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Stochastic modelling in the asset-liability management of life insurance companies

A scenario-based approach for interest rate risk management

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Kurzfassung

Im quantitativen Risikomanagement von Lebensversicherungsunternehmen kommt dem Begriff des Asset-Liability Managements durch das von langen Vertragslaufzeiten geprägte Versicherungsgeschäft und der damit stark in Wechselwirkung stehenden langfristigen Kapitalveranlagung besondere Bedeutung zu. Die Komplexität des Geschäfts, das breite Konkurrenzumfeld und weitreichende regulatorische Anforderungen erfordern dafür von Versicherungsunternehmen die Implementierung komplexer stochastischer Modelle zur Gewinnung wichtiger Informationen für strategische Entscheidungen und zur Absicherung von Risiken. Im Rahmen dieser Arbeit beleuchten wir die Rolle und Notwendigkeit stochastischer Szenarienmodellierung im Asset-Liability Management von Lebensversicherern und stellen ein stochastisches Mehrperiodenmodell zur Absicherung des Zinsrisikos von klassischen Lebensversicherungsportfolios unter Berücksichtigung der damit zusammenhängenden nationalen rechtlichen Anforderungen vor. Wir betrachten dafür zwei miteinander interagierende Modellansätze für die zeitliche Entwicklung von Vermögenswerten (Assets) und Versicherungsverbindlichkeiten (Liabilities) und leiten daraus die zeitabhängigen Zinssensitivitäten des Kapitalanlageportfolios und des Versicherungsvertragsportfolios ab. Darauf basierend formulieren wir eine dynamische Duration-Matching-Optimierungsstrategie zur Immunisierung eines Lebensversicherungsbestandes gegen adverse Entwicklungen im ökonomischen Zinsumfeld und damit zur Steigerung der finanziellen Stabilität und Profitabilität eines Lebensversicherungsunternehmens.

Abstract

In the quantitative risk management of life insurance companies, the concept of asset-liability management is of particular importance due to the long contract terms of the life insurance business and the strongly interrelated long-term investment of capital. The complexity of the business, the broad competitive environment and far-reaching regulatory requirements demand that insurance companies implement complex stochastic models to hedge risks and obtain important information for strategic decisions. In this thesis, we highlight the role and necessity of stochastic scenario modelling in the asset-liability management of life insurers and present a stochastic multi-period model for interest rate risk management of traditional life insurance portfolios, taking into account the associated national legal requirements. We consider two interacting modelling approaches for the development of assets and liabilities over time and derive the time-dependent interest rate sensitivities of the investment portfolio and the insurance contract portfolio. Based on this, we formulate a dynamic duration matching optimisation strategy to immunise a life insurance portfolio against adverse developments in the economic interest rate environment and thus to increase the financial stability and profitability of a life insurance company.

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Eidesstattliche Erklärung

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Contents

1. Introduction

The traditional life insurance business is characterized by long contract terms and a strong dependence on developments in the interest rate environment on the capital markets. Compared to non-life insurance, life insurance companies are exposed to a higher liquidity and interest rate risk, particularly due to the long-term liabilities to policyholders typical of life insurance and the generally high sums insured. Even with the in Austria legally regulated maximum guaranteed interest rate for new contracts in traditional life insurance being set to 0.00%, insurers are often required to earn high guaranteed interest obligations from existing contracts by investing the premiums collected in suitable financial assets. At the same time, the investment strategy must be designed in such a way that, despite the inhomogeneous distribution of cash flows from the insurance business over time (premiums, benefits, etc.), appropriate liquidity is always available to meet the contractual obligations. In addition, changes in the interest rate environment can have a significant impact on the profitability and solvency of a life insurance company due to the high interest rate sensitivity of its liabilities, provided that this change does not affect investments to a similar extent. With inadequate risk measures, this can lead to a mismatch of assets and liabilities, and scenarios can arise in which the fair value of the investments is no longer sufficient to cover the liabilities. These risks have increased in importance in recent years due to the challenging conditions on the capital markets, such as the long-lasting phase of low interest rates and the negative developments on the stock markets during the 2009 financial crisis or the COVID 19 pandemic.

To analyse and manage such risks appropriately, life insurance companies must adopt suitable strategies and measures as part of their asset-liability management (ALM). Within the ALM framework and the quantitative risk management of life insurance companies, stochastic scenario and optimisation models in particular are indispensable tools for analysing sources of risk and obtaining information for strategic corporate planning. Consequently, the topic of stochastic simulation of both assets and liabilities has been widely treated in insurance literature, we refer to Di Francesco/Simonella [5], Führer [7], Gerstner at al. [10], Jaquemod at al. [12], Koller [17] and the references therein. In this thesis, we aim to define and derive a multi-period stochastic ALM model to analyse and manage the interest rate risk in traditional life insurance business that is consistent with the corresponding Austrian legal regulations for life insurance. We first define a model for the simulation of capital market scenarios and the capital investment of a life insurance company, whereby we cover a stochastic short rate model for the underlying yield curve as well as market models for equity investments, fixed rate bonds and a money market investment possibility. Particular emphasis is placed on taking the payout structure into account when modelling fixed rate bonds in order to analyse and derive the interest rate sensitivity of such fixed income securities on this basis. Second, we present a Markov chain-based modelling approach to simulate endowment insurance contract portfolios, taking into account the policyholders' contractual options guaranteed by national law. This modelling approach provides the flexibility to be adapted to other types of life insurance contracts without great difficulty. We derive a fair value for both the assets and the liabilities of the insurance company and analyse the interest rate sensitivity of both values using the Fisher-Weil duration as a sensitivity measure. Last, we introduce a dynamic two-stage duration matching optimisation strategy in order to immunise the insurance company against the risk of a shortfall in the assets covering the liabilities due to a change in the yield curve. As part of a periodic restructuring of the asset portfolio, the two-stage approach takes into account not only the minimisation of interest rate risk but also the insurance company's responsibility to maximise the expected investment return of its capital investments, subject to restrictions. For the individual model components, we address their relevant mathematical properties and place particular emphasis on a flexible definition of the model in order to capture the stochastic dynamics of both the capital market and the insurance contract portfolio and enable the simulation of a large number of different scenarios within this framework.

This thesis is structured as follows: In Chapter 2, we present the necessary mathematical foundations, primarily the important concepts of financial and actuarial mathematics, which are required for the main model. In Chapter 3, we discuss the concept of asset-liability management in the field of life insurance, focusing in particular on the role of stochastic modelling. In addition, the legal framework in connection with asset-liability management covered by Austrian insurance law and the extent to which these regulations must be taken into account in stochastic models are discussed. Chapter 4 forms the centrepiece of this thesis and covers the derivation of the models for both assets and liabilities as well as the interest rate risk management optimisation approach. Chapter 5 shortly covers the model implementation aspect of Euler discretization for numerical simulation of stochastic differential equation and presents the numerical results for a selected scenario. Finally, in chapter 6, we review the model and discuss possible extensions of the modelling approach. The implementation of the model in Python can be found in Appendix B.

2. Mathematical foundations and notation

This chapter aims to provide the essential definitions and tools related to stochastic processes, stochastic analysis and financial mathematics relevant for this thesis. We will concentrate primarily on results and mathematical properties that are directly relevant to the model in Chapter 4 and thus quietly assume certain mathematical principles, such as the existence of stochastic integrals, for example. The sections on stochastic processes and stochastic calculus are based on Brzezniak/Zastawniak [2], Karatzas/Shreve [15] and Shreve [26]. With regard to interest rate models, we follow the results in Brigo/Mercurio [1] and Desmettre/Korn [4].

2.1. Stochastic calculus and stochastic processes

For this chapter and this thesis in general, we will operate on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. For that, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathbb{P}$ reflects the real-world probability measure. Let further $\mathbb{F} := \{\mathcal{F}_t, t \in [0, \infty)\}$ be a filtration, that is an increasing sequence of σ-algebras with $\mathcal{F}_t \subseteq \mathcal{F}$ and

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad s \leq t.$$

For any given point in time t, the corresponding σ-algebra $\mathcal{F}_t$ represents the information that is available and can be observed up to and including time t.

An important probability distribution that is later used to determine distribution properties in the model is the log-normal distribution:

Definition 2.1.1 (Log-normal distribution)**.** Let Z be a standard normally distributed random variable and consider $\mu \in \mathbb{R}$, $\sigma > 0$. Then the random variable

$$X = e^{\mu + \sigma Z}$$

is *log-normally distributed* with parameters μ and σ^2, denoted by

$$X \sim Log\mathcal{N}(\mu, \sigma^2).$$

Without proof, we present expressions for the expected value and the variance of log-normally distributed random variables:

Lemma 2.1.2. *Let $X \sim Log\mathcal{N}(\mu, \sigma^2)$. Then the following expressions hold:*

$$\mathbb{E}[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{2.1}$$

$$\mathbb{V}\text{ar}(X) = \exp\left(2\mu + \sigma^2\right)\Big(\exp\left(\sigma^2\right) - 1\Big) \tag{2.2}$$

One of the most important stochastic processes in financial mathematics and a central component in the modelling of many other stochastic processes is Brownian motion:

Definition 2.1.3 (Standard Brownian motion)**.** An $\mathbb{R}$-valued stochastic process $W = \big(W(t)\big)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a *standard Brownian motion* or *standard Wiener process*, if the following conditions hold:

(a) $W(0) = 0$ a.s.,

(b) $W(t) - W(s)$ is $\mathcal{F}_s$-measurable for all $s < t$ (Measurability of past increments),

(c) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $s < t$ (Independence of future increments),

(d) $W(t) - W(s)$ has the same distribution as $W(t-s)$ for all $s < t$ (Stationarity of the distribution of increments),

(e) $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ (Normal distribution of increments),

(f) W has continuous paths, that is $[0,\infty) \ni t \mapsto W(t)(\omega)$ is continuous for every $\omega \in \Omega$.

Lemma 2.1.4 (Properties of stochastic integrals)**.** *Consider $T \in \mathbb{R}^+$. Let $W = \big(W(t)\big)_{t\geq 0}$ be a standard Brownian motion and let $X = \big(X(t)\big)_{t\geq 0}$, $Y = \big(Y(t)\big)_{t\geq 0}$ be two stochastic processes adapted to the natural filtration $\big(\mathcal{F}_t^W\big)_{t\geq 0}$ of the Brownian motion such that*

$$\mathbb{E}\Big[\int_0^T X(t)^2\,dt\Big] < \infty, \tag{2.3}$$

$$\mathbb{E}\Big[\int_0^T Y(t)^2\,dt\Big] < \infty \tag{2.4}$$

hold. Then, the following properties of the stochastic integral with respect to the Brownian motion W hold for $0 \leq a < b \leq T$:

1. *Linearity of the stochastic integral:*

$$\int_a^b \alpha\, X(t) + \beta\, Y(t)\, dW(t) = \alpha \int_a^b X(t)\, dW(t) + \beta \int_a^b Y(t)\, dW(t), \tag{2.5}$$

2. *Itô isometry:*

$$\mathbb{E}\Big[\Big(\int_0^T X(t)\, dW(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T X(t)^2\, dt\Big], \tag{2.6}$$

3. *Normal distribution for deterministic integrands:*
If $X = \big(X(t)\big)_{t\geq 0}$ is deterministic, then $\int_0^T X(t)\, dW(t)$ is normally distributed with

$$\int_0^T X(t)\, dW(t) \sim \mathcal{N}\Big(0,\ \int_0^T X(t)^2\, dt\Big). \tag{2.7}$$

Proof. See Shreve [26]. □

For the definition of stochastic differential equations and their solutions, we follow Karatzas/Shreve [15]:

Definition 2.1.5 (Stochastic differential equation)**.** A *one-dimensional stochastic differential equation (SDE)* is an equation of the form

$$dX(t) = \alpha\big(t, X(t)\big)\, dt + \sigma\big(t, X(t)\big)\, dW(t), \tag{2.8}$$

with initial value $X(0) = X_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, where $\big(W(s)\big)_{s\geq 0}$ is a standard Brownian motion and $\alpha(t, x)$ and $\sigma(t, x)$ are real-valued Borel-measurable functions, called *drift* and *diffusion*, respectively. By definition of the stochastic differential, equation (2.8) is equivalent to the stochastic integral equation

$$X(t) = X(0) + \int_0^t \alpha\big(s, X(s)\big)\, ds + \int_0^t \sigma\big(s, X(s)\big)\, dW(s), \quad 0 \leq t < \infty. \tag{2.9}$$

An $\mathbb{R}$-valued stochastic process $X = \big(X(t)\big)_{t\geq 0}$ on the given probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a strong solution of equation (2.8) if it has the following properties:

(i) X is adapted to the filtration $\mathbb{F}$;

(ii) $\mathbb{P}[X(0) = X_0] = 1$;

(iii) $\mathbb{P}\Big[\int_0^t \big|\alpha\big(s, X(s)\big)\big| + \sigma\big(s, X(s)\big)^2\, ds < \infty\Big] = 1$;

(iv) the integral equation (2.9) holds almost surely for all $t \geq 0$.

A strong solution X of (2.8) is called *unique* if any other strong solution $\tilde{X} = \big(\tilde{X}(t)\big)_{t\geq 0}$ is indistinguishable from X, that is

$$\mathbb{P}\big[X(t) = \tilde{X}(t) \text{ for all } 0 \leq t < \infty\big] = 1.$$

If the drift and diffusion functions fulfil certain conditions, the uniqueness of a solution of the SDE can be guaranteed:

Theorem 2.1.6. *Consider an SDE with drift $\alpha(t, x)$ and diffusion $\sigma(t, x)$ and assume that a strong solution $X = \big(X(t)\big)_{t\geq 0}$ exists for the SDE. Suppose that the coefficient functions $\alpha(t, x)$ and $\sigma(t, x)$ are Lipschitz-continuous in x, i.e. there exists a positive constant K such that for all $x, y \in \mathbb{R}$ and $t \geq 0$ it holds that*

$$\big|\alpha(t, x) - \alpha(t, y)\big| + \big|\sigma(t, x) - \sigma(t, y)\big| \leq K|x - y|. \tag{2.10}$$

Then the solution X of the SDE is unique.

Proof. See Karatzas/Shreve [15]. □

A fundamental result in stochastic calculus and a useful tool for solving stochastic differential equations is Itô's Lemma. We will here present the one-dimensional version of the Itô formula for Itô processes based on Shreve [26]:

Definition 2.1.7 (Itô process)**.** Let $\big(W(t)\big)_{t\geq 0}$ be a Brownian motion and let $\big(\mathcal{F}_t\big)_{t\geq 0}$ be an associated filtration. An *Itô process* is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)\, dW(u) + \int_0^t \Theta(u)\, du,$$

where $X(0)$ is non-random and $\big(\Delta(u)\big)_{u\geq 0}$ and $\big(\Theta(u)\big)_{u\geq 0}$ are stochastic processes adapted to $\big(\mathcal{F}_t\big)_{t\geq 0}$.

Theorem 2.1.8 (Itô formula for Itô processes)**.** *Let $X = \big(X(t)\big)_{t\geq 0}$ be an Itô process and let $f(t,x)$ be a function such that the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous. Then it holds for every $T \geq 0$ that*

$$\begin{aligned} f\big(T, X(T)\big) = f\big(0, X(0)\big) &+ \int_0^T f_t\big(t, X(t)\big)\, dt + \int_0^T f_x\big(t, X(t)\big)\, dX(t) \\ &+ \frac{1}{2} \int_0^T f_{xx}\big(t, X(t)\big)\, d[X,X](t). \end{aligned}$$

where the quadratic variation $[X,X](t)$ of the Itô process X is given as

$$[X,X](t) = \int_0^t \Delta^2(u)\, du.$$

Proof. See Shreve [26]. □

Remark. The Itô formula is often rewritten in differential notation as

$$df\big(t, X(t)\big) = f_t\big(t, X(t)\big)\, dt + f_x\big(t, X(t)\big)\, dX(t) + \frac{1}{2} f_{xx}\big(t, X(t)\big)\, dX(t)\, dX(t).$$

2.1.1. Markov chains

The modelling approach for the insurance contract portfolio in this model is based on a specific type of stochastic process called Markov chain. We will here define this stochastic process and introduce the model-relevant properties based on the results in Brzezniak/Zastawniak [2] and Koller [17].

Definition 2.1.9. Consider a finite or countable set $\mathcal{S}$ and an index set $I \subseteq \mathbb{N}_0$. A stochastic process $X = (X_k)_{k\in I}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *discrete-time Markov chain*, if for all $n \geq 1$, $k_1 < \cdots < k_{n+1} \in I$ and arbitrary $i_1, \ldots, i_{n+1} \in \mathcal{S}$ with $\mathbb{P}[X_{k_1} = i_1, \ldots, X_{k_n} = i_n] > 0$ the following holds true:

$$\mathbb{P}[X_{k_{n+1}} = i_{n+1} \mid X_{k_1} = i_1, \ldots, X_{k_n} = i_n] = \mathbb{P}[X_{k_{n+1}} = i_{n+1} \mid X_{k_n} = i_n]. \tag{2.11}$$

The set $\mathcal{S}$ is called the *state space* of the Markov chain.

From the definition one can see that the defining property of a Markov chain is that if the Markov chain is in state $i \in \mathcal{S}$ after n steps, the conditional distribution of the next state only depends on the current state i and not on the positions of the Markov chain before the n-th step. This property also refers to the 'memoryless property' of a stochastic process.

Definition 2.1.10 (Transition probability, transition matrix)**.** Let $X = (X_k)_{k\in I}$ be a discrete-time Markov chain with state space $\mathcal{S}$. Then, for $i, j \in \mathcal{S}$, $k, \ell \in I$ with $k \leq \ell$ and $\mathbb{P}[X_k = i] > 0$,

$$p_{ij}(k,\ell) := \mathbb{P}\big[X_\ell = j \;\big|\; X_k = i\big] \tag{2.12}$$

is called the *transition probability* from state i at time step k to state j at time step ℓ. If $\mathbb{P}[X_k = i] = 0$, then $p_{ij}(k,\ell) := 0$.
The matrix

$$P_X(k,\ell) := \big(p_{ij}(k,\ell)\big)_{(i,j)\in S\times S} \tag{2.13}$$

is called the *transition matrix* from k to ℓ of the Markov chain X.

As the total sum of transition probabilities from a state i to all other states must equal 1, an arbitrary transition matrix P fulfils

$$\sum_{j\in\mathcal{S}} P_{i,j} = 1 \tag{2.14}$$

for all $i \in \mathcal{S}$, that is, the sum of all rows of the transition matrix is 1. A square matrix with non-negative entries fulfilling (2.14) is called *stochastic matrix*, i.e. every transition matrix of a Markov chain is also a stochastic matrix.

A fundamental theorem in the theory of Markov chains is the following theorem of Chapman and Kolmogorov, which enables an efficient calculation of the transition probabilities:

Theorem 2.1.11 (Chapman-Kolmogorov equation)**.** *Let $X = (X_k)_{k\in I}$ be a discrete-time Markov chain with state space $\mathcal{S}$. For $k \leq \ell \leq m \in I$ and $i, j \in \mathcal{S}$ with $\mathbb{P}[X_k = i] > 0$, the following holds:*

$$p_{ij}(k,m) = \sum_{h\in S} p_{ih}(k,\ell)\, p_{hj}(\ell,m), \tag{2.15}$$

$$P_X(k,m) = P_X(k,\ell)\, P_X(\ell,m). \tag{2.16}$$

Proof. See Brzezniak/Zastawniak [2]. □

2.2. Financial mathematics

Definition 2.2.1. A *zero-coupon bond with maturity T (T-bond)* is a financial contract that guarantees its holder a payment of one monetary unit at maturity T with no intermediate payments. We denote with $P(t,T)$ the value or price of a zero-coupon bond with maturity T at time $t \leq T$.

Definition 2.2.2 (Forward rate, Short rate)**.** Let $P(t,T)$ be the price of a zero-coupon bond with maturity $T \geq t$ and assume that $T \mapsto \ln P(t,T)$ is differentiable with respect to T for all $0 \leq t \leq T$. Then

$$f(t,T) := -\frac{\partial}{\partial T} \ln\big(P(t,T)\big) \tag{2.17}$$

is called the *forward rate* in T at time t and

$$r(t) := f(t,t) \tag{2.18}$$

is called the *short rate* at time t.

The approach to modelling the risk-free yield curve in this thesis is to stochastically model the evolution of the short rate over time and then derive the yield curve accordingly. Given a stochastic short rate process $r = \big(r(s)\big)_{s\geq 0}$, the price of a T-bond at time $t < T$ is given as

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds} \,\middle|\, \mathcal{F}_t \right]. \tag{2.19}$$

To derive the yield curve from the zero-coupon bond prices, we first consider the following types of spot interest rates (cf. Brigo/Mercurio [1]):

Definition 2.2.3 (Spot interest rates)**.** The *simply-compounded spot interest rate* at time t for maturity T is the constant rate $L(t,T)$ at which $P(t,T)$ units of currency at time t need to be invested to produce one unit of currency at time T, when accruing occurs proportionally to the investment time:

$$L(t,T) := \frac{1 - P(t,T)}{P(t,T)(T-t)}.$$

The *periodically-compounded spot interest rate* at time t for maturity T is the constant rate $Y(t,T)$ at which $P(t,T)$ units of currency at time t need to be invested to produce one unit of currency at time T, when reinvesting the obtained amounts once every time period (e.g. month or year):

$$Y(t,T) := \frac{1}{P(t,T)^{1/(T-t)}} - 1.$$

Definition 2.2.4 (Term structure of interest rates)**.** The *term structure of interest rates* or *yield curve* at time t is the graph of the function

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t+1 \\ Y(t,T) & T > t+1. \end{cases}$$

In terms of stochastic modelling approaches for short rate processes, we restrict ourselves to the following type of affine short rate models:

Definition 2.2.5 (Affine short rate model)**.** A short rate model, in which the short rate $r(t)$ evolves according to

$$dr(t) = \big(\nu(t)\, r(t) + \eta(t)\big)\, dt + \sqrt{\gamma(t)\, r(t) + \delta(t)}\, dW(t), \tag{2.20}$$

where $\nu(t)$, $\eta(t)$, $\gamma(t)$ and $\delta(t)$ are deterministic functions such that the stochastic differential equation has a unique solution, is called an *affine short rate model.*

In Desmettre/Korn [4] it is shown that affine short rate models have the favourable property that the zero-coupon bond prices in (2.19) can be expressed in closed form:

Theorem 2.2.6. *For an affine short rate model, zero-coupon bond prices are given by*

$$P(t,T) = e^{-B(t,T)\, r(t) + A(t,T)}, \tag{2.21}$$

where $A(t,T)$ and $B(t,T)$ are the unique solutions of the system of ordinary differential equations

$$\frac{\partial}{\partial t} B(t,T) + \nu(t)\, B(t,T) - \frac{1}{2}\,\gamma(t)\, B(t,T)^2 + 1 = 0, \qquad B(T,T) = 0, \tag{2.22}$$

$$\frac{\partial}{\partial t} A(t,T) - \eta(t)\, B(t,T) + \frac{1}{2}\,\delta(t)\, B(t,T)^2 = 0, \qquad A(T,T) = 0. \tag{2.23}$$

In this case, the price of a T-bond satisfies the stochastic differential equation

$$dP(t,T) = P(t,T)\Big(r(t)\, dt - B(t,T)\sqrt{\gamma(t)\, r(t) + \delta(t)}\, dW(t) \Big).$$

Proof. See Desmettre/Korn [4]. □

2.2.1. Interest rate sensitivity and duration

A very important concept for this thesis is the concept of duration to measure the interest rate sensitivity of the present value of a given cash flow pattern. First, we follow the presentation in Koller [18] for the derivation of the *Macaulay duration* and then define the concept in a more general matter based on Peterson [24] in order for it to be applicable to the model in this thesis, leading to the *Fisher-Weil duration*.

Consider a cash flow pattern $\mathcal{B} = (CF_k)_{k \in \mathbb{N}_0}$ of annual future cash flows. Such a pattern could represent the future cash flows from holding a fixed rate bond or the future expected cash flows from an insurance contract. The theoretical value or price $\pi_t(\mathcal{B})$ of the future cash flows at a time $t \geq 0$ corresponds to the present value of the future cash flows calculated with an underlying risk-free yield curve and is given as

$$\pi_t(\mathcal{B}) = \sum_{k=0}^{\infty} CF_k \cdot P(t,k),$$

where $P(t,k)$ is the price of a k-year zero-coupon bond at time t, see Definition 2.2.1. In the case of a flat yield curve with an annual interest rate i, the price at $t = 0$ simplifies to

$$\pi_0(\mathcal{B})(i) = \sum_{k=0}^{\infty} \frac{CF_k}{(1+i)^k}.$$

From this formula, it can be seen that the price of $\mathcal{B}$ is highly dependent on the value of the interest rate i. The Macaulay duration $d(\mathcal{B})$ aims to measure how sensitive $\pi_t(\mathcal{B})$ reacts to a parallel shift of the yield curve. Thus, deriving the price with respect to the interest rate yields

$$\frac{\partial}{\partial i}\pi_0(\mathcal{B})(i) = -\frac{1}{1+i}\sum_{k=0}^{\infty}\frac{k \cdot CF_k}{(1+i)^k}.$$

and dividing this derivative by the price of $\mathcal{B}$, one obtains a relative measure for the interest rate sensitivity,

$$\frac{\frac{\partial}{\partial i}\pi_0(\mathcal{B})(i)}{\pi_0(\mathcal{B})} = -\frac{1}{1+i}\underbrace{\left(\frac{\sum_{k=0}^{\infty} k \cdot CF_k \cdot (1+i)^{-k}}{\sum_{k=0}^{\infty} CF_k \cdot (1+i)^{-k}}\right)}_{=:d(\mathcal{B})},$$

see Koller [18]. The quantity $d(\mathcal{B})$ is called the *Macaulay duration* of $\mathcal{B}$.

The concept of duration for flat yield curves motivates a more general approach for situations in which the yield curve is not flat and in which the discount rates are given by corresponding zero coupon bond prices as described above. This leads to the following extended concept of duration as a measure for interest rate sensitivity, see Peterson [24]:

Definition 2.2.7 (Fisher-Weil duration)**.** Consider $t \geq 0$, a time sequence $(t_k)_{k\in\mathbb{N}_0}$ with $t_k \geq t$ and a sequence $\mathcal{B} = \left(CF_{t_k}\right)_{k\in\mathbb{N}_0}$ of future cash flows with CF_{t_k} occurring at time t_k. The *Fisher-Weil duration* of $\mathcal{B}$ at time t is defined as

$$d_t(\mathcal{B}) := \frac{\sum_{k=0}^{\infty}(t_k - t) \cdot CF_{t_k} \cdot P(t, t_k)}{\sum_{k=0}^{\infty} CF_{t_k} \cdot P(t, t_k)}, \tag{2.24}$$

where $P(t, t_k)$ is the price of a zero-coupon bond at time t with maturity t_k.

The defining formula implicates that the Fisher-Weil duration, as well as the Macaulay duration, is a measure in units of time. In particular, if the time grid $(t_k)_{k\in\mathbb{N}_0}$ corresponds to time in years, the unit of the Fisher-Weil duration is also years.

Remark. Rewriting formula (2.24), the duration can also be represented as

$$d_t(\mathcal{B}) = \sum_{k=0}^{\infty}(t_k - t) \cdot \frac{CF_{t_k} \cdot P(t, t_k)}{\pi_t(\mathcal{B})}.$$

This sum corresponds to the weighted average of the durations from t to the payment time t_k of the k-th cash flow, where the weights are the proportion of the discounted k-th cash flow to the total present value $\pi_t(\mathcal{B})$ of the cash flow pattern.

For a set of cash flow patterns, for example a portfolio consisting of multiple bonds or a portfolio of life insurance contracts, the common duration is given as the weighted average of the durations of the single cash flow patterns. To see this, it follows for two cash flow patterns $\mathcal{B}_1 = \left(CF^1_{t_k}\right)_{k\in\mathbb{N}_0}$ and $\mathcal{B}_2 = \left(CF^2_{t_k}\right)_{k\in\mathbb{N}_0}$ that

$$\begin{aligned} d_t(\mathcal{B}_1+\mathcal{B}_2) &= \frac{\sum_{k=0}^{\infty}(t_k-t)\cdot\left(CF^1_{t_k}+CF^2_{t_k}\right)\cdot P(t,t_k)}{\pi_t(\mathcal{B}_1+\mathcal{B}_2)} \\ &= \frac{\sum_{k=0}^{\infty}(t_k-t)\cdot CF^1_{t_k}\cdot P(t,t_k)}{\pi_t(\mathcal{B}_1+\mathcal{B}_2)} + \frac{\sum_{k=0}^{\infty}(t_k-t)\cdot CF^2_{t_k}\cdot P(t,t_k)}{\pi_t(\mathcal{B}_1+\mathcal{B}_2)} \\ &= \underbrace{\frac{\pi_t(\mathcal{B}_1)}{\pi_t(\mathcal{B}_1+\mathcal{B}_2)}}_{=:w_1}\cdot d_t(\mathcal{B}_1) + \underbrace{\frac{\pi_t(\mathcal{B}_2)}{\pi_t(\mathcal{B}_1+\mathcal{B}_2)}}_{=:w_2}\cdot d_t(\mathcal{B}_2), \end{aligned}$$

whereby the weights w_1 and w_2 correspond to the proportions of the respective cash flow pattern values to the total value of the combined pattern. The general case for n cash flow patterns $\mathcal{B}_1,\dots,\mathcal{B}_n$ follows by induction, yielding

$$d_t\Big(\sum_{i=1}^{n}\mathcal{B}_i\Big) = \sum_{i=1}^{n} w_i\cdot d_t(\mathcal{B}_i) \tag{2.25}$$

with weights

$$w_i := \frac{\pi_t(\mathcal{B}_i)}{\pi_t\Big(\sum_{i=1}^{n}\mathcal{B}_i\Big)}.$$

2.3. Linear algebra

To model the stochastic dependency structure of the capital market model, we will need the concept of Cholesky decomposition of Hermitian, positive semi-definite matrices.

Definition 2.3.1 (Hermitian matrix, Positive semi-definite matrix)**.** Let $n\in\mathbb{N}$. A matrix $C\in\mathbb{C}^{n\times n}$ is called a *Hermitian matrix* if it is equal to its own conjugate transpose, that is

$$c_{ij} = \overline{c_{ji}}, \quad i,j = 1,\dots,n.$$

A Hermitian matrix $C\in\mathbb{C}^{n\times n}$ is called *positive semi-definite*, if

$$x^*Cx \geq 0 \text{ for all } x\in\mathbb{C}^n\backslash\{0\},$$

where x^* is the conjugate transpose of x.

The following theorem states that Hermitian and positive semi-definite matrices can always be decomposed in a specific way:

Theorem 2.3.2 (Cholesky decomposition)**.** *Let $n\in\mathbb{N}$ and $C\in\mathbb{C}^{n\times n}$ be Hermitian. Then C is positive semi-definite if and only if there exists a lower triangular matrix $A\in\mathbb{C}^{n\times n}$ with non-negative diagonal entries such that*

$$C = AA^*.$$

If C is positive definite, the diagonal entries are positive and A is unique. If C is real, A may be taken to be real.

Proof. See Horn/Johnson [11]. □

3. Asset-liability management in life insurance

In life insurance, the term 'asset-liability management' (ALM) generally refers to all processes for the optimal matching of technical provisions (liabilities) and investments (assets) and thus ranges from mathematical modelling approaches and far-reaching economic theory to holistic approaches for corporate management. This chapter aims to illustrate the economic necessity of asset-liability management in life insurance and to place the stochastic modelling approach of this thesis within the broad scope of ALM. Furthermore, the legal framework for life insurance companies with respect to ALM is discussed and the regulatory requirements, which are taken into account in the stochastic model in Chapter 4, are summarized.

3.1. Motivation and introductory example

As discussed in Chapter 1, the nature of life insurance business exposes life insurers to an increased interest rate and liquidity risk. To illustrate the importance of ALM processes and scenario-based models to manage such risks, we consider the following introductory illustrative example. It should be noted here that this is a simplified example that does not take into account the many investment possibilities of modern financial markets, nor insurance-typical features such as broadly diversified insurance policy portfolios. For the example, we consider a portfolio of $1,000$ simultaneously issued whole life insurance policies, 10 years after policy inception, with all insured persons assumed to still be alive. We assume a flat yield curve with an annual actuarial interest rate and risk-free bond valuation rate of 2.50%. In addition, the following assumptions apply to the insurance contracts:

- Sum insured per contract: 100 EUR;
- Age of policyholders at contract inception: 40 years;
- Annual premium payments in advance within 30 years from contract inception;
- No cancellation during the contract period.

