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Optimal complexity of standard and goal-oriented adaptive FEM for general second-order linear elliptic PDEs

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Kurzfassung

Adaptive Finite-Element-Methoden (AFEMn) sind ein unverzichtbares Werkzeug für die effiziente numerische Simulation von partiellen Differentialgleichungen (PDG) mit einer Vielzahl von Anwendungen, insbesondere im Ingenieurwesen. Das Ziel besteht darin, eine zuverlässige numerische Approximation der unbekannten Lösung mit minimalen Rechenkosten zu berechnen. Diese Dissertation entwickelt adaptive Algorithmen zur effizienten Lösung von unsymmetrischen linearen elliptischen PDG zweiter Ordnung. Dazu verfeinern wir die adaptive Prozedur von Standard-AFEM: Berechne die Lösung der diskreten Formulierung und schätze den Approximationsfehler, markiere Dreiecke mit größerem Fehlerbeitrag und führe eine Verfeinerung des Gitters damit durch. Hierzu verwenden wir einen geschalteten iterativen Löser im `SOLVE`-Modul des adaptiven Algorithmus, der sich aus einer kontraktiven Symmetrisierung und der anschließenden Lösung dieses symmetrischen Problems mittels algebraischem Löser zusammensetzt. Hierbei ist es entscheidend, dass das Verfahren den Fehler in der zugehörigen Sobolev-Norm kontrahiert und der Löser lineare Komplexität aufweist. Die Abbruchkriterien werden so formuliert, dass sie die verschiedenen Fehlerkomponenten von Diskretisierung, Symmetrisierung und Algebra ausbalancieren. Diese Strategie garantiert volle R-lineare Konvergenz des Fehlers, d.h. im Wesentlichen Kontraktion eines geeigneten Quasi-Fehlers in jedem Schritt des Algorithmus. Eine hinreichend kleine Wahl der Parameter garantiert auch optimale Konvergenzraten bezüglich der Freiheitsgrade und Rechenkosten. Darüber hinaus wird die Methode auf zielorientierte adaptive Algorithmen erweitert, die die effiziente Berechnung eines Funktionalwertes der Lösung der PDG ermöglichen. Die Arbeit umfasst folgende Hauptbeiträge:

Zur iterativen Lösung der linearen Gleichungssysteme von symmetrischen PDG präsentieren wir ein neuartiges geometrisches Mehrgitterverfahren, welches robust in Bezug auf den Polynomgrad $p \geq 1$ und die (lokale) Netzweite h kontrahiert. Darüber hinaus wird bewiesen, dass der eingebaute algebraische Fehlerschätzer hp-robust äquivalent zum algebraischen Fehler ist und die Anwendung des Mehrgitterverfahrens auf symmetrische PDG optimale Komplexität garantiert.

Zweitens wird gezeigt, dass das kontraktive inexakte Lösungsverfahren für eine unsymmetrische PDG zu optimaler Komplexität des Algorithmus führt. Eine neue Beweisstrategie ermöglicht es, die bisherigen Einschränkungen an die Parameter in vorherigen Arbeiten abzuschwächen. Zudem wird der übliche Beweisschritt über eine (Quasi-)Pythagoras-Identität durch eine verallgemeinerte Quasi-Orthogonalität ersetzt. Insgesamt ebnet die neue Beweisstrategie den Weg für Erweiterungen der Analyse auf allgemeine inf-sup-stabile Probleme jenseits von Energieminimierungs-Problemen.

Schließlich wird ein zielorientierter adaptiver Algorithmus analysiert, der die effiziente Berechnung einer Zielgröße ermöglicht, die von der Lösung $u^\star$ einer unsymmetrischen PDG abhängt. Es wird eine zielorientierte adaptive iterativ symmetrisierte Finite-Element-Methode präsentiert und analysiert. Es wird gezeigt, dass der vorgeschlagene Algorithmus volle R-lineare Konvergenz und optimale Konvergenzraten hinsichtlich sowohl der Freiheitsgrade als auch der Rechenkosten garantiert.

Abstract

Adaptive finite element methods (AFEMs) have become an indispensable tool for efficient numerical simulations of partial differential equations (PDEs). Such methods successfully cover a wide range of applications, in particular, in engineering. However, what one strives to achieve in practice is computing a *reliable* numerical approximation of the unavailable solution at the lowest possible *computational cost* and therewith *time*. In this thesis, adaptive mesh-refining algorithms are developed for the efficient solution of nonsymmetric linear elliptic PDEs of second order. Standard AFEM employs the feedback loop: given a computational mesh, solve the discrete problem yielding an approximation, estimate its error, and mark certain elements for mesh-refinement. A centerpiece of this thesis consists of embedding a nested procedure in the `SOLVE` module including a contractive symmetrization and a contractive algebraic linear solver such that the error contracts in the PDE-related norm. Furthermore, it is crucial that the iterative linear solver is of linear complexity. Suitable stopping criteria are then formulated to equilibrate the error components (coming from discretization, symmetrization, and algebraic solver). We show that such an adaptive strategy leads to *full R-linear convergence* of the error, i.e., essentially a contraction of an appropriate quasi-error in every step of the adaptive algorithm. Moreover, a sufficiently small choice of adaptivity parameters guarantees optimal convergence rates with respect to the computational cost, i.e., *optimal complexity*. Furthermore, we show that this approach can be extended to *goal-oriented adaptive algorithms*, where the quantity of interest is a functional value of the PDE solution. Overall, the thesis comprises the following main contributions.

First, we design an optimal local multigrid method for the iterative solution of the discrete systems arising from the finite element discretization of symmetric second-order linear elliptic diffusion problems. We show that the iterative solver contracts the algebraic error robustly with respect to the polynomial degree $p \geq 1$ and the (local) mesh size h. This is achieved by an overlapping additive Schwarz smoother. Moreover, embedding the solver into the AFEM framework for symmetric PDEs, we prove that this leads to the optimal convergence rate with respect to the overall cost.

Second, we show that the proposed combined symmetrization-algebra procedure leads to a contractive inexact solver for nonsymmetric problems. The resulting AFEM algorithm is shown to be of optimal complexity. Initially, the analysis requires several fine-tuned parameters. However, a redesign of the proofs via a summability criterion for R-linear convergence allows us to relax such restrictions. Moreover, the usual proof via a (quasi-)Pythagorean identity is replaced by a generalized notion of quasi-orthogonality. Importantly, this paves the way towards extending the analysis to general inf-sup stable problems beyond the energy minimization setting.

Finally, we consider the problem of efficiently computing a quantity of interest depending on the solution of a general second-order linear elliptic, yet nonsymmetric PDE. To this end, we propose a goal-oriented adaptive iteratively symmetrized finite element method (GOAISFEM) by combining the previous approaches for nonsymmetric problems. We show that this algorithm guarantees full R-linear convergence and, thus, allows for the proof of optimal convergence rates with respect to both degrees of freedom and total computational cost.

Danksagung

Diese Dissertation wäre ohne die Unterstützung vieler Menschen nicht möglich gewesen. Zunächst möchte ich meinem Betreuer Prof. Dr. Dirk Praetorius danken, von dem ich die letzten Jahre sehr viel lernen durfte, nicht nur wissenschaftlich sondern auch menschlich. Seine Geduld und seine außerordentliche Fähigkeit, komplexe Sachverhalte einfach, anschaulich und verständlich zu erklären, haben mir die Arbeit sehr erleichtert. Dankbar bin ich auch für die vielen Wortwitze und Kaffeepausen, die die Arbeit so angenehm gemacht haben. Ich bin immens dankbar für die Freiheit, die ich in meiner Arbeit hatte, die stetige Begleitung (selbst mit Krücken) auf diesem Weg und für die vielen Möglichkeiten, mich weiterzuentwickeln und über mich selbst hinauszugehen.

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Mit dieser Dissertation geht eine unvergessliche Studienzeit zu Ende. Ich bin dankbar für meine Studienkolleginnen und -kollegen sowie meine Freunde, die mich in den letzten Jahren begleitet haben. Ein besonderer Dank gilt meiner Familie, die mich immer unterstützt hat und mir diesen Bildungsweg erst ermöglicht hat, und meiner Freundin Franziska.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Wien, am 11. Juni 2024

Julian Streitberger

Contents

1 Introduction

The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing.

— George Polya, 1954

1.1 Motivation

Partial differential equations (PDEs) are a fundamental tool in science and engineering. They are used to model the behavior of continuous systems in physics including heat transfer, fluid flow, wave propagation, electromagnetism, and extend to other areas, such as biology, chemistry, and materials science. This thesis focuses on developing efficient numerical algorithms for linear elliptic PDEs.

Writing the PDE in the so-called variational formulation in a function space $\mathcal{X}$ allows us to employ the finite element method (FEM) as a discretization method. The core idea of FEM is to replace the inherent infinite-dimensional space $\mathcal{X}$ by a finite-dimensional subspace $\mathcal{X}_\ell$ of dimension $N := \dim \mathcal{X}_\ell$. For instance, consider a computational domain $\Omega \subseteq \mathbb{R}^d$ for $d \geq 1$ and a given a mesh $\mathcal{T}_\ell$ consisting of intervals/triangles/tetrahedra $T \in \mathcal{T}_\ell$ of size h_T that cover Ω. Then, the globally continuous and piecewise polynomial functions on $\mathcal{T}_\ell$ of degree at most $p \in \mathbb{N}$ are typical examples for a finite-dimensional subspace $\mathcal{X}_\ell$. Then, a system of algebraic equations of size $N \times N$ to compute the coefficients of the simple functions arises, which allows to compute an approximation $u_\ell^\star$ to the exact solution $u^\star$. Since the exact solution $u^\star$ and hence the error $|||u^\star - u_\ell^\star|||$ in the PDE-related norm is not available in general, the quality of this numerical approximation can only be assessed by *a posteriori* error estimation. This means a computable quantity relying only on the given data, the generated mesh $\mathcal{T}_\ell$, and the computed approximation $u_\ell^\star$ is used to measure the approximation quality of the discretization scheme. In practice, the *a posteriori* error estimator should be an upper bound to the error, i.e., *reliable*, such that if an adaptive algorithm drives the error estimator to zero, the error also converges to zero. Furthermore, the construction of the error estimator should be local, i.e., computed on each triangle T of the mesh $\mathcal{T}_\ell$, and, thus, can (at least heuristically) detect singularities. Then, this technique can drive an adaptive mesh-refinement process by selecting only a subset of elements that dominate a particular portion of the full error. For a sufficiently smooth solution $u^\star$, the approximation error is uniformly distributed, hence choosing all elements for refinement leads to optimal convergence rates $-p/d$ of the error with respect to the so-called number of degrees of freedom $\dim \mathcal{X}_\ell$ which is equivalent (up to a multiplicative constant depending on the polynomial degree p) to the number of triangles $\#\mathcal{T}_\ell$ in the mesh $\mathcal{T}_\ell$. However, if the exact solution $u^\star$ suffers from reduced regularity due to singularities stemming from the geometry of the domain or from low regularity of the data and the coefficients of the PDE, then uniform mesh refinement leads to a reduced order of convergence. For instance, consider a boundary value problem with exact solution given in Figure 1.1. Here, $u^\star$ has a singularity at the reentrant corner of the domain and, thus, we cannot expect optimal convergence rate for a sequence $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ obtained by uniform mesh-refinement as depicted in Figure 1.2. Indeed, in Figure 1.3, we see that a uniform mesh-refinement strategy leads to the suboptimal convergence rate $-1/3$ with respect to the number

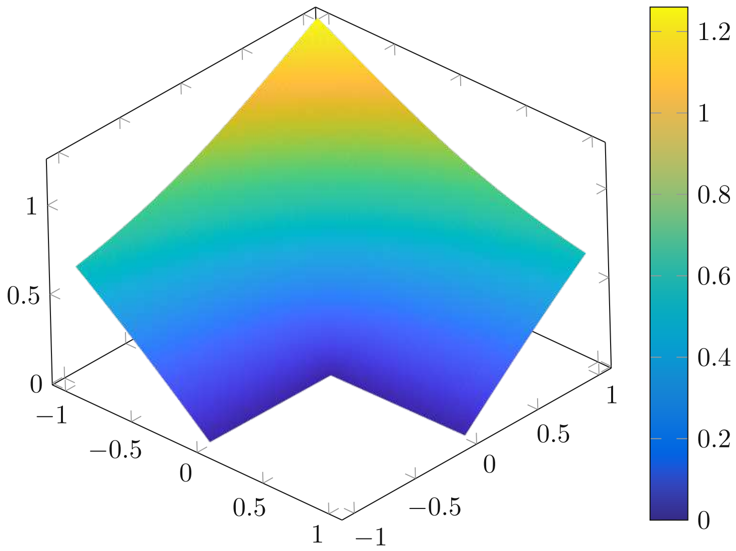


Figure 1.1: Solution $u^\star$ with reentrant-corner singularity at the origin.

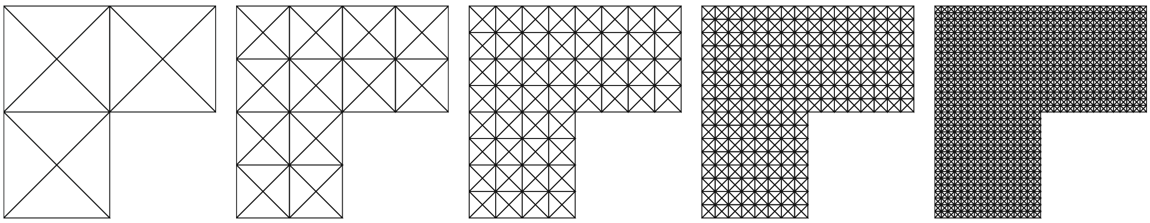


Figure 1.2: Sequence of meshes obtained by uniform mesh-refinement up to $\#\mathcal{T}_5 = 3072$.

of degrees of freedom compared to the optimal convergence rate $-p/2$ for any polynomial degree p in the adaptive algorithm. Due to the singularity, the error is thus not uniformly distributed, but rather concentrated in the vicinity of the singularity. A better choice is thus to refine only certain elements in order to get a better resolution and better capturing of the behavior of the solution $u^\star$ while also being more frugal with the computational resources. This allows to restore the optimal convergence rate of the error and, thus, emphasizes the importance of adaptive mesh-refinement strategies as in Figure 1.4.

In practice, users are not primarily interested in optimal error decay with respect to the degrees of freedom but rather in optimal decay with respect to the overall computation time to obtain an approximation with error below a user-given tolerance (thanks to the *a posteriori* error estimator). However, the solution of large linear systems becomes computationally expensive and prevents quasi-optimal computational run-time, i.e., optimal run-time up to a fixed multiplicative constant, of the adaptive algorithm. Therefore, one requires iterative algebraic solvers with linear complexity per step (i.e., the run-time grows linearly proportional to the size of the linear system) embedded into the AFEM loop. Additionally, we require that the algebraic solver contracts the error in the PDE-given norm, i.e., the error is reduced by a fixed uniform portion in each step of the solver. Moreover, apt stopping criteria are needed to prevent a surplus in the solution of the linear system.

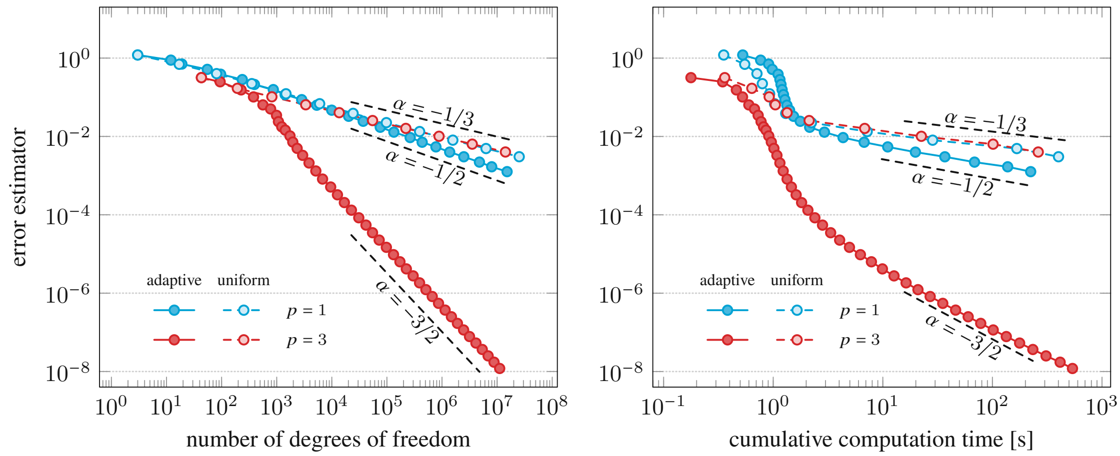


Figure 1.3: Convergence history plot of an *a posteriori* error estimator with respect to $\dim \mathcal{X}_\ell$ and the cumulative computation time.

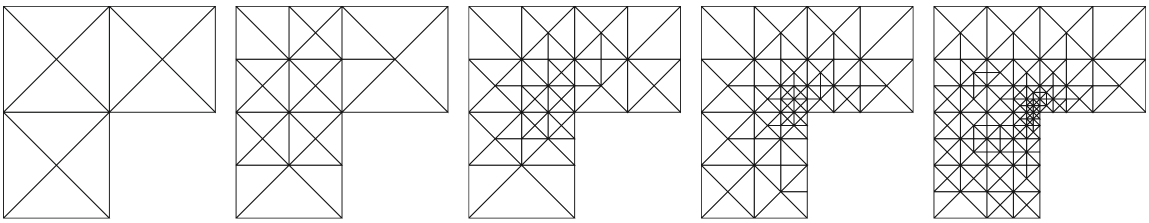


Figure 1.4: Sequence of meshes obtained by adaptive mesh-refinement up to $\#\mathcal{T}_5 = 219$.

While optimal contractive solvers for lowest-order FEM discretizations of symmetric linear elliptic PDEs are well-established, the treatment within AFEM of nonsymmetric discrete problems remained open due to a missing link between the generalized contraction in a discrete vector norm and the PDE-given norm.

In many applications, the cost-effective approximation of a certain goal value $G(u^\star)$ of the solution $u^\star$ is the main focus. This leads to the so-called goal-oriented adaptive FEM which employs a certain duality approach.

The development and analysis of standard and goal-oriented adaptive FEM for general linear elliptic PDEs by means of nested iterative solvers is the main focus of this thesis. To this end, first, we present an optimal local multigrid (MG) method for a symmetric linear elliptic PDE which is robust both in the local mesh-size h and the polynomial degree p and is embedded into an adaptive algorithm. Subsequently, we employ a nested iterative solver consisting of a symmetrization and an algebraic solver, e.g., the proposed optimal multigrid method. This results in the presentation of AFEM and GOAFEM algorithms with quasi-optimal computation time for general second-order linear elliptic PDEs with nested components as illustrated in Figure 1.5: The discretization of a nonsymmetric PDE in weak formulation leads to a nonsymmetric linear system with unavailable exact solution $u_\ell^\star$ on the mesh level ℓ. Hence, we employ a symmetrization loop to obtain a symmetric linear system with unavailable exact solution $u_\ell^{k,\star}$ in the k-th iteration of the symmetrization loop. Since the exact

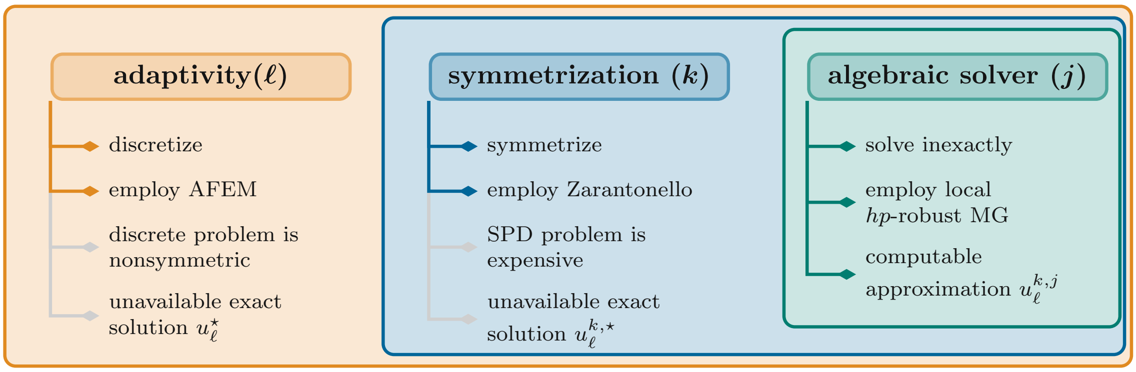


Figure 1.5: Modules of the adaptive algorithm with nested iterative solver for nonsymmetric PDEs.

solution of symmetric and positive definite linear systems is computationally expensive with growing size of the linear system, we employ an algebraic solver to obtain an approximation $u_\ell^{k,j}$ to the exact solution $u_\ell^{k,\star}$ of the symmetrized system. Altogether, we obtain a computable approximation $u_\ell^{k,j}$ to the exact solution $u_\ell^\star$.

Although the presentation in this thesis is restricted to linear nonsymmetric PDEs, the new theory also improves existing results for nonlinear PDEs with strongly monotone and Lipschitz continuous nonlinearity.

Outline of the introduction. In this first chapter, we provide a brief overview of standard FEM, adaptive FEM and goal-oriented adaptive FEM. More precisely, starting from a symmetric model problem and a detailed look at the idea of FEM in Section 1.2, we present the state-of-the-art theory of AFEM in Section 1.4 with a comprehensive study of a quasi-optimal AFEM algorithm with exact solve. Section 1.5 extends the previously presented ideas with the embedding of an algebraic solver opening the door to optimal convergence rates with respect to the overall computational cost. In Section 1.6, we extend the result to a goal-oriented framework and discuss the cost-optimal approximation for the given quantity of interest. Finally, we conclude this chapter with a detailed statement highlighting the main contributions of this thesis in Section 1.7.

1.2 Model problem and classic FEM approach

Throughout the thesis, we employ standard notation of Lebesgue and Sobolev spaces and their norms. Additionally, we abbreviate $A \lesssim B$ if there exists a generic constant $C > 0$ such that $A \leq C\,B$ with C independent of the mesh size h. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. This thesis considers a general second-order linear elliptic PDE on a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$. In this first chapter, we explain the concept and the historic development for a prototypical symmetric linear elliptic PDE and postpone the treatment of nonsymmetric linear elliptic PDEs to the main chapters of the thesis. Let $\boldsymbol{A} \in \big[L^\infty(\Omega)\big]^{d\times d}_{\mathrm{sym}}$ be a symmetric and uniformly positive definite diffusion matrix, i.e., there exist constants $\alpha_-, \alpha_+ > 0$ such that

$$\alpha_- \, |\xi|^2 \leq \boldsymbol{A}(x)\,\xi \cdot \xi \leq \alpha_+ \, |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and almost every } x \in \Omega. \tag{1.1}$$

Let $f \in L^2(\Omega)$ and $\boldsymbol{f} \in \left[L^2(\Omega)\right]^d$ be given right-hand sides. The model problem seeks the solution $u^\star : \Omega \to \mathbb{R}$ to

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) = f - \operatorname{div}(\boldsymbol{f}) \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Introducing the Sobolev space $\mathcal{X} := H^1_0(\Omega)$ equipped with the usual $H^1(\Omega)$-seminorm $\|\cdot\|_{\mathcal{X}} = \|\nabla(\cdot)\|_{L^2(\Omega)}$, a multiplication of (1.2) with a test function $v \in \mathcal{X}$, integration over the domain, and integration by parts together with the use of the Dirichlet boundary condition leads to the weak formulation: Find $u^\star \in \mathcal{X}$ such that

$$a(u^\star, v) := \int_\Omega \boldsymbol{A}\nabla u^\star \cdot \nabla v \,\mathrm{d}x = \int_\Omega \big(f\,v + \boldsymbol{f}\cdot\nabla v\big)\,\mathrm{d}x =: F(v) \quad \text{for all } v \in \mathcal{X}. \tag{1.3}$$

Owing to (1.1), the bilinear form $a(\,\cdot\,,\,\cdot\,)$ is continuous and elliptic to ensure existence of a solution to (1.3), i.e., there exist constants $C_{\mathrm{cnt}} = \alpha_+, C_{\mathrm{ell}} = \alpha_- > 0$ such that

$$a(v,w) \le C_{\mathrm{cnt}}\,\|v\|_{\mathcal{X}}\,\|w\|_{\mathcal{X}} \quad \text{and} \quad a(v,v) \ge C_{\mathrm{ell}}\,\|v\|^2_{\mathcal{X}} \quad \text{for all } v,w \in \mathcal{X}. \tag{1.4}$$

Therefore, $a(\cdot,\cdot)$ defines a scalar product on $\mathcal{X}$ with norm $|||\cdot||| := a(\cdot,\cdot)^{1/2}$ and the Lax–Milgram lemma [LM54] guarantees existence and uniqueness of a weak solution $u^\star$ to (1.3), see, e.g., [Eva10, Section 6.2].

A triangulation $\mathcal{T}_H$ of Ω into compact simplices is called conforming if the intersection of any two simplices is either empty or a common vertex or a common edge or a common face etc. Given such a conforming triangulation $\mathcal{T}_H$ of Ω, we define the finite element space and a fixed polynomial degree $p \in \mathbb{N}$, we define the finite element subspace

$$\mathcal{X}_H := \{v_H \in \mathcal{X} : \forall T \in \mathcal{T}_H, v_H|_T \text{ is a polynomial of total degree at most } p\}. \tag{1.5}$$

Then, the Lax–Milgram lemma also guarantees existence and uniqueness of the solution $u^\star_H \in \mathcal{X}_H$ to the discrete formulation

$$a(u^\star_H, v_H) = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{1.6}$$

Moreover, $u^\star_H$ is the quasi-best approximation of $u^\star$ in the sense of the Céa-type estimate

$$\|u^\star - u^\star_H\|_{\mathcal{X}} \le C_{\text{Céa}} \min_{v_H \in \mathcal{X}_H} \|u^\star - v_H\|_{\mathcal{X}} \quad \text{with} \quad C_{\text{Céa}} := C_{\mathrm{cnt}}/C_{\mathrm{ell}} > 0. \tag{Céa}$$

Given a basis $\{\varphi_1, \dots, \varphi_N\}$ of $\mathcal{X}_H$ with $N := \dim \mathcal{X}_H$, the finite element solution $u^\star_H$ to (1.6) is a linear combination of the basis functions, i.e., $u^\star_H = \sum_{j=1}^N x_j\,\varphi_j$. The coefficients x_j are determined by the solution of the linear system $Ax = L$ where $A_{jk} = a(\varphi_k, \varphi_j)$ and $L_k = F(\varphi_k)$ for $1 \le j,k \le N$.

1.3 State-of-the-art and outlook

We start with a review of the state-of-the-art literature and discuss the inherent open questions treated in this thesis. Adaptive algorithms have been investigated since the late 1970s with pioneering works proposing adaptive mesh-refinement in [BR78; ZR79; GKZB83]. Since adaptive mesh-refinement strategies do not drive the maximal mesh-size to zero, their convergence is not covered by the classical *a priori* error analysis. While convergence in 1D was already proved in [BV84], it took almost two decades until the works [Dör96; MNS00; Vee02; DK08] paved the way for the further development by the first rigorous plain convergence proofs of AFEMs for $d \geq 2$. The notion of nonlinear approximation classes [BDD04] provided a formal statement of optimal convergence rates and enabled the first proof of optimal convergence rates for standard adaptive FEMs in [Ste07; CKNS08]. This laid the groundwork for various generalizations of quasi-optimal convergence rates to nonresidual error estimators in [CN12], the p-Laplacian [BDK12], quasi-linear problems with strongly monotone nonlinearity [GMZ12], nonsymmetric problems [FFP14; BHP17], and discontinuous Galerkin methods in [BN10] and [KG18] for plain convergence of these methods with fewer restrictions on the penalty parameter. The work [CFPP14] presents an axiomatic framework for optimal convergence rates with respect to the degrees of freedom and summarizes the mentioned and a variety of unmentioned earlier references.

Early works in the direction of *optimal complexity* originate from adaptive wavelet methods [CDD01; CDD03] and the idea was later adopted for AFEM in [Ste07; CG12]. Driven by the interest in AFEMs for nonlinear problems [CW17; GHPS18; HW20b; HW20a; DFTW20], recent papers [GHPS21; HPW21; HPSV21] aimed to combine linearization and algebraic iterations into a nested adaptive algorithm. Following this idea, the algorithmic decision for either mesh refinement or linearization or algebraic solver step is steered by *a posteriori*-based stopping criteria with suitable stopping parameters in the spirit of [AGL13; EV13]. The literature [GHPS21; HPW21] covers full linear convergence for energy contractive solvers with no restrictions on the solver parameter and [GHPS21] requires sufficiently small solver parameter for a norm-contractive solver even for full R-linear convergence, i.e., contraction in each step of the algorithm up to a multiplicative constant. Moreover, [GHPS21; HPW21] prove optimal computational cost of the adaptive algorithm under the assumption that the arising linear systems can be solved directly in linear complexity. The work [HPSV21] considers a coupling of the linearization and algebraic solver to prove optimal complexity with restrictions on the involved solver parameters. More precisely, full R-linear convergence therein requires the algebraic solver parameter to be small with respect to the linearization parameter and the linearization parameter to be small with respect to the marking parameter. The proofs essentially rely on two building blocks: First, a contractive solver in the PDE-related norm and, second, a Pythagorean identity for the exact solutions. Both cornerstones remained open for nonsymmetric problems and are now treated in the course of this thesis. In essence, optimal complexity follows as soon as full R-linear convergence of AFEM with inexact solver and optimal rates of AFEM with exact solver (for sufficiently small marking parameter) have been established; see, e.g., [BIM+24a].

1.4 Adaptive FEM with exact solver

This section introduces the modules `SOLVE–ESTIMATE–MARK–REFINE` constituting the adaptive algorithm depicted in Figure 1.6.

For a first illustration of the concepts of adaptivity, we suppose that the `SOLVE` module employs a direct solver to the linear system associated with (1.6). Nevertheless, we emphasize that the application of a direct solver prevents the adaptive algorithm from attaining optimal complexity. Figure 1.7 displays an overview over the connections between the necessary requirements (called toolbox) and the results in Section 1.4–1.5.

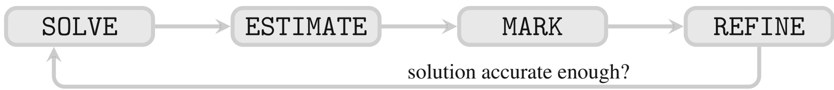


Figure 1.6: Modules of the standard AFEM algorithm with exact solver.

1.4.1 The module ESTIMATE

Since the exact solution $u^\star$ and, thus, the discrete error $\|u^\star - u_H^\star\|_{\mathcal{X}}$ is not available, we resort to an *a posteriori* error estimator $\eta_H(u_H^\star)$, which is reliable and efficient, i.e., there exist constants $C_{\mathrm{rel}}, C_{\mathrm{eff}} > 0$ such that

$$\|u^\star - u_H^\star\|_{\mathcal{X}} \le C_{\mathrm{rel}}\, \eta_H(u_H^\star) \quad \text{and} \quad \eta_H(u_H^\star) \le C_{\mathrm{eff}} \left[\|u^\star - u_H^\star\|_{\mathcal{X}}^2 + \mathrm{osc}_H^2\right]^{1/2} \tag{1.7}$$

Here, osc_H denotes the so-called data oscillations and, for a constant diffusion $\boldsymbol{A}$, it is a measure how good the data f can be approximated by piecewise polynomials. In the case of a nonconstant diffusion coefficient, the data oscillations include a volume term measuring how well $f + \mathrm{div}(\boldsymbol{A}\nabla v_H)$ can be approximated by piecewise polynomials and a boundary term measuring how well the normal jump $[\![\boldsymbol{A}\,\nabla v_H \cdot \boldsymbol{n}]\!]$ can be approximated by edgewise polynomials. Therefore, if the oscillations vanish, the error estimator is equivalent to the error. For an early reference on *a posteriori* techniques, we refer to [Ver89]. A possible approach to obtain an error estimator is by means of a residual-based ansatz, where the residual measures to which extent the discrete solution $u_H^\star$ satisfies the PDE. Given $v_H \in \mathcal{X}_H$ and $\mathcal{U}_H \subseteq \mathcal{T}_H$, the associated local contributions to the residual-based error estimator $\eta_H(\mathcal{U}_H; v_H) := \big(\sum_{T\in\mathcal{U}_H} \eta_H(T; v_H)^2\big)^{1/2}$ read

$$\eta_\ell(T; v_H)^2 := |T|^{2/d} \|f + \mathrm{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\|_{L^2(T)}^2 + |T|^{1/d} \|[\![\boldsymbol{A}\nabla v_H - \boldsymbol{f}]\!]\cdot\boldsymbol{n}\|_{L^2(\partial T\cap\Omega)}^2 \quad \text{for all } T \in \mathcal{T}_H. \tag{1.8}$$

In the case of $\mathcal{U}_H = \mathcal{T}_H$ above, we abbreviate $\eta_H(v_H) := \eta_H(\mathcal{T}_H; v_H)$. We stress that the definition of the estimator in (1.8) additionally requires that $\boldsymbol{A}|_T \in \big[W^{1,\infty}(T)\big]^{d\times d}_{\mathrm{sym}}$ and $\boldsymbol{f}|_T \in \big[H^1(T)\big]^d$ for all $T \in \mathcal{T}_0$, where $\mathcal{T}_0$ is an initial supertriangulation of Ω. From the definition (1.8), we see that, first, this quantity is computed locally on each triangle $T \in \mathcal{T}_H$ and, second, acts as an indicator of which elements cause a larger contribution to the global error. Thus, the estimator provides a means to steer a local mesh-refinement strategy. Then, it is well-known that the residual *a posteriori* estimator from (1.8) satisfies the following *axioms of adaptivity* from [CFPP14]. These properties ensure optimal convergence rates of the adaptive algorithm with respect to the number of degrees of freedom.

Proposition 1.1 (axioms of adaptivity [CFPP14, Section 6.1])**.** *There exist constants $C_{\mathrm{stab}}, C_{\mathrm{rel}}, C_{\mathrm{drel}}, C_{\mathrm{mon}} > 0$, and $0 < q_{\mathrm{red}} < 1$ such that the following properties are satisfied for any conforming triangulation $\mathcal{T}_H$ and any conforming refinement $\mathcal{T}_h$ of $\mathcal{T}_H$, any subset $\mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h$, corresponding Galerkin solutions $u_H^\star \in \mathcal{X}_H$, $u_h^\star \in \mathcal{X}_h$ to (1.6) and arbitrary $v_H \in \mathcal{X}_H$, $v_h \in \mathcal{X}_h$.*

(A1) *stability.* $\quad |\eta_h(\mathcal{U}_H, v_h) - \eta_H(\mathcal{U}_H, v_H)| \le C_{\mathrm{stab}}\, \|v_h - v_H\|_{\mathcal{X}}.$

(A2) *reduction.* $\quad \eta_h(\mathcal{T}_h \backslash \mathcal{T}_H, v_H) \le q_{\mathrm{red}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, v_H).$

(A3) *reliability.* $\quad \|u^\star - u_H^\star\|_{\mathcal{X}} \le C_{\mathrm{rel}}\, \eta_H(u_H^\star).$

Toolbox:

Properties of the algebraic solver $\Psi_H : \mathcal{X}_H \to \mathcal{X}_H$:
(C1) contraction: $|||u_H^\star - \Psi(u_H)||| \le q_{\rm ctr}\, |||u_H^\star - u_H|||$
(C2) a posteriori error control: $\frac{1-q_{\rm ctr}}{q_{\rm ctr}} |||u_H^\star - \Psi(u_H)||| \le |||\Psi(u_H) - u_H||| \le (1+q_{\rm ctr}) |||u_H^\star - u_H|||$

Properties of mesh refinement:
(R1) splitting estimate: $\#\mathcal{T}_h \le C_{\rm child}\, \#\mathcal{T}_H$
(R2) overlay estimate: $\#(\mathcal{T}_H \oplus \mathcal{T}_h) \le \#\mathcal{T}_H + \#\mathcal{T}_h - \#\mathcal{T}_0$
(R3) closure estimate: $\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm closure} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$

Properties of Dörfler marking (D):
Almost-minimal-cardinality set $\mathcal{M}_H \subseteq \mathcal{T}_H$ with $\theta\, \eta(u_H^\star)^2 \le \eta_H(\mathcal{M}_H; u_H^\star)^2$

Properties of the space $\mathcal{X}_H$:
nestedness: $\mathcal{X}_H \subseteq \mathcal{X}_h$ for $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$
(A4) quasi-orthogonality: $\sum_{\ell'=\ell}^{\ell+N} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 \le C_{\rm orth} (N+1)^{1-\delta} |||u^\star - u_\ell^\star|||^2$ for $\ell, N \in \mathbb{N}_0$.

Properties of the estimator $\eta_H : \mathcal{X}_H \to \mathbb{R}_{\ge 0}$:
(A1) stability: $|\eta_h(\mathcal{T}_h \cap \mathcal{T}_H, v_h) - \eta_H(\mathcal{T}_h \cap \mathcal{T}_H, v_H)| \le C_{\rm stab} |||v_h - v_H|||$
(A2) reduction: $\eta_h(\mathcal{T}_h \setminus \mathcal{T}_H, v_H) \le q_{\rm red}\, \eta_H(\mathcal{T}_H, v_H)$
(A3) reliability: $|||u^\star - u_H^\star||| \le C_{\rm rel}\, \eta_H(u_H^\star)$
(A3$^+$) discrete reliability: $|||u_h^\star - u_H^\star||| \le C_{\rm drel}\, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h, u_H^\star)$

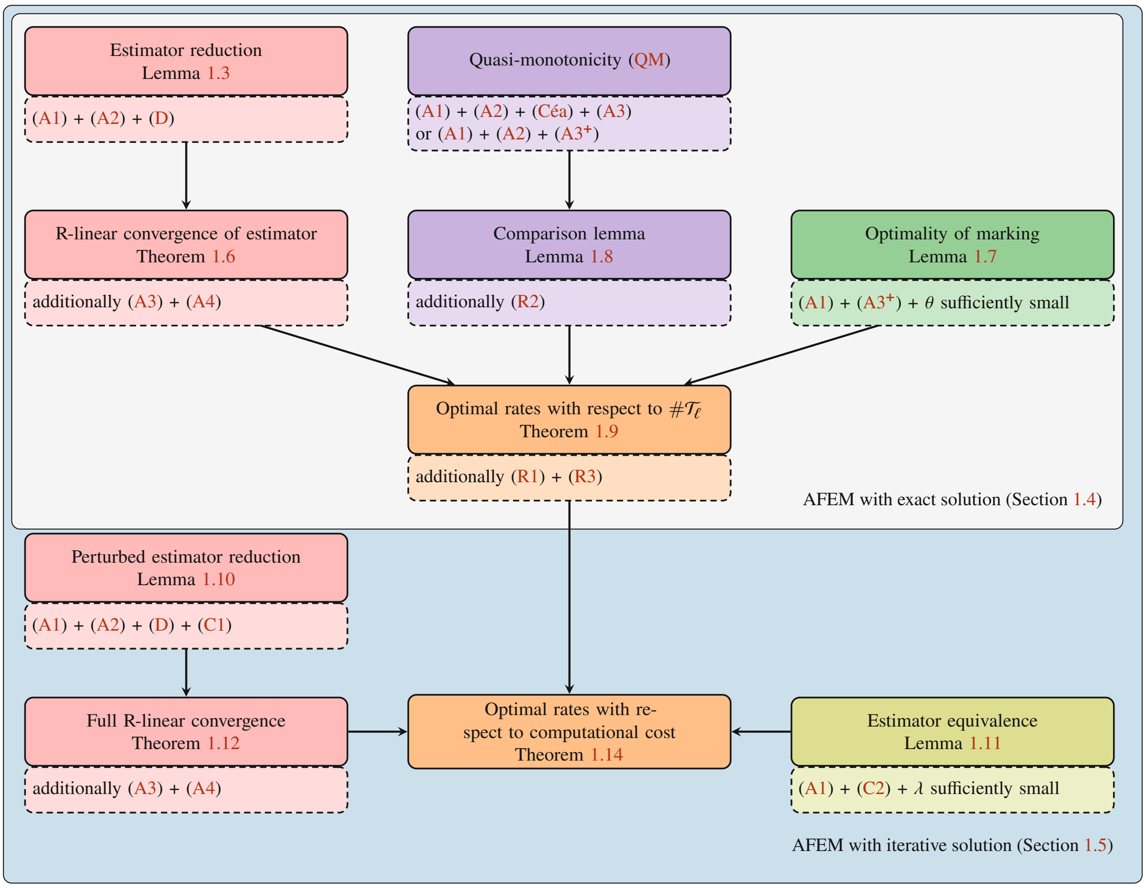


Figure 1.7: Overview of the connections between the assumptions and the results in Section 1.4–1.5.

(A3$^+$) *discrete reliability.* $\|u_h^\star - u_H^\star\|_\mathcal{X} \le C_{\mathrm{drel}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, u_H^\star)$.

(QM) *quasi-monotonicity.* $\eta_h(u_h^\star) \le C_{\mathrm{mon}}\, \eta_H(u_H^\star)$.

The constant C_{rel} depends only on uniform shape-regularity (defined in (1.9) below) of $\mathcal{T}_H$ and the dimension d, while C_{stab} and C_{drel} additionally depend on the polynomial degree p. The constant $q_{\mathrm{red}} := 2^{-1/(2d)}$ holds for bisection-based refinement rules in $\mathbb{R}^d$ and the constant C_{mon} can be bounded by $C_{\mathrm{mon}} \le \min\{1 + C_{\mathrm{stab}}(1 + C_{\mathrm{Céa}})C_{\mathrm{rel}}\,,\ 1 + C_{\mathrm{stab}}\, C_{\mathrm{drel}}\}$. □

1.4.2 The module `MARK`

In practical applications (see the experiment in Figure 1.3), it is desirable to mark a small set of elements $\mathcal{M}_H \subseteq \mathcal{T}_H$ whose corresponding error contributions are estimated to be a given $0 < \theta \le 1$ portion of the full error, see [Dör96]. In particular, the error estimator on the marked elements controls the error estimator contribution of the nonmarked elements. The marking strategy consists in using the following Dörfler criterion: Determine a set $\mathcal{M}_H$ of almost minimal (up to the multiplicative constant $C_{\mathrm{mark}} \ge 1$) cardinality with

$$
\begin{aligned}
\#\mathcal{M}_H &\le C_{\mathrm{mark}} \min_{\mathcal{U}_\ell^\star \in \mathbb{M}_\ell[\theta, u_\ell^\star]} \#\mathcal{U}_\ell^\star, \quad \text{where} \\
\mathbb{M}_H[\theta, u_H^\star] &:= \{\mathcal{U}_H \subseteq \mathcal{T}_H : \theta\, \eta(u_H^\star)^2 \le \eta_H(\mathcal{U}_H; u_H^\star)^2\}.
\end{aligned} \tag{D}
$$

An optimal implementation determining a set of minimal cardinality with $C_{\mathrm{mark}} = 1$ in linear complexity $\mathcal{O}(N)$ (in average time) employs a partitioning strategy as proposed in [PP20]. Importantly, the choice of the Dörfler criterion in the marking step allows for contraction of the error estimator from (1.8) through mesh refinement.

1.4.3 The module `REFINE`

Let $\mathbb{T}$ be a countably infinite set of regular triangulations $\mathcal{T}_H \in \mathbb{T}$ called admissible triangulations. We suppose that $\mathtt{refine}(\cdot,\cdot)$ is a fixed mesh-refinement strategy such that $\mathcal{T}_h := \mathtt{refine}(\mathcal{T}_H, \mathcal{M}_H) \in \mathbb{T}$ is obtained from $\mathcal{T}_H \in \mathbb{T}$ by refinement of at least the marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, i.e., there holds $\mathcal{M}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_h$. For any two admissible meshes $\mathcal{T}_H, \mathcal{T}_h \in \mathbb{T}$, we write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ can be obtained from $\mathcal{T}_H$ by a finite number of mesh-refinement steps. Throughout the thesis, we assume that all admissible meshes stem from a common initial mesh $\mathcal{T}_0 \in \mathbb{T}$, i.e., it holds that $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$. This thesis employs newest-vertex bisection (NVB) as a refinement strategy and illustrates this procedure for $d = 2$. For a nondegenerate triangle, newest-vertex bisection introduces the edge between the midpoint of the refinement edge and the opposite vertex. This leads to subtriangles of equal area and, thus, is a binary refinement rule. To refine a conforming triangulation $\mathcal{T}_H \in \mathbb{T}$, the NVB algorithm employs successive steps of newest-vertex bisection of $T \in \mathcal{T}_H$ until all hanging nodes have been removed. Depending on the selection of the refinement edges, it takes at most three bisections to refine the triangle T and ensure that the midpoints of the marked edges have become new vertices; see Figure 1.8. Then, newest-vertex bisection (even for any $d \ge 2$) satisfies the following properties [Dör96; Ste07; CKNS08; Ste08; KPP13; GSS14].

Proposition 1.2 (properties of mesh refinement)**.** *There exist constants $C_{\mathrm{child}}, C_{\mathrm{closure}} > 0$ such that the following properties hold:*

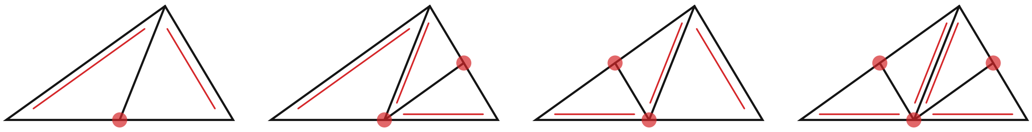


Figure 1.8: Newest-vertex bisection (NVB) of a triangle $T \subset \mathbb{R}^2$ introduces the edge between the midpoint (indicated by the red dots) of the refinement edge (indicated by the red lines) and the opposite vertex. NVB refines T into two children T', T'' with equal area $|T'| = |T''| = |T|/2$ and, therefore, is a binary refinement rule.

(R1) *splitting estimate.* *For all $\mathcal{T}_H \in \mathbb{T}$ and marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, the refinement $\mathcal{T}_h \coloneqq \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ satisfies*

$$\#\mathcal{T}_h \le C_{\rm child}\, \#\mathcal{T}_H.$$

(R2) *overlay estimate.* *For all $\mathcal{T}_H, \mathcal{T}_h \in \mathbb{T}$, there exists a common refinement $\mathcal{T}_H \oplus \mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H) \cap \mathbb{T}(\mathcal{T}_h)$ with*

$$\#(\mathcal{T}_H \oplus \mathcal{T}_h) \le \#\mathcal{T}_H + \#\mathcal{T}_h - \#\mathcal{T}_0.$$

(R3) *closure estimate.* *Let $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ be an arbitrary sequence of successively refined meshes, i.e., for all $\ell \in \mathbb{N}_0$, there holds that $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ for some set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$. Then, we have that*

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm closure} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \quad \text{for all } \ell \in \mathbb{N}.$$

In FEM analysis, the constants often depend on the fact that the shape of the simplices does not deteriorate under mesh refinement to ensure that neighboring elements are not too different in size. We say that $\mathcal{T}_H \in \mathbb{T}$ is uniformly γ-shape regular if

$$0 < \gamma \coloneqq \inf_{T \in \mathcal{T}_H} \frac{\sup\{r > 0 \colon \exists x \in T, B_r(x) \subset T\}}{\operatorname{diam}(T)} < \infty. \tag{1.9}$$

An important property of NVB refinement is that it ensures that all meshes $\mathcal{T}_H \in \mathbb{T}$ are uniformly γ-shape regular. Henceforth, we will only work with conforming γ-shape regular meshes.

1.4.4 Adaptive algorithm with exact solve

With the modules from Section 1.4.1–1.4.3, we are able to present a first adaptive algorithm employing an exact solution of the discrete problem.

Algorithm 1A: AFEM

Input: Initial triangulation $\mathcal{T}_0 \in \mathbb{T}$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$.

For all $\ell = 0, 1, 2, \dots,$ repeat the following steps (i)–(iv):

(i) `SOLVE`: Compute the exact discrete solution $u_\ell^\star \in \mathcal{X}_\ell$ to (1.6).

(ii) `ESTIMATE`: Compute the refinement indicators $\eta_\ell(T; u_\ell^\star)$ from (1.8) for all $T \in \mathcal{T}_\ell$.

(iii) MARK: Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ satisfying the Dörfler marking criterion (D).

(iv) REFINE: Generate the refined triangulation $\mathcal{T}_{\ell+1} \coloneqq \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ by employing NVB.

Output: Sequence of successively refined triangulations $\mathcal{T}_\ell$, corresponding exact solutions $u_\ell^\star \in \mathcal{X}_\ell$, and residual error estimators $\eta_\ell(u_\ell^\star)$ for all $\ell \in \mathbb{N}_0$.

1.4.5 Quasi-optimality of Algorithm 1A

In the following subsection, we investigate the properties of the Algorithm 1A and show that the adaptive algorithm guarantees optimal convergence rates with respect to the number of simplices. We start with the estimator reduction due to the Dörfler marking criterion (D).

Lemma 1.3 (estimator reduction [CFPP14, Lemma 4.7])**.** *Suppose that the estimator satisfies stability* (A1) *and reduction* (A2)*. Let* $0 < \theta \leq 1$ *and suppose the use of Dörfler marking* (D)*. Then, we have estimator reduction, i.e., there exist* $0 < q_{\mathrm{est}} < 1$ *and* $C_{\mathrm{est}} > 0$ *such that*

$$\eta_h(u_h^\star) \leq q_{\mathrm{est}}\, \eta_H(u_H^\star) + C_{\mathrm{stab}}\, \|u_h^\star - u_H^\star\|_{\mathcal{X}} \quad \text{for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } \mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H). \tag{1.10}$$

The contraction constant reads $q_{\mathrm{est}} = \big[1 - (1 - q_{\mathrm{red}})\,\theta\big]^{1/2}$ *with* q_{red} *from* (A2). □

We see that the contraction constant q_{est} by definition satisfies $q_{\mathrm{est}} \to 1$ as $\theta \to 0$, so that the contraction deteriorates for tiny marking parameter θ. A first important consequence of estimator reduction (1.10) is (plain) convergence of the error estimator $\eta_\ell(u_\ell^\star) \to 0$ as $\ell \to \infty$. Moreover, reliability (A3) asserts error convergence $\|u^\star - u_\ell^\star\|_{\mathcal{X}} \to 0$ as $\ell \to \infty$, i.e., the computed solution $u_\ell^\star$ converges to the exact solution $u^\star$.

Corollary 1.4 (plain convergence [CFPP14, Corollary 4.8])**.** *Let* $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ *be a sequence of meshes* $\mathcal{T}_\ell \in \mathbb{T}$ *for all* $\ell \geq 0$ *satisfying estimator reduction* (1.10) *from Lemma 1.3 and* $\lim_{\ell\to\infty} \|u_{\ell+1}^\star - u_\ell^\star\|_{\mathcal{X}} = 0$*. Then, there holds estimator convergence*

$$\lim_{\ell\to\infty} \eta_\ell(u_\ell^\star) = 0 \tag{1.11}$$

and reliability (A3) *assures error convergence*

$$\lim_{\ell\to\infty} \|u^\star - u_\ell^\star\|_{\mathcal{X}} = 0. \quad \square \tag{1.12}$$

Additional to the axioms above, there holds a generalized quasi-orthogonality from [Fei22, Equation 8], which generalizes the Pythagorean identity for symmetric PDEs. This property links the error between two consecutive Galerkin solutions $u_{\ell+1}^\star, u_\ell^\star$ to the error between the exact solution and the Galerkin solution.

Proposition 1.5 (validity of quasi-orthogonality [Fei22, Equation 8])**.** *There exist* $C_{\mathrm{orth}} > 0$ *and* $0 < \delta \leq 1$ *such that the following holds: For any sequence* $(\mathcal{X}_\ell)_{\ell\in\mathbb{N}_0}$ *of nested finite-dimensional subspaces* $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H_0^1(\Omega)$*, the corresponding Galerkin solutions* $u_\ell^\star \in \mathcal{X}_\ell$ *to* (1.6) *satisfy*

(A4) *quasi-orthogonality.* $\sum_{\ell'=\ell}^{\ell+N} \|u^\star_{\ell'+1} - u^\star_{\ell'}\|^2_{\mathcal{X}} \le C_{\mathrm{orth}} (N+1)^{1-\delta} \|u^\star - u^\star_\ell\|^2_{\mathcal{X}}$ *for all* $\ell, N \in \mathbb{N}_0$.

Here, C_{orth} *and* δ *depend only on the dimension* d*, the elliptic bilinear form* $a(\cdot,\cdot)$*, and the chosen norm* $\|\cdot\|_{\mathcal{X}}$*, but are independent of the spaces* $\mathcal{X}_\ell$*.* □

Indeed, symmetric PDEs (like in (1.2)) allow for the Pythagorean identity

$$|||u^\star - u^\star_{\ell+1}|||^2 + |||u^\star_{\ell+1} - u^\star_\ell|||^2 = |||u^\star - u^\star_\ell|||^2 \quad \text{for all } \ell \in \mathbb{N}_0 \tag{1.13}$$

resulting in (A4) with $C_{\mathrm{orth}} = C^2_{\mathrm{cont}}/C^2_{\mathrm{ell}}$ and $\delta = 1$. However, for the later consideration of nonsymmetric PDEs, (1.13) fails, while (A4) remains valid. The generalized quasi-orthogonality (A4) allows us to improve the estimator reduction property in (1.10) to R-linear convergence of the estimator, i.e., quasi-contraction.

Theorem 1.6: R-linear convergence of the estimator [BFM⁺23, Theorem 4]

Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ *be a sequence of meshes with* $\mathcal{T}_\ell \in \mathbb{T}$*. Suppose estimator reduction (1.10) from Lemma 1.3, reliability (A3), and generalized quasi-orthogonality (A4). Then, there holds R-linear convergence of the estimator, i.e., there exist constants* $C_{\mathrm{lin}} \ge 1$ *and* $0 < q_{\mathrm{lin}} < 1$ *such that*

$$\eta_{\ell+N}(u^\star_{\ell+N}) \le C_{\mathrm{lin}}\, q^N_{\mathrm{lin}}\, \eta_\ell(u^\star_\ell) \quad \textit{for all } \ell, N \in \mathbb{N}_0. \tag{1.14}$$

The constants C_{lin} *and* q_{lin} *depend only on* C_{rel}*,* C_{orth} *from (A3) and (A4) and on* q_{est}*,* C_{est} *from Lemma 1.3.* □

So far, we have investigated linear convergence of the error estimator only. However, this convergence can be arbitrarily slow in theory. To formalize the idea of convergence rates, we abbreviate the set of all admissible meshes with at most $N \in \mathbb{N}_0$ more elements compared to the initial mesh $\mathcal{T}_0$ by $\mathbb{T}_N := \{\mathcal{T}_H \in \mathbb{T} : \#\mathcal{T}_H - \#\mathcal{T}_0 \le N\}$. For arbitrary $s > 0$, we introduce the nonlinear approximation class by

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{T}_H\in\mathbb{T}_N} \big[(N+1)^s\, \eta_H(u^\star_H)\big] \in [0,\infty]. \tag{1.15}$$

If $\|u^\star\|_{\mathbb{A}_s} < \infty$, then one can show that there exists a sequence of (not necessarily nested) optimal meshes $(\widehat{\mathcal{T}}_\ell)_{\ell\in\mathbb{N}_0}$ such that the corresponding error estimators $\widehat{\eta}_\ell(\widehat{u}^\star_\ell)$ decay with rate s, i.e., they satisfy $\widehat{\eta}_\ell = \mathcal{O}\big((\#\widehat{\mathcal{T}}_\ell)^{-s}\big)$. In the following, we want to illustrate that the sequence of computed meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 1A leads to decay of the output error estimator with rate s as well. One key observation is that not only the Dörfler marking (D) does imply R-linear convergence but that also the converse implication is true; the result essentially goes back to the seminal work [Ste07].

Lemma 1.7 (optimality of Dörfler marking (D) [CFPP14, Proposition 4.12])**.** *Suppose that the estimator satisfies stability (A1) and discrete reliability (A3⁺). Then, for all* $0 < \theta < \theta_{\mathrm{opt}} := (1 + C^2_{\mathrm{stab}}\, C^2_{\mathrm{drel}})^{-1}$*, there exists some* $0 < q_\theta < 1$ *such that*

$$\Big(\eta_h(u^\star_h) \le q_\theta \eta_H(u^\star_H) \Longrightarrow \theta\, \eta^2_H \le \eta(\mathcal{T}_H \setminus \mathcal{T}_h)^2\Big) \quad \textit{for all } \mathcal{T}_H \in \mathbb{T} \textit{ and } \mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H). \tag{1.16}$$

The constant q_θ depends only on $C_{\mathrm{stab}}, C_{\mathrm{drel}}$ from (A1) and (A3⁺), and θ. □

A key ingredient in the proof of optimal convergence of the error estimator is the following comparison lemma, which bounds the number of nonrefined elements in terms of the approximation class.

Lemma 1.8 (comparison lemma [CFPP14, Lemma 4.14])**.** *Suppose that the mesh refinement satisfies the overlay property (R2) and that the estimator satisfies quasi-monotonicity (QM) with constant C_{mon}. Let $\mathcal{T}_H \in \mathbb{T}$ such that the associated estimator fulfills $\eta_H(u_H^\star) > 0$ and let $0 < q < 1$. Then, for all $s > 0$ with $\|u^\star\|_{\mathbb{A}_s} < \infty$ there exists a refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with*

$$\eta_h(u_h^\star) \le q\,\eta_H(u_H^\star) \quad \text{and} \quad \#\mathcal{T}_h - \#\mathcal{T}_H \le C_{\mathrm{mon}}^{1/s}\, q^{-1/s}\, \|u^\star\|_{\mathbb{A}_s}^{1/s}\, \eta_H(u_H^\star)^{1/s}. \quad \square \tag{1.17}$$

Theorem 1.9: Optimal convergence rates of Algorithm 1A [CFPP14, Proposition 4.15]

Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be the meshes generated by Algorithm 1A. Suppose the mesh refinement properties (R1)–(R3) and that the estimator satisfies (A1)–(A3⁺) and the generalized quasi-orthogonality (A4). Recall θ_{opt} from Lemma 1.7. Then, for all $0 < \theta < \theta_{\mathrm{opt}} < 1$, there holds convergence of the estimator at optimal rate with respect to the number of elements, i.e., for all $s > 0$, there exist constants $c_{\mathrm{opt}}, C_{\mathrm{opt}} > 0$ such that

$$c_{\mathrm{opt}}\, \|u^\star\|_{\mathbb{A}_s} \le \sup_{\ell\in\mathbb{N}_0} \big[(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s\, \eta_\ell\big] \le C_{\mathrm{opt}}\, \|u^\star\|_{\mathbb{A}_s}. \tag{1.18}$$

The constant $c_{\mathrm{opt}} > 0$ depends only on $\#\mathcal{T}_0$, C_{stab} from (A1), C_{child} from (R1), and s; the constant $C_{\mathrm{opt}} > 0$ depends only on C_{mon} from (QM), C_{child} and C_{closure} from (R1) and (R3), $C_{\mathrm{mark}} \ge 1$ from (D), C_{lin} and q_{lin} from (1.14), q_θ from (1.16), and on s.

The upper bound in (1.18) guarantees that the estimator sequence generated by Algorithm 1A converges with rate $s > 0$ if a decay with rate s is possible along a sequence of optimal meshes. The lower bound in (1.18) means that the theoretically attainable optimal rate can be estimated by the convergence rate of the computed estimator sequence. In summary, the adaptive algorithm converges at any possible algebraic rate $s > 0$ and, thus, Algorithm 1A is indeed rate-optimal.

1.5 Adaptive FEM with inexact solver

The proof of optimal convergence rates with respect to the computational time relies on the fact that each module in Figure 1.6 can be realized in linear complexity, i.e., the invested work for each step of the algorithm is of order $\mathcal{O}(\#\mathcal{T}_\ell)$. Since a direct solver, in general, does not satisfy this assumption, we will embed an iterative algebraic solver into Algorithm 1A.

1.5.1 The module SOLVE

Suppose that we are given an iterative algebraic solver to treat the problem (1.3). Denote its iteration step by the function $\Psi_H : \mathcal{X}_H \to \mathcal{X}_H$, i.e., given an approximation $u_H \in \mathcal{X}_H$ of the exact solution $u_H^\star$ to (1.6), the solver returns an improved approximation $\Psi_H(u_H)$. Furthermore, we assume that the algebraic solver is contractive in the PDE-related norm $|||\cdot|||$ towards $u_H^\star$, i.e., there exists a constant $0 < q_{\rm ctr} < 1$ independent of $\mathcal{X}_H$ such that

$$|||u_H^\star - \Psi(u_H)||| \le q_{\rm ctr}\, |||u_H^\star - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H. \tag{C1}$$

The triangle inequality then assures a computable *a posteriori* control of the algebraic solver error

$$\frac{1-q_{\rm ctr}}{q_{\rm ctr}}\, |||u_H^\star - \Psi(u_H)||| \le |||\Psi(u_H) - u_H||| \le (1+q_{\rm ctr})\, |||u_H^\star - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H. \tag{C2}$$

Therefore, the algebraic solver error $|||u_H^\star - \Psi(u_H)|||$ is controlled by the computable error $|||\Psi(u_H) - u_H|||$ of two consecutive iterates. For a stopping parameter $\lambda > 0$, it is reasonable to stop the algebraic solver for the minimal index such that error by the inexact solution is controlled by a fixed λ-portion of the discretization error (similar to the Dörfler marking criterion (D)). Given a stopping parameter $\lambda > 0$, this means that we stop once there holds

$$|||\Psi(u_H) - u_H||| \le \lambda\, \eta_H(\Psi(u_H)). \tag{1.19}$$

In fact, our analysis only needs a weaker form of contraction (C1) with $C\, q_{\rm ctr}^k$ replacing $q_{\rm ctr}^k$ after k iterations of the algebraic solver with a uniform constant $C \ge 1$ independent of $\mathcal{X}_H$, together with the additional assumption of *a posteriori* control similar to (C2).

For symmetric positive definite systems, many solver options are available, e.g., methods that only need information on the given matrix. Examples of such methods include the Jacobi, Gauss–Seidel, and symmetric successive over-relaxation (SSOR) methods, as mentioned in [Kel95; Var00; Saa03]. More advanced algebraic methods include algebraic multigrid methods, which can be found in [BMR85; RS87; NN12; KA21], or domain decomposition methods with a divide-and-conquer ansatz in, e.g., [DW90; QV99; TW05; DJN15; GZ22]. Finally, we emphasize the importance of having $q_{\rm ctr}$ robust in the discretization parameters h and p to fully exploit the convergence rates of AFEM. Therefore, we resort to optimal geometric multigrid solvers or optimally preconditioned conjugate gradient methods and refer to Section 1.7.1 for a detailed presentation. Finally, it is worth mentioning that the described solvers can be adapted to serve as preconditioners for conjugate gradient methods [HS52], effectively accelerating the convergence speed of the solver.

1.5.2 Adaptive algorithm with iterative solve

Recall the `MARK` and `REFINE` modules from Section 1.4. We propose an adaptive algorithm with an iterative solver, where, due to the stopping criterion of the algebraic solver in (1.19), the `SOLVE` and `ESTIMATE` module are merged.

Algorithm 1B: AFEM with contractive solver

Input: Initial mesh $\mathcal{T}_0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$, solver-stopping parameter $\lambda > 0$, and an initial guess $u_0^0 \in \mathcal{X}_0$.

