DISSERTATION

Refined entropy method for Fokker-Planck equations with confinement

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Kurzfassung

Diese Arbeit widmet sich der Analyse der Fokker–Planck-Gleichungen aus der kinetischen Gastheorie. Diese Gleichungen sind hilfreich bei der Untersuchung des kollektiven Verhaltens von Vielteilchensystemen in verschiedenen Bereichen (z. B. Physik, Biologie, Elektrotechnik und Sozialwissenschaften). Wir untersuchen das Langzeitverhalten und hypoelliptische Regularisierungseigenschaften von Lösungen. Im Speziellen untersuchen wir die kinetischen Fokker–Planck-Gleichungen, das Vlasov–Poisson–Fokker–Planck-System und die relativistische kinetische Fokker–Planck-Gleichung. Für jede der betrachteten Gleichungen gibt es ein eindeutiges globales Gleichgewicht (oder einen stationären Zustand). Aufgrund der dissipativen Struktur dieser Gleichungen wird erwartet, dass die Lösungen im Laufe der Zeit gegen das entsprechende globale Gleichgewicht konvergieren. Wir beweisen diese Konvergenz und erhalten explizite und konstruktive Schätzungen der Konvergenzraten in Abhängigkeit vom Ausgangsdatum und den in den Gleichungen vorkommenden Parametern. Untersuchungen über das Konvergenzverhalten zum Gleichgewicht sind für Anwendungen in der Physik (z. B. Gleichgewichtsprozesse, numerische Simulationen) unerlässlich. Sie geben Auskunft über die Zeitskala der Konvergenz zum Gleichgewicht und damit über das qualitative Verhalten der Modelle sowie deren Gültigkeit. Unsere Beweistechnik basiert auf der Modifikation von Entropie-Entropie-Dissipationsansätzen, Hypokoerziviätsmethoden und der Konstruktion geeigneter Lyapunov-Funktionale.

Die Arbeit besteht aus vier Kapiteln. In Kapitel 1 wollen wir die Physik hinter den Fokker–Planck-Gleichungen, ihre Herleitung und ihr Langzeitverhalten vorstellen. In Kapitel 2 wird die kinetische Fokker–Planck-Gleichung mit einem Einschließungspotential analysiert. Wir entwickeln eine modifizierte Entropiemethode, mit der wir hypoelliptische Regularität von Lösungen und deren exponentielle Konvergenz zum stationären Zustand in einem gewichteten H^1-Raum mit expliziten konstruktiven Raten beweisen können. In Kapitel 3 wird für das nichtlineare Vlasov–Poisson–Fokker–Planck-System die Wohlgestelltheit, die hypoelliptische Regularität von Lösungen und deren Konvergenz zum stationären Zustand bewiesen. In Kapitel 4 wird die relativistische kinetische Fokker–Planck-Gleichung untersucht.

Abstract

This thesis is devoted to the analysis of the Fokker-Planck equations coming from the kinetic theory of gases. These equations arise in the study of the collective behavior of many-particle systems in various fields (e.g., physics, biology, electrical engineering, and social sciences). We study long time behavior and hypoelliptic regularizing properties. The main models of interest are *the kinetic Fokker-Planck equations, the Vlasov-Poisson-Fokker-Planck system,* and *the relativistic kinetic Fokker-Planck equation.* For each equation there is a unique global equilibrium (or steady state). Because of the dissipative structure of these equations, the solutions are expected to converge to the corresponding global equilibrium as time goes infinity. We prove this convergence and obtain explicit and constructive estimates on rates of convergence, in terms of the initial datum and the parameters appearing in the equations. Studies on the trend to equilibrium are essential for applications in physics (e.g., equilibration process, numerical simulations). They give information on the time scale for relaxation to equilibrium, and therefore on the qualitative behavior of the models as well as on their validity. The technique of our proofs is based on modification of entropy-entropy dissipation approaches, hypocoercivity methods, and construction of suitable Lyapunov functionals.

The thesis has four chapters. In Chapter 1, we aim to introduce the physics behind the Fokker-Planck equations, their derivation and long time behavior. Chapter 2 analyze the kinetic Fokker-Planck equation with a confining potential. We develop a modified entropy method which lets us prove exponential decay of solutions to the steady state in a weighted H^1 space with explicit constructive rates as well as hypoelliptic regularity. Chapter 3 is devoted to establish well-posedness, hypoelliptic regularity, and convergence to the steady state for the nonlinear Vlasov-Poisson-Fokker-Planck system. Chapter 4 studies the relativistic kinetic Fokker-Planck equation.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am Datum

Gayrat Toshpulatov

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Contents

Chapter 1

Introduction

This thesis is about the long time behavior and hypoelliptic regularizing properties of Fokker-Planck equations which arise in the kinetic theory of gases. This introduction is my own review of the background and the current literature. We first discuss how one can study many-particle systems statistically using the kinetic theory of gases and the partial differential equations in this field. Next, we explain how the Fokker-Planck equations are derived to study the time evolution of many-particle systems. After this we show that the Fokker-Planck equations have steady solutions and the solution is supposed to converge to them as time goes infinity. Lastly we briefly describe the contents of each of the chapters.

1.1 Many-particle system

Consider the time evolution of $N >> 1$ particles of a gas (or any system made up of a large number of particles) in some domain of $\mathbb{R}^d$. If we follow classical mechanics, the position $x_i(t) \in \mathbb{R}^d$ and the velocity $v_i(t) \in \mathbb{R}^d$ of particle i, $i \in \{1, ..., N\}$, at time t satisfy the following Newton's equations

$$\begin{cases} \dfrac{dx_i(t)}{dt} = v_i(t), \\ m_i \dfrac{dv_i(t)}{dt} = F + \displaystyle\sum_{j=1,\, j\neq i}^{N} F_{ij}, \end{cases} \tag{1.1}$$

where m_i is the mass of particle i. F_{ij} denotes the force acting on particle i due to particle j, and F is an external force. The sum $F + \sum_{j=1,\, j\neq i}^{N} F_{ij} \in \mathbb{R}^d$ means the total force exerted on particle i. To get information about the time evolution of the particles we need to solve these equations, but we immediately face serious difficulties: A typical gas contains approximately $N \approx 10^{23}$ particles and solving $N \approx 10^{23}$ coupled differential equations is extremely difficult. Secondly, the system (1.1) is deterministic and so we get different solutions for different initial conditions. Before thinking about a solution, we need to know the precise position and the velocity of each particle at $t = 0$. This is practically impossible. Therefore, it is generally impossible to solve the system (1.1). Moreover, experimental evidences show that the trajectories of (1.1) are extremely complicated and unstable with respect to minor changes of the initial conditions. However, the *macroscopic* or *statistical* properties (e.g., density, pressure, average velocity, temperature) of a gas are generally stable and largely independent of the precise initial conditions.

The kinetic theory (more generally nonequilibrium statistical mechanic), introduced by Maxwell, Boltzmann, Gibbs and others at the end of the nineteenth century, describes a gas or any system made up of a large number of particles from a statistical point of view forgetting about the individual trajectories. It makes the link between the fundamental laws of mechanics which describe the motion of the particles and the time evolution of the macroscopic properties of the system.

The calculations of the macroscopic properties of a gas requires the knowledge of the *distribution function* of the particles. Hence, the main object in kinetic theory is the distribution function $f = f(t, x, v) \geq 0$ in the particle phase space. For an infinitesimal volume $dxdv$ around the point (x, v) of phase space, $f(t, x, v)dxdv$ represents the number of particles in the volume element $dxdv$. This distribution function allows to compute the macroscopic quantities. For example, the *local density* $n = n(t, x)$ is defined by

$$n(t,x) := \int_{\mathbb{R}^d} f(t,x,v)dv.$$

$n(t, x)dx$ means the number of particles in the element dx. The *total number N of the particles* in the gas is

$$N = \int_{\mathbb{R}^{2d}} f(t,x,v)dxdv.$$

Similarly, the *average velocity* $u(t, x)$ and the *average temperature* $T = T(t, x)$ are defined by

$$n(t,x)u(t,x) := \int_{\mathbb{R}^d} vf(t,x,v)dv,$$

$$n(t,x)(|u(t,x)|^2 + dT(t,x)) := \int_{\mathbb{R}^d} |v|^2 f(t,x,v)dv.$$

One of the important problems in kinetic theory is to derive a reasonable equation for the time evolution of f. If we assume all particles have the same mass m and if we neglect the interaction between particles (i.e., the forces F_{ij} are zero for all $i, j \in \{1, ..., N\}$), then one can easily obtain the following equation for f

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{1}{m} F \cdot \nabla_v f = 0 \\ f_{|t=0} = f_0. \end{cases} \tag{1.2}$$

If the binary interactions between the particles are not described individually and if each particle generate a global interaction force, then F can be generated by the gas particles and computed by the distribution function f. For example, in plasma physics, the distribution function f of a dilute plasma (which is a gas of charged particles) satisfies (1.2) with the force

$$F = -\nabla_x(V + \phi), \tag{1.3}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is defined by $-\Delta_x \phi = \int_{\mathbb{R}^d} f dv$ and it means the self-consistent electrostatic potential. $V : \mathbb{R}^d \to \mathbb{R}$ is a given function which means an external electrostatic potential. The equation (1.2) with the force defined in (1.3) is called the *Vlasov-Poisson system.*

In reality, the particles do interact and collide with each other. In this case deriving an equation for f is challenging. The equation generally should take the following form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{1}{m} F \cdot \nabla_v f = Q[f], \\ f_{|t=0} = f_0, \end{cases} \tag{1.4}$$

where the operator $Q[f]$ presents the effect of the collisions. Depending on the interaction, various forms of $Q[f]$ have been suggested, for example, the Fokker-Planck, Boltzmann, Landau, and BGK collision operators. These collision operators allow that (1.4) admit several local conservation laws: mass, momentum, and energy. We present Boltzmann's collision operator which is fundamental in kinetic theory: In 1872 Ludwig Boltzmann was able to derive the following operator for single atomic dilute gases (assuming that elastic binary collisions happen)

$$Q[f](t,x,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(v-v_*,\sigma)[f(t,x,v')f(t,x,v'_*) - f(t,x,v)f(t,x,v_*)]dv_* d\sigma, \quad (1.5)$$

where $B : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$ is a given non-negative collision kernel, $v' = v - \sigma[(v-v_*)\cdot\sigma]$, $v'_* = v_* + \sigma[(v-v_*)\cdot\sigma]$. For a detailed presentation of the various models and their derivation from the fundamental laws of physics, we refer to the books [21, 7, 24, 19].

In the following we explain how to derive the Fokker-Planck equations using the Brownian motion theory.

1.2 Derivation of Fokker-Planck equations

A Fokker-Planck equation was introduced by Fokker [13] and Planck [20] to describe the Brownian motion of particles. The Fokker-Planck equation usually describes the time evolution of the distribution function f for a small subsystem of particles: As we discussed in the section above let us consider the time evolution of $N >> 1$ particles in some domain of $\mathbb{R}^d$. Assume $\tilde{N}$ particles with $\tilde{N} << N$ are considerable larger and heavier than the other particles. If f is the distribution function of these $\tilde{N}$ particles, then the time evolution of f is often described by Fokker-Planck equations. In the following we explain with examples.

Let one particle of mass m be immersed in a dense fluid. We call it $\mathfrak{B}-$particle (Brownian particle). Let the $\mathfrak{B}-$particle be larger and heavier than the fluid particles. If the fluid has $N >> 1$ particles, then the $\mathfrak{B}-$particle and the fluid particles form a system of $N+1$ particles. Following classical mechanics we can write $N+1$ coupled differential equation as in (1.1) to describe the time evolution of the $\mathfrak{B}-$particle and the fluid particles, but we are more interested in the dynamics of the $\mathfrak{B}-$particle. Since there is a friction force $-\theta v(t)$, $\theta > 0$, in the fluid by Stokes' law, the position $x(t) \in \mathbb{R}^d$ and the velocity $v(t) \in \mathbb{R}^d$ of the $\mathfrak{B}-$particle satisfy the following Newton's equations (similar to (1.1))

$$\begin{cases} \dfrac{dx(t)}{dt} = v(t), \\ m\dfrac{dv(t)}{dt} = -\theta v(t) + F + \displaystyle\sum_{j=1}^{N} F_j, \end{cases} \quad (1.6)$$

where F_j denotes the force acting on the $\mathfrak{B}-$particle due to the fluid particle j and F is an external force. As the fluid is very dense, the $\mathfrak{B}-$particle suffers a large number of collisions with the fluid particles in a very short time. Hence, the collisions with the $\mathfrak{B}-$particle are so frequent and irregular that we can not speak of separate collisions. Since the $\mathfrak{B}-$particle is very heavy and large compared to the fluid particles, the effect of each collision with a fluid particle produce a tiny deflection or *fluctuation* of $\mathfrak{B}-$particle. Therefore, the total force $\sum_{j=1}^{N} F_j$ of the fluid particle acting on $\mathfrak{B}-$particle is erratic and fluctuates a lot. Since it is practically impossible to compute $\sum_{j=1}^{N} F_j$, we replace it with a stochastic force $A(t)$ (or stochastic process) which depends on some probability law. In this way we convert (1.6) into a stochastic differential equation [16, 18]

$$\begin{cases} \dfrac{dx(t)}{dt} = v(t), \\ m\dfrac{dv(t)}{dt} = -\theta v(t) + F + A(t). \end{cases} \tag{1.7}$$

This equation means the position $x(t)$ and the velocity $v(t)$ are now stochastic processes and depend on the probability law of $A(t)$. In the literature (1.7) is called *the Langevin equation* [14, 8, 9, 22]. To proceed further, one has to make reasonable quantitative assumptions on the stochastic force $A(t)$. The first assumption is its average (expectation) is zero

$$\mathbb{E}[A(t)] = 0, \quad \forall t \geq 0$$

because the fluctuations cancel each other on average. Next we assume

$$\mathbb{E}[A(t_1) \otimes A(t_2)] = 2\theta k_B T \delta_0(t_1 - t_2) I,$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix, $\delta_0(t)$ is the delta function, $k_B > 0$ is Boltzmann's constant, and $T > 0$ is the temperature. This condition expresses that the fluid particles that strike the $\mathfrak{B}-$particle and give rise to the stochastic force $A(t)$ behave independently of one another except when they act at the same time. Finally, we require that all higher-order moments can be expressed in terms of the second moments. More precisely, one usually assumes that $\frac{1}{\sqrt{2\theta k_B T}} A(t)$ is distributed according to a Gaussian law, which is connected to the law of large numbers of probability theory. $\frac{1}{\sqrt{2\theta k_B T}} A(t)$ is also called *white noise* and formally denoted as the time derivative of the Brownian motion $B(t)$, i.e., $\frac{1}{\sqrt{2\theta k_B T}} A(t) =: \frac{dB(t)}{dt}$.

Let $f = f(t, x, v)$ be the probability density that the $\mathfrak{B}-$particle has coordinates x, v at time t. Because of the assumption on $A(t)$, we can easily pass from the stochastic differential equation (1.7) to a deterministic evolution equation for f using Kolmogorov's forward equation [16, Section 8], [18, Section 2]. This leads to the famous *kinetic Fokker-Planck equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{1}{m} F \cdot \nabla_v f = \dfrac{\theta}{m} \mathrm{div}_v \left(\dfrac{k_B T}{m} \nabla_v f + vf \right), & x, v \in \mathbb{R}^d, t > 0, \\ f_{|t=0} = f_0, \end{cases} \tag{1.8}$$

where f_0 denotes the probability density of the $\mathfrak{B}-$particle at $t = 0$. Note that (1.8) is a particular case of (1.4) with the collision operator $Q[f] = \frac{\theta}{m} \mathrm{div}_v \left(vf + \frac{k_B T}{m} \nabla_v f \right)$.

Instead of a single particle, we can consider a system containing $\tilde{N}$ number of $\mathfrak{B}-$particles with the same mass m which are immersed in a dense fluid. We assume $\mathfrak{B}-$particles are larger and heavier than the fluid particles. We use the notation $f = f(t, x, v)$ to denote the joint distribution function of these $\mathfrak{B}-$particles, i.e., $f(t, x, v)dxdv$ is the probable number of $\mathfrak{B}-$particles lying in the volume element $dxdv$ around (x, v). This implies $\int_{\mathbb{R}^{2d}} f dxdv = \tilde{N}$. Let the fluid have $N >> 1$ particles. If $\mathfrak{B}-$particles are sufficiently dilute compared to the fluid particles (i.e., $\tilde{N} << N$), then the collisions between $\mathfrak{B}-$particles are negligible. We only need to take into account the collisions between $\mathfrak{B}-$particles and the fluid particles. Similarly these collisions can be described by (1.7) for each $\mathfrak{B}-$particle, and we can show that the Fokker-Planck equation (1.8) remains valid for the joint distribution function f. As a concrete example, we consider a dilute plasma in a thermal bath or reservoir. To deal with the collision effects of the plasma with the particles of the reservoir, one considers the position and the velocities of each individual particle of the plasma as a stochastic process and use the Langevin equation (1.7) with the force F defined in (1.3). Hence, we obtain the following equation, called the *Vlasov-Poisson-Fokker-Planck system*, for the distribution function f of the plasma

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \dfrac{1}{m}(\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \dfrac{\theta}{m} \mathrm{div}_v \left(\dfrac{k_B T}{m} \nabla_v f + v f \right), \quad x, v \in \mathbb{R}^d, \ t > 0 \\ -\Delta_x \phi = \displaystyle\int_{\mathbb{R}^d} f dv, \quad f_{|t=0} = f_0. \end{cases} \tag{1.9}$$

The kinetic Fokker-Planck equation (1.8) is a special case of the following *generalized Fokker-Planck equation*

$$\begin{cases} \partial_t f = \displaystyle\sum_{i,j=1}^d \partial^2_{\xi_i \xi_j} (a_{ij} f) - \sum_{i=1}^d \partial_{\xi_i} (b_i f) = \mathrm{div}_\xi \left(D \nabla_\xi f + \tilde{b} f \right), \quad \xi \in \mathbb{R}^d, \ t > 0, \\ f_{|t=0} = f_0, \end{cases} \tag{1.10}$$

where $b = b(\xi) = \begin{pmatrix} b_1(\xi) \\ \vdots \\ b_d(\xi) \end{pmatrix}$ is a vector function, $D = D(\xi) = \{a_{ij}(\xi)\}_{i,j=1}^d$ is a symmetric positive semi-definite matrix function, and $\tilde{b} := -b + \mathrm{div}_\xi D$. (1.10) describes the time evolution of distribution functions of, for example, *diffusion* processes and stochastic differential equations, see [18, Chapter 2].

We note that we obtained (1.8) following the laws of classical mechanics. This equation is inconsistent with relativistic mechanics because it has infinite speed of propagation: if the particles are initially in a compact region (i.e., $f_0(x, v)$ has a compact support with respect to x), then, after any short time $t > 0$, we can find particles everywhere with non-zero probability (i.e., $f(t, x, v) > 0$). If we study the dynamics of particles following relativistic mechanics, we need to replace the equations (1.6) and (1.7) with relativistic ones. This case was studied in [12, 4] by developing the relativistic theory of Brownian motions. The following *relativistic Fokker-Planck equation* was obtained

$$\begin{cases} \partial_t f + \dfrac{p}{m\sqrt{1 + \frac{|p|^2}{m^2 c^2}}} \cdot \nabla_x f + F \cdot \nabla_p f = \theta \mathrm{div}_p (k_B T m D(p) \nabla_p f + p f), \quad x, p \in \mathbb{R}^d, \ t > 0, \\ f_{|t=0} = f_0, \end{cases} \tag{1.11}$$

where p means the momentum, c is the vacuum speed of light, and $D(p) = \dfrac{I + \frac{p \otimes p}{m^2 c^2}}{\sqrt{1 + \frac{|p|^2}{m^2 c^2}}} \in \mathbb{R}^{d \times d}$ is the relativistic diffusion matrix. If we formally let $c \to \infty$, then we obtain (1.8) with $p = mv$.

1.3 Long time behavior of Fokker-Planck equations

Let us consider the generalized Fokker-Planck equation (1.10) with smooth D and $\tilde{b}$. We look for a steady solution $f_\infty = f_\infty(\xi)$, which is a probability density satisfying

$$\mathrm{div}_\xi \left(D \nabla_\xi f_\infty + \tilde{b} f_\infty \right) = 0.$$

It was proven in [11, Theorem 1] that a probability density f_∞ is a steady state if and only if

$$\tilde{b} = -D\nabla_\xi \log f_\infty + G,$$

where $G = G(\xi) \in \mathbb{R}^d$ satisfies

$$\operatorname{div}_\xi(Gf_\infty) = 0, \quad \forall\, \xi \in \mathbb{R}^d. \tag{1.12}$$

Let us assume that there is a steady state f_∞ and it is given as $f_\infty(\xi) = c_E e^{-E(\xi)}$ for some smooth function E. We assume $E(\xi)$ grows fast enough as $|\xi| \to \infty$ so that $e^{-E} \in L^1(\mathbb{R}^d)$. Here $c_E := \left(\int_{\mathbb{R}^d} e^{-E} d\xi\right)^{-1}$ yields $\int_{\mathbb{R}^d} f_\infty d\xi = 1$. Then (1.10) can be written as

$$\begin{cases} \partial_t f = \operatorname{div}_\xi \left(D[\nabla_\xi f + f\nabla_\xi E] + Gf\right), & \xi \in \mathbb{R}^d,\ t > 0, \\ f_{|t=0} = f_0. \end{cases} \tag{1.13}$$

One of the important property of (1.13) is that it makes a certain *entropy functional* decrease, and this entropy achieves its minimum value for some subfamily of Gaussian distributions: Let $\psi \in C[0,\infty) \cap C^4(0,\infty)$ satisfy $\psi(1) = \psi'(1) = 0$, $\psi'' > 0$ and $(\psi''')^2 \le \frac{1}{2}\psi''\psi^{IV}$ on $(0,\infty)$. The most typical examples of such functions are $\psi(s) = s\log s - s + 1$ and $\psi(s) = (s-1)^2$. We define a functional on the space of probability densities

$$f \mapsto \mathrm{H}[f] := \int_{\mathbb{R}^d} \psi\left(\frac{f}{f_\infty}\right) f_\infty d\xi \ge 0. \tag{1.14}$$

If $f = f(t,\xi)$ is a (well-behaved) solution to (1.13), then integration by parts and (1.12) show that

$$\frac{d}{dt}\mathrm{H}[f(t)] = -\int_{\mathbb{R}^d} \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla_\xi^T \left(\frac{f(t)}{f_\infty}\right) D\nabla_\xi \left(\frac{f(t)}{f_\infty}\right) f_\infty d\xi.$$

Since D is positive semi-definite, $\frac{d}{dt}\mathrm{H}[f(t)]$ is nonpositive and so $\mathrm{H}[f(t)]$ is a decreasing function of $t \ge 0$. Because of this decay one may conjecture that this functional decays to its minimum (which is zero) as $t \to \infty$. Since this minimum is attained at $f = f_\infty$, one can argue that the solution $f(t)$ converges to f_∞ as $t \to \infty$. An interesting problem is now to prove (or disprove) this convergence and to obtain explicit and constructive estimates on the rate of convergence. When the matrix $D = D(\xi)$ is positive definite for all $\xi \in \mathbb{R}^d$, this problem has been comprehensively studied, see [5, 6, 2, 1]. One of the well-known conditions which provides the exponential decay of the solution to the steady state is called *the Bakry-Emery condition* (see (1.15) below), we state it for simplicity when D is a constant matrix:

Theorem 1.3.1 ([1, Theorem 2.6]). *Assume* $\mathrm{H}[f_0] < \infty$, $D \in \mathbb{R}^{d\times d}$ *is positive definite, and*

$$\exists \lambda > 0 \ \textit{such that}\ D\frac{\partial^2 E}{\partial \xi^2}D - \frac{1}{2}\left(\frac{\partial G}{\partial \xi} + \left(\frac{\partial G}{\partial \xi}\right)^T\right) \ge \lambda D, \ \ \forall \xi \in \mathbb{R}^d. \tag{1.15}$$

Then

$$\mathrm{H}[f(t)] \le e^{-2\lambda t}\mathrm{H}[f_0]. \tag{1.16}$$

Sketch of the proof. We have seen that $\frac{d}{dt}\mathrm{H}[f(t)]$ is non-positive. We define the *dissipation functional (or Fisher information)*

$$I[f(t)] := -\frac{d}{dt}\mathrm{H}[f(t)] = \int_{\mathbb{R}^d} \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla_\xi^T \left(\frac{f(t)}{f_\infty}\right) D\nabla_\xi \left(\frac{f(t)}{f_\infty}\right) f_\infty d\xi \ge 0. \tag{1.17}$$

The key idea of Bakry and Emery is to estimate the time derivative of $I[f(t)]$. Under the assumptions of the theorem they proved that

$$\frac{d}{dt}I[f(t)] \leq -2\lambda I[f(t)]$$

holds with λ from (1.15). Integrating this inequality from (t,∞) and using the convergence $\frac{d}{dt}\mathrm{H}[f(t)] \to 0$ and $\mathrm{H}[f(t)] \to 0$ as $t \to \infty$ (whose proofs are nontrivial), we obtain

$$\frac{d}{dt}\mathrm{H}[f(t)] \leq -2\lambda \mathrm{H}[f(t)].$$

Then Gronwall's lemma implies (1.16). As $\mathrm{H}[f]$ is zero if and only if $f = f_\infty$, it tell us that $f(t)$ converges to f_∞ exponentially as $t \to \infty$. □

We now return to the kinetic Fokker-Planck equation (1.8). First, we consider the very simple homogeneous case, i.e., we assume $F = 0$ and f does not depend on x :

$$\begin{cases} \partial_t f(t,v) = \frac{\theta}{m}\mathrm{div}_v\left(\frac{k_BT}{m}\nabla_v f(t,v) + vf(t,v)\right), & v\in\mathbb{R}^d,\ t>0,\\ f(0,v) = f_0(v). \end{cases} \tag{1.18}$$

This equation is a special case of (1.13) with $\xi = v$, $D = \frac{\theta k_BT}{m^2}I$, $E = \frac{m|v|^2}{2k_BT}$, and $G = 0$. We can check they satisfies condition (1.15) with $\lambda = \frac{\theta}{m}$. Hence, Theorem 1.3.1 implies that the solution $f(t,v)$ converges to the steady state $M(v) := \left(2\pi\frac{k_BT}{m}\right)^{-d/2} e^{-\frac{m|v|^2}{2k_BT}}$ exponentially.

One can also similarly use Theorem 1.3.1 to obtain an exponential decay result for the relativistic homogeneous Fokker-Planck equation

$$\begin{cases} \partial_t f(t,p) = \theta\mathrm{div}_p(k_BTmD(p)\nabla_p f(t,p) + pf(t,p)), & p\in\mathbb{R}^d,\ t>0,\\ f(0,p) = f_0(p). \end{cases}$$

with $D(p)$ as in (1.11). We discuss this equation with more details in Chapter 4.

Next, we consider the full kinetic Fokker-Planck equation (1.8) with the force

$$F = -\nabla_x V(x),$$

where $V:\mathbb{R}^d\to\mathbb{R}$ is a given smooth function such that $e^{-\frac{1}{k_BT}V}\in L^1(\mathbb{R}^d)$. This assumption ensures that there is a steady state $f_\infty = c_V e^{-\frac{1}{k_BT}V}M(v)$ with $c_V = \left(\int_{\mathbb{R}^d} e^{-\frac{1}{k_BT}V}dx\right)^{-1}$. We denote $\xi := \begin{pmatrix} x\\ v\end{pmatrix}\in\mathbb{R}^{2d}$, $E(\xi) := \frac{1}{k_BT}V(x) + \frac{m|v|^2}{2k_BT}$. Then the kinetic Fokker-Planck equation (1.8) can be written in the form of (1.13)

$$\partial_t f = \mathrm{div}_\xi(D[\nabla_\xi f + f\nabla_\xi E] + Gf)$$

with

$$D = \begin{pmatrix} 0 & 0\\ 0 & \frac{\theta k_BT}{m^2}I\end{pmatrix}\in\mathbb{R}^{2d\times 2d} \quad\text{and}\quad G = \begin{pmatrix} v\\ -\frac{1}{m}\nabla_x V\end{pmatrix}\in\mathbb{R}^{2d}. \tag{1.19}$$

Since the matrix D in (1.19) is positive semi-definite, Theorem 1.3.1 unfortunately does apply for the kinetic Fokker-Planck equation (1.8) and so it does not allow to conclude the convergence $f(t)\to f_\infty$ as $t\to\infty$. Similarly, we can write the

Vlasov-Poisson-Fokker-Planck system and the relativistic kinetic Fokker-Planck equation in the form of (1.13) with positive semi-definite matrices D. Hence, we also face the same problem for these equations. However, the functional $\mathrm{H}[f(t)]$ defined in (1.14) still decreases under the evolution of the solution $f(t)$. Hence, we can still expect the convergence $f(t) \to f_\infty$ as $t \to \infty$. The degeneracy of D makes proving this convergence challenging and requires to develop new techniques and methods. This is the main goal of this thesis. We study the long time behavior of the kinetic Fokker-Planck equation (1.8), the Vlasov-Poisson-Fokker-Planck (1.9), and the relativistic kinetic Fokker-Planck equation (1.11). We provide a new method to establish exponential decay of solutions to their corresponding steady states for a wide class of potentials V. Our method relies on the construction of Lyapunov functionals by modifying the entropy functional. We provide a formula to estimate easily the exponential decay rate depending on the parameters of the equations. Moreover, our method lets us obtain estimates on the hypoelliptic regularization of $f(t)$.

1.4 Outline of the thesis

The rest of the thesis is divided into three parts:

Chapter 2: The kinetic Fokker-Planck equation

This chapter contains a work in collaboration with Prof. Anton Arnold (TU Wien) and is taken from [3]. This paper was published in *Journal of Statistical Physics.*

We consider the kinetic Fokker-Planck equation (1.8) with the force $F = -\nabla_x V(x)$. By changing V to mV and denoting $\sigma := \frac{\theta k_B T}{m^2}$ and $\nu := \frac{\theta}{m}$, (1.8) can be written as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \nu \mathrm{div}_v(vf) + \sigma \Delta_v f, \;\; x,\, v \in \mathbb{R}^d,\, t > 0 \\ f_{|t=0} = f_0 \in L^1(\mathbb{R}^{2d}) \end{cases} . \tag{1.20}$$

The unique normalized steady state is given by

$$f_\infty(x,v) = c_V e^{-\frac{\nu}{\sigma}[V(x)+\frac{|v|^2}{2}]}, \;\; x, v \in \mathbb{R}^d,$$

where c_V is a positive constant such that $\int_{\mathbb{R}^{2d}} f_\infty(x,v)dxdv = 1$.

We develop a modified entropy method to establish the large-time convergence towards the unique global equilibrium for the kinetic Fokker-Planck equation (1.20) with a non-quadratic confinement potentials V. We introduce Lyapunov functionals with non-constant weight matrices in the dissipation functional (a generalized Fisher information). We establish exponential convergence in a weighted H^1-norm with rates that become sharp in the case of quadratic potentials. The presented method also gives new hypoelliptic regularization results for the kinetic Fokker-Planck equation (from a weighted L^2-space to a weighted H^1-space).

In the following we explain the main ideas in Chapter 2. As we discussed in the previous section, when the diffusion matrix D is positive definite, the Bakry-Emery method (or entropy method) lets us prove that the solution of (1.13) converges to the steady state as $t \to \infty$. The key idea of Bakry-Emery is to estimate the time derivative of the dissipation functional (see (1.17) with $\psi(s) = (s-1)^2$)

$$I[f(t)] = 2\int_{\mathbb{R}^d} \nabla_\xi^T \left(\frac{f(t)}{f_\infty}\right) D \nabla_\xi \left(\frac{f(t)}{f_\infty}\right) f_\infty d\xi$$

and to obtain

$$\frac{d}{dt}I[f(t)] \leq -2\lambda I[f(t)], \;\; \forall\, t > 0 \tag{1.21}$$

for some $\lambda > 0$. In the case of the kinetic Fokker-Planck equation (1.20), the diffusion matrix (see (1.19))

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \sigma I \end{pmatrix} \in \mathbb{R}^{2d\times 2d}$$

is positive semi-definite. Hence, in general, (1.21) does not hold for the dissipation function $I[f(t)]$ of the kinetic Fokker-Planck equation. Since the problem comes from the positive semi-definite matrix D, we introduce a modified dissipation functional (a generalized Fisher information) with a non-constant, positive definite matrix P :

$$S[f(t)] := 2\int_{\mathbb{R}^{2d}} \nabla^T_{x,v}\left(\frac{f}{f_\infty}\right) P(t,x) \nabla_{x,v}\left(\frac{f}{f_\infty}\right) f_\infty dx dv.$$

We note this functional coincides with $I[f(t)]$ if $P = D$. Our goal is to choose a suitable matrix P so that $S[f(t)]$ satisfies the following differential inequality (like the dissipation functional satisfied for non-degenerate equations):

$$\frac{d}{dt}S[f(t)] \leq -2\lambda S[f(t)], \quad \forall\, t > 0 \tag{1.22}$$

for some $\lambda > 0$. We compute

$$\frac{d}{dt}S[f(t)] = -2\int_{\mathbb{R}^{2d}} \nabla^T_{x,v}\left(\frac{f}{f_\infty}\right) \left\{QP + PQ^T\right\} \nabla_{x,v}\left(\frac{f}{f_\infty}\right) f_\infty dx dv + \text{some terms},$$

where $Q = Q(x) := \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}$. We observe that, if $\frac{\partial^2 V(x)}{\partial x^2}$ is positive definite, then $Q(x)$ is positive stable. This lets us construct a positive definite matrix P such that

$$QP + PQ^T \geq 2\mu P$$

holds for some constant $\mu > 0$. More precisely, the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + 2aI \end{pmatrix}$$

with a suitable constant $a > 0$ satisfies this inequality. This let us estimate

$$\frac{d}{dt}S[f(t)] \leq -2\mu S[f(t)] + \text{some terms}.$$

The next step is to control "some terms" in the estimates above. This can be done under some growth assumptions on V, and we obtain (1.22) with some $\lambda \in (0, \mu]$. Then Grönwall's lemma implies

$$S[f(t)] \leq e^{-2\lambda t} S[f_0], \quad \forall\, t \geq 0. \tag{1.23}$$

We also show that $S[f(t)]$ is equivalent to a weighted H^1-norm

$$\int_{\mathbb{R}^{2d}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} \left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv$$
$$+ \int_{\mathbb{R}^{2d}} \nabla^T_v\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv,$$

where $\alpha_0 := \inf_{x\in\mathbb{R}^d} \alpha(x)$ and $\alpha(x)$ is the smallest eigenvalue of $\frac{\partial^2 V(x)}{\partial x^2}$. This equivalence and (1.23) leads one of the main results of Chapter 2, which is the exponential decay of the solution in the weighted H^1-norm:

$$\int_{\mathbb{R}^{2d}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv$$
$$+ \int_{\mathbb{R}^{2d}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv$$
$$\leq Ce^{-2\lambda t} \left[\int_{\mathbb{R}^{2d}} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} \left|\nabla_x \left(\frac{f_0}{f_\infty}\right)\right|^2 f_\infty dxdv \right.$$
$$\left. + \int_{\mathbb{R}^{2d}} \nabla_v^T \left(\frac{f_0}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v \left(\frac{f_0}{f_\infty}\right) f_\infty dxdv\right],$$

for some $C > 0$. We also provide a formula to estimate the decay rate λ explicitly and we also check that it is sharp when V is quadratic polynomial. Next, we choose a time dependent matrix

$$P = P(t,x) := \begin{pmatrix} 2\varepsilon^3 t^3 I & \varepsilon^2 t^2 I \\ \varepsilon^2 t^2 I & 2\varepsilon t I + t(\frac{\partial^2 V}{\partial x^2} + aI) \end{pmatrix},$$

where ε and a are some suitable constants. We prove that, with this matrix P, the modified dissipation functional $S[t, f(t)]$ is decreasing for $t \in (0, t_0]$, $t_0 > 0$. This lets us obtain the following hypoelliptic regularity estimates:

$$\int_{\mathbb{R}^{2d}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv \leq \frac{C_1}{t^3} \int_{\mathbb{R}^{2d}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv$$

and

$$\int_{\mathbb{R}^{2d}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv$$
$$\leq \frac{C_2}{t} \int_{\mathbb{R}^{2d}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv$$

for all $t \in (0, t_0]$ and some constants $C_1 > 0$ and $C_2 > 0$.

In our arguments above, we assumed for simplicity that $\frac{\partial^2 V(x)}{\partial x^2}$ is positive definite. Our results actually hold for a larger class of confining potentials V. For example, V can be any polynomial which grows as $|x|^{2k}$, $k \geq 1$ when $|x| \to \infty$.

Perspectives. As we discussed in Section 1.3, the kinetic Fokker-Planck equation is a special case of

$$\begin{cases} \partial_t f = \mathrm{div}_\xi \left(D[\nabla_\xi f + f\nabla_\xi E] + Gf\right), \ \ \xi \in \mathbb{R}^d, \ t > 0, \\ f_{|t=0} = f_0. \end{cases}$$

Another special case of this equation is the Fokker-Planck equation of the *generalized Lagevin process* [17, 15].

Hence, as a next problem one could extend our modified entropy method to this general equation. The construction of a suitable matrix P plays important role in our method.

For this general equation, one can compute the time derivative of the modified dissipation function and get the following term

$$\frac{d}{dt}S[f(t)] = -2\int_{\mathbb{R}^d} \nabla_\xi^T\left(\frac{f}{f_\infty}\right)\left\{QP + PQ^T\right\}\nabla_\xi\left(\frac{f}{f_\infty}\right) f_\infty d\xi + \text{some terms},$$

where Q is some matrix depending on D, G, and E. This expression may help to construct P : if Q is positive stable, there exists a positive definite matrix P such that $QP + PQ^T \geq \mu P$, $\mu > 0$.

Chapter 3: The Vlasov-Poisson-Fokker-Planck system

This chapter contains my own results and is taken from [23]. This paper was published in *Kinetic and Related Models.*

We study the well-posedness and long time behavior of the nonlinear Vlasov-Poisson-Fokker-Planck system (1.9) with a confining potential V in the whole space. By changing $V + \phi$ to $m(V + \phi)$ and denoting $\sigma := \frac{\theta k_B T}{m^2}$ and $\nu := \frac{\theta}{m}$, (1.9) can be written as

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - (\nabla_x V + \nabla_x\phi)\cdot\nabla_v f = \nu \operatorname{div}_v(vf) + \sigma\Delta_v f, \quad x, v\in\mathbb{R}^d,\ t > 0 \\ -\Delta_x\phi = \int_{\mathbb{R}^d} f dv, \quad f_{|t=0} = f_0. \end{cases} \tag{1.24}$$

If V grows fast enough as $|x| \to \infty$, the system has a unique normalized *steady state* or *global equilibrium*

$$f_\infty(x, v) = \rho_\infty(x)M(v),$$

where

$$\rho_\infty(x) := \frac{e^{-\frac{\nu}{\sigma}[V(x)+\phi_\infty(x)]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]}dx'}, \qquad M(v) := \frac{e^{-\frac{\nu}{\sigma}|v|^2/2}}{(2\pi\sigma/\nu)^{d/2}},$$

and ϕ_∞ is a solution of the Poisson-Boltzmann-Emden equation

$$-\Delta_x\phi_\infty(x) = \frac{e^{-\frac{\nu}{\sigma}[V(x)+\phi_\infty(x)]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]}dx'}.$$

We establish existence and uniqueness of mild solutions, hypoelliptic regularization, and exponential decay of solutions to the steady state. Our results hold for a wide class of external potentials V and the estimates on the rate of convergence are explicit and constructive. New short and long time estimates for the semigroup of the linearized system and fixed point arguments play an important role in our analysis.

In the following we explain our results with more details. Let $h := \dfrac{f - f_\infty}{f_\infty}$, $\psi := \phi - \phi_\infty$ and $h_0 := \dfrac{f_0 - f_\infty}{f_\infty}$. Then, we write the system (1.24) as

$$\begin{cases} \partial_t h + v\cdot\nabla_x h - \nabla_x(V + \phi_\infty)\cdot\nabla_v h + v\cdot\nabla_x\psi - \sigma\Delta_v h + \nu v\cdot\nabla_v h = \nabla_x\psi\cdot(\nabla_v h - \frac{\nu}{\sigma}vh) \\ -\Delta_x\psi = \int_{\mathbb{R}^d} h f_\infty dv, \quad h_{|t=0} = h_0. \end{cases} \tag{1.25}$$

It is obvious that the existence of a unique solution $f(t)$ to (1.24) and the convergence $f(t) \to f_\infty$ as $t \to \infty$ are, respectively, equivalent to the existence of a unique solution $h(t)$ to (1.25) and the convergence $h(t) \to 0$ as $t \to \infty$.

The term $\nabla_x\psi\cdot\nabla_v h - \frac{\nu}{\sigma}v\cdot\nabla_x\psi h$ appearing on the right hand side of (1.25) is nonlinear. If we drop it, we obtain the linearized Vlasov-Poisson-Fokker-Planck system around the steady state f_∞

$$\begin{cases}\partial_t h + v\cdot\nabla_x h - \nabla_x(V+\phi_\infty)\cdot\nabla_v h + v\cdot\nabla_x\psi - \sigma\Delta_v h + \nu v\cdot\nabla_v h = 0\\ -\Delta_x\psi = \displaystyle\int_{\mathbb{R}^d} h f_\infty dv, \quad h_{|t=0} = h_0.\end{cases} \tag{1.26}$$

We define an operator

$$Kh := v\cdot\nabla_x h - \nabla_x(V+\phi_\infty)\cdot\nabla_v h + v\cdot\nabla_x\psi - \sigma\Delta_v h + \nu v\cdot\nabla_v h.$$

Since $\nabla_x\psi$ can be expressed by h as

$$\nabla_x\psi = \frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} * \int_{\mathbb{R}^d} h f_\infty dv,$$

we consider K as an operator acting only on h. With this operator (1.26) can be written as

$$\begin{cases}\partial_t h + Kh = 0, \ \ t>0\\ h_{|t=0} = h_0.\end{cases}$$

We first study this linearized system in dimension $d\geq 3$. We prove that K generates a C_0 semigroup e^{-tK} on $L^2(\mathbb{R}^{2d}, f_\infty)$ and has hypoelliptic regularizing properties. More precisely, we obtain the following estimates in some weighted fractional Sobolev spaces

$$||e^{-tK}h_0||_{H_x^\alpha(\mathbb{R}^{2d},f_\infty)} \leq C(1+t^{-\frac{3\alpha}{2}})e^{-\lambda_1 t}||h_0||_{L^2(\mathbb{R}^{2d},f_\infty)}, \ \ \forall t>0, \ \ \alpha\in[0,1],$$

and

$$||e^{-tK}h_0||_{H_v^1(\mathbb{R}^{2d},f_\infty)} \leq C(1+t^{-\frac{1}{2}})e^{-\lambda_1 t}||h_0||_{L^2(\mathbb{R}^{2d},f_\infty)}, \ \ \forall t>0.$$

By defining an operator

$$R[h] := \nabla_x\psi\cdot\nabla_v h - \frac{\nu}{\sigma}v\cdot\nabla_x\psi h,$$

the nonlinear system (1.25) can be written as

$$\begin{cases}\partial_t h + Kh = R[h], \ \ t>0\\ h_{|t=0} = h_0.\end{cases}$$

Then the Duhamel principle suggests to convert this nonlinear system to an integral equation

$$h(t) = e^{-tK}h_0 + \int_0^t e^{-(t-s)K}R[h(s)]ds. \tag{1.27}$$

We mention that a function h satisfying (1.27) is called a *mild solutions* to (1.25). Using the properties of e^{-tK} and fixed point arguments we prove that there is a unique solution to this integral equation in $C\left([0,\infty); H_x^\alpha(\mathbb{R}^6, f_\infty)\right)\cap C\left([0,\infty); H_v^1(\mathbb{R}^6, f_\infty)\right)$. We also prove that the solution converges to 0 as $t\to\infty$.

Perspectives. The Vlasov-Poisson-Fokker-Planck system usually models the time evolution of a plasma. As particles of a plasma are charged, besides the electrostatic field, these particles often generate a magnetic field. The correct model is then

$$\begin{cases}\partial_t f + v\cdot\nabla_x f - (\nabla_x V + E + v\times B)\cdot\nabla_v f = \nu\mathrm{div}_v(vf) + \sigma\Delta_v f, \ \ t>0\\ \partial_t E = \nabla_x\times B + \int_{\mathbb{R}^d} vf dv, \ \ \partial_t B + \nabla_x\times E = 0, \ \ t>0\\ -\mathrm{div}_x E = \displaystyle\int_{\mathbb{R}^d} f dv, \ \ \mathrm{div}_x B = 0\\ f_{|t=0} = f_0, \ E_{|t=0} = E_0, \ B_{|t=0} = B_0.\end{cases}$$

One can check that, if $B = 0$, this system coincides with (1.24). The steady state of (1.24) is also the steady state for this system. Proving the convergence of the solution to the steady state as $t \to \infty$ is currently challenging for us, because we are lacking some important estimates for the magnetic field B to apply our method.

Chapter 4: The relativistic kinetic Fokker-Planck equation
This chapter contains results obtained in collaboration with Prof. Anton Arnold and we shall submit it soon for publication.

We study the long time behavior of the relativistic Fokker-Planck equation (1.11) with the force $F = -\nabla_x V(x)$. For simplicity, we set all physical constants to unity $m = c = \theta = k_B = T = 1$. Therefore, we shall consider the normalized equation

$$\begin{cases} \partial_t f + \dfrac{p}{\sqrt{1+|p|^2}} \cdot \nabla_x f - \nabla_x V \cdot \nabla_p f = \operatorname{div}_p(D(p)\nabla_p f + pf), \quad x, p \in \mathbb{R}^d,\ t > 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.28}$$

with $D(p) = \dfrac{I + p \otimes p}{p_0}$.

We prove the exponential time decay of solutions towards the global equilibrium in the weighted L^2 and H^1 spaces. Moreover, we prove that the associated semigroup of the equation has hypoelliptic regularizing properties and we obtain explicit rates on this regularization.

Our first result is the exponential decay of the solution to the steady state in the weighted $L^2(\mathbb{R}^{2d}, f_\infty)$, and its proof is based on the well-known L^2-hypocoercivity method [10]: Let $h := \dfrac{f - f_\infty}{f_\infty}$. Then (1.28) can be written as

$$\begin{cases} \partial_t h + \dfrac{p}{p_0} \cdot \nabla_x h - \nabla_x V \cdot \nabla_p h = \dfrac{1}{f_\infty}\operatorname{div}_p(D\nabla_p h f_\infty) \\ h_{|t=0} = \frac{f_0 - f_\infty}{f_\infty}. \end{cases} \tag{1.29}$$

We define a Hilbert space

$$\mathcal{H} := \left\{ h \in L^2(\mathbb{R}^{2d}, f_\infty) : \int_{\mathbb{R}^{2d}} h f_\infty dx dp = 0 \right\}$$

with the scalar product $\langle h, g\rangle := \int_{\mathbb{R}^{2d}} h_1 h_2 f_\infty dx dp$ and the norm $||h||_{L^2(\mathbb{R}^{2d}, f_\infty)} = \sqrt{\langle h, h\rangle}$. We can present (1.29) as

$$\begin{cases} \partial_t h + \mathrm{T}h = \mathrm{L}h, \ \ t > 0 \\ h_{|t=0} = h_0 \end{cases}$$

with

$$\mathrm{T}h := \frac{p}{p_0} \cdot \nabla_x h - \nabla_x V \cdot \nabla_p h \quad \text{and} \quad \mathrm{L}h := \frac{1}{f_\infty}\operatorname{div}_p(D\nabla_p h f_\infty).$$

We define

$$\Pi h = \Pi h(x) := \int_{\mathbb{R}^d} h(x, p')M(p')dp', \quad h \in \mathcal{H}$$

and

$$\mathrm{A}h := (\mathrm{I} + (\mathrm{T}\Pi)^*\mathrm{T}\Pi)^{-1}(\mathrm{T}\Pi)^* h, \ \ h \in \mathcal{H}.$$

Using these operators we define the functional

$$\mathrm{H}_\delta[h] := \frac{1}{2}||h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} + \delta\langle \mathrm{A}h, h\rangle, \ \ h \in \mathcal{H}, \ \ \delta > 0.$$

Following the ideas of [10] we show that, under some growth assumption V, the functional $\mathrm{H}_\delta[h(t)]$ is decreasing and satisfies

$$\frac{d}{dt}\mathrm{H}_\delta[h(t)] \le -2\lambda \mathrm{H}_\delta[h(t)], \quad \forall t > 0$$

for some $\lambda > 0$. As $\mathrm{H}_\delta[\cdot]$ is equivalent to $||\cdot||^2_{L^2(\mathbb{R}^{2d},f_\infty)}$, this inequality and Grönwall's inequality yield

$$||h(t)||_{L^2(\mathbb{R}^{2d},f_\infty)} \le Ce^{-\lambda t}||h_0||_{L^2(\mathbb{R}^{2d},f_\infty)}, \quad \forall t \ge 0.$$

Since (1.29) is degenerate parabolic, it is natural to expect some hypoelliptic regularizing properties. We prove that the equation has such properties in a weighted Sobolev space $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ associated to the norm

$$||h||^2_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)} := \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp + \int_{\mathbb{R}^{2d}} \frac{1}{V_0^3(x)p_0^3}\nabla_x^T h\left(I - \frac{p\otimes p}{p_0^2}\right)\nabla_x h f_\infty dxdp + \int_{\mathbb{R}^{2d}} \frac{1}{V_0(x)p_0}\nabla_p^T h(I + p\otimes p)\nabla_p h f_\infty dxdp,$$

where $V_0(x) := \sqrt{1+|\nabla_x V(x)|^2}$ and $p_0 := \sqrt{1+|p|^2}$. Let $P = P(x,p) \in \mathbb{R}^{2d\times 2d}$ be a symmetric, positive definite matrix depending on the variables x, $p \in \mathbb{R}^d$. We define

$$\mathrm{S}_P[h] := \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp.$$

We define a functional

$$\mathrm{E}[h] := ||h||^2_{L^2(\mathbb{R}^{2d},f_\infty)} + \mathrm{H}_\delta[h] + \mathrm{S}_P[h].$$

We choose

$$P = P(x,p) := \begin{pmatrix} \frac{2\varepsilon^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{\varepsilon^2}{V_0^2 p_0^2}I \\ \frac{\varepsilon^2}{V_0^2p_0^2}I & \frac{2\varepsilon}{V_0 p_0}(I + p\otimes p)\end{pmatrix}$$

where ε is a suitable positive constant. We prove that $\mathrm{E}[h]$ is a Lyapunov functional and satisfies

$$\frac{d}{dt}\mathrm{E}[h(t)] \le -2\Lambda \mathrm{E}[h(t)], \quad \forall t > 0$$

for some $\Lambda > 0$. Grönwall's inequality yields

$$\mathrm{E}[h(t)] \le e^{-2\Lambda t}\mathrm{E}[h_0].$$

As $\mathrm{E}[\cdot]$ is equivalent to $||\cdot||^2_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)}$, we obtain the exponential decay of the solution in $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$

$$||h(t)||_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)} \le Ce^{-\Lambda t}||h_0||_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)}.$$

Next, we choose a matrix which depends not only on x and p but also on time t :

$$P = P(t,x,p) := \begin{pmatrix} \frac{2\varepsilon^3 t^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{\varepsilon^2 t^2}{V_0^2 p_0^2}I \\ \frac{\varepsilon^2 t^2}{V_0^2p_0^2}I & \frac{2\varepsilon t}{V_0 p_0}(I + p\otimes p)\end{pmatrix}.$$

We show that $\mathrm{E}[h(t)]$ with this matrix is a decreasing function of $t \in (0, t_0]$, $t_0 > 0$. This helps us to prove the following hypoelliptic estimates

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0^3(x)p_0^3}\nabla_x^T\left(\frac{f(t)}{f_\infty}\right)\left(I - \frac{p\otimes p}{p_0^2}\right)\nabla_x\left(\frac{f(t)}{f_\infty}\right) f_\infty dxdp \le \frac{C_1}{t^3}\int_{\mathbb{R}^{2d}}\left(\frac{f(t)-f_\infty}{f_\infty}\right)^2 f_\infty dxdp$$

and

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0(x)p_0} \nabla_p^T \left(\frac{f(t)}{f_\infty}\right) (I + p \otimes p) \nabla_p \left(\frac{f(t)}{f_\infty}\right) f_\infty dx dp \le \frac{C_2}{t} \int_{\mathbb{R}^{2d}} \left(\frac{f(t) - f_\infty}{f_\infty}\right)^2 f_\infty dx dp$$

for all $t \in (0, t_0]$.

All of our results hold for a large class of potentials, e.g., $V(x) = |x|^k,\ k > 1$.

Perspectives. One can consider the relativistic versions of the Vlasov-Poisson (Maxwell)-Fokker-Planck system [4]. Similar arguments as in Chapter 3 and Chapter 4 can be used to study the well-posedness and long time behavior of this system.

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Chapter 2

The kinetic Fokker-Planck equation

2.1 Introduction

This chapter is devoted to the study of the long time behavior of the kinetic Fokker-Planck equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \nu \mathrm{div}_v(vf) + \sigma \Delta_v f, \quad x,\, v \in \mathbb{R}^n,\, t > 0 \\ f_{|t=0} = f_0 \in L^1(\mathbb{R}^{2n}) \end{cases} \tag{2.1}$$

describing the time evolution of the phase space probability density $f(t,x,v)$, e.g. in a plasma [31]. Applications range from plasma physics [29, 13] to stellar dynamics [17, 18]. Here $V = V(x)$ is a given smooth, bounded below confinement potential for the system, and $\nu > 0$, $\sigma > 0$ denote the friction and diffusion parameters, respectively. This equation is associated with the Langevin stochastic differential equation

$$\begin{cases} dx_t = v_t dt \\ dv_t = -\nu v_t dt - \nabla V(x_t) dt + \sqrt{2\sigma} dB_t, \end{cases}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$ with covariance $\langle B_t, B_{t'} \rangle = \delta_{t-t'}$.
Since the equation conserves mass, i.e.,

$$\int_{\mathbb{R}^{2n}} f(t,x,v)dxdv = \int_{\mathbb{R}^{2n}} f_0(x,v)dxdv, \quad t \geq 0,$$

we shall always assume (without restriction of generality) that $\int_{\mathbb{R}^{2n}} f_0(x,v)dxdv = 1$. The unique normalized steady state of (2.1) is given by

$$f_\infty(x,v) = c_V e^{-\frac{\nu}{\sigma}[V(x) + \frac{|v|^2}{2}]}, \quad x, v \in \mathbb{R}^n, \tag{2.2}$$

where c_V is a positive constant such that $\int_{\mathbb{R}^{2n}} f_\infty(x,v)dxdv = 1$. The following equation is also considered as the kinetic Fokker-Planck equation:

$$\partial_t h + v \cdot \nabla_x h - \nabla_x V \cdot \nabla_v h = \sigma \Delta_v h - \nu v \cdot \nabla_v h, \quad x,\, v \in \mathbb{R}^n,\, t > 0, \tag{2.3}$$

and to switch from (2.1) to (2.3) it suffices to set $h := f/f_\infty$.

It was shown in [22] that, if $V \in C^\infty(\mathbb{R}^n)$, (2.3) generates a C^∞ regularizing contraction semigroup in $L^2(\mathbb{R}^d, f_\infty) := \{g \colon \mathbb{R}^d \to \mathbb{R} : g \text{ is measurable and } \int_{\mathbb{R}^d} g^2 f_\infty dx dv < \infty\}$, $d = 2n$. For well-posedness with non-smooth potentials, we refer to [32, Theorem 6, Theorem 7].

The long time behavior and exponential convergence of the solution to the steady state has been studied and there are various results: in [19], algebraic decay was proved for potentials that are asymptotically quadratic (as $|x| \to \infty$) and for initial conditions that are bounded below and above by Gaussians. The authors used logarithmic Sobolev inequalities and entropy methods. In [24], exponential decay was obtained also for faster growing potentials and more general initial conditions. That proof is based on hypoellipticity techniques. By using hypoelliptic methods, Villani proved exponential convergence results in $H^1(\mathbb{R}^d, f_\infty) := \{g \in L^2(\mathbb{R}^d, f_\infty) : |\nabla g| \in L^2(\mathbb{R}^d, f_\infty)\}$ [32, Theorem 35] and in $L^2(\mathbb{R}^d, f_\infty)$ [32, Theorem 37]. The main conditions in Villani's theorems above, as well as in [20, 9, 10, 33, 14, 15], are the validity of the Poincaré inequality (2.5) and the criterion

$$\exists\, C \geq 0 \ : \quad \left|\left|\frac{\partial^2 V(x)}{\partial x^2}\right|\right| \leq C(1 + |\nabla V(x)|), \quad \forall x \in \mathbb{R}^n, \tag{2.4}$$

where $\left|\left|\frac{\partial^2 V(x)}{\partial x^2}\right|\right|$ denotes the Frobenius norm of $\frac{\partial^2 V(x)}{\partial x^2}$.
When $\frac{\partial^2 V}{\partial x^2}$ is bounded, Villani also proved that the solution converges to the steady state exponentially in the logarithmic entropy [32, Theorem 39]. This result was extended in [16] to potentials V satisfying a weighted log-Sobolev inequality and the condition that $V^{-2\eta}\frac{\partial^2 V}{\partial x^2}$ is bounded for some $\eta \geq 0$. Even though Villani's result allows for a general class of potentials, the growth condition (2.4) is not satisfied by potentials with singularities. This type of potentials, such as Lennard-Jones type interactions with confinement, are considered in [10] and their method relies on an explicit construction of a Lypunov function and Gamma calculus. In [20], Dolbeault, Mouhot, and Schmeiser developed a method to get exponential decay in L^2 for a large class of linear kinetic equations, including (2.1). Their method was also used to study the long time behavior of (2.1) when the potential V is zero or grows slowly as $|x| \to \infty$, see [11, 12]. Based on a probabilistic coupling method, Eberle, Guillin, and Zimmer [21] obtained an exponential decay result in Wasserstein distance.

The associated semigroup of the kinetic Fokker-Planck equation has instantaneous regularizing properties which is called *hypoellipticity* [26]. This hypoelliptic regularization is obvious when the confining potential V is zero or quadratic as the fundamental solution can be explicitly computed (see [28], [26]). For potentials such that $\frac{\partial^2 V}{\partial x^2}$ is bounded, Hérau [23] obtained short time estimates for a $L^2(\mathbb{R}^d, f_\infty) \to H^1(\mathbb{R}^d, f_\infty)$ regularization by constructing a suitable Lyapunov functional. Based on interpolation inequalities and a system of differential inequalities, Villani [32, Appendix A.21] extended Hérau's result for potentials satisfying (2.4).

We provide a new method to establish exponential decay of the solution to the steady state in $H^1(\mathbb{R}^d, f_\infty)$ for a wide class of potentials: Our method extends [32, 1, 3] by allowing for more general Lyapunov functionals. Generalizing the previous approaches, the weight matrix in the dissipation functional (a generalized Fisher information) may now depend on x and v. This leads to a new criterion on the potential V. For this entropy method we need the time derivative of the dissipation functional, but we also provide its (x, v)–pointwise analog, in the spirit of the *Gamma calculus* [9]. We provide a formula to estimate easily the exponential decay rate depending on the parameters of the equation, the constants appearing in the Poincaré inequality (2.5) and the growth condition on the potential (see

(2.6) below). As a test of the effectiveness of our method, we show that our estimate on the decay rate is sharp when the potential is a quadratic polynomial. Moreover, our method lets us obtain estimates on the hypoelliptic regularization for potentials that are more general than in [23].

The organization of this paper is as follows. In Section 2, we define the assumptions on the potential, state the main results, and present concrete examples of such potentials. In Section 3, we present the intuition and explain our method. Section 4 contains important lemmas about matrix inequalities which are important to construct suitable Lyapunov functionals. The final section presents the proof of the main results.

2.2 Main results

We make the following assumptions.

Assumption 2.2.1. *There exists a constant $C_{PI} > 0$ such that the Poincaré inequality*

$$\int_{\mathbb{R}^{2n}} h^2 f_\infty dxdv - \left(\int_{\mathbb{R}^{2n}} hf_\infty dxdv\right)^2 \leq \frac{1}{C_{PI}} \int_{\mathbb{R}^{2n}} (|\nabla_x h|^2 + |\nabla_v h|^2) f_\infty dxdv \tag{2.5}$$

holds for all $h \in H^1(\mathbb{R}^d, f_\infty)$.