The assumptions for mortality in this example are derived from the smoothed unisex life table for Austria for the years 2020–2022, see Table A.1. In order to cover the technical provision with assets, the insurance company holds an asset portfolio consisting of the following two types of fixed rate bonds in equal proportions:

- Fixed rate bond 1:
  - Face value: 1 EUR

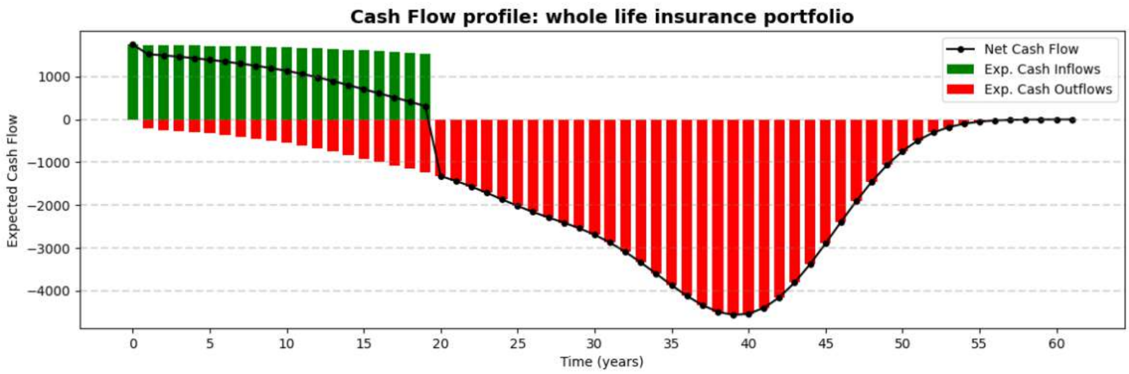
  - Term to maturity: 25 years
  - Annual coupon payments with coupon rate of 3.5% of the face value at the end of each year

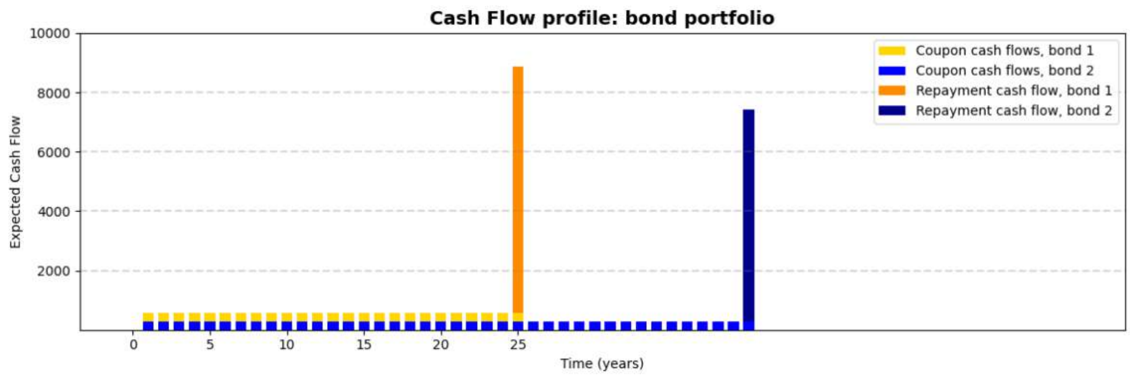
- Fixed rate bond 2:
  - Face value: 1 EUR
  - Term to maturity: 40 years
  - Annual coupon payments with coupon rate of 4.0% of the face value at the end of each year

The asset portfolio is assumed to consist of a sufficient number of such bonds to exceed the present value of the insurance obligations (provision for future contractual cash flows) by 5%. These assumptions result in the following present values for the asset portfolio and the insurance portfolio:

Present value: Assets ($\mathcal{A}$)	**Present value: Liabilities ($\mathcal{L}$)**
$19,667.37$ EUR[1]	$-18,730.83$ EUR

If we consider the expected future cash flows of the insurance portfolio and the investment portfolio, we obtain the following profiles:





[1]The asset portfolio consists of 8,304 units of bond 1 with a unit price of 1.184 EUR and 7,144 units of bond 2 with a unit price of 1.376 EUR.

Figure 3.1.: Introductory example: Cash flow profiles of assets and liabilities

Here, it can be seen immediately that the cash flow profiles differ significantly and that the expected cash flows of the insurance portfolio are not replicated by the asset portfolio. Such differences in cash flow profiles lead to a liquidity risk where the insurance company may be forced to liquidate part of its asset portfolio off schedule — and at possibly disadvantageous market conditions — in order to meet the insurance benefits in case of higher liquidity needs. This may result in financial losses, or the insurance company may not be able to meet its obligations from the insurance contracts at all.

To assess and measure the arising interest rate risk in this example, we consider the duration of the present values of both assets and liabilities, see Definition 2.2.7, as a measure of their interest rate sensitivity:

Asset duration ($D^{\mathcal{A}}$)	**Liability duration** ($D^{\mathcal{L}}$)
20.59	61.55

The risk arising from the significantly higher interest rate sensitivity of liabilities becomes clear when we consider the dependency of the present value on changes in the underlying annual interest rate for assets and liabilities, respectively:

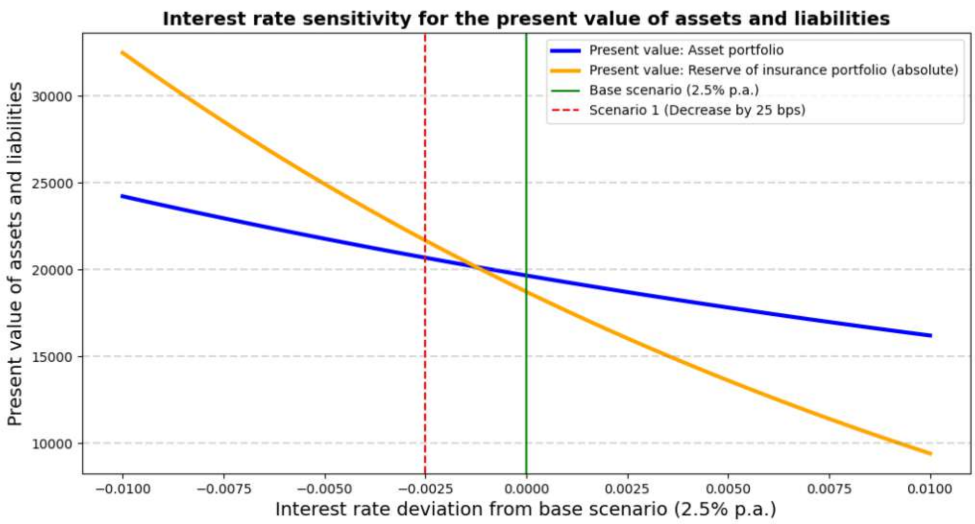


Figure 3.2.: Introductory example: Dependency of the present values of assets and liabilities on interest rate changes

As can be seen from Figure 3.2, the present value of the insurance portfolio reacts significantly more strongly to shifts in the yield curve, and already a reduction of the interest rate by 25 basis points (scenario 1) results in the present value of the liabilities exceeding the present value of the assets by almost 5%. The insurance company is therefore forced to add additional investments to the asset portfolio to cover the actuarial provisions. On the one hand, this reduces equity and on the other hand, this may lead to the insurer's insolvency.

The mismatch between assets and liabilities in the example underlines that a lack of ALM measures can lead to risks that threaten the existence of a life insurance company. It is therefore necessary for life insurance companies to implement appropriate and effective asset-liability management processes to measure, monitor and, most importantly, manage these risks. Such ALM processes must be adapted to the characteristics of a company, such as new business, the lapse behaviour of policyholders, any options inherent in the insurance contracts, as well as individual company objectives. After successful implementation, they promote the financial stability and crisis resistance of a life insurance company and also contribute to remaining competitive on the market and gaining the trust of existing and potential policyholders.

3.2. The role of stochastic modelling in asset-liability management

Although the substantive scope of asset-liability management is not uniformly defined in both legal literature and insurance economics literature, modern definitions often elevate ALM to the level of a holistic management approach for the responsible governance of life insurance companies. The Society of Actuaries (SOA), for example, defines ALM in Luckner et al. [21] as follows:

> "ALM is the practice of managing a business so that decisions and actions taken with respect to assets and liabilities are coordinated. ALM can be defined as the ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities to achieve an organization's financial objectives, given the organization's risk tolerances and other constraints. ALM is relevant to, and critical for, the sound management of the finances of any organization that invests to meet its future cash flow needs and capital requirements."

Even though this definition does not address the characteristics of life insurance, it illustrates that ALM is a comprehensive process in which several key operating departments of an insurance company (management, asset management, actuarial department, risk management, etc.) must be involved in close cooperation. Some authors, see Cottin/Kurz [3] for example, even go one step further and see asset-liability management not only as an element of risk management and a way to achieve business goals, but also as a central component of the life insurance business. In any case, this makes it clear that mathematical methods such as stochastic modelling and optimisation are important components of an overall ALM concept, however, these models alone do not constitute a comprehensive concept themselves.

In the former understanding of asset-liability management, before it was further developed into the aforementioned modern concept, the focus lay on techniques for matching the capital investment strategy to the technical provisions. In these sequential approaches, the insurance business with its liabilities was largely assumed to be fixed and the aim was purely to match and optimise the investment of capital, see Führer [7]. This resulted in the direction of action 'Liabilities control Assets'. In Cottin/Kurz [3] this point of view is extended to three different, complementary approaches for ALM in modern concepts:

1. Assets are managed on the basis of liabilities.
2. Assets and liabilities are managed simultaneously, taking into account possible interactions and dependencies.
3. The insurance company is managed as a whole on the basis of ongoing analysis and systematic restructuring of assets and liabilities.

The first approach corresponds to the original understanding of ALM described above. It considers liabilities as a consequence of the business activity of a life insurance company on the one hand and assets as a control and optimisation variable on the other. This approach, even though it is now embedded in more comprehensive concepts, continues to be of great importance for short- and medium-term ALM, both in practice and in the regulatory context. The second and third approaches each provide for simultaneous management of assets and liabilities, with approach 2 here reflecting more of an actuarial view, while approach 3 amounts to the modern holistic view of corporate management discussed above. The simultaneous management also includes the direction of action 'Assets control Liabilities' in ALM, which a priori appears to be less relevant in practice than its reversal. Nevertheless, this direction of action must not be neglected in an overall concept, as it plays a key role, for example, in strategies for adjusting new business to the capital market situation and developing new insurance products.

At first, stochastic scenario modelling and stochastic optimisation as specific quantitative ALM methods can mainly be associated with the first two approaches, which then merge into an overall concept within approach 3. For models with a short- and medium-term time horizon, which includes the main model of this thesis, approach 1 is particularly relevant, since obligations of existing contracts cannot be influenced by the insurer during this period, or can only be influenced to a small extent. Therefore, it is practicably realistic to consider these as largely fixed variables in the model. This leaves the capital investment strategy as the control variable for interest rate and liquidity risk, which can be adjusted and optimised accordingly. Moreover, within this time horizon, more detailed modelling approaches — such as partial models for sub-portfolios or more comprehensive capital market models — can be applied and specific optimisation methods that require detailed models (such as duration matching, see Section 4.3) can be considered. For approach 2, in contrast, the focus is on more general stochastic models with longer time horizons, within which matching of actuarial liabilities to investments is also a practically realistic option. Less suitable for this purpose are concrete optimisation models, but rather models for strategic long-term planning and information gathering for strategy decisions. In addition, stochastic models with time projection horizons often require significant computational resources, so that such models usually have to resort to simpler modelling approaches and simplifications, which also limits predictability. Of course, since stochastic models that are in use within approaches 1 and 2 serve not only to achieve their specific goals (capital investment optimisation, new business strategy planning, etc.) but also help to gain valuable information for corporate management, they implicitly contribute to fulfilling the goals of approach 3 as well. Stochastic models that are developed solely for the scope of approach 3 can also be useful, although the degree of simplification generally needs to be increased considerably here in order to obtain meaningful forecasts with practical computational power.

With regard to the role of stochastic modelling as a tool for fulfilling the core tasks of an ALM system, Cottin/Kurz [3] initially differentiate these tasks between matching, modelling and management. AL matching comprises the specific matching of investments and insurance liabilities (approaches 1 and 2) and therefore plays a role primarily for specific (partial) insurance contract portfolios. It thus primarily fulfils the task of securing short-term liquidity and hedging against adverse changes in the interest rate environment. AL modelling deals with deterministic or stochastic forecasts of future business or capital market developments and offers possibilities for analysing correlations between assets and liabilities and the quantification of risks. The central model of this thesis covers both a matching and a modelling aspect and can thus be assigned to these two core tasks. The management task focuses on KPI-based decision support (approach 3). The central goals of this task are the long-term assurance of solvency and the value- and profit-oriented alignment of the company in the sense of the shareholders, stakeholders and policyholders. Stochastic models, also preceding models of AL-matching and AL-modelling, can be used here in a supportive way for the acquisition of corresponding information and key figures.

3.3. Regulatory framework for ALM

In addition to the described economic necessity for life insurers to implement control and steering measures within the framework of asset-liability management, Austrian insurance law also contains regulatory requirements for the implementation of suitable ALM processes. The main legal sources with regard to ALM are the Insurance Supervision Act 2016 (Versicherungsaufsichtsgesetz 2016 — VAG 2016), the Directive 2009/138/EC of the European Parliament and of the Council (Solvency II Directive) and the Insurance Undertakings Investment Regulation (Versicherungsunternehmen Kapitalanlageverordnung — VU-KAV) of the Austrian Financial Market Authority (FMA). These legal sources explicitly oblige insurance companies to asset-liability management and their requirements must be taken into account in corresponding deterministic or stochastic models. The explicit regulations on ALM contained therein as well as directly related regulations are summarized in Section 3.3.1. As the Solvency II Directive was implemented in national law (VAG 2016), this section only refers directly to the laws mentioned and not to the directive itself.

Additionally, legal requirements that are not directly connected to ALM, but implicitly require ALM measures or need to be taken into account in a stochastic model, are discussed in Section 3.3.2. These mainly include legally regulated optional rights of policyholders under the Insurance Contract Act (Versicherungsvertragsgesetz — VersVG), various regulations of the FMA and further requirements in the VAG 2016. Among the FMA regulations, in addition to the VU-KAV, the Maximum Interest Rate Regulation for Insurance Undertakings (Versicherungsunternehmen-Höchstzinssatzverordnung — VU-HZV) and the Life Insurance Profit Sharing Regulation (Lebensversicherung-Gewinnbeteiligungsverordnung — LV-GBV) are discussed in more detail due to their relevance for ALM modelling.

3.3.1. Explicit legal requirements for ALM

The key legal requirements for asset-liability management can be found in the section on the risk management system in the VAG 2016. Pursuant to Art. 110 para. 1 VAG 2016, this risk management system must compose all the required strategies, processes and reporting procedures necessary to identify, measure, monitor, manage and report the risks:

1. at an individual and at an aggregated level, to which they are or could be exposed; and
2. the interdependencies between those risks.

Article 110 para. 2 VAG 2016 further stipulates that the risk management system must be integrated into the organizational structure and decision-making processes of an insurance undertaking and must in any case cover the following areas:

1. underwriting and reserving;
2. **asset-liability management**;
3. **investment**, in particular derivatives and similar commitments;
4. **liquidity and concentration risk management**;
5. operational risk management;
6. reinsurance and other **risk mitigation techniques**.

Here, an implementation of ALM processes is explicitly required for the first time, but a specification of specific mathematical or non-mathematical methods is not to be found in the VAG 2016. The required scope of ALM concepts is explained in more detail in Art. 260 (1) lit b of Commission Delegated Regulation (EU) 2015/35 supplementing the Solvency II Directive and includes:

1. the structural mismatch between assets and liabilities and in particular the duration mismatch of those assets and liabilities;
2. any dependency between risks of different asset and liability classes;
3. any dependency between the risks of different insurance or reinsurance obligations;
4. any off-balance sheet exposures of the undertaking;
5. the effect of relevant risk-mitigating techniques on asset-liability management.

Despite this specification, however, the implementation of an ALM concept and the selection of the methods used is — under the general restriction of being 'effective' — essentially left to the companies themselves. Nevertheless, these regulations already indicate that the implementation of stochastic models is a necessary part of asset-liability management and the risk management system, as the purposes of such models described in Section 3.2 are well compatible with the legal requirements. This applies both to micro-level models for risk measurement and control (approaches 1 and 2, page 16) and to macro-level models to support decision-making processes (approach 3, page 16).

The governance system under Solvency II, and subordinately also the risk management system, must be set up in such a way that it is proportionate to the nature, scale and complexity of the business operations (Art. 107 para. 2 VAG 2016). Art. 107 para. 4 VAG 2016 further requires insurance undertakings to use appropriate and proportionate systems, procedures and resources in order to ensure the continuity and regularity in the performance of their activities. It can therefore be concluded that a high degree of individuality is expected from stochastic ALM models. Consequently, this means that such models must take into account the characteristic, individual features of a life insurance company and its portfolio and capital structure, and that models that are too general would tend to be judged inadequate by the Financial Market Authority. This conclusion is legally backed up in Art. 2 VU-KAV by the fact that insurance companies are explicitly required to use appropriate assumptions and factors when modelling ALM.

The additional provisions on ALM in Art. 110 VAG 2016 primarily relate to the calculation of technical provisions, sensitivity assessments of provisions and eligible own funds downstream of ALM, and the adjustment options (matching adjustment, volatility adjustment) of the relevant risk-free interest rate term structure under Solvency II, which will not be discussed in more detail here. It is noted, however, that the application of the aforementioned matching adjustment requires the insurance company to define a portfolio of assets (bonds and other assets) with similar cash flow characteristics in order to cover the best estimate of the associated portfolio of insurance obligations (Art. 166 para. 1 no 1 VAG 2016). In addition, the expected cash flows of the allocated asset portfolio must replicate all future cash flows of the portfolio of insurance obligations (Art. 166 para 1 no 3 VAG 2016). The fulfilment of these requirements is thus in any case preceded by the application of cash flow matching methods within the framework of a deterministic or stochastic ALM model.
Complementary to the risk management system, the requirements for capital investment in the VAG 2016 and the VU-KAV contain further explicit requirements for ALM: Pursuant to Art. 124 para. 1 no 3 VAG 2016, assets held to cover technical provisions under Solvency II must be invested in a manner that is appropriate to the nature and duration of the insurance liabilities. These assets must also be invested in in the best interests of all policyholders and beneficiaries, taking into account any disclosed policy objectives. This specifically calls for an alignment of investments with insurance liabilities, which also gives legal significance to the impact direction 'Liabilities control Assets' as described in the preceding section above. What is to be understood by the 'nature' of insurance liabilities is not described in more detail, but it is only reasonable to assume that the liquidity requirements

and interest rate sensitivity of the liabilities are also targeted here. This is emphasised by the fact that Art. 7 VU-KAV requires a comparison of the interest rate sensitivities of assets and liabilities as part of the development of the company's own risk indicators, and Art. 9 VU-KAV requires insurance companies to ensure that adequate liquid funds must always be available at the time of maturity for all obligations assumed (including those from the investment of capital) in order to fully meet them. Thus, stochastic optimisation methods such as cash flow matching or duration matching — the latter corresponds to the core optimisation problem in the model of this thesis — are of particular importance here.

For completeness of explicit legal regulations on ALM, it should be noted that Art. 3 VU-KAV sets out the requirements of the FMA for the creation and implementation of internal company policies on ALM processes and investment objectives, taking into account obligations arising from insurance contracts and other liabilities, risk, return, time horizon and liquidity requirements. However, as these provisions do not have a direct impact on the implementation of the stochastic model, they will not be discussed in more detail.

In view of these legal requirements for coordinated management of assets and liabilities, however, it should be noted that this does not necessarily mean that the underlying risk factors, such as interest rate sensitivity, have to be perfectly balanced. Kruse/Schaumlöffel [20] argue in this matter that insurance companies can rather also consciously allow mismatches between investments and insurance liabilities, provided that these are compatible with the company's risk strategy and are also sufficiently documented and justified in the risk management guidelines and policies mentioned above.

3.3.2. Additional legal provisions relevant to the model

In addition to the legal provisions that explicitly require the implementation of ALM measures, there are a number of legal requirements that implicitly affect asset-liability management. This section takes a closer look in particular at the regulations that have a significant impact on the stochastic modelling approach in Chapter 4.

Regulatory profit participation in life insurance

The Austrian Life Insurance Profit-sharing Regulation (Lebensversicherung-Gewinnbeteiligungsverordnung — LV-GBV) stipulates that, for life insurance contracts that are eligible for profit participation, life insurance companies must allocate at least 85% of a defined minimum assessment basis, including any direct bonuses, to the provision for profit-dependent premium reimbursement per financial year (Art. 3 LV-GBV). The minimum assessment basis is defined in Art. 4 LV-GBV. In general, policyholders only participate in profits of an insurance company and not in losses, as the provision for profit-dependent premium reimbursement is not reduced in case of a negative minimum assessment basis. Nonetheless, an amendment to the LV-GBV in 2021 now allows insurance companies to take into account overfunding or negative minimum assessment basis from previous years when determining the current minimum assessment basis.

Statutory maximum interest rate for life insurance contracts

The maximum interest rate for the calculation of technical provisions (pursuant to Chapter 7 of the Insurance Supervision Act 2016) set out in Art. 2 para. 1 of the Maximum Interest Rate Regulation for Insurance Undertakings (VU-HZV) of the FMA is currently fixed with 0.00%, which is the result of a gradual reduction from 4% in 1994 to 0% since 2022. This is thus also the maximum possible interest rate guaranteed to the policyholder at the time the contract is concluded for new life insurance contracts.

Legal requirements for capital investment

The VAG 2016 and the VU-KAV contain numerous requirements regarding the structure and management of investments: Under the prudent person principle, insurance undertakings may only invest in assets and instruments whose risks they can properly identify, measure, monitor, manage, control and report (Art. 124 para. 1 no 1 VAG 2016). These must be invested in such a way that the security, quality, liquidity and profitability of the entire portfolio is guaranteed (Art. 124 para. 1 no 2 VAG 2016). With regard to the permissible financial instruments, the following is specified in Art. 124 para. 1 nos 5 to 6 VAG 2016:

- Derivative financial instruments are permitted if they contribute to reducing risks or facilitating efficient portfolio management;
- Short selling is not permitted;
- Investment and assets which are not admitted to trading on a regulated financial market shall be kept to prudent levels.

Supplementing this provision, with regard to the use of derivative financial instruments to reduce risks, insurance companies must at all times be able to provide quantitative evidence of their risk-reducing effect for the entire period of usage and prove that no additional substantial risks arise from the use of derivatives (Art. 11 para. 1 no 1 VU-KAV). In addition, speculative investments with high leverage, short holding periods, high transaction frequency or within the context of arbitrage strategies as well as the use of short put options shall only be kept at a prudent level (Art. 11 para. 2 VU-KAV).

With regard to the composition of the investment portfolio, Art. 124 para. 1 no 7 VAG 2016 requires that assets are appropriately mixed and diversified in order to avoid excessive dependencies on a particular asset, issuer, group of companies or geographical region. To this end, insurance undertakings must define and implement a suitable limit system with quantitative investment limits for all relevant concentration risks in accordance with Art. 6 para. 1 VU-KAV.

Cancellation rights of policyholders and fully paid-up insurance

In life insurance, policyholders have the right to terminate an insurance contract with continuous premium payments at any time for the end of the current insurance period

(Art. 165 para. 1 VersVG). For covered risks for which the insurer is certain to be liable, this right of termination may also be exercised in case of a single premium payment (Art. 165 para 2 VersVG). For the latter type of contracts, where the payment of the agreed sum by the insurer is certain, the surrender value attributable to the insurance must be refunded to the policyholder in the event of termination (Art. 176 para. 1 VersVG). Pursuant to Art. 176 para. 3 VersVG, this surrender value is to be calculated as the current value of the insurance at the end of the current insurance period, in accordance with the accepted rules of actuarial mathematics on the basis of the calculation principles of the premium calculation, taking into account possible outstanding premiums. The insurance company is entitled to a deduction from the refund ('cancellation deduction') if this is agreed and appropriate (Art. 176 para. 4 VersVG). Further regulations on deductions from the surrender value (such as for acquisition costs), which are set out in Art. 176 para. 4 to 5 VersVG, are not taken into account in the model and are therefore not discussed here in more detail.
In addition to the right of termination, the policyholder is entitled at any time to demand the conversion of the insurance into a fully paid-up insurance for the end of the current insurance period (Art. 173 para. 1 VersVG). As a result, the agreed lump sum or annuity amount is replaced by the amount calculated according to the accepted rules of actuarial mathematics on the basis of the calculation principles of the premium calculation, whereby outstanding premiums are also taken into account here (Art. 173 para. 2 VersVG).

4. The model

In this chapter, we present the underlying stochastic model for the development of assets and liabilities over time and the ALM approach for the management of interest rate risk. The stochastic model consists of two main sub-models: An asset model and a liability model, which are defined and described separately, although dependencies between the sub-models are taken into account at the appropriate point. With regard to the mathematical properties of each model component, we will focus in particular on those that are essential for understanding the model's dynamics and the calibration of the model, as well as on properties that are crucial for the objective of interest rate risk optimisation.

First, we will introduce the modelling concepts for the financial market and the possible investment instruments in which the insurance company can invest. For the asset processes, we choose stochastic modelling approaches in continuous time based on established financial mathematical models because, although their numerical treatment requires more computational resources, central results of stochastic analysis can be applied to derive probabilistic properties and closed-form representations of the corresponding stochastic processes.

Second, we will present the modelling approach for the insurance portfolio, which is based on a discrete-time Markov chain model. For this purpose, as well as for the optimisation within the interest rate risk management approach, we consider a time discretization $\mathcal{T}$ that is given by a grid of equidistant points in time $0 = t_0 < t_1 < \cdots < t_N = T$, where T corresponds to the maturity of the insurance contracts of the portfolio. For the further course of this thesis, we assume annual time steps for the projection of the insurance portfolio and the rebalancing of the investment portfolio and define the model accordingly.

Third, we describe the dynamic multi-period ALM optimisation approach of this model to manage the interest rate risk to which the insurer is exposed. This approach comprises a periodic adjustment of the asset portfolio as a result of a two-stage non-linear optimisation problem.

4.1. Asset model

For this model, we consider a simplified capital market with a stochastically modelled risk-free yield curve, market models for equities and fixed rate bonds, and a risk-free money-market account. We start with the modelling approach of the risk-free yield curve, then present the models for the stochastic processes for the evolution of the money-market account, the bond prices and the stock price development. Following a risk-based approach such as described in Brigo/Mercurio [1] or Gerstner et al. [10], all sub-models are defined under the real-world probability measure $\mathbb{P}$. The consideration of stochastic dependencies between the respective processes and model-relevant mathematical properties of the model components are discussed as well. In addition, restrictions on the insurance company's investment portfolio resulting from practical considerations and the legal requirements discussed in Chapter 3 are covered in this section.

4.1.1. Interest rate term structure model

The risk-free interest rate term structure in this thesis is modelled using the Vasicek short rate model. The Vasicek model is a one-factor affine short rate model (see Definition 2.2.5) that describes interest rate movements driven by a single source of market risk, where the short rate evolves as an Ornstein-Uhlenbeck process with constant coefficients. The model was first presented in Vasicek [28] and has been widely treated in mathematical theory, see Brigo/Mercurio [1] or Desmettre/Korn [4], for example. The dynamics of the short rate in the Vasicek model are given by the stochastic differential equation

$$dr(t) = \kappa\big(\theta - r(t)\big)\,dt + \sigma\,dW^r(t), \quad r(0) = r_0 \tag{4.1}$$

with $\kappa > 0$ and $\theta, \sigma \geq 0$ and a one-dimensional standard Brownian motion W^r and is therefore of the affine form (2.20) with the constant parameter functions

$$\begin{aligned} \nu(t) &= -\kappa, \\ \eta(t) &= \kappa\,\theta, \\ \gamma(t) &= 0, \\ \delta(t) &= \sigma^2. \end{aligned}$$

Rather than defining the short rate dynamics under a risk-neutral probability measure $\mathbb{Q}$, we define the dynamics under the real-world measure $\mathbb{P}$ and assume a corresponding risk premium to be zero, such that the short rate dynamics under $\mathbb{Q}$ and $\mathbb{P}$ coincide. This implies that the corresponding Brownian motion for the short rate and the Brownian motions underlying the stock price dynamics, see Section 4.1.4, can be simulated under the same probability measure, with the short rate model still following risk-neutral dynamics and thus being suitable for the pricing of fixed rate bonds and the calculation of present values of future cash flows.

The Vasicek model has the mean-reverting property, which means that the process tends to drift towards its long-term mean θ. This can easily be seen from equation (4.1) above as, for $\kappa > 0$, the drift coefficient $\kappa(\theta - r(t))$ is negative as long as the short rate is above θ and positive as long as the short rate is below θ, resulting in r being pushed to the level θ on average at every point in time. The parameter κ in this case specifies the speed of the mean reversion effect, as increasing values of κ result in a higher tendency of r towards the level of θ. However, the speed of mean reversion κ must be considered in relation to the volatility σ, which indicates the degree of random noise in the model. A high volatility σ can even overcompensate the mean reverting effect, as shown in Desmettre/Korn [4]. Generally, this mean reverting property is favourable for modelling interest rate structures and can be economically justified by the fact that in phases of high interest rates, economy is slowed down and interest rates fall as a result, and in phases of low interest rates, the demand for funds increases, forcing interest rates to rise.

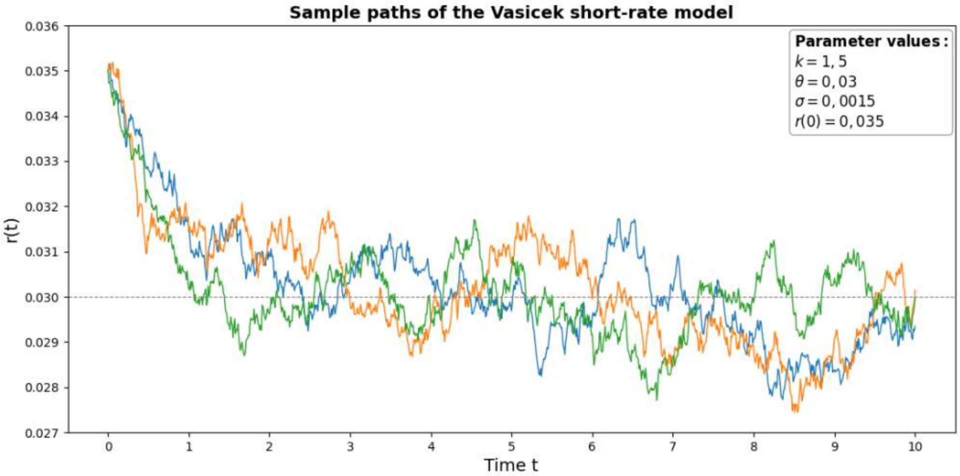


Figure 4.1.: Sample paths of the short rate in the Vasicek model

Using Itô's lemma, we can obtain the solution of the defining stochastic differential equation (4.1), to then derive the essential distributional properties of $r(t)$:

Theorem 4.1.1. *The unique solution of equation (4.1) is given as*

$$r(t) = r(s)\,e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}\,dW^r(u), \quad 0 \le s \le t. \tag{4.2}$$

Proof. To solve equation (4.1) we will use variation of constants, see Brigo/Mercurio [1]. We assume $\theta = 0$ and $\sigma = 0$ and consider the ordinary differential equation

$$\frac{dr}{dt} = -\kappa\, r(t).$$

The solution of this ordinary differential equation is $Ce^{-\kappa t}$ with some arbitrary constant $C \in \mathbb{R}$. To find the solution of the original equation we set

$$r(t) = Y(t)\,e^{-\kappa t} \tag{4.3}$$

for some unknown stochastic process Y. Rearranging (4.3) and applying Itô's Lemma (Theorem 2.1.8) with $f(t,r) = r\,e^{\kappa t}$, $\alpha\big(t, r(t)\big) = \kappa\big(\theta - r(t)\big)$ and $\sigma\big(t, r(t)\big) = \sigma$ yields

$$\begin{aligned} dY(t) = df(t, r(t)) &= \left(\kappa\, r(t)\, e^{\kappa t} + \kappa\big(\theta - r(t)\big)e^{\kappa t}\right) dt + \sigma e^{\kappa t}\, dW^r(t) \\ &= \kappa\,\theta\, e^{\kappa t}\, dt + \sigma e^{\kappa t}\, dW^r(t). \end{aligned}$$

Integrating this equation, we obtain, for each $0 \le s \le t$,

$$Y(t) = Y(s) + \kappa\,\theta \int_s^t e^{\kappa u}\,du + \sigma \int_s^t e^{\kappa u}\,dW^r(u)$$

and changing $Y(t)$ back to $r(t)\,e^{\kappa t}$ yields

$$r(t) = r(s)\,e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma \int_s^t e^{-\kappa(t-u)}\,dW^r(u).$$

The uniqueness of the solution immediately follows from Theorem 2.1.6 as the coefficient functions $\alpha(t,x)$ and $\sigma(t,x)$ are Lipschitz-continuous in x. □

From the explicit representation of r in (4.2), the following distributional properties of the short rate can be derived:

Corollary 4.1.2. *For $t \ge 0$, the short rate $r(t)$ in the Vasicek model is normally distributed with*

$$r(t) \sim \mathcal{N}\left(r_0\,e^{-\kappa t} + \theta\,(1 - e^{-\kappa t}),\; \frac{\sigma^2}{2\kappa}\,(1 - e^{-2\kappa t})\right). \tag{4.4}$$

Conditional on $\mathcal{F}_s$, it holds for $s \le t$:

$$\begin{aligned}\mathbb{E}\big[r(t)|\mathcal{F}_s\big] &= r(s)\,e^{-\kappa(t-s)} + \theta\,(1 - e^{-\kappa(t-s)}),\\ \mathbb{V}\mathrm{ar}\big(r(t)|\mathcal{F}_s\big) &= \frac{\sigma^2}{2\kappa}\,(1 - e^{-2\kappa(t-s)}).\end{aligned}$$

Proof. The closed-form representation (4.2) with $s = 0$ yields

$$r(t) = r(0)\,e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{-\kappa(t-u)}\,dW^r(u), \quad t \ge 0.$$

Thus, since $X(u) := e^{-\kappa(t-u)}$ is deterministic and fulfils (2.3), it follows from the properties of the stochastic integral with respect to Brownian motion, see Lemma 2.1.4, and the invariance under linear transformation of the normal distribution that

$$\sigma \int_0^t e^{-\kappa(t-u)}\,dW^r(u) \sim \mathcal{N}\left(0,\; \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right)$$

and consequently

$$r(t) \sim \mathcal{N}\left(r_0\,e^{-\kappa t} + \theta\,(1 - e^{-\kappa t}),\; \frac{\sigma^2}{2\kappa}\,(1 - e^{-2\kappa t})\right).$$

For the conditional expectation, it follows

$$\begin{aligned}\mathbb{E}\big[r(t)|\mathcal{F}_s\big] &= \mathbb{E}\left[r(s)e^{-\kappa(t-s)} + \theta\big(1 - e^{-\kappa(t-s)}\big) + \sigma \int_s^t e^{-\kappa(t-u)} dW^r(u) \,\Big|\, \mathcal{F}_s\right]\\ &= \mathbb{E}\left[r(s)e^{-\kappa(t-s)} \,\Big|\, \mathcal{F}_s\right] + \theta\big(1 - e^{-\kappa(t-s)}\big) + \sigma\,\mathbb{E}\left[\int_s^t e^{-\kappa(t-u)} dW^r(u) \,\Big|\, \mathcal{F}_s\right]\\ &\overset{(*)}{=} r(s)e^{-\kappa(t-s)} + \theta\big(1 - e^{-\kappa(t-s)}\big) + \sigma\,\underbrace{\mathbb{E}\left[\int_s^t e^{-\kappa(t-u)} dW^r(u)\right]}_{=0}\\ &= r(s)e^{-\kappa(t-s)} + \theta\big(1 - e^{-\kappa(t-s)}\big).\end{aligned}$$

Equality $(*)$ holds because $r(s)$ is $\mathcal{F}_s$ measurable and the stochastic integral $\int_s^t e^{-\kappa(t-u)} dW^r(u)$ is independent of $\mathcal{F}_s$ with expected value equal to zero, see Definition 2.1.3 and Lemma 2.1.4.