For all $\ell = 0, 1, 2, \dots$, repeat the following steps (i)–(iv):

(i) `SOLVE & ESTIMATE`: For all $k = 1, 2, 3, \dots$, repeat (a)–(b) until

$$|||u_\ell^k - u_\ell^{k-1}||| \le \lambda\, \eta_\ell(u_\ell^k). \tag{1.20}$$

(a) Compute $u_\ell^k := \Psi_\ell(u_\ell^{k-1})$ by one step of the contractive algebraic solver.
(b) Compute the refinement indicators $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$.

(ii) Upon termination of the iterative solver, define the index $\underline{k}[\ell] := k \in \mathbb{N}$.

(iii) `MARK`: Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k}}]$ satisfying (D) for $u_\ell^\star$ replaced by $u_\ell^{\underline{k}}$ with up to the factor C_{mark} minimal cardinality.

(iv) `REFINE`: Generate $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and employ nested iteration $u_{\ell+1}^0 := u_\ell^{\underline{k}}$.

Compared to Algorithm 1A (with mesh level ℓ only), the indices associated with Algorithm 1B are pairs (ℓ, k) and R-linear convergence (1.14) from Lemma 1.6 needs to be adapted to the new lexicographic ordering. To this end, we define the index set

$$\mathcal{Q} := \{(\ell, k) : u_\ell^k \text{ appears in Algorithm 1B}\}$$

and the lexicographic ordering

$$(\ell', k') \le (\ell, k) \iff u_{\ell'}^{k'} \text{ appears not later than } u_\ell^k \quad \text{for all } (\ell, k), (\ell', k') \in \mathcal{Q}.$$

The total step counter $|\cdot, \cdot|$ reads

$$|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} : (\ell', k') \le (\ell, k)\} = k + \sum_{\ell'=0}^{\ell-1} \underline{k}[\ell'] \quad \text{for all } (\ell, k) \in \mathcal{Q}.$$

The subsequent presentation employs certain stopping indices for the mesh level ℓ and the algebraic solver counter k defined by

$$\begin{aligned} \underline{\ell} &:= \sup\{\ell \in \mathbb{N}_0 : (\ell, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \\ \underline{k}[\ell] &:= \sup\{k \in \mathbb{N} : (\ell, k) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\} \quad \text{whenever } (\ell, 0) \in \mathcal{Q}, \end{aligned}$$

These definitions are consistent with Algorithm 1B and in $\underline{k}[\ell]$, we often omit the ℓ dependence whenever it is clear from the context, i.e., $\underline{k}$ abbreviates $\underline{k}[\ell]$ and $u_\ell^{\underline{k}}$ abbreviates $u_\ell^{\underline{k}[\ell]}$ for a given $\ell \in \mathbb{N}$ with $(\ell, \underline{k}) = (\ell, \underline{k}[\ell]) \in \mathcal{Q}$.

1.5.3 Quasi-optimality of Algorithm 1B

We start with a perturbed version of the estimator reduction property from Lemma 1.3 for the inexact final iterates $u_\ell^{\underline{k}}$.

Lemma 1.10 (perturbed estimator reduction for inexact solver [BFM+23, Equation (36)])**.** *Recall the constant $0 < q_{\mathrm{est}} < 1$ from Lemma 1.3. Suppose that the estimator satisfies stability (A1) and reduction (A2). Let $0 < \theta \le 1$ and suppose the use of Dörfler marking (D) with $u_\ell^\star$ replaced by $u_\ell^{\underline{k}}$. Then, we have estimator reduction*

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k}}) \le q_{\mathrm{est}}\, \eta_\ell(u_\ell^{\underline{k}}) + 2\, C_{\mathrm{stab}}\, \|u_{\ell+1}^\star - u_\ell^{\underline{k}}\|_{\mathcal{X}} \quad \textit{for all } (\ell+1, \underline{k}) \in \mathcal{Q}. \quad \square \tag{1.21}$$

The proof of optimal convergence rates for the inexact Algorithm 1B follows by the optimality result for AFEM with exact solver in Theorem 1.9 and the following perturbation argument with sufficiently small solver parameter λ. The latter ensures that the inexact final iterates $u_\ell^{\underline{k}}$ are sufficiently close to the exact discrete solution $u_\ell^\star$ leading to an equivalence of the inexact error estimator $\eta_\ell(u_\ell^{\underline{k}})$ to the exact error estimator $\eta_\ell(u_\ell^\star)$. Hence, Dörfler marking for some appropriate θ_{mark} and $\eta_\ell(u_\ell^\star)$ implies Dörfler marking for θ and $\eta_\ell(u_\ell^{\underline{k}})$.

Lemma 1.11 (estimator equivalence [CFPP14, Lemma 7.4])**.** *Suppose that the estimator satisfies stability (A1). Then, for all $0 < \theta \le 1$, $0 < \lambda < \lambda^\star \coloneqq \min\{1, C_{\mathrm{cont}}\, C_{\mathrm{ell}}^{-1}\, C_{\mathrm{stab}}^{-1}\,(1-q_{\mathrm{ctr}})/q_{\mathrm{ctr}}\}$, and all $(\ell,\underline{k}) \in \mathcal{Q}$, there holds equivalence*

$$[1-\lambda/\lambda^\star]\eta_\ell(u_\ell^{\underline{k}}) \le \eta_\ell(u_\ell^\star) \le [1+\lambda/\lambda^\star]\eta_\ell(u_\ell^{\underline{k}}). \tag{1.22}$$

Moreover, with θ sufficiently small to guarantee $0 < \theta_{\mathrm{mark}} \coloneqq (\theta^{1/2} + \lambda/\lambda^\star)^2\,(1-\lambda/\lambda^\star)^{-2} < 1$, there holds the following implication for any $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$

$$\theta_{\mathrm{mark}}\,\eta_\ell(u_\ell^\star)^2 \le \eta_\ell(\mathcal{R}_\ell; u_\ell^\star)^2 \quad\Longrightarrow\quad \theta\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 \le \eta_\ell(\mathcal{R}_\ell; u_\ell^{\underline{k},\underline{j}})^2. \tag{1.23}$$

This estimator equivalence ensures linear convergence of the estimator similarly to Theorem 1.6 with the final iterates $u_\ell^{\underline{k}}$ replacing $u_\ell^\star$ provided that the solver-stopping parameter λ is sufficiently small. However, we can even prove full R-linear convergence (i.e., quasi-contraction in each step of the algorithm) of some quasi-error consisting of discretization and algebraic solver error for any $\lambda > 0$. This will turn out to be the key ingredient in the proof of optimal complexity of Algorithm 1B.

Theorem 1.12: Full R-linear convergence of the quasi-error [BFM⁺23, Theorem 7]

Suppose that the estimator satisfies the axioms (A1)–(A3) and suppose quasi-orthogonality (A4). Let the adaptivity parameters $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, solver-stopping parameter $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Then, Algorithm 1B guarantees R-linear convergence of the quasi-error

$$\mathrm{H}_\ell^k \coloneqq \|u_\ell^\star - u_\ell^k\|_{\mathcal{X}} + \eta_\ell(u_\ell^k), \tag{1.24}$$

i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that

$$\mathrm{H}_\ell^k \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|}\,\mathrm{H}_{\ell'}^{k'} \quad \text{for all } (\ell',k'),(\ell,k) \in \mathcal{Q} \text{ with } |\ell',k'| \le |\ell,k|. \tag{1.25}$$

The constants C_{lin} and q_{lin} depend only on C_{stab}, q_{red}, C_{rel}, C_{orth}, from (A1)–(A3) and (A4), $C_{\mathrm{Céa}}$ from (Céa), C_{cont} and C_{ell} from (1.4), q_{ctr} from (C1), θ, and on λ. □

Full R-linear convergence asserts quasi-contraction independent of the algorithmic decision for an algebraic solver step (i.e., an increase of k) or local mesh-refinement (i.e., an increase of the mesh level ℓ). A first important corollary of full linear convergence (1.25) in Theorem 1.12 assures that the rates with respect to the degrees of freedom $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ and the convergence rates with respect to the computational cost indeed coincide provided that each module of the adaptive algorithm is realized in linear complexity. We shortly comment on the cost of the modules in the following.

(a) Each solver step of an optimal multigrid method in Algorithm 1B(i) can be performed in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, if smoothing is done according to the grading of the mesh [WZ17; IMPS24].

The same remark is valid for the preconditioned CG method with optimal additive Schwarz or BPX preconditioner [CNX12], where each solver step can be realized via successive updates in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations.

(b) The Dörfler marking strategy (D) in Algorithm 1B(ii) can be realized in linear complexity $\mathcal{O}(\#\mathcal{T}_\ell)$; see [Ste07] for $C_{\mathrm{mark}} = 2$ and [PP20] for $C_{\mathrm{mark}} = 1$.

(c) Local mesh refinement (including mesh closure) in Algorithm 1B(iv) of $\mathcal{T}_\ell$ by bisection can be realized in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations; see, e.g., [BDD04; Ste07].

The adaptive algorithm depends on the full history of algorithmic decisions, i.e., the computation of u_ℓ^k and $\mathcal{T}_\ell$ depends on all previously computed $u_{\ell'}^{k'}$ and $\mathcal{T}_{\ell'}$. Since each step can be realized in linear complexity, the overall computational cost until step $(\ell, k) \in \mathcal{Q}$, i.e., until (and including) the computation of u_ℓ^k, is thus proportional to

$$\mathtt{work}(\ell, k) := \sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \le |\ell,k|}} \#\mathcal{T}_{\ell'}. \tag{1.26}$$

Corollary 1.13 (rates = complexity [BIM⁺24a, Corollary 4.2])**.** *For $s > 0$, full convergence* (1.25) *yields*

$$M(s) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \, \mathsf{H}_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \mathtt{work}(\ell,k)^s \, \mathsf{H}_\ell^k \le C_{\mathrm{cost}}(s)\, M(s), \tag{1.27}$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_{lin} and q_{lin} from (1.25)*, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \le s_0$.* □

The second main result of this section combines the rate-optimality (1.18) from Theorem 1.9, full linear convergence (1.25) from Theorem 1.12, and a geometric series argument. The theorem shows optimal complexity of Algorithm 1B, i.e., optimal convergence rates with respect to the computational cost from (1.26), hence, the cumulative computation time.

Theorem 1.14: Optimal rates of Algorithm 1B with respect to computational cost

Recall $\lambda^\star$ and θ_{mark} from Lemma 1.11. Suppose that the estimator satisfies the axioms (A1)*–*(A3⁺) *and suppose quasi-orthogonality* (A4)*. Let θ and λ be sufficiently small in the sense that*

$$0 < \lambda < \lambda^\star \quad \text{and} \quad 0 < \theta_{\mathrm{mark}} = \frac{(\theta^{1/2} + \lambda/\lambda^\star)^2}{(1-\lambda/\lambda^\star)^2} < \theta_{\mathrm{opt}} := (1 + C_{\mathrm{stab}}^2 C_{\mathrm{drel}}^2)^{-1} < 1. \tag{1.28}$$

Then, Algorithm 1B guarantees, for all $s > 0$, that

$$c_{\mathrm{opt}} \, \|u^\star\|_{\mathbb{A}_s} \le \sup_{(\ell,k)\in\mathcal{Q}} \mathtt{work}(\ell,k)^s \, \mathsf{H}_\ell^k \le \widetilde{C}_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s}, \mathsf{H}_0^0\}. \tag{1.29}$$

The constant $c_{\mathrm{opt}} > 0$ depends only on C_{stab} from (A1)*, C_{child} from* (R1)*, and s; while the constant $\widetilde{C}_{\mathrm{opt}}$ depends only on C_{opt} from* (1.18)*, q_{ctr} from* (C1)*, C_{lin} and q_{lin} from* (1.25)*, and on s.*

1.6 Goal-oriented adaptive FEM

Rather than the exact solution $u^\star$ itself to (1.3), many practical applications aim to approximate a functional value $G(u^\star)$ of a linear continuous functional $G\colon \mathcal{X} \to \mathbb{R}$ at quasi-optimal computational cost. Clearly, any approximation $u_H \in \mathcal{X}_H$ of $u^\star$ leads to the following error estimate

$$|G(u^\star) - G(u_H)| \le \|G\|_{\mathcal{X}'} \, \|u^\star - u_H\|_{\mathcal{X}} \quad \text{where} \quad \|G\|_{\mathcal{X}'} \coloneqq \sup_{v \in \mathcal{X}\setminus\{0\}} \frac{|G(v)|}{\|v\|_{\mathcal{X}}}.$$

Thus, the approximation quality is limited by the quality of the approximation u_H of $u^\star$. While this estimate is linear in the approximation error, a duality technique from [GS02] allows to essentially obtain a quadratic estimate instead. To this end, let $z^\star$ be the unique solution to the so-called dual problem with given data $g \in L^2(\Omega)$ and $\boldsymbol{g} \in \left[L^2(\Omega)\right]^d$,

$$a(v, z^\star) = G(v) \coloneqq \int_\Omega g\, v + \boldsymbol{g} \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in \mathcal{X}. \tag{1.30}$$

Then, the Lax–Milgram lemma applies and guarantees existence and uniqueness of $z^\star$. Any approximation $z_H \in \mathcal{X}_H$ to $z^\star$ then leads to

$$\begin{aligned} G(u^\star) - G(u_H) = G(u^\star - u_H) &\overset{(1.30)}{=} a(u^\star - u_H, z^\star) \\ &\overset{(1.6)}{=} a(u^\star - u_H, z^\star - z_H) + \left[F(z_H) - a(u_H, z_H)\right]. \end{aligned}$$

Hence, the definition of the discrete goal $G_H(u_H, z_H) \coloneqq G(u_H) + \left[F(z_H) - a(u_H, z_H)\right]$ permits the goal-error estimate

$$|G(u^\star) - G_H(u_H, z_H)| = |a(u^\star - u_H, z^\star - z_H)| \overset{(1.4)}{\le} C_{\mathrm{cont}} \, \|u^\star - u_H\|_{\mathcal{X}} \, \|z^\star - z_H\|_{\mathcal{X}} \tag{1.31}$$

Thus, the goal-error can be estimated by the product of the two approximation errors and, in this sense, the estimate is quadratic.

1.6.1 State-of-the-art and outlook

The first rigorous quasi-optimality results for goal-oriented AFEM with linear goals were established in the seminal work [MS09] for the Poisson model problem. The work by [MS09] employs two possible sets of marked elements, one for reducing the error in the primal variable u and another for the dual variable z. The set of minimal cardinality is then chosen as the set of marked elements. However, since error reduction is only guaranteed for one term of the product, [BET11] proposes a weighted marking strategy and verifies an improved contraction compared to the marking criterion in [MS09], whereas optimal convergence rates remained open. The work [FGH+16] proved linear convergence and optimal convergence rates for symmetric problems and for both, the marking strategy of [MS09] and that of [BET11]. Notably, the analysis of [FGH+16] required new ideas beyond those of [MS09; BET11] for the proof of linear convergence, since the earlier works exploited monotonicity of the energy error and oscillations, which might fail if the diffusion coefficient is not constant. Finally, [FPZ16] extended this analysis to general second-order linear elliptic PDEs, while [HP16] only proved contraction but did not address optimal rates. In particular, [FPZ16]

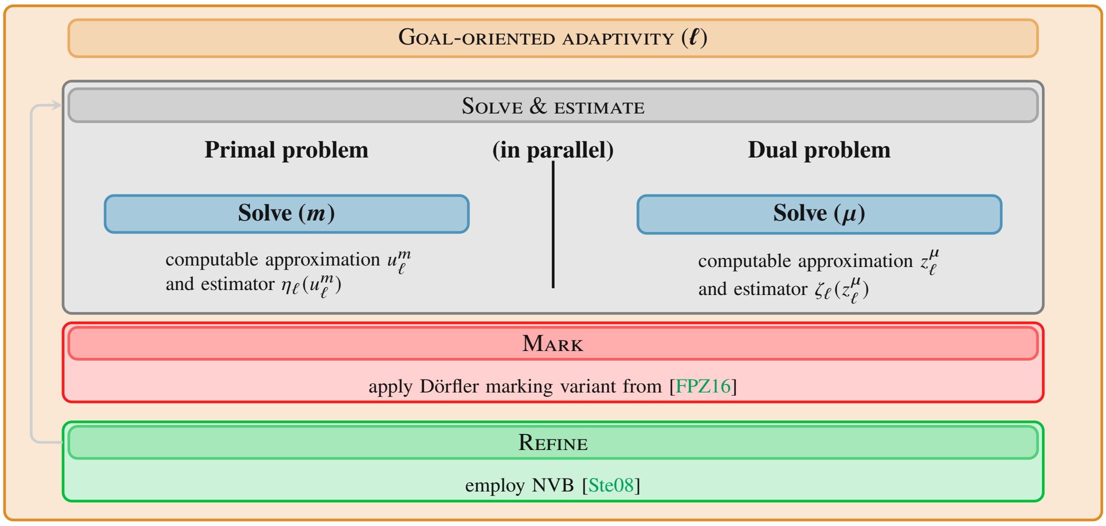


Figure 1.9: Nested loops of the goal-oriented adaptive algorithm with inexact solver.

proposes a novel marking strategy based on [MS09] that guarantees optimal convergence of the estimator product and is computationally favorable compared to earlier approaches. In addition, [BIP21] shows optimal convergence of a goal-oriented AFEM with a nonlinear quadratic goal, while [BBI+22] shows optimal convergence for semilinear problems with a linear goal functional. Another strategy includes a dual-weighted residual approach in the works [ELW19; ELW20; DBR21].

In the framework of GOAFEM, optimal complexity was first established in [MS09] for the Poisson model problem and sufficiently small adaptivity parameters. [BGIP23] proves optimal complexity for symmetric linear elliptic PDEs with a linear goal functional. Again, we note that the proof of full R-linear convergence in [BGIP23] is structurally different to those of [MS09; BET11; FGH+16; FPZ16] and explicitly based on tail-summability to avoid any constraint on λ. For nonsymmetric linear elliptic PDEs [FPZ16] proved optimal convergence of the estimator product. However, the analysis strongly relies on an exact solution of the arising discrete systems (through quasi-monotonicity of the exact error estimator) and thus optimal complexity is out of reach with this approach if arbitrary λ is targeted. Overall, optimal complexity for nonsymmetric problems with a linear goal functional, remained an open question, which we address in this thesis.

1.6.2 Product estimator structure in module `ESTIMATE`

In the following, we revisit the modules of the adaptive algorithm from Figure 1.6 and explain similarities and difference due to the nonlinear product structure of the upper bound in (1.31). The nested components of the proposed algorithm are illustrated in Figure 1.9. The `SOLVE` module is essentially the same as in Section 1.5.1 but consists of the simultaneous computation of the approximations u_ℓ^m and z_ℓ^μ for iteration counters m and μ instead of the single solver loop with iteration counter k. While the primal error estimator $\eta_H(\cdot)$ is given in (1.8), we define the local contributions to the dual error estimator $\zeta_H(\cdot)$, for all $T \in \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$, by

$$\zeta_\ell(T; v_H)^2 := |T|^{2/d} \, \|g + \operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{g})\|_{L^2(T)}^2 + |T|^{1/d} \, \|[\![(\boldsymbol{A}\nabla v_H - \boldsymbol{g}) \cdot \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2. \quad (1.32)$$

We again emphasize that the well-definiteness of the estimator in (1.32) requires that $\boldsymbol{g}|_T \in \big[H^1(T)\big]^d$ for all $T \in \mathcal{T}_0$. Then, it is well-known that the estimators η_H and ζ_H satisfy the following extended set of axioms.

Proposition 1.15 (goal-oriented axioms of adaptivity [CFPP14, Section 6.1])**.** *The error estimators η_H from (1.8) and ζ_H from (1.32) satisfy the following properties with constants $C_{\rm stab}, C_{\rm rel}, C_{\rm drel}, C_{\rm mon} > 0$ and $0 < q_{\rm red} < 1$ for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any conforming refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with the corresponding Galerkin solutions $u_H^\star \in \mathcal{X}_H, u_h^\star \in \mathcal{X}_h$ to (1.6), $z_H \in \mathcal{X}_H, z_h^\star \in \mathcal{X}_h$ to (1.30) and arbitrary $v_H \in \mathcal{X}_H$, $v_h \in \mathcal{X}_h$.*

(GOA1) ***stability:*** $|\eta_h(\mathcal{U}_H; v_h) - \eta_H(\mathcal{U}_H; v_H)| + |\zeta_h(\mathcal{U}_H; v_h) - \zeta_H(\mathcal{U}_H; v_H)| \le C_{\rm stab}\, \|v_h - v_H\|_{\mathcal{X}}$.

(GOA2) ***reduction:*** $\eta_h(\mathcal{T}_h\backslash\mathcal{T}_H; v_H) \le q_{\rm red}\, \eta_H(\mathcal{T}_H\backslash\mathcal{T}_h; v_H)$ *and* $\zeta_h(\mathcal{T}_h\backslash\mathcal{T}_H; v_H) \le q_{\rm red}\zeta_H(\mathcal{T}_H\backslash\mathcal{T}_h; v_H)$.

(GOA3) ***reliability:*** $\|u^\star - u_H^\star\|_{\mathcal{X}} \le C_{\rm rel}\, \eta_H(u_H^\star)$ *and* $\|z^\star - z_H^\star\|_{\mathcal{X}} \le C_{\rm rel}\, \zeta_H(z_H^\star)$.

(GOA3⁺) ***discrete reliability:*** $\|u_h^\star - u_H^\star\|_{\mathcal{X}} \le C_{\rm drel}\, \eta_H(\mathcal{T}_H\backslash\mathcal{T}_h, u_H^\star)$ *and*
$\|z_h^\star - z_H^\star\|_{\mathcal{X}} \le C_{\rm drel}\, \zeta_H(\mathcal{T}_H\backslash\mathcal{T}_h, z_H^\star)$.

(GOQM) ***quasi-monotonicity:*** $\eta_h(u_h^\star) \le C_{\rm mon}\, \eta_H(u_H^\star)$ *and* $\zeta_h(z_h^\star) \le C_{\rm mon}\, \zeta_H(z_H^\star)$. □

Reliability (GOA3) and stability (GOA1) verify

$$\begin{aligned}\|u^\star - u_H\|_{\mathcal{X}} &\le \max\{C_{\rm rel}, 1 + C_{\rm stab}\, C_{\rm rel}\}\, \big[\eta_H(u_H) + \|u_H^\star - u_H\|_{\mathcal{X}}\big],\\ \|z^\star - z_H\|_{\mathcal{X}} &\le \max\{C_{\rm rel}, 1 + C_{\rm stab}\, C_{\rm rel}\}\, \big[\zeta_H(z_H) + \|z_H^\star - z_H\|_{\mathcal{X}}\big].\end{aligned}$$

In combination with the estimate (1.31), we finally conclude for $C_{\rm goal} := C_{\rm cont}\, \max\{C_{\rm rel}^2, \big(1 + C_{\rm stab}\, C_{\rm rel}\big)^2\}$ the reliable *goal-error estimate*

$$|G(u^\star) - G_H(u_H, z_H)| \le C_{\rm goal}\, \big[\eta_H(u_H) + \|u_H^\star - u_H\|_{\mathcal{X}}\big]\, \big[\zeta_H(z_H) + \|z_H^\star - z_H\|_{\mathcal{X}}\big]. \tag{1.33}$$

Therefore, the goal-oriented adaptive algorithm needs to drive down the product of the primal and dual quasi-error rather than the primal error alone. Moreover, we note that the quasi-orthogonality from (A4) extends to the GOAFEM setting due to ellipticity of $a(\cdot\,,\cdot)$ and, thus, inf-sup stability.

Proposition 1.16 (validity of quasi-orthogonality [Fei22, Equation (8)])**.** *For any sequence $(\mathcal{X}_\ell)_{\ell\in\mathbb{N}_0}$ of nested discrete subspaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset \mathcal{X}$, there holds*

(GOA4) ***quasi-orthogonality:*** *There exist constants $C_{\rm orth} > 0$ and $0 < \delta < 1$ such that the corresponding Galerkin solutions $u_\ell^\star, z_\ell^\star \in \mathcal{X}_\ell$ to (1.6) and (1.30) satisfy, for all $\ell, N \in \mathbb{N}_0$,*

$$\sum_{\ell'=\ell}^{\ell+N} \|u_{\ell'+1}^\star - u_{\ell'}^\star\|_{\mathcal{X}}^2 \le C_{\rm orth}\, (N+1)^{1-\delta}\, \|u^\star - u_\ell^\star\|_{\mathcal{X}}^2, \tag{1.34a}$$

$$\sum_{\ell'=\ell}^{\ell+N} \|z_{\ell'+1}^\star - z_{\ell'}^\star\|_{\mathcal{X}}^2 \le C_{\rm orth}\, (N+1)^{1-\delta}\, \|z^\star - z_\ell^\star\|_{\mathcal{X}}^2. \tag{1.34b}$$

The constants $C_{\rm orth}$ and δ depend only on the dimension d, the elliptic bilinear form $a(\cdot,\cdot)$, and the chosen norm $\|\cdot\|_{\mathcal{X}}$, but are independent of the spaces $\mathcal{X}_\ell$. □

1.6.3 Combined marking in module MARK

A first marking strategy for goal-oriented AFEM was proposed in [MS09] that reads as follows: First, determine sets $\mathcal{M}_\ell^\eta, \mathcal{M}_\ell^\zeta \subseteq \mathcal{T}_\ell$ that satisfy the Dörfler marking criterion (D) with common parameter θ for the respective estimator in the upper index. Then, choose $\mathcal{M}_\ell = \mathcal{M}_\ell^\eta$ if $\mathcal{M}_\ell^\eta$ is the set with smaller cardinality and $\mathcal{M}_\ell = \mathcal{M}_\ell^\zeta$ otherwise. The work [FPZ16] extended this strategy and proposed to choose $\mathcal{M}_\ell \subseteq \mathcal{M}_\ell^\eta \cup \mathcal{M}_\ell^\zeta$ such that $\#\mathcal{M}_\ell \le C_{\rm mark}\,\#\mathcal{M}_\ell^{\rm min}$ and $\mathcal{M}_\ell^{\rm min} \subseteq \mathcal{M}_\ell$ with $C_{\rm mark} \ge 1$ and $\mathcal{M}_\ell^{\rm min} \in \{\mathcal{M}_\ell^\eta, \mathcal{M}_\ell^\zeta\}$ being the set of smaller cardinality. Then, $C_{\rm mark} = 1$ corresponds to the case, where the marking strategies from [MS09] and [FPZ16] coincide. However, $C_{\rm mark} = 2$ leads to the empirical observation that the resulting extended marking strategy from [FPZ16] is computationally superior to the criterion in [MS09].

1.6.4 Goal-oriented AFEM algorithm

We can now present an extension of Algorithm 1B with inexact solver to goal-oriented AFEM.

Algorithm 1C: Goal-oriented AFEM with contractive solver

Input: Initial mesh $\mathcal{T}_0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$, solver-stopping parameter $\lambda > 0$, and an initial guess $u_0^0, z_0^0 \in \mathcal{X}_0$.

For all $\ell = 0, 1, 2, \dots$, repeat the following steps (i)–(v):

(i) `SOLVE & ESTIMATE (PRIMAL)`: For all $m = 1, 2, 3, \dots$, repeat (a)–(b) until

$$|||u_\ell^m - u_\ell^{m-1}||| \le \lambda\, \eta_\ell(u_\ell^m). \tag{1.35}$$

  (a) Compute $u_\ell^m := \Psi_\ell(u_\ell^{m-1})$ with one step of the contractive solver.

  (b) Compute the refinement indicators $\eta_\ell(T, u_\ell^m)$ for all $T \in \mathcal{T}_\ell$.

(ii) `SOLVE & ESTIMATE (DUAL)`: For all $\mu = 1, 2, 3, \dots$, repeat (a)–(b) until

$$|||z_\ell^\mu - z_\ell^{\mu-1}||| \le \lambda\, \zeta_\ell(z_\ell^\mu). \tag{1.36}$$

  (a) Compute $z_\ell^\mu := \Psi_\ell(z_\ell^{\mu-1})$ with one step of the contractive solver.

  (b) Compute the refinement indicators $\zeta_\ell(T, z_\ell^\mu)$ for all $T \in \mathcal{T}_\ell$.

(iii) Upon termination of the iterative solvers, define the indices $\underline{m}[\ell] := m \in \mathbb{N}$, $\underline{\mu}[\ell] := \mu$, and $\underline{k}[\ell] := \max\{\underline{m}[\ell], \underline{\mu}[\ell]\} \in \mathbb{N}$.

(iv) `MARK`: Determine sets $\overline{\mathcal{M}}_\ell^u \in \mathbb{M}_\ell^u[\theta, u_\ell^{\underline{m}}]$ and $\overline{\mathcal{M}}_\ell^z \in \mathbb{M}_\ell^z[\theta, z_\ell^{\underline{\mu}}]$ satisfying the Dörfler criterion (D) for $u_\ell^{\underline{m}}$ resp. $z_\ell^{\underline{\mu}}$ with up to the factor $C_{\rm mark}$ minimal cardinality. Then, define the set of marked elements $\mathcal{M}_\ell := \mathcal{M}_\ell^u \cup \mathcal{M}_\ell^z$, where $\mathcal{M}_\ell^u \subseteq \overline{\mathcal{M}}_\ell^u$ and $\mathcal{M}_\ell^z \subseteq \overline{\mathcal{M}}_\ell^z$ satisfy $\#\mathcal{M}_\ell^u = \#\mathcal{M}_\ell^z = \min\{\#\overline{\mathcal{M}}_\ell^u, \#\overline{\mathcal{M}}_\ell^z\}$.

(v) `REFINE`: Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and define $u_{\ell+1}^0 := u_\ell^{\underline{m}}$ and $z_{\ell+1}^0 := z_\ell^{\underline{\mu}}$.

The analysis of Algorithm 1C is more involved compared to Algorithm 1B since the computation of the algebraic solver loops are decoupled and, therefore, one solver might stop earlier than the other. To this end, we define the full index set $\mathcal{Q} := \mathcal{Q}^u \cup \mathcal{Q}^z$ consisting of

$$\mathcal{Q}^u := \{(\ell, m) \in \mathbb{N}_0^2 \colon u_\ell^m \text{ is used in Algorithm 1C}\},$$
$$\mathcal{Q}^z := \{(\ell, \mu) \in \mathbb{N}_0^2 \colon z_\ell^\mu \text{ is used in Algorithm 1C}\}.$$

For the goal-oriented algorithm, additionally to the primal quasi-error H_ℓ^m from (1.24), we define the dual quasi error by

$$\mathsf{Z}_\ell^\mu := \|z_\ell^\star - z_\ell^\mu\|_{\mathcal{X}} + \zeta_\ell(u_\ell^\mu) \quad \text{for all } (\ell, \mu) \in \mathcal{Q}^z, \tag{1.37}$$

These definitions extend to the full index set $\mathcal{Q}$ by $\mathsf{H}_\ell^k := \mathsf{H}_\ell^{\underline{m}}$ for all $(\ell, k) \in \mathcal{Q}$ with $1 \le \underline{m}[\ell] \le k \le \underline{k}[\ell]$ and analogously for the dual quasi-error Z_ℓ^k. With this extension, we are able to formulate full linear convergence for the quasi-error product.

Theorem 1.17: Full R-linear convergence of the quasi-error product [BGIP23, Theorem 6]

Suppose that the estimators η from (1.8) and ζ from (1.32) satisfy the axioms (GOA1)–(GOA3) and suppose quasi-orthogonality (GOA4). Let $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$, and $u_0^0, z_0^0 \in \mathcal{X}_0$ be arbitrary. Then, Algorithm 1C guarantees R-linear convergence of the quasi-error product i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that

$$\mathsf{H}_\ell^k\, \mathsf{Z}_\ell^k \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k| - |\ell',k'|}\, \mathsf{H}_{\ell'}^{k'}\, \mathsf{Z}_{\ell'}^{k'} \quad \textit{for all } (\ell', k'), (\ell, k) \in \mathcal{Q} \textit{ with } |\ell', k'| \le |\ell, k|. \quad \square \tag{1.38}$$

The constants C_{lin} and q_{lin} depend only on C_{stab}, q_{red}, C_{rel} from (GOA1)–(GOA3), C_{orth} from (GOA4), C_{cont} and C_{ell} from (1.4), q_{ctr}, θ, and on λ.

Hence, optimal complexity can be established for Algorithm 1C with the quasi-error product $\mathsf{H}_\ell^k\, \mathsf{Z}_\ell^k$ replacing the primal quasi-error H_ℓ^k in (1.29) from Theorem 1.14 but with an upper bound only.

Theorem 1.18: Optimal complexity of Algorithm 1C [BGIP23, Theorem 8]

Recall $\lambda^\star$ and θ_{mark} from Lemma 1.11. Suppose that the estimators η from (1.8) and ζ from (1.32) satisfy the axioms (GOA1)–(GOA3) and suppose quasi-orthogonality (GOA4). Let θ and λ be sufficiently small in the sense that

$$0 < \lambda < \lambda^\star \quad \textit{and} \quad 0 < \theta_{\mathrm{mark}} = \frac{(\theta^{1/2} + \lambda/\lambda^\star)^2}{(1 - \lambda/\lambda^\star)^2} < \theta_{\mathrm{opt}} := (1 + C_{\mathrm{stab}}^2\, C_{\mathrm{drel}}^2)^{-1} < 1. \tag{1.39}$$

Then, Algorithm 1B guarantees, for all $s + t > 0$, that

$$\sup_{(\ell,k) \in \mathcal{Q}} \texttt{work}(\ell, k)^{s+t}\, \mathsf{H}_\ell^k\, \mathsf{Z}_\ell^k \le C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t},\, \mathsf{H}_0^0\, \mathsf{Z}_0^0\}. \tag{1.40}$$

> *The constant C_{opt} depends only on C_{opt} from (1.18), q_{ctr} from (C1), C_{lin} and q_{lin} from (1.25), and on s.*

1.7 Main contributions of the thesis and outline

The remainder of the thesis consists of four chapters. In the following, we give an overview over the scientific contributions of each of the chapters. We emphasize that all numerical experiments are openly available with reproducible MATLAB scripts under https://www.tuwien.at/mg/asc/praetorius/software/mooafem.

1.7.1 *hp*-robust multigrid solver for linear elliptic PDEs

M. Innerberger, A. Miraçi, D. Praetorius, and J. Streitberger. hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs. ESAIM Math. Model. Numer. Anal.*,* 58(1):247–272*,* 2024. DOI: 10.1051/m2an/2023104

In the framework of adaptive FEM, multilevel methods stand out as excellent and natural candidates for algebraic solvers due to the available hierarchy of meshes. Among the most notable techniques in this category, these are (geometric) multigrid methods [BL11; Hac85; BHM00; Osw94] and together with domain decomposition methods [DW90; QV99; TW05; DJN15; GZ22]. In the subsequent discussion, we provide a concise explanation of the first method tailored to symmetric second-order linear elliptic PDEs.

Geometric multilevel solvers represent some of the most efficient and versatile linear solvers. The fundamental concept of multigrid solvers involves capturing complementary components of the algebraic error through a hierarchy of meshes. To transfer information from mesh to mesh, interpolation and restriction operators transfer geometric data between the meshes in the hierarchy. At each level of the hierarchy, a set of simple iterations, termed smoothing, is employed to suppress high-oscillatory contributions to the error. On the coarsest mesh in the hierarchy, the matrix is small enough to permit a direct solve with constant computational effort

An iteration of the considered algebraic solver comprises a cycle, e.g., a V-cycle iterating from the finest mesh all meshes are visited in descending order until the coarsest mesh (with pre-smoothing steps), solves directly at the coarsest level, and then successively revisits the meshes to the finest one (with post-smoothing steps). An important feature of geometric multigrid methods is their intrinsic independence of mesh size (referred to as h-robustness), i.e., the contraction factor q_{ctr} from (C1) does not deteriorate with smaller local mesh size h. Additionally, the geometric multigrid solver discussed below is also p-robust, indicating that q_{ctr} is also independent of the polynomial degree p.

Previous work [MPV21] presents a p-robust geometric multigrid solver with a built-in algebraic error estimator for *a posteriori* steering of the algebraic solver. However, the patchwise smoothing in every vertex relative to every level used in this method causes a linear dependence of the contraction factor on the number of levels. In this chapter, we present a geometric multigrid method that overcomes this dependence and is, therefore, robust in both the mesh-size h and the polynomial degree p. To achieve this, we only use local lowest-order smoothing on patches that have changed in the mesh-refinement step in the intermediate levels, and we use patchwise higher-order smoothing only on the finest level. Furthermore, the solver involves only one post-smoothing step, requires no pre-smoothing, no symmetrization of the procedure, and employs optimal step sizes on the error-correction stage. More explicitly, Chapter 2 contains following main results:

(I) The solver iterates and the built-in algebraic error estimator η_{alg} are connected by

$$|||u_L^\star - \Psi(v_L)|||^2 \le |||u_L^\star - v_L|||^2 - \eta_{\mathrm{alg}}(v_L)^2 \quad \text{for all } v_L \in \mathcal{X}_L,$$

where $|||\cdot|||$ denotes the PDE-induced energy norm. For the details, we refer to equation (2.12) in Theorem 2.4.

(II) The solver contracts the error, i.e., there exists a uniform constant $0 < q_{\mathrm{ctr}} < 1$ such that

$$|||u_L^\star - \Psi(v_L)||| \le q_{\mathrm{ctr}}\, |||u_L^\star - v_L||| \quad \text{for all } v_L \in \mathcal{X}_L.$$

The precise statement is given in (2.13) in Theorem 2.4. In particular, q_{ctr} is independent of the polynomial degree p, the number of mesh levels L, and the sequence of computed meshes $\mathcal{T}_1, \dots, \mathcal{T}_L$.

(III) Finally, equation (2.14) in Theorem 2.4 states that the built-in estimator is a two-sided bound of the algebraic error, i.e., there exists $C_{\mathrm{rel}} > 1$ such that

$$\eta_{\mathrm{alg}}(v_L) \le |||u_L^\star - v_L||| \le C_{\mathrm{rel}}\, \eta_{\mathrm{alg}}(v_L) \quad \text{for all } v_L \in \mathcal{X}_L.$$

(IV) The application of the hp-robust solver in the context of adaptive FEM leads to optimal complexity of the adaptive algorithm; see Theorem 2.7.

We highlight some key features of the algebraic solver in the following and refer to Section 2.4 for the details and more numerical experiments involving jumps in the diffusion coefficient $\boldsymbol{A}$.

Optimal complexity of the solver. Figure 1.10 illustrates that the proposed solver is indeed of linear complexity and after 10^6 degrees of freedom is even faster than the MATLAB built-in optimized solver `mldivide`. Figure 1.10 (right) verifies that the relative computation time per degree of freedom is constant for the proposed solver and, hence, the solver cost is of order $\mathcal{O}(\#\mathcal{T}_L)$.

Optimality of AFEM with the hp-robust multigrid method. Figure 1.11 displays that an embedding of the multigrid solver into Algorithm 1B from Section 1.5 leads to optimal convergence rates with respect to both the number of degrees of freedom and the cumulative computation time.

1.7.2 Optimal complexity of AFEM for nonsymmetric linear elliptic PDEs

M. Brunner, M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger, and P. Heid. Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs. IMA J. Numer. Anal., 44(3):1560–1596, 2024. DOI: `10.1093/imanum/drad039`

M. Brunner, M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger, and P. Heid. Corrigendum to: Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs. IMA J. Numer. Anal., 44(3):1903–1909, 2024. DOI: `10.1093/imanum/drad103`

Recent works [GHPS21; HPW21; HPSV21] have proven optimal complexity for energy minimization problems, especially for symmetric linear elliptic PDEs. However, for nonsymmetric linear elliptic PDEs, the question of optimal complexity remained open due to the lack of an algebraic solver that contracts in the equivalent energy norm of the principal part of the PDE. A possible approach in line with developments in AFEM for nonlinear problems employs a coupling of a symmetrization (e.g., the Zarantonello iteration, which is a Richardson-type iteration stemming

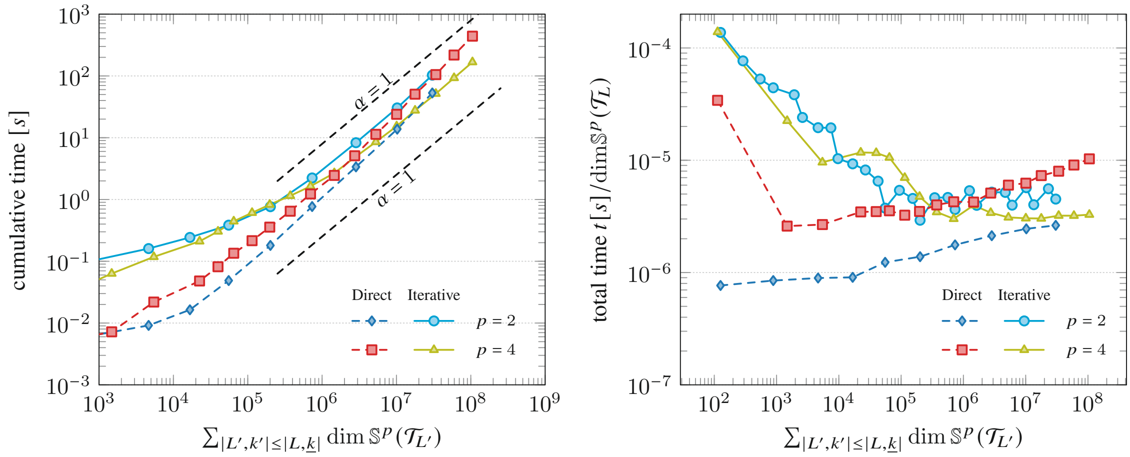


Figure 1.10: Left: History plot of the cumulative computation time of the hp-robust multigrid solver compared to the MATLAB built-in solver `mldivide`. Right: History plot of the relative computation time per degree of freedom.

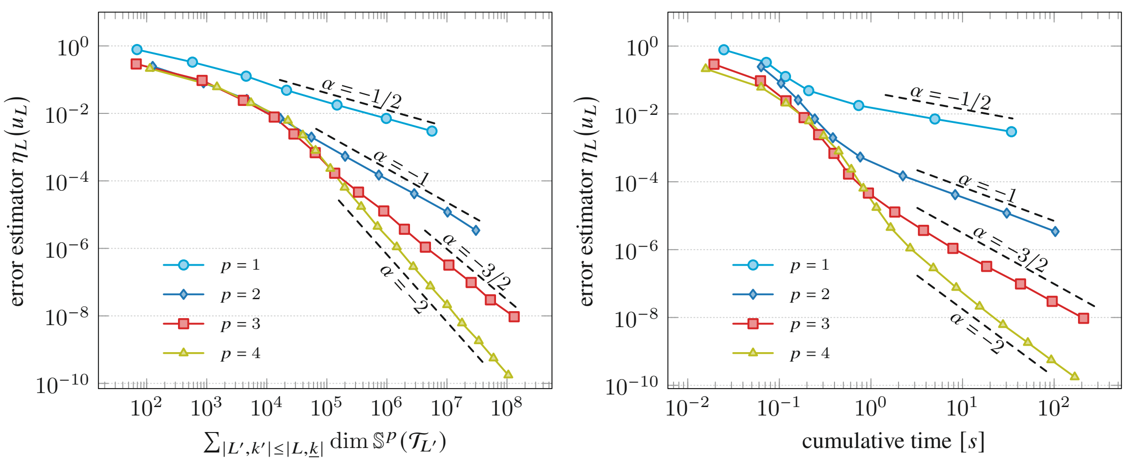


Figure 1.11: Left: Convergence history plot of the error estimator on the final iterates u_L of the multigrid solver with respect to the number of degrees of freedom. Right: Convergence history plot of the error estimator over the cumulative computation time.

from the state-of-the-art proof of the Lax–Milgram lemma) and an optimal algebraic solver. This combination leads to a contractive solver itself except for the final iterate of this inexact Zarantonello iteration. This presents a challenge as it seemingly does not fit into the framework for symmetric PDEs from Section 1.5. However, perturbation arguments in Chapter 3 allow to show the following main results:

(I) Theorem 3.7 guarantees full linear convergence provided that the algebraic solver parameter λ_{alg} is sufficiently small (to ensure contraction of the coupling) and the symmetrization parameter λ_{sym} is arbitrary but satisfies the natural assumption $0 < \lambda_{\mathrm{sym}} \leq 1$. Moreover, the full linear convergence in this chapter is restricted to a sufficiently large mesh level $\ell \geq \ell_0$ (with *a priori* unknown $\ell_0 \in \mathbb{N}_0$) since the proof involves a quasi-Pythagorean estimate replacing the Pythagorean identity for symmetric PDEs. The proof is based on contraction of an equivalent quasi-error quantity and thus leads to restriction on the solver parameters.

(II) Full linear convergence implies that the convergence rates with respect to the number of degrees of freedom and with respect to the overall computational cost are equivalent in Corollary 3.8.

(III) Theorem 3.9 states optimal complexity for sufficiently small adaptivity parameters using optimal convergence rates with respect to the number of degrees of freedom, the estimator equivalence in the spirit of (1.22), and full linear convergence.

Again, we want to briefly illustrate the results with numerical experiments and refer to Section 3.6 for an in-depth analysis of the involved contraction constants, the upper bound for the restriction on the algebraic solver parameter, and further optimality experiments.

Optimality of AFEM with the inexact Zarantonello iteration. In Figure 1.12, we see that the proposed nested iteration leads to optimal convergence rates with respect to both the number of degrees of freedom and the cumulative computation time for several polynomial degrees p.

Linear complexity of the inexact Zarantonello iteration. In Figure 1.13, we see that the proposed inexact Zarantonello iteration is indeed of linear complexity and outperforms the MATLAB built-in direct solver.

1.7.3 On full linear convergence and optimal complexity of AFEM

P. Bringmann, M. Feischl, A. Miraçi, D. Praetorius, and J. Streitberger. On full linear convergence and optimal complexity of adaptive FEM with inexact solver, 2023. *arXiv*: `2311.15738`, *submitted to Comput. Math. Appl.*

In this chapter, we review the progress of the state-of-the-art analysis concerning optimal convergence rates with respect to the cumulative computational cost. Improving the analysis from Chapter 3, we present a novel proof of full linear convergence for AFEM with nested iterative solvers under weaker assumption compared to the previous results in [BIM+24a]. Chapter 4 contains the following main results:

(I) Section 4.3 reviews the recent progress in the field of AFEM with exact solver, with contractive algebraic solver in Section 4.4, and nested iterative solver in Section 4.5 as well as an outlook on nonlinear PDEs in Section 4.6.

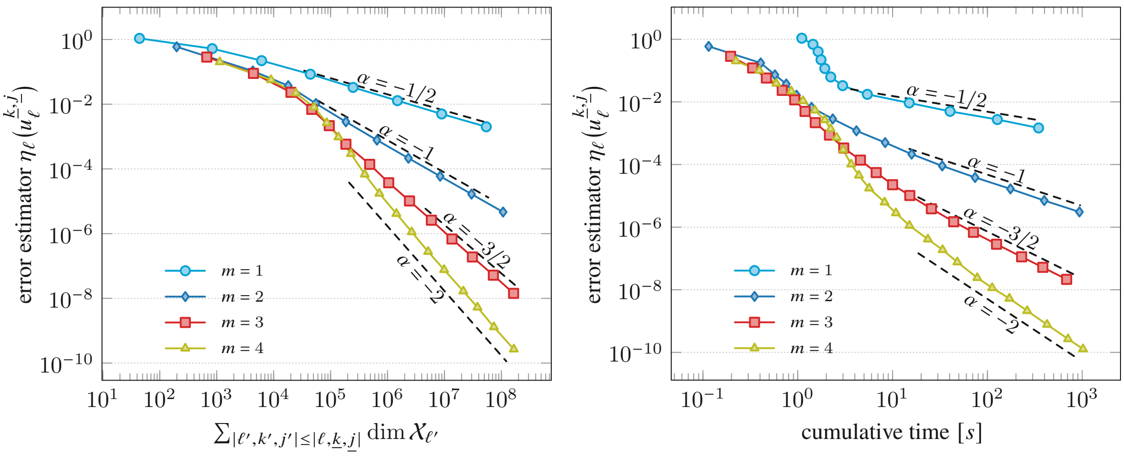


Figure 1.12: Left: Convergence history plot of the error estimator over the overall computational cost. Right: Convergence history plot of the error estimator with respect to the cumulative computation time.

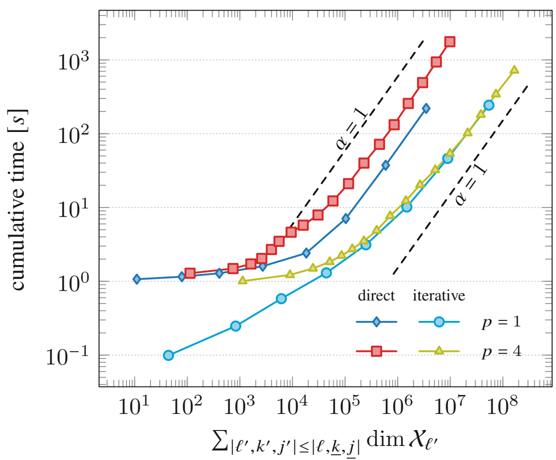


Figure 1.13: History plot of the cumulative computation time of the nested Zarantonello iteration with algebraic solver compared to the MATLAB built-in backslash solver.

(II) Unlike earlier works [GHPS21; HPW21], the proof of full R-linear convergence with iterative solver in Theorem 4.7 does not require any additional constraints on the solver parameter. Moreover, the proof relies on the novel quasi-orthogonality from [Fei22] and is therefore not restricted to energy minimization problems but is applied to nonsymmetric linear elliptic PDEs and paves the way for further extension to more general inf-sup stable problems.

(III) Contrary to [BIM⁺24a], the proof of full R-linear convergence for nested iterative solvers in Theorem 4.15 is not based on a contraction argument but on a summability argument. This novel proof strategy eliminates the need for a sufficiently large mesh level (i.e., $\ell_0 = 0$) and relaxes the constraint on the parameters from [HPSV21; BIM⁺24a]. More precisely, the full R-linear convergence in [HPSV21] requires λ_{alg} to be small with respect to the linearization parameter λ_{lin} and the latter to be small with respect to θ. Moreover, the paper [BIM⁺24a] still needs λ_{alg} to be small with respect to q_{ctr}, q_{sym}, and $\lambda_{\mathrm{sym}} \leq 1$. With the new proof, we show that this requirement can be replaced by λ_{alg} being small with respect to q_{ctr} and q_{sym} and the product $\lambda_{\mathrm{alg}}\, \lambda_{\mathrm{sym}}$ being sufficiently small.

(IV) We show that the equivalence of the rates with respect to the degrees of freedom and with respect to the overall computational cost from Corollary 1.13 is still valid for a suboptimal algebraic solver in Corollary 4.11.

In the numerical experiments in Section 4.7, we investigate optimality for several adaptivity parameters and analyze favorable choices of the solver stopping parameters. Thereby, we show that also larger parameters are feasible and, in practice, favorable. We want to illustrate the contributions with the following experiment:

Optimality of AFEM with nested iterative solver and several stopping parameters. In Figure 1.14, we see optimal convergence rates for polynomial degrees $p = 2$ and several stopping parameters λ_{alg} and λ_{sym}.

1.7.4 Optimal complexity of GOAFEM for nonsymmetric linear elliptic PDEs

P. Bringmann, M. Brunner, D. Praetorius, and J. Streitberger. Optimal complexity of goal-oriented adaptive FEM for nonsymmetric linear elliptic PDEs, 2023. *arXiv*: `2312.00489`*, submitted to J. Numer. Math.*

This chapter is devoted to the cost-effective approximation of the linear quantity of interest $G(u^\star)$. For symmetric linear elliptic PDEs, a goal-oriented adaptive algorithm with optimal complexity has been presented in [BGIP23]. However, for nonsymmetric PDEs, due to the additional symmetrization loop and lack of a Pythagorean identity, the analysis does not directly transfer to nonsymmetric PDEs. Instead, the proof strategy from [BFM⁺23] based on summability is required due to the nonlinear product structure achieved by the combined quasi-error product and the resulting nonlinear remainder term in the proof of full linear convergence. The main contributions of Chapter 5 read as follows:

(I) Algorithm 5A presents a novel GOAFEM approach with nested iterative solvers for the numerical solution of nonsymmetric linear elliptic PDEs and a linear quantity of interest.

(II) Theorem 5.10 asserts full linear convergence for the combined quasi-error product by proving contraction up to a summable remainder term. Therefore, we again only require that the

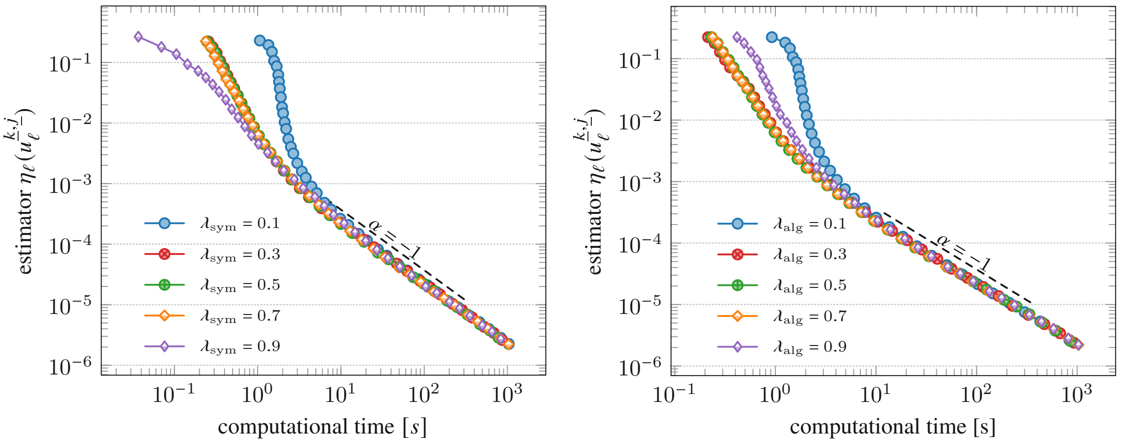


Figure 1.14: Left: Convergence history plot of the error estimator with respect to the cumulative computation time for fixed $\theta = 0.3$ in both experiments and $\lambda_{alg} = 0.7$ with several symmetrization stopping parameter. Right: Convergence history plot of the error estimator with respect to the cumulative computation time for fixed $\lambda_{sym} = 0.7$ with several algebraic solver stopping parameter.

algebraic solver parameter λ_{alg} is sufficiently small and that the product $\lambda_{alg}\,\lambda_{sym}$ is sufficiently small.

(III) Theorem 5.15 ensures optimal complexity of the proposed GOAFEM algorithm provided that all involved adaptivity parameters are sufficiently small.

In the numerical experiments in Section 5.7, we investigate optimality of the proposed GOAFEM algorithm and analyze favorable choices of the solver stopping parameters with respect to the cumulative computational cost.

Optimality of GOAFEM with nested iterative solver and large solver-stopping parameters. In Figure 1.15, that the algorithm leads to optimal decay rates $-p$ for the estimator product and the goal error with respect to the number of degrees of freedom and the overall computation time for several polynomial degrees p.

1.8 Additional scientific contributions

This section highlights additional scientific contributions beyond the scope of this thesis.

1.8.1 Parameter-robust full linear convergence and optimal complexity of AFEM for nonlinear PDEs

A. Miraçi, D. Praetorius, and J. Streitberger. Parameter-robust full linear convergence and optimal complexity of adaptive iteratively linearized FEM for strongly monotone nonlinear PDEs, 2024. *arXiv*: 2401.17778, *submitted to Math. Comp.*

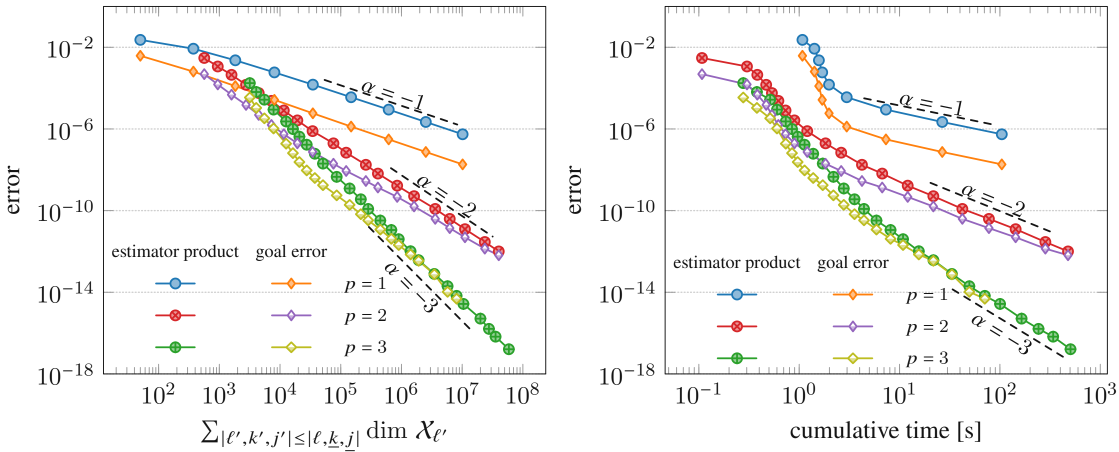


Figure 1.15: Left: Convergence history plot of the error estimator and the goal error with respect to the number of degrees of freedom for fixed $\theta = 0.5$ and $\lambda_{\text{sym}} = \lambda_{\text{alg}} = 0.7$. Right: Convergence history plot of the error estimator and the goal error with respect to the cumulative computation time.

In this work, we consider a nonlinear elliptic partial differential equation with a scalar nonlinearity $\mu \in C^1(\mathbb{R}_{\geq 0})$. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ where $d \geq 1$ and right-hand sides $f \in L^2(\Omega)$ and $\boldsymbol{f} \in \left[L^2(\Omega)\right]^d$, we seek the solution $u^\star$ to

$$-\operatorname{div}\left(\mu(|\nabla u^\star|^2)\nabla u^\star\right) = f - \operatorname{div} \boldsymbol{f} \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega. \tag{1.41}$$

We exploit the inherent energy structure of this equation to develop and analyze an adaptive algorithm with a novel parameter-free stopping criterion that steers the algebraic solver. We show that the new stopping criterion leads to an equivalence of norm error and energy difference and a uniformly bounded number of algebraic solver steps, which simplifies the proof of full R-linear convergence. Therefore, full R-linear convergence of the quasi-error is guaranteed for arbitrary adaptivity parameters and, thus, the analysis overcomes even the weaker parameter restriction of [BFM+23]. Finally, we show that sufficiently small adaptivity parameters assert optimal complexity of the adaptive algorithm. The numerical experiments highlight the new stopping criterion together with the application of the parameter-free Kačanov iteration as linearization method.

1.8.2 Cost-optimal AFEM for semilinear elliptic PDEs

M. Brunner, D. Praetorius, and J. Streitberger. Cost-optimal adaptive FEM with linearization and algebraic solver for semilinear elliptic PDEs, 2024. *arXiv*: *2401.06486*, *submitted to Numer. Math.*

In this work, we consider the semilinear elliptic PDE

$$-\operatorname{div}\big(\boldsymbol{A}\,\nabla u^\star\big) + b(u^\star) = f - \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega \tag{1.42}$$

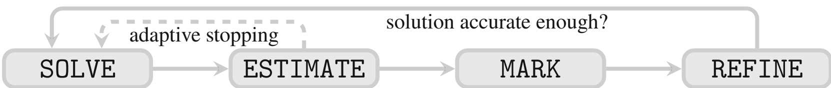
with a uniformly elliptic diffusion matrix $\boldsymbol{A} \in \left[L^\infty(\Omega)\right]^{d\times d}_{\text{sym}}$ and monotone nonlinearity $b\colon \Omega \to \mathbb{R}$. We assume that the problem (1.42) fits into the framework of the Browder–Minty theorem, i.e., the associated operator $\mathcal{A}$ is strongly monotone and *locally* Lipschitz continuous. Therefore, the Lipschitz constant of $\mathcal{A}$ depends on the considered functions and the adaptive algorithm consisting of a coupling of iterative linearization and algebraic solver needs to ensure that the norm of all computed iterates remains bounded. This is achieved by enforcing that sufficiently many algebraic solver steps have been carried out and it is shown that the additional cost of this procedure is negligible. Finally, the uniform boundedness of all iterates allows the proof of full R-linear convergence for arbitrary adaptivity parameters (at the expense of ensuring sufficiently many solver steps). Finally, optimal complexity is shown for sufficiently small adaptivity parameters.

2 *hp*-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs

The Sections 2.1–2.5 of this chapter correspond to the publication:
M. Innerberger, A. Miraçi, D. Praetorius, and J. Streitberger. hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs. *ESAIM Math. Model. Numer. Anal.*, 58(1):247–272, 2024. DOI: 10.1051/m2an/2023104

2.1 Introduction

Numerical schemes for PDEs aim at approximating the solution $u^\star$ of the weak formulation with an error below a certain tolerance at minimal computational cost. Since the accuracy is spoiled by singularities, e.g., in given data or domain geometry, adaptive finite element methods (AFEMs) employ the loop



to obtain a sequence of meshes $\mathcal{T}_L$ that resolve such singularities. For a large class of problems, it is known that AFEM is *rate-optimal*, i.e., one can construct an estimator $\eta_L(u_L^\star)$ from the exact Galerkin solution $u_L^\star$ for the discretization error $|||u^\star - u_L^\star|||$ that decreases with the largest possible rate with respect to the number elements in $\mathcal{T}_L$; see, e.g., the seminal works [Dör96; MNS00; BDD04; Ste07; CKNS08] or the abstract overview [CFPP14] for h-adaptive FEM with fixed polynomial degree p.

In practice, the SOLVE module may become computationally expensive (in contrast to all other modules) when employing a direct solver; see, e.g., [PP20; GHPS21; IP23] for a discussion of implementational aspects. Thus, usually, an iterative solver is employed to compute an approximation u_L of $u_L^\star$ on each level, and the exact Galerkin solution $u_L^\star$ is not available. The question of whether the approximations u_L converge with optimal rate with respect to the *overall computational cost* was already treated in the seminal work [Ste07] under some *realistic* assumptions about an abstract iterative solver. The recent work [GHPS21] employs nested iterations and an adaptive stopping criterion to steer a uniformly contractive iterative solver, linking the SOLVE and ESTIMATE module in the above scheme by an inner loop. Then, it is shown that even the *full sequence of iterates* converges with optimal rates with respect to the overall computational cost. For this reason, the design of algebraic solvers that are uniformly contractive and robust with respect to the discretization parameters is of utmost importance.

The hierarchical structure of AFEM and the very nature of the arising linear systems suggest to use a *multilevel* solver; see, e.g., [Hac85; BMR85; BPS86; BDY88; BPX90; Zha92; Rüd93a; Osw94]. Different adaptive methods integrating a multilevel solver are possible; see, e.g., [BB87] for generating local meshes, and [Rüd93b] for a fully adaptive multigrid method that steers the local refinement process. In the context of AFEM, the adaptively constructed hierarchy of locally refined meshes calls for suitable *local* solvers. We refer to [CNX12] for a multilevel preconditioner on a mesh hierarchy consisting of one bisection in each step and [HWZ12; WZ17] for multiplicative multigrid methods, all of which are robust with respect to the mesh size h. Though these works allow for higher-order FEM, an analytic and numerical study on the behavior with increasing polynomial degree was not presented. This aspect is treated, e.g., in [Pav94; SMPZ08; AMV18; BF22], which design iterative solvers that are robust with respect to the polynomial degree p on various types of polyhedral meshes. The recent own work [MPV21] proposes a p-robust geometric multigrid which comes with a *built-in algebraic error estimator* $\eta_{\text{alg}}(u_L)$, which is suited perfectly for *a posteriori* steering (i.e., adaptive termination) of the algebraic solver. However, the employed patchwise smoothing associated to every vertex and every level causes a linear dependence on the number of adaptive mesh levels L.

