell's equations with piecewise smooth coefficients?

Maxwell problem

Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected and bounded domain with smooth and simply connected boundary Γ and outer normal unit vector $\mathbf{n}$.

For a given wavenumber $k \in \mathbb{R}$ with $|k| \geq 1$, a given right-hand side $\mathbf{f}$ and a given tangent field $\mathbf{g}$, we look for a solution $\mathbf{u}$ of the equation

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{u} - k^2 \varepsilon \mathbf{u} = \mathbf{f} \quad \text{in } \Omega,$$
$$\mu^{-1} \operatorname{curl} \mathbf{u} \times \mathbf{n} - ik\mathbf{u}_T = \mathbf{g} \quad \text{on } \Gamma,$$

where μ is the magnetic permeability, ε is the electric permittivity, i is the imaginary unit and $\mathbf{u}_T := \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$.

We assume that the tensor fields μ and ε are real-valued, symmetric positive definite and piecewise smooth in Ω.

A shift theorem for vector fields

As a consequence of the seminal work [1] every divergence-free vector field $\mathbf{v}$ on the considered domain Ω can be written as

$$\mathbf{v} = \operatorname{curl} \mathbf{R}\mathbf{v} + \mathbf{K}\mathbf{v},$$

where $\mathbf{R}$ and $\mathbf{K}$ are pseudodifferential operators of orders -1 and $-\infty$, respectively. In essence, $\mathbf{R}$ is a **right-inverse to the curl**-operator. The operators $\mathbf{R}$ and $\mathbf{K}$ are essential for the proof of the subsequent theorem, which generalizes the main result of [5].

Theorem 1

Assume that Ω is decomposed into smooth subdomains $\mathcal{G}_1, \ldots, \mathcal{G}_n$, and let ν be a real-valued SPD tensor field that is piecewise smooth and discontinuous only across subdomain interfaces. For $\ell \in \mathbb{N}_0$ let $\mathbf{v} \in \mathrm{H}(\operatorname{curl}, \Omega)$ with $\operatorname{curl} \mathbf{v}|_{\mathcal{G}_i} \in \mathbf{H}^\ell(\mathcal{G}_i)$ as well as $\nu\mathbf{v} \in \mathrm{H}(\operatorname{div}, \Omega)$ with $\operatorname{div} \nu\mathbf{v}|_{\mathcal{G}_i} \in \mathbf{H}^\ell(\mathcal{G}_i)$.

If $\nu\mathbf{v} \cdot \mathbf{n} = h$ on Γ for some $h \in \mathrm{H}^{\ell+\frac{1}{2}}(\Gamma)$, then there exists a decomposition $\mathbf{v} = \mathbf{z} + \nabla\varphi$ such that

$$\sum_{i=1}^n \|\mathbf{z}\|_{\mathbf{H}^{\ell+1}(\mathcal{G}_i)} \lesssim \sum_{i=1}^n \|\operatorname{curl} \mathbf{v}\|_{\mathbf{H}^\ell(\mathcal{G}_i)}, \qquad \|\varphi\|_{\mathrm{H}^{\ell+2}(\Omega)} \lesssim \|h\|_{\mathrm{H}^{\ell+\frac{1}{2}}(\Gamma)} + \sum_{i=1}^n \|\operatorname{div} \nu\mathbf{v}\|_{\mathbf{H}^\ell(\mathcal{G}_i)},$$

and $(\nu\mathbf{v}, \nabla\xi)_{\mathbf{L}^2(\Omega)} = 0$ for all $\xi \in \mathrm{H}^1(\Omega)$. As a consequence, $\mathbf{v}|_{\mathcal{G}_i} \in \mathbf{H}^{\ell+1}(\mathcal{G}_i)$ for all $\mathcal{G}_i$.

A similar result holds if the boundary condition $\nu\mathbf{v} \cdot \mathbf{n} = h$ is replaced by $\mathbf{v}_T = \mathbf{g}$ for a tangent field $\mathbf{g} \in \mathbf{H}^{\ell+\frac{1}{2}}(\Gamma)$.

References

[1] M. Costabel and A. McIntosh, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains*, Math. Z. **265** (2010), no. 2, 297–320.

[2] J. M. Melenk and S. Sauter, *Wavenumber-explicit hp-FEM analysis for Maxwell's equations with impedance boundary conditions*, Found. Comp. Math., To appear.

