Converting nonlocality into contextuality (and back)

http://tph.tuwien.ac.at/~svozil/publ/2024-QIP24-pres.pdf

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(3) ”Hardy-Cabello-Type” ones, such as TIFS and TITS, as exposed already by the KS “bug” (1965, DOI 10.1007/978-3-0348-9259-9_19) two years before their “major paper”, which is a TIFS; their Γ_1 is a TITS by an extension of their previous “bug” TIFS;

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(5) Boole-Bell type violations of classical inequalities stemming from non-independent, non-separable quantum properties – those violate classical predictions relative to the assumption of classical independent existence — cf Froissart (1981, DOI 10.1007/BF02903286) and Pitowsky (1986, DOI 10.1063/1.527066); eg, CHSH (4 disconnected contexts) or intertwining contexts (aka orthonormal bases) Svozil (2001, DOI 10.48550/arXiv.quant-ph/0012066) Specker bug, Klyashko (2008, DOI 10.1103/PhysRevLett.101.020403) pentagon/gram/house;

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(5) GHZ Mermin type parity type proofs within a single context (more on this later).

Are there more?

Are there more? Please let us know!

(1) versus (5)

From a structural (algebraic) point of view

Kochen-Specker type (1)

and GHZ Mermin type (5)

are VERY different!

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- Those (mutually orthogonal) orthogonal projection operators can be expressed in terms of the dyadic products of elements of an orthonormal basis aka context.

Challenges to find joint eigensystem for mutually commuting degenerate observables

A technical problem arises if the mutually commuting operators of the observables are all degenerate. For the sake of an example take, for instance, the two hermitian matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

commute, and yet, none of their respective eigenvalues coincide: indeed, the eigensystem of the first matrix consist of separable vectors $\left(1, \pm 1, 0, 0\right)$ and $\left(0, 0, 1, \pm 1\right)$ while the eigenvectors of the second matrix $\left(1, 0, 0, \pm 1\right)$ and $\left(0, 1, \pm 1, 0\right)$ are all nonseparable. In such cases, finding their respective unique context can be rather tedious, although constructively feasible, as it involves finding simultaneous eigenvectors for all these commuting operators.

Solution: Matrix pencils that are linear combinations (coherent superpositions) of respective matrices

Mutually commuting normal operators (such as Hermitian or unitary operators that commute with their respective adjoints) $A_1, \ldots, A_l$ share common projection operators.

Solution: diagonalize the matrix pencil that is a linear combination of the operator matrices:

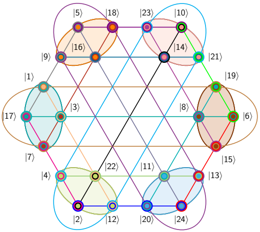
$$P = \sum_{i=1}^{l} a_i A_i,$$

where a_i are scalars (for our purposes, real numbers). As P commutes with $A_1, \ldots, A_l$, they share a common set of projection operators. Moreover, since the scalar parameters a_i can be adjusted, and in particular, can be identified with Kronecker delta functions δ_{ij}, and as P commutes with each operator A_j for $1 \leq j \leq l$, P and A_j share a common set of projection operators.

Case study I: Matrix pencil of the Peres-Mermin square

$$\begin{pmatrix} \sigma_z \mathbb{1}_2 & \mathbb{1}_2 \sigma_z & \sigma_z \sigma_z \\ \mathbb{1}_2 \sigma_x & \sigma_x \mathbb{1}_2 & \sigma_x \sigma_x \\ \sigma_z \sigma_x & \sigma_x \sigma_z & \sigma_y \sigma_y \end{pmatrix}$$

matrix pencils	eigenvalues			
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\mathbb{1}_2 + b\mathbb{1}_2\sigma_z + c\sigma_z\sigma_z$	$\vert 7\rangle = (0,1,0,0)^\intercal$	$\vert 3\rangle = (0,0,1,0)^\intercal$	$\vert 1\rangle = (0,0,0,1)^\intercal$	$\vert 17\rangle = (1,0,0,0)^\intercal$
$a\mathbb{1}_2\sigma_x + b\sigma_x\mathbb{1}_2 + c\sigma_x\sigma_x$	$\vert 20\rangle = (-1,-1,1,1)^\intercal$	$\vert 13\rangle = (-1,1,-1,1)^\intercal$	$\vert 11\rangle = (1,-1,-1,1)^\intercal$	$\vert 24\rangle = (1,1,1,1)^\intercal$
$a\sigma_z\sigma_x + b\sigma_x\sigma_z + c\sigma_y\sigma_y$	$\vert 21\rangle = (1,1,-1,1)^\intercal$	$\vert 14\rangle = (1,-1,1,1)^\intercal$	$\vert 23\rangle = (-1,1,1,1)^\intercal$	$\vert 10\rangle = (-1,-1,-1,1)^\intercal$
$a\sigma_z\mathbb{1}_2 + b\mathbb{1}_2\sigma_x + c\sigma_z\sigma_x$	$\vert 12\rangle = (-1,1,0,0)^\intercal$	$\vert 4\rangle = (0,0,1,1)^\intercal$	$\vert 2\rangle = (0,0,-1,1)^\intercal$	$\vert 22\rangle = (1,1,0,0)^\intercal$
$a\mathbb{1}_2\sigma_z + b\sigma_x\mathbb{1}_2 + c\sigma_x\sigma_z$	$\vert 15\rangle = (-1,0,1,0)^\intercal$	$\vert 8\rangle = (0,1,0,1)^\intercal$	$\vert 6\rangle = (0,-1,0,1)^\intercal$	$\vert 19\rangle = (1,0,1,0)^\intercal$
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\sigma_z + b\sigma_x\sigma_x + c\sigma_y\sigma_y$	$\vert 5\rangle = \vert\Psi_-\rangle = (0,1,-1,0)^\intercal$	$\vert 18\rangle = \vert\Phi_+\rangle = (1,0,0,1)^\intercal$	$\vert 16\rangle = \vert\Phi_-\rangle = (1,0,0,-1)^\intercal$	$\vert 9\rangle = \vert\Psi_+\rangle = (0,1,1,0)^\intercal$