In the present work, we modify the solver from [MPV21] and overcome this dependence for locally refined meshes: we only apply *local* lowest-order smoothing on patches which change in the refinement step on intermediate levels, whereas a patchwise (and hence parallelizable) higher-order smoothing on all patches of the finest level is applied. This solver only needs one post-smoothing step, requires no symmetrization of the procedure (see also [DHM⁺21]), and, in particular, has no tunable parameters since it utilizes optimal step-sizes on the error-correction stage. As the main result of the present work, we show that the proposed solver *uniformly contracts* the algebraic error $|||u_L^\star - u_L|||$. Moreover, it comes with a built-in estimator $\eta_{\text{alg}}(u_L)$, which is shown to be *equivalent* to the algebraic error $|||u_L^\star - u_L|||$. Throughout, all involved estimates are robust in the discretization parameters h and p.

As one potential application, we formulate an AFEM algorithm in the spirit of [GHPS21] that naturally embeds the multigrid solver and leverages the solver's built-in algebraic error estimator $\eta_{\text{alg}}(u_L)$ to stop the solver as soon as the discretization and algebraic error are comparable. Adapting the arguments of [GHPS21], we prove that, for fixed polynomial degree p, the AFEM algorithm guarantees optimal convergence rates with respect to overall computational cost.

Using the open-source object-oriented 2D MATLAB code MooAFEM [IP23], we present a detailed numerical study of both the algebraic solver and the adaptive algorithm, including higher-order experiments and jumping coefficients.

The outline of this chapter reads as follows: Section 2.2 first poses the model problem and introduces some notation. Then, we state the proposed multigrid solver (Algorithm 2A) and formulate our main results on hp-robust contraction (Theorem 2.4) and algebraic error control (Corollary 2.5). As a potential application, Section 2.3 formulates an AFEM algorithm (Algorithm 2B) which employs nested iteration and an adaptive stopping criterion for the iterative solver using the built-in *a posteriori* estimator for the algebraic error. Theorem 2.7 proves optimal computational complexity of the proposed AFEM algorithm. After we confirm the theoretical results by numerical examples in Section 2.4, we present proofs of the main results in Section 2.5. For better readability, we precede these proofs with three subsections presenting their core arguments: geometric properties of the meshes $\mathcal{T}_L$, an hp-robust stable decomposition combining a local lowest-order multilevel stable decomposition from [WZ17] with a one-level p-robust decomposition from [SMPZ08], and a strengthened Cauchy–Schwarz inequality in the spirit of [CNX12; HWZ12].

2.2 *hp*-robust multigrid solver

In this section, we formulate the model problem, the proposed geometric multigrid method, and the main results, while the proofs are postponed to Section 2.5.

2.2.1 Model problem

For $d \in \{1, 2, 3\}$, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with polygonal boundary $\partial\Omega$. Given $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$, we consider the second-order linear elliptic diffusion problem

$$\begin{aligned} -\operatorname{div}(\boldsymbol{A}\nabla u^\star) &= f - \operatorname{div}\boldsymbol{f} && \text{in } \Omega, \\ u^\star &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ is the symmetric and uniformly positive definite diffusion coefficient. More precisely, given a conforming simplicial triangulation $\mathcal{T}_H$ of Ω into compact simplices, we have $\boldsymbol{A}|_T \in [W^{1,\infty}(T)]^{d\times d}$ for all $T \in \mathcal{T}_H$. For $x \in \Omega$ we denote the maximal and minimal eigenvalue of $\boldsymbol{A}(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ by $\lambda_{\max}(\boldsymbol{A}(x))$ and $\lambda_{\min}(\boldsymbol{A}(x))$, respectively, and define $\Lambda_{\max} \coloneqq \operatorname{ess\,sup}_{x\in\Omega} \lambda_{\max}(\boldsymbol{A}(x))$ as well as $\Lambda_{\min} \coloneqq \operatorname{ess\,inf}_{x\in\Omega} \lambda_{\min}(\boldsymbol{A}(x))$. With $\langle \cdot\,,\,\cdot \rangle_\omega$ denoting the usual $L^2(\omega)$-scalar product for a measurable subset $\omega \subseteq \Omega$, the weak formulation of (2.1) seeks $u^\star \in \mathcal{X} \coloneqq H^1_0(\Omega)$ solving

$$\langle\!\langle u^\star\,,\, v \rangle\!\rangle_\Omega \coloneqq \langle \boldsymbol{A}\nabla u^\star\,,\, \nabla v\rangle_\Omega = \langle f\,,\, v\rangle_\Omega + \langle \boldsymbol{f}\,,\, \nabla v\rangle_\Omega \eqqcolon F(v) \quad \text{for all } v \in \mathcal{X}. \tag{2.2}$$

We note that $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle_\Omega$ is a scalar product and the induced semi-norm $|||u|||^2_\Omega \coloneqq \langle\!\langle u\,,\,u \rangle\!\rangle_\Omega$ is an equivalent norm on $\mathcal{X}$. Therefore, the Lax–Milgram lemma yields existence and uniqueness of the weak solution $u^\star \in \mathcal{X}$. For $\omega = \Omega$, we omit the index ω throughout.

To discretize (2.2), denote for a polynomial degree $p \geq 1$ and a triangle $T \in \mathcal{T}_H$ the space of all polynomials on T of degree at most p with $\mathbb{P}^p(T)$ and define

$$\mathbb{S}^q(\mathcal{T}_H) \coloneqq \{v_H \in C(\Omega) \colon v_H|_T \in \mathbb{P}^q(T) \text{ for all } T \in \mathcal{T}_H\} \quad \text{with } q \in \{1, p\}. \tag{2.3}$$

With the definition $\mathcal{X}^p_H \coloneqq \mathbb{S}^p_0(\mathcal{T}_H) \coloneqq \mathbb{S}^p(\mathcal{T}_H) \cap H^1_0(\Omega)$, the discrete problem consists of finding $u^\star_H \in \mathcal{X}^p_H$ such that

$$\langle\!\langle u^\star_H\,,\, v_H \rangle\!\rangle = F(v_H) \quad \text{for all } v_H \in \mathcal{X}^p_H. \tag{2.4}$$

Clearly, the formulation of the discrete problem (2.4) hinges on the choice of the mesh $\mathcal{T}_H$, which directly influences the quality of $u^\star_H$ as an approximation of $u^\star$. Note that (2.4) can be rewritten as a symmetric and positive definite linear system by introducing a basis of $\mathcal{X}^p_H$. However, we opt to work instead with the functional basis-independent description.

2.2.2 Mesh and space hierarchy

We suppose that the refinement strategy in the module `REFINE` is newest vertex bisection (NVB); see, e.g., [Tra97; Ste08] and Figure 2.1 for an illustration in 2D. Let $\mathcal{T}_0$ be the conforming initial mesh. We refer to [Ste08] for NVB with admissible $\mathcal{T}_0$ and $d \geq 2$, to [KPP13] for NVB with general $\mathcal{T}_0$ for

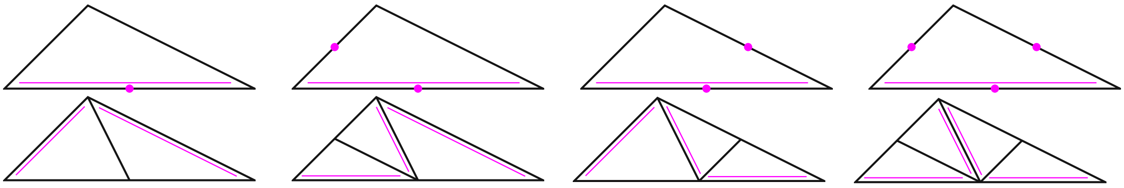


Figure 2.1: Schematic of 2D NVB refinement pattern: For each triangle $T \in \mathcal{T}$, there is one fixed *refinement edge* E_T indicated by the extra pink line. The pink dots indicate edges that are marked for refinement. If an element is marked for refinement, at least its refinement edge is marked for refinement (top). Iterated bisection refines a marked element into 2, 3, or 4 children (bottom).

$d = 2$, and to the recent work [DGS23] for NVB with general $\mathcal{T}_0$ in any dimension. Throughout, we suppose that $\mathcal{T}_0$ is admissible. In the 1D case, [AFF+15] splits each element into two children of half-length and additionally ensures that any two neighboring elements have uniformly comparable diameter. Let $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ be the set of all refinements of $\mathcal{T}_0$ that can be obtained by arbitrarily many steps of NVB.

From now on, suppose that we are given a sequence $\{\mathcal{T}_\ell\}_{\ell=0}^L \subset \mathbb{T}$ of successively refined triangulations, i.e., for all $\ell = 1, \dots, L$, it holds that $\mathcal{T}_\ell = \texttt{REFINE}(\mathcal{T}_{\ell-1}, \mathcal{M}_{\ell-1})$ is the coarsest conforming triangulation obtained by NVB, where all marked elements $\mathcal{M}_{\ell-1} \subseteq \mathcal{T}_{\ell-1}$ have been refined by (at least) one bisection. We note that NVB-refinement generates meshes that are uniformly γ-shape regular, i.e.,

$$\max_{\ell=0,\dots,L} \max_{T \in \mathcal{T}_\ell} \frac{\operatorname{diam}(T)}{|T|^{1/d}} \le \gamma < \infty, \tag{2.5a}$$

and

$$\max_{\ell=0,\dots,L} \max_{T \in \mathcal{T}_\ell} \max_{\substack{T' \in \mathcal{T}_\ell \\ T \cap T' \neq \emptyset}} \frac{\operatorname{diam}(T)}{\operatorname{diam}(T')} \le \gamma < \infty, \tag{2.5b}$$

where γ depends only on $\mathcal{T}_0$ and is, in particular, independent of L and the meshes $\mathcal{T}_1, \dots, \mathcal{T}_L$; see, e.g., [Ste08, Theorem 2.1] for $d \ge 2$ or [AFF+15] for $d = 1$. We note that (2.5a) implies (2.5b) for $d \ge 2$, while (2.5a) is trivial with $\gamma = 1$ and independent of (2.5b) for $d = 1$. In addition, we define the quasi-uniformity constant

$$C_{\mathrm{qu}} := \min\{\operatorname{diam}(T)/\operatorname{diam}(T') : T, T' \in \mathcal{T}_0\} \in (0, 1]. \tag{2.6}$$

For each mesh $\mathcal{T}_\ell$, let $\mathcal{V}_\ell$ denote the set of vertices. Given a vertex $z \in \mathcal{V}_\ell$, we denote by $\mathcal{T}_{\ell,z} := \{T \in \mathcal{T}_\ell : z \in T\}$ the patch of elements of $\mathcal{T}_\ell$ that share the vertex z. The corresponding (open) patch subdomain is denoted by $\omega_{\ell,z} := \operatorname{interior}(\bigcup_{T \in \mathcal{T}_{\ell,z}} T)$ and its size by $h_{\ell,z} := \max_{T \in \mathcal{T}_{\ell,z}} h_T := \max_{T \in \mathcal{T}_{\ell,z}} |T|^{1/d}$. Finally, we denote by $\mathcal{V}_\ell^+$ the set of new vertices in $\mathcal{T}_\ell$ and the pre-existing vertices of $\mathcal{T}_{\ell-1}$ whose associated patches have shrunk in size in the refinement step ℓ, i.e.,

$$\mathcal{V}_0^+ := \mathcal{V}_0 \quad \text{and} \quad \mathcal{V}_\ell^+ := \mathcal{V}_\ell \setminus \mathcal{V}_{\ell-1} \cup \{z \in \mathcal{V}_\ell \cap \mathcal{V}_{\ell-1} : \omega_{\ell,z} \neq \omega_{\ell-1,z}\} \quad \text{for } \ell \ge 1.$$

While this notation is used in the analysis of the solver below, the presentation of Algorithm 2A is more compact with the abbreviation $\mathcal{N}_\ell = \mathcal{V}_\ell^+$ for $\ell = 1, \dots, L-1$ and $\mathcal{N}_L := \mathcal{V}_L^+$ for $p = 1$ and

$\mathcal{N}_L := \mathcal{V}_L$ otherwise, where we recall that $p \in \mathbb{N}$ is the fixed polynomial degree of the FEM ansatz functions.

From the definition of the discrete FEM spaces (2.3) and NVB-refinement, we see that there holds nestedness

$$\mathcal{X}_0^1 \subseteq \mathcal{X}_1^1 \subseteq \cdots \subseteq \mathcal{X}_{L-1}^1 \subseteq \mathcal{X}_L^p. \tag{2.7}$$

Furthermore, we require the local spaces

$$\mathcal{X}_{\ell,z}^q := \mathbb{S}_0^q(\mathcal{T}_{\ell,z}) \quad \text{for all vertices } z \in \mathcal{V}_\ell \text{ and } q \in \{1, p\}, \tag{2.8}$$

where we use $q = 1$ for $\ell = 0, \ldots, L-1$ and $q = p$ for $\ell = L$; see Figure 2.2 for the illustration of the degrees of freedom for $p = 2$.

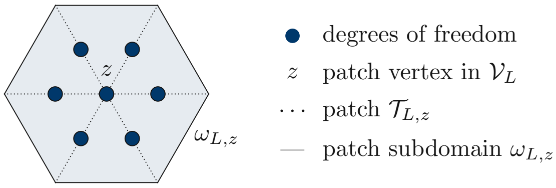


Figure 2.2: Illustration of degrees of freedom ($p = 2$) for the space $\mathcal{X}_{L,z}^p$ associated to the patch $\mathcal{T}_{L,z}$.

2.2.3 Multigrid solver

In the following, we introduce a local geometric multigrid method, which will serve as iterative solver within the `SOLVE` module of an adaptive FEM algorithm. Each full step of the proposed multigrid method can be mathematically described by an iteration operator $\Psi\colon \mathcal{X}_L^p \to \mathcal{X}_L^p$, i.e., given the current approximation $u_L \in \mathcal{X}_L^p$, the solver generates the new iterate $\Psi(u_L) \in \mathcal{X}_L^p$.

The main ingredients in the solver construction are an inexpensive global residual solve on $\mathcal{T}_0$ and local residual solves on all patches $\mathcal{T}_{\ell,z}$ for $z \in \mathcal{V}_\ell^+$ on the intermediate levels $\ell = 1, \ldots, L-1$ and all patches on the finest level $\mathcal{T}_L$ when $p > 1$. For ease of notation, we define the algebraic residual functional $R_L\colon \mathcal{X}_L^p \to \mathbb{R}$ by

$$v_L \in \mathcal{X}_L^p \mapsto R_L(v_L) := F(v_L) - \langle\!\langle u_L\,,\, v_L \rangle\!\rangle = \langle\!\langle u_L^\star - u_L\,,\, v_L \rangle\!\rangle \in \mathbb{R}. \tag{2.9}$$

To construct the new iterate $\Psi(u_L)$, levelwise residual liftings of the algebraic error are added to the current approximation u_L. The same levelwise residual liftings are used to define an *a posteriori* error estimator $\eta_{\mathrm{alg}}(u_L)$ for the algebraic error, i.e., the solver comes with a built-in estimator.

Algorithm 2A: One step of the optimal local multigrid solver

Input: Current approximation $u_L \in \mathcal{X}_L^p$, meshes $\{\mathcal{T}_\ell\}_{\ell=0}^L$, polynomial degree $p \in \mathbb{N}$.
Solver step: Perform the following steps (i)–(ii):

(i) **Global lowest-order residual problem on the coarsest level**:

- Compute $\rho_0 \in \mathcal{X}_0^1$ by solving

$$\langle\!\langle \rho_0 \,,\, v_0 \rangle\!\rangle = R_L(v_0) \quad \text{for all } v_0 \in \mathcal{X}_0^1. \tag{2.10}$$

- Define step-size $\lambda_0 \coloneqq 1$.
- Initialize algebraic lifting $\sigma_0 \coloneqq \lambda_0 \rho_0$ and *a posteriori* estimator $\zeta_0^2 \coloneqq |||\lambda_0 \rho_0|||^2$.

(ii) **Local residual-update:** For all $\ell = 1, \ldots, L$, do the following steps, where $q = 1$ for $\ell = 1, \ldots, L-1$ and $q = p$ for $\ell = L$:

- For all $z \in \mathcal{N}_\ell$, compute $\rho_{\ell,z} \in \mathcal{X}_{\ell,z}^q$ by solving

$$\langle\!\langle \rho_{\ell,z} \,,\, v_{\ell,z} \rangle\!\rangle = R_L(v_{\ell,z}) - \langle\!\langle \sigma_{\ell-1} \,,\, v_{\ell,z} \rangle\!\rangle \quad \text{for all } v_{\ell,z} \in \mathcal{X}_{\ell,z}^q. \tag{2.11}$$

- Define the line-search step-size $s_\ell \coloneqq (R_L(\rho_\ell) - \langle\!\langle \sigma_{\ell-1} \,,\, \rho_\ell \rangle\!\rangle)/|||\rho_\ell|||^2$, with $\rho_\ell \coloneqq \sum_{z \in \mathcal{N}_\ell} \rho_{\ell,z}$ and the understanding that $0/0 \coloneqq 0$ if $\rho_\ell = 0$, and

$$\lambda_\ell \coloneqq \begin{cases} s_\ell & \text{if } s_\ell \le d+1 \ \text{ or } \big[\ell = L \text{ and } p > 1\big], \\ (d+1)^{-1} & \text{otherwise.} \end{cases}$$

- Update $\sigma_\ell \coloneqq \sigma_{\ell-1} + \lambda_\ell \rho_\ell$ and $\zeta_\ell^2 \coloneqq \zeta_{\ell-1}^2 + \lambda_\ell \sum_{z \in \mathcal{N}_\ell} |||\rho_{\ell,z}|||^2$.

Output: Improved approximation $\Psi(u_L) \coloneqq u_L + \sigma_L \in \mathcal{X}_L^p$ and associated *a posteriori* estimator $\eta_{\mathrm{alg}}(u_L) \coloneqq \eta_{\mathrm{alg}}$ of the algebraic error.

Remark 2.1 (Construction of the new iterate)**.** *The construction of $\Psi(u_L)$ from u_L by Algorithm 2A can be seen as one iteration of a V-cycle multigrid with no pre- and one post-smoothing step, and a step-size at the error correction stage. The smoother on each level is additive Schwarz associated to patch subdomains where the local problems* (2.11) *are defined. This is equivalent to diagonal Jacobi smoothing for $p = 1$ (e.g., on intermediate levels) and block-Jacobi smoothing for $p > 1$ (e.g., on the finest level). The choice and use of the step-sizes λ_ℓ in Algorithm 2A(ii) comes from a line-search approach; see, e.g., [MPV21,* Lemma 4.3*] and one of the earlier works [Hei88]. However, if the step-size from the line-search is too large, we use instead a fixed damping parameter offsetting the $d+1$ patch overlaps. We note that this case never occurred in practice in any of our numerical experiments.*

Remark 2.2 (Computational effort and speed of convergence)**.** *We note that we apply a patchwise Cholesky factorization on the finest level. Hence, the computational effort for the local residual solve on the finest mesh $\mathcal{T}_L$ in dependence on the polynomial degree p is of order $\mathcal{O}(p^{3d} \# \mathcal{T}_L)$. The presented algorithm is a linear method. One could symmetrize the procedure by adding one pre-smoothing step to define a preconditioner in the hope of accelerating convergence with the help of conjugate gradients. However, in our experience, the patchwise pre-smoothing typically did not yield considerable algebraic error decrease; see, e.g. [DHM+21], while still doubling the number of smoothing operations of a V-cycle. The remaining steps needed to compute the new approximation stem from classical multigrid solvers (such as intergrid operators). We stress that the overall effort does not depend on the number of levels L.*

Remark 2.3 (Nested iterations)**.** *In the context of adaptive FEM, the solver does not start from an arbitrary initial guess on each newly-refined mesh but from the final approximation of the previous level (see Algorithm 2B below). This will ensure a posteriori error control in each step after initialization as well as optimal computational cost. From the algebraic solver perspective, such an approach can be seen as a full multigrid method over the evolving hierarchy of meshes whose number of cycles is determined by the adaptive stopping criterion.*

2.2.4 Main result

This subsection formulates the main results regarding the iterative solver stating the *contraction* of the multigrid solver and *reliability* of the built-in *a posteriori* estimator of the algebraic error. Both results hold *robustly* in the number of levels L and the polynomial degree p.

Theorem 2.4

Let $u_L^\star \in \mathcal{X}_L^p$ be the (unknown) finite element solution of (2.4) and let $v_L \in \mathcal{X}_L^p$ be arbitrary. Let $\Psi(v_L) \in \mathcal{X}_L^p$ and $\eta_{\mathrm{alg}}(v_L)$ be generated by Algorithm 2A. Then, the solver iterates and the estimator are connected by

$$|||u_L^\star - \Psi(v_L)|||^2 \leq |||u_L^\star - v_L|||^2 - \eta_{\mathrm{alg}}(v_L)^2. \tag{2.12}$$

Moreover, the solver contracts the error, i.e., there exists $0 < q_{\mathrm{ctr}} < 1$ such that

$$|||u_L^\star - \Psi(v_L)||| \leq q_{\mathrm{ctr}}\, |||u_L^\star - v_L|||. \tag{2.13}$$

Finally, the estimator is a two-sided bound of the algebraic error, i.e., there exists $C_{\mathrm{rel}} > 1$ such that

$$\eta_{\mathrm{alg}}(v_L) \leq |||u_L^\star - v_L||| \leq C_{\mathrm{rel}}\, \eta_{\mathrm{alg}}(v_L). \tag{2.14}$$

The contraction and reliability constants q_{ctr} and C_{rel} depend only on the space dimension d, the γ-shape regularity (2.5), the quasi-uniformity constant C_{qu} from (2.6), $\Lambda_{\max}/\Lambda_{\min}$, and $\max_{T\in\mathcal{T}_L} \|\operatorname{div}(\boldsymbol{A})\|_{L^\infty(T)}/\Lambda_{\min}$. In particular, q_{ctr} is independent of the polynomial degree p, the number of mesh levels L, and the meshes $\mathcal{T}_1, \dots, \mathcal{T}_L$.

Corollary 2.5. *The reliability of the estimator in (2.14) is equivalent to the solver contraction (2.13). In particular, this also yields that*

$$|||u_L^\star - \Psi(v_L)||| \leq q_{\mathrm{ctr}}\, C_{\mathrm{rel}}\, \eta_{\mathrm{alg}}(v_L). \tag{2.15}$$

Remark 2.6. *We note that (2.12) holds with equality whenever the step-size criterion $s_\ell \leq d+1$ in Algorithm 2A(ii) is fulfilled and the construction is thus done by optimal-line search. In such a case, which was always satisfied in all our numerical tests, a Pythagoras identity in the spirit of [MPV21, Theorem 4.7] yielding exact algebraic error decrease is obtained.*

2.3 Application to adaptive FEM with inexact solver

Given a coarse mesh $\mathcal{T}_0$, we use an adaptive finite element method (AFEM) to generate locally refined meshes $\{\mathcal{T}_L\}_{L\in\mathbb{N}}$ tailored to the behavior of the sought solution. In the spirit of [GHPS21], Algorithm 2B presents such an approach with an adaptively stopped iterative solver, where Step (Ii) exploits the built-in *a posteriori* estimator of the geometric multigrid solver from Section 2.2.

While we note that the present Algorithm 2B and the corresponding Theorem 2.7 are restricted to fixed polynomial degree p, the inclusion of the proposed hp-robust iterative solver into the hp-adaptive FEM algorithm of [CNSV17] remains for future research, since the mathematical understanding of hp-adaptive FEM is still widely open.

Algorithm 2B: AFEM with iterative solver

Input: Initial mesh $\mathcal{T}_0$, polynomial degree $p \in \mathbb{N}$, adaptivity parameters $0 < \theta \le 1$, $C_{\rm mark} \ge 1$, and $\lambda_{\rm alg} > 0$, initial guess $u_0^0 := 0$.
Adaptive loop: repeat the following steps (I)–(III) for all $L = 0, 1, 2, \dots$:

(I) `SOLVE & ESTIMATE`: repeat the following steps (i)–(iii) for all $k = 1, 2, 3, \dots$:

(i) Do one step of the algebraic solver to obtain $u_L^k \in \mathcal{X}_L^p$ and an associated *a posteriori* estimator $\eta_{\rm alg}(u_L^{k-1})$ for the algebraic error

$$[u_L^k, \eta_{\rm alg}(u_L^{k-1})] := \texttt{SOLVE}(u_L^{k-1}, \{\mathcal{T}_\ell\}_{\ell=0}^L, p).$$

(ii) Compute *a posteriori* indicators for the elementwise discretization error

$$\{\eta_L(T, u_L^k)\}_{T\in\mathcal{T}_L} := \texttt{ESTIMATE}(u_L^k, \mathcal{T}_L).$$

(iii) If $\eta_{\rm alg}(u_L^{k-1}) \le \lambda_{\rm alg}\eta_L(u_L^k)$, terminate the k-loop, set the index $\underline{k}[L] := k$ and define $u_L := u_L^{\underline{k}[L]}$.

(II) `MARK`: Determine a set of marked elements $\mathcal{M}_L \subseteq \mathcal{T}_L$ of (up to the multiplicative constant $C_{\rm mark}$) minimal cardinality that satisfies

$$\theta\,\eta_L(u_L)^2 \le \sum_{T\in\mathcal{M}_L} \eta_L(T, u_L)^2.$$

(III) `REFINE`: Generate the new mesh $\mathcal{T}_{L+1} := \texttt{REFINE}(\mathcal{M}_L, \mathcal{T}_L)$ and define $u_{L+1}^0 := u_L$.

Output: Sequences of successively refined triangulations $\mathcal{T}_L$, discrete approximations u_L and corresponding error estimators $(\eta_L(u_L), \eta_{\rm alg}(u_L))$.

Mesh-refinement is steered by the discretization error estimator. For all $T \in \mathcal{T}_H$, let $\eta_H(T;\cdot)\colon \mathcal{X}_H^p \to \mathbb{R}_{\ge 0}$ be the local contributions of the standard residual error estimator defined by

$$\eta_H^2(T; v_H) := h_T^2\|f + \operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\|_T^2 + h_T\|[\![\boldsymbol{A}\nabla v_H - \boldsymbol{f}]\!]\cdot\boldsymbol{n}\|_{\partial T\cap\Omega}^2, \tag{2.16}$$

where $\|\cdot\|_\omega$ denote the appropriate $L^2(\omega)$-norms. We define

$$\eta_H(\mathcal{U}_H; v_H) := \Big(\sum_{T\in\mathcal{U}_H} \eta_H(T; v_H)^2\Big)^{1/2} \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \text{ and } v_H \in \mathcal{X}_H^p.$$

To abbreviate notation, let $\eta_H(v_H) := \eta_H(\mathcal{T}_H; v_H)$.

One important consequence of Theorem 2.4 is optimal convergence of Algorithm 2B with respect to computational complexity. To formulate this mathematically, we define the ordered set

$$\mathcal{Q} := \{(L,k) \in \mathbb{N}_0^2 \colon \text{index tuple } (L,k) \text{ is used in Algorithm 2B and } 1 \le k \le \underline{k}[L]\}.$$

On $\mathcal{Q}$, we define the ordering $\le$ by

$$(L',k') \le (L,k) \quad \Longleftrightarrow \quad u_{L'}^{k'} \text{ is computed earlier than or equal to } u_L^k \text{ in Algorithm 2B.}$$

Furthermore, we introduce the total step counter $|\cdot,\cdot|$, defined for all $(L,k) \in \mathcal{Q}$, by

$$|L,k| := \#\{(L',k') \in \mathcal{Q} \colon (L',k') \le (L,k)\}.$$

Before we state the theorem, we introduce the notion of approximation classes. For $s > 0$, define

$$\|u\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N(\mathcal{T}_0)} \big(|||u^\star - u_{\rm opt}^\star||| + \eta_{\rm opt}(u_{\rm opt}^\star) \big) \Big), \tag{2.17}$$

with Galerkin solution $u_{\rm opt}^\star$ and estimator $\eta_{\rm opt}$ on the optimal triangulation $\mathcal{T}_{\rm opt} \in \mathbb{T}_N(\mathcal{T}_0)$, where $\mathbb{T}_N(\mathcal{T}_0) := \{\mathcal{T}_H \in \mathbb{T}(\mathcal{T}_0) \colon \#\mathcal{T}_H - \#\mathcal{T}_0 \le N\}$. By reliability (A3) of the estimator, see, e.g., [CFPP14], the sum on the right-hand side of (2.17) is equivalent to $\eta_{\rm opt}(u_{\rm opt}^\star)$. If $\|u\|_{\mathbb{A}_s} < \infty$, then we say that rate s is possible.

In [GHPS21], it is shown that in the case of a contractive solver, convergence rates with respect to degrees of freedom are equivalent to convergence rates with respect to computational complexity. We abbreviate with $\operatorname{cost}(L,k)$ the total costs of Algorithm 2B defined by

$$\operatorname{cost}(L,k) := \sum_{\substack{(L',k') \in \mathcal{Q} \\ (L',k') \le (L,k)}} \#\mathcal{T}_{L'}.$$

Theorem 2.7

Let $\{\mathcal{T}_L\}_{L \in \mathbb{N}_0}$ be the sequence generated by Algorithm 2B and define the quasi-error by

$$\Delta_L^k := |||u^\star - u_L^k||| + \eta_L(u_L^k) \quad \text{for all } (L,k) \in \mathcal{Q}.$$

Then, for all parameters $0 < \theta \le 1$ and $\lambda_{\rm alg} > 0$, it holds that

$$\sup_{(L,k) \in \mathcal{Q}} (\#\mathcal{T}_L)^s \Delta_L^k \simeq \sup_{(L,k) \in \mathcal{Q}} \operatorname{cost}(L,k)^s\, \Delta_L^k \quad \text{and} \quad \Delta_L^k \to 0 \text{ as } |L,k| \to \infty. \tag{2.18}$$

Furthermore, there exist $0 < \theta^\star \le 1$, and $\lambda_{\rm alg}^\star > 0$ such that, for sufficiently small parameters $0 < \theta < \theta^\star$ and $0 < \lambda_{\rm alg}/\theta < \lambda_{\rm alg}^\star$, and for all $s > 0$, it holds that

$$c_{\rm opt}\, \|u\|_{\mathbb{A}_s} \le \sup_{(L,k) \in \mathcal{Q}} \operatorname{cost}(L,k)^s\, \Delta_L^k \le C_{\rm opt} \max\{\|u\|_{\mathbb{A}_s}, \Delta_0^0\}. \tag{2.19}$$

The constants $c_{\rm opt}, C_{\rm opt} > 0$ depend only on the polynomial degree p, the initial triangulation $\mathcal{T}_0$, $\Lambda_{\max}/\Lambda_{\min}$, $\max_{T \in \mathcal{T}_L} \|\operatorname{div}(\boldsymbol{A})\|_{L^\infty(T)}/\Lambda_{\min}$, the rate s, the ratios $\theta/\theta^\star$ and $\lambda_{\rm alg}/(\theta\lambda_{\rm alg}^\star)$, *and the properties of newest vertex bisection. In particular, this proves the equivalence*

$$\|u\|_{\mathbb{A}_s} < \infty \quad \Longleftrightarrow \quad \sup_{(L,k)\in Q} \operatorname{cost}(L,k)^s\, \Delta_L^k < \infty, \tag{2.20}$$

which proves optimal complexity of Algorithm 2B.

Remark 2.8. *We note that in [GHPS21, Theorem 8], the constant $c_{\mathrm{opt}} > 0$ additionally depends on the stopping index $\underline{L}$ in the case the algorithm terminates after a finite number of mesh levels $\underline{L} < \infty$ or the estimator satisfies $\eta_L(u_L) = 0$. The refined analysis in the recent work [BIM+24a] removes this dependence.*

Remark 2.9. *We note that it is also possible to use the same stopping criterion for the algebraic solver as in [GHPS21, Algorithm 2]. However, since the multigrid solver from Algorithm 2A has a built-in estimator for the algebraic error, we opt for its choice within Algorithm 2B instead.*

Proof of Theorem 2.7. We show that Algorithm 2B satisfies the requirements of [GHPS21, Theorem 4 and Theorem 8]. First note that the standard residual error estimator from (2.16) satisfies the axioms of adaptivity from [CFPP14] and thus satisfies the assumptions (A1)–(A4) from [GHPS21, Theorem 8]. Furthermore, newest vertex bisection satisfies the assumptions (R1)–(R3) from [GHPS21, Section 2.2]. For the present setting, the conditions (C1) and (C2) from [GHPS21, Section 2.5] coincide and are satisfied.

Tracing the role of the stopping criterion for the case (C1) in the proof of [GHPS21, Theorem 4], one sees that the stopping criterion needs to guarantee that, for all $(L,k) \in Q$,

$$\begin{aligned} |||u_L^k - u_L^{k-1}||| &\le \lambda_1\, \eta_L(u_L^k) && \text{if } u_L^k = u_L, \\ \eta_L(u_L^k) &\le \lambda_2^{-1}\, |||u_L^k - u_L^{k-1}||| && \text{else,} \end{aligned} \tag{2.21}$$

for some $\lambda_1, \lambda_2 > 0$. The upper bound in (2.14) in Theorem 2.4 as well as contraction (2.13) show that, for all $(L,k) \in Q$, our stopping criterion in Algorithm 2B Step (Iiii) leads for $u_L^k = u_L$ to

$$|||u_L^k - u_L^{k-1}||| \overset{(2.13)}{\le} (1+q_{\mathrm{ctr}})\, |||u_L^\star - u_L^{k-1}||| \overset{(2.14)}{\le} C_{\mathrm{rel}}(1+q_{\mathrm{ctr}})\, \eta_{\mathrm{alg}}(u_L^{k-1}) \le \lambda_{\mathrm{alg}} C_{\mathrm{rel}}\, (1+q_{\mathrm{ctr}})\, \eta_L(u_L^k).$$

For the remaining case, the contraction (2.13) leads to

$$|||u_L^\star - u_L^k||| \overset{(2.13)}{\le} q_{\mathrm{ctr}}\, |||u_L^\star - u_L^{k-1}||| \le q_{\mathrm{ctr}}\, |||u_L^\star - u_L^k||| + q_{\mathrm{ctr}}\, |||u_L^k - u_L^{k-1}|||.$$

This implies

$$|||u_L^\star - u_L^k||| \le \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\, |||u_L^k - u_L^{k-1}|||. \tag{2.22}$$

The not met stopping criterion in Algorithm 2B(Iiii), the lower bound in (2.14), and (2.22) show

$$\begin{aligned} \eta_L(u_L^k) < \lambda_{\mathrm{alg}}^{-1}\, \eta_{\mathrm{alg}}(u_L^{k-1}) &\overset{(2.14)}{\le} \lambda_{\mathrm{alg}}^{-1}\, |||u_L^\star - u_L^{k-1}||| \le \lambda_{\mathrm{alg}}^{-1}\, \big(|||u_L^\star - u_L^k||| + |||u_L^k - u_L^{k-1}|||\big) \\ &\overset{(2.22)}{\le} \lambda_{\mathrm{alg}}^{-1}\, \Big(1 + \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\Big)\, |||u_L^k - u_L^{k-1}||| \\ &= \lambda_{\mathrm{alg}}^{-1}\, \Big(\frac{1}{1-q_{\mathrm{ctr}}}\Big)\, |||u_L^k - u_L^{k-1}|||. \end{aligned}$$

Overall, (2.21) is satisfied with

$$\lambda_1 = C_{\mathrm{rel}}\,(1 + q_{\mathrm{ctr}})\,\lambda_{\mathrm{alg}} \quad \text{and} \quad \lambda_2 = (1 - q_{\mathrm{ctr}})\,\lambda_{\mathrm{alg}},$$

and [GHPS21, Theorem 4] proves full linear convergence, so that, in particular, (2.18) is fulfilled (see the proof of [GHPS21, Theorem 8] or [BIM+24a, Corollary 4.2]).

The lower bound in (2.19) follows as in [GHPS21, Theorem 8] or [BIM+24a, Theorem 4.3]. For the upper bound in (2.19), [GHPS21, Theorem 8] requires that

$$0 < \lambda_1/\theta < \lambda_{\mathrm{opt}} \coloneqq (1 - q_{\mathrm{ctr}})/(q_{\mathrm{ctr}}\,C_{\mathrm{stab}})$$

and

$$0 < \theta' \coloneqq \frac{\theta + \lambda_1/\lambda_{\mathrm{opt}}}{1 - \lambda_1/\lambda_{\mathrm{opt}}} < \theta_{\mathrm{opt}} \coloneqq (1 + C_{\mathrm{stab}}^2\,C_{\mathrm{drel}}^2)^{-1},$$

where C_{stab} is the stability constant from (A1) and C_{drel} is the constant from discrete reliability (A4); see, e.g., [GHPS21]. We define

$$\lambda_{\mathrm{alg}}^{\star} \coloneqq \frac{\lambda_{\mathrm{opt}}}{C_{\mathrm{rel}}\,(1 + q_{\mathrm{ctr}})},$$

and $\lambda_{\mathrm{alg}}/\theta < \lambda_{\mathrm{alg}}^{\star}$ thus implies $\lambda_1/\theta = C_{\mathrm{rel}}\,(1 + q_{\mathrm{ctr}})\,\lambda_{\mathrm{alg}}/\theta < \lambda_{\mathrm{opt}}$. Finally, we choose $\theta^{\star}$ such that any $0 < \widetilde{\theta} \le \theta^{\star}$ satisfies $\frac{2\,\widetilde{\theta}}{1-\widetilde{\theta}} < \theta_{\mathrm{opt}}$. Then, $0 < \theta < \theta^{\star}$ yields $\theta' = \frac{\theta+\lambda_1/\lambda_{\mathrm{opt}}}{1-\lambda_1/\lambda_{\mathrm{opt}}} < \frac{2\,\theta}{1-\theta} < \theta_{\mathrm{opt}}$ and optimal cost in Theorem 2.7 follows directly from [GHPS21, Theorem 8]. □

2.4 Numerical experiments

This section investigates the numerical performance of the proposed multigrid solver of Algorithm 2A and the adaptive Algorithm 2B. The MATLAB implementation of the multigrid solver is embedded into the MooAFEM framework from [IP23]. Throughout, we choose the marking parameter $\theta = 0.5$ in the adaptive Algorithm 2B and $\boldsymbol{f} = (0, 0)^{\top}$. We introduce the following test case:

- *L-shaped domain.* Let $\Omega = (-1, 1)^2 \setminus \big([0, 1] \times [-1, 0]\big)$ with right-hand side $f = 1$ and $\boldsymbol{A} = \boldsymbol{I}$.

2.4.1 Contraction and performance of local multigrid solver

We confirm numerically our main results from Theorem 2.4. In order to study the algebraic solver and its built-in estimator with respect to different polynomial degrees, we take $\lambda_{\mathrm{alg}} = 10^{-5}$ in Algorithm 2B, thus *oversolving* the algebraic problem. Moreover, we stop the adaptive algorithm once the final mesh consists of 10^6 degrees of freedom. Note that thanks to Corollary 2.5 proving the equivalence of the reliability of the algebraic error estimator with the contraction of the algebraic solver, we indeed only need to investigate numerically the existence of the p-robust bound on the contraction of the solver. In Figure 2.3 (left), we present the maximal contraction factors on each level L of the adaptive algorithm from Algorithm 2B. We see that the contraction factors are robust in the polynomial degree p with an upper bound of about 0.7 in all our experiments. In Figure 2.3 (right), we see that on a fixed number of levels (L=10) even for higher-order polynomials their behavior is clustered around similar values. Moreover, from a purely solver-centric perspective, we see that the solver variant which employs higher-order smoothing also on the intermediate levels (and not only on the finest one) as studied in [MPV21] only leads to slight improvements of the contraction constants. Adapting the arguments of [MPV21], this modified construction can be guaranteed to be contractive with p-robust, but linearly L-dependent contraction bound on the algebraic error. However, this degradation with increasing L is not seen in practice, provided that the patchwise smoothing is done *everywhere* for *level* $L = 1$ (as new degrees of freedom are added on all patches when the polynomial degree is $p > 1$) and *local* patchwise smoothing is employed in the remaining levels. We present a comparison of the resulting contraction factors of this approach to Algorithm 2A for a fixed number of level ($L = 10$) in Figure 2.3(right).

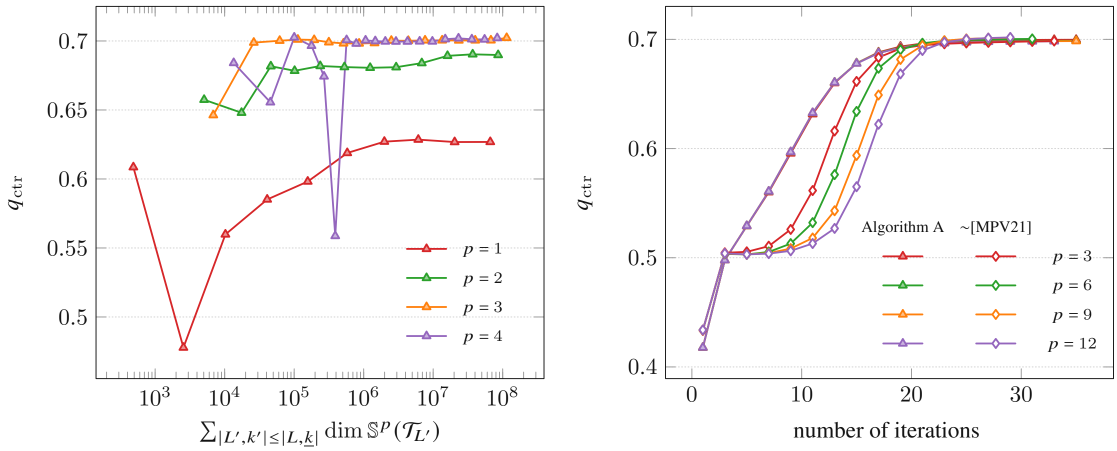


Figure 2.3: *Contraction of the algebraic solver.* History plot of the contraction factors q_{ctr} from (2.13) for various polynomial degrees p with parameter $\lambda_{\mathrm{alg}} = 10^{-5}$ for the presented polynomial hierarchy from (2.7) in the adaptive algorithm from Algorithm 2B stopping once the final mesh consists of 10^6 degrees of freedom (left) and the comparison with polynomial hierarchy motivated by [MPV21] with localized smoothing for a fixed number of levels $L = 10$ (right).

2.4.2 Optimality of the adaptive algorithm

We take $\lambda_{\mathrm{alg}} = 0.1$ in Algorithm 2B and study the decrease of the discretization error estimator $\eta_L(u_L)$, both in terms of number of degrees of freedom and timing. We remark that the error estimator $\eta_L(u_L)$ on the final iterates is equivalent to the quasi-error Δ_L. After a pre-asymptotic phase, we see in Figure 2.4 for different polynomial degrees p that the optimal convergence rate $-p/2$ is recovered both with respect to number of degrees of freedom and computational time, and the singularity at the reentrant corner $(0, 0)$ is resolved through local mesh refinement. Furthermore, Figure 2.5 shows that the proposed multigrid solver behaves faster than the built-in direct solver (MATLAB backslash operator) concerning the time per dof. The displayed timings include the setup of the linear system, the time for the solver module, computation of estimator, and mesh refinement. Overall, the numerical experiments in Figure 2.5 validate the linear complexity of the suggested local multigrid solver from Algorithm 2A.

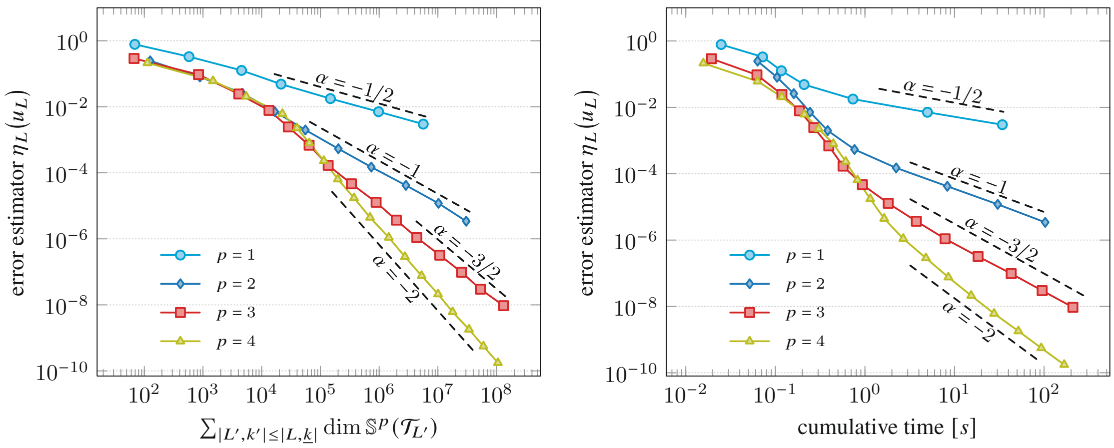


Figure 2.4: *Optimality of AFEM on L-shape.* The convergence history plot of the discretization error estimator $\eta_L(u_L)$ with respect to the total computational cost (left) and the cumulative computational time (right).

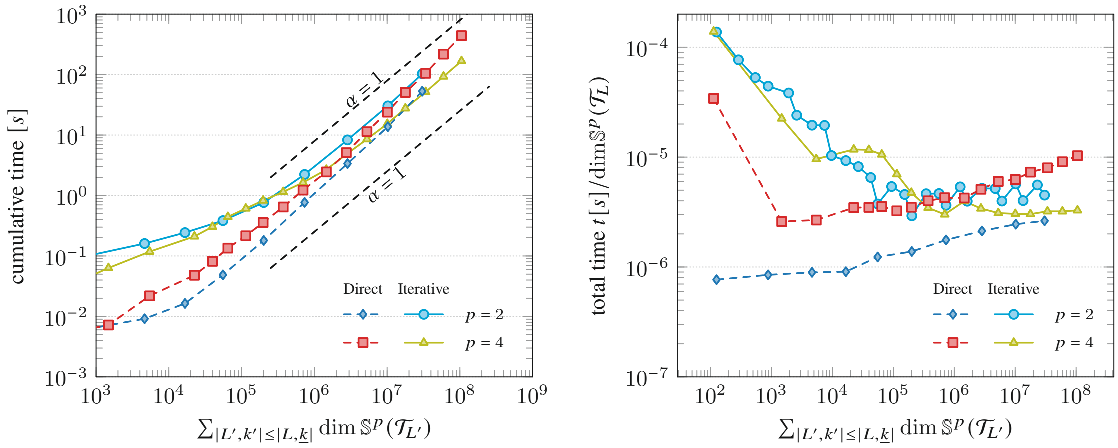


Figure 2.5: *Optimality of the local multigrid solver.* History plot of the cumulative computational time and the relative computational time per degree of freedom for the polynomial degrees $p = 1$ and $p = 4$. We compare the overall time with the direct solve (square) to the overall time of the AFEM algorithm with the multigrid solver (diamond). In particular, the displayed times include setup, marking, and mesh refinement.

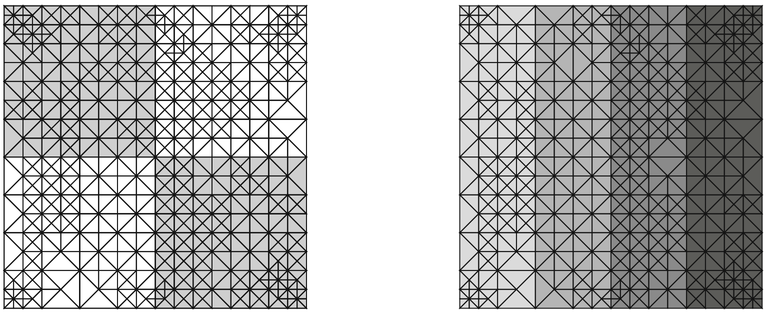


Figure 2.6: Adaptively refined meshes. Left: checkerboard diffusion with $k = 1$, polynomial degree $p = 1$ and $\#\mathcal{T}_8 = 603$. Right: stripe diffusion with $k = 2$, $p = 1$ and $\#\mathcal{T}_8 = 753$ (right).

2.4.3 Numerical performance and insights for jumping coefficients

We consider two additional test cases with jumps in the diffusion coefficient:

- *Checkerboard.* Let $\Omega = (0, 1)^2$ be the unit square and $\boldsymbol{A}$ the 2×2 checkerboard diffusion with values 1 (white) and 10^k (grey) for fixed $k = 1, 2, 3$, see Figure 2.6 (left).
- *Striped diffusion.* Let $\Omega = (0, 1)^2$ be the unit square split into 2^k stripes for $k = 1, 2, 3$. The value of $\boldsymbol{A}$ on the j-th stripe is 10^{j-1} with $j \in \{1, \dots, 2^k\}$, see Figure 2.6(right).

In Table 2.1, we see the optimal convergence of the discretization estimator with the optimal rate $-1/2$ for $p = 1$ as well as -1 for $p = 2$ for both diffusion coefficients regardless of the jump size. We stress that the discontinuity in the diffusion coefficient does not affect the optimality of the proposed adaptive algorithm and the iteration numbers remain uniformly bounded as displayed in Table 2.2.

Both test cases exhibit singularities due to jumps in the diffusion coefficient; however, the jump can be much higher for two neighboring elements in the checkerboard case. In this case, near the cross point $(1/2, 1/2)$, the jump is of order 10^k from one element to the next, which coincides with the jump from the highest to the lowest value of $\boldsymbol{A}$ on the *whole* domain. For the striped test case, the jump between two neighboring elements belonging to different “stripes” is of order 10, even if the *global* jump in the diffusion (for non-neighboring elements) is of order 10^{2^k-1}.

This gives us the tools to observe numerically if the performance of our method only depends on *local* jumps in the diffusion coefficient.

2.5 Proofs

Below we present proofs of intermediate results leading to our main Theorem 2.4 of L- and p-robust contraction of the multigrid solver and the L- and p-robust two-sided bound of the algebraic error by the built-in *a posteriori* estimator. We emphasize that this result improves the recent work [MPV21] by removing the L-dependence. From an *algorithmic* point of view, this is done by applying *local* smoothing only on patches which change in the refinement step on lowest-order levels instead of on *every* patch as was the case in [MPV21]. From an *analysis* point of view, L-robustness is achieved thanks to the strengthened Cauchy–Schwarz inequality on bisection-generated meshes (Proposition 2.16) building on the property that the levelwise overlap of the smoothed patches

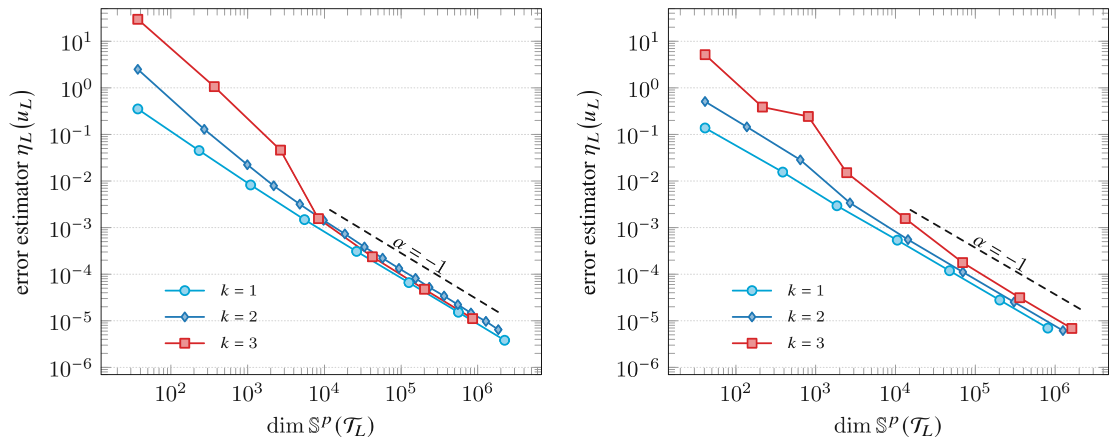


Figure 2.7: *Optimality of AFEM for jumping diffusion.* The convergence history plot of the discretization error estimator $\eta_L(u_L)$ for polynomial degree $p = 2$ with respect to the total computational cost for the checkerboard diffusion (left) and the stripe diffusion (right).

	Checkerboard		Stripe	
	$p = 1$	$p = 2$	$p = 1$	$p = 2$
$k = 1$	-0.4961	-0.9877	-0.4956	-1.0116
$k = 2$	-0.4960	-0.9946	-0.4969	-0.9670
$k = 3$	-0.4960	-0.9826	-0.5095	-0.9766

Table 2.1: Mean value of experimental convergence rates of the discretization error estimator $\eta_L(u_L)$ over the cumulative cost in a $\log\log$-plot for polynomial degrees $p = 1, 2$ and diffusion coefficient numbers $k = 1, 2, 3$.

	Checkerboard		Stripe	
	$p = 1$	$p = 2$	$p = 1$	$p = 2$
$k = 1$	1	1.0455 (mean), 2 (max)	1	1.0455 (mean), 2 (max)
$k = 2$	1	2.3261 (mean), 5 (max)	1	1.0417 (mean), 2 (max)
$k = 3$	1	1.1818 (mean), 3 (max)	1	1.0833 (mean), 2 (max)

Table 2.2: Mean and maximal iteration numbers for polynomial degrees $p = 1, 2$ and diffusion coefficient numbers $k = 1, 2, 3$.

stays uniformly bounded. The next essential ingredient to prove the main result is an hp-stable decomposition on bisection generated meshes (Proposition 2.14), then one combines the results carefully together with the simple but crucial observation of uniform boundedness in the number of overlapping patches for a fixed level (Lemma 2.10) and bounds on the step-sizes and the levelwise solver update (Lemma 2.11).

2.5.1 Auxiliary results

We start with the simple observation that the number of overlapping patches is uniformly bounded.

Lemma 2.10 (Finite patch overlap)**.** *For all $T \in \mathcal{T}_\ell$, there holds*

$$\#(\mathcal{V}_\ell \cap T) = d + 1. \tag{2.23}$$

Therefore, for all $q \in \mathbb{N}$, it holds that

$$|||\sum_{z\in\mathcal{V}_\ell} v_{\ell,z}|||^2 \le (d+1) \sum_{z\in\mathcal{V}_\ell} |||v_{\ell,z}|||^2 \quad \textit{for all } v_{\ell,z} \in \mathcal{X}^q_{\ell,z}. \tag{2.24}$$

Similar arguments show that

$$\Big\|\nabla \sum_{z\in\mathcal{V}_\ell} v_{\ell,z}\Big\|^2 \le (d+1) \sum_{z\in\mathcal{V}_\ell} \|\nabla v_{\ell,z}\|^2 \quad \textit{for all } v_{\ell,z} \in \mathcal{X}^q_{\ell,z}. \tag{2.25}$$

Proof. The overlap (2.23) is clear from the geometry of the elements in the mesh. For all $\ell = 0, \dots, L$, the discrete Cauchy–Schwarz inequality and (2.23) lead to

$$|||\sum_{z\in\mathcal{V}_\ell} v_{\ell,z}|||^2 = \sum_{T\in\mathcal{T}_\ell} |||\sum_{z\in\mathcal{V}_\ell\cap T} v_{\ell,z}|||_T^2 \le (d+1) \sum_{z\in\mathcal{V}_\ell} |||v_{\ell,z}|||^2.$$

This concludes the proof. □

Next, we present bounds on the step-size and the levelwise solver update.

Lemma 2.11. *For all $\ell \in \{1, \dots, L\}$, we have*

$$|||\lambda_\ell \rho_\ell|||^2 \le \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2. \tag{2.26}$$

Moreover, we have upper and lower bounds for the step-sizes,

$$\frac{1}{d+1} \le \lambda_\ell \le d+1 \quad \textit{for all } \ell = 1, \dots, L-1 \quad \textit{and} \quad \frac{1}{d+1} \le \lambda_L. \tag{2.27}$$

Proof. **Step 1:** Proof of (2.26) if $\ell = L$ or $(R_L(\rho_\ell) - \langle\!\langle \sigma_{\ell-1}, \rho_\ell \rangle\!\rangle)/|||\rho_\ell|||^2 \le d+1$ for $\ell \in \{1, \dots, L-1\}$. From Step (ii) of Algorithm 2A, we have that $\lambda_\ell = (R_L(\rho_\ell) - \langle\!\langle \sigma_{\ell-1}, \rho_\ell \rangle\!\rangle)/|||\rho_\ell|||^2$

and thus

$$|||\lambda_\ell \rho_\ell|||^2 = \lambda_\ell \frac{R_L(\rho_\ell) - \langle\!\langle \sigma_{\ell-1}\,,\, \rho_\ell \rangle\!\rangle}{|||\rho_\ell|||^2} |||\rho_\ell|||^2 = \lambda_\ell \sum_{z \in \mathcal{V}_\ell^+} \big(R_L(\rho_{\ell,z}) - \langle\!\langle \sigma_{\ell-1}\,,\, \rho_{\ell,z} \rangle\!\rangle \big) \stackrel{(2.11)}{=} \lambda_\ell \sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2.$$

Step 2: Proof of (2.26) in the remaining cases. We use the finite overlap of the patches in Lemma 2.10 to obtain

$$|||\lambda_\ell \rho_\ell|||^2 = \frac{\lambda_\ell}{d+1} |||\rho_\ell|||^2 \stackrel{(2.24)}{\le} \frac{\lambda_\ell}{d+1} (d+1) \sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 = \lambda_\ell \sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2.$$

Step 3: Proof of (2.27). For $\ell \in \{1, \dots, L-1\}$, the upper bound is guaranteed by definition of λ_ℓ. The lower bound for $\ell \in \{1, \dots, L\}$ is trivial if $\lambda_\ell = 1/(d+1)$. Otherwise, it is a consequence of the finite patch overlap:

$$\lambda_\ell = \frac{R_L(\rho_\ell) - \langle\!\langle \sigma_{\ell-1}\,,\, \rho_\ell \rangle\!\rangle}{|||\rho_\ell|||^2} \stackrel{(2.11)}{=} \frac{\sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2}{|||\rho_\ell|||^2} \stackrel{(2.24)}{\ge} \frac{1}{d+1}.$$

This concludes the proof. □

In the next two subsections, we combine existing results from the literature to obtain a multilevel hp-robust stable decomposition and a strengthened Cauchy–Schwarz inequality for our setting of bisection-generated meshes. These will be crucial for the proofs of Theorem 2.4 and Corollary 2.5 in Subsection 2.5.4 below.

2.5.2 Multilevel *hp*-robust stable decomposition on NVB-generated meshes

We start by recalling the one-level p-robust stable decomposition from Section 3.4 and Section 4.3 in [SMPZ08] for $d = 2$ and $d = 3$, respectively.

Lemma 2.12 (p-robust one level decomposition)**.** *Let $v_L \in \mathcal{X}_L^p$. Then, there exists a decomposition*

$$v_L = v_L^1 + \sum_{z \in \mathcal{V}_L} v_{L,z}^p \quad \textit{with } v_L^1 \in \mathcal{X}_L^1 \textit{ and } v_{L,z}^p \in \mathcal{X}_{L,z}^p, \tag{2.28}$$

which is stable in the sense of

$$\|\nabla v_L^1\|^2 + \sum_{z \in \mathcal{V}_L} \|\nabla v_{L,z}^p\|^2 \le C_{\mathrm{OL}}^2 \|\nabla v_L\|^2. \tag{2.29}$$

The constant C_{OL} depends only on the space dimension d, the γ-shape regularity (2.5)*, and the quasi-uniformity constant C_{qu} from* (2.6).

Similarly, we recall the *local* multilevel decomposition for piecewise affine functions proven in [WZ17, Lemma 3.1]. In order to present this stable decomposition in a form that is more suitable for our forthcoming analysis, we add a short proof for completeness.

Lemma 2.13 (h-robust local multilevel decomposition for lowest-order functions)**.** *Let $v_L^1 \in \mathcal{X}_L^1$. Then, there exists a decomposition*

$$v_L^1 = \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z}^1 \quad \textit{with } v_{\ell,z}^1 \in \mathcal{X}_{\ell,z}^1, \tag{2.30}$$

which is stable in the sense of

$$\sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} \|\nabla v_{\ell,z}^1\|^2 \le C_{\mathrm{ML}}^2 \|\nabla v_L^1\|^2. \tag{2.31}$$

The constant C_{ML} depends only on the space dimension d, the γ-shape regularity (2.5)*, and the quasi-uniformity constant C_{qu} from* (2.6)*.*

Proof. Let $v_L^1 \in \mathcal{X}_L^1$. Define $w_\ell^1 := (\Pi_\ell - \Pi_{\ell-1})v_L^1$ for $\ell \in \{0,\dots,L\}$, where $\Pi_{-1} := 0$ and Π_ℓ is the projection to $\mathcal{X}_\ell^1$ from [WZ17, Section 3]. From [WZ17, Lemma 3.1], it holds that $w_\ell^1 \in \operatorname{span}\{\varphi_{\ell,z} : z \in \mathcal{V}_\ell^+\}$ with $\varphi_{\ell,z}$ being the $\mathbb{S}^1(\mathcal{T}_\ell)$ hat-function at vertex $z \in \mathcal{V}_\ell$. We decompose $w_\ell^1 = \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z}^1$ with $v_{\ell,z}^1 := w_\ell^1(z)\varphi_{\ell,z} \in \mathcal{X}_{\ell,z}^1$ and thus obtain

$$v_L^1 = \sum_{\ell=0}^{L} (\Pi_\ell - \Pi_{\ell-1})v_L^1 = \sum_{\ell=0}^{L} w_\ell^1 = \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z}^1. \tag{2.32}$$

For fixed ℓ and $z \in \mathcal{V}_\ell^+$, the equivalence of norms on finite-dimensional spaces proves

$$\begin{aligned} \|v_{\ell,z}^1\|_{\omega_{\ell,z}} &\le \sum_{T \in \mathcal{T}_{\ell,z}} \|w_\ell^1(z)\varphi_{\ell,z}\|_T \\ &\le \sum_{T \in \mathcal{T}_{\ell,z}} \|w_\ell^1\|_{L^\infty(T)} |T|^{1/2} \lesssim \sum_{T \in \mathcal{T}_{\ell,z}} \|w_\ell^1\|_T \simeq \|w_\ell^1\|_{\omega_{\ell,z}}, \end{aligned} \tag{2.33}$$

where the hidden constants depend only on γ-shape regularity (2.5). To obtain stability of the decomposition (2.32), we use an inverse inequality on the patches and the stability proved in [WZ17, Lemma 3.7]:

$$\begin{aligned} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} \|\nabla v_{\ell,z}^1\|^2 &\lesssim \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} h_{\ell,z}^{-2} \|v_{\ell,z}^1\|_{\omega_{\ell,z}}^2 \overset{(2.33)}{\lesssim} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} h_{\ell,z}^{-2} \|w_\ell^1\|_{\omega_{\ell,z}}^2 \\ &= \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} h_{\ell,z}^{-2} \|(\Pi_\ell - \Pi_{\ell-1})v_L^1\|_{\omega_{\ell,z}}^2 \overset{\text{[WZ17]}}{\lesssim} \|\nabla v_L^1\|^2. \end{aligned}$$

This concludes the proof. □

The combination of the two previous lemmas, done similarly in [MPV20, Proposition 7.6] for a *non-local* and hence not h-robust solver, leads to the following hp-robust decomposition.