Further, using the identity $\mathbb{V}\mathrm{ar}(r(t)|\mathcal{F}_s) = \mathbb{E}[r(t)^2|\mathcal{F}_s] - \mathbb{E}[r(t)|\mathcal{F}_s]^2$ for conditional variance, it holds that

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big(r(t)\big|\mathcal{F}_s\big) = \quad & \mathbb{E}\Bigg[\bigg(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big) + \sigma\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg)^2 \,\Bigg|\, \mathcal{F}_s\Bigg] \\
& - \mathbb{E}\Bigg[r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big) + \sigma\int_s^t e^{-\kappa(t-u)} dW^r(u) \,\Bigg|\, \mathcal{F}_s\Bigg]^2 \\
= \quad & \mathbb{E}\Bigg[\Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)^2 \,\Bigg|\, \mathcal{F}_s\Bigg] \\
& + \mathbb{E}\Bigg[2\,\Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)\cdot\sigma\int_s^t e^{-\kappa(t-u)} dW^r(u) \,\Bigg|\, \mathcal{F}_s\Bigg] \\
& + \mathbb{E}\Bigg[\sigma^2\bigg(\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg)^2 \,\Bigg|\, \mathcal{F}_s\Bigg] - \Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)^2 \\
\overset{(**)}{=} \quad & \Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)^2 \\
& + 2\,\Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)\cdot\sigma\,\underbrace{\mathbb{E}\bigg[\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg]}_{=0} \\
& + \sigma^2\,\mathbb{E}\Bigg[\bigg(\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg)^2\Bigg] - \Big(r(s)e^{-\kappa(t-s)} + \theta\big(1-e^{-\kappa(t-s)}\big)\Big)^2 \\
= \quad & \sigma^2\,\mathbb{E}\Bigg[\bigg(\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg)^2\Bigg]
\end{aligned}$$

Equation $(**)$ above again follows from the $\mathcal{F}_s$-measurability of $r(s)$ and from the independence of $\int_s^t e^{-\kappa(t-u)} dW^r(u)$ of $\mathcal{F}_s$. Using Itô isometry, see Lemma 2.1.4, yields

$$\begin{aligned}
\sigma^2\,\mathbb{E}\Bigg[\bigg(\int_s^t e^{-\kappa(t-u)} dW^r(u)\bigg)^2\Bigg] &= \sigma^2\,\mathbb{E}\bigg[\int_s^t e^{-2\kappa(t-u)} du\bigg] \\
&= \sigma^2 \int_s^t e^{-2\kappa(t-u)} du \\
&= \frac{\sigma^2}{2\kappa}\Big(1 - e^{-2\kappa(t-s)}\Big).
\end{aligned}$$

□

We will now derive a closed form representation of the integrated short rate $\int_s^t r(u)\,du$, which will be relevant for the calculation of zero-coupon bond prices and the distributional properties of the bank account evolution in Section 4.1.2:

Lemma 4.1.3. *Let $\big((r(u)\big)_{u\geq 0}$ be the Vasicek short rate process. Then, for $0 \leq s < t$, the integrated short rate can be written as*

$$\int_s^t r(u)\,du = r(s)\,\frac{1-e^{-\kappa(t-s)}}{k} + \theta\left((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\right) + \sigma\int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa}\,dW^r(v). \tag{4.5}$$

Furthermore, the integrated short rate is normally distributed,

$$\int_s^t r(u)\,du \sim \mathcal{N}\big(\tilde{\mu}(s,t),\ \tilde{\sigma}^2(s,t)\big), \tag{4.6}$$

with

$$\tilde{\mu}(s,t) := \theta(t-s) + \big(r_0 - \theta\big)\frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa}, \tag{4.7}$$

$$\tilde{\sigma}^2(s,t) := \frac{\sigma^2}{\kappa^2}(t-s) + \frac{\sigma^2}{2\kappa^3}\Big(\big(1-e^{-\kappa(t-s)}\big)^2\big(1-e^{-2\kappa s}\big) + 4e^{-\kappa(t-s)} - e^{-2\kappa(t-s)} - 3\Big). \tag{4.8}$$

Proof. For $0 \leq s < t$, it follows from the representation (4.2) of the Vasicek short rate $r(u)$ that

$$\begin{aligned}
\int_s^t r(u)\,du &= \int_s^t \left(r(s)e^{-\kappa(u-s)} + \theta\Big(1-e^{-\kappa(u-s)}\Big) + \sigma\int_s^u e^{-\kappa(u-v)}\,dW^r(v)\right)du \\
&= r(s)\int_s^t e^{-\kappa(u-s)}\,du + \theta\int_s^t 1-e^{-\kappa(u-s)}\,du + \sigma\int_s^t\int_s^u e^{-\kappa(u-v)}\,dW^r(v)\,du \\
&\overset{(*)}{=} r(s)\int_s^t e^{-\kappa(u-s)}\,du + \theta\int_s^t 1-e^{-\kappa(u-s)}\,du + \sigma\int_s^t\int_v^t e^{-\kappa(u-v)}\,du\,dW^r(v) \\
&= r(s)\frac{1-e^{-\kappa(t-s)}}{\kappa} + \theta\Big((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\Big) + \sigma\int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa}\,dW^r(v).
\end{aligned}$$

In $(*)$, the order of integration can be changed by applying the Fubini-Tonelli theorem due to the non-negativity of the integrand. Since $r(s)$ and $\int_s^t \frac{1}{k}(1-e^{-\kappa(t-v)})\,dW^r(v)$ are stochastically independent normally distributed random variables, see Corollary 4.1.2 and Lemma 2.1.4, where the independence follows immediately from property (c) in Definition 2.1.3 of Brownian motion, the integrated short rate $\int_s^t r(u)\,du$ itself is also normally distributed.

Furthermore, it follows from the linearity of the expected value and Lemma 2.1.4 that

$$\begin{aligned}
\mathbb{E}\Big[\int_s^t r(u)\,du\Big] &= \mathbb{E}\Big[r(s)\frac{1-e^{-\kappa(t-s)}}{\kappa} + \theta\Big((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\Big) \\
&\quad + \sigma \int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa}\, dW^r(v)\Big] \\
&= \mathbb{E}\big[r(s)\big]\frac{1-e^{-\kappa(t-s)}}{\kappa} + \theta\Big((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\Big) \\
&\quad + \sigma\, \underbrace{\mathbb{E}\Big[\int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa} dW^r(v)\Big]}_{=0} \\
&= \Big(r_0 e^{-\kappa s} + \theta\big(1-e^{-\kappa s}\big)\Big)\frac{1-e^{-\kappa(t-s)}}{\kappa} + \theta\Big((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\Big) \\
&= \theta(t-s) + (r_0 - \theta)\frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa}.
\end{aligned}$$

For the variance, similar to the proof of Corollary 4.1.2, it holds

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\Big(\int_s^t r(u)\,du\Big) &= \mathbb{V}\mathrm{ar}\Big(r(s)\frac{1-e^{-\kappa(t-s)}}{\kappa} + \theta\Big((t-s) - \frac{1-e^{-\kappa(t-s)}}{\kappa}\Big) \\
&\quad + \sigma \int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa}\, dW^r(v)\Big) \\
&\overset{(**)}{=} \Big(\frac{1-e^{-\kappa(t-s)}}{\kappa}\Big)^2 \mathbb{V}\mathrm{ar}\big(r(s)\big) + \sigma^2\, \mathbb{V}\mathrm{ar}\Big(\int_s^t \frac{1-e^{-\kappa(t-v)}}{\kappa}\, dW^r(v)\Big) \\
&= \Big(\frac{1-e^{-\kappa(t-s)}}{\kappa}\Big)^2 \frac{\sigma^2}{2\kappa}\Big(1-e^{-2\kappa s}\Big) + \sigma^2 \int_s^t \Big(\frac{1-e^{-\kappa(t-v)}}{\kappa}\Big)^2 dv \\
&= \frac{\sigma^3}{2\kappa^3}\big(1-e^{-\kappa(t-s)}\big)^2\big(1-e^{-2\kappa s}\big) \\
&\quad + \frac{\sigma^2}{\kappa^2}\Big((t-s) - \frac{2}{\kappa}\big(1-e^{-\kappa(t-s)}\big) + \frac{1}{2\kappa}\big(1-e^{-2\kappa(t-s)}\big)\Big) \\
&= \frac{\sigma^2}{\kappa^2}(t-s) \\
&\quad + \frac{\sigma^2}{2\kappa^3}\Big(\big(1-e^{-\kappa(t-s)}\big)^2\big(1-e^{-2\kappa s}\big) + 4e^{-\kappa(t-s)} - e^{-2\kappa(t-s)} - 3\Big).
\end{aligned}$$

Here, similar to the derivation of the normal distribution of the integrated short rate above, equation $(**)$ holds because $r(s)$ and $\int_s^t \frac{1}{\kappa}(1-e^{-\kappa(t-v)})\, dW^r(v)$ are independent. □

Based on the previous lemma, we can now present a closed-form representation of zero-coupon bond prices $P(t,T)$ in this model. Using the zero-coupon bond prices, the term structure of interest rates can be derived directly using Definition 2.2.4. As the Vasicek short rate model is an affine short rate model, this representation can be derived in the following form according to Theorem 2.2.6:

Theorem 4.1.4. *Let* $\big(r(s)\big)_{s\geq 0}$ *be the Vasicek short rate process. Then, the price of a zero-coupon bond at time* t *with maturity* T *is given as*

$$P(t,T) = e^{-B(t,T)\,r(t)+A(t,T)}, \tag{4.9}$$

with

$$\begin{aligned} B(t,T) &:= \frac{1}{\kappa}\Big(1-e^{-\kappa(T-t)}\Big), \\ A(t,T) &:= \Big(\theta - \frac{\sigma^2}{2\kappa^2}\Big)\big(B(t,T)-T+t\big) - \frac{\sigma^2}{4\kappa}B(t,T)^2. \end{aligned}$$

Proof. Consider $0 \leq t \leq T$. Given the Vasicek short rate process $\big(r(s)\big)_{s\geq 0}$, it holds

$$P(t,T) = \mathbb{E}\Big[e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}_t\Big].$$

Using the representation (4.5) of the integrated short rate in Lemma 4.1.3, it follows from the properties of conditional expectation that

$$\begin{aligned} \mathbb{E}\Big[e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}_t\Big] &= \mathbb{E}\Big[\exp\Big(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\Big((T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}\Big) \\ &\qquad - \sigma\int_t^T \frac{1-e^{-\kappa(T-v)}}{\kappa}\,dW^r(v)\Big) \,\Big|\, \mathcal{F}_t\Big] \\ &= \exp\Big(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\Big((T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}\Big)\Big) \\ &\qquad \cdot \mathbb{E}\Big[\exp\Big(-\sigma\int_t^T \frac{1-e^{-\kappa(T-v)}}{\kappa}\,dW^r(v)\Big)\Big] \\ &= \exp\Big(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\Big((T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}\Big)\Big) \\ &\qquad \cdot \exp\Big(\frac{\sigma^2}{2\kappa^2}\Big((T-t) - \frac{2}{\kappa}\big(1-e^{-\kappa(T-t)}\big) + \frac{1}{2\kappa}\big(1-e^{-2\kappa(T-t)}\big)\Big)\Big) \end{aligned}$$

Here, we use that $r(t)$ is $\mathcal{F}_t$-measurable and that the stochastic integral with respect to W^r is independent of $\mathcal{F}_t$ and normally distributed, see Lemma 2.1.4.

Defining $A(t,T)$ and $B(t,T)$ as in the theorem, we can simplify this expression as

$$\begin{aligned}
& \exp\Big(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\Big((T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}\Big)\Big) \\
& \cdot \exp\Big(\frac{\sigma^2}{2\kappa^2}\Big((T-t) - \frac{2}{\kappa}\big(1-e^{-\kappa(T-t)}\big) + \frac{1}{2\kappa}\big(1-e^{-2\kappa(T-t)}\big)\Big)\Big) \\
= \quad & \exp\Big(-r(t)B(t,T) - \theta\Big((T-t) - B(t,T)\Big) \\
& + \frac{\sigma^2}{2\kappa^2}\Big((T-t) - 2B(t,T) + \frac{1}{2\kappa}\big(1-e^{-2\kappa(T-t)}\big)\Big)\Big) \\
= \quad & \exp\Big(-r(t)B(t,T) + \Big(\theta - \frac{\sigma^2}{2\kappa^2}\Big)\Big(B(t,T) - T + t\Big) \\
& - \frac{\sigma^2}{4\kappa}\underbrace{\Big(\frac{2}{\kappa}B(t,T) - \frac{1}{\kappa^2}\big(1-e^{-2\kappa(T-t)}\big)\Big)}_{=B(t,T)^2}\Big) \\
= \quad & e^{-B(t,T)\,r(t)+A(t,T)}.
\end{aligned}$$

□

Remark. By computing the partial derivatives, one can immediately verify that the given functions $A(t,T)$ and $B(t,T)$ in Theorem 4.1.4 are the unique solutions of the system of differential equations in (2.22) and (2.23) from Theorem 2.2.6.

By combining the results of this theorem and the distributional properties of $r(t)$ from Lemma 4.1.2, the distribution of future zero-coupon bond prices can be derived directly, which will later be used in particular for the calculation of distributional properties of fixed rate bond prices. Since $r(t)$ is normally distributed according to (4.4) and $A(t,T)$, $B(t,T)$ are deterministic functions, it follows that $P(t,T)$ is log-normally distributed. More precisely, by denoting

$$\hat{\mu}(t) := r_0\, e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) \tag{4.10}$$

$$\hat{\sigma}^2(t) := \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right), \tag{4.11}$$

whereby $r(t) \sim \mathcal{N}\big(\hat{\mu}(t), \hat{\sigma}^2(t)\big)$, it follows that

$$P(t,T) \sim Log\mathcal{N}\Big(A(t,T) - B(t,T)\,\hat{\mu}(t),\ B(t,T)^2\,\hat{\sigma}^2(t)\Big). \tag{4.12}$$

Based on the formulas for the expected value and variance of the log-normal distribution, see Lemma 2.1.2, we obtain

$$\mathbb{E}\big[P(t,T)\big] = e^{A(t,T)-B(t,T)\,\hat{\mu}(t)+\frac{1}{2}B(t,T)^2\hat{\sigma}^2(t)}, \tag{4.13}$$

$$\mathbb{V}\mathrm{ar}\big(P(t,T)\big) = e^{2\big(A(t,T)-B(t,T)\,\hat{\mu}(t)\big)+B(t,T)^2\,\hat{\sigma}^2(t)}\,\Big(e^{B(t,T)^2\hat{\sigma}^2(t)} - 1\Big). \tag{4.14}$$

For the conditional expectation of future zero-coupon bond prices, the following lemma holds:

Lemma 4.1.5. *Let* $\big((r(u)\big)_{u\geq 0}$ *be the Vasicek short rate process and consider* $0 \leq s \leq t < T$. *The conditional expectation of* $P(t,T)$ *given* $\mathcal{F}_s$ *is given as*

$$\mathbb{E}\big[P(t,T)\,\big|\,\mathcal{F}_s\big] = \exp\bigg(A(t,T) - B(t,T)\,\Big(r(s)\,e^{-\kappa(t-s)} + \theta\,\big(1 - e^{-\kappa\,(t-s)}\big)\Big) + \frac{B(t,T)^2\,\sigma^2\,(1-e^{-2\kappa(t-s)})}{4\kappa}\bigg). \tag{4.15}$$

Proof. With the representation of $P(t,T)$ in (4.9), the representation of $r(t)$ in (4.2) and Lemma 2.1.2, it follows that

$$\begin{aligned}
\mathbb{E}\big[P(t,T)\,\big|\,\mathcal{F}_s\big] =& \mathbb{E}\Big[\exp\Big(-B(t,T)\,r(t) + A(t,T)\Big)\,\Big|\,\mathcal{F}_s\Big] \\
=& \exp\big(A(t,T)\big)\cdot\mathbb{E}\bigg[\exp\bigg(-B(t,T)\cdot\Big(r(s)\,e^{-\kappa(t-s)} + \theta\,\big(1-e^{-\kappa(t-s)}\big) \\
&+ \sigma\int_s^t e^{-\kappa(t-u)}\,dW^r(u)\Big)\bigg)\,\bigg|\,\mathcal{F}_s\bigg] \\
\overset{(*)}{=}& \exp\bigg(A(t,T) - B(t,T)\Big(r(s)\,e^{-\kappa(t-s)} + \theta\,\big(1-e^{-\kappa(t-s)}\big)\Big)\bigg) \\
&\cdot\mathbb{E}\bigg[\underbrace{\exp\bigg(-B(t,T)\,\sigma\int_s^t e^{-\kappa(t-u)}\,dW^r(u)\bigg)}_{\sim Log\mathcal{N}\left(0,\,B(t,T)^2\cdot\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa(t-s)}\right)\right)}\bigg] \\
=& \exp\bigg(A(t,T) - B(t,T)\Big(r(s)\,e^{-\kappa(t-s)} + \theta\,\big(1-e^{-\kappa(t-s)}\big)\Big) \\
&+ \frac{B(t,T)^2\,\sigma^2\,(1-e^{-2\kappa(t-s)})}{4\kappa}\bigg).
\end{aligned}$$

Equation $(*)$ above holds due to the $\mathcal{F}_s$-measurability of $r(s)$ and the independence of the stochastic integral of $\mathcal{F}_s$ as a result of the defining property (c) of Brownian motion, see Definition 2.1.3. □

Remark. From the normal distribution property of the short rate it follows that, for positive κ and σ, the short rate $r(t)$ in the Vasicek model can also take on negative values for each time t with probability

$$\mathbb{P}\big(r(t) < 0\big) = \Phi\left(-\frac{r_0 e^{-\kappa t} + \theta\big(1-e^{-\kappa t}\big)}{\sqrt{\frac{\sigma^2}{2\kappa}\big(1-e^{-2\kappa t}\big)}}\right) > 0. \tag{4.16}$$

Here, Φ is the cumulative distribution function of the standard normal distribution. The fact that the short rate can become negative is often cited as a drawback of the Vasicek

model in the literature. This point of view can especially be traced back to a pre-financial crisis perspective in which negative interest rates were considered unrealistic. But given the fact that negative interest rates became reality in the recent past, this approach was consciously chosen and has the advantage that, via a suitable choice of parameters, scenarios with negative interest rates can also be modelled and investigated. If, however, negative yield curves are not desired, the model parameters can be chosen such that the probabilities of negative short rates given in (4.16) become negligibly small for all t.
For a very similar model that explicitly excludes negative short rates, we refer here to the affine Cox-Ingersoll-Ross short rate model (CIR model). The short rate dynamics under the CIR model are given by the stochastic differential equation

$$dr(t) = \kappa\big(\theta - r(t)\big)\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad r(0) = r_0,$$

with positive constants κ, θ, σ and a one-dimensional standard Brownian motion W. By satisfying the so-called Feller condition $2\kappa\theta > \sigma^2$, it can even be guaranteed that the short rate in the CIR model is strictly positive, see Brigo/Mercurio [1].

To conclude this section, we will present one more result regarding the yield curve (see Definition 2.2.4) resulting from the zero-coupon bond prices at a time t. Desmettre/Korn [4] show that the yield curve derived from the Vasicek model can take four different shapes depending on the model parameters:

Theorem 4.1.6 (Vasicek yield curve shape)**.** *Assume $\kappa > 0$. Then the following statements about the shape of the yield curve in the Vasicek model hold:*

(a) If $\sigma > 0$, then:

$$\begin{aligned} \textit{The yield curve is normal} \quad &\Leftrightarrow \quad r(t) \le \theta - \frac{3\sigma^2}{4\kappa^2} \\ \textit{The yield curve is inverted} \quad &\Leftrightarrow \quad r(t) \ge \theta \\ \textit{The yield curve is humped} \quad &\Leftrightarrow \quad \theta - \frac{3\sigma^2}{4\kappa^2} < r(t) < \theta. \end{aligned}$$

Here, a normal yield curve is strictly monotonically increasing and bounded from above, an inverted yield curve is strictly monotonically decreasing and bounded from below, and a humped yield curve is a curve with exactly one maximum and no minimum in the interval $(0, \infty)$.

(b) If $\sigma = 0$, then the yield curve is flat (constant) for $r(t) = \theta$, normal for $r(t) < \theta$ and inverted for $r(t) > \theta$.

Proof. For a proof of this theorem and a detailed treatment of all cases, see Desmettre/Korn [4]. □

4.1.2. Cash model

To represent the riskless investment option in this asset model, we consider a bank account whose value development is driven by the development of the modelled risk-free short rate:

Definition 4.1.7 (Bank account)**.** Let $C(t)$ be the value of a bank account with $C(0) = 1$. The bank account evolves according to the ordinary differential equation

$$dC(t) = r(t)\, C(t)\, dt, \quad C(0) = 1, \tag{4.17}$$

where $r(t)$ is the short rate modelled in (4.1) in the previous section.

Solving the defining ordinary differential equation for the evolution of the bank account, it follows that

$$C(t) = e^{\int_0^t r(s)\, ds}, \quad C(0) = 1. \tag{4.18}$$

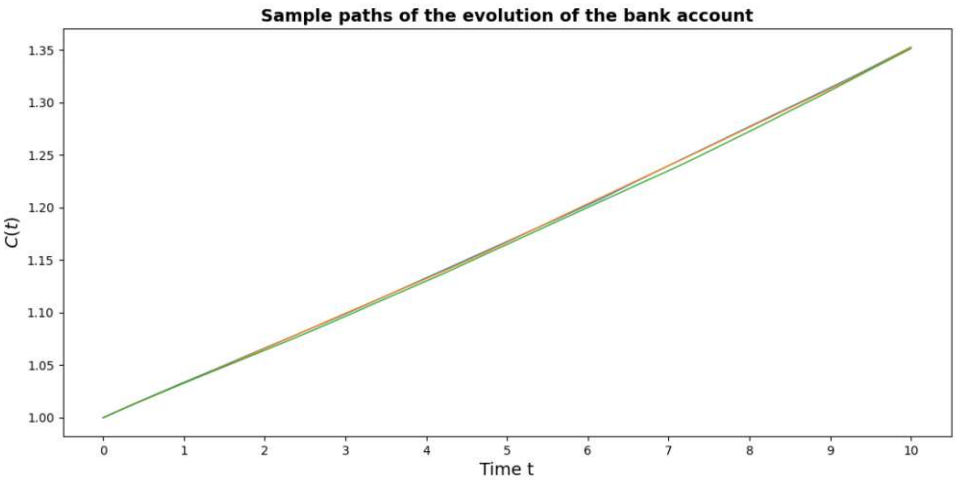


Figure 4.2.: Sample paths of the bank account development given the Vasicek short rate paths in Figure 4.1

From the distributional properties of the integrated short rate in Lemma 4.1.3, it follows that

$$\int_0^t r(s)\, ds \sim \mathcal{N}\big(\tilde{\mu}(0,t),\ \tilde{\sigma}^2(0,t)\big),$$

with $\tilde{\mu}(0,t)$ and $\tilde{\sigma}^2(0,t)$ as defined in Lemma 4.1.3. Thus, for $t \geq 0$, using Lemma 2.1.2, $C(t)$ is log-normally distributed with

$$\begin{aligned} \mathbb{E}\big[C(t)\big] = &\quad \exp\left(\tilde{\mu}(0,t) + \frac{\tilde{\sigma}^2(0,t)}{2}\right) \quad (4.19) \\ = &\quad \exp\left(\frac{r_0-\theta}{\kappa} - \frac{3\sigma^2}{4\kappa^3} + \Big(\theta + \frac{\sigma^2}{2\kappa^2}\Big)t + \Big(\frac{\sigma^2}{\kappa^3} - \frac{r_0-\theta}{\kappa}\Big)e^{-\kappa t} - \frac{\sigma^2}{4\kappa^3}e^{-2\kappa t}\right) \end{aligned}$$

and

$$\begin{aligned}\mathbb{V}\text{ar}\big(C(t)\big) =&\ \exp\Big(2\tilde{\mu}(0,t)+\tilde{\sigma}^2(0,t)\Big)\Big(\exp\big(\tilde{\sigma}^2(0,t)\big)-1\Big) \\ =&\ \exp\bigg(\frac{2(r_0-\theta)}{\kappa}-\frac{3\sigma^2}{2\kappa^3}+\Big(2\theta+\frac{\sigma^2}{\kappa^2}\Big)t+\Big(\frac{2\sigma^2}{\kappa^3}-\frac{2(r_0-\theta)}{\kappa}\Big)e^{-\kappa t}-\frac{\sigma^2}{2\kappa^3}e^{-2\kappa t}\bigg) \\ &\cdot\bigg(\exp\Big(\frac{\sigma^2}{\kappa^2}t+\frac{\sigma^2}{2\kappa^3}\Big(4e^{-\kappa t}-e^{-2\kappa t}-3\Big)\Big)-1\bigg).\end{aligned} \tag{4.20}$$

The return for an investment in the bank account for a given time period $[t_{k-1},t_k)$ of the model time grid $\mathcal{T}$ is defined as

$$R_k^C := \frac{C(t_k)-C(t_{k-1})}{C(t_{k-1})} = \exp\bigg(\int_{t_{k-1}}^{t_k} r(s)\,ds\bigg)-1,\quad k=1,\dots,N. \tag{4.21}$$

From Lemma 4.1.3, the proof of Theorem 4.1.4 and the fact that the time increments of the underlying time grid $\mathcal{T}$ equal one year, i.e. $(t_k-t_{k-1})=1$, it immediately follows for the conditional expectation of the bank account return R_k^C, given $\mathcal{F}_{t_{k-1}}$, that

$$\mathbb{E}\Big[R_k^C\,\Big|\,\mathcal{F}_{t_{k-1}}\Big] = \mathbb{E}\bigg[\exp\bigg(\int_{t_{k-1}}^{t_k} r(s)\,ds\bigg)\,\bigg|\,\mathcal{F}_{t_{k-1}}\bigg]-1 \tag{4.22}$$

$$= \exp\bigg(\frac{1-e^{-\kappa}}{\kappa}\cdot\Big(r(t_{k-1})-\theta-\frac{\sigma^2}{\kappa^2}\Big)+\frac{\sigma^2}{4\kappa^3}\Big(1-e^{-2\kappa}\Big)+\frac{\sigma^2}{2\kappa^2}+\theta\bigg)-1. \tag{4.23}$$

4.1.3. Bond model

As investment instruments from fixed-income markets, we consider a total of $n_B \in \mathbb{N}$ different fixed rate bonds $B_1,\dots,B_{n_B}$ with different coupon rates and maturities. For the i-th bond, we denote with $T_i^B \in \mathbb{N}$ the maturity of the bond in years and with $c_i^B \in [0,1]$ the constant coupon rate as the rate of interest paid on the bond's face value N_i^B at fixed points in time. We also denote with

$$T^B_{\mathbf{max}} := \max\big\{T_i^B \,\big|\, 1 \le i \le n_B\big\}$$

the maximum maturity of all bonds. We further assume the following simplifications for the bond model:

1. The face value of all bonds is equal to 1, that is

$$N_i^B = 1,\quad i=1,\dots,n_B. \tag{4.24}$$

   Since the price B_i of an arbitrary bond with face value N_i and coupon rate c_i is linear in its face value, as can be seen from the bond's pricing formula (4.27) below, B_i/N_i equals the price of a bond with the same coupon rate and face value 1. Under the assumption that all financial assets are arbitrarily divisible, see Section 4.1.6, this assumption is not a restrictive simplification.

2. All bonds have equal future coupon payment dates until their individual maturity and the coupon payment dates are consistent with the model time grid $\mathcal{T}$. Therefore, formally, we consider the time grid $0 < t_1^c < \cdots < t_{k_B}^c \leq T_{\mathbf{max}}^B$ of k_B equidistant coupon payment times with

$$\left\{t_j^c \,|\, t_j^c \leq T\right\} \subseteq \mathcal{T}, \tag{4.25}$$

and assume that at every coupon date $t_j^c \in \{t_1^c, \ldots, t_{k_B}^c\}$, each bond $B_i \in \{B_1, \ldots, B_{n_B}\}$ pays a coupon of

$$c_i \cdot N_i^B \cdot \mathbb{1}_{\{t_j^c \leq T_i^B\}} = c_i \cdot \mathbb{1}_{\{t_j^c \leq T_i^B\}}.$$

Here,

$$\mathbb{1}_{\{t_j^c \leq T_i^B\}} = \begin{cases} 1 & \text{if } \; t_j^c \leq T_i^B \\ 0 & \text{if } \; t_j^c > T_i^B \end{cases}$$

denotes the indicator function, which ensures that the i-th bond only pays coupons until its maturity T_i^B. As the price of a fixed rate bond drops at each of its coupon payment dates by the amount of the coupon paid, as can be seen from the fixed rate bond pricing formula (4.26) below, condition (4.25) guarantees that if a bond is held for a time period $[t_k, t_{k+1})$, no coupons are paid within this period and thus no such drops in the fixed rate bond's value occur.

3. We assume that the probability of default is zero for all bond issuers. In particular, we only consider high-rated bonds in this model for which default risk can be neglected for simplification.

We assume that the fixed rate bonds in this model are only held for one period and that the set of available bonds is 'renewed' after every time period. That is, for every point in time t_k of the model time grid $\mathcal{T}$, the bonds available for investment have the same properties (i.e. term to maturity, coupon rate and future coupon payment times). Thus, at t_k, the maturity of the i-th bond equals $t_k + T_i^B$ and the future coupons after t_k are paid at the coupon payment times $\{t_k + t_1^c, \ldots, t_k + t_{k_B}^c\}$, accordingly. This yields that the bonds' durations, see (4.61), do not significantly change between different times of the time grid as their structures of future cash flows remain the same. As the main objective of this model is to investigate optimal asset portfolio compositions to manage the interest rate risk, see Section 4.3, we require that the investment opportunities for the insurance company in terms of available assets and their characteristics are equal for each time period.

In this thesis, we model the price of each bond as the theoretical financial mathematical price corresponding to the present value of the bond's future cash flows under the risk-free yield curve modelled in Section 4.1.1. Very generally, following the notation in Kellerhals [16], the price of a fixed rate bond at time t with maturity T is given as

$$B(t, T) = \sum_{u \in (t, T]} c(u) \cdot P(t, u), \tag{4.26}$$

where $c(u)$ denotes the cash flows received by the bond holder at time u. With the above assumptions and notation, the price of the i-th bond at an arbitrary time $t \leq T$ is of the form

$$B_i(t) = \sum_{j=1}^{k_B} c_i \, P(t, t + t_j^c) + P(t, t + T_i^B). \tag{4.27}$$

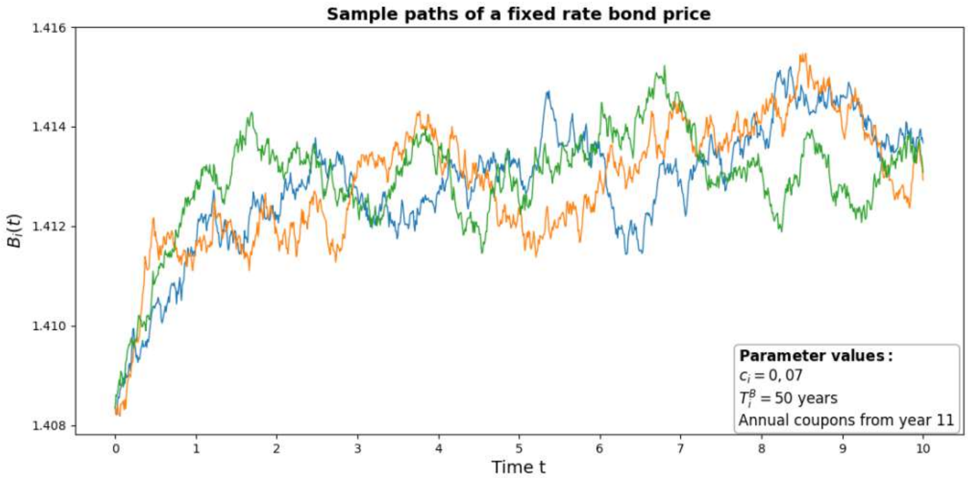


Figure 4.3.: Sample paths of a fixed rate bond price given the Vasicek short rate paths in Figure 4.1

Note that, as $P(t, T)$ is continuous in both arguments t and T as a consequence of the continuity of the short rate process r, it follows that $B_i(t)$ is also continuous.

To analyse the distributional properties of a coupon bond price and, in particular, to derive a closed formula for the variance of a coupon bond price, we consider the following lemma:

Lemma 4.1.8. *Consider $0 \leq t < s < u$ and consider two zero-coupon bond prices $P(t, s)$ and $P(t, u)$ given by (4.9). Then, it holds for the covariance of $P(t, s)$ and $P(t, u)$ that*

$$\mathbb{C}\mathrm{ov}\Big(P(t,s), P(t,u)\Big) = e^{A(t,s)+A(t,u)-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big)+\frac{\hat{\sigma}^2(t)}{2}\big(B(t,s)^2+B(t,u)^2\big)} \cdot \Big(e^{\hat{\sigma}^2(t)B(t,s)B(t,u)} - 1\Big), \tag{4.28}$$

where $\hat{\mu}(t)$ and $\hat{\sigma}^2(t)$ are given by (4.10) and (4.11), respectively.