Sufficient conditions on the potential appearing in f_∞ so that the Poincaré inequality holds, e.g. the Bakry-Emery criterion, are presented in [8, Chapter 4].

Assumption 2.2.2. *There are constants $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ such that the following $\mathbb{R}^{m\times m}$ matrix, $m := n(n+1)$,*

$$\begin{pmatrix} \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V(x))}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V(x))}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V(x))}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V(x))}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V(x))}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V(x))}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right) \end{pmatrix} \tag{2.6}$$

is positive semi-definite for all $x \in \mathbb{R}^n$, where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

Roughly speaking, Assumption 2.2.2 essentially means that the second order derivatives of V control the third order ones. It implies that $\frac{\partial^2 V(x)}{\partial x^2} + cI$ is positive semi-definite for all $x \in \mathbb{R}^n$, and hence the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$ are uniformly bounded from below. We note that, in contrast to the Bakry-Emery strategy [7], the eigenvalues here may take negative values.

Let $\alpha(x) \in \mathbb{R}$ denote the smallest eigenvalue of $\frac{\partial^2 V(x)}{\partial x^2}$ at $x \in \mathbb{R}^n$. Then the following condition implies Assumption 2.2.2. For its proof see Appendix 2.6.1.

Assumption 2.2'. *There are constants $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ such that $\frac{\partial^2 V(x)}{\partial x^2} + cI$ is positive semi-definite and*[1]

$$-\sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c)I \leq \frac{\partial^2(\partial_{x_i}V(x))}{\partial x^2} \leq \sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c)I \tag{2.7}$$

[1] For two matrices A and $B \in \mathbb{R}^{n\times n}$, $A \geq B$ means that $A - B$ is positive semi-definite.

for all $x \in \mathbb{R}^n$ *and* $i \in \{1, ..., n\}$.

We denote

$$\alpha_0 := \inf_{x \in \mathbb{R}^n} \alpha(x) \tag{2.8}$$

and assume in the sequel that $\alpha_0 > -\infty$. Hence Assumption 2.2.2 can only hold for some $c \geq -\alpha_0$.

In the following results, we require that $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2n}, f_\infty)$ which implies $f_0 \in L^1(\mathbb{R}^{2d})$ because of the Hölder inequality $\int_{\mathbb{R}^{2n}} f_0 dxdv \leq \sqrt{(\int_{\mathbb{R}^{2n}} \frac{f_0^2}{f_\infty} dxdv)(\int_{\mathbb{R}^{2n}} f_\infty dxdv)}$ and $\int_{\mathbb{R}^{2n}} f_\infty dxdv = 1$. We now state our first result, i.e. exponential decay of a functional that is a linear combination of the weighted L^2-norm and a Fisher information-type functional:

Theorem 2.2.3. *Let V be a C^∞ potential in $\mathbb{R}^n$ satisfying Assumptions 2.2.1 and 2.2.2. Let C_{PI}, c, τ, and α_0 be the constants in (2.5), (2.6), and (2.8). Suppose the initial data f_0 satisfies $\frac{f_0}{f_\infty} \in H^1(\mathbb{R}^{2n}, f_\infty)$ and $\int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f_0}{f_\infty}\right) \frac{\partial^2 V}{\partial x^2} \nabla_v \left(\frac{f_0}{f_\infty}\right) f_\infty dxdv < \infty$. Then there are explicitly computable constants $C > 0$ and $\lambda > 0$ (independent of f_0) such that the solution $f(t)$ of (2.1) satisfies*

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv \\
&\quad + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv \\
&\leq C e^{-2\lambda t} \Bigg[\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f_0}{f_\infty}\right)\right|^2 f_\infty dxdv \\
&\quad + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f_0}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0) I\right) \nabla_v \left(\frac{f_0}{f_\infty}\right) f_\infty dxdv\Bigg]
\end{aligned} \tag{2.9}$$

for all $t \geq 0$. Moreover, we have:

(a) if $\alpha_0 > \frac{\nu^2}{4}$, $c \leq -\frac{\nu^2}{4}$, then $2\lambda = \nu - \tau$;

(b) if $c = -\alpha_0 = -\frac{\nu^2}{4}$, then $2\lambda = \nu - \tau - \varepsilon$ for any $\varepsilon \in (0, \nu - \tau)$;

(c) if $c > -\frac{\nu^2}{4}$, $c + 2\alpha_0 > \frac{\nu^2}{4}$, then

$$2\lambda = \begin{cases} \nu - \tau - \dfrac{c + \frac{\nu^2}{4}}{\sqrt{c + \alpha_0}} & \text{if } \nu - \tau \geq A_1^{-1} + \frac{c + \frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} \\ \dfrac{(\nu - \tau)\sqrt{c + \alpha_0} - (c + \frac{\nu^2}{4})(\sqrt{1 + s_1^2} - s_1)}{\sqrt{c + \alpha_0} + A_1 s_1 (c + \frac{\nu^2}{4})} & \text{if } \nu - \tau < A_1^{-1} + \frac{c + \frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} \end{cases},$$

$$\text{where} \quad A_1 := \frac{1 + \frac{\nu^2}{4} + c + \alpha_0 + \sqrt{(\frac{\nu^2}{4} + c + \alpha_0 - 1)^2 + \nu^2}}{2\sigma C_{PI}},$$

$$s_1 := \begin{cases} \frac{A_1^2 (c + \frac{\nu^2}{4})^2 - c - \alpha_0}{2A_1 (c + \frac{\nu^2}{4})\sqrt{c+\alpha_0}} & \text{if } (\nu - \tau) A_1 = 2 \\ \frac{1}{\nu - \tau} \left[\left|\frac{(\nu-\tau)A_1 - 1}{(\nu-\tau)A_1 - 2}\right| \sqrt{\frac{(c+\frac{\nu^2}{4})^2}{c+\alpha_0} + 2(\nu-\tau)A_1^{-1} - (\nu-\tau)^2} - \frac{c + \frac{\nu^2}{4}}{((\nu-\tau)A_1 - 2)\sqrt{c+\alpha_0}}\right] & \text{if } (\nu - \tau) A_1 \neq 2 \end{cases};$$

(d) *if* $c > -\frac{\nu^2}{4}$, $c + 2\alpha_0 \le \frac{\nu^2}{4}$, *then*

$$2\lambda = \begin{cases} \nu - \tau - \sqrt{\nu^2 - 4\alpha_0} & \text{if } \nu - \tau \ge A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0} \\ \dfrac{\nu - \tau - \sqrt{\nu^2 - 4\alpha_0}(\sqrt{1+s_2^2} - s_2)}{1 + A_2 s_2 \sqrt{\nu^2 - 4\alpha_0}} & \text{if } \nu - \tau < A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0} \end{cases},$$

where $A_2 := \dfrac{1 + \frac{\nu^2}{2} - \alpha_0 + \sqrt{(\frac{\nu^2}{2} - \alpha_0 - 1)^2 + \nu^2}}{2\sigma C_{PI}}$,

$$s_2 := \begin{cases} \frac{A_2^2(\nu^2 - 4\alpha_0) - 1}{2A_2\sqrt{\nu^2 - 4\alpha_0}} & \text{if } (\nu - \tau)A_2 = 2 \\ \frac{1}{\nu - \tau}\left[\left|\frac{(\nu-\tau)A_2 - 1}{(\nu-\tau)A_2 - 2}\right| \sqrt{\nu^2 - 4\alpha_0 + 2(\nu-\tau)A_2^{-1} - (\nu-\tau)^2} - \frac{\sqrt{\nu^2 - 4\alpha_0}}{((\nu-\tau)A_2 - 2)}\right] & \text{if } (\nu - \tau)A_2 \ne 2 \end{cases};$$

(e) *if* $V(x)$ *is a quadratic polynomial of* x *and* $\frac{\partial^2 V}{\partial x^2}$ *is positive definite, then Assumptions 2.2.1 and 2.2.2 are satisfied with* $\tau = 0$, $c = -\alpha_0$ *(this rules out the conditions in the case of (c)). Moreover, the decay rates* λ *in (a) and (d) are sharp and, in the case of (d),* $\nu \ge A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0}$ *holds and so* $2\lambda = \nu - \sqrt{\nu^2 - 4\alpha_0}$. *In the case of (b), the decay rate* $2\lambda = \nu - \varepsilon$ *is sharp in the sense that (2.9) holds with the rate* $2\lambda = \nu - \varepsilon$ *for any small fixed* $\varepsilon \in (0, \nu)$, *but it does not hold with the rate* $2\lambda = \nu$.

Remark 2.2.4. 1. *It is possible to make weaker regularity hypothesis on the potential* V, *but we maintain the assumption that* $V \in C^\infty$ *to keep the presentation simple.*

2. *Since* $\frac{\partial^2 V}{\partial x^2} + (1 - \alpha_0)I \ge I$, (2.9) *implies that the solution converges exponentially to the steady state in* $H^1(\mathbb{R}^{2n}, f_\infty)$. *If the eigenvalues of* $\frac{\partial^2 V}{\partial x^2}$ *are uniformly bounded, then* (2.9) *is equivalent to the exponential decay of the solution to the steady state in* $H^1(\mathbb{R}^{2n}, f_\infty)$. *Due to the Poincaré inequality* (2.5), *the* L^2-*term on the right hand side of* (2.9) *could be omitted.*

3. *If* V *satisfies Assumption 2.2.2 with some constants* $c \in \mathbb{R}$ *and* $\tau \in [0, \nu)$, *then* V *also satisfies Assumption 2.2.2 with any* $\tilde{c} \ge c$ *and* $\tilde{\tau} \in [\tau, \nu)$. *Therefore, these constants are not unique. But the exponential decay rate* λ *obtained in Theorem 2.2.3 depends on the choice of* c *and* τ. *To obtain a better rate, one has to optimize* $\lambda = \lambda(c, \tau)$ *with respect to all* c *and* τ *satisfying Assumption 2.2.2.*

4. *In Theorem 2.2.3* (*b*), *the constant* C *in* (2.9) *depends on* ε, *and* $C = C(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$.

5. *The highest exponential rate is* $\frac{\nu}{2}$ *which can be attained by the quadratic potentials* V *with* $\frac{\partial^2 V}{\partial x^2} \ge \frac{\nu^2}{4} I$.

When V is a quadratic polynomial as in Theorem 2.2.3 (e), we prove the following sharp estimates.

Proposition 2.2.5. *Let* V *be a quadratic polynomial and* $\frac{\partial^2 V}{\partial x^2}$ *be positive definite. Let* $\alpha_0 > 0$ *be the smallest eigenvalue of* $\frac{\partial^2 V}{\partial x^2}$, *then*[2]

$$\sup_{1 \ne \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}t}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases} \quad \text{as } t \to \infty. \tag{2.10}$$

[2] For functions $\varphi = \varphi(t)$ and $\phi = \phi(t)$, $\varphi \asymp \phi$ as $t \to \infty$ means $\varphi = \mathcal{O}(\phi)$ and $\phi = \mathcal{O}(\varphi)$ as $t \to \infty$.

We shall use this proposition to prove the sharpness of the decay rates in Theorem 2.2.3 (e). When V is a quadratic polynomial and $-\alpha_0 = -\frac{\nu^2}{4} =: c$, Theorem 2.2.3 (e) shows that the decay in (2.9) can be $e^{-(\nu-\varepsilon)t}$ for any small fixed $\varepsilon \in (0,\nu)$, but it can not be $e^{-\nu t}$. In this case, it is natural to expect a decay between $e^{-\nu t}$ and $e^{-(\nu-\varepsilon)t}$: Proposition 2.2.5 shows that this is indeed the case for the square of the L^2-norm, with the decay $(1+t)^2 e^{-\nu t}$. But an analogous extension of this result for the functional on the left hand side of (2.9) (i.e., to replace the term $Ce^{-(\nu-\varepsilon)t}$ with $C(1+t)^2e^{-\nu t}$) has not been obtained so far.

Remark 2.2.6. *Under assumptions of Proposition 2.2.5, we can construct special solutions $f_s(t)$ (see [3, Section 6]) which satisfy*

$$\frac{||f_s(t)/f_\infty - 1||_{L^2(\mathbb{R}^d,f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d,f_\infty)}} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu-\sqrt{\nu^2-4\alpha_0}}{2}t}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases} \quad \text{as} \quad t \to \infty.$$

Our next result is about the estimates on the hypoelliptic regularization.

Theorem 2.2.7. *Assume V is a C^∞ potential on $\mathbb{R}^n$ and there are constants $c \in \mathbb{R}$ and $\tau \geq 0$ such that the matrix (2.6) is positive semi-definite for all $x \in \mathbb{R}^n$. Suppose the initial data f_0 satisfies $\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv < \infty$. Then, for any $t_0 > 0$, there are explicitly computable constants $C_1 = C_1(t_0) > 0$ and $C_2 = C_2(t_0) > 0$ (independent of f_0) such that the inequalities*

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv \leq \frac{C_1}{t^3} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv \quad (2.11)$$

and

$$\int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv \leq \frac{C_2}{t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv \quad (2.12)$$

hold for all $t \in (0, t_0]$.

In Theorem 2.2.3 we assumed that the initial data f_0/f_∞ is in $H^1(\mathbb{R}^d, f_\infty)$. If we use the estimates in Theorem 2.2.7, this condition can be relaxed:

Corollary 2.2.8. *Let V be a C^∞ potential in $\mathbb{R}^n$ satisfying Assumptions 2.2.1 and 2.2.2. Suppose the initial data f_0 satisfies $\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv < \infty$. Then, for any $t_0 > 0$, there is an explicitly computable constant $C = C(t_0) > 0$ (independent of f_0) such that*

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right) \left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) \nabla_v \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv \leq Ce^{-2\lambda t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dxdv \quad (2.13)$$

holds for all $t \geq t_0$ with λ defined in Theorem 2.2.3.

Remark 2.2.9. *1. In contrast to Theorem 2.2.3, Theorem 2.2.7 holds even if the Poincaré inequality (2.5) is not satisfied by f_∞. Also, τ can be larger than ν.*

2. The exponents of t in (2.11) and (2.12) are optimal when V is a quadratic polynomial (see [33, Appendix A]).

To illustrate our result, we present concrete examples of potentials V satisfying our Assumption 2.2.1 and Assumption 2.2.2:

Example 2.2.10 (Polynomial confining potentials)**.** *a*) As mentioned in Theorem 2.2.3, if $V(x) = \frac{x^T M^{-1} x}{2} + p\cdot x + q,\ x \in \mathbb{R}^n$ with a positive definite covariance matrix $M^{-1} \in \mathbb{R}^{n\times n}$, a constant vector $p \in \mathbb{R}^n$ and a constant $q \in \mathbb{R}$, the convergence rate is

$$\lambda = \begin{cases} \frac{\nu}{2}, & \text{if } \alpha_0 > \frac{\nu^2}{4} & \text{(case (a))} \\ \frac{\nu-\varepsilon}{2}, & \text{if } \alpha_0 = \frac{\nu^2}{4}, \text{ for any } \varepsilon \in (0,\nu) & \text{(case (b))} \\ \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}, & \text{if } \alpha_0 < \frac{\nu^2}{4} & \text{(case (d))} \end{cases},$$

and it is sharp for $\alpha_0 \neq \frac{\nu^2}{4}$, where α_0 is the smallest eigenvalue of M^{-1} (see Theorem 2.3 (*e*)).

b) More generally, we consider potentials of the form

$$V(x) = r|x|^{2k} + V_0(x)$$

where $r > 0$, $k \in \mathbb{N}$ and $V_0 \colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $j < 2k$. Since we have already considered quadratic potentials, we assume $k \geq 2$. V satisfies the Poincaré inequality (2.5); this can be proven, for example, by showing that V satisfies one of the sufficient conditions given in [6, Corollary 1.6]. Concerning Assumption 2.2.2' we have

$$r\frac{\partial^2 |x|^{2k}}{\partial x^2} = 2kr|x|^{2k-2}I + 2k(2k-2)r|x|^{2k-4}\begin{pmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_1x_2 & x_2^2 & \dots & x_2x_n \\ \dots & \dots & \dots & \dots \\ x_1x_n & x_2x_n & \dots & x_n^2 \end{pmatrix} \geq 2kr|x|^{2k-2}I.$$

Since V_0 has degree $j < 2k$, there is a constant $A > 0$ such that

$$-A(1+|x|^{2k-3})I \leq \frac{\partial^2 V_0(x)}{\partial x^2} \leq A(1+|x|^{2k-3})I.$$

Therefore, we can estimate

$$\frac{\partial^2 V(x)}{\partial x^2} \geq \left(2kr|x|^{2k-2} - A|x|^{2k-3} - A\right) I. \tag{2.14}$$

We also observe that there exists a positive constant B such that

$$-B(1+|x|^{2k-3})I \leq \frac{\partial^2(\partial_{x_i} V(x))}{\partial x^2} \leq B(1+|x|^{2k-3})I$$

for all $i \in \{1, ..., n\}$. (2.14) shows that the smallest eigenvalue of $\frac{\partial^2 V(x)}{\partial x^2}$ satisfies $\alpha(x) \geq 2kr|x|^{2k-2} - A|x|^{2k-3} - A$. Since $2kr|x|^{2k-2} - A|x|^{2k-3} - A$ grows faster than $B(1+|x|^{2k-3})$ as $|x| \to \infty$, there are constants c and $\tau \in [0,\nu)$ such that (2.7) is satisfied. Thus, Theorem 2.2.3 applies to this type of potentials. In particular, it applies to double-well potentials of the form $V(x) = r_1|x|^4 - r_2|x|^2,\ \ r_1, r_2 > 0$.

Remark 2.2.11. *1. Our decay and regularization results above extend those of [23], where a stronger assumption, i.e. $\partial^2_{x_i x_j} V \in \bigcap_{p=1}^{\infty} W^{p,\infty}(\mathbb{R}^n)$ for all $i, j \in \{1, ..., n\}$, was made. By contrast, we did not require the boundedness of the second and higher derivatives of V.*

2. Most of the previous works on the exponential convergence $f(t) \to f_\infty$ as $t \to \infty$ (e.g. [32, 20, 9, 10, 33, 14, 15]) used the growth condition (2.4) to get some weighted Poincaré type inequalities (see [32, Lemma A.24]), which are crucial in these works – and additional to the Poincaré inequality (2.5). Our technique is rather different, based on construction of appropriate state dependent matrices and state dependent matrix inequalities so that the (modified) dissipation functional (see (2.20) below) decays exponentially.

3. Most of the previous methods for proving the exponential convergence do not give an accurate decay rate, λ is typically much too small there (see [32, Section 7.2], [20, Section 1.4]). For example, in [32, Section 7.2], the exponential decay rate $\lambda = \frac{1}{40}$ was obtained for $V(x) = \frac{|x|^2}{2}$ and $\nu = \sigma = 1$. Since our decay rates are sharp for quadratic potentials, in this setting, the true rate $\lambda = \frac{1}{2}$ is given by Theorem 2.2.3 (a) and (e).

2.3 Modified entropy methods

We first consider the following degenerate and non-symmetric Fokker-Planck equation [2, 1]:

$$\begin{cases} \partial_t f = \operatorname{div}(D\nabla f + (D+R)\nabla E f), \ \xi \in \mathbb{R}^d, \, t > 0, \\ f(t=0) = f_0 \in L^1_+(\mathbb{R}^d), \ \int_{\mathbb{R}^d} f_0 \, d\xi = 1 \end{cases} \tag{2.15}$$

where $D \in \mathbb{R}^{d\times d}$ is a constant, symmetric, positive semi-definite ($\operatorname{rank}(D) < d$) matrix, $R \in \mathbb{R}^{d\times d}$ is a constant skew-symmetric matrix. $E : \mathbb{R}^d \to \mathbb{R}$ is a function which only depends on the state variable ξ. We assume that E is confining (i.e. $E(\xi) \to \infty$ for $|\xi| \to \infty$) and smooth enough so that (2.15) has a unique and smooth solution. If E grows fast enough, (2.15) has a normalized steady state $f_\infty = c_E e^{-E}$, $c_E > 0$. The weak maximum principle for degenerate parabolic equations [25] can be applied to (2.15) and we can prove that $f(t,\xi) \ge 0$ for all $t > 0$, $\xi \in \mathbb{R}^d$. The divergence structure implies that the initial mass is conserved and $f(t,\cdot)$ describes the evolution of a probability density

$$\int_{\mathbb{R}^d} f(t,\xi)d\xi = \int_{\mathbb{R}^d} f_0(\xi)d\xi = 1, \ \ \forall t \ge 0.$$

We are interested in the large-time behavior of the solution, in particular, when $\operatorname{rank}(D)$ is less than the dimension d. When D is positive definite ($\operatorname{rank}(D) = d$), the large time behavior and exponential convergence have been studied comprehensively (see [7], [4], [2]). One of the well-know conditions which provides the exponential decay of the solution to the steady state is called *the Bakry-Emery condition* (see (2.16) below) leading to:

Theorem 2.3.1 ([2, Theorem 2.6]). *Assume* $\int_{\mathbb{R}^d} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty d\xi < \infty$ *and*

$$\exists \lambda > 0 \ \text{ such that } \frac{\partial^2 E}{\partial \xi^2}(I + RD^{-1}) + \left(\frac{\partial^2 E}{\partial \xi^2}(I + RD^{-1})\right)^T \ge \lambda D^{-1}, \ \ \forall \xi \in \mathbb{R}^d. \tag{2.16}$$

Then

$$\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \le e^{-2\lambda t} \int_{\mathbb{R}^d} \left(\frac{f_0}{f_\infty} - 1\right)^2 f_\infty d\xi.$$

To prove the theorem above, one considers the time derivative of the L^2-norm and we see that it decreases

$$\frac{d}{dt}\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi = -2\int_{\mathbb{R}^d} \nabla^T\left(\frac{f}{f_\infty}\right) D\nabla\left(\frac{f}{f_\infty}\right) f_\infty d\xi =: -I(f(t)|f_\infty) \le 0. \tag{2.17}$$

$I(f(t)|f_\infty)$ is called the dissipation functional and since D is positive definite it vanishes if and only if $f = f_\infty$. It can be proven that, under the Bakry-Emery condition,

$$\frac{d}{dt}I(f(t)|f_\infty) \le -2\lambda I(f(t)|f_\infty). \tag{2.18}$$

Integrating this inequality from (t, ∞) and using the convergences $I(f(t)|f_\infty) \to 0$ and $\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \to 0$ as $t \to \infty$, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \le -2\lambda\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \tag{2.19}$$

and, by Grönwall's lemma, we get the desired result.

When D is only positive semi-definite, i.e. rank$(D) < d$, one observes that $I(f(t)|f_\infty)$ may vanish for certain probability densities $f \ne f_\infty$. Hence the inequalities (2.18) and (2.19) will not hold in general. Since the above problems stem from the singularity of D, one can modify the dissipation function and define a modified dissipation functional (see also [1, 3])

$$S(f) := 2\int_{\mathbb{R}^d} \nabla_\xi^T\left(\frac{f}{f_\infty}\right) P(\xi)\nabla_\xi\left(\frac{f}{f_\infty}\right) f_\infty d\xi \tag{2.20}$$

where $P : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a symmetric positive definite matrix which will be chosen later. Extending the approach of [1, 3], we allow the matrix P here to depend on $\xi \in \mathbb{R}^d$. Our goal is to derive a differential inequality similar to (2.18) (like the dissipation functional satisfied for non-degenerate equations), i.e.

$$\frac{d}{dt}S(f(t)) \le -2\lambda S(f(t)), \tag{2.21}$$

for some $\lambda > 0$ and a "good" choice of the matrix P. If this holds true, we would obtain

$$S(f(t)) \le S(f_0)e^{-2\lambda t}.$$

If we can choose such $P = P(\xi) \ge \eta I$ for some $\eta > 0$ and all $\xi \in \mathbb{R}^d$, under the validity of the Poincaré inequality (2.5) for $f_\infty(\xi) = c_E e^{-E(\xi)}$ (where $\begin{pmatrix} x \\ v \end{pmatrix}$ in (2.5) is replaced with ξ) we have

$$\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \le \frac{1}{C_{PI}}\int_{\mathbb{R}^d} \left|\nabla_\xi\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty d\xi \le \frac{1}{2C_{PI}\eta}S(f(t)),$$

which implies the exponential decay of the L^2-norm

$$\int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi \le \frac{1}{2C_{PI}\eta}S(f_0)e^{-2\lambda t}.$$

More generally, since the quadratic entropy is also a decreasing function of time t, instead of proving (2.21), we can consider the functional

$$\begin{aligned}\Phi(f(t)) &:= \gamma \int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi + S(f(t)) \\ &= \gamma \int_{\mathbb{R}^d} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty d\xi + 2\int_{\mathbb{R}^d} \nabla^T \left(\frac{f}{f_\infty}\right) P(\xi) \nabla \left(\frac{f}{f_\infty}\right) f_\infty d\xi \end{aligned} \tag{2.22}$$

and choose a suitable parameter $\gamma \geq 0$ and a matrix P such that

$$\frac{d\Phi(f(t))}{dt} \leq -2\lambda \Phi(f(t)) \leq 0 \tag{2.23}$$

for some $\lambda > 0$. This idea and method were successfully applied in [3] to (2.15) when the potential E is quadratic.

We shall apply this method to the kinetic Fokker-Planck equation with non-quadratic $V(x)$. First, we denote $\xi := \begin{pmatrix} x \\ v \end{pmatrix} \in \mathbb{R}^{2n}$, $E(\xi) := \frac{\nu}{\sigma}[V(x) + \frac{|v|^2}{2}]$, $f_\infty = e^{-E}$. Then the kinetic Fokker-Planck equation (2.1) can be written in the form of (2.15),

$$\partial_t f = \mathrm{div}_\xi (D\nabla_\xi f + (D+R)\nabla_\xi E f) \tag{2.24}$$

with

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \sigma I \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \quad \text{and} \quad R = \frac{\sigma}{\nu}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{2.25}$$

The rank of the diffusion matrix D is $n < d = 2n$. Thus, (2.1) is both non-symmetric and degenerate and the arguments above apply to the equation.

We will develop a modified entropy method. We will choose $\xi-$dependent matrix P in the modified dissipation functional (2.20) so that (2.23) holds and $\lambda > 0$ is as large as possible.

We also mention that when the potential E is quadratic in (2.15), the question about the long time behavior can be reduced to an ODE problem:

Theorem 2.3.2. *Let $0 \neq D \in \mathbb{R}^{d\times d}$ be positive semi-definite, $R \in \mathbb{R}^{d\times d}$ be skew-symmetric and $\mathbb{R}^d \ni \xi \to E(\xi) = \frac{\xi^T K^{-1}\xi}{2}$ for some positive definite matrix K. Assume $(D+R)K^{-1}$ is positive stable and there is no non-trivial subspace of $\mathrm{Ker} D$ which is invariant under $K^{-1}(D-R)$. If f is the solution of (2.15) and $\xi(t) \in \mathbb{R}^d$ is the solution of the ODE $\dot{\xi}(t) = -K^{-\frac{1}{2}}(D+R)K^{-\frac{1}{2}}\xi$ with initial datum $\xi(0) = \xi_0$, then*

$$\sup_{1 \neq \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} = \sup_{0\neq \xi_0 \in \mathbb{R}^d} \frac{||\xi(t)||_2}{||\xi_0||_2}, \quad t \geq 0. \tag{2.26}$$

Proof. See [5, Theorem 3.4]. □

One consequence of Theorem 2.3.2 is that the decay estimate of the ODE-solution carries over to the corresponding Fokker-Planck equation.

2.4 The choice of the matrix P

For future reference (in the proof of Theorem 2.2.7) we shall now also allow the matrix P to be time dependent. Hence we shall next consider the generalized functional

$$S(t,f) := 2\int_{\mathbb{R}^d} \nabla_\xi^T \left(\frac{f}{f_\infty}\right) P(t,\xi)\nabla_\xi \left(\frac{f}{f_\infty}\right) f_\infty d\xi.$$

The following lemmas will play a crucial role in our arguments.

Lemma 2.4.1. *Let $P : [0,\infty)\times\mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ be smooth and f be the solution of (2.1), then*

$$\begin{aligned}\frac{d}{dt}S(t,f(t)) = -4\sigma \int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u\right\} f_\infty dxdv - 2\int_{\mathbb{R}^{2n}} u^T \left\{QP + PQ^T\right\} u f_\infty dxdv \\ - 2\int_{\mathbb{R}^{2n}} u^T \left\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v - \partial_t] P\right\} u f_\infty dxdv, \quad (2.27)\end{aligned}$$

where $u := \nabla_{x,v}\left(\frac{f}{f_\infty}\right)$, $Q = Q(x) := \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I\end{pmatrix}$, and $[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot \nabla_v - \sigma\Delta_v - \partial_t]$ denotes a scalar differential operator that is applied to each element of the matrix $P = P(t,x,v)$.

Proof. We denote $u_1 := \nabla_x\left(\frac{f}{f_\infty}\right)$, $u_2 := \nabla_v\left(\frac{f}{f_\infty}\right)$, then u_1 and u_2 satisfy

$$\partial_t u_1 = \sigma\Delta_v u_1 - \nu\sum_{i=1}^n v_i\partial_{v_i}u_1 + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u_1 - \sum_{i=1}^n v_i\partial_{x_i}u_1 + \frac{\partial^2 V}{\partial x^2}u_2,$$

$$\partial_t u_2 = \sigma\Delta_v u_2 - \nu\sum_{i=1}^n v_i\partial_{v_i}u_2 + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u_2 - \sum_{i=1}^n v_i\partial_{x_i}u_2 - u_1 - \nu u_2.$$

These equations can be written with respect to $u = \begin{pmatrix} u_1 \\ u_2\end{pmatrix}$:

$$\partial_t u = \sigma\Delta_v u - \nu\sum_{i=1}^n v_i\partial_{v_i}u + \sum_{i=1}^n \partial_{x_i}V\partial_{v_i}u - \sum_{i=1}^n v_i\partial_{x_i}u - Q^T u.$$

It allows us to compute the time derivative of the modified dissipation functional

$$\begin{aligned}\frac{d}{dt}S(t,f(t)) &= 4\int_{\mathbb{R}^{2n}} u^T P\partial_t u f_\infty dxdv + 2\int_{\mathbb{R}^{2n}} u^T\partial_t P u f_\infty dxdv \\ &= 4\sigma\int_{\mathbb{R}^{2n}} u^T P\Delta_v u f_\infty dxdv - 4\nu\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{v_i}u v_i f_\infty dxdv \\ &+ 4\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{v_i}u\partial_{x_i}V f_\infty dxdv - 4\sum_{i=1}^n\int_{\mathbb{R}^{2n}} u^T P\partial_{x_i}u v_i f_\infty dxdv \\ &- 2\int_{\mathbb{R}^{2n}} u^T\{QP + PQ^T\}u f_\infty dxdv + 2\int_{\mathbb{R}^{2n}} u^T\partial_t P u f_\infty dxdv. \quad (2.28)\end{aligned}$$

First, we consider the term in the second line of (2.28) and use $\partial_{v_i} f_\infty = -\frac{\nu}{\sigma} v_i f_\infty$:

$$\begin{aligned} &4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial^2_{v_i v_i} u f_\infty dx dv - 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u v_i f_\infty dx dv \\ &\qquad = -4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} \partial_{v_i} u^T P \partial_{v_i} u f_\infty dx dv - 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv. \end{aligned} \tag{2.29}$$

By integrating by parts the last term of (2.29) we obtain

$$\begin{aligned} &- 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv \\ &\qquad = 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv + 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial^2_{v_i v_i} P) u f_\infty dx dv \\ &\qquad\qquad - 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) u v_i f_\infty dx dv \end{aligned}$$

and we find

$$\begin{aligned} - 4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) \partial_{v_i} u f_\infty dx dv &= 2\sigma \int_{\mathbb{R}^{2n}} u^T (\Delta_v P) u f_\infty dx dv \\ &\quad - 2\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (v_i \partial_{v_i} P) u f_\infty dx dv. \end{aligned}$$

If we use this equality in (2.29), we get

$$\begin{aligned} &4\sigma \int_{\mathbb{R}^{2n}} u^T P \Delta_v u f_\infty dx dv - 4\nu \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P v_i \partial_{v_i} u f_\infty dx dv \\ &= -4\sigma \sum_{i=1}^n \int_{\mathbb{R}^{2n}} (\partial_{v_i} u)^T P \partial_{v_i} u f_\infty dx dv - 2 \int_{\mathbb{R}^{2n}} u^T \{[\nu v \cdot \nabla_v - \sigma \Delta_v] P\} u f_\infty dx dv. \end{aligned} \tag{2.30}$$

Next, we integrate by parts in the terms in the third line of (2.28):

$$\begin{aligned} &4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u \partial_{x_i} V f_\infty dx dv \\ &\qquad = -4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{v_i} u \partial_{x_i} V f_\infty dx dv - 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{v_i} P) u \partial_{x_i} V f_\infty dx dv \\ &\qquad\qquad + \frac{4\nu}{\sigma} \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P u \partial_{x_i} V v_i f_\infty dx dv, \end{aligned} \tag{2.31}$$

$$\begin{aligned} &- 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{x_i} u v_i f_\infty dx dv \\ &\qquad = 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P \partial_{x_i} u v_i f_\infty dx dv + 4 \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T (\partial_{x_i} P) u v_i f_\infty dx dv \\ &\qquad\qquad - \frac{4\nu}{\sigma} \sum_{i=1}^n \int_{\mathbb{R}^{2n}} u^T P u \partial_{x_i} V v_i f_\infty dx dv. \end{aligned} \tag{2.32}$$

(2.31) and (2.32) show that the third line of (2.28) equals

$$-2\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T(\partial_{v_i}P)u\partial_{x_i}V f_\infty dxdv + 2\sum_{i=1}^{n}\int_{\mathbb{R}^{2n}} u^T(\partial_{x_i}P)uv_i f_\infty dxdv$$
$$= -2\int_{\mathbb{R}^{2n}} u^T\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x]P\}uf_\infty dxdv. \quad (2.33)$$

Combining (2.28), (2.30), and (2.33) we obtain the statement (2.27). □

Remark 2.4.2. *We give now a (formal) generalization of the above result (2.27) to Markovian evolution equations using the* Gamma calculus, *see, e.g., [8, 9, 10]:*

First, let L be the generator of some Markovian evolution on $\mathbb{R}^d$ with corresponding invariant measure $f_\infty d\xi$. Let $P = P(\xi)$ be a smooth matrix function (but it does not have to be symmetric or positive definite). We define the first order bilinear form

$$\Gamma^P(g,h) := \nabla_\xi g^T P \nabla_\xi h$$

and

$$\Gamma_2^P(g,h) := \frac{1}{2}\left(L\Gamma^P(g,h) - \Gamma^P(Lg,h) - \Gamma^P(g,Lh)\right).$$

For a solution $h(t)$ of $\partial_t h = Lh$, these definitions give

$$\frac{d}{dt}\Gamma^P(h,h) = \Gamma^P(Lh,h) + \Gamma^P(h,Lh) = -2\Gamma_2^P(h,h) + L\Gamma^P(h,h), \qquad \forall \xi\in\mathbb{R}^d. \quad (2.34)$$

We use Γ^P to define the modified dissipation functional

$$S(f) := 2\int_{\mathbb{R}^d}\Gamma^P(h,h)\, f_\infty d\xi \qquad \text{with } h = \frac{f}{f_\infty}.$$

We obtain by integrating (2.34):

$$\frac{d}{dt}S(f(t)) = -4\int_{\mathbb{R}^d}\Gamma_2^P(h,h)\, f_\infty d\xi\,, \quad (2.35)$$

where we used that $\int_{\mathbb{R}^d} L\Gamma^P(h,h)\, f_\infty d\xi = 0$.

In particular, let L be the generator of the kinetic Fokker-Planck equation (2.3), and we recall that $\xi := \begin{pmatrix} x \\ v \end{pmatrix}$. Then, a straightforward (but lengthy) computation shows that

$$2\Gamma_2^P(h,h) = 2\sigma\sum_{i=1}^{n}(\partial_{v_i}u)^T P\partial_{v_i}u + u^T\left(QP + PQ^T\right)u + u^T(LP)u$$
$$+ 2\sigma\sum_{i=1}^{n}(\partial_{v_i}u)^T(\partial_{v_i}P)u + 4\sigma\sum_{i=1}^{n}u^T(\partial_{v_i}P)\partial_{v_i}u.$$

One can check (by integrating by parts the term $4\sigma\int_{\mathbb{R}^d}\sum_{i=1}^n u^T(\partial_{v_i}P)\partial_{v_i}uf_\infty d\xi$ in the right hand side of (2.35)) that (2.35) coincides with (2.27). Hence, (2.35) reproduces (2.27). But in contrast to (2.27), the preceding statement (2.34) is local in ξ and therefore stronger.

The key question for using the modified entropy dissipation functional $S(f)$ is how to choose the matrix P. To determine P we shall need the following algebraic result:

Lemma 2.4.3. *For any fixed matrix $\mathcal{Q} \in \mathbb{R}^{d\times d}$, let $\mu := \min\{\mathrm{Re}(\beta) : \beta$ is an eigenvalue of $\mathcal{Q}\}$. Let $\{\beta_m : 1 \le m \le m_0\}$ be all the eigenvalues of $\mathcal{Q}$ with $\mu = \mathrm{Re}(\beta)$, only counting their geometric multiplicity.*

(a) If β_m is non-defective for all $m \in \{1, ..., m_0\}$, then there exists a symmetric, positive definite matrix $P \in \mathbb{R}^{d\times d}$ with

$$\mathcal{Q}P + P\mathcal{Q}^T \ge 2\mu P.$$

(b) If β_m is defective for at least one $m \in \{1, ..., m_0\}$, then for any $\varepsilon > 0$ there exists a symmetric, positive definite matrix $P(\varepsilon) \in \mathbb{R}^{d\times d}$ with

$$\mathcal{Q}P(\varepsilon) + P(\varepsilon)\mathcal{Q}^T \ge 2(\mu - \varepsilon)P(\varepsilon).$$

Proof. See [3, Lemma 4.3]. □

We consider the matrix function

$$Q(x) := \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}, \quad x \in \mathbb{R}^n, \tag{2.36}$$

which appears in (2.27). We want to construct a symmetric positive definite matrix $P(x)$ such that $Q(x)P(x) + P(x)Q^T(x)$ is positive definite and

$$Q(x)P(x) + P(x)Q^T(x) \ge 2\mu P(x)$$

for some $\mu > 0$ and for all $x \in \mathbb{R}^n$. We recall

$$\alpha(x) := \min_{i\in\{1,..,n\}} \left\{ \alpha_i(x) \ : \ \alpha_i(x) \ \text{ is an eigenvalue of } \frac{\partial^2 V(x)}{\partial x^2} \right\},$$

$$\alpha_0 := \inf_{x\in\mathbb{R}^n} \alpha(x),$$

$$\mu := \inf_{x\in\mathbb{R}^n,\, i\in\{1,..,n\}} \{\mathrm{Re}(\beta_i(x)) : \beta_i(x) \text{ is an eigenvalue of } Q(x)\}.$$

Lemma 2.4.4. *1) The matrix $Q(x)$ is positive stable at any fixed $x \in \mathbb{R}^n$, if and only if $\frac{\partial^2 V(x)}{\partial x^2}$ is positive definite.*
2) Let $\frac{\partial^2 V(x)}{\partial x^2}$ be positive definite for some $x \in \mathbb{R}^n$. Then:

(a) If $\alpha_0 > \frac{\nu^2}{4}$, then $\mu = \frac{\nu}{2}$ and there exists a symmetric positive definite matrix $P(x)$ such that

$$Q(x)P(x) + P(x)Q^T(x) = 2\mu P(x).$$

(b) If $0 < \alpha_0 < \frac{\nu^2}{4}$, then $\mu = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}$ and there exists a symmetric positive definite matrix $P(x)$ such that

$$Q(x)P(x) + P(x)Q^T(x) \ge 2\mu P(x).$$

(c) If $\alpha_0 = \frac{\nu^2}{4}$, then $\mu = \frac{\nu}{2}$ and, for any $\varepsilon \in (0, \nu)$, there exists a symmetric positive definite matrix $P(x, \varepsilon)$ such that

$$Q(x)P(x,\varepsilon) + P(x,\varepsilon)Q^T(x) \ge (2\mu - \varepsilon)P(x,\varepsilon).$$

Proof. Part 1) Let x be any point of $\mathbb{R}^n$, we compute the eigenvalues $\beta(x)$ of $Q(x)$. If $\beta(x) \neq 0$ we have the condition

$$\begin{aligned}\det(Q(x)-\beta(x)I) &= \begin{vmatrix} -\beta(x)I & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & (\nu-\beta(x))I \end{vmatrix} \\ &= \frac{1}{(\beta(x))^n} \begin{vmatrix} -\beta(x)I & 0 \\ -\frac{\partial^2 V(x)}{\partial x^2} & -\frac{\partial^2 V(x)}{\partial x^2} + \beta(x)(\nu-\beta(x))I \end{vmatrix} \\ &= (-1)^n \det\left(-\frac{\partial^2 V(x)}{\partial x^2} + \beta(x)(\nu-\beta(x))I\right) = 0.\end{aligned}$$

Let $\alpha_i(x) \in \mathbb{R}$, $i \in \{1, ..., n\}$ denote the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$, then the above eigenvalue condition reads

$$\prod_{i=1}^{n} (\beta^2(x) - \nu\beta(x) + \alpha_i(x)) = 0.$$

Hence the non-zero eigenvalues of $Q(x)$ are

$$\beta_i^{\pm}(x) = \begin{cases} \frac{\nu \pm \sqrt{\nu^2 - 4\alpha_i(x)}}{2}, & \text{if} \quad \nu^2 \geq 4\alpha_i(x) \\ \frac{\nu \pm \mathbf{i}\sqrt{4\alpha_i(x) - \nu^2}}{2}, & \text{if} \quad \nu^2 < 4\alpha_i(x) \end{cases}, \; i \in \{1, ..., n\}, \tag{2.37}$$

where $\mathbf{i} = \sqrt{-1}$. Moreover, $\beta(x) = 0$ can be an eigenvalue of $Q(x)$ iff one of the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$ is zero. This shows that $Q(x)$ is positive stable (i.e., the eigenvalues $\beta_i(x)$ have positive real part) iff $\frac{\partial^2 V(x)}{\partial x^2} > 0$.

For Part 2) we shall construct matrices $P(x)$, which relies on the proof of Lemma 2.4.3 (Lemma 4.3 in [3]).

(a) Let $\alpha_0 > \frac{\nu^2}{4}$. In this case, because of (2.37) the matrix $Q(x)$ is positive stable and $\mu = \frac{\nu}{2} > 0$. We define the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} \end{pmatrix},$$

and for this choice, it is easy to check that

$$Q(x)P(x) + P(x)Q^T(x) = \nu P(x) = 2\mu P(x).$$

To make sure that $P(x)$ is positive definite, we compute the eigenvalues $\eta(x)$ of $P(x)$ at each $x \in \mathbb{R}^n$: For $\eta(x) \neq 2$ we have the condition

$$\begin{aligned}\det(P(x)-\eta(x)I) &= \begin{vmatrix} (2-\eta(x))I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I \end{vmatrix} \\ &= \frac{1}{(2-\eta(x))^n} \begin{vmatrix} (2-\eta(x))I & 0 \\ \nu I & (2-\eta(x))\left(2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I\right) - \nu^2 I \end{vmatrix} \\ &= \det\left((2-\eta(x))\left(2\frac{\partial^2 V(x)}{\partial x^2} - \eta(x)I\right) - \nu^2 I\right) = 0.\end{aligned}$$

$\eta(x) = 2$ is not an eigenvalue of $P(x)$ and so the eigenvalues of $P(x)$ satisfy

$$\prod_{i=1}^{n} \left(\eta^2(x) - (2 + 2\alpha_i(x))\eta(x) + 4\alpha_i(x) - \nu^2\right) = 0.$$

We conclude that the eigenvalues are

$$\eta_i^{\pm}(x) = 1 + \alpha_i(x) \pm \sqrt{(\alpha_i(x) + 1)^2 - (4\alpha_i(x) - \nu^2)}, \; i \in \{1, ..., n\}.$$

Since we assumed $\alpha_i(x) \geq \alpha(x) \geq \alpha_0 > \frac{\nu^2}{4}$ for all $i \in \{1, ..., n\}$, the eigenvalues are positive and satisfy

$$\eta := \inf_{x\in\mathbb{R}^n,\, i\in\{1,...,n\}} \eta_i^{\pm}(x) = 1 + \alpha_0 - \sqrt{(\alpha_0 + 1)^2 - (4\alpha_0 - \nu^2)} > 0.$$

Thus, $P(x)$ is positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$.

$(b) - (c)$ Let $0 < \alpha_0 \leq \frac{\nu^2}{4}$. Then (2.37) shows $\mu = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}$. Let $\varepsilon > 0$ be a fixed small number. We define

$$\omega := \begin{cases} \alpha_0, & \text{if } \alpha_0 < \frac{\nu^2}{4} \\ \alpha_0 - \frac{\varepsilon^2}{4}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \end{cases}$$

and consider the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + (\nu^2 - 4\omega)I \end{pmatrix}.$$

We compute its eigenvalues $\eta(x)$ by a similar computation as above:

$$\eta_i^{\pm}(x) = 1 + \zeta_i(x) \pm \sqrt{(\zeta_i(x) + 1)^2 - (4\zeta_i(x) - \nu^2)}, \tag{2.38}$$

where $\zeta_i(x) := \alpha_i(x) + \frac{\nu^2}{2} - 2\omega > \frac{\nu^2}{4}$. We also have

$$\eta := \inf_{x\in\mathbb{R}^n,\, i\in\{1,...,n\}} \eta_i^{\pm}(x) = 1 + \alpha_0 + \frac{\nu^2}{2} - 2\omega - \sqrt{(\alpha_0 + \frac{\nu^2}{2} - 2\omega - 1)^2 + \nu^2} > 0.$$

Thus, $P(x)$ is positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$. Then we compute

$$Q(x)P(x) + P(x)Q^T(x)$$
$$= (\nu - \sqrt{\nu^2 - 4\omega})P(x) + \sqrt{\nu^2 - 4\omega} \begin{pmatrix} 2I & (\nu + \sqrt{\nu^2 - 4\omega})I \\ (\nu + \sqrt{\nu^2 - 4\omega})I & 2\frac{\partial^2 V}{\partial x^2} + \sqrt{\nu^2 - 4\omega}(\nu + \sqrt{\nu^2 - 4\omega})I \end{pmatrix}. \tag{2.39}$$

Since $\frac{\partial^2 V}{\partial x^2} \geq \omega I$, the second matrix in the last line of (2.39) is bounded below by

$$\begin{pmatrix} 2I & (\nu + \sqrt{\nu^2 - 4\omega})I \\ (\nu + \sqrt{\nu^2 - 4\omega})I & 2\omega + \sqrt{\nu^2 - 4\omega}(\nu + \sqrt{\nu^2 - 4\omega})I \end{pmatrix}$$
$$= \begin{pmatrix} 2I & (\nu + \sqrt{\nu^2 - 4\omega})I \\ (\nu + \sqrt{\nu^2 - 4\omega})I & \frac{1}{2}(\nu + \sqrt{\nu^2 - 4\omega})^2 I \end{pmatrix} \geq 0.$$

Consequently, we get

$$Q(x)P(x) + P(x)Q^T(x) \geq (\nu - \sqrt{\nu^2 - 4\omega})P(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

□

Lemma 2.4.4 shows that, if $\frac{\partial^2 V(x)}{\partial x^2}$ is not positive definite at some $x \in \mathbb{R}^n$ (and hence $\alpha_0 \leq 0$), then $Q(x)$ is not positive stable. In this case, it is not possible to find a positive constant μ and a positive definite matrix $P(x)$ such that $Q(x)P(x)+P(x)Q^T(x) \geq \mu P(x)$. If α_0 is just finite and not necessarily positive, we have the following modified inequality.

Lemma 2.4.5. *Let $\alpha_0 > -\infty$. Then there exist $\gamma \geq 0$, $\delta \in [0,\nu)$, and a symmetric positive definite matrix function $P(x)$ such that*

$$Q(x)P(x) + P(x)Q^T(x) + \gamma D \geq (\nu - \delta)P(x), \quad \forall x \in \mathbb{R}^d, \tag{2.40}$$

where $D = \begin{pmatrix} 0 & 0 \\ 0 & \sigma I \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$ is the matrix defined in (2.24).

Proof. Let $a \geq 0$ be any constant such that $a + \alpha_0 > \frac{\nu^2}{4}$. We consider the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + 2aI \end{pmatrix}.$$

In analogy to (2.38) we find its eigenvalues as

$$\eta_i^\pm(x) = 1 + \zeta_i(x) \pm \sqrt{(\zeta_i(x)+1)^2 - (4\zeta_i(x) - \nu^2)},$$

where $\zeta_i(x) := \alpha_i(x)+a \geq a+\alpha_0 > \frac{\nu^2}{4}$, and $\alpha_i(x) \in \mathbb{R},\ i \in \{1,...,n\}$ denote the eigenvalues of $\frac{\partial^2 V(x)}{\partial x^2}$. We also have

$$\eta := \inf_{x\in\mathbb{R}^n, i\in\{1,\dots,n\}} \eta_i^\pm(x) = \frac{4(a+\alpha_0 - \frac{\nu^2}{4})}{1+a+\alpha_0+\sqrt{(a+\alpha_0-1)^2+\nu^2}} > 0. \tag{2.41}$$

Thus, $P(x)$ is uniformly positive definite and $P(x) \geq \eta I$ for all $x \in \mathbb{R}^n$.

Next we compute

$$QP + PQ^T + \gamma D = \nu P + \begin{pmatrix} 0 & 2aI \\ 2aI & (2\nu a + \gamma\sigma)I \end{pmatrix}$$

$$= (\nu - \delta)P + \begin{pmatrix} 2\delta I & (\nu\delta + 2a)I \\ (\nu\delta+2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma)I \end{pmatrix}, \tag{2.42}$$

where $\delta \in [0,\nu)$ will be chosen later. We compute the (real) eigenvalues θ of the symmetric matrix

$$\begin{pmatrix} 2\delta I & (\nu\delta + 2a)I \\ (\nu\delta+2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma)I \end{pmatrix} \tag{2.43}$$

which appears in (2.42):

For $\theta(x) \neq 2\delta$ we have the condition

$$\begin{vmatrix} (2\delta-\theta)I & (\nu\delta+2a)I \\ (\nu\delta+2a)I & \delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I \end{vmatrix}$$

$$= \frac{1}{(2\delta-\theta)^n} \begin{vmatrix} (2\delta-\theta)I & 0 \\ (\nu\delta+2a)I & (2\delta-\theta)\left(\delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I\right) - (\nu\delta+2a)^2 I \end{vmatrix}$$

$$= \left|(2\delta-\theta)\left(\delta(2\frac{\partial^2 V}{\partial x^2} + 2aI) + (2\nu a + \gamma\sigma - \theta)I\right) - (\nu\delta+2a)^2 I\right|$$

$$= \prod_{i=1}^n \left(\theta^2 - \theta\left[2\delta(\alpha_i(x)+a) + 2\delta + 2\nu a + \gamma\sigma\right] + 4\delta^2(\alpha_i(x) + a - \nu^2/4) + 2\delta\gamma\sigma - 4a^2\right) = 0.$$

Let us consider the following equations with $i \in \{1, ..., n\}$:

$$\theta^2 - \theta[2\delta(\alpha_i(x) + a) + 2\delta + 2\nu a + \gamma\sigma] + [4\delta^2(\alpha_i(x) + a - \nu^2/4) + 2\delta\gamma\sigma - 4a^2] = 0, \quad (2.44)$$

and we shall show that they have non-negative solutions for an appropriate choice of δ and γ. To this end we see first that

$$2\delta(\alpha_i(x) + a) + 2\delta + 2\nu a + \gamma\sigma \geq 2\delta(\alpha_0 + a) + 2\delta \geq \frac{\delta\nu^2}{2} + 2\delta \geq 0.$$

Next, we choose

$$\delta = \delta(a, \gamma) := \frac{1}{\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}} \left[\sqrt{\left(\frac{\gamma\sigma}{4\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}} \right)^2 + a^2} - \frac{\gamma\sigma}{4\sqrt{a + \alpha_0 - \frac{\nu^2}{4}}} \right] \geq 0, \quad (2.45)$$

which satisfies

$$4\delta^2(a + \alpha_0 - \frac{\nu^2}{4}) + 2\delta\gamma\sigma - 4a^2 = 0. \quad (2.46)$$

Hence, the last term of (2.44) satisfies

$$4\delta^2(\alpha_i(x) + a - \frac{\nu^2}{4}) + 2\delta\gamma\sigma - 4a^2 \geq 4\delta^2(a + \alpha_0 - \frac{\nu^2}{4}) + 2\delta\gamma\sigma - 4a^2 = 0$$

for all $i \in \{1, ..., n\}$. Therefore, the quadratic equations (2.44) have non-negative coefficients and so their solutions, i.e. the eigenvalues of (2.43), are non-negative. Consequently, we get (2.40).

We note that δ from (2.45) satisfies, for any fixed $a > \frac{\nu}{4} - \alpha_0$, $\delta(a, \gamma) \to 0$ as $\gamma \to \infty$. Hence, choosing γ large enough, we have $\delta \in [0, \nu)$. □

Remark 2.4.6. *If $\alpha_0 > 0$, we can take $\gamma = 0$ in Lemma 2.4.5. This follows by choosing in the proof of Lemma 2.4.5*

$$a = \begin{cases} 0, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\varepsilon^2}{2}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ \frac{\nu^2 - 4\alpha_0}{2}, & \text{if } 0 < \alpha_0 < \frac{\nu^2}{4} \end{cases}, \quad \delta = \begin{cases} 0, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\varepsilon}{\sqrt{2}}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ \sqrt{\nu^2 - 4\alpha_0}, & \text{if } 0 < \alpha_0 < \frac{\nu^2}{4} \end{cases},$$

with any $\varepsilon \in (0, \nu)$. Therefore, Lemma 2.4.5 includes the second part of Lemma 2.4.4. However, if $\alpha_0 \leq 0$, we have to choose $\gamma > 0$.

2.5 Proofs

2.5.1 Proof of Theorem 2.2.3

Proof. We denote $u_1 := \nabla_x \left(\frac{f}{f_\infty} \right)$, $u_2 := \nabla_v \left(\frac{f}{f_\infty} \right)$, and $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. We consider the modified dissipation functional

$$S(f(t)) = 2 \int_{\mathbb{R}^{2n}} u^T(t) P u(t) f_\infty dx dv$$

for some symmetric positive definite matrix $P = P(x, v) \in \mathbb{R}^{2n\times 2n}$. By Lemma 2.4.1 (for a t-independent matrix P) we have

$$\frac{d}{dt}S(f(t)) = -4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv - 2\int_{\mathbb{R}^{2n}} u^T \left\{ QP + PQ^T \right\} u f_\infty dx dv$$
$$- 2\int_{\mathbb{R}^{2n}} u^T \left\{ [\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P \right\} u f_\infty dx dv, \quad (2.47)$$

with $Q(x) = \begin{pmatrix} 0 & I \\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I \end{pmatrix}$. Let $c \in \mathbb{R}$ and $\tau \in [0, \nu)$ are the constants such that Assumption 2.2.2 is satisfied. Since (2.6) is positive semi-definite, $\frac{\partial^2 V(x)}{\partial x^2} + cI$ is also positive semi-definite and so $\frac{\partial^2 V(x)}{\partial x^2} \geq -cI$ for all $x \in \mathbb{R}^n$. We define the matrix P depending on the constant c.

Case (a)**:**

Assume $c \leq -\frac{\nu^2}{4}$, $\alpha_0 > \frac{\nu^2}{4}$. By Lemma 2.4.4 $(2a)$ and by its proof, the matrix $P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} \end{pmatrix}$ satisfies

$$Q(x)P(x) + P(x)Q^T(x) = \nu P(x) \quad \text{and} \quad P(x) \geq \eta I$$

for all $x \in \mathbb{R}^n$ and $\eta := 1 + \alpha_0 - \sqrt{(\alpha_0 + 1)^2 - (4\alpha_0 - \nu^2)} > 0$. For this choice of the matrix P,

$$[\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P(x) = \begin{pmatrix} 0 & 0 \\ 0 & -2\frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix}. \quad (2.48)$$

Then (2.47) can be written as

$$\frac{d}{dt}S(f(t)) = -4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv$$
$$- 2\nu \int_{\mathbb{R}^{2n}} u^T P u f_\infty dx dv + 4\int_{\mathbb{R}^{2n}} u^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix} u f_\infty dx dv$$
$$= -4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv - \nu S(f(t)) + 4\int_{\mathbb{R}^{2n}} u^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix} u f_\infty dx dv. \quad (2.49)$$

We shall now consider each term of this equation. First we compute

$$S(f(t)) = 2\int_{\mathbb{R}^{2n}} \left\{ 2|u_1|^2 + 2\nu u_1 \cdot u_2 + 2u_2^T \frac{\partial^2 V}{\partial x^2} u_2 \right\} f_\infty dx dv$$
$$= 4\int_{\mathbb{R}^{2n}} |u_1 + \frac{\nu}{2} u_2|^2 f_\infty dx dv + 4\int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) u_2 f_\infty dx dv$$
$$\geq 4\int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) u_2 f_\infty dx dv. \quad (2.50)$$

Then

$$
\begin{aligned}
&4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv \\
&= 4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} \left(2|\partial_{v_i} u_1|^2 + 2\nu \partial_{v_i} u_1 \cdot \partial_{v_i} u_2 + 2(\partial_{v_i} u_2)^T \frac{\partial^2 V}{\partial x^2} \partial_{v_i} u_2 \right) \right\} f_\infty dx dv \\
&= 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} |\partial_{v_i} u_1 + \frac{\nu}{2} \partial_{v_i} u_2|^2 \right\} f_\infty dx dv \\
&+ 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) \partial_{v_i} u_2 \right\} f_\infty dx dv \\
&\geq 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) \partial_{v_i} u_2 \right\} f_\infty dx dv. \qquad (2.51)
\end{aligned}
$$

Now we consider the last term in (2.49)

$$
\begin{aligned}
&4 \int_{\mathbb{R}^{2n}} u^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix} u f_\infty dx dv = 4 \int_{\mathbb{R}^{2n}} u_2^T \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} u_2 f_\infty dx dv \\
&= 4 \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j=1}^{n} u_{2,i} v \cdot \nabla_x V_{ij} u_{2,j} \right\} f_\infty dx dv = 4 \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j,k=1}^{n} u_{2,i} v_k V_{ijk} u_{2,j} \right\} f_\infty dx dv \\
&= -\frac{4\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j,k=1}^{n} u_{2,i} V_{ijk} u_{2,j} (\partial_{v_k} f_\infty) \right\} dx dv = \frac{4\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j,k=1}^{n} \partial_{v_k} (u_{2,i} u_{2,j}) V_{ijk} \right\} f_\infty dx dv \\
&= \frac{4\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j,k=1}^{n} (\partial_{v_k} u_{2,i}) u_{2,j} V_{ijk} + u_{2,i} (\partial_{v_k} u_{2,j}) V_{ijk} \right\} f_\infty dx dv \\
&= \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i,j,k=1}^{n} (\partial_{v_k} u_{2,i}) u_{2,j} V_{ijk} \right\} f_\infty dx dv = \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{k=1}^{n} (\partial_{v_k} u_2)^T \frac{\partial^2 (\partial_{x_k} V)}{\partial x^2} u_2 \right\} f_\infty dx dv, \\
&\qquad (2.52)
\end{aligned}
$$

where we integrated by parts and used $\partial_{v_k} f_\infty = -\frac{\nu}{\sigma} v_k f_\infty$ and the notations $u_{2,i} := \partial_{v_i}\left(\frac{f}{f_\infty}\right)$, $V_{ij} := \partial^2_{x_i x_j} V$, $V_{ijk} := \partial^3_{x_i x_j x_k} V$. By (2.49), (2.51), (2.52), and (2.50) we obtain

$$\begin{aligned}
\frac{d}{dt}S(f(t)) + (\nu-\tau)S(f(t)) &\le -\tau S(f(t)) \\
&- 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) \partial_{v_i} u_2 \right\} f_\infty dx dv \\
&+ \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2 \right\} f_\infty dx dv \\
&\le -4\tau \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) u_2 f_\infty dx dv \\
&- 8\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) \partial_{v_i} u_2 \right\} f_\infty dx dv \\
&+ \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2 \right\} f_\infty dx dv
\end{aligned}$$

$$\begin{aligned}
= -\frac{8\sigma}{\nu} \sum_{i=1}^n \int_{\mathbb{R}^{2n}} \left\{ \nu (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) \partial_{v_i} u_2 - (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2 \right\} f_\infty dx dv \\
- \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \frac{\tau\nu}{2\sigma} u_2^T \left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \right) u_2 f_\infty dx dv.
\end{aligned}$$

The right hand side of this inequality is a quadratic polynomial with respect to $\partial_{v_i} u_2$, $i \in \{1, ..., n\}$, and u_2. The corresponding matrix of this quadratic polynomial is

$$\begin{pmatrix}
\nu\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V)}{\partial x^2} \\
0 & \nu\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V)}{\partial x^2} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \nu\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V)}{\partial x^2} \\
-\frac{1}{2}\frac{\partial^2(\partial_{x_1} V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V)}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I\right)
\end{pmatrix}. \tag{2.53}$$

The assumption $\frac{\partial^2 V}{\partial x^2} - \frac{\nu^2}{4} I \ge \frac{\partial^2 V}{\partial x^2} + cI$ and the Assumption 2.2.2 imply that (2.53) is positive semi-definite.