Proposition 2.14 (hp-robust local multilevel decomposition)**.** *Let $v_L \in \mathcal{X}_L^p$. Then, there exist $v_0 \in \mathcal{X}_0^1$, $v_{\ell,z} \in \mathcal{X}_{\ell,z}^1$, and $v_{L,z} \in \mathcal{X}_{L,z}^p$ such that*

$$v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z \in \mathcal{V}_L} v_{L,z}. \tag{2.34}$$

and this decomposition is stable in the sense of

$$|||v_0|||^2 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} |||v_{\ell,z}|||^2 + \sum_{z \in \mathcal{V}_L} |||v_{L,z}|||^2 \le C_{\mathrm{SD}}^2 \, |||v_L|||^2. \tag{2.35}$$

The constant $C_{\mathrm{SD}} \ge 1$ depends only on the space dimension d, γ-shape regularity (2.5)*, the quasi-uniformity constant C_{qu} from* (2.6)*, and the ratio of $\Lambda_{\max}$ and $\Lambda_{\min}$.*

Proof. Let $v_L \in \mathcal{X}_L^p$. We begin with the decomposition of v_L by (2.28), then continue with the further decomposition of the lowest-order contribution v_L^1 in a multilevel way (2.30):

$$\begin{aligned} v_L &\overset{(2.28)}{=} v_L^1 + \sum_{z \in \mathcal{V}_L} v_{L,z}^p \overset{(2.30)}{=} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z}^1 + \sum_{z \in \mathcal{V}_L} v_{L,z}^p \\ &= \sum_{z \in \mathcal{V}_0} v_{0,z}^1 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z}^1 + \sum_{z \in \mathcal{V}_L^+} v_{L,z}^1 + \sum_{z \in \mathcal{V}_L} v_{L,z}^p. \end{aligned}$$

By defining $v_0 := \sum_{z \in \mathcal{V}_0} v_{0,z}^1 \in \mathcal{X}_0^1$, $v_{\ell,z} := v_{\ell,z}^1 \in \mathcal{X}_{\ell,z}^1$ for $z \in \mathcal{V}_\ell^+$ and $1 \le \ell \le L-1$, and $v_{L,z} := v_{L,z}^1 + v_{L,z}^p \in \mathcal{X}_{L,z}^p$ for $z \in \mathcal{V}_L^+$ and $v_{L,z} := v_{L,z}^p \in \mathcal{X}_{L,z}^p$ for $z \in \mathcal{V}_L \setminus \mathcal{V}_L^+$, we obtain the decomposition (2.34). It remains to show that this decomposition is stable (2.35). First, we have for the coarsest level that

$$\|\nabla v_0\|^2 \overset{(2.25)}{\le} (d+1) \sum_{z \in \mathcal{V}_0} \|\nabla v_{0,z}^1\|^2.$$

For the finest level, it holds that

$$\begin{aligned} \sum_{z \in \mathcal{V}_L} \|\nabla v_{L,z}\|^2 &\le \sum_{z \in \mathcal{V}_L \setminus \mathcal{V}_L^+} \|\nabla v_{L,z}^p\|^2 + 2 \sum_{z \in \mathcal{V}_L^+} \big(\|\nabla v_{L,z}^1\|^2 + \|\nabla v_{L,z}^p\|^2\big) \\ &\le (d+1) \sum_{z \in \mathcal{V}_L^+} \|\nabla v_{L,z}^1\|^2 + (d+1) \sum_{z \in \mathcal{V}_L} \|\nabla v_{L,z}^p\|^2. \end{aligned}$$

A combination of the two estimates shows that

$$
\begin{aligned}
&\|\nabla v_0\|^2 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \|\nabla v_{\ell,z}\|^2 + \sum_{z\in\mathcal{V}_L} \|\nabla v_{L,z}\|^2 \\
&\quad\le (d+1)\Big(\sum_{z\in\mathcal{V}_0} \|\nabla v^1_{0,z}\|^2 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \|\nabla v^1_{\ell,z}\|^2 + \sum_{z\in\mathcal{V}_L^+} \|\nabla v^1_{L,z}\|^2 + \sum_{z\in\mathcal{V}_L} \|\nabla v^p_{L,z}\|^2 \Big) \\
&\quad\le (d+1) \sum_{\ell=0}^{L} \sum_{z\in\mathcal{V}_\ell^+} \|\nabla v^1_{\ell,z}\|^2 + (d+1) \sum_{z\in\mathcal{V}_L} \|\nabla v^p_{L,z}\|^2 \\
&\quad\overset{(2.31)}{\le} C_{\mathrm{ML}}^2\,(d+1)\,\|\nabla v^1_L\|^2 + (d+1) \sum_{z\in\mathcal{V}_L} \|\nabla v^p_{L,z}\|^2 \\
&\quad\overset{(2.29)}{\le} \max\{1, C_{\mathrm{ML}}^2\}\, C_{\mathrm{OL}}^2\,(d+1)\|\nabla v_L\|^2.
\end{aligned}
$$

Hence, the decomposition (2.34) is stable with $(C'_{\mathrm{SD}})^2 := \max\{1, C_{\mathrm{ML}}^2\} C_{\mathrm{OL}}^2 (d+1)$ with respect to the $H^1(\Omega)$-seminorm. Taking into account the variations of the diffusion coefficient $\boldsymbol{A}$, we obtain (2.35) with the stability constant $C_{\mathrm{SD}} := C'_{\mathrm{SD}}\Lambda_{\max}/\Lambda_{\min}$. □

2.5.3 Strengthened Cauchy–Schwarz inequality on NVB-generated meshes

The following results are proved in the spirit of [HWZ12; CNX12]. Note that the setting of this work is similar to [HWZ12], and unlike [CNX12], the underlying adaptive meshes of the space hierarchy are not restricted to one bisection per level.

For analysis purposes, we introduce a sequence of uniformly refined triangulations indicated by $\{\widehat{\mathcal{T}}_j\}_{j=0}^M$ such that $\widehat{\mathcal{T}}_{j+1} := \mathtt{refine}(\widehat{\mathcal{T}}_j, \widehat{\mathcal{T}}_j)$ and $\widehat{\mathcal{T}}_0 = \mathcal{T}_0$, where `refine` enforces one bisection per element. According to [Ste08], admissibility of $\mathcal{T}_0$ ensures that indeed each element $T \in \widehat{\mathcal{T}}_j$ is bisected only once into two children $T', T'' \in \widehat{\mathcal{T}}_{j+1}$. In the following, we will indicate the equivalent notation to Section 2.2 on uniform triangulations $\widehat{\mathcal{T}}_j$ with a hat, e.g., $\widehat{\mathcal{X}}^1_j$ is the equivalent of $\mathcal{X}^1_\ell$ on the uniformly refined mesh $\widehat{\mathcal{T}}_j$. The connection of the uniformly refined meshes and their adaptively generated counterpart requires further notation. For a given level $0 \le \ell \le L$ and a given node $z \in \mathcal{V}_\ell$, we define the generation $g_{\ell,z}$ of the patch by the maximum number of times an element of the patch has been bisected

$$
g_{\ell,z} := \max_{T\in\mathcal{T}_{\ell,z}} \log_2(|T_0|/|T|) \in \mathbb{N}_0, \tag{2.36}
$$

where $T_0 \in \mathcal{T}_0$ denotes the unique ancestor element of $T \in \mathcal{T}_\ell$. Define the maximal generation $M = \max_{z\in\mathcal{V}_L} g_{L,z}$.

First, we present the following result for uniformly refined meshes and then exploit this for our setting of adaptively refined meshes.

Lemma 2.15 (Strengthened Cauchy–Schwarz on nested uniform meshes)**.** *Let $0 \le i \le j \le M$, and $\widehat{u}_i \in \widehat{\mathcal{X}}^1_i$ as well as $\widehat{v}_j \in \widehat{\mathcal{X}}^1_j$. Then, it holds that*

$$
\langle\!\langle \widehat{u}_i\,,\, \widehat{v}_j \rangle\!\rangle \le \widehat{C}_{\mathrm{SCS}}\, \delta^{j-i}\, \widehat{h}_j^{-1} \big\|\nabla \widehat{u}_i\big\|\, \big\|\widehat{v}_j\big\|, \tag{2.37}
$$

where $\delta = 2^{-1/2}$ *and* $\widehat{C}_{\mathrm{SCS}} > 0$ *depends only on the domain* Ω*, the initial triangulation* $\mathcal{T}_0$, $\Lambda_{\max}$, $\max_{T\in\widehat{\mathcal{T}}_M} \|\operatorname{div}(\boldsymbol{A})\|_{L^\infty(T)}$*, and* γ*-shape regularity from* (2.5).

Proof. We begin by splitting the domain Ω into elementwise components, applying integration by parts, and using the Cauchy–Schwarz inequality. Note that the restriction of $\widehat{u}_i$ to any element $T \in \widehat{\mathcal{T}}_i$ is an affine function, and hence the second derivatives vanish. Thus, it holds with the outer normal $\boldsymbol{n}$ to ∂T that

$$\begin{aligned}
\langle\!\langle \widehat{u}_i\,,\,\widehat{v}_j\rangle\!\rangle &= \sum_{T\in\widehat{\mathcal{T}}_i} \int_T \boldsymbol{A}\nabla\widehat{u}_i\cdot\nabla\widehat{v}_j\,\mathrm{d}x\\
&= \sum_{T\in\widehat{\mathcal{T}}_i} \Big(-\int_T \operatorname{div}(\boldsymbol{A}\nabla\widehat{u}_i)\widehat{v}_j\,\mathrm{d}x + \int_{\partial T}\boldsymbol{A}\nabla\widehat{u}_i\cdot\boldsymbol{n}\,\widehat{v}_j\,\mathrm{d}x\Big)\\
&\le \sum_{T\in\widehat{\mathcal{T}}_i}\Big(\big\|(\operatorname{div}\boldsymbol{A})\cdot\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)} + \big\|\boldsymbol{A}\nabla\widehat{u}_i\big\|_{L^2(\partial T)}\big\|\widehat{v}_j\big\|_{L^2(\partial T)}\Big).
\end{aligned}$$

Due to $\boldsymbol{A}\in W^{1,\infty}(T)$, the fact that $\widehat{u}_i, \widehat{v}_j$ are piecewise affine, a discrete trace inequality, and $\widehat{h}_i^{-1}\gtrsim 1$, we get

$$\begin{aligned}
\langle\!\langle \widehat{u}_i\,,\,\widehat{v}_j\rangle\!\rangle &\lesssim \sum_{T\in\widehat{\mathcal{T}}_i}\Big(\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)} + \big\|\nabla\widehat{u}_i\big\|_{L^2(\partial T)}\big\|\widehat{v}_j\big\|_{L^2(\partial T)}\Big)\\
&\lesssim \sum_{T\in\widehat{\mathcal{T}}_i}\Big(\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)} + \big(\widehat{h}_i^{-1/2}\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big)\,\big(\widehat{h}_i^{-1/2}\big\|\widehat{v}_j\big\|_{L^2(T)}\big)\Big)\\
&= \sum_{T\in\widehat{\mathcal{T}}_i}\big(1+\widehat{h}_i^{-1}\big)\,\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)}\\
&\lesssim \sum_{T\in\widehat{\mathcal{T}}_i}\widehat{h}_i^{-1}\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)}.
\end{aligned}$$

Moreover, note that due to uniform refinement, we have the equivalence $\delta^{j-i} = (2^{-1/2})^{j-i} \simeq (\widehat{h}_j/\widehat{h}_i)^{1/2}$ and $\widehat{h}_j \le \widehat{h}_i$. Using the last equation multiplied by $1 = \widehat{h}_j^{1/2}\,\widehat{h}_j^{-1/2}$, we derive that

$$\begin{aligned}
\langle\!\langle \widehat{u}_i\,,\,\widehat{v}_j\rangle\!\rangle &\lesssim \sum_{T\in\widehat{\mathcal{T}}_i}\Big(\frac{\widehat{h}_j}{\widehat{h}_i}\Big)^{1/2}\widehat{h}_i^{-1/2}\,\widehat{h}_j^{-1/2}\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)}\\
&\lesssim \sum_{T\in\widehat{\mathcal{T}}_i}\delta^{j-i}\,\widehat{h}_j^{-1}\,\big\|\nabla\widehat{u}_i\big\|_{L^2(T)}\big\|\widehat{v}_j\big\|_{L^2(T)} \le \widehat{h}_j^{-1}\delta^{j-i}\big\|\nabla\widehat{u}_i\big\|_{L^2(\Omega)}\big\|\widehat{v}_j\big\|_{L^2(\Omega)}.
\end{aligned}$$

This concludes the proof. □

The last result enables us to tackle the setting of adaptively refined meshes.

Proposition 2.16 (Strengthened Cauchy–Schwarz on nested adaptive meshes)**.** *Consider levelwise functions* $v_\ell = \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z}^1 \in \mathcal{X}_\ell^1$ *with* $v_{\ell,z}^1 \in \mathcal{X}_{\ell,z}^1$ *for all* $1\le\ell\le L-1$*. Then, it holds that*

$$\sum_{\ell=1}^{L-1}\sum_{k=1}^{\ell-1}\langle\!\langle v_k\,,\,v_\ell\rangle\!\rangle \le C_{\mathrm{SCS}}\Big(\sum_{k=1}^{L-2}\sum_{w\in\mathcal{V}_k^+}|\!|\!|v_{k,w}^1|\!|\!|^2\Big)^{1/2}\Big(\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|\!|\!|v_{\ell,z}^1|\!|\!|^2\Big)^{1/2}, \tag{2.38}$$

> *where* $C_{\rm SCS} > 0$ *depends only on* Ω*, the initial triangulation* $\mathcal{T}_0$*, the ratio* $\Lambda_{\max}/\Lambda_{\min}$, $\max_{T\in\mathcal{T}_L}\|\operatorname{div}(\boldsymbol{A})\|_{L^\infty(T)}/\Lambda_{\min}$*, and* γ*-shape regularity* (2.5).

Proof. Let $M \in \mathbb{N}$. The proof consists of five steps.

Step 1. First note that, for any $0 < \delta < 1$ and $x_i, y_i > 0$ with $0 \le i \le M$, there holds

$$\sum_{i=0}^{M}\sum_{j=i}^{M}\delta^{j-i}x_i y_j \le \frac{1}{1-\delta}\Big(\sum_{i=0}^{M}x_i^2\Big)^{1/2}\Big(\sum_{j=0}^{M}y_j^2\Big)^{1/2}. \tag{2.39}$$

To see this, we change the summation order accordingly and use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}\sum_{i=0}^{M}\sum_{j=i}^{M}\delta^{j-i}x_i y_j &= \sum_{i=0}^{M}\sum_{m=0}^{M-i}\delta^{m}x_i y_{m+i} = \sum_{m=0}^{M}\sum_{i=0}^{M-m}\delta^{m}x_i y_{m+i}\\ &\le \sum_{m=0}^{M}\delta^m\Big[\Big(\sum_{i=0}^{M-m}x_i^2\Big)^{1/2}\Big(\sum_{i=0}^{M-m}y_{m+i}^2\Big)^{1/2}\Big] \le \Big(\sum_{m=0}^{M}\delta^m\Big)\Big(\sum_{i=0}^{M}x_i^2\Big)^{1/2}\Big(\sum_{j=0}^{M}y_j^2\Big)^{1/2}.\end{aligned}$$

The geometric series then proves the claim (2.39).

Step 2. Let $z \in \mathcal{V}_L$ and $0 \le j \le M$ and recall the patch generation $g_{\ell,z}$ from (2.36). We introduce the set

$$\mathcal{L}_{\underline{\ell},\overline{\ell}}(z,j) \coloneqq \{\ell \in \{\underline{\ell},\dots,\overline{\ell}\} : z \in \mathcal{V}_\ell^+ \text{ and } g_{\ell,z} = j\} \quad \text{with } 0 \le \underline{\ell} \le \overline{\ell} \le L. \tag{2.40}$$

This set allows to track how large the levelwise overlap of patches with the same generation is. Crucially, the cardinality of these sets is uniformly bounded by

$$\max_{\substack{z\in\mathcal{V}_L\\ 0\le j\le M}} \#(\mathcal{L}_{0,L}(z,j)) \le C_{\rm lev} < \infty; \tag{2.41}$$

see, e.g., [WC06, Lemma 3.1] in the two-dimensional setting with arguments that transfer to three dimensions. The constant $C_{\rm lev}$ solely depends on γ-shape regularity (2.5).

Step 3. We introduce a way to reorder the patch contributions by generations (2.36). Note that, for any $0 \le j \le M$, $1 \le \ell \le L-1$, and $z \in \mathcal{V}_\ell^+$ such that $g_{\ell,z} = j$, the patch contribution $v_{\ell,z}^1 \in \mathcal{X}_{\ell,z}^1$ also belongs to $\widehat{\mathcal{X}}_j^1$. Once the generation constraint is introduced, one can shift the perspective from summing over "adaptive" levels and associated vertices to summing over "uniform" vertices and *only* the (finitely many, cf. (2.41)) levels where each vertex satisfies the generation constraint, i.e., for $0 \le \underline{\ell} \le \overline{\ell} \le L$ and $0 \le j \le M$, the two following sets coincide

$$\begin{aligned}&\{(\ell,z) \in \mathbb{N}_0 \times \mathcal{V}_L : \ell \in \{\underline{\ell},\dots,\overline{\ell}\}, z \in \mathcal{V}_\ell^+ \text{ with } g_{\ell,z} = j\}\\ &= \{(\ell,z) \in \mathbb{N}_0 \times \mathcal{V}_L : z \in \widehat{\mathcal{V}}_j, \ell \in \mathcal{L}_{\underline{\ell},\overline{\ell}}(z,j)\}.\end{aligned} \tag{2.42}$$

Step 4. According to γ-shape regularity (2.5), all elements in the patch have comparable size depending on $C_{\rm qu}$ from (2.6). If $g_{\ell,z} = j$, (at least) one element $T^\star \in \mathcal{T}_{\ell,z}$ satisfies $T^\star \in \widehat{\mathcal{T}}_j$ and it follows that $\widehat{h}_j \simeq |T^\star|^{1/d} \simeq |\omega_{\ell,z}|^{1/d} \simeq h_{\ell,z}$. In particular, there exists $C_{\rm eq} > 0$ such that

$$\widehat{h}_j^{-1} \le C_{\rm eq} h_{\ell,z}^{-1}. \tag{2.43}$$

Step 5. We proceed to prove the main estimate (2.38). The central feature of the following approach is to introduce *additional* sums over the generations with generation constraints, i.e., there holds for every admissible ℓ, k, that

$$\langle\!\langle v_k\,,\,v_\ell\rangle\!\rangle = \sum_{z\in\mathcal{V}_\ell^+}\sum_{w\in\mathcal{V}_k^+}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle = \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle$$
$$= \sum_{j=0}^{M}\sum_{i=0}^{j}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle + \sum_{j=0}^{M}\sum_{i=j+1}^{M}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle.$$

We abbreviate the terms as $S_1(\ell,k)$ and $S_2(\ell,k)$, respectively. A change of the summation of order i and j yields for $S_1(\ell,k)$ that

$$S_1(\ell,k) = \sum_{i=0}^{M}\sum_{j=i}^{M}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle.$$

Summing $S_2(\ell,k)$ over all ℓ and k and changing the order of summation, we obtain

$$\sum_{\ell=1}^{L-1}\sum_{k=1}^{\ell-1}S_2(\ell,k) = \sum_{j=0}^{M}\sum_{i=j+1}^{M}\sum_{k=1}^{L-2}\sum_{\ell=k+1}^{L-1}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\langle\!\langle v_{k,w}^1\,,\,v_{\ell,z}^1\rangle\!\rangle.$$

Combining these two identities with (2.42), we see that

$$\sum_{\ell=1}^{L-1}\sum_{k=1}^{\ell-1}\Big(S_1(\ell,k)+S_2(\ell,k)\Big) = \sum_{i=0}^{M}\sum_{j=i}^{M}\sum_{\ell=1}^{L-1}\Big\langle\!\!\Big\langle \sum_{w\in\widehat{\mathcal{V}}_i}\sum_{k\in\mathcal{L}_{1,\ell-1}(w,i)} v_{k,w}^1\,,\,\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}} v_{\ell,z}^1\Big\rangle\!\!\Big\rangle$$
$$+\sum_{j=0}^{M}\sum_{i=j+1}^{M}\sum_{k=1}^{L-2}\Big\langle\!\!\Big\langle \sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}} v_{k,w}^1\,,\,\sum_{z\in\widehat{\mathcal{V}}_j}\sum_{\ell\in\mathcal{L}_{k+1,L-1}(z,j)} v_{\ell,z}^1\Big\rangle\!\!\Big\rangle.$$

We define the last two terms as S_1 and S_2, respectively. Since the second term S_2 is treated in the same way, we only present detailed estimations of the first term S_1. The strengthened Cauchy–Schwarz inequality (2.37) for functions defined on uniform meshes followed by the patch overlap (2.24) leads to

$$S_1 \le \widehat{C}_{\mathrm{SCS}}\sum_{i=0}^{M}\sum_{j=i}^{M}\delta^{j-i}\sum_{\ell=1}^{L-1}\Big((d+1)\sum_{w\in\widehat{\mathcal{V}}_i}\Big\|\sum_{k\in\mathcal{L}_{1,\ell-1}(w,i)}\nabla v_{k,w}^1\Big\|^2\Big)^{1/2}\sum_{\substack{z\in\mathcal{V}_\ell^+\\ g_{\ell,z}=j}}\widehat{h}_j^{-1}\,\|v_{\ell,z}^1\|.$$

The identity (2.42) and the finite levelwise overlap (2.41) show

$$\sum_{w\in\widehat{\mathcal{V}}_i}\Big\|\sum_{k\in\mathcal{L}_{1,\ell-1}(w,i)}\nabla v_{k,w}^1\Big\|^2 \le \sum_{k=1}^{\ell-1}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\#(\mathcal{L}_{1,\ell-1}(w,i))\big\|\nabla v_{k,w}^1\big\|^2 \le C_{\mathrm{lev}}\sum_{k=1}^{L-2}\sum_{\substack{w\in\mathcal{V}_k^+\\ g_{k,w}=i}}\|\nabla v_{k,w}^1\|^2.$$

The equivalence of mesh sizes from (2.43) and a Poincaré-inequality prove

$$\sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \widehat{h}_j^{-1} \|v_{\ell,z}^1\| \leq C_{\mathrm{eq}} C_{\mathrm{P}} \sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \|\nabla v_{\ell,z}^1\|.$$

A combination of (2.42) with (2.25) and (2.41), followed again by (2.42), yields

$$\Big(\sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \|\nabla v_{\ell,z}^1\|\Big)^2 = \Big(\sum_{z\in\widehat{\mathcal{V}}_j} \sum_{\ell\in\mathcal{L}_{1,L-1}(z,j)} \|\nabla v_{\ell,z}^1\|\Big)^2 \leq (d+1)\, C_{\mathrm{lev}} \sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \|\nabla v_{\ell,z}^1\|^2.$$

Thus, we obtain the bound

$$\sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \|\nabla v_{\ell,z}^1\| \leq (d+1)^{1/2} C_{\mathrm{lev}}^{1/2} \Big(\sum_{\ell=1}^{L-1} \sum_{\substack{z\in\mathcal{V}_\ell^+ \\ g_{\ell,z}=j}} \|\nabla v_{\ell,z}^1\|^2\Big)^{1/2}$$

Combining all estimates, together with the geometric series bound (2.39), confirms

$$S_1 \leq \widehat{C}_{\mathrm{SCS}}\, (d+1)\, C_{\mathrm{lev}}\, C_{\mathrm{eq}}\, C_{\mathrm{P}} \frac{1}{1-\delta} \Big(\sum_{k=1}^{L-2} \sum_{w\in\mathcal{V}_k^+} \|\nabla v_{k,w}^1\|^2\Big)^{1/2} \Big(\sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \|\nabla v_{\ell,z}^1\|^2\Big)^{1/2}.$$

Finally, the result (2.38) is obtained after summing together with the analogous estimations coming from the remaining term S_2 and taking into consideration the variations of the diffusion coefficient $\boldsymbol{A}$ so that the result holds for the energy norm. This concludes the proof. □

2.5.4 Proof of the main results

For the sake of a concise presentation, we only consider the case $p > 1$. The case $p = 1$ is already covered in the literature [CNX12; WZ17] and follows from our proof with only minor modifications.

Proof of Theorem 2.4, connection of solver and estimator (2.12). The proof consists of two steps.

Step 1. We show that there holds the identity

$$\begin{aligned} &|\!|\!| \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell |\!|\!|^2 - 2\langle\!\langle u_L^\star - v_L\,,\, \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \rangle\!\rangle \\ &\qquad = -|\!|\!|\rho_0|\!|\!|^2 + \sum_{\ell=1}^{L-1} |\!|\!|\lambda_\ell \rho_\ell|\!|\!|^2 - 2\sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |\!|\!|\rho_{\ell,z}|\!|\!|^2. \end{aligned} \tag{2.44}$$

Indeed, note that $\sigma_\ell = \sum_{k=0}^{\ell} \lambda_k \rho_k$. By definition of the local lowest-order problems in (2.10) and

(2.11) as well as the definition of $\rho_\ell = \sum_{z\in\mathcal{V}_\ell^+} \rho_{\ell,z}$, we have

$$\begin{aligned}\Big\langle\!\!\Big\langle u_L^\star - v_L\,, \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big\rangle\!\!\Big\rangle &\overset{(2.9)}{=} R_L(\rho_0) + \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} R_L(\rho_{\ell,z}) \overset{(2.10)}{=} |||\rho_0|||^2 + \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} R_L(\rho_{\ell,z}) \\ &\overset{(2.11)}{=} |||\rho_0|||^2 + \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} \Big(|||\rho_{\ell,z}|||^2 + \langle\!\langle \sigma_{\ell-1}\,,\, \rho_{\ell,z} \rangle\!\rangle \Big) \\ &= |||\rho_0|||^2 + \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} \Big(|||\rho_{\ell,z}|||^2 + \sum_{k=0}^{\ell-1} \langle\!\langle \lambda_k \rho_k\,,\, \rho_{\ell,z} \rangle\!\rangle \Big) \\ &= |||\rho_0|||^2 + \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 + \sum_{\ell=1}^{L-1} \sum_{k=0}^{\ell-1} \langle\!\langle \lambda_k \rho_k\,,\, \lambda_\ell \rho_\ell \rangle\!\rangle.\end{aligned}$$

Thus, by expanding the square, we have

$$\begin{aligned}||| \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell |||^2 - 2 \Big\langle\!\!\Big\langle u_L^\star - v_L\,, \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big\rangle\!\!\Big\rangle &= \sum_{\ell=0}^{L-1} |||\lambda_\ell \rho_\ell|||^2 - 2|||\rho_0|||^2 - 2 \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 \\ &= -|||\rho_0|||^2 + \sum_{\ell=1}^{L-1} |||\lambda_\ell \rho_\ell|||^2 - 2 \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2.\end{aligned}$$

This proves the identity (2.44).

Step 2. Recall that $\Psi(v_L) = v_L + \sigma_L = v_L + \sigma_{L-1} + \lambda_L \rho_L$. By definition of R_L in (2.9) and the choice of λ_L in Algorithm 2A, we have

$$\begin{aligned}|||u_L^\star - \Psi(v_L)|||^2 &= |||u_L^\star - (v_L + \sigma_{L-1})|||^2 - 2\,\lambda_L\, \langle\!\langle u_L^\star - (v_L + \sigma_{L-1})\,,\, \rho_L \rangle\!\rangle + |||\lambda_L \rho_L|||^2 \\ &= |||u_L^\star - (v_L + \sigma_{L-1})|||^2 - 2\lambda_L \Big(R_L(\rho_L) - \langle\!\langle \sigma_{L-1}\,,\, \rho_L \rangle\!\rangle \Big) + \lambda_L \sum_{z\in\mathcal{V}_L} |||\rho_{L,z}|||^2 \\ &\overset{(2.11)}{=} |||u_L^\star - \Big(v_L + \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big)|||^2 - \lambda_L \sum_{z\in\mathcal{V}_L} |||\rho_{L,z}|||^2.\end{aligned}$$

For the first term it holds that

$$\begin{aligned}|||u_L^\star - \Big(v_L + \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big)|||^2 &= |||u_L^\star - v_L|||^2 + ||| \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell |||^2 - 2 \Big\langle\!\!\Big\langle u_L^\star - v_L\,, \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big\rangle\!\!\Big\rangle \\ &\overset{(2.44)}{=} |||u_L^\star - v_L|||^2 - |||\rho_0|||^2 + \sum_{\ell=1}^{L-1} |||\lambda_\ell \rho_\ell|||^2 - 2 \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 \\ &\overset{(2.26)}{\le} |||u_L^\star - v_L|||^2 - |||\rho_0|||^2 - \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2.\end{aligned}$$

Combining the last two estimates with the definition of $\eta_{\mathrm{alg}}(v_L)$ in Algorithm 2A, we obtain

$$\begin{aligned} |||u_L^\star - \Psi(v_L)|||^2 &\le |||u_L^\star - v_L|||^2 - |||\rho_0|||^2 - \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 - \lambda_L \sum_{z \in \mathcal{V}_L} |||\rho_{L,z}|||^2 \\ &= |||u_L^\star - v_L|||^2 - \eta_{\mathrm{alg}}(v_L)^2. \end{aligned}$$

This concludes the proof of (2.12). □

Proof of Theorem 2.4, lower bound in (2.14). The relation between the solver and the estimator given in (2.12) shows that $\eta_{\mathrm{alg}}(v_L) \le |||u_L^\star - v_L|||$. □

Proof of Corollary 2.5, equivalence of (2.13) *and* (2.14). We prove that the solver contraction (2.13) is equivalent to the upper bound of (2.14).

First, suppose that (2.13) holds. Then, we proceed similarly as in the proof of (2.12) to obtain

$$\begin{aligned} |||u_L^\star - v_L|||^2 &= |||u_L^\star - \Psi(v_L)|||^2 - |||\sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell|||^2 + 2 \Big\langle\!\Big\langle u_L^\star - v_L\,, \sum_{\ell=0}^{L-1} \lambda_\ell \rho_\ell \Big\rangle\!\Big\rangle + \lambda_L \sum_{z \in \mathcal{V}_L} |||\rho_{L,z}|||^2 \\ &\overset{(2.44)}{=} |||u_L^\star - \Psi(v_L)|||^2 + |||\rho_0|||^2 - \sum_{\ell=1}^{L-1} |||\lambda_\ell \rho_\ell|||^2 + 2 \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z \in \mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 + \lambda_L \sum_{z \in \mathcal{V}_L} |||\rho_{L,z}|||^2 \\ &\overset{(2.13)}{\le} q_{\mathrm{ctr}}^2\, |||u_L^\star - v_L|||^2 + 2\, \eta_{\mathrm{alg}}(v_L)^2. \end{aligned}$$

Rearranging this estimate proves the upper bound in (2.14) with $C_{\mathrm{rel}}^2 = 2/(1 - q_{\mathrm{ctr}}^2) > 1$.

Second, suppose the upper bound in (2.14). Then, it follows that

$$|||u_L^\star - \Psi(v_L)|||^2 \overset{(2.12)}{\le} |||u_L^\star - v_L|||^2 - \eta_{\mathrm{alg}}(v_L)^2 \overset{(2.14)}{\le} |||u_L^\star - v_L|||^2 - C_{\mathrm{rel}}^{-2}\, |||u_L^\star - v_L|||^2.$$

This verifies the solver contraction (2.13) for $q_{\mathrm{ctr}}^2 = 1 - C_{\mathrm{rel}}^{-2} \in (0,1)$ and concludes the equivalence proof. □

Proof of Theorem 2.4, upper bound in (2.14). We use the stable decomposition of Proposition 2.14 on the algebraic error $u_L^\star - v_L \in \mathcal{X}_L^p$ to obtain $v_0 \in \mathcal{X}_0^1$, $v_{\ell,z} \in \mathcal{X}_{\ell,z}^1$ and $v_{L,z} \in \mathcal{X}_{L,z}^p$ such that

$$u_L^\star - v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z \in \mathcal{V}_L} v_{L,z}$$

$$\text{and} \quad |||v_0|||^2 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} |||v_{\ell,z}|||^2 + \sum_{z \in \mathcal{V}_L} |||v_{L,z}|||^2 \le C_{\mathrm{SD}}^2\, |||u_L^\star - v_L|||^2. \qquad (2.45)$$

Note that $\sigma_\ell = \sum_{k=0}^{\ell} \lambda_k \rho_k$ for all $\ell = 0, \dots, L$; see Algorithm 2A. We use (2.45) to develop

$$\begin{aligned}
|||u_L^\star - v_L|||^2 &= \Big\langle\!\Big\langle u_L^\star - v_L \,,\, v_0 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z\in\mathcal{V}_L} v_{L,z} \Big\rangle\!\Big\rangle \\
&\overset{(2.9)}{\underset{(2.10)}{=}} \langle\!\langle \rho_0 \,,\, v_0 \rangle\!\rangle + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} R_L(v_{\ell,z}) + \sum_{z\in\mathcal{V}_L} R_L(v_{L,z}) \\
&\overset{(2.11)}{=} \langle\!\langle \rho_0 \,,\, v_0 \rangle\!\rangle + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \big(\langle\!\langle \rho_{\ell,z} \,,\, v_{\ell,z} \rangle\!\rangle + \langle\!\langle \sigma_{\ell-1} \,,\, v_{\ell,z} \rangle\!\rangle \big) \\
&\qquad + \sum_{z\in\mathcal{V}_L} \big(\langle\!\langle \rho_{L,z} \,,\, v_{L,z} \rangle\!\rangle + \langle\!\langle \sigma_{L-1} \,,\, v_{L,z} \rangle\!\rangle \big).
\end{aligned}$$

Expanding $\sigma_\ell = \rho_0 + \sum_{k=1}^{\ell} \lambda_k \rho_k$ and rearranging the terms finally leads to

$$\begin{aligned}
|||u_L^\star - v_L|||^2 &= \Big\langle\!\Big\langle \rho_0 \,,\, v_0 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z\in\mathcal{V}_L} v_{L,z} \Big\rangle\!\Big\rangle + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \langle\!\langle \rho_{\ell,z} \,,\, v_{\ell,z} \rangle\!\rangle \\
&\quad + \sum_{z\in\mathcal{V}_L} \langle\!\langle \rho_{L,z} \,,\, v_{L,z} \rangle\!\rangle + \sum_{\ell=1}^{L-1} \sum_{k=1}^{\ell-1} \Big\langle\!\Big\langle \lambda_k \rho_k \,,\, \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z} \Big\rangle\!\Big\rangle + \sum_{k=1}^{L-1} \Big\langle\!\Big\langle \lambda_k \rho_k \,,\, \sum_{z\in\mathcal{V}_L} v_{L,z} \Big\rangle\!\Big\rangle.
\end{aligned}$$

Note that, until this point, only equalities are used. In the following, we will estimate each of the constituting terms of the algebraic error using Young's inequality in the form $ab \le (\alpha/2)\, a^2 + (2\alpha)^{-1}\, b^2$ with $\alpha = 4C_{\mathrm{SD}}^2$, the strengthened Cauchy–Schwarz inequality, and patch overlap arguments as done in the proof of Lemma 2.10. Using the fact that $\lambda_0 = 1$ and the decomposition of the error $u_L^\star - v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z\in\mathcal{V}_L} v_{L,z}$, we see that the first term yields

$$\Big\langle\!\Big\langle \rho_0 \,,\, v_0 + \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z} + \sum_{z\in\mathcal{V}_L} v_{L,z} \Big\rangle\!\Big\rangle \overset{(2.45)}{=} \langle\!\langle \rho_0 \,,\, u_L^\star - v_L \rangle\!\rangle \le \frac{1}{2}\, |||\lambda_0 \rho_0|||^2 + \frac{1}{2}\, |||u_L^\star - v_L|||^2.$$

For the second term, we obtain that

$$\begin{aligned}
\sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} \langle\!\langle \rho_{\ell,z} \,,\, v_{\ell,z} \rangle\!\rangle &\le 2C_{\mathrm{SD}}^2 \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 + \frac{1}{8C_{\mathrm{SD}}^2} \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} |||v_{\ell,z}|||^2 \\
&\overset{(2.27)}{\le} 2C_{\mathrm{SD}}^2\, (d+1) \sum_{\ell=1}^{L-1} \lambda_\ell \sum_{z\in\mathcal{V}_\ell^+} |||\rho_{\ell,z}|||^2 + \frac{1}{8C_{\mathrm{SD}}^2} \sum_{\ell=1}^{L-1} \sum_{z\in\mathcal{V}_\ell^+} |||v_{\ell,z}|||^2,
\end{aligned}$$

and similarly for the third term

$$\sum_{z\in\mathcal{V}_L} \langle\!\langle \rho_{L,z} \,,\, v_{L,z} \rangle\!\rangle \overset{(2.27)}{\le} 2\, C_{\mathrm{SD}}^2\, (d+1)\, \lambda_L \sum_{z\in\mathcal{V}_L} |||\rho_{L,z}|||^2 + \frac{1}{8C_{\mathrm{SD}}^2} \sum_{z\in\mathcal{V}_L} |||v_{L,z}|||^2.$$

For the fourth term, we have

$$\sum_{\ell=1}^{L-1}\sum_{k=1}^{\ell-1}\left\langle\!\!\left\langle \lambda_k\rho_k\,,\, \sum_{z\in\mathcal{V}_\ell^+} v_{\ell,z}\right\rangle\!\!\right\rangle \overset{(2.38)}{\leq} C_{\rm SCS}\Big(\sum_{k=1}^{L-2}\sum_{w\in\mathcal{V}_k^+}|||\lambda_k\rho_{k,w}|||^2\Big)^{1/2}\Big(\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|||v_{\ell,z}|||^2\Big)^{1/2}$$
$$\leq 2\,C_{\rm SCS}^2\,C_{\rm SD}^2\sum_{k=0}^{L-2}\sum_{w\in\mathcal{V}_k^+}|||\lambda_k\rho_{k,w}|||^2 + \frac{1}{8C_{\rm SD}^2}\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|||v_{\ell,z}|||^2$$
$$\overset{(2.27)}{\leq} 2C_{\rm SCS}^2\,C_{\rm SD}^2\,(d+1)\sum_{k=0}^{L-2}\lambda_k\sum_{w\in\mathcal{V}_k^+}|||\rho_{k,w}|||^2 + \frac{1}{8C_{\rm SD}^2}\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|||v_{\ell,z}|||^2.$$

Finally, to treat the last term where higher-order terms appear together with a sum over levels, we proceed similarly as in [CNX12, Proof of Theorem 4.8] and obtain

$$\sum_{k=1}^{L-1}\left\langle\!\!\left\langle \lambda_k\rho_k\,,\, \sum_{z\in\mathcal{V}_L} v_{L,z}\right\rangle\!\!\right\rangle = \sum_{z\in\mathcal{V}_L}\left\langle\!\!\left\langle \sum_{k=1}^{L-1}\lambda_k\rho_k\,,\, v_{L,z}\right\rangle\!\!\right\rangle$$
$$\leq 2C_{\rm SD}^2\sum_{z\in\mathcal{V}_L}|||\sum_{k=1}^{L-1}\lambda_k\rho_k|||^2_{\omega_{L,z}} + \frac{1}{8C_{\rm SD}^2}\sum_{z\in\mathcal{V}_L}|||v_{L,z}|||^2.$$

For the first term of the last bound, we have that

$$\sum_{z\in\mathcal{V}_L}|||\sum_{k=1}^{L-1}\lambda_k\rho_k|||^2_{\omega_{L,z}} \lesssim |||\sum_{k=1}^{L-1}\lambda_k\rho_k|||^2 = \sum_{k=1}^{L-1}|||\lambda_k\rho_k|||^2 + 2\sum_{\ell=1}^{L-1}\sum_{k=1}^{\ell-1}\langle\!\langle\lambda_k\rho_k\,,\,\lambda_\ell\rho_\ell\rangle\!\rangle$$
$$\overset{(2.38)}{\leq}\sum_{k=1}^{L-1}|||\lambda_k\rho_k|||^2 + 2\,C_{\rm SCS}\Big(\sum_{k=1}^{L-2}\sum_{w\in\mathcal{V}_k^+}|||\lambda_k\rho_{k,w}|||^2\Big)^{1/2}\Big(\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|||\lambda_\ell\rho_{\ell,z}|||^2\Big)^{1/2}$$
$$\underset{(2.26)}{\overset{(2.27)}{\leq}}\big(1+2\,C_{\rm SCS}\,(d+1)\big)\Big(\sum_{\ell=1}^{L-1}\lambda_\ell + \sum_{z\in\mathcal{V}_\ell^+}|||\rho_{\ell,z}|||^2\Big).$$

Summing all the estimates of the algebraic error components and defining the constant $C_{\rm rel}^2 := \max\{1/2, C_{\rm SD}^2\,(d+1)\,\big(2 + C_{\rm SCS}^2 + 2\,C_{\rm SCS}\,(d+1)^{1/2}\big)\}$, we see that

$$|||u_L^\star - v_L|||^2 \leq \frac{1}{2}|||\lambda_0\rho_0|||^2 + \frac{1}{2}|||u_L^\star - v_L|||^2 + 4C_{\rm rel}^2\Big(\sum_{\ell=1}^{L-1}\lambda_\ell\sum_{z\in\mathcal{V}_\ell^+}|||\rho_{\ell,z}|||^2 + \lambda_L\sum_{z\in\mathcal{V}_L}|||\rho_{L,z}|||^2\Big)$$
$$+\frac{1}{4C_{\rm SD}^2}\Big(\sum_{\ell=1}^{L-1}\sum_{z\in\mathcal{V}_\ell^+}|||v_{\ell,z}|||^2 + \sum_{z\in\mathcal{V}_L}|||v_{L,z}|||^2\Big)$$
$$\overset{(2.45)}{\leq} 4C_{\rm rel}^2\,\eta_{\rm alg}(v_L)^2 + \frac{3}{4}|||u_L^\star - v_L|||^2.$$

After rearranging the terms, we finally obtain that

$$|||u_L^\star - v_L|||^2 \leq C_{\rm rel}^2\,\eta_{\rm alg}(v_L)^2. \tag{2.46}$$

This proves the upper bound of (2.14) and thus concludes the proof of Theorem 2.4. □

3 Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs

The sections 3.1–3.7 of this chapter correspond to the publication:
M. Brunner, M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger, and P. Heid. Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs. *IMA J. Numer. Anal.*, 44(3):1560–1596, 2024. DOI: 10.1093/imanum/drad039
Unfortunately, there is a minor flaw in the original manuscript [BIM+24a] that, however, required major adjustments of the analysis. Thus, we present the modified version with the corrections proposed in:
M. Brunner, M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger, and P. Heid. Corrigendum to: Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs. *IMA J. Numer. Anal.*, 44(3):1903–1909, 2024. DOI: 10.1093/imanum/drad103

3.1 Introduction

The mathematical understanding of optimal adaptivity for finite element methods (AFEMs) has reached a high level of maturity; see, e.g., [BDD04; Ste07; CKNS08; KS11; CN12; FFP14; CFPP14] for some contributions to linear PDEs. While the focus is usually on optimal convergence rates with respect to the degrees of freedom [BDD04; CKNS08; KS11; CN12; FFP14; CFPP14], the cumulative nature of adaptivity should rather ask for optimal convergence rates with respect to the overall computational cost, i.e., the overall elapsed computational time. This, usually called *optimal complexity*, has been thoroughly analyzed for adaptive wavelet methods [CDD01; CDD03] and it has also been addressed in the seminal work [Ste07] on AFEM for the Poisson model problem. Recent works [GHPS21; HPW21; HPSV21] considered optimal complexity for energy minimization problems and, in particular, for symmetric linear elliptic PDEs. In contrast to this, optimal complexity for nonsymmetric linear elliptic PDEs remained an open question due to the lack of a contractive algebraic solver that is compatible with the variational structure of the PDE. Closing this gap is the topic of the present work. While the canonical candidate for solving the nonsymmetric discrete systems would be GMRES, we take a different path that is motivated by up-to-date proofs of the Lax–Milgram lemma and closely related to the Richardson iteration used in the context of optimal adaptive wavelet methods. Some comments on the challenges presented by GMRES and related future work are given below.

As a model problem, we consider the nonsymmetric second-order linear elliptic PDE

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + cu^\star = f - \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

on a polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, where $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ is a symmetric and

uniformly positive definite diffusion matrix, $\boldsymbol{b} \in [L^\infty(\Omega)]^d$ is a convection coefficient, $c \in L^\infty(\Omega)$ is a reaction coefficient, and $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$ are the given data.
With $b(u, v) \coloneqq \langle \boldsymbol{A}\nabla u \,,\, \nabla v\rangle_\Omega + \langle \boldsymbol{b} \cdot \nabla u + cu \,,\, v\rangle_\Omega$ and $F(v) \coloneqq \langle f \,,\, v\rangle_\Omega + \langle \boldsymbol{f} \,,\, \nabla v\rangle_\Omega$, where $\langle \cdot \,,\, \cdot\rangle_\Omega$ denotes the usual $L^2(\Omega)$-scalar product, the weak formulation of (3.1) reads:

$$\text{Find } u^\star \in \mathcal{X} \coloneqq H^1_0(\Omega) \quad \text{such that} \quad b(u^\star, v) = F(v) \quad \text{for all } v \in \mathcal{X}. \tag{3.2}$$

To ensure the existence and uniqueness of $u^\star \in H^1_0(\Omega)$, we assume that the bilinear form $b(\cdot, \cdot)$ is continuous and elliptic on $H^1_0(\Omega)$ so that the Lax–Milgram lemma applies.

To discretize (3.2), we employ a conforming finite element method based on a conforming simplicial triangulation $\mathcal{T}_\ell$ of Ω and a fixed polynomial degree $m \in \mathbb{N}$. With

$$\mathcal{X}_\ell \coloneqq \{v_\ell \in H^1_0(\Omega) \mid v_\ell|_T \text{ is a polynomial of degree} \leq m, \text{ for all } T \in \mathcal{T}_\ell\}, \tag{3.3}$$

the finite element formulation reads:

$$\text{Find } u^\star_\ell \in \mathcal{X}_\ell \quad \text{such that} \quad b(u^\star_\ell, v_\ell) = F(v_\ell) \quad \text{for all } v_\ell \in \mathcal{X}_\ell. \tag{3.4}$$

Existence and uniqueness of $u^\star_\ell$ follow again from the Lax–Milgram lemma. Note that (3.4) leads to a *nonsymmetric*, yet *positive definite* linear system of equations. To derive an optimal nonsymmetric algebraic solver, we follow the constructive proof of the Lax–Milgram lemma and reduce the discrete formulations (3.4) to symmetric problems by employing the so-called Zarantonello symmetrization (sometimes referred to as Banach–Picard fixed-point iteration). To this end, we define the bilinear form associated with the principal part of the PDE by

$$a(u, v) \coloneqq \langle \boldsymbol{A}\nabla u \,,\, \nabla v\rangle_\Omega \quad \text{for all } u, v \in \mathcal{X}. \tag{3.5}$$

Note that $a(\cdot, \cdot)$ is continuous and elliptic on $\mathcal{X}$ and consult Section 3.2 for details. For a given damping parameter $\delta > 0$, define the Zarantonello mapping $\Phi_\ell(\delta; \cdot) \colon \mathcal{X}_\ell \to \mathcal{X}_\ell$ by

$$a(\Phi_\ell(\delta; u_\ell), v_\ell) = a(u_\ell, v_\ell) + \delta\big[F(v_\ell) - b(u_\ell, v_\ell)\big] \quad \text{for all } v_\ell \in \mathcal{X}_\ell; \tag{3.6}$$

see [Zar60] or [Zei90a, Section 25.4]. The Riesz–Fischer theorem (and also the Lax–Milgram lemma) proves existence and uniqueness of $\Phi_\ell(\delta; u_\ell) \in \mathcal{X}_\ell$, i.e., the Zarantonello operator is well-defined. In particular, $u^\star_\ell = \Phi(\delta; u^\star_\ell)$ is the only fixed point of $\Phi(\delta; \cdot)$ for any $\delta > 0$. Moreover, choosing δ suitably small will lead to a contractive method to approximate $u^\star_\ell$ in the spirit of the Banach fixed point theorem with respect to the $a(\cdot, \cdot)$-induced energy norm $|||v||| \coloneqq a(v, v)^{1/2}$. At this point, it thus remains to treat a symmetric, positive definite (SPD) linear system of equations corresponding to (3.6), that can be solved iteratively in practice for instance by the use of either a conjugate gradient (CG) method with an optimal preconditioner, see e.g., [CNX12], or an optimal geometric multigrid (MG) solver, see e.g., [WZ17; IMPS24].

The proposed adaptive strategy of this work, hereafter referred to as AISFEM, begins with the initial guess $u_0^{0,0} \coloneqq u_0^{0,\underline{j}} \coloneqq u_0^{0,\star} \coloneqq 0 \in \mathcal{X}_0$ associated to a coarse mesh $\mathcal{T}_0$. Finite element approximations $u_\ell^{k,j} \in \mathcal{X}_\ell$ are successively computed, where $\ell \in \mathbb{N}_0$ is the mesh-refinement index of the ℓ-th adaptively refined mesh. More precisely, $u_\ell^{k,j}$ is obtained after j algebraic solver steps in the k-th step of the Zarantonello symmetrization approximating the unique $u_\ell^{k,\star} \coloneqq \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}}) \in \mathcal{X}_\ell$,

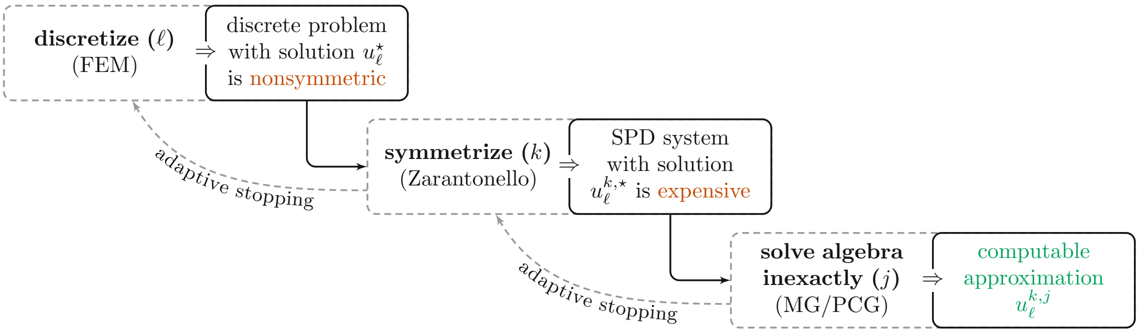


Figure 3.1: Schematic view of the AISFEM algorithm components.

where $u_\ell^{k-1,\underline{j}} \in \mathcal{X}_\ell$ denotes the final approximation of $u_\ell^{k-1,\star}$ when the algebraic solver is *adaptively* terminated. In particular, our analysis provides stopping criteria for the algebraic solver as well as the (perturbed) Zarantonello symmetrization. We give a schematic view of our approach in Figure 3.1; see Algorithm 3A in Section 3.3 below for the formal statement.

Overall, the adaptive strategy thus leads to a triple index set

$$\mathcal{Q} := \{(\ell, k, j) \in \mathbb{N}_0^3 \mid u_\ell^{k,j} \text{ is used by the AISFEM Algorithm 3A}\}, \tag{3.7}$$

equipped with the natural lexicographic order $|\cdot,\cdot,\cdot|$. This enables us to present the main contributions of this work: First, in the spirit of [GHPS21; HPSV21], we prove that the quasi-error

$$\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}) \quad \text{for all } (\ell, k, j) \in \mathcal{Q}, \tag{3.8}$$

which is the sum of the overall error plus the algebraic solver error plus the residual error estimator, is linearly convergent with respect to the order of $\mathcal{Q}$, i.e., $|\ell', k', j'| < |\ell, k, j|$ means that $u_{\ell'}^{k',j'}$ is computed earlier than $u_\ell^{k,j}$ within the (sequential) adaptive loop and $|\ell, k, j| - |\ell', k', j'| \in \mathbb{N}_0$ is the overall number of discretization, symmetrization, and algebraic solver steps in between. In explicit terms, Theorem 3.7 proves the existence of constants $C_{\text{lin}} > 0$ and $0 < q_{\text{lin}} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ such that, for all $(\ell, k, j), (\ell', k', j') \in \mathcal{Q}$ with $|\ell, k, j| > |\ell', k', j'|$ and $\ell' \geq \ell_0$, there holds that

$$\Delta_\ell^{k,j} \leq C_{\text{lin}} q_{\text{lin}}^{|\ell,k,j| - |\ell',k',j'|} \Delta_{\ell'}^{k',j'}. \tag{3.9}$$

The threshold level $\ell_0 \in \mathbb{N}_0$ arises from the lack of Galerkin orthogonality with respect to the $a(\cdot,\cdot)$-induced energy norm leading to a more involved analysis. Second, as shown in Corollary 3.8, this implies that, for any $s > 0$, there holds the equivalence

$$\sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \, \Delta_\ell^{k,j} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \, \Delta_\ell^{k,j} < \infty. \tag{3.10}$$

The interpretation of (3.10) is that the AISFEM algorithm leads to algebraic convergence rate $s > 0$ with respect to the degrees of freedom (finite left-hand side) if and only if it leads to algebraic convergence rate s with respect to the overall computational cost (finite right-hand side), i.e., with respect to the computational time. Third, extending available results from the literature [CN12; FFP14; BHP17], Theorem 3.9 proves that, for sufficiently small adaptivity parameters, the proposed algorithm has optimal complexity (which follows from optimal rates with respect to the degrees of freedom and (3.10)). Finally, we admit that the proposed strategy hinges crucially on the appropriate (sufficiently small) choice of the Zarantonello parameter $\delta > 0$ in (3.6) as well as on the parameter $\lambda_{\mathrm{alg}} > 0$ in the stopping criterion for the algebraic solver in Algorithm 3A(i.b.II) below. If these parameters are chosen too large, the proposed method may fail to converge. Besides this restriction, linear convergence (3.9) is guaranteed for any choice of the other adaptivity parameters $\lambda_{\mathrm{sym}}, \theta, C_{\mathrm{mark}}$ (see Algorithm 3A below).

Outline

The remainder of the chapter is organized as follows. Section 3.2 focuses on the setting and underlying assumptions. In Section 3.3, we present the AISFEM algorithm in full detail and highlight some of its properties. The main results of this work are presented in Section 3.4, the proofs of which are given in Section 3.5. Numerical experiments in Section 3.6 underline the theoretical results, before the short Section 3.7 concludes our results and outlines future work. Throughout, $A \lesssim B$ denotes $A \leq c\, B$ with a generic constant $c > 0$ that is independent of the discretization, but may depend on all problem parameters. Moreover, $A \simeq B$ abbreviates $A \lesssim B \lesssim A$.

3.2 Preliminaries

In this section, we state all prerequisites to formulate the AISFEM algorithm (Algorithm 3A in Section 3.3 below). In particular, we collect the contraction properties of the Zarantonello symmetrization, the algebraic solver, the mesh-refinement strategy, and the required properties of the *a posteriori* error estimator.

3.2.1 Abstract formulation of the model problem

According to the Rellich compactness theorem [KJF77, Theorem 5.8.2], $\langle \mathcal{K} u\,,\, v\rangle \coloneqq \langle \boldsymbol{b} \cdot \nabla u + cu\,,\, v\rangle_\Omega$ defines a compact linear operator $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}'$, where we recall that $\mathcal{X}' = H^{-1}(\Omega)$ is the dual space of $\mathcal{X} = H^1_0(\Omega)$. With this notation, the weak formulation (3.2) takes the more abstract form

$$b(u^\star, v) = a(u^\star, v) + \langle \mathcal{K} u^\star\,,\, v\rangle = F(v) \quad \text{for all } v \in \mathcal{X}. \tag{3.11}$$

Since $b(\cdot,\cdot)$ is continuous and elliptic on $\mathcal{X}$, i.e., there exists $\alpha_0 > 0$ such that

$$\alpha_0\, \|u\|^2_{\mathcal{X}} \leq b(u,u) \quad \text{for all } u \in \mathcal{X}, \tag{3.12}$$

a simple compactness argument proves that also the principal part $a(\cdot,\cdot)$ is elliptic, i.e., there exists $\alpha_0' > 0$ such that

$$\alpha_0'\, \|u\|^2_{\mathcal{X}} \leq a(u,u) \quad \text{for all } u \in \mathcal{X}; \tag{3.13}$$

see, e.g. [BHP17, Remark 3]. In particular, $a(\cdot,\cdot)$ is a scalar product on $\mathcal{X}$ and the $a(\cdot,\cdot)$-induced energy norm $|||v|||^2 = a(v,v)$ is an equivalent norm on $\mathcal{X}$, i.e., $|||v||| \simeq \|v\|_{\mathcal{X}}$ for all $v \in \mathcal{X}$.

Consequently, $b(\cdot,\cdot)$ is also elliptic and continuous with respect to $|||\cdot|||$, i.e., there exist (in practice unknown) constants $0 < \alpha \le L < \infty$ such that

$$\alpha\, |||u|||^2 \le b(u,u) \quad \text{and} \quad |b(u,v)| \le L\, |||u|||\, |||v||| \quad \text{for all } u, v \in \mathcal{X}. \tag{3.14}$$

While this setting already guarantees the Céa-type quasi-optimality of Galerkin solutions $u_\ell^\star \in \mathcal{X}_\ell \subset \mathcal{X}$ to (3.4), i.e.,

$$|||u^\star - u_\ell^\star||| \le C_{\text{Céa}} \min_{v_\ell \in \mathcal{X}_\ell} |||u^\star - v_\ell||| \quad \text{with} \quad C_{\text{Céa}} \coloneqq L/\alpha, \tag{3.15}$$

we recall from [BHP17, Theorem 20] that adaptivity improves the constant $C_{\text{Céa}}$ in the Céa-type estimate (3.15): If $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ and $|||u^\star - u_\ell^\star||| \to 0$ as $\ell \to \infty$, then (3.15) holds with a constant $1 \le C_\ell \le L/\alpha$ and $C_\ell \to 1$ as $\ell \to \infty$.

Remark 3.1. *The contractive Zarantonello symmetrization and hence the results of this work hold in an abstract framework beyond that of the introduction in Section 3.1. More precisely, the analysis allows for an abstract separable Hilbert space $\mathcal{X}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_{\mathcal{X}}$ and a weak formulation (3.11), where $a(\cdot,\cdot)$ is a Hermitian and continuous sesquilinear form on $\mathcal{X}$ and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}'$ is a compact linear operator such that $b(\cdot,\cdot)$ is elliptic and continuous on $\mathcal{X}$. Provided that a contractive algebraic solver is used (see Section 3.2.5), the analysis thus also applies to other boundary conditions (e.g., mixed Dirichlet–Neumann–Robin instead of homogeneous Dirichlet boundary conditions used in the introduction).*

3.2.2 Mesh refinement

From now on, let $\mathcal{T}_0$ be a given conforming triangulation of $\Omega \subset \mathbb{R}^d$ with $d \ge 1$ which is admissible in the sense of [Ste08] for $d \ge 3$. For mesh refinement, we employ newest vertex bisection (NVB); see [AFF+15] for $d = 1$, [Ste08] for $d \ge 2$ and [KPP13] for $d = 2$ with non-admissible $\mathcal{T}_0$. For each triangulation $\mathcal{T}_H$ and marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, let $\mathcal{T}_h \coloneqq \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ be the coarsest conforming triangulation where all $T \in \mathcal{M}_H$ have been refined, i.e., $\mathcal{M}_H \subseteq \mathcal{T}_H \backslash \mathcal{T}_h$. We write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ results from $\mathcal{T}_H$ by finitely many steps of refinement and, for $N \in \mathbb{N}_0$, we write $\mathcal{T}_h \in \mathbb{T}_N(\mathcal{T}_H)$ if $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ and $\#\mathcal{T}_h - \#\mathcal{T}_H \le N$. To abbreviate notation, let $\mathbb{T} \coloneqq \mathbb{T}(\mathcal{T}_0)$. Throughout, each triangulation $\mathcal{T}_H \in \mathbb{T}$ is associated with a finite-dimensional finite element space $\mathcal{X}_H \subset \mathcal{X}$, see (3.3), and refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ implies nestedness $\mathcal{X}_H \subseteq \mathcal{X}_h \subset \mathcal{X}$.

Within the setting of AFEM, we will work with a hierarchy $\{\mathcal{T}_\ell\}_{\ell \in \mathbb{N}_0}$ generated by NVB refinements from the initial mesh $\mathcal{T}_0$.

3.2.3 *A posteriori* error estimator and axioms of adaptivity

For $\mathcal{T}_H \in \mathbb{T}$, let

$$\eta_H(T;\cdot)\colon \mathcal{X}_H \to \mathbb{R}_{\ge 0} \quad \text{for all } T \in \mathcal{T}_H \tag{3.16}$$

be the local contributions of some computable error estimator. We define

$$\eta_H(\mathcal{U}_H; v_H) \coloneqq \Big(\sum_{T \in \mathcal{U}_H} \eta_H(T; v_H)^2 \Big)^{1/2} \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \text{ and } v_H \in \mathcal{X}_H.$$

To abbreviate notation, let $\eta_H(v_H) := \eta_H(\mathcal{T}_H; v_H)$. Furthermore, we suppose that η_H satisfies the following *axioms of adaptivity* from [CFPP14] with constants $C_{\mathrm{stab}}, C_{\mathrm{rel}}, C_{\mathrm{drel}} > 0$ and $0 < q_{\mathrm{red}} < 1$ only depending on the dimension d, the polynomial degree m, and shape regularity of $\mathcal{T}_0$:

(A1) **stability:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, all $v_h \in \mathcal{X}_h$ and all $v_H \in \mathcal{X}_H$, and every $\mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h$, it holds that

$$|\eta_h(\mathcal{U}_H, v_h) - \eta_H(\mathcal{U}_H, v_H)| \le C_{\mathrm{stab}} \, |||v_h - v_H|||.$$

(A2) **reduction:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, and all $v_H \in \mathcal{X}_H$, it holds that

$$\eta_h(\mathcal{T}_h \setminus \mathcal{T}_H, v_H) \le q_{\mathrm{red}} \, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h, v_H).$$

(A3) **reliability:** For all $\mathcal{T}_H \in \mathbb{T}$, the exact solutions $u^\star \in \mathcal{X}$ of (3.2) and $u_H^\star \in \mathcal{X}_H$ of (3.4) satisfy that

$$|||u^\star - u_H^\star||| \le C_{\mathrm{rel}} \, \eta_H(u_H^\star).$$

(A4) **discrete reliability:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, the corresponding exact discrete solutions satisfy that

$$|||u_h^\star - u_H^\star||| \le C_{\mathrm{drel}} \, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h, u_H^\star).$$

We note that these axioms (A1)–(A4) are satisfied for the standard residual error estimators; see Section 3.6 below for the model problem (3.1) from the introduction.

3.2.4 Contractive Zarantonello symmetrization

Recall $0 < \alpha \le L$ from (3.14). It is well known [Zei90b, Section 25.4] that the Zarantonello mapping $\Phi_H(\delta; \cdot)$ introduced in (3.6) is a contraction for sufficiently small $\delta > 0$, i.e., for $0 < \delta < 2\alpha/L^2$. Indeed, for all $u_H, w_H \in \mathcal{X}_H$, there holds

$$|||\Phi_H(\delta; u_H) - \Phi_H(\delta; w_H)||| \le q[\delta] \, |||u_H - w_H||| \quad \text{with} \quad q[\delta] := 1 - \delta(2\alpha - \delta L^2) < 1. \tag{3.17}$$

Theoretically, $\delta^\star := \alpha/L^2$ minimizes the expression in (3.17) resulting in $q[\delta^\star] = 1 - \alpha^2/L^2$; see, e.g., [HW20b].