Proof. Let $0 \leq t < s < u$ be arbitrary. With the closed form (4.9) of a zero-coupon bond price, the bilinearity of covariance and the identity $\mathbb{C}\mathrm{ov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ for covariance, it follows that

$$\begin{aligned}\mathbb{C}\text{ov}\Big(P(t,s),P(t,u)\Big) &= \mathbb{C}\text{ov}\Big(e^{-r(t)B(t,s)+A(t,s)},e^{-r(t)B(t,u)+A(t,u)}\Big)\\ &= e^{A(t,s)+A(t,u)}\,\mathbb{C}\text{ov}\Big(e^{-r(t)B(t,s)},e^{-r(t)B(t,u)}\Big)\\ &= e^{A(t,s)+A(t,u)}\bigg(\mathbb{E}\left[e^{-r(t)\big(B(t,s)+B(t,u)\big)}\right]\\ &\quad -\mathbb{E}\Big[e^{-r(t)B(t,s)}\Big]\mathbb{E}\Big[e^{-r(t)B(t,u)}\Big]\bigg)\end{aligned}$$

Since $r(t)\sim\mathcal{N}\big(\hat{\mu}(t),\,\hat{\sigma}^2(t)\big)$, see Lemma 4.1.2, it follows that

$$e^{-r(t)\big(B(t,s)+B(t,u)\big)}\sim Log\mathcal{N}\Big(-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big),\;\hat{\sigma}^2(t)\big(B(t,s)+B(t,u)\big)^2\Big)$$

and further, using Lemma 2.1.2,

$$\mathbb{E}\left[e^{-r(t)\big(B(t,s)+B(t,u)\big)}\right]=\exp\left(-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big)+\frac{\hat{\sigma}^2(t)\big(B(t,s)+B(t,u)\big)^2}{2}\right).$$

Analogously, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-r(t)B(t,s)}\right] &= \exp\left(-\hat{\mu}(t)B(t,s)+\frac{\hat{\sigma}^2(t)B(t,s)^2}{2}\right)\\ \mathbb{E}\left[e^{-r(t)B(t,u)}\right] &= \exp\left(-\hat{\mu}(t)B(t,u)+\frac{\hat{\sigma}^2(t)B(t,u)^2}{2}\right)\end{aligned}$$

and further

$$\begin{aligned}&\mathbb{E}\left[e^{-r(t)\big(B(t,s)+B(t,u)\big)}\right]-\mathbb{E}\left[e^{-r(t)B(t,s)}\right]\mathbb{E}\left[e^{-r(t)B(t,u)}\right]\\ &=\exp\left(-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big)+\frac{\hat{\sigma}^2(t)\big(B(t,s)+B(t,u)\big)^2}{2}\right)\\ &\quad-\exp\left(-\hat{\mu}(t)B(t,s)+\frac{\hat{\sigma}^2(t)B(t,s)^2}{2}-\hat{\mu}(t)B(t,u)+\frac{\hat{\sigma}^2(t)B(t,u)^2}{2}\right)\\ &=\exp\Big(-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big)\Big)\\ &\quad\cdot\left(\exp\left(\frac{\hat{\sigma}^2(t)}{2}\big(B(t,s)+B(t,u)\big)^2\right)-\exp\left(\frac{\hat{\sigma}^2(t)}{2}\big(B(t,s)^2+B(t,u)^2\big)\right)\right)\\ &=\exp\left(-\hat{\mu}(t)\big(B(t,s)+B(t,u)\big)+\frac{\hat{\sigma}^2(t)}{2}\big(B(t,s)^2+B(t,u)^2\big)\right)\\ &\quad\cdot\Big(\exp\Big(\hat{\sigma}^2(t)B(t,s)B(t,u)\Big)-1\Big)\end{aligned}$$

which concludes the proof. □

Now we can present closed formulas for the expected value and the variance of the i-th coupon bond price $B_i(t)$. For the expected value, it follows immediately from the linearity of the expected value and the form of the pricing formula (4.27) of a coupon bond price that

$$\mathbb{E}\big[B_i(t)\big] = \mathbb{E}\bigg[\sum_{j=1}^{k_B} c_i\, P(t, t+t_j^c) + P(t, t+T_i^B)\bigg] \tag{4.29}$$

$$= \sum_{j=1}^{k_B} c_i\, \mathbb{E}\big[P(t, t+t_j^c)\big] + \mathbb{E}\big[P(t, t+T_i^B)\big], \tag{4.30}$$

where $\mathbb{E}\big[P(t,u)\big]$ can be computed by (4.13) for $t < u$.
For the variance of $B_i(t)$, using the identity for the variance of a sum of random variables

$$\mathbb{V}\mathrm{ar}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbb{V}\mathrm{ar}(X_i) + 2 \sum_{i<j} \mathbb{C}\mathrm{ov}(X_i, X_j),$$

it holds that

$$\mathbb{V}\mathrm{ar}\big(B_i(t)\big) = \mathbb{V}\mathrm{ar}\bigg(\sum_{j=1}^{k_B} c_i\, P(t, t+t_j^c) + P(t, t+T_i^B)\bigg) \tag{4.31}$$

$$\begin{aligned} &= \sum_{j=1}^{k_B} c_i^2\, \mathbb{V}\mathrm{ar}\big(P(t, t+t_j^c)\big) + \mathbb{V}\mathrm{ar}\big(P(t, t+T_i^B)\big) \\ &\quad + \sum_{j=1}^{k_B} \sum_{k=j+1}^{k_B} 2\, c_i^2\, \mathbb{C}\mathrm{ov}\Big(P(t, t+t_j^c),\, P(t, t+t_k^c)\Big) \\ &\quad + \sum_{j=1}^{k_B} 2\, c_i\, \mathbb{C}\mathrm{ov}\Big(P(t, t+t_j^c),\, P(t, t+T_i^B)\Big), \end{aligned} \tag{4.32}$$

where $\mathbb{V}\mathrm{ar}\big(P(t,u)\big)$ is given by (4.14) for $t < u$ and $\mathbb{C}\mathrm{ov}\big(P(t,s), P(t,u)\big)$ can be computed using Lemma 4.1.8 for $t < s < u$.

Remark. According to the third assumption for the bond model, we do not consider bond issuer default risk in this model. However, this model extension could be implemented by scaling the expected future cash flows of the bonds with corresponding (time-dependent or constant) default probabilities. For an approach to pricing defaultable coupon bonds using a stochastic process that models the default of a bond issuer, we refer to Wüthrich/Merz [29].

We conclude this section with a derivation of expected returns for investments in fixed rate bonds. If a bond is held for a given time period $[t_{k-1}, t_k)$ of length 1, we define the return of the i-th bond as

$$R_k^{B_i} := \frac{\hat{B}_i(t_k) - B_i(t_{k-1})}{B_i(t_{k-1})} = \frac{\hat{B}_i(t_k)}{B_i(t_{k-1})} - 1, \quad k = 1, \dots, N, \tag{4.33}$$

where $\hat{B}_i(t_k)$ denotes the value of the i-th bond if bought at time t_{k-1} and held until t_k. From the bond pricing formula (4.27) and the assumption in (4.25) that within the period $[t_{k-1}, t_k)$ no coupons are paid, it follows that $\hat{B}_i(t_k)$ is given as

$$\hat{B}_i(t_k) = \sum_{j=1}^{k_B} c_i\, P(t_k, t_k + t_j^c - 1) + P(t_k, t_k + T_i^B - 1). \tag{4.34}$$

The conditional expected return of $R_k^{B_i}$ given $\mathcal{F}_{t_{k-1}}$, using the $\mathcal{F}_{t_{k-1}}$-measurability of $B_i(t_{k-1})$, is given as

$$\begin{aligned}
\mathbb{E}\Big[R_k^{B_i} \,\Big|\, \mathcal{F}_{t_{k-1}}\Big] &= \mathbb{E}\Big[\frac{\hat{B}_i(t_k)}{B_i(t_{k-1})} - 1 \,\Big|\, \mathcal{F}_{t_{k-1}}\Big] &(4.35)\\
&= \frac{\mathbb{E}\big[\hat{B}_i(t_k) \,\big|\, \mathcal{F}_{t_{k-1}}\big]}{B_i(t_{k-1})} - 1 &(4.36)\\
&= \frac{1}{B_i(t_{k-1})} \Bigg(\sum_{j=1}^{k_B} c_i\, \mathbb{E}\big[P(t_k, t_k + t_j^c - 1) \,\big|\, \mathcal{F}_{t_{k-1}}\big] &(4.37)\\
&\qquad + \mathbb{E}\big[P(t_k, t_k + T_i^B - 1) \,\big|\, \mathcal{F}_{t_{k-1}}\big]\Bigg) - 1, &(4.38)
\end{aligned}$$

where the conditional expectation of a zero-coupon bond price is given as in Lemma 4.1.5.

4.1.4. Stock price model

To model a stock market and provide the possibility to invest in equities, we consider a total of $n_E \in \mathbb{N}$ different equities, whose price processes S_i, $i = 1, \ldots, n_E$, under the real-world measure each follow a geometric Brownian motion, given by the stochastic differential equations

$$dS_i(t) = \mu_i\, S_i(t)\, dt + \sigma_i\, S_i(t)\, dW_i^S(t), \quad 0 \le t,\ i = 1, \ldots, n_E, \tag{4.39}$$

with drift parameters $\mu_i \in \mathbb{R}$, volatility parameters $\sigma_i \ge 0$ and correlated standard Brownian motions $W_i^S(t)$, $i = 1, \ldots, n_E$. This choice of a geometric Brownian motion for the equity price dynamics is widely known from the famous Black-Scholes model (see Jeanblanc et al. [13] for example). We assume that $S_i(0) = 1$ for all $i = 1, \ldots, n_E$, meaning that the price of each equity starts with initial value 1. Since we assume that all investment instruments are arbitrarily divisible, this assumption is not restrictive to the overall model.

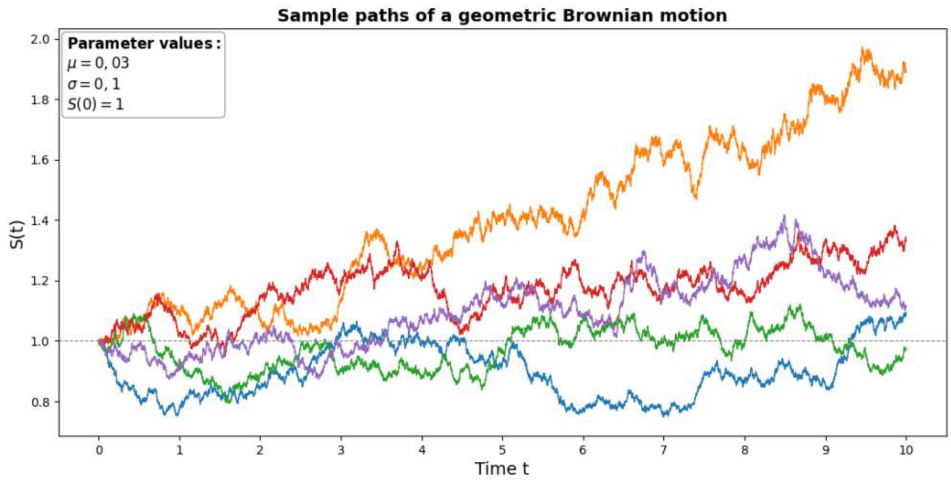


Figure 4.4.: Sample paths of geometric Brownian motion

In a similar way to the Vasicek model, Itô's Lemma can be applied to obtain the solution of the defining stochastic differential equation. To increase readability, we omit the subscripts in the solution:

Theorem 4.1.9. *The unique solution of equation (4.39) is given as*

$$S(t) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}, \quad 0 \leq t. \tag{4.40}$$

Proof. Applying Itô's Lemma (Theorem 2.1.8) with $f(t,s) = \ln S$, $\alpha(t, S(t)) = \mu S(t)$ and $\sigma(t, S(t)) = \sigma S(t)$ yields

$$d \ln S(t) = df\big(t, S(t)\big) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW(t).$$

Integrating the obtained equation, we get

$$\begin{aligned} \ln S(t) &= \ln S(0) + \int_0^t \mu - \frac{\sigma^2}{2}\, ds + \int_0^t \sigma\, dW(s) \\ &= \ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t), \end{aligned}$$

and applying the exponential function yields the given solution in (4.40). Uniqueness of the solution follows from Theorem 2.1.6 and the Lipschitz-continuity of $\alpha(t,x)$ and $\sigma(t,x)$. □

From the closed form (4.40) we can derive distributional properties of $S(t)$:

Theorem 4.1.10. *Let $S = (S(t))_{t \geq 0}$ be a geometric Brownian motion given by (4.40). Then, $S(t)$ is log-normally distributed for every $t \geq 0$ with expected value and variance* *given by*

$$\mathbb{E}\big[S(t)\big] = S(0)\, e^{\mu t},$$
$$\mathbb{V}\mathrm{ar}\big(S(t)\big) = S(0)^2\, e^{2\mu t}\Big(e^{\sigma^2 t} - 1\Big).$$

Conditional on $\mathcal{F}_s$, it holds for $0 \leq s \leq t$ that

$$\mathbb{E}\big[S(t)|\mathcal{F}_s\big] = S(0)\, e^{\mu t + \sigma W(s)}\, e^{-\sigma^2 s/2}. \tag{4.41}$$

Proof. Since $W(t) \sim \mathcal{N}(0,t)$, see Definition 2.1.3, it holds that

$$\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t) \sim \mathcal{N}\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t,\ \sigma^2 t\bigg)$$

and further

$$\exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\bigg) \sim Log\mathcal{N}\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t,\ \sigma^2 t\bigg).$$

From the properties of the log-normal distribution, see Lemma 2.1.2, it follows that

$$\begin{aligned}\mathbb{E}\big[S(t)\big] &= \mathbb{E}\bigg[S(0)\ \exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\bigg)\bigg]\\ &= S(0)\ \exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \frac{\sigma^2 t}{2}\bigg)\\ &= S(0)\, e^{\mu t}\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big(S(t)\big) &= \mathbb{V}\mathrm{ar}\bigg(S(0)\ \exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\bigg)\bigg)\\ &= S(0)^2\ \exp\bigg(2\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma^2 t\bigg)\Big(\exp(\sigma^2 t) - 1\Big)\\ &= S(0)^2\, e^{2\mu t}\Big(e^{\sigma^2 t} - 1\Big).\end{aligned}$$

For the conditional expectation, given $s \leq t$, it holds that

$$\begin{aligned}\mathbb{E}\big[S(t)|\mathcal{F}_s\big] &= \mathbb{E}\bigg[S(0)\ \exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W(t)\bigg)\bigg|\mathcal{F}_s\bigg]\\ &= S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t}\, \mathbb{E}\Big[e^{\sigma W(t)}\Big|\mathcal{F}_s\Big]\\ &= S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t}\, \mathbb{E}\Big[e^{\sigma\left(W(t) - W(s)\right)}\, e^{\sigma W(s)}\Big|\mathcal{F}_s\Big]\\ &\overset{(*)}{=} S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t}\, \mathbb{E}\Big[e^{\sigma\left(W(t) - W(s)\right)}\Big]\, \mathbb{E}\Big[e^{\sigma W(s)}\Big|\mathcal{F}_s\Big]\\ &\overset{(**)}{=} S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t}\, e^{\frac{(t-s)\sigma^2}{2}}\, e^{\sigma W(s)}\\ &= S(0)\, e^{\mu t + \sigma W(s)}\ e^{-\sigma^2 s/2}.\end{aligned}$$

In equality $(*)$ we used the independence of $\big(W(t)-W(s)\big)$ and $W(s)$, see Definition 2.1.3. Equality $(**)$ follows from the properties of conditional expectation and the facts that $e^{\sigma(W(t)-W(s))}$ is independent of $\mathcal{F}_s$ and $e^{\sigma W(s)}$ is $\mathcal{F}_s$-measurable.

□

From the distribution properties of $S(t)$ above it follows directly that, for any $t_j < t_k$ of the model's time grid $\mathcal{T}$,

$$\frac{S(t_k)}{S(t_j)} \sim Log\mathcal{N}\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)(t_k - t_j),\ \sigma^2(t_k - t_j)\bigg).$$

and further, for any time period $[t_{k-1}, t_k)$ of length 1, i.e. $(t_k - t_{k-1}) = 1$, the expected value and the variance for the return of the underlying asset simplify to

$$\begin{aligned}\mathbb{E}\left[\frac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}\right] &= \mathbb{E}\left[\frac{S(t_k)}{S(t_{k-1})} - 1\right] = e^{\mu} - 1,\\ \mathbb{V}\mathrm{ar}\left(\frac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}\right) &= e^{2\mu}\Big(e^{\sigma^2} - 1\Big).\end{aligned}$$

These relations provide easy interpretability of the trend parameter μ and the volatility parameter σ in the model, which not least helps to select suitable parameter values for specific scenarios. By choosing $\sigma = 0$, even deterministic scenarios can be generated since the random-driven component of the Brownian motion vanishes. Moreover, these relationships are helpful for the statistical estimation of the parameters and the calibration to real market values.

Also, from the independence of the increments of Brownian motion for non-overlapping time intervals (see Definition 2.1.3), it follows that for all $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \cdots < t_n \le T$, the random variables

$$\frac{S(t_1)}{S(t_0)}, \frac{S(t_2)}{S(t_1)}, \ldots, \frac{S(t_n)}{S(t_{n-1})}$$

and thus the returns for non-overlapping intervals are independent.

Remark. From the explicit representation (4.40) of the price process and the continuity of the paths of Brownian motion, it follows that the paths of S are also continuous. This property of a stock price model is often criticized, Desmettre/Korn [4] refer here for example to the possibility of strong price fluctuations and price jumps which are implausible under the Black-Scholes assumption. Also, the Black-Scholes model does not take into account the decrease of a share price by the amount of a dividend after the ex-dividend date. However, since dividends are not included in this model, it is assumed for simplicity that any dividends are reinvested immediately, which means that there is no cash payout and there are no dividend price jumps in the process. Moreover, in the optimisation we do not consider path-dependent properties, but only the stock prices at finitely many equidistant points in time, making pathwise continuity of the price process a not too restrictive assumption.

Remark (Extension of the model to real estate markets). Even though real estate investments are not explicitly included in this model, this stock market model is methodologically not limited to equities or equity funds. To cover real estate, we could introduce an additional stochastic price process of a real estate investment funds via the stochastic differential equation

$$dR(t) = \mu^R \, R(t) \, dt + \sigma^R \, R(t) \, dW^R(t) \tag{4.42}$$

with an suitable choice of parameters. Here, however, the corresponding extended stochastic dependency structure arising from the consideration of real estate investments must be taken into account, see Section 4.1.5.

To conclude this section, we derive the conditional expectation of the stock price returns. Analogue to the returns of the bank account and the fixed rate bond prices, we define the i-th stock return for a time period $[t_{k-1}, t_k)$ as

$$R_k^{S_i} := \frac{S_i(t_k) - S_i(t_{k-1})}{S_i(t_{k-1})} = \frac{S_i(t_k)}{S_i(t_{k-1})} - 1. \tag{4.43}$$

For the conditional expectation of the stock return, using the $\mathcal{F}_{t_{k-1}}$-measurability of $S(t_{k-1})$ as well as (4.41) in Theorem 4.1.10 above, it follows that

$$\mathbb{E}\Big[R_k^{S_i} \,\Big|\, \mathcal{F}_{t_{k-1}}\Big] = \mathbb{E}\left[\frac{S_i(t_k)}{S_i(t_{k-1})} - 1 \,\middle|\, \mathcal{F}_{t_{k-1}}\right] \tag{4.44}$$

$$= \frac{\mathbb{E}\big[S_i(t_k) \,\big|\, \mathcal{F}_{t_{k-1}}\big]}{S_i(t_{k-1})} - 1 \tag{4.45}$$

$$= \exp\Big(\mu(t_k - t_{k-1})\Big) - 1. \tag{4.46}$$

4.1.5. Stochastic dependency modelling within the asset model

By linking the stochastic processes for the short rate and the asset price processes together within the asset model, we consider a situation with multiple sources of risk. This situation requires the consideration of stochastic dependency structures and correlations between the corresponding processes. To model this stochastic dependency structure, we can restrict ourselves to the process for the short rate r and the processes for the stock prices S_i, each of whose volatility is driven by a Brownian motion. Since the processes of the bond prices and the bank account are derived from the short rate process and therefore do not include a separate source of risk, they need not be considered here.

For this purpose, the stochastic differential equations in (4.1) and (4.39) defining the corresponding processes r and $S_1, \ldots, S_{n_E}$ will not be considered separately, but will be modelled as the following system of correlated SDEs:

$$\begin{bmatrix} dS_1 \\ \vdots \\ dS_{n_E} \\ dr \end{bmatrix} = \begin{bmatrix} \mu_1 S_1 \\ \vdots \\ \mu_{n_E} S_{n_E} \\ \kappa(\theta - r) \end{bmatrix} dt + \begin{bmatrix} \sigma_1 S_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \sigma_{n_E} S_{n_E} & 0 \\ 0 & \cdots & 0 & \sigma \end{bmatrix} \begin{bmatrix} dW_1^S \\ \vdots \\ dW_{n_E}^S \\ dW^r \end{bmatrix}. \tag{4.47}$$

Here,

$$W := \begin{bmatrix} W_1^S \\ \vdots \\ W_{n_E}^S \\ W^r \end{bmatrix}$$

is a $(n_E + 1)$-dimensional column vector of correlated Brownian motions with correlation matrix

$$\hat{C} := \begin{pmatrix} 1 & \rho_{1,2}^S & \cdots & \rho_{1,n_E}^S & \rho_1^{S,r} \\ \rho_{1,2}^S & 1 & \cdots & \rho_{2,n_E}^S & \rho_2^{S,r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1,n_E}^S & \rho_{2,n_E}^S & \cdots & 1 & \rho_{n_E}^{S,r} \\ \rho_1^{S,r} & \rho_2^{S,r} & \cdots & \rho_{n_E}^{S,r} & 1 \end{pmatrix}.$$

The correlation coefficients in $\hat{C}$ are given by

$$\rho_{i,j}^S := \mathbb{C}\mathrm{orr}\Big(W_i^S(t), W_j^S(t)\Big) \in [-1, 1],$$
$$\rho_i^{S,r} := \mathbb{C}\mathrm{orr}\Big(W_i^S(t), W^r(t)\Big) \in [-1, 1],$$

with $i, j = 1, \ldots, n_E$ and $0 < t \leq T$.

In order to construct and simulate such a multidimensional Brownian motion, it should be noted first that, as a correlation matrix, $\hat{C}$ is symmetric with real entries (hence also Hermitian) and positive semi-definite, see Horn/Johnson [11]. For this model, we demand that $\hat{C}$ is even positive definite, otherwise this would imply that there is a linear dependency among the columns of $\hat{C}$ and one of the Brownian motions could be represented as a linear combination of the others and therefore be eliminated. Thus, we can consider the Cholesky decomposition

$$\hat{C} = AA^\top$$

of $\hat{C}$ according to Theorem 2.3.2. Since we assume that $\hat{C}$ is positive definite with real valued entries, A is unique and has positive diagonal entries. We can now use this decomposition of $\hat{C}$ to transform the SDE system (4.47) into a SDE system containing a vector of independent Brownian motions. As shown in Oosterlee/Grzelak [22] by a generalization of a construction of a two-dimensional correlated Brownian motion, we can construct the $(n_E + 1)$-dimensional Brownian motion W as

$$W = A\hat{W}$$

with a $(n_E + 1)$-dimensional Brownian motion $\hat{W} = \left(\hat{W}_1, \ldots, \hat{W}_{n_E+1}\right)^\top$, whose components are independent, one-dimensional Brownian motions and can therefore be simulated component-wise. Thus, we can write the system in (4.47) as

$$\begin{bmatrix} dS_1 \\ \vdots \\ dS_{n_E} \\ dr \end{bmatrix} = \begin{bmatrix} \mu_1 S_1 \\ \vdots \\ \mu_{n_E} S_{n_E} \\ \kappa(\theta - r) \end{bmatrix} dt + \begin{bmatrix} \sigma_1 S_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \sigma_{n_E} S_{n_E} & 0 \\ 0 & \cdots & 0 & \sigma \end{bmatrix} A\, d\hat{W}. \tag{4.48}$$

and obtain a system of SDEs with the required correlation structure given by $\hat{C}$.

4.1.6. Asset allocation, diversification and portfolio composition constraints

In order to cover the actuarial reserves resulting from the insurance portfolio, the insurance company is able to invest in the above modelled financial instruments of a risk-free cash account, n_E different kinds of equities and n_B different kinds of fixed rate bonds, whereby we suppose that financial markets are appropriately liquid, i.e. all assets can be bought and sold at any time, and that all assets are arbitrarily divisible. We consider the model's time mesh $\mathcal{T}$ with $0 = t_0 < t_1 < \cdots < t_N = T$ and define for each $k = 0, \ldots, N-1$ the vector

$$\lambda_k := \left(\lambda_k^C, \lambda_{k,1}^S, \ldots, \lambda_{k,n_E}^S, \lambda_{k,1}^B, \ldots, \lambda_{k,n_B}^B\right) \tag{4.49}$$

of portfolio weights for each asset class, describing the composition of the asset portfolio at time t_k. Here, λ_k^C, $\lambda_{k,i}^S$ and $\lambda_{k,j}^B$ are the proportions of the portfolio's total value that is invested in cash, the i-th type of equity and the j-th type of fixed rate bond at t_k, respectively. We further assume that the investment portfolio is only rebalanced at the time steps of the underlying time mesh, implying that the portfolio composition vectors λ_k remain unchanged within the corresponding time period $[t_k, t_{k+1})$. We also define with

$$\lambda_k^S := \sum_{i=1}^{n_E} \lambda_{k,i}^S,$$
$$\lambda_k^B := \sum_{j=1}^{n_B} \lambda_{k,j}^B$$

the total weights of equities and fixed rate bonds in the composition of the overall portfolio at t_k. Since we do not consider other investment possibilities in this capital market model, the following budget constraint must hold for each $k = 0, \dots, N-1$:

$$\lambda_k^C + \lambda_k^S + \lambda_k^B = \lambda_k^C + \sum_{i=1}^{n_E} \lambda_{k,i}^S + \sum_{j=1}^{n_B} \lambda_{k,j}^B = 1. \tag{4.50}$$

Since portfolio diversification and concentration risk minimization are not within the scope of this model, we assume that all modelled financial instruments are already sufficiently diversified and that the legal requirements in terms of asset diversification, see page 21, are fulfilled at any time. In terms of the modelled assets, we will thus assume entire equity funds or baskets of equities instead of individual equities and bond portfolios instead of bonds issued by the same issuer. For the model itself, this assumption will be taken into account accordingly in the choice of model parameters. For example, in terms of the modelled equities, the model parameters can be chosen in such a way that the dynamics of the price processes correspond to those of broadly diversified equity funds rather than individual stocks, implying a choice of lower trend parameters μ_i, but also lower volatility parameters σ_i.

We additionally place the following conditions on the composition of the portfolio and the asset weights based on economic and legal requirements, which are later included as constraints in the interest risk management optimisation problem:

- Since the Austrian Insurance Supervision Act 2016 (VAG 2016) explicitly prohibits short selling with respect to capital investment in life insurance companies, see page 21, the condition of non-negativity of all weights at all times $t_k,\ k = 0, \dots, N-1$, is required, implying

$$\lambda_k^C,\ \lambda_{k,i}^S,\ \lambda_{k,j}^B \geq 0, \quad i = 1, \dots, n_E,\ j = 1, \dots, n_B. \tag{4.51}$$

- As investments in equities are by nature associated with higher returns but also higher volatility than investments in high-rated bonds, we introduce lower and upper limits $0 \leq \alpha_S^{\min} < \alpha_S^{\max} < 1$ for the proportion of equities in the portfolio in order to keep the overall risk of the portfolio at a prudent and controllable level and to fulfil the obligation to invest in the best interest of all policyholders and beneficiaries, as described on page 19. Accordingly, we consider the condition

$$\alpha_S^{\min} \leq \lambda_k^S \leq \alpha_S^{\max}, \quad k = 0, \dots, N-1 \tag{4.52}$$

  for the total weight of equities in the asset portfolio. This condition also reflects the implementation of a suitable quantitative limit system for relevant concentration risks in the model, see page 21. Since we assume that all modelled assets are already sufficiently diversified and can thus neglect concentration risk within asset classes, no tighter restrictions are placed on the weights within asset classes.

- Similar to the upper condition, we introduce a lower bound $\alpha_C > 0$ for the proportion of the total assets that is held as cash in the bank account, yielding the condition

$$\lambda_k^C \geq \alpha_C, \quad k = 0, \dots, N-1. \tag{4.53}$$

  This should not least provide the insurance company with a liquidity safety buffer in the event of short-term or unforeseen payments. From (4.50) it follows that $\alpha_S^{\min}$ and α_C have to fulfil

$$\alpha_S^{\min} + \alpha_C \leq 1.$$

- We introduce restructuring limits for the rebalancing of the investment portfolio in order to be able to cover specific investment strategies and to regulate how much the portfolio composition may change at each rebalancing. The latter is insofar important as, on real financial markets, any change in the portfolio composition involves transaction costs, which such limits intend to keep below a conscience level. Actual asset transaction costs are not included in the model, as Di Francesco/Simonella [5] point out that a consideration of such costs significantly increases the computational time of such stochastic scenario models, but does not affect the results to a substantial extent. Therefore, we follow the approach in Di Francesco/Simonella [5] and consider the turnover conditions

$$\left|\lambda_k^C - \lambda_{k-1}^C\right| \leq \delta_\lambda, \tag{4.54}$$
$$\left|\lambda_{k,i}^S - \lambda_{k-1,i}^S\right| \leq \delta_\lambda, \quad i = 1, \dots, n_E, \tag{4.55}$$
$$\left|\lambda_{k,j}^B - \lambda_{k-1,j}^B\right| \leq \delta_\lambda, \quad j = 1, \dots, n_B, \tag{4.56}$$

  with an upper bound $\delta_\lambda \in (0,1)$ for the extent an asset weight may change when the portfolio is rebalanced.

4.1.7. Asset portfolio value and interest rate sensitivity

With the given price processes of the assets the insurance company is able to invest in and the constraints on the investment portfolio, we can now derive a recursive formula for the value of the assets at a given time as well as an expression for the interest rate sensitivity of the asset portfolio for the points in time of the underlying time grid $\mathcal{T}$.
To first derive the return of the insurance company's asset portfolio, consider the given returns R_k^C, $R_k^{B_i}$ and $R_k^{S_i}$ for the single asset types as defined in (4.21), (4.33) and (4.43), respectively. The investment return of the entire asset portfolio for the k-th period $[t_{k-1}, t_k)$, depending on the portfolio composition λ_{k-1} within the period, is given as the weighted average of the single asset returns, see Korn [19], yielding

$$R_k^A(\lambda_{k-1}) := \lambda_{k-1}^C \cdot R_k^C + \sum_{i=1}^{n_E} \lambda_{k-1,i}^S \cdot R_k^{S_i} + \sum_{j=1}^{n_B} \lambda_{k-1,j}^B \cdot R_k^{B_j}. \tag{4.57}$$

Consequently, the conditional expectation of the asset portfolio return $R_k^A(\lambda_{k-1})$, given $\mathcal{F}_{t_{k-1}}$, equals

$$\mathbb{E}\Big[R_k^A(\lambda_{k-1})\,\Big|\,\mathcal{F}_{t_{k-1}}\Big] = \lambda_{k-1}^C\,\mathbb{E}\Big[R_k^C\,\Big|\,\mathcal{F}_{t_{k-1}}\Big] + \sum_{i=1}^{n_E}\lambda_{k-1,i}^S\,\mathbb{E}\Big[R_k^{S_i}\,\Big|\,\mathcal{F}_{t_{k-1}}\Big] + \sum_{j=1}^{n_B}\lambda_{k-1,j}^B\,\mathbb{E}\Big[R_k^{B_j}\,\Big|\,\mathcal{F}_{t_{k-1}}\Big], \tag{4.58}$$

with the conditional expectations of the single assets being given by (4.22), (4.35) and (4.44), respectively.

Let now $\mathcal{A}_k$ denote the value of the asset portfolio at time t_k. Note that, as all assets are evaluated using market prices and not book values, $\mathcal{A}_k$ corresponds to the market value of the asset portfolio. We start with an initial value of the asset portfolio at $t_0 = 0$ given by

$$\mathcal{A}_0 := (1+\epsilon_0)\mathcal{L}_0 \tag{4.59}$$

with $\mathcal{L}_0$ denoting the value of the liabilities at $t_0 = 0$ defined by (4.116) and a coverage factor $\epsilon_0 \geq 0$ which determines the equity or surplus of the insurance company at the beginning of the projection. Using the derived investment return above, the value of the assets can be calculated recursively as

$$\mathcal{A}_k = \mathcal{A}_{k-1}\big(1 + R_k^A(\lambda_{k-1})\big) - \sum_{i=1}^{n_L}\Delta A_k^i, \quad k = 1,\ldots,N. \tag{4.60}$$

Here, ΔA_k^i denotes the cash flows that occur for the i-th insurance contract at time t_k as defined in Section 4.2 below and thus $\sum_{i=1}^{n_L}\Delta A_k^i$ equals the total sum of cash flows from the entire insurance contract portfolio occurring at time t_k.

Duration of the asset portfolio

In order to measure the interest rate sensitivity of the asset portfolio's market value, we limit ourselves to the Fisher-Weil duration of that part of the portfolio that is made up of bonds. The reason for this is that, on the one hand, investment portfolios in the traditional life insurance sector are typically largely made up of bonds and, on the other hand, the concept of duration is unsuitable for equities as equities are not dependent on interest rates in the same way and to the same extent as bonds are. The value of the money market account also does not change as a result of a shift in the yield curve, as this only influences the future development of the money market account.

For the i-th bond with maturity T_i^B, coupon rate c_i and price $B_i(t)$ given by (4.27), let $d_t(B_i)$ denote the Fisher-Weil duration of the bond at time $t < T_i^B$. Applying Definition 2.2.7 in Section 2.2.1 to the defined bond model, this duration is given as

$$d_t(B_i) = \frac{1}{B_i(t)}\cdot\left(\sum_{j=1}^{k_B} t_j\cdot c_i\,P(t,t+t_j^c) + T_i^B\cdot P(t,t+T_i^B)\right). \tag{4.61}$$

Applying (2.25), the duration $D_k^{\mathcal{A}}(\lambda_k)$ of the entire bond portfolio given the portfolio composition λ_k, which will be called *asset duration* at time $t_k \in \mathcal{T}$, corresponds to the weighted average of the durations of the single bonds described above, which yields

$$D_k^{\mathcal{A}}(\lambda_k) := \sum_{i=1}^{n_B} \frac{\lambda_{k,i}^B}{\lambda_k^B} \cdot d_{t_k}(B_i), \tag{4.62}$$

where the weight for the i-th bond $\lambda_{k,i}^B / \lambda_k^B$ equals the proportion of the i-th bond on the total bond portfolio at time t_k.

4.2. Liability model

For the liability model in this thesis, we cover the life insurance contract type of endowment insurance. Thus, we deal with insurance contracts with a benefit payment in the event of a premature death of the insured person as well as if a contractually agreed age is reached. With payments to the policyholders or beneficiaries in the event of both death and survival, the expected benefit cash flows are distributed over the entire contract term in contrast to other contract types such as pure endowment contracts, which naturally results in a more complex interest rate risk.