[3] J. M. Melenk and D. Wörgötter, *Wavenumber-explicit regularity by decomposition for Maxwell's equations in piecewise smooth media*, Work in progress.

[4] J. Schöberl, *Finite Element Software NETGEN/NGSolve version 6.2.*, https://ngsolve.org/.

[5] C. Weber, *Regularity Theorems for Maxwell's Equations*, Math. Meth. in Appl. Sci. **3** (1981), 523–536.

Wavenumber-explicit estimates

While Theorem 1 leads to an essential **finite regularity shift**, the subsequent theorem provides wavenumber-explicit analytic regularity shifts for the considered Maxwell problem.

Theorem 2

Let Ω be decomposed into subdomains $\mathcal{G}_1, \ldots, \mathcal{G}_n$ and suppose that the boundary Γ and all subdomain interfaces are analytic. Furthermore, suppose that the coefficients μ and ε are piecewise analytic on Ω, and that $\mathbf{f} \in \mathrm{H}(\operatorname{div}, \Omega)$ is piecewise analytic as well. Then, if the given tangent field $\mathbf{g}$ is analytic, the solution $\mathbf{u}$ of the considered Maxwell problem satisfies

$$\sum_{i=1}^n \|\mathbf{u}\|_{\mathbf{H}^\ell(\mathcal{G}_i)} \lesssim \left(|k|^{-1} + \|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + |k|\,\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\right) A^\ell (\ell + |k|)^\ell,$$

for all $\ell \in \mathbb{N}_0$, where $A > 0$ and the hidden constant depend on $\mathbf{f}$, $\mathbf{g}$, μ, ε and the geometry, but are independent of ℓ and the wavenumber k. As a consequence, $\mathbf{u}$ is piecewise analytic.

Theorem 2 is a key ingredient in the "regularity splitting" of solutions of the Maxwell problem. This regularity splitting is crucial for the proof of quasi-optimality of finite element approximations under the **scale resolution condition**

$$|hk/p| \quad \text{suffiently small and} \quad p \gtrsim \log|k|.$$

Regularity splitting

Theorem 1 and Theorem 2 are essential for the proof of the following splitting result. For simplicity, we consider only the case $\mathbf{g} = 0$ and $\operatorname{div} \mathbf{f} = 0$. Furthermore, let $C_k > 0$ be the stability constant of the Maxwell problem, i.e., the smallest number such that

$$\|\operatorname{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + |k|\,\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + |k|^{1/2}\|\mathbf{u}_T\|_{\mathbf{L}^2(\Gamma)} \leq C_k \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

The subsequent theorem requires that C_k grows at most algebraically in k.

Theorem 3

Under the hypothesises of Theorem 2, let $\mathbf{u}$ be the solution of the considered Maxwell problem subject to $\mathbf{g} = 0$ and a divergence-free and piecewise regular $\mathbf{f}$. In addition, assume that $C_k \leq C|k|^\theta$ for some $C > 0$ and $\theta \in \mathbb{R}$. Then, $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{u}_{\mathrm{H}^2} + \mathbf{u}_{\mathcal{A}}$ with

$$\sum_{i=1}^n \|\mathbf{u}_{\mathbf{H}^2}\|_{\mathbf{H}^2(\mathcal{G}_i)} \lesssim |k|^{-1} \sum_{i=1}^n \|\mathbf{f}\|_{\mathbf{H}^1(\mathcal{G}_i)} \quad \text{and} \quad \sum_{i=1}^n \|\mathbf{u}_{\mathcal{A}}\|_{\mathbf{H}^\ell(\mathcal{G}_i)} \lesssim |k|^\lambda M^{\ell+1} (\ell + |k|)^\ell$$

for all $\ell \in \mathbb{N}_0$. Furthermore, $\lambda \in \mathbb{R}$ depends only C and θ, and $M > 0$ depends on the geometry, μ, ε and $\mathbf{f}$.

Based on this theorem we can employ techniques from [2] to conclude **quasi-optimality** of finite element approximations under the conditions $|hk/p|$ sufficiently small and $p \gtrsim \log|k|$.