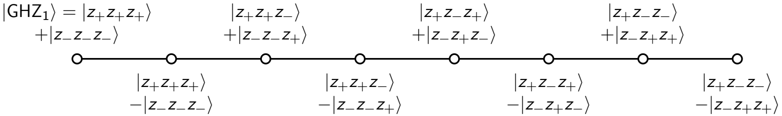


$24 - 24 \supset 18 - 9$ (Kochen-Specker)

Case study II: Matrix pencil of the Greenberger-Horne-Zeilinger-Mermin argument

$$a\sigma_x\sigma_x\sigma_x + b\sigma_y\sigma_y\sigma_x + c\sigma_y\sigma_x\sigma_y + d\sigma_x\sigma_y\sigma_y$$

$$\begin{aligned}
\pm(a-b-c-d) &: |\mathrm{GHZ}_{1,2}\rangle = \frac{1}{\sqrt{2}}\left(|z_+z_+z_+\rangle \pm |z_-z_-z_-\rangle\right) = \frac{1}{\sqrt{2}}\left(1,0,0,0,0,0,0,\pm 1\right)^\intercal ,\\
\pm(a-b+c+d) &: |\mathrm{GHZ}_{3,4}\rangle = \frac{1}{\sqrt{2}}\left(|z_+z_+z_-\rangle \pm |z_-z_-z_+\rangle\right) = \frac{1}{\sqrt{2}}\left(0,1,0,0,0,0,\pm 1,0\right)^\intercal ,\\
\pm(a+b-c+d) &: |\mathrm{GHZ}_{5,6}\rangle = \frac{1}{\sqrt{2}}\left(|z_+z_-z_+\rangle \pm |z_-z_+z_-\rangle\right) = \frac{1}{\sqrt{2}}\left(0,0,1,0,0,\pm 1,0,0\right)^\intercal ,\\
\pm(a+b+c-d) &: |\mathrm{GHZ}_{7,8}\rangle = \frac{1}{\sqrt{2}}\left(|z_+z_-z_-\rangle \pm |z_-z_+z_+\rangle\right) = \frac{1}{\sqrt{2}}\left(0,0,0,1,\pm 1,0,0,0\right)^\intercal ,
\end{aligned}$$



Case study III: Matrix pencil of two-partite Greenberger-Horne-Zeilinger argument

$$a(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z) + b\sigma_x\sigma_x + c\sigma_z\sigma_z$$

$$\begin{aligned}
-a-b-c : |\Psi_-\rangle &= \frac{1}{\sqrt{2}}\left(|z_+z_-\rangle - |z_-z_+\rangle\right) = \frac{1}{\sqrt{2}}\left(0,1,-1,0\right)^{\intercal},\\
a+b-c : |\Psi_+\rangle &= \frac{1}{\sqrt{2}}\left(|z_+z_-\rangle + |z_-z_+\rangle\right) = \frac{1}{\sqrt{2}}\left(0,1,1,0\right)^{\intercal},\\
a-b+c : |\Phi_-\rangle &= \frac{1}{\sqrt{2}}\left(|z_+z_+\rangle - |z_-z_-\rangle\right) = \frac{1}{\sqrt{2}}\left(1,0,0,-1\right)^{\intercal},\\
-a+b+c : |\Phi_+\rangle &= \frac{1}{\sqrt{2}}\left(|z_+z_+\rangle + |z_-z_-\rangle\right) = \frac{1}{\sqrt{2}}\left(1,0,0,1\right)^{\intercal},
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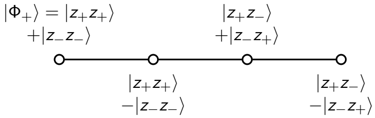

Thank you for your attention!