3.2.5 Contractive algebraic solver

We assume that we have at hand an iterative algebraic solver with iteration step $\Psi_H : \mathcal{X}' \times \mathcal{X}_H \to \mathcal{X}_H$. This means, given a linear and continuous functional $G \in \mathcal{X}'$ and an approximation $w_H \in \mathcal{X}_H$ of the unique solution $w_H^\star \in \mathcal{X}_H$ to

$$a(w_H^\star, v_H) = G(v_H) \quad \text{for all } v_H \in \mathcal{X}_H, \tag{3.18}$$

the algebraic solver returns an improved $\Psi_H(G; w_H) \in \mathcal{X}_H$ in the sense that there exists a constant $0 < q_{\mathrm{ctr}} < 1$, which is independent of G and $\mathcal{X}_H$, such that

$$|||w_H^\star - \Psi_H(G; w_H)||| \le q_{\mathrm{ctr}} \, |||w_H^\star - w_H|||. \tag{3.19}$$

To simplify notation when the right-hand side G is complicated or lengthy (as for the Zarantonello iteration (3.6)), we shall write $\Psi_H(w_H^\star;\cdot)$ instead of $\Psi_H(G;\cdot)$, even though $w_H^\star$ is unknown and will never be computed.

In the framework of AFEM, possible examples for such contractive solvers include optimally preconditioned conjugate gradient methods or optimal geometric multigrid methods, see, e.g., [CNX12] or [WZ17], respectively, for approaches focused on lowest-order discretizations and [IMPS24] for an optimal multigrid method which is also robust with respect to the polynomial degree.

3.3 Completely adaptive algorithm

In the following, we formulate an inexact adaptive iteratively symmetrized finite element method (AISFEM) in the spirit of [HPSV21]. For ease of presentation, we make the following conventions: Algorithm 3A defines certain terminal indices $\underline{\ell}$, $\underline{k}[\ell]$, $\underline{j}[\ell,k]$, indicated by underlining. We shall omit the arguments of $\underline{k}$ and $\underline{j}$ if these are clear from the context, e.g., we simply write

$$u_\ell^{k,\underline{j}} := u_\ell^{k,\underline{j}[\ell,k]} \quad \text{and} \quad u_\ell^{\underline{k},\underline{j}} := u_\ell^{\underline{k}[\ell],\underline{j}[\ell,\underline{k}[\ell]]}, \quad \text{etc.}$$

A similar convention will be used for triple indices, e.g., $(\ell,k,\underline{j}) = (\ell,k,\underline{j}[\ell,k])$, etc.

Algorithm 3A: adaptive iteratively symmetrized finite element method (AISFEM)

Input: Initial triangulation $\mathcal{T}_0$, initial guess $u_0^{0,0} := u_0^{0,\underline{j}} := 0$, marking parameters $0 < \theta \le 1$ and $C_{\mathrm{mark}} \ge 1$, solver parameters $\lambda_{\mathrm{sym}}, \lambda_{\mathrm{alg}} > 0$, and damping parameter $\delta > 0$.
Loop: For $\ell = 0, 1, 2, \dots$, repeat the following steps (i)–(iv):

- (i) For all $k = 1, 2, 3, \dots$, repeat the following steps (a)–(d):
  - (a) Define $u_\ell^{k,0} := u_\ell^{k-1,\underline{j}}$ and, for purely theoretical reasons, $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$.
  - (b) For all $j = 1, 2, 3, \dots$ repeat the following steps (I)–(II):
    - (I) Compute $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$ and $\eta_\ell(T; u_\ell^{k,j})$ for all $T \in \mathcal{T}_\ell$.
    - (II) Terminate j-loop if $|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\mathrm{alg}} \big[\lambda_{\mathrm{sym}} \eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|||\big]$.
  - (c) Upon termination of the j-loop, define $\underline{j}[\ell,k] := j$.
  - (d) Terminate k-loop if $|||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \le \lambda_{\mathrm{sym}}\, \eta_\ell(u_\ell^{k,\underline{j}})$.
- (ii) Upon termination of the k-loop, define $\underline{k}[\ell] := k$.
- (iii) Determine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of up to the constant C_{mark} minimal cardinality satisfying $\theta\, \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 \le \eta_\ell(\mathcal{M}_\ell; u_\ell^{\underline{k},\underline{j}})^2$.
- (iv) Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and define $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{j}} := u_{\ell+1}^{0,\star} := u_\ell^{\underline{k},\underline{j}}$.

Output: Discrete approximations $u_\ell^{k,j}$ and corresponding error estimators $\eta_\ell(u_\ell^{k,j})$.

Remark 3.2. *To give an interpretation of the stopping criteria in Step* (i.b.II) *and Step* (i.d) *of Algorithm 3A, we note the following: Since the algebraic solver is contractive* (3.19)*, the term* $|||u_\ell^{k,j} - u_\ell^{k,j-1}|||$ *provides a posteriori error control of the algebraic error* $|||u_\ell^{k,\star} - u_\ell^{k,j}|||$*, i.e.,*

$$|||u_\ell^{k,\star} - u_\ell^{k,j}||| \le \frac{q_{\rm ctr}}{1 - q_{\rm ctr}} |||u_\ell^{k,j} - u_\ell^{k,j-1}|||.$$

Moreover, for sufficiently small $\lambda_{\rm alg} > 0$ *and ongoing Zarantonello iterations, also the perturbed Zarantonello symmetrization is a contraction; see Lemma 3.10 below. With the same reasoning as for the algebraic solver, the term* $|||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k-1},\underline{j}}||| = |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{k,0}|||$ *thus provides a posteriori error control of the symmetrization error* $|||u_\ell^\star - u_\ell^{k,\star}||| \approx |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||$ *(at least if* $1 \le k < \underline{k}[\ell]$*). With this understanding and the interpretation that the error estimator* $\eta_\ell(u_\ell^{k,j})$ *controls the discretization error* $|||u^\star - u_\ell^\star|||$ *(which is indeed true for* $u_\ell^{k,j} = u_\ell^{\underline{k},\underline{j}}$*), the heuristics behind the stopping criteria is as follows: We stop the algebraic solver in Algorithm 3A*(i.b.II) *provided that the algebraic error* $|||u_\ell^{k,\star} - u_\ell^{k,j}|||$ *is of the level of the discretization error plus the symmetrization error. Moreover, we stop the (perturbed) Zarantonello symmetrization in Algorithm 3A*(i.d) *provided that the symmetrization error* $|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||$ *is of the level of the discretization error. Up to the factors* $\lambda_{\rm alg}$ *and* $\lambda_{\rm sym}$*, this ensures that all three error sources of* $|||u^\star - u_\ell^{\underline{k},\underline{j}}|||$ *are equibalanced.*

For the analysis of Algorithm 3A, we recall that the set $\mathcal{Q}$ from (3.7) is given by

$$\mathcal{Q} := \{(\ell, k, j) \in \mathbb{N}_0^3 \colon u_\ell^{k,j} \text{ is used in Algorithm 3A}\}.$$

Together with this set, we define

$$\begin{aligned}
\underline{\ell} &:= \sup\{\ell \in \mathbb{N}_0 \colon (\ell, 0, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, && && \text{(3.20a)}\\
\underline{k}[\ell] &:= \sup\{k \in \mathbb{N}_0 \colon (\ell, k, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, && \text{whenever } (\ell, 0, 0) \in \mathcal{Q}, && \text{(3.20b)}\\
\underline{j}[\ell, k] &:= \sup\{j \in \mathbb{N}_0 \colon (\ell, k, j) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, && \text{whenever } (\ell, k, 0) \in \mathcal{Q}. && \text{(3.20c)}
\end{aligned}$$

Note that these definitions are consistent with that of Algorithm 3A, but also cover the cases that the ℓ-loop, the k-loop, or the j-loop in the algorithm do not terminate, respectively. We note that formally $\#\mathcal{Q} = \infty$ and hence either $\underline{\ell} = \infty$ or $\underline{k}[\underline{\ell}] = \infty$ or $\underline{j}[\underline{\ell}, \underline{k}[\underline{\ell}]] = \infty$, where the latter case is excluded by Lemma 3.3.

On $\mathcal{Q}$, we define a total order by

$$(\ell', k', j') \le (\ell, k, j) \quad \Longleftrightarrow \quad u_{\ell'}^{k',j'} \text{ is computed in Algorithm 3A not later than } u_\ell^{k,j}.$$

Furthermore, we introduce the total step counter $|\cdot,\cdot,\cdot|$, defined for all $(\ell, k, j) \in \mathcal{Q}$, by

$$|\ell, k, j| := \#\{(\ell', k', j') \in \mathcal{Q} \colon (\ell', k', j') \le (\ell, k, j)\} \in \mathbb{N}_0. \tag{3.21}$$

Our first observation is that the algebraic solver in the innermost loop of Algorithm 3A always terminates.

Lemma 3.3. *Independently of the adaptivity parameters θ, λ_{sym}, and λ_{alg}, the j-loop of Algorithm 3A always terminates, i.e., $\underline{j}[\ell, k] < \infty$ for all $(\ell, k, 0) \in \mathcal{Q}$.*

Proof. Let $(\ell, k, 0) \in \mathcal{Q}$. We argue by contradiction and assume that the stopping criterion in Algorithm 3A(i.b.II) always fails and hence $\underline{j}[\ell, k] = \infty$. By assumption (3.19), the algebraic solver is contractive and hence convergent with limit $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$. Moreover, by failure of the stopping criterion in Algorithm 3A(i.b.II), we thus obtain that

$$\eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}||| \lesssim |||u_\ell^{k,j} - u_\ell^{k,j-1}||| \xrightarrow{j\to\infty} 0.$$

This yields $|||u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}||| = 0$. Consequently, $u_\ell^{k-1,\underline{j}}$ is a fixed point of $\Phi_\ell(\delta; \cdot)$, cf. Algorithm 3A(i.a), and hence $u_\ell^{k-1,\underline{j}} = u_\ell^\star$ by uniqueness of the fixed point. In particular, the initial guess $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}} = u_\ell^{k,\star}$ is already the exact solution of the linear Zarantonello system and hence the algebraic solver guarantees that $u_\ell^{k,j} = u_\ell^{k,\star}$ for all $j \in \mathbb{N}_0$. Consequently, the stopping criterion in Algorithm 3A(i.b.II) will be satisfied for $j = 1$. This contradicts our assumption, and hence we conclude that $\underline{j}[\ell, k] < \infty$. □

Remark 3.4. *For the mathematical tractability, we formulated Algorithm 3A in a way that $\#\mathcal{Q} = \infty$. Any practical implementation will aim to provide a sufficiently accurate approximation $u_\ell^{k,j}$ in finite time. More precisely, Algorithm 3A will then be terminated after Algorithm 3A*(i.b.II) *if*

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) + |||u_{\underline{\ell}}^{\underline{k},\underline{j}} - u_{\underline{\ell}}^{\underline{k}-1,\underline{j}}||| + |||u_{\underline{\ell}}^{\underline{k},\underline{j}} - u_{\underline{\ell}}^{\underline{k},\underline{j}-1}||| \le \tau \tag{3.22}$$

where $\tau > 0$ is a user-specified tolerance. For $\tau = 0$, finite termination yields that $u_{\underline{\ell}}^{\underline{k},\underline{j}} = u^\star$ with $\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) = 0$. To see this, note that (3.22) *implies $u_{\underline{\ell}}^{\underline{k},\star} = u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^{\underline{k},\underline{j}-1}$ and $u_{\underline{\ell}}^\star = u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^{\underline{k}-1,\underline{j}}$ by uniqueness of the fixed point of the contractive solver and the contractive Zarantonello symmetrization, respectively. Finally, the first summand in* (3.22) *states $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = \eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) = 0$ and hence $u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^\star = u^\star$ by reliability* (A3) *of the estimator.*

Remark 3.5. *Up to the algebraic stopping criterion in Algorithm 3A*(i.b.II)*, the AISFEM algorithm coincides with the adaptive algorithm from [HPSV21], where the (perturbed) Zarantonello iteration is employed for an adaptive iteratively linearized finite element method for the solution of an energy minimization problem with strongly monotone nonlinearity in the corresponding Euler–Lagrange equations. However, the present analysis is much more refined than that of [HPSV21]:*

(i) To guarantee full linear convergence, [HPSV21, Theorem 4] requires θ sufficiently small, λ_{sym} sufficiently small with respect to θ, and λ_{alg} sufficiently small with respect to λ_{sym}. In contrast, the present analysis proves full linear convergence for arbitrary $0 < \theta \le 1$ and $0 < \lambda_{\mathrm{sym}} \le 1$, and only requires λ_{alg} to be sufficiently small to preserve the contraction of the perturbed Zarantonello iteration (see Lemma 3.10 below in comparison to [HPSV21, Lemma 6]).

(ii) Despite the linear model problem, our analytical setting is more involved: the compact perturbation in (3.11) *prevents the use of energy arguments that guarantee a Pythagorean-type identity in terms of the energy error (see, e.g., [HPSV21; HPW21]). Instead, we first need to exploit a priori convergence of Algorithm 3A (see Lemma 3.12) to deduce a quasi-Pythagorean*

estimate in Lemma 3.13, which then allows proving linear convergence (Theorem 3.7). As a consequence (and beyond the results of [HPSV21]), this finally yields that, for arbitrary θ and λ_{sym}, the convergence rates with respect to the number of the degrees of freedom and with respect to the overall computational work coincide (Corollary 3.8).

The following proposition provides a computable upper bound for the energy error $|||u^\star - u_\ell^{k,j}|||$. Since Algorithm 3A follows the structure of [HPSV21, Algorithm 1], the proof can be obtained analogously to [HPSV21, Proposition 2] and is thus omitted here.

Proposition 3.6 (reliable error control)**.** *Suppose that the estimator satisfies (A1) and (A3). Then, for all $(\ell, k, j) \in \mathcal{Q}$, it holds that*

$$|||u^\star - u_\ell^{k,j}||| \le C'_{\mathrm{rel}} \begin{cases} \eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}||| \\ \qquad + |||u_\ell^{k,j} - u_\ell^{k,j-1}||| & \text{if } 1 \le k \le \underline{k}[\ell] \text{ and } 1 \le j < \underline{j}[\ell,k], \\ \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| & \text{if } 1 \le k \le \underline{k}[\ell] \text{ and } j = \underline{j}[\ell,k], \\ \eta_\ell(u_\ell^{\underline{k},\underline{j}}) & \text{if } k = \underline{k}[\ell] \text{ and } j = \underline{j}[\ell,\underline{k}], \\ \eta_{\ell-1}(u_{\ell-1}^{\underline{k},\underline{j}}) & \text{if } \ell > 0 \text{ and } k = 0. \end{cases} \tag{3.23}$$

The constant $C'_{\mathrm{rel}} > 0$ depends only on C_{rel}, C_{stab}, q_{ctr}, λ_{alg}, q_{sym}, and λ_{sym}.

3.4 Main results

In the following, we formulate the main results of the present work. We refer to Section 3.5 for the proofs and Section 3.6 for numerical experiments, which underline these theoretical results. First, recall from (3.17) that a sufficiently small parameter $\delta > 0$ ensures contraction of the Zarantonello mapping and hence

$$|||u_\ell^\star - u_\ell^{k,\star}||| \le q_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \text{for all } (\ell,k,0) \in \mathcal{Q} \tag{3.24}$$

with $0 < q_{\mathrm{sym}} < 1$. The following theorem states full linear convergence of the quasi-error.

Theorem 3.7: full linear convergence of AISFEM

Suppose that $\delta > 0$ is sufficiently small and that the estimator satisfies (A1)–(A3). Choose $\lambda_{\mathrm{alg}}^\star > 0$ depending only on q_{ctr} from (3.19) and q_{sym} from (3.24) such that

$$0 < \overline{q}_{\mathrm{sym}} := \frac{q_{\mathrm{sym}} + 2\,\frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}^\star}{1 - 2\,\frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}^\star} < 1. \tag{3.25}$$

Then, for arbitrary $0 < \theta \le 1$ and $0 < \lambda_{\mathrm{sym}} \le 1$, there exists $0 < \lambda'_{\mathrm{alg}} \le \lambda^\star_{\mathrm{alg}}$ such that Algorithm 3A, for all $0 < \lambda_{\mathrm{alg}} \le \lambda'_{\mathrm{alg}}$, guarantees full linear convergence: There exist constants $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le \underline{\ell}$ such that the quasi-error

$$\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}) \quad \textit{for all } (\ell,k,j) \in \mathcal{Q} \tag{3.26}$$

satisfies that, for all $(\ell,k,j),(\ell',k',j') \in Q$ *with* $|\ell,k,j| > |\ell',k',j'|$ *and* $\ell' \geq \ell_0$,

$$\Delta_\ell^{k,j} \leq C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\, \Delta_{\ell'}^{k',j'}. \tag{3.27}$$

The constants C_{lin} *and* q_{lin} *depend only on* C_{stab}, C_{rel}, q_{red}, q_{sym}, q_{ctr}, θ, λ_{sym}, λ_{alg}, *and* $C_{\text{Céa}} = L/\alpha$, *while the index* ℓ_0 *depends on* $u^\star$ *and the sequence* $(u_\ell^\star)_{\ell\in\mathbb{N}_0}$.

While the proof of Theorem 3.7 is postponed to Section 3.5.5, we shall immediately prove the following important consequence of Theorem 3.7: Algorithm 3A guarantees that rates with respect to the number of degrees of freedom coincide with rates with respect to the overall computational cost.

Corollary 3.8. *Let* $s > 0$. *Under the assumptions of Theorem 3.7, the output of Algorithm 3A guarantees that*

$$M(s) := \sup_{\substack{(\ell,k,j)\in Q\\ \ell\geq\ell_0}} (\#\mathcal{T}_\ell)^s\, \Delta_\ell^{k,j} \leq \sup_{\substack{(\ell,k,j)\in Q\\ \ell\geq\ell_0}} \Big(\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \leq \frac{C_{\mathrm{lin}}}{(1-q_{\mathrm{lin}}^{1/s})^s}\, M(s). \tag{3.28}$$

This yields the equivalence

$$\sup_{(\ell,k,j)\in Q} (\#\mathcal{T}_\ell)^s\, \Delta_\ell^{k,j} < \infty \quad\Longleftrightarrow\quad \sup_{(\ell,k,j)\in Q} \Big(\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\leq|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} < \infty. \tag{3.29}$$

Proof. The lower bound in (3.28) is obvious. To prove the upper bound, without loss of generality, we may assume that $M(s) < \infty$. By definition of $M(s)$, it follows that

$$\#\mathcal{T}_{\ell'} \leq M(s)^{1/s} [\Delta_{\ell'}^{k',j'}]^{-1/s} \quad \text{for } (\ell',k',j') \in Q \text{ with } \ell' \geq \ell_0. \tag{3.30}$$

For $|\ell,k,j| \geq |\ell',k',j'|$ and $\ell' \geq \ell_0$, full linear convergence (3.27) can be rewritten as

$$[\Delta_{\ell'}^{k',j'}]^{-1/s} \leq C_{\mathrm{lin}}^{1/s} [q_{\mathrm{lin}}^{1/s}]^{|\ell,k,j|-|\ell',k',j'|}\, [\Delta_\ell^{k,j}]^{-1/s}. \tag{3.31}$$

The geometric series yields that

$$\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} \#\mathcal{T}_{\ell'} \overset{(3.30)}{\leq} M(s)^{1/s} \sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} [\Delta_{\ell'}^{k',j'}]^{-1/s} \overset{(3.31)}{\leq} M(s)^{1/s}\, C_{\mathrm{lin}}^{1/s} \frac{1}{1-q_{\mathrm{lin}}^{1/s}}\, [\Delta_\ell^{k,j}]^{-1/s}.$$

Rearranging this estimate, we see that

$$\Big(\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \leq M(s)\, C_{\mathrm{lin}} \frac{1}{(1-q_{\mathrm{lin}}^{1/s})^s}.$$

Taking the supremum over all $(\ell, k, j) \in Q$ with $\ell \geq \ell_0$, we prove the second estimate in (3.28). Moreover,

$$Q \backslash \{(\ell, k, j) \in Q : \ell \geq \ell_0\} = \{(\ell, k, j) \in Q : \ell < \ell_0\} \quad \text{is finite,}$$

i.e., the sets over which we compute the suprema in (3.28)–(3.29) differ only by finitely many index triples. This and (3.28) thus prove the equivalence in (3.29). □

To present our second main result on quasi-optimal computational cost, we first introduce the notion of approximation classes. For $\mathcal{T} \in \mathbb{T}$ and $s > 0$, define

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T})} \coloneqq \sup_{N \in \mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T})} \big[|||u^\star - u^\star_{\mathrm{opt}}||| + \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \big] \Big), \tag{3.32}$$

with $u^\star_{\mathrm{opt}}$ and η_{opt} denoting the exact discrete solution and the estimator on the optimal triangulation $\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T})$, respectively. When (3.32) is finite, this means that a decrease of the error plus estimator with rate s is possible along optimal meshes obtained by refining $\mathcal{T}$.

Theorem 3.9: optimal computational complexity

Suppose that $\delta > 0$ is sufficiently small and that the estimator satisfies (A1)–(A4). Let $0 < \theta < \theta^\star \coloneqq (1 + C_{\mathrm{stab}}^2 C_{\mathrm{drel}}^2)^{-1} < 1$. Define $\lambda^\star_{\mathrm{sym}} \coloneqq \min\{1, C_{\mathrm{alg}}^{-1} C_{\mathrm{stab}}^{-1}\}$, where

$$C_{\mathrm{alg}} \coloneqq \frac{1}{1 - q_{\mathrm{sym}}} \Big(\frac{2\, q_{\mathrm{ctr}}}{1 - q_{\mathrm{ctr}}} \lambda^\star_{\mathrm{alg}} + q_{\mathrm{sym}} \Big). \tag{Calg}$$

Choose $0 < \lambda_{\mathrm{sym}} < \lambda^\star_{\mathrm{sym}}$ sufficiently small such that

$$0 < \theta_{\mathrm{mark}} \coloneqq \Big(\frac{\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}}{1 - \lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}} \Big)^2 < \theta^\star. \tag{3.33}$$

Then, for any $0 < \lambda_{\mathrm{alg}} \leq \lambda'_{\mathrm{alg}}$ with $\lambda'_{\mathrm{alg}} > 0$ from Theorem 3.7, Algorithm 3A guarantees, for all $s > 0$, that

$$c_{\mathrm{opt}} \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \leq \sup_{(\ell,k,j) \in Q} \Big(\sum_{\substack{(\ell',k',j') \in Q \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j}, \tag{3.34a}$$

$$\sup_{\substack{(\ell,k,j) \in Q \\ \ell \geq \ell_0}} \Big(\sum_{\substack{(\ell',k',j') \in Q \\ |\ell',k',j'| \leq |\ell,k,j| \\ \ell' \geq \ell_0}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \leq C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\}. \tag{3.34b}$$

where $\ell_0 \in \mathbb{N}$ is the index from Theorem 3.7. The constant $c_{\mathrm{opt}} > 0$ depends only on $C_{\mathrm{Céa}} = L/\alpha$, C_{stab}, C_{rel}, s, and the use of NVB refinement; the constant $C_{\mathrm{opt}} > 0$ depends only on C_{stab}, C_{drel}, C_{mark}, $C_{\mathrm{Céa}} = L/\alpha$, C'_{rel}, C_{lin}, q_{lin}, $\#\mathcal{T}_{\ell_0}$, q_{red}, λ_{sym}, q_{sym}, θ, s, and the use of NVB refinement.

In particular, this proves the equivalence

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k,j)\in Q} \Big(\sum_{\substack{(\ell',k',j')\in Q \\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} < \infty, \tag{3.35}$$

which yields optimal complexity of Algorithm 3A.

The proof is postponed to Section 3.5.6.

3.5 Proofs

3.5.1 Contraction of perturbed Zarantonello symmetrization

Recall that for $\delta < 2\,\delta^\star = 2\,\alpha/L^2$, the Zarantonello mapping is a contraction (3.17). However, Algorithm 3A does not compute $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ exactly, but relies on an approximation $u_\ell^{k,\underline{j}} \approx u_\ell^{k,\star}$. The next lemma states that, for a sufficiently small stopping parameter $\lambda_{\mathrm{alg}} > 0$ in Algorithm 3A, the Zarantonello symmetrization remains a contraction under this perturbation (up to the final iteration). Its proof essentially follows along the lines of [HPSV21, Lemma 6]. However, the present work considers a stopping criterion of the algebraic solver in Algorithm 3A(i.b.II) which allows to choose λ_{alg} independently of λ_{sym}.

Lemma 3.10. *Let $\lambda_{\mathrm{alg}}^\star > 0$ and $0 < \overline{q}_{\mathrm{sym}} < 1$ as in Theorem 3.7. Then, for all stopping parameters $0 < \lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star$ and $\lambda_{\mathrm{sym}} > 0$, it holds that*

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le \overline{q}_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \textit{for all } (\ell,k,\underline{j}) \in Q \textit{ with } 1 \le k < \underline{k}[\ell]. \tag{3.36}$$

Moreover, for $k = \underline{k}[\ell]$, it holds that

$$|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \le q_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}\,\lambda_{\mathrm{sym}}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \quad \textit{for all } (\ell,\underline{k},\underline{j}) \in Q. \tag{5.1$^+$}$$

Proof. Let $(\ell,k,\underline{j}) \in Q$ and suppose first that $1 \le k < \underline{k}[\ell]$. By using the triangle inequality and the contraction (3.24) of the unperturbed Zarantonello iteration, we obtain that

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(3.24)}{\le} q_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}|||. \tag{3.37}$$

It remains to treat the algebraic error term and to show that it is sufficiently contractive. We use the contraction (3.19) of the algebraic solver, i.e.,

$$|||u_\ell^{k,\star} - u_\ell^{k,j}||| \le q_{\mathrm{ctr}}\, |||u_\ell^{k,\star} - u_\ell^{k,j-1}||| \quad \text{for all } (\ell,k,j) \in Q \text{ with } j \ge 1, \tag{3.38}$$

the met algebraic stopping criterion in Algorithm 3A(i.b.II), and the not met stopping criterion in Algorithm 3A(i.d) to obtain that

$$\begin{aligned} |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| &\overset{(3.38)}{\le} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\, |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{\text{(i.b.II)}}{\le} \lambda_{\mathrm{alg}}\, \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \big[\lambda_{\mathrm{sym}}\,\eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}|||\big] \\ &\overset{\text{(i.d)}}{<} 2\,\lambda_{\mathrm{alg}}\, \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\, |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \le 2\,\lambda_{\mathrm{alg}}\, \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \big[|||u_\ell^\star - u_\ell^{k,\underline{j}}||| + |||u_\ell^\star - u_\ell^{k-1,\underline{j}}|||\big]. \end{aligned}$$

Combining the last estimate with (3.37) and rearranging the terms lead us to

$$|||u_\ell^\star - u_\ell^{k,j}||| \le \frac{q_{\rm sym} + 2\,\lambda_{\rm alg}\,\frac{q_{\rm ctr}}{1-q_{\rm ctr}}}{1 - 2\,\lambda_{\rm alg}\,\frac{q_{\rm ctr}}{1-q_{\rm ctr}}}\,|||u_\ell^\star - u_\ell^{k-1,j}||| \overset{(3.25)}{\le} \overline{q}_{\rm sym}\,|||u_\ell^\star - u_\ell^{k-1,j}|||.$$

This concludes the proof of (3.36).

Now suppose that $k = \underline{k}[\ell]$. By the met algebraic stopping criterion in Algorithm 3A(i.b.II) followed by the met stopping criterion of the Zarantonello iteration in Algorithm 3A(i.d), we obtain that

$$|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \overset{\text{(i.b.II)}}{\le} \lambda_{\rm alg}\,\big[\lambda_{\rm sym}\,\eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,j}|||\big] \overset{\text{(i.d)}}{\le} 2\,\lambda_{\rm alg}\,\lambda_{\rm sym}\,\eta_\ell(u_\ell^{k,j}).$$

Together with the contraction (3.38) of the algebraic solver, this yields that

$$|||u_\ell^{k,\star} - u_\ell^{k,j}||| \overset{(3.38)}{\le} \frac{q_{\rm ctr}}{1-q_{\rm ctr}}\,|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \frac{2\,q_{\rm ctr}}{1-q_{\rm ctr}}\,\lambda_{\rm alg}\,\lambda_{\rm sym}\,\eta_\ell(u_\ell^{k,j}). \tag{3.39}$$

By contraction (3.24) of the unperturbed Zarantonello iteration, we obtain that

$$\begin{aligned} |||u_\ell^\star - u_\ell^{k,j}||| &\le |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| \\ &\overset{(3.24)}{\le} q_{\rm sym}\,|||u_\ell^\star - u_\ell^{k-1,j}||| + \frac{2\,q_{\rm ctr}}{1-q_{\rm ctr}}\,\lambda_{\rm alg}\,\lambda_{\rm sym}\,\eta_\ell(u_\ell^{k,j}). \end{aligned}$$

This concludes also the proof of (5.1$^+$). □

An important consequence of the contraction (3.36) of the perturbed Zarantonello iteration is that $\underline{k}[\underline{\ell}] = \infty$ implies that the exact solution is already discrete $u^\star = u_{\underline{\ell}}^\star \in \mathcal{X}_{\underline{\ell}}$.

> **Lemma 3.11.** *Suppose that the estimator satisfies stability (A1) and reliability (A3), and that the perturbed Zarantonello iteration is contractive (3.36). Then, $\underline{\ell} < \infty$ implies that $\underline{k}[\underline{\ell}] = \infty$ as well as $u^\star = u_{\underline{\ell}}^\star$ with $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$.*

Proof. Since $\underline{j}[\underline{\ell}, \underline{k}] < \infty$ by virtue of Lemma 3.3, it follows for $\underline{\ell} < \infty$ that $\underline{k}[\underline{\ell}] = \infty$ and hence by the not met stopping criterion in Algorithm 3A(i.d) that

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^{k,\underline{j}}) < \lambda_{\rm sym}^{-1}\,|||u_{\underline{\ell}}^{k,\underline{j}} - u_{\underline{\ell}}^{k-1,\underline{j}}||| \quad \text{for all } k \in \mathbb{N}.$$

Since the perturbed Zarantonello iteration is convergent (see Lemma 3.10) with limit $u_{\underline{\ell}}^\star$ (and thus $(u_{\underline{\ell}}^{k,\underline{j}})_{k\in\mathbb{N}_0}$ is a Cauchy sequence), we infer that

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) \overset{(A1)}{\le} \eta_{\underline{\ell}}(u_{\underline{\ell}}^{k,\underline{j}}) + C_{\rm stab}\,|||u_{\underline{\ell}}^\star - u_{\underline{\ell}}^{k,\underline{j}}||| \xrightarrow{k\to\infty} 0.$$

This proves $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$, whence with reliability (A3), we conclude $u_{\underline{\ell}}^\star = u^\star$. □

3.5.2 A priori convergence

For general second-order linear elliptic PDEs, an *a priori* convergence result is required to ensure that there holds a quasi-Pythagorean estimate; see Lemma 3.13 below.

Lemma 3.12 (a priori convergence). *With $\underline{\ell} \in \mathbb{N}_0 \cup \{\infty\}$ from (3.20), define the discrete limit space $\mathcal{X}_\infty \coloneqq operatornameclosure\big(\bigcup_{\ell=0}^{\underline{\ell}} \mathcal{X}_\ell\big)$. Then, there exists $u_\infty^\star \in \mathcal{X}_\infty$ such that*

$$b(u_\infty^\star, v_\infty) = F(v_\infty) \quad \textit{for all } v_\infty \in \mathcal{X}_\infty, \tag{3.40}$$

and it holds that

$$\|u_\infty^\star - u_\ell^\star\| \to 0 \quad \textit{as } \ell \to \underline{\ell}. \tag{3.41}$$

In particular, this implies $u_{\underline{\ell}}^\star = u_\infty^\star$ if $\underline{\ell} < \infty$. Moreover, with $C_{\text{Céa}} = L/\alpha$ from (3.15), there holds the Céa-type estimate

$$|||u_\infty^\star - u_\ell^\star||| \le C_{\text{Céa}} \min_{v_\ell \in \mathcal{X}_\ell} |||u_\infty^\star - v_\ell||| \quad \textit{for all } \ell \in \mathbb{N}_0 \textit{ with } \ell \le \underline{\ell}. \tag{3.42}$$

Moreover, reliability (A3) implies that

$$|||u_\infty^\star - u_\ell^\star||| \le (C_{\text{Céa}} + 1)\, C_{\text{rel}}\, \eta_\ell(u_\ell^\star) \quad \textit{for all } \ell \in \mathbb{N}_0 \textit{ with } \ell \le \underline{\ell}. \tag{3.43}$$

Proof. Existence and uniqueness of $u_\infty^\star$ follow from the Lax–Milgram lemma. Since $u_\ell^\star \in \mathcal{X}_\ell \subseteq \mathcal{X}_\infty$ is a Galerkin approximation of $u_\infty^\star$, the Céa lemma (3.15) holds with $u^\star$ being replaced by $u_\infty^\star$, and the definition of $\mathcal{X}_\infty$ proves that

$$|||u_\infty^\star - u_\ell^\star||| \le C_{\text{Céa}} \min_{v_\ell \in \mathcal{X}_\ell} |||u_\infty^\star - v_\ell||| \xrightarrow{\ell \to \underline{\ell}} 0.$$

Reliability (3.43) follows from the triangle inequality, nestedness of spaces $\mathcal{X}_\ell \subseteq \mathcal{X}_\infty$, and the Céa lemma (3.15), since

$$|||u_\infty^\star - u_\ell^\star||| \le |||u^\star - u_\infty^\star||| + |||u^\star - u_\ell^\star||| \overset{(3.15)}{\le} (C_{\text{Céa}}{+}1)\, |||u^\star - u_\ell^\star||| \overset{(A3)}{\le} (C_{\text{Céa}}{+}1)\, C_{\text{rel}}\, \eta_\ell(u_\ell^\star).$$

This concludes the proof. □

3.5.3 Quasi-Pythagorean estimate

While symmetric PDEs satisfy a Pythagorean identity in the energy norm (with $\varepsilon = 0$ and $\ell_0 = 0$ in (3.44) below), the situation is more involved for nonsymmetric PDEs. The following result generalizes [BHP17, Lemma 18] by considering general $v_\ell \in \mathcal{X}_\ell$ and by additionally proving the lower bound in (3.44). Moreover, it is given here in terms of the *a priori* limit $u_\infty^\star$. Although the proof follows essentially that of [BHP17], we include it for the sake of completeness.

Lemma 3.13 (quasi-Pythagorean estimate). *Recall the a priori limit $u_\infty^\star \in \mathcal{X}_\infty$ from Lemma 3.12 and the compact linear operator $\mathcal{K}$ from Section 3.2.1. Then, for all $0 < \varepsilon < 1$, there exists an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le \underline{\ell}$ such that, for all $\ell_0 \le \ell \le \underline{\ell}$,*

$$\frac{1}{1+\varepsilon} |||u_\infty^\star - v_\ell|||^2 \le |||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2 \le \frac{1}{1-\varepsilon} |||u_\infty^\star - v_\ell|||^2 \quad \textit{for all } v_\ell \in \mathcal{X}_\ell. \tag{3.44}$$

Proof. The proof is split into four steps.

Step 1. If $\underline{\ell} < \infty$, Lemma 3.12 proves that $u_\infty^\star = u_{\underline{\ell}}^\star$. We choose $\ell_0 = \underline{\ell}$ and obtain that (3.44) holds with equality and $\varepsilon = 0$, since $\ell = \underline{\ell}$ and hence $u_\infty^\star = u_\ell^\star$. Consequently, (3.44) holds also for all $0 < \varepsilon < 1$. Therefore, it only remains to prove (3.44) for $\underline{\ell} = \infty$.

Step 2. Suppose $\underline{\ell} = \infty$. Let $\ell \in \mathbb{N}_0$ and $v_\ell \in \mathcal{X}_\ell$. The limit formulation (3.40) yields

$$|||u_\infty^\star - v_\ell|||^2 = |||u_\infty^\star|||^2 + |||v_\ell|||^2 - 2\,\Re a(u_\infty^\star, v_\ell) \overset{(3.40)}{=} |||u_\infty^\star|||^2 + |||v_\ell|||^2 - 2\,\Re\big[F(v_\ell) - \langle \mathcal{K}u_\infty^\star\,,\, v_\ell\rangle\big]. \tag{3.45}$$

Analogously, from the discrete formulation (3.4) and the linearity of $\mathcal{K}$, we obtain that

$$\begin{aligned} |||u_\ell^\star - v_\ell|||^2 &= |||u_\ell^\star|||^2 + |||v_\ell|||^2 - 2\,\Re a(u_\ell^\star, v_\ell) \\ &\overset{(3.4)}{=} |||u_\ell^\star|||^2 + |||v_\ell|||^2 - 2\,\Re\big[F(v_\ell) - \langle \mathcal{K}u_\ell^\star\,,\, v_\ell\rangle\big] \\ &= |||u_\ell^\star|||^2 + |||v_\ell|||^2 - 2\Re\big[F(v_\ell) - \langle \mathcal{K}u_\infty^\star\,,\, v_\ell\rangle + \langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, v_\ell\rangle\big] \end{aligned} \tag{3.46}$$

as well as

$$F(u_\ell^\star) \overset{(3.4)}{=} a(u_\ell^\star, u_\ell^\star) + \langle \mathcal{K}u_\ell^\star\,,\, u_\ell^\star\rangle = |||u_\ell^\star|||^2 + \langle \mathcal{K}u_\ell^\star\,,\, u_\ell^\star\rangle. \tag{3.47}$$

For $v_\ell = u_\ell^\star$, we see that

$$\begin{aligned} |||u_\infty^\star - u_\ell^\star|||^2 &\overset{(3.45)}{=} |||u_\infty^\star|||^2 + |||u_\ell^\star|||^2 - 2\Re\big[F(u_\ell^\star) - \langle \mathcal{K}u_\infty^\star\,,\, u_\ell^\star\rangle\big] \\ &\overset{(3.47)}{=} |||u_\infty^\star|||^2 - |||u_\ell^\star|||^2 + 2\Re\langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star\rangle. \end{aligned} \tag{3.48}$$

Summing (3.46) and (3.48), we obtain that

$$\begin{aligned} &|||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2 \\ &\quad= |||u_\infty^\star|||^2 + |||v_\ell|||^2 - 2\Re\big[F(v_\ell) - \langle \mathcal{K}u_\infty^\star\,,\, v_\ell\rangle - \langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star - v_\ell\rangle\big] \\ &\quad\overset{(3.45)}{=} |||u_\infty^\star - v_\ell|||^2 + 2\Re\langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star - v_\ell\rangle. \end{aligned} \tag{3.49}$$

Step 3. We recall from [BHP17, Lemma 17] that the convergence (3.41) of Lemma 3.12 yields that

$$e_\ell := \begin{cases} \dfrac{u_\infty^\star - u_\ell^\star}{|||u_\infty^\star - u_\ell^\star|||} & \text{if } u_\infty^\star \neq u_\ell^\star, \\ 0 & \text{otherwise} \end{cases}$$

defines a weakly convergent sequence in $\mathcal{X}_\infty$ with $e_\ell \rightharpoonup 0$ as $\ell \to \infty$. We recall that compact operators turn weak convergence into norm convergence. With the operator norm $|||\phi|||' := \sup_{v \in \mathcal{X}_\infty \setminus \{0\}} |\phi(v)|/|||v|||$ of $\phi \in \mathcal{X}_\infty'$, it thus follows that

$$|\langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star - v_\ell\rangle| \le |||\mathcal{K}e_\ell|||'\, |||u_\infty^\star - u_\ell^\star|||\, |||u_\ell^\star - v_\ell||| \quad \text{and} \quad |||\mathcal{K}e_\ell|||' \xrightarrow{\ell \to \infty} 0.$$

Given $\varepsilon > 0$, this provides an index $\ell_0 \in \mathbb{N}$ such that $|||\mathcal{K}e_\ell|||' \le \varepsilon$ for all $\ell \ge \ell_0$ and hence

$$\begin{aligned} 2\,|\Re\langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star - v_\ell\rangle| &\le 2\varepsilon\, |||u_\infty^\star - u_\ell^\star|||\, |||u_\ell^\star - v_\ell||| \\ &\le \varepsilon\big[|||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2\big]. \end{aligned} \tag{3.50}$$

Step 4. Rearranging the identity (3.49) and estimating the compact perturbation via (3.50), we obtain that

$$
\begin{aligned}
|||u_\infty^\star - v_\ell|||^2 &\overset{(3.49)}{=} |||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2 - 2\Re\langle \mathcal{K}(u_\infty^\star - u_\ell^\star)\,,\, u_\ell^\star - v_\ell\rangle \\
&\overset{(3.50)}{\leq} (1+\varepsilon)\big[|||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2\big].
\end{aligned}
$$

This proves the lower estimate in (3.44), and the upper estimate is proved analogously. □

3.5.4 Auxiliary contraction estimates

The following lemma extends [GHPS21, Lemma 10] to the present setting with a quasi-Pythagorean estimate.

Lemma 3.14 (combined discretization-symmetrization error)**.** *Recall the a priori limit $u_\infty^\star \in \mathcal{X}_\infty$ from Lemma 3.12. Suppose that the estimator satisfies (A1)–(A3). Let $\lambda_{\rm alg}^\star > 0$ and $0 < \overline{q}_{\rm sym} < 1$ be as in Theorem 3.7. Let $0 < \theta \leq 1$ and $0 < \lambda_{\rm sym} \leq 1$. Then, there exists $0 < \lambda_{\rm alg}' \leq \lambda_{\rm alg}^\star$ such that for all $0 < \lambda_{\rm alg} \leq \lambda_{\rm alg}'$ the following holds: There exists an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \leq \underline{\ell}$ and scalars $\nu > 0$ and $0 < q_{\rm lin} < 1$ such that*

$$
\Lambda_\ell^k := \big[\, |||u_\infty^\star - u_\ell^{k,\underline{j}}|||^2 + \nu\, \eta_\ell(u_\ell^{k,\underline{j}})^2\,\big]^{1/2} \quad \text{for all } (\ell,k,\underline{j}) \in \mathcal{Q} \tag{3.51}
$$

satisfies

$$
\Lambda_\ell^{k+1} \leq q_{\rm lin}\,\Lambda_\ell^k \qquad \text{for all } (\ell,k+1,\underline{j}) \in \mathcal{Q} \text{ with } \ell \geq \ell_0 \text{ and } k+1 < \underline{k}[\ell], \tag{3.52a}
$$

$$
\Lambda_{\ell+1}^0 \leq q_{\rm lin}\,\Lambda_\ell^{\underline{k}-1} \qquad \text{for all } (\ell+1,0,0) \in \mathcal{Q} \text{ with } \ell \geq \ell_0. \tag{3.52b}
$$

Proof. Let $0 < \varepsilon < 1$ as well as $\nu, \omega > 0$ be free parameters to be fixed below. The proof consists of seven steps, where most of the work is necessary to prove (3.52b).

Step 1. Lemma 3.13 provides an index $\ell_0 = \ell_0(\varepsilon)$ such that for all $\ell_0 \leq \ell \leq \underline{\ell}$ the quasi-Pythagorean estimate (3.44) holds true. For $(\ell, k+1, \underline{j}) \in \mathcal{Q}$ with $\ell_0 \leq \ell$, we get that

$$
\begin{aligned}
(\Lambda_\ell^{k+1})^2 &= |||u_\infty^\star - u_\ell^{k+1,\underline{j}}|||^2 + \nu\,\eta_\ell(u_\ell^{k+1,\underline{j}})^2 \\
&\overset{(3.44)}{\leq} (1+\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\,|||u_\ell^\star - u_\ell^{k+1,\underline{j}}|||^2 + \nu\,\eta_\ell(u_\ell^{k+1,\underline{j}})^2.
\end{aligned} \tag{3.53}
$$

Analogously, for $(\ell+1,0,0) \in \mathcal{Q}$ with $\ell \geq \ell_0$, nested iteration $u_{\ell+1}^{0,0} = u_{\ell+1}^{0,j} = u_\ell^{\underline{k},\underline{j}}$ shows that

$$
\begin{aligned}
(\Lambda_{\ell+1}^0)^2 &= |||u_\infty^\star - u_\ell^{\underline{k},\underline{j}}|||^2 + \nu\,\eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 \\
&\overset{(3.44)}{\leq} (1+\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\,|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||^2 + \nu\,\eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2.
\end{aligned} \tag{3.54}
$$

Step 2. Define $C_1 := 6\,(1 + C_{\text{Céa}})^2\, C_{\rm rel}^2$ and $C_2 := 6\,(1 + C_{\text{Céa}})^2\, C_{\rm rel}^2\, C_{\rm stab}^2$. Then, stability (A1) and reliability (3.43) prove that, for all $v_\ell \in \mathcal{X}_\ell$,

$$
\begin{aligned}
3\,|||u_\infty^\star - u_\ell^\star|||^2 &\overset{(3.43)}{\leq} 3\,(1 + C_{\text{Céa}})^2\, C_{\rm rel}^2\, \eta_\ell(u_\ell^\star)^2 \\
&\overset{(A1)}{\leq} 6\,(1 + C_{\text{Céa}})^2\, C_{\rm rel}^2\, \eta_\ell(v_\ell)^2 + 6\,(1 + C_{\text{Céa}})^2\, C_{\rm rel}^2\, C_{\rm stab}^2\, |||u_\ell^\star - v_\ell|||^2 \\
&= C_1\,\eta_\ell(v_\ell)^2 + C_2\,|||u_\ell^\star - v_\ell|||^2.
\end{aligned} \tag{3.55}
$$

Step 3. For $(\ell, k+1, \underline{j}) \in \mathcal{Q}$ with $\ell \geq \ell_0$ and $k+1 < \underline{k}[\ell]$, contraction (3.36) of the perturbed Zarantonello iteration proves that

$$|||u_\ell^{k+1,\underline{j}} - u_\ell^{k,\underline{j}}||| \leq |||u_\ell^\star - u_\ell^{k+1,\underline{j}}||| + |||u_\ell^\star - u_\ell^{k,\underline{j}}||| \overset{(3.36)}{\leq} (1 + \overline{q}_{\mathrm{sym}})\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||.$$

Define $C_3 \coloneqq (1+\overline{q}_{\mathrm{sym}})^2$. Using this with the not met stopping criterion in Algorithm 3A(i.d) for $(\ell, k+1, \underline{j}) \in \mathcal{Q}$ with $k+1 < \underline{k}[\ell]$ shows that

$$\eta_\ell(u_\ell^{k+1,\underline{j}})^2 \overset{\text{(i.d)}}{<} \lambda_{\mathrm{sym}}^{-2} |||u_\ell^{k+1,\underline{j}} - u_\ell^{k,\underline{j}}|||^2 \leq C_3 \lambda_{\mathrm{sym}}^{-2}\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2. \tag{3.56}$$

In this case, we are thus led to

$$\begin{aligned}
(\Lambda_\ell^{k+1})^2 &\overset{(3.53)}{\leq} (1-2\varepsilon)\, |||u_\infty^\star - u_\ell^\star|||^2 + 3\varepsilon\, |||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\, |||u_\ell^\star - u_\ell^{k+1,\underline{j}}|||^2 + \nu\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 \\
&\overset{(3.55)}{\leq} (1-2\varepsilon)\, |||u_\infty^\star - u_\ell^\star|||^2 + (\nu + \varepsilon\, C_1)\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 + (1 + \varepsilon(1+C_2))\, |||u_\ell^\star - u_\ell^{k+1,\underline{j}}|||^2 \\
&\overset{(3.36)}{\leq} (1-2\varepsilon)\, |||u_\infty^\star - u_\ell^\star|||^2 + (\nu + \varepsilon\, C_1)\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 + (1 + \varepsilon(1+C_2))\, \overline{q}_{\mathrm{sym}}^2\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2 \\
&\overset{(3.56)}{\leq} (1-2\varepsilon)\, |||u_\infty^\star - u_\ell^\star|||^2 + \big[(\nu + \varepsilon\, C_1) C_3 \lambda_{\mathrm{sym}}^{-2} + (1 + \varepsilon(1+C_2))\, \overline{q}_{\mathrm{sym}}^2\big]\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2.
\end{aligned}$$

Provided that

$[I]$ $\overline{q}_{\mathrm{sym}}^2 + \nu C_3 \lambda_{\mathrm{sym}}^{-2} + \varepsilon\, [C_1 C_3 \lambda_{\mathrm{sym}}^{-2} + (1+C_2)\, \overline{q}_{\mathrm{sym}}^2] \leq 1 - 2\varepsilon$,

the quasi-Pythagorean estimate (3.44) proves that

$$(\Lambda_\ell^{k+1})^2 \leq (1-2\varepsilon)\, \big[|||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2\big] \overset{(3.44)}{\leq} \frac{1-2\varepsilon}{1-\varepsilon}\, |||u_\infty^\star - u_\ell^{k,\underline{j}}|||^2 \leq \frac{1-2\varepsilon}{1-\varepsilon}\, (\Lambda_\ell^k)^2.$$

Up to the choice of the parameters ε and ν, this proves (3.52a) for any $0 < \lambda_{\mathrm{alg}} \leq \lambda_{\mathrm{alg}}^\star$.

Step 4. For $(\ell+1, 0, 0) \in \mathcal{Q}$, stability (A1), reduction (A2), and the Dörfler marking in Algorithm 3A(iii) yield that

$$\begin{aligned}
\eta_{\ell+1}(u_\ell^{k,\underline{j}})^2 &= \eta_{\ell+1}(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell; u_\ell^{k,\underline{j}})^2 + \eta_{\ell+1}(\mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell; u_\ell^{k,\underline{j}})^2 \\
&\overset{(A1)}{=} \eta_\ell(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell; u_\ell^{k,\underline{j}})^2 + \eta_{\ell+1}(\mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell; u_\ell^{k,\underline{j}})^2 \\
&\overset{(A2)}{\leq} \eta_\ell(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell; u_\ell^{k,\underline{j}})^2 + q_{\mathrm{red}}^2\, \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}; u_\ell^{k,\underline{j}})^2 \\
&= \eta_\ell(u_\ell^{k,\underline{j}})^2 - (1 - q_{\mathrm{red}}^2)\, \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}; u_\ell^{k,\underline{j}})^2 \\
&\leq \eta_\ell(u_\ell^{k,\underline{j}})^2 - (1 - q_{\mathrm{red}}^2)\, \eta_\ell(\mathcal{M}_\ell; u_\ell^{k,\underline{j}})^2 \leq [1 - (1 - q_{\mathrm{red}}^2)\, \theta]\, \eta_\ell(u_\ell^{k,\underline{j}})^2 \eqqcolon q_\theta\, \eta_\ell(u_\ell^{k,\underline{j}})^2,
\end{aligned} \tag{3.57}$$

where $0 < q_\theta < 1$ by definition.

Step 5. Let $(\ell+1, 0, 0) \in \mathcal{Q}$. With stability (A1), we infer from (5.1⁺) that

$$\begin{aligned}
\eta_\ell(u_\ell^{k,\underline{j}}) &\overset{(A1)}{\leq} \eta_\ell(u_\ell^{k-1,\underline{j}}) + C_{\mathrm{stab}}\, |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \\
&\overset{(5.1^+)}{\leq} \eta_\ell(u_\ell^{k-1,\underline{j}}) + C_{\mathrm{stab}}(1 + q_{\mathrm{sym}})\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + \frac{2\, q_{\mathrm{ctr}}}{1 - q_{\mathrm{ctr}}}\, C_{\mathrm{stab}}\, \lambda_{\mathrm{alg}} \lambda_{\mathrm{sym}}\, \eta_\ell(u_\ell^{k,\underline{j}}).
\end{aligned}$$

For sufficiently small $0 < \lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star$ with, e.g., $\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,C_{\mathrm{stab}}\,\lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}} \le 1/2$, we thus derive

$$\eta_\ell(u_\ell^{k,j}) \le \frac{1}{1-\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,C_{\mathrm{stab}}\,\lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}}}\,\eta_\ell(u_\ell^{k-1,j}) + \frac{C_{\mathrm{stab}}(1+q_{\mathrm{sym}})}{1-\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,C_{\mathrm{stab}}\,\lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}}}\,|||u_\ell^\star - u_\ell^{k-1,j}|||.$$

Define $C_4 := C_{\mathrm{stab}}^2(1+q_{\mathrm{sym}})^2$ and $C(\lambda) := \big[1-\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,C_{\mathrm{stab}}\,\lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}}\big]^{-2}$ with $\lambda = \lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}}$, where we already note that $C(\lambda) \to 1$ is (strictly) monotonically decreasing as $\lambda \to 0$. Stability (A1) and the Young inequality in the form $(a+b)^2 \le (1+\omega)a^2 + (1+\omega^{-1})b^2$ for $a, b \in \mathbb{R}$ and $\omega > 0$ show that

$$\eta_\ell(u_\ell^{k,j})^2 \le C(\lambda)\,\big[(1+\omega)\;\eta_\ell(u_\ell^{k-1,j})^2 + (1+\omega^{-1})\,C_4\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\big]. \tag{3.58}$$

Step 6. For $(\ell+1,0,0) \in \mathcal{Q}$ with $\ell \ge \ell_0$, we have that

$$\begin{aligned}
(\Lambda_{\ell+1}^0)^2 &\overset{(3.54)}{\le} (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\,|||u_\ell^\star - u_\ell^{k,j}|||^2 + 3\varepsilon\,|||u_\infty^\star - u_\ell^\star|||^2 + \nu\,\eta_{\ell+1}(u_\ell^{k,j})^2\\
&\overset{(3.57)}{\le} (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\,|||u_\ell^\star - u_\ell^{k,j}|||^2 + 3\varepsilon\,|||u_\infty^\star - u_\ell^\star|||^2 + \nu q_\theta\,\eta_\ell(u_\ell^{k,j})^2\\
&\overset{(3.55)}{\le} (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + (1+\varepsilon)\,|||u_\ell^\star - u_\ell^{k,j}|||^2 + \varepsilon\,C_2\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\\
&\qquad + \varepsilon\,C_1\,\eta_\ell(u_\ell^{k-1,j})^2 + \nu q_\theta\,\eta_\ell(u_\ell^{k,j})^2\\
&\overset{(5.1^+)}{\le} (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + \big[\varepsilon\,C_2 + (1+\varepsilon)^2 q_{\mathrm{sym}}^2\big]\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\\
&\qquad + \varepsilon\,C_1\,\eta_\ell(u_\ell^{k-1,j})^2 + \Big[q_\theta + \nu^{-1}\,(1+\varepsilon)(1+\varepsilon^{-1})\,\Big(\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\Big)^2\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\Big]\,\nu\,\eta_\ell(u_\ell^{k,j})^2.
\end{aligned}$$

With $C_\varepsilon := (1+\varepsilon)(1+\varepsilon^{-1})\,\big(\frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\big)^2$, we get

$$\begin{aligned}
(\Lambda_{\ell+1}^0)^2 &\le (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + \big[\varepsilon\,C_2 + (1+\varepsilon)^2 q_{\mathrm{sym}}^2\big]\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\\
&\qquad + \varepsilon\,C_1\,\eta_\ell(u_\ell^{k-1,j})^2 + \big[q_\theta + C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\big]\,\nu\,\eta_\ell(u_\ell^{k,j})^2.\\
&\overset{(3.58)}{\le} (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2 + \big[\varepsilon\,C_2 + (1+\varepsilon)^2 q_{\mathrm{sym}}^2\big]\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\\
&\qquad + \varepsilon\,C_1\,\eta_\ell(u_\ell^{k-1,j})^2\\
&\qquad + \big[q_\theta + C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\big]\,\nu\,C(\lambda)\,\big[(1+\omega)\,\eta_\ell(u_\ell^{k-1,j})^2 + (1+\omega^{-1})\,C_4\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\big]\\
&= (1-2\varepsilon)\,|||u_\infty^\star - u_\ell^\star|||^2\\
&\qquad + \Big(\varepsilon\,C_2 + (1+\varepsilon)^2 q_{\mathrm{sym}}^2 + (1+\omega^{-1})\,C_4\,C(\lambda)\,\big[q_\theta + C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\big]\,\nu\Big)\,|||u_\ell^\star - u_\ell^{k-1,j}|||^2\\
&\qquad + \Big((1+\omega)C(\lambda)\,\big[q_\theta + C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\big] + \varepsilon\,C_1\nu^{-1}\Big)\,\nu\,\eta_\ell(u_\ell^{k-1,j})^2.
\end{aligned}$$

Provided that

$$[II]\quad (1+\varepsilon)^2 q_{\mathrm{sym}}^2 + C(\lambda)\,(1+\omega^{-1})\,C_4\,\big[q_\theta + C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\big]\,\nu + \varepsilon\,C_2 \le 1-2\varepsilon,$$

$$[III]\quad C(\lambda)\,\Big[(1+\omega)q_\theta + (1+\omega)C_\varepsilon\nu^{-1}\,\lambda_{\mathrm{alg}}^2\lambda_{\mathrm{sym}}^2\Big] + \varepsilon C_1\nu^{-1} \le 1-2\varepsilon,$$

the quasi-Pythagorean estimate (3.44) shows that

$$(\Lambda_{\ell+1}^0)^2 \le (1-2\varepsilon)\big[|||u_\infty^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 + \nu\,\eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2\big]$$
$$\overset{(3.44)}{\le} \frac{1-2\varepsilon}{1-\varepsilon}\,|||u_\infty^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 + (1-2\varepsilon)\nu\,\eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2 \le \frac{1-2\varepsilon}{1-\varepsilon}\,(\Lambda_\ell^{\underline{k}-1})^2.$$

This proves (3.52b) up to the choice of the parameters ω, ν, and ε in the following step.

Step 7. A suitable choice of the parameters ω, ν, and ε can be obtained as follows:

- first, we choose ω such that $(1+\omega)q_\theta < 1$;
- second, we choose ν such that $\overline{q}_{\rm sym}^2 + \nu C_3 \lambda_{\rm sym}^{-2} < 1$ and $q_{\rm sym}^2 + (1+\omega^{-1})C_4\nu \le 1$;
- third, we choose $\varepsilon > 0$ sufficiently small such that
  - $\overline{q}_{\rm sym}^2 + \nu C_3\lambda_{\rm sym}^{-2} + \varepsilon\,[C_1C_3\lambda_{\rm sym}^{-2} + (1+C_2)\,\overline{q}_{\rm sym}^2] \le 1-2\varepsilon$,
  - $(1+\varepsilon)^2 q_{\rm sym}^2 + (1+\omega^{-1})C_4 q_\theta \nu + \varepsilon\, C_2 < 1-2\varepsilon$,
  - $(1+\omega)q_\theta + \varepsilon C_1 \nu^{-1} < 1-2\varepsilon$;

  in particular, constraint $[I]$ from Step 3 is satisfied;
- finally, we note that $C(\lambda) \longrightarrow 1$ monotonically as $\lambda = \lambda_{\rm alg}\lambda_{\rm sym} \longrightarrow 0$. Hence, we can choose $0 < \lambda_{\rm alg}' \le \min\big\{\lambda_{\rm alg}^\star, \frac{1-q_{\rm ctr}}{4q_{\rm ctr}C_{\rm stab}}\,\lambda_{\rm sym}^{-1}\big\}$ sufficiently small such that also the constraints $[II]$ and $[III]$ from Step 6 are satisfied for all $0 < \lambda_{\rm alg} \le \lambda_{\rm alg}'$.

This concludes the proof with $q_{\rm lin}^2 := \dfrac{1-2\varepsilon}{1-\varepsilon} < 1$ for any $0 < \lambda_{\rm alg} \le \lambda_{\rm alg}'$. □

3.5.5 Proof of Theorem 3.7

The proof is split into five steps. For $(\ell,k,j) \in \mathcal{Q}$, we consider

$$\begin{aligned}\Delta_\ell^{k,j} &= |||u_\infty^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}),\\ \Lambda_\ell^k &\overset{(3.51)}{=} \big[\,|||u_\infty^\star - u_\ell^{k,\underline{j}}|||^2 + \nu\,\eta_\ell(u_\ell^{k,\underline{j}})^2\,\big]^{1/2},\end{aligned} \tag{3.59}$$

where, compared with (3.26), the quasi-error $\Delta_\ell^{k,j}$ has been redefined. Later, we shall conclude that indeed $u_\infty^\star = u^\star$ so that both definitions coincide.

Step 1. In the first step, we prove that

$$\Delta_\ell^{k,j} \lesssim |||u_\ell^{k,\star} - u_\ell^{k,j-1}||| \quad \text{for all } (\ell,k,j) \in \mathcal{Q} \text{ with } 1 \le k \le \underline{k}[\ell] \text{ and } 1 \le j < \underline{j}[\ell,k]. \tag{3.60}$$

Together with reliability (3.43) and stability (A1), the definition of $\Delta_\ell^{k,j}$ shows that

$$\begin{aligned}\Delta_\ell^{k,j} &\overset{(3.59)}{=} |||u_\infty^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j})\\ &\le |||u_\infty^\star - u_\ell^\star||| + |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j})\\ &\overset{(3.43)}{\le} (C_{\rm Céa}+1)C_{\rm rel}\,\eta_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j})\\ &\overset{(A1)}{\lesssim} \eta_\ell(u_\ell^{k,j}) + |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}|||.\end{aligned}$$

The contraction of the (unperturbed) Zarantonello iteration (3.24) proves that

$$\begin{aligned}|\!|\!|u_\ell^\star - u_\ell^{k,j}|\!|\!| \le |\!|\!|u_\ell^\star - u_\ell^{k,\star}|\!|\!| + |\!|\!|u_\ell^{k,\star} - u_\ell^{k,j}|\!|\!| &\overset{(3.24)}{\le} \frac{q_{\mathrm{sym}}}{1-q_{\mathrm{sym}}}\,|\!|\!|u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}|\!|\!| + |\!|\!|u_\ell^{k,\star} - u_\ell^{k,j}|\!|\!| \\ &\lesssim |\!|\!|u_\ell^{k,\star} - u_\ell^{k,j}|\!|\!| + |\!|\!|u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|\!|\!|.\end{aligned}$$

Furthermore, the contraction of the algebraic solver (3.38) proves that

$$|\!|\!|u_\ell^{k,\star} - u_\ell^{k,j}|\!|\!| \overset{(3.38)}{\le} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,|\!|\!|u_\ell^{k,j} - u_\ell^{k,j-1}|\!|\!|.$$

Combining the last three estimates with the not met stopping criterion of the algebraic solver in Algorithm 3A(i.b.II) for $1 \le j < \underline{j}[\ell,k]$, we conclude that

$$\Delta_\ell^{k,j} \lesssim \eta_\ell(u_\ell^{k,j}) + |\!|\!|u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|\!|\!| + |\!|\!|u_\ell^{k,j} - u_\ell^{k,j-1}|\!|\!| \overset{\text{(i.b.II)}}{\lesssim} |\!|\!|u_\ell^{k,j} - u_\ell^{k,j-1}|\!|\!|.$$

Finally, the triangle inequality and the contraction (3.38) imply (3.60).