Thus we have obtained

$$\frac{d}{dt}S(f(t)) + (\nu-\tau)S(f(t)) \le 0$$

and by Grönwall's lemma

$$S(f(t)) \le e^{-(\nu-\tau)t} S(f_0). \tag{2.54}$$

The estimate $P(x) \ge \eta I$ and the Poincaré inequality (2.5) imply

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1 \right)^2 f_\infty dx dv \le \frac{1}{2C_{PI}\eta} S(f(t)) \le \frac{1}{2C_{PI}\eta} e^{-(\nu-\tau)t} S(f_0). \tag{2.55}$$

The matrix inequalities (see Lemma 2.6.1 in Appendix 2.6.2)

$$\frac{1}{1+\alpha_0+\sqrt{(1-\alpha_0)^2+\nu^2}}P \leq \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I \end{pmatrix} \leq \frac{1+\alpha_0+\sqrt{(1-\alpha_0)^2+\nu^2}}{4\alpha_0-\nu^2}P \tag{2.56}$$

show that $S(f(t))$ is equivalent to the functional

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \nabla_v^T\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv.$$

This equivalence, and (2.55) let us obtain (2.9).

Case (b)**:**

Assume $c=-\alpha_0=-\frac{\nu^2}{4}$. Then by Lemma 2.4.4 (2$c$), for any $\varepsilon\in(0,\nu-\tau)$, the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2}+\varepsilon^2 I \end{pmatrix}$$

satisfies

$$Q(x)P(x)+P(x)Q^T(x) \geq (\nu-\varepsilon)P(x) \ \text{ and } \ P(x)\geq \eta I \tag{2.57}$$

for all $x\in\mathbb{R}^n$ and $\eta := 1+\frac{\nu^2+2\varepsilon^2}{4}-\sqrt{(\frac{\nu^2+2\varepsilon^2}{4}-1)^2+\nu^2}>0$. With this matrix we have

$$\begin{aligned} S(f(t)) &= 4\int_{\mathbb{R}^{2n}} |u_1+\frac{\nu}{2}u_2|^2 f_\infty dxdv + 4\int_{\mathbb{R}^{2n}} u_2^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)u_2 f_\infty dxdv \\ &\geq 4\int_{\mathbb{R}^{2n}} u_2^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)u_2 f_\infty dxdv, \end{aligned} \tag{2.58}$$

$$\begin{aligned} &4\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u)^T P\partial_{v_i}u\right\}f_\infty dxdv \\ &\qquad = 8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n|\partial_{v_i}u_1+\frac{\nu}{2}\partial_{v_i}u_2|^2\right\}f_\infty dxdv \\ &\qquad + 8\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)\partial_{v_i}u_2\right\}f_\infty dxdv, \end{aligned} \tag{2.59}$$

and by using (2.48), $\partial_{v_i}f_\infty=-\frac{\nu}{\sigma}v_i f_\infty$:

$$\begin{aligned} &-2\int_{\mathbb{R}^{2n}} u^T\left\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x+\nu v\cdot\nabla_v-\sigma\Delta_v]P\right\}uf_\infty dxdv \\ &\qquad = \frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv. \end{aligned} \tag{2.60}$$

(2.47), (2.57), (2.58), (2.59), (2.60), and similar estimates as for Case a) show that

$$\begin{aligned} &\frac{d}{dt}S(f(t))+(\nu-\tau-\varepsilon)S(f(t)) \\ &\leq -\frac{8\sigma}{\nu}\sum_{i=1}^n\int_{\mathbb{R}^{2n}}\left\{\nu(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)\partial_{v_i}u_2-(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv \\ &\qquad -\frac{8\sigma}{\nu}\int_{\mathbb{R}^{2n}}\frac{\tau\nu}{2\sigma}u_2^T\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right)u_2 f_\infty dxdv. \end{aligned}$$

The right hand side of this inequality is a quadratic polynomial with respect to $\partial_{v_i} u_2$, $i \in \{1, ..., n\}$, and u_2. The corresponding matrix of this quadratic polynomial is

$$\begin{pmatrix} \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1}V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2}V)}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n}V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I\right) \end{pmatrix}. \tag{2.61}$$

Because of $\frac{\partial^2 V}{\partial x^2}+\frac{2\varepsilon^2-\nu^2}{4}I > \frac{\partial^2 V}{\partial x^2}+cI$ and Assumption 2.2.2, (2.61) is positive definite and we get

$$\frac{d}{dt}S(f(t)) + (\nu-\tau-\varepsilon)S(f(t)) \le 0$$

and by Grönwall's lemma

$$S(f(t)) \le e^{-(\nu-\tau-\varepsilon)t}S(f_0). \tag{2.62}$$

Similar to (2.55), we have

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dxdv \le \frac{1}{2C_{PI}\eta}S(f(t)) \le \frac{1}{2C_{PI}\eta}e^{-(\nu-\tau-\varepsilon)t}S(f_0). \tag{2.63}$$

The functional

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \nabla_v^T\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv$$

and $S(f(t))$ are equivalent because of (see Lemma 2.6.1 in Appendix 2.6.2)

$$\begin{aligned}\frac{1}{1+\frac{\nu^2+2\varepsilon^2}{4}+\sqrt{\left(1-\frac{\nu^2+2\varepsilon^2}{4}\right)^2+\nu^2}}P \le \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\end{pmatrix} \\ \le \frac{1+\frac{\nu^2+2\varepsilon^2}{4}+\sqrt{\left(1-\frac{\nu^2+2\varepsilon^2}{4}\right)^2+\nu^2}}{2\varepsilon^2}P.\end{aligned} \tag{2.64}$$

This equivalence, and (2.63) imply (2.9).

Case (c) **and** (d)**, exponential decay:**

Assume $c > -\frac{\nu^2}{4}$. For some $\gamma \ge 0$ to be chosen later, we consider the functional

$$\begin{aligned}\Phi(f(t)) :=& \gamma\int_{\mathbb{R}^{2n}} \left(\frac{f}{f_\infty}-1\right)^2 f_\infty dxdv + S(f(t)) \\ =& \gamma\int_{\mathbb{R}^{2n}} \left(\frac{f}{f_\infty}-1\right)^2 f_\infty dxdv + 2\int_{\mathbb{R}^{2n}} u^T P u f_\infty dxdv.\end{aligned} \tag{2.65}$$

Using (2.17) and (2.47) its time derivative reads

$$\frac{d\Phi(f(t))}{dt} = -4\sigma\int_{\mathbb{R}^{2n}} \left\{\sum_{i=1}^n (\partial_{v_i}u)^T P \partial_{v_i}u\right\} f_\infty dxdv - 2\int_{\mathbb{R}^{2n}} u^T\left\{QP+PQ^T+\gamma D\right\}uf_\infty dxdv$$

$$- 2 \int_{\mathbb{R}^{2n}} u^T \left\{ [\nabla_x V \cdot \nabla_v - v \cdot \nabla_x + \nu v \cdot \nabla_v - \sigma \Delta_v] P \right\} u f_\infty dx dv. \tag{2.66}$$

Let a, to be chosen later, be any number such that $a \geq c + \frac{\nu^2}{4} > 0$ and $a + \alpha_0 > \frac{\nu^2}{4}$. We consider the matrix

$$P(x) := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V(x)}{\partial x^2} + 2aI \end{pmatrix}. \tag{2.67}$$

Then, by Lemma 2.4.5 we have

$$Q(x)P(x) + P(x)Q^T(x) + \gamma D \geq (\nu - \delta) P(x), \quad \forall x \in \mathbb{R}^d, \tag{2.68}$$

with a constant δ defined in (2.45). If γ is large enough, (2.45) shows that $\delta \in (0, \nu - \tau)$.

The choice of the matrix P in (2.67), (2.66), and (2.68) lets us estimate

$$\begin{aligned} \frac{d\Phi(f(t))}{dt} &\leq -4\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^n (\partial_{v_i} u)^T P \partial_{v_i} u \right\} f_\infty dx dv \\ &\quad - (\nu - \delta) S(f(t)) + 4 \int_{\mathbb{R}^{2n}} u^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 (v \cdot \nabla_x V)}{\partial x^2} \end{pmatrix} u f_\infty dx dv. \end{aligned} \tag{2.69}$$

Similar computations as for Case (a) as well as (2.58) (but with $\varepsilon^2 = 2a$) lead to

$$\begin{aligned} &\frac{d}{dt}\Phi(f(t)) + (\nu - \delta - \tau) S(f(t)) \\ &\leq -\frac{8\sigma}{\nu} \sum_{i=1}^n \int_{\mathbb{R}^{2n}} \left\{ \nu (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + \frac{4a - \nu^2}{4} I \right) \partial_{v_i} u_2 - (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2 \right\} f_\infty dx dv \\ &\qquad - \frac{8\sigma}{\nu} \int_{\mathbb{R}^{2n}} \frac{\tau \nu}{2\sigma} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + \frac{4a - \nu^2}{4} I \right) u_2 f_\infty dx dv. \end{aligned}$$

The two integrands of the right hand side are together a quadratic polynomial of $\partial_{v_i} u_2$, $i \in \{1, ..., n\}$, and u_2, and its corresponding matrix is

$$\begin{pmatrix} \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{4a-\nu^2}{4} I\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V)}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{4a-\nu^2}{4} I\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V)}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{4a-\nu^2}{4} I\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V)}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V)}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V)}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2} + \frac{4a-\nu^2}{4} I\right) \end{pmatrix}. \tag{2.70}$$

Because of $a - \frac{\nu^2}{4} \geq c$ and Assumption 2.2.2, the matrix (2.70) is positive semi-definite, thus, we have

$$\frac{d}{dt}\Phi(f(t)) + (\nu - \tau - \delta) S(f(t)) \leq 0. \tag{2.71}$$

The estimate $P(x) \geq \eta I$ ($\eta > 0$ defined in (2.41)) and the Poincaré inequality (2.5) imply

$$\int_{\mathbb{R}^{2n}} \left(\frac{f}{f_\infty} - 1 \right)^2 f_\infty dx dv \leq \frac{1}{2\eta C_{PI}} S(f(t))$$

and so

$$\frac{1}{1 + \frac{\gamma}{2\eta C_{PI}}} \Phi(f(t)) \leq S(f(t)).$$

This estimate and (2.71) let us conclude

$$\frac{d}{dt}\Phi(f(t)) + 2\lambda\Phi(f(t)) \le 0 \tag{2.72}$$

for

$$2\lambda = \frac{\nu - \tau - \delta}{1 + \frac{\gamma}{2\eta C_{PI}}} > 0. \tag{2.73}$$

By Grönwall's lemma we obtain

$$\Phi(f(t)) \le e^{-2\lambda t}\Phi(f_0). \tag{2.74}$$

One can check that (see Lemma 2.6.1 in Appendix 2.6.2)

$$\begin{aligned}\frac{1}{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}P \le \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\end{pmatrix} \\ \le \frac{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{4(a+\alpha_0)-\nu^2}P. \end{aligned}\tag{2.75}$$

Hence, $S(f(t))$ is equivalent to the functional

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x \left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dx dv + \int_{\mathbb{R}^{2n}} \nabla_v^T \left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dx dv.$$

Subsequently, $\Phi(f(t))$ and the functional on the left hand side of (2.9) are equivalent. This equivalence and (2.74) let us obtain (2.9).

Case (c) **and** (d)**, estimated decay rate:**

Next, we shall estimate λ from (2.73) explicitly, and we shall choose the parameters a and γ such that λ is (rather) large. By (2.41) and (2.46), $\eta = \eta(a)$ and $\delta = \delta(a,\gamma)$ are functions of $a \in [c + \frac{\nu^2}{4}, \infty) \bigcap (\frac{\nu^2}{4} - \alpha_0, \infty)$ and $\gamma \in [0,\infty)$. Since $\delta > 0$, and η is monotonically increasing up to 2, we have the following uniform estimate and choice of the decay rate:

$$2\lambda := \sup_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty),\,\gamma\ge 0} \frac{\nu-\tau-\delta(a,\gamma)}{1+\frac{\gamma}{2\eta(a)C_{PI}}} \le \sup_{\gamma\ge 0}\frac{\nu-\tau}{1+\frac{\gamma}{4C_{PI}}} \le \nu-\tau.$$

Next, we shall estimate this supremum (in fact it is a maximum). First we introduce a new variable $s := \frac{\gamma\sigma}{4a\sqrt{a+\alpha_0-\frac{\nu^2}{4}}} \in [0,\infty)$, then

$$\delta(a,\gamma) = \frac{a}{\sqrt{a+\alpha_0-\frac{\nu^2}{4}}}(\sqrt{1+s^2}-s).$$

With the notations $A(a) := \frac{1+a+\alpha_0+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{2\sigma C_{PI}} > 0$ and $B(a) := \frac{a}{\sqrt{a+\alpha_0-\frac{\nu^2}{4}}} > 0$, we have

$$2\lambda = \max_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty),\, s\ge 0} \frac{\nu-\tau-B(a)(\sqrt{1+s^2}-s)}{1+A(a)B(a)s}.$$

Next, we shall fix the parameter a. To estimate λ as accurately as possible, we choose a as the argmin of $B(a)$ such that $\nu-\tau-B(a)(\sqrt{1+s^2}-s)$ is maximal with respect to a. The minimal value of $B(a)$ is

$$\min_{a\in[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty)} B(a)=\begin{cases} B(a_1)=\dfrac{c+\frac{\nu^2}{4}}{\sqrt{c+\alpha_0}} & \text{if } \ c+2\alpha_0>\dfrac{\nu^2}{4} \\ B(a_2)=\sqrt{\nu^2-4\alpha_0} & \text{if } \ c+2\alpha_0\le\dfrac{\nu^2}{4}\end{cases},$$

and this minimum is attained at $a_1:=c+\frac{\nu^2}{4}$ if $c+2\alpha_0>\frac{\nu^2}{4}$ (i.e. in Case (c)), and $a_2:=2(\frac{\nu^2}{4}-\alpha_0)$ if $c+2\alpha_0\le\frac{\nu^2}{4}$ (i.e. in Case (d)).

If $c+2\alpha_0>\frac{\nu^2}{4}$, then $c>-\alpha_0$ and so a varies in

$$[c+\frac{\nu^2}{4},\infty)\bigcap(\frac{\nu^2}{4}-\alpha_0,\infty)=[c+\frac{\nu^2}{4},\infty)=[a_1,\infty).$$

Since $A(a)$ is increasing, both $A(a)$ and $B(a)$ attain their minimal values at a_1. Thus, a_1 is optimal, i.e.

$$\max_a\frac{\nu-\tau-B(a)(\sqrt{1+s^2}-s)}{1+A(a)B(a)s}=\frac{\nu-\tau-B(a_1)(\sqrt{1+s^2}-s)}{1+A(a_1)B(a_1)s}.$$

If $c+2\alpha_0\le\frac{\nu^2}{4}$, $a_2=2(\frac{\nu^2}{4}-\alpha_0)$ may not be optimal as $A(a)$ does not attain its minimum at this point, i.e.

$$\max_a\frac{\nu-\tau-B(a)(\sqrt{1+s^2}-s)}{1+A(a)B(a)s}\ge\frac{\nu-\tau-B(a_2)(\sqrt{1+s^2}-s)}{1+A(a_2)B(a_1)s}.$$

But it is the optimal choice when $s=0$ and so it gives a good approximation if s is small. From now on we assume that a is fixed as

$$a:=\begin{cases} a_1=c+\frac{\nu^2}{4} & \text{if } \ c+2\alpha_0>\frac{\nu^2}{4} \\ a_2=2(\frac{\nu^2}{4}-\alpha_0) & \text{if } \ c+2\alpha_0\le\frac{\nu^2}{4}\end{cases}. \tag{2.76}$$

Note that this choice is independent of s.
Let $\Lambda(a,s):=\dfrac{\nu-\tau-B(a)(\sqrt{1+s^2}-s)}{1+A(a)B(a)s}$ and we seek its maximum with respect to $s\in[0,\infty)$. We compute

$$\begin{aligned}&\partial_s\Lambda(a,s)\\ &=\frac{B(a)}{(1+A(a)B(a)s)^2\sqrt{s^2+1}}\Big([1-(\nu-\tau-B(a))A(a)]\sqrt{s^2+1}-A(a)B(a)(\sqrt{s^2+1}-1)-s\Big).\end{aligned} \tag{2.77}$$

If $1-(\nu-\tau-B(a))A(a)\le 0$, then $\partial_s\Lambda(a,s)\le 0$ which implies that $\Lambda(a,s)$ is a decreasing function of s and the maximum in $[0,\infty)$ is attained at $s=0$.

If $1-(\nu-\tau-B(a))A(a)>0$, then $\partial_s\Lambda(a,0)=B(a)[1-(\nu-\tau-B(a))A(a)]>0$ and $\Lambda(a,s)$ is increasing in a neighborhood of $s=0$. We also see $\partial_s\Lambda(a,s)$ is negative if s is large enough (since $\nu-\tau>0$). This means that $\Lambda(a,s)$ starts to grow at $s=0$ and it decreases as $s \to \infty$. Therefore, there is a point in $(0, \infty)$ at which $\Lambda(a, s)$ takes its maximum. Setting $\partial_s \Lambda(a, s) = 0$ we obtain

$$[1 - (\nu - \tau)A(a)]\sqrt{s^2 + 1} - s + A(a)B(a) = 0.$$

It has only one solution in $(0, \infty)$ given by

$$s(a) = \begin{cases} \frac{A^2(a)B^2(a)-1}{2A(a)B(a)} & \text{if } (\nu-\tau)A(a) = 2 \\ \frac{1}{\nu-\tau}\left[\left|\frac{(\nu-\tau)A(a)-1}{(\nu-\tau)A(a)-2}\right| \sqrt{B^2(a) + 2(\nu-\tau)A^{-1}(a) - (\nu-\tau)^2} - \frac{B(a)}{(\nu-\tau)A(a)-2}\right] & \text{if } (\nu-\tau)A(a) \neq 2 \end{cases} \tag{2.78}$$

and at this point $\Lambda(a, s)$ attains its maximum with respect to s.

Considering the computations above, we conclude that the decay rate can be estimated by:

$$2\lambda = \begin{cases} \nu - \tau - B(a) & \text{if } \nu - \tau \geq A^{-1}(a) + B(a) \\ \frac{\nu-\tau-B(a)(\sqrt{1+s^2(a)}-s(a))}{1+A(a)B(a)s(a)} & \text{if } \nu - \tau < A^{-1}(a) + B(a) \end{cases}, \tag{2.79}$$

where two cases correspond to the two cases discussed after (2.77). Moreover, a and $s(a)$ are defined in (2.76) and (2.78), respectively. If we denote $A_1 := A(a_1)$, $A_2 := A(a_2)$, $s_1 := s(a_1)$ and $s_2 := s(a_2)$ and take into account that $B(a_1) = \frac{c+\frac{\nu^2}{4}}{\sqrt{c+\alpha_0}}$ and $B(a_2) = \sqrt{\nu^2 - 4\alpha_0}$, we obtain the explicit decay rates stated in the theorem.

Case (e):

Let $V(x)$ be a quadratic function of x and $\frac{\partial^2 V}{\partial x^2}$ be positive definite. Then, $\frac{\partial^2(\partial_{x_i} V)}{\partial x^2}$ are zero matrices for all $i \in \{1, ..., n\}$. Thus, V satisfies Assumption 2.2.2 with $\tau = 0$, $-c = \alpha_0 > 0$.

If $\alpha_0 < \frac{\nu^2}{4}$, then $c + 2\alpha_0 = \alpha_0 < \frac{\nu^2}{4}$ which falls into Case (d). The constant in the Poincaré inequality (2.5) equals $C_{PI} = \frac{\nu}{\sigma}\min\{1, \alpha_0\}$ (see [4]). It lets us compute A_2^{-1} explicitly:

$$A_2^{-1} = \frac{2\nu \min\{1, \alpha_0\}}{1 + \frac{\nu^2}{2} - \alpha_0 + \sqrt{(\frac{\nu^2}{2} - \alpha_0 - 1)^2 + \nu^2}}.$$

In Appendix 2.6.3 we prove the following inequality:

$$\nu \geq A_2^{-1} + \sqrt{\nu^2 - 4\alpha_0}. \tag{2.80}$$

Thus Case (d) implies

$$\lambda = \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}. \tag{2.81}$$

If $\alpha_0 \geq \frac{\nu^2}{4}$, the decay rate is explicit by Case (a) and Case (b) :

$$\lambda = \begin{cases} \frac{\nu}{2} & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ \frac{\nu-\varepsilon}{2} & \text{if } \alpha_0 = \frac{\nu^2}{4}, \text{ for any } \varepsilon \in (0, \nu) \end{cases}. \tag{2.82}$$

We now prove that the decay rates in (2.81) and (2.82) are sharp: From Corollary 2.2.8

$$\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty} - 1\right)^2 f_\infty dx dv \leq C e^{-2\lambda t} \int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty} - 1\right)^2 \left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2 + 1\right) f_\infty dx dv, \quad \forall t \geq t_0$$

holds with the same λ given in (2.81) and (2.82). Since $\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|+1$ is constant, this estimate implies

$$\sup_{1\neq \frac{f_0}{f_\infty}\in L^2(\mathbb{R}^d,f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d,f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d,f_\infty)}} \leq \tilde{C}e^{-\lambda t}, \quad \forall t\geq t_0 \tag{2.83}$$

for some constant $\tilde{C}>0$. On the one hand this means that the estimated decay rate λ can not be larger than the (true) decay rate of the propagator norm given on the left hand side of (2.83). On the other hand, Proposition 2.2.5 gives the sharp decay rates for this propagator norm. The decay rates in (2.81) and (2.82) coincide with the ones in Proposition 2.2.5 except in the case of $\alpha_0 = \frac{\nu^2}{4}$. Thus, the exponential decay rates in Case (a) and Case (d) are sharp. When $\alpha_0 = \frac{\nu^2}{4}$, Proposition 2.2.5 provides the sharp decay $(1+t)e^{-\frac{\nu}{2}t}$ for the propagator norm. Hence, (2.9) can hold with rates $\lambda = \frac{\nu-\varepsilon}{2}$ for any small fixed $\varepsilon\in(0,\nu)$, but it does not hold for $\varepsilon=0$. □

2.5.2 Proof of Proposition 2.2.5

Proof of Proposition 2.2.5. Let V be a quadratic polynomial and $\frac{\partial^2 V}{\partial x^2} =: M^{-1}\in\mathbb{R}^{n\times n}$ be positive definite. Then there are $x_0\in\mathbb{R}^n$ and $\mathcal{C}\in\mathbb{R}$ such that $V(x) = \frac{(x-x_0)^T M^{-1}(x-x_0)}{2} + \mathcal{C}$, $\forall x\in\mathbb{R}^n$. Since the change $x\to x+x_0$ does not affect the supremum in (2.10) and only the gradient of V appears in (2.1), without loss of generality we assume that $x_0=0$ and $\mathcal{C}=0$.

Step 1, reformulation as an ODE-problem:

To this end we use Theorem 2.3.2. We check the conditions of this theorem for the kinetic Fokker-Planck equation. With the notation $\xi = \begin{pmatrix} x \\ v\end{pmatrix}$, we write

$$E(\xi) = \frac{\nu}{\sigma}\left(V(x)+\frac{|v|^2}{2}\right) = \frac{\nu}{\sigma}\left(\frac{x^T M^{-1}x}{2}+\frac{|v|^2}{2}\right) = \frac{1}{2}\xi^T\begin{pmatrix}\frac{\nu}{\sigma}M^{-1} & 0\\ 0 & \frac{\nu}{\sigma}I\end{pmatrix}\xi = \frac{\xi^T K^{-1}\xi}{2} \tag{2.84}$$

with $K^{-1} := \frac{\nu}{\sigma}\begin{pmatrix} M^{-1} & 0\\ 0 & I\end{pmatrix}$. From (2.25) we see that $\mathrm{Ker}D = \{(\psi,0)^T : \psi\in\mathbb{R}^n\}$. Let $(\psi,0)^T\in\mathrm{Ker}D$, then its image under $K^{-1}(D-R)$ is

$$K^{-1}(D-R)\begin{pmatrix}\psi\\0\end{pmatrix} = \begin{pmatrix}0 & M^{-1}\\ -I & \nu I\end{pmatrix}\begin{pmatrix}\psi\\0\end{pmatrix} = \begin{pmatrix}0\\-\psi\end{pmatrix}$$

and it is in $\mathrm{Ker}D$ iff $\psi=0$. Therefore, there is no non-trivial $K^{-1}(D-R)-$invariant subspace of $\mathrm{Ker}D$. Next we compute the eigenvalues β of $K^{-1/2}(D+R)K^{-1/2} = \begin{pmatrix}0 & -M^{-1/2}\\ M^{-1/2} & \nu I\end{pmatrix}$:

$$\begin{vmatrix}-\beta I & -M^{-1/2}\\ M^{-1/2} & (\nu-\beta)I\end{vmatrix} = \begin{vmatrix}-\beta I & 0\\ M^{-1/2} & (\nu-\beta)I - \beta^{-1}M^{-1}\end{vmatrix}$$
$$= \det(\beta(\beta-\nu)I + M^{-1}) = \prod_{i=1}^n(\beta^2-\nu\beta+\alpha_i) = 0,$$

where α_i, $i \in \{1, ..., n\}$ denote the eigenvalues of M^{-1}. By solving the latter equation, we find that the eigenvalues of $K^{-1/2}(D+R)K^{-1/2}$ are $\beta_i^- = \frac{\nu - \sqrt{\nu^2 - 4\alpha_i}}{2}$, $\beta_i^+ = \frac{\nu + \sqrt{\nu^2 - 4\alpha_i}}{2}$, $i \in \{1, ..., n\}$. If $\alpha_0 > 0$ is the smallest eigenvalue of M^{-1}, then

$$\mu := \min_i \{\mathsf{Re}(\beta_i) : \beta_i \text{ is an eigenvalue of } K^{-1/2}(D+R)K^{-1/2}\} = \begin{cases} \frac{\nu}{2} & \text{if } \alpha_0 \geq \frac{\nu^2}{4} \\ \frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2} & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases}.$$

Hence μ is positive, so $K^{-1/2}(D+R)K^{-1/2}$ and $(D+R)K^{-1}$ are positive stable. Therefore, Theorem 2.3.2 applies to the kinetic Fokker-Planck equation.

Step 2, decay rates of the ODE-solution:

We consider the ODE

$$\dot{\xi}(t) = -K^{-1/2}(D+R)K^{-1/2}\xi$$

with the initial data $\xi(0) = \xi_0$. Since $K^{-1/2}(D+R)K^{-1/2}$ is positive stable, the solution $\xi(t)$ is stable. To quantify the decay rate, we continue to analyze the eigenvalues of $K^{-1/2}(D+R)K^{-1/2}$. Let m_i be the multiplicity of $\alpha_i > 0$ as an eigenvalue of M^{-1} (now the α_i with $i \in \{1, ..., \tilde{n}\}$ are labeled without multiplicity). Since M^{-1} is symmetric, there are linearly independent eigenvectors $\psi_{ij} \in \mathbb{R}^n$, $j \in \{1, ..., m_i\}$ of M^{-1} corresponding to α_i. Then we can check that the vectors

$$\begin{pmatrix} -\frac{\alpha_i^{1/2}}{\beta_i^-}\psi_{ij} \\ \psi_{ij} \end{pmatrix} \in \mathbb{R}^{2n}, \ j \in \{1, ..., m_i\} \tag{2.85}$$

are linearly independent eigenvectors of $K^{-1/2}(D+R)K^{-1/2}$ corresponding to β_i^-, $i \in \{1, ..., \tilde{n}\}$. Moreover, these vectors form a basis of the space of eigenvectors corresponding to β_i^-. Similarly, the vectors

$$\begin{pmatrix} -\frac{\alpha_i^{1/2}}{\beta_i^+}\psi_{ij} \\ \psi_{ij} \end{pmatrix} \in \mathbb{R}^{2n}, \ j \in \{1, ..., m_i\}. \tag{2.86}$$

satisfy the same property for β_i^+.

If $\alpha_i \neq \frac{\nu^2}{4}$ for all $i \in \{1, ..., \tilde{n}\}$ (i.e., $\beta_i^- \neq \beta_i^+$), the algebraic multiplicities of β_i^- and β_i^+ are equal to m_i. Then β_i^- (resp. β_i^+) has m_i eigenvectors given by (2.85) (resp. (2.86)). Thus, the geometric multiplicities of β_i^- and β_i^+ also equal m_i. In particular, $K^{-1/2}(D+R)K^{-1/2}$ is diagonalizable.

If $\alpha_{i_0} = \frac{\nu^2}{4}$ for some $i_0 \in \{1, ..., \tilde{n}\}$, then the algebraic multiplicity of $\beta_{i_0}^- = \beta_{i_0}^+ = \frac{\nu}{2}$ equals $2m_{i_0}$. Since the vectors (2.85) and (2.86) coincide in this case, the geometric multiplicity of $\frac{\nu}{2}$ equals m_{i_0}. Thus, in this case, $\frac{\nu}{2}$ is a defective[3] eigenvalue of $K^{-1/2}(D+R)K^{-1/2}$ with the corresponding eigenvectors

$$\begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix} \in \mathbb{R}^{2n}, \ j \in \{1, ..., m_{i_0}\}. \tag{2.87}$$

By solving the following linear system (with respect to ξ)

$$K^{-1/2}(D+R)K^{-1/2}\xi - \frac{\nu}{2}\xi = \begin{pmatrix} -\frac{\nu}{2}I & -M^{-1/2} \\ M^{-1/2} & \frac{\nu}{2}I \end{pmatrix}\xi = \begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix}, \quad \xi \in \mathbb{R}^{2d},$$

[3]An eigenvalue is *defective* if its geometric multiplicity is strictly less than its algebraic multiplicity.

we find that the solution $\xi = \begin{pmatrix} 0 \\ \frac{2}{\nu}\psi_{i_0 j} \end{pmatrix}$ is a generalized eigenvector of $\frac{\nu}{2}$ corresponding to the eigenvector $\begin{pmatrix} -\psi_{i_0 j} \\ \psi_{i_0 j} \end{pmatrix}$. Since $\psi_{i_0 j}$, $j \in \{1, ..., m_{i_0}\}$ are linearly independent, the vectors

$$\begin{pmatrix} 0 \\ \frac{2}{\nu}\psi_{i_0 j} \end{pmatrix}, \ j \in \{1, ..., m_{i_0}\} \tag{2.88}$$

form a set of linearly independent generalized eigenvectors of $\frac{\nu}{2}$. Since the vectors in (2.87) and (2.88) are linearly independent and their total number equals $2m_{i_0}$ (which is the algebraic multiplicity of $\frac{\nu}{2}$), we conclude that each eigenvector of $\frac{\nu}{2}$ has only one generalized eigenvector. Therefore, all Jordan blocks associated to $\frac{\nu}{2}$ have the same size 2×2. In particular, if $\alpha_0 = \frac{\nu^2}{4}$, then the eigenvalue $\mu = \frac{\nu}{2}$ is defective and the maximal size of the Jordan blocks associated to $\frac{\nu}{2}$ is 2.

Then, the classical stability theory for ODEs shows that

$$\sup_{1 \neq \frac{f_0}{f_\infty} \in L^2(\mathbb{R}^d, f_\infty)} \frac{||f(t)/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}}{||f_0/f_\infty - 1||_{L^2(\mathbb{R}^d, f_\infty)}} = \sup_{0 \neq \xi_0 \in \mathbb{R}^d} \frac{||\xi(t)||_2}{||\xi_0||_2} \asymp \begin{cases} e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 > \frac{\nu^2}{4} \\ (1+t)e^{-\frac{\nu}{2}t}, & \text{if } \alpha_0 = \frac{\nu^2}{4} \\ e^{-\frac{\nu - \sqrt{\nu^2 - 4\alpha_0}}{2}t}, & \text{if } \alpha_0 < \frac{\nu^2}{4} \end{cases}$$

as $t \to \infty$.

□

Remark 2.5.1. *With the eigenvalues of $C := (D+R)K^{-1}$ (see (2.24), (2.84)) obtained at the end of Step 1 in the above proof, the sharpness of the decay rate μ in the cases 1 and 3 of (2.10) would also follow from [3, Theorem 6.1].*

2.5.3 Proof of Theorem 2.2.7 and Corollary 2.2.8

Proof of Theorem 2.2.7. **Step 1, an auxiliary inequality:**

As we assume the matrix (2.6) is positive semi-definite, then the following submatrices of (2.6) are positive semi-definite:

$$Y_k := \begin{pmatrix} \nu\left(\frac{\partial^2 V}{\partial x^2} + cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_k} V)}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_k} V)}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V}{\partial x^2} + cI\right) \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \quad k \in \{1, ..., n\}.$$

Letting $\delta > 0$, we consider

$$X_\delta := \begin{pmatrix} I & \delta I \\ \delta I & \delta^2 I \end{pmatrix} \otimes \left(\frac{\partial^2 V}{\partial x^2} + cI\right) = \begin{pmatrix} \frac{\partial^2 V}{\partial x^2} + cI & \delta\frac{\partial^2 V}{\partial x^2} + \delta cI \\ \delta\frac{\partial^2 V}{\partial x^2} + \delta cI & \delta^2\frac{\partial^2 V}{\partial x^2} + \delta^2 cI \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

X_δ is positive semi-definite as it is the Kronecker product [27, Corollary 4.2.13] of two positive semi-definite matrices. Hence, we have for all $k \in \{1, ..., n\}$:

$$\begin{aligned} \mathrm{Tr}(X_\delta^{1/2} Y_k X_\delta^{1/2}) &= \mathrm{Tr}(X_\delta Y_k) \\ &= (\nu + \delta^2\frac{\tau\nu}{2\sigma})\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)^2\right] - \delta\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2} + cI\right)\frac{\partial^2(\partial_{x_k} V)}{\partial x^2}\right] \geq 0. \end{aligned}$$

This implies

$$\frac{2\sigma\nu+\delta^2\tau\nu}{2\sigma\delta}\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right]\geq \mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}\right] \tag{2.89}$$

and by minimizing the constant on the left hand side of (2.89) with respect to δ (i.e., by choosing $\delta=\sqrt{\frac{2\sigma}{\tau}}$), we obtain

$$\sqrt{\frac{2\tau\nu^2}{\sigma}}\mathrm{Tr}\left[\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)^2\right]\geq \mathrm{Tr}\left[\left(\frac{\partial^2 V(x)}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V(x))}{\partial x^2}\right]\quad \text{for all}\ \ x\in\mathbb{R}^n. \tag{2.90}$$

Step 2, growth estimate for the r.h.s. of (2.11), (2.12):

We denote $u_1:=\nabla_x\left(\frac{f(t)}{f_\infty}\right)$, $u_2:=\nabla_v\left(\frac{f(t)}{f_\infty}\right)$, and $u:=\begin{pmatrix}u_1\\u_2\end{pmatrix}$. Since $\frac{f(t)}{f_\infty}-1$ satisfies

$$\begin{aligned}\partial_t\left(\frac{f(t)}{f_\infty}-1\right)=-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)+\nabla_xV\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)+\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)\\ -\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\end{aligned}$$

and by integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dxdv=-2\sigma\int_{\mathbb{R}^{2n}}|u_2|^2 f_\infty dxdv. \tag{2.91}$$

Next, we compute (with $||\cdot||$ denoting the Frobenius norm)

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv\\ &\qquad=2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\partial_t\left(\frac{f(t)}{f_\infty}-1\right)\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv\\ &=2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)+\nabla_xV\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv\\ &\quad+2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)-\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv.\end{aligned} \tag{2.92}$$

Integrating by parts with respect to v, we obtain

$$\begin{aligned}2\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)\left[\sigma\Delta_v\left(\frac{f(t)}{f_\infty}-1\right)-\nu v\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv\\ =-2\sigma\int_{\mathbb{R}^{2n}}|u_2|^2\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv.\end{aligned} \tag{2.93}$$

Next, we work on the term in the second line of (2.92):

$$
\begin{aligned}
2\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\left[-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)\right]\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv \\
= \int_{\mathbb{R}^{2n}} \left(-v\cdot\nabla_x\left(\frac{f(t)}{f_\infty}-1\right)^2+\nabla_x V\cdot\nabla_v\left(\frac{f(t)}{f_\infty}-1\right)^2\right)\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv \\
= \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2\left[v\cdot\nabla_x\left(\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty\right)-\nabla_x V\cdot\nabla_v\left(\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2 f_\infty\right)\right] dxdv \\
= \int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2 v\cdot\nabla_x\left(\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2\right) f_\infty dxdv \\
= \frac{2\sigma}{\nu}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right) u_2\cdot\nabla_x\left(\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2\right) f_\infty dxdv \\
= \frac{2\sigma}{\nu}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\sum_{k=1}^n u_{2,k}\partial_{x_k}\left(\left|\left|\frac{\partial^2 V}{\partial x^2}+cI\right|\right|^2\right) f_\infty dxdv \\
= \frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n u_{2,k}\sum_{i,j=1}^n (\partial^2_{x_i x_j}V+\delta_{ij}c)\partial^2_{x_i x_j}(\partial_{x_k}V)\right\} f_\infty dxdv, \quad (2.94)
\end{aligned}
$$

where we integrated by parts twice, and used $-\frac{\nu}{\sigma}vf_\infty=\nabla_v f_\infty$ and the notations

$$
u_{2,k}:=\partial_{v_k}\left(\frac{f(t)}{f_\infty}\right) \quad \text{and} \quad \delta_{ij}:=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}.
$$

Using the identity

$$
\sum_{i,j=1}^n (\partial^2_{x_i x_j}V+\delta_{ij}c)\partial^2_{x_i x_j}(\partial_{x_k}V)=\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}\right],
$$

the estimate (2.90), and the discrete Hölder inequality, (2.94) can be estimated as

$$
\begin{aligned}
\frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{i,j,k=1}^n u_{2,k}(\partial^2_{x_i x_j}V+\delta_{ij}c)\partial^2_{x_i x_j}(\partial_{x_k}V)\right\} f_\infty dxdv \\
= \frac{4\sigma}{\nu}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n u_{2,k}\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\frac{\partial^2(\partial_{x_k}V)}{\partial x^2}\right]\right\} f_\infty dxdv \\
\le 4\sqrt{2\sigma\tau}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)\left\{\sum_{k=1}^n |u_{2,k}|\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right]\right\} f_\infty dxdv \\
\le 4\sqrt{2\sigma\tau n}\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)|u_2|\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right] f_\infty dxdv \\
\le \sigma\int_{\mathbb{R}^{2n}} |u_2|^2\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right] f_\infty dxdv \\
+8\tau n\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right] f_\infty dxdv. \quad (2.95)
\end{aligned}
$$

Combining the equations from (2.92) to (2.95) and the identity

$$\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2=\mathrm{Tr}\left[\left(\frac{\partial^2 V}{\partial x^2}+cI\right)^2\right],$$

we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\\ &\leq -\sigma\int_{\mathbb{R}^{2n}}|u_2|^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv+8\tau n\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv.\end{aligned} \tag{2.96}$$

(2.96) can be reformulated as

$$\begin{aligned}&\frac{d}{dt}\left(e^{-8\tau nt}\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv\right)\\ &\qquad\qquad\leq -\sigma e^{-8\tau nt}\int_{\mathbb{R}^{2n}}|u_2|^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2 f_\infty dxdv.\end{aligned} \tag{2.97}$$

Step 3, $t-$dependent functional Ψ:

In order to prove the short-time regularization of (2.11) and (2.12) we introduce now an auxiliary functional that depends explicitly on time. Our strategy is the generalization of the approach in [32, Theorem A.12], [23, Theorem 1.1], [3, Theorem 4.8].

For $t\in(0,t_0]$, we consider the following functional

$$\Psi(t,f(t)):=\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left(\gamma_1 e^{-8\tau nt}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right)f_\infty dxdv+\int_{\mathbb{R}^{2n}}u^T P u f_\infty dxdv, \tag{2.98}$$

with the $t-$ and $x-$dependent matrix in $\mathbb{R}^{2n\times 2n}$,

$$P=P(t,x):=\begin{pmatrix}2\varepsilon^3t^3I & \varepsilon^2t^2I\\ \varepsilon^2t^2I & 2\varepsilon tI+t(\frac{\partial^2 V}{\partial x^2}+cI)\end{pmatrix}. \tag{2.99}$$

ε, γ_1, and γ_2 are positive constants which we shall fix later. We note that, for all $t\in(0,t_0]$,

$$P(t,x)\geq\begin{pmatrix}\varepsilon^3t^3I & 0\\ 0 & t(\frac{\partial^2 V}{\partial x^2}+cI)+\varepsilon tI\end{pmatrix}>\begin{pmatrix}\varepsilon^3t^3I & 0\\ 0 & t(\frac{\partial^2 V}{\partial x^2}+cI)\end{pmatrix}\geq 0 \tag{2.100}$$

as $\frac{\partial^2 V}{\partial x^2}+cI$ is positive semi-definite. Thus, $\Psi(t,f(t))$ is non-negative and satisfies

$$\begin{aligned}\Psi(t,f(t))\geq&\int_{\mathbb{R}^{2n}}\left(\frac{f(t)}{f_\infty}-1\right)^2\left(\gamma_1 e^{-8\tau nt}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right)f_\infty dxdv\\ &+\varepsilon^3t^3\int_{\mathbb{R}^{2n}}|u_1|^2 f_\infty dxdv+t\int_{\mathbb{R}^{2n}}u_2^T\left(\frac{\partial^2 V}{\partial x^2}+(c+\varepsilon)I\right)u_2 f_\infty dxdv.\end{aligned} \tag{2.101}$$

Our goal is to show that $\Psi(t,f(t))$ decreases. To this end we estimate the time derivative of the second term in (2.98). First, (2.27) yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2n}} u^T P u f_\infty dxdv \\ = -2\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u)^T P\partial_{v_i}u\right\} f_\infty dxdv - \int_{\mathbb{R}^{2n}} u^T\left\{QP+PQ^T-\partial_t P\right\} u f_\infty dxdv \\ - \int_{\mathbb{R}^{2n}} u^T\left\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v]P\right\} u f_\infty dxdv,\end{aligned} \tag{2.102}$$

with $Q=\begin{pmatrix}0 & I\\ -\frac{\partial^2 V(x)}{\partial x^2} & \nu I\end{pmatrix}$. We consider each terms of (2.102). Because of (2.100), the first term can be estimated as

$$\begin{aligned}-2\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u)^T P\partial_{v_i}u\right\} f_\infty dxdv \\ \le -2t\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\partial_{v_i}u_2\right\} f_\infty dxdv.\end{aligned} \tag{2.103}$$

For the third term of (2.102) we have

$$[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v]P = \begin{pmatrix}0 & 0\\ 0 & -t\frac{\partial^2(v\cdot\nabla_x V)}{\partial x^2}\end{pmatrix}$$

and using $vf_\infty = -\frac{\sigma}{\nu}\nabla_v f_\infty$ yields

$$\begin{aligned}-\int_{\mathbb{R}^{2n}} u^T\left\{[\nabla_x V\cdot\nabla_v - v\cdot\nabla_x + \nu v\cdot\nabla_v - \sigma\Delta_v]P\right\} u f_\infty dxdv \\ = \frac{2t\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\} f_\infty dxdv.\end{aligned} \tag{2.104}$$

For the second term of (2.102) we compute

$$\begin{aligned}-\int_{\mathbb{R}^{2n}} u^T\left\{QP+PQ^T-\partial_t P\right\} u f_\infty dxdv \\ = -\int_{\mathbb{R}^{2n}}\begin{pmatrix}u_1\\u_2\end{pmatrix}^T\begin{pmatrix}0 & (t-2\varepsilon^3t^3)\left(\frac{\partial^2V}{\partial x^2}+cI\right)\\ (t-2\varepsilon^3t^3)\left(\frac{\partial^2V}{\partial x^2}+cI\right) & (-1+2\nu t-2\varepsilon^2t^2)\left(\frac{\partial^2V}{\partial x^2}+cI\right)\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} f_\infty dxdv \\ -\int_{\mathbb{R}^{2n}}\begin{pmatrix}u_1\\u_2\end{pmatrix}^T\begin{pmatrix}2\varepsilon^2t^2(1-3\varepsilon)I & [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I\\ [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I & [2c\varepsilon^2t^2+4\varepsilon\nu t-2\varepsilon]I\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} f_\infty dxdv.\end{aligned} \tag{2.105}$$

Using the estimates

$$\begin{aligned}-(t-2\varepsilon^3t^3)\int_{\mathbb{R}^{2n}} u_1^T\left(\frac{\partial^2V}{\partial x^2}+cI\right)u_2 f_\infty dxdv \\ \le \varepsilon^3t^2|1-2\varepsilon^3t^2|\int_{\mathbb{R}^{2n}}|u_1|^2 f_\infty dxdv + \frac{|1-2\varepsilon^3t^2|}{4\varepsilon^3}\int_{\mathbb{R}^{2n}}|u_2|^2\left|\left|\frac{\partial^2V}{\partial x^2}+cI\right|\right|^2 f_\infty dxdv\end{aligned}$$

and

$$
\begin{aligned}
&-(-1+2\nu t-2\varepsilon^2t^2)\int_{\mathbb{R}^{2n}} u_2^T\left(\frac{\partial^2 V}{\partial x^2}+cI\right)u_2 f_\infty dxdv \\
&\qquad\qquad \le |1-2\nu t+2\varepsilon^2t^2|\int_{\mathbb{R}^{2n}}|u_2|^2\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\| f_\infty dxdv,
\end{aligned}
$$

we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2n}} u^T\left\{QP+PQ^T-\partial_t P\right\}uf_\infty dxdv \\
&\quad\le \int_{\mathbb{R}^{2n}}|u_2|^2\left[\frac{|1-2\varepsilon^3t^2|}{2\varepsilon^3}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+|1-2\nu t+2\varepsilon^2t^2|\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\right]f_\infty dxdv \\
&-\int_{\mathbb{R}^{2n}}\begin{pmatrix}u_1\\u_2\end{pmatrix}^T\begin{pmatrix}2\varepsilon^2t^2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|)I & [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I\\ [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I & [2c\varepsilon^2t^2+4\varepsilon\nu t-2\varepsilon]I\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}f_\infty dxdv.
\end{aligned}
\tag{2.106}
$$

We fix $\varepsilon=\varepsilon(t_0)>0$ so that the element in the upper left corner of the matrix in (2.106) is positive for $t>0$; more precisely we require

$$
1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|>0 \quad \text{for all} \quad t\in[0,t_0]. \tag{2.107}
$$

Then, the matrix in the last line of (2.106) can be estimated as

$$
\begin{aligned}
&\begin{pmatrix}2\varepsilon^2t^2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|)I & [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I\\ [2c\varepsilon^3t^3+\nu\varepsilon^2t^2+2(1-\varepsilon)\varepsilon t]I & [2c\varepsilon^2t^2+4\varepsilon\nu t-2\varepsilon]I\end{pmatrix} \\
&\qquad\qquad\ge\begin{pmatrix}0 & 0\\ 0 & [2c\varepsilon^2t^2+4\varepsilon\nu t-2\varepsilon]I-\frac{[2c\varepsilon^2t^2+\nu\varepsilon t+2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|)}I\end{pmatrix}.
\end{aligned}
$$

Using this matrix inequality, we obtain from (2.106):

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2n}} u^T\left\{QP+PQ^T-\partial_t P\right\}uf_\infty dxdv \\
&\quad\le \int_{\mathbb{R}^{2n}}|u_2|^2\left[\frac{|1-2\varepsilon^3t^2|}{2\varepsilon^3}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+|1-2\nu t+2\varepsilon^2t^2|\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\right. \\
&\qquad\left.-2c\varepsilon^2t^2-4\varepsilon\nu t+2\varepsilon+\frac{[2c\varepsilon^2t^2+\nu\varepsilon t+2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|)}\right]f_\infty dxdv.
\end{aligned}
\tag{2.108}
$$

(2.102), (2.103), (2.104), and (2.108) show that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^{2n}}u^TPuf_\infty dxdv &\le -2t\sigma\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\left(\frac{\partial^2 V}{\partial x^2}+cI\right)\partial_{v_i}u_2\right\}f_\infty dxdv \\
&\quad+\frac{2t\sigma}{\nu}\int_{\mathbb{R}^{2n}}\left\{\sum_{i=1}^n(\partial_{v_i}u_2)^T\frac{\partial^2(\partial_{x_i}V)}{\partial x^2}u_2\right\}f_\infty dxdv \\
&+\int_{\mathbb{R}^{2n}}|u_2|^2\left[\frac{|1-2\varepsilon^3t^2|}{2\varepsilon^3}\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+|1-2\nu t+2\varepsilon^2t^2|\left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|\right. \\
&\qquad\left.-2c\varepsilon^2t^2-4\varepsilon\nu t+2\varepsilon+\frac{[2c\varepsilon^2t^2+\nu\varepsilon t+2(1-\varepsilon)]^2}{2(1-3\varepsilon-\varepsilon|1-2\varepsilon^2t^2|)}\right]f_\infty dxdv.
\end{aligned}
$$

As the matrix (2.6) is positive semi-definite, we have

$$\begin{aligned}
&- 2t\sigma \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u_2)^T \left(\frac{\partial^2 V}{\partial x^2} + cI \right) \partial_{v_i} u_2 \right\} f_\infty dx dv \\
&\qquad + \frac{2t\sigma}{\nu} \int_{\mathbb{R}^{2n}} \left\{ \sum_{i=1}^{n} (\partial_{v_i} u_2)^T \frac{\partial^2 (\partial_{x_i} V)}{\partial x^2} u_2 \right\} f_\infty dx dv \\
&\leq \tau t \int_{\mathbb{R}^{2n}} u_2^T \left(\frac{\partial^2 V}{\partial x^2} + cI \right) u_2 f_\infty dx dv \leq \tau t \int_{\mathbb{R}^{2n}} |u_2|^2 \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\| f_\infty dx dv.
\end{aligned}$$

Subsequently,

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^{2n}} u^T P u f_\infty dx dv \\
&\leq \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\|^2 + (|1 - 2\nu t + 2\varepsilon^2 t^2| + \tau t) \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\| \right. \\
&\qquad \left. -2c\varepsilon^2 t^2 - 4\varepsilon\nu t + 2\varepsilon + \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1 - 3\varepsilon - \varepsilon|1 - 2\varepsilon^2 t^2|)} \right] f_\infty dx dv. \qquad (2.109)
\end{aligned}$$

Step 4, decay of the functional Ψ:

We estimate the time derivative of (2.98): Combining (2.91), (2.97), and (2.109) yield

$$\begin{aligned}
\frac{d}{dt}\Psi(t, f(t)) \leq &- \int_{\mathbb{R}^{2n}} |u_2|^2 \left[\left(\sigma e^{-8\tau n t}\gamma_1 - \frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \right) \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\|^2 \right. \\
&\qquad -(|1 - 2\nu t + 2\varepsilon^2 t^2| + \tau t) \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\| \\
&\qquad \left. +2\sigma\gamma_2 + 2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon - \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1 - 3\varepsilon - \varepsilon|1 - 2\varepsilon^2 t^2|)} \right] f_\infty dx dv. \qquad (2.110)
\end{aligned}$$

We fix $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\begin{aligned}
&\left(\sigma e^{-8\tau n t}\gamma_1 - \frac{|1-2\varepsilon^3 t^2|}{2\varepsilon^3} \right) \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\|^2 - (|-1 + 2\nu t - 2\varepsilon^2 t^2| + \tau t) \left\| \frac{\partial^2 V}{\partial x^2} + cI \right\| \\
&\qquad + 2\sigma\gamma_2 + 2c\varepsilon^2 t^2 + 4\varepsilon\nu t - 2\varepsilon - \frac{[2c\varepsilon^2 t^2 + \nu\varepsilon t + 2(1-\varepsilon)]^2}{2(1 - 3\varepsilon - \varepsilon|1 - 2\varepsilon^2 t^2|)} \geq 0 \qquad (2.111)
\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, t_0]$. We recall that we have fixed $\varepsilon = \varepsilon(t_0)$ so that (2.107) holds, which makes the above denominator positive. The existence of such $\gamma_1 > 0$ and $\gamma_2 > 0$ can be proven by the following arguments: We can consider the left hand side of (2.111) as a quadratic polynomial of $\left\| \frac{\partial^2 V}{\partial x^2} + cI \right\| \in [0, \infty)$. As time t varies in a bounded interval $[0, t_0]$, the terms containing t are bounded. Therefore, we can choose large values for $\gamma_1 = \gamma_1(t_0)$ and $\gamma_2 = \gamma_2(t_0)$ so that this quadratic polynomial is non-negative for all $t \in [0, t_0]$.

Consequently, we obtain that

$$\frac{d}{dt}\Psi(t, f(t)) \leq 0.$$

Hence $\Psi(t,f(t))$ is decreasing and

$$\Psi(t,f(t)) \le \Psi(0,f_0) \quad \text{for all} \quad t\in[0,t_0]. \tag{2.112}$$

(2.101) and (2.112) show that

$$\int_{\mathbb{R}^{2n}} |u_1|^2 f_\infty dxdv \le \frac{1}{\varepsilon^3 t^3}\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty}-1\right)^2 \left(\gamma_1 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dxdv, \tag{2.113}$$

$$\int_{\mathbb{R}^{2n}} |u_2|^2 f_\infty dxdv \le \frac{1}{\varepsilon t}\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty}-1\right)^2 \left(\gamma_1 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dxdv, \tag{2.114}$$

and

$$\int_{\mathbb{R}^{2n}} u_2^T\left(\frac{\partial^2 V}{\partial x^2}+cI\right)u_2 f_\infty dxdv \le \frac{1}{t}\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty}-1\right)^2 \left(\gamma_1 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2\right) f_\infty dxdv. \tag{2.115}$$

It is clear that there is a positive constant C such that

$$\gamma_1 \left\|\frac{\partial^2 V}{\partial x^2}+cI\right\|^2+\gamma_2 \le C\left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2+1\right). \tag{2.116}$$

(2.113), a proper linear combination of (2.114) and (2.115), and (2.116) imply the claimed estimates (2.11), (2.12). □

Proof of Corollary 2.2.8. Theorem 2.2.3 and Theorem 2.2.7 show that, for $t\ge t_0>0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left(\frac{f(t)}{f_\infty}-1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x\left(\frac{f(t)}{f_\infty}\right)\right|^2 f_\infty dxdv \\
&\qquad + \int_{\mathbb{R}^{2n}} \nabla_v^T\left(\frac{f(t)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t)}{f_\infty}\right) f_\infty dxdv \\
&\quad \le Ce^{-2\lambda(t-t_0)}\left[\int_{\mathbb{R}^{2n}} \left(\frac{f(t_0)}{f_\infty}-1\right)^2 f_\infty dxdv + \int_{\mathbb{R}^{2n}} \left|\nabla_x\left(\frac{f(t_0)}{f_\infty}\right)\right|^2 f_\infty dxdv\right. \\
&\qquad \left. + \int_{\mathbb{R}^{2n}} \nabla_v^T\left(\frac{f(t_0)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t_0)}{f_\infty}\right) f_\infty dxdv\right]
\end{aligned} \tag{2.117}$$

holds with the constant C and the rate λ given in Theorem 2.2.3. Using (2.11) and (2.12) at $t=t_0$, we get

$$\int_{\mathbb{R}^{2n}} \left|\nabla_x\left(\frac{f(t_0)}{f_\infty}\right)\right|^2 f_\infty dxdv \le \frac{C_1}{t_0^3}\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty}-1\right)^2\left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2+1\right) f_\infty dxdv \tag{2.118}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \nabla_v^T\left(\frac{f(t_0)}{f_\infty}\right)\left(\frac{\partial^2 V}{\partial x^2}+(1-\alpha_0)I\right)\nabla_v\left(\frac{f(t_0)}{f_\infty}\right) f_\infty dxdv \\
&\qquad \le \frac{C_2}{t_0}\int_{\mathbb{R}^{2n}} \left(\frac{f_0}{f_\infty}-1\right)^2\left(\left\|\frac{\partial^2 V}{\partial x^2}\right\|^2+1\right) f_\infty dxdv.
\end{aligned} \tag{2.119}$$

Combining (2.117), (2.118), and (2.119), we obtain (2.13). □

2.6 Appendix

2.6.1 Proof that Assumption 2.2.2' implies Assumption 2.2.2

Assume Assumption 2.2.2' is satisfied. Let $(u_1, u_2, ..., u_{n+1})^T$ be any vector in $\mathbb{R}^{n(n+1)}$, where u_i is a vector in $\mathbb{R}^n$ for all $i \in \{1, ..., n+1\}$. We compute the quadratic form of the matrix (2.6)

$$\begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{n+1} \end{pmatrix}^T \begin{pmatrix} \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & 0 & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V(x))}{\partial x^2} \\ 0 & \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & \dots & 0 & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V(x))}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \nu\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V(x))}{\partial x^2} \\ -\frac{1}{2}\frac{\partial^2(\partial_{x_1} V(x))}{\partial x^2} & -\frac{1}{2}\frac{\partial^2(\partial_{x_2} V(x))}{\partial x^2} & \dots & -\frac{1}{2}\frac{\partial^2(\partial_{x_n} V(x))}{\partial x^2} & \frac{\tau\nu}{2\sigma}\left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{n+1} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \left\{ \nu u_i^T \left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) u_i - u_i^T \frac{\partial^2(\partial_{x_i} V(x))}{\partial x^2} u_{n+1} \right\} + \frac{\tau\nu}{2\sigma} u_{n+1}^T \left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) u_{n+1}.$$

To show that (2.6) is positive semi-definite, it is enough to show the quadratic form above is non-negative. Assumption 2.2.2' implies

$$\left| u_i^T \frac{\partial^2(\partial_{x_i} V(x))}{\partial x^2} u_{n+1} \right| \le |u_i||u_{n+1}| \sqrt{\frac{2\tau\nu^2}{n\sigma}}(\alpha(x)+c) \le \nu(\alpha(x)+c)|u_i|^2 + \frac{\tau\nu}{2n\sigma}(\alpha(x)+c)|u_{n+1}|^2.$$

Therefore, we get the desired result

$$\sum_{i=1}^{n} \left\{ \nu u_i^T \left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) u_i - u_i^T \frac{\partial^2(\partial_{x_i} V(x))}{\partial x^2} u_{n+1} + \frac{\tau\nu}{2n\sigma} u_{n+1}^T \left(\frac{\partial^2 V(x)}{\partial x^2} + cI\right) u_{n+1} \right\}$$
$$\ge \sum_{i=1}^{n} \left\{ \nu u_i^T \left(\frac{\partial^2 V(x)}{\partial x^2} - \alpha(x)I\right) u_i + \frac{\tau\nu}{2n\sigma} u_{n+1}^T \left(\frac{\partial^2 V(x)}{\partial x^2} - \alpha(x)I\right) u_{n+1} \right\} \ge 0.$$

□

2.6.2 Matrix inequalities for Section 5.1

Lemma 2.6.1. *Let $\alpha_0 > -\infty$ be the constant defined by (2.8), $a \in \mathbb{R}$ be some constant such that $a + \alpha_0 > \frac{\nu^2}{4}$, and $P := \begin{pmatrix} 2I & \nu I \\ \nu I & 2\frac{\partial^2 V}{\partial x^2} + 2aI \end{pmatrix}$. Then*

$$c_1 P \le \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I \end{pmatrix} \le c_2 P \tag{2.120}$$

holds with $c_1 := \frac{1}{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}} > 0, \quad c_2 := \frac{a+\alpha_0+1+\sqrt{(a+\alpha_0-1)^2+\nu^2}}{4(a+\alpha_0)-\nu^2} > 0.$

Proof. We consider, for some $k \in \mathbb{R}$ to be chosen later as $\frac{1}{2c_{1,2}}$,

$$A := P - 2k \begin{pmatrix} I & 0 \\ 0 & \frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I \end{pmatrix}$$
$$= \begin{pmatrix} 2(1-k)I & \nu I \\ \nu I & 2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I \end{pmatrix}.$$

We check the (real) eigenvalues η of the symmetric matrix A (depending on k). It is easy to check that $\eta = 2(1-k)$ is not an eigenvalue of A. If $\eta \neq 2(1-k)$, then we have the condition

$$\det(A-\eta I) = \begin{vmatrix} 2(1-k)I - \eta I & \nu I \\ \nu I & 2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I \end{vmatrix}$$

$$= \frac{1}{(2(1-k)-\eta)^n} \times$$

$$\begin{vmatrix} 2(1-k)I - \eta I & 0 \\ \nu I & (2(1-k)-\eta)\left[2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I\right] - \nu^2 I \end{vmatrix}$$

$$= \det\left((2(1-k)-\eta)\left[2(1-k)\left(\frac{\partial^2 V}{\partial x^2} + (1-\alpha_0)I\right) + 2(a+\alpha_0-1)I - \eta I\right] - \nu^2 I\right) = 0.$$

If α_i, $i \in \{1,...,n\}$ are the eigenvalues of $\frac{\partial^2 V}{\partial x^2}$, then the eigenvalues η of A satisfy

$$\prod_{i=1}^{n} \left(\eta^2 - 2\eta[(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1] + 4(1-k)^2(\alpha_i - \alpha_0 + 1) \right.$$
$$\left. + 4(1-k)(a+\alpha_0-1) - \nu^2\right) = 0. \quad (2.121)$$

Right inequality of (2.120): From (2.121), we see that A is positive semi-definite (i.e., all $\eta \geq 0$) if the following three conditions hold:

$$1-k \geq 0, \qquad \text{(due to the first minor of } A) \quad (2.122)$$

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \geq 0, \ \ \forall i \in \{1,...,n\}, \quad (2.123)$$

$$4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a+\alpha_0-1) - \nu^2 \geq 0, \ \ \forall i \in \{1,...,n\}. \quad (2.124)$$

We set

$$k := \frac{1}{2c_2} > 0.$$

Then, (2.122) holds:

$$1-k = \frac{\sqrt{(a+\alpha_0-1)^2+\nu^2} - (a+\alpha_0-1)}{2} > 0. \quad (2.125)$$

Using $\alpha_i \geq \alpha_0$ for all $i \in \{1,...,n\}$ we see that (2.123) also holds:

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \geq 2(1-k) + a + \alpha_0 - 1 = \sqrt{(a+\alpha_0-1)^2+\nu^2} > 0.$$

To verify (2.124) we estimate using $\alpha_i \geq \alpha_0$ for all $i \in \{1,...,n\}$ and (2.125)

$$4(1-k)^2(\alpha_i - \alpha_0 + 1) + 4(1-k)(a+\alpha_0-1) - \nu^2$$
$$\geq 4(1-k)^2 + 4(1-k)(a+\alpha_0-1) - \nu^2 = 0.$$

Therefore, for k defined in (2.125), A is positive semi-definite. Hence, the inequality on the right hand side of (2.120) holds.