The central modelling approach for the insurance portfolio is based on a discrete-time Markov chain model with a finite state space and time-dependent transition probabilities. This model approach has the advantage that important information necessary for asset-liability management, such as the development of actuarial provisions over time or future expected cash flows, can be efficiently calculated using the Chapman-Kolmogorov equations.

We start with a general description of the Markov model and the key underlying assumptions. Then, the sub-models for mortality, surrender and policyholder contractual options are presented, from which the transition probabilities for the Markov model are derived. We hereby take into account the relevant regulatory provisions for life insurance contracts under Austrian law that were discussed in Chapter 3. Finally, based on the model, the profiles of the expected future cash flows from the insurance portfolio are derived and the interest rate sensitivity of the portfolio is determined.

4.2.1. Markov chain model and general assumptions

To model the insurance contract portfolio and analyse the interest rate risk of the contractual liabilities, we consider an insurance portfolio of n_L homogeneous insurance contracts whose evolution over time is modelled at individual contract level by pairwise independent Markov chains (see Definition 2.1.9). Since the consideration of contract portfolios consisting of identical contracts represents a simplification compared to real portfolios in practice, it would be more precise at this point to speak of a model point instead of an actual portfolio. However, since the central objective of the model is to analyse and manage interest rate risk and as model point aggregation is a common practice in life insurance

mathematics, this approach is followed accordingly.
As stated at the beginning of the chapter, we define the Markov chains over the N-period time grid $\mathcal{T} = \{t_0, \dots, t_N\}$ with

$$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$$

and annual time increments. The time horizon $t_N = T$ denotes the time of the contracts' maturity and is assumed to be integer-valued as a consequence of the annual time increments. After T, all insurance policies have ended and the insurance company is no longer exposed to any insurance risk. Further, we denote with

$$\mathcal{I} := \{0, \dots, N\}$$

the indices of the time grid, which will be the index set for the Markov chains.

Remark. The main reason for annual time increments in this model is the sub-model for mortality, which is defined based on an underlying mortality table with annual time steps. For remaining model components, we aim to define them in a more general way such that the model can be adapted to arbitrary time grids by adapting the mortality model accordingly.

As the state space for the Markov chain model, we consider the set

$$\mathcal{S} := \{*, \dagger, \blacktriangledown, \lozenge_1, \dots, \lozenge_N\} \tag{4.63}$$

with the following states the policyholders can enter during the term of the contract:

- $*$: Active state: prior to contract maturity, the policyholder pays regular premiums; upon maturity, a state-dependent survival benefit is paid out to the policyholder and the contract ends,
- $\dagger$: Decease of the insured person: the death benefit is paid out to the policyholder or the beneficiaries at the end of the period in which the insured person dies and the contract ends,
- $\blacktriangledown$: Policy surrender: the surrender benefit is paid out to the policyholder at the end of the period in which the policyholder surrenders from the contract and the contract ends,
- $\lozenge_j$: Waiver of premium: the policyholder exercises the option to convert the policy into a fully paid-up insurance within the period $[t_{j-1}, t_j)$; after exercising the option, the payment of premiums is suspended until the end of contract; upon maturity, a state-dependent survival benefit is paid out to the policyholder and the contract ends.

The defined state space covers in particular the legally guaranteed options of a policyholder described in Section 3.3.
Given these preparatory measures, we can now define the insurance contract portfolio as a set of n_L Markov chains

$$X^j := \left(X_k^j\right)_{k \in \mathcal{I}}, \quad j = 1, \dots, n_L, \tag{4.64}$$

where $X_k^j,\ k = 0, \dots, N$, describes the state of the j-th policyholder at time $t_k \in \mathcal{T}$. These Markov chains will also be referred to as *life insurance policies* for this chapter. For the beginning of the projection at $t_0 = 0$, we assume that all policyholders are active, that is

$$\mathbb{P}\big[X_0^j = *\big] = 1, \quad j = 1, \dots, n_L. \tag{4.65}$$

Furthermore, we make the following assumptions which remain valid for the entire model:

(A1) The Markov chain processes modelling the insurance policies are independent of the random processes and random variables of the economic model discussed in Section 4.1.

(A2) All insurance policies are independent and have identical structure, i.e. the Markov chains $X^j = \big(X_k^j\big)_{k\in\mathcal{I}}$ are identically distributed and pairwise independent.

We also define the following parameters which apply equally to all insurance policies of the modelled portfolio:

- $x_0 \in \mathbb{N}$: Age of the policyholders at $t_0 = 0$,
- $i_{\min} \geq 0$: Contractually agreed annual actuarial interest rate,
- $\Psi_0^* \geq 0$: Actuarial reserve per insurance contract at $t_0 = 0$, see (4.86),
- $\Gamma_0^* \geq 0$: Bonus account value for profit participation per insurance contract at $t_0 = 0$, see (4.89),
- $\beta \in [0.85, 1]$: Profit participation factor: Proportion of investment gains that the insurance company passes on to policyholders,
- $\Lambda_\dagger \geq 0$: Guaranteed benefit paid in the event of death,
- $\Lambda_* \geq 0$: Guaranteed endowment benefit paid if contract maturity is reached.

By including the value of the actuarial reserve and a bonus account at the beginning of the projection t_0, the model is defined in a way that does not require that t_0 corresponds to the actual starting date of the contracts. Also, note that the profit participation factor is required to be bigger or equal to 85% in accordance with the Austrian legal provisions described in Section 3.3.

We denote with $p_{ij}(k,\ell)$ the time-dependent transition probability from state i at time t_k to state j at time t_ℓ as defined in (2.12) and with $P_X(k,\ell)$ the corresponding transition matrix between t_k and t_ℓ as defined in (2.13). The specific numerical values for the transition probabilities will be defined and described in more detail in the subsections below. For brevity, we denote the one-period transition probabilities with

$$p_{ij}(k) := p_{ij}(k, k+1), \quad k = 0, \dots, N-1,$$

and the one-period transition matrix with

$$P_X(k) := P_X(k, k+1), \quad k = 0, \dots, N-1.$$

Before deriving $P_X(k)$, we will shortly summarize the main dynamics of the state transition. Firstly, the states for surrender $\blacktriangledown$ and decease $\dagger$ are absorbing states, which means that these states can not be left once they have been entered, implying

$$
\begin{aligned}
p_{\blacktriangledown\blacktriangledown}(k) &= 1, \quad k = 0, \dots, N-1, \\
p_{\dagger\dagger}(k) &= 1, \quad k = 0, \dots, N-1.
\end{aligned}
$$

Secondly, by definition, state $\Diamond_j$ can only be reached if the policyholder exercises the premium waiver option within the period $[t_{j-1}, t_j)$, thus implying

$$p_{*\Diamond_j}(k) = 0$$

for $j \neq k+1$. Additionally, we require

$$p_{\Diamond_j\Diamond_j}(k) = 1, \quad j > k,$$

to ensure that the transition matrix $P_X(k)$ is a stochastic matrix, see Section 2.1.1 . Note, however, that state $\Diamond_j$ cannot be reached before time t_k for $j > k$.
Thirdly, it is not possible to become active again after exercising the premium waiver option, that is, for any $j = 1, \dots, N$,

$$p_{\Diamond_j *}(k) = 0, \quad k = 0, \dots, N-1.$$

Remark. The third characteristic stated above does not necessarily have to apply to contracts in practice, as contracts may well include a policyholder's right to change back to the active state once exercising the premium waiver option. But since such a right is not guaranteed by Austrian life insurance law and is also not very common in practise, it is not included in this model. In principle, however, an extension of the model by such a right would be possible through a corresponding extension of the state space, for example by considering the state space

$$\mathcal{S} \cup \left\{ *_{i,j} \;\middle|\; i < j,\ i = 1, \dots, N-1,\ j = 2, \dots, N \right\},$$

whereby $*_{i,j}$ denotes a state where the policyholder exercises the premium waiver option in the i-th period and reactivates in the j-th period.

Based on these model characteristics and the given state space, the one-period transition matrix of the Markov chain for the period $[t_k, t_{k+1})$ is given as

$$
P_X(k) = \tag{4.66}
$$

$$
\begin{array}{c|cccccccccc}
\text{i,j} & * & \blacktriangledown & \dagger & \Diamond_1 & \dots & \Diamond_k & \Diamond_{k+1} & \Diamond_{k+2} & \dots & \Diamond_N \\
\hline
* & p_{**}(k) & p_{*\blacktriangledown}(k) & p_{*\dagger}(k) & 0 & \dots & 0 & p_{*\Diamond_{k+1}}(k) & 0 & \dots & 0 \\
\blacktriangledown & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\dagger & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\Diamond_1 & 0 & p_{\Diamond_1\blacktriangledown}(k) & p_{\Diamond_1\dagger}(k) & p_{\Diamond_1\Diamond_1}(k) & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\
\Diamond_k & 0 & p_{\Diamond_k\blacktriangledown}(k) & p_{\Diamond_k\dagger}(k) & 0 & 0 & p_{\Diamond_k\Diamond_k}(k) & 0 & 0 & \dots & 0 \\
\Diamond_{k+1} & 0 & \dots & \dots & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\
\Diamond_{k+2} & \vdots & & & & & \vdots & 0 & \ddots & & \vdots \\
\vdots & \vdots & & & & & \vdots & \vdots & & \ddots & \vdots \\
\Diamond_N & 0 & \dots & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 1
\end{array}
.
$$

The described Markov chain model can be illustrated graphically as a time-dependent weighted graph, where the vertices represent the states and the edges represent the corresponding transition probabilities. For the model here, this so-called transition diagram for period $[t_k, t_{k+1})$ with $j \leq k$ and $i > k+1$ can be displayed as

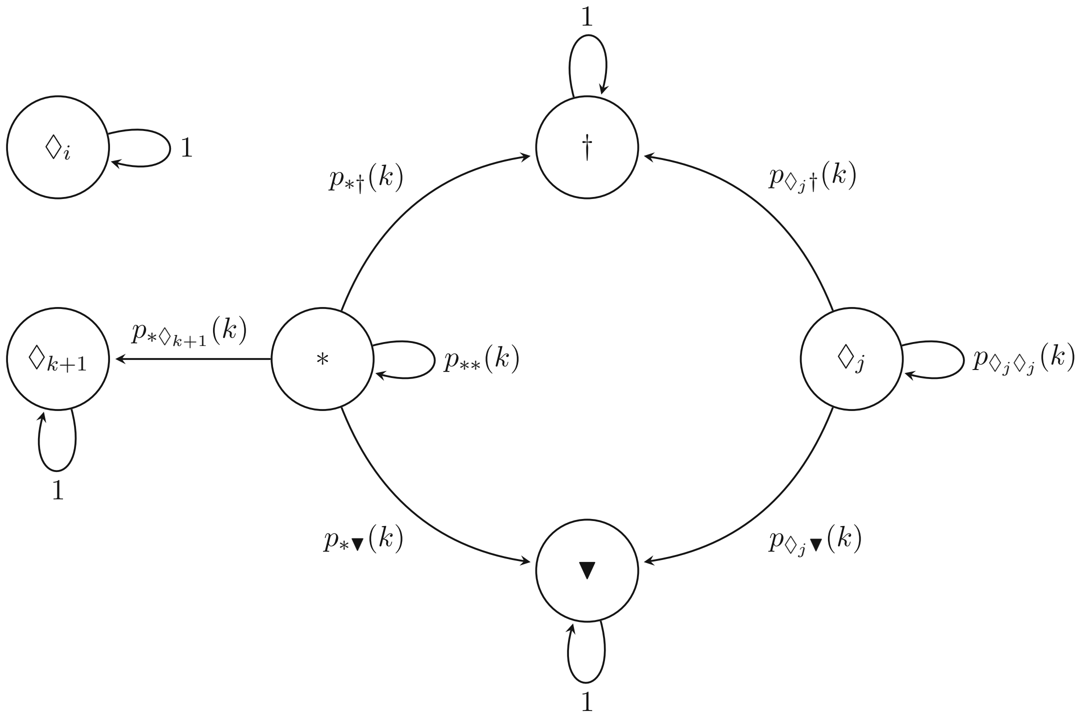


Figure 4.5.: Transition diagram of the Markov chain model

4.2.2. Transition probabilities of the Markov chain model

Given the general structure of the Markov chain model, we will now derive the one-period transition probabilities given in (4.66) based on the underlying models described below. We will first focus on the mortality model and the models for the probabilities of surrender and exercise of the premium waiver option separately and then combine the models for the derivation of the conditional transition probabilities. In this section, we will only derive the one-period transition probabilities, as the multi-period transition probabilities can then be calculated using the Chapman-Kolmogorov equations, see Theorem 2.1.11.

Mortality model

For the consideration of mortality, a deterministic, biometric approach is applied in this thesis, where the probabilities of survival and death of a policyholder of a fixed age are derived from an underlying life table. For the numerical results in Chapter 5 in particular, we will use the smoothed unisex life table for Austria for the years 2020–2022, see Table A.1. We will not go into the deduction of such probabilities here and refer to standard life insurance literature such as Gerber [8] and Ortmann [23].

Using standard notation of life insurance mathematics, we denote with q_x the probability that a policyholder of age x dies within one year. This probability is directly provided by the life table. The counter-probability $p_x := 1 - q_x$ thus denotes the probability that a policyholder of age x survives at least one year.
By applying this to the Markov model and assuming that the probability of death is independent of whether the premium waiver option has been exercised or not, it yields for the transition probabilities that

$$p_{*\dagger}(k) = q_{x_0+k}, \tag{4.67}$$
$$p_{\Diamond_j\dagger}(k) = q_{x_0+k}, \ j \leq k, \tag{4.68}$$

with x_0 being a policyholder's age at $t_0 = 0$.

Surrender and premium waiver model

To model the probabilities that a policyholder surrenders or exercises the premium waiver option, we follow the approaches in Di Francesco/Simonella [5] and Férnandez et al. [6] and link the probabilities to the excess return that the insurance company offers to policyholders over an investment return of a benchmark investment opportunity. The benchmark for this model, as proposed in Férnandez et al. [6], is the investment in the risk-free money market account defined in Section 4.1.2. Following this approach, the benchmark return for the k-th period $[t_{k-1}, t_k)$ is given as R_k^C as defined in (4.21) with its conditional expectation given by (4.22).

A policyholder's return from the life insurance contract for the k-th period is given by $i_{\min} + \max\left(\beta \cdot (R_k^A - i_{\min}), 0\right)$, see Section 4.2.3, with the insurance company's asset return R_k^A for the k-th period as defined in (4.57) and $i_{\min}$ and β as defined in Section 4.2.1. Thus, the corresponding excess return compared to the benchmark is given as

$$\Delta R_k := \max\Big(i_{\min} + \max\left(\beta \cdot (R_k^A - i_{\min}), 0\right) - R_k^C, 0\Big), \quad k = 1, \ldots, N. \tag{4.69}$$

Following Di Francesco/Simonella [5], we introduce K threshold intervals for ΔR_k as

$$\begin{aligned} I_1 &:= [0, \xi_1), \\ I_2 &:= [\xi_1, \xi_2), \\ &\vdots \\ I_K &:= [\xi_{K-1}, \infty) \end{aligned}$$

with $0 < \xi_1 < \cdots < \xi_{K-1} < \infty$. The probabilities for surrender and the exercise of the premium waiver option now depend on ΔR_k and the period k. If the excess return ΔR_k falls into the threshold interval I_j, the conditional probabilities that a an active policyholder exercises the premium waiver option or surrenders within $[t_k, t_{k+1})$, given the survival of the policyholder, are denoted by q_{jk}^{PW} and q_{jk}^{S}, respectively. We do not differentiate in the surrender probabilities as to whether the policyholder is active or in the premium waiver state. The given probabilities are specified and pre-defined in the following table:

	Period				
	$[t_0, t_1)$	$[t_1, t_2)$	$[t_2, t_3)$	$\dots$	$[t_{N-1}, t_N)$
Intervals					
I_1	$q_{10}^S,\, q_{10}^{PW}$	$q_{11}^S,\, q_{11}^{PW}$	$q_{12}^S,\, q_{12}^{PW}$	$\dots$	$q_{1N-1}^S,\, q_{1N-1}^{PW}$
I_2	$q_{20}^S,\, q_{20}^{PW}$	$q_{21}^S,\, q_{21}^{PW}$	$q_{22}^S,\, q_{22}^{PW}$	$\dots$	$q_{2N-1}^S,\, q_{2N-1}^{PW}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
I_K	$q_{K0}^S,\, q_{K0}^{PW}$	$q_{K1}^S,\, q_{K1}^{PW}$	$q_{K2}^S,\, q_{K2}^{PW}$	$\dots$	$q_{KN-1}^S,\, q_{KN-1}^{PW}$

Table 4.1.: Surrender and premium waiver option probabilities

A necessary condition that has to be fulfilled for all $i = 1, \dots, K$ and $k = 0, \dots, N-1$ is

$$q_{ik}^S + q_{ik}^{PW} \leq 1.$$

Additionally, a reasonable but not necessary restriction on the choice of these probabilities is that, for $i > j$, the probabilities fulfil

$$\begin{aligned} q_{ik}^S &\leq q_{jk}^S, \\ q_{ik}^{PW} &\leq q_{jk}^{PW}, \end{aligned}$$

as this implies that the higher the return from the insurance contract, the lower the probabilities that a policyholder exercises one of the given options.
Note that, as ΔR_0 cannot be defined from (4.69), the model requires an initial choice for the threshold interval for $t_0 = 0$, which should reflect the policyholders' initial expectation of the excess return.

One-period transition probabilities

Combining the models for mortality and the policyholder options above, the time-dependent one-period transition probabilities for the k-th period $[t_{k-1}, t_k)$, provided that $\Delta R_{k-1} \in I_i$, are given as

$$p_{**}(k) = p_{x_0+k} \cdot \left(1 - q_{ik}^{S} - q_{ik}^{PW}\right), \tag{4.70}$$

$$p_{*\Diamond_j}(k) = \begin{cases} p_{x_0+k} \cdot q_{ik}^{PW} & \text{if } \ j = k+1 \\ 0 & \text{else,} \end{cases} \tag{4.71}$$

$$p_{*\blacktriangledown}(k) = p_{x_0+k} \cdot q_{ik}^{S}, \tag{4.72}$$

$$p_{*\dagger}(k) = q_{x_0+k} = 1 - p_{x_0+k}, \tag{4.73}$$

$$p_{\blacktriangledown\blacktriangledown}(k) = 1, \tag{4.74}$$

$$p_{\dagger\dagger}(k) = 1, \tag{4.75}$$

$$p_{\Diamond_j\Diamond_j}(k) = \begin{cases} p_{x_0+k} \cdot \left(1 - q_{ik}^{S}\right) & \text{if } \ j \leq k \\ 1 & \text{else,} \end{cases} \tag{4.76}$$

$$p_{\Diamond_j\blacktriangledown}(k) = \begin{cases} p_{x_0+k} \cdot q_{ik}^{S} & \text{if } \ j \leq k \\ 0 & \text{else,} \end{cases} \tag{4.77}$$

$$p_{\Diamond_j\dagger}(k) = \begin{cases} q_{x_0+k} & \text{if } \ j \leq k \\ 0 & \text{else.} \end{cases} \tag{4.78}$$

All remaining one-period transition probabilities are zero.

4.2.3. Contract cash flows and profit participation mechanism

To model the cash flows arising from the insurance contracts, the Markov model is extended by corresponding policy functions. First, we define these functions for the cash flows arising solely from the insurance contract without profit participation and then we define the profit participation mechanism and the policy functions extended by the cash flows arising from this mechanism. In terms of the cash flows themselves, we focus on premium and benefit cash flows, whereby premiums are paid at the beginning and benefits (endowment, death and surrender benefits) are paid at the end of a corresponding time period. Acquisition or administration expenses from the insurance contracts are not included in this model. Following the notation and results in Koller [17] and Koller [18], for $i, j \in \mathcal{S}$ and $k = 0, \ldots, N$ we define the policy functions

- $a_i^{Pre}(k)$: Cash flow at time t_k if the insured is in state i at time t_k,

and for $k = 0, \ldots, N-1$ we define the policy functions

- $a_{ij}^{Post}(k)$: Cash flow at the end of the period $[t_k, t_{k+1})$ if the insured switches from state i at time t_k to state j at time t_{k+1}.

We will only describe the policy functions that are non-trivial, i.e. all policy functions that are not explicitly mentioned below are constant zero.

Premium cash flows and endowment benefit

If the insured person is active at the beginning of the k-th period, a possibly time-dependent annual premium $P_k \geq 0$ is paid at the beginning of the period. If a policyholder is in state $\Diamond_j$, i.e. has exercised the premium waiver option, no premiums are paid. We assume that P_k only reflects the savings premium and that the risk premium is paid at the beginning of the contract in the form of a single premium. As the share of the risk premium in the total premium is generally small and therefore does not have a major influence on the present value or the duration of future cash flows, this approach is considered sufficient. However, as the savings portion of a constant periodically paid premium decreases with the increasing age of a policyholder as the risk of death increases, see Gerber [8], a time-dependent premium pattern can be applied to reflect this aspect.

In case the insured person reaches maturity, i.e. the insured person is alive at time $t_N = T$ and the contract has not terminated before, a guaranteed endowment benefit Λ_* is paid at time t_N. If the policyholder has exercised the premium waiver option within the contract term, this benefit is reduced accordingly so that the amount of the reduced benefit is consistent with the non-payment of premiums after exercising the option. The legal requirements in this case, see Section 3.3, require that the reduced benefit is calculated according to the accepted rules of actuarial mathematics and is consistent with the premium calculation principles used. As we do not cover such principles in this thesis, we take this fact into account by introducing deduction factors

$$\gamma_i^{PW} \in [0,1], \ i = 1, \dots, N,$$

that are dependent on the period in which the option is exercised and assume that these factors reflect the reduction appropriately.

These considerations result in the policy functions

$$a_*^{Pre}(k) = \begin{cases} -P_k, & k = 0, \dots, N-1 \\ \Lambda_*, & k = N, \end{cases} \tag{4.79}$$

$$a_{\Diamond_j}^{Pre}(k) = \begin{cases} 0, & k = 0, \dots, N-1 \\ (1-\gamma_j^{PW})\Lambda_* & k = N. \end{cases} \tag{4.80}$$

Death and surrender benefit cash flows

If the insured is active or in the premium waiver state at time t_k and dies within the period $[t_k, t_{k+1})$, the guaranteed death benefit sum $\Lambda_\dagger$ is paid at the end of the period, yielding for the policy functions

$$a_{*\dagger}^{Post}(k) = \Lambda_\dagger, \quad k = 0, \dots, N-1, \tag{4.81}$$
$$a_{\Diamond_j \dagger}^{Post}(k) = \Lambda_\dagger, \quad k = 0, \dots, N-1. \tag{4.82}$$

Note that due to the assumption that the entire risk premium for the contract is paid as a single premium at the inception of the policy, the death benefit is not reduced if a policyholder exercises the premium waiver option.

In case of a policyholder's surrender, the policyholder is paid a time- and state-dependent surrender value, which is calculated in accordance with the legal requirements as the current value of the insurance at the end of the respective insurance period, taking into account an appropriate cancellation deduction $\gamma^S \in [0,1]$. The current value at $t_k \in \mathcal{T}$ for a given state $i \in \mathcal{S}$ corresponds to the prospective reserve $V_i(t_k, A)$ as defined in (4.104) in Section 4.2.4 below and it follows for the policy functions that

$$a_{*\blacktriangledown}^{Post}(k) = (1-\gamma^S) \cdot V_*(t_{k+1}, A), \quad k = 0, \dots, N-1, \tag{4.83}$$
$$a_{\Diamond_j \blacktriangledown}^{Post}(k) = (1-\gamma^S) \cdot V_{\Diamond_j}(t_{k+1}, A), \quad k = 0, \dots, N-1. \tag{4.84}$$

As legal provisions require the deduction factor to be appropriate, a restriction to $\gamma^S \in [0,\, 0.05]$ is considered more adequate.

Remark. For consistency, the policy functions involving the premium waiver states, namely (4.80), (4.82) and (4.84), are defined for all k including $k < j$. However, as state $\Diamond_j$ cannot be reached before t_j by definition, the assigned values of the corresponding policy function values for $k < j$ have no actual relevance.

Profit participation mechanism

As described in Section 3.3, in the case of profit participation contracts, life insurance companies are obliged to allocate a minimum of 85% of the minimum assessment basis to the provision for profit-dependent premium reimbursement. This mechanism is modelled here in a simplified way by linking the profit participation to an excess interest rate compared to the contractually guaranteed minimum interest rate $i_{\min}$. For a given period $[t_{k-1}, t_k)$, the interest rate of return offered to policyholders for their capital, including the participation on the insurance company's gains from capital investments, is given by

$$z_k := i_{\min} + \max\left(\beta \cdot (R_k^A - i_{\min}),\, 0\right). \tag{4.85}$$

Thus, policyholders partly participate in the excess return of the insurance company's investment return R_k^A over the minimum rate $i_{\min}$ in the extent of $\beta \geq 0.85$, if this excess return is positive. Otherwise, the policyholders receive the minimum interest rate $i_{\min}$. In order to derive the resulting profit participation cash flows for an insurance policy, we follow the approach in Gerstner et al. [9] and Gerstner et. al [10] and apply this to the underlying Markov model. We first introduce the actuarial reserve

$$\Psi^i = \left(\Psi_k^i\right)_{k \in \mathcal{I}}, \quad i \in \{*, \Diamond_1, \dots, \Diamond_N\}, \tag{4.86}$$

where Ψ_k^i denotes the value of the actuarial reserve at time $t_k \in \mathcal{T}$ for a policy in state i and corresponds to the accumulated premiums paid before t_k, accrued with the actuarial interest rate $i_{\min}$. If $t_0 = 0$ corresponds to the contract start date of the insurance contracts, which the model a priori does not demand, then $\Psi_0^* = 0$. In the case the contract start lies before t_0 and premiums have been paid before, the initial value Ψ_0^* as a model parameter should reflect these accumulated and accrued premiums prior to t_0.

Given the initial value Ψ_0^*, the state-dependent actuarial reserves develop deterministically according to the following recursions:

$$\Psi_k^* = \left(\Psi_{k-1}^* + P_{k-1}\right) \cdot (1 + i_{\min}), \quad k = 1, \dots, N, \tag{4.87}$$

$$\Psi_k^{\Diamond_j} = \begin{cases} 0, & k = 0, \dots, j-1 \\ \left(\Psi_{k-1}^* + P_{k-1}\right) \cdot (1 + i_{\min}), & k = j, \\ \Psi_{k-1}^{\Diamond_j} \cdot (1 + i_{\min}), & k = j+1, \dots, N. \end{cases} \tag{4.88}$$

Remark. As mentioned above, if t_0 does not correspond to the time of the insurance contracts' start, then Ψ_0^* depends on the contract term that has already elapsed between the start of the contracts and t_0 as well as the premiums that have already been paid before t_0. To keep the number of model parameters small, we do not make any assumptions or introduce further parameters for the elapsed time and past premiums, but rather work with the initial value Ψ_0^* as a single parameter. Nevertheless, given the information on the number of years n between the contract start and t_0 as well as the historically paid premiums $P_{-n}, \dots, P_{-1}$ for these past years, the calculation of the initial value Ψ_0^* is straightforward as it is easy to see from the recursive definition of Ψ_k^* in (4.87) that

$$\Psi_0^* = \sum_{j=1}^{n} P_{-j} \, (1 + i_{\min})^j.$$

Next, we introduce state-dependent bonus accounts

$$\Gamma^i = \left(\Gamma_k^i\right)_{k \in \mathcal{I}}, \quad i \in \{*, \Diamond_1, \dots, \Diamond_N\}, \tag{4.89}$$

where Γ_k^i reflects the sum of all bonuses allocated to an insurance policy up to time $t_k \in \mathcal{T}$ if the insured is in state i at time t_k. With the given initial value Γ_0^* for the bonus accounts at $t_0 = 0$ as a model parameter, their values are recursively defined by

$$\Gamma_k^* = (1 + z_k) \cdot \Gamma_{k-1}^* + (z_k - i_{\min}) \cdot \left(P_{k-1} + \Psi_{k-1}^*\right) \quad k = 1, \dots, N, \tag{4.90}$$

$$\Gamma_k^{\Diamond_j} = \begin{cases} 0, & k = 0, \dots, j-1 \\ (1 + z_k) \cdot \Gamma_{k-1}^* + (z_k - i_{\min}) \cdot \left(P_{k-1} + \Psi_{k-1}^*\right), & k = j, \\ (1 + z_k) \cdot \Gamma_{k-1}^{\Diamond_j} + (z_k - i_{\min}) \cdot \Psi_{k-1}^{\Diamond_j}, & k = j+1, \dots, N. \end{cases} \tag{4.91}$$

For each time step, the preceding bonus account value is compounded with the policyholder return z_k and increased by the excess interest above the guaranteed interest rate on the preceding actuarial reserve and the last premium, provided that the policyholder was active at t_{k-1} and paid the premium P_{k-1}. Note that state $\Diamond_j$ can only be reached if the insured changes from the active state $*$ to state $\Diamond_j$ within the time period $[t_{j-1}, t_j)$, hence the policyholder's bonus account value at t_{j-1} equals Γ^*_{j-1} and also the premium P_{j-1} was still paid, leading to (4.91) above.

Remark. Similar to the initial value for the actuarial reserve Ψ^*_0, if the contracts started before t_0, the initial value for the bonus account Γ^*_0 depends on the past asset returns, for which we do not make any assumptions and therefore include it as a given model parameter. If t_0 corresponds to the contracts' starting date, then we require $\Gamma^*_0 = 0$.

If a benefit (death, endowment or surrender) is paid, the corresponding value of the bonus account is also paid out in addition to the guaranteed benefits described above, which leads to the following extended policy functions:

$$\hat{a}^{Pre}_{*}(k) = \begin{cases} -P_k, & k = 0, \dots, N-1 \\ \Lambda_* + \Gamma^*_N, & k = N, \end{cases} \tag{4.92}$$

$$\hat{a}^{Pre}_{\Diamond_j}(k) = \begin{cases} 0, & k = 0, \dots, N-1 \\ (1 - \gamma^{PW}_j)\Lambda_* + \Gamma^{\Diamond_j}_N & k = N, \end{cases} \tag{4.93}$$

$$\hat{a}^{Post}_{*\dagger}(k) = \Lambda_\dagger + \Gamma^*_k, \quad k = 0, \dots, N-1, \tag{4.94}$$

$$\hat{a}^{Post}_{\Diamond_j\dagger}(k) = \Lambda_\dagger + \Gamma^{\Diamond_j}_k, \quad k = 0, \dots, N-1, \tag{4.95}$$

$$\hat{a}^{Post}_{*\blacktriangledown}(k) = (1 - \gamma^S) \cdot \left(V_*(t_{k+1}, A) + \Gamma^*_k\right), \quad k = 0, \dots, N-1, \tag{4.96}$$

$$\hat{a}^{Post}_{\Diamond_j\blacktriangledown}(k) = (1 - \gamma^S) \cdot \left(V_{\Diamond_j}(t_{k+1}, A) + \Gamma^{\Diamond_j}_k\right), \quad k = 0, \dots, N-1. \tag{4.97}$$

Note that, unlike the initial policy functions above, these policy functions are random variables as at a time t_k, the future values of the bonus accounts Γ^i_ℓ, $\ell > k$, are dependent on future asset returns and are thus uncertain.

4.2.4. Prospective reserve, liability value and interest rate sensitivity

Given the liability model's structure, we are now able to derive the prospective reserve, the liability value and the duration of an insurance contract in the portfolio. The results on single contracts will then be applied to derive the liability value and the duration of the full contract portfolio at a given time $t_k \in \mathcal{T}$.

To formalise the derivation of future expected contractual cash flows from the life insurance portfolio, following Koller [17], we define the following indicator functions:

Definition 4.2.1. Let $(X_n)_{n\geq 0}$ be a discrete-time Markov chain talking values in a countable set $\mathcal{S}$. For $j \in \mathcal{S}$, the indicator function for the Markov chain at step k is defined as

$$\mathbb{I}_j(k)(\omega) := \begin{cases} 1 & \text{if } X_k(\omega) = j \\ 0 & \text{if } X_k(\omega) \neq j \end{cases} \tag{4.98}$$

For $i, j \in \mathcal{S}$, let $N_{ij}(k)$ denote the number of jumps from state i to state j up to time step k, that is

$$N_{ij}(k)(\omega) := \#\big\{\ell \in \{1, \dots, k\} : X_{\ell-1}(\omega) = i \wedge X_\ell(\omega) = j\big\}. \tag{4.99}$$

Now, we can define the stochastic cash flow profile of accumulated contractual cash flows of a life insurance policy without profit participation:

Definition 4.2.2. For a life insurance policy $X = (X_k)_{k\in\mathcal{I}}$ with state space $\mathcal{S}$ and policy functions a_i^{Pre} and a_{ij}^{Post}, let $\big(A(k)\big)_{k\in\mathcal{I}} \in L^2(\Omega, \mathcal{F}, \mathbb{P})^{N+1}$ be the random process such that $A(k)$ represents the accumulated cash flows (premium and benefit payments) induced by the life insurance policy up to and including time t_k. We will call A the *cash flow profile* of X. Further, let $A_{ij}(k)$ denote the accumulated cash flows up to time t_k induced by transitions from state i to j and let $A_j(k)$ denote the accumulated cash flows up to time t_k by being in state j.

For a given cash flow profile, the cash flows occurring at time $t_k,\ k > 0$, consequently correspond to the increment $\Delta A(k) := A(k) - A(k-1)$, which can be written as

$$\Delta A(k)(\omega) = \sum_{j\in S} \Delta A_j(k)(\omega) + \sum_{i,j\in S} \Delta A_{ij}(k)(\omega) \tag{4.100}$$

with

$$\Delta A_j(k)(\omega) = \mathbb{I}_j(k)(\omega) \cdot a_j^{Pre}(k), \tag{4.101}$$

$$\Delta A_{ij}(k)(\omega) = \Delta N_{ij}(k)(\omega) \cdot a_{ij}^{Post}(k-1), \tag{4.102}$$

where $\Delta N_{ij}(k) := N_{ij}(k) - N_{ij}(k-1)$. For $t_0 = 0$, as all policyholders are active, we have $A(0) = P_0$ and define $\Delta A(0) := -P_0$.