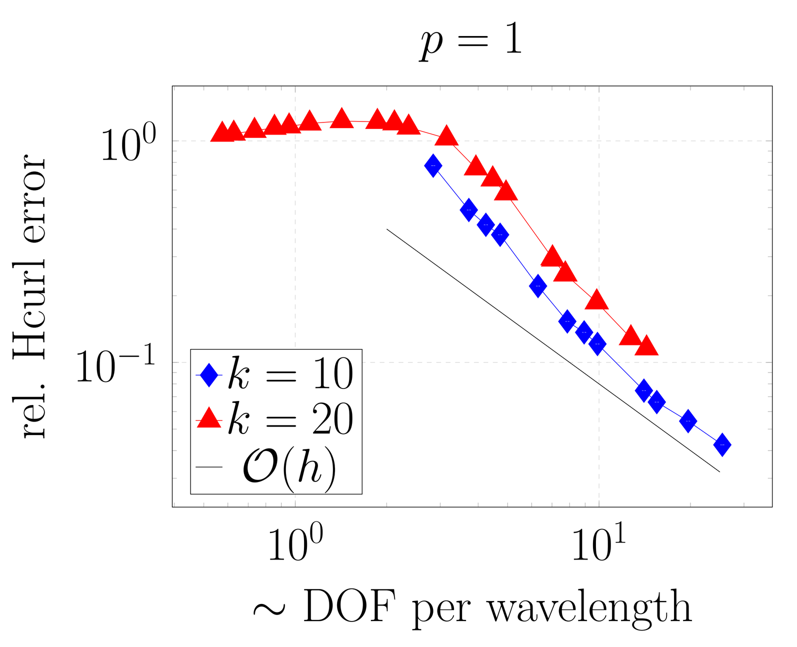
Numerical experiments

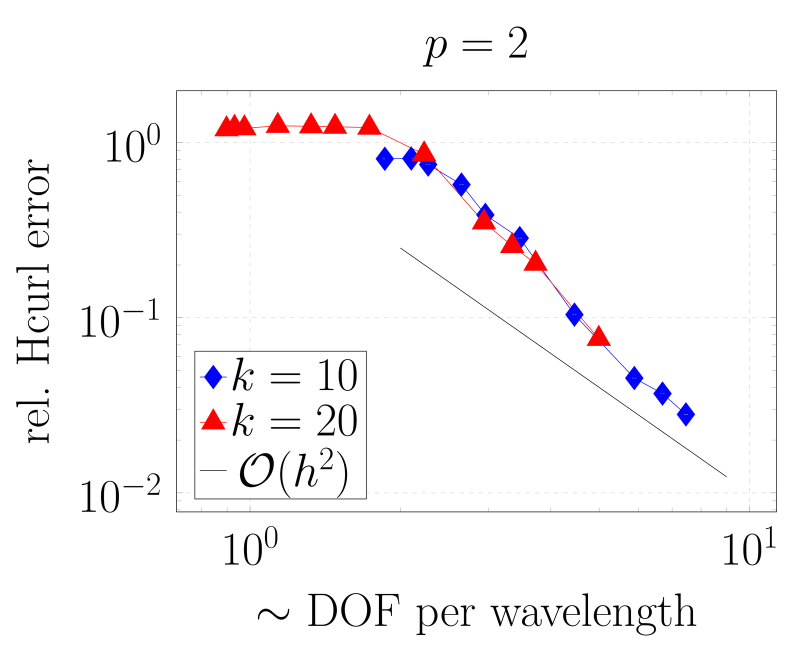
We consider

- Domain $\Omega := B_1(0) \subseteq \mathbb{R}^3$,
- Subdomains $\mathcal{G}_1 := B_{1/2}(0)$ and $\mathcal{G}_2 := B_1(0) \setminus \mathcal{G}_1$,
- Right-hand side $\mathbf{f}(x, y, z) = (z, 0, 0)^T$, boundary data $\mathbf{g} = 0$,
- In outer subdomain $\mathcal{G}_2$ we set $\mu = \varepsilon = \mathrm{I}$, and in the inner ball $\mathcal{G}_1$ we choose

$$\mu^{-1} := \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \varepsilon := \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

- Computations based on Nédélec type-II elements of degree p using NGSolve [4],
- Exact solutions unknown, we computed reference solutions by higher order methods.





For $p = 1$, the plot indicates an increasing gap between the finite element solution and best approximation for rising k. For $p = 2$, the gap does hardly increase between $k = 10$ and $k = 20$. That means, for $p = 2$ the **pollution effect is weakened!**