Left inequality of (2.120): Similarly, A is negative semi-definite if the following three conditions hold:

$$1-k \leq 0, \quad (2.126)$$

$$(1-k)(\alpha_i - \alpha_0 + 2) + a + \alpha_0 - 1 \leq 0, \ \ \forall i \in \{1,...,n\}, \quad (2.127)$$

$$4(1-k)^2(\alpha_i-\alpha_0+1)+4(1-k)(a+\alpha_0-1)-\nu^2\geq 0, \quad \forall i\in\{1,...,n\}. \tag{2.128}$$

Setting

$$k:=\frac{1}{2c_1}>0$$

we find

$$1-k=\frac{-\sqrt{(a+\alpha_0-1)^2+\nu^2}-(a+\alpha_0-1)}{2}<0 \tag{2.129}$$

and

$$(1-k)(\alpha_i-\alpha_0+2)+a+\alpha_0-1\leq 2(1-k)+a+\alpha_0-1=-\sqrt{(a+\alpha_0-1)^2+\nu^2}<0.$$

Finally, we check using $\alpha_i\geq\alpha_0$ for all $i\in\{1,...,n\}$ and (2.129)

$$4(1-k)^2(\alpha_i-\alpha_0+1)+4(1-k)(a+\alpha_0-1)-\nu^2\geq 4(1-k)^2+4(1-k)(a+\alpha_0-1)-\nu^2=0.$$

Therefore, for k defined in (2.129), A is negative semi-definite. Hence, the inequality on the left hand side of (2.120) holds. □

Remark 2.6.2. *Lemma 2.6.1 proves the following matrix inequalities from Section 5.1:*

(a) If $a=0$ and $\alpha_0>\frac{\nu^2}{4}$, then (2.120) is the matrix inequality (2.56).

(b) If $a=\frac{\varepsilon^2}{2}$ and $\alpha_0=\frac{\nu^2}{4}$, then (2.120) is the matrix inequality (2.64).

(c) (2.120) coincides with the matrix inequality (2.75).

2.6.3 Proof of inequality (2.80)

We recall the assumption $\alpha_0<\frac{\nu^2}{4}$. We first rewrite

$$A_2^{-1}=\frac{2\nu\min\{1,\alpha_0\}}{1+\frac{\nu^2}{2}-\alpha_0+\sqrt{(\frac{\nu^2}{2}-\alpha_0-1)^2+\nu^2}}$$

$$=\frac{4\min\{1,\alpha_0\}}{\nu+2(1-\alpha_0)\nu^{-1}+\sqrt{(\nu^2-4\alpha_0)+4(\alpha_0+1)^2\nu^{-2}}}$$

and

$$\nu-\sqrt{\nu^2-4\alpha_0}=\frac{4\alpha_0}{\nu+\sqrt{\nu^2-4\alpha_0}}.$$

Then (2.80) is equivalent to

$$\frac{\alpha_0}{\nu+\sqrt{\nu^2-4\alpha_0}}\geq\frac{\min\{1,\alpha_0\}}{\nu+2(1-\alpha_0)\nu^{-1}+\sqrt{(\nu^2-4\alpha_0)+4(\alpha_0+1)^2\nu^{-2}}}. \tag{2.130}$$

If $\min\{1,\alpha_0\}=\alpha_0$, then (2.130) is true because of

$$\nu+2(1-\alpha_0)\nu^{-1}+\sqrt{(\nu^2-4\alpha_0)+4(\alpha_0+1)^2\nu^{-2}}>\nu+\sqrt{\nu^2-4\alpha_0}.$$

If $\min\{1,\alpha_0\}=1$, then (2.130) is equivalent to

$$\alpha_0\nu-2\alpha_0(\alpha_0-1)\nu^{-1}+\alpha_0\sqrt{(\nu^2-4\alpha_0)+4(\alpha_0+1)^2\nu^{-2}}\geq\nu+\sqrt{\nu^2-4\alpha_0},$$

or equivalently

$$(\alpha_0 - 1)(\nu^2 - 2\alpha_0)\nu^{-1} + \alpha_0\sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0 + 1)^2\nu^{-2}} \geq \sqrt{\nu^2 - 4\alpha_0}.$$

The last inequality holds since

$$(\alpha_0 - 1)(\nu^2 - 2\alpha_0)\nu^{-1} \geq (\alpha_0 - 1)(\nu^2 - 4\alpha_0)\nu^{-1} \geq 0$$

and

$$\alpha_0\sqrt{(\nu^2 - 4\alpha_0) + 4(\alpha_0 + 1)^2\nu^{-2}} > \sqrt{\nu^2 - 4\alpha_0}.$$

These two cases show that inequality (2.80) holds.

□

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Chapter 3

The Vlasov-Poisson-Fokker-Planck system

3.1 Introduction

This chapter is devoted to the study of well-posedness and long time behavior of the nonlinear Vlasov-Poisson-Fokker-Planck system

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - (\nabla_x V + \nabla_x\phi)\cdot\nabla_v f = \nu \mathrm{div}_v(vf) + \sigma\Delta_v f, \quad x,v\in\mathbb{R}^d,\ t>0 \\ -\Delta_x\phi = \int_{\mathbb{R}^d} f dv, \quad f_{|t=0} = f_0. \end{cases} \tag{3.1}$$

The system is one of the fundamental models in plasma physics, for the derivation and applications we refer to [18, 19, 45, 48]. The variables $t\ge 0$, $x\in\mathbb{R}^d$, and $v\in\mathbb{R}^d$, respectively, stand for time, position, and velocity. The first unknown $f = f(t,x,v)\ge 0$ describes the evolution of the phase space probability density of charged particles. The second unknown $\phi = \phi(t,x)$ determines the self-consistent *repulsive* electrostatic potential. Because of the Poisson equation in (3.1) we have

$$\nabla_x\phi = \frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} * \int_{\mathbb{R}^d} f dv,$$

where $|\mathbb{S}^{d-1}|$ is the area of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. $V = V(x)$ is a given external electrostatic confinement potential (i.e., $V(x)\to+\infty$ as $|x|\to\infty$). The operator $v\cdot\nabla_x - (\nabla_x V + \nabla_x\phi)\cdot\nabla_v$ is the transport operator. The Fokker-Planck operator $\nu\mathrm{div}_v(v\,\cdot) + \sigma\Delta_v$ describes the collision effects of particles and the interaction with the environment. $\nu>0$ and $\sigma>0$ denote respectively the friction and diffusion parameters.

In the literature there exists also the Vlasov-Poisson-Fokker-Planck system with the self-consistent *attractive* electrostatic potential, and it is widely used in stellar physics [42]. For that case the self-consistent electrostatic potential ϕ is defined by a change of sign in the Poisson equation. In this paper, we only consider the repulsive case.

We mention that, if there is an interaction with a fixed background of positive charges in a plasma, then the self-consistent electrostatic potential ϕ is defined by $-\Delta_x\phi = \int_{\mathbb{R}^d} f dv - n$, where $n = n(x)$ is a given non-negative function which describes the background density (e.g. of ions). In this case, we can still write the system in the form of (3.1) by replacing ϕ with $\tilde\phi$ defined by $-\Delta_x\tilde\phi = \int_{\mathbb{R}^d} f dv$ and replacing V with $\tilde V := V + V_0$, where V_0 is defined by $\Delta_x V_0 = n$. Therefore, without loss of generality, we assume $n = 0$.

The system has several properties following standard physical consideration. First, the Fokker-Planck operator *acts only on the velocity* v. It reflects the physical fact that collisions are localized in space.

Whenever $f(t,x,v)$ is a (well-behaved) solution, we have *conservation of mass*

$$\int_{\mathbb{R}^{2d}} f(t,x,v)dxdv = \int_{\mathbb{R}^{2d}} f_0(x,v)dxdv, \quad \forall t > 0.$$

Therefore, without loss of generality, we shall assume $f_0 \geq 0$ and $\int_{\mathbb{R}^{2d}} f_0(x,v)dxdv = 1$.

If V grows fast enough as $|x| \to \infty$, the system has a unique normalized *steady state* or *global equilibrium* [24, 28]

$$f_\infty(x,v) = \rho_\infty(x)M(v),$$

where

$$\rho_\infty(x) := \frac{e^{-\frac{\nu}{\sigma}[V(x)+\phi_\infty(x)]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]}dx'}, \qquad M(v) := \frac{e^{-\frac{\nu}{\sigma}|v|^2/2}}{(2\pi\sigma/\nu)^{d/2}},$$

and ϕ_∞ is a solution of the Poisson-Boltzmann-Emden equation [6]

$$-\Delta_x \phi_\infty(x) = \frac{e^{-\frac{\nu}{\sigma}[V(x)+\phi_\infty(x)]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]}dx'}. \tag{3.2}$$

The system is *dissipative* in the sense that the following relative entropy or free energy functional decreases under time-evolution of f [10, 23]: let H be a functional defined on the space of probability densities by

$$f \to \mathrm{H}[f] := \int_{\mathbb{R}^{2d}} f \ln \frac{f}{f_\infty} dxdv + \int_{\mathbb{R}^d} |\nabla_x \phi - \nabla_x \phi_\infty|^2 dx,$$

with ϕ given by the Poisson equation $-\Delta_x \phi = \int_{\mathbb{R}^d} f dv$. By the Csiszár-Kullback-Pinsker inequality [20]

$$\mathrm{H}[f] \geq \frac{1}{2}||f - f_\infty||^2_{L^1(\mathbb{R}^{2d})} + \int_{\mathbb{R}^d} |\nabla_x \phi - \nabla_x \phi_\infty|^2 dx \geq 0,$$

the minimum of H is zero and it is attained at f_∞ (i.e., $\mathrm{H}[f] \geq 0$ for all probability densities f and $\mathrm{H}[f_\infty] = 0$). If $f = f(t,x,v)$ solves (3.1) and has sufficient smoothness and decay properties (as $|(x,v)^T| \to \infty$), we have

$$\frac{d}{dt}\mathrm{H}[f(t)] \leq 0.$$

This decay of the functional H reminds us of the famous Boltzmann $H-$theorem stated for the Boltzmann equation [17]. This similarity is expected since the Fokker-Planck operator can be considered as a linear variant of Boltzmann's collision operator [17, 49].

On the basis of the decay of the functional H, one can expect that $\mathrm{H}[f(t)]$ decreases to its minimum (which is zero) as $t \to \infty$. Since this minimum is attained at f_∞, one can argue that $f(t)$ converges to the equilibrium distribution f_∞ as $t \to \infty$. Clearly, before proving this convergence, we first need to establish the well-posedness of the system (3.1). Therefore, we get an important problem: to prove existence and uniqueness of the solution f, then to prove the convergence $f(t) \to f_\infty$ as $t \to \infty$.

If we have a reasonable solution f and if it satisfies some a-priori bounds, using the decay of the functional H above and compactness tools, we can prove that $f(t)$ does indeed converge to f_∞ as $t \to \infty$ [13, 23]. But this method based on compactness gives no information on the rate of convergence and it is non-constructive. We are interested in the study of rates of convergence and we want to derive constructive bounds for this convergence, because explicit and constructive estimates are essential for applications in physics (e.g. equilibration process, numerical simulations).

There are many works dealing with this problem. When the system (3.1) does not have a confining potential (i.e., $V = 0$) existence, uniqueness and asymptotic behavior have been studied comprehensively: Degond considered the frictionless system (i.e., $\nu = 0$) in [21] and showed global existence of classical solutions in dimension $d \leq 2$, see also [40, 41]. The long time behavior of the frictionless system was studied in [16, 38, 14]. With non-zero friction (i.e., $\nu > 0$), global existence of classical solutions in dimension $d \leq 2$ and local in time existence in dimension $d \geq 3$ were obtained by Victory and O'Dwyer in [48]. Bouchut [11] proved global existence of classical solutions in dimension $d = 3$. He also showed in [12] that the system has smoothing properties. Then global existence of weak solutions in dimension $d = 3$ was studied in [15, 47]. Using the micro-macro strategy Hwang and Jang [37] obtained exponential decay in a close-to-equilibrium regime. There are recent studies [8, 46] on torus (i.e., $x \in \mathbb{T}^d$) concerning the long time behavior and Landau damping in a weak collisional regime, i.e. if ν and σ are sufficiently small.

When the equation has a non-zero confining potential V, there are only few studies: When the self-consistent interaction is sufficiently small (i.e., the non-linear term $\nabla_x\phi\cdot\nabla_v f$ is replaced with $\varepsilon\nabla_x\phi\cdot\nabla_v f$ and ε is sufficiently small) and $\partial^2_{x_i x_j} V \in \bigcap_{p=1}^\infty W^{p,\infty}$ for all $i, j \in \{1, ..., d\}$, Hérau and Thomann [34] proved a global existence result in dimensions $d = 2$ and $d = 3$. They also showed that the solution converges to the steady state exponentially. Their proof relies on the hypocoercive and hypoelliptic properties of the linear kinetic Fokker-Planck equation obtained in [33, 32] and a fixed point argument. Recently, Abddala, Dolbeault et al. [2] studied the linearized Vlasov-Poisson-Fokker-Planck system around the steady state f_∞. They used a hypocoercive method developed in [25, 26] and proved exponential stability of the linearized system.

In this paper, we shall improve these previous results when there is a non-zero potential V. For the full system (3.1) with a non-zero potential V and the parameters $\nu > 0$, $\sigma > 0$ we shall prove existence, uniqueness and convergence $f(t) \to f_\infty$ as $t \to \infty$ for a wide class of potentials V. Moreover, our decay rates are explicit and constructive.

The organization of this paper is as follows. In Section 2 we present the assumptions on the potential, define functional spaces and state the main results. In Section 3 we show existence and regularity of the steady state, establish Poincaré type inequalities and gather some estimates for the Poisson equation. Section 4 contains the analysis of the linearized Vlasov-Poisson-Fokker-Planck system: existence, uniqueness, exponential stability, and hypoelliptic regularity. The final section presents some estimates on the semigroup of the linearized system and the proof of the main results concerning the nonlinear Vlasov-Poisson-Fokker-Planck system.

3.2 Setting and main results

We make the following assumptions on the external potential V.

Assumption 3.2.1. *(A1) $V \in C^\infty(\mathbb{R}^d)$ is bounded from below,*

$$e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d) \quad \textit{and} \quad |\nabla_x V| e^{-\frac{\nu}{\sigma}V} \in L^r(\mathbb{R}^d),\ r > d.$$

(A2) There exists a constant $c_1 > 0$ such that

$$\left|\left|\frac{\partial^2 V(x)}{\partial x^2}\right|\right|_F \leq c_1(1 + |\nabla_x V(x)|), \quad \forall x \in \mathbb{R}^d, \tag{3.3}$$

where $||\cdot||_F$ denotes the Frobenius norm.

(A3) There exists a constant $\kappa_1 > 0$ such that the Poincaré inequality

$$\int_{\mathbb{R}^d} h^2 e^{-\frac{\nu}{\sigma}V} dx - \left(\int_{\mathbb{R}^d} h e^{-\frac{\nu}{\sigma}V} dx\right)^2 \leq \kappa_1 \int_{\mathbb{R}^d} |\nabla_x h|^2 e^{-\frac{\nu}{\sigma}V} dx \tag{3.4}$$

holds for all h with $\int_{\mathbb{R}^d} h^2 e^{-\frac{\nu}{\sigma}V} dx < \infty$ and $\int_{\mathbb{R}^d} |\nabla_x h|^2 e^{-\frac{\nu}{\sigma}V} dx < \infty$.

There are a lot of studies and sufficient conditions implying the Poincaré inequality (3.4). For example, if V is uniformly convex (Bakry-Emery criterion) or $\liminf_{|x|\to\infty} \left(a|\nabla V(x)|^2 - \Delta V(x)\right) > 0$ for some $a \in (0,1)$, then the Poincaré inequality holds, for more information see [7, Chapter 4], [5]. We note that Assumption 3.2.1 includes the potentials V considered in [2, 34]. It is possible to make weaker regularity hypothesis on the potential V, but we maintain the assumption that $V \in C^\infty$ to keep the presentation simple. We note that the potentials of the form

$$V(x) = r|x|^k + V_0(x),$$

where $r > 0$, $k > 1$ and $V_0 \colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $j < k$, satisfy our assumptions. In particular, it includes the double-well potentials of the form $V(x) = r_1|x|^4 - r_2|x|^2$, $r_1, r_2 > 0$.

We define the following weighted spaces

$$L^2(\mathbb{R}^{2d}, f_\infty) := \left\{g : \mathbb{R}^{2d} \to \mathbb{R} : \int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv < \infty\right\}$$

and

$$H^1(\mathbb{R}^{2d}, f_\infty) := \left\{g \in L^2(\mathbb{R}^{2d}, f_\infty) : \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_v g|^2 f_\infty dx dv < \infty\right\}.$$

The corresponding norms are

$$||g||_{L^2(\mathbb{R}^{2d}, f_\infty)} := \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv}$$

and

$$||g||_{H^1(\mathbb{R}^{2d}, f_\infty)} := \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_v g|^2 f_\infty dx dv}.$$

We define the generalized Sobolev space or the Bessel potential space [43, Section V.3], [1, Section 1.2.6]

$$\mathscr{L}^p_\alpha(\mathbb{R}^d) := \{g : \mathbb{R}^d \to \mathbb{R} : \ (1 - \Delta_x)^{\frac{\alpha}{2}} g \in L^p(\mathbb{R}^d)\}, \quad 1 < p < \infty, \quad \alpha \in \mathbb{R},$$

where $(1 - \Delta_x)^{\frac{\alpha}{2}} g := \mathcal{F}^{-1}\left((1 + 4\pi^2|\xi|^2)^{\frac{\alpha}{2}} \mathcal{F} g\right)$ with the Fourier transform $\mathcal{F}g(\xi) := \int_{\mathbb{R}^d} g(x) e^{-2\pi x\cdot\xi} dx$. The norm on $\mathscr{L}^p_\alpha(\mathbb{R}^d)$ is

$$||g||_{\mathscr{L}^p_\alpha(\mathbb{R}^d)} := ||(1 - \Delta_x)^{\frac{\alpha}{2}} g||_{L^p(\mathbb{R}^d)}.$$

For $\alpha \in \mathbb{N}$, $\mathscr{L}^p_\alpha(\mathbb{R}^d)$ coincides with the usual Sobolev space $W^{\alpha,p}(\mathbb{R}^d)$, $1 < p < \infty$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $1 < p_1, p_2 < \infty$, $\theta \in (0,1)$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ and $p = (1-\theta)p_1 + \theta p_2$, then $\mathscr{L}^p_\alpha(\mathbb{R}^d)$ is the complex interpolation space [9, Chapter 6] between $\mathscr{L}^{p_1}_{\alpha_1}(\mathbb{R}^d)$ and $\mathscr{L}^{p_2}_{\alpha_2}(\mathbb{R}^d)$, i.e., $\mathscr{L}^p_\alpha(\mathbb{R}^d) = \left(\mathscr{L}^{p_1}_{\alpha_1}(\mathbb{R}^d), \mathscr{L}^{p_2}_{\alpha_2}(\mathbb{R}^d)\right)_{[\theta]}$.

We define a weighted fractional Sobolev space

$$H^\alpha_x(\mathbb{R}^{2d}, f_\infty) := \{g \in L^2(\mathbb{R}^{2d}, f_\infty) : (1-\Delta_x)^{\frac{\alpha}{2}}(g f_\infty^{1/2}) \in L^2(\mathbb{R}^{2d})\}, \quad \alpha \in [0,1].$$

The corresponding norm is

$$||g||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} := ||(1-\Delta_x)^{\frac{\alpha}{2}}(g f_\infty^{1/2})||_{L^2(\mathbb{R}^{2d})}.$$

By the Plancherel theorem

$$||g||_{H^0_x(\mathbb{R}^{2d}, f_\infty)} = ||g||_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

and

$$||g||_{H^1_x(\mathbb{R}^{2d}, f_\infty)} = \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_x(g f_\infty^{1/2})|^2 dxdv}. \tag{3.5}$$

We also define

$$H^1_v(\mathbb{R}^{2d}, f_\infty) := \{g \in L^2(\mathbb{R}^{2d}, f_\infty) : |\nabla_v g| \in L^2(\mathbb{R}^{2d}, f_\infty)\}$$

with the norm

$$||g||_{H^1_v(\mathbb{R}^{2d}, f_\infty)} := \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v g|^2 f_\infty dxdv}.$$

Let $h := \dfrac{f - f_\infty}{f_\infty}$, $\psi := \phi - \phi_\infty$ and $h_0 := \dfrac{f_0 - f_\infty}{f_\infty}$. Then, we write the system (3.1) as

$$\begin{cases} \partial_t h + v \cdot \nabla_x h - \nabla_x(V + \phi_\infty) \cdot \nabla_v h + v \cdot \nabla_x \psi - \sigma \Delta_v h + \nu v \cdot \nabla_v h = \nabla_x \psi \cdot (\nabla_v h - \frac{\nu}{\sigma} v h) \\ -\Delta_x \psi = \displaystyle\int_{\mathbb{R}^d} h f_\infty dv, \quad h_{|t=0} = h_0. \end{cases} \tag{3.6}$$

We note that $\int_{\mathbb{R}^{2d}} f_0 dxdv = \int_{\mathbb{R}^{2d}} f_\infty dxdv = 1$ implies

$$\int_{\mathbb{R}^{2d}} h_0 f_\infty dxdv = 0.$$

It is obvious that the existence of a unique solution $f(t)$ to (3.1) and the convergence $f(t) \to f_\infty$ as $t \to \infty$ are respectively equivalent to the existence of a unique solution $h(t)$ to (3.6) and the convergence $h(t) \to 0$ as $t \to \infty$.

The term $\nabla_x \psi \cdot \nabla_v h - \frac{\nu}{\sigma} v \cdot \nabla_x \psi h$ appearing on the right hand side of (3.6) is nonlinear. If we drop it, we obtain the linearized Vlasov-Poisson-Fokker-Planck system around the steady state f_∞

$$\begin{cases} \partial_t h + v \cdot \nabla_x h - \nabla_x(V + \phi_\infty) \cdot \nabla_v h + v \cdot \nabla_x \psi - \sigma \Delta_v h + \nu v \cdot \nabla_v h = 0 \\ -\Delta_x \psi = \displaystyle\int_{\mathbb{R}^d} h f_\infty dv, \quad h_{|t=0} = h_0. \end{cases} \tag{3.7}$$

We first study this linearized system in dimension $d \geq 3$. We prove that the linearized system (3.7) is well-posed in $C\left([0,\infty); L^2(\mathbb{R}^{2d}, f_\infty)\right)$ and has regularizing properties which is called *hypoellipticity* [36]. More precisely, even if the initial data h_0 is in $L^2(\mathbb{R}^{2d}, f_\infty)$, the solution $h(t)$ is in $H^1(\mathbb{R}^{2d}, f_\infty)$ for $t > 0$, and we obtain short time estimates for this $L^2(\mathbb{R}^{2d}, f_\infty) \to H^1(\mathbb{R}^{2d}, f_\infty)$ regularization. We also prove that the solutions of (3.7) decay exponentially to zero as $t \to \infty$ in $H^1(\mathbb{R}^{2d}, f_\infty)$:

Theorem 3.2.2 **(The linerized Vlasov-Poisson-Fokker-Planck system).** *Let $d \geq 3$ and $h_0 \in L^2(\mathbb{R}^{2d}, f_\infty)$.*

(i) Let $V \in C^\infty(\mathbb{R}^d)$ be bounded from below and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$. Then, the system (3.7) admits a unique mild solution

$$h \in C\left([0,\infty); L^2(\mathbb{R}^{2d}, f_\infty)\right)$$

and

$$|\nabla_x \psi| \in C\left([0,\infty); L^{\frac{pd}{d-2}}(\mathbb{R}^d)\right), \quad \forall p \in (1,2].$$

(ii) Let the assumptions (A1) and (A2) hold. Then, for any $t_0 > 0$, there are explicitly computable constants $C_1 > 0$ and $C_2 > 0$ (independent of h_0) such that

$$\int_{\mathbb{R}^{2d}} |\nabla_x h(t)|^2 f_\infty dxdv \leq \frac{C_1}{t^3} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dxdv \tag{3.8}$$

and

$$\int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dxdv \leq \frac{C_2}{t} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dxdv \tag{3.9}$$

hold for all $t \in (0, t_0]$.

(iii) Let $\int_{\mathbb{R}^{2d}} h_0 f_\infty dxdv = 0$, the assumptions (A1), (A2) and (A3) hold. Then, there are explicitly computable constants $\lambda > 0$, $C_3 > 0$ and $C_4 > 0$ (independent of h_0) such that

$$||h(t)||_{H^1(\mathbb{R}^{2d}, f_\infty)} \leq C_3 e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \tag{3.10}$$

and

$$||\nabla_x \psi(t)||_{L^{\frac{pd}{d-2}}(\mathbb{R}^d)} + ||\nabla_x \psi(t)||_{W^{1,\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C_4 e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \tag{3.11}$$

hold for all $t \geq t_0 > 0$ and $p \in (1,2]$.

Remark 3.2.3. *1. We think that Theorem 3.2.2(ii) is the first regularity result for the linearized system. This can be considered as a generalization of the regularity results for the linear kinetic Fokker-Planck equation [32, Theorem 1.1], [50, Theorem A.8].*

2. Theorem 3.2.2(iii) extends the work of Abddala, Dolbeault et al. in [2, Theorem 1], since they obtained the exponential decay only in $L^2(\mathbb{R}^{2d}, f_\infty)$.

3. Theorem 3.2.2 holds for any parameters $\nu > 0$ and $\sigma > 0$. It can be obtained when $x \in \mathbb{T}^d$ by similar computation. Hence, Theorem 3.2.2 extends the result of Landau damping for the linearized system in [46, Theorem 3.1] where ν and σ are required to be small.

Next, we pass to the nonlinear Vlasov-Poisson- Fokker-Planck system in dimension $d = 3$. We define operators

$$Kh := v \cdot \nabla_x h - \nabla_x (V + \phi_\infty) \cdot \nabla_v h + v \cdot \nabla_x \psi - \sigma \Delta_v h + \nu v \cdot \nabla_v h$$

and

$$R[h] := \nabla_x \psi \cdot \nabla_v h - \frac{\nu}{\sigma} v \cdot \nabla_x \psi h.$$

Since $\nabla_x \psi$ can be expressed by h as

$$\nabla_x \psi = \frac{1}{|\mathbb{S}^{d-1}|} \frac{x}{|x|^d} * \int_{\mathbb{R}^d} h f_\infty dv,$$

we consider K and R as operators acting only on h. It shows that K is linear and R is nonlinear with respect to h. Then the linearized system can be written as

$$\partial_t h + Kh = 0,$$

while the nonlinear system (3.6) can be written as

$$\partial_t h + Kh = R[h].$$

From Theorem 3.2.2 (i) we obtain that K generates a C_0 semigroup e^{-tK} on $L^2(\mathbb{R}^{2d}, f_\infty)$. Then the Duhamel principle suggests to convert this nonlinear system to an integral equation

$$h(t) = e^{-tK} h_0 + \int_0^t e^{-(t-s)K} R[h(s)] ds. \tag{3.12}$$

We mention that a function h satisfying (3.12) is called a *mild solutions* to (3.6), see [44, Section 6.1]. Using the properties of e^{-tK} and fixed point arguments we show that there is a unique solution to this integral equation:

Theorem 3.2.4 (Local well-posedness). *Let $d = 3$, $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, the assumptions (A1) and (A2) hold. Then, for every $h_0 \in H_x^\alpha(\mathbb{R}^6, f_\infty) \cap H_v^1(\mathbb{R}^6, f_\infty)$, there is a $t_{max} \in (0, \infty]$ such that (3.6) has a unique mild solution*

$$h \in C\left([0, t_{max}); H_x^\alpha(\mathbb{R}^6, f_\infty)\right) \cap C\left([0, t_{max}); H_v^1(\mathbb{R}^6, f_\infty)\right)$$

and

$$|\nabla_x \psi| \in C\left([0, t_{max}); \mathscr{L}_\alpha^6(\mathbb{R}^3)\right).$$

Moreover, if $t_{max} < \infty$, then at least one of the limits

$$\lim_{t \nearrow t_{max}} ||h(t)||_{H_x^\alpha(\mathbb{R}^{2d}, f_\infty)} \quad \text{and} \quad \lim_{t \nearrow t_{max}} ||h(t)||_{H_v^1(\mathbb{R}^{2d}, f_\infty)}$$

is infinite.

If the initial data h_0 is small (i.e. f_0 is close to f_∞), then there is a unique global solution and it decays exponentially:

Theorem 3.2.5 (Global well-posedness). *Let $d = 3$, $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, the assumptions (A1), (A2) and (A3) hold. Let $h_0 \in H_x^\alpha(\mathbb{R}^6, f_\infty) \cap H_v^1(\mathbb{R}^6, f_\infty)$, $\int_{\mathbb{R}^6} h_0 f_\infty dx dv = 0$,*

$$||h_0||_{H_x^\alpha(\mathbb{R}^6, f_\infty)} \leq \delta_1 \quad \text{and} \quad ||h_0||_{H_v^1(\mathbb{R}^6, f_\infty)} \leq \delta_2$$

for explicitly computable constants δ_1, $\delta_2 > 0$ (given in the proof). Then (3.6) has a unique global mild solution

$$h \in C\left([0, \infty); H_x^\alpha(\mathbb{R}^6, f_\infty)\right) \cap C\left([0, \infty); H_v^1(\mathbb{R}^6, f_\infty)\right)$$

and

$$|\nabla_x \psi| \in C\left([0, \infty); \mathscr{L}_\alpha^6(\mathbb{R}^3)\right).$$

Moreover, for any $\lambda_1 \in (0, \lambda)$, there are explicitly computable constants $C_5 > 0$, $C_6 > 0$ and $C_7 > 0$ (independent of h_0, but depending on δ_1 and δ_2) such that

$$||h(t)||_{H_x^\alpha(\mathbb{R}^6, f_\infty)} \le C_5 e^{-\lambda_1 t},$$

$$||h(t)||_{H_v^1(\mathbb{R}^6, f_\infty)} \le C_6 e^{-\lambda_1 t}$$

and

$$||\nabla_x \psi(t)||_{\mathscr{L}_\alpha^6(\mathbb{R}^3)} \le C_7 e^{-\lambda_1 t}$$

hold, where λ is the decay rate obtained for the linearized system in Theorem 3.2.2.

In Theorem 3.2.5 we assume the initial data $h_0 = \frac{f_0 - f_\infty}{f_\infty}$ is in a neighborhood of zero in $H_x^\alpha(\mathbb{R}^{2d}, f_\infty)) \cap H_v^1(\mathbb{R}^{2d}, f_\infty)$, and the radius of this neighborhood can be estimated explicitly. We have to make this assumption because of the difficulties coming from the nonlinearity of the system. The smallness of $\frac{f_0 - f_\infty}{f_\infty}$ is a common assumption to study various nonlinear kinetic equations (e.g. [3, 8, 46, 37, 39, 22, 29]) and in some cases this assumptions is necessary. It is often required the smallness of $\frac{f_0 - f_\infty}{f_\infty}$ in more regular Sobolev spaces, for example in $H^s(\mathbb{R}^{2d}, f_\infty)$ with $s \ge d$. While we require this assumption in a larger and less regular space (i.e., $H_x^\alpha(\mathbb{R}^{2d}, f_\infty)) \cap H_v^1(\mathbb{R}^{2d}, f_\infty))$ and so our result improves the previous works in this respect. It would be interesting to generalize our results for arbitrary large initial data, away from the steady state. Yet, this extension is not within reach so far and will be a matter of further study.

We believe that Theorem 3.2.4 and Theorem 3.2.5 are the first well-posedness and exponential stability results for a large class of potentials V (i.e, the potentials satisying Assumption 3.2.1). Most of the previous results were obtained when $V = 0$. Our results hold for any parameters $\nu > 0$ and $\sigma > 0$, they do not need to be small or large as in [31, 8, 46].

3.3 Preliminaries

3.3.1 Steady state

In this section, we show that there is a unique solution to (3.2) and we establish some regularity estimates in the Sobolev spaces.

Lemma 3.3.1 ([24, Section 2])**.** *Let $d \ge 3$ and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$. Then (3.2) has a unique solution $\phi_\infty \ge 0$ such that*

$$\phi_\infty \in L^{\frac{d}{d-2},\infty}(\mathbb{R}^d) \quad and \quad |\nabla \phi_\infty| \in L^{\frac{d}{d-1},\infty}(\mathbb{R}^d),$$

where $L^{p,\infty}(\mathbb{R}^d) := \left\{g \in L^1_{loc}(\mathbb{R}^d) : \sup_{\lambda > 0} \left[\lambda^p meas(\{x \in \mathbb{R}^d : \ g(x) > \lambda\})\right] < \infty\right\}, p > 1.$

As we have the existence by Lemma 3.3.1, we next establish some regularity for ϕ_∞. The important tool here is the Hardy-Littlewood-Sobolev inequality:

Theorem 3.3.2 ([27, Theorem 7.25])**.** *Let $p, q \in (1, \infty)$, $a \in (0, d)$ such that $\frac{1}{q} - \frac{1}{p} + \frac{a}{d} = 0$. There exists a constant $C_{HLS} > 0$ such that, for all $g \in L^p(\mathbb{R}^d)$,*

$$\left|\left| g * \frac{1}{|x|^{d-a}} \right|\right|_{L^q(\mathbb{R}^d)} \le C_{HLS} ||g||_{L^p(\mathbb{R}^d)}.$$

Lemma 3.3.3. *Let $d \geq 3$, V be bounded from below and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$. Then the solution ϕ_∞ of (3.2) satisfies*

$$\phi_\infty \in W^{2,q}(\mathbb{R}^d)$$

for all $q \in (\frac{d}{d-2}, \infty)$. Moreover, if $V \in C^1(\mathbb{R}^d)$ and $|\nabla_x V| e^{-\frac{\nu}{\sigma}V} \in L^r(\mathbb{R}^d)$ for some $r \in (\frac{d}{d-2}, \infty)$, then

$$\phi_\infty \in W^{3,r}(\mathbb{R}^d).$$

In particular, if $r > d$, then $\phi_\infty \in W^{2,\infty}(\mathbb{R}^d)$.

Proof. The boundedness of V from below, $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$ and $\phi_\infty \geq 0$ imply that

$$e^{-\frac{\nu}{\sigma}[V+\phi_\infty]} \in L^p(\mathbb{R}^d), \quad \forall p \in [1, \infty] \tag{3.13}$$

and so

$$-\Delta_x \phi_\infty = \frac{e^{-\frac{\nu}{\sigma}[V+\phi_\infty]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]} dx'} \in L^p(\mathbb{R}^d), \quad \forall p \in [1, \infty]. \tag{3.14}$$

We present ϕ_∞ as

$$\phi_\infty = \frac{1}{(d-2)|\mathbb{S}^{d-1}|} \frac{1}{|x|^{d-2}} * \left(\frac{e^{-\frac{\nu}{\sigma}[V+\phi_\infty]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]} dx'} \right).$$

(3.13) and the Hardy-Littlewood-Sobolev inequality show

$$\phi_\infty \in L^q(\mathbb{R}^d), \quad \forall q \in \left(\frac{d}{d-2}, \infty \right). \tag{3.15}$$

(3.14) and (3.15) yield $-\Delta_x \phi_\infty + \phi_\infty \in L^q(\mathbb{R}^d)$ for all $q \in (\frac{d}{d-2}, \infty)$. Thus, the elliptic regularity [27, Section 7.2 and Section 7.3] shows

$$\phi_\infty \in W^{2,q}(\mathbb{R}^d), \quad \forall q \in \left(\frac{d}{d-2}, \infty \right).$$

In particular, by the Sobolev embedding theorem

$$\phi_\infty \in L^\infty(\mathbb{R}^d), \quad |\nabla_x \phi_\infty| \in L^\infty(\mathbb{R}^d). \tag{3.16}$$

We use the bootstrap argument. Because of the assumption $|\nabla_x V| e^{-\frac{\nu}{\sigma}V} \in L^r(\mathbb{R}^d)$ and (3.16), we have

$$-\Delta_x(\partial_{x_i}\phi_\infty) = -\frac{\frac{\nu}{\sigma}(\partial_{x_i}V + \partial_{x_i}\phi_\infty) e^{-\frac{\nu}{\sigma}[V+\phi_\infty]}}{\int_{\mathbb{R}^d} e^{-\frac{\nu}{\sigma}[V(x')+\phi_\infty(x')]} dx'} \in L^r(\mathbb{R}^d)$$

for all $i \in \{1, ..., d\}$. Again using the elliptic regularity we obtain $\partial_{x_i}\phi_\infty \in W^{2,r}(\mathbb{R}^d)$. If $r > d$, then the Sobolev embedding theorem provides that $\phi_\infty \in W^{2,\infty}(\mathbb{R}^d)$.

□

3.3.2 Poincaré type inqualities

In this section, we present some sufficient conditions on the potential V such that ρ_∞ satisfies Poincaré type-inequalities.

Lemma 3.3.4. *Let V be bounded from below, $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$ and $e^{-\frac{\nu}{\sigma}V}$ satisfy the Poincaré inequality (3.4). Then, there exists a positive constant κ_2 such that*

$$\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv - \left(\int_{\mathbb{R}^{2d}} h f_\infty dxdv\right)^2 \leq \kappa_2 \int_{\mathbb{R}^{2d}} (|\nabla_x h|^2 + |\nabla_v h|^2) f_\infty dxdv \tag{3.17}$$

holds for all $h \in H^1(\mathbb{R}^{2d}, f_\infty)$.

Proof. ϕ_∞ is bounded by Lemma 3.3.3. Then the Holley-Stroock perturbation argument [35] implies that ρ_∞ satisfies the Poincaré inequality

$$\int_{\mathbb{R}^d} h^2 \rho_\infty dx - \left(\int_{\mathbb{R}^d} h\rho_\infty dx\right)^2 \leq \kappa_1' \int_{\mathbb{R}^d} |\nabla_x h|^2 \rho_\infty dx$$

for some constant $\kappa_1' > 0$. Since the Gaussian distribution $M(v) = \frac{e^{-\frac{\nu}{\sigma}|v|^2/2}}{(2\pi\sigma/\nu)^{d/2}}$ satisfies the Poincaré inequality, [7, Proposition 4.3.1] shows that $f_\infty = \rho_\infty M$ satisfies (3.17). □

Lemma 3.3.5. *(i) Let V satisfy the assumptions (A1) and (A2). Then there exist $\kappa_3 > 0$ and $\kappa_4 > 0$ such that, for all $g \in H^1(\mathbb{R}^d, \rho_\infty)$,*

$$\int_{\mathbb{R}^d} g^2 \left|\left|\frac{\partial^2(V+\phi_\infty)}{\partial x^2}\right|\right|_F^2 \rho_\infty dx \leq \kappa_3 \left(\int_{\mathbb{R}^d} g^2\rho_\infty dx + \int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_\infty dx\right), \tag{3.18}$$

$$\int_{\mathbb{R}^d} g^2 |\nabla_x(V+\phi_\infty)|^2 \rho_\infty dx \leq \kappa_4 \left(\int_{\mathbb{R}^d} g^2\rho_\infty dx + \int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_\infty dx\right). \tag{3.19}$$

(ii) There exist $\kappa_4' > 0$ such that, for all $g \in H^1(\mathbb{R}^d, M)$,

$$\int_{\mathbb{R}^d} |v|^2 g^2 M dv \leq \kappa_4' \left(\int_{\mathbb{R}^d} g^2 M dv + \int_{\mathbb{R}^d} |\nabla_v g|^2 M dv\right). \tag{3.20}$$

Proof. (i) We first prove that there exists $c_2 > 0$ such that

$$\frac{\nu}{\sigma}\left|\left|\frac{\partial^2(V(x)+\phi_\infty(x))}{\partial x^2}\right|\right|_F \leq c_2\left(1 + \frac{\nu}{\sigma}|\nabla_x (V(x)+\phi_\infty(x))|\right), \quad \forall x \in \mathbb{R}^d. \tag{3.21}$$

Lemma 3.3.3 provides $\phi_\infty \in W^{2,\infty}(\mathbb{R}^d)$. Then, (3.21) follows by (3.3) and the following estimates:

$$\begin{aligned}
\left|\left|\frac{\partial^2(V+\phi_\infty)}{\partial x^2}\right|\right|_F &\leq \left|\left|\frac{\partial^2\phi_\infty}{\partial x^2}\right|\right|_F + \left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F \\
&\leq \left|\left|\frac{\partial^2\phi_\infty}{\partial x^2}\right|\right|_F + c_1(1+|\nabla_x V|) \\
&\leq \left|\left|\frac{\partial^2\phi_\infty}{\partial x^2}\right|\right|_F + c_1|\nabla_x\phi_\infty| + c_1 + c_1|\nabla_x(V+\phi_\infty)| \\
&\leq c_2\left(\frac{\sigma}{\nu} + |\nabla_x(V+\phi_\infty)|\right),
\end{aligned}$$

where $c_2 := \max\left\{c_1, \frac{\nu}{\sigma}\left|\left|\, ||\frac{\partial^2\phi_\infty}{\partial x^2}||_F + c_1|\nabla_x\phi_\infty| + c_1\right|\right|_{L^\infty(\mathbb{R}^d)}\right\}$.
Then, (3.21) and [50, Lemma A.24] (by replacing V with $\frac{\nu}{\sigma}[V+\phi_\infty]$) provide (3.18) and (3.19).

(ii) The proof follows from [50, Lemma A.24] (by replacing V with $|v|^2/2$). □

Lemma 3.3.6. *Let V satisfy the assumptions (A1) and (A2). Then there is a positive constant κ_5 such that, for all $g \in H^1_x(\mathbb{R}^d, f_\infty)$,*

$$||g||_{H^1_x(\mathbb{R}^{2d},f_\infty)} \leq \kappa_5 \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv}. \tag{3.22}$$

Proof. Using (3.5) we estimate

$$||g||^2_{H^1_x(\mathbb{R}^{2d},f_\infty)} = \int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} \left| \nabla_x g f_\infty^{1/2} - \frac{\nu}{2\sigma} \nabla_x (V + \phi_\infty) g f_\infty^{1/2} \right|^2 dx dv$$

$$\leq \int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + 2 \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv + \frac{\nu^2}{2\sigma^2} \int_{\mathbb{R}^{2d}} g^2 |\nabla_x (V + \phi_\infty)|^2 f_\infty dx dv.$$

Applying (3.19) to the last term we obtain

$$\begin{aligned} ||g||^2_{H^1_x(\mathbb{R}^{2d},f_\infty)} &\leq \left(1 + \frac{\kappa_4 \nu^2}{2\sigma^2}\right) \int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + \left(2 + \frac{\kappa_4 \nu^2}{2\sigma^2}\right) \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv \\ &\leq \left(2 + \frac{\kappa_4 \nu^2}{2\sigma^2}\right) \left(\int_{\mathbb{R}^{2d}} g^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_x g|^2 f_\infty dx dv \right). \end{aligned}$$

Thus, (3.22) holds with $\kappa_5 := \sqrt{2 + \frac{\kappa_4 \nu^2}{2\sigma^2}}$. □

3.3.3 The Poisson equation

In this section, we present some estimates for the Poisson equation

$$-\Delta_x \psi = \int_{\mathbb{R}^d} h f_\infty dv. \tag{3.23}$$

We define $L^p(\mathbb{R}^{2d}, f_\infty) := \left\{ g : \mathbb{R}^{2d} \to \mathbb{R} : \int_{\mathbb{R}^{2d}} |g|^p f_\infty dx dv < \infty \right\}$ with the norm

$$||g||_{L^p(\mathbb{R}^{2d},f_\infty)} := \left(\int_{\mathbb{R}^{2d}} |g|^p f_\infty dx dv \right)^{1/p}.$$

Lemma 3.3.7. *Let V be bounded from below and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$.*

(i) Let $p \in [1,2]$ and $h \in L^p(\mathbb{R}^{2d}, f_\infty)$. Then

$$\left|\left| \int_{\mathbb{R}^d} h f_\infty dv \right|\right|_{L^p(\mathbb{R}^d)} \leq ||\rho_\infty||^{1-\frac{1}{p}}_{L^\infty(\mathbb{R}^d)} ||h||_{L^p(\mathbb{R}^{2d},f_\infty)}. \tag{3.24}$$

(ii) Let $h \in L^2(\mathbb{R}^{2d}, f_\infty)$. Then, we have $h \in L^p(\mathbb{R}^{2d}, f_\infty)$ for all $p \in [1,2]$ and

$$||h||_{L^p(\mathbb{R}^{2d},f_\infty)} \leq ||h||_{L^2(\mathbb{R}^{2d},f_\infty)}. \tag{3.25}$$

(iii) Let $\alpha \in [0,1]$ and $h \in H^\alpha_x(\mathbb{R}^{2d}, f_\infty)$. If V satisfies the assumption (A2), then there is a constant $\mathcal{A} > 0$ (independent of h) such that

$$\left|\left| \int_{\mathbb{R}^d} h f_\infty dv \right|\right|_{\mathscr{L}^2_\alpha(\mathbb{R}^d)} \leq \mathcal{A} ||h||_{H^\alpha_x(\mathbb{R}^{2d},f_\infty)}. \tag{3.26}$$

Proof. The assumptions on V provide that f_∞ is well-defined and bounded.

(i) If $p=1$, (3.24) follows by

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} hf_\infty dv\right| dx \leq \int_{\mathbb{R}^{2d}} |h| f_\infty dvdx.$$

If $p\in(1,2]$, the Hölder inequality implies

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} hf_\infty dv\right|^p dx &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(hf_\infty^{1/p})f_\infty^{1-1/p}dv\right|^p dx\\ &\leq \int_{\mathbb{R}^d}\left[\left(\int_{\mathbb{R}^d}|h|^p f_\infty dv\right)^{1/p}\left(\int_{\mathbb{R}^d} f_\infty dv\right)^{1-1/p}\right]^p dx \leq ||\rho_\infty||^{p-1}_{L^\infty(\mathbb{R}^d)}||h||^p_{L^p(\mathbb{R}^{2d},f_\infty)}.\end{aligned}$$

(ii) The Hölder inequality and $\int_{\mathbb{R}^{2d}} f_\infty dxdv = 1$ show

$$||h||^p_{L^2(\mathbb{R}^{2d},f_\infty)} = \left(\int_{\mathbb{R}^{2d}}(|h|^p f_\infty^{\frac{p}{2}})^{\frac{2}{p}}dxdv\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^{2d}} f_\infty dxdv\right)^{1-\frac{p}{2}} \geq \int_{\mathbb{R}^{2d}}|h|^p f_\infty dxdv.$$

(iii) If $\alpha=0$, then the Plancherel theorem and (3.24) with $p=2$ yield (3.26) with $\mathcal{A} := ||\rho_\infty||^{\frac{1}{2}}_{L^\infty(\mathbb{R}^d)}$. If $\alpha=1$, by the Plancherel theorem

$$\left|\left|\int_{\mathbb{R}^d} hf_\infty dv\right|\right|^2_{\mathscr{L}^2_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} hf_\infty dv\right|^2 dx + \int_{\mathbb{R}^d}\left|\nabla_x\int_{\mathbb{R}^d} hf_\infty dv\right|^2 dx. \tag{3.27}$$

We estimate the second term on the right using $\nabla_x f_\infty^{1/2} = -\frac{\nu}{2\sigma}\nabla_x(V+\phi_\infty)f_\infty^{1/2}$

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\nabla_x\int_{\mathbb{R}^d} hf_\infty dv\right|^2 dx &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\nabla_x(hf_\infty^{1/2})f_\infty^{1/2}dv - \frac{\nu}{2\sigma}\int_{\mathbb{R}^d} h\nabla_x(V+\phi_\infty)f_\infty dv\right|^2 dx\\ &\leq 2\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\nabla_x(hf_\infty^{1/2})f_\infty^{1/2}dv\right|^2 dx + \frac{\nu^2}{2\sigma^2}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} h\nabla_x(V+\phi_\infty)f_\infty dv\right|^2 dx.\end{aligned} \tag{3.28}$$

We estimate the first integral in (3.28)

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\nabla_x(hf_\infty^{1/2})f_\infty^{1/2}dv\right|^2 dx &\leq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\nabla_x(hf_\infty^{1/2})|^2 dv\right)\left(\int_{\mathbb{R}^d} f_\infty dv\right)dx\\ &\leq ||\rho_\infty||_{L^\infty}\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2 dxdv.\end{aligned}$$

We estimate the second integral in (3.28)

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} h\nabla_x(V+\phi_\infty)f_\infty dv\right|^2 dx &\leq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(hf_\infty^{1/2})^2|\nabla_x(V+\phi_\infty)|^2 dv\right)\left(\int_{\mathbb{R}^d} f_\infty dv\right)dx\\ &= \int_{\mathbb{R}^{2d}}(hf_\infty^{1/2})^2|\nabla_x(V+\phi_\infty)|^2\rho_\infty dxdv.\end{aligned}$$

These estimates yield

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\nabla_x\int_{\mathbb{R}^d} hf_\infty dv\right|^2 dx \leq{}& 2||\rho_\infty||_{L^\infty}\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2 dxdv\\ &+ \frac{\nu^2}{2\sigma^2}\int_{\mathbb{R}^{2d}}(hf_\infty^{1/2})^2|\nabla_x(V+\phi_\infty)|^2\rho_\infty dxdv.\end{aligned}$$

Applying (3.19) to the last term we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\nabla_x\int_{\mathbb{R}^d}hf_\infty dv\right|^2dx &\le 2||\rho_\infty||_{L^\infty}\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2dxdv\\ &\quad+\frac{\kappa_4\nu^2}{2\sigma^2}\int_{\mathbb{R}^{2d}}h^2f_\infty\rho_\infty dxdv+\frac{\kappa_4\nu^2}{2\sigma^2}\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2\rho_\infty dxdv\\ &\le\frac{\kappa_4||\rho_\infty||_{L^\infty}\nu^2}{2\sigma^2}\int_{\mathbb{R}^{2d}}h^2f_\infty dxdv+\left(2||\rho_\infty||_{L^\infty}+\frac{\kappa_4||\rho_\infty||_{L^\infty}\nu^2}{2\sigma^2}\right)\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2dxdv.\end{aligned}\tag{3.29}$$

(3.27), (3.24) with $p=2$ and (3.29) provide

$$\begin{aligned}\left|\left|\int_{\mathbb{R}^d}hf_\infty dv\right|\right|^2_{\mathscr{L}^2_1(\mathbb{R}^d)} &\le\left(||\rho_\infty||_{L^\infty}+\frac{\kappa_4||\rho_\infty||_{L^\infty}\nu^2}{2\sigma^2}\right)\int_{\mathbb{R}^{2d}}h^2f_\infty dxdv\\ &\quad+\left(2||\rho_\infty||_{L^\infty}+\frac{\kappa_4||\rho_\infty||_{L^\infty}\nu^2}{2\sigma^2}\right)\int_{\mathbb{R}^{2d}}|\nabla_x(hf_\infty^{1/2})|^2dxdv\\ &\le\left(2||\rho_\infty||_{L^\infty}+\frac{\kappa_4||\rho_\infty||_{L^\infty}\nu^2}{2\sigma^2}\right)||h||^2_{H^1_x(\mathbb{R}^{2d},f_\infty)}.\end{aligned}$$

This estimate proves (3.26) when $\alpha=1$. By interpolation it holds for all $\alpha\in(0,1)$. □

Lemma 3.3.8. *Let $d\ge 3$, V be bounded from below and $e^{-\frac{\nu}{\sigma}V}\in L^1(\mathbb{R}^d)$. Let $h\in L^2(\mathbb{R}^{2d},f_\infty)$ and ψ satisfy (3.23). Then*

(i) There is a positive constant θ_1 such that, for all $p\in(1,2]$,

$$||\nabla_x\psi||_{L^{\frac{pd}{d-p}}(\mathbb{R}^d)}\le\theta_1||h||_{L^p(\mathbb{R}^{2d},f_\infty)}\le\theta_1||h||_{L^2(\mathbb{R}^{2d},f_\infty)}.\tag{3.30}$$

(ii) If $h\in H^\alpha_x(\mathbb{R}^{2d},f_\infty)$ for some $\alpha\in[0,1]$ and V satisfies the assumption (A2), then there is a positive constant θ_2 such that

$$||\nabla_x\psi||_{\mathscr{L}_\alpha^{\frac{2d}{d-2}}(\mathbb{R}^d)}\le\theta_2||h||_{H^\alpha_x(\mathbb{R}^{2d},f_\infty)}.\tag{3.31}$$

Moreover, if $d=3$ and $\alpha\in(\frac12,1]$, then there is a positive constant θ_3 such that

$$||\nabla_x\psi||_{L^\infty(\mathbb{R}^3)}\le\theta_2||h||_{H^\alpha_x(\mathbb{R}^6,f_\infty)}.\tag{3.32}$$

Proof. (i) Applying the Hardy-Littlewood-Sobolev inequality to the right hand sight of

$$|\nabla_x\psi|=\frac{1}{|\mathbb{S}^{d-1}|}\left|\frac{x}{|x|^d}*\int_{\mathbb{R}^d}hf_\infty dv\right|\le\frac{d}{|\mathbb{S}^{d-1}|}\frac{1}{|x|^{d-1}}*\left|\int_{\mathbb{R}^d}hf_\infty dv\right|,$$

we obtain that there is $C>0$ such that

$$||\nabla_x\psi||_{L^{\frac{pd}{d-p}}(\mathbb{R}^d)}\le C\left|\left|\int_{\mathbb{R}^d}hf_\infty dv\right|\right|_{L^p(\mathbb{R}^d)}$$

holds for all $p\in(1,2]$. Then, (3.24) and (3.25) implies (3.30).

(ii) (3.23) shows

$$-(1-\Delta_x)^{\frac{\alpha}{2}}\Delta_x\psi=-\Delta_x\left((1-\Delta_x)^{\frac{\alpha}{2}}\psi\right)=(1-\Delta_x)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d}hf_\infty dv.$$

Applying the Hardy-Littlewood-Sobolev inequality with $p = 2$ to the right hand sight of

$$|\nabla_x \left((1-\Delta_x)^{\frac{\alpha}{2}}\psi\right)| = \frac{1}{|\mathbb{S}^{d-1}|}\left|\frac{x}{|x|^d} * \left[(1-\Delta_x)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d} hf_\infty dv\right]\right|$$
$$\leq \frac{d}{|\mathbb{S}^{d-1}|}\frac{1}{|x|^{d-1}} * \left|(1-\Delta_x)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d} hf_\infty dv\right|,$$

we get

$$\left|\left|\nabla_x\left((1-\Delta_x)^{\frac{\alpha}{2}}\psi\right)\right|\right|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = ||\nabla_x\psi||_{\mathscr{L}_\alpha^{\frac{2d}{d-2}}(\mathbb{R}^d)}$$
$$\leq C\left|\left|(1-\Delta_x)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d} hf_\infty dv\right|\right|_{L^2(\mathbb{R}^d)} = C\left|\left|\int_{\mathbb{R}^d} hf_\infty dv\right|\right|_{\mathscr{L}_\alpha^2(\mathbb{R}^d)}.$$

Then, (3.31) follows by (3.26).

Let $d = 3$ and $\alpha \in (\frac{1}{2}, 1]$. Since $\frac{2d\alpha}{d-2} = 6\alpha > 3$ for $\alpha \in (1/2, 1]$, the Sobolev embedding [1, Theorem 1.2.4] provides (3.32).

□

3.4 The Linearized Vlasov-Poisson-Fokker-Planck system

In this section, we analyze the linearized Vlasov-Poisson-Fokker-Planck system (3.7). We first show existence and uniqueness.