Prospective reserve of an insurance contract

With the given setting, we can now calculate the prospective reserve of the insurance portfolio, defined as the present value of the future expected contractual cash flows induced by the insurance contracts given the information at a certain point in time. The prospective reserve does not fully reflect the present value of all cash flows induced by a contract as future profit participation cash flows are not included, but it corresponds to the benefit paid to a policyholder in the event of surrender, see (4.83) and (4.84).

Definition 4.2.3. With the notation and assumptions above, the *prospective value of future cash flows* at a time $t_k \in \mathcal{T}$ of an insurance policy $X = (X_k)_{k\in\mathcal{I}}$ with cash flow profile $A = \big(A(k)\big)_{k\in\mathcal{I}}$ is defined as

$$V^+(t_k, A) := \sum_{i\in\mathcal{S}} \Delta A_i(k) + \sum_{\ell=k+1}^{N} P(t_k, t_\ell)\, \Delta A(\ell). \tag{4.103}$$

The *prospective reserve* at time $t_k \in \mathcal{T}$ for a given state $j \in \mathcal{S}$ is defined as

$$V_j^+(t_k, A) := \mathbb{E}\big[V^+(t_k, A) \,\big|\, X_k = j\big]. \tag{4.104}$$

By definition, $V_j^+(t_k, A)$ corresponds to the present value of expected future contractual cash flows occurring after t_k, including any cash flows that occur at time t_k resulting from the policyholder being in state j at that time. Deriving the prospective reserve by direct calculation and discounting of the expected future cash flows yields the following formula for $V_j^+(t_k, A)$:

Lemma 4.2.4. *Given an arbitrary state $j \in \mathcal{S}$ and time $t_k \in \mathcal{T}$, it holds for the prospective reserve that*

$$\begin{aligned} V_j^+(t_k, A) = a_j^{Pre}(k) + \sum_{\ell=k+1}^{N} \mathbb{E}\big[P(t_k, t_\ell)\big] \cdot \Bigg(&\sum_{g \in \mathcal{S}} a_g^{Pre}(\ell) \cdot p_{jg}(k, \ell) \\ &+ \sum_{g,h \in \mathcal{S}} a_{gh}^{Post}(\ell - 1) \cdot p_{jg}(k, \ell - 1) \cdot p_{gh}(\ell - 1) \Bigg), \end{aligned} \tag{4.105}$$

where $\mathbb{E}\big[P(t_k, t_\ell)\big]$ is given as in (4.13).

Proof. From the independence of the stochastic short rate process r and the Markov chain X, see assumption (A1) on page 52, as well as linearity of expectation, it follows that

$$\begin{aligned} V_j^+(t_k, A) &= \mathbb{E}\big[V^+(t_k, A) \,\big|\, X_k = j\big] \\ &= \mathbb{E}\Bigg[\sum_{i \in \mathcal{S}} \Delta A_i(k) + \sum_{\ell=k+1}^{N} P(t_k, t_\ell)\, \Delta A(\ell) \,\Bigg|\, X_k = j\Bigg] \\ &= \sum_{i \in \mathcal{S}} \mathbb{E}\big[\Delta A_i(k) \,\big|\, X_k = j\big] + \sum_{\ell=k+1}^{N} \mathbb{E}\big[P(t_k, t_\ell)\big] \cdot \mathbb{E}\big[\Delta A(\ell) \,\big|\, X_k = j\big]. \end{aligned}$$

By the definition of $\Delta A_i(k)$, the first sum above simplifies to

$$\begin{aligned} \sum_{i \in \mathcal{S}} \mathbb{E}\big[\Delta A_i(k) \,\big|\, X_k = j\big] &= \sum_{i \in \mathcal{S}} a_i^{Pre}(k) \cdot \mathbb{E}\big[\mathbb{I}_i(k) \,\big|\, X_k = j\big] \\ &= a_j^{Pre}(k) \end{aligned}$$

Further, it follows from (4.100) that

$$\begin{aligned} \mathbb{E}\big[\Delta A(\ell) \,\big|\, X_k = j\big] &= \mathbb{E}\Bigg[\sum_{g \in \mathcal{S}} \Delta A_g(\ell) + \sum_{g,h \in \mathcal{S}} \Delta A_{gh}(\ell) \,\Bigg|\, X_k = j\Bigg] \\ &= \sum_{g \in \mathcal{S}} \mathbb{E}\big[\Delta A_g(\ell) \,\big|\, X_k = j\big] + \sum_{g,h \in \mathcal{S}} \mathbb{E}\big[\Delta A_{gh}(\ell) \,\big|\, X_k = j\big] \\ &= \sum_{g \in \mathcal{S}} a_g^{Pre}(\ell) \cdot \mathbb{E}\big[\mathbb{I}_g(\ell) \,\big|\, X_k = j\big] + \sum_{g,h \in \mathcal{S}} a_{gh}^{Post}(\ell - 1) \cdot \mathbb{E}\big[\Delta N_{gh}(\ell) \,\big|\, X_k = j\big]. \end{aligned}$$

By the definitions of $\mathbb{I}_g(\ell)$ and $\Delta N_{gh}(\ell)$, it is easy to see that for $\ell > k$ it holds

$$\mathbb{E}\big[\mathbb{I}_g(\ell)\,\big|\,X_k = j\big] = p_{jg}(k,\ell),$$
$$\mathbb{E}\big[\Delta N_{gh}(\ell)\,\big|\,X_k = j\big] = p_{jg}(k,\ell-1)\cdot p_{gh}(\ell-1).$$

Thus, combining the results, it yields that

$$V_j^+(t_k, A) = a_j^{Pre}(k) + \sum_{\ell=k+1}^{N} \mathbb{E}\big[P(t_k, t_\ell)\big]\cdot\Bigg(\sum_{g\in\mathcal{S}} a_g^{Pre}(\ell)\cdot p_{jg}(k,\ell) + \sum_{g,h\in\mathcal{S}} a_{gh}^{Post}(\ell-1)\cdot p_{jg}(k,\ell-1)\cdot p_{gh}(\ell-1)\Bigg).$$

□

The multi-period transition probabilities occurring in formula (4.105) for $V_j^+(t_k, A)$ can be efficiently calculated using the Chapman-Kolmogorov equation, since it follows inductively from Theorem 2.1.11 that, for $\ell > k$,

$$P_X(k,\ell) = \prod_{m=k}^{\ell-1} P_X(m, m+1) = \prod_{m=k}^{\ell-1} P_X(m), \tag{4.106}$$

where the one-period transition probabilities in $P_X(m)$ are given as defined in Section 4.2.2 above.

Note that, since the prospective reserve reduced by a deduction is paid to the policyholder in the event of surrender, it follows from (4.105) that for the calculation of $V_j^+(t_k, A)$, the prospective reserves for $V_i^+(t_{k+1}, A), \ldots, V_i^+(t_N, A)$ need to be calculated in advance for all $i \in \mathcal{S}$. This makes it necessary to calculate the prospective reserves backward in time, starting with the boundary conditions

$$V_*^+(t_N, A) = \Lambda_* \tag{4.107}$$
$$V_{\Diamond_j}^+(t_N, A) = \left(1 - \gamma_j^{PW}\right)\Lambda_*, \;\; j = 1, \ldots, N \tag{4.108}$$
$$V_\dagger^+(t_N, A) = 0 \tag{4.109}$$
$$V_\blacktriangledown^+(t_N, A) = 0. \tag{4.110}$$

Remark. A more efficient way to calculate the prospective reserves for a discrete Markov chain model is the application of a system of coupled backward recursions called *Thiele's difference equations*. As shown in Koller [17], the prospective reserves for the given discrete time Markov model satisfy the recursions

$$V_i^+(t_k, A) = a_i^{Pre}(k) + \sum_{j\in\mathcal{S}} \mathbb{E}\big[P(t_k, t_{k+1})\big]\, p_{ij}(k)\left(a_{ij}^{Post}(k) + V_j^+(t_{k+1}, A)\right). \tag{4.111}$$

The boundary conditions for the recursions are the same as in (4.107) – (4.110) above. Nevertheless, the direct calculation approach of Lemma 4.2.4 is more suitable in the context of this thesis as it includes the individual expected future cash flows and will therefore be the basis for deriving the liability value and duration of an insurance contract.

Single policy liability value and insurance contract portfolio liability value

The fair value of the liability that arises from holding an insurance contract corresponds to the present value of all future cash flows that are induced by the contract. Having defined the prospective reserve of a contract at a given time and state covering all contractual cash flows, we will now extend this by the expected cash flows from the profit participation mechanism described in Section 4.2.3. The liability value as well as the duration for the entire portfolio has to be estimated at every time step of the model at which the asset portfolio is rebalanced to minimize the interest rate risk.
By adapting the resulting formula for the prospective reserve in (4.105) and replacing the initial policy functions $a_i^{Pre}(k)$ and $a_{ij}^{Post}(k)$ with the expected extended policy functions $\hat{a}_i^{Pre}(k)$ and $\hat{a}_{ij}^{Post}(k)$ given the information at t_k, we define the liability value $\mathcal{L}_k^j$ at time $t_k \in \mathcal{T}$ of an insurance policy, given the policyholder is in state j, as

$$\mathcal{L}_k^j := \sum_{\ell=k+1}^{N} P(t_k, t_\ell) \cdot \Bigg(\sum_{g \in \mathcal{S}} \mathbb{E}\big[\hat{a}_g^{Pre}(\ell) \,\big|\, \mathcal{F}_{t_k}\big] \cdot p_{jg}(k, \ell) + \sum_{g,h \in \mathcal{S}} \mathbb{E}\big[\hat{a}_{gh}^{Post}(\ell-1) \,\big|\, \mathcal{F}_{t_k}\big] \cdot p_{jg}(k, \ell-1) \cdot p_{gh}(\ell-1) \Bigg). \tag{4.112}$$

Note that, in contrast to the prospective reserve, we hereby only include cash flows that occur strictly after t_k, thus the first term $a_j^{Pre}(k)$ from (4.105) is not included. Also, $P(t_k, t_\ell)$ is $\mathcal{F}_{t_k}$-measurable and, due to the properties of conditional expectation, it holds almost surely that

$$\mathbb{E}\big[P(t_k, t_\ell) \,\big|\, \mathcal{F}_{t_k}\big] = P(t_k, t_\ell).$$

In particular, when $\mathcal{L}_k^j$ is evaluated at time t_k, the zero coupon bond prices $P(t_k, t_\ell)$, $\ell > k$, are not random any more as the short rate value $r(t_k)$ is known at that time.

To derive the liability value in this form, for each $t_k \in \mathcal{T}$, it is necessary to estimate the future bonus account values

$$\Gamma_n^i, \quad n > k, \; i \in \{*, \Diamond_1, \dots, \Diamond_N\},$$

which essentially reduces to estimating the expected future policyholder returns z_n, $n > k$, i.e.

$$\mathbb{E}\big[i_{\min} + \max\big(\beta \cdot (R_n^A - i_{\min}), 0\big) \,\big|\, \mathcal{F}_{t_k}\big], \quad n > k, \; i \in \{*, \Diamond_1, \dots, \Diamond_N\}.$$

For this estimation, we employ a Monte Carlo integration approach. Given the information at t_k, especially the information on $r(t_k)$, $S_1(t_k), \dots, S_{n_E}(t_k)$, the portfolio composition λ_{k-1} and the bonus account values Γ_k^i, $i \in \{*, \Diamond_1, \dots, \Diamond_N\}$, we simulate N_Γ observations of the future asset returns

$$R_{k+1}^A(\lambda_{k-1}), \dots, R_N^A(\lambda_{k-1})$$

under the assumption that the portfolio composition is held constant as λ_{k-1} as this reflects the insurance company's investment strategy before the new portfolio restructuring at t_k takes place. Given the simulated returns, the expected future values of $z_n,\ n > k$, are estimated as

$$\bar{z}_n := \frac{1}{N_\Gamma} \sum_{\ell=1}^{N_\Gamma} \Big(i_{\min} + \max\big(\beta \cdot (R^A_{n,\ell} - i_{\min}),\, 0\big) \Big),$$

where $R^A_{n,\ell}$ denotes the ℓ-th realization of R^A_n from the Monte Carlo simulation. The estimations for the future bonus account values $\Gamma^i_n,\ n > k$, are then derived using the estimations of $z_n,\ n > k$, and the recursive definition of the bonus accounts in (4.90) and (4.91).

Having the single contract liability values, in order to compute the liability value of the entire contract portfolio, information about the structure of the insurance portfolio over time is required, i.e. the number of policyholders in each state of $\mathcal{S}$ at each point in time of the underlying time grid $\mathcal{T}$. For that, we introduce the random variables

$$\ell^j_k := \sum_{i=1}^{n_L} \mathbb{1}_{\left\{X^i_k = j\right\}}, \quad j \in \mathcal{S},\ k \in \mathcal{I}, \tag{4.113}$$

counting the number of policyholders in state $j \in \mathcal{S}$ at time $t_k \in \mathcal{T}$, and the vectors

$$\ell_k := \big(\ell^*_k, \ell^\blacktriangledown_k, \ell^\dagger_k, \ell^{\Diamond_1}_k, \ldots, \ell^{\Diamond_N}_k\big), \quad k \in \mathcal{I}, \tag{4.114}$$

describing the structure of the insurance portfolio at time $t_k \in \mathcal{T}$. For these portfolio structure vectors, it obviously holds for every $k \in \mathcal{I}$ that

$$\sum_{j \in \mathcal{S}} \ell^j_k = n_L.$$

Given the structure of the insurance contract portfolio at a time $t_k \in \mathcal{T}$, the liability value $\mathcal{L}_k$ of the full contract portfolio at t_k corresponds to the sum of the liability values of the single contracts and thus is defined as

$$\mathcal{L}_k = \sum_{j \in \mathcal{S}} \ell^j_k \cdot \mathcal{L}^j_k. \tag{4.115}$$

Remark. As we assume that all n_L insurance policies are in the active state at the beginning of the projection, we have the initial portfolio structure given as

$$\ell_0 = (n_L, 0, \ldots, 0)$$

and the liability value of the contract portfolio at $t_0 = 0$ given as

$$\mathcal{L}_0 = n_L \cdot \mathcal{L}^*_0. \tag{4.116}$$

Single policy duration and insurance contract portfolio duration

To investigate the interest rate sensitivity of an insurance contract liability at a given time, we derive the Fisher-Weil duration of the expected future cash flows at a given time using Definition 2.2.7 and the given formula for $\mathcal{L}_k^j$ above. Analogous to the derivation of the duration of a fixed rate bond in Section 4.1.7, weighting the expected future cash flows with the remaining term until their payment times, the Fisher-Weil duration $D_{k,j}^{\mathcal{L}}$ of a single insurance policy at time $t_k \in \mathcal{T}$, provided the insured is in state j, is given as

$$D_{k,j}^{\mathcal{L}} := \frac{1}{\mathcal{L}_k^j} \cdot \left(\sum_{\ell=k+1}^{N} (t_\ell - t_k) \cdot P(t_k, t_\ell) \cdot \left(\sum_{g \in \mathcal{S}} \mathbb{E}\left[\hat{a}_g^{Pre}(\ell) \,\middle|\, \mathcal{F}_{t_k}\right] \cdot p_{jg}(k, \ell) \right.\right. \\ \left.\left. + \sum_{g,h \in \mathcal{S}} \mathbb{E}\left[\hat{a}_{gh}^{Post}(\ell - 1) \,\middle|\, \mathcal{F}_{t_k}\right] \cdot p_{jg}(k, \ell - 1) \cdot p_{gh}(\ell - 1) \right)\right). \tag{4.117}$$

Using (2.25), the duration of the full contract portfolio $D_k^{\mathcal{L}}$ at time $t_k \in \mathcal{T}$, which we will call the *liability duration at* t_k, is given as the weighed sum of the state-dependent durations defined above. The weights at a given time equal the contributions of the liability values of all contracts in state j to the liability value of the full contract portfolio, which yields

$$D_k^{\mathcal{L}} := \sum_{j \in \mathcal{S}} \frac{\ell_k^j \cdot \mathcal{L}_k^j}{\mathcal{L}_k} D_{k,j}^{\mathcal{L}}. \tag{4.118}$$

4.3. Two-stage stochastic duration matching for interest rate risk management

Applying the models for the assets and liabilities defined in the sections above, the simplified market value balance sheet of the insurance company at a time $t_k \in \mathcal{T}$ is given accordingly:

Assets	**Liabilities and Equity**
– Market value of the asset portfolio $\mathcal{A}_k$	– Present value of the insurance liabilities $\mathcal{L}_k$
	– Equity capital

The shareholders' equity capital at time t_k equals

$$\mathcal{A}_k - \mathcal{L}_k$$

as the company's total assets minus its total liabilities. Both the market values of assets and liabilities are highly dependent on the interest rate environment and may react very differently to yield curve movements, with the asset duration $D_k^{\mathcal{A}}$ and the liability duration $D_k^{\mathcal{L}}$ as the quantifying risk measures for these sensitivities. In a typical cash flow profile of an endowment insurance contract, the timing of the benefit cash outflows to policyholders tends to have a higher weighting towards the end of the contract term due to the endowment benefit payments at maturity. Thus, they are discounted more heavily in the calculation of the liability value than premium cash inflows at earlier stages of the contract term. In

the event of an upward shift of the underlying yield curve, the present value of future cash outflows increases more steeply than than the present value of future cash inflows decreases, leading to an overall increase in the liability value. At the same time, the prices of fixed rate bonds given by (4.27) fall due to the shift in yield curve as future coupon and maturity cash flows are discounted more strongly. In case of a downward shift of the yield curve, the effects move in the respective opposite direction. If there is a material mismatch between the sensitivities of asset and liability values to changes in the yield curve, a shortfall in the coverage of liabilities by the assets and further an increased risk of insolvency can be the result.
The main objective in this model to manage the risk discussed is therefore to minimise the mismatch between the interest rate sensitivities of assets and liabilities using a duration matching approach. In Führer [7], this approach is described as an immunisation strategy to immunise the technical provisions against the interest rate risk and to structure the asset portfolio in such a way that the changes in the value of assets and liabilities due to interest rate shifts are as parallel as possible. Referring to the discussion in Chapter 3, we therefore deal with a situation in which the liabilities are given as a constant factor and optimise the capital investment, resulting in the direction of impact 'Liabilities control Assets'.

The general idea of a duration matching approach is to construct a portfolio of assets whose duration matches that of the liabilities, as described in Führer [7] or Koller [17]. In Di Francesco/Simonella [5], this concept is extended by additionally requiring the asset return to match the return of a separately modelled benchmark return. The ALM approach in this thesis comprises a dynamic investment portfolio rebalancing at every time step $t_k \in \mathcal{T}$ as the result of a two-stage optimisation procedure. In the first stage, the optimisation objective is to obtain an asset portfolio composition that minimises the difference between the asset and liability duration under the constraints described in Section 4.1.6. Given the minimal possible gap between the duration of assets and liabilities, the objective of the second stage is to derive a portfolio composition with maximized expected return, such that the asset-liability duration gap lies within a specified accepted deviation from the minimum gap of stage one. This approach thus covers two key aspects of a life insurance company with regard to capital investment: Firstly, the interest rate risk minimisation of the first stage reflects the obligation of a life insurance company to invest the assets covering the technical provisions in a manner that is appropriate to the nature and duration of the insurance liabilities, see Section 3.3.1. Secondly, the objective of the second stage serves the interests of policyholders and shareholders to obtain a competitive return on their capital.

Remark. As duration matching is primarily a short- to medium-term ALM method, it may not be reasonable to solve the optimisation problem up to the maturity of the insurance contracts as the stochastic projection and optimisation require high computational capacities and the results for longer time horizons are less meaningful due to the underlying model uncertainty. Therefore, it may be appropriate to introduce an optimisation horizon $\hat{N} \leq N$ and to consider the dynamic multi-periodic ALM problem only for the time steps $t_0, \ldots, t_{\hat{N}}$.

At this point it is necessary to emphasise that, as discussed in Chapter 3, duration matching is by no means a comprehensive solution to ALM. The approach described here primarily aims to investigate portfolio structures that minimise the difference between the asset and liability duration and the development of such optimal portfolios over time.

4.3.1. First stage ALM optimisation

At a time step $t_k,\ k > 0$, the first objective is to minimise the interest rate risk. Given the liability duration $D_k^{\mathcal{L}}$ and the durations of the individual types of fixed rate bonds at t_k, the first stage problem consists of finding an asset portfolio composition vector λ_k such that the duration gap between assets and liabilities is minimised. Thus, we consider the following non-linear optimisation problem with constraints:

Optimisation problem 1

$$\min_{\lambda_k} \quad \left|D_k^{\mathcal{A}}(\lambda_k) - D_k^{\mathcal{L}}\right| \tag{4.119}$$

$$\text{subject to} \quad \begin{cases} \lambda_k^C,\ \lambda_{k,i}^S,\ \lambda_{k,j}^B \geq 0,\ i = 1, \dots, n_E,\ j = 1, \dots, n_B, \\ \lambda_k^C + \lambda_k^S + \lambda_k^B = 1, \\ \alpha_S^{\min} \leq \lambda_k^S \leq \alpha_S^{\max}, \\ \lambda_k^C \geq \alpha_C, \\ \left|\lambda_k^C - \lambda_{k-1}^C\right| \leq \delta_\lambda, \\ \left|\lambda_{k,i}^S - \lambda_{k-1,i}^S\right| \leq \delta_\lambda,\ i = 1, \dots, n_E, \\ \left|\lambda_{k,j}^B - \lambda_{k-1,j}^B\right| \leq \delta_\lambda,\ j = 1, \dots, n_B, \end{cases} \tag{4.120}$$

Given a solution $\tilde{\lambda}_k$ of the optimisation problem above, the minimal attainable duration gap between assets and liabilities, given the constraints, is denoted by

$$\Delta D_k := \left|D_k^{\mathcal{A}}(\tilde{\lambda}_k) - D_k^{\mathcal{L}}\right|. \tag{4.121}$$

Remark. The solution of Optimisation problem 1 is not necessarily unique, but there always exists a solution that minimises the objective (4.119) and fulfils the given constraints. To see this, consider the set $\Theta \subseteq \mathbb{R}^{n_E+n_B+1}$ of all portfolio composition vectors that fulfil (4.120). The set Θ is non-empty, as $\lambda_{k-1} \in \Theta$, as well as a compact subset of the euclidean space $\mathbb{R}^{n_E+n_B+1}$. Further, it is easy to see that the minimisation objective depends continuously on λ_k, implying that the set

$$\left\{\left|D_k^{\mathcal{A}}(\lambda_k) - D_k^{\mathcal{L}}\right| : \lambda_k \in \Theta\right\}$$

as the image of a compact set under the continuous function

$$\begin{cases} \Theta & \to \quad \mathbb{R} \\ \lambda_k & \mapsto \quad \left|D_k^{\mathcal{A}}(\lambda_k) - D_k^{\mathcal{L}}\right| \end{cases}$$

is itself compact in $\mathbb{R}$, see Kaltenbäck [14]. This allows the application of the Extreme value theorem, which guarantees the existence of a corresponding minimum.

4.3.2. Second stage ALM optimisation

Let the minimal duration gap ΔD_k at a time t_k be given from the first optimisation stage. In the second stage we take into account that, in addition to managing the interest rate risk, it is also the responsibility of an insurance company to design its investment strategy in such a way that a competitive investment return can be achieved. Thus, we do not only accept portfolio compositions that minimise the duration gap, but also compositions with an asset-liability duration gap within a specified range from the minimal gap ΔD_k. Hence, we introduce a maximum accepted absolute deviation $\epsilon \geq 0$ from the minimal gap and accept portfolio composition vectors with a duration gap within the interval

$$\Delta D_k^{\epsilon} := \Big[\Delta D_k - \epsilon,\, \Delta D_k + \epsilon\Big].$$

Here, we choose this boundary to be constant for all time periods. However, as the liability duration of the contract portfolio decreases over the term of the contracts, a time-dependent definition of this maximum deviation could be a useful extension to the model.

Within this restriction, we search for a portfolio vector that provides the maximised expected return of the asset portfolio for the subsequent time period. Formally, this leads to the following problem for the second stage:

Optimisation problem 2

$$\max_{\lambda_k} \quad \mathbb{E}\Big[R_{k+1}^{A}(\lambda_k)\,\Big|\,\mathcal{F}_{t_k}\Big] \tag{4.122}$$

$$\text{subject to} \quad \begin{cases} \left|D_k^{\mathcal{A}}(\lambda_k) - D_k^{\mathcal{L}}\right| \in \Delta D_k^{\epsilon}, \\ \text{constraints in (4.120).} \end{cases} \tag{4.123}$$

Both optimisation problems are solved numerically in the implementation. The obtained solution of this optimisation problem is then chosen as the asset portfolio composition for the period $[t_k, t_{k+1})$.
Note that, as the duration is measured in units of years, see Section 2.2.1, the duration gap is also measured in years. Therefore, the choice of an absolute maximum deviation rather than a relative deviation is considered appropriate for this optimisation approach. The approach of choosing a maximum relative deviation $\tilde{\epsilon}$ from the minimal duration gap ΔD_k has the disadvantage that if the liability duration at a given time t_k can be perfectly or almost perfectly replicated by an appropriate asset portfolio, i.e. $\Delta D_k \approx 0$, the relative interval around the minimal duration gap

$$\Big[\Delta D_k\,(1-\tilde{\epsilon}),\, \Delta D_k\,(1+\tilde{\epsilon})\Big]$$

gets very small or even reduces to the single element ΔD_k itself, making the second stage of optimisation redundant as the first constraint in (4.123) forces the solution to identical or almost identical to the solution obtained in the first optimisation stage.

Analogous to the remark for stage one, it is easy to see that a solution of the optimisation problem exists. Also, note that the parameter ϵ significantly influences the individual importance of the two respective optimisation stages in the overall optimisation process, as with $\epsilon = 0$, the focus lies on an optimal duration matching, whereas with $\epsilon \to \infty$, the focus is solely on a maximisation of the expected investment return.

5. Implementation and numerical results

In this concluding chapter, we briefly discuss the mathematical background for numerical path simulation of continuous-time asset processes that is applied for the implementation of this model and present some numerical results of the model for a chosen scenario. The model is implemented in Python and the code can be found in Appendix B.

5.1. Path simulation of stochastic differential equations

In order to model the stochastic differential equations from the asset model in Chapter 4 numerically, the method of *Euler* (or *Euler-Maruyama*) *discretization* is applied in this thesis to simulate the corresponding paths of the SDEs. We will start with the Euler discretization of a Brownian motion and then proceed to the simulation of general stochastic differential equations, based on Seydel [25].

5.1.1. Euler discretization for Brownian motion

Let $\Delta t > 0$ be a constant time increment, $N \in \mathbb{N}$ and $W = (W(t))_{t \geq 0}$ be a standard Brownian motion. For a discrete time grid $\{t_j = j \cdot \Delta t \mid j = 0, \ldots, N\}$ and a given time t_k of this time grid, the value of $W(t_k)$ can be written as the sum of its past increments:

$$W(t_k) = \sum_{j=1}^{k} \underbrace{\big(W(t_j) - W(t_{j-1})\big)}_{=:\Delta W_j},$$

with $\Delta W_j \sim \mathcal{N}(0, \Delta t)$ by the property of a Brownian motion that its increments over disjoint increments are independent and normally distributed (see Definition 2.1.3). Thus, by simulating N independent normally distributed random variables

$$Z_j \sim \mathcal{N}(0, \Delta t), \quad j = 1, \ldots, N,$$

a standard Brownian motion can be simulated via the recursive formula

$$\begin{aligned} W(t_0) &= 0 \\ W(t_j) &= W(t_{j-1}) + Z_j, \quad j = 1, \ldots, N. \end{aligned}$$

Based on simulated one-dimensional standard Brownian motions, n-dimensional Brownian motions with independent components can then easily be simulated by simulating each of the n components separately as a one-dimensional Brownian motions using the upper recursion. This n-dimensional Brownian motion can then be transformed according to Section 4.1.5 to obtain a simulation of a correlated n-dimensional Brownian.

5.1.2. Euler discretization for stochastic differential equations

Given the time grid $\{t_j = j \cdot \Delta t \mid j = 0, \dots, N\}$ with constant time increment Δt, a general stochastic differential equation

$$dX(t) = \alpha\big(t, X(t)\big)\, dt + \sigma\big(t, X(t)\big)\, dW(t), \quad X(t_0) = X_0, \tag{5.1}$$

can be simulated by combining the discretization

$$\Delta X(t_k) = \alpha\big(t_k, X(t_k)\big)\, \Delta t + \sigma\big(t_k, X(t_k)\big)\, \Delta W(t_k)$$

and the approximated standard Brownian motion, more precisely, its increments, from the section before. Thus, an SDE of the form (5.1) can be simulated using the Euler discretization by an implementation of the recursion

$$\begin{aligned} &X(t_0) = X_0 \\ &X(t_j) = X(t_{j-1}) + \alpha\big(t_j, X(t_j)\big)\, \Delta t + \sigma\big(t_j, X(t_j)\big)\, \Delta W(t_j), \quad j = 1, \dots, N. \end{aligned}$$

5.2. Numerical results

For the numerical results, we consider an insurance portfolio of $n_L = 1000$ homogenous endowment insurance contracts at their inception time with a contract maturity of $T = 20$ years. The asset portfolio is dynamically rebalanced at every time step with regard to the optimisation problem in Section 4.3 in order to offset the corresponding interest rate risk. For the presented scenario, the model was parametrised as follows:

Asset model parameters

- Short rate model parameters:
  - $r_0 = 0.03$
  - $\kappa = 1.2$
  - $\theta = 0.02535$
  - $\sigma = 0.004$
- Bond model parameters:
  - $n_B = 5$ fixed rate bonds:

Fixed rate bond B_i	Coupon rate c_i	Bond maturity T_i^B
B_1	0.027	5 years
B_2	0.027	10 years
B_3	0.028	15 years
B_4	0.028	20 years
B_5	0.026	25 years

  - Annual coupon payments with $k_B = 25$ coupon payment times

$$\left\{t_j^c \,|\, j = 1, \dots, k_B\right\} = \{1, \dots, 25\}.$$

- Stock price model parameters:
  - $n_E = 1$
  - $\mu_1 = 0.09060$
  - $\sigma_1 = 0.15140$
- Correlation matrix:

$$\hat{C} = \begin{pmatrix} 1 & \rho_1^{S,r} \\ \rho_1^{S,r} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}$$

- Initial portfolio composition λ_0 at $t_0 = 0$:

$$\lambda_0 = \left(\lambda_0^C,\ \lambda_{0,1}^S,\ \lambda_{0,1}^B,\ \lambda_{0,2}^B,\ \lambda_{0,3}^B,\ \lambda_{0,4}^B,\ \lambda_{0,5}^B\right)$$
$$= \left(0.03,\ 0.08,\ 0.02,\ 0.05,\ 0.16,\ 0.17,\ 0.49\right)$$

- Initial surplus of assets over liabilities:
  - $\epsilon_0 = 0.05$
- Asset portfolio composition constraints:
  - Equity weight boundaries: $\left[\alpha_S^{\min},\ \alpha_S^{\max}\right] = \left[0.02,\ 0.12\right]$
  - Cash weight boundary: $\alpha_C = 0.02$
  - Single weight turnover boundary: $\delta_\lambda = 0.06$

Liability model parameters

- Initial policyholder age: $x_0 = 45$
- Contractually guaranteed interest rate: $i_{\min} = 0.00$
- Initial actuarial reserve: $\Psi_0 = 0$
- Initial bonus account value: $\Gamma_0^* = 0$
- Profit participation factor: $\beta = 0.90$
- Guaranteed benefits: $\Lambda_* = \Lambda_\dagger = 30,000$ EUR
- Premiums: Combination of single premium at $t_0 = 0$ and recurring annual premium payments until contract maturity:
  - $P_0 = 12,595$ EUR

  - $P_k = 595$ EUR, $k = 1, \dots, N-1$
- Surrender deduction factor: $\gamma^S = 0.05$
- Premium waiver deduction factors:

i	γ_i^{PW}	i	γ_i^{PW}	i	γ_i^{PW}	i	γ_i^{PW}
1	0.44	6	0.31	11	0.20	16	0.08
2	0.42	7	0.29	12	0.17	17	0.06
3	0.39	8	0.27	13	0.15	18	0.04
4	0.36	9	0.24	14	0.13	19	0.02
5	0.34	10	0.22	15	0.10	20	0.00

- Surrender and premium waiver probabilities:
  - Threshold intervals:

$$
\begin{aligned}
I_1 &= \big[0,\, 0.01\big) \\
I_2 &= \big[0.01,\, 0.02\big) \\
I_3 &= \big[0.02,\, \infty\big)
\end{aligned}
$$

  - Premium waiver probabilities:

Periods	0 – 4	5 – 9	10 – 15	15 – 18	19
Intervals					
I_1	0.013	0.016	0.019	0.022	0.000
I_2	0.007	0.010	0.013	0.016	0.000
I_3	0.005	0.008	0.011	0.014	0.000

  - Surrender probabilities:

Periods	0 – 4	5 – 9	10 – 15	15 – 18	19
Intervals					
I_1	0.0065	0.0080	0.0095	0.0110	0.0000
I_2	0.0035	0.0050	0.0065	0.0080	0.0000
I_3	0.0025	0.0040	0.0055	0.0070	0.0000

- Accepted deviation from optimal duration gap: $\epsilon = 0.3$

In total, 1000 independent simulations of the multi-period scenario were generated, of which we will now present the average results. Here, we focus on investigating the development of the asset portfolio structure, the value of assets and liabilities and their durations over time.

In terms of the optimised asset portfolio composition, the expectation is that in the earlier stages of the insurance contract term, bonds with a higher duration are more heavily weighted in order to match the high liability duration at contract inception, and with increasing time, bonds with a lower duration increase in their weights as the liability duration is increasing the closer the contracts come to the contract maturity T.