Step 2. Next, we show that

$$\Delta_\ell^{k,\underline{j}} \lesssim \Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j) \in \mathcal{Q}, \tag{3.61}$$

which is trivial for $j = \underline{j}[\ell,k]$. To deal with $j = \underline{j}[\ell,k]-1$, note that the definition of $\Delta_\ell^{k,j}$ shows that

$$\begin{aligned}\Delta_\ell^{k,\underline{j}} &\overset{(3.59)}{=} |\!|\!|u_\infty^\star - u_\ell^{k,\underline{j}}|\!|\!| + |\!|\!|u_\ell^{k,\star} - u_\ell^{k,\underline{j}}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}}) \\ &\le |\!|\!|u_\infty^\star - u_\ell^{k,\underline{j}-1}|\!|\!| + |\!|\!|u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}|\!|\!| + 2\,|\!|\!|u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}}).\end{aligned}$$

Stability (A1) and the algebraic solver contraction (3.38) lead us to

$$\begin{aligned}2\,|\!|\!|u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}}) &\overset{(A1)}{\le} (2 + C_{\mathrm{stab}})\,|\!|\!|u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}-1}) \\ &\overset{(3.38)}{\le} (2 + C_{\mathrm{stab}})(1 + q_{\mathrm{ctr}})\,|\!|\!|u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}-1}).\end{aligned}$$

Combining the last two estimates verifies (3.61) for $j = \underline{j}[\ell,k]-1$, i.e.,

$$\Delta_\ell^{k,\underline{j}} \lesssim |\!|\!|u_\infty^\star - u_\ell^{k,\underline{j}-1}|\!|\!| + |\!|\!|u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}|\!|\!| + \eta_\ell(u_\ell^{k,\underline{j}-1}) \overset{(3.59)}{=} \Delta_\ell^{k,\underline{j}-1}. \tag{3.62}$$

We prove the remaining case $j < \underline{j}[\ell,k]-1$ by (3.60) from Step 1 and the algebraic solver contraction (3.38), i.e.,

$$\Delta_\ell^{k,\underline{j}} \overset{(3.62)}{\lesssim} \Delta_\ell^{k,\underline{j}-1} \overset{(3.60)}{\lesssim} |\!|\!|u_\ell^{k,\star} - u_\ell^{k,\underline{j}-2}|\!|\!| \overset{(3.38)}{\le} q_{\mathrm{ctr}}^{(\underline{j}[\ell,k]-2)-j}\,|\!|\!|u_\ell^{k,\star} - u_\ell^{k,j}|\!|\!| \le \Delta_\ell^{k,j}.$$

This concludes the proof of (3.61).

Step 3. In this step, we prove that

$$\Lambda_\ell^0 \simeq \Delta_\ell^{0,0} = \Delta_\ell^{0,\underline{j}} \quad \text{and} \quad \Lambda_\ell^k \lesssim \Delta_\ell^{k,\underline{j}} \overset{(3.61)}{\lesssim} \Delta_\ell^{k,0} \lesssim \Lambda_\ell^{k-1} \text{ for all } (\ell,k,\underline{j}) \in \mathcal{Q} \text{ with } k \ge 1. \tag{3.63}$$

Together with $u_\ell^{0,\star} = u_\ell^{0,\underline{j}} = u_\ell^{0,0}$, the definition of Λ_ℓ^0 and $\Delta_\ell^{0,0}$ proves that $\Lambda_\ell^0 \simeq \Delta_\ell^{0,0} = \Delta_\ell^{0,\underline{j}}$ as well as $\Lambda_\ell^k \lesssim \Delta_\ell^{k,\underline{j}}$ for all $(\ell,k,\underline{j}) \in \mathcal{Q}$, where the hidden constants depend only on ν. Together with (3.61) from Step 2, it thus only remains to prove $\Delta_\ell^{k,0} \lesssim \Lambda_\ell^{k-1}$ for $k \geq 1$.

To this end, let $(\ell,k,\underline{j}) \in \mathcal{Q}$ with $k \geq 1$. From contraction (3.24) of the unperturbed Zarantonello symmetrization and nested iteration $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}}$, we get that

$$\begin{aligned}\Delta_\ell^{k,0} &= |||u_\infty^\star - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}}) \\ &\overset{(3.24)}{\leq} |||u_\infty^\star - u_\ell^{k-1,\underline{j}}||| + (1+q_{\rm sym})\,|||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}}).\end{aligned}$$

The Céa lemma (3.42) proves that

$$|||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \leq |||u_\infty^\star - u_\ell^\star||| + |||u_\infty^\star - u_\ell^{k-1,\underline{j}}||| \overset{(3.42)}{\lesssim} |||u_\infty^\star - u_\ell^{k-1,\underline{j}}|||.$$

Combining the last two estimates, we arrive at

$$\Delta_\ell^{k,0} \lesssim |||u_\infty^\star - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}}) \simeq \Lambda_\ell^{k-1}.$$

This concludes the proof of (3.63).

Step 4. In this step, we prove that

$$\sum_{j'=j}^{\underline{j}[\ell,k]} \Delta_\ell^{k,j'} \lesssim \Delta_\ell^{k,\underline{j}} + \Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j) \in \mathcal{Q}. \tag{3.64}$$

According to the right-hand side of (3.64), it remains to consider the sum for $j' = j+1,\dots,\underline{j}[\ell,k]-1$. With (3.60) and contraction (3.38) of the algebraic solver, we get that

$$\sum_{j'=j+1}^{\underline{j}[\ell,k]-1} \Delta_\ell^{k,j'} \overset{(3.60)}{\lesssim} \sum_{j'=j+1}^{\underline{j}[\ell,k]-1} |||u_\ell^{k,\star} - u_\ell^{k,j'-1}||| \overset{(3.38)}{\leq} |||u_\ell^{k,\star} - u_\ell^{k,j}||| \sum_{j'=j}^{\underline{j}[\ell,k]-2} q_{\rm ctr}^{j'-j}.$$

With the geometric series and $|||u_\ell^{k,\star} - u_\ell^{k,j}||| \leq \Delta_\ell^{k,j}$, this concludes the proof of (3.64).

Step 5. For $(\ell,k,j) \in \mathcal{Q}$ with $\ell \geq \ell_0$, the preceding steps show that

$$\begin{aligned}\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\(\ell',k',j')>(\ell,k,j)}} \Delta_{\ell'}^{k',j'} &= \sum_{j'=j+1}^{\underline{j}[\ell,k]} \Delta_\ell^{k,j'} + \sum_{\substack{(\ell',k',0)\in\mathcal{Q}\\(\ell',k',0)>(\ell,k,0)}} \sum_{j'=0}^{\underline{j}[\ell',k']} \Delta_{\ell'}^{k',j'} \\ &\overset{(3.64)}{\lesssim} \big[\Delta_\ell^{k,\underline{j}} + \Delta_\ell^{k,j}\big] + \sum_{\substack{(\ell',k',0)\in\mathcal{Q}\\(\ell',k',0)>(\ell,k,0)}} \big[\Delta_{\ell'}^{k',\underline{j}} + \Delta_{\ell'}^{k',0}\big] \overset{(3.61)}{\lesssim} \Delta_\ell^{k,j} + \sum_{\substack{(\ell',k',0)\in\mathcal{Q}\\(\ell',k',0)>(\ell,k,0)}} \Delta_{\ell'}^{k',0}.\end{aligned}$$

With linear convergence (3.52) of Λ_ℓ^k from Lemma 3.14 and the geometric series, we thus see that

$$\begin{aligned}\sum_{\substack{(\ell',k',0)\in\mathcal{Q}\\(\ell',k',0)>(\ell,k,0)}} \Delta_{\ell'}^{k',0} &= \sum_{k'=k+1}^{\underline{k}[\ell]} \Delta_\ell^{k',0} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \Delta_{\ell'}^{k',0} \overset{(3.63)}{\lesssim} \sum_{k'=k}^{\underline{k}[\ell]-1} \Lambda_\ell^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']-1} \Lambda_{\ell'}^{k'} \\ &\overset{(3.52)}{\lesssim} \Lambda_\ell^k + \Lambda_{\ell+1}^0 \leq 2\Lambda_\ell^k \overset{(3.63)}{\lesssim} \Delta_\ell^{k,\underline{j}} \overset{(3.61)}{\lesssim} \Delta_\ell^{k,j}.\end{aligned}$$

Altogether, this proves that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\(\ell',k',j')>(\ell,k,j)}} \Delta_{\ell'}^{k',j'} \le C_{\mathrm{sum}}\,\Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j)\in\mathcal{Q}.$$

According to basic calculus (see, e.g., [CFPP14, Lemma 4.9]), this is equivalent to linear convergence with respect to the lexicographic order on $\mathcal{Q}$, i.e., for all $(\ell,k,j),(\ell',k',j')\in\mathcal{Q}$ with $|\ell',k',j'|\le|\ell,k,j|$ and $\ell'\ge\ell_0$, it holds that

$$\Delta_\ell^{k,j} \le C_{\mathrm{lin}}\,q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\,\Delta_{\ell'}^{k',j'},$$

where the constants $C_{\mathrm{lin}}>0$ and $0<q_{\mathrm{lin}}<1$ depend only on C_{sum}. This also yields that

$$\begin{aligned}|\!|\!| u^\star-u_\ell^\star |\!|\!| &\overset{(A3)}{\lesssim} \eta_\ell(u_\ell^\star) \overset{(A1)}{\lesssim} \eta_\ell(u_\ell^{k,j}) + |\!|\!| u_\ell^\star-u_\ell^{k,j}|\!|\!| \le \eta_\ell(u_\ell^{k,j}) + |\!|\!| u_\infty^\star-u_\ell^{k,j}|\!|\!| + |\!|\!| u_\infty^\star-u_\ell^\star|\!|\!| \\ &\overset{(3.42)}{\lesssim} \eta_\ell(u_\ell^{k,j}) + |\!|\!| u_\infty^\star-u_\ell^{k,j}|\!|\!| \to 0 \quad \text{as } |\ell,k,j|\to\infty\end{aligned}$$

and hence $u_\infty^\star=u^\star$. In particular, the definitions of $\Delta_\ell^{k,j}$ from (3.26) and (3.59) coincide. Overall, we thus conclude the proof of linear convergence (3.27). □

3.5.6 Proof of Theorem 3.9

The proof of Theorem 3.9 requires the following auxiliary lemma stating that the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ of the inexact but available final iterate of Algorithm 3A is equivalent to the error estimator $\eta_\ell(u_\ell^\star)$ of the (unknown) exact solution $u_\ell^\star$. While the statement is similar to [HPSV21, Lemma 7], the present proof provides a minor clarification of the involved constant.

Lemma 3.15. *Recall $C_{\mathrm{alg}}>0$ from (Calg) and $\lambda_{\mathrm{alg}}^\star>0$ from Theorem 3.7. Then, for all $0<\theta\le 1$, $0<\lambda_{\mathrm{alg}}\le\lambda_{\mathrm{alg}}^\star$, $0<\lambda_{\mathrm{sym}}<\lambda_{\mathrm{sym}}^\star=\min\{1,C_{\mathrm{stab}}^{-1}C_{\mathrm{alg}}^{-1}\}$, and all $(\ell,\underline{k},\underline{j})\in\mathcal{Q}$, it holds that*

$$|\!|\!| u_\ell^\star-u_\ell^{\underline{k},\underline{j}}|\!|\!| \le C_{\mathrm{alg}}\,\lambda_{\mathrm{sym}}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}). \tag{3.65}$$

Moreover, there holds equivalence

$$\big[1-\lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\big]\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \le \eta_\ell(u_\ell^\star) \le \big[1+\lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\big]\eta_\ell(u_\ell^{\underline{k},\underline{j}}). \tag{3.66}$$

Proof. The proof consists of two steps.

Step 1. Recall from (5.1⁺) that

$$|\!|\!| u_\ell^\star-u_\ell^{\underline{k},\underline{j}}|\!|\!| \overset{(5.1^+)}{\le} q_{\mathrm{sym}}\,|\!|\!| u_\ell^\star-u_\ell^{\underline{k}-1,\underline{j}}|\!|\!| + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}\,\lambda_{\mathrm{sym}}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}).$$

The stopping criterion in Algorithm 3A(i.d) proves that

$$|\!|\!| u_\ell^\star-u_\ell^{\underline{k}-1,\underline{j}}|\!|\!| \le |\!|\!| u_\ell^\star-u_\ell^{\underline{k},\underline{j}}|\!|\!| + |\!|\!| u_\ell^{\underline{k},\underline{j}}-u_\ell^{\underline{k}-1,\underline{j}}|\!|\!| \overset{\text{(i.d)}}{\le} |\!|\!| u_\ell^\star-u_\ell^{\underline{k},\underline{j}}|\!|\!| + \lambda_{\mathrm{sym}}\eta_\ell(u_\ell^{\underline{k},\underline{j}}).$$

Combining these estimates with $0 < \lambda_{\rm alg} \le \lambda_{\rm alg}^\star$, we prove (3.65), since

$$|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \le \frac{1}{1-q_{\rm sym}}\Big(\frac{2\,q_{\rm ctr}}{1-q_{\rm ctr}}\,\lambda_{\rm alg} + q_{\rm sym}\Big)\,\lambda_{\rm sym}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \le C_{\rm alg}\lambda_{\rm sym}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}).$$

Step 2. With the definition $\lambda_{\rm sym}^\star = \min\{1, C_{\rm stab}^{-1}C_{\rm alg}^{-1}\}$, stability (A1) and (3.65) show

$$\begin{aligned}\eta_\ell(u_\ell^\star) &\overset{(A1)}{\le} \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + C_{\rm stab}\,|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \overset{(3.65)}{\le} \big[1 + C_{\rm stab}\,C_{\rm alg}\lambda_{\rm sym}\big]\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})\\ &\le \big[1+\lambda_{\rm sym}/\lambda_{\rm sym}^\star\big]\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}).\end{aligned}$$

If $0 < \lambda_{\rm sym} < \lambda_{\rm sym}^\star$, the analogous argument also proves the converse inequality

$$\big[1-\lambda_{\rm sym}/\lambda_{\rm sym}^\star\big]\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \le \eta_\ell(u_\ell^\star). \tag{3.67}$$

This concludes the proof. □

Proof of Theorem 3.9. It is sufficient to show that

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \lesssim \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\Delta_\ell^{k,j}, \tag{3.68a}$$

$$\sup_{\substack{(\ell,k,j)\in\mathcal{Q}\\ \ell\ge\ell_0}} (\#\mathcal{T}_\ell)^s\,\Delta_\ell^{k,j} \lesssim \max\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\}. \tag{3.68b}$$

Then, (3.34b) follows from (3.68b) and Corollary 3.8. We split the proof into six steps.

Step 1. We first show (3.68a) for the case $\underline{\ell} = \infty$. Algorithm 3A ensures that $\#\mathcal{T}_\ell \to \infty$ as $\ell \to \infty$. We recall that in NVB refinement an element is split into at least two but at most $C_{\rm child}$ child elements. In particular, for all $\ell \ge 0$, we have that

$$\#\mathcal{T}_{\ell+1} \le C_{\rm child}\,\#\mathcal{T}_\ell. \tag{3.69}$$

For any given $N \in \mathbb{N}$, we can argue similarly as in the proof of [CFPP14, Proposition 4.15]. Choose the maximal index $\ell' \in \mathbb{N}_0$ such that $\#\mathcal{T}_{\ell'} - \#\mathcal{T}_0 \le N$. The maximality of ℓ' leads us to

$$N+1 < \#\mathcal{T}_{\ell'+1} - \#\mathcal{T}_0 + 1 \le \#\mathcal{T}_{\ell'+1} \overset{(3.69)}{\le} C_{\rm child}\,\#\mathcal{T}_{\ell'}. \tag{3.70}$$

Since $\mathcal{T}_{\ell'} \in \mathbb{T}_N(\mathcal{T}_0)$, we have that

$$\min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T}_0)} \big[|||u^\star - u_{\rm opt}^\star||| + \eta_{\rm opt}(u_{\rm opt}^\star)\big] \le |||u^\star - u_{\ell'}^\star||| + \eta_{\ell'}(u_{\ell'}^\star), \tag{3.71}$$

and stability (A1) and the Céa lemma (3.15) show, for $(\ell', k', j') \in \mathcal{Q}$, that

$$\begin{aligned}|||u^\star - u_{\ell'}^\star||| + \eta_{\ell'}(u_{\ell'}^\star) &\overset{(A1)}{\le} |||u^\star - u_{\ell'}^\star||| + \eta_{\ell'}(u_{\ell'}^{k',j'}) + C_{\rm stab}\,|||u_{\ell'}^\star - u_{\ell'}^{k',j'}|||\\ &\le (1+C_{\rm stab})|||u^\star - u_{\ell'}^\star||| + \eta_{\ell'}(u_{\ell'}^{k',j'}) + C_{\rm stab}\,|||u^\star - u_{\ell'}^{k',j'}|||\\ &\overset{(3.15)}{\le} \big(C_{\rm Céa}\,(1+C_{\rm stab}) + C_{\rm stab}\big)\,|||u^\star - u_{\ell'}^{k',j'}||| + \eta_{\ell'}(u_{\ell'}^{k',j'})\\ &\le \big(C_{\rm Céa}\,(1+C_{\rm stab}) + C_{\rm stab}\big)\,\Delta_{\ell'}^{k',j'}.\end{aligned} \tag{3.72}$$

A combination of the previous estimates leads us to

$$(N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T}_0)} \big[|||u^\star - u^\star_{\rm opt}||| + \eta_{\rm opt}(u^\star_{\rm opt})\big] \overset{(3.71)}{\le} (N+1)^s \big[|||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}(u^\star_{\ell'})\big]$$
$$\overset{(3.70)}{\le} C^s_{\rm child}\,(\#\mathcal{T}_{\ell'})^s \big[|||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}(u^\star_{\ell'})\big] \overset{(3.72)}{\lesssim} (\#\mathcal{T}_{\ell'})^s \Delta^{k',j'}_{\ell'} \le \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\Delta^{k,j}_\ell.$$

Finally, taking the supremum over all N yields the sought result

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \lesssim \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\Delta^{k,j}_\ell.$$

Step 2. We proceed to show (3.68a) for the case $\underline{\ell} < \infty$. Recall from Lemma 3.11 that $\eta_{\underline{\ell}}(u^\star_{\underline{\ell}}) = 0$ and $u^\star_{\underline{\ell}} = u^\star$. Without loss of generality, we may assume $\underline{\ell} > 0$, since otherwise $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} = 0$. Combined with reliability (A3), this yields that

$$\begin{aligned}\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} &\overset{(3.32)}{=} \sup_{N\in\mathbb{N}_0}\Big((N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T}_0)} \big[|||u^\star - u^\star_{\rm opt}||| + \eta_{\rm opt}(u^\star_{\rm opt})\big]\Big)\\ &\overset{(A3)}{\le} (1+C_{\rm rel}) \sup_{0\le N<\#\mathcal{T}_{\underline{\ell}}-\#\mathcal{T}_0}\Big((N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T}_0)} \eta_{\rm opt}(u^\star_{\rm opt})\Big).\end{aligned} \tag{3.73}$$

We argue as in Step 1 above: Let $0 \le N < \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0$. Choose the maximal index $0 \le \ell' < \underline{\ell}$ with $\#\mathcal{T}_{\ell'} - \#\mathcal{T}_0 \le N$. Arguing along the lines of (3.70)–(3.72), we see that

$$\sup_{0\le N<\#\mathcal{T}_{\underline{\ell}}-\#\mathcal{T}_0}\Big((N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T}_0)} \eta_{\rm opt}(u^\star_{\rm opt})\Big) \lesssim \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\Delta^{k,j}_\ell.$$

Combining this with (3.73), we conclude the lower bound (3.68a) also in this case.

Step 3. We prove (3.68b) for $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} < \infty$, since the result becomes trivial if $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} = \infty$. First, we show that for all $\ell' \ge \ell_0$ with $(\ell'+1, 0, 0) \in \mathcal{Q}$, there exists $\mathcal{R}_{\ell'} \subseteq \mathcal{T}_{\ell'}$ such that

$$\#\mathcal{R}_{\ell'} \lesssim \|u^\star\|^{1/s}_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}\,(\Delta^{0,\underline{j}}_{\ell'+1})^{-1/s} \quad\text{and}\quad \theta_{\rm mark}\eta_{\ell'}(u^\star_{\ell'})^2 \le \eta_{\ell'}(\mathcal{R}_{\ell'}, u^\star_{\ell'})^2. \tag{3.74}$$

Since $0 < \theta_{\rm mark} = (\theta^{1/2} + \lambda_{\rm sym}/\lambda^\star_{\rm sym})^2\,(1-\lambda_{\rm sym}/\lambda^\star_{\rm sym})^{-2} < \theta^\star$, and because there holds (A4), [CFPP14, Lemma 4.14] ensures, for all $\ell' \ge \ell_0$, the existence of a set $\mathcal{R}_{\ell'} \subseteq \mathcal{T}_{\ell'}$ satisfying

$$\#\mathcal{R}_{\ell'} \lesssim \|u^\star\|^{1/s}_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}\,\eta_{\ell'}(u^\star_{\ell'})^{-1/s} \quad\text{and}\quad \theta_{\rm mark}\eta_{\ell'}(u^\star_{\ell'})^2 \le \eta_{\ell'}(\mathcal{R}_{\ell'}, u^\star_{\ell'})^2. \tag{3.75}$$

Since $\lambda_{\rm sym}/\lambda^\star_{\rm sym} < 1$ by assumption, the estimator equivalence (3.66) shows that

$$\big[1 - \lambda_{\rm sym}/\lambda^\star_{\rm sym}\big]\,\eta_{\ell'}(u^{\underline{k},\underline{j}}_{\ell'}) \le \eta_{\ell'}(u^\star_{\ell'}), \tag{3.76}$$

which leads us to

$$\#\mathcal{R}_{\ell'} \overset{(3.75)}{\lesssim} \|u^\star\|^{1/s}_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}\,\eta_{\ell'}(u^{\underline{k},\underline{j}}_{\ell'})^{-1/s}.$$

Moreover, thanks to nested iteration, Step 3 of the proof of Theorem 3.7, Step 3 of the proof of Lemma 3.14, and reliability (3.23) of Proposition 3.6, there holds that

$$\begin{aligned}\Delta_{\ell'+1}^{0,\underline{j}} &\overset{(3.63)}{\simeq} \Lambda_{\ell'+1}^{0} = \big[|||u^\star - u_{\ell'}^{\underline{k},\underline{j}}|||^2 + \nu\,\eta_{\ell'+1}(u_{\ell'}^{\underline{k},\underline{j}})^2\big]^{1/2} \\ &\overset{(3.57)}{\lesssim} \big[|||u^\star - u_{\ell'}^{\underline{k},\underline{j}}|||^2 + \eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}})^2\big]^{1/2} \overset{(3.23)}{\lesssim} \eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}}).\end{aligned} \tag{3.77}$$

By summarizing the last two estimates, we obtain (3.74).

Step 4. For $(\ell'+1,0,0) \in \mathcal{Q}$ with $\ell' \geq \ell_0$, we show that

$$\#\mathcal{M}_{\ell'} \leq C_{\rm mark}\,\#\mathcal{R}_{\ell'} \tag{3.78}$$

with the constant $C_{\rm mark} \geq 1$ from Algorithm 3A. Recall the definition

$$\theta_{\rm mark} \overset{(3.33)}{=} \Big(\frac{\theta^{1/2} + \lambda_{\rm sym}/\lambda_{\rm sym}^\star}{1-\lambda_{\rm sym}/\lambda_{\rm sym}^\star}\Big)^2 \quad \text{with} \quad \lambda_{\rm sym}^\star = \min\{1, C_{\rm alg}^{-1}C_{\rm stab}^{-1}\}.$$

This shows that

$$\begin{aligned}|||u_{\ell'}^\star - u_{\ell'}^{\underline{k},\underline{j}}||| &\overset{(3.65)}{\leq} C_{\rm alg}\,\lambda_{\rm sym}\,\eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}}) \\ &\leq C_{\rm stab}^{-1}\frac{\lambda_{\rm sym}}{\lambda_{\rm sym}^\star}\eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}}) = C_{\rm stab}^{-1}\Big(\theta_{\rm mark}^{1/2}\big[1-\lambda_{\rm sym}/\lambda_{\rm sym}^\star\big] - \theta^{1/2}\Big)\eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}}).\end{aligned} \tag{3.79}$$

Now, we can estimate

$$\begin{aligned}\theta_{\rm mark}^{1/2}\big[1-\lambda_{\rm sym}/\lambda_{\rm sym}^\star\big]\,\eta_{\ell'}(u_{\ell'}^{\underline{k'},\underline{j'}}) &\overset{(3.66)}{\leq} \theta_{\rm mark}^{1/2}\eta_{\ell'}(u_{\ell'}^\star) \overset{(3.75)}{\leq} \eta_{\ell'}(\mathcal{R}_{\ell'}, u_{\ell'}^\star) \\ &\overset{(A1)}{\leq} \eta_{\ell'}(\mathcal{R}_{\ell'}, u_{\ell'}^{\underline{k'},\underline{j'}}) + C_{\rm stab}\,|||u_{\ell'}^\star - u_{\ell'}^{\underline{k'},\underline{j'}}||| \\ &\overset{(3.79)}{\leq} \eta_{\ell'}(\mathcal{R}_{\ell'}, u_{\ell'}^{\underline{k'},\underline{j'}}) + \Big(\theta_{\rm mark}^{1/2}\big[1-\lambda_{\rm sym}/\lambda_{\rm sym}^\star\big] - \theta^{1/2}\Big)\eta_{\ell'}(u_{\ell'}^{\underline{k'},\underline{j'}}).\end{aligned}$$

Rearranging the terms, we obtain that $\mathcal{R}_{\ell'}$ from Step 3 satisfies the Dörfler marking criterion of Algorithm 3A(iii) with the same parameter θ, i.e., there holds

$$\theta\,\eta_{\ell'}(u_{\ell'}^{\underline{k'},\underline{j'}})^2 \leq \eta_{\ell'}(\mathcal{R}_{\ell'}, u_{\ell'}^{\underline{k'},\underline{j'}})^2. \tag{3.80}$$

Hence, quasi-minimality of the set of marked elements $\mathcal{M}_{\ell'}$ implies (3.78).

Step 5. Consider the case $(\ell,k,j) \in \mathcal{Q}$ with $\ell \geq \ell_0$. Full linear convergence from Theorem 3.7 yields that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} (\Delta_{\ell'}^{k',j'})^{-1/s} \overset{(3.27)}{\lesssim} (\Delta_\ell^{k,j})^{-1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} (q_{\rm lin}^{1/s})^{|\ell,k,j|-|\ell',k',j'|} \lesssim (\Delta_\ell^{k,j})^{-1/s}. \tag{3.81}$$

Recall that NVB refinement satisfies the mesh-closure estimate, i.e., there holds that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\mathrm{mesh}} \sum_{\ell'=0}^{\ell-1} \#\mathcal{M}_{\ell'} \quad \text{for all } \ell \ge 0, \tag{3.82}$$

where $C_{\mathrm{mesh}} > 1$ depends only on $\mathcal{T}_0$. Thus, for $(\ell, k, j) \in \mathcal{Q}$ with $\ell > \ell_0$, we have by the mesh-closure estimate (3.82), optimality of Dörfler marking (3.78), and full linear convergence (3.81) that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} \overset{(3.82)}{\lesssim} \sum_{\ell'=\ell_0}^{\ell-1} \#\mathcal{M}_{\ell'} \overset{(3.78)}{\lesssim} \sum_{\ell'=\ell_0}^{\ell-1} \#\mathcal{R}_{\ell'} \overset{(3.74)}{\lesssim} \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} \sum_{\ell'=\ell_0}^{\ell-1} (\Delta_{\ell'+1}^{0,\underline{j}})^{-1/s}$$
$$\le \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\le|\ell,k,j| \\ \ell'\ge\ell_0}} (\Delta_{\ell'}^{k',j'})^{-1/s} \overset{(3.81)}{\lesssim} \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} (\Delta_\ell^{k,j})^{-1/s}.$$

Rearranging the terms and noting that $\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1 \le 2\,(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0})$, we obtain that

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_\ell^{k,j} \lesssim \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} \quad \text{for} \quad \ell > \ell_0.$$

Trivially, full linear convergence (3.27) proves that

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_{\ell_0}^{k,j} = \Delta_{\ell_0}^{k,j} \overset{(3.27)}{\lesssim} \Delta_{\ell_0}^{0,0} \quad \text{for} \quad \ell = \ell_0.$$

We recall from [BHP17, Lemma 22] that for all $\mathcal{T}_H \in \mathbb{T}$ and all $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, it holds that

$$\#\mathcal{T}_h - \#\mathcal{T}_H + 1 \le \#\mathcal{T}_h \le \#\mathcal{T}_H\,(\#\mathcal{T}_h - \#\mathcal{T}_H + 1). \tag{3.83}$$

Overall, we have thus shown that

$$(\#\mathcal{T}_\ell)^s \Delta_\ell^{k,j} \overset{(3.83)}{\lesssim} (\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_\ell^{k,j} \lesssim \max\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\}$$

for all $(\ell, k, j) \in \mathcal{Q}$ with $\ell \ge \ell_0$. This concludes the proof of the upper bound in (3.68b) and hence that of (3.34).

Step 6. We prove the equivalence in (3.35) by combining the steps above. Recall that

$$\mathcal{Q} \backslash \{(\ell, k, j) \in \mathcal{Q} : \ell \ge \ell_0\} = \{(\ell, k, j) \in \mathcal{Q} : \ell < \ell_0\} \quad \text{is finite}$$

and that $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} < \infty$ is equivalent to $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} < \infty$. Thus, the claim follows immediately by the equivalence in (3.34). This concludes the proof. □

3.6 Numerical experiments

We consider the model problem (3.1) from the introduction. The MATLAB implementation of the following experiments is embedded into the open source software package MooAFEM from [IP23]. In the following, Algorithm 3A employs the optimal local hp-robust multigrid method from [IMPS24] as algebraic solver and the standard residual error estimator η_ℓ. Given $T \in \mathcal{T}_\ell$ and $v_\ell \in \mathcal{X}_\ell$, the local contribution of η_ℓ reads

$$\eta_\ell(T; v_\ell)^2 := h_T^2\,\|-\operatorname{div}(\boldsymbol{A}\,\nabla v_\ell - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_\ell + c\,v_\ell - f\|_{L^2(T)}^2 + h_T\,\|[\![(\boldsymbol{A}\,\nabla v_\ell - \boldsymbol{f})\cdot\boldsymbol{n}]\!]\|_{L^2(\partial T\cap\Omega)}^2.$$

It is well-known (see, e.g., [CFPP14, Section 6.1]) that η_ℓ satisfies the axioms (A1)–(A4) from Section 3.2.3.

3.6.1 Diffusion-convection-reaction on L-shaped domain

In this subsection, we consider the problem (3.1) on the L-shaped domain $\Omega = (-1,1)^2 \setminus \big([0,1] \times [-1,0]\big) \subset \mathbb{R}^2$ with coefficients $\boldsymbol{A}(x) = \mathrm{Id}$, $\boldsymbol{b}(x) = x$, and $c(x) = 1$, and right-hand side $f(x) = 1$, i.e.,

$$-\Delta u^\star(x) + x \cdot \nabla u^\star(x) + u^\star(x) = 1 \quad \text{for } x \in \Omega \quad \text{subject to} \quad u^\star(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Optimality of AISFEM

We first display the optimality of Algorithm 3A with respect to the computational cost stated in Theorem 3.9 using the equivalence $\#\mathcal{T}_\ell \simeq \dim \mathcal{X}_\ell$. Numerically, we test with the parameters $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.1$, $\delta = 0.5$, and $\theta = 0.5$ and, unless stated explicitly, the stopping criterion $\dim \mathcal{X}_\ell > 10^7$. Note that both the total error and the algebraic error are unknown in all practical purposes. Therefore, we cannot study the decay of the quasi-error, but rather consider the equivalent error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \simeq \Delta_\ell^{\underline{k},\underline{j}}$. Figure 3.2 shows that the proposed algorithm achieves optimal rates $-m/2$ for several polynomial degrees m both with respect to the computational costs and the elapsed computational time after a short preasymptotic phase.

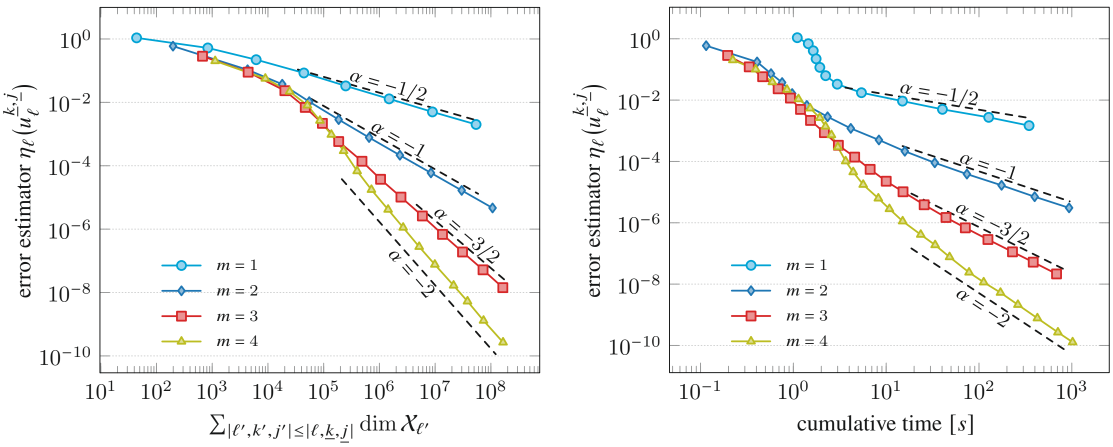


Figure 3.2: *Optimality of AISFEM for the diffusion-convection-reaction problem on the L-shaped domain from Section 3.6.1.* Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ over the computational costs (left) and the elapsed computational time (right) for different polynomial degrees m.

Optimality of the iteratively symmetrized solver

Optimality of AISFEM is possible when the inherent symmetrization and algebraic procedures are treated efficiently. In Figure 3.3, we present the time required for our iteratively symmetrized solver compared to the MATLAB built-in direct solver (backslash) of the linear system related to (3.4). We note that the displayed timings are comparing the direct solve time itself with the remaining time (including the setup of the Zarantonello system, computation of the error estimator, and mesh refinement). Hence, the presented numbers favor the built-in direct solver over the MATLAB-implemented multigrid code. Nevertheless, the combination of the Zarantonello symmetrization with the optimal local multigrid solver from [IMPS24] appears to be of comparable speed to the built-in direct solver with the observation that as the dimension of the linear system increases, the backslash performance begins to degrade. Moreover, Figure 3.4 shows that the iteration numbers of the solver remain uniformly bounded in the levels for various choices of the parameters λ_{sym} and θ. Note that when λ_{sym} decreases, a higher accuracy of the Zarantonello symmetrization is required. Therefore, the iteration numbers are expected to increase with smaller λ_{sym} as seen in Figure 3.4 (left). Moreover, the iteration numbers are also expected to increase as θ becomes larger. This is due to the aggressive refinement leading to hierarchies of low numbers of levels but with considerable increase in the dimension of the linear systems. This may lead to the conclusion that θ should be chosen very small in order to have fewer iterations per level, but studying the *cumulative* solver steps in Figure 3.4 (right) shows that this is not the best strategy.

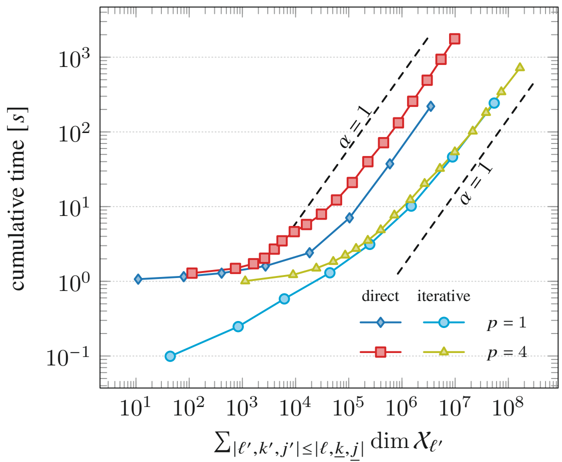


Figure 3.3: *Optimality of the combined iterative solver for the diffusion-convection-reaction problem on the L-shaped domain from Section 3.6.1.* Cumulative time for the direct solve and AISFEM over the computational costs.

Parameter study of AISFEM

We now investigate which parameters yield the best contraction in the iteratively symmetrized steps 3A(ii)–(iii). Since the parameters depend on the contraction factors q_{ctr} from (3.19) and q_{sym} from (3.24), we study a setting where the exact discrete solution $u_\ell^\star$ to (3.4) and the exact Zarantonello solution $u_\ell^{k,\star}$ to (3.6) are computed. Then, we compute $q_{\mathrm{ctr}}(\ell, k, j)$ for $(\ell, k, j) \in \mathcal{Q}$ and define the level-wise contraction factors $q_{\mathrm{ctr}}(\ell)$ as the maximum over all $q_{\mathrm{ctr}}(\ell, k, j)$ for fixed $\ell \in \mathbb{N}_0$ and analogously for q_{sym}. From now on, we fix the polynomial degree $m = 2$ and the parameters $\lambda_{\mathrm{alg}} = 10^{-2}$ for the numerical experiments. We investigate the behavior of the combined solver for various choices of $\lambda_{\mathrm{sym}} \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ and $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Figure 3.6 shows upper bounds $\lambda_{\mathrm{alg}} < \overline{\lambda}_{\mathrm{alg}}^\star = (1 - q_{\mathrm{sym}})(1 - q_{\mathrm{alg}})/(4\, q_{\mathrm{alg}})$ (see the implicit definition of $\overline{\lambda}_{\mathrm{alg}}^\star$ in (3.25)) and Figure 3.5 displays contraction factors $q_{\mathrm{sym}} \approx 1/2$ and $\overline{q}_{\mathrm{sym}} \approx 1/2$, independently of the

λ_{sym} \ θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10^{-1}	533	470	402	424	497	608	801	971	1513
10^{-2}	3084	1878	1566	1482	1524	1624	1869	2485	4266
10^{-3}	6543	4490	3478	2831	2894	3371	3826	4729	6956
10^{-4}	10791	6621	5211	4381	4475	4777	5979	7398	10901

Table 3.1: *Optimal selection of parameters with respect to the computational costs for the experiment from Section 3.6.1.* For the comparison, we consider the weighted costs $\big[\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \sum_{|\ell',k',j'|\le|\ell,\underline{k},\underline{j}|} \dim \mathcal{X}_{\ell'}\big]$ with stopping criterion $\eta_\ell(u_\ell^{\underline{k},\underline{j}}) < 10^{-5}$ for various choices of λ_{sym} and θ with the optimal choice highlighted in color.

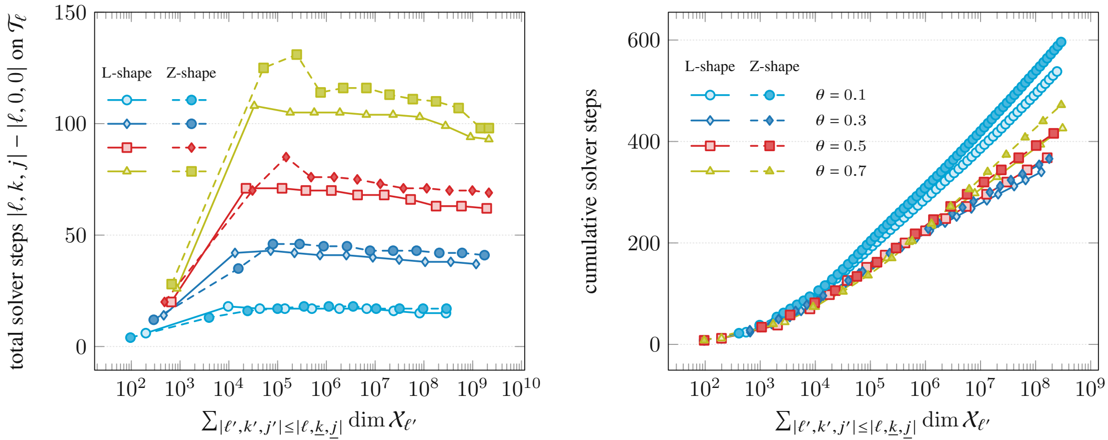


Figure 3.4: *Uniform bound on the iteration numbers for the diffusion-convection-reaction problem on the L-shaped domain from Section 3.6.1 and the strong convection problem on the Z-shaped domain from Section 3.6.2.* Number of total solver steps $|\ell,k,j| - |\ell,0,0|$ on the level ℓ for various selections of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and the cumulative solver steps for different marking parameter θ with fixed $\lambda_{\text{sym}} = 0.1$ (right).

choice of θ and λ_{sym}. Note that q_{sym} being close to $\overline{q}_{\mathrm{sym}}$ means that the perturbed, i.e., iteratively symmetrized, Zarantonello step is of comparable performance to the unperturbed Zarantonello iteration. Moreover, Table 3.1 shows that the optimal combination of the parameters with respect to the computational costs is $\theta = 0.3$ and $\lambda_{\mathrm{sym}} = 10^{-1}$. Furthermore, it appears that the choice of θ has a stronger impact on the costs than the selection of λ_{sym}.

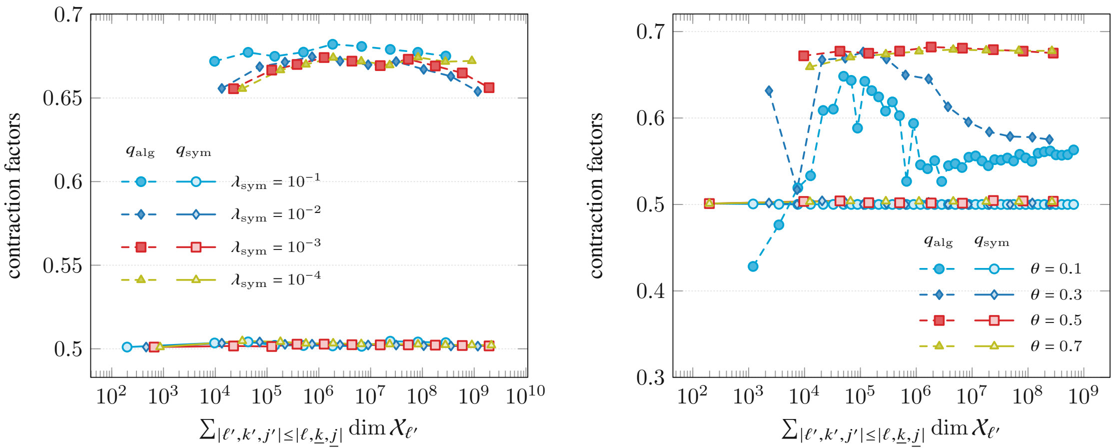


Figure 3.5: *Uniform contraction of the iterative solver for the diffusion-convection-reaction problem on the L-shaped domain from Section 3.6.1.* Experimental contraction factors q_{alg}, q_{sym} and $\overline{q}_{\mathrm{sym}}$ for various choices of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\mathrm{sym}} = 0.1$ (right).

3.6.2 Strong convection on Z-shaped domain

In this subsection, we consider the problem (3.1) on the Z-shaped domain $\Omega = (-1,1)^2 \setminus \mathrm{conv}\{(0,0),(-1,0),(-1,-1)\} \subset \mathbb{R}^2$ with coefficients $\boldsymbol{A}(x) = \mathrm{Id}$ and $\boldsymbol{b}(x) = (5,5)^\top$, and right-hand side $f(x) = 1$, i.e.,

$$-\Delta u^\star(x) + (5,5)^\top \cdot \nabla u^\star(x) = 1 \quad \text{for } x \in \Omega \quad \text{and} \quad u^\star(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Figure 3.7 shows that even for a strong convection combined with a strong singularity at the origin, the adaptive algorithm recovers the optimal convergence rates $-m/2$ for several polynomial degrees m both with respect to the cumulative costs and computational time. In Figure 3.4 we see that the number of solver steps per level ℓ behaves similarly to the diffusion-convection-reaction problem on the L-shape from Section 3.6.1 with an increase due to the stronger singularity. Furthermore, Figure 3.8 displays upper bounds on $\lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star < \overline{\lambda}_{\mathrm{alg}}^\star$ and the contraction factor $\overline{q}_{\mathrm{sym}} \approx 1/2$ (after an initial phase of reduced contraction) for the perturbed Zarantonello system in (3.25).

3.7 Conclusion and future work

In this work, we have developed and analyzed an adaptive finite element method for nonsymmetric second-order linear elliptic PDEs (3.1). From a conceptual point of view, the crucial assumption is

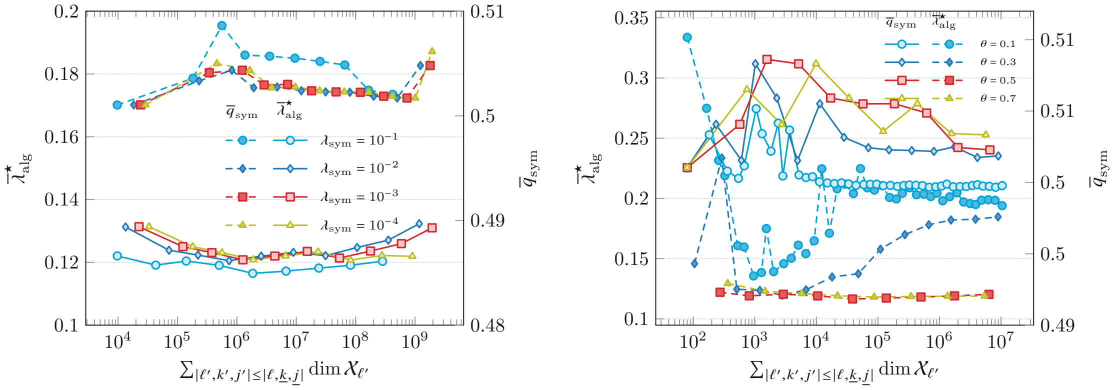


Figure 3.6: Computed upper bounds for $\lambda^{\star}_{\mathrm{alg}} < \overline{\lambda}^{\star}_{\mathrm{alg}}$ for various choices of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\mathrm{sym}} = 0.1$ (right), where we emphasize the double scaling of the y-axis for $\lambda^{\star}_{\mathrm{alg}}$ resp. $\overline{q}_{\mathrm{sym}}$ in both figures.

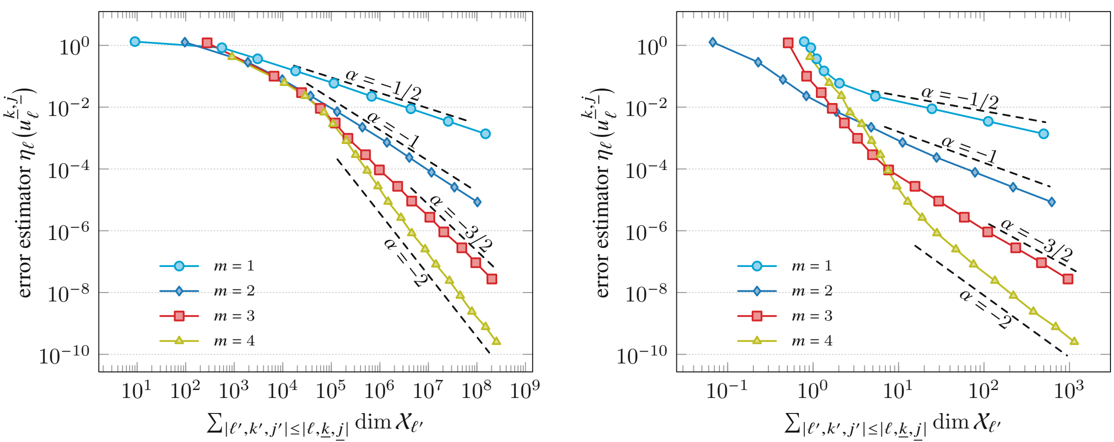


Figure 3.7: *Optimality of AISFEM for the strong convection problem on the Z-shaped domain from Section 3.6.2.* Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ over the computational cost (left) and the elapsed computational time (right).

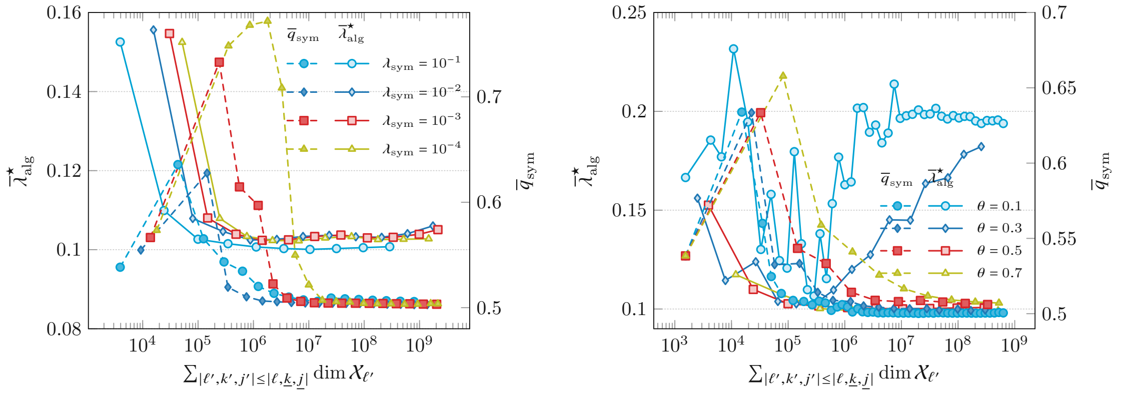


Figure 3.8: *Uniform contraction of the combined solver for the strong convection problem on the Z-shaped domain from Section 3.6.2.* Contraction factor $\overline{q}_{\rm sym}$ and computed upper bound for $\lambda^\star_{\rm alg} < \overline{\lambda}^\star_{\rm alg}$ for various symmetrization stopping parameter $\lambda_{\rm sym}$ with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\rm sym} = 0.1$ (right), where we emphasize the double scaling of the y-axis for $\lambda^\star_{\rm alg}$ resp. $\overline{q}_{\rm sym}$ in both figures.

that the weak formulation takes the form

$$b(u^\star, v) := a(u^\star, v) + \langle \mathcal{K} u^\star \,,\, v\rangle = F(v) \quad \text{for all } \; v \in \mathcal{X}, \tag{3.84}$$

where $F \in \mathcal{X}'$ is a linear and continuous functional, $a(\cdot,\cdot)$ is a symmetric, continuous, and elliptic bilinear form on $\mathcal{X}$, and $\mathcal{K}\colon \mathcal{X} \to \mathcal{X}'$ is a compact operator such that the bilinear form $b(\cdot,\cdot)$ is still elliptic on $\mathcal{X}$. Let $|||\cdot|||$ denote the $a(\cdot,\cdot)$-induced energy norm. For the discrete formulation

$$b(u^\star_\ell, v_\ell) = F(v_\ell) \quad \text{for all } \; v_\ell \in \mathcal{X}_\ell, \tag{3.85}$$

we require an (abstract) inexact iterative solver with iteration map given by $\overline{\Phi}_\ell(F;\cdot)\colon \mathcal{X}_\ell \to \mathcal{X}_\ell$ that contracts the *error* in the energy norm, i.e.,

$$|||u^\star_\ell - \overline{u}^{k+1}_\ell||| \le \overline{q}_{\rm sym}\, |||u^\star_\ell - \overline{u}^k_\ell||| \quad \text{with } \overline{u}^{k+1}_\ell := \overline{\Phi}_\ell(F;\overline{u}^k_\ell) \text{ for all } k \in \mathbb{N}, \tag{3.86}$$

where the contraction constant $0 < \overline{q}_{\rm sym} < 1$ is independent of $\overline{u}^0_\ell \in \mathcal{X}_\ell$. Under such assumptions and with the usual residual *a posteriori* error estimator $\eta_\ell(\cdot)$ (satisfying the abstract assumptions (A1)–(A4)) on nested conforming discrete spaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset \mathcal{X}$, the present work proves that the analysis from [GHPS21] can be generalized from symmetric PDEs (with $\mathcal{K} = 0$) to the general formulation (3.84): Restricting Algorithm 3A to the outer ℓ-loop (for mesh refinement) and the inner k-loop (for the solver associated to $\overline{\Phi}_\ell$), we obtain a simplified index set

$$\overline{\mathcal{Q}} := \{(\ell,k) \in \mathbb{N}_0^2 \mid \overline{u}^k_\ell \text{ is computed by the simplified algorithm}\} \tag{3.87}$$

together with the canonical step counter $|\ell,k| \in \mathbb{N}_0$ on $\overline{\mathcal{Q}}$ defined analogously to (3.21). Then, Lemma 3.11 (lucky non-termination of the solver), Lemma 3.12 (*a priori* convergence), Lemma 3.13 (quasi-Pythagorean estimate), and Lemma 3.14 (contraction of weighted discretization and solver error) hold verbatim (and the proof of Lemma 3.13 indeed relies on the compactness of $\mathcal{K}$) if we replace $u_\ell^{k,\underline{j}}$ in the given proofs by $\overline{u}_\ell^k$ in the current solver setting. Therefore, we obtain full linear convergence in the spirit of Theorem 3.7: For arbitrary adaptivity parameters $0 < \theta \le 1$ and $\lambda_{\mathrm{sym}} > 0$, there exist constants $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ such that

$$\overline{\Delta}_\ell^k \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|}\, \overline{\Delta}_{\ell'}^{k'} \quad \text{for all } (\ell',k'),(\ell,k) \in \overline{Q} \text{ with } |\ell',k'| \le |\ell,k| \text{ and } \ell' \ge \ell_0, \tag{3.88}$$

where $\overline{\Delta}_\ell^k := |||u^\star - \overline{u}_\ell^k||| + \eta_\ell(\overline{u}_\ell^k)$ denotes the corresponding quasi-error. In particular, also Corollary 3.8 holds verbatim with Q replaced by $\overline{Q}$ and $\Delta_\ell^{k,j}$ replaced by $\overline{\Delta}_\ell^k$, i.e., convergence rates with respect to the number of degrees of freedom coincide with rates with respect to the overall computational cost. Finally, it is easy to check that also Theorem 3.9 holds verbatim and proves that, for sufficiently small adaptivity parameters $0 < \theta \ll 1$ and $0 < \lambda_{\mathrm{sym}} \ll 1$ in the sense of (3.33), it holds that

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} < \infty \quad\Longleftrightarrow\quad \sup_{(\ell,k)\in Q} \Big(\sum_{\substack{(\ell',k')\in\overline{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \,\overline{\Delta}_\ell^k < \infty, \tag{3.89}$$

which yields optimal complexity of the simplified algorithm.

In the current analysis, the combined Zarantonello symmetrization with a contractive SPD algebraic solver is used as one solver module to guarantee (3.86) for $\overline{u}_\ell^k := u_\ell^{k,\underline{j}}$ (see Lemma 3.10, where contraction, however, only holds for $1 \le k < \underline{k}[\ell]$), leading to all results being formulated over the triple index set $Q \subset \mathbb{N}_0^3$ (see Section 3.3–3.4).

We note that another choice for solving the arising nonsymmetric FEM systems would be preconditioned GMRES (see, e.g., [SS86; Saa03]), where an optimal preconditioner for the symmetric part would be employed. Then, it is well-known from the field-of-value analysis (see, e.g., [Sta97]) that the algebraic solver would satisfy a *generalized* contraction for the algebraic residual (in a discrete vector norm). However, the link between the algebraic residual and the functional setting appears to be open. Moreover, the *a posteriori* error control of the algebraic error for such a GMRES solver is still to be developed.

While these questions are left for future work, we already note some results that can be achieved along the arguments of [GHPS21]: If the solver $\overline{\Phi}_\ell(F;\cdot)$ provides iterates $(\overline{u}_\ell^k)_{k\in\mathbb{N}_0}$ satisfying only the generalized contraction

$$|||u_\ell^\star - \overline{u}_\ell^k||| \le C_{\mathrm{ctr}}\, \overline{q}_{\mathrm{sym}}^k\, |||u_\ell^\star - \overline{u}_\ell^0||| \quad \text{for all } k \in \mathbb{N} \tag{3.90}$$

together with the *a posteriori* error control

$$|||u_\ell^\star - \overline{u}_\ell^k||| \le C'_{\mathrm{ctr}}\, |||u_\ell^k - \overline{u}_\ell^{k-1}||| \quad \text{for all } k \in \mathbb{N}, \tag{3.91}$$

where $C_{\mathrm{ctr}}, C'_{\mathrm{ctr}} > 0$ and $0 < \overline{q}_{\mathrm{sym}} < 1$ are given constants independently of $\overline{u}_\ell^0 \in \mathcal{X}_\ell$, then full linear convergence (3.88) can be proved for all $0 < \theta \le 1$ under the additional assumption that λ_{sym} has to be sufficiently small. However, the proof of full linear convergence (3.88) for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda_{\mathrm{sym}} > 0$ is open, while optimal complexity (3.89) for sufficiently small $0 < \theta < 1$ and λ_{sym} in the sense of (3.33) remains valid (even with the same proof).

4 On full linear convergence and optimal complexity of adaptive FEM with inexact solver

The Sections 4.1–4.7 of this chapter correspond to the publication:
P. Bringmann, M. Feischl, A. Miraçi, D. Praetorius, and J. Streitberger. On full linear convergence and optimal complexity of adaptive FEM with inexact solver, 2023. arXiv: 2311.15738

4.1 Introduction

Over the past three decades, the mathematical understanding of adaptive finite element methods (AFEMs) has matured; see, e.g., [Dör96; MNS00; BDD04; Ste07; CKNS08; CN12; FFP14] for linear elliptic PDEs, [Vee02; DK08; BDK12; GMZ12] for certain nonlinear PDEs, and [CFPP14] for an axiomatic framework summarizing the earlier references. In most of the cited works, the focus is on *(plain) convergence* [Dör96; MNS00; Vee02; DK08; GMZ12] and optimal convergence rates with respect to the number of degrees of freedom, i.e., *optimal rates*, [BDD04; CKNS08; CN12; BDK12; GMZ12; FFP14].

The adaptive feedback loop strives to approximate the unknown and possibly singular exact PDE solution $u^\star$ on the basis of *a posteriori* error estimators and adaptive mesh refinement strategies. Employing AFEM with *exact solver*, detailed in Algorithm 4A below, generates a sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ of successively refined meshes together with the corresponding finite element solutions $u_\ell^\star \approx u^\star$ and error estimators $\eta_\ell(u_\ell^\star)$ by iterating

$$\boxed{\texttt{solve}} \longrightarrow \boxed{\texttt{estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}} \tag{4.1}$$

A key argument in the analysis of (4.1) in [CKNS08] and succeeding works for symmetric PDEs consists in showing *linear convergence* of the quasi-error

$$\Delta_\ell^\star \le q_{\rm lin}\, \Delta_{\ell-1}^\star \quad \text{with} \quad \Delta_\ell^\star := \big[|||u^\star - u_\ell^\star|||^2 + \gamma\, \eta_\ell(u_\ell^\star)^2 \big]^{1/2} \quad \text{for all } \ell \in \mathbb{N}, \tag{4.2}$$

where $0 < q_{\rm lin}, \gamma < 1$ depend only on the problem setting and the marking parameter θ. Here, $|||\cdot|||$ is the PDE-induced energy norm providing a Pythagorean identity of the form

$$|||u^\star - u_{\ell+1}^\star|||^2 + |||u_{\ell+1}^\star - u_\ell^\star|||^2 = |||u^\star - u_\ell^\star|||^2 \quad \text{for all } \ell \in \mathbb{N}_0. \tag{4.3}$$

An extension of the analysis to nonsymmetric linear PDEs can be done by relaxing the Pythagorean identity to a quasi-Pythagorean estimate in [CN12; FFP14; BHP17]. However, this comes at the expense that either the initial mesh has to be sufficiently fine [CN12], or (4.2) only holds for

$\ell \geq \ell_0 \in \mathbb{N}_0$ [BHP17], or (4.2) holds in the general form (4.5) below, where the constants depend on the adaptively generated meshes [FFP14]. The later work [CFPP14] showed that a *tail-summability* of the estimator sequence

$$\sum_{\ell'=\ell+1}^{\infty} \eta_{\ell'}(u_{\ell'}^\star) \leq C_{\rm lin}' \eta_\ell(u_\ell^\star) \quad \text{for all } \ell \in \mathbb{N}_0 \tag{4.4}$$

or, equivalently, *R-linear convergence*

$$\eta_\ell(u_\ell^\star) \leq C_{\rm lin} q_{\rm lin}^{\ell-\ell'} \, \eta_{\ell'}(u_{\ell'}^\star) \quad \text{for all } \; \ell \geq \ell' \geq 0, \tag{4.5}$$

with $0 < q_{\rm lin} < 1$ and $C_{\rm lin}, C_{\rm lin}' > 0$, suffices to prove convergence. Additionally, a sufficiently small marking parameter θ leads to optimal rates in the sense of [Ste07; CKNS08]. This can be stated in terms of approximation classes [BDD04; Ste08; CKNS08] by mathematically guaranteeing the largest possible convergence rate $s > 0$ with

$$\sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star) < \infty. \tag{4.6}$$

However, due to the incremental nature of adaptivity, the mathematical question on optimal convergence rates should rather refer to the overall computational cost (resp. the cumulative computation time). This, coined as *optimal complexity* in the context of adaptive wavelet methods [CDD01; CDD03], was later adopted for AFEM in [Ste07; CG12]. Therein, optimal complexity is guaranteed for AFEM with *inexact* solver, provided that the computed iterates u_ℓ^k are sufficiently close to the (unavailable) exact discrete solutions $u_\ell^\star$. This theoretical result requires that the algebraic error is controlled by the discretization error multiplied by a sufficiently small solver-stopping parameter λ. However, numerical experiments in [CG12] indicate that also moderate choices of the stopping parameter suffice for optimal complexity. Hence, the interrelated stopping criterion led to a combined solve-estimate module in the adaptive algorithm

$$\boxed{\texttt{solve \& estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}} \tag{4.7}$$

Driven by the interest in AFEMs for nonlinear problems [EV13; CW17; GHPS18; HW20b; HW20a], recent papers [GHPS21; HPW21; HPSV21] aimed to combine linearization and algebraic iterates into a nested adaptive algorithm. Following the latter, the algorithmic decision for either mesh refinement or linearization or algebraic solver step is steered by *a-posteriori*-based stopping criteria with suitable stopping parameters. This allows to balance the error components and compute the inexact approximations $u_\ell^k \approx u_\ell^\star$ given by a contractive solver with iteration counter $k = 1, \dots, \underline{k}[\ell]$ on the mesh $\mathcal{T}_\ell$, and $|\ell, k| \in \mathbb{N}_0$ denotes the lexicographic order of the sequential loop (4.7); see Algorithm 4B below.

Due to an energy identity (coinciding with (4.3) for symmetric linear PDEs), the works [GHPS21; HPW21] prove full R-linear convergence for the quasi-error $\Delta_\ell^k := \big[|||u^\star - u_\ell^k|||^2 + \gamma \, \eta_\ell(u_\ell^k)^2 \big]^{1/2}$ with respect to the lexicographic ordering $|\cdot,\cdot|$, i.e.,

$$\Delta_\ell^k \leq C_{\rm lin} q_{\rm lin}^{|\ell,k|-|\ell',k'|} \, \Delta_{\ell'}^{k'} \quad \text{for all } \; (\ell',k'), (\ell,k) \in \mathcal{Q} \text{ with } |\ell',k'| \leq |\ell,k|, \tag{4.8}$$

which is guaranteed for arbitrary marking parameter θ and stopping parameter λ (with constants $C_{\rm lin} > 0$ and $0 < q_{\rm lin} < 1$ depending on θ and λ). Moreover, [GHPS21] proves that full R-linear convergence is also the key argument for optimal complexity in the sense that it ensures, for all $s > 0$,

$$M(s) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \le C_{\mathrm{cost}}(s)\, M(s), \tag{4.9}$$

where $C_{\mathrm{cost}}(s) > 1$ depends only on s, C_{lin}, and q_{lin}. Since all modules of AFEM with inexact solver as displayed in (4.7) can be implemented at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$, the equivalence (4.9) means that the quasi-error Δ_ℓ^k decays with rate s over the number of elements $\#\mathcal{T}_\ell$ if and only if it decays with rate s over the related *overall* computational work (and hence total computation time).

In essence, optimal complexity of AFEM with inexact solver thus follows from a perturbation argument (by taking the stopping parameter λ sufficiently small) as soon as full linear convergence (4.8) of AFEM with inexact solver and optimal rates of AFEM with exact solver (for sufficiently small θ) have been established; see, e.g., [CFPP14; GHPS21].