3.4.1 Existence and uniqueness

We write the linearized system (3.7) as

$$\begin{cases}\partial_t h + Kh = 0\\ h_{|t=0} = h_0,\end{cases}$$

where $Kh := v\cdot\nabla_x h - \nabla_x(V+\phi_\infty)\cdot\nabla_v h + v\cdot\nabla_x\psi - \sigma\Delta_v h + \nu v\cdot\nabla_v h$. Clearly, K depends on $\nabla_x\psi$. But we consider K as an operator acting only on h, since $\nabla_x\psi$ can be expressed by h as

$$\nabla_x\psi = \frac{1}{|\mathbb{S}^{d-1}|}\frac{x}{|x|^d} * \int_{\mathbb{R}^d} hf_\infty dv.$$

Theorem 3.4.1. *Let $V \in C^\infty(\mathbb{R}^d)$ be bounded from below and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$. Then K generates a C_0 semigroup e^{-tK} on $L^2(\mathbb{R}^{2d}, f_\infty)$. In particular, for any $h_0 \in L^2(\mathbb{R}^{2d}, f_\infty)$, the linearized system (3.7) has a unique mild solution*

$$h \in C\left([0,\infty); L^2(\mathbb{R}^{2d}, f_\infty)\right)$$

and

$$|\nabla_x\psi| \in C\left([0,\infty); L^{\frac{pd}{d-2}}(\mathbb{R}^d)\right), \quad \forall p \in (1,2].$$

Proof. By Lemma 3.3.3, we have $\phi_\infty \in W^{2,q}(\mathbb{R}^d)$ for all $q \in (\frac{d}{d-2}, \infty)$, and so $\phi_\infty \in C^1(\mathbb{R}^d)$. Since we have $V \in C^\infty(\mathbb{R}^d)$, we can show $\phi_\infty \in C^\infty(\mathbb{R}^d)$ by a bootstrap argument, see [27, Theorem 5.20]. We consider the following equation without the Poisson equation

$$\begin{cases}\partial_t h + v\cdot\nabla_x h - \nabla_x(V+\phi_\infty)\cdot\nabla_v h - \sigma\Delta_v h + \nu v\cdot\nabla_v h = 0\\ h_{|t=0} = h_0.\end{cases} \tag{3.33}$$

Let $Lh := v \cdot \nabla_x h - \nabla_x(V + \phi_\infty) \cdot \nabla_v h - \sigma \Delta_v h + \nu v \cdot \nabla_v h$. Then (3.33) can be written as

$$\begin{cases} \partial_t h + Lh = 0 \\ h_{|t=0} = h_0. \end{cases}$$

Since we have $V + \phi_\infty \in C^\infty(\mathbb{R}^d)$, [30, Section 5.2] shows that L generates a C^∞ regularizing contraction semigroup in $L^2(\mathbb{R}^{2d}, f_\infty)$. K differs from L in the term $v \cdot \nabla_x \psi$ coming from the Poisson equation. (3.30) with $p = \frac{2d}{d+2}$ shows

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |v \cdot \nabla_x \psi|^2 f_\infty dx dv &\leq \left(\int_{\mathbb{R}^d} |v|^2 M(v) dv \right) ||\rho_\infty||_{L^\infty} \int_{\mathbb{R}^d} |\nabla_x \psi|^2 dx \\ &\leq \theta_1^2 ||\rho_\infty||_{L^\infty} \left(\int_{\mathbb{R}^d} |v|^2 M(v) dv \right) \int_{\mathbb{R}^{2d}} h^2 f_\infty dx dv. \end{aligned}$$

Therefore, we can consider $h \to v \cdot \nabla_x \psi$ as a bounded operator from $L^2(\mathbb{R}^{2d}, f_\infty)$ to $L^2(\mathbb{R}^{2d}, f_\infty)$. This implies that K is a bounded perturbation of L in $L^2(\mathbb{R}^{2d}, f_\infty)$. Then [44, Chapter 3] provides that K generates a C_0 semigroup e^{-tK} on $L^2(\mathbb{R}^{2d}, f_\infty)$. Also, $e^{-tK} h_0 \in C\left([0,\infty); L^2(\mathbb{R}^{2d}, f_\infty)\right)$ is the unique mild solution to (3.7) by [44, Chapter 4]. Then (3.30) implies that the absolute value of

$$\nabla_x \psi(t) = \frac{1}{|\mathbb{S}^{d-1}|} \frac{x}{|x|^d} * \int_{\mathbb{R}^d} e^{-tK} h_0 \, f_\infty dv$$

is in $L^{\frac{pd}{d-2}}(\mathbb{R}^d)$ for all $p \in (1, 2]$. Moreover, $||\nabla_x \psi(t)||_{L^{\frac{pd}{d-2}}(\mathbb{R}^d)}$ is continuous function of t as $e^{-tK} h_0 \in C\left([0,\infty); L^2(\mathbb{R}^{2d}, f_\infty)\right)$.

□

3.4.2 Exponential stability in $H^1(\mathbb{R}^{2d}, f_\infty)$

In this subsection, we shall construct a Lypunov functional for the linearized system (3.7). This functional will help us to show that the solutions of (3.7) are exponentially stable in $H^1(\mathbb{R}^{2d}, f_\infty)$.

We introduce a norm

$$||h||^2 := \int_{\mathbb{R}^{2d}} h^2 f_\infty dx dv + \int_{\mathbb{R}^d} |\nabla_x \psi|^2 dx.$$

Lemma 3.4.2. *Let h be the solution of (3.7). Then, for all $t > 0$,*

$$\frac{d}{dt} ||h(t)||^2 = -2\sigma \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dx dv.$$

In particular, we have $||h(t)|| \leq ||h_0||$ for all $t \geq 0$.

Proof. First, we compute

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} h^2 f_\infty dx dv &= 2 \int_{\mathbb{R}^{2d}} h \partial_t h f_\infty dx dv \\ &= -2 \int_{\mathbb{R}^{2d}} \left(v \cdot \nabla_x h - \nabla_x (V + \phi_\infty) \cdot \nabla_v h \right) h f_\infty dx dv \\ &\quad + 2 \int_{\mathbb{R}^{2d}} (\sigma \Delta_v h - \nu v \cdot \nabla_v h) h f_\infty dx dv - 2 \int_{\mathbb{R}^{2d}} v \cdot \nabla_x \psi h f_\infty dx dv. \end{aligned}$$

Integrating by parts we get

$$-2\int_{\mathbb{R}^{2d}}(v\cdot\nabla_x h-\nabla_x(V+\phi_\infty)\cdot\nabla_v h)\,hf_\infty dxdv$$
$$=-\int_{\mathbb{R}^{2d}}\left(v\cdot\nabla_x h^2-\nabla_x(V+\phi_\infty)\cdot\nabla_v h^2\right)f_\infty dxdv=0,$$

and

$$2\int_{\mathbb{R}^{2d}}(\sigma\Delta_v h-\nu v\cdot\nabla_v h)hf_\infty dxdv=-2\sigma\int_{\mathbb{R}^{2d}}|\nabla_v h|^2 f_\infty dxdv.$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}}h^2 f_\infty dxdv=-2\sigma\int_{\mathbb{R}^{2d}}|\nabla_v h|^2 f_\infty dxdv-2\int_{\mathbb{R}^{2d}}v\cdot\nabla_x\psi hf_\infty dxdv. \tag{3.34}$$

Secondly, we compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla_x\psi|^2dx&=2\int_{\mathbb{R}^d}\nabla_x(\partial_t\psi)\cdot\nabla_x\psi dx\\&=-2\int_{\mathbb{R}^d}\psi\Delta_x(\partial_t\psi)dx=2\int_{\mathbb{R}^d}\psi\left(\int_{\mathbb{R}^d}\partial_t hf_\infty dv\right)dx\\&=2\int_{\mathbb{R}^{2d}}\psi\left[\nabla_x(V+\phi_\infty)\cdot\nabla_v h-v\cdot\nabla_x h+\sigma\Delta_v h-\nu v\cdot\nabla_v h-v\cdot\nabla_x\psi\right]f_\infty dxdv\\&=2\int_{\mathbb{R}^{2d}}v\cdot\nabla_x\psi hf_\infty dxdv-2\int_{\mathbb{R}^{2d}}\psi\nabla_x\psi\cdot vf_\infty dxdv=2\int_{\mathbb{R}^{2d}}v\cdot\nabla_x\psi hf_\infty dxdv,\end{aligned} \tag{3.35}$$

where we integrated by parts and used $\nabla_x f_\infty=-\frac{\nu}{\sigma}\nabla_x(V+\phi_\infty)f_\infty$ and $\nabla_v f_\infty=-\frac{\nu}{\sigma}vf_\infty$. (3.34) and (3.35) provide the claimed equality. □

Let $P\in\mathbb{R}^{2n\times 2n}$ be a constant, symmetric, positive definite matrix. We define

$$\mathrm{S}_P[h]:=\int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v(h+\psi)\end{pmatrix}^T P\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v(h+\psi)\end{pmatrix}f_\infty dxdv$$
$$=\int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v h\end{pmatrix}^T P\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v h\end{pmatrix}f_\infty dxdv.$$

Lemma 3.4.3. *Let h be the solution of (3.7). Then, for all $t>0$,*

$$\begin{aligned}\frac{d}{dt}\mathrm{S}_P[h(t)]=&-2\sigma\int_{\mathbb{R}^{2d}}\left\{\sum_{i=1}^d\begin{pmatrix}\nabla_x(\partial_{v_i}h)\\\nabla_v(\partial_{v_i}h)\end{pmatrix}^T P\begin{pmatrix}\nabla_x(\partial_{v_i}h)\\\nabla_v(\partial_{v_i}h)\end{pmatrix}\right\}f_\infty dxdv\\&-\int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v h\end{pmatrix}^T\{QP+PQ^T\}\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v h\end{pmatrix}f_\infty dxdv\\&-2\int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x(h+\psi)\\\nabla_v h\end{pmatrix}^T P\begin{pmatrix}\nabla_x\partial_t\psi\\0\end{pmatrix}f_\infty dxdv,\end{aligned} \tag{3.36}$$

where $Q=Q(x):=\begin{pmatrix}0 & I\\-\frac{\partial^2(V(x)+\phi_\infty(x))}{\partial x^2} & \nu I\end{pmatrix}$.

Proof. Since ψ does not depends on v, we write (3.7) as

$$\partial_t(h+\psi) = -v\cdot\nabla_x(h+\psi) + \nabla_x(V+\phi_\infty)\cdot\nabla_v(h+\psi) \\ + \sigma\Delta_v(h+\psi) - \nu v\cdot\nabla_v(h+\psi) + \partial_t\psi,$$

with $-\Delta_x\psi = \int_{\mathbb{R}^d} hf_\infty dv, \quad h_{|t=0} = h_0$. We denote $u := \begin{pmatrix}\nabla_x(h+\psi)\\ \nabla_v(h+\psi)\end{pmatrix}$, $u_1 := \nabla_x(h+\psi)$, $u_2 := \nabla_v(h+\psi)$. Then u_1 and u_2 satisfy

$$\partial_t u_1 = \sigma\Delta_v u_1 - \nu\sum_{i=1}^d v_i\partial_{v_i}u_1 + \sum_{i=1}^d \partial_{x_i}(V+\phi_\infty)\partial_{v_i}u_1 - \sum_{i=1}^d v_i\partial_{x_i}u_1 + \frac{\partial^2(V+\phi_\infty)}{\partial x^2}u_2 + \nabla_x\partial_t\psi,$$

$$\partial_t u_2 = \sigma\Delta_v u_2 - \nu\sum_{i=1}^d v_i\partial_{v_i}u_2 + \sum_{i=1}^d \partial_{x_i}(V+\phi_\infty)\partial_{v_i}u_2 - \sum_{i=1}^d v_i\partial_{x_i}u_2 - u_1 - \nu u_2.$$

These equations can be written with respect to $u = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$:

$$\partial_t u = \sigma\Delta_v u - \nu\sum_{i=1}^d v_i\partial_{v_i}u + \sum_{i=1}^d \partial_{x_i}(V+\phi_\infty)\partial_{v_i}u - \sum_{i=1}^d v_i\partial_{x_i}u - Q^T u + \begin{pmatrix}\nabla_x\partial_t\psi\\ 0\end{pmatrix}.$$

It allows us to compute the time derivative

$$\frac{d}{dt}\mathrm{S}_P[h(t)] = 2\int_{\mathbb{R}^{2d}} u^T P\partial_t u f_\infty dxdv$$

$$= 2\sigma\int_{\mathbb{R}^{2d}} u^T P\Delta_v u f_\infty dxdv - 2\nu\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{v_i}uv_i f_\infty dxdv$$

$$+ 2\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{v_i}u\partial_{x_i}(V+\phi_\infty)f_\infty dxdv - 2\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{x_i}uv_i f_\infty dxdv$$
$$- \int_{\mathbb{R}^{2d}} u^T\{QP+PQ^T\}uf_\infty dxdv - 2\int_{\mathbb{R}^{2d}} u^T P\begin{pmatrix}\nabla_x\partial_t\psi\\ 0\end{pmatrix} f_\infty dxdv. \quad (3.37)$$

First, we consider the term in the second line of (3.37) and use $\partial_{v_i}f_\infty = -\frac{\nu}{\sigma}v_i f_\infty$:

$$2\sigma\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial^2_{v_iv_i}uf_\infty dxdv - 2\nu\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{v_i}uv_i f_\infty dxdv$$
$$= -2\sigma\sum_{i=1}^d\int_{\mathbb{R}^{2d}} \partial_{v_i}u^T P\partial_{v_i}uf_\infty dxdv. \quad (3.38)$$

Next, we consider the terms in the third line of (3.37):

$$2\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{v_i}u\partial_{x_i}(V+\phi_\infty)f_\infty dxdv$$
$$= -2\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T P\partial_{v_i}u\partial_{x_i}(V+\phi_\infty)f_\infty dxdv + \frac{2\nu}{\sigma}\sum_{i=1}^d\int_{\mathbb{R}^{2d}} u^T Pu\partial_{x_i}(V+\phi_\infty)v_i f_\infty dxdv$$
$$(3.39)$$

and

$$
\begin{aligned}
&-2\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{x_i} u v_i f_\infty dxdv \\
&\quad = 2\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{x_i} u v_i f_\infty dxdv - \frac{2\nu}{\sigma}\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} u^T P u\partial_{x_i}(V+\phi_\infty) v_i f_\infty dxdv. \quad (3.40)
\end{aligned}
$$

(3.39) and (3.40) show that the third line of (3.37) equals to zero. Combining (3.37) and (3.38), we obtain the statement (3.36). □

Lemma 3.4.4. *Let h be the solution of (3.7) and ψ be the solution of $-\Delta_x\psi = \int_{\mathbb{R}^d} hf_\infty dv$. Then, for all $t>0$,*

$$
\int_{\mathbb{R}^{2d}} |\nabla_x\partial_t\psi|^2 f_\infty dxdv \le \frac{\sigma^2||\rho_\infty||^2_{L^\infty}}{\nu^2}\int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv. \quad (3.41)
$$

Proof. We compute

$$
\begin{aligned}
-\Delta_x(\partial_t\psi) &= \int_{\mathbb{R}^d} \partial_t h f_\infty dv \\
&= \int_{\mathbb{R}^d} [-v\cdot\nabla_x h + \nabla_x(V+\phi_\infty)\cdot\nabla_v h + \sigma\Delta_v h - \nu v\cdot\nabla_v h - v\cdot\nabla_x\psi] f_\infty dv \\
&= \int_{\mathbb{R}^d} [-v\cdot\nabla_x h + \nabla_x(V+\phi_\infty)\cdot\nabla_v h] f_\infty dv \\
&= \int_{\mathbb{R}^d} [-v\cdot\nabla_x h + \frac{\nu}{\sigma} v\cdot\nabla_x(V+\phi_\infty) h] f_\infty dv \\
&= -\mathrm{div}_x \int_{\mathbb{R}^d} vhf_\infty dv = -\frac{\sigma}{\nu}\mathrm{div}_x\int_{\mathbb{R}^d}\nabla_v h f_\infty dv,
\end{aligned}
$$

where we integrated by parts and used $\nabla_v f_\infty = -\frac{\nu}{\sigma} v f_\infty$. It lets us compute

$$
\begin{aligned}
\int_{\mathbb{R}^{2d}} |\nabla_x\partial_t\psi|^2 f_\infty dxdv &= \int_{\mathbb{R}^d} |\nabla_x\partial_t\psi|^2\rho_\infty dx \le ||\rho_\infty||_{L^\infty}\int_{\mathbb{R}^d}|\nabla_x\partial_t\psi|^2 dx \\
&= -||\rho_\infty||_{L^\infty}\int_{\mathbb{R}^d}\partial_t\psi\Delta_x(\partial_t\psi)dx \\
&= -\frac{\sigma||\rho_\infty||_{L^\infty}}{\nu}\int_{\mathbb{R}^{2d}}\partial_t\psi\,\mathrm{div}_x(\nabla_v h f_\infty)dxdv \\
&= \frac{\sigma||\rho_\infty||_{L^\infty}}{\nu}\int_{\mathbb{R}^{2d}}\nabla_x\partial_t\psi\cdot\nabla_v h f_\infty dxdv \\
&\le \frac{1}{2}\int_{\mathbb{R}^{2d}}|\nabla_x\partial_t\psi|^2 f_\infty dxdv + \frac{\sigma^2||\rho_\infty||^2_{L^\infty}}{2\nu^2}\int_{\mathbb{R}^{2d}}|\nabla_v h|^2 f_\infty dxdv.
\end{aligned}
$$

By simplifying this inequality we get (3.41). □

Let $\gamma>0$. We consider a functional

$$
\begin{aligned}
&\mathrm{E}[h] := \gamma||h||^2 + \mathrm{S}_P[h] \\
&= \gamma\left[\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + \int_{\mathbb{R}^d}|\nabla_x\psi|^2 dx\right] + \int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x(h+\psi)\\ \nabla_v h\end{pmatrix}^T P\begin{pmatrix}\nabla_x(h+\psi)\\ \nabla_v h\end{pmatrix} f_\infty dxdv.
\end{aligned}
$$

It is clear that E depends on the parameter γ and the matrix P, we will fix them later. We show that E is equivalent to the H^1-norm.

Lemma 3.4.5. *Let V be bounded from below and $e^{-\frac{\nu}{\sigma}V} \in L^1(\mathbb{R}^d)$. Let p_1 and p_2 be the smallest and the largest eigenvalues of P, respectively. Then, for all $h \in H^1(\mathbb{R}^{2d}, f_\infty)$,*

$$\frac{\mathrm{E}[h]}{\max\left\{\gamma + \theta_1^2(\gamma + 2p_2||\rho_\infty||_{L^\infty}),\ 2p_2\right\}} \leq ||h||^2_{H^1(\mathbb{R}^{2d}, f_\infty)} \leq \frac{\mathrm{E}[h]}{\min\left\{\gamma,\ \frac{\gamma p_1}{\gamma + p_1||\rho_\infty||_{L^\infty}}\right\}}, \tag{3.42}$$

where θ_1 is the constants appearing in (3.30). Moreover, if $e^{-\frac{\nu}{\sigma}V}$ satisfies the Poincaré inequality (3.4) and $\int_{\mathbb{R}^{2d}} h f_\infty dxdv = 0$, then

$$\mathrm{S}_P[h] \leq \mathrm{E}[h] \leq \frac{p_1 + \gamma\kappa_2}{p_1}\mathrm{S}_P[h], \tag{3.43}$$

where κ_2 is the constants appearing in (3.17).

Proof. As P is positive definite, we have $0 < p_1 \leq p_2$. We estimate $\mathrm{E}[h]$ from above by using $P \leq p_2 I$:

$$\begin{aligned}
\mathrm{E}[h] \leq \gamma &\left[\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + \int_{\mathbb{R}^d} |\nabla_x \psi|^2 dx\right] \\
&+ p_2 \left[\int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv\right] \\
&\leq \gamma \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + (\gamma + 2p_2||\rho_\infty||_{L^\infty}) \int_{\mathbb{R}^d} |\nabla_x\psi|^2 dx \\
&+ 2p_2 \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dxdv + p_2 \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv.
\end{aligned}$$

We use (3.30) with $p = \frac{2d}{d+2}$ to get

$$\mathrm{E}[h] \leq \max\{\gamma + \theta_1^2(\gamma + 2p_2||\rho_\infty||_{L^\infty}), 2p_2\}||h||^2_{H^1(\mathbb{R}^{2d}, f_\infty)}. \tag{3.44}$$

We estimate $\mathrm{E}[h]$ from below by using $p_1 I \leq P$

$$\begin{aligned}
\mathrm{E}[h] \geq &\gamma \left[\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + \int_{\mathbb{R}^d} |\nabla_x\psi|^2 dx\right] \\
&+ p_1 \left[\int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv\right].
\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv &\geq \frac{\gamma}{\gamma + p_1||\rho_\infty||_{L^\infty}} \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dxdv \\
&\quad - \frac{\gamma}{p_1||\rho_\infty||_{L^\infty}} \int_{\mathbb{R}^{2d}} |\nabla_x\psi|^2 f_\infty dxdv \\
&\geq \frac{\gamma}{\gamma + p_1|\rho_\infty||_{L^\infty}} \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dxdv - \frac{\gamma}{p_1} \int_{\mathbb{R}^d} |\nabla_x\psi|^2 dx.
\end{aligned}$$

Using the last two estimates

$$\begin{aligned}
\mathrm{E}[h] &\geq \gamma \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + \frac{\gamma p_1}{\gamma + p_1||\rho_\infty||_{L^\infty}} \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dxdv + p_1 \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv \\
&\geq \min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1||\rho_\infty||_{L^\infty}}\right\} \left[\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv\right] \\
&= \min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1||\rho_\infty||_{L^\infty}}\right\} ||h||^2_{H^1(\mathbb{R}^{2d}, f_\infty)}.
\end{aligned} \tag{3.45}$$

(3.44) and (3.45) provide (3.42).

We now prove (3.43). The definition of E implies $\mathrm{S}_P[h] \leq \mathrm{E}[h]$. Since $P \geq p_1 I$, we have

$$\mathrm{S}_P[h] \geq p_1 \left[\int_{\mathbb{R}^{2d}} |\nabla_x (h+\psi)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v (h+\psi)|^2 f_\infty dxdv \right].$$

Using the Poincaré inequality (3.17) and $\int_{\mathbb{R}^{2d}} h f_\infty dxdv = 0$

$$\mathrm{S}_P(h) \geq \frac{p_1}{\kappa_2} \left[\int_{\mathbb{R}^{2d}} (h+\psi)^2 f_\infty dxdv - \left(\int_{\mathbb{R}^{2d}} \psi f_\infty dxdv \right)^2 \right].$$

The Hölder inequality and $-\Delta_x \psi = \int_{\mathbb{R}^d} h f_\infty dv$ imply

$$\begin{aligned} \int_{\mathbb{R}^{2d}} (h+\psi)^2 f_\infty dxdv - \left(\int_{\mathbb{R}^{2d}} \psi f_\infty dxdv \right)^2 &\geq \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + 2 \int_{\mathbb{R}^{2d}} h\psi f_\infty dxdv \\ &= \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv - 2 \int_{\mathbb{R}^d} \psi \Delta_x \psi dx \\ &= \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdv + 2 \int_{\mathbb{R}^d} |\nabla_x \psi|^2 dx \geq ||h||^2. \end{aligned}$$

Thus, $\mathrm{S}_P[h] \geq \frac{p_1}{\kappa_2} ||h||^2$ and (3.43) follows. □

We now prove the main result of this subsection.

Theorem 3.4.6. *Let h be the solution of (3.7) with an initial data $h_0 \in H^1(\mathbb{R}^{2d}, f_\infty)$ such that $\int_{\mathbb{R}^{2d}} h_0 f_\infty dxdv = 0$. Let the assumptions (A1), (A2) and (A3) hold. Then, there exist a positive constant γ and a constant, symmetric, positive definite matrix P such that*

$$\frac{d}{dt} \mathrm{E}[h(t)] \leq -2\lambda \mathrm{E}[h(t)], \quad t \geq 0 \tag{3.46}$$

holds for some $\lambda > 0$ depending γ and P. In particular,

$$\mathrm{E}[h(t)] \leq e^{-2\lambda t} \mathrm{E}[h_0], \quad t \geq 0.$$

Proof. Lemma 3.4.2 and Lemma 3.4.3 show that

$$\begin{aligned} \frac{d}{dt} \mathrm{E}[h(t)] = -2\sigma\gamma \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv \\ - 2\sigma \int_{\mathbb{R}^{2d}} \left\{ \sum_{i=1}^d \begin{pmatrix} \nabla_x (\partial_{v_i} h) \\ \nabla_v (\partial_{v_i} h) \end{pmatrix}^T P \begin{pmatrix} \nabla_x (\partial_{v_i} h) \\ \nabla_v (\partial_{v_i} h) \end{pmatrix} \right\} f_\infty dxdv \\ - \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x (h+\psi) \\ \nabla_v h \end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix} \nabla_x (h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv \\ - 2 \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x (h+\psi) \\ \nabla_v h \end{pmatrix}^T P \begin{pmatrix} \partial_t \nabla_x \psi \\ 0 \end{pmatrix} f_\infty dxdv. \end{aligned} \tag{3.47}$$

We choose the matrix $P := \begin{pmatrix} \varepsilon^3 I & \varepsilon^2 I \\ \varepsilon^2 I & 2\varepsilon I \end{pmatrix}$ with $\varepsilon > 0$ which will be fixed later. It is easy to check that P is positive definite. We denote $W := V + \phi_\infty$. Then, we compute

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv \\
&= \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T \begin{pmatrix} -2\varepsilon^2 I & \varepsilon^3 \frac{\partial^2 W}{\partial x^2} - (\nu\varepsilon^2 + 2\varepsilon) I \\ \varepsilon^3 \frac{\partial^2 W}{\partial x^2} - (\nu\varepsilon^2 + 2\varepsilon) I & 2\varepsilon^2 \frac{\partial^2 W}{\partial x^2} - 4\nu\varepsilon I \end{pmatrix} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv \\
&= -2\varepsilon^2 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + 2\int_{\mathbb{R}^{2d}} \nabla_x^T(h+\psi) \left(\varepsilon^3 \frac{\partial^2 W}{\partial x^2} - (\nu\varepsilon^2 + 2\varepsilon) I\right) \nabla_v h f_\infty dxdv \\
&\qquad + \int_{\mathbb{R}^{2d}} \nabla_v^T h \left(2\varepsilon^2 \frac{\partial^2 W}{\partial x^2} - 4\nu\varepsilon I\right) \nabla_v h f_\infty dxdv. \quad (3.48)
\end{aligned}
$$

We work on the terms which contain $\frac{\partial^2 W}{\partial x^2}$. We use the Hölder inequality and (3.18) to get

$$
\begin{aligned}
2\varepsilon^3 \int_{\mathbb{R}^{2d}} \nabla_x^T(h+\psi) \frac{\partial^2 W}{\partial x^2} \nabla_v h f_\infty dxdv &\le 2\varepsilon^3 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)| \left|\left|\frac{\partial^2 W}{\partial x^2}\right|\right|_F |\nabla_v h| f_\infty dxdv \\
&\le \varepsilon^2 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \varepsilon^4 \int_{\mathbb{R}^{2d}} \left|\left|\frac{\partial^2 W}{\partial x^2}\right|\right|_F^2 |\nabla_v h|^2 f_\infty dxdv \\
\le \varepsilon^2 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \varepsilon^4 \kappa_3 \int_{\mathbb{R}^{2d}} &|\nabla_v h|^2 f_\infty dxdv + \varepsilon^4 \kappa_3 \sum_{i=1}^d \int_{\mathbb{R}^{2d}} |\nabla_x(\partial_{v_i} h)|^2 f_\infty dxdv.
\end{aligned} \quad (3.49)
$$

Similarly,

$$
\begin{aligned}
2\varepsilon^2 \int_{\mathbb{R}^{2d}} \nabla_v^T h \frac{\partial^2 W}{\partial x^2} \nabla_v h f_\infty dxdv &\le 2\varepsilon^2 \int_{\mathbb{R}^{2d}} |\nabla_v h| \left|\left|\frac{\partial^2 W}{\partial x^2}\right|\right|_F |\nabla_v h| f_\infty dxdv \\
&\le \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv + \varepsilon^4 \int_{\mathbb{R}^{2d}} \left|\left|\frac{\partial^2 W}{\partial x^2}\right|\right|_F^2 |\nabla_v h|^2 f_\infty dxdv \\
&\le (1 + \varepsilon^4 \kappa_3) \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv + \varepsilon^4 \kappa_3 \sum_{i=1}^d \int_{\mathbb{R}^{2d}} |\nabla_x(\partial_{v_i} h)|^2 f_\infty dxdv. \quad (3.50)
\end{aligned}
$$

Combining (3.48), (3.49) and (3.50) we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv \\
&\qquad \le 2\varepsilon^4 \kappa_3 \sum_{i=1}^d \int_{\mathbb{R}^{2d}} |\nabla_x(\partial_{v_i} h)|^2 f_\infty dxdv \\
&+ \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T \begin{pmatrix} -\varepsilon^2 I & -(\nu\varepsilon^2 + 2\varepsilon) I \\ -(\nu\varepsilon^2 + 2\varepsilon) I & (1 - 4\nu\varepsilon + 2\varepsilon^4 \kappa_3) I \end{pmatrix} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv.
\end{aligned} \quad (3.51)
$$

Next, we estimate the last term of (3.47):

$$
\begin{aligned}
&-2\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T P \begin{pmatrix} \nabla_x \partial_t \psi \\ 0 \end{pmatrix} f_\infty dxdv \\
&\qquad = -2\varepsilon^3 \int_{\mathbb{R}^{2d}} \nabla_x(h+\psi)\cdot \nabla_x\partial_t\psi f_\infty dxdv - 2\varepsilon^2 \int_{\mathbb{R}^{2d}} \nabla_v h \cdot \nabla_x \partial_t \psi f_\infty dxdv \\
&\qquad\qquad \le \varepsilon^4 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \varepsilon^2 \int_{\mathbb{R}^{2d}} |\nabla_x\partial_t\psi|^2 f_\infty dxdv \\
&\qquad\qquad\quad + 2\varepsilon^2 \sqrt{\int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv}\sqrt{\int_{\mathbb{R}^{2d}} |\nabla_x \partial_t\psi|^2 f_\infty dxdv} \\
&\le \varepsilon^4 \int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dxdv + \varepsilon^2 \left(\frac{\sigma^2 ||\rho_\infty||^2_{L^\infty}}{\nu^2} + 2\frac{\sigma ||\rho_\infty||_{L^\infty}}{\nu} \right) \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dxdv,
\end{aligned}
\tag{3.52}
$$

where we used the Hölder inequality and (3.41). We gather (3.51) and (3.52) to estimate (3.47)

$$
\begin{aligned}
\frac{d}{dt}\mathrm{E}[h(t)] \le -2\sigma \int_{\mathbb{R}^{2d}} &\left\{ \sum_{i=1}^d \begin{pmatrix} \nabla_x(\partial_{v_i} h) \\ \nabla_v(\partial_{v_i} h) \end{pmatrix}^T \begin{pmatrix} (\varepsilon^3 - \frac{\varepsilon^4 \kappa_3}{\sigma}) I & \varepsilon^2 I \\ \varepsilon^2 I & 2\varepsilon I \end{pmatrix} \begin{pmatrix} \nabla_x(\partial_{v_i} h) \\ \nabla_v(\partial_{v_i} h) \end{pmatrix} \right\} f_\infty dxdv \\
&- \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T P_1 \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dxdv,
\end{aligned}
\tag{3.53}
$$

where

$$
P_1 := \begin{pmatrix} (\varepsilon^2 - \varepsilon^4) I & (\nu\varepsilon^2 + 2\varepsilon) I \\ (\nu\varepsilon^2 + 2\varepsilon) I & (2\gamma\sigma - 1 + 4\nu\varepsilon - \varepsilon^2 (\frac{\sigma^2 ||\rho_\infty||^2_{L^\infty}}{\nu^2} + \frac{2\sigma ||\rho_\infty||_{L^\infty}}{\nu}) - 2\varepsilon^4 \kappa_3) I \end{pmatrix}.
$$

We choose $\gamma > 0$ and $\varepsilon > 0$ such that the matrices in the first and second lines of (3.53) satisfy

$$
\begin{pmatrix} (\varepsilon^3 - \frac{\varepsilon^4 \kappa_3}{\sigma}) I & \varepsilon^2 I \\ \varepsilon^2 I & 2\varepsilon I \end{pmatrix} \ge 0 \quad \text{and} \quad P_1 > 0.
$$

It is possible to choose such γ and ε, for example, if ε is small enough and γ is large enough, then the conditions above are satisfied. Moreover, there is $\tilde{\lambda} = \tilde{\lambda}(\varepsilon, \gamma) > 0$ such that

$$
P_1 \ge 2\tilde{\lambda} P.
$$

Using this estimate in (3.53) we get

$$
\frac{d}{dt}\mathrm{E}[h(t)] \le -2\tilde{\lambda}\mathrm{S}_P[h(t)]. \tag{3.54}
$$

By using (3.43)

$$
\frac{d}{dt}\mathrm{E}[h(t)] \le -2\lambda \mathrm{E}[h(t)] \quad \text{for all } t \ge 0
$$

with $\lambda := \tilde{\lambda}\dfrac{p_1}{p_1 + \gamma\kappa_2}$. Then, the Grönwall inequality implies

$$
\mathrm{E}[h(t)] \le e^{-2\lambda t}\mathrm{E}[h_0].
$$

□

Remark 3.4.7.

1. We note that γ and P such that Theorem 3.4.6 holds are not unique. But the decay rate λ depends on γ and P. To get a better rate, one has to optimize $\lambda = \lambda(\gamma, P)$ with respect to γ and P.

2. The Poincaré inequality is essential to get the inequality (3.46) and so the exponential decay. When the Poincaré inequality is not valid, we only get the bound $\mathrm{E}[h(t)] \leq \mathrm{E}[h_0]$ by (3.54).

3.4.3 Hypoelliptic regularity

In this section we show that, for any initial data $h_0 \in L^2(\mathbb{R}^{2d}, f_\infty)$, the solution $h(t)$ of the linearized equation is in $H^1(\mathbb{R}^{2d}, f_\infty)$ for all $t > 0$.

Theorem 3.4.8. *Let h be the solution of (3.7) with an initial data $h_0 \in L^2(\mathbb{R}^{2d}, f_\infty)$. Under the assumptions (A1) and (A2), for any $t_0 > 0$, there are explicitly computable constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\int_{\mathbb{R}^{2d}} |\nabla_x h(t)|^2 f_\infty dx dv \leq \frac{C_1}{t^3} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv \tag{3.55}$$

and

$$\int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dx dv \leq \frac{C_2}{t} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv \tag{3.56}$$

hold for all $t \in (0, t_0]$.

Proof. In order to prove the short-time regularization of (3.55) and (3.56) we consider now the functional E with a matrix P which depends explicitly on time t, i.e.

$$P = P(t) := \begin{pmatrix} \varepsilon^3 t^3 I & \varepsilon^2 t^2 I \\ \varepsilon^2 t^2 I & 2\varepsilon t I \end{pmatrix}.$$

We shall fix $\varepsilon > 0$ later. It is easy to check

$$P(t) \geq \begin{pmatrix} \frac{\varepsilon^3 t^3}{3} I & 0 \\ 0 & \frac{\varepsilon t}{2} I \end{pmatrix} \tag{3.57}$$

which implies that $P(t)$ is positive definite for all $t > 0$. Our goal is to show that $\mathrm{E}[h(t)]$ decreases. To this end we compute the time derivative of $\mathrm{E}[h(t)]$. We follow the proofs of Lemma 3.4.2 and Lemma 3.4.3 to compute the time derivative of E, but we need to take into account that P depends on time t :

$$\begin{aligned} \frac{d}{dt}\mathrm{E}[h(t)] = &-2\sigma\gamma \int_{\mathbb{R}^{2d}} |\nabla_v h|^2 f_\infty dx dv \\ &- 2\sigma \int_{\mathbb{R}^{2d}} \left\{ \sum_{i=1}^d \begin{pmatrix} \nabla_x(\partial_{v_i} h) \\ \nabla_v(\partial_{v_i} h) \end{pmatrix}^T P \begin{pmatrix} \nabla_x(\partial_{v_i} h) \\ \nabla_v(\partial_{v_i} h) \end{pmatrix} \right\} f_\infty dx dv \\ &- \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dx dv \\ &- 2\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T P \begin{pmatrix} \partial_t \nabla_x \psi \\ 0 \end{pmatrix} f_\infty dx dv + \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_v h \end{pmatrix}^T \partial_t P \begin{pmatrix} \nabla_x h \\ \nabla_v h \end{pmatrix} f_\infty dx dv. \end{aligned}$$

We estimate the terms on the right as (3.49)-(3.52) (where we need to replace ε to εt) and obtain

$$\frac{d}{dt}\mathrm{E}[h(t)] \leq -2\sigma \int_{\mathbb{R}^{2d}} \left\{ \sum_{i=1}^{d} \begin{pmatrix} \nabla_x(\partial_{v_i}h) \\ \nabla_v(\partial_{v_i}h) \end{pmatrix}^T \begin{pmatrix} (\varepsilon^3 t^3 - \frac{\varepsilon^4 t^4 \kappa_3}{\sigma})I & \varepsilon^2 t^2 I \\ \varepsilon^2 t^2 I & 2\varepsilon t I \end{pmatrix} \begin{pmatrix} \nabla_x(\partial_{v_i}h) \\ \nabla_v(\partial_{v_i}h) \end{pmatrix} \right\} f_\infty dx dv$$
$$- \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix}^T [P_1 - \partial_t P] \begin{pmatrix} \nabla_x(h+\psi) \\ \nabla_v h \end{pmatrix} f_\infty dx dv, \quad (3.58)$$

where

$$P_1 := \begin{pmatrix} (\varepsilon^2 t^2 - \varepsilon^4 t^4)I & (\nu\varepsilon^2 t^2 + 2\varepsilon t)I \\ (\nu\varepsilon^2 t^2 + 2\varepsilon t)I & \left(2\gamma\sigma - 1 + 4\nu\varepsilon t - \varepsilon^2 t^2 (\frac{||\rho_\infty||^2_{L^\infty}\sigma^2}{\nu^2} + \frac{2||\rho_\infty||_{L^\infty}\sigma}{\nu}) - 2\varepsilon^4 t^4 \kappa_3\right) I \end{pmatrix}.$$

Since $\partial_t P = \begin{pmatrix} 3\varepsilon^3 t^2 I & 2\varepsilon^2 t I \\ 2\varepsilon^2 t I & 2\varepsilon I \end{pmatrix}$, we have

$$P_1 - \partial_t P$$
$$= \begin{pmatrix} ([\varepsilon^2 - 3\varepsilon^3]t^2 - \varepsilon^4 t^4)I & (\nu\varepsilon^2 t^2 + 2[\varepsilon - \varepsilon^2]t)I \\ (\nu\varepsilon^2 t^2 + 2[\varepsilon - \varepsilon^2]t)I & \left(2\gamma\sigma - 1 - 2\varepsilon + 4\nu\varepsilon t - \varepsilon^2 t^2 (\frac{||\rho_\infty||^2_{L^\infty}\sigma^2}{\nu^2} + \frac{2||\rho_\infty||_{L^\infty}\sigma}{\nu}) - 2\varepsilon^4 t^4 \kappa_3\right) I \end{pmatrix}.$$

We choose γ and ε such that, for all $t \in [0, t_0]$, the matrices in the first and the second lines of (3.58) are positive semi-definite, i.e.

$$\begin{pmatrix} (\varepsilon^3 t^3 - \frac{\varepsilon^4 t^4 \kappa_3}{\sigma})I & \varepsilon^2 t^2 I \\ \varepsilon^2 t^2 I & 2\varepsilon t I \end{pmatrix} \geq 0, \quad P_1 - \partial_t P \geq 0.$$

It is possible to choose such γ and ε, for example, if γ is large and ε is small enough, then these inequalities hold. Then we get

$$\frac{d}{dt}\mathrm{E}[h(t)] \leq 0, \quad t \in (0, t_0].$$

This yields that $\mathrm{E}[h(t)]$ is decreasing in $[0, t_0]$. $\mathrm{E}[h(t=0)] = \gamma||h_0||^2$ and (3.30) with $p = \frac{2d}{d+2}$ show

$$\mathrm{E}[h(t)] \leq \gamma||h_0||^2 \leq \gamma(1+\theta_1^2) \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv, \quad \forall t \in [0, t_0]. \quad (3.59)$$

On the other hand, we have by (3.57) that

$$\mathrm{E}[h(t)] \geq \gamma \int_{\mathbb{R}^{2d}} h^2(t) f_\infty dx dv + \gamma \int_{\mathbb{R}^{2d}} |\nabla_x \psi(t)|^2 dx$$
$$+ \frac{\varepsilon^3 t^3}{3} \int_{\mathbb{R}^{2d}} |\nabla_x h(t) + \nabla_x \psi(t)|^2 f_\infty dx dv + \frac{\varepsilon t}{2} \int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dx dv.$$

If we use the estimate

$$\int_{\mathbb{R}^{2d}} |\nabla_x(h+\psi)|^2 f_\infty dx dv \geq \frac{1}{2} \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dx dv - \int_{\mathbb{R}^{2d}} |\nabla_x \psi|^2 f_\infty dx dv$$
$$\geq \frac{1}{2} \int_{\mathbb{R}^{2d}} |\nabla_x h|^2 f_\infty dx dv - ||\rho_\infty||_{L^\infty} \int_{\mathbb{R}^d} |\nabla_x \psi|^2 dv,$$

we get

$$\mathrm{E}[h(t)] \geq \gamma \int_{\mathbb{R}^{2d}} |h(t)|^2 f_\infty dxdv + \left(\gamma - \frac{||\rho_\infty||_{L^\infty}\varepsilon^3 t^3}{3}\right) \int_{\mathbb{R}^{2d}} |\nabla_x \psi(t)|^2 dx$$
$$+ \frac{\varepsilon^3 t^3}{6} \int_{\mathbb{R}^{2d}} |\nabla_x h(t)|^2 f_\infty dxdv + \frac{\varepsilon t}{2} \int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dxdv.$$

If we take γ large enough so that $\gamma - \frac{||\rho_\infty||_{L^\infty}\varepsilon^3 t_0^3}{3} \geq 0$, then

$$\mathrm{E}[h(t)] \geq \frac{\varepsilon^3 t^3}{6} \int_{\mathbb{R}^{2d}} |\nabla_x h(t)|^2 f_\infty dxdv + \frac{\varepsilon t}{2} \int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dxdv, \qquad \forall t \in (0, t_0]. \quad (3.60)$$

(3.59) and (3.60) show that the statement of the theorem holds with constants $C_1 := \frac{6\gamma(1+\theta_1^2)}{\varepsilon^3}$ and $C_2 := \frac{2\gamma(1+\theta_1^2)}{\varepsilon}$. □

Now we are ready to prove Theorem 3.2.2 concerning the linearized Vlasov-Poissson-Fokker-Planck equation.

Proof of Theorem 3.2.2. The proofs of Theorem 3.2.2 (i) and Theorem 3.2.2 (ii) follow from Theorem 3.4.1 and Theorem 3.4.8.

We use Theorem 3.4.6 to prove Theorem 3.2.2 (iii). Let E be the functional in Theorem 3.4.6. Then, (3.46) can be written as

$$\frac{d}{dt}\left(e^{2\lambda t}\mathrm{E}[h(t)]\right) \leq 0.$$

For any $t > t_0$, we integrate this inequality in $[t_0, t]$ to get

$$\mathrm{E}[h(t)] \leq e^{-2\lambda(t-t_0)}\mathrm{E}[h(t_0)]. \quad (3.61)$$

By (3.43) we have

$$\mathrm{E}[h(t_0)] \leq \frac{p_1 + \gamma\kappa_2}{p_1}\mathrm{S}_P[h(t_0)]$$
$$\leq \frac{p_2(p_1+\gamma\kappa_2)}{p_1}\left[\int_{\mathbb{R}^{2d}} |\nabla_x h(t_0) + \nabla_x\psi(t_0)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h(t_0)|^2 f_\infty dxdv\right]$$
$$\leq \frac{p_2(p_1+\gamma\kappa_2)}{p_1}\left[2\int_{\mathbb{R}^{2d}} |\nabla_x h(t_0)|^2 f_\infty + \int_{\mathbb{R}^{2d}} |\nabla_v h(t_0)|^2 f_\infty dxdv\right]$$
$$+2||\rho_\infty||_{L^\infty}\frac{p_2(p_1+\gamma\kappa_2)}{p_1}\int_{\mathbb{R}^{2d}} |\nabla_x\psi(t_0)|^2 dxdv,$$

where p_1 and p_2 are the smallest and the largest eigenvalues of the matrix P which we defined in the proof of Theorem 3.4.6. Because of (3.30) with $p = \frac{2d}{d+2}$ and the Poincaré inequality (3.17), the integral $\int_{\mathbb{R}^d} |\nabla_x\psi(t_0)|^2 dx$ is bounded by $\int_{\mathbb{R}^{2d}} |\nabla_x h(t_0)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h(t_0)|^2 f_\infty dxdv$. Thus, there is a constant $C > 0$ such that

$$\mathrm{E}[h(t_0)] \leq C\left[\int_{\mathbb{R}^{2d}} |\nabla_x h(t_0)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h(t_0)|^2 f_\infty dxdv\right],$$

and by Theorem 3.4.8

$$\mathrm{E}[h(t_0)] \leq C\max\{C_1 t_0^{-3}, C_2 t_0^{-1}\}\int_{\mathbb{R}^{2d}} h_0^2 f_\infty dxdv. \quad (3.62)$$

If we combine (3.61), (3.62) and (3.42), we obtain

$$\int_{\mathbb{R}^{2d}} h^2(t) f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_x h(t)|^2 f_\infty dxdv + \int_{\mathbb{R}^{2d}} |\nabla_v h(t)|^2 f_\infty dxdv$$
$$\leq \frac{\mathrm{E}[h(t)]}{\min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1 ||\rho_\infty||_{L^\infty}}\right\}} \leq \frac{e^{-2\lambda(t-t_0)}\mathrm{E}[h(t_0)]}{\min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1 ||\rho_\infty||_{L^\infty}}\right\}}$$
$$\leq \frac{Ce^{2\lambda t_0}\max\{C_1 t_0^{-3}, C_2 t_0^{-1}\}}{\min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1 ||\rho_\infty||_{L^\infty}}\right\}} e^{-2\lambda t} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dxdv.$$

This proves (3.10) with the constant $C_3 := \sqrt{\frac{Ce^{2\lambda t_0}\max\{C_1 t_0^{-3}, C_2 t_0^{-1}\}}{\min\left\{\gamma, \frac{\gamma p_1}{\gamma + p_1 ||\rho_\infty||_{L^\infty}}\right\}}}$.

We now prove (3.11). We have from (3.30)

$$||\nabla_x \psi(t)||_{L^{\frac{pd}{d-p}}(\mathbb{R}^d)} \leq \theta_1 ||h(t)||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq \theta_1 ||h(t)||_{H^1(\mathbb{R}^{2d}, f_\infty)}, \quad \forall p \in (1,2].$$

The relation $\mathscr{L}_1^{\frac{2d}{d-2}}(\mathbb{R}^d) = W^{1,\frac{2d}{d-2}}(\mathbb{R}^d)$ (see [1]) and (3.31) with $\alpha = 1$ show

$$||\nabla_x \psi(t)||_{W^{1,\frac{2d}{d-2}}(\mathbb{R}^d)} \leq \theta_2 ||h(t)||_{H^1_x(\mathbb{R}^{2d}, f_\infty)} \leq \theta_2 ||h(t)||_{H^1(\mathbb{R}^{2d}, f_\infty)}$$

for all $t > 0$. These estimates and (3.10) imply

$$||\nabla_x \psi(t)||_{L^{\frac{pd}{d-p}}(\mathbb{R}^d)} + ||\nabla_x \psi(t)||_{W^{1,\frac{2d}{d-2}}(\mathbb{R}^d)} \leq (\theta_1 + \theta_2)||h(t)||_{H^1(\mathbb{R}^{2d}, f_\infty)}$$
$$\leq C_3(\theta_1 + \theta_2) e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

for all $t \geq t_0$. This proves (3.11) with the constant $C_4 := C_3(\theta_1 + \theta_2)$.

□

3.5 The nonlinear Vlasov-Poisson-Fokker-Planck system in 3D

In this section we work on the nonlinear Vlasov-Poisson-Fokker-Planck system (3.6) in dimension $d = 3$. We mention that we get (3.6) from (3.1) by taking $h := \frac{f}{f_\infty} - 1$ and $\psi := \phi - \phi_\infty$. To prove Theorem 3.2.4 and Theorem 3.2.5, we will consider the integral version of the system (3.12), and then apply a fixed point argument to find a candidate for a solution, then prove existence, uniqueness and stability.

3.5.1 Semigroup estimates

We recall that the linearized system (3.7) can be written as

$$\partial_t h + Kh = 0$$

with the operator $Kh := v \cdot \nabla_x h - \nabla_x(V + \phi_\infty) \cdot \nabla_v h + v \cdot \nabla_x \psi - \sigma \Delta_v h + \nu v \cdot \nabla_v h$. By Theorem 3.4.1 K generates a C_0 semigroup e^{tK} on $L^2(\mathbb{R}^{2d}, f_\infty)$. We define a subspace of $L^2(\mathbb{R}^{2d}, f_\infty)$

$$\mathcal{H} := \{g \in L^2(\mathbb{R}^{2d}, f_\infty) : \int_{\mathbb{R}^{2d}} g f_\infty dxdv = 0\}.$$

Here the norm of $\mathcal{H}$ is the norm of $L^2(\mathbb{R}^{2d}, f_\infty)$. Let $h_0 \in \mathcal{H}$, then $h(t) = e^{-tK}h_0,\ t > 0$ is the solution of (3.7) and it is in $H^1(\mathbb{R}^{2d}, f_\infty)$ by Theorem 3.2.2 (ii). By integrating (3.7), we obtain

$$\int_{\mathbb{R}^{2d}} e^{-tK} h_0 f_\infty dx dv = \int_{\mathbb{R}^{2d}} h_0 f_\infty dx dv = 0.$$

Therefore, e^{-tK} maps $\mathcal{H}$ into $\mathcal{H} \cap H^1(\mathbb{R}^{2d}, f_\infty)$ for all $t > 0$.

We will need the following estimates on e^{-tK}.

Lemma 3.5.1. *Let $d \geq 3$, $\alpha \in [0,1]$, $t_0 > 0$, the assumption (A1) and (A2) hold. There are positive constants $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$ and $\mathscr{C}_5$ such that*

(i)

$$||e^{-tK}h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_1 ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \ \ \forall t \in [0, t_0], \ \ \forall h_0 \in L^2(\mathbb{R}^{2d}, f_\infty). \quad (3.63)$$

(ii)

$$||e^{-tK}h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_2 ||h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)}, \ \ \forall t \in [0, t_0], \ \ \forall h_0 \in H^\alpha_x(\mathbb{R}^{2d}, f_\infty). \quad (3.64)$$

(iii)

$$||e^{-tK}h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_3 (1 + t^{-\frac{3\alpha}{2}}) ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \ \ \forall t \in (0, t_0], \quad \forall h_0 \in L^2(\mathbb{R}^{2d}, f_\infty). \quad (3.65)$$

(iv)

$$||e^{-tK}h_0||_{H^1_v(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_4 ||h_0||_{H^1_v(\mathbb{R}^{2d}, f_\infty)}, \ \ \forall t \in [0, t_0], \ \ \forall h_0 \in H^1_v(\mathbb{R}^{2d}, f_\infty). \quad (3.66)$$

(v)

$$||e^{-tK}h_0||_{H^1_v(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_5 (1 + t^{-\frac{1}{2}}) ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \ \ \forall t \in (0, t_0], \quad \forall h_0 \in L^2(\mathbb{R}^{2d}, f_\infty). \quad (3.67)$$

Proof. (i) Lemma 4.4.1 shows that

$$||e^{-tK}h_0|| \leq ||h_0||, \quad t \geq 0.$$

The inequality $||e^{-tK}h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq ||e^{-tK}h_0||$ and (3.30) with $p = \frac{2d}{d-2}$ yield

$$||e^{-tK}h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq \sqrt{1 + \theta_1^2} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad t \geq 0. \quad (3.68)$$

Thus, (3.63) holds with the constant $\mathscr{C}_1 := \sqrt{1 + \theta_1^2}$.

(ii) Since e^{-tK} generates a C_0 semigroup on $L^2(\mathbb{R}^{2d}, f_\infty)$ and $e^{-tK}h_0 \in H^1(\mathbb{R}^{2d}, f_\infty)$ for all $t > 0$, (see Theorem 3.2.2 (ii)), we conclude e^{-tK} also generates a C_0 semigroup on $H^\alpha_x(\mathbb{R}^{2d}, f_\infty)$ (see [4, Theorem 0.1]). By the semigroup property [44, Theorem 1.2.2] there exist constants $\omega \geq 0$ and $C \geq 1$ such that

$$||e^{-tK}h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq C e^{\omega t} ||h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)}$$

for all $t \geq 0$. This implies that (3.64) holds with $\mathscr{C}_2 := C e^{\omega t_0}$.

(iii) (3.65) coincides with (3.63) when $\alpha = 0$. We prove (3.65) when $\alpha = 1$. By Theorem 3.2.2 (ii) we have, for all $h_0 \in \mathcal{H}$,

$$\int_{\mathbb{R}^{2d}} |\nabla_x e^{-tK} h_0|^2 f_\infty dx dv \le \frac{C_1}{t^3} \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv, \quad t \in (0, t_0].$$

We add the square of (3.68) to this estimate to get

$$\int_{\mathbb{R}^{2d}} |e^{-tK} h_0|^2 f_\infty dx dv + \int_{\mathbb{R}^{2d}} |\nabla_x e^{-tK} h_0|^2 f_\infty dx dv \le \left(1 + \theta_1^2 + \frac{C_1}{t^3}\right) \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv$$
$$\le \max\{1 + \theta_1^2, C_1\}(1 + t^{-\frac{3}{2}})^2 \int_{\mathbb{R}^{2d}} h_0^2 f_\infty dx dv, \quad \forall t \in (0, t_0].$$

Then by (3.22)

$$||e^{-tK} h_0||_{H_x^1(\mathbb{R}^{2d}, f_\infty)} \le \kappa_5 \sqrt{\max\{1 + \theta_1^2, C_1\}} (1 + t^{-\frac{3}{2}}) ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

This proves (3.65) when $\alpha = 1$. The complete proof follows by interpolation.

The proofs of (iv) and (v) follow by similar arguments which we did in (ii) and (iii). $\square$

Lemma 3.5.2. *Let $d \ge 3$, the assumption (A1), (A2) and (A3) hold. Let $\alpha \in [0, 1]$, $\lambda > 0$ be the constant appearing in Theorem 3.2.2 (iii) and $\lambda_1 \in (0, \lambda)$. There are positive constants $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ such that*

(i)

$$||e^{-tK} h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \le \mathcal{C}_1 e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t \ge 0, \quad \forall h_0 \in \mathcal{H}. \tag{3.69}$$

(ii)

$$||e^{-tK} h_0||_{H_x^\alpha(\mathbb{R}^{2d}, f_\infty)} \le \mathcal{C}_2 e^{-\lambda t} ||h_0||_{H_x^\alpha(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t \ge 0, \quad \forall h_0 \in \mathcal{H} \cap H_x^\alpha(\mathbb{R}^{2d}, f_\infty). \tag{3.70}$$

(iii)

$$||e^{-tK} h_0||_{H_x^\alpha(\mathbb{R}^{2d}, f_\infty)} \le \mathcal{C}_3 (1 + t^{-\frac{3\alpha}{2}}) e^{-\lambda_1 t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t > 0, \quad \forall h_0 \in \mathcal{H}. \tag{3.71}$$

(iv)

$$||e^{-tK} h_0||_{H_v^1(\mathbb{R}^{2d}, f_\infty)} \le \mathcal{C}_4 e^{-\lambda t} ||h_0||_{H_v^1(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t \ge 0, \quad \forall h_0 \in \mathcal{H} \cap H_v^1(\mathbb{R}^{2d}, f_\infty). \tag{3.72}$$

(v)

$$||e^{-tK} h_0||_{H_v^1(\mathbb{R}^{2d}, f_\infty)} \le \mathcal{C}_5 (1 + t^{-\frac{1}{2}}) e^{-\lambda_1 t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t > 0, \quad \forall h_0 \in \mathcal{H}. \tag{3.73}$$

Proof. (i) Let $t_0 > 0$. Theorem 3.2.2 (iii) implies, for all $h_0 \in \mathcal{H}$,

$$||e^{-tK} h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \le ||e^{-tK} h_0||_{H^1(\mathbb{R}^{2d}, f_\infty)} \le C_3 e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad t \ge t_0.$$

We combine this inequality and (3.63) to get

$$||e^{-tK} h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \le \max\{C_3, \mathscr{C}_1\} e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

for all $t \ge 0$. This inequality implies (3.69) with the constant $\mathcal{C}_1 := \max\{C_3, \mathscr{C}_1\}$.

(ii) (3.70) coincides with (3.69) when $\alpha = 0$. Let $\alpha = 1$. (3.64) lets us write for $t \in [0, t_0]$

$$||e^{-tK} h_0||_{H^1_x(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_2 e^{\lambda t_0} e^{-\lambda t} ||h_0||_{H^1_x(\mathbb{R}^{2d}, f_\infty)}.$$

For $t \geq t_0$, we use (3.22) and Theorem 3.2.2 (ii) to get

$$||e^{-tK} h_0||_{H^1_x(\mathbb{R}^{2d}, f_\infty)} \leq \kappa_5 ||e^{-tK} h_0||_{H^1(\mathbb{R}^{2d}, f_\infty)}$$

$$\leq \kappa_5 C_3 e^{-\lambda t} ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq \kappa_5 C_3 e^{-\lambda t} ||h_0||_{H^1_x(\mathbb{R}^{2d}, f_\infty)}.$$

The combination of these estimates imply (3.70) with $\mathcal{C}_2 := \max\{\mathscr{C}_2 e^{\lambda t_0}, \kappa_5 C_3\}$ when $\alpha = 1$. The case of $\alpha \in (0, 1)$ follows by interpolation.

(iii) By (3.65) we have

$$||e^{-tK} h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq \mathscr{C}_3 e^{\lambda_1 t_0} e^{-\lambda_1 t} (1 + t^{-\frac{3\alpha}{2}}) ||h_0||_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t \in (0, t_0]. \tag{3.74}$$

For $t \geq t_0$, we have by (3.70)

$$||e^{-tK} h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq \mathcal{C}_2 e^{-\lambda t} ||h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)}$$

Since $\lambda > \lambda_1 > 0$, there is a constant $C > 0$ depending t_0, λ_1 and λ such that

$$\mathcal{C}_2 e^{-\lambda t} \leq C(1 + t^{-\frac{3\alpha}{2}}) e^{-\lambda_1 t}, \quad \forall t \geq t_0.$$

Thus, we obtain

$$||e^{-tK} h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)} \leq C(1 + t^{-\frac{3\alpha}{2}}) e^{-\lambda_1 t} ||h_0||_{H^\alpha_x(\mathbb{R}^{2d}, f_\infty)}, \quad \forall t \geq t_0. \tag{3.75}$$

(3.74) and (3.75) show that (3.71) holds with the constant $\mathcal{C}_3 := \max\{C, \mathscr{C}_3 e^{\lambda_1 t_0}\}$.

The proofs of (iv) and (v) follow by similar arguments as we did in (iii) and (iv).

□

3.5.2 Local well-posedness

In this subsection we prove Theorem 3.2.4 i.e., the existence of a unique solution to (3.12) in a (possible short) time interval. We use ψ_h below to denote the solution of $-\Delta_x \psi = \int_{\mathbb{R}^d} h f_\infty dv$.

Proof of Theorem 3.2.4 . Let $t_0 > 0$ be a fixed constant as in Lemma 3.5.1. For a given $h_0 \in H^\alpha_x(\mathbb{R}^6, f_\infty) \cap H^1_v(\mathbb{R}^6, f_\infty)$ we define a mapping

$$F : C\left([0,\tau]; H^\alpha_x(\mathbb{R}^6, f_\infty)\right) \cap C\left([0,\tau]; H^1_v(\mathbb{R}^6, f_\infty)\right) \to C\left([0,\tau]; H^\alpha_x(\mathbb{R}^6, f_\infty)\right) \cap C\left([0,\tau]; H^1_v(\mathbb{R}^6, f_\infty)\right)$$

by

$$F[h] = e^{-tK} h_0 + \int_0^t e^{-(t-s)K} \left(\nabla_x \psi_h \cdot \nabla_v h - \frac{\nu}{\sigma} v \cdot \nabla_x \psi_h h\right) ds, \quad t \in [0, \tau],$$

where $\tau \in (0, t_0]$ will be fixed later. We want to show that F has a unique fixed point if τ is small enough.