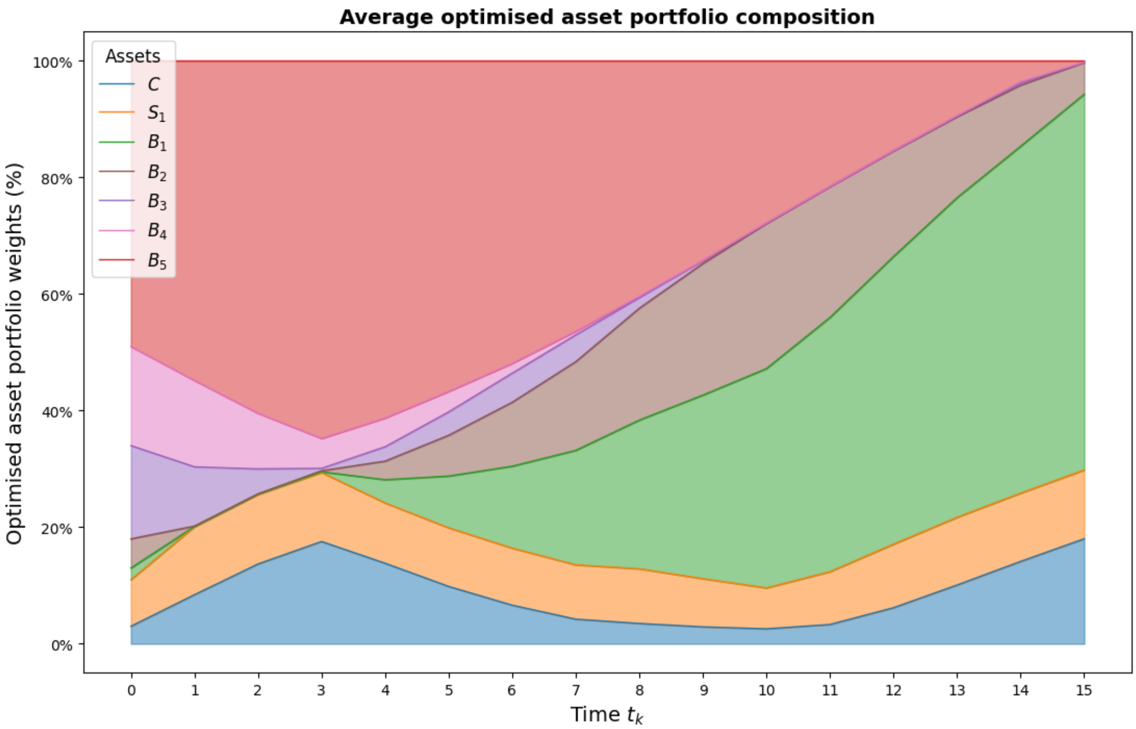


Figure 5.1.: Average optimal asset portfolio composition over time

In Figure 5.1, which shows the average development of the optimised portfolio composition over time, it can be clearly seen that this is the case in the sense that the bond with the highest duration, namely bond B_5, makes the largest contribution to the portfolio within the first time periods, with a maximum average weighting of 64,7% at time t_3. With increasing time and consequently an ongoing decrease in the liability duration, bonds with a lower duration, in particular the bonds B_1 and B_2, increase in their weights significantly to compensate for the decreasing but still high duration share of the asset portfolio through bond B_5. This development can be seen even more clearly in the following Figure 5.2, showing the optimised single asset weights over time for all available assets.

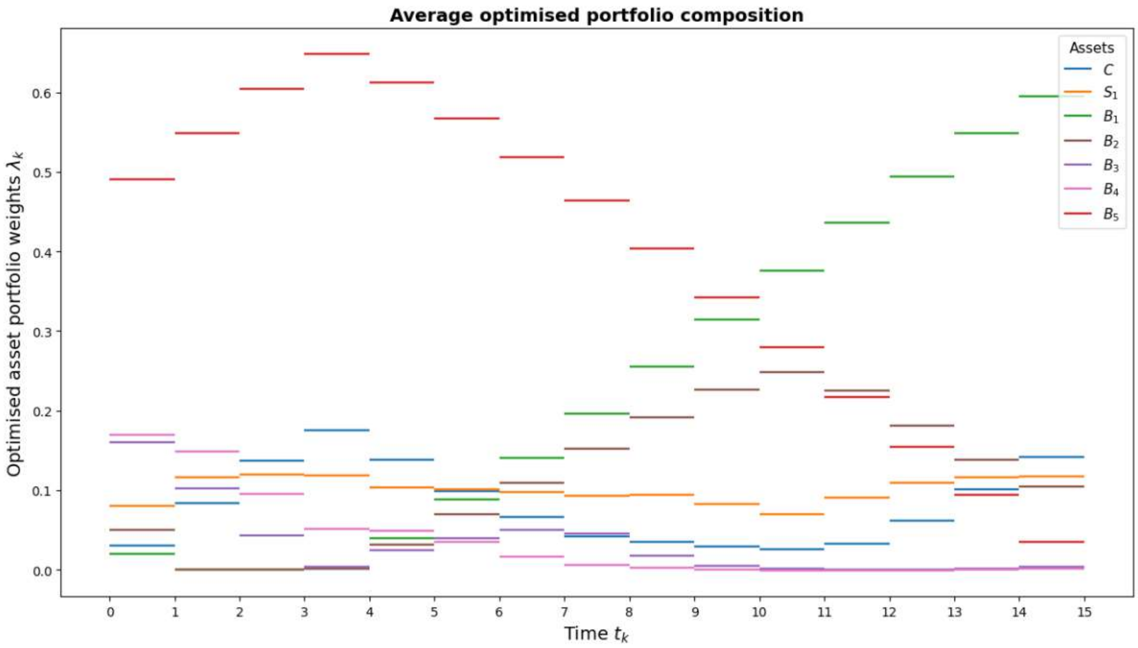


Figure 5.2.: Average optimal asset weights over time

It is observable that the fulfilment of the optimisation problem in this scenario is mainly covered by a time-dependent combination of bonds B_1, B_2 and B_5, as already described above. Bonds B_3 and B_4 with medium duration play a very subordinate role with almost continuously falling weights, which are approximately zero for both bonds from time t_{10} onwards. The weight of the equity investment S_1 remains relatively stable around 10%.

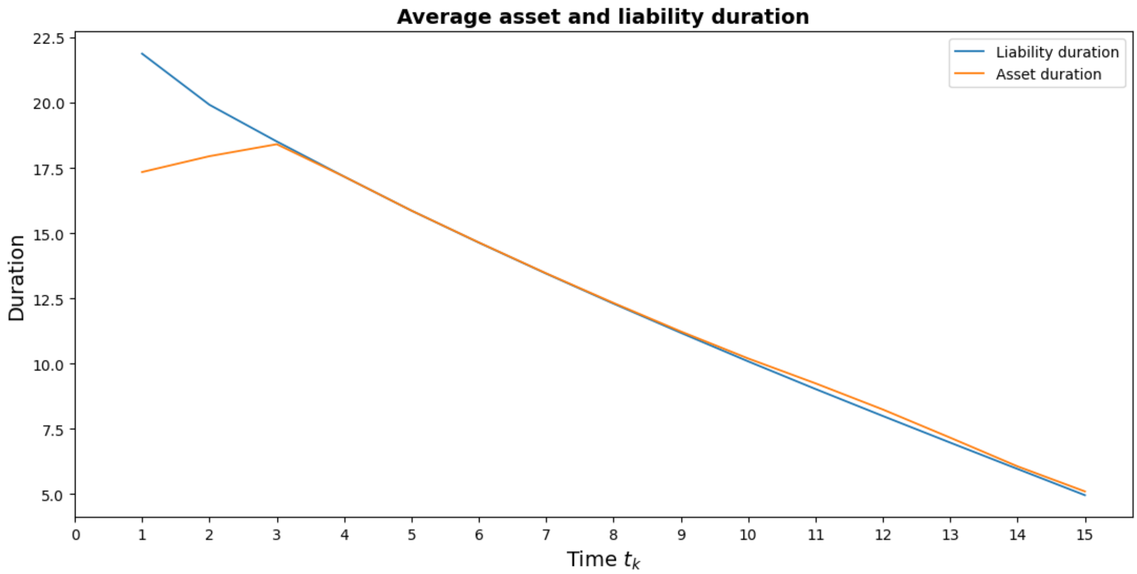


Figure 5.3.: Average asset and liability duration over time

Due to the initial choice λ_0 of the asset composition, the liability duration is not matched at the beginning of the projection, see Figure 5.3, leading to a gap between the asset and liability duration. Thus, within the first periods, a gradual development of the asset weights towards a portfolio whose duration replicates that of the liabilities can be observed. The speed of this convergence of the initial portfolio towards a duration replicating portfolio is here determined by the turnover constraint, only allowing the weights to change up to the specified extent δ_λ from one time step to the next. Within the first three time periods $[t_0, t_3]$, see Figure 5.2, the weights of the bonds $B_1, \ldots, B_4$ decrease while the weight of bond B_5 rises, leading to an increase in the asset duration. From t_3 onwards, the liability duration can be matched by the corresponding asset portfolio within the accepted deviation ϵ from the perfect duration match. For this scenario, the initial portfolio composition was consciously chosen not to be optimal in order to be able to see the convergence towards an optimal portfolio.

In terms of the development of the average asset and liability value over time, Figure 5.4 shows that both are steadily increasing on average within the projection horizon.

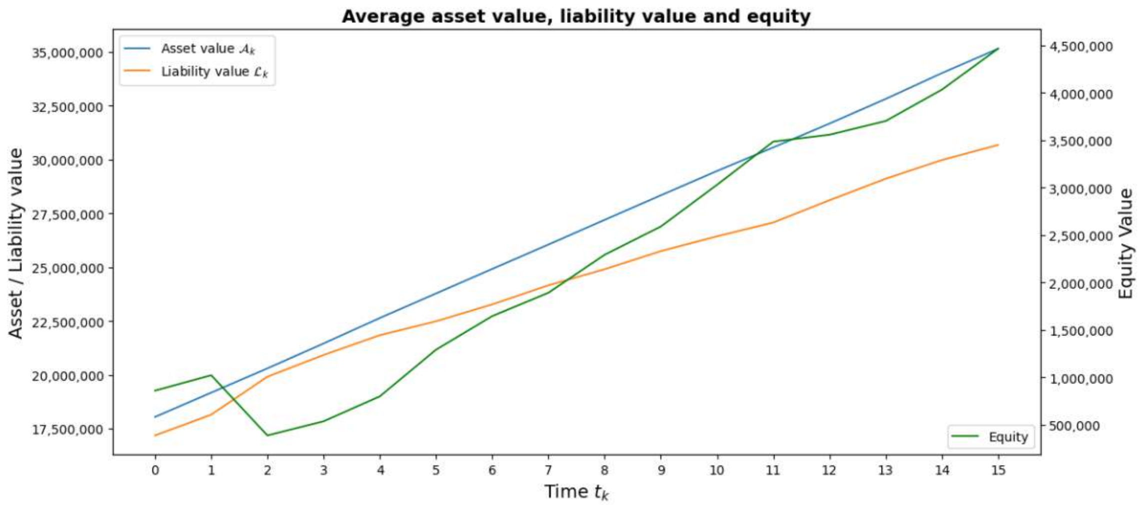


Figure 5.4.: Average asset and liability value and equity over time

Even though the average value of equity as the difference between the asset value and the liability value is positive for all time steps, scenarios can occur in which the value of equity can become negative, which could be interpreted as a form of default of the insurance company. Even with a matched duration of assets and liabilities leading to a similar sensitivity of their values in case of changes in the yield curve due to the short rate dynamics, the liability value, unlike the asset value, cannot decrease in case of a negative investment return due to the profit-sharing mechanism in (4.90) and (4.91). Thus, high negative investment returns within a certain period, i.e. $R_k^A < 0$, can force the value of the asset portfolio to fall below the value of the liabilities, leading to a negative equity at the time. Additionally, as there is a gap between the asset and liability duration within the first periods as described above, an exposure to an increased interest rate risk is still given at the beginning. For the generated simulations in the presented scenario, this was the case for 14 scenarios in total, leading to a probability of default of 1.4%.

To conclude, we illustrate the development of the average insurance contract portfolio structure over all simulations for the given scenario, i.e. the average number of insured per state over time, in Figure 5.5:

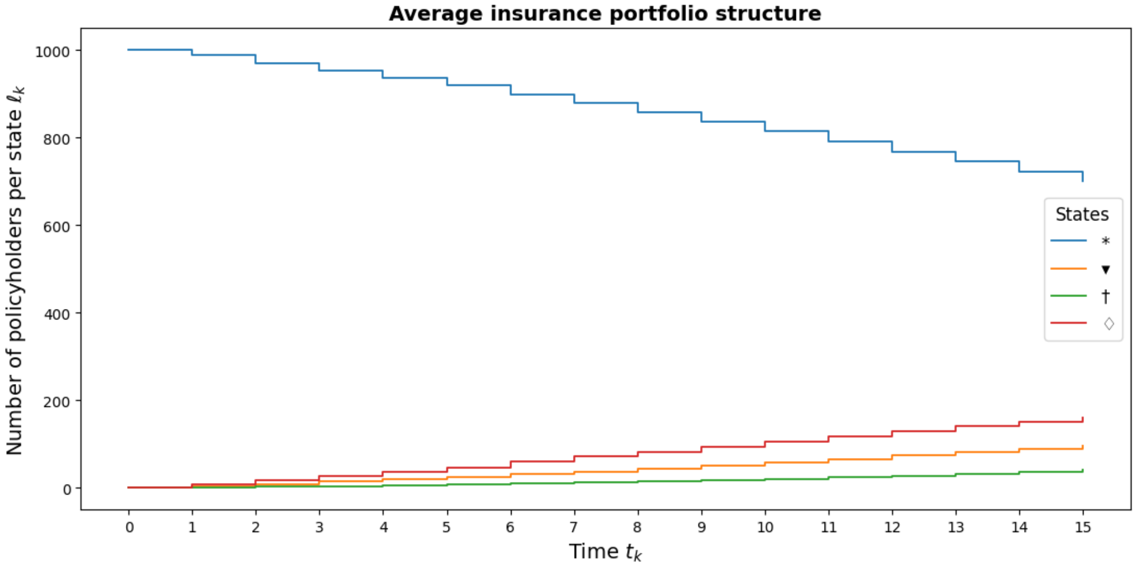


Figure 5.5.: Average number of insured per state over time

Comparing the number of policyholders exercising the premium waiver option to the number of policyholders exercising the surrender option reflects the parametrisation of this scenario, in which the premium waiver probabilities were chosen to be twice as high as the surrender probabilities. The number of deceased persons in this scenario has a minor effect as the chosen age and contract term imply low probabilities of decease in general. For scenarios where the insured are older at the start of the projection, this effect has a greater impact.

6. Conclusion and future scope

In the course of this thesis, we studied a scenario-based stochastic modelling approach for interest rate risk management in life insurance. We started by discussing the methodological framework of stochastic modelling in the area of asset-liability management and the legal requirements of Austrian insurance law that significantly influence the realisation of ALM measures and the implementation of stochastic ALM models. Taking this legal framework into account, we have derived interacting models for assets and liabilities and formulated a two-stage optimisation problem with the objective of immunising an insurance company against the underlying interest rate risk by dynamically restructuring the investment portfolio.

Due to the complexity of modern financial markets and the broad spectrum of life insurance products, we had to restrict ourselves to aspects considered essential for the purpose of the model when deriving the individual model components. This, of course, opens up a range of possibilities for model extensions or flexibilisations, some of which have already been mentioned at appropriate points in the thesis. Especially for the capital market model, in terms of the modelled financial instruments the insurance company can invest in, we restricted ourselves to equities, fixed rate bonds and a money market account. Consequently, extending the model by including further financial products and incorporating their risk-reducing effects into the interest rate risk optimisation problem offers interesting options for further analysis. First and foremost, floating-rate notes, interest rate swaps or other derivative financial instruments permitted for life insurance companies can be mentioned here as financial instruments that would meaningfully complement the model.
For the liability model, we covered the life insurance type of endowment insurance contracts as these contracts are associated with a more complex interest rate risk due to the distribution of expected benefit cash flows over the entire contract term and were therefore considered as particularly relevant. This, however, makes it a sensible approach to model and analyse other contract types such as whole life or pure endowment contracts or even insurance portfolios consisting of different contract types. With regard to benefit payments, it would be interesting to move away from the assumption of one-off benefit payments towards benefit payments in the form of fixed-term or life annuities, as this has a particular influence on the duration and thus the interest rate sensitivity of the insurance portfolio. To conclude, the possible consideration of reinsurance contracts with their risk-reducing effects, their influence on the structure of future expected cash flows and consequently their impact on the interest rate sensitivity of the liabilities should also be mentioned here.

A. Tables

Age in years x	Probability of dying between ages x to $x+1$ q_x	Number surviving to age x l_x	Number dying between ages x to $x+1$ d_x	Expectation of life at age x e_x
0	0.00277	100000	277	81.36
1	0.00020	99723	20	80.58
2	0.00014	99703	14	79.60
3	0.00011	99689	11	78.61
4	0.00009	99678	9	77.62
5	0.00008	99669	8	76.63
6	0.00008	99660	8	75.63
7	0.00007	99653	7	74.64
8	0.00006	99646	6	73.64
9	0.00005	99640	5	72.65
10	0.00005	99634	5	71.65
11	0.00007	99629	7	70.66
12	0.00009	99622	9	69.66
13	0.00012	99614	12	68.67
14	0.00016	99601	16	67.68
15	0.00021	99585	21	66.69
16	0.00025	99565	25	65.70
17	0.00029	99540	29	64.72
18	0.00032	99511	32	63.74
19	0.00035	99479	35	62.76
20	0.00037	99444	37	61.78
21	0.00038	99407	38	60.80
22	0.00038	99369	37	59.82
23	0.00037	99332	36	58.85
24	0.00036	99295	35	57.87
25	0.00035	99260	35	56.89
26	0.00035	99225	35	55.91
27	0.00037	99190	36	54.93
28	0.00039	99154	39	53.95
29	0.00042	99115	41	52.97

x	q_x	l_x	d_x	e_x
30	0.00045	99074	45	51.99
31	0.00048	99029	48	51.01
32	0.00052	98981	51	50.04
33	0.00055	98930	54	49.06
34	0.00058	98876	58	48.09
35	0.00062	98819	61	47.12
36	0.00066	98757	65	46.15
37	0.00071	98692	70	45.18
38	0.00076	98622	75	44.21
39	0.00082	98547	81	43.24
40	0.00089	98466	87	42.28
41	0.00096	98379	94	41.31
42	0.00104	98285	103	40.35
43	0.00114	98182	112	39.39
44	0.00124	98071	122	38.44
45	0.00136	97949	133	37.49
46	0.00149	97816	145	36.54
47	0.00163	97671	159	35.59
48	0.00179	97512	175	34.65
49	0.00198	97337	192	33.71
50	0.00219	97145	212	32.77
51	0.00242	96933	235	31.84
52	0.00270	96698	261	30.92
53	0.00300	96437	289	30.00
54	0.00334	96147	321	29.09
55	0.00371	95827	356	28.19
56	0.00412	95471	393	27.29
57	0.00457	95078	435	26.40
58	0.00508	94643	481	25.52
59	0.00565	94163	532	24.65
60	0.00631	93630	591	23.78
61	0.00706	93039	657	22.93
62	0.00790	92382	730	22.09
63	0.00880	91652	807	21.26
64	0.00974	90845	885	20.45
65	0.01070	89960	963	19.64
66	0.01166	88997	1038	18.85
67	0.01265	87960	1113	18.07
68	0.01374	86847	1194	17.29
69	0.01501	85654	1286	16.53

x	q_x	l_x	d_x	e_x
70	0.01652	84368	1394	15.77
71	0.01834	82974	1522	15.03
72	0.02044	81452	1665	14.30
73	0.02274	79787	1814	13.59
74	0.02515	77973	1961	12.89
75	0.02761	76012	2099	12.21
76	0.03006	73913	2222	11.54
77	0.03263	71691	2339	10.89
78	0.03555	69352	2466	10.24
79	0.03907	66886	2613	9.60
80	0.04342	64273	2791	8.97
81	0.04883	61482	3002	8.35
82	0.05544	58480	3242	7.75
83	0.06334	55238	3499	7.18
84	0.07262	51739	3758	6.63
85	0.08340	47981	4002	6.11
86	0.09575	43980	4211	5.62
87	0.10969	39769	4362	5.16
88	0.12519	35407	4433	4.74
89	0.14225	30974	4406	4.34
90	0.16084	26568	4273	3.98
91	0.18093	22295	4034	3.65
92	0.20245	18261	3697	3.35
93	0.22529	14564	3281	3.07
94	0.24934	11283	2813	2.82
95	0.27450	8470	2325	2.58
96	0.30110	6145	1850	2.37
97	0.32892	4295	1413	2.18
98	0.35800	2882	1032	2.00
99	0.38835	1850	719	1.84
100	0.41997	1132	475	1.69
101	0.45285	656	297	1.56
102	0.48700	359	175	1.43
103	0.52241	184	96	1.32
104	0.55909	88	49	1.21
105	0.59704	39	23	1.12
106	0.63625	16	10	1.03
107	0.67673	6	4	0.97
108	0.71848	2	1	0.95
109	0.76149	1	0	1.09
110	0.80576	0	0	1.97

Table A.1.: Smoothed mortality table (Unisex) for Austria, 2020–2022, Source: Statistics Austria [27]

B. Model implementation in Python

We here present the implementation code of the stochastic scenario-based model of Chapter 4 using the Python programming language. The code is structured in the following modules:

- `Model_parameters.py`
  Specification of all required model parameters.
- `Asset_Model.py`
  Specification of all functionalities of the asset model as described in Section 4.1.
- `Liability_Model.py`
  Specification of all functionalities of the liability model as described in Section 4.2.
- `ALM_simulation.py`
  Main functionality for the stochastic scenario modelling and asset portfolio optimisation.

`Model_parameters.py`:

```
import numpy as np
import pandas as pd

### General model parameters
N = 20
T_grid = list(range(N+1))
N_hat = 15
T_grid_hat = list(range(N_hat+1))
num_sim = 1000

param_general = {
  'N': N,
  'T_grid': T_grid,
  'N_hat': N_hat,
  'T_grid_hat': T_grid_hat,
  'num_sim': num_sim
}

### Asset model parameters
# short rate model parameters
r0_r = 0.03
kappa_r = 1.2
theta_r = 0.02535
sigma_r = 0.004

# bond model parameters
n_B = 5
T_bonds = [5,10,15,20,25]
```

```
c_bonds = [0.027, 0.027, 0.028, 0.028, 0.026]
t_c = list(range(1,max(T_bonds)+1))

# equity model parameters
n_E = 1
mu_S = [0.09060036]
sigma_S = [0.15140469]

# general asset model parameters
dt = 1/250
C_hat = np.array([
[1.0, -0.1],
[-0.1, 1.0]
])
lambda_0 = [0.03, 0.08, 0.02, 0.05, 0.16, 0.17, 0.49]
epsilon_0 = 0.05
alpha_S_min = 0.02
alpha_S_max = 0.12
alpha_C = 0.02
delta_lambda = 0.06

param_assets = {
  'r0_r': r0_r,
  'kappa_r': kappa_r,
  'theta_r': theta_r,
  'sigma_r': sigma_r,
  'n_B': n_B,
  'T_bonds': T_bonds,
  'c_bonds': c_bonds,
  't_c': t_c,
  'n_E': n_E,
  'mu_S': mu_S,
  'sigma_S': sigma_S,
  'dt': dt,
  'C_hat': C_hat,
  'lambda_0': lambda_0,
  'epsilon_0': epsilon_0,
  'alpha_S_min': alpha_S_min,
  'alpha_S_max': alpha_S_max,
  'alpha_C': alpha_C,
  'delta_lambda': delta_lambda
}

### Liability model parameters
n_L = 1000
x_0 = 45
i_min = 0.00
Psi_0 = 0.0
Gamma_0 = 0.0
beta = 0.90
Lambda_d = 30000.0
Lambda_a = 30000.0
K = 3
xi = [0, 0.01, 0.02]
Delta_R_0 = 0.015
```

```
premiums = pd.read_excel("model_parameters.xlsx", sheet_name = 'premiums',
  ↪  index_col=0).loc[:,'premiums'].to_list()
gamma_S = 0.05
gamma_PW = pd.read_excel("model_parameters.xlsx", sheet_name = 'gamma_PW',
  ↪  index_col=0).loc[:,'gamma_PW'].to_list()
N_Gamma = 500
dt_Gamma_estimation = 1/100
epsilon = 0.3

param_liabilities = {
    'n_L': n_L,
    'x_0': x_0,
    'i_min': i_min,
    'Psi_0': Psi_0,
    'Gamma_0': Gamma_0,
    'beta': beta,
    'Lambda_d': Lambda_d,
    'Lambda_a': Lambda_a,
    'K': K,
    'xi': xi,
    'Delta_R_0': Delta_R_0,
    'premiums': premiums,
    'gamma_S': gamma_S,
    'gamma_PW': gamma_PW,
    'N_Gamma': N_Gamma,
    'dt_Gamma_estimation': dt_Gamma_estimation,
    'epsilon': epsilon
}

### Liability model data
# mortality table
mort_table = pd.read_csv("Geglaettete_Sterbetafel_2022.csv", delimiter = '
  ↪  ;', decimal = ',')

# probability tables for surrender and premium waiver probabilities
prob_table_surr = pd.read_excel("model_parameters.xlsx", sheet_name = '
  ↪  Surrender', skiprows=2, index_col=0)
prob_table_pw = pd.read_excel("model_parameters.xlsx", sheet_name = '
  ↪  Premium_waiver', skiprows=2, index_col=0)

data_liabilities = {
    'mort_table': mort_table,
    'prob_table_surr': prob_table_surr,
    'prob_table_pw': prob_table_pw
}
```

Asset_Model.py:

```
import numpy as np
import pandas as pd

def simulate_corr_brownian_motion(param_assets, param_general):
"""
Simulates n-dimensional Brownian motion up to time T with given
```

```
  ↪ correlation structure C_hat.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:return W:              List of (n_E+1) data frames containing simulations
  ↪  of correlated brownian motions
"""

N_hat = param_general['N_hat']
num_sim = param_general['num_sim']
dt = param_assets['dt']
C_hat = param_assets['C_hat']
n_E = param_assets['n_E']

W_S = list()
for _ in range(n_E):
W_S.append(pd.DataFrame(0, index=[i * dt for i in range(0, int(N_hat / dt)
  ↪  + 1)], columns = [str(i) for i in range(num_sim)]))
W_r = pd.DataFrame(0, index=[i * dt for i in range(0, int(N_hat / dt) + 1)
  ↪ ], columns = [str(i) for i in range(num_sim)])

# Calculation of the Cholesky decompositon of C_hat
A = np.linalg.cholesky(C_hat)

for n in range(num_sim):
# Simulate n-dimensional Brownian motion with independent components
W_hat = np.zeros((int(n_E+1),int(N_hat/dt+1)))
for i in range(n_E+1):
W_hat[i,:] = np.cumsum(np.concatenate(([0], np.random.normal(0, np.sqrt(dt
  ↪ ), int(N_hat/dt)))))

# Simulation of correlated Brownian motion
W = np.dot(A, W_hat)
for i in range(n_E):
W_S[i][str(n)] = W[i,:].T
W_r[str(n)] = W[-1,:].T

W = {'W_S': W_S,
  'W_r': W_r}

return W

def simulate_short_rate(param_assets, param_general, W_r):
"""
Simulates paths for the short rate using the Vasicek model.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:param W_r:             Brownian motion paths for short rate
:return r:                Data frame containing simulations of short rate
  ↪ paths
"""

N_hat = param_general['N_hat']
num_sim = param_general['num_sim']
```

```
dt = param_assets['dt']
r0 = param_assets['r0_r']
kappa = param_assets['kappa_r']
theta = param_assets['theta_r']
sigma = param_assets['sigma_r']

r = pd.DataFrame(0., index=[i * dt for i in range(0, int(N_hat / dt) + 1)], columns = [str(i) for i in range(num_sim)])
r.iloc[0] = r0

dW_r = W_r.diff().iloc[1:]
for i in range(1,int(N_hat/dt)+1):
r.iloc[i] = r.iloc[i-1] + kappa * (theta - r.iloc[i-1]) * dt + sigma * dW_r.iloc[i-1]

return r


def simulate_bank_account(param_assets, param_general, r):
"""
Simulates paths for the bank account using the short rate paths.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:param r:               Short rate paths
:return C:              Data frame containing simulations of bank account value paths
"""

N_hat = param_general['N_hat']
num_sim = param_general['num_sim']
dt = param_assets['dt']

C = pd.DataFrame(0., index=[i * dt for i in range(0, int(N_hat / dt) + 1)], columns = [str(i) for i in range(num_sim)])
C.iloc[0] = 1

for i in range(1,int(N_hat/dt)+1):
C.iloc[i] = C.iloc[i-1] + C.iloc[i - 1] * r.iloc[i - 1] * dt

return C

def simulate_stocks(param_assets, param_general, W_S):
"""
Simulates paths for stock prices following geometric Brownian motions.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:param W_S:             List of Brownian motion paths for stock prices
:return S:              List of Data frames containing simulations of stock prices
"""

N_hat = param_general['N_hat']
dt = param_assets['dt']
```

```python
n_E = param_assets['n_E']
mu = param_assets['mu_S']
sigma = param_assets['sigma_S']

S = list()

for j in range(n_E):
S.append(np.exp((sigma[j] * W_S[j]).add([(mu[j] - (sigma[j]**2)/2) * i * dt for i in range(0, int(N_hat / dt) + 1)], axis=0)))

return S

def simulate_bonds(param_assets, param_general, r):
"""
Simulates paths for the fixed rate bond prices using the short rate paths.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:param r:               Short rate paths
:return B:              List of Data frames containing simulations of fixed rate bond price paths
"""

num_sim = param_general['num_sim']
n_B = param_assets['n_B']
T_bonds = param_assets['T_bonds']
c_bonds = param_assets['c_bonds']
t_c = np.array(param_assets['t_c'])

# Cash flow profiles of all bonds
cfs = pd.DataFrame(0., index = sorted(set(t_c) | set(T_bonds)), columns = [str(i) for i in range(n_B)])

for i in range(n_B):
cfs.loc[t_c[t_c <= T_bonds[i]],str(i)] = c_bonds[i]
cfs.loc[[T_bonds[i]], str(i)] = cfs.loc[[T_bonds[i]], str(i)] + 1

B = [pd.DataFrame(0., index=param_general['T_grid_hat'], columns = [str(i) for i in range(num_sim)]) for _ in range(n_B)]

for t in param_general['T_grid_hat']:
zcb_prices = np.transpose(np.array([np.exp(-B_helper(param_assets, t, t+t_j) * r[r.index == t].values + A_helper(param_assets, t, t+t_j)).tolist()[0] for t_j in list(cfs.index)]))
for j in range(n_B):
B[j].loc[t,:] = np.dot(zcb_prices[:, :len(cfs.index[cfs.index <= T_bonds[j]])], np.array(cfs.loc[cfs.index <= T_bonds[j], str(j)]))

return B

# functions A(t,T) and B(t,T) for the zero coupon price formula
def B_helper(param_assets, t, T):
kappa = param_assets['kappa_r']
if t > T:
raise TypeError('t must be smaller than T.')
```

```
return (1 / kappa) * (1 - np.exp(-kappa * (T - t)))

def A_helper(param_assets, t, T):
kappa = param_assets['kappa_r']
theta = param_assets['theta_r']
sigma = param_assets['sigma_r']
if t > T:
raise TypeError('Error: t must be smaller than T.')
return (theta - ((sigma ** 2) / (2 * (kappa ** 2)))) * (B_helper(
  ↪ param_assets, t, T) - T + t) - ((sigma ** 2) / (4 * kappa)) * (
  ↪ B_helper(param_assets, t, T) ** 2)

def P(param_assets, t, T, r):
"""
Calculates the zero coupon bond price at time t with maturity T under the
  ↪ Vasicek short rate model.
:param param_assets: asset parameters
:param t: time point
:param T: maturity
:param r: short rate paths
:return: zero coupon bond price at time t with maturity T
"""
if t > T:
return [0]*r.shape[1]
return np.exp(-B_helper(param_assets, t, T)*r[r.index == t].values +
  ↪ A_helper(param_assets, t, T)).tolist()[0]

def E_P_t_T_s(param_assets, t, T, s, r_s):
"""
Calculates the conditional expectation of P(t,T) given F_s for s <= t < T.
:param param_assets: asset parameters
:param t: time point
:param T: maturity
:param s: time point
:param r_s: short rate paths at time s
:return: conditional expectation of P(t,T) given F_s
"""
kappa = param_assets['kappa_r']
theta = param_assets['theta_r']
sigma = param_assets['sigma_r']
if t > T:
raise TypeError('t must be smaller than T.')
if s > t:
raise TypeError('s must be smaller than or equal to t.')
return np.exp(A_helper(param_assets, t, T) - B_helper(param_assets, t, T)
  ↪ *(r_s*np.exp(-kappa*(t-s)) + theta*(1-np.exp(-kappa*(t-s)))) + ((
  ↪ B_helper(param_assets, t, T)**2)*(sigma**2)*(1-np.exp(-2*kappa*(t-s))
  ↪ ))/(4*kappa))


def portfolio_return(k, lambda_k, C, S, B, r, param_assets):
"""
Calculates the return of the portfolio for the k-th time period.

:param k:                Time point
```

```
:param lambda_k:        Portfolio weights at time k
:param C:               Bank account values
:param S:               Stock prices
:param B:               Bond prices
:param r:               Short rate paths
:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:return:                Portfolio return at time k
"""

T_bonds = param_assets['T_bonds']
c_bonds = param_assets['c_bonds']
t_c = param_assets['t_c']
n_B = param_assets['n_B']

R_C = (C[C.index == k].sum().sum() - C[C.index == k-1].sum().sum()) / C[C.index == k-1].sum().sum()
R_S = [(S[j][S[j].index == k].sum().sum() - S[j][S[j].index == k-1].sum().sum()) / S[j][S[j].index == k-1].sum().sum() for j in range(len(S))]

# bond cash flow profiles
cfs = pd.DataFrame(0., index=sorted(set(t_c) | set(T_bonds)), columns=[str(i) for i in range(param_assets['n_B'])])

for i in range(n_B):
cfs.loc[[x for x in t_c if x<= T_bonds[i]], str(i)] = c_bonds[i]
cfs.loc[[T_bonds[i]], str(i)] = cfs.loc[[T_bonds[i]], str(i)] + 1

R_B = []
zcb_prices_shift = np.transpose(np.array([np.exp(-B_helper(param_assets, 1, t_j) * r[r.index == k].values + A_helper(param_assets, 1, t_j)).tolist()[0] for t_j in list(cfs.index)]))
for j in range(n_B):
R_B.append((np.dot(zcb_prices_shift[:, :len(cfs.index[cfs.index <= T_bonds[j]])],np.array(cfs.loc[cfs.index <= T_bonds[j], str(j)])) / B[j][B[j].index == k-1].sum().sum() - 1).sum())

R_k = lambda_k[0] * R_C + sum([lambda_k[j+1] * R_S[j] for j in range(len(S))]) + sum([lambda_k[j+1+len(S)] * R_B[j] for j in range(len(B))])
return R_k


def Gamma_estimation_corr_brownian_motion(param_assets, param_liabilities, param_general):
"""
Simulates n-dimensional Brownian motion up to time T with given correlation structure C_hat.

:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:return W:                List of (n_E+1) data frames containing simulations of correlated brownian motions
"""

N = param_general['N']
```

```
N_hat = param_general['N_hat']
C_hat = param_assets['C_hat']
n_E = param_assets['n_E']
N_Gamma = param_liabilities['N_Gamma']
dt = param_liabilities['dt_Gamma_estimation']

W_Gamma = []

# Calculation of the Cholesky decompositon of C_hat
A = np.linalg.cholesky(C_hat)

for k in range(N_hat+1):
W_S = []
for _ in range(n_E):
W_S.append(pd.DataFrame(0, index=[i * dt for i in range(0, int((N-k) / dt)
 ↪ + 1)], columns = [str(i) for i in range(N_Gamma)]))
W_r = pd.DataFrame(0, index=[i * dt for i in range(0, int((N-k) / dt) + 1)
 ↪ ], columns = [str(i) for i in range(N_Gamma)])

for n in range(N_Gamma):
W_hat = np.zeros((int(n_E+1),int((N-k)/dt+1)))

for i in range(n_E+1):
W_hat[i,:] = np.cumsum(np.concatenate(([0], np.random.normal(0, np.sqrt(dt
 ↪ ), int((N-k)/dt)))))

W = np.dot(A, W_hat)
for i in range(n_E):
W_S[i][str(n)] = W[i,:].T
W_r[str(n)] = W[-1,:].T

for i in range(n_E):
W_S[i] = W_S[i].loc[range(N-k+1)]

W = {'W_S': W_S,
  'W_r': W_r}
W_Gamma.append(W)

return W_Gamma

def duration_bonds(k, r_k, param_assets):
"""
Calculates the duration of the bonds at time k.

:param k:                 Time point
:param r_k:               Short rate at time k
:param param_assets:      dictionary containing asset parameters
:param param_general:     dictionary containing general model parameters
:return:                  List of bond durations at time k
"""

T_bonds = param_assets['T_bonds']
c_bonds = param_assets['c_bonds']
t_c = param_assets['t_c']
n_B = param_assets['n_B']
```

```

durations = []

# bond cash flow profiles
cfs = pd.DataFrame(0., index=sorted(set(t_c) | set(T_bonds)), columns=[str(i) for i in range(param_assets['n_B'])])

for i in range(n_B):
cfs.loc[[x for x in t_c if x <= T_bonds[i]], str(i)] = c_bonds[i]
cfs.loc[[T_bonds[i]], str(i)] = cfs.loc[[T_bonds[i]], str(i)] + 1

for i in range(n_B):
weighted_cfs = sum([E_P_t_T_s(param_assets, k, k + j, k, r_k) * cfs.iloc[j-1,i] * j for j in [x for x in cfs.index if x <= T_bonds[i]]])
price = sum([E_P_t_T_s(param_assets, k, k + j, k, r_k) * cfs.iloc[j-1,i] for j in [x for x in cfs.index if x <= T_bonds[i]]])
durations.append(weighted_cfs / price)

return durations

def expected_one_period_returns(k, r_k, param_assets, bond_prices):
"""
Calculates the expected one-period returns of the assets at time k.

:param k:               Time point
:param r_k:             Short rate at time k
:param param_assets:    dictionary containing asset parameters
:param param_general:   dictionary containing general model parameters
:return:                List of expected one-period returns of the assets at time k
"""

mu_S = param_assets['mu_S']
n_E = param_assets['n_E']
T_bonds = param_assets['T_bonds']
c_bonds = param_assets['c_bonds']
t_c = param_assets['t_c']
n_B = param_assets['n_B']

exp_returns = []

# expected return of bank account
exp_returns.append(np.exp((1-np.exp(-param_assets['kappa_r']))/param_assets['kappa_r']*(r_k-param_assets['theta_r']-(param_assets['sigma_r']**2)/(param_assets['kappa_r']**2))+(param_assets['sigma_r']**2)/(4*(param_assets['kappa_r']**3))*(1-np.exp(-2*param_assets['kappa_r']))+(param_assets['sigma_r']**2)/(2*param_assets['kappa_r']**2)+param_assets['theta_r'])-1)

# expected returns of stocks
for j in range(n_E):
exp_returns.append(np.exp(mu_S[j])-1)

# expected returns of bonds
cfs = pd.DataFrame(0., index=sorted(set(t_c) | set(T_bonds)), columns=[str
```

```
  ↪ (i) for i in range(param_assets['n_B'])])

for i in range(n_B):
cfs.loc[[x for x in t_c if x <= T_bonds[i]], str(i)] = c_bonds[i]
cfs.loc[[T_bonds[i]], str(i)] = cfs.loc[[T_bonds[i]], str(i)] + 1
exp_returns.append(sum([E_P_t_T_s(param_assets, k+1, k+j, k, r_k) * cfs.
  ↪ iloc[j - 1, i] for j in [x for x in cfs.index if x <= T_bonds[i]]])/
  ↪ bond_prices[i] - 1)

return exp_returns
```