In this paper, we present a novel proof of full linear convergence (4.8) with contractive solver that, unlike [GHPS21; HPW21], avoids the Pythagorean identity (4.3), but relies only on the quasi-orthogonality from [CFPP14] (even in its generalized form from [Fei22]). The latter is known to be sufficient and necessary for linear convergence (4.5) in the presence of exact solvers [CFPP14]. In particular, this opens the door to proving optimal complexity for AFEM beyond symmetric energy minimization problems. Moreover, problems exhibiting additional difficulties such as nonsymmetric linear elliptic PDEs, see [BHI+23], or nonlinear PDEs, see [HPSV21], ask for more intricate (nested) solvers that treat iterative symmetrization/linearization together with solving the arising linear SPD systems. This leads to computed approximations $u_\ell^{k,j} \approx u_\ell^\star$ with symmetrization/linearization iteration counter $k = k[\ell]$ and algebraic solver index $j = j[\ell, k]$. The new proof of full linear convergence allows to improve the analysis of [BHI+23; HPSV21] by relaxing the choice of the solver-stopping parameters. Additionally, in the setting of [BHI+23], we are able to show that the full linear convergence holds from the arbitrary *initial* mesh onwards instead of the *a priori* unknown and possibly large mesh threshold level $\ell_0 > 0$. In particular, unlike the previous works [CN12; FFP14; BHP17; BHI+23] that employ a quasi-Pythagorean identity, the new analysis shows that the constants in the full R-linear convergence are independent of $\mathcal{T}_0$ and/or the sequence of adaptively generated meshes and therefore *a priori* fixed. Furthermore, the new analysis does not only improve the state-of-the-art theory of full linear convergence leading to optimal complexity, but also allows the choice of larger solver-stopping parameters which also leads to a better numerical performance in experiments.

The remainder of this work is structured as follows: As a model problem, Section 4.2 formulates a general second-order linear elliptic PDE together with the validity of the so-called *axioms of adaptivity* from [CFPP14] and the quasi-orthogonality from [Fei22]. In Section 4.3, AFEM with exact solver (4.1) is presented in Algorithm 4A and, for completeness and easier readability of the later sections, Theorem 4.4 summarizes the proof of R-linear convergence (4.5) from [CFPP14; Fei22]. Section 4.4 focuses on AFEM with inexact contractive solver (4.7) detailed in Algorithm 4B. The main contribution is the new and more general proof of full R-linear convergence of Theorem 4.7. Corollary 4.11 proves the important equivalence (4.9). The case of AFEM with *nested* contractive solvers, which are useful for nonlinear or nonsymmetric problems, is treated in Section 4.5 by presenting Algorithm 4C from [BHI+23] and improving its main result in Theorem 4.15. In Section 4.6, we discuss the impact of the new analysis on AFEM for nonlinear PDEs. We show that Theorem 4.15 applies also to the setting from [HPSV21], namely strongly monotone PDEs with scalar nonlinearity. Numerical experiments and remarks are discussed in-depth in Section 4.7, where the impact of the adaptivity parameters on the overall cost is investigated empirically.

Throughout the proofs, the notation $A \lesssim B$ abbreviates $A \le CB$ for some positive constant $C > 0$ whose dependencies are clearly presented in the respective theorem and $A \simeq B$ abbreviates $A \lesssim B \lesssim A$.

4.2 General second-order linear elliptic PDEs

On a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \ge 1$, we consider the PDE

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b} \cdot \nabla u^\star + c u^\star = f - \operatorname{div} \boldsymbol{f} \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega, \tag{4.10}$$

where $\boldsymbol{A}, \boldsymbol{b}, c \in L^\infty(\Omega)$ and $\boldsymbol{f}, f \in L^2(\Omega)$ with, for almost every $x \in \Omega$, positive definite $\boldsymbol{A}(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\boldsymbol{b}(x), \boldsymbol{f}(x) \in \mathbb{R}^d$, and $c(x), f(x) \in \mathbb{R}$. With $\langle \cdot\,,\,\cdot \rangle_{L^2(\Omega)}$ denoting the usual $L^2(\Omega)$-scalar product, we suppose that the PDE fits into the setting of the Lax–Milgram lemma, i.e., the bilinear forms

$$a(u,v) := \langle \boldsymbol{A}\nabla u\,,\,\nabla v\rangle_{L^2(\Omega)} \quad \text{and} \quad b(u,v) := a(u,v) + \langle \boldsymbol{b}\cdot\nabla u + cu\,,\,v\rangle_{L^2(\Omega)} \tag{4.11}$$

are continuous and elliptic on $H^1_0(\Omega)$. Then, indeed, $a(\cdot,\cdot)$ is a scalar product and $|||u||| := a(u,u)^{1/2}$ defines an equivalent norm on $H^1_0(\Omega)$. Moreover, the weak formulation

$$b(u^\star, v) = F(v) := \langle f\,,\,v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}\,,\,\nabla v\rangle_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega) \tag{4.12}$$

admits a unique solution $u^\star \in H^1_0(\Omega)$. Let $0 < C_{\mathrm{ell}} \le C_{\mathrm{bnd}}$ denote the boundedness and ellipticity constant of $b(\cdot,\cdot)$ with respect to $|||\cdot|||$, i.e., there holds

$$C_{\mathrm{ell}}\, |||v|||^2 \le b(v,v) \quad \text{and} \quad |b(v,w)| \le C_{\mathrm{bnd}}\, |||v|||\, |||w||| \quad \text{for all } v, w \in \mathcal{X}.$$

Let $\mathcal{T}_0$ be an initial conforming triangulation of $\Omega \subset \mathbb{R}^d$ into compact simplices. The mesh refinement employs newest-vertex bisection (NVB). We refer to [Ste08] for NVB with admissible $\mathcal{T}_0$ and $d \ge 2$, to [KPP13] for NVB with general $\mathcal{T}_0$ for $d = 2$, and to the recent work [DGS23] for NVB with general $\mathcal{T}_0$ in any dimension $d \ge 2$. For $d = 1$, we refer to [AFF+15]. For each triangulation $\mathcal{T}_H$ and $\mathcal{M}_H \subseteq \mathcal{T}_H$, let $\mathcal{T}_h := \mathtt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ be the coarsest conforming refinement of $\mathcal{T}_H$ such that at least all elements $T \in \mathcal{M}_H$ have been refined, i.e., $\mathcal{M}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_h$. To abbreviate notation, we write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ can be obtained from $\mathcal{T}_H$ by finitely many steps of NVB and, in particular, $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$.

For each $\mathcal{T}_H \in \mathbb{T}$, we consider conforming finite element spaces

$$\mathcal{X}_H := \{v_H \in H^1_0(\Omega) \,:\, v_H|_T \text{ is a polynomial of total degree} \le p \text{ for all } T \in \mathcal{T}_H\}, \tag{4.13}$$

where $p \in \mathbb{N}$ is a fixed polynomial degree. We note that $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ yields nestedness $\mathcal{X}_H \subseteq \mathcal{X}_h$ of the corresponding discrete spaces.

Given $\mathcal{T}_H \in \mathbb{T}$, there exists a unique Galerkin solution $u_H^\star \in \mathcal{X}_H$ solving

$$b(u_H^\star, v_H) = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{4.14}$$

Moreover, there holds the following Céa lemma

$$|||u^\star - u_H^\star||| \le C_{\text{Céa}} \min_{v_H \in \mathcal{X}_H} |||u^\star - v_H||| \quad \text{with a constant } 1 \le C_{\text{Céa}} \le C_{\text{bnd}}/C_{\text{ell}}, \tag{4.15}$$

where $C_{\text{Céa}} \to 1$ as adaptive mesh-refinement progresses [BHP17, Theorem 20].

We consider the residual error estimator $\eta_H(\cdot)$ defined, for $T \in \mathcal{T}_H$ and $v_H \in \mathcal{X}_H$, by

$$\begin{aligned}\eta_H(T, v_H)^2 := |T|^{2/d}\, \| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_H + c\, v_H - f\|_{L^2(T)}^2 \\ + |T|^{1/d}\, \| [\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f})\cdot n]\!] \|_{L^2(\partial T \cap \Omega)}^2,\end{aligned} \tag{4.16a}$$

where $[\![\cdot]\!]$ denotes the jump over $(d-1)$-dimensional faces. Clearly, the well-posedness of (4.16a) requires more regularity of $\boldsymbol{A}$ and $\boldsymbol{f}$ than stated above, e.g., $\boldsymbol{A}|_T, \boldsymbol{f}|_T \in W^{1,\infty}(T)$ for all $T \in \mathcal{T}_0$. To abbreviate notation, we define, for all $\mathcal{U}_H \subseteq \mathcal{T}_H$ and all $v_H \in \mathcal{X}_H$,

$$\eta_H(v_H) := \eta_H(\mathcal{T}_H, v_H) \quad \text{with} \quad \eta_H(\mathcal{U}_H, v_H) := \Big(\sum_{T\in\mathcal{U}_H} \eta_H(T, v_H)^2\Big)^{1/2}. \tag{4.16b}$$

From [CFPP14], we recall that the error estimator satisfies the following properties.

Proposition 4.1 (axioms of adaptivity)**.** *There exist constants $C_{\text{stab}}, C_{\text{rel}}, C_{\text{drel}}, C_{\text{mon}} > 0$, and $0 < q_{\text{red}} < 1$ such that the following properties are satisfied for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any conforming refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with the corresponding Galerkin solutions $u_H^\star \in \mathcal{X}_H$, $u_h^\star \in \mathcal{X}_h$ to (4.14) and arbitrary $v_H \in \mathcal{X}_H$, $v_h \in \mathcal{X}_h$.*

(A1) *stability.* $|\eta_h(\mathcal{T}_h \cap \mathcal{T}_H, v_h) - \eta_H(\mathcal{T}_h \cap \mathcal{T}_H, v_H)| \le C_{\text{stab}}\, |||v_h - v_H|||$.

(A2) *reduction.* $\eta_h(\mathcal{T}_h \backslash \mathcal{T}_H, v_H) \le q_{\text{red}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, v_H)$.

(A3) *reliability.* $|||u^\star - u_H^\star||| \le C_{\text{rel}}\, \eta_H(u_H^\star)$.

(A3⁺) *discrete reliability.* $|||u_h^\star - u_H^\star||| \le C_{\text{drel}}\, \eta_H(\mathcal{T}_H \backslash \mathcal{T}_h, u_H^\star)$.

(QM) *quasi-monotonicity.* $\eta_h(u_h^\star) \le C_{\text{mon}}\, \eta_H(u_H^\star)$.

The constant C_{rel} depends only on uniform shape regularity of all meshes $\mathcal{T}_H \in \mathbb{T}$ and the dimension d, while C_{stab} and C_{drel} additionally depend on the polynomial degree p. The constant q_{red} reads $q_{\text{red}} := 2^{-1/(2d)}$ for bisection-based refinement rules in $\mathbb{R}^d$ and the constant C_{mon} can be bounded by $C_{\text{mon}} \le \min\{1 + C_{\text{stab}}(1 + C_{\text{Céa}})C_{\text{rel}}\,,\ 1 + C_{\text{stab}}\, C_{\text{drel}}\}$. □

In addition to the estimator properties in Proposition 4.1, we recall the following quasi-orthogonality result from [Fei22] as one cornerstone of the improved analysis in this chapter.

Proposition 4.2 (validity of quasi-orthogonality)**.** *There exist $C_{\text{orth}} > 0$ and $0 < \delta \le 1$ such that the following holds: For any sequence $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H_0^1(\Omega)$ of nested finite-dimensional subspaces, the corresponding Galerkin solutions $u_\ell^\star \in \mathcal{X}_\ell$ to (4.14) satisfy*

(A4) *quasi-orthogonality.* $\displaystyle\sum_{\ell'=\ell}^{\ell+N} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 \le C_{\text{orth}}(N+1)^{1-\delta}\, |||u^\star - u_\ell^\star|||^2$ *for all* $\ell, N \in \mathbb{N}_0$.

Here, C_{orth} and δ depend only on the dimension d, the elliptic bilinear form $b(\cdot,\cdot)$, and the chosen *norm $|||\cdot|||$, but are independent of the spaces $\mathcal{X}_\ell$.* □

Remark 4.3. *Quasi-orthogonality (A4) is a generalization of the Pythagorean identity (4.3) for symmetric problems. Indeed, if $\boldsymbol{b} = 0$ in (4.10) and $a(\cdot,\cdot) := b(\cdot,\cdot)$ is a scalar product, the Galerkin method for nested subspaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H^1_0(\Omega)$ guarantees (4.3). Thus, the telescopic series proves (A4) with $C_{\rm orth} = 1$ and $\delta = 1$. We highlight that [Fei22] proves (A4) even for more general linear problems and Petrov–Galerkin discretizations.*

A closer look at the proofs of R-linear convergence in Section 4.3–4.5 below reveals that they rely only on the properties (A1), (A2), (A3), (A4), and (QM), but not on (A3$^+$), the Céa lemma (4.15), or linearity of the PDE. Hence, Algorithms 4A, 4B, and 4C and the corresponding Theorems 4.4, 4.7, and 4.15 apply beyond the linear problem (4.10); see Section 4.6 for a nonlinear PDE.

4.3 AFEM with exact solution

To outline the new proof strategy, we first consider the standard adaptive algorithm, where the arising Galerkin systems (4.14) are solved exactly (see, e.g., [CKNS08]).

Algorithm 4A: AFEM with exact solver

Given an initial mesh $\mathcal{T}_0$ and adaptivity parameters $0 < \theta \leq 1$ and $C_{\rm mark} \geq 1$, iterate the following steps for all $\ell = 0, 1, 2, 3, \ldots$:

(i) **Solve:** Compute the exact solution $u_\ell^\star \in \mathcal{X}_\ell$ to (4.14).

(ii) **Estimate:** Compute the refinement indicators $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$.

(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^\star] := \{\mathcal{U}_\ell \subseteq \mathcal{T}_\ell \colon \theta\eta_\ell(u_\ell^\star)^2 \leq \eta_\ell(\mathcal{U}_\ell, u_\ell^\star)^2\}$ satisfying the Dörfler marking criterion with almost minimal cardinality

$$\#\mathcal{M}_\ell \leq C_{\rm mark} \min_{\mathcal{U}_\ell^\star \in \mathbb{M}_\ell[\theta, u_\ell^\star]} \#\mathcal{U}_\ell^\star. \tag{4.17}$$

(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

The following theorem asserts convergence of Algorithm 4A in the spirit of [CFPP14] and the proof given below essentially summarizes the arguments from [Fei22]. It will, however, be the starting point for the later generalizations, i.e., for the adaptive algorithms below with inexact solvers.

Theorem 4.4: R-linear convergence of Algorithm 4A

Let $0 < \theta \leq 1$ and $C_{\rm mark} \geq 1$ be arbitrary. Then, Algorithm 4A guarantees R-linear convergence of the estimators $\eta_\ell(u_\ell^\star)$, i.e., there exist constants $0 < q_{\rm lin} < 1$ and $C_{\rm lin} > 0$ such that

$$\eta_{\ell+n}(u_{\ell+n}^\star) \leq C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell(u_\ell^\star) \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.18}$$

Remark 4.5. *For vanishing convection $\boldsymbol{b} = 0$ in (4.10) and $a(\cdot,\cdot) := b(\cdot,\cdot)$, [CKNS08] proves linear convergence of the quasi-error (4.2). Together with reliability (A3), this yields R-linear convergence* *of the estimator sequence*

$$\eta_{\ell+n}(u^\star_{\ell+n}) \le \frac{(C_{\mathrm{rel}}^2+\gamma)^{1/2}}{\gamma^{1/2}}\, q_{\mathrm{ctr}}^n\, \eta_\ell(u^\star_\ell) \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.19}$$

In this sense, Theorem 4.4 is weaker than linear convergence (4.2) from [CKNS08], but provides a direct proof of R-linear convergence even if $b(\cdot,\cdot) \neq a(\cdot,\cdot)$. Moreover, while the proof of (4.2) crucially relies on the Pythagorean identity (4.3), the works [FFP14; BHP17] extend the analysis to the general second-order linear elliptic PDE (4.10) using

$$\forall 0<\varepsilon<1\, \exists \ell_0 \in \mathbb{N}_0\, \forall \ell \ge \ell_0: \quad |||u^\star - u^\star_{\ell+1}|||^2 \le |||u^\star - u^\star_\ell|||^2 - \varepsilon\, |||u^\star_{\ell+1} - u^\star_\ell|||^2. \tag{4.20}$$

However, the index ℓ_0 depends on the exact solution $u^\star$ and on the sequence of exact discrete solutions $(u^\star_\ell)_{\ell\in\mathbb{N}_0}$. Moreover, $\ell_0 = 0$ requires sufficiently fine $\mathcal{T}_0$ in [CN12; BHP17] while the constants in (4.18) depend on $u^\star$ and the sequence $(u^\star_\ell)_{\ell\in\mathbb{N}_0}$ in [FFP14]. In contrast to that, R-linear convergence (4.18) from Theorem 4.4 holds with $\ell_0 = 0$ and any initial mesh $\mathcal{T}_0$, and the constants are independent of $u^\star$ and $(u^\star_\ell)_{\ell\in\mathbb{N}_0}$.

The proof of Theorem 4.4 relies on the following elementary lemma that extends arguments implicitly found for the estimator sequence in [Fei22] but that will be employed for certain quasi-errors in the present chapter. Its proof is found in Appendix 4.A.

Lemma 4.6 (tail summability criterion)**.** *Let $(a_\ell)_{\ell\in\mathbb{N}_0}, (b_\ell)_{\ell\in\mathbb{N}_0}$ be scalar sequences in $\mathbb{R}_{\ge 0}$. With given constants $0<q<1$, $0<\delta<1$, and $C_1, C_2 > 0$, suppose that*

$$a_{\ell+1} \le q a_\ell + b_\ell, \quad b_{\ell+N} \le C_1\, a_\ell, \ \text{and} \ \sum_{\ell'=\ell}^{\ell+N} b_\ell^2 \le C_2\,(N+1)^{1-\delta}\, a_\ell^2 \ \text{for all } \ell, N \in \mathbb{N}_0. \tag{4.21}$$

Then, $(a_\ell)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent, i.e., there exist $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ with

$$a_{\ell+n} \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^n\, a_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.22}$$

Proof of Theorem 4.4. We employ Lemma 4.6 for the sequences defined by $a_\ell = \eta_\ell(u^\star_\ell)$ and $b_\ell := C_{\mathrm{stab}}\, |||u^\star_{\ell+1} - u^\star_\ell|||$. First, we note that

$$|||u^\star_{\ell''} - u^\star_{\ell'}||| \overset{\text{(A3)}}{\lesssim} \eta_{\ell''}(u^\star_{\ell''}) + \eta_{\ell'}(u^\star_{\ell'}) \overset{\text{(QM)}}{\lesssim} \eta_\ell(u^\star_\ell) \quad \text{for all } \ell, \ell', \ell'' \in \mathbb{N}_0 \text{ with } \ell \le \ell' \le \ell''. \tag{4.23}$$

In particular, this proves $b_{\ell+N} \lesssim a_\ell$ for all $\ell, N \in \mathbb{N}_0$. Moreover, quasi-orthogonality (A4) and reliability (A3) show

$$\sum_{\ell'=\ell}^{\ell+N} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 \le C_{\mathrm{orth}} C_{\mathrm{rel}}^2\,(N+1)^{1-\delta}\, \eta_\ell(u^\star_\ell)^2 \quad \text{for all } \ell, N \in \mathbb{N}_0. \tag{4.24}$$

In order to verify (4.21), it thus only remains to prove the perturbed contraction of a_ℓ. To this end, let $\ell \in \mathbb{N}_0$. Then, stability (A1) and reduction (A2) show

$$\eta_{\ell+1}(u^\star_\ell)^2 \le \eta_\ell(\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell, u^\star_\ell)^2 + q_{\mathrm{red}}^2 \eta_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}, u^\star_\ell)^2 = \eta_\ell(u^\star_\ell)^2 - (1-q_{\mathrm{red}}^2)\,\eta_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}, u^\star_\ell)^2.$$

Moreover, Dörfler marking (4.17) and refinement of (at least) all marked elements lead to

$$\theta\eta_\ell(u_\ell^\star)^2 \overset{(4.17)}{\le} \eta_\ell(\mathcal{M}_\ell, u_\ell^\star)^2 \le \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}, u_\ell^\star)^2.$$

The combination of the two previously displayed formulas results in

$$\eta_{\ell+1}(u_\ell^\star) \le q_\theta\, \eta_\ell(u_\ell^\star) \quad \text{with} \quad 0 < q_\theta := \big[1 - (1 - q_{\rm red}^2)\theta\big]^{1/2} < 1.$$

Finally, stability (A1) thus leads to the desired estimator reduction estimate

$$\eta_{\ell+1}(u_{\ell+1}^\star) \le q_\theta\, \eta_\ell(u_\ell^\star) + C_{\rm stab}\, |||u_{\ell+1}^\star - u_\ell^\star||| \quad \text{for all } \ell \in \mathbb{N}_0. \tag{4.25}$$

Altogether, all the assumptions (4.21) are satisfied and Lemma 4.6 concludes the proof. □

4.4 AFEM with contractive solver

Let $\Psi_H : \mathcal{X}_H \to \mathcal{X}_H$ be the iteration mapping of a uniformly contractive solver, i.e.,

$$|||u_H^\star - \Psi_H(v_H)||| \le q_{\rm ctr}\, |||u_H^\star - v_H||| \quad \text{for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } v_H \in \mathcal{X}_H. \tag{4.26}$$

The following algorithm is thoroughly analyzed in [GHPS21] under the assumption that the problem is symmetric (and hence the Pythagorean identity (4.3) holds).

Algorithm 4B: AFEM with contractive solver

Given an initial mesh $\mathcal{T}_0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$, a solver-stopping parameter $\lambda > 0$, and an initial guess $u_0^0 \in \mathcal{X}_0$, iterate the following steps for all $\ell = 0, 1, 2, 3, \ldots$:

(i) **Solve & Estimate:** For all $k = 1, 2, 3, \ldots$, repeat (a)–(b) until

$$|||u_\ell^k - u_\ell^{k-1}||| \le \lambda\, \eta_\ell(u_\ell^k). \tag{4.27}$$

(a) Compute $u_\ell^k := \Psi_\ell(u_\ell^{k-1})$ with one step of the contractive solver.

(b) Compute the refinement indicators $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$.

(ii) Upon termination of the iterative solver, define the index $\underline{k}[\ell] := k \in \mathbb{N}$.

(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k}}]$ satisfying (4.17) with $u_\ell^\star$ replaced by $u_\ell^{\underline{k}}$.

(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and employ nested iteration $u_{\ell+1}^0 := u_\ell^{\underline{k}}$.

The sequential nature of Algorithm 4B gives rise to the countably infinite index set

$$\mathcal{Q} := \big\{(\ell, k) \in \mathbb{N}_0^2 \;:\; u_\ell^k \in \mathcal{X}_\ell \text{ is defined in Algorithm 4B}\big\} \tag{4.28}$$

together with the lexicographic ordering

$$(\ell', k') \le (\ell, k) \quad :\Longleftrightarrow \quad u_{\ell'}^{k'} \text{ is defined not later than } u_\ell^k \text{ in Algorithm 4B} \tag{4.29}$$

and the total step counter

$$|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} \,:\, (\ell', k') \le (\ell, k)\} \in \mathbb{N}_0 \quad \text{for all } (\ell, k) \in \mathcal{Q}.$$

Defining the stopping indices

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 \,:\, (\ell, 0) \in \mathcal{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \tag{4.30a}$$

$$\underline{k}[\ell] := \sup\{k \in \mathbb{N}_0 \,:\, (\ell, k) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell, 0) \in \mathcal{Q}, \tag{4.30b}$$

we note that these definitions are consistent with that of Algorithm 4B(ii). We abbreviate $\underline{k} = \underline{k}[\ell]$, whenever the index ℓ is clear from the context, e.g., $u_\ell^{\underline{k}} := u_\ell^{\underline{k}[\ell]}$ or $(\ell, \underline{k}) = (\ell, \underline{k}[\ell])$.

As $\mathcal{Q}$ is an infinite set, the typical case is $\underline{\ell} = \infty$ and $\underline{k}[\ell] < \infty$ for all $\ell \in \mathbb{N}_0$, whereas $\underline{\ell} < \infty$ implies that $\underline{k}[\underline{\ell}] = \infty$, i.e., non-termination of the iterative solver on the mesh $\mathcal{T}_{\underline{\ell}}$. The following theorem states convergence of Algorithm 4B. In particular, it shows that $\underline{\ell} < \infty$ implies $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$ and consequently $u^\star = u_{\underline{\ell}}^\star$ by reliability (A3).

Theorem 4.7: full R-linear convergence of Algorithm 4B

Let $0 < \theta \le 1$, $C_{\mathrm{mark}} \ge 1$, $\lambda > 0$, and $u_0^0 \in \mathcal{X}_0$ be arbitrary. Then, Algorithm 4B guarantees R-linear convergence of the modified quasi-error

$$\mathsf{H}_\ell^k := |||u_\ell^\star - u_\ell^k||| + \eta_\ell(u_\ell^k), \tag{4.31}$$

i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that

$$\mathsf{H}_\ell^k \le C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|}\, \mathsf{H}_{\ell'}^{k'} \quad \text{for all } (\ell', k'), (\ell, k) \in \mathcal{Q} \text{ with } |\ell', k'| \le |\ell, k|. \tag{4.32}$$

Remark 4.8. *Unlike [GHPS21] (and [CKNS08]), Theorem 4.7 and its proof employ the quasi-error H_ℓ^k from (4.31) instead of $\Delta_\ell^k := \big[|||u^\star - u_\ell^k|||^2 + \gamma\,\eta_\ell(u_\ell^k)^2\big]^{1/2}$ analogous to (4.2). We note that stability (A1) and reliability (A3) yield $\Delta_\ell^k \lesssim \mathsf{H}_\ell^k$, while the converse estimate follows from the Céa lemma (4.15).*

Remark 4.9. *The work [GHPS21] extends the ideas of [CKNS08] (that proves (4.2) for AFEM with exact solver) and of [FP18] (that extends (4.2) to the final iterates for AFEM with contractive solver). For the scalar product $b(\cdot,\cdot) = a(\cdot,\cdot)$ and arbitrary stopping parameters $\lambda > 0$, it shows that the quasi-error Δ_ℓ^k from Remark 4.8 satisfies contraction*

$$\Delta_\ell^k \le q_{\mathrm{ctr}}\, \Delta_\ell^{k-1} \quad \text{for all } (\ell, k) \in \mathcal{Q} \text{ with } 0 < k < \underline{k}[\ell], \tag{4.33a}$$

$$\Delta_{\ell+1}^0 \le q_{\mathrm{ctr}}\, \Delta_\ell^{\underline{k}-1} \quad \text{for all } (\ell, \underline{k}) \in \mathcal{Q} \tag{4.33b}$$

with contraction constant $0 < q_{\mathrm{ctr}} < 1$, along the approximations $u_\ell^k \in \mathcal{X}_\ell$ generated by Algorithm 4B. The proof of (4.33) can be generalized similarly to Remark 4.5, see [BHI+23]: With the quasi-Pythagorean estimate (4.20), the contraction (4.33) transfers to general second-order linear elliptic PDEs (4.10) under the restriction that (4.33b) holds only for all $\ell \ge \ell_0$, where $\ell_0 \in \mathbb{N}_0$ exists, but is unknown in practice. While, as noted in Remark 4.5, contraction (4.33) implies full R-linear convergence (4.32), the proof of Theorem 4.7 works under much weaker assumptions than that of [GHPS21] and covers the PDE (4.10) with $\ell_0 = 0$.

The proof of Theorem 4.7 relies on Lemma 4.6 and the following elementary result essentially taken from [CFPP14, Lemma 4.9]. Its proof is found in Appendix 4.A.

Lemma 4.10 (tail summability vs. R-linear convergence). *Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ be a scalar sequence in $\mathbb{R}_{\geq 0}$ and $m > 0$. Then, the following statements are equivalent:*

(i) ***tail summability:*** *There exists a constant $C_m > 0$ such that*

$$\sum_{\ell'=\ell+1}^{\infty} a_{\ell'}^m \leq C_m\, a_\ell^m \quad \text{for all } \ell \in \mathbb{N}_0. \tag{4.34}$$

(ii) ***R-linear convergence:*** *There holds (4.22) with certain $0 < q_{\rm lin} < 1$ and $C_{\rm lin} > 0$.*

Before proving Theorem 4.7, we want to briefly summarize its proof strategy. First, we show that the estimator reduction together with the contraction (4.33) of the algebraic solver leads to tail-summability of a weighted quasi-error on the mesh level ℓ. Second, we show that the quasi-error from (4.31) is contractive in the algebraic solver index k and is stable in the nested iteration. Finally, we combine these two steps to prove R-linear convergence of the quasi-error (4.32).

Proof of Theorem 4.7. The proof is split into two steps.

Step 1 (tail summability with respect to ℓ). Let $\ell \in \mathbb{N}$ with $(\ell+1, \underline{k}) \in \mathcal{Q}$. Algorithm 4B guarantees nested iteration $u_{\ell+1}^0 = u_\ell^{\underline{k}}$ and $\underline{k}[\ell] \geq 1$. This and contraction of the algebraic solver (4.26) show

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}}||| \overset{(4.26)}{\leq} q_{\rm ctr}^{\underline{k}[\ell+1]}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \leq q_{\rm ctr}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| \tag{4.35}$$

As in the proof of Theorem 4.4, one obtains the estimator reduction

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k}}) \overset{(4.25)}{\leq} q_\theta\, \eta_\ell(u_\ell^{\underline{k}}) + C_{\rm stab}\, |||u_{\ell+1}^{\underline{k}} - u_\ell^{\underline{k}}||| \overset{(4.35)}{\leq} q_\theta\, \eta_\ell(u_\ell^{\underline{k}}) + (q_{\rm ctr}+1) C_{\rm stab}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k}}|||. \tag{4.36}$$

Choosing $0 < \gamma \leq 1$ with $0 < q_{\rm ctr} := \max\{q_{\rm ctr} + (q_{\rm ctr}+1) C_{\rm stab}\gamma\,,\, q_\theta\} < 1$, the combination of (4.35)–(4.36) reads

$$\begin{aligned} a_{\ell+1} &:= |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}}||| + \gamma\, \eta_{\ell+1}(u_{\ell+1}^{\underline{k}}) \leq q_{\rm ctr}\, \big[|||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \gamma\, \eta_\ell(u_\ell^{\underline{k}})\big] \\ &\leq q_{\rm ctr}\, \big[|||u_\ell^\star - u_\ell^{\underline{k}}||| + \gamma\, \eta_\ell(u_\ell^{\underline{k}})\big] + q_{\rm ctr}\, |||u_{\ell+1}^\star - u_\ell^\star||| =: q_{\rm ctr} a_\ell + b_\ell. \end{aligned} \tag{4.37}$$

Moreover, estimate (4.23) from the proof of Theorem 4.4 and stability (A1) prove that

$$|||u_{\ell''}^\star - u_{\ell'}^\star||| \overset{(4.23)}{\lesssim} \eta_\ell(u_\ell^\star) \overset{(A1)}{\lesssim} |||u_\ell^\star - u_\ell^{\underline{k}}||| + \eta_\ell(u_\ell^{\underline{k}}) \simeq a_\ell \text{ for } \ell \leq \ell' \leq \ell'' \leq \underline{\ell} \text{ with } (\ell, \underline{k}) \in \mathcal{Q}, \tag{4.38}$$

which yields $b_{\ell+N} \lesssim a_\ell$ for all $0 \leq \ell \leq \ell + N \leq \underline{\ell}$ with $(\ell, \underline{k}) \in \mathcal{Q}$. As in (4.24), we see

$$\begin{aligned} \sum_{\ell'=\ell}^{\ell+N} b_{\ell'}^2 &\simeq \sum_{\ell'=\ell}^{\ell+N} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 \overset{(A4)}{\lesssim} (N+1)^{1-\delta}\, |||u^\star - u_\ell^\star|||^2 \overset{(A3)}{\lesssim} (N+1)^{1-\delta}\, \eta_\ell(u_\ell^\star)^2 \\ &\overset{(A1)}{\lesssim} (N+1)^{1-\delta}\, \big[\eta_\ell(u_\ell^{\underline{k}}) + |||u_\ell^\star - u_\ell^{\underline{k}}|||\big]^2 \simeq (N+1)^{1-\delta}\, a_\ell^2 \text{ for all } 0 \leq \ell \leq \ell + N < \underline{\ell}. \end{aligned} \tag{4.39}$$

Hence, the assumptions (4.21) are satisfied and Lemma 4.6 concludes tail summability (or equivalently R-linear convergence by Lemma 4.10) of $\mathsf{H}_{\ell}^{\underline{k}} \simeq a_\ell$, i.e.,

$$\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathsf{H}_{\ell'}^{\underline{k}} \lesssim \mathsf{H}_{\ell}^{\underline{k}} \quad \text{for all } \ 0 \le \ell < \underline{\ell}. \tag{4.40}$$

Step 2 (tail summability with respect to ℓ and k). First, for $0 \le k < k' < \underline{k}[\ell]$, the failure of the termination criterion (4.27) and contraction of the solver (4.26) prove that

$$\mathsf{H}_{\ell}^{k'} \overset{(4.27)}{\lesssim} |||u_\ell^\star - u_\ell^{k'}||| + |||u_\ell^{k'} - u_\ell^{k'-1}||| \overset{(4.26)}{\lesssim} |||u_\ell^\star - u_\ell^{k'-1}||| \overset{(4.26)}{\lesssim} q_{\mathrm{ctr}}^{k'-k}\, |||u_\ell^\star - u_\ell^{k}||| \overset{(4.31)}{\le} q_{\mathrm{ctr}}^{k'-k}\, \mathsf{H}_{\ell}^{k}.$$

Second, for $(\ell, \underline{k}) \in \mathcal{Q}$, it holds that

$$\begin{aligned} \mathsf{H}_{\ell}^{\underline{k}} &\overset{(\mathrm{A1})}{\lesssim} |||u_\ell^\star - u_\ell^{\underline{k}}||| + \eta_\ell(u_\ell^{\underline{k}-1}) + |||u_\ell^{\underline{k}} - u_\ell^{\underline{k}-1}||| \\ &\le \mathsf{H}_{\ell}^{\underline{k}-1} + 2\,|||u_\ell^\star - u_\ell^{\underline{k}}||| \overset{(4.26)}{\le} (1 + 2\,q_{\mathrm{ctr}})\, \mathsf{H}_{\ell}^{\underline{k}-1} \quad \text{for all } (\ell, \underline{k}) \in \mathcal{Q}. \end{aligned}$$

Hence, we may conclude

$$\mathsf{H}_{\ell}^{k'} \lesssim q_{\mathrm{ctr}}^{k'-k}\, \mathsf{H}_{\ell}^{k} \quad \text{for all } 0 \le k \le k' \le \underline{k}[\ell]. \tag{4.41}$$

With $|||u_{\ell+1}^\star - u_\ell^\star||| \lesssim a_\ell \simeq \mathsf{H}_{\ell}^{\underline{k}}$ from (4.23), stability (A1) and reduction (A2) show

$$\mathsf{H}_{\ell+1}^{0} = |||u_{\ell+1}^\star - u_\ell^{\underline{k}}||| + \eta_{\ell+1}(u_\ell^{\underline{k}}) \le \mathsf{H}_{\ell}^{\underline{k}} + |||u_{\ell+1}^\star - u_\ell^\star||| \lesssim \mathsf{H}_{\ell}^{\underline{k}} \quad \text{for all } (\ell, \underline{k}) \in \mathcal{Q}. \tag{4.42}$$

Overall, the geometric series proves tail summability (4.34) via

$$\begin{aligned} \sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| > |\ell,k|}} \mathsf{H}_{\ell'}^{k'} &= \sum_{k'=k+1}^{\underline{k}[\ell]} \mathsf{H}_{\ell}^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathsf{H}_{\ell'}^{k'} \\ &\overset{(4.41)}{\lesssim} \mathsf{H}_{\ell}^{k} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathsf{H}_{\ell'}^{0} \overset{(4.42)}{\lesssim} \mathsf{H}_{\ell}^{k} + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathsf{H}_{\ell'}^{\underline{k}} \overset{(4.40)}{\lesssim} \mathsf{H}_{\ell}^{k} + \mathsf{H}_{\ell}^{\underline{k}} \overset{(4.41)}{\lesssim} \mathsf{H}_{\ell}^{k} \ \text{for all } \ (\ell, k) \in \mathcal{Q}. \end{aligned}$$

Since $\mathcal{Q}$ is countable and linearly ordered, Lemma 4.10 concludes the proof of (4.32) □

The following comments on the computational cost of implementations of standard finite element methods underline the importance of full linear convergence (4.32).

- ***Solve & Estimate.*** One solver step of an optimal multigrid method can be performed in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, if smoothing is done according to the grading of the mesh [WZ17; IMPS24]. Instead, one step of a multigrid method on $\mathcal{T}_\ell$, where smoothing is done on all levels and all vertex patches needs $\mathcal{O}(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'})$ operations. The same remark is valid for the preconditioned CG method with optimal additive Schwarz or BPX preconditioner [CNX12]. One solver step can be realized via successive updates in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, while $\mathcal{O}\big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\big)$ is faced if the preconditioner does not respect the grading of the mesh hierarchy.

- ***Mark.*** The Dörfler marking strategy (4.17) can be realized in linear complexity $\mathcal{O}(\#\mathcal{T}_\ell)$; see [Ste07] for $C_{\rm mark} = 2$ and [PP20] for $C_{\rm mark} = 1$.
- ***Refine.*** Local mesh refinement (including mesh closure) of $\mathcal{T}_\ell$ by bisection can be realized in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations; see, e.g., [BDD04; Ste07].

Since the adaptive algorithm depends on the full history of algorithmic decisions, the overall computational cost until step $(\ell, k) \in \mathcal{Q}$, i.e., until (and including) the computation of u_ℓ^k, is thus proportionally bounded by

$$\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \le \texttt{cost}(\ell,k) \le \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \sum_{\ell''=0}^{\ell'} \#\mathcal{T}_{\ell''}.$$

Here, the lower bound corresponds to the case that all steps of Algorithm 4B are done at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$. The upper bound corresponds to the case that *solve* & *estimate*, *mark*, and *refine* are performed at linear cost $\mathcal{O}(\#\mathcal{T}_\ell)$, while a suboptimal solver leads to cost $\mathcal{O}(\sum_{\ell''=0}^{\ell} \#\mathcal{T}_{\ell''})$ for each mesh $\mathcal{T}_\ell$. In any case, the following corollary shows that full R-linear convergence guarantees that convergence rates with respect to the number of degrees of freedom $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ and with respect to the overall computational cost $\texttt{cost}(\ell,k)$ coincide even for a suboptimal solver. Moreover, the corollary shows that there exists a bound $s_0 > 0$ such that all rates $0 < s \le s_0$ are possible.

Corollary 4.11 (rates = complexity)**.** *For $s > 0$, full R-linear convergence (4.32) yields*

$$M(s) \coloneqq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \mathsf{H}_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \sum_{\ell''=0}^{\ell'} \#\mathcal{T}_{\ell''} \Big)^s \mathsf{H}_\ell^k \le C_{\rm cost}(s)\, M(s), \tag{4.43}$$

where the constant $C_{\rm cost}(s) > 0$ depends only on $C_{\rm lin}$, $q_{\rm lin}$, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \le s_0$ with $s_0 = \infty$ if $\underline{\ell} < \infty$.

The last corollary is an immediate consequence of the following elementary lemma for $a_{|\ell,k|} \coloneqq \mathsf{H}_\ell^k$ and $t_{|\ell,k|} \coloneqq \#\mathcal{T}_\ell$.

Lemma 4.12 (rates = complexity criterion)**.** *Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ and $(t_\ell)_{\ell\in\mathbb{N}_0}$ be sequences in $\mathbb{R}_{\ge 0}$ such that*

$$a_{\ell+n} \le C_1 q^n a_\ell \quad \text{and} \quad t_{\ell+1} \le C_2\, t_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.44}$$

Then, for all $s > 0$, there holds

$$M(s) \coloneqq \sup_{\ell\in\mathbb{N}_0} t_\ell^s\, a_\ell \le \sup_{\ell\in\mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \sum_{\ell''=0}^{\ell'} t_{\ell''} \Big)^s a_\ell \le C_{\rm cost}(s)\, M(s), \tag{4.45}$$

where the constant $C_{\rm cost}(s) > 0$ depends only on C_1, q, and s. Moreover, there exists $s_0 > 0$ depending only on C_2 and q such that $M(s) < \infty$ for all $0 < s \le s_0$.

Proof. By definition, it holds that

$$t_\ell \le M(s)^{1/s}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

This, assumption (4.44), and the geometric series prove that

$$\sum_{\ell''=0}^{\ell'} t_{\ell''} \le M(s)^{1/s} \sum_{\ell''=0}^{\ell'} a_{\ell''}^{-1/s} \overset{(4.44)}{\le} M(s)^{1/s}\, C_1^{1/s} a_{\ell'}^{-1/s} \sum_{\ell''=0}^{\ell'} (q^{1/s})^{\ell'-\ell''}$$
$$\le M(s)^{1/s} \frac{C_1^{1/s}}{1-q^{1/s}}\, a_{\ell'}^{-1/s} \quad \text{for all } \ell' \in \mathbb{N}_0.$$

A further application of (4.44) and the geometric series prove that

$$\sum_{\ell'=0}^{\ell} a_{\ell'}^{-1/s} \overset{(4.44)}{\le} C_1^{1/s} a_\ell^{-1/s} \sum_{\ell'=0}^{\ell} (q^{1/s})^{\ell-\ell'} \le \frac{C_1^{1/s}}{1-q^{1/s}}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

The combination of the two previously displayed formulas results in

$$\sum_{\ell'=0}^{\ell} \sum_{\ell''=0}^{\ell'} t_{\ell''} \le \Big(\frac{C_1^{1/s}}{1-q^{1/s}}\Big)^2 M(s)^{1/s}\, a_\ell^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

Rearranging this estimate, we conclude the proof of (4.43). It remains to verify $M(s) < \infty$ for some $s > 0$. Note that (4.44) guarantees that

$$0 \le t_\ell \le C_2\, t_{\ell-1} \le C_2^\ell\, t_0 \quad \text{for all } \ell \in \mathbb{N}.$$

Moreover, R-linear convergence (4.44) yields that

$$0 \le a_\ell \overset{(4.44)}{\le} C_1 q^\ell\, a_0 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Multiplying the two previously displayed formulas, we see that

$$t_\ell^s\, a_\ell \le (C_2^s q)^\ell C_1\, t_0^s\, a_0 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Note that the right-hand side is uniformly bounded, provided that $s > 0$ guarantees $C_2^s q \le 1$. This concludes the proof with $s_0 \coloneqq \log(1/q)/\log(C_2)$. □

With full linear convergence (4.32), the following theorem from [GHPS21, Theorem 8] can be applied and thus Algorithm 4B guarantees optimal convergence rates with respect to the overall computational cost in the case of sufficiently small adaptivity parameters θ and λ. To formalize achievable convergence rate $s > 0$, we introduce the notion of nonlinear approximation classes [BDD04; Ste07; CKNS08; CFPP14]

$$\|u^\star\|_{\mathbb{A}_s} \coloneqq \sup_{N\in\mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N} \eta_{\rm opt}(u^\star_{\rm opt}) \Big),$$

where $\eta_{\rm opt}(u^\star_{\rm opt})$ is the estimator for the (unavailable) exact Galerkin solution $u^\star_{\rm opt}$ on an optimal $\mathcal{T}_{\rm opt} \in \mathbb{T}_N \coloneqq \{\mathcal{T}_H \in \mathbb{T} \colon \#\mathcal{T}_H - \#\mathcal{T}_0 \le N\}$.

Theorem 4.13: optimal complexity of Algorithm 4B, [GHPS21, Theorem 8]

Suppose that the estimator satisfies the axioms of adaptivity (A1), (A2), (A3$^+$), and suppose that quasi-orthogonality (A4) holds. Suppose that the parameters θ and λ are chosen such

$$0 < \lambda < \lambda^\star = \min\Big\{1, \frac{1-q_{\rm ctr}}{q_{\rm ctr}} C_{\rm stab}^{-1}\Big\} \quad \text{and} \quad 0 < \frac{(\theta^{1/2} + \lambda/\lambda^\star)^2}{(1-\lambda/\lambda^\star)^2} < \theta^\star := \big(1 + C_{\rm stab}^2\, C_{\rm drel}^2\big)^{-1}.$$

Then, Algorithm 4B guarantees for all $s > 0$ that

$$c_{\rm opt} \|u^\star\|_{\mathbb{A}_s} \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \, \mathsf{H}_\ell^k \le C_{\rm opt} \, \max\{\|u^\star\|_{\mathbb{A}_s}, \mathsf{H}_0^0\}.$$

The constant $c_{\rm opt} > 0$ depends only on $C_{\rm stab}$, the use of NVB refinement, and s, while $C_{\rm opt} > 0$ depends only on $C_{\rm stab}$, $q_{\rm red}$, $C_{\rm drel}$, $C_{\rm lin}$, $q_{\rm lin}$, $\#\mathcal{T}_0$, λ, $q_{\rm ctr}$, θ, s, and the use of NVB refinement. □

Remark 4.14. *Considering the nonsymmetric model problem (4.10), a natural candidate for the solver is the generalized minimal residual method (GMRES) with optimal preconditioner for the symmetric part. Another alternative would be to consider an optimal preconditioner for the symmetric part and apply a conjugate gradient method to the normal equations (CGNR). However, for both approaches, a posteriori error estimation and contraction in the PDE-related energy norm are still open. Instead, [BHI$^+$23] follows the constructive proof of the Lax–Milgram lemma to derive a contractive solver. Its convergence analysis, as given in [BHI$^+$23], is improved in the following Section 4.5.*

4.5 AFEM with nested contractive solvers

While contractive solvers for SPD systems are well-understood in the literature, the recent work [BHI$^+$23] presents contractive solvers for the nonsymmetric variational formulation (4.14) that essentially fit into the framework of Section 4.4 and allow for the numerical analysis of AFEM with optimal complexity. To this end, the proof of the Lax–Milgram lemma as proposed by Zarantonello [Zar60] is exploited algorithmically (while the original proof [LM54] relies on the Hahn–Banach separation theorem): For $\delta > 0$, we consider the Zarantonello mapping $\Phi_H(\delta;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ defined by

$$a(\Phi_H(\delta;u_H), v_H) = a(u_H, v_H) + \delta\big[F(v_H) - b(u_H,v_H)\big] \quad \text{for all } u_H, v_H \in \mathcal{X}_H. \tag{4.46}$$

Since $a(\cdot,\cdot)$ is a scalar product, $\Phi_H(\delta;u_H) \in \mathcal{X}_H$ is well-defined. Moreover, for any $0 < \delta < 2\alpha/L^2$ and $0 < q_{\rm sym} := [1 - \delta(2\alpha - \delta L^2)]^{1/2} < 1$, this mapping is contractive, i.e.,

$$|||u_H^\star - \Phi_H(\delta;u_H)||| \le q_{\rm sym}\, |||u_H^\star - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H; \tag{4.47}$$

see also [HW20b; HW20a]. Note that (4.46) corresponds to a linear SPD system. For this, we employ a uniformly contractive algebraic solver with iteration function $\Psi_H(u_H^\sharp;\cdot) : \mathcal{X}_H \to \mathcal{X}_H$ to approximate the solution $u_H^\sharp := \Phi_H(\delta;u_H)$ to the SPD system (4.46), i.e.,

$$|||u_H^\sharp - \Psi_H(u_H^\sharp; w_H)||| \le q_{\rm ctr}\, |||u_H^\sharp - w_H||| \quad \text{for all } w_H \in \mathcal{X}_H \text{ and all } \mathcal{T}_H \in \mathbb{T}, \tag{4.48}$$

where $0 < q_{\rm ctr} < 1$ depends only on $a(\cdot,\cdot)$, but is independent of $\mathcal{X}_H$. Clearly, no knowledge of $u_H^\sharp$ is needed to compute $\Psi_H(u_H^\sharp; w_H)$ but only that of the corresponding right-hand side $a(u_H^\sharp,\cdot)\colon \mathcal{X}_H \to \mathbb{R}$; see, e.g., [CNX12; WZ17; IMPS24].

Algorithm 4C: AFEM with nested contractive solvers

Given an initial mesh $\mathcal{T}_0$, the Zarantonello parameter $\delta > 0$, adaptivity parameters $0 < \theta \le 1$ and $C_{\rm mark} \ge 1$, solver-stopping parameters $\lambda_{\rm sym}, \lambda_{\rm alg} > 0$, and an initial guess $u_0^{0,0} := u_0^{0,\underline{j}} \in \mathcal{X}_0$, iterate the following steps (i)–(iv) for all $\ell = 0, 1, 2, 3, \dots$:

(i) **Solve & estimate:** For all $k = 1, 2, 3, \dots$, repeat the following steps (a)–(c) until

$$|||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \le \lambda_{\rm sym}\, \eta_\ell(u_\ell^{k,\underline{j}}). \tag{4.49}$$

(a) Define $u_\ell^{k,0} := u_\ell^{k-1,\underline{j}}$ and, only as a theoretical quantity, $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$.

(b) **Inner solver loop:** For all $j = 1, 2, 3, \dots$, repeat the steps (I)–(II) until

$$|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\rm alg}\, \big[\lambda_{\rm sym}\eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|||\big]. \tag{4.50}$$

(I) Compute one step of the contractive SPD solver $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$, where $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}}) \in \mathcal{X}_\ell$ is only a theoretical quantity.

(II) Compute the refinement indicators $\eta_\ell(T, u_\ell^{k,j})$ for all $T \in \mathcal{T}_\ell$.

(c) Upon termination of the inner solver loop, define the index $\underline{j}[\ell,k] := j \in \mathbb{N}$.

(ii) Upon termination of the outer solver loop, define the index $\underline{k}[\ell] := k \in \mathbb{N}$.

(iii) **Mark:** Determine a set $\mathcal{M}_\ell \in \mathbb{M}_\ell[\theta, u_\ell^{\underline{k},\underline{j}}]$ satisfying (4.17) with $u_\ell^\star$ replaced by $u_\ell^{\underline{k},\underline{j}}$.

(iv) **Refine:** Generate $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ and define $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{j}} := u_\ell^{\underline{k},\underline{j}}$.

Extending the index notation from Section 4.4, we define the triple index set

$$\mathcal{Q} := \{(\ell,k,j) \in \mathbb{N}_0^3 \;:\; u_\ell^{k,j} \text{ is used in Algorithm 4C}\}$$

together with the lexicographic ordering

$$(\ell',k',j') \le (\ell,k,j) \quad :\Longleftrightarrow \quad u_{\ell'}^{k',j'} \text{ is defined not later than } u_\ell^{k,j} \text{ in Algorithm 4C}$$

and the total step counter

$$|\ell,k,j| := \#\{(\ell',k',j') \in \mathcal{Q} \;:\; (\ell',k',j') \le (\ell,k,j)\} \in \mathbb{N}_0 \quad \text{for } (\ell,k,j) \in \mathcal{Q}. \tag{4.51}$$

Moreover, we define the stopping indices

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 \;:\; (\ell,0,0) \in \mathcal{Q}\} \;\in \mathbb{N}_0 \cup \{\infty\}, \tag{4.52a}$$
$$\underline{k}[\ell] := \sup\{k \in \mathbb{N}_0 \;:\; (\ell,k,0) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell,0,0) \in \mathcal{Q}, \tag{4.52b}$$
$$\underline{j}[\ell,k] := \sup\{j \in \mathbb{N}_0 \;:\; (\ell,k,j) \in \mathcal{Q}\} \in \mathbb{N} \cup \{\infty\}, \quad \text{whenever } (\ell,k,0) \in \mathcal{Q}. \tag{4.52c}$$

First, these definitions are consistent with those of Algorithm 4C(i.a.II) and Algorithm 4C(ii). Second, there holds indeed $\underline{j}[\ell,k] < \infty$ for all $(\ell,k,0) \in \mathcal{Q}$; see [BHI+23, Lemma 3.2]. Third, $\underline{\ell} < \infty$ yields $\underline{k}[\underline{\ell}] = \infty$ and $\eta_{\underline{\ell}}(u^\star_{\underline{\ell}}) = 0$ with $u^\star_{\underline{\ell}} = u^\star$; see [BHI+23, Lemma 5.2].

The following theorem improves [BHI+23, Theorem 4.1] in the sense that, first, we prove R-linear convergence for all $\ell \geq \ell_0 = 0$, while $\ell_0 \in \mathbb{N}$ is unknown in practice and depends on $u^\star$ and the non-accessible sequence $(u^\star_\ell)_{\ell\in\mathbb{N}_0}$ in [BHI+23], and, second, [BHI+23] requires severe restrictions on λ_{alg} beyond (4.53) below. We note that (4.53) is indeed satisfied, if the algebraic system is solved exactly, i.e., $\lambda_{\mathrm{alg}} = 0$, so that Theorem 4.15 is a consistent generalization of Theorem 4.7.

Theorem 4.15: full R-linear convergence of Algorithm 4C

Let $0 < \theta \leq 1$, $C_{\mathrm{mark}} \geq 1$, $\lambda_{\mathrm{sym}}, \lambda_{\mathrm{alg}} > 0$, and $u_0^{0,0} \in \mathcal{X}_0$. With $q_\theta := [1-(1-q_{\mathrm{red}}^2)\theta]^{1/2}$, suppose that

$$0 < \frac{q_{\mathrm{sym}} + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\lambda_{\mathrm{alg}}}{1 - \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\lambda_{\mathrm{alg}}} =: \overline{q}_{\mathrm{sym}} < 1 \quad \text{and} \quad \lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}} < \frac{(1-q_{\mathrm{ctr}})(1-q_{\mathrm{sym}})(1-q_\theta)}{8\,q_{\mathrm{ctr}}C_{\mathrm{stab}}}. \tag{4.53}$$

Then, Algorithm 4C guarantees R-linear convergence of the quasi-error

$$\mathrm{H}_\ell^{k,j} := |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j}), \tag{4.54}$$

i.e., there exist constants $0 < q_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} > 0$ such that

$$\mathrm{H}_\ell^{k,j} \leq C_{\mathrm{lin}} q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\,\mathrm{H}_{\ell'}^{k',j'} \quad \text{for all } (\ell',k',j'),(\ell,k,j) \in \mathcal{Q} \text{ with } |\ell',k',j'| \leq |\ell,k,j|. \tag{4.55}$$

As proven for Corollary 4.11 in Section 4.4, an immediate consequence of full linear convergence (and the geometric series) is that convergence rates with respect to the number of degrees of freedom and with respect to the overall computational cost coincide.

Corollary 4.16 (rates = complexity)**.** *For $s > 0$, full R-linear convergence (4.55) yields*

$$\begin{aligned} M(s) := \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\mathrm{H}_\ell^{k,j} &\leq \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j|}} \sum_{\substack{(\ell'',k'',j'')\in\mathcal{Q} \\ |\ell'',k'',j''|\leq|\ell',k',j'|}} \#\mathcal{T}_{\ell''} \Big)^s \mathrm{H}_\ell^{k,j} \\ &\leq C_{\mathrm{cost}}(s)\,M(s), \end{aligned} \tag{4.56}$$

where the constant $C_{\mathrm{cost}}(s) > 0$ depends only on C_{lin}, q_{lin}, and s. Moreover, there exists $s_0 > 0$ such that $M(s) < \infty$ for all $0 < s \leq s_0$. □

The proof of Theorem 4.15 requires the following lemma (essentially taken from [BHI+23]). It deduces the contraction of the inexact Zarantonello iteration with computed iterates $u_\ell^{k,\underline{j}} \approx u_\ell^{k,\star}$ from the exact Zarantonello iteration. For the inexact iteration, the linear SPD system (4.46) is solved with the contractive algebraic solver (4.48), i.e., $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ and $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}, u_\ell^{k,j-1})$ guarantee

$$|||u_\ell^\star - u_\ell^{k,\star}||| \leq q_{\mathrm{sym}}\,|||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \text{for all } (\ell,k,j) \in \mathcal{Q} \text{ with } k \geq 1. \tag{4.57}$$

We emphasize that contraction is only guaranteed for $0 < k < \underline{k}[\ell]$ in (4.58) below, while the final iteration $k = \underline{k}[\ell]$ leads to a perturbed contraction (4.59) thus requiring additional treatment in the later analysis. The proof of Lemma 4.17 is given in Appendix 4.A.

Lemma 4.17 (contraction of inexact Zarantonello iteration)**.** *Under the assumptions of Theorem 4.15, the inexact Zarantonello iteration used in Algorithm 4C satisfies*

$$|||u_\ell^\star - u_\ell^{k,j}||| \le \overline{q}_{\rm sym}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \textit{for all } (\ell,k,\underline{j}) \in \mathcal{Q} \textit{ with } 1 \le k < \underline{k}[\ell] \tag{4.58}$$

as well as

$$|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \le q_{\rm sym}\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| + \frac{2\,q_{\rm ctr}}{1-q_{\rm ctr}}\,\lambda_{\rm alg}\lambda_{\rm sym}\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \quad \textit{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}. \tag{4.59}$$

The building blocks of the proof of Theorem 4.15 are the following: First, we show that a suitably weighted quasi-error involving the final iterates of the inexact Zarantonello iteration is tail-summable in the mesh level index ℓ. Second, we show that the quasi-errors are tail-summable in the Zarantonello index k and, third, in the algebraic solver index j and are stable in the nested iteration. Finally, combining these ideas leads to tail-summability with respect to the total step counter.