We define

$$||h||_{\tau,1} := \sup_{t \in [0,\tau]} \{||h(t)||_{H^\alpha_x(\mathbb{R}^6, f_\infty)}\},$$

$$||h||_{\tau,2} := \sup_{t \in [0,\tau]} \{||h(t)||_{H^1_v(\mathbb{R}^6, f_\infty)}\}.$$

We use

$$||h||_\tau := \max\{||h||_{\tau,1}, ||h||_{\tau,2}\}$$

as a norm in $C\left([0,\tau]; H_x^\alpha(\mathbb{R}^6, f_\infty)\right) \cap C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$. (3.32) shows that $\nabla_x\psi_h(t)$ is bounded for all $t \geq 0$. Hence

$$\nabla_x\psi_h(t) \cdot \nabla_v h(t) - \frac{\nu}{\sigma} v \cdot \nabla_x\psi_h(t)h(t) \in L^2(\mathbb{R}^6, f_\infty)$$

for all $t \in [0,\tau]$. Using the Hölder inequality, (3.32) and (3.20)

$$\begin{aligned} &||\nabla_x\psi_h(t) \cdot \nabla_v h(t) - \nu/\sigma v \cdot \nabla_x\psi_h(t)h(t)||_{L^2(\mathbb{R}^6,f_\infty)} \\ &\qquad \leq \sqrt{2||\nabla_x\psi_h(t)||^2_{L^\infty} \int_{\mathbb{R}^6} \left(|\nabla_v h(t)|^2 + \nu^2/\sigma^2|v|^2|h^2(t)\right) f_\infty dx dv} \\ &\qquad\qquad \leq C_R ||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ||h(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \end{aligned} \tag{3.76}$$

for some constant $C_R > 0$. We estimate $F[h]$ in $C\left([0,\tau]; H_x^\alpha(\mathbb{R}^6, f_\infty)\right)$:

$$\begin{aligned} ||F[h(t)]||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} &\leq ||e^{-tK}h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \\ &\quad + \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v h - \nu/\sigma v \cdot \nabla_x\psi_h h\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ds. \end{aligned} \tag{3.77}$$

(3.65) and (3.76) let us estimate

$$\begin{aligned} &\int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v h - \nu/\sigma v \cdot \nabla_x\psi_h h\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ds \\ &\leq \mathscr{C}_3 \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}}) ||\nabla_x\psi_h(s) \cdot \nabla_v h(s) - \nu/\sigma v \cdot \nabla_x\psi_h(s)h(s)||_{L^2(\mathbb{R}^6,f_\infty)} ds \\ &\qquad \leq \mathscr{C}_3 C_R \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}}) ||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ||h(s)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds \\ &\qquad\qquad \leq \mathscr{C}_3 C_R ||h||_{\tau,1} ||h||_{\tau,2} \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}}) ds. \end{aligned} \tag{3.78}$$

Then, (3.64), (3.77) and (3.78) provide

$$||F[h]||_{\tau,1} \leq \mathscr{C}_2 ||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} + \mathscr{C}_3 C_R ||h||_{\tau,1} ||h||_{\tau,2} \sup_{t\in[0,\tau]} \left\{ \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}}) ds \right\}. \tag{3.79}$$

We estimate $F[h]$ in $C\left([0,\tau], H_v^1(\mathbb{R}^6, f_\infty)\right)$:

$$\begin{aligned} ||F[h(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} &\leq ||e^{-tK}h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} \\ &\quad + \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v h - \nu/\sigma v \cdot \nabla_x\psi_h h\right)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds. \end{aligned} \tag{3.80}$$

(3.67) and (3.76) let us estimate

$$\begin{aligned} &\int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h(s) \cdot \nabla_v h(s) - \nu/\sigma v \cdot \nabla_x\psi_h(s)h(s)\right)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds \\ &\leq \mathscr{C}_5 \int_0^t (1+(t-s)^{-\frac{1}{2}}) ||\nabla_x\psi_h(s) \cdot \nabla_v h(s) - \nu/\sigma v \cdot \nabla_x\psi_h(s)h(s)||_{L^2(\mathbb{R}^6,f_\infty)} ds \end{aligned}$$

$$\le \mathscr{C}_5 C_R \int_0^t (1+(t-s)^{-\frac{1}{2}})||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||h(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds$$
$$\le \mathscr{C}_5 C_R ||h||_{\tau,1}||h||_{\tau,2}(t+2\sqrt{t}), \tag{3.81}$$

where we used

$$\int_0^t (1+(t-s)^{-\frac{1}{2}})ds = t+2\sqrt{t}.$$

(3.80), (3.81) and (3.66) show

$$||F[h(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} \le \mathscr{C}_4 ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \mathscr{C}_5 C_R ||h||_1 ||h||_2 (t+2\sqrt{t}),$$

and so

$$||F[h]||_{\tau,2} \le \mathscr{C}_4 ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \mathcal{C}_5 C_R ||h||_{\tau,1}||h||_{\tau,2}(\tau+2\sqrt{\tau}). \tag{3.82}$$

Let $h,g \in C\left([0,\tau];H_x^\alpha(\mathbb{R}^6,f_\infty)\right) \cap C\left([0,\tau];H_v^1(\mathbb{R}^6,f_\infty)\right)$. We consider

$$F[h]-F[g] = \int_0^t e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v(h-g) - \nu/\sigma v\cdot\nabla_x\psi_h(h-g)\right)ds$$
$$+\int_0^t e^{-(t-s)K}\left((\nabla_x\psi_h-\nabla_x\psi_g)\cdot\nabla_v g - \nu/\sigma v\cdot(\nabla_x\psi_h-\nabla_x\psi_g)g\right)ds.$$

As we did in (3.76), we can show by using the Hölder inequality, (3.32) and (3.20) that

$$||\nabla_x\psi_h(t)\cdot\nabla_v(h(t)-g(t)) - \nu/\sigma v\cdot\nabla_x\psi_h(t)(h(t)-g(t))||_{L^2(\mathbb{R}^6,f_\infty)}$$
$$\le C_R ||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||h(t)-g(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \tag{3.83}$$

and

$$||(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))\cdot\nabla_v g(t) - \nu/\sigma v\cdot(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))g(t)||_{L^2(\mathbb{R}^6,f_\infty)}$$
$$\le C_R ||h(t)-g(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||g(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \tag{3.84}$$

hold for all $t \in [0,\tau]$. Using (3.67), (3.83) and (3.84) we estimate $F[h]-F[g]$ in $C\left([0,\tau];H_v^1(\mathbb{R}^6,f_\infty)\right)$:

$$||F[h(t)]-F[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)}$$
$$\le \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v(h-g)-\nu/\sigma v\cdot\nabla_x\psi_h(h-g)\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds$$
$$+\int_0^t ||e^{-(t-s)K}\left((\nabla_x\psi_h-\nabla_x\psi_g)\cdot\nabla_v g-\nu/\sigma v\cdot(\nabla_x\psi_h-\nabla_x\psi_g)g\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds$$
$$\le \mathscr{C}_5 C_R \int_0^t (1+(t-s)^{-\frac{1}{2}})||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||h(s)-g(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds$$
$$+\mathscr{C}_5 C_R \int_0^t (1+(t-s)^{-\frac{1}{2}})||h(s)-g(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||g(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds.$$

This shows that

$$||F[h(t)]-F[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} \le \mathscr{C}_5 C_R ||h||_{\tau,1}||h-g||_{\tau,2}\int_0^t (1+(t-s)^{-\frac{1}{2}})ds$$
$$+\mathscr{C}_5 C_R ||h-g||_{\tau,1}||g||_{\tau,2}\int_0^t (1+(t-s)^{-\frac{1}{2}})ds.$$

Therefore, we get

$$||F[h(t)]-F[g(t)]||_{\tau,2} \leq \mathscr{C}_5 C_R ||h||_{\tau,1}||h-g||_{\tau,2}(\tau+2\sqrt{\tau})+\mathscr{C}_5 C_R||h-g||_{\tau,1}||g||_{\tau,2}(\tau+2\sqrt{\tau}). \tag{3.85}$$

Similarly, we estimate $F[h]-F[g]$ in $C\left([0,\tau];H_x^\alpha(\mathbb{R}^6,f_\infty)\right)$ using (3.65), (3.83) and (3.84):

$$\begin{aligned}
&||F[h(t)]-F[g(t)]||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \\
&\qquad\leq \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v(h-g)-\nu/\sigma v\cdot\nabla_x\psi_h(h-g)\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}ds \\
&\quad+\int_0^t ||e^{-(t-s)K}\left((\nabla_x\psi_h-\nabla_x\psi_g)\cdot\nabla_v g-\nu/\sigma v\cdot(\nabla_x\psi_h-\nabla_x\psi_g)g\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}ds \\
&\qquad\leq \mathscr{C}_3 C_R\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||h(s)-g(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds \\
&\qquad\quad+\mathscr{C}_3 C_R\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})||h(s)-g(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||g(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds.
\end{aligned}$$

This shows that

$$\begin{aligned}
||F[h(t)]-F[g(t)]||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} &\leq \mathscr{C}_3 C_R||h||_{\tau,1}||h-g||_{\tau,2}\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds \\
&\quad+\mathscr{C}_3 C_R||h-g||_{\tau,1}||g||_{\tau,2}\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds.
\end{aligned}$$

We take the supremum in time

$$\begin{aligned}
||F[h]-F[g]||_{\tau,1} &\leq \mathscr{C}_3 C_R||h||_{\tau,1}||h-g||_{\tau,2}\sup_{t\in[0,\tau]}\left\{\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds\right\} \\
&\quad+\mathscr{C}_3 C_R||h-g||_{\tau,1}||g||_{\tau,2}\sup_{t\in[0,\tau]}\left\{\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds\right\}.
\end{aligned} \tag{3.86}$$

Let $r:=2\max\{\mathscr{C}_2||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)},\mathscr{C}_4||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}\}$. We choose a small $\tau\in(0,t_0]$ such that

$$\frac{r}{2}+\mathscr{C}_3 C_R r^2\sup_{t\in[0,\tau]}\left\{\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds\right\}\leq r,\qquad \frac{r}{2}+\mathcal{C}_5 C_R r^2(\tau+2\sqrt{\tau})\leq r.$$

Then, (3.79) and (3.82) show that

$$||F[h]||_\tau\leq r \quad\text{for}\quad ||h||_\tau\leq r.$$

We choose a smaller $\tau\in(0,t_0]$ such that

$$2\mathscr{C}_5 C_R r(\tau+2\sqrt{\tau})<\frac{1}{2},\quad 2\mathscr{C}_3 C_R r\sup_{t\in[0,\tau]}\left\{\int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})ds\right\}<\frac{1}{2},$$

then (3.85) and (3.86) show

$$||F[h]-F[g]||_\tau\leq\frac{1}{2}||h-g||_\tau \quad\text{for}\quad ||h||_\tau\leq r,\ ||g||_\tau\leq r.$$

By the well known contraction principle F has a unique fixed point in $C\left([0,\tau], H_x^\alpha(\mathbb{R}^6, f_\infty)\right) \cap C\left([0,\tau], H_v^1(\mathbb{R}^6, f_\infty)\right)$. This fixed point is the desired solution of the integral equation

$$h(t) = e^{-tK} h_0 + \int_0^t e^{-(t-s)K} \left(\nabla_x \psi_h \cdot \nabla_v h - v \cdot \nabla_x \psi_h h\right) ds, \quad t \in [0,\tau].$$

We note that τ depends on $||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}$ and $||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}$. From what we have just proved it follows that if h is a mild solution on the interval $[0,\tau]$ it can be extended on the interval $[0,\tau+\tau_1]$ with $\tau_1 \in (0,t_0]$ by defining on $[\tau,\tau+\tau_1]$, $h(t) = h_1(t)$ where $h_1(t)$ is the solution of

$$h_1(t) = e^{-(t-\tau)K} h(\tau) + \int_\tau^t e^{-(t-s)K} \left(\nabla_x \psi_{h_1}(s) \cdot \nabla_v h_1(s) - v \cdot \nabla_x \psi_{h_1}(s) h_1(s)\right) ds.$$

Moreover, $\tau_1 \in (0,t_0]$ depends on $||h(\tau)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}$ and $||h(\tau)||_{H_v^1(\mathbb{R}^6,f_\infty)}$.

Let $[0,t_{max})$ be the maximal interval of existence of the solution. If $t_{max} < \infty$ then at least one of the limits

$$\lim_{t \nearrow t_{max}} ||h(t)||_{H_x^\alpha(\mathbb{R}^{2d},f_\infty)} \quad \text{and} \quad \lim_{t \nearrow t_{max}} ||h(t)||_{H_v^1(\mathbb{R}^{2d},f_\infty)}$$

is infinite. Otherwise there is a sequence $t_n \nearrow t_{max}$, $n \in \mathbb{N}$, such that $||h(t_n)||_{H_x^\alpha(\mathbb{R}^{2d},f_\infty)}$ and $||h(t_n)||_{H_v^1(\mathbb{R}^{2d},f_\infty)}$ is bounded. This would imply what we have just proved that for each t_n, near enough to t_{max}, the solution h on the interval $[0,t_n]$ can be extended to the interval $[0,t_n+\delta]$, where $\delta \in (0,t_0]$ is independent of t_n and hence h can be extended beyond t_{max}. This contradicts the definition of t_{max}.

Then (3.31) completes the proof. □

3.5.3 Global well-posedness and exponential stability

Let $\lambda > 0$ be the constant appearing in Theorem 3.2.2 (iii), $\lambda_1 \in (0,\lambda)$ and $\alpha \in [0,1]$. We define

$$X := \left\{ h \in C\left([0,\infty); H_x^\alpha(\mathbb{R}^6, f_\infty)\right) : \sup_{t\geq 0} \left\{ e^{\lambda_1 t} ||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \right\} < \infty \right\},$$

$$Y := \left\{ h \in C\left([0,\infty); H_v^1(\mathbb{R}^6, f_\infty)\right) : \sup_{t\geq 0} \left\{ e^{\lambda_1 t} ||h(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \right\} < \infty \right\}$$

with the norms

$$||h||_X := \sup_{t\geq 0} \left\{ e^{\lambda_1 t} ||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \right\},$$

$$||h||_Y := \sup_{t\geq 0} \left\{ e^{\lambda_1 t} ||h(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \right\}.$$

We denote

$$I_1 := \sup_{t\geq 0} \left\{ \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}}) e^{-\lambda_1 s} ds \right\}, \quad I_2 := \sup_{t\geq 0} \left\{ \int_0^t (1+(t-s)^{-\frac{1}{2}}) e^{-\lambda_1 s} ds \right\}.$$

Here I_2 is finite, but I_1 is finite if $\alpha \in (0,\frac{2}{3})$.

We need the following Grönwall type inequality.

Lemma 3.5.3. *Let a and b be positive constants, and $y : [0,\infty) \to [0,\infty)$ be a continuous function satisfying*

$$y(t) \le a + b\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1 s}y(s)ds, \quad \forall\, t \in [0,\infty).$$

Then there is a positive constant Λ depending only on λ_1 such that

$$y(t) \le (a + abI_2)e^{b^2\Lambda}, \quad \forall\, t \in [0,\infty). \tag{3.87}$$

Proof. We observe

$$\begin{aligned} y(t) &\le a + b\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1 s}y(s)ds \\ &\le a + ab\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1 s}ds \\ &\quad + b^2\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1 s}\left[\int_0^s (1+(s-\tau)^{-\frac{1}{2}})e^{-\lambda_1\tau}y(\tau)d\tau\right]ds. \end{aligned}$$

By Fubini's theorem for computing multiple integrals, we obtain

$$\begin{aligned} y(t) &\le a + ab\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1 s}ds \\ &\quad + b^2\int_0^t e^{-2\lambda_1\tau}y(\tau)\left[\int_\tau^t (1+(t-s)^{-\frac{1}{2}})(1+(s-\tau)^{-\frac{1}{2}})e^{-\lambda_1(s-\tau)}ds\right]d\tau. \end{aligned} \tag{3.88}$$

The integral in the brackets can be written as

$$\begin{aligned} &\int_\tau^t (1+(t-s)^{-\frac{1}{2}})(1+(s-\tau)^{-\frac{1}{2}})e^{-\lambda_1(s-\tau)}ds \\ &= \int_0^{t-\tau} (1+(t-\tau-s)^{-\frac{1}{2}})(1+s^{-\frac{1}{2}})e^{-\lambda_1 s}ds \\ &= \int_0^{t-\tau} (1+(t-\tau-s)^{-\frac{1}{2}}+s^{-\frac{1}{2}}+(t-\tau-s)^{-\frac{1}{2}}s^{-\frac{1}{2}})e^{-\lambda_1 s}ds. \end{aligned} \tag{3.89}$$

We show this integral is bounded by a constant depending only on λ_1. We first compute

$$\int_0^{t-\tau} e^{-\lambda_1 s}ds = \frac{1}{\lambda_1}(1-e^{-\lambda_1(t-\tau)}) \le \frac{1}{\lambda_1}$$

and

$$\int_0^{t-\tau} (t-\tau-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}e^{-\lambda_1 s}ds \le \int_0^{t-\tau} (t-\tau-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}ds = \arcsin\frac{2s-(t-\tau)}{t-\tau}\Big|_0^{t-\tau} = \pi.$$

If $t-\tau \le 1$, then

$$\int_0^{t-\tau} s^{-\frac{1}{2}}e^{-\lambda_1 s}ds \le \int_0^{t-\tau} s^{-\frac{1}{2}}ds = 2(t-\tau)^{\frac{1}{2}} \le 2. \tag{3.90}$$

If $t-\tau > 1$, then

$$\int_0^1 s^{-\frac{1}{2}}e^{-\lambda_1 s}ds \le \int_0^1 s^{-\frac{1}{2}}ds = 2 \tag{3.91}$$

and

$$\int_1^{t-\tau} s^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le \int_1^{t-\tau} e^{-\lambda_1 s} ds = \frac{e^{-\lambda_1}(1-e^{-\lambda_1(t-\tau-1)})}{\lambda_1} \le \frac{e^{-\lambda_1}}{\lambda_1}. \tag{3.92}$$

(3.90), (3.91), and (3.92) show

$$\int_0^{t-\tau} s^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le 2 + \frac{e^{-\lambda_1}}{\lambda_1}, \quad \forall\, t-\tau \ge 0.$$

Similarly, if $t-\tau \le 1$, then

$$\int_0^{t-\tau} (t-\tau-s)^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le \int_0^{t-\tau} (t-\tau-s)^{-\frac{1}{2}} ds = 2(t-\tau)^{\frac{1}{2}} \le 2. \tag{3.93}$$

If $t-\tau > 1$, then

$$\int_0^{t-\tau-1} (t-\tau-s)^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le \int_0^{t-\tau-1} e^{-\lambda_1 s} ds \le \frac{1}{\lambda_1} \tag{3.94}$$

and

$$\int_{t-\tau-1}^{t-\tau} (t-\tau-s)^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le \int_{t-\tau-1}^{t-\tau} (t-\tau-s)^{-\frac{1}{2}} ds = 2. \tag{3.95}$$

(3.90), (3.91), and (3.92) show

$$\int_0^{t-\tau} (t-\tau-s)^{-\frac{1}{2}} e^{-\lambda_1 s} ds \le 2 + \frac{1}{\lambda_1}, \quad \forall\, t-\tau \ge 0.$$

The estimates above shows that the integral in (3.89) is bounded by $\frac{e^{-\lambda_1}+2}{\lambda_1} + \pi + 4$. Then we get from (3.88)

$$y(t) \le a + abI_2 + b^2 \left(\frac{e^{-\lambda_1}+2}{\lambda_1} + \pi + 4 \right) \int_0^t e^{-2\lambda_1 \tau} y(\tau) d\tau.$$

The Grönwall inequality yields

$$y(t) \le (a+abI_2) e^{b^2 (\frac{e^{-\lambda_1}+2}{\lambda_1} + \pi + 4) \int_0^t e^{-2\lambda_1 \tau} d\tau} \le (a+abI_2) e^{\frac{b^2}{2\lambda_1}(\frac{e^{-\lambda_1}+2}{\lambda_1}+\pi+4)}.$$

This proves (3.87) with the constant $\Lambda := \frac{1}{2\lambda_1}(\frac{e^{-\lambda_1}+2}{\lambda_1} + \pi + 4)$. □

Lemma 3.5.4. *Let $h_0 \in H_x^\alpha(\mathbb{R}^6, f_\infty) \cap H_v^1(\mathbb{R}^6, f_\infty)$, the assumptions (A1), (A2) and (A3) hold. Then, for any $h \in X$, there is a unique $w \in X \cap Y$ satisfying*

$$w(t) = e^{-tK} h_0 + \int_0^t e^{-(t-s)K} \left(\nabla_x \psi_h(s) \cdot \nabla_v w(s) - v \cdot \nabla_x \psi_h(s) w(s) \right) ds, \quad \forall t \ge 0. \tag{3.96}$$

Proof. Let $\tau > 0$. We define a mapping

$$G : C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right) \to C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$$

by

$$G[w] = e^{-tK} h_0 + \int_0^t e^{-(t-s)K} \left(\nabla_x \psi_h \cdot \nabla_v w - \nu/\sigma v \cdot \nabla_x \psi_h w \right) ds, \quad t \in [0,\tau].$$

w is a solution of (3.96) if and only if it is a fixed point of G. We first show that, if $\tau > 0$ is small enough, then there is a unique fixed point of G in $C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$. We use the norm $||h||_{\tau,2} := \sup_{t\in[0,\tau]}\left\{||h(t)||_{H_v^1(\mathbb{R}^6,f_\infty)}\right\}$ in $C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$. As we assume $h \in X$, (3.32) shows that $\nabla_x\psi_h(t)$ is bounded for all $t \geq 0$. Therefore,

$$\nabla_x\psi_h(t)\cdot\nabla_v w(t) - \nu/\sigma v\cdot\nabla_x\psi_h(t)w(t) \in L^2(\mathbb{R}^6, f_\infty)$$

for all $w \in C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$ and $t \geq 0$. Using the Hölder inequality, (3.32) and (3.20)

$$\begin{aligned}
&||\nabla_x\psi_h(t)\cdot\nabla_v w(t) - \nu/\sigma v\cdot\nabla_x\psi_h(t)w(t)||_{L^2(\mathbb{R}^6,f_\infty)} \\
&\qquad\leq \sqrt{2||\nabla_x\psi_h(t)||^2_{L^\infty}\int_{\mathbb{R}^6}\left(|\nabla_v w(t)|^2 + \nu^2/\sigma^2|v|^2|w^2(t)\right)f_\infty dxdv} \\
&\qquad\qquad\leq C_R||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||w(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \qquad (3.97)
\end{aligned}$$

for some constant $C_R > 0$. Also, we can check by integration by parts that

$$\nabla_x\psi_h(t)\cdot\nabla_v w(t) - \nu/\sigma v\cdot\nabla_x\psi_h(t)w(t) \in \mathcal{H} \qquad (3.98)$$

for all $t \geq 0$.

We estimate $G[w]$ in $C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$:

$$\begin{aligned}
||G[w(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} &\leq ||e^{-tK}h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} \\
&\quad + \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v w - v\cdot\nabla_x\psi_h w\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds. \qquad (3.99)
\end{aligned}$$

(3.73), (3.98) and (3.97) let us estimate the second term on the right side of (3.99)

$$\begin{aligned}
&\int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h(s)\cdot\nabla_v w(s) - \nu/\sigma v\cdot\nabla_x\psi_h(s)w(s)\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds \\
&\leq \mathcal{C}_5\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}||\nabla_x\psi_h(s)\cdot\nabla_v w(s) - \nu/\sigma v\cdot\nabla_x\psi_h(s)w(s)||_{L^2(\mathbb{R}^6,f_\infty)}ds \\
&\quad\leq \mathcal{C}_5 C_R\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||w(s)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds \qquad (3.100) \\
&\quad\leq \mathcal{C}_5 C_R e^{-\lambda_1 t}||h||_X||w||_{\tau,2}\int_0^t(1+(t-s)^{-\frac{1}{2}})ds = \mathcal{C}_5 C_R e^{-\lambda_1 t}||h||_X||w||_{\tau,2}(t+2\sqrt{t}),
\end{aligned}$$

where we used

$$\int_0^t(1+(t-s)^{-\frac{1}{2}})ds = t + 2\sqrt{t}.$$

The estimates above show

$$||G[w(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} \leq ||e^{-tK}h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \mathcal{C}_5 C_R e^{-\lambda_1 t}||h||_X||w||_{\tau,2}(t+2\sqrt{t}),$$

and so

$$||G[w]||_{\tau,2} \leq ||e^{-tK}h_0||_{\tau,2} + \mathcal{C}_5 C_R||h||_X||w||_{\tau,2}(\tau+2\sqrt{\tau}), \qquad (3.101)$$

Let $r := 2||e^{-tK}h_0||_{\tau,2}$. If τ is small enough so that

$$\mathcal{C}_5 C_R||h||_X(\tau+2\sqrt{\tau}) \leq \frac{1}{2}, \qquad (3.102)$$

then (3.101) shows that

$$||G[w]||_{\tau,2} \leq r \quad \text{for any} \quad ||w||_{\tau,2} \leq r. \tag{3.103}$$

Similar computations show

$$||G[w] - G[u]||_{\tau,2} \leq \mathcal{C}_5 C_R ||h||_X ||w-u||_{\tau,2}(\tau + 2\sqrt{\tau})$$

for all $w, u \in C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$. If τ satisfies (3.102), then

$$||G[w] - G[u]||_{\tau,2} \leq \frac{1}{2}||w-u||_{\tau,2}. \tag{3.104}$$

We fix τ so that (3.102) holds. Then, (3.103) and (3.104) shows that F has a fixed point w in $C\left([0,\tau]; H_v^1(\mathbb{R}^6, f_\infty)\right)$. We note that τ only depends on the product $\mathcal{C}_4 C_R ||h||_X < \infty$. From what we have just proved it follows that if w is a solution of (3.96) on the interval $[0,\tau]$, it can be extended to the interval $[0,2\tau]$ by defining on $[\tau, 2\tau]$, $w(t) = w_1(t)$ where w_1 is the solution of the integral equation

$$w_1(t) = e^{-(t-\tau)K} w(\tau) + \int_\tau^t e^{-(t-s)K} \left(\nabla_x \psi_h \cdot \nabla_v w_1 - \nu/\sigma v \cdot \nabla_x \psi_h w_1\right) ds, \quad t \in [\tau, 2\tau].$$

After that, we extend this solution to the interval $[0, 3\tau]$ and so on. Thus, we can prove that there is a unique global in time solution.

Next, we show that the solution w is in Y :

$$||w(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \leq ||e^{-tK} h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \int_0^t ||e^{-(t-s)K}\left(\nabla_x \psi_h \cdot \nabla_v w - \nu/\sigma v \cdot \nabla_x \psi_h w\right)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds.$$

Then, (3.72) and similar estimate as we did in (3.100) show that

$$\begin{aligned} ||w(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} &\leq \mathcal{C}_4 e^{-\lambda_1 t} ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} \\ &+ \mathcal{C}_5 C_R \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1 (t-s)} ||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ||w(s)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds \\ &\leq \mathcal{C}_4 e^{-\lambda_1 t} ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \mathcal{C}_5 C_R ||h||_X e^{-\lambda_1 t} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) ||w(s)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds. \end{aligned}$$

It yields

$$e^{\lambda_1 t} ||w(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \leq \mathcal{C}_4 ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} + \mathcal{C}_5 C_R ||h||_X \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1 s} e^{\lambda_1 s} ||w(s)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds.$$

Using Lemma 3.5.3 we obtain

$$e^{\lambda_1 t} ||w(t)||_{H_v^1(\mathbb{R}^6,f_\infty)} \leq \mathcal{C}_4 (1 + \mathcal{C}_5 C_R I_2 ||h||_X) e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2} ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}$$

After taking the supremum in time

$$||w||_Y \leq \mathcal{C}_4 (1 + \mathcal{C}_5 C_R I_2 ||h||_X) e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2} ||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} < \infty. \tag{3.105}$$

We show that the solution w is also in X :

$$||w(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \leq ||e^{-tK}h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} + \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v w - \nu/\sigma v \cdot \nabla_x\psi_h w\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ds. \quad (3.106)$$

(3.71) and (3.97) let us estimate

$$\int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v w - \nu/\sigma v \cdot \nabla_x\psi_h w\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} ds$$
$$\leq \mathcal{C}_3 \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1(t-s)}||\nabla_x\psi_h \cdot \nabla_v w - \nu/\sigma v \cdot \nabla_x\psi_h w||_{L^2(\mathbb{R}^6,f_\infty)} ds$$
$$\leq \mathcal{C}_3 C_R \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1(t-s)}||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||w(s)||_{H_v^1(\mathbb{R}^6,f_\infty)} ds \quad (3.107)$$
$$\leq \mathcal{C}_3 C_R e^{-\lambda_1 t}||h||_X||w||_Y \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1 s} ds.$$

Then, (3.106), (3.70) and (3.107) provide

$$e^{\lambda_1 t}||w(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} \leq \mathcal{C}_2||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} + \mathcal{C}_3 C_R||h||_X||w||_Y \int_0^t (1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1 s} ds.$$

We take the supremum in time and use (3.105)

$$||w||_X \leq \mathcal{C}_2||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} + \mathcal{C}_3\mathcal{C}_4 C_R I_1(||h||_X + \mathcal{C}_5 C_R I_2||h||_X^2)e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2}||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)} < \infty. \quad (3.108)$$

Finally, (3.105) and (3.108) show that $w \in X \cap Y$. □

Proof of Theorem 3.2.5 . We construct a solution to (3.12) with a fixed point argument, and therefore we define the mapping $U : X \to X$ such that $U[h]$ (which is the value of U at h) is the solution of

$$U[h(t)] = e^{-tK}h_0 + \int_0^t e^{-(t-s)K}\left(\nabla_x\psi_h(s) \cdot \nabla_v U[h(s)] - \nu/\sigma v \cdot \nabla_x\psi_h(s)U[h(s)]\right) ds. \quad (3.109)$$

Lemma 3.5.4 provides that, for any $h \in X$, there is a unique $U[h] \in X \cap Y$ which satisfies (3.109). Therefore, this mapping is well-defined. $h \in X$ solves (3.12) if and only if $h = U[h]$. We will show that U has a unique fixed point in X.

The estimates (3.105) and (3.108) provide

$$||U[h]||_Y \leq \mathcal{C}_4(1 + \mathcal{C}_5 C_R I_2||h||_X)e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2}||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}, \quad (3.110)$$

$$||U[h]||_X \leq \mathcal{C}_2||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} + \mathcal{C}_3\mathcal{C}_4 C_R I_1(||h||_X + \mathcal{C}_5 C_R I_2||h||_X^2)e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2}||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}. \quad (3.111)$$

For $h, g \in X$, we have

$$U[h] - U[g] = \int_0^t e^{-(t-s)K}\left(\nabla_x\psi_h \cdot \nabla_v(U[h]-U[g]) - \nu/\sigma v \cdot \nabla_x\psi_h(U[h]-U[g])\right) ds$$
$$+ \int_0^t e^{-(t-s)K}\left((\nabla_x\psi_h - \nabla_x\psi_g) \cdot \nabla_v U[g] - \nu/\sigma v \cdot (\nabla_x\psi_h - \nabla_x\psi_g)U[g]\right) ds.$$

As we did in (3.97), we can show by using the Hölder inequality, (3.32) and (3.20) that

$$||\nabla_x\psi_h(t)\cdot\nabla_v(U[h(t)]-U[g(t)])-\nu/\sigma v\cdot\nabla_x\psi_h(t)(U[h(t)]-U[g(t)])||_{L^2(\mathbb{R}^6,f_\infty)} \leq C_R||h(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[h(t)]-U[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} \quad (3.112)$$

and

$$||(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))\cdot\nabla_vU[g(t)]-\nu/\sigma v\cdot(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))U[g(t)]||_{L^2(\mathbb{R}^6,f_\infty)} \leq C_R||h(t)-g(t)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)} \quad (3.113)$$

hold for all $t\geq 0$. Integrating by parts we can check

$$\nabla_x\psi_h(t)\cdot\nabla_v(U[h(t)]-U[g(t)])-\nu/\sigma v\cdot\nabla_x\psi_h(t)(U[h(t)]-U[g(t)])\in\mathcal{H} \quad (3.114)$$

and

$$(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))\cdot\nabla_vU[g(t)]-\nu/\sigma v\cdot(\nabla_x\psi_h(t)-\nabla_x\psi_g(t))U[g(t)]\in\mathcal{H} \quad (3.115)$$

for all $t\geq 0$. Using (3.73), (3.114), (3.115), (3.112) and (3.113) we estimate $U[h]-U[g]$ in Y :

$$\begin{aligned}
&||U[h(t)]-U[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)}\\
&\leq\int_0^t||e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v(U[h]-U[g])-\nu/\sigma v\cdot\nabla_x\psi_h(U[h]-U[g])\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&+\int_0^t||e^{-(t-s)K}\left((\nabla_x\psi_h-\nabla_x\psi_g)\cdot\nabla_vU[g]-\nu/\sigma v\cdot(\nabla_x\psi_h-\nabla_x\psi_g)U[g]\right)||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&\leq\mathcal{C}_5C_R\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[h(s)]-U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&\quad+\mathcal{C}_5C_R\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1(t-s)}||h(s)-g(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds.
\end{aligned}$$

This shows that

$$\begin{aligned}
&e^{\lambda_1t}||U[h(t)]-U[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)}\\
&\quad\leq\mathcal{C}_5C_R||h||_X\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1s}e^{\lambda_1s}||U[h(s)]-U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&\qquad\qquad+\mathcal{C}_5C_R||h-g||_X||U[g]||_Y\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1s}ds\\
&\quad\leq\mathcal{C}_5C_R||h||_X\int_0^t(1+(t-s)^{-\frac{1}{2}})e^{-\lambda_1s}e^{\lambda_1s}||U[h(s)]-U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&\qquad\qquad+\mathcal{C}_5C_RI_2||h-g||_X||U[g]||_Y.
\end{aligned}$$

We apply Lemma 3.5.3 to this inequality

$$\begin{aligned}
&e^{\lambda_1t}||U[h(t)]-U[g(t)]||_{H_v^1(\mathbb{R}^6,f_\infty)}\\
&\qquad\leq\mathcal{C}_5C_RI_2(1+\mathcal{C}_5C_RI_2||h||_X)e^{\mathcal{C}_5^2C_R^2\Lambda||h||_X^2}||h-g||_X||U[g]||_Y.
\end{aligned}$$

We take the supremum in time to get

$$||U[h]-U[g]||_Y\leq\mathcal{C}_5C_RI_2(1+\mathcal{C}_5C_RI_2||h||_X)e^{\mathcal{C}_5^2C_R^2\Lambda||h||_X^2}||h-g||_X||U[g]||_Y.$$

The estimate on $||U[g]||_Y$ in (3.110) shows

$$||U[h]-U[g]||_Y \\ \leq \mathcal{C}_4\mathcal{C}_5C_RI_2(1+\mathcal{C}_5C_RI_2||h||_X)(1+\mathcal{C}_5C_RI_2||g||_X)e^{\mathcal{C}_5^2C_R^2\Lambda(||h||_X^2+||g||_X^2)}||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}||h-g||_X \tag{3.116}$$

Using (3.71), (3.112) and (3.113) we estimate $U[h]-U[g]$ in X :

$$\begin{aligned}
&||U[h(t)]-U[g(t)]||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}\\
&\leq \int_0^t ||e^{-(t-s)K}\left(\nabla_x\psi_h\cdot\nabla_v(U[h]-U[g])-\nu/\sigma v\cdot\nabla_x\psi_h(U[h]-U[g])\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}ds\\
&+\int_0^t ||e^{-(t-s)K}\left((\nabla_x\psi_h-\nabla_x\psi_g)\cdot\nabla_vU[g]-\nu/\sigma v\cdot(\nabla_x\psi_h-\nabla_x\psi_g)U[g]\right)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}ds\\
&\leq \mathcal{C}_3C_R\int_0^t(1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1(t-s)}||h(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[h(s)]-U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds\\
&\quad+\mathcal{C}_3C_R\int_0^t(1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1(t-s)}||h(s)-g(s)||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}||U[g(s)]||_{H_v^1(\mathbb{R}^6,f_\infty)}ds.
\end{aligned}$$

This shows that

$$\begin{aligned}
e^{\lambda_1 t}||U[h(t)]-U[g(t)]||_{H_x^\alpha(\mathbb{R}^6,f_\infty)} &\leq \mathcal{C}_3C_R||h||_X||U[h]-U[g]||_Y\int_0^t(1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1 s}ds\\
&+\mathcal{C}_3C_R||h-g||_X||U[g]||_Y\int_0^t(1+(t-s)^{-\frac{3\alpha}{2}})e^{-\lambda_1 s}ds.
\end{aligned}$$

We take the supremum in time and obtain

$$||U[h]-U[g]||_X\leq \mathcal{C}_3C_RI_1||h||_X||U[h]-U[g]||_Y+\mathcal{C}_3C_RI_1||h-g||_X||U[g]||_Y.$$

The estimates (3.110) and (3.116) provide

$$\begin{aligned}
&||U[h]-U[g]||_X\\
&\leq\Big[\mathcal{C}_3\mathcal{C}_4\mathcal{C}_5C_R^2I_1I_2(||h||_X+\mathcal{C}_5C_RI_2||h||_X^2)(1+\mathcal{C}_5C_RI_2||g||_X)e^{\mathcal{C}_5^2C_R^2\Lambda(||h||_X^2+||g||_X^2)}\\
&\quad+\mathcal{C}_3\mathcal{C}_4C_RI_1(1+\mathcal{C}_5C_RI_2||g||_X)e^{\mathcal{C}_5^2C_R^2\Lambda||g||_X^2}\Big]||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}||h-g||_X
\end{aligned}\tag{3.117}$$

Let $r>0$, then there exist $\delta_1=\delta_1(r)>0$ and $\delta_2=\delta_2(r)>0$ such that

$$\begin{cases}\mathcal{C}_2\delta_1+\mathcal{C}_3\mathcal{C}_4C_RI_1(r+\mathcal{C}_5C_RI_2r^2)e^{\mathcal{C}_5^2C_R^2\Lambda r^2}\delta_2\leq r\\ \Big[\mathcal{C}_3\mathcal{C}_4\mathcal{C}_5C_R^2I_1I_2r(1+\mathcal{C}_5C_RI_2r)^2e^{2\mathcal{C}_5^2C_R^2\Lambda r^2}+\mathcal{C}_3\mathcal{C}_4C_RI_1(1+\mathcal{C}_5C_RI_2r)e^{\mathcal{C}_5^2C_R^2\Lambda r^2}\Big]\delta_2<1.\end{cases}\tag{3.118}$$

If h_0 satisfies

$$||h_0||_{H_x^\alpha(\mathbb{R}^6,f_\infty)}\leq\delta_1\quad\text{and}\quad||h_0||_{H_v^1(\mathbb{R}^6,f_\infty)}\leq\delta_2,\tag{3.119}$$

then (3.111), (3.117) and (3.118) show

$$||U[h]||_X\leq r\quad\text{for all}\quad||h||_X\leq r$$

and

$$||U[h]-U[g]||_X\leq\epsilon||h-g||_X,\qquad\text{for all}\quad||h||_X\leq r,\ ||g||_X\leq r,$$

where $\epsilon := \left[\mathcal{C}_3\mathcal{C}_4\mathcal{C}_5 C_R^2 I_1 I_2 r(1+\mathcal{C}_5 C_R I_2 r)^2 e^{2\mathcal{C}_5^2 C_R^2 \Lambda r^2} + \mathcal{C}_3\mathcal{C}_4 C_R I_1 (1+\mathcal{C}_5 C_R I_2 r) e^{\mathcal{C}_5^2 C_R^2 \Lambda r^2}\right] \delta_2 \in (0,1)$. Then, the contraction principle yields that U has a unique fixed point $h \in X$ such that $||h||_X \leq r$. Moreover, (3.110) shows the fixed point h is also in Y and

$$||h||_Y \leq \mathcal{C}_4(1+\mathcal{C}_5 C_R I_2 ||h||_X) e^{\mathcal{C}_5^2 C_R^2 \Lambda ||h||_X^2} ||h_0||_{H_v^1(\mathbb{R}^6, f_\infty)} \leq \mathcal{C}_4(1+\mathcal{C}_5 C_R I_2 r) e^{\mathcal{C}_5^2 C_R^2 \Lambda r^2} \delta_2.$$

Therefore, Theorem 3.2.5 holds with $C_5 := r$, $C_6 := \mathcal{C}_4(1+\mathcal{C}_5 C_R I_2 r) e^{\mathcal{C}_5^2 C_R^2 \Lambda r^2} \delta_2$ and $C_7 := \theta_2 r$ by (3.31).

Here, r can be any positive number and there always exist $\delta_1 = \delta_1(r) > 0$ and $\delta_2 = \delta_2(r) > 0$ such that (3.118) holds. To have the condition (3.119) with larger δ_1 and δ_2, we fix $r > 0$ so that $\delta_1 = \delta_1(r) > 0$ and $\delta_2 = \delta_2(r) > 0$ are as large as possible. □

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Chapter 4

The relativistic kinetic Fokker-Planck equation

4.1 Introduction

In this chapter, we study the long time behavior of the relativistic, spatially inhomogeneous Fokker-Planck equation [22, 2]

$$\begin{cases} \partial_t f + \dfrac{p}{m\sqrt{1+\frac{|p|^2}{m^2c^2}}} \cdot \nabla_x f - q\nabla_x V(x) \cdot \nabla_p f = \mathrm{div}_p(\sigma D(p)\nabla_p f + \nu p f), \quad x,\, p \in \mathbb{R}^d,\ t>0 \\ f_{|t=0} = f_0. \end{cases} \tag{4.1}$$

This kinetic model describes the time evolution of a system with a large number of particles (e.g. in a plasma) undergoing diffusion and friction. The unknown $f = f(t,x,p) \geq 0$ represents the evolution of the phase space probability density of particles. The left hand side is the transport operator with force field $-\nabla_x V(x)$, while the right hand side describes the diffusion of particles and the interaction with the environment. The positive physical constants denoted by m, c, q, σ, and ν are respectively the particle mass, the vacuum speed of light, the particle charge, diffusion and friction coefficients. $D(p)$ is the relativistic diffusion matrix given by

$$D(p) = \frac{I + \frac{p\otimes p}{m^2c^2}}{\sqrt{1+\frac{|p|^2}{m^2c^2}}} \in \mathbb{R}^{d\times d},$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix and $\otimes$ denotes the Kronecker product.

Equation (4.1) has several properties following standard physical considerations. Whenever $f(t,x,p)$ is a (well-behaved) solution of (4.1), one has *global conservation of mass*

$$\int_{\mathbb{R}^{2d}} f(t,x,p)\,dx dp = \int_{\mathbb{R}^{2d}} f_0(x,p)\,dx dp, \quad \forall t \geq 0. \tag{4.2}$$

Therefore, without loss of generality, we shall assume $f_0 \geq 0$ and $\displaystyle\int_{\mathbb{R}^{2d}} f_0(x,p)\,dxdp = 1$. If V grows fast enough, (4.1) has a unique normalized *steady state* or *global equilibrium* [2, Section 3.4] given by

$$f_\infty(x,p) = \rho_\infty(x) M(p), \tag{4.3}$$

where

$$\rho_\infty(x) := \frac{e^{-\frac{mq\nu}{\sigma}V(x)}}{\int_{\mathbb{R}^d} e^{-\frac{qm\nu}{\sigma}V(x')}dx'}, \quad M(p) := \frac{e^{-\frac{mc\nu}{\sigma}\sqrt{m^2c^2+|p|^2}}}{\int_{\mathbb{R}^d} e^{-\frac{mc\nu}{\sigma}\sqrt{m^2c^2+|p'|^2}}dp'}.$$

(4.1) is *dissipative* in the sense that the relative entropy or free energy functional decreases [2, Section 3.3]: let H be a functional defined on the space of probability densities by

$$f \mapsto H[f] := \int_{\mathbb{R}^{2d}} f \ln \frac{f}{f_\infty} dxdp$$

(f is not necessarily the solution). We note that $H[f_\infty] = 0$ and $H[f] \geq \frac{1}{2}||f - f_\infty||^2_{L^1(\mathbb{R}^{2d})}$ by the Csiszár-Kullback-Pinsker inequality [15]. Hence, the minimum of H is zero and it is attained at f_∞. If $f = f(t,x,p)$ is a smooth solution of (4.1), we have

$$\frac{d}{dt}H[f(t)] \leq 0.$$

This decay of the functional H is a version of Boltzmann's $H-$theorem stated for the Boltzmann equation [12, 30].

On the basis of the decay of the functional H, one can guess that $H[f(t)]$ decreases to its minimum (which is zero) as $t \to \infty$. Since this minimum is obtained at f_∞, one can conjecture that $f(t)$ converges to the equilibrium distribution f_∞ as $t \to \infty$. We shall therefore tackle the interesting problem which is to prove (or disprove) that solutions of (4.1) converge towards this equilibrium as $t \to \infty$ and to estimate the convergence rate. We are interested in the study of rates of convergence and we want to derive constructive bounds for this convergence. Such explicit and constructive estimates are essential for applications in physics (e.g., equilibration process, numerical simulations).

Equation (4.1) was introduced in [22, Eq.(47)] and [2, Eq.(8)] as a relativistic generalization of the classical kinetic Fokker-Planck equation [13, 14, 29]

$$\begin{cases} \partial_t f + \frac{p}{m} \cdot \nabla_x f - q\nabla_x V \cdot \nabla_p f = \operatorname{div}_p(\sigma \nabla_p f + \nu p f), \quad x,\, p \in \mathbb{R}^d,\ t > 0 \\ f_{|t=0} = f_0 \end{cases}. \tag{4.4}$$

This classical equation can be obtained from (4.1) by formally taking the Newtonian limit $c \to \infty$. In [21], this formal limit was justified in the sense that solutions of (4.1) converge to the solutions of (4.4) in L^1 as $c \to \infty$. The equation (4.4) is inconsistent with relativistic mechanics because it has infinite speed of propagation: if the particles are initially in a compact region (i.e. $f_0(x,p)$ has compact support with respect to x and p), then, after any short time $t > 0$, we can find particles everywhere with non-zero probability (i.e. $f(t,x,p) > 0$), see [31, Appendix A.22]. This property contradicts the law of special relativity that particles can not move faster than light. While Equation (4.1) is compatible with this physical law as it exhibits finite speed of propagation w.r.t. the x variable [2, Section 3.2]. Note, however, that the degenerate parabolicity of (4.1) does entail infinite speed of propagation w.r.t. $p \in \mathbb{R}^d$.

While the equation (4.1) is compatible with this physical law as it exhibits finite speed of propagation [2, Section 3.2].

The classical equation (4.4) has been studied comprehensively: well-posedness and hypoelliptic regularity were obtained in [28, 24, 31]. The long time behavior of (4.4) was studied in [25] for fast growing potentials. By using hypocoercivity methods, Villani proved exponential convergence results in [16, 31]. This result was extended in [8] for potentials with singularities. In [18], Dolbeault, Mouhot, and Schmeiser developed a method to obtain exponential decay in L^2 for a large class of linear kinetic equations, and, as an application, an exponential decay in L^2 was proven for (4.4). Their method was also used to study the long time behavior of (4.4) when the potential V is zero or grows slowly as $|x| \to \infty$, see [9, 10]. Based on a probabilistic coupling method, Eberle, Guillin, and Zimmer [19] obtained an exponential decay result in Wasserstein distance. We also refer the recent work [5] where sharper exponential rates were obtained using a modified entropy method.

Concerning the relativistic equation (4.1), there are few studies: global existence and uniqueness were proven in [2]. The long time behavior of spatially homogeneous solutions of (4.1) was studied [3, 21], where the authors used logarithmic Sobolev inequalities and entropy methods [4, 27]. When (4.1) is supplemented with periodic boundary conditions (i.e. $x \in \mathbb{T}^d$) and $V = 0$, exponential decay of solutions to the steady state was proven in [11] by using the hypocoercive method developed by Villani [31].

In this paper, we shall improve these previous results when there is a non-zero potential V. For the full system (4.1) with a non-zero potential V we shall prove the exponential convergence $f(t) \to f_\infty$ as $t \to \infty$ for a wide class of potentials V. Our rates on this convergence are explicit and constructive. We show that, although the equation is degenerate parabolic, the equation has instantaneous regularizing properties which is called *hypoellipticity* [26]. We provide explicit rates on this regularization. We believe our results are the first convergence and regularity results for (4.1) with a non-zero potentials V.

The organization of this paper is as follows. In Section 2, we define the assumptions on the potential and state the main results. Section 3 and 4 are devoted to prove the convergence $f(t) \to f_\infty$ as $t \to \infty$ in the weighted L^2 and Sobolev spaces. We study regularity properties of the equation in Section 5.

4.2 Setting and main result

We use the notations

$$V_0(x) := \sqrt{1 + |\nabla_x V|^2} \quad \text{and} \quad p_0(p) := \sqrt{1 + |p|^2}.$$

For simplicity, we set all physical constants to unity $m = c = q = \sigma = \nu = 1$. Therefore, we shall consider the normalized equation

$$\begin{cases} \partial_t f + \dfrac{p}{p_0} \cdot \nabla_x f - \nabla_x V \cdot \nabla_p f = \operatorname{div}_p(D(p)\nabla_p f + pf), \quad x, p \in \mathbb{R}^d, \ t > 0 \\ f_{|t=0} = f_0 \end{cases} \tag{4.5}$$

with $D(p) = \dfrac{I + p \otimes p}{p_0}$. We define weighted spaces $L^2(\mathbb{R}^d, \rho_\infty)$, $L^2(\mathbb{R}^d, M)$, and $L^2(\mathbb{R}^{2d}, f_\infty)$ as the Lebesgue spaces associated, respectively, to the norms

$$||g||_{L^2(\mathbb{R}^d,\rho_\infty)} := \sqrt{\int_{\mathbb{R}^d} g^2 \rho_\infty dx}, \quad ||g||_{L^2(\mathbb{R}^d,M)} := \sqrt{\int_{\mathbb{R}^d} g^2 M dp},$$

and

$$||g||_{L^2(\mathbb{R}^{2d},f_\infty)} := \sqrt{\int_{\mathbb{R}^{2d}} g^2 f_\infty dx dp}.$$

We note that $M(p) = \frac{e^{-\sqrt{1+|p|^2}}}{\int_{\mathbb{R}^d} e^{-\sqrt{1+|p'|^2}}dp'}$ gives rise to a the following Poincaré inequality [21, Theorem 3]: there is a positive constant κ_1 such that

$$\int_{\mathbb{R}^d} h^2 M dp - \left(\int_{\mathbb{R}^d} hM dp\right)^2 \leq \frac{1}{\kappa_1} \int_{\mathbb{R}^d} \nabla_p^T h D \nabla_p h M dp \tag{4.6}$$

holds for all $h \in L^2(\mathbb{R}^d, M)$ with $\int_{\mathbb{R}^d} \nabla_p^T h D \nabla_p h M dp < \infty$.

We shall assume that $\rho_\infty(x) = \frac{e^{-V(x)}}{\int_{\mathbb{R}^d} e^{-V(x')}dx'}$ also gives rise to a Poincaré inequality. Also, we shall assume some growth conditions on V :

Assumption 4.2.1. *i) Let $V \in C^2(\mathbb{R}^d)$ be such that $e^{-V} \in L^1(\mathbb{R}^d)$, and there exists a constant $\kappa_2 > 0$ such that the Poincaré inequality*

$$\int_{\mathbb{R}^d} h^2 \rho_\infty dx - \left(\int_{\mathbb{R}^d} h\rho_\infty dx\right)^2 \leq \frac{1}{\kappa_2} \int_{\mathbb{R}^d} |\nabla_x h|^2 \rho_\infty dx \tag{4.7}$$

holds for all $h \in L^2(\mathbb{R}^d, \rho_\infty)$ with $|\nabla_x h| \in L^2(\mathbb{R}^d, \rho_\infty)$.

ii) There exist constants $c_1 > 0$, $c_2 \in [0, 1)$, and $c_3 > 0$ such that

$$\Delta_x V(x) \leq c_1 + \frac{c_2}{2}|\nabla_x V(x)|^2, \quad \left\|\frac{\partial^2 V(x)}{\partial x^2}\right\|_F \leq c_3(1 + |\nabla_x V(x)|), \quad \forall x \in \mathbb{R}^d, \tag{4.8}$$

where $\left\|\frac{\partial^2 V(x)}{\partial x^2}\right\|_F := \sqrt{\sum_{i,j=1}^d (\partial_{x_i x_j} V(x))^2}$ *is the Frobenius norm of* $\frac{\partial^2 V(x)}{\partial x^2}$.

There are a lot of studies and sufficient conditions implying the Poincaré inequality (4.7). For example, if V is uniformly convex (Bakry-Emery criterion) or if

$$\liminf_{|x|\to\infty} \left(a|\nabla V(x)|^2 - \Delta V(x)\right) > 0$$

for some $a \in (0, 1)$, then the Poincaré inequality (4.7) holds. For more information see [6], [7, Chapter 4]. We note that the potentials of the form

$$V(x) = r|x|^{2k} + \tilde{V}(x),$$

where $r > 0$, $k > 1$ and $\tilde{V}\colon \mathbb{R}^d \to \mathbb{R}$ is a polynomial of degree $j < 2k$, satisfy our assumptions.

We now state our first result:

Theorem 4.2.2 (Exponential decay in $L^2(\mathbb{R}^{2d}, f_\infty)$). *Let $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2d}, f_\infty)$ and V satisfy Assumption 4.2.1. Then there are explicitly computable constants $C_1 > 0$ and $\lambda > 0$ (independent of f_0) such that*

$$\left\|\frac{f(t) - f_\infty}{f_\infty}\right\|_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq C_1 e^{-\lambda t} \left\|\frac{f_0 - f_\infty}{f_\infty}\right\|_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

holds for all $t \geq 0$.

Theorem 4.2.2 shows that the solution $\frac{f(t)-f_\infty}{f_\infty}$ converges exponentially to zero in $L^2(\mathbb{R}^{2d}, f_\infty)$ as $t \to \infty$. Next we want to obtain this convergence result in a more regular space. Hence, we define the following weighted Sobolev space $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ associated to the norm

$$\begin{aligned} ||h||^2_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)} &:= \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp + \int_{\mathbb{R}^{2d}} \frac{1}{V_0^3(x)p_0^3} \nabla_x^T h \left(I - \frac{p\otimes p}{p_0^2}\right) \nabla_x h f_\infty dxdp \\ &\quad + \int_{\mathbb{R}^{2d}} \frac{1}{V_0(x)p_0} \nabla_p^T h (I + p\otimes p) \nabla_p h f_\infty dxdp \\ &= \int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp + \int_{\mathbb{R}^{2d}} \frac{1}{V_0^3(x)p_0^3} \left(|\nabla_x h|^2 - \frac{|p\cdot\nabla_x h|^2}{p_0^2}\right) f_\infty dxdp \\ &\quad + \int_{\mathbb{R}^{2d}} \frac{1}{V_0(x)p_0} (|\nabla_p h|^2 + |p\cdot \nabla_p h|^2) f_\infty dxdp. \end{aligned} \tag{4.9}$$

This norm is well-defined since the matrices $\frac{1}{V_0^3 p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right)$ and $\frac{1}{V_0 p_0}(I + p\otimes p)$ are positive definite for all $x, p \in \mathbb{R}^d$. Clearly, $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty) \subset L^2(\mathbb{R}^{2d}, f_\infty)$.

Our second result shows that the solution $\frac{f(t)-f_\infty}{f_\infty}$ converges exponentially to zero in $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ as $t \to \infty$:

Theorem 4.2.3 (**Exponential decay in** $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$)**.** *Let* $\frac{f_0}{f_\infty} \in \mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ *and* V *satisfy Assumption 4.2.1. Then there are constants* $C_2 > 0$ *and* $\Lambda > 0$ *(independent of* f_0*) such that*

$$\left|\left|\frac{f(t)-f_\infty}{f_\infty}\right|\right|_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)} \le C_2 e^{-\Lambda t} \left|\left|\frac{f_0-f_\infty}{f_\infty}\right|\right|_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)}$$

holds for all $t \ge 0$.

Our next result is about the estimates on the hypoelliptic regularization:

Theorem 4.2.4 (**Hypoelliptic regularity from** $L^2(\mathbb{R}^{2d}, f_\infty)$ **to** $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$)**.** *Assume* $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2d}, f_\infty)$ *and that there exists a constant* $c_3 > 0$ *such that*

$$\left|\left|\frac{\partial^2 V(x)}{\partial x^2}\right|\right|_F \le c_3(1 + |\nabla_x V(x)|), \quad \forall x \in \mathbb{R}^d.$$

Then, for any $t_0 > 0$, *there are explicitly computable constants* $C_3 > 0$ *and* $C_4 > 0$ *(independent of* f_0*) such that*

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0^3(x)p_0^3} \nabla_x^T \left(\frac{f(t)}{f_\infty}\right) \left(I - \frac{p\otimes p}{p_0^2}\right) \nabla_x \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdp \le \frac{C_3}{t^3} \int_{\mathbb{R}^{2d}} \left(\frac{f(t)-f_\infty}{f_\infty}\right)^2 f_\infty dxdp \tag{4.10}$$

and

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0(x)p_0} \nabla_p^T \left(\frac{f(t)}{f_\infty}\right) (I + p\otimes p) \nabla_p \left(\frac{f(t)}{f_\infty}\right) f_\infty dxdp \le \frac{C_4}{t} \int_{\mathbb{R}^{2d}} \left(\frac{f(t)-f_\infty}{f_\infty}\right)^2 f_\infty dxdp \tag{4.11}$$

hold for all $t \in (0, t_0]$. *In particular,*

$$\left|\left|\frac{f(t)-f_\infty}{f_\infty}\right|\right|_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)} \le \frac{(C_3 + C_4 t_0^2)^{1/2}}{t^{3/2}} \left|\left|\frac{f_0-f_\infty}{f_\infty}\right|\right|_{L^2(\mathbb{R}^{2d},f_\infty)} \tag{4.12}$$

holds for all $t \in (0, t_0]$.

Theorem 4.2.4 shows that, for any initial data $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2d}, f_\infty)$, the solution $\frac{f(t)}{f_\infty} \in \mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ for any time $t > 0$. Compared to Theorem 4.2.2 and Theorem 4.2.3, we do not require the validity of a Poincaré inequality in Theorem 4.2.4. It is also important to note that the regularization rates for the x derivative and the p derivative are not the same: the regularization rate in the p derivative is faster, as it also is for the classical kinetic Fokker-Planck equation [24, 31, 5]. This difference is expected since (4.5) can be considered as a transport equation with respect to the x variable and as a parabolic equation with respect to the p variable.

In Theorem 4.2.3 we assumed that the initial data f_0/f_∞ is in $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$. If we use the regularity estimates from Theorem 4.2.4, this condition can be relaxed:

Corollary 4.2.5. *Let $\frac{f_0}{f_\infty} \in L^2(\mathbb{R}^{2d}, f_\infty)$ and V satisfies Assumption 4.2.1. Then, for any $t_0 > 0$, there is explicitly computable constant $C_5 > 0$ (independent of f_0) such that*

$$\left\| \frac{f(t) - f_\infty}{f_\infty} \right\|_{\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)} \leq C_5 e^{-\Lambda t} \left\| \frac{f_0 - f_\infty}{f_\infty} \right\|_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

holds for all $t \geq t_0 > 0$, where $\Lambda > 0$ is the constant appearing in Theorem 4.2.3.

Remark 4.2.6. *If one considers (4.5) on a torus as done in [11], our results also hold in this setting since the method which we use can be adapted without difficulty.*

4.3 Exponential convergence in L^2

4.3.1 The first Lyapunov functional

Let us consider the relativistic homogeneous Fokker-Planck equation

$$\begin{cases} \partial_t \varrho = \operatorname{div}_p(D \nabla_p \varrho + p\varrho), \quad p \in \mathbb{R}^d, \quad t > 0, \\ \varrho_{|t=0} = \varrho_0. \end{cases}$$

This equation is a case of (4.5) when we do not have dependence on x and $V = 0$. The unique normalized global equilibrium for this equation is $M(p) = \dfrac{e^{-\sqrt{1+|p|^2}}}{\int_{\mathbb{R}^d} e^{-\sqrt{1+|p'|^2}} dp'}$. The convergence

$$\varrho(t) \to M \quad \text{as} \quad t \to \infty$$

can be easily proven using the Poincaré inequality (4.6):

$$\begin{aligned} \frac{d}{dt} \left\| \frac{\varrho(t) - M}{M} \right\|^2_{L^2(\mathbb{R}^d, M)} &= -2 \int_{\mathbb{R}^d} \nabla_p^T \left(\frac{\varrho(t)}{M} \right) D \nabla_p \left(\frac{\varrho(t)}{M} \right) M dp \\ &\leq -2\kappa_1 \left\| \frac{\varrho(t) - M}{M} \right\|^2_{L^2(\mathbb{R}^d, M)}, \quad \forall t > 0. \end{aligned}$$

By Grönwall's lemma we obtain the exponential decay

$$\left\| \frac{\varrho(t) - M}{M} \right\|_{L^2(\mathbb{R}^d, M)} \leq e^{-\kappa_1 t} \left\| \frac{\varrho_0 - M}{M} \right\|_{L^2(\mathbb{R}^d, M)}, \quad \forall t > 0.$$

On the contrary, we do not obtain easily such exponential decay for the relativistic, spatially inhomogeneous Fokker-Planck equation (4.5). As the Fokker-Planck operator on the right hand side of (4.5) acts only on the variable p, we only have

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}}\left|\left|\frac{f(t)-f_\infty}{f_\infty}\right|\right|^2_{L^2(\mathbb{R}^{2d},f_\infty)} = -2\int_{\mathbb{R}^{2d}}\nabla_p^T\left(\frac{f(t)}{f_\infty}\right)D\nabla_p\left(\frac{f(t)}{M}\right)f_\infty dxdp \le 0. \quad (4.13)$$

The integral on the right hand side only gives information on the $p-$derivative and it is lacking information on the $x-$derivatives. Hence, in general, the integral on the right hand side of (4.13) is not bigger than $2\lambda\left|\left|\frac{f(t)-f_\infty}{f_\infty}\right|\right|^2_{L^2(\mathbb{R}^{2d},f_\infty)}$ for some $\lambda > 0$.