Liability_Model.py:

```
from functools import reduce
from Asset_Model import *

def P_X(k, l, I, param_liabilities, data_liabilities, param_general):
"""
Calculates the transition matrix P_X(k,l) for given k, l and threshold
  ↪ interval I for surrender/premium waiver probabilities.

:param k: k in P_X(k,l)
:param l: l in P_X(k,l)
:param I: threshold interval for surrender/premium waiver probabilities
:return: transition probability P_X(k,l)
"""

N = param_general['N']
x_0 = param_liabilities['x_0']
K = param_liabilities['K']
mort_table = data_liabilities['mort_table']
prob_table_surr = data_liabilities['prob_table_surr']
prob_table_pw = data_liabilities['prob_table_pw']

assert type(k) == int, "Parameter 'k' must be an integer."
assert k < N, "Parameter 'k' must be less than 'N'."
assert type(I) == int, "Parameter 'I' must be an integer."
assert I <= K, "Parameter 'I' must be less or equal to 'K'."

assert type(l) == int, "Parameter 'l' must be an integer."
assert l >= k & l <= N, "Parameter 'l' must be greater or equal to 'k' and
  ↪  less or equal to 'N'."

states = ['a', 's', 'd'] + ["pw_" + str(i+1) for i in range(N)]  # state
  ↪ space

if l == k:
return pd.DataFrame(np.eye(len(states)), index=states, columns=states)

P = []
for j in range(l-k):
P_j = pd.DataFrame(np.zeros((len(states), len(states))), index=states,
  ↪ columns=states)
P_j.loc['a', 'a'] = (1-mort_table.loc[x_0+k+j, 'q_x']) * (1-
```

```
↪ prob_table_surr.loc[I, k+j] - prob_table_pw.loc[I, k+j])
P_j.loc['a', 'pw_'+str(k+j+1)] = (1-mort_table.loc[x_0+k+j, 'q_x']) * prob_table_pw.loc[I, k+j]
P_j.loc['a', 's'] = (1-mort_table.loc[x_0+k+j, 'q_x']) * prob_table_surr.loc[I, k+j]
P_j.loc['a', 'd'] = mort_table.loc[x_0+k+j, 'q_x']
P_j.loc['s', 's'] = 1
P_j.loc['d', 'd'] = 1
for i in range(N):
if i < k:
P_j.loc['pw_'+str(i+1), 'pw_'+str(i+1)] = (1-mort_table.loc[x_0+k+j, 'q_x']) * (1-prob_table_surr.loc[I, k+j])
P_j.loc['pw_'+str(i+1), 's'] = (1-mort_table.loc[x_0+k+j, 'q_x']) * prob_table_surr.loc[I, k+j]
P_j.loc['pw_' + str(i + 1), 'd'] = mort_table.loc[x_0 + k + j, 'q_x']
else:
P_j.loc['pw_' + str(i + 1), 'pw_' + str(i + 1)] = 1
P.append(P_j)
P = reduce(lambda P_i, P_j: P_i.dot(P_j), P)

return P

def one_step_insurance_portfolio_simulation(k, ell_old, I, param_liabilities, data_liabilities, param_general, r, Gamma, param_assets):
"""
Simulates the insurance contract portfolio.

:param k: time step
:param ell_old: preceding contract portfolio structure vectors at time t_k for each simulation
:param I: threshold interval for surrender/premium waiver probabilities
:param param_liabilities: liability model parameters
:param data_liabilities: liability model data
:param param_general: general model parameters
:param r: short rate model parameters
:param Gamma: state-dependent bonus accounts at time t_k-1
:return: portfolio structure vectors at time t_k+1
"""

N = param_general['N']
states = ['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(N)]  # state space

Delta_Nij = np.zeros((N+3, N+3))
P = P_X(k, k+1, I, param_liabilities, data_liabilities, param_general)
for i in range(N+3):
if ell_old[i] == 0:
continue
transition = np.random.choice(states, size = int(ell_old[i]), p = P.iloc[i])
Delta_Nij[i,:] = list({key: np.count_nonzero(transition == key) for key in states}.values())

ell_new = list(np.sum(Delta_Nij, axis = 0))
```

```
### calculation of cash flows
# prospective reserve at t_k+1
V_res = list(V_prospective_reserve(k+1, r[r.index == k+1].sum().sum(),
  ↪ param_liabilities, param_general, I, data_liabilities, param_assets).
  ↪ loc[:, str(k+1)])
# payout function
a = a_Post(k, V_res, param_liabilities, param_general, Gamma)
cash_flow = (pd.DataFrame(Delta_Nij, index = states, columns = states) * a
  ↪ ).sum().sum()
return [ell_new, cash_flow]

def calculation_actuarial_reserve(param_liabilities, param_general):
"""
Calculates the actuarial reserve Psi for all relevant states.

:param param_liabilities: liability model parameters
:param param_general: general model parameters
:return: actuarial reserve Psi
"""

N = param_general['N']
Psi_0 = param_liabilities['Psi_0']
premiums = param_liabilities['premiums']
i_min = param_liabilities['i_min']

Psi = []
Psi_a = [Psi_0]
for k in range(1, N+1):
Psi_a.append((Psi_a[k-1] + premiums[k-1]) * (1 + i_min))
Psi.append(Psi_a)

for j in range(1, N+1):
Psi_pw_j = np.zeros(j).tolist()
Psi_pw_j.append(Psi_a[j])
for k in range(j+1, N+1):
Psi_pw_j.append(Psi_pw_j[k-1] * (1 + i_min))
Psi.append(Psi_pw_j)
Psi = pd.DataFrame(Psi, index=['a'] + ["pw_" + str(i + 1) for i in range(N
  ↪ )], columns=[str(i) for i in range(N+1)])
return Psi

def a_Pre(k, param_liabilities, param_general, Gamma = None):
"""
Calculates the prior payout function matrix at t_k.

:param param_liabilities: liability model parameters
:param param_general: general model parameters
:return: payout function vector
"""

N = param_general['N']

assert k <= N, "Parameter 'k' must smaller or equal to N."
```

```
if Gamma is None:
Gamma = list(np.zeros((N+1)))

states = ['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(N)]

if k < N:
return pd.DataFrame(np.array([-param_liabilities['premiums'][k]] + [0 for
  ↪ i in range(N+2)]).T, index=states)
else:
return pd.DataFrame(np.array([(param_liabilities['Lambda_a'] + Gamma[0])]
  ↪ + [0,0] + [(1-param_liabilities['gamma_PW'][i]) * param_liabilities['
  ↪ Lambda_a'] + Gamma[1:][i] for i in range(N)]).T, index=states)

def a_Post(k, V_k_1, param_liabilities, param_general, Gamma = None):
"""
Calculates the posterior payout function matrix at t_k.

:param V_k_1: Vector of state-dependent prospective reserves at k+1
:param param_liabilities: liability model parameters
:param param_general: general model parameters
:param Gamma: List of state-dependent bonus accounts at k for states a,
  ↪ pw_1, ..., pw_N (length = N+1)
:return: payout function matrix
"""
N = param_general['N']

assert k < N, "Parameter 'k' must smaller than N."

if Gamma is None:
Gamma = list(np.zeros((N+1)))

states = ['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(N)]
a = pd.DataFrame(np.zeros((N+3, N+3)), index=states, columns=states)

a.loc[:, 'd'] = [(param_liabilities['Lambda_d'] + Gamma[0])] + [0,0] + [(
  ↪ param_liabilities['Lambda_d'] + val) for val in Gamma[1:]]
a.loc[:, 's'] = [(1-param_liabilities['gamma_S']) * (V_k_1[0] + Gamma[0])]
  ↪  + [0,0] + [(1-param_liabilities['gamma_S']) * (V_k_1[-N:][i] + val)
  ↪ for i, val in enumerate(Gamma[1:])]
return a

def V_prospective_reserve(k, r_k, param_liabilities, param_general, I,
  ↪ data_liabilities, param_assets):
"""
Calculates the future state-dependent prospective reserves at time t_k.

:param k: time step
:param r_k: short rate at time t_k
:param param_liabilities: liability model parameters
:param param_general: general model parameters
:param I: threshold interval for surrender/premium waiver probabilities
:return: future state-dependent prospective reserves at time t_k
"""

states = ['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(
```

```
  ↪ param_general['N'])]
# boundary conditions
V = [[param_liabilities['Lambda_a']] + [0,0] + [(1-param_liabilities['
  ↪ gamma_PW'][i]) * param_liabilities['Lambda_a'] for i in range(
  ↪ param_general['N'])]]

for j in range(param_general['N']-1, k-1, -1):
V_j = []
for i in range(param_general['N']+3):

# a_ij * V_j for all j according to Thiele difference equation
tmp = [a + b for a,b in zip(list(a_Post(j, V[-1], param_liabilities,
  ↪ param_general).iloc[i,:]),V[-1])]

P = list(P_X(j, j+1, I, param_liabilities, data_liabilities, param_general
  ↪ ).iloc[i,:])
V_j.append(a_Pre(j, param_liabilities, param_general).iloc[i,:].sum() +
  ↪ E_P_t_T_s(param_assets, j, j+1, k, r_k) * sum([a * b for a,b in zip(
  ↪ tmp,P)]))
V.append(V_j)

V = V[::-1]
df = pd.DataFrame(V).transpose()
df.index = states
df.columns = [str(i) for i in range(k, param_general['N']+1)]
return df

def Gamma_recursion(k, Gamma_old, Psi, z_k, param_liabilities,
  ↪ param_general):
"""
Calculates the state-dependent bonus accounts at time t_k.

:param k: time step
:param Gamma_old: preceding state-dependent bonus accounts at time t_k-1
:param Psi: actuarial reserve in all relevant states
:param z_k: state-dependent cash flows at time t_k
:param param_liabilities: liability model parameters
:param param_general: general model parameters
:return: state-dependent bonus accounts at time t_k
"""

i_min = param_liabilities['i_min']
premiums = param_liabilities['premiums']
N = param_general['N']

Gamma_active = (1+z_k)*Gamma_old[0] + (z_k - i_min)*(premiums[k-1] + Psi.
  ↪ iloc[0,k-1])
Gamma = [Gamma_active] + [(1+z_k)*Gamma_old[i] + (z_k - i_min)*Psi.iloc[i,
  ↪ k-1] for i in range(1, k)] + [Gamma_active] + [0 for i in range(k+1,N
  ↪ +1)]
return Gamma


def Gamma_estimation(k, W_Gamma, r_k, lambda_k, Gamma_k, Psi,
  ↪ param_liabilities, param_assets, param_general):
```

```
"""
Estimates the state-dependent bonus accounts at future time steps.

:param k: time step
:param W_Gamma: Brownian motion for bonus account estimation
:param r_k: short rate at time t_k
:param lambda_k: portfolio weights for estimation
:param Gamma_k: state-dependent bonus accounts at time t_k
:param Psi: actuarial reserve in all relevant states
:param param_liabilities: liability model parameters
:param param_assets: asset model parameters
:param param_general: general model parameters
:return: estimates for future state-dependent bonus accounts
"""

T_bonds = param_assets['T_bonds']
c_bonds = param_assets['c_bonds']
t_c = np.array(param_assets['t_c'])
dt = param_liabilities['dt_Gamma_estimation']
n_B = param_assets['n_B']

W_S = W_Gamma[k]['W_S']
W_r = W_Gamma[k]['W_r']

r = pd.DataFrame(0., index=W_r.index, columns=W_r.columns)
r.iloc[0] = r_k

dW_r = W_r.diff().iloc[1:]
for i in range(1, len(r.index)):
r.iloc[i] = r.iloc[i - 1] + param_assets['kappa_r'] * (param_assets['
  ↪ theta_r'] - r.iloc[i - 1]) * dt + param_assets['sigma_r'] * dW_r.iloc
  ↪ [i - 1]

C = pd.DataFrame(0., index=r.index, columns=r.columns)
C.iloc[0] = 1

for i in range(1, len(r.index)):
C.iloc[i] = C.iloc[i - 1] + C.iloc[i - 1] * r.iloc[i - 1] * dt

C = C.loc[W_S[0].index]
C_ret = ((C.shift(-1)-C).div(C)).iloc[:-1]

# bond cash flow profiles
cfs = pd.DataFrame(0., index=sorted(set(t_c) | set(T_bonds)), columns=[str
  ↪ (i) for i in range(param_assets['n_B'])])

for i in range(n_B):
cfs.loc[[x for x in t_c if x<= T_bonds[i]], str(i)] = c_bonds[i]
cfs.loc[[T_bonds[i]], str(i)] = cfs.loc[[T_bonds[i]], str(i)] + 1

B_ret = [pd.DataFrame(0., index=list(range(param_general['N']-k)), columns
  ↪ =r.columns) for _ in range(n_B)]

for t in list(B_ret[0].index):
zcb_prices = np.transpose(np.array([np.exp(-B_helper(param_assets, t, t +
```

```
↪ t_j) * r[r.index == t].values + A_helper(param_assets, t, t + t_j)).
↪ tolist()[0] for t_j in list(cfs.index)]))
zcb_prices_shift =  np.transpose(np.array([np.exp(-B_helper(param_assets,
↪ t+1, t + t_j) * r[r.index == t+1].values + A_helper(param_assets, t
↪ +1, t + t_j)).tolist()[0] for t_j in list(cfs.index)]))
for j in range(n_B):
B_ret[j].loc[t, :] = (pd.DataFrame(np.dot(zcb_prices_shift[:, :len(cfs.
↪ index[cfs.index <= T_bonds[j]])],np.array(cfs.loc[cfs.index <=
↪ T_bonds[j], str(j)]))).div(pd.DataFrame(np.dot(zcb_prices[:, :len(cfs
↪ .index[cfs.index <= T_bonds[j]])],np.array(cfs.loc[cfs.index <=
↪ T_bonds[j], str(j)]))))-1)[0].values

S = list()
S_ret = list()
for j in range(param_assets['n_E']):
S.append(np.exp(
(param_assets['sigma_S'][j] * W_S[j]).add([(param_assets['mu_S'][j] - (
↪ param_assets['sigma_S'][j] ** 2) / 2) * i for i in range(0,
↪ param_general['N'] - k + 1)],
axis=0)))
S_ret.append(((S[j].shift(-1) - S[j]).div(S[j])).iloc[:-1])

Gamma_estimate = [Gamma_k]

for j in range(param_general['N']-k):
ret_j = lambda_k[0] * C_ret.iloc[j]
for df, weight in zip(S_ret, [lambda_k[j] for j in range(1,param_assets['
↪ n_E']+1)]):
ret_j = ret_j.add(df.iloc[j] * weight)
for df, weight in zip(B_ret, lambda_k[-param_assets['n_B']:]):
ret_j = ret_j.add(df.iloc[j] * weight)

z_j_sim = [param_liabilities['i_min'] + max(param_liabilities['beta'] * (
↪ R_k-param_liabilities['i_min']),0) for R_k in ret_j]
z_j_estimate = sum(z_j_sim) / len(z_j_sim)
Gamma_estimate.append(Gamma_recursion(k+j+1, Gamma_estimate[-1], Psi,
↪ z_j_estimate, param_liabilities, param_general))

Gamma_estimate = pd.DataFrame(Gamma_estimate).transpose()
Gamma_estimate.index = ['a'] + ["pw_" + str(i + 1) for i in range(
↪ param_general['N'])]
Gamma_estimate.columns = [str(i) for i in range(k, param_general['N']+1)]
return Gamma_estimate

def liability_value_duration(k, I, Gamma_estimate, r_k, param_liabilities,
↪  data_liabilities, param_general, param_assets, scope = 'all'):
"""
Calculates the state-dependent liability values at time t_k.

:param k: time step
:param I: threshold interval for surrender/premium waiver probabilities
:param Gamma_estimate: estimates for future state-dependent bonus accounts
:param r_k: short rate at time t_k
:param param_liabilities: liability model parameters
:param data_liabilities: liability model data
```

```
:param param_general: general model parameters
:param param_assets: asset model parameters
:param scope: scope of calculation ('all' liability value and duration, 'liability value' for calculation of liability value only)
:return: state-dependent liability values at time t_k
"""

assert scope in ['all', 'liability value'], "Parameter 'scope' must be either 'all' or 'liability value'."

N = param_general['N']
states = ['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(N)]
L = pd.DataFrame(np.zeros((N+3, 1)), index = states, columns = ['L_i'])
D_L = pd.DataFrame(np.zeros((N+3, 1)), index = states, columns = ['D_L_i'])
V_pro = V_prospective_reserve(k, r_k, param_liabilities, param_general, I, data_liabilities, param_assets)

for l in range(k+1, N+1):
P_kl = P_X(k, l, I, param_liabilities, data_liabilities, param_general)
P_k_l_1 = P_X(k, l - 1, I, param_liabilities, data_liabilities, param_general)
P_l_1 = P_X(l - 1, l, I, param_liabilities, data_liabilities, param_general)
a_Pre_l = a_Pre(l, param_liabilities, param_general, list(Gamma_estimate[str(l)]))
a_Post_l_1 = a_Post(l-1, list(V_pro[str(l)]), param_liabilities, param_general, list(Gamma_estimate[str(l-1)]))

for j in range(N+3):
L.loc[states[j], 'L_i'] = L.loc[states[j], 'L_i'] + E_P_t_T_s(param_assets, k, l, k, r_k)*sum([a * b for a,b in zip(list(a_Pre_l[0]), list(P_kl.iloc[j,:]))])
L.loc[states[j], 'L_i'] = L.loc[states[j], 'L_i'] + E_P_t_T_s(param_assets, k, l, k, r_k)*sum([sum([a * pkl_1 * pl for a, pkl_1, pl in zip(list(a_Post_l_1.iloc[:,h]), list(P_k_l_1.iloc[j,:]), list(P_l_1.iloc[:,h]))]) for h in range(N+3)])
if scope == 'all':
D_L.loc[states[j], 'D_L_i'] = D_L.loc[states[j], 'D_L_i'] + (l-k)*E_P_t_T_s(param_assets, k, l, k, r_k)*sum([a * b for a,b in zip(list(a_Pre_l[0]), list(P_kl.iloc[j,:]))])
D_L.loc[states[j], 'D_L_i'] = D_L.loc[states[j], 'D_L_i'] + (l-k)*E_P_t_T_s(param_assets, k, l, k, r_k)*sum([sum([a * pkl_1 * pl for a, pkl_1, pl in zip(list(a_Post_l_1.iloc[:,h]), list(P_k_l_1.iloc[j,:]), list(P_l_1.iloc[:,h]))]) for h in range(N+3)])

D_L['D_L_i'] = D_L['D_L_i'].div(L['L_i']).fillna(0)
return [L, D_L]
```

ALM_simulation.py:

```
from scipy.optimize import minimize
import bisect
import time
```

```
import warnings
from Asset_Model import *
from Liability_Model import *
from Model_parameters import *


def main_ALM_simulation_optimisation(param_general, param_assets, param_liabilities, data_liabilities):
"""
Main function for the ALM simulation with optimisation of the asset portfolio
:param param_general: General model parameters
:param param_assets: Asset model parameters
:param param_liabilities: Liability model parameters
:param data_liabilities: Data for liability model
:return: simulation results
"""

N = param_general['N']
num_sim = param_general['num_sim']
i_min = param_liabilities['i_min']
beta = param_liabilities['beta']
Gamma_0 = param_liabilities['Gamma_0']
T_grid_hat = param_general['T_grid_hat']

# Simulation of asset processes for all simulation paths
W = simulate_corr_brownian_motion(param_assets, param_general)
r = simulate_short_rate(param_assets, param_general, W['W_r'])
C = simulate_bank_account(param_assets, param_general, r)
S = simulate_stocks(param_assets, param_general, W['W_S'])
B = simulate_bonds(param_assets, param_general, r)

# Reduction of asset processes to the relevant time grid
r = r.loc[T_grid_hat]
C = C.loc[T_grid_hat]
for i in range(len(S)):
S[i] = S[i].loc[T_grid_hat]
for j in range(len(B)):
B[j] = B[j].loc[T_grid_hat]

# Simulation of the Brownian motion for the estimation of the future Gamma_k for all simulation paths
W_Gamma = Gamma_estimation_corr_brownian_motion(param_assets, param_liabilities, param_general)

# Initialisation of main variables
ell = [[] for _ in range(N_hat + 1)]
lambda_A = [[] for _ in range(N_hat + 1)]
A = [[] for _ in range(N_hat + 1)]
L = [[] for _ in range(N_hat + 1)]
D_L = [[] for _ in range(N_hat)]
D_A = [[] for _ in range(N_hat)]

# Calculation of actuarial reserve Psi
Psi = calculation_actuarial_reserve(param_liabilities, param_general)
```

```
# Simulation of the ALM model for all simulation paths
for i in range(num_sim):
# Simulation asset paths
ri = r[str(i)].to_frame()
Ci = C[str(i)].to_frame()
Si = [S[j][str(i)].to_frame() for j in range(len(S))]
Bi = [B[j][str(i)].to_frame() for j in range(len(B))]

# Initialisation of insurance portfolio, asset portfolio composition and Gamma at t_0
ell[0].append([param_liabilities['n_L']] + list(np.zeros(N + 2)))
lambda_A[0].append(lambda_0)
Gamma = []
Gamma.append([Gamma_0] + [0 for _ in range(N)])

# Simulation of future profit participation account values
Gamma_estimate = Gamma_estimation(0, W_Gamma, param_assets['r0_r'], lambda_0, Gamma[-1], Psi, param_liabilities, param_assets, param_general)

# Liability value at t_0
L[0].append(param_liabilities['n_L'] * liability_value_duration(0, bisect.bisect_right(xi, Delta_R_0), Gamma_estimate, param_assets['r0_r'], param_liabilities, data_liabilities, param_general, param_assets, 'liability value')[0].loc['a'].sum())

# Asset portfolio value at t_0
A[0].append(L[0][-1] * (1 + param_assets['epsilon_0']))

# Simulation of time steps up to optimisation horizon N_hat
for k in range(1, N_hat + 1):
# Calculation of preceding return and z_k
R_C_k = (Ci[Ci.index == k].sum().sum() - Ci[Ci.index == k - 1].sum().sum()) / Ci[Ci.index == k - 1].sum().sum()
R_A_k = portfolio_return(k, lambda_A[k - 1][-1], Ci, Si, Bi, ri, param_assets)
Delta_R_k = max(max(i_min, beta * R_A_k) - R_C_k, 0)
z_k = i_min + max(beta * (R_A_k-i_min), 0)

# Calculation of new profit participation account values and I (threshold interval)
Gamma.append(Gamma_recursion(k, Gamma[-1], Psi, z_k, param_liabilities, param_general))
if k == 1:
I = bisect.bisect_right(xi, Delta_R_0)
else:
I = bisect.bisect_right(xi, Delta_R_k)

# Simulation of new insurance portfolio structure
[ell_k, cf_Post] = one_step_insurance_portfolio_simulation(k - 1, ell[k - 1][-1], I, param_liabilities, data_liabilities, param_general, ri, Gamma[k - 1], param_assets)
ell[k].append(ell_k)
```

```
# Calculation of cash flows at t_k / Updating the asset values at t_k
cf_Pre = sum([l * cf for l, cf in zip(ell_k, list(a_Pre(k, param_liabilities, param_general, Gamma[k]).iloc[:, 0]))])
A[k].append(A[k - 1][-1] * (1 + R_A_k) - cf_Post - cf_Pre)

# Simulation of future profit participation account values
Gamma_estimate = Gamma_estimation(k, W_Gamma, ri[ri.index == k].sum().sum(), lambda_A[k - 1][-1], Gamma[-1], Psi, param_liabilities, param_assets, param_general)

# Calculation of state-dependent liability values at and portfolio liability value at t_k
[L_k_i, D_L_k_i] = liability_value_duration(k, I, Gamma_estimate, ri[ri.index == k].sum().sum(), param_liabilities, data_liabilities, param_general, param_assets, 'all')
L_k = np.dot(ell[k][-1], list(L_k_i['L_i'].values))
L[k].append(L_k)

# Calculation of liability duration at t_k
D_L_k = sum([(ell_k_j * L_k_j / L_k) * D_k_j for ell_k_j, L_k_j, D_k_j in zip(ell[k][-1], list(L_k_i['L_i'].values), list(D_L_k_i['D_L_i'].values))])
D_L[k - 1].append(D_L_k)

### Optimisation of the asset portfolio composition at t_k
# Calculation of asset durations at t_k
D_B = duration_bonds(k, ri[ri.index == k].sum().sum(), param_assets)

# First stage of optimisation
def objective_function_stage1(lambda_opt, D_L_k, D_B, n_B):
return abs(np.multiply(lambda_opt[-n_B:], D_B).sum() / sum(lambda_opt[-n_B:]) - D_L_k)

def constraint1(lambda_opt):
return sum(lambda_opt) - 1

def constraint2(lambda_opt, alpha_S_max, n_E):
return alpha_S_max - lambda_opt[1:n_E + 1]

def constraint3(lambda_opt, alpha_S_min, n_E):
return lambda_opt[1:n_E + 1] - alpha_S_min

def constraint4(lambda_opt, alpha_C):
return lambda_opt[0] - alpha_C

def constraint5(lambda_opt):
return lambda_opt

def constraint6(lambda_opt, lambda_old, delta_lambda):
return delta_lambda - abs(lambda_opt - lambda_old)

constraints = [{'type': 'eq', 'fun': constraint1},
{'type': 'ineq', 'fun': constraint2,
  'args': (param_assets['alpha_S_max'], param_assets['n_E'])},
{'type': 'ineq', 'fun': constraint3,
```

```
  'args': (param_assets['alpha_S_min'], param_assets['n_E'])},
{'type': 'ineq', 'fun': constraint4, 'args': (param_assets['alpha_C'],)},
{'type': 'ineq', 'fun': constraint5},
{'type': 'ineq', 'fun': constraint6,
  'args': (np.array(lambda_A[k - 1][-1]), param_assets['delta_lambda'])}]

stage_1_result = minimize(objective_function_stage1, np.array(lambda_A[k - 1][-1]), method='SLSQP', args=(D_L_k, np.array(D_B), param_assets['n_B']), constraints=constraints)
min_duration_gap = stage_1_result.fun
optimised_lambda_stage1 = [round(i, 10) for i in list(stage_1_result.x)]

# stage 2 optimisation
bond_prices_k = [Bi[j].loc[k].values[0] for j in range(len(Bi))]
exp_returns_k = expected_one_period_returns(k, ri[ri.index == k].sum().sum(), param_assets, bond_prices_k)

def objective_function_stage2(lambda_opt, exp_returns_k):
return -sum(np.multiply(lambda_opt, exp_returns_k))

def constraint7(lambda_opt, min_duration_gap, epsilon, D_B, D_L_k, n_B):
return min_duration_gap + epsilon - abs(np.multiply(lambda_opt[-n_B:], D_B).sum() / sum(lambda_opt[-n_B:]) - D_L_k)

def constraint8(lambda_opt, min_duration_gap, epsilon, D_B, D_L_k, n_B):
return abs(np.multiply(lambda_opt[-n_B:], D_B).sum() / sum(lambda_opt[-n_B:]) - D_L_k) - min_duration_gap - epsilon

constraints = constraints + [
{'type': 'eq', 'fun': constraint7,
  'args': (min_duration_gap, param_assets['epsilon_0'], np.array(D_B), D_L_k, param_assets['n_B'])},
{'type': 'eq', 'fun': constraint8,
  'args': (min_duration_gap, param_assets['epsilon_0'], np.array(D_B), D_L_k, param_assets['n_B'])}]


stage_2_result = minimize(objective_function_stage2, np.array(optimised_lambda_stage1), method='trust-constr', args=(np.array(exp_returns_k)), constraints=constraints)
optimised_lambda_stage2 = [round(i, 10) for i in list(stage_2_result.x)]
lambda_A[k].append(optimised_lambda_stage2)
D_A[k - 1].append(np.multiply(np.array(optimised_lambda_stage2)[-param_assets['n_B']:], np.array(D_B)).sum() / sum(np.array(optimised_lambda_stage2)[-n_B:]))

lambda_results = [pd.DataFrame(np.transpose(np.array(lambda_A[k])), index=['C'] + ['S' + str(i + 1) for i in range(param_assets['n_E'])] + ['B' + str(i + 1) for i in range(param_assets['n_B'])]) for k in range(N_hat + 1)]
ell_results = [pd.DataFrame(np.transpose(np.array(ell[k])), index=['a', 's', 'd'] + ["pw_" + str(i + 1) for i in range(N)]) for k in range(N_hat + 1)]
A_results = pd.DataFrame(np.array(A), index=T_grid_hat)
L_results = pd.DataFrame(np.array(L), index=T_grid_hat)
```

```
    Equity = A_results - L_results
    D_L_results = pd.DataFrame(np.array(D_L), index=T_grid_hat[1:])
    D_A_results = pd.DataFrame(np.array(D_A), index=T_grid_hat[1:])

    return [lambda_results, ell_results, A_results, L_results, Equity,
     ↪ D_L_results, D_A_results]


# run the model
[lambda_results, ell_results, A_results, L_results, Equity, D_L_results,
 ↪ D_A_results] = main_ALM_simulation_optimisation(param_general,
 ↪ param_assets, param_liabilities, data_liabilities)
```

List of Figures

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