The idea to overcome this difficulty is to construct an appropriate Lyapunov functional which is equivalent to the L^2-norm and satisfies a Grönwall type differential inequality under the evolution of the solution. A method in Hilbert spaces was introduced by Dolbeault, Mouhot and Schmeiser in [17, 18] for proving exponential stability for a large class of linear kinetic models confined by an external potential. We will apply this method for (4.5) and it is crucial to obtain our results. In the following we explain this method. We consider linear kinetic equations of the form

$$\partial_t f + \mathrm{T}f = \mathrm{L}f, \;\; t>0 \quad (4.14)$$

in a Hilbert space $\mathcal{H}$ with an initial data $f_{|t=0} = f_0 \in \mathcal{H}$. Here, T and L are closed linear operators such that $\mathrm{L}-\mathrm{T}$ generates the strongly continuous semigroup $e^{(\mathrm{L}-\mathrm{T})t}$ on $\mathcal{H}$. Let I be the identity operator, Π be the orthogonal projection on the null space $\mathcal{N}(\mathrm{L})$ of L, the domains of T and L are denoted by $\mathcal{D}(\mathrm{T})$ and $\mathcal{D}(\mathrm{L})$ respectively. We define the operator

$$\mathrm{A}f := (\mathrm{I}+(\mathrm{T}\Pi)^*\mathrm{T}\Pi)^{-1}(\mathrm{T}\Pi)^*f, \quad f\in\mathcal{H}$$

and a functional

$$\mathrm{H}_\delta[f] := \frac{1}{2}||f||^2 + \delta\langle \mathrm{A}f, f\rangle, \quad f\in\mathcal{H}, \quad \delta>0, \quad (4.15)$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathcal{H}$, and $||\cdot||$ denotes the norm on $\mathcal{H}$ associated with the scalar product. We assume the following conditions are satisfied:

- (*microscopic coercivity*) L is symmetric and there exists $\lambda_m > 0$ such that

$$-\langle \mathrm{L}f, f\rangle \ge \lambda_m||(\mathrm{I}-\Pi)f||^2 \quad \text{for all} \quad f\in\mathcal{D}(\mathrm{L}). \quad (4.16)$$

- (*macroscopic coercivity*) T is skew symmetric and there exists $\lambda_M > 0$ such that

$$||\mathrm{T}\Pi f||^2 \ge \lambda_M||\Pi f||^2 \quad \text{for all} \quad f\in\mathcal{H} \;\; \text{with} \;\; \Pi f\in\mathcal{D}(\mathrm{T}). \quad (4.17)$$

- (*parabolic macroscopic dynamics*)

$$\Pi\mathrm{T}\Pi = 0. \quad (4.18)$$

- (*boundedness of auxiliary operators*) The operators $\mathrm{AT}(\mathrm{I}-\Pi)$ and AL are bounded, and there exists a constant $C_M > 0$ such that, for all $f\in\mathcal{H}$,

$$||\mathrm{AT}(\mathrm{I}-\Pi)f|| + ||\mathrm{AL}|| \le C_M||(\mathrm{I}-\Pi)f||. \quad (4.19)$$

We define

$$\delta_0 := \min\left\{2, \lambda_m, \frac{4\lambda_m\lambda_M}{4\lambda_M + C_M^2(1+\lambda_M)}\right\}. \quad (4.20)$$

Under the validity of these conditions and for $\delta\in(0,\delta_0)$, one can show that H_δ is a Lypunov functional for (4.14) and it decays exponentially:

Theorem 4.3.1 ([18, Theorem 2]). *Assume* (4.16)-(4.19) *are satisfied and* $\delta \in (0, \delta_0)$. *Then,*

i) H *and* $||\cdot||^2$ *are equivalent, more precisely,*

$$\frac{2-\delta}{4}||f||^2 \leq \mathrm{H}_\delta[f] \leq \frac{2+\delta}{4}||f||^2 \quad \text{for all } f \in \mathcal{H}.$$

ii) There exists a positive constant λ, *which is computable in terms of* λ_m, λ_M *and* C_M, *such that, for any initial data* $f_0 \in \mathcal{H}$,

$$\frac{d}{dt}\mathrm{H}[e^{(\mathrm{L}-\mathrm{T})t} f_0] \leq -2\lambda \mathrm{H}[e^{(\mathrm{L}-\mathrm{T})t} f_0], \quad t > 0.$$

In particular, we have

$$||e^{(\mathrm{L}-\mathrm{T})t} f_0|| \leq \sqrt{\frac{2+\delta}{2-\delta}} e^{-\lambda t}||f_0|| \quad \text{for all } t \geq 0. \tag{4.21}$$

This method has been successfully applied to study the long time behavior of various linear kinetic models, see [18, 9, 20, 1]. In particular, in [18, Theorem 10], the exponential convergence $f(t) \to f_\infty$ in $L^2(\mathbb{R}^{2d}, f_\infty)$ as $t \to \infty$ was proven for the classical kinetic Fokker-Planck equation (2.1).

4.3.2 Weighted Poincaré inequalities and an elliptic regularity result

In this section we consider the elliptic equation

$$u(x) - \frac{a}{\rho_\infty(x)}\mathrm{div}_x(\nabla_x u(x)\rho_\infty(x)) = w(x), \qquad x \in \mathbb{R}^d, \tag{4.22}$$

where u is unknown, a is a positive constant, and w is a given function. We will establish some regularity estimates for this equation in $L^2(\mathbb{R}^d, \rho_\infty)$:

Theorem 4.3.2. *Let* $w \in L^2(\mathbb{R}^d, \rho_\infty)$ *and* $\int_{\mathbb{R}^d} w\rho_\infty dx = 0$. *Assume that the potential* V *satifies Assumption 4.2.1. Then, there are positive constants* C_1 *and* C_2 *such that*

$$\int_{\mathbb{R}^d} |\nabla_x u|^2 |\nabla_x V|^2 \rho_\infty dx \leq C_1 \int_{\mathbb{R}^d} w^2 \rho_\infty dx, \tag{4.23}$$

$$\int_{\mathbb{R}^d} \left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2 \rho_\infty dx \leq C_2 \int_{\mathbb{R}^d} w^2 \rho_\infty dx. \tag{4.24}$$

To prove Theorem 4.3.2, we need the weighted Poincaré inequalities (4.25) and (4.26) below. We mention that these inequalities were obtained in [18] in a general setting, but we provide proofs for being self-contained.

Lemma 4.3.3. *Assume that Assumption 4.2.1 holds. Then*

i) There exists $\kappa_3 > 0$ *such that*

$$\int_{\mathbb{R}^d} h^2 |\nabla_x V|^2 \rho_\infty dx \leq \frac{1}{\kappa_3} \int_{\mathbb{R}^d} |\nabla_x h|^2 \rho_\infty dx \tag{4.25}$$

holds for all $h \in L^2(\mathbb{R}^d, \rho_\infty)$ *with* $|\nabla_x h| \in L^2(\mathbb{R}^d, \rho_\infty)$ *and* $\int_{\mathbb{R}^d} h\rho_\infty dx = 0$.

ii) There exists $\kappa_4 > 0$ *such that*

$$\int_{\mathbb{R}^d} h^2(1+|\nabla_x V|^2)|\nabla_x V|^2 \rho_\infty dx \le \frac{1}{\kappa_4}\int_{\mathbb{R}^d} |\nabla_x h|^2(1+|\nabla_x V|^2)\rho_\infty dx \tag{4.26}$$

holds for all $h \in L^2(\mathbb{R}^d, \rho_\infty)$ *with* $|\nabla_x h|(1+|\nabla_x V|) \in L^2(\mathbb{R}^d, \rho_\infty)$ *and* $\int_{\mathbb{R}^d} h\rho_\infty dx = 0$.

Proof. *i)* By the identity $\sqrt{\rho_\infty}\nabla_x h = \nabla_x(h\sqrt{\rho_\infty}) + \frac{h\sqrt{\rho_\infty}}{2}\nabla_x V$ and integrating by parts

$$\int_{\mathbb{R}^d} |\nabla_x h|^2 \rho_\infty dx \ge \frac{1}{4}\int_{\mathbb{R}^d} h^2|\nabla_x V|^2\rho_\infty dx + \int_{\mathbb{R}^d} h\sqrt{\rho_\infty}\nabla_x(h\sqrt{\rho_\infty})\cdot\nabla_x V dx$$
$$= \frac{1}{4}\int_{\mathbb{R}^d} h^2|\nabla_x V|^2\rho_\infty dx - \frac{1}{2}\int_{\mathbb{R}^d} h^2\Delta_x V\rho_\infty dx.$$

This estimate and the first condition in (4.8) show

$$\int_{\mathbb{R}^d} |\nabla_x h|^2\rho_\infty dx \ge \frac{1-c_2}{4}\int_{\mathbb{R}^d} h^2|\nabla_x V|^2\rho_\infty dx - \frac{c_1}{2}\int_{\mathbb{R}^d} h^2\rho_\infty dx.$$

Then, (4.7) lets us obtain (4.25) with the constant $\kappa_3 := \frac{(1-c_2)(c_1+2\kappa_2)}{8\kappa_2}$.

ii) We recall $V_0 := \sqrt{1+|\nabla_x V|^2}$. Let $\bar{h} := \int_{\mathbb{R}^d} hV_0\rho_\infty dx$, then by (4.25)

$$\int_{\mathbb{R}^d} (hV_0 - \bar{h})^2|\nabla_x V|^2\rho_\infty dx \le \frac{1}{\kappa_3}\int_{\mathbb{R}^d} |\nabla_x(hV_0)|^2\rho_\infty dx.$$

This leads

$$\int_{\mathbb{R}^d} h^2V_0^2|\nabla_x V|^2\rho_\infty dx \le \frac{1}{\kappa_3}\int_{\mathbb{R}^d} |\nabla_x(hV_0)|^2\rho_\infty dx + 2\bar{h}\int_{\mathbb{R}^d} hV_0|\nabla_x V|^2\rho_\infty dx. \tag{4.27}$$

Next, we estimate the terms on the right hand side of (4.27):

$$\int_{\mathbb{R}^d} |\nabla_x(hV_0)|^2\rho_\infty dx = \int_{\mathbb{R}^d}\left|\nabla_x hV_0 + h\frac{\partial^2 V}{\partial x^2}\frac{\nabla_x V}{V_0}\right|^2\rho_\infty dx$$
$$\le 2\int_{\mathbb{R}^d} |\nabla_x h|^2V_0^2\rho_\infty dx + 2\int_{\mathbb{R}^d} h^2\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F^2\frac{|\nabla_x V|^2}{V_0^2}\rho_\infty dx$$
$$\le 2\int_{\mathbb{R}^d} |\nabla_x h|^2V_0^2\rho_\infty dx + 4c_3^2\int_{\mathbb{R}^d} h^2|\nabla_x V|^2\rho_\infty dx$$
$$\le 2\int_{\mathbb{R}^d} |\nabla_x h|^2V_0^2\rho_\infty dx + \frac{4c_3^2}{\kappa_3}\int_{\mathbb{R}^d} |\nabla_x h|^2\rho_\infty dx, \tag{4.28}$$

where we used the second condition in (4.8) and (4.25). By the Hölder inequality and (4.7)

$$|\bar{h}| \le ||V_0||_{L^2(\mathbb{R}^d,\rho_\infty)}||h||_{L^2(\mathbb{R}^d,\rho_\infty)} \le \frac{1}{\sqrt{\kappa_2}}||V_0||_{L^2(\mathbb{R}^d,\rho_\infty)}||\nabla_x h||_{L^2(\mathbb{R}^d,\rho_\infty)}.$$

We note here $||V_0||_{L^2(\mathbb{R}^d,\rho_\infty)}$ is finite, because the first condition in (4.8) yields:

$$\int_{\mathbb{R}^d} |\nabla_x V|^2\rho_\infty dx = \int_{\mathbb{R}^d} \Delta_x V\rho_\infty dx \le c_1 + \frac{c_2}{2}\int_{\mathbb{R}^d} |\nabla_x V|^2\rho_\infty dx,$$

hence $\int_{\mathbb{R}^d} |\nabla_x V|^2 \rho_\infty dx \leq \frac{2c_1}{2-c_2}$. Then, the Hölder inequality shows

$$2\bar{h}\int_{\mathbb{R}^d} hV_0|\nabla_x V|^2\rho_\infty dx \leq \frac{2||V_0||^4_{L^2(\mathbb{R}^d,\rho_\infty)}}{\kappa_2}\int_{\mathbb{R}^d}|\nabla_x h|^2\rho_\infty dx + \frac{1}{2}\int_{\mathbb{R}^d} h^2V_0^2|\nabla_x V|^2\rho_\infty dx. \tag{4.29}$$

(4.27), (4.28), and (4.29) yield

$$\int_{\mathbb{R}^d} h^2V_0^2|\nabla_x V|^2\rho_\infty dx \leq \int_{\mathbb{R}^d}(4\kappa_2^{-1}||V_0||^4_{L^2(\mathbb{R}^d,\rho_\infty)} + 8c_3^2\kappa_3^{-1} + 4V_0^2)|\nabla_x h|^2\rho_\infty dx.$$

Therefore, we obtain (4.26) with $\kappa_4^{-1} := 4\kappa_2^{-1}||\mathcal{V}||^4_{L^2(\mathbb{R}^d,\rho_\infty)} + 8c_3^2\kappa_3^{-1} + 4$.

□

Proof of Theorem 4.3.2. Multiplying (4.22) by ρ_∞ and integrating by parts we obtain

$$\int_{\mathbb{R}^d} u\rho_\infty dx = \int_{\mathbb{R}^d} w\rho_\infty dx = 0.$$

We multiply (4.22) by $u\rho_\infty$ and integrate by parts

$$\int_{\mathbb{R}^d} u^2\rho_\infty dx + a\int_{\mathbb{R}^d}|\nabla_x u|^2\rho_\infty dx = \int_{\mathbb{R}^d} uw\rho_\infty dx.$$

The Hölder inequality provides

$$\int_{\mathbb{R}^d} u^2\rho_\infty dx + 2a\int_{\mathbb{R}^d}|\nabla_x u|^2\rho_\infty dx \leq \int_{\mathbb{R}^d} w^2\rho_\infty dx. \tag{4.30}$$

We start proving (4.23): we multiply (4.22) by $u|\nabla_x V|^2\rho_\infty$ and integrate by parts

$$\begin{aligned}\int_{\mathbb{R}^d} u^2|\nabla_x V|^2\rho_\infty dx &+ a\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx \\ &= \int_{\mathbb{R}^d} wu|\nabla_x V|^2\rho_\infty dx - a\int_{\mathbb{R}^d} u\nabla_x u\cdot\nabla_x(|\nabla_x V|^2)\rho_\infty dx. \end{aligned}\tag{4.31}$$

Using the Hölder inequality we estimate the terms on the right hand side of (4.31)

$$\int_{\mathbb{R}^d} wu|\nabla_x V|^2\rho_\infty dx \leq \frac{1}{2\delta}\int_{\mathbb{R}^d} w^2\rho_\infty dx + \frac{\delta}{2}\int_{\mathbb{R}^d} u^2|\nabla_x V|^4\rho_\infty dx \tag{4.32}$$

and

$$\begin{aligned}-a\int_{\mathbb{R}^d} u\nabla_x u\cdot\nabla_x(|\nabla_x V|^2)\rho_\infty dx &= -2a\int_{\mathbb{R}^d} u\nabla_x u\cdot\left(\frac{\partial^2 V}{\partial x^2}\nabla_x V\right)\rho_\infty dx \\ &\leq 2a\int_{\mathbb{R}^d}|u||\nabla_x u|\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F|\nabla_x V|\rho_\infty dx \\ &\leq \varepsilon\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx + \frac{a^2}{\varepsilon}\int_{\mathbb{R}^d} u^2\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F^2\rho_\infty dx, \end{aligned}\tag{4.33}$$

where $\delta > 0$, $\varepsilon > 0$. (4.31), (4.32), and (4.33) show that

$$\begin{aligned}\int_{\mathbb{R}^d} u^2|\nabla_x V|^2\rho_\infty dx &+ (a-\varepsilon)\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx \\ &\leq \frac{1}{2\delta}\int_{\mathbb{R}^d} w^2\rho_\infty dx + \frac{\delta}{2}\int_{\mathbb{R}^d} u^2|\nabla_x V|^4\rho_\infty dx + \frac{a^2}{\varepsilon}\int_{\mathbb{R}^d} u^2\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F^2\rho_\infty dx. \end{aligned}\tag{4.34}$$

The Poincaré inequality (4.26) and (4.30) imply

$$\int_{\mathbb{R}^d} u^2|\nabla_x V|^4\rho_\infty dx \le \frac{1}{\kappa_4}\int_{\mathbb{R}^d}|\nabla_x u|^2(1+|\nabla_x V(x)|^2)\rho_\infty dx$$
$$\le \frac{1}{2a\kappa_4}\int_{\mathbb{R}^d} w^2\rho_\infty dx + \frac{1}{\kappa_4}\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx. \quad (4.35)$$

To estimate the last term in (4.34), we use the second condition in (4.8), (4.30), and the Poincaré inequality (4.25)

$$\int_{\mathbb{R}^d} u^2\left|\left|\frac{\partial^2 V}{\partial x^2}\right|\right|_F^2\rho_\infty dx \le 2c_3^2\int_{\mathbb{R}^d} u^2(1+|\nabla_x V|^2)\rho_\infty dx$$
$$\le 2c_3^2\left(\int_{\mathbb{R}^d} w^2\rho_\infty dx + \frac{1}{\kappa_3}\int_{\mathbb{R}^d}|\nabla_x u|^2\rho_\infty dx\right) \le 2c_3^2\left(1+\frac{1}{2a\kappa_3}\right)\int_{\mathbb{R}^d} w^2\rho_\infty dx. \quad (4.36)$$

(4.34), (4.35), and (4.36) show that

$$\int_{\mathbb{R}^d} u^2|\nabla_x V|^2\rho_\infty dx + (a-\varepsilon-\frac{\delta}{2\kappa_4})\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx$$
$$\le \left[\frac{1}{2\delta}+\frac{\delta}{4a\kappa_4}+\frac{2c_3^2a^2}{\varepsilon}\left(1+\frac{1}{2a\kappa_3}\right)\right]\int_{\mathbb{R}^d} w^2\rho_\infty dx. \quad (4.37)$$

We choose δ and ε such that $a-\varepsilon-\frac{\delta}{2\kappa_4}>0$. Then, (4.37) shows that (4.23) holds with $C_1 := \frac{1}{a-\varepsilon-\frac{\delta}{2\kappa_4}}\left[\frac{1}{2\delta}+\frac{\delta}{4a\kappa_4}+\frac{2c_3^2a^2}{\varepsilon}\left(1+\frac{1}{2a\kappa_3}\right)\right]$.

Next, we prove (4.24): We integrate by parts

$$\int_{\mathbb{R}^d}\left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2\rho_\infty dx = \sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^2_{x_ix_j}u\,\partial^2_{x_ix_j}u\,\rho_\infty dx$$
$$= -\sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^3_{x_ix_jx_j}u\,\partial_{x_i}u\,\rho_\infty dx + \sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^2_{x_ix_j}u\,\partial_{x_i}u\,\partial_{x_j}V\rho_\infty dx$$
$$= \sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^2_{x_jx_j}u\,\partial^2_{x_ix_i}u\,\rho_\infty dx - \sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^2_{x_jx_j}u\,\partial_{x_i}u\,\partial_{x_i}V\rho_\infty dx + \sum_{i,j=1}^d\int_{\mathbb{R}^d}\partial^2_{x_ix_j}u\,\partial_{x_i}u\,\partial_{x_j}V\rho_\infty dx$$
$$= \int_{\mathbb{R}^d}|\Delta_x u|^2\rho_\infty dx - \int_{\mathbb{R}^d}\Delta_x u\nabla_x u\cdot\nabla_x V\rho_\infty dx + \int_{\mathbb{R}^d}\nabla_x^T u\frac{\partial^2 u}{\partial x^2}\nabla_x V\rho_\infty dx. \quad (4.38)$$

We multiply (4.22) by $\Delta_x u$ to get

$$a|\Delta_x u|^2 = \Delta_x u(u-w) + a\Delta_x u\nabla_x u\cdot\nabla_x V. \quad (4.39)$$

We use (4.39) and the Hölder inequality to estimate (4.38)

$$\int_{\mathbb{R}^d}\left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2\rho_\infty dx = a^{-1}\int_{\mathbb{R}^d}\Delta_x u(u-w)\rho_\infty dx + \int_{\mathbb{R}^d}\nabla_x^T u\frac{\partial^2 u}{\partial x^2}\nabla_x V\rho_\infty dx$$
$$\le a^{-1}\sqrt{\int_{\mathbb{R}^d}|\Delta_x u|^2\rho_\infty dx}\sqrt{\int_{\mathbb{R}^d}(u-w)^2\rho_\infty dx}$$
$$+\sqrt{\int_{\mathbb{R}^d}\left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2\rho_\infty dx}\sqrt{\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx}.$$

This inequality and $|\Delta_x u|^2 \le d \left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2$ show that

$$\sqrt{\int_{\mathbb{R}^d} \left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_F^2 \rho_\infty dx} \le a^{-1}\sqrt{d\int_{\mathbb{R}^d}(u-w)^2\rho_\infty dx} + \sqrt{\int_{\mathbb{R}^d}|\nabla_x u|^2|\nabla_x V|^2\rho_\infty dx}. \tag{4.40}$$

Finally, (4.40), (4.30), and (4.23) yield (4.24).

□

4.3.3 Proof of Theorem 4.2.2

Let $h := \frac{f-f_\infty}{f_\infty}$. Then (4.5) can be written as

$$\begin{cases} \partial_t h + \frac{p}{p_0}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h = \frac{1}{f_\infty}\mathrm{div}_p(D\nabla_p h f_\infty) \\ h_{|t=0} = \frac{f_0-f_\infty}{f_\infty}. \end{cases} \tag{4.41}$$

We shall apply Theorem 4.3.1 to (4.41). To do that, we first define a proper Hilbert space

$$\mathcal{H} := \left\{h \in L^2(\mathbb{R}^{2d}, f_\infty) : \int_{\mathbb{R}^{2d}} h f_\infty dx dp = 0\right\}$$

with the scalar product $\langle h, g\rangle := \int_{\mathbb{R}^{2d}} h_1 h_2 f_\infty dx dp$ and the norm $||h||_{L^2(\mathbb{R}^{2d},f_\infty)} = \sqrt{\langle h, h\rangle}$. We note that if $h(t)$ is the solution of (4.41), then the conservation of mass (4.2) shows $h(t) \in \mathcal{H}$ for all $t \ge 0$. We can present (4.41) in the form of (4.14) with

$$\mathrm{T}h := \frac{p}{p_0}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h \tag{4.42}$$

and

$$\mathrm{L}h := \frac{1}{f_\infty}\mathrm{div}_p(D\nabla_p h f_\infty). \tag{4.43}$$

T can be defined in $C_0^1(\mathbb{R}^{2d})$, and L can be defined in the space of continuous functions which have compact support and continuous second order derivatives with respect to p. These operators can be extended using the Friederichs extension, but we omit details concerning domain issues and extensions as we need only properties that apply to solutions of the evolution problem (4.41).

We define

$$\Pi h = \Pi h(x) := \int_{\mathbb{R}^d} h(x,p')M(p')dp', \quad h \in \mathcal{H}. \tag{4.44}$$

It is easy to check that Π is a symmetric operator in $\mathcal{H}$ and $\Pi\circ\Pi = \Pi$.

In the following proposition we show that the operators defined in (4.42)-(4.44) satisfy the conditions (4.16)-(4.18):

Proposition 4.3.4. *Assume that Assumption 4.2.1 holds. Then we have*

i) T *and* L *are, respectively, skew-symmetric and symmetric operators in* $\mathcal{H}$.

ii) Π *is the orthogonal projection on the null space* $\mathcal{N}(\mathrm{L})$ *of* L. *Microscopic coercivity* (4.16) *holds with* $\lambda_m = \kappa_1$, *where* κ_1 *is the constant appearing in the Poincaré inequality* (4.6).

iii) Macroscopic coercivity (4.17) *holds with* $\lambda_M = \kappa_2 \left(\int_{\mathbb{R}^d} \frac{1}{(1+|p|^2)^{3/2}} M dp \right)^{-1}$, *where* κ_2 *is the constant in the Poincaré inequality* (4.7).

iv) $\Pi \mathrm{T} \Pi = 0$.

Proof. *i*) Let h, $g \in \mathcal{H}$ be smooth functions with compact support. The equations

$$\nabla_x f_\infty = -\nabla_x V f_\infty, \qquad \nabla_p f_\infty = -\frac{p}{p_0} f_\infty,$$

and integration by parts yield

$$\langle \mathrm{T}h, g \rangle = \int_{\mathbb{R}^{2d}} \left(\frac{p}{p_0} \cdot \nabla_x h - \nabla_x V \cdot \nabla_p h \right) g f_\infty dx dp$$
$$= -\int_{\mathbb{R}^{2d}} \left(\frac{p}{p_0} \cdot \nabla_x g - \nabla_x V \cdot \nabla_p g \right) h f_\infty dx dp = -\langle h, \mathrm{T}g \rangle.$$

Then, integrating by parts we show that L is symmetric:

$$\langle \mathrm{L}h, g \rangle = \int_{\mathbb{R}^{2d}} \mathrm{div}_p (D \nabla_p h f_\infty) g dx dp = -\int_{\mathbb{R}^{2d}} \nabla_p^T h D \nabla_p g f_\infty dx dp = \langle h, \mathrm{L}g \rangle. \tag{4.45}$$

ii) As $D = D(p)$ is positive definite for all $p \in \mathbb{R}^d$, (4.45) implies

$$\langle \mathrm{L}h, h \rangle = -\int_{\mathbb{R}^{2d}} \nabla_p^T h D \nabla_p h f_\infty dx dp \leq 0.$$

This shows that $\mathrm{L}h$ vanishes if h is constant with respect to p, in particular $\mathrm{L}\Pi h = 0$. Moreover, the Poincaré inequality (4.6) shows

$$-\langle \mathrm{L}h, h \rangle \geq \kappa_1 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h^2 M dp - \left(\int_{\mathbb{R}^d} h M dp \right)^2 \right) \rho_\infty dx$$
$$= \kappa_1 \int_{\mathbb{R}^{2d}} (h - \Pi h)^2 f_\infty dx dp = \kappa_1 ||(\mathrm{I} - \Pi) h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

This justifies that Π is the orthogonal projection on the null space $\mathcal{N}(\mathrm{L})$ of L.

iii) Using $\mathrm{T}\Pi h = \frac{p}{p_0} \cdot \nabla_x \Pi h$ we compute

$$||\mathrm{T}\Pi h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} = \sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \frac{p_i p_j}{p_0} \partial_{x_i} \Pi h \, \partial_{x_j} \Pi h f_\infty dx dp$$
$$= -\sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \partial_{p_i} p_0 \partial_{x_i} \Pi h \, \partial_{x_j} \Pi h \partial_{p_j} f_\infty dx dp$$
$$= \sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \partial^2_{p_i p_j} p_0 \partial_{x_i} \Pi h \, \partial_{x_j} \Pi h f_\infty dx dp = \int_{\mathbb{R}^{2d}} \nabla_x^T \Pi h \frac{\partial^2 p_0}{\partial p^2} \nabla_x \Pi h f_\infty dx dp. \tag{4.46}$$

We have

$$\frac{\partial^2 p_0}{\partial p^2} = \frac{1}{p_0} \left(I - \frac{p \otimes p}{p_0^2} \right) \geq \frac{1}{p_0^3} I, \tag{4.47}$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix. (4.46) and (4.47) yield

$$||\mathrm{T}\Pi h||^2_{L^2(\mathbb{R}^{2d},f_\infty)} \geq \left(\int_{\mathbb{R}^d} \frac{1}{p_0^3} M dp\right) \int_{\mathbb{R}^d} |\nabla_x \Pi h|^2 \rho_\infty dx.$$

Then, the Poincaré inequality (4.7) provides the claimed result.

iv) Using $\nabla_p M = -\dfrac{p}{p_0} M$ and integrating by parts with respect to p

$$\Pi\mathrm{T}\Pi h = \int_{\mathbb{R}^d} \frac{p}{p_0} \cdot \nabla_x \Pi h M dp = -\int_{\mathbb{R}^d} \nabla_x \Pi h \cdot \nabla_p M dp = 0.$$

□

We now show that the condition (4.19) holds.

Lemma 4.3.5. *Assume Assumption 4.2.1 holds. Then, the operators* $\mathrm{AT}(1-\Pi)$ *and* AL *are bounded, and there exists a constant* $C_M > 0$ *such that, for all* $h \in \mathcal{H}$,

$$||\mathrm{AT}(\mathrm{I}-\Pi)h||_{L^2(\mathbb{R}^{2d},f_\infty)} + ||\mathrm{AL}||_{L^2(\mathbb{R}^{2d},f_\infty)} \leq C_M ||(\mathrm{I}-\Pi)h||_{L^2(\mathbb{R}^{2d},f_\infty)}.$$

Proof. **Step 1, boundedness of** $\mathrm{AT}(\mathrm{I}-\Pi)$**:**
The operator $\mathrm{AT}(\mathrm{I}-\Pi)$ is bounded if and only if its adjoint

$$[\mathrm{AT}(\mathrm{I}-\Pi)]^* = -(\mathrm{I}-\Pi)\mathrm{T}^2\Pi[\mathrm{I} + (\mathrm{T}\Pi)^*(\mathrm{T}\Pi)]^{-1}$$

is bounded. Let $h \in \mathcal{H}$ and $g := (\mathrm{I} + (\mathrm{T}\Pi)^*\mathrm{T}\Pi)^{-1}h$. Since Π is self-adjoint and T is skew-symmetric, we have

$$[\mathrm{AT}(\mathrm{I}-\Pi)]^* h = -(\mathrm{I}-\Pi)\mathrm{T}^2\Pi g.$$

We compute

$$\mathrm{T}^2\Pi g = \frac{p^T}{p_0}\frac{\partial^2 \Pi g}{\partial x^2}\frac{p}{p_0} - \nabla_x^T V \frac{\partial^2 p_0}{\partial p^2}\nabla_x \Pi g = \sum_{i,j=1}^d \frac{p_i p_j}{p_0^2}\partial^2_{x_i x_j}\Pi g - \sum_{i,j=1}^d \partial_{x_i} V \partial^2_{p_i p_j} p_0 \partial_{x_j}\Pi g. \tag{4.48}$$

We note that $\displaystyle\int_{\mathbb{R}^d} \frac{p_i p_j}{p_0^2} M dp = \int_{\mathbb{R}^d} \partial^2_{p_i p_j} p_0 M dp = 0$ if $i \neq j$, and we denote $a := \displaystyle\int_{\mathbb{R}^d} \frac{p_i^2}{p_0^2} M dp = \frac{1}{d}\int_{\mathbb{R}^d} \frac{|p|^2}{1+|p|^2} M dp$. Then, using (4.48) we compute

$$\Pi\mathrm{T}^2\Pi g = a\Delta_x \Pi g - a\nabla_x V \cdot \nabla_x \Pi g = \frac{a}{\rho_\infty}\mathrm{div}_x(\nabla_x \Pi g \rho_\infty), \tag{4.49}$$

hence

$$h = g + (\mathrm{T}\Pi)^*\mathrm{T}\Pi g = g - \Pi\mathrm{T}^2\Pi g = g - \Pi\mathrm{T}^2\Pi h = a\Delta_x \nabla_x \Pi g - \nabla_x V \cdot \nabla_x \Pi g.$$

Applying the operator Π to this equation we get

$$\Pi g - \frac{a}{\rho_\infty}\mathrm{div}_x(\nabla_x \Pi g \rho_\infty) = \Pi h.$$

Then, by Theorem 4.3.2 we have

$$\int_{\mathbb{R}^d} |\nabla_x \Pi g|^2 |\nabla_x V|^2 \rho_\infty dx \leq C_1 \int_{\mathbb{R}^d} (\Pi h)^2 \rho_\infty dx, \quad \int_{\mathbb{R}^d} \left|\left| \frac{\partial^2 \Pi g}{\partial x^2} \right|\right|_F^2 \rho_\infty dx \leq C_2 \int_{\mathbb{R}^d} (\Pi g)^2 \rho_\infty dx. \tag{4.50}$$

Using (4.48) we estimate

$$\begin{aligned} ||[\mathrm{AT}(\mathrm{I}-\Pi)]^* h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} &= ||(\mathrm{I}-\Pi)\mathrm{T}^2 \Pi g||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq ||\mathrm{T}^2 \Pi g||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} \\ &= \int_{\mathbb{R}^{2d}} \left(\frac{p^T}{p_0} \frac{\partial^2 \Pi g}{\partial x^2} \frac{p}{p_0} - \nabla_x^T V(x) \frac{\partial^2 p_0}{\partial p^2} \nabla_x \Pi g \right)^2 f_\infty dx dp \\ &\leq 2 \int_{\mathbb{R}^{2d}} \left(\frac{p^T}{p_0} \frac{\partial^2 \Pi g}{\partial x^2} \frac{p}{p_0} \right)^2 f_\infty dx dp + 2 \int_{\mathbb{R}^{2d}} \left(\nabla_x^T V(x) \frac{\partial^2 \sqrt{1+|p|^2}}{\partial p^2} \nabla_x \Pi g \right)^2 f_\infty dx dp. \end{aligned} \tag{4.51}$$

Using the Hölder inequality and (4.50) we estimate the last two terms of (4.51):

$$2 \int_{\mathbb{R}^{2d}} \left(\frac{p^T}{p_0} \frac{\partial^2 \Pi g}{\partial x^2} \frac{p}{p_0} \right)^2 f_\infty dx dp \leq 2 \int_{\mathbb{R}^{2d}} \frac{|p|^2}{1+|p|^2} \left|\left| \frac{\partial^2 \Pi g}{\partial x^2} \right|\right|_F^2 f_\infty dx dp \leq a d C_2 \int_{\mathbb{R}^d} (\Pi h)^2 \rho_\infty dx, \tag{4.52}$$

$$\begin{aligned} 2 \int_{\mathbb{R}^{2d}} \left(\nabla_x^T V(x) \frac{\partial^2 p_0}{\partial p^2} \nabla_x \Pi g \right)^2 f_\infty dx dp &\leq 2 \int_{\mathbb{R}^{2d}} \left|\left| \frac{\partial^2 p_0}{\partial p^2} \right|\right|_F |\nabla_x V|^2 |\nabla_x \Pi g|^2 f_\infty dx dp \\ &\leq K_1 \int_{\mathbb{R}^d} (\Pi h)^2 \rho_\infty dx, \end{aligned} \tag{4.53}$$

where $K_1 := 2C_1 \left(\int_{\mathbb{R}^d} \left|\left| \frac{\partial^2 p_0}{\partial p^2} \right|\right|_F M dp \right)$.
(4.51), (4.52), and (4.53) show that

$$\begin{aligned} ||[\mathrm{AT}(\mathrm{I}-\Pi)]h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} &= ||[\mathrm{AT}(\mathrm{I}-\Pi)]^* h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} \\ &\leq (adC_2 + K_1) ||\Pi h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq (adC_2 + K_1) ||h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)}. \end{aligned}$$

This shows that $\mathrm{AT}(\mathrm{I}-\Pi)$ is bounded. Moreover, replacing h with $(\mathrm{I}-\Pi)h$ and using $(\mathrm{I}-\Pi)^2 = (\mathrm{I}-\Pi)$ we obtain

$$||[\mathrm{AT}(\mathrm{I}-\Pi)]h||_{L^2(\mathbb{R}^{2d}, f_\infty)} \leq \sqrt{adC_2 + K_1} ||(\mathrm{I}-\Pi)h||_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

Step 2, boundedness of AL:
Let $h \in \mathcal{H}$ and $g := \mathrm{AL}h$. Then

$$(\mathrm{T}\Pi)^*(\mathrm{L}h) = g + (\mathrm{T}\Pi)^*(\mathrm{T}\Pi) g \quad \Longleftrightarrow \quad g = -\Pi\mathrm{T}(\mathrm{L}h) + \Pi\mathrm{T}^2\Pi g.$$

This shows that $g = \Pi g$. Using (4.49) we obtain

$$g - \frac{a}{\rho_\infty} \mathrm{div}_x(\nabla_x g \rho_\infty) = -\Pi\mathrm{T}(\mathrm{L}h). \tag{4.54}$$

Integrating by parts we find

$$\begin{aligned} \Pi\mathrm{T}(\mathrm{L}h) &= \int_{\mathbb{R}^d} \left[\frac{p}{p_0} \cdot \nabla_x(\mathrm{L}h) - \nabla_x V \cdot \nabla_p(\mathrm{L}h) \right] M dp \\ &= \int_{\mathbb{R}^d} \left[\frac{p}{p_0} \cdot \nabla_x(\mathrm{L}h) - \nabla_x V \cdot \frac{p}{p_0}(\mathrm{L}h) \right] M dp = \frac{1}{\rho_\infty} \mathrm{div}_x \left[\int_{\mathbb{R}^d} \frac{p}{p_0} (\mathrm{L}h) f_\infty dp \right]. \end{aligned} \tag{4.55}$$

Then, for $k \in \{1, ..., d\}$ and $p = (p_1, ..., p_d)^T$, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{p_k}{p_0}(\mathrm{L}h) f_\infty dp &= \int_{\mathbb{R}^d} \frac{p_k}{p_0} \mathrm{div}_p(D\nabla_p h f_\infty) dp \\ &= -\int_{\mathbb{R}^d} \nabla_p^T \left(\frac{p_k}{p_0}\right) D\nabla_p h f_\infty dp = -\int_{\mathbb{R}^d} \left[D\nabla_p \left(\frac{p_k}{p_0}\right)\right] \cdot \nabla_p h f_\infty dp \\ &= -\int_{\mathbb{R}^d} \frac{1}{p_0^2} \partial_{p_k} h f_\infty dp = -\int_{\mathbb{R}^d} \left(\frac{p_k}{p_0^3} + \frac{2p_k}{p_0^4}\right) h f_\infty dp. \end{aligned} \tag{4.56}$$

(4.54), (4.55), and (4.56) show that

$$g - \frac{a}{\rho_\infty} \mathrm{div}_x(\nabla_x g \rho_\infty) = \frac{1}{\rho_\infty} \mathrm{div}_x \left[\int_{\mathbb{R}^d} \left(\frac{p}{p_0^3} + \frac{2p}{p_0^4}\right) h f_\infty dp\right]. \tag{4.57}$$

We multiply this equation by $g\rho_\infty$ and integrate by parts

$$\begin{aligned}\int_{\mathbb{R}^d} g^2 \rho_\infty dx + a \int_{\mathbb{R}^d} \nabla_x^T g \nabla_x g \rho_\infty dx &= -\int_{\mathbb{R}^d} \nabla_x g \cdot \left[\int_{\mathbb{R}^d} \left(\frac{p}{p_0^3} + \frac{2p}{p_0^4}\right) hM dp\right] \rho_\infty dx \\ &\le \varepsilon \int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_\infty dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \left(\frac{p}{p_0^3} + \frac{2p}{p_0^4}\right) hM dp\right|^2 \rho_\infty dx,\end{aligned}$$

where $\varepsilon > 0$ is small enough so that $a - \varepsilon$ is positive. Then by the Hölder inequality

$$\int_{\mathbb{R}^d} g^2 \rho_\infty dx + (a - \varepsilon) \int_{\mathbb{R}^d} \nabla_x^T g \nabla_x g \rho_\infty dx \le \frac{1}{4\varepsilon} \left(\int_{\mathbb{R}^d} \left|\frac{p}{p_0^3} + \frac{2p}{p_0^4}\right|^2 M dp\right) \int_{\mathbb{R}^{2d}} h^2 f_\infty dx dp.$$

This equation implies

$$||g||_{L^2(\mathbb{R}^{2d}, f_\infty)} = ||\mathrm{AL}h||_{L^2(\mathbb{R}^{2d}, f_\infty)} \le K_2 ||h||_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

with $K_2 := \sqrt{\frac{1}{4\varepsilon} \int_{\mathbb{R}^d} \left|\frac{p}{p_0^3} + \frac{2p}{p_0^4}\right|^2 M dp}$. This implies that $\mathrm{AL}h$ is bounded. Moreover, replacing h with $(\mathrm{I} - \Pi)h$ in the equation above and using $\mathrm{L}\Pi = 0$, we obtain

$$||\mathrm{AL}h||_{L^2(\mathbb{R}^{2d}, f_\infty)} \le K_2 ||(\mathrm{I} - \Pi)h||_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

□

Proof of Theorem 4.2.2. Let f be the solution of (4.5). Then $h := \dfrac{f - f_\infty}{f_\infty}$ satisfies

$$\partial_t h + \mathrm{T}h = \mathrm{L}h, \qquad h_{|t=0} = h_0, \tag{4.58}$$

where T and L operators defined in (4.42) and (4.43), respectively. If Π is defined as in (4.44), Proposition 4.3.4 and Lemma 4.3.5 show that these operators satisfy the conditions (4.16)-(4.19). Therefore, Theorem 4.3.1 holds for (4.58), and (4.21) provides the claimed result. □

4.4 Exponential convergence in $\mathscr{H}^1$

In this section, we shall study the long time behavior of (4.41) in $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$. To this end we construct another Lyapunov functional (rather than H_δ which was used in Theorem 4.2.2).

4.4.1 Preliminaries

Lemma 4.4.1. *Let h be the solution of (4.41). Then, for all $t > 0$,*

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp = -2\int_{\mathbb{R}^{2d}} \nabla_p^T h D \nabla_p h f_\infty dxdp.$$

In particular, we have $||h(t)||_{L^2(\mathbb{R}^{2d},f_\infty)} \leq ||h_0||_{L^2(\mathbb{R}^{2d},f_\infty)}$ for all $t \geq 0$.

Proof. We integrate by parts and use $\nabla_p f_\infty = -\frac{p}{p_0} f_\infty$ and $\nabla_x f_\infty = -\nabla_x V f_\infty$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp &= 2\int_{\mathbb{R}^{2d}} h\partial_t h f_\infty dxdp \\ &= -2\int_{\mathbb{R}^{2d}} \left(\frac{p}{p_0}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h\right) h f_\infty dxdp \\ &\quad + 2\int_{\mathbb{R}^{2d}} \operatorname{div}_p(D\nabla_p h f_\infty) h dxdp \\ &= -2\int_{\mathbb{R}^{2d}} \nabla_p^T h D\nabla_p h f_\infty dxdp.\end{aligned}$$

□

Let $P = P(x,p) \in \mathbb{R}^{2d\times 2d}$ be a symmetric, positive definite matrix depending on the variables x, $p \in \mathbb{R}^d$. We define

$$\mathrm{S}_P[h] := \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp. \tag{4.59}$$

For $i,j \in \{1,...,d\}$, we use the notation

$$\delta_{ij} := \begin{cases}1, & \text{if} \quad i=j\\ 0, & \text{if} \quad i\neq j\end{cases}.$$

Lemma 4.4.2. *Let h be the solution of (4.41). Then, for all $t > 0$,*

$$\begin{aligned}\frac{d}{dt}\mathrm{S}_P[h(t)] = &-2\int_{\mathbb{R}^{2d}} \left\{\sum_{i,j=1}^d \begin{pmatrix}\nabla_x(\partial_{p_i}h)\\ \nabla_p(\partial_{p_i}h)\end{pmatrix}^T P \begin{pmatrix}\nabla_x(\partial_{p_j}h)\\ \nabla_p(\partial_{p_j}h)\end{pmatrix} a_{ij}\right\} f_\infty dxdp \\ &+ 2\int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix}0\\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp \\ &- \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp \\ + \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T &\left\{\sum_{i=1}^d \left(\frac{p_i}{p_0}\partial_{x_i}P - \partial_{x_i}V\partial_{p_i}P\right) + \sum_{i,j=1}^d \frac{1}{f_\infty}\partial_{p_j}(\partial_{p_i}P a_{ij} f_\infty)\right\} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp,\end{aligned} \tag{4.60}$$

where $Q = Q(x,p) := \begin{pmatrix}0 & \frac{1}{p_0}(I - \frac{p\otimes p}{p_0^2})\\ -\frac{\partial^2 V}{\partial x^2} & I - \frac{d}{p_0}(I - \frac{p\otimes p}{p_0^2})\end{pmatrix}$ *and* $a_{ij} := \dfrac{\delta_{ij} + p_ip_j}{p_0}$ *(which are the elements of* $D = D(p) = \dfrac{I + p\otimes p}{p_0}$*).*

Proof. We write (4.41) as

$$
\begin{aligned}
\partial_t h &= \frac{1}{f_\infty}\mathrm{div}_p[D\nabla_p h f_\infty] - \frac{p}{p_0}\cdot\nabla_x h + \nabla_x V\cdot\nabla_p h \\
&= \sum_{i,j=1}^d a_{ij}\partial^2_{p_ip_j}h - \sum_{i,j=1}^d a_{ij}\frac{p_i}{p_0}\partial_{p_j}h + \sum_{i,j=1}^d \partial_{p_i}a_{ij}\partial_{p_j}h - \frac{p}{p_0}\cdot\nabla_x h + \nabla_x V\cdot\nabla_p h \\
&= \sum_{i,j=1}^d a_{ij}\partial^2_{p_ip_j}h - p\cdot\nabla_p h + \frac{dp}{p_0}\cdot\nabla_p h - \frac{p}{p_0}\cdot\nabla_x h + \nabla_x V\cdot\nabla_p h, \qquad (4.61)
\end{aligned}
$$

where we used

$$
\sum_{i=1}^d a_{ij}\frac{p_i}{p_0} = p_j, \quad \sum_{i=1}^d \partial_{p_i}a_{ij} = \frac{dp_j}{p_0}. \qquad (4.62)
$$

We denote $u := \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}$, $u_1 := \nabla_x h$, $u_2 := \nabla_p h$. We get from (4.61) that u_1 and u_2 satisfy

$$
\partial_t u_1 = \sum_{i,j=1}^d a_{ij}\partial^2_{p_ip_j}u_1 - \sum_{j=1}^d p_j\partial_{p_j}u_1 + \sum_{j=1}^d \frac{dp_j}{p_0}\partial_{p_j}u_1 - \sum_{j=1}^d \frac{p_j}{p_0}\partial_{x_j}u_1 + \sum_{j=1}^d \partial_{x_j}V\partial_{p_j}u_1 + \frac{\partial^2 V}{\partial x^2}u_2,
$$

$$
\begin{aligned}
\partial_t u_2 = &\sum_{i,j=1}^d a_{ij}\partial^2_{p_ip_j}u_2 - \sum_{j=1}^d p_j\partial_{p_j}u_2 + \sum_{j=1}^d \frac{dp_j}{p_0}\partial_{p_j}u_2 - \sum_{j=1}^d \frac{p_j}{p_0}\partial_{x_j}u_2 + \sum_{j=1}^d \partial_{x_j}V\partial_{p_j}u_2 \\
&+ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h - \left(I - \frac{d}{p_0}\left(I - \frac{p\otimes p}{p_0^2}\right)\right)u_2 - \frac{1}{p_0}\left(I - \frac{p\otimes p}{p_0^2}\right)u_1.
\end{aligned}
$$

These equations can be written with respect to $u = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$:

$$
\begin{aligned}
\partial_t u = &\sum_{i,j=1}^d a_{ij}\partial^2_{p_ip_j}u - \sum_{j=1}^d p_j\partial_{p_j}u + \sum_{j=1}^d \frac{dp_j}{p_0}\partial_{p_j}u - \sum_{j=1}^d \frac{p_j}{p_0}\partial_{x_j}u + \sum_{j=1}^d \partial_{x_j}V\partial_{p_j}u \\
&- Q^T u + \begin{pmatrix}0\\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix}.
\end{aligned}
$$

It allows us to compute the time derivative

$$
\begin{aligned}
\frac{d}{dt}\mathrm{S}_P[h(t)] = 2\int_{\mathbb{R}^{2d}} u^T P\partial_t u f_\infty dxdp \\
= 2\sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} u^T P\partial^2_{p_ip_j}u a_{ij} f_\infty dxdp \\
- 2\sum_{j=1}^d \int_{\mathbb{R}^{2d}} u^T P\partial_{p_j}u p_j f_\infty dxdp + 2d\sum_{j=1}^d \int_{\mathbb{R}^{2d}} u^T P\partial_{p_j}u\frac{p_j}{p_0} f_\infty dxdp \\
- 2\sum_{j=1}^d \int_{\mathbb{R}^{2d}} u^T P\partial_{x_j}u\frac{p_j}{p_0} f_\infty dxdp + 2\sum_{j=1}^d \int_{\mathbb{R}^{2d}} u^T P\partial_{p_j}u\partial_{x_j}V f_\infty dxdp \\
- \int_{\mathbb{R}^{2d}} u^T\{QP + PQ^T\}u f_\infty dxdp + 2\int_{\mathbb{R}^{2d}} u^T P\begin{pmatrix}0\\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp. \qquad (4.63)
\end{aligned}
$$

First, we consider the term in the second line of (4.63), and we integrate by parts using (4.62) and $\partial_{p_i} f_\infty = -\frac{p_i}{p_0} f_\infty$:

$$
\begin{aligned}
2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial^2_{p_i p_j} u a_{ij} f_\infty dxdp &= -2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{p_i} u)^T P \partial_{p_j} u a_{ij} f_\infty dxdv \\
&- 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u \partial_{p_i} a_{ij} f_\infty dxdp + 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u a_{ij} \frac{p_i}{p_0} f_\infty dxdp \\
- 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T \partial_{p_i} P \partial_{p_j} u a_{ij} f_\infty dxdp &= -2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{p_i} u)^T P \partial_{p_j} u a_{ij} f_\infty dxdv \\
&- 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u \frac{dp_j}{p_0} f_\infty dxdp + 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u p_j f_\infty dxdp \\
&- 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T \partial_{p_i} P \partial_{p_j} u a_{ij} f_\infty dxdp. \qquad (4.64)
\end{aligned}
$$

We now compute the last integral in (4.64) by integrating by parts

$$
\begin{aligned}
&- 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T \partial_{p_i} P \partial_{p_j} u a_{ij} f_\infty dxdp \\
&\quad = 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{p_j} u)^T \partial_{p_i} P u a_{ij} f_\infty dxdp + 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T \partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) u dxdp.
\end{aligned}
$$

Since P is symmetric and $a_{ij} = a_{ji}$, this equation implies

$$
- 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T \partial_{p_i} P \partial_{p_j} u a_{ij} f_\infty dxdp = \int_{\mathbb{R}^{2d}} u^T \left(\sum_{i,j=1}^{d} \frac{1}{f_\infty} \partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) \right) u f_\infty dxdp. \tag{4.65}
$$

(4.63), (4.64) and (4.65) show that the sum of the terms in the second and third lines of (4.63) equals

$$
\begin{aligned}
&2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial^2_{p_i p_j} u a_{ij} f_\infty dxdp \\
&\qquad - 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u p_j f_\infty dxdp + 2d\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P \partial_{p_j} u p_j \frac{1}{p_0} f_\infty dxdp \\
&= -2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{p_i} u)^T P \partial_{p_j} u a_{ij} f_\infty dxdv + \int_{\mathbb{R}^{2d}} u^T \left(\sum_{i,j=1}^{d} \frac{1}{f_\infty} \partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) \right) u f_\infty dxdp.
\end{aligned} \tag{4.66}
$$

We consider the term in the fourth line of (4.63)

$$
\begin{aligned}
-2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{x_j}u\frac{p_j}{p_0}f_\infty dxdp + 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{p_j}u\partial_{x_j}V f_\infty dxdp \\
= 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{x_j}u)^T Pu\frac{p_j}{p_0}f_\infty dxdp + 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T\partial_{x_j}Pu\frac{p_j}{p_0}f_\infty dxdp \\
-2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} (\partial_{p_j}u)^T Pu\partial_{x_j}V f_\infty dxdp - 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T\partial_{p_j}Pu\partial_{x_j}V f_\infty dxdp.
\end{aligned}
$$

Since P is symmetric, we get

$$
\begin{aligned}
-2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{x_j}u\frac{p_j}{p_0}f_\infty dxdp + 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T P\partial_{p_j}u\partial_{x_j}V f_\infty dxdp \\
= \sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} u^T\left(\partial_{x_j}P\frac{p_j}{p_0} - \partial_{p_j}P\partial_{x_j}V\right)uf_\infty dxdp. \quad (4.67)
\end{aligned}
$$

(4.63), (4.66) and (4.67) yield the claimed equality. □

4.4.2 The second Lyapunov functional

Let H_δ and S_P be, respectively, the functionals defined in (4.15) and (4.59). Let $\gamma > 0$. We define the functional

$$\mathrm{E}[h] := \gamma||h||^2_{L^2(\mathbb{R}^{2d},f_\infty)} + \mathrm{H}_\delta[h] + \mathrm{S}_P[h].$$

It is clear that E depends on the parameters γ, δ and the matrix P. Let δ_0 be given in (4.20) and choose any $\delta \in (0,\delta_0)$. Then the decay estimates of Theorem 4.3.1 holds for the relativistic Fokker-Planck equation (4.41). Our goal is to choose $\gamma > 0$ and a suitable matrix P so that $\mathrm{E}[h]$ is equivalent to $||h||^2_{\mathscr{H}^1(\mathbb{R}^{2d},f_\infty)}$ and satisfies a Grönwall inequality (see (4.70) below) for the solution h of (4.41). We choose

$$P = P(x,p) := \begin{pmatrix} \frac{2\varepsilon^3}{V_0^3p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{\varepsilon^2}{V_0^2p_0^2}I \\ \frac{\varepsilon^2}{V_0^2p_0^2}I & \frac{2\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix}, \quad (4.68)$$

where ε is a positive constant which will be fixed later. We note the matrices $I - \frac{p\otimes p}{p_0^2}$ and $I + p\otimes p$ are positive definite and $\left(I - \frac{p\otimes p}{p_0^2}\right)^{-1} = I + p\otimes p$. This helps to check that P is positive definite for all x, $p \in \mathbb{R}^d$ and

$$0 < \begin{pmatrix} \frac{\varepsilon^3}{V_0^3p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix} \le P \le \begin{pmatrix} \frac{3\varepsilon^3}{V_0^3p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{3\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix}. \quad (4.69)$$

We now state the main result of this section.

Theorem 4.4.3. *Let Assumption 4.2.1 hold and h be the solution of (4.41) with initial data $h_0 \in \mathscr{H}^1(\mathbb{R}^{2d},f_\infty)$ such that $\int_{\mathbb{R}^{2d}} h_0 f_\infty dxdp = 0$. If $\varepsilon > 0$ in (4.68) is small enough, then*

$$\frac{d}{dt}\mathrm{E}[h(t)] \le -2\Lambda\mathrm{E}[h(t)], \quad \forall\, t > 0 \quad (4.70)$$

holds for a positive constant Λ *(independent of* h_0*). In particular,*

$$\mathrm{E}[h(t)] \le e^{-2\Lambda t}\mathrm{E}[h_0], \quad \forall\, t \ge 0. \tag{4.71}$$

Proof. Theorem 4.3.1 provides

$$\frac{d}{dt}\mathrm{H}_\delta[h(t)] \le -2\lambda \mathrm{H}_\delta[h(t)] \le -\frac{\lambda(2-\delta)}{2}\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp.$$

This estimate, Lemma 4.4.1, and Lemma 4.4.2 show that

$$\begin{aligned}\frac{d}{dt}\mathrm{E}[h(t)] \le& -\frac{\lambda(2-\delta)}{2}\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp - 2\gamma \int_{\mathbb{R}^{2d}} \nabla_p^T h D \nabla_p h f_\infty dxdp \\ & - 2\int_{\mathbb{R}^{2d}}\left\{\sum_{i,j=1}^d \begin{pmatrix}\nabla_x(\partial_{p_i}h)\\ \nabla_p(\partial_{p_i}h)\end{pmatrix}^T P \begin{pmatrix}\nabla_x(\partial_{p_j}h)\\ \nabla_p(\partial_{p_j}h)\end{pmatrix} a_{ij}\right\} f_\infty dxdp \\ & + 2\int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix} 0 \\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp \\ & - \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T \{QP + PQ^T\} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp \\ + \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T & \left\{\sum_{i=1}^d \left(\frac{p_i}{p_0}\partial_{x_i}P - \partial_{x_i}V\partial_{p_i}P\right) + \sum_{i,j=1}^d \frac{1}{f_\infty}\partial_{p_j}(\partial_{p_i}P a_{ij} f_\infty)\right\}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp.\end{aligned} \tag{4.72}$$

Step 1, estimates on the second order derivatives:

We first consider the term in the third line of (4.72):

$$\begin{aligned} 2\int_{\mathbb{R}^{2d}} & \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix} 0 \\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp \\ &= 2\varepsilon^2 \sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \frac{1}{V_0^2p_0^2}\nabla_x h\cdot \nabla_p a_{ij}\partial^2_{p_ip_j}h f_\infty dxdp \\ &\quad + 4\varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \frac{1}{V_0p_0}\nabla_p^T h(I+p\otimes p)\nabla_p a_{ij}\partial^2_{p_ip_j}h f_\infty dxdp. \end{aligned} \tag{4.73}$$

Let $w := \begin{pmatrix}\nabla_p(\partial_{p_1}h)\\ \vdots \\ \nabla_p(\partial_{p_d}h)\end{pmatrix} \in \mathbb{R}^{d^2}$ and $z := \begin{pmatrix}\nabla_x h\cdot\nabla_p a_{11}\\ \nabla_x h\cdot\nabla_p a_{12}\\ \vdots \\ \nabla_x h\cdot\nabla_p a_{dd}\end{pmatrix} \in \mathbb{R}^{d^2}$. Using the the relation $(D\otimes D)^{-1} = D^{-1}\otimes D^{-1} > 0$ (see [23, Corollary 4.2.11]) and applying (4.100) (from Appendix) to w and z, we obtain

$$\begin{aligned} 2\varepsilon^2 \sum_{i,j=1}^d \int_{\mathbb{R}^{2d}} \frac{1}{V_0^2p_0^2}\nabla_x h\cdot\nabla_p a_{ij}\partial^2_{p_ip_j}h f_\infty dxdp &= 2\varepsilon^2\int_{\mathbb{R}^{2d}}\frac{1}{V_0^2p_0^2} z\cdot w f_\infty dxdp \\ \le \frac{\varepsilon^3}{\eta}\int_{\mathbb{R}^{2d}} \frac{1}{V_0^3p_0^3} z^T D^{-1}\otimes D^{-1} z f_\infty dxdp &+ \varepsilon\eta\int_{\mathbb{R}^{2d}}\frac{1}{V_0p_0} w^T D\otimes D w f_\infty dxdp, \end{aligned} \tag{4.74}$$

where $\eta > 0$ will be fixed later. We use $D^{-1} = p_0\left(I - \frac{p\otimes p}{p_0^2}\right)$ and (4.101) (from Appendix) to estimate

$$\begin{aligned}
z^T D^{-1} \otimes D^{-1} z &= \sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right)(\nabla_x h \cdot \nabla_p a_{lj})(\nabla_x h \cdot \nabla_p a_{ki}) \\
&= \nabla_x^T h \left\{ \sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right) \nabla_p a_{lj} \otimes \nabla_p a_{ki} \right\} \nabla_x h \\
&\le d \nabla_x^T h \left(I - \frac{p\otimes p}{p_0^2}\right) \nabla_x h.
\end{aligned}$$

(4.74) and the last estimate imply

$$\begin{aligned}
&2\varepsilon^2 \sum_{i,j=1}^{d} \int_{\mathbb{R}^{2d}} \frac{1}{V_0^2 p_0^2} \nabla_x h \cdot \nabla_p a_{ij} \partial^2_{p_i p_j} h f_\infty dx dp \\
&\quad \le \frac{\varepsilon^3 d}{\eta} \int_{\mathbb{R}^{2d}} \frac{1}{V_0^3 p_0^3} \nabla_x^T h \left(I - \frac{p\otimes p}{p_0^2}\right) \nabla_x h f_\infty dx dp + \varepsilon\eta \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} w^T D \otimes D w f_\infty dx dp,
\end{aligned} \tag{4.75}$$

We now work on the last term of (4.73). We define

$$z_1 := \begin{pmatrix} \nabla_p^T h (I + p\otimes p) \nabla_p a_{11} \\ \nabla_p^T h (I + p\otimes p) \nabla_p a_{12} \\ \vdots \\ \nabla_p^T h (I + p\otimes p) \nabla_p a_{dd} \end{pmatrix} \in \mathbb{R}^{d^2}.$$

Similar to (4.74), we estimate

$$\begin{aligned}
&4\varepsilon \sum_{i,j=1}^{d} \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} \nabla_p^T h (I + p\otimes p) \nabla_p a_{ij} \partial^2_{p_i p_j} h f_\infty dx dp = 4\varepsilon \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} z_1 \cdot w f_\infty dx dp \\
&\quad \le \frac{2\varepsilon}{\eta} \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} z_1^T D^{-1} \otimes D^{-1} z_1 f_\infty dx dp + 2\varepsilon\eta \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} w^T D \otimes D w f_\infty dx dp.
\end{aligned} \tag{4.76}$$

(4.105) (from Appendix) lets us estimate

$$\begin{aligned}
&z_1^T D^{-1} \otimes D^{-1} z_1 \\
&= \sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right)(\nabla_p^T h (I + p\otimes p) \nabla_p a_{lj})(\nabla_p^T h (I + p\otimes p) \nabla_p a_{ki}) \\
&= \nabla_p^T h \left\{ \sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right) ((I + p\otimes p) \nabla_p a_{lj}) \otimes ((I + p\otimes p) \nabla_p a_{ki}) \right\} \nabla_p h \\
&\le d \nabla_p^T h \left(I + p\otimes p\right) \nabla_p h.
\end{aligned}$$

(4.76) and the last estimate imply

$$\begin{aligned}
&4\varepsilon \sum_{i,j=1}^{d} \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} \nabla_p^T h (I + p\otimes p) \nabla_p a_{ij} \partial^2_{p_i p_j} h f_\infty dx dp = 4\varepsilon \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} z_1 \cdot w f_\infty dx dp \\
&\quad \le \frac{2\varepsilon d}{\eta} \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} \nabla_p^T h \left(I + p\otimes p\right) \nabla_p h f_\infty dx dp + 2\varepsilon\eta \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} w^T D \otimes D w f_\infty dx dp.
\end{aligned} \tag{4.77}$$

Then (4.73), (4.75), and (4.77) imply

$$
\begin{aligned}
2\int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P \begin{pmatrix}0\\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp \\
\leq \int_{\mathbb{R}^{2d}} \begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T \begin{pmatrix}\frac{\varepsilon^3 d}{\eta V_0^3 p_0^3}(I-\frac{p\otimes p}{p_0^2}) & 0\\ 0 & \frac{2\varepsilon d}{\eta V_0 p_0}(I+p\otimes p)\end{pmatrix}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp \\
+3\varepsilon\eta\int_{\mathbb{R}^{2d}}\frac{1}{V_0p_0}w^T D\otimes Dwf_\infty dxdp. \quad (4.78)
\end{aligned}
$$

Let $u := \begin{pmatrix}\nabla_x(\partial_{p_1}h)\\ \nabla_p(\partial_{p_1}h)\\ \vdots \\ \nabla_x(\partial_{p_d}h)\\ \nabla_p(\partial_{p_d}h)\end{pmatrix} \in \mathbb{R}^{2d^2}$. and $\tilde{P} := D\otimes P = \begin{pmatrix}a_{11}P & \cdots & a_{1d}P\\ \cdots & \cdots & \cdots \\ a_{1d}P & \cdots & a_{dd}P\end{pmatrix} \in \mathbb{R}^{2d^2\times 2d^2}$.
Then we can write

$$
-2\int_{\mathbb{R}^{2d}}\left\{\sum_{i,j=1}^d \begin{pmatrix}\nabla_x(\partial_{p_i}h)\\ \nabla_p(\partial_{p_i}h)\end{pmatrix}^T P\begin{pmatrix}\nabla_x(\partial_{p_j}h)\\ \nabla_p(\partial_{p_j}h)\end{pmatrix} a_{ij}\right\} f_\infty dxdp = -2\int_{\mathbb{R}^{2d}} u^T\tilde{P}uf_\infty dxdp. \quad (4.79)
$$

Since P and D are positive definite, $D\otimes P$ is also positive definite, see [23, Corollary 4.2.13]. Moreover, P can be written as a sum of two positive semi-definite matrices:

$$
P = \begin{pmatrix}\frac{2\varepsilon^3}{V_0^3p_0^3}(I-\frac{p\otimes p}{p_0^2}) & \frac{\varepsilon^2}{V_0^2p_0^2}I\\ \frac{\varepsilon^2}{V_0^2p_0^2}I & \frac{\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \frac{\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix}.
$$

This implies

$$
\tilde{P} = D\otimes P \geq D\otimes \begin{pmatrix}0 & 0\\ 0 & \frac{\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix}.
$$

This inequality and (4.79) show that

$$
\begin{aligned}
-2\int_{\mathbb{R}^{2d}}\left\{\sum_{i,j=1}^d \begin{pmatrix}\nabla_x(\partial_{p_i}h)\\ \nabla_p(\partial_{p_i}h)\end{pmatrix}^T P\begin{pmatrix}\nabla_x(\partial_{p_j}h)\\ \nabla_p(\partial_{p_j}h)\end{pmatrix} a_{ij}\right\} f_\infty dxdp \\
\leq -2\int_{\mathbb{R}^{2d}} u^T D\otimes\begin{pmatrix}0 & 0\\ 0 & \frac{\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix} uf_\infty dxdp = -2\varepsilon\int_{\mathbb{R}^{2d}}\frac{1}{V_0}w^T D\otimes Dwf_\infty dxdp. \quad (4.80)
\end{aligned}
$$

We choose $\eta \in (0, \frac{2}{3}]$ so that $2 - \frac{3\eta}{p_0} \geq 0$ for all $p \in \mathbb{R}^d$. Then (4.78) and (4.80) yield

$$
\begin{aligned}
&-2\int_{\mathbb{R}^{2d}}\left\{\sum_{i,j=1}^{d}\begin{pmatrix}\nabla_x(\partial_{p_i}h)\\ \nabla_p(\partial_{p_i}h)\end{pmatrix}^T P\begin{pmatrix}\nabla_x(\partial_{p_j}h)\\ \nabla_p(\partial_{p_j}h)\end{pmatrix}a_{ij}\right\} f_\infty dxdp\\
&\qquad\qquad + 2\int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P\begin{pmatrix}0\\ \sum_{i,j=1}^d \nabla_p a_{ij}\partial^2_{p_ip_j}h\end{pmatrix} f_\infty dxdp\\
&\qquad \leq \int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T\begin{pmatrix}\frac{\varepsilon^3 d}{\eta V_0^3 p_0^3}(I-\frac{p\otimes p}{p_0^2}) & 0\\ 0 & \frac{2\varepsilon d}{\eta V_0 p_0}(I+p\otimes p)\end{pmatrix}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp\\
&\qquad\qquad - \int_{\mathbb{R}^{2d}}\varepsilon\left(2-\frac{3\eta}{p_0}\right)\frac{1}{V_0}w^T D\otimes D w f_\infty dxdp\\
&\qquad \leq \int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T\begin{pmatrix}\frac{\varepsilon^3 d}{\eta V_0^3 p_0^3}(I-\frac{p\otimes p}{p_0^2}) & 0\\ 0 & \frac{2\varepsilon d}{\eta V_0 p_0}(I+p\otimes p)\end{pmatrix}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp. \qquad (4.81)
\end{aligned}
$$

Step 2, Grönwall type inequality:

(4.72) and (4.81) show

$$
\frac{d}{dt}\mathrm{E}[h(t)] \leq -\frac{\lambda(2-\delta)}{2}\int_{\mathbb{R}^{2d}} h^2 f_\infty dxdp - \int_{\mathbb{R}^{2d}}\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix}^T P_1\begin{pmatrix}\nabla_x h\\ \nabla_p h\end{pmatrix} f_\infty dxdp, \qquad (4.82)
$$

where

$$
\begin{aligned}
P_1 &:= 2\gamma\begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} + (QP+PQ^T) - \sum_{i=1}^d\left(\frac{p_i}{p_0}\partial_{x_i}P - \partial_{x_i}V\partial_{p_i}P\right) - \sum_{i,j=1}^d\frac{1}{f_\infty}\partial_{p_j}(\partial_{p_i}Pa_{ij}f_\infty)\\
&\qquad - \begin{pmatrix}\frac{\varepsilon^3 d}{\eta V_0^3 p_0^3}(I-\frac{p\otimes p}{p_0^2}) & 0\\ 0 & \frac{2\varepsilon d}{\eta V_0 p_0}(I+p\otimes p)\end{pmatrix}. \qquad (4.83)
\end{aligned}
$$

The first two terms can be rewritten as

$$
2\gamma\begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} + QP + PQ^T =
$$

$$
\begin{pmatrix}\frac{2\varepsilon^2}{V_0^2p_0^3}(I-\frac{p\otimes p}{p_0^2}) & -\frac{\varepsilon^2}{V_0^2p_0^3}(I-\frac{p\otimes p}{p_0^2})(\frac{2\varepsilon}{V_0}\frac{\partial^2 V}{\partial x^2}+dI) + (\frac{\varepsilon^2}{V_0^2p_0^2}+\frac{2\varepsilon}{V_0p_0^2})I\\ -\frac{\varepsilon^2}{V_0^2p_0^3}(\frac{2\varepsilon}{V_0}\frac{\partial^2 V}{\partial x^2}+dI)(I-\frac{p\otimes p}{p_0^2}) + (\frac{\varepsilon^2}{V_0^2p_0^2}+\frac{2\varepsilon}{V_0p_0^2})I & -\frac{2\varepsilon^2}{V_0^2p_0^2}\frac{\partial^2 V}{\partial x^2} + (\frac{4\varepsilon}{V_0}+2\gamma)\frac{I+p\otimes p}{p_0} - \frac{4\varepsilon d}{V_0p_0^2}I\end{pmatrix}.
$$

Then, using Lemma 4.6.4 (from Appendix) for the forth term of (4.83) and Lemma 4.6.5 (from Appendix) for the third term of (4.83), we can show that there exist constants $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4 > 0$ such that

$$
P_1 \geq \begin{pmatrix}X & Y^T\\ Y & Z\end{pmatrix}, \qquad (4.84)
$$

where

$$
X := \left(1-\theta_3\varepsilon - \frac{\theta_1\varepsilon}{V_0} - \frac{\varepsilon d}{2\eta V_0}\right)\frac{2\varepsilon^2}{V_0^2p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right),
$$

$$
Y := -\frac{\varepsilon^2}{V_0^2p_0^3}\left(\frac{2\varepsilon}{V_0}\frac{\partial^2 V}{\partial x^2}+dI\right)\left(I-\frac{p\otimes p}{p_0^2}\right) + \left(\frac{\varepsilon}{V_0}+2\right)\frac{\varepsilon}{V_0p_0^2}I,
$$

$$Z := -\frac{2\varepsilon^2}{V_0^2 p_0^2}\frac{\partial^2 V}{\partial x^2} + \left(\frac{(4-2\theta_2)\varepsilon}{V_0} + 2\gamma - 2\theta_4\varepsilon - \frac{2\varepsilon d}{\eta V_0}\right)\frac{I + p\otimes p}{p_0} - \frac{4\varepsilon d}{V_0 p_0^2} I.$$

We choose a sufficiently small $\varepsilon > 0$ such that

$$1 - \theta_3\varepsilon - \frac{\theta_1\varepsilon}{V_0(x)} - \frac{\varepsilon d}{2\eta V_0(x)} > \frac{1}{2}$$

for all $x \in \mathbb{R}^d$. It is possible to choose such $\varepsilon > 0$ because $\frac{1}{V_0(x)}$ is uniformly bounded for $x \in \mathbb{R}^d$. Then we have

$$X \geq \frac{\varepsilon^2}{V_0^2 p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right) > 0 \tag{4.85}$$

for all $x, p \in \mathbb{R}^d$. Since the elements of the matrix $\frac{1}{V_0}\frac{\partial^2 V}{\partial x^2}$ are bounded (due to Assumption (4.8)) and $\frac{1}{p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right) \leq \frac{1}{p_0^2} I$, if we (possibly) choose $\varepsilon > 0$ even smaller, we have

$$\frac{\varepsilon}{V_0 p_0^2} I \leq Y \leq \frac{3\varepsilon}{V_0 p_0^2} I \tag{4.86}$$

for all $x, p \in \mathbb{R}^d$. Similarly, since the elements of the matrix $\frac{1}{V_0}\frac{\partial^2 V}{\partial x^2}$ are bounded and $\frac{1}{p_0^2} \leq \frac{1}{p_0}(I + p\otimes p)$, if we (possibly) choose $\varepsilon > 0$ even smaller,we have

$$Z \geq \frac{2\gamma - 1}{p_0}(I + p\otimes p) > 0 \tag{4.87}$$

for all $x, p \in \mathbb{R}^d$. (4.85), (4.86), and (4.87) show that, if $\varepsilon > 0$ is small enough and γ is large enough, then $\begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix}$ is positive definite and there is a constant $C > 0$ such that

$$\begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix} \geq CP. \tag{4.88}$$

We fix $\varepsilon > 0$ such that this condition holds. Then (4.82), (4.84), and (4.88) imply

$$\frac{d}{dt}\mathrm{E}[h(t)] \leq -\frac{\lambda(2-\delta)}{2}\int_{\mathbb{R}^{2d}} h^2 f_\infty dx dp - C\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix}^T P \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix} f_\infty dx dp.$$

$\int_{\mathbb{R}^{2d}} h^2 f_\infty dx dp$ and $\mathrm{H}_\delta[h]$ are equivalent by Theorem 4.3.1. Hence, from the equation above we conclude that there is a constant $\Lambda > 0$ such that (4.70) holds. □

We now ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. H_δ is equivalent to the square of the L^2-norm by Theorem 4.3.1 (i). This fact and the inequalities (4.69) show that E is equivalent to the $\mathscr{H}^1-$norm. Then the proof follows from (4.71). □

4.5 Hypoelliptic regularity

In this section, we prove Theorem 4.2.4, i.e., we show that, for any initial data $h_0 \in L^2(\mathbb{R}^{2d}, f_\infty)$, the solution $h(t)$ of (4.41) is in $\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ for all $t > 0$. Then, we shall prove Corollary 4.2.5.

Proof of Theorem 4.2.4. Let $h := \dfrac{f - f_\infty}{f_\infty}$. Then h solves (4.41). We define a functional

$$\mathcal{E}[h] := \gamma ||h||^2_{L^2(\mathbb{R}^{2d}, f_\infty)} + \mathrm{S}_P[h],$$

where $\mathrm{S}_P[h]$ is defined in (4.59). In order to prove the short-time regularization of (4.10) and (4.11) we consider this functional with a matrix P which depends not only on x and p but also on time t, i.e.

$$P = P(t,x,p) := \begin{pmatrix} \frac{2\varepsilon^3 t^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{\varepsilon^2 t^2}{V_0^2 p_0^2} I \\ \frac{\varepsilon^2 t^2}{V_0^2 p_0^2} I & \frac{2\varepsilon t}{V_0 p_0}(I + p\otimes p) \end{pmatrix},$$

where $\varepsilon > 0$ will be fixed later. Compared to (4.68), ε was replaced by εt. It is easy to check

$$\begin{pmatrix} \frac{\varepsilon^3 t^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{\varepsilon t}{V_0 p_0}(I + p\otimes p) \end{pmatrix} \le P, \tag{4.89}$$

which implies that $P(t,x,p)$ is positive definite for all $t > 0$ and $x, p \in \mathbb{R}^d$. Our goal is to show that $\mathcal{E}[h(t)]$ decreases. To this end we compute the time derivative of $\mathcal{E}[h(t)]$. We follow the proofs of Lemma 4.4.1 and Lemma 4.4.2 to compute the time derivative of $\mathcal{E}[h(t)]$, but now we need to take into account that P depends on time t :

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}[h(t)] = &-2\gamma \int_{\mathbb{R}^{2d}} \nabla_p^T h D \nabla_p h f_\infty dx dp \\
&- 2\int_{\mathbb{R}^{2d}} \left\{ \sum_{i,j=1}^d \begin{pmatrix} \nabla_x(\partial_{p_i} h) \\ \nabla_p(\partial_{p_i} h) \end{pmatrix}^T P \begin{pmatrix} \nabla_x(\partial_{p_j} h) \\ \nabla_p(\partial_{p_j} h) \end{pmatrix} a_{ij} \right\} f_\infty dx dp \\
&+ 2\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix}^T P \begin{pmatrix} 0 \\ \sum_{i,j=1}^d \nabla_p a_{ij} \partial^2_{p_i p_j} h \end{pmatrix} f_\infty dx dp \\
&- \int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix}^T \{QP + PQ^T - \partial_t P\} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix} f_\infty dx dp \\
+\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix}^T &\left\{ \sum_{i=1}^d \left(\frac{p_i}{p_0} \partial_{x_i} P - \partial_{x_i} V \partial_{p_i} P \right) + \sum_{i,j=1}^d \frac{1}{f_\infty} \partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) \right\} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix} f_\infty dx dp.
\end{aligned} \tag{4.90}$$

We estimate the terms on the right as in (4.74)-(4.81) (where we need to replace ε to εt) and obtain

$$\frac{d}{dt}\mathcal{E}[h(t)] \le -\int_{\mathbb{R}^{2d}} \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix}^T P_2 \begin{pmatrix} \nabla_x h \\ \nabla_p h \end{pmatrix} f_\infty dx dp, \tag{4.91}$$

where

$$\begin{aligned}
P_2 := 2\gamma \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + (QP + PQ^T - \partial_t P) - \sum_{i=1}^d \left(\frac{p_i}{p_0} \partial_{x_i} P - \partial_{x_i} V \partial_{p_i} P \right) - \sum_{i,j=1}^d \frac{1}{f_\infty} \partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) \\
- \begin{pmatrix} \frac{\varepsilon^3 t^3 d}{\eta V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2\varepsilon t d}{\eta V_0 p_0}(I + p\otimes p) \end{pmatrix}.
\end{aligned} \tag{4.92}$$

The first two terms can be rewritten as

$$
\begin{aligned}
&2\gamma \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + QP + PQ^T - \partial_t P \\
&= \begin{pmatrix} (1-\frac{3\varepsilon}{V_0})\frac{2\varepsilon^2 t^2}{V_0^2 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & -\frac{\varepsilon^2 t^2}{V_0^2 p_0^3}(I-\frac{p\otimes p}{p_0^2})(\frac{2\varepsilon t}{V_0}\frac{\partial^2 V}{\partial x^2} + dI) + (\frac{\varepsilon t - 2\varepsilon}{V_0}+2)\frac{\varepsilon t}{V_0 p_0^2} I \\ -\frac{\varepsilon^2 t^2}{V_0^2 p_0^3}(\frac{2\varepsilon t}{V_0}\frac{\partial^2 V}{\partial x^2} + dI)(I - \frac{p\otimes p}{p_0^2}) + (\frac{\varepsilon t-2\varepsilon}{V_0}+2)\frac{\varepsilon t}{V_0 p_0^2} I & -\frac{2\varepsilon^2 t^2}{V_0^2 p_0^2}\frac{\partial^2 V}{\partial x^2} + (\frac{4\varepsilon t - 2\varepsilon}{V_0} + 2\gamma)\frac{I+p\otimes p}{p_0} - \frac{4\varepsilon t d}{V_0 p_0^2} I \end{pmatrix},
\end{aligned}
$$

Lemma 4.6.4 for the fifth term of (4.92) and Lemma 4.6.5 for the forth term of (4.92) (where we need to replace ε to εt) show that there exist constants θ_1, θ_2, $\theta_3\,\theta_4 > 0$ such that

$$
P_2 \geq \begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix} \tag{4.93}
$$

where

$$
\begin{aligned}
X &:= \left(1 - \frac{3\varepsilon}{V_0} - \theta_3 \varepsilon t - \frac{\theta_1 \varepsilon t}{V_0} - \frac{\varepsilon t d}{2\eta V_0}\right)\frac{2\varepsilon^2 t^2}{V_0^2 p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right), \\
Y &:= -\frac{\varepsilon^2 t^2}{V_0^2 p_0^3}\left(\frac{2\varepsilon t}{V_0}\frac{\partial^2 V}{\partial x^2} + dI\right)\left(I - \frac{p\otimes p}{p_0^2}\right) + \left(\frac{\varepsilon t - 2\varepsilon}{V_0} + 2\right)\frac{\varepsilon t}{V_0 p_0^2} I, \\
Z &:= -\frac{2\varepsilon^2 t^2}{V_0^2 p_0^2}\frac{\partial^2 V}{\partial x^2} + \left(\frac{(4-2\theta_2)\varepsilon t - 2\varepsilon}{V_0} + 2\gamma - 2\theta_4 \varepsilon t - \frac{2\varepsilon t d}{\eta V_0}\right)\frac{I + p\otimes p}{p_0} - \frac{4\varepsilon t d}{V_0 p_0^2} I.
\end{aligned}
$$

We choose a sufficiently small $\varepsilon > 0$ such that

$$
1 - \frac{3\varepsilon}{V_0} - \theta_3\varepsilon t - \frac{\theta_1 \varepsilon t}{V_0} - \frac{\varepsilon t d}{2\eta V_0} > \frac{1}{2}
$$

for all $x \in \mathbb{R}^d$, $t \in [0, t_0]$. It is possible to choose such $\varepsilon_0 > 0$ because $\frac{1}{V_0(x)}$ is bounded for $x \in \mathbb{R}^d$ and t varies in a bounded interval. Then we have

$$
X \geq \frac{\varepsilon^2 t^2}{V_0^2 p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right) \geq 0 \tag{4.94}
$$

for all $x, p \in \mathbb{R}^d$, $t \in [0, t_0]$. Since the elements of the matrix $\frac{1}{V_0}\frac{\partial^2 V}{\partial x^2}$ are uniformly bounded by Assumption (4.8) and $\frac{1}{p_0^3}\left(I - \frac{p\otimes p}{p_0^2}\right) \leq \frac{1}{p_0^2} I$, if we (possibly) choose $\varepsilon > 0$ even smaller, then we have

$$
\frac{\varepsilon t}{V_0 p_0^2} I \leq Y \leq \frac{3\varepsilon t}{V_0 p_0^2} I \tag{4.95}
$$

for all $x, p \in \mathbb{R}^d$, $t \in [0, t_0]$. Similarly, since the elements of the matrix $\frac{1}{V_0}\frac{\partial^2 V}{\partial x^2}$ are bounded and $\frac{1}{p_0^2} \leq \frac{1}{p_0}(I + p\otimes p)$, if we choose smaller $\varepsilon > 0$, we have

$$
Z \geq \frac{2\gamma - 1}{p_0}(I + p\otimes p) > 00 \tag{4.96}
$$

for all $x, p \in \mathbb{R}^d$, $t \in [0, t_0]$. (4.94), (4.95), and (4.96) show that, if $\varepsilon > 0$ is small enough, then

$$
\begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix} \geq 0. \tag{4.97}
$$

We fix $\varepsilon > 0$ such that this condition holds. Then (4.91), (4.93), and (4.97) imply

$$
\frac{d}{dt}\mathcal{E}[h(t)] \leq 0.
$$

This yields that $\mathcal{E}[h(t)]$ is decreasing in $[0, t_0]$ and therefore

$$\mathcal{E}[h(t)] \leq \mathcal{E}[h(0)] = \gamma ||h_0||^2_{L^2(\mathbb{R}^{2d}, f_\infty)}, \quad \forall\, t \in [0, t_0]. \tag{4.98}$$

Moreover, we have by (4.89) that

$$\mathcal{E}[h(t)] \geq \varepsilon^3 t^3 \int_{\mathbb{R}^{2d}} \frac{1}{V_0^3 p_0^3} \nabla_x^T h(t) \left(I - \frac{p \otimes p}{p_0^2} \right) \nabla_x h(t) f_\infty dx dp \\ + \varepsilon t \int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} \nabla_p^T h(t) (I + p \otimes p) \nabla_p h(t) f_\infty dx dp. \tag{4.99}$$

(4.98) and (4.99) show that

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0^3 p_0^3} \nabla_x^T h(t) \left(I - \frac{p \otimes p}{p_0^2} \right) \nabla_x h(t) f_\infty dx dp \leq \frac{\gamma}{\varepsilon^3 t^3} ||h_0||^2_{L^2(\mathbb{R}^{2d}, f_\infty)}$$

and

$$\int_{\mathbb{R}^{2d}} \frac{1}{V_0 p_0} \nabla_p^T h(t) (I + p \otimes p) \nabla_p h(t) f_\infty dx dp \leq \frac{\gamma}{\varepsilon t} ||h_0||^2_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

Hence, (4.10) and (4.11) hold with constants $C_3 := \frac{\gamma}{\varepsilon^3}$ and $C_4 := \frac{\gamma}{\varepsilon}$. (4.12) follows easily by adding these estimates. □

Proof of Corollary 4.2.5. Let $t_0 > 0$. Theorem 4.2.4 and Theorem 4.2.3 show that $\frac{f(t_0)}{f_\infty} \in \mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)$ and

$$\left\| \frac{f(t) - f_\infty}{f_\infty} \right\|_{\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)} \leq C_2 e^{-\Lambda(t - t_0)} \left\| \frac{f(t_0) - f_\infty}{f_\infty} \right\|_{\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)}$$

holds for all $t \geq t_0 > 0$ with the constant C_2 and the rate Λ given in Theorem 4.2.3. Using (4.12) at $t = t_0$, we get

$$\left\| \frac{f(t) - f_\infty}{f_\infty} \right\|_{\mathscr{H}^1(\mathbb{R}^{2d}, f_\infty)} \leq \frac{C_2 (C_3 + C_4 t_0^2)^{1/2} e^{\Lambda t_0}}{t_0^{3/2}} e^{-\Lambda t} \left\| \frac{f_0 - f_\infty}{f_\infty} \right\|_{L^2(\mathbb{R}^{2d}, f_\infty)}.$$

This proves the claimed estimate with the constant $C_5 := \frac{C_2 (C_3 + C_4 t_0^2)^{1/2} e^{\Lambda t_0}}{t_0^{3/2}}$. □

4.6 Appendix

Lemma 4.6.1. *Let $A \in \mathbb{R}^{d \times d}$ be a symmetric, positive definite matrix. For any $u, v \in \mathbb{R}^d$, we have*

$$2u^T A v \leq u^T A u + v^T A v \quad \text{and} \quad 2u \cdot v \leq u^T A u + v^T A^{-1} v. \tag{4.100}$$

Proof. Since the matrices $A \in \mathbb{R}^{d \times d}$ and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ are positive semi-definite, their Kronecker product $\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}$ is also positive semi-definite, see [23, Corollary 4.2.13]. Hence,

$$u^T A u + v^T A v - 2u^T A v = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq 0.$$

The second inequality follows by replacing v with $A^{-1} v$. □

Lemma 4.6.2. *Let* $a_{ij} := \frac{\delta_{ij}+p_ip_j}{p_0}$, $p \in \mathbb{R}^d$. *Then*

$$\sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right) \nabla_p a_{lj} \otimes \nabla_p a_{ki} \leq d\left(I - \frac{p\otimes p}{p_0^2}\right) \tag{4.101}$$

holds for all $p \in \mathbb{R}^d$.

Proof. We compute the element which is in the intersection of the m^{th} row and the n^{th} column

$$\sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right) \partial_{p_m} a_{ki} \partial_{p_n} a_{lj}$$
$$= \sum_{l,i=1}^{d} p_0^2 \left(\sum_{j=1}^{d}\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)\partial_{p_n} a_{lj}\right)\left(\sum_{k=1}^{d}\left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)\partial_{p_m} a_{ki}\right).$$

We first compute the sums in the brackets

$$\sum_{j=1}^{d}\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)\partial_{p_n} a_{lj} \tag{4.102}$$
$$= \sum_{j=1}^{d}\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)\left(\frac{p_l\delta_{nj}+p_j\delta_{nl}}{p_0} - \frac{(\delta_{lj}+p_lp_j)p_n}{p_0^3}\right)$$
$$= \frac{p_l\delta_{ni}+p_i\delta_{nl}}{p_0} - \frac{(\delta_{li}+p_lp_i)p_n}{p_0^3} - \frac{p_i}{p_0^2}\left(\frac{p_lp_n+|p|^2\delta_{nl}}{p_0} - \frac{(p_l+p_l|p|^2)p_n}{p_0^3}\right)$$
$$= \frac{p_l\delta_{ni}+p_i\delta_{nl}}{p_0} - \frac{(\delta_{li}+p_lp_i)p_n}{p_0^3} - \frac{p_ip_lp_n+p_i|p|^2\delta_{nl}}{p_0^3} + \frac{p_lp_ip_n}{p_0^3}$$
$$= \frac{p_l\delta_{ni}+p_i\delta_{nl}}{p_0} - \frac{(\delta_{li}+p_lp_i)p_n}{p_0^3} - \frac{p_i|p|^2\delta_{nl}}{p_0^3}$$
$$= \frac{p_l\delta_{ni}}{p_0} - \frac{(\delta_{li}+p_lp_i)p_n}{p_0^3} + \frac{p_i\delta_{nl}}{p_0^3}$$
$$= \frac{p_l\delta_{ni}}{p_0} - \frac{p_n\delta_{li}}{p_0^3} + \frac{p_i\delta_{nl}}{p_0^3} - \frac{p_lp_ip_n}{p_0^3}. \tag{4.103}$$

Similarly, we can show

$$\sum_{k=1}^{d}\left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)\partial_{p_m} a_{ki} = \frac{p_i\delta_{ml}}{p_0} - \frac{p_m\delta_{li}}{p_0^3} + \frac{p_l\delta_{mi}}{p_0^3} - \frac{p_lp_ip_m}{p_0^3}. \tag{4.104}$$

Next, we sum the product of (4.102) and (4.104) with respect to i and l

$$\sum_{i=1}^{d}\left(\frac{p_l\delta_{ni}}{p_0} - \frac{p_n\delta_{li}}{p_0^3} + \frac{p_i\delta_{nl}}{p_0^3} - \frac{p_lp_ip_n}{p_0^3}\right)\left(\frac{p_i\delta_{ml}}{p_0} - \frac{p_m\delta_{li}}{p_0^3} + \frac{p_l\delta_{mi}}{p_0^3} - \frac{p_lp_ip_m}{p_0^3}\right)$$

$$= \sum_{l=1}^{d}\left[\frac{p_l}{p_0}\left(\frac{p_n\delta_{ml}}{p_0} - \frac{p_m\delta_{ln}}{p_0^3} + \frac{p_l\delta_{mn}}{p_0^3} - \frac{p_lp_np_m}{p_0^3}\right) - \frac{p_n}{p_0^3}\left(\frac{p_l\delta_{ml}}{p_0} - \frac{p_m}{p_0^3} + \frac{p_l\delta_{ml}}{p_0^3} - \frac{p_l^2p_m}{p_0^3}\right)\right.$$
$$\left. + \left(\frac{\delta_{nl}}{p_0^3} - \frac{p_lp_n}{p_0^3}\right)\left(\frac{|p|^2\delta_{ml}}{p_0} - \frac{p_mp_l}{p_0^3} + \frac{p_lp_m}{p_0^3} - \frac{p_lp_m|p|^2}{p_0^3}\right)\right]$$

$$= \left(\frac{p_n p_m}{p_0^2} - \frac{p_m p_n}{p_0^4} + \frac{|p|^2\delta_{mn}}{p_0^4} - \frac{p_n p_m |p|^2}{p_0^4}\right) - \left(\frac{p_n p_m}{p_0^4} - \frac{d p_m p_n}{p_0^6} + \frac{p_m p_n}{p_0^6} - \frac{p_n p_m |p|^2}{p_0^6}\right)$$

$$+ \left(\frac{|p|^2\delta_{mn}}{p_0^4} - \frac{p_n p_m |p|^2}{p_0^6}\right) - \left(\frac{p_n p_m |p|^2}{p_0^4} - \frac{p_n p_m |p|^4}{p_0^6}\right)$$

$$= \frac{2|p|^2\delta_{mn}}{p_0^4} + \frac{(d-2)p_m p_n}{p_0^6} - \frac{p_n p_m |p|^2}{p_0^4} - \frac{p_n p_m |p|^2}{p_0^6} + \frac{p_n p_m |p|^4}{p_0^6}$$

$$= \frac{2\delta_{mn}}{p_0^2} - \frac{2\delta_{mn}}{p_0^4} + \frac{(d-2)p_m p_n}{p_0^6} - \frac{p_n p_m}{p_0^2} + \frac{p_n p_m}{p_0^4} + \frac{p_m p_n}{p_0^2} - \frac{3 p_m p_n}{p_0^4} + \frac{2 p_m p_n}{p_0^6}$$

$$= \frac{2\delta_{mn}}{p_0^2} - \frac{2\delta_{mn}}{p_0^4} + \frac{d p_m p_n}{p_0^6} - \frac{2 p_m p_n}{p_0^4} = \frac{2}{p_0^2}(\delta_{mn} - \frac{p_n p_m}{p_0^2}) - \frac{2}{p_0^4}(\delta_{mn} - \frac{p_n p_m}{p_0^2}) + \frac{(d-2)p_m p_n}{p_0^6}.$$

This shows

$$\sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right) \nabla_p a_{lj} \otimes \nabla_p a_{ki}$$

$$= 2\left(I - \frac{p\otimes p}{p_0^2}\right) - \frac{2}{p_0^2}\left(I - \frac{p\otimes p}{p_0^2}\right) + \frac{(d-2)p\otimes p}{p_0^4}.$$

The claimed inequality follows from

$$\frac{(d-2)p\otimes p}{p_0^4} < \frac{d-2}{p_0^2} I \leq (d-2)\left(I - \frac{p\otimes p}{p_0^2}\right).$$

□

Lemma 4.6.3. *Let* $a_{ij} := \frac{\delta_{ij}+p_i p_j}{p_0}$, $p \in \mathbb{R}^d$. *Then*

$$\sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right)((I+p\otimes p)\nabla_p a_{lj}) \otimes ((I+p\otimes p)\nabla_p a_{ki}) \leq d(I+p\otimes p) \tag{4.105}$$

holds for all $p \in \mathbb{R}^d$.

Proof. We compute the element which is in the intersection of the m^{th} row and the n^{th} column

$$\sum_{k,l,i,j=1}^{d} p_0^2 \left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right)(\partial_{p_n} a_{lj} + p_n p\cdot\nabla_p a_{lj})(\partial_{p_m} a_{ki} + p_m p\cdot\nabla_p a_{ki}) =$$

$$\sum_{l,i=1}^{d} p_0^2 \left(\sum_{j=1}^{d}\left(\delta_{ij} - \frac{p_i p_j}{p_0^2}\right)(\partial_{p_n} a_{lj} + p_n p\cdot\nabla_p a_{lj})\right)\left(\sum_{k=1}^{d}\left(\delta_{kl} - \frac{p_k p_l}{p_0^2}\right)(\partial_{p_m} a_{ki} + p_m p\cdot\nabla_p a_{ki})\right).$$

We want to compute the sums in the brackets. We first compute

$$p\cdot\nabla_p a_{lj} = \sum_{r=1}^{d} p_r \left(\frac{p_l\delta_{rj} + p_j\delta_{rl}}{p_0} - \frac{(\delta_{lj}+p_l p_j)p_r}{p_0^3}\right) = \frac{2p_l p_j}{p_0} - \frac{(\delta_{lj}+p_l p_j)|p|^2}{p_0^3}$$

and so

$$\partial_{p_n} a_{lj} + p_n(p\cdot\nabla_p a_{lj}) = \frac{p_l\delta_{nj}+p_j\delta_{nl}}{p_0} - \frac{(\delta_{lj}+p_lp_j)p_n}{p_0^3} + \frac{2p_lp_jp_n}{p_0} - \frac{(\delta_{lj}+p_lp_j)p_n|p|^2}{p_0^3}$$
$$= \frac{p_l\delta_{nj}+p_j\delta_{nl}-\delta_{lj}p_n}{p_0} + \frac{p_lp_jp_n}{p_0}.$$

This help us to compute

$$\sum_{j=1}^d \left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)(\partial_{p_n}a_{lj}+p_np\cdot\nabla_p a_{lj}) = \sum_{j=1}^d \left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)\left(\frac{p_l\delta_{nj}+p_j\delta_{nl}-\delta_{lj}p_n}{p_0} + \frac{p_lp_jp_n}{p_0}\right)$$
$$= \frac{p_l\delta_{ni}+p_i\delta_{nl}-\delta_{li}p_n}{p_0} + \frac{p_lp_ip_n}{p_0} - \frac{p_i}{p_0^2}\left(\frac{|p|^2\delta_{nl}}{p_0} + \frac{p_lp_n|p|^2}{p_0}\right) = \frac{p_l\delta_{ni}-\delta_{li}p_n}{p_0} + \frac{(\delta_{nl}+p_lp_n)p_i}{p_0^3}.$$

Similarly, we compute

$$\sum_{k=1}^d \left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)(\partial_{p_m}a_{ki}+p_mp\cdot\nabla_p a_{ki}) = \frac{p_i\delta_{ml}-\delta_{li}p_m}{p_0} + \frac{(\delta_{mi}+p_ip_m)p_l}{p_0^3}.$$

We sum the product of the last two equations with respect to i and j

$$\sum_{l,i=1}^d p_0^2 \left(\sum_{j=1}^d \left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)(\partial_{p_n}a_{lj}+p_np\cdot\nabla_p a_{lj})\right)\left(\sum_{k=1}^d \left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)(\partial_{p_m}a_{ki}+p_mp\cdot\nabla_p a_{ki})\right)$$
$$= \sum_{l,i=1}^d p_0^2\left(\frac{p_l\delta_{ni}-\delta_{li}p_n}{p_0} + \frac{(\delta_{nl}+p_lp_n)p_i}{p_0^3}\right)\left(\frac{p_i\delta_{ml}-\delta_{li}p_m}{p_0} + \frac{(\delta_{mi}+p_ip_m)p_l}{p_0^3}\right)$$
$$= \sum_{l=1}^d p_0^2\left[\frac{p_l}{p_0}\left(\frac{p_n\delta_{ml}-\delta_{ln}p_m}{p_0} + \frac{(\delta_{mn}+p_np_m)p_l}{p_0^3}\right) - \frac{p_n}{p_0}\left(\frac{p_l\delta_{ml}-p_m}{p_0} + \frac{(\delta_{ml}+p_lp_m)p_l}{p_0^3}\right)\right.$$
$$\left. + \frac{(\delta_{nl}+p_lp_n)}{p_0^3}\left(\frac{|p|^2\delta_{ml}-p_lp_m}{p_0} + \frac{(p_m+|p|^2p_m)p_l}{p_0^3}\right)\right]$$
$$= \frac{(\delta_{mn}+p_np_m)|p|^2}{p_0^2} + (d-1)p_mp_n - \frac{p_np_m+p_np_m|p|^2}{p_0^2} + \frac{|p|^2\delta_{mn}}{p_0^2} + \frac{p_np_m|p|^2}{p_0^2}$$
$$= d(\delta_{mn}+p_np_m) - (d-2)\delta_{mn} - \frac{2(\delta_{mn}+p_np_m)}{p_0^2}.$$

This shows

$$\sum_{k,l,i,j=1}^d p_0^2\left(\delta_{kl} - \frac{p_kp_l}{p_0^2}\right)\left(\delta_{ij} - \frac{p_ip_j}{p_0^2}\right)((I+p\otimes p)\nabla_p a_{lj})\otimes((I+p\otimes p)\nabla_p a_{ki})$$
$$= d(I+p\otimes p) - (d-2)I - \frac{2(I+p\otimes p)}{p_0^2} \le d(I+p\otimes p).$$

□

Lemma 4.6.4. *Let P be the matrix defined in (4.68). Then there are constant $\theta_1 > 0$ and $\theta_2 > 0$ such that*

$$\sum_{i,j=1}^d \frac{1}{f_\infty}\partial_{p_j}(\partial_{p_i}Pa_{ij}f_\infty) \le \begin{pmatrix} \frac{2\theta_1\varepsilon^3}{V_0^3p_0^3}(I-\frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2\theta_2\varepsilon}{V_0p_0}(I+p\otimes p)\end{pmatrix}, \quad \forall x, p\in\mathbb{R}^d.$$

Proof. We have

$$\sum_{i,j=1}^{d} \frac{1}{f_\infty}\partial_{p_j}(\partial_{p_i} P a_{ij} f_\infty) = \sum_{i,j=1}^{d} \partial_{p_i p_j} P a_{ij} + \sum_{i=1}^{d} \partial_{p_i} P \sum_{j=1}^{d}\left(\partial_{p_j} a_{ij} - \frac{a_{ij} p_j}{p_0}\right)$$

Since $a_{ij} = \frac{\delta_{ij}+p_i p_j}{p_0}$, we have

$$\sum_{j=1}^{d}\left(\partial_{p_j} a_{ij} - \frac{a_{ij} p_j}{p_0}\right) = \sum_{j=1}^{d}\left(\frac{p_i + \delta_{ij} p_j}{p_0} - \frac{\delta_{ij}+p_i p_j}{p_0^3} p_j - \frac{(\delta_{ij}+p_i p_j)p_j}{p_0^2}\right) = \left(\frac{d}{p_0} - 1\right) p_i.$$

We denote $\varepsilon_1 = \varepsilon_1(x) := \frac{\varepsilon}{V_0(x)} > 0$ which is uniformly bounded for $x \in \mathbb{R}^d$. Then

$$\partial_{p_i} P = \begin{pmatrix} \frac{-6\varepsilon_1^3 p_i}{p_0^5} I - \frac{2\varepsilon_1^3}{p_0^5}\partial_{p_i}(p\otimes p) + \frac{10\varepsilon_1^3 p_i}{p_0^7} p\otimes p & -\frac{2\varepsilon_1^2 p_i}{p_0^4} I \\ -\frac{2\varepsilon_1^2 p_i}{p_0^4} I & -\frac{2\varepsilon_1 p_i}{p_0^3}(I + p\otimes p) + \frac{2\varepsilon_1}{p_0}\partial_{p_i}(p\otimes p) \end{pmatrix}.$$

The last two equations show

$$\sum_{i=1}^{d} \partial_{p_i} P \sum_{j=1}^{d}\left(\partial_{p_j} a_{ij} - \frac{a_{ij} p_j}{p_0}\right) = \left(\frac{d}{p_0} - 1\right)\sum_{i=1}^{d} \partial_{p_i} P p_i$$

$$= \left(\frac{d}{p_0} - 1\right)\begin{pmatrix} \frac{-6\varepsilon_1^3 |p|^2}{p_0^5} I - \frac{4\varepsilon_1^3}{p_0^5} p\otimes p + \frac{10\varepsilon_1^3 |p|^2}{p_0^7} p\otimes p & -\frac{2\varepsilon_1^2 |p|^2}{p_0^4} I \\ -\frac{2\varepsilon_1^2 |p|^2}{p_0^4} I & -\frac{2\varepsilon_1 |p|^2}{p_0^3}(I + p\otimes p) + \frac{4\varepsilon_1}{p_0} p\otimes p \end{pmatrix}$$

$$= \left(1 - \frac{d}{p_0}\right)\begin{pmatrix} \frac{6\varepsilon_1^3}{p_0^3}(I - \frac{p\otimes p}{p_0^2}) + \frac{10\varepsilon_1^3 p\otimes p}{p_0^7} - \frac{6\varepsilon_1^3}{p_0^5} I & \frac{2\varepsilon_1^2}{p_0^2} I - \frac{2\varepsilon_1^2}{p_0^4} I \\ \frac{2\varepsilon_1^2}{p_0^2} I - \frac{2\varepsilon_1^2}{p_0^4} I & -\frac{2\varepsilon_1}{p_0}(I + p\otimes p) + \frac{4\varepsilon_1}{p_0} I - \frac{2\varepsilon_1}{p_0^3}(I + p\otimes p) \end{pmatrix}.$$

Since $\left|1 - \frac{d}{p_0}\right|$ is uniformly bounded for all $p \in \mathbb{R}^d$, this computation shows that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\sum_{i=1}^{d} \partial_{p_i} P \sum_{j=1}^{d}\left(\partial_{p_j} a_{ij} - \frac{a_{ij} p_j}{p_0}\right) \leq \begin{pmatrix} \frac{2C_1\varepsilon_1^3}{p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2C_2\varepsilon_1}{p_0}(I + p\otimes p) \end{pmatrix}, \quad \forall\, x, p \in \mathbb{R}^d. \tag{4.106}$$

Next, using the computation above for $\partial_{p_i} P$ we compute

$$\partial^2_{p_j p_i} P = \begin{pmatrix} X_{ij} & -\frac{2\varepsilon_1^2 \delta_{ij}}{p_0^4} I + \frac{8\varepsilon_1^2 p_i p_j}{p_0^6} I \\ -\frac{2\varepsilon_1^2 \delta_{ij}}{p_0^4} I + \frac{8\varepsilon_1^2 p_i p_j}{p_0^6} I & Y_{ij} \end{pmatrix},$$

where

$$X_{ij} := -\frac{2\varepsilon_1^3[3\delta_{ij} I + \partial^2_{p_i p_j}(p\otimes p)]}{p_0^5} + \frac{10\varepsilon_1^3[3p_i p_j I + p_j \partial_{p_i}(p\otimes p) + p_i \partial_{p_j}(p\otimes p) + \delta_{ij} p\otimes p]}{p_0^7} - \frac{70\varepsilon_1^3 p_i p_j p\otimes p}{p_0^9},$$

$$Y_{ij} := -2\varepsilon_1\left(\frac{\delta_{ij}}{p_0^3} - \frac{3p_i p_j}{p_0^5}\right)(I + p\otimes p) - \frac{2\varepsilon_1[p_i \partial_{p_j}(p\otimes p) + p_j \partial_{p_i}(p\otimes p)]}{p_0^3} + \frac{2\varepsilon_1}{p_0}\partial^2_{p_i p_j}(p\otimes p)$$

The identities

$$\sum_{i,j=1}^{d} \delta_{ij}\partial^2_{p_ip_j}(p\otimes p) = 2I, \quad \sum_{i,j=1}^{d} p_ip_j\partial^2_{p_ip_j}(p\otimes p) = 2p\otimes p, \quad \sum_{i=1}^{d} p_i\partial_{p_i}(p\otimes p) = 2p\otimes p$$

will be used in the following computations:

$$X := \sum_{i,j=1}^{d} X_{ij}a_{ij} = -\sum_{i,j=1}^{d} \frac{2\varepsilon_1^3[3\delta_{ij}I + \delta_{ij}\partial^2_{p_ip_j}(p\otimes p) + 3\delta_{ij}p_ip_jI + p_ip_j\partial^2_{p_ip_j}(p\otimes p)]}{p_0^6}$$

$$+\sum_{i,j=1}^{d} \frac{10\varepsilon_1^3[3\delta_{ij}p_ip_jI + \delta_{ij}p_j\partial_{p_i}(p\otimes p) + \delta_{ij}p_i\partial_{p_j}(p\otimes p) + \delta_{ij}p\otimes p]}{p_0^8}$$

$$+\sum_{i,j=1}^{d} \frac{10\varepsilon_1^3 p_ip_j[3p_ip_jI + p_j\partial_{p_i}(p\otimes p) + p_i\partial_{p_j}(p\otimes p) + \delta_{ij}p\otimes p]}{p_0^8}$$

$$-\sum_{i,j=1}^{d} \frac{70\varepsilon_1^3[\delta_{ij} + p_ip_j]p_ip_jp\otimes p}{p_0^{10}}$$

$$= -\frac{2\varepsilon_1^3[3(d-1+p_0^2)I + 2(I+p\otimes p)]}{p_0^6} + \frac{10\varepsilon_1^3[3|p|^2p_0^2I + 4p_0^2p\otimes p + (d-1+p_0^2)p\otimes p]}{p_0^8}$$

$$-\frac{70\varepsilon_1^3|p|^2p\otimes p}{p_0^8} = \frac{24\varepsilon_1^3}{p_0^4}\left(I - \frac{p\otimes p}{p_0^2}\right) - \frac{\varepsilon_1^3(28+6d)}{p_0^6}I + \frac{\varepsilon_1^3(60+10d)}{p_0^8}p\otimes p,$$

$$Y := \sum_{i,j=1}^{d} Y_{ij}a_{ij} = -\sum_{i,j=1}^{d} 2\varepsilon_1\left(\frac{\delta_{ij}}{p_0^4} + \frac{\delta_{ij}p_ip_j}{p_0^4} - \frac{3\delta_{ij}p_ip_j}{p_0^6} - \frac{3p_i^2p_j^2}{p_0^6}\right)(I+p\otimes p)$$

$$-\sum_{i,j=1}^{d} \frac{2\varepsilon_1[\delta_{ij}p_i\partial_{p_j}(p\otimes p) + \delta_{ij}p_j\partial_{p_i}(p\otimes p)] + 2\varepsilon_1 p_ip_j[p_i\partial_{p_j}(p\otimes p) + p_j\partial_{p_i}(p\otimes p)]}{p_0^4}$$

$$+\sum_{i,j=1}^{d} \frac{2\varepsilon_1}{p_0^2}[\delta_{ij}\partial^2_{p_ip_j}(p\otimes p) + p_ip_j\partial^2_{p_ip_j}(p\otimes p)]$$

$$= \left(\frac{4\varepsilon_1}{p_0^2} - \frac{2\varepsilon_1(d+2)}{p_0^4}\right)(I+p\otimes p) - \frac{8\varepsilon_1}{p_0^2}p\otimes p + \frac{4\varepsilon_1}{p_0^2}(I+p\otimes p)$$

$$= \frac{8\varepsilon_1}{p_0^2}I - \frac{2\varepsilon_1(d+2)}{p_0^4}(I+p\otimes p),$$

$$Z := \sum_{i,j=1}^{d} \left(-\frac{2\varepsilon_1^2\delta_{ij}}{p_0^4}I + \frac{8\varepsilon_1^2p_ip_j}{p_0^6}I\right)a_{ij}$$

$$= -\sum_{i,j=1}^{d} \left(\frac{2\varepsilon_1^2\delta_{ij}}{p_0^5}I - \frac{8\varepsilon_1^2\delta_{ij}p_ip_j}{p_0^7}I + \frac{2\varepsilon_1^2\delta_{ij}p_ip_j}{p_0^5}I - \frac{8\varepsilon_1^2p_i^2p_j^2}{p_0^7}I\right)$$

$$= \frac{6\varepsilon_1^2}{p_0^3}I - \frac{2\varepsilon_1^2(d+3)}{p_0^5}I.$$

According to our notations, we have

$$\sum_{i,j=1}^{d} \partial_{p_i p_j} P a_{ij} = \begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}$$
$$= \begin{pmatrix} \frac{24\varepsilon_1^3}{p_0^4}(I - \frac{p\otimes p}{p_0^2}) - \frac{\varepsilon_1^3(28+6d)}{p_0^6} I + \frac{\varepsilon_1^3(60+10d)}{p_0^8} p\otimes p & \frac{6\varepsilon_1^2}{p_0^3} I - \frac{2\varepsilon_1^2(d+3)}{p_0^5} I \\ \frac{6\varepsilon_1^2}{p_0^3} I - \frac{2\varepsilon_1^2(d+3)}{p_0^5} I & \frac{8\varepsilon_1}{p_0^2} I - \frac{2\varepsilon_1(d+2)}{p_0^4}(I + p\otimes p) \end{pmatrix}.$$

The explicit computations above show that the elements of X and Y respectively decay faster than the elements of $\frac{\varepsilon_1^3}{p_0^3}(I - \frac{p\otimes p}{p_0^2})$ and $\frac{\varepsilon_1}{p_0^2}(I + p\otimes p)$ as $|p| \to \infty$. The elements of Z decay faster than the elements of $\frac{\varepsilon_1^2}{p_0^2} I$. Therefore, there are constants $C_1' > 0$ and $C_2' > 0$ such that

$$\sum_{i,j=1}^{d} \partial_{p_i p_j} P a_{ij} \le \begin{pmatrix} \frac{2C_1'\varepsilon_1^3}{p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2C_2'\varepsilon_1}{p_0}(I + p\otimes p) \end{pmatrix}, \quad \forall p \in \mathbb{R}^d.$$

This estimate and (4.106) provide the claimed result. □

Lemma 4.6.5. *Let $P = P(x,p)$ be the matrix defined in (4.68). Assume there exists a constant $c_3 > 0$ such that*

$$\left\|\frac{\partial^2 V(x)}{\partial x^2}\right\|_F \le c_3(1 + |\nabla_x V(x)|), \quad \forall\, x \in \mathbb{R}^d. \tag{4.107}$$

Then there are constant $\theta_3 > 0$ and $\theta_4 > 0$ such that

$$\sum_{i=1}^{d} \left(\frac{p_i}{p_0}\partial_{x_i} P - \partial_{x_i} V \partial_{p_i} P\right) \le \begin{pmatrix} \frac{2\theta_3\varepsilon^3}{V_0^2 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2\theta_4\varepsilon}{p_0}(I + p\otimes p) \end{pmatrix}, \quad \forall\, x,\, p \in \mathbb{R}^d.$$

Proof. We compute

$$\sum_{i=1}^{d} \frac{p_i}{p_0}\partial_{x_i} P = \left[\sum_{i=1}^{d} \frac{\nabla_x V \cdot \nabla_x(\partial_{x_i} V) p_i}{V_0^2 p_0}\right] \begin{pmatrix} -\frac{6\varepsilon^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{-2\varepsilon^2}{V_0^2 p_0^2} I \\ \frac{-2\varepsilon^2}{V_0^2 p_0^2} I & \frac{-2\varepsilon}{V_0 p_0}(I + p\otimes p) \end{pmatrix}$$
$$= \left[\frac{1}{V_0^2 p_0}\nabla_x^T V \frac{\partial^2 V}{\partial x^2} p\right] \begin{pmatrix} -\frac{6\varepsilon^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & \frac{-2\varepsilon^2}{V_0^2 p_0^2} I \\ \frac{-2\varepsilon^2}{V_0^2 p_0^2} I & \frac{-2\varepsilon}{V_0 p_0}(I + p\otimes p) \end{pmatrix}. \tag{4.108}$$

Since (4.107) implies $\left\|\frac{\partial^2 V}{\partial x^2}\right\|_F \le \sqrt{2} c_3 V_0$, we have

$$\left|\frac{1}{V_0^2 p_0}\nabla_x^T V \frac{\partial^2 V}{\partial x^2} p\right| \le \frac{1}{V_0^2 p_0}|\nabla_x V| \left\|\frac{\partial^2 V}{\partial x^2}\right\|_F |p| \le \sqrt{2} c_3, \quad \forall\, x,\, p \in \mathbb{R}^d.$$

This uniform bound and (4.108) show that there are constants $C_1 > 0$ and C_2 such that

$$\sum_{i=1}^{d} \frac{p_i}{p_0}\partial_{x_i} P \le \begin{pmatrix} \frac{2C_1\varepsilon^3}{V_0^3 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2C_2\varepsilon}{V_0 p_0}(I + p\otimes p) \end{pmatrix}$$
$$\le \begin{pmatrix} \frac{2C_1\varepsilon^2}{V_0^2 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2C_2\varepsilon}{p_0}(I + p\otimes p) \end{pmatrix}, \quad \forall\, x,\, p \in \mathbb{R}^d. \tag{4.109}$$

Next, we compute

$$-\sum_{i=1}^{d}\partial_{x_i}V\partial_{p_i}P =$$
$$\sum_{i=1}^{d}\begin{pmatrix} 2\varepsilon^3\left[\frac{5\partial_{x_i}Vp_i}{V_0^3p_0^5}(I-\frac{p\otimes p}{p_0^2})-\frac{\partial_{x_i}V}{V_0^3p_0^5}(2p_iI-\partial_{p_i}(p\otimes p))\right] & \frac{2\varepsilon^2\partial_{x_i}Vp_i}{V_0^2p_0^4}I \\ \frac{2\varepsilon^2\partial_{x_i}Vp_i}{V_0^2p_0^4}I & \frac{2\partial_{x_i}\varepsilon Vp_i}{V_0p_0^3}(I+p\otimes p)-\frac{2\varepsilon\partial_{x_i}V}{V_0p_0}\partial_{p_i}(p\otimes p)\end{pmatrix}. \tag{4.110}$$

We denote $\tilde{p}_i := \begin{pmatrix} p_1 \\ \vdots \\ p_{i-1} \\ p_i - 1 \\ p_{i+1} \\ \vdots \\ p_d \end{pmatrix}$ and $\bar{p}_i := \begin{pmatrix} p_1 \\ \vdots \\ p_{i-1} \\ p_i + 1 \\ p_{i+1} \\ \vdots \\ p_d \end{pmatrix}$ for $i \in \{1,...,d\}$. Let $E_i \in \mathbb{R}^{d\times d}$ denote the matrix whose element in the intersection of the $i-$th column and the $i-$th row equals 1 and other elements are zero. Then one can check

$$\frac{1}{p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right)-\frac{1}{p_0^5}(2p_iI-\partial_{p_i}(p\otimes p)) = \frac{1}{p_0^5}E_i+\frac{1}{p_0^5}(|\tilde{p}_i|^2I-\tilde{p}_i\otimes\tilde{p}_i)\geq 0$$

and

$$\frac{1}{p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right)+\frac{1}{p_0^5}(2p_iI-\partial_{p_i}(p\otimes p)) = \frac{1}{p_0^5}E_i+\frac{1}{p_0^5}(|\bar{p}_i|^2I-\bar{p}_i\otimes\bar{p}_i)\geq 0.$$

From these equations we obtain

$$-\frac{1}{p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right)\leq\frac{1}{p_0^5}(2p_iI-\partial_{p_i}(p\otimes p))\leq\frac{1}{p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right).$$

Using these inequalities and the fact that $\left|\frac{\partial_{x_i}V}{V_0}\right|$ and $\left|\frac{p_i}{p_0}\right|$ are bounded for all $x,p\in\mathbb{R}^d$, we conclude that there is a constant $C_1'>0$ such that

$$2\varepsilon^3\left[\frac{5\partial_{x_i}Vp_i}{V_0^3p_0^5}\left(I-\frac{p\otimes p}{p_0^2}\right)-\frac{\partial_{x_i}V}{V_0^3p_0^5}(2p_iI-\partial_{p_i}(p\otimes p))\right]\leq\frac{2C_1'\varepsilon^3}{V_0^2p_0^3}\left(I-\frac{p\otimes p}{p_0^2}\right),\quad\forall x,p\in\mathbb{R}^d. \tag{4.111}$$

The inequalities

$$I+p\otimes p-\partial_{p_i}(p\otimes p)=\tilde{p}_i\otimes\tilde{p}_i+I-E_i\geq 0$$

and

$$I+p\otimes p+\partial_{p_i}(p\otimes p)=\bar{p}_i\otimes\bar{p}_i+I-E_i\geq 0$$

imply

$$-(I+p\otimes p)\leq\partial_{p_i}(p\otimes p)\leq I+p\otimes p.$$

Using these inequalities and the fact that $\left|\frac{\partial_{x_i}V}{V_0}\right|$ and $\left|\frac{p_i}{p_0}\right|$ are bounded for all $x,p\in\mathbb{R}^d$, we conclude that there is a constant $C_2'>0$ such that

$$\frac{2\varepsilon\partial_{x_i}Vp_i}{V_0p_0^3}(I+p\otimes p)-\frac{2\varepsilon\partial_{x_i}V}{V_0p_0}\partial_{p_i}(p\otimes p)\leq\frac{2C_2'\varepsilon}{p_0}(I+p\otimes p). \tag{4.112}$$

(4.110), (4.111), and (4.112) show that there are constants $C_1'' > 0$ and $C_2'' > 0$ such that

$$-\sum_{i=1}^{d} \partial_{x_i} V \partial_{p_i} P \leq \begin{pmatrix} \frac{2C_1''\varepsilon^3}{V_0^2 p_0^3}(I - \frac{p\otimes p}{p_0^2}) & 0 \\ 0 & \frac{2C_2''\varepsilon}{p_0}(I + p \otimes p) \end{pmatrix}, \quad \forall\, x,\, p \in \mathbb{R}^d.$$

This inequality and (4.109) yield the claimed estimate. □

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