Proof of Theorem 4.15. The proof is split into six steps. The first four steps follow the proof of Theorem 4.7 using

$$\mathsf{H}_\ell^k := |||u_\ell^\star - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}}) \quad \text{for all } (\ell,k,\underline{j}) \in \mathcal{Q}. \tag{4.60}$$

By contraction of the algebraic solver (4.48) as well as the stopping criteria for the algebraic solver (4.50) and for the symmetrization (4.49), it holds that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(4.48)}{\lesssim} |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{(4.50)}{\lesssim} \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \overset{(4.49)}{\lesssim} \eta_\ell(u_\ell^{k,\underline{j}}) \le \mathsf{H}_\ell^k.$$

In particular, this proves equivalence

$$\mathsf{H}_\ell^{\underline{k}} \le \mathsf{H}_\ell^{\underline{k}} + |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| = \mathsf{H}_\ell^{\underline{k},\underline{j}} \lesssim \mathsf{H}_\ell^{\underline{k}} \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}. \tag{4.61}$$

Step 1 (auxiliary estimates & estimator reduction). For $(\ell,\underline{k},\underline{j}) \in \mathcal{Q}$, nested iteration $u_\ell^{\underline{k},0} = u_\ell^{\underline{k}-1,\underline{j}}$ and $\underline{j}[\ell,\underline{k}] \ge 1$ yield

$$|||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| \overset{(4.48)}{\le} q_{\rm ctr}^{\underline{j}[\ell,\underline{k}]}\, |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},0}||| \le q_{\rm ctr}\, |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k}-1,\underline{j}}|||. \tag{4.62}$$

From this, we obtain that

$$\begin{aligned} |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| &\le |||u_\ell^\star - u_\ell^{\underline{k},\star}||| + |||u_\ell^{\underline{k},\star} - u_\ell^{\underline{k},\underline{j}}||| \\ &\overset{(4.62)}{\le} (1+q_{\rm ctr})\, |||u_\ell^\star - u_\ell^{\underline{k},\star}||| + q_{\rm ctr}\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \\ &\overset{(4.57)}{\le} \big[(1+q_{\rm ctr})q_{\rm sym} + q_{\rm ctr}\big]\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \le 3\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||. \end{aligned} \tag{4.63}$$

For $(\ell+1, \underline{k}, \underline{j}) \in \mathcal{Q}$, contraction of the inexact Zarantonello iteration (4.58), nested iteration $u_{\ell+1}^{0,\underline{j}} = u_\ell^{\underline{k},\underline{j}}$, and $\underline{k}[\ell+1] \geq 1$, show that

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| \overset{(4.58)}{\leq} q_{\mathrm{sym}}^{\underline{k}[\ell+1]-1}\, |||u_{\ell+1}^\star - u_{\ell+1}^{0,\underline{j}}||| \leq |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{4.64}$$

The combination of the previous two displayed formulas shows

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| \overset{(4.63)}{\leq} 3\, |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| \overset{(4.64)}{\leq} 3\, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{4.65}$$

Analogous arguments to (4.36) in the proof of Theorem 4.4 establish

$$\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \overset{(4.36)}{\leq} q_\theta\, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + C_{\mathrm{stab}}\, |||u_{\ell+1}^{\underline{k},\underline{j}} - u_\ell^{\underline{k},\underline{j}}||| \overset{(4.65)}{\leq} q_\theta\, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) + 4C_{\mathrm{stab}}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}|||. \tag{4.66}$$

Step 2 (tail summability with respect to ℓ). With $\lambda := \lambda_{\mathrm{alg}}\lambda_{\mathrm{sym}}$, we define

$$\gamma := \frac{q_\theta(1-q_{\mathrm{sym}})}{4\,C_{\mathrm{stab}}}, \quad C(\gamma,\lambda) := 1 + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\frac{\lambda}{\gamma}, \quad \text{and} \quad \alpha := \frac{\lambda}{\gamma} \overset{(4.53)}{<} \frac{(1-q_{\mathrm{ctr}})(1-q_\theta)}{2\,q_{\mathrm{ctr}} q_\theta}.$$

By definition, it follows that

$$C(\gamma,\lambda) = 1 + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\alpha < 1 + \frac{1-q_\theta}{q_\theta} = 1/q_\theta.$$

This ensures that

$$q_\theta C(\gamma,\lambda) < 1 \quad \text{as well as} \quad q_{\mathrm{sym}} + 4\,C_{\mathrm{stab}} C(\gamma,\lambda)\,\gamma < q_{\mathrm{sym}} + \frac{4\,C_{\mathrm{stab}}}{q_\theta}\,\gamma = 1. \tag{4.67}$$

With contraction of the inexact Zarantonello iteration (4.59), Step 1 proves

$$\begin{aligned}
|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| + \gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) &\overset{(4.59)}{\leq} q_{\mathrm{sym}}\, |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k}-1,\underline{j}}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \\
&\overset{(4.64)}{\leq} q_{\mathrm{sym}}\, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) \\
&\overset{(4.66)}{\leq} \big(q_{\mathrm{sym}} + 4\,C_{\mathrm{stab}}\, C(\gamma,\lambda)\,\gamma\big)\, |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}||| + q_\theta\, C(\gamma,\lambda)\,\gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \\
&\leq q_{\mathrm{ctr}}\, \big[|||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})\big] \quad \text{for all } (\ell+1,\underline{k},\underline{j}) \in \mathcal{Q},
\end{aligned} \tag{4.68}$$

where (4.67) ensures the bound

$$0 < q_{\mathrm{ctr}} := \max\big\{q_{\mathrm{sym}} + 4\,C_{\mathrm{stab}}\, C(\gamma,\lambda)\,\gamma\,,\; q_\theta\, C(\gamma,\lambda)\big\} < 1. \tag{4.69}$$

Altogether, we obtain

$$\begin{aligned}
a_{\ell+1} := |||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| + \gamma\,\eta_{\ell+1}(u_{\ell+1}^{\underline{k},\underline{j}}) &\overset{(4.68)}{\leq} q_{\mathrm{ctr}}\, \big[|||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{k},\underline{j}})\big] + q_{\mathrm{ctr}}\, |||u_{\ell+1}^\star - u_\ell^\star||| \\
&=: q_{\mathrm{ctr}}\, a_\ell + b_\ell \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q},
\end{aligned}$$

which corresponds to (4.37) in the case of a single contractive solver (with $u_\ell^{\underline{k},\underline{j}}$ replacing $u_\ell^{\underline{k}}$ in (4.37)). Together with (4.38)–(4.39) (with $u_\ell^{\underline{k},\underline{j}}$ replacing $u_\ell^{\underline{k}}$), the assumptions (4.21) of Lemma 4.6 are satisfied. Therefore, Lemma 4.6 proves tail summability

$$\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathsf{H}_{\ell'}^{\underline{k}} \overset{(4.60)}{\simeq} \sum_{\ell'=\ell+1}^{\underline{\ell}-1} \big[|||u_{\ell'}^\star - u_{\ell'}^{\underline{k},\underline{j}}||| + \gamma\, \eta_{\ell'}(u_{\ell'}^{\underline{k},\underline{j}}) \big]$$
$$\lesssim |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + \gamma\, \eta_\ell(u_\ell^{\underline{k},\underline{j}}) \overset{(4.60)}{\simeq} \mathsf{H}_\ell^{\underline{k}} \quad \text{for all } (\ell, \underline{k}, \underline{j}) \in \mathcal{Q}.$$

Step 3 (auxiliary estimates). First, we employ (4.63) to deduce

$$\begin{aligned} \mathsf{H}_\ell^{\underline{k}} &\overset{(A1)}{\lesssim} |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| + \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}}) \overset{(4.60)}{\le} \mathsf{H}_\ell^{\underline{k}-1} + 2\,|||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| \\ &\overset{(4.63)}{\le} \mathsf{H}_\ell^{\underline{k}-1} + 8\,|||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \le 9\,\mathsf{H}_\ell^{\underline{k}-1} \quad \text{for all } (\ell, \underline{k}, \underline{j}) \in \mathcal{Q}. \end{aligned} \tag{4.70}$$

Second, for $0 \le k < k' < \underline{k}[\ell]$, the failure of the stopping criterion for the inexact Zarantonello symmetrization (4.49) and contraction (4.58) prove that

$$\mathsf{H}_\ell^{k'} \overset{(4.49)}{\lesssim} |||u_\ell^\star - u_\ell^{k',\underline{j}}||| + |||u_\ell^{k',\underline{j}} - u_\ell^{k'-1,\underline{j}}||| \overset{(4.58)}{\lesssim} |||u_\ell^\star - u_\ell^{k'-1,\underline{j}}||| \overset{(4.58)}{\lesssim} \overline{q}_{\mathrm{sym}}^{\,k'-k}\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||. \tag{4.71}$$

Moreover, for $k < k' = \underline{k}[\ell]$, we combine (4.70) with (4.71) to get

$$\mathsf{H}_\ell^{k} \overset{(4.70)}{\lesssim} \mathsf{H}_\ell^{\underline{k}[\ell]-1} \overset{(4.71)}{\lesssim} \overline{q}_{\mathrm{sym}}^{\,(\underline{k}[\ell]-1)-k}\, |||u_\ell^\star - u_\ell^{k,\underline{j}}||| \simeq \overline{q}_{\mathrm{sym}}^{\,\underline{k}[\ell]-k}\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||. \tag{4.72}$$

The combination of (4.71)–(4.72) proves that

$$\mathsf{H}_\ell^{k'} \lesssim \overline{q}_{\mathrm{sym}}^{\,k'-k}\, |||u_\ell^\star - u_\ell^{k,\underline{j}}||| \lesssim \overline{q}_{\mathrm{sym}}^{\,k'-k}\, \mathsf{H}_\ell^{k} \quad \text{for all } \; (\ell, 0, 0) \in \mathcal{Q} \text{ with } 0 \le k \le k' \le \underline{k}[\ell], \tag{4.73}$$

where the hidden constant depends only on C_{stab}, λ_{sym}, and $\overline{q}_{\mathrm{sym}}$. Third, we recall

$$|||u_\ell^\star - u_{\ell-1}^\star||| \overset{(4.23)}{\lesssim} \eta_{\ell-1}(u_{\ell-1}^\star) \overset{(A1)}{\lesssim} \eta_{\ell-1}(u_{\ell-1}^{\underline{k},\underline{j}}) + |||u_{\ell-1}^\star - u_{\ell-1}^{\underline{k},\underline{j}}||| = \mathsf{H}_{\ell-1}^{\underline{k}}.$$

Together with nested iteration $u_{\ell-1}^{\underline{k},\underline{j}} = u_\ell^{0,\underline{j}}$, this yields that

$$\mathsf{H}_\ell^0 = |||u_\ell^\star - u_{\ell-1}^{\underline{k},\underline{j}}||| + \eta_\ell(u_{\ell-1}^{\underline{k},\underline{j}}) \le |||u_\ell^\star - u_{\ell-1}^\star||| + \mathsf{H}_{\ell-1}^{\underline{k}} \lesssim \mathsf{H}_{\ell-1}^{\underline{k}} \quad \text{for all } (\ell, 0, 0) \in \mathcal{Q}. \tag{4.74}$$

Step 4 (tail summability with respect to *ℓ* and *k*). The auxiliary estimates from Step 3 and the geometric series prove that

$$\begin{aligned} \sum_{\substack{(\ell',k',\underline{j}) \in \mathcal{Q} \\ |\ell',k',\underline{j}| > |\ell,k,\underline{j}|}} \mathsf{H}_{\ell'}^{k'} &= \sum_{k'=k+1}^{\underline{k}[\ell]} \mathsf{H}_\ell^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell]} \mathsf{H}_{\ell'}^{k'} \overset{(4.73)}{\lesssim} \mathsf{H}_\ell^{k} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathsf{H}_{\ell'}^{0} \\ &\overset{(4.74)}{\lesssim} \mathsf{H}_\ell^{k} + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathsf{H}_{\ell'}^{\underline{k}} \lesssim \mathsf{H}_\ell^{k} + \mathsf{H}_\ell^{\underline{k}} \overset{(4.73)}{\lesssim} \mathsf{H}_\ell^{k} \quad \text{for all } (\ell, k, \underline{j}) \in \mathcal{Q}. \end{aligned} \tag{4.75}$$

Step 5 (auxiliary estimates). Recall $\mathsf{H}_\ell^k \le \mathsf{H}_\ell^{k,\underline{j}}$ from (4.61). For $j = 0$ and $k = 0$, the definition $u_\ell^{0,0} := u_\ell^{0,\underline{j}} := u_\ell^{0,\star}$ leads to $\mathsf{H}_\ell^{0,0} = \mathsf{H}_\ell^0$. For $k \ge 1$, nested iteration $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}}$ and contraction of the Zarantonello iteration (4.57) imply

$$|||u_\ell^{k,\star} - u_\ell^{k,0}||| \le |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \overset{(4.57)}{\le} (q_{\mathrm{sym}} + 1)\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \le 2\,\mathsf{H}_\ell^{k-1}.$$

Therefore, we derive that

$$\mathsf{H}_\ell^{k,0} \le 3\,\mathsf{H}_\ell^{(k-1)_+} \quad \text{for all } (\ell,k,0) \in \mathcal{Q}, \quad \text{where } (k-1)_+ := \max\{0, k-1\}. \tag{4.76}$$

For any $0 \le j < j' < \underline{j}[\ell,k]$, the contraction of the Zarantonello iteration (4.57), the contraction of the algebraic solver (4.48), and the failure of the stopping criterion for the algebraic solver (4.50) prove

$$\begin{aligned}
\mathsf{H}_\ell^{k,j'} &\le |||u_\ell^\star - u_\ell^{k,\star}||| + 2\,|||u_\ell^{k,\star} - u_\ell^{k,j'}||| + \eta_\ell(u_\ell^{k,j'}) \\
&\overset{(4.57)}{\lesssim} |||u_\ell^{k,j'} - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,j'}||| + \eta_\ell(u_\ell^{k,j'}) \\
&\overset{(4.48)}{\lesssim} |||u_\ell^{k,j'} - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,j'} - u_\ell^{k,j'-1}||| + \eta_\ell(u_\ell^{k,j'}) \\
&\overset{(4.50)}{\lesssim} |||u_\ell^{k,j'} - u_\ell^{k,j'-1}||| \overset{(4.48)}{\lesssim} |||u_\ell^{k,\star} - u_\ell^{k,j'-1}||| \overset{(4.48)}{\lesssim} q_{\mathrm{ctr}}^{j'-j}\, |||u_\ell^{k,\star} - u_\ell^{k,j}||| \lesssim q_{\mathrm{ctr}}^{j'-j}\, \mathsf{H}_\ell^{k,j}.
\end{aligned}$$

For $j' = \underline{j}[\ell,k]$, it follows that

$$\mathsf{H}_\ell^{k,\underline{j}} \overset{(A1)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}-1} + |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{(4.48)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}-1} + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}||| \overset{(4.54)}{\le} 2\,\mathsf{H}_\ell^{k,\underline{j}-1} \lesssim q_{\mathrm{ctr}}^{\underline{j}[\ell,k]-j}\, \mathsf{H}_\ell^{k,j}.$$

The combination of the previous two displayed formulas results in

$$\mathsf{H}_\ell^{k,j'} \lesssim q_{\mathrm{ctr}}^{j'-j}\, \mathsf{H}_\ell^{k,j} \quad \text{for all } (\ell,k,0) \in \mathcal{Q} \quad \text{with} \quad 0 \le j \le j' \le \underline{j}[\ell,k], \tag{4.77}$$

where the hidden constant depends only on q_{sym}, λ_{sym}, q_{ctr}, λ_{alg}, and C_{stab}.

Step 6 (tail summability with respect to ℓ, k, and j). Finally, we observe that

$$\begin{aligned}
\sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| > |\ell,k,j|}} \mathsf{H}_{\ell'}^{k',j'} &= \sum_{j'=j+1}^{\underline{j}[\ell,k]} \mathsf{H}_\ell^{k,j'} + \sum_{k'=k+1}^{\underline{k}[\ell]} \sum_{j'=0}^{\underline{j}[\ell,k']} \mathsf{H}_\ell^{k',j'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \sum_{j'=0}^{\underline{j}[\ell',k']} \mathsf{H}_{\ell'}^{k',j'} \\
&\overset{(4.77)}{\lesssim} \mathsf{H}_\ell^{k,j} + \sum_{k'=k+1}^{\underline{k}[\ell]} \mathsf{H}_\ell^{k',0} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell]} \mathsf{H}_{\ell'}^{k',0} \overset{(4.76)}{\lesssim} \mathsf{H}_\ell^{k,j} + \sum_{\substack{(\ell',k',\underline{j}) \in \mathcal{Q} \\ |\ell',k',\underline{j}| > |\ell,k,\underline{j}|}} \mathsf{H}_{\ell'}^{k'} \\
&\overset{(4.75)}{\lesssim} \mathsf{H}_\ell^{k,j} + \mathsf{H}_\ell^{k} \overset{(4.61)}{\lesssim} \mathsf{H}_\ell^{k,j} + \mathsf{H}_\ell^{k,\underline{j}} \overset{(4.77)}{\lesssim} \mathsf{H}_\ell^{k,j} \quad \text{for all } (\ell,k,j) \in \mathcal{Q}.
\end{aligned}$$

Since $\mathcal{Q}$ is countable and linearly ordered, Lemma 4.10 concludes the proof of (4.55). □

The final theorem, following from [BIM+24b, Theorem 4.3], states that for sufficiently small adaptivity parameters θ, λ_{sym}, and λ_{alg}, Algorithm 4C achieves optimal complexity.

Theorem 4.18: Optimal complexity of Algorithm 4C, [BIM+24b, Theorem 4.3]

Suppose that the estimator satisfies the axioms of adaptivity (A1)–(A3+) and suppose quasi-orthogonality (A4) holds. Suppose full R-linear convergence from Theorem 4.15. Define the constants $\theta^\star$, $\lambda_{\mathrm{sym}}^\star$ by

$$\theta^\star \coloneqq \big(1 + C_{\mathrm{stab}}^2\, C_{\mathrm{drel}}^2\big)^{-1},$$
$$\lambda_{\mathrm{sym}}^\star \coloneqq \min\{1, C_{\mathrm{stab}}^{-1}\, C_{\mathrm{alg}}^{-1}\} \quad \text{with} \quad C_{\mathrm{alg}} \coloneqq \frac{1}{1 - q_{\mathrm{sym}}}\Big(\frac{2\, q_{\mathrm{ctr}}}{1 - q_{\mathrm{ctr}}}\, \lambda_{\mathrm{alg}}^\star + q_{\mathrm{sym}}\Big).$$

Suppose that the constants θ, λ_{sym}, and λ_{alg} are sufficiently small in the sense that, additionally to (4.53), there holds

$$0 < \lambda_{\mathrm{sym}} < \lambda_{\mathrm{sym}}^\star \quad \text{and} \quad 0 < \frac{\big(\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\big)^2}{\big(1 - \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\big)^2} < \theta^\star < 1.$$

Then, Algorithm 4C guarantees for all $s > 0$

$$c_{\mathrm{opt}} \|u^\star\|_{\mathbb{A}_s} \le \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathsf{H}_\ell^{k,j} \le C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s}, \mathsf{H}_0^{0,0}\}.$$

The constant $c_{\mathrm{opt}} > 0$ depends only on C_{stab}, the use of NVB refinement, and s, while $C_{\mathrm{opt}} > 0$ depends only on C_{stab}, q_{red}, C_{drel}, C_{lin}, q_{lin}, $\#\mathcal{T}_0$, λ_{sym}, q_{sym}, λ_{alg}, q_{ctr}, θ, s, and the use of NVB refinement. □

4.6 Application to strongly monotone nonlinear PDEs

In the previous sections, the particular focus was on general second-order linear elliptic PDEs (4.10). However, the results also apply to nonlinear PDEs with strongly monotone and Lipschitz-continuous nonlinearity as considered, e.g., in [GMZ11; GMZ12; CW17; GHPS18; HW20b; HW20a; GHPS21; HPSV21; HPW21; HW22; HMRV23; MV23] to mention only some recent works.

Given a nonlinearity $\boldsymbol{A}\colon \mathbb{R}^d \to \mathbb{R}^d$, we consider the nonlinear elliptic PDE

$$-\operatorname{div}\big(\boldsymbol{A}(\nabla u^\star)\big) = f - \operatorname{div} \boldsymbol{f} \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \partial\Omega. \tag{4.78}$$

We define the nonlinear operator $\mathcal{A}\colon H_0^1(\Omega) \to H^{-1}(\Omega) \coloneqq H_0^1(\Omega)^*$ via $\mathcal{A}u \coloneqq \langle \boldsymbol{A}(\nabla u), \nabla(\cdot)\rangle_{L^2(\Omega)}$, where we suppose that the $L^2(\Omega)$ scalar product on the right-hand side is well-defined. Then, the weak formulation of (4.78) reads

$$\langle \mathcal{A}u^\star, v\rangle = F(v) \coloneqq \langle f, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v\rangle_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega), \tag{4.79}$$

where $\langle \cdot, \cdot\rangle$ on the left-hand side denotes the duality brackets on $H^{-1}(\Omega) \times H_0^1(\Omega)$.

Let $a(\cdot,\cdot)$ be an equivalent scalar product on $H_0^1(\Omega)$ with induced norm $|||\cdot|||$. Suppose that $\mathcal{A}$ is strongly monotone and Lipschitz continuous, i.e., there exist $0 < \alpha \le L$ such that, for all $u, v, w \in H_0^1(\Omega)$,

$$\alpha\, |||u - v|||^2 \le \langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \quad \text{and} \quad \langle \mathcal{A}u - \mathcal{A}v, w\rangle \le L\, |||u - v|||\, |||w|||. \tag{4.80}$$

Under these assumptions, the Zarantonello theorem [Zar60] (or main theorem on strongly monotone operators [Zei90b, Section 25.4]) yields existence and uniqueness of the solution $u^\star \in H^1_0(\Omega)$ to (4.79). For $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{X}_H \subseteq H^1_0(\Omega)$ from (4.13), it also applies to the discrete setting and yields existence and uniqueness of the discrete solution $u_H^\star \in \mathcal{X}_H$ to

$$\langle \mathcal{A}u_H^\star\,,\, v_H\rangle = F(v_H) \quad \text{for all } v_H \in \mathcal{X}_H, \tag{4.81}$$

which is quasi-optimal in the sense of the Céa lemma (4.15).

As already discussed in Section 4.5, the proof of the Zarantonello theorem relies on the Banach fixed-point theorem: For $0 < \delta < 2\alpha/L^2$, define $\Phi_H(\delta;\cdot)\colon \mathcal{X}_H \to \mathcal{X}_H$ via

$$a(\Phi_H(\delta;u_H), v_H) = a(u_H, v_H) + \delta\big[F(v_H) - \langle \mathcal{A}(u_H)\,,\, v_H\rangle\big] \quad \text{for all } u_H, v_H \in \mathcal{X}_H. \tag{4.82}$$

Since $a(\cdot,\cdot)$ is a scalar product, $\Phi_H(\delta;u_H) \in \mathcal{X}_H$ is well-defined. Moreover, for $0 < \delta < 2\alpha/L^2$ and $0 < q_{\mathrm{sym}} \coloneqq [1 - \delta(2\alpha - \delta L^2)]^{1/2} < 1$, this mapping is a contraction, i.e.,

$$|||u_H^\star - \Phi_H(\delta;u_H)||| \le q_{\mathrm{sym}}\, |||u_H^\star - u_H||| \quad \text{for all } u_H \in \mathcal{X}_H; \tag{4.83}$$

see also [HW20b; HW20a]. Analogously to Section 4.5, the variational formulation (4.82) leads to a linear SPD system for which we employ a uniformly contractive solver (4.48). Overall, we note that for the nonlinear PDE (4.78), the natural AFEM loop consists of

- discretization via a conforming triangulation $\mathcal{T}_\ell$ (leading to the non-computable solution $u_\ell^\star$ to the discrete nonlinear system (4.81)),
- iterative linearization (giving rise to the solution $u_\ell^{k,\star} = \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ of the large-scale discrete SPD system (4.82) obtained by linearizing (4.81) in $u_\ell^{k-1,\underline{j}}$),
- and an algebraic solver (leading to computable approximations $u_\ell^{k,j} \approx u_\ell^{k,\star}$).

Thus, the natural AFEM algorithm takes the form of Algorithm 4C in Section 4.5.

So far, the only work analyzing convergence of such a full adaptive loop for the numerical solution of (4.78) is [HPSV21], which uses the Zarantonello approach (4.82) for linearization and a preconditioned CG method with optimal additive Schwarz preconditioner for solving the arising SPD systems. Importantly and contrary to the present work, the adaptivity parameters θ, λ_{sym}, and λ_{alg} in [HPSV21] must be sufficiently small to guarantee full linear convergence and optimal complexity, while even plain convergence for arbitrary θ, λ_{sym}, and λ_{alg} is left open. Instead, the present work proves full R-linear convergence at least for arbitrary θ and λ_{sym} and the milder constraint (4.53) on λ_{alg}.

To apply the analysis from Section 4.5, it only remains to check the validity of Proposition 4.1 and Proposition 4.2. The following result provides the analogue of Proposition 4.1 for scalar nonlinearities. Note that, first, the same assumptions are made in [HPSV21] and, second, only the proof of stability (A1) (going back to [GMZ12]) is restricted to scalar nonlinearities and lowest-order discretizations, i.e., $p = 1$ in (4.13).

Proposition 4.19 (see, e.g., [GMZ12, Section 3.2] or [CFPP14, Section 10.1])**.** *Suppose that* $A(\nabla u) = a(|\nabla u|^2)\nabla u$, *where* $a \in C^1(\mathbb{R}_{\geq 0})$ *satisfies*

$$\alpha(t-s) \leq a(t^2)t - a(s^2)s \leq \frac{L}{3}(t-s) \quad \text{for all } t \geq s \geq 0. \tag{4.84}$$

Then, there holds (4.80) *for* $|||v||| := \|\nabla v\|_{L^2(\Omega)}$ *and the standard residual error estimator* (4.16) *for lowest-order elements* $p = 1$ *(with* $A\nabla v_H$ *understood as* $A(\nabla v_H)$ *and* $b = 0 = c$*) satisfies stability* (A1)*, reduction* (A2)*, reliability* (A3)*, discrete reliability* (A3$^+$)*, and quasi-monotonicity* (QM) *from Proposition 4.1.* □

Under the same assumptions as in Proposition 4.19, quasi-orthogonality (A4) is satisfied. For the convenience of the reader, we include a sketch of the proof.

Proposition 4.20. *Under the assumptions of Proposition 4.19 and for any sequence of nested finite-dimensional subspaces* $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset H_0^1(\Omega)$*, the corresponding Galerkin solutions* $u_\ell^\star \in \mathcal{X}_\ell$ *to* (4.81) *satisfy quasi-orthogonality* (A4) *with* $\delta = 1$ *and* $C_{\mathrm{orth}} = L/\alpha$*, i.e.,*

$$\sum_{\ell'=\ell}^{\infty} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 \leq \frac{L}{\alpha} |||u^\star - u_\ell|||^2 \quad \text{for all } \ell \in \mathbb{N}_0. \tag{4.85}$$

Sketch of proof. One can prove that the energy

$$E(v) := \frac{1}{2}\int_\Omega \int_0^{|\nabla v(x)|^2} a(t)\,\mathrm{d}t\,\mathrm{d}x - F(v) \quad \text{for all } v \in H_0^1(\Omega)$$

is Gâteaux-differentiable with $\mathrm{d}E(v) = \mathcal{A}v - F$. Then, elementary calculus (see, e.g., [GHPS18, Lemma 5.1] or [HW20a, Lemma 2]) yields the equivalence

$$\frac{\alpha}{2}|||u_H^\star - v_H|||^2 \leq E(v_H) - E(u_H^\star) \leq \frac{L}{2}|||u_H^\star - v_H|||^2 \text{ for all } \mathcal{T}_H \in \mathbb{T} \text{ and all } v_H \in \mathcal{X}_H. \tag{4.86}$$

In particular, we see that $u_H^\star$ is the unique minimizer to

$$E(u_H^\star) = \min_{v_H \in \mathcal{X}_H} E(v_H), \tag{4.87}$$

and (4.86)–(4.87) also hold for $u^\star$ and $H_0^1(\Omega)$ replacing $u_H^\star$ and $\mathcal{X}_H$, respectively.

From this and the telescopic sum, we infer that

$$\begin{aligned}\frac{\alpha}{2}\sum_{\ell'=\ell}^{\ell+N} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 &\overset{(4.86)}{\leq} \sum_{\ell'=\ell}^{\ell+N}\big[E(u_{\ell'}^\star) - E(u_{\ell'+1}^\star)\big] = E(u_\ell^\star) - E(u_{\ell+N+1}^\star)\\ &\overset{(4.87)}{\leq} E(u_\ell^\star) - E(u^\star) \overset{(4.86)}{\leq} \frac{L}{2}|||u^\star - u_\ell^\star|||^2 \quad \text{for all } \ell, N \in \mathbb{N}_0.\end{aligned}$$

Since the right-hand side is independent of N, we conclude the proof for $N \to \infty$. □

Thus, full R-linear convergence from Theorem 4.15 and optimal complexity from Theorem 4.18 apply also to the nonlinear PDE (4.78) under the assumptions on the nonlinearity from Proposition 4.19. Unlike [HPSV21], we can guarantee full R-linear convergence (4.55) for arbitrary θ, arbitrary λ_{sym}, and a weaker constraint (4.53) on λ_{alg}. As in [HPSV21, Theorem 5], optimal complexity follows if the adaptivity parameters are sufficiently small.

Remark 4.21. *The cost-optimal numerical solution of nonlinear PDEs is widely open beyond the case of strongly monotone and Lipschitz continuous nonlinearities considered here. We stress that this problem class even excludes the p-Laplacian, for which linear convergence [DK08] and optimal convergence rates [BDK12] are known under the constraint of the exact solution of the arising nonlinear discrete systems. Convergent linearization strategies for the p-Laplacian are the topic of recent research [DFTW20; BDS23; Hei23]. However, optimal complexity appears to be still out of reach. Nevertheless, the present work could outline potential strategies also in this respect.*

4.7 Numerical experiment

The following numerical experiment employs the MATLAB software package MooAFEM from [IP23].[1] On the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1) \times [-1,0)$, we consider

$$-\Delta u^\star + \boldsymbol{b} \cdot \nabla u^\star + u^\star = 1 \text{ in } \Omega \quad \text{and} \quad u^\star = 0 \text{ on } \partial\Omega \quad \text{with} \quad \boldsymbol{b}(x) = x; \tag{4.88}$$

see Figure 4.1 for the geometry and some adaptively generated meshes.

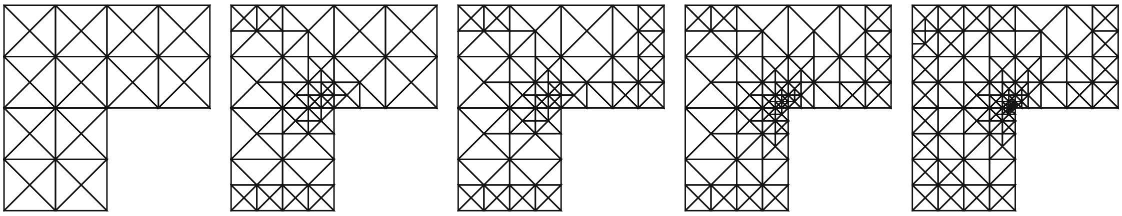


Figure 4.1: Illustration of the initial triangulation $\mathcal{T}_0$ and the sequence of adaptively generated meshes $\mathcal{T}_0, \dots, \mathcal{T}_4$ for the experiment (4.88).

Optimality of Algorithm 4C with respect to large solver-stopping parameters λ_{sym} and λ_{alg}. We choose $\delta = 0.5$, $\theta = 0.3$, and the polynomial degree $p = 2$. Figure 4.2 presents the convergence rates for fixed $\lambda_{\mathrm{alg}} = 0.7$ and several symmetrization parameters $\lambda_{\mathrm{sym}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. We observe that Algorithm 4C obtains the optimal convergence rate -1 with respect to the number of degrees of freedom and the cumulative computational time for any selection of λ_{sym}. Moreover, the same holds true for fixed $\lambda_{\mathrm{sym}} = 0.7$ and any choice of the algebraic solver parameter $\lambda_{\mathrm{alg}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ as depicted in Figure 4.3. Table 4.1 illustrates the weighted cumulative computational time of Algorithm 4C and shows that a smaller marking parameter $\theta = 0.3$ in combination with larger solver-stopping parameters λ_{sym} and λ_{alg} is favorable. Furthermore, Figure 4.5 shows that Algorithm 4C guarantees optimal convergence rates $-p/2$ for several polynomial degrees p with fixed $\delta = 0.5$, marking parameter $\theta = 0.3$, and moderate $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.7$.

Optimality of Algorithm 4C with respect to large marking parameter θ. We choose the polynomial degree $p = 2$, $\delta = 0.5$, and solver-stopping parameters $\lambda_{\mathrm{alg}} = \lambda_{\mathrm{sym}} = 0.7$. Figure 4.4 shows that also for moderate marking parameters θ, Algorithm 4C guarantees optimal convergence rates with respect to the number of degrees of freedom and the cumulative computational time. Moreover, we observe that a very small as well as a large choice of θ lead to a worse performance.

[1]The experiments accompanying this paper will be provided under https://www.tuwien.at/mg/asc/praetorius/software/mooafem.

$\cdot 10^{-4}$	$\theta = 0.1$					$\theta = 0.3$					$\theta = 0.5$				
λ_{alg} \ λ_{sym}	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.1	64.5	64.6	54.7	55.6	54.8	27.1	20.7	20.3	20.3	20.3	25.5	20.5	20.5	20.9	20.6
0.3	63.8	56.2	55.0	54.7	55.1	24.0	20.2	19.3	19.2	19.1	21.8	20.9	21.2	21.5	21.8
0.5	56.4	56.5	55.7	55.1	55.2	21.6	19.1	19.1	18.3	17.7	19.2	18.3	17.7	17.8	17.7
0.7	56.6	55.9	55.6	55.7	54.4	21.0	19.2	18.7	17.7	17.9	17.5	18.1	18.6	18.0	17.6
0.9	57.4	55.3	55.3	55.2	55.2	21.1	19.3	18.5	17.8	17.8	17.5	17.8	18.5	18.1	17.9
	$\theta = 0.7$					$\theta = 0.8$					$\theta = 0.9$				
0.1	36.2	33.4	25.8	25.7	25.8	45.8	43.1	36.1	31.3	31.3	63.5	68.6	60.8	44.6	44.2
0.3	27.4	28.0	29.5	30.2	30.9	34.3	37.1	36.7	40.4	43.2	48.4	54.7	53.5	56.1	69.7
0.5	23.8	21.5	21.0	21.5	23.1	34.2	27.4	25.9	25.8	29.6	47.1	35.9	41.9	44.6	46.4
0.7	23.0	21.0	21.7	22.1	23.3	28.9	25.9	27.0	31.0	30.0	40.0	36.3	40.7	45.6	49.8
0.9	22.9	21.0	21.8	22.1	23.0	28.8	26.3	27.0	31.0	29.8	40.7	36.4	40.6	45.5	49.8

Table 4.1: Optimal selection of parameters with respect to the computational costs for experiment (4.88) with $p = 2$ and $\delta = 0.5$. For the comparison, we consider the weighted cumulative time $\big[\eta_\ell(u_\ell^{\underline{k},\underline{j}}) \sum_{|\ell',k',j'| \le |\ell,\underline{k},\underline{j}|} \mathrm{time}(\ell')\big]$ (values in 10^{-4}) with stopping criterion $\eta_\ell(u_\ell^{\underline{k},\underline{j}}) < 5 \cdot 10^{-5}$ for various choices of λ_{sym}, λ_{alg}, and θ. In each θ-block, we mark in yellow the best choice per column, in blue the best choice per row, and in green when both choices coincide. The best choices for λ_{alg} and λ_{sym} are observed for $\theta = 0.3$ and $\theta = 0.5$.

4.A Proofs of Lemma 4.6, Lemma 4.10, and Lemma 4.17

Proof of Lemma 4.6. The proof is split into four steps.

Step 1. We consider the perturbed contraction of $(a_\ell)_{\ell \in \mathbb{N}_0}$ from (4.21). By induction on n, we see with the empty sum understood (as usual) as zero that

$$a_{\ell+n} \le q^n a_\ell + \sum_{j=1}^{n} q^{n-j} b_{\ell+j-1} \quad \text{for all } \ell, n \in \mathbb{N}_0.$$

From this and the geometric series, we infer that

$$a_{\ell+n} \le q^n a_\ell + C_1 \Big(\sum_{j=1}^{n} q^{n-j} \Big) a_\ell \le \Big(q^n + \frac{C_1}{1-q} \Big) a_\ell =: C_3\, a_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.89}$$

Step 2. Next, we note that the perturbed contraction of $(a_\ell)_{\ell \in \mathbb{N}_0}$ from (4.21) and the Young inequality with sufficiently small $\varepsilon > 0$ ensure

$$0 < \kappa := (1+\varepsilon)\, q^2 < 1 \quad \text{and} \quad a_{\ell+1}^2 \overset{(4.21)}{\le} \kappa\, a_\ell^2 + (1+\varepsilon^{-1})\, b_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

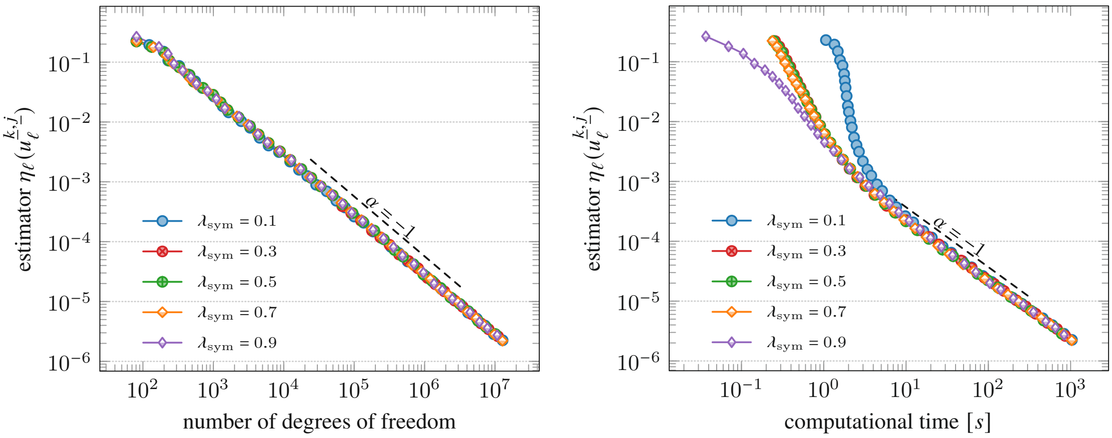


Figure 4.2: Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computational time (right) for experiment (4.88) with $p = 2$ and $\delta = 0.5$ for several symmetrization parameters $\lambda_{sym} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed algebraic solver parameter $\lambda_{alg} = 0.7$ and marking parameter $\theta = 0.3$.

This and the summability of $(b_\ell)_{\ell\in\mathbb{N}_0}$ from (4.21) guarantee

$$\sum_{\ell'=\ell+1}^{\ell+N} a_{\ell'}^2 = \sum_{\ell'=\ell}^{\ell+N-1} a_{\ell'+1}^2 \overset{(4.21)}{\leq} \kappa \sum_{\ell'=\ell}^{\ell+N-1} a_{\ell'}^2 + (1+\varepsilon^{-1})C_2\, N^{1-\delta}\, a_\ell^2.$$

Rearranging the estimate, we arrive at

$$\sum_{\ell'=\ell}^{\ell+N} a_{\ell'}^2 \leq 1 + \frac{\kappa + (1+\varepsilon^{-1})C_2\, N^{1-\delta}}{1-\kappa}\, a_\ell^2 =: D_N\, a_\ell^2 \quad \text{for all } \ell, N \in \mathbb{N}_0, \tag{4.90}$$

where we note that $1 \leq D_N \simeq N^{1-\delta}$ as $N \to \infty$. In the following, we prove that this already guarantees that (4.90) holds with an N-independent constant (instead of the constant D_N growing with N); see also Lemma 4.10.

Step 3. We show by mathematical induction on n that (4.90) implies

$$a_{\ell+n}^2 \leq \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big) \sum_{\ell'=\ell}^{\ell+n} a_{\ell'}^2 \quad \text{for all } \ell, n \in \mathbb{N}_0. \tag{4.91}$$

Note that (4.91) holds for all $\ell \in \mathbb{N}_0$ and $n = 0$ (with the empty product interpreted as 1). Hence, we may suppose that (4.91) holds for all $\ell \in \mathbb{N}_0$ and up to $n \in \mathbb{N}_0$. Then,

$$a_{\ell+(n+1)}^2 = a_{(\ell+1)+n}^2 \overset{(4.91)}{\leq} \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big) \sum_{\ell'=\ell+1}^{(\ell+1)+n} a_{\ell'}^2 = \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big)\Big(\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2 - a_\ell^2\Big)$$
$$\overset{(4.90)}{\leq} \Big(\prod_{j=1}^{n}(1-D_j^{-1})\Big)\Big(\sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2 - D_{n+1}^{-1} \sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2\Big) = \Big(\prod_{j=1}^{n+1}(1-D_j^{-1})\Big) \sum_{\ell'=\ell}^{\ell+(n+1)} a_{\ell'}^2.$$

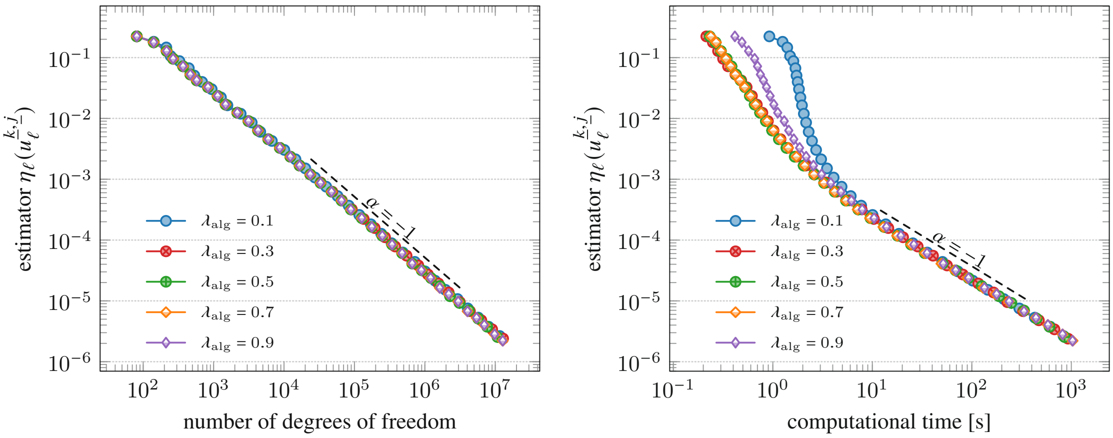


Figure 4.3: Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computational time (right) for experiment (4.88) with $p = 2$ and $\delta = 0.5$ for several algebraic solver parameters $\lambda_{\mathrm{alg}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed symmetrization parameter $\lambda_{\mathrm{sym}} = 0.7$ and marking parameter $\theta = 0.3$.

This concludes the proof of (4.91).

Step 4. From (4.90)–(4.91), we infer that

$$a_{\ell+n}^2 \le \Big(\prod_{j=1}^{n}(1 - D_j^{-1})\Big) D_n\, a_\ell^2 \quad \text{for all } \ \ell, n \in \mathbb{N}. \tag{4.92}$$

Note that

$$M_n := \log\Big[\Big(\prod_{j=1}^{n}(1 - D_j^{-1})\Big) D_n\Big] = \sum_{j=1}^{n} \log(1 - D_j^{-1}) + \log D_n.$$

With $1 - x \le \exp(-x)$ for all $0 < x < 1$, it follows for $x = D_j^{-1}$ that

$$M_n \le \log D_n - \sum_{j=1}^{n} D_j^{-1} \simeq (1-\delta)\ \log n - \sum_{j=1}^{n} \frac{1}{j^{1-\delta}} \xrightarrow{n\to\infty} -\infty,$$

since $\log n \le \sum_{j=1}^n (1/j)$. Fix $n_0 \in \mathbb{N}$ such that $M_{n_0} < 0$. It follows from (4.92) that

$$a_{\ell+in_0}^2 \le q_0^i\, a_\ell^2 \quad \text{for all } \ \ell, i \in \mathbb{N}_0, \quad \text{where } \ 0 < q_0 := \exp(M_{n_0}) < 1. \tag{4.93}$$

Let $\ell \in \mathbb{N}_0$. For general $n \in \mathbb{N}_0$, choose $i, j \in \mathbb{N}$ with $j < n_0$ such that $n = in_0 + j$. With (4.93) and quasi-monotonicity (4.89) of a_ℓ, we derive

$$a_{\ell+n}^2 = a_{(\ell+j)+in_0}^2 \overset{(4.93)}{\le} q_0^i\, a_{\ell+j}^2 \overset{(4.89)}{\le} C_3^2\, q_0^i\, a_\ell^2 = C_3^2\, q_0^{-j/n_0} q_0^{n/n_0} a_\ell^2 \le (C_3^2/q_0)\,(q_0^{1/n_0})^n a_\ell^2.$$

This completes the proof of (4.22) with $C_{\mathrm{lin}} := C_3^2/q_0 > 0$ and $0 < q_{\mathrm{lin}} := q_0^{1/n_0} < 1$. □

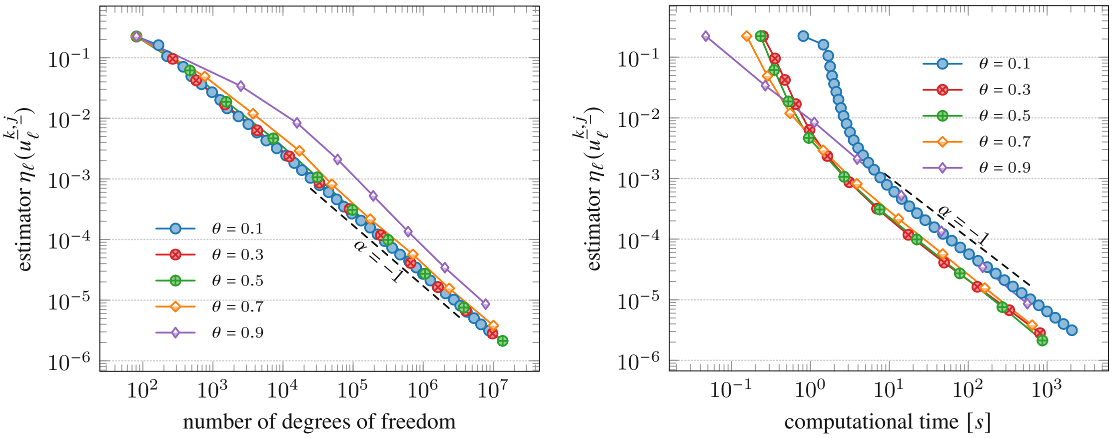


Figure 4.4: Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and the computational time (right) for experiment (4.88) with $p = 2$ and $\delta = 0.5$ for several Dörfler marking parameters $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and fixed solver-stopping parameters $\lambda_{\text{sym}} = \lambda_{\text{alg}} = 0.7$.

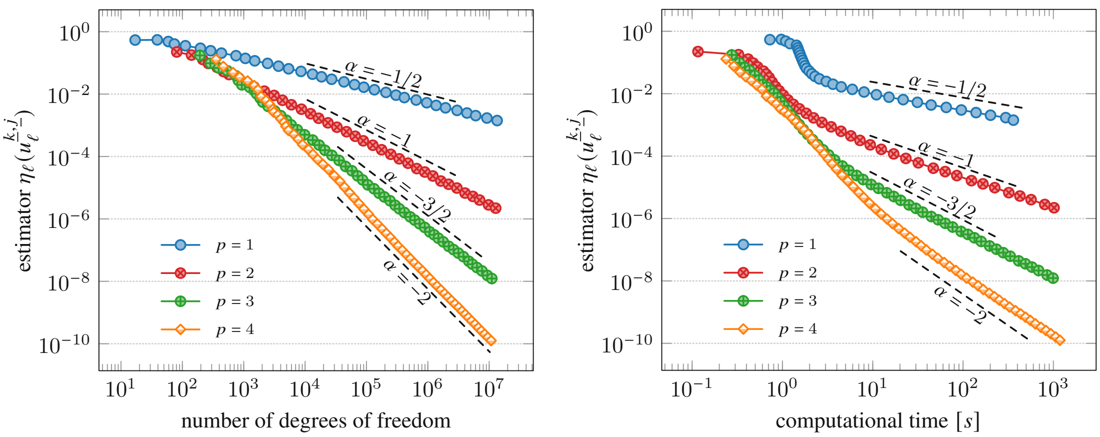


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Figure 4.5: Convergence history plot of the error estimator with respect to the number of degrees of freedom (left) and with respect to the overall computational time (right) for experiment (4.88) with $\delta = 0.5$ for several polynomial degrees $p = 1, 2, 3, 4$, and fixed marking parameter $\theta = 0.3$ and solver-stopping parameters $\lambda_{\text{sym}} = \lambda_{\text{alg}} = 0.7$.

Proof of Lemma 4.10. First, observe that $(a_\ell)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent in the sense of (ii) if and only if $(a_\ell^m)_{\ell\in\mathbb{N}_0}$ is R-linearly convergent in the sense of (ii) with C_{lin} replaced by C_{lin}^m and q_{lin} replaced by q_{lin}^m. Therefore, we may restrict to $m = 1$.

The implication (ii) $\Longrightarrow$ (i) follows from the geometric series, i.e.,

$$\sum_{\ell'=\ell+1}^{\infty} a_{\ell'} \overset{\text{(ii)}}{\le} C a_\ell \sum_{\ell'=\ell+1}^{\infty} q^{\ell'-\ell} = \frac{Cq}{1-q} a_\ell \quad \text{for all } \ell \in \mathbb{N}_0.$$

Conversely, (i) yields that

$$(C_1^{-1}+1) \sum_{\ell'=\ell+1}^{\infty} a_{\ell'} \overset{\text{(i)}}{\le} a_\ell + \sum_{\ell'=\ell+1}^{\infty} a_{\ell'} = \sum_{\ell'=\ell}^{\infty} a_{\ell'} \quad \text{for all } \ell \in \mathbb{N}_0.$$

Inductively, this leads to

$$a_{\ell+n} \le \sum_{\ell'=\ell+n}^{\infty} a_{\ell'} \overset{\text{(i)}}{\le} \frac{1}{(C_1^{-1}+1)^n} \sum_{\ell'=\ell}^{\infty} a_{\ell'} \overset{\text{(i)}}{\le} \frac{1+C_1}{(C_1^{-1}+1)^n} a_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0.$$

This proves (ii) with $C_{\mathrm{lin}} := 1 + C_1$ and $q_{\mathrm{lin}} := (C_1^{-1}+1)^{-1}$. □

Proof of Lemma 4.17. Let $(\ell, k, \underline{j}) \in \mathcal{Q}$ with $k \ge 1$. Contraction of the Zarantonello iteration (4.46) proves

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(4.46)}{\le} q_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}|||.$$

From the termination criterion of the algebraic solver (4.50), we see that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \le \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{(4.50)}{\le} \frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}} \big[\lambda_{\mathrm{sym}} \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}|||\big].$$

With the termination criterion of the inexact Zarantonello iteration (4.49), it follows that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(4.49)}{\le} \frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}} \begin{cases} \lambda_{\mathrm{sym}} \eta_\ell(u_\ell^{k,\underline{j}}) & \text{for } k = \underline{k}[\ell], \\ |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| & \text{for } 1 \le k < \underline{k}[\ell]. \end{cases}$$

For $k = \underline{k}[\ell]$, the preceding estimates prove (4.59). For $k < \underline{k}[\ell]$, it follows that

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le q_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + \frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}} \big[|||u_\ell^\star - u_\ell^{k,\underline{j}}||| + |||u_\ell^\star - u_\ell^{k-1,\underline{j}}|||\big].$$

Provided that $\frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}} < 1$, this proves

$$|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \le \frac{q_{\mathrm{sym}} + \frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}}}{1 - \frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}} \lambda_{\mathrm{alg}}} |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \overset{(4.53)}{=} \overline{q}_{\mathrm{sym}}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}|||,$$

which is (4.58). This concludes the proof. □

5 Optimal complexity of goal-oriented adaptive FEM for nonsymmetric linear elliptic PDEs

The Sections 5.1–5.8 of this chapter correspond to the publication:
P. Bringmann, M. Brunner, D. Praetorius, and J. Streitberger. Optimal complexity of goal-oriented adaptive FEM for nonsymmetric linear elliptic PDEs, 2023. arXiv: 2312.00489

5.1 Introduction

Adaptive finite element methods (AFEMs) are a cornerstone in the numerical solution of partial differential equations (PDEs). The abundant literature emphasizes significant progress and manifests a matured understanding of the topic; see, e.g., [Dör96; MNS00; BDD04; Ste07; CKNS08; KS11; CN12; FFP14; CFPP14] for linear elliptic PDEs.

The variational formulation of a *nonsymmetric* second-order linear elliptic PDE with bilinear form $b(\cdot,\cdot)$ and right-hand side functional F on the Sobolev space $\mathcal{X} \coloneqq H^1_0(\Omega)$ seeks a weak solution $u^\star$ to

$$b(u^\star, v) = F(v) \quad \text{for all} \quad v \in \mathcal{X}. \tag{5.1}$$

While standard AFEM aims at an efficient approximation of the solution $u^\star \in \mathcal{X}$, goal-oriented AFEM (GOAFEM) strives only to approximate a *quantity of interest* $G(u^\star)$; see [BR01; BR03; EEHJ95; GS02] for early prominent contributions. However, to accurately approximate $G(u^\star)$ for a continuous linear *goal functional* $G\colon \mathcal{X} \to \mathbb{R}$, following the generic approach $G(u_H) \approx G(u^\star)$ leads to convergence rates determined by the error of the approximation $u_H \approx u^\star$ to the *primal problem* (5.1). Instead, GOAFEM adopts a duality technique by additionally approximating $z_H \approx z^\star \in \mathcal{X}$ solving the *dual problem*

$$b(v, z^\star) = G(v) \quad \text{for all } v \in \mathcal{X}. \tag{5.2}$$

Following [GS02], a discrete approximation $G_H(u_H, z_H) \approx G(u^\star)$ enables the control of the error for any $u_H, z_H \in \mathcal{X}$ by

$$|G(u^\star) - G_H(u_H, z_H)| \le |b(u^\star - u_H, z^\star - z_H)| \le L\, |||u^\star - u_H|||\, |||z^\star - z_H|||, \tag{5.3}$$

where $L > 0$ is the continuity constant of $b(\cdot,\cdot)$ with respect to the energy norm $|||\cdot|||$; see Section 5.2 for details. As seen in (5.3), this approach allows to add the convergence rates of the primal and dual problem. Moreover, it is not necessary – and may even lead to unnecessary computational expense – to compute approximations $u_H \approx u^\star$ and $z_H \approx z^\star$ across the entire domain with the same accuracy. Instead, a careful marking of elements for refinement enables a considerable reduction of the computational costs and makes GOAFEM highly relevant in both practical applications and mathematical research.

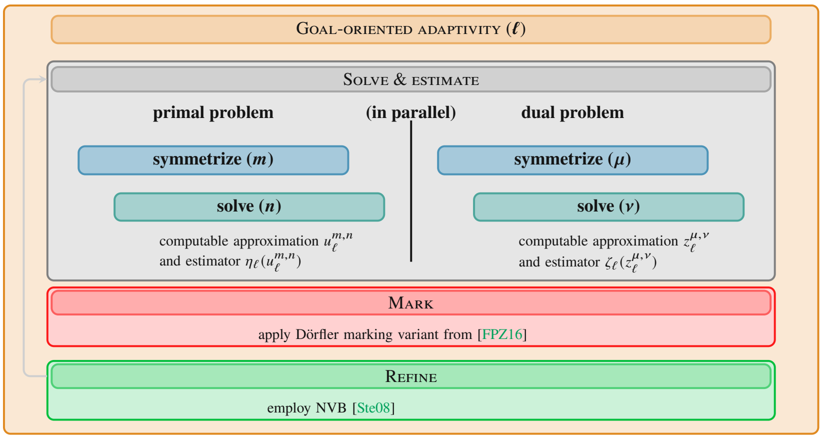

Figure 5.1: Schematic overview of the GOAISFEM algorithm with nested symmetrization and inexact solver.

First rigorous convergence results of GOAFEM are found in [MS09; BET11; FGH⁺16; FPZ16; HP16], recent contributions in this context include [BIP21; BBI⁺22] and for a dual weighted-residual approach see, e.g., [ELW19; ELW20; DBR21]. The works [MS09; FGH⁺16; FPZ16; BIP21; BBI⁺22] focus on optimal convergence rates with respect to the degrees of freedom. However, the cumulative nature of adaptivity calls for optimal convergence rates with respect to the total computational effort, i.e., the overall computational time. Coined as *optimal complexity* initially for wavelet-based discretizations [CDD01; CDD03], this notion was later adopted for AFEM with contributions including, e.g., [Ste07; CG12; GHPS21; BIM⁺24a]. In the setting of GOAFEM, optimal complexity was established first in [MS09] for the Poisson problem and sufficiently small adaptivity parameters, and extended to a general second-order *symmetric* linear elliptic PDE with uniformly contractive algebraic solver in [BGIP23]. Since uniform contraction with respect to the PDE-related energy norm for nonsymmetric algebraic solvers such as GMRES is still open, as a remedy, the proof of the Lax–Milgram lemma motivates the application of an iterative symmetrization [BIM⁺24a]. This results in a sequence of symmetric algebraic systems that allow the application of optimal algebraic solvers, e.g., [WZ17; CNX12; IMPS24]. Figure 5.1 illustrates the nested structure of the resulting goal-oriented adaptive iteratively symmetrized finite element method (GOAISFEM). The detailed Algorithm 5A is presented in Section 5.3 below. Table 5.1 displays the notation of the associated indices and quasi-error quantities, which are equivalent to the total error.

The first challenge in the analysis of the GOAISFEM algorithm consists of the nonlinear product structure attained by the combined quasi-error product as displayed in Table 5.1. The resulting nonlinear remainder term significantly complicates the proof compared to treating only the primal problem as in [BIM⁺24a] and requires the application of a novel proof strategy from [BFM⁺23] that only utilizes summability of the remainder, denoted as *tail-summability* throughout. The second challenge arises from the combination of the primal and dual marking leading to a merged marked set. Thereby, either only the primal or only the dual estimator is guaranteed to satisfy the estimator reduction property. Since the estimator belongs to the quasi-error, this also leads to a failure of contraction for one of the two involved quasi-errors. While [BGIP23] solves this issue in the symmetric case, the additional symmetrization loop results in a more involved situation at hand.

iteration	mesh refinement		symmetrization		algebraic solver			
	running	final	running	final	running	final	index set	quasi-error
primal	ℓ	$\underline{\ell}$	m	$\underline{m}$	n	$\underline{n}$	Q^u	$\mathsf{H}_\ell^{m,n}$
dual	ℓ	$\underline{\ell}$	μ	$\underline{\mu}$	ν	$\underline{\nu}$	Q^z	$\mathsf{Z}_\ell^{\mu,\nu}$
combined	ℓ	$\underline{\ell}$	k	$\underline{k} = \max\{\underline{m}, \underline{\mu}\}$	j	$\underline{j} = \max\{\underline{n}, \underline{\nu}\}$	$Q = Q^u \cup Q^z$	$\mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j}$

Table 5.1: Iteration counters and quasi-errors for the GOAISFEM algorithm. We note that for the combination of the index sets, the quasi-errors are extended to the full index set by the last available quasi-error. We refer to Section 5.3 for details on the iteration counters and index sets and to the beginning of Section 5.5 for a detailed description of the quasi-errors and their extension to the full index set Q.

Adapting the novel approach of the tail-summability criterion from [BFM+23] enables the proof of full linear convergence and optimal complexity for the nonlinear quasi-error product in this paper. The analysis employs the generalized quasi-orthogonality from [Fei22] to remedy the lack of a Pythagorean identity for nonsymmetric problems.

Our main result asserts full linear convergence of the quasi-error product $\mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j}$ with respect to the total step counter $|\cdot,\cdot,\cdot|$ (measuring the total solver steps in the index set). Therein, we allow for an arbitrary symmetrization stopping parameter λ_{sym} and only require a small algebraic solver parameter λ_{alg} such that the product $\lambda_{\mathrm{sym}}\,\lambda_{\mathrm{alg}}$ is sufficiently small. More precisely, Theorem 5.10 states that there exist constants $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ such that, for all $(\ell,k,j), (\ell',k',j') \in Q$ with $|\ell',k',j'| \le |\ell,k,j|$,

$$\mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\, \mathsf{H}_{\ell'}^{k',j'}\, \mathsf{Z}_{\ell'}^{k',j'}.$$

Note that, unlike [BIM+24a], where full linear convergence is guaranteed only for sufficiently large $\ell \ge \ell_0$, the current result is stronger in the sense that the result holds for $\ell_0 = 0$ owing to a generalized quasi-orthogonality from [Fei22]. An immediate consequence of full linear convergence and the geometric series in Corollary 5.14 states that the rates with respect to the degrees of freedom coincide with the rates with respect to the cumulative computational work (i.e., computational time), i.e., for all $r > 0$, there holds

$$\sup_{(\ell,k,j)\in Q} (\#\mathcal{T}_\ell)^r\, \mathsf{H}_\ell^{k,j} \mathsf{Z}_\ell^{k,j} \le \sup_{(\ell,k,j)\in Q} \Big(\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^r \mathsf{H}_\ell^{k,j} \mathsf{Z}_\ell^{k,j} \le C_{\mathrm{cost}} \sup_{(\ell,k,j)\in Q} (\#\mathcal{T}_\ell)^r\, \mathsf{H}_\ell^{k,j} \mathsf{Z}_\ell^{k,j}$$

along the sequence of meshes $\mathcal{T}_\ell$ generated by the GOAISFEM algorithm. The second main result of Theorem 5.15 proves that, for sufficiently small adaptivity parameters and any achievable rates $s,t > 0$ of the primal resp. dual problem (stated in terms of nonlinear approximation classes), the algorithm guarantees optimal complexity, i.e.,

$$\sup_{(\ell,k,j)\in Q} \Big(\sum_{\substack{(\ell',k',j')\in Q\\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^{s+t} \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \le C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t}, \mathsf{H}_0^{0,0}\, \mathsf{Z}_0^{0,0}\}.$$

This means the convergence of the algorithm attains the optimal rate $s + t$ with respect to the overall computational work, where $\|u^\star\|_{\mathbb{A}_s} < \infty$ means that $u^\star$ can be approximated at rate s (along a sequence of unavailable optimal meshes) and likewise for $z^\star$.

The remaining parts of the paper are organized as follows. The preliminary Section 5.2 introduces the model problem, the assumptions on the solvers, and the axioms of adaptivity from [CFPP14], including the general quasi-orthogonality from [Fei22]. Following the algorithm in Section 5.3 and its contraction properties in Section 5.4, Section 5.5 presents full linear convergence as the first main result of this paper. This allows to prove optimal complexity in Section 5.6 as the second main result, which is underlined by the numerical experiments in Section 5.7 including a thorough investigation of the adaptivity parameters. The paper concludes with a summary in Section 5.8.

5.2 Setting

In this section, we introduce the problem and explain the key components needed to design the adaptive algorithm in Section 5.3.

5.2.1 Continuous model problem

Let $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a polygonal Lipschitz domain. Given right-hand sides $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$, we consider a general second-order linear elliptic PDE

$$-\operatorname{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + c\,u^\star = f - \operatorname{div}(\boldsymbol{f}) \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega, \tag{5.4}$$

with a pointwise symmetric and positive definite diffusion matrix $\boldsymbol{A} \in \big[L^\infty(\Omega)\big]^{d\times d}_{\mathrm{sym}}$, a convection coefficient $\boldsymbol{b} \in \big[L^\infty(\Omega)\big]^d$, and a reaction coefficient $c \in L^\infty(\Omega)$. For well-definedness of the *a posteriori* error estimator in Section 5.2.6 below, we additionally require that $\boldsymbol{A}|_T \in \big[W^{1,\infty}(T)\big]^{d\times d}_{\mathrm{sym}}$ and $\boldsymbol{f}|_T \in \big[H^1(T)\big]^d$ for all $T \in \mathcal{T}_0$, where $\mathcal{T}_0$ is an initial triangulation that subdivides Ω into compact simplices. Let $\langle \cdot\,,\,\cdot\rangle$ denote the $L^2(\Omega)$-scalar product. With the principal part $a(u,v) \coloneqq \langle \boldsymbol{A}\nabla u\,,\,\nabla v\rangle$, the variational formulation of (5.4) seeks a solution $u^\star \in \mathcal{X} \coloneqq H^1_0(\Omega)$ to the so-called *primal problem*

$$b(u^\star, v) \coloneqq a(u^\star, v) + \langle \boldsymbol{b}\cdot\nabla u^\star + c\,u^\star\,,\,v\rangle = \langle f\,,\,v\rangle + \langle \boldsymbol{f}\,,\,\nabla v\rangle \eqqcolon F(v) \quad \text{for all } v \in \mathcal{X}. \tag{5.5}$$

We suppose that the bilinear form $b(\cdot,\cdot)$ from (5.5) is continuous and elliptic with respect to the norm $\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$, i.e., there exist constants $L', \alpha' > 0$ such that

$$b(u,v) \leq L'\,\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}} \quad \text{and} \quad b(v,v) \geq \alpha'\,\|v\|^2_{\mathcal{X}} \quad \text{for all } u, v \in \mathcal{X}. \tag{5.6}$$

Then, the Lax–Milgram lemma proves existence and uniqueness of the solution $u^\star$ to (5.5). An elementary compactness argument shows that (5.6) implies ellipticity of the principal part $a(\cdot\,,\,\cdot)$ and thus $a(\cdot\,,\,\cdot)$ is a scalar product on $\mathcal{X}$ with induced energy norm $a(\cdot\,,\,\cdot)^{1/2} \eqqcolon |||\cdot||| \simeq \|\cdot\|_{\mathcal{X}}$, cf. [BHP17, Remark 3]. Therefore, $b(\cdot\,,\,\cdot)$ is also continuous and elliptic with respect to $|||\cdot|||$, i.e., there exist constants $L, \alpha > 0$ such that

$$b(u,v) \leq L\,|||u|||\,|||v||| \quad \text{and} \quad b(v,v) \geq \alpha\,|||v|||^2 \quad \text{for all } u, v \in \mathcal{X}. \tag{5.7}$$

In the present paper, we suppose that the *quantity of interest* G is linear and reads for given data $g \in L^2(\Omega)$ and $\boldsymbol{g} \in \left[L^2(\Omega)\right]^d$,

$$G(v) := \int_\Omega \left(g\, v + \boldsymbol{g} \cdot \nabla v\right) \mathrm{d}x.$$

In order to guarantee well-definedness of the error estimator in Section 5.2.6 below, we suppose $\boldsymbol{g}|_T \in \left[H^1(T)\right]^d$ for all initial simplices $T \in \mathcal{T}_0$. In view of the continuity and coercivity of $b(\,\cdot\,,\,\cdot\,)$, the Lax–Milgram lemma yields existence and uniqueness of the solution $z^\star \in \mathcal{X}$ of the so-called *dual problem*: Find $z^\star \in \mathcal{X}$ such that

$$b(v, z^\star) = G(v) \quad \text{for all } v \in \mathcal{X}. \tag{5.8}$$

5.2.2 Finite element discretization and discrete goal

For a polynomial degree $p \in \mathbb{N}$ and a conforming simplicial triangulation $\mathcal{T}_H$ of Ω, the discrete ansatz space reads

$$\mathcal{X}_H := \{v_H \in \mathcal{X} : \forall T \in \mathcal{T}_H,\ v_H|_T \text{ is a polynomial of total degree} \leq p\}. \tag{5.9}$$

Since $\mathcal{X}_H \subset \mathcal{X}$ is conforming, the Lax–Milgram lemma ensures the existence and uniqueness of primal and dual discrete solutions $u_H^\star, z_H^\star \in \mathcal{X}_H$ satisfying

$$b(u_H^\star, v_H) = F(v_H) \quad \text{and} \quad b(v_H, z_H^\star) = G(v_H) \quad \text{for all } v_H \in \mathcal{X}_H. \tag{5.10}$$

It is well-known that conforming FEMs are quasi-optimal, i.e., there hold Céa-type estimates with constant $C_{\text{Céa}} = L/\alpha$

$$|||u^\star - u_H^\star||| \leq C_{\text{Céa}} \min_{v_H \in \mathcal{X}_H} |||u^\star - v_H||| \quad \text{and} \quad |||z^\star - z_H^\star||| \leq C_{\text{Céa}} \min_{v_H \in \mathcal{X}_H} |||z^\star - v_H|||. \tag{5.11}$$

For arbitrary approximations $u_H, z_H, \in \mathcal{X}_H$ the linearity of the quantity of interest G as well as the primal and the dual problem (5.1) and (5.2) show that

$$\begin{aligned} G(u^\star) - G(u_H) = G(u^\star - u_H) &\overset{(5.2)}{=} b(u^\star - u_H, z^\star) \\ &\overset{(5.1)}{=} b(u^\star - u_H, z^\star - z_H) + \left[F(z_H) - b(u_H, z_H)\right]. \end{aligned}$$

The definition of the discrete goal quantity by $G_H(u_H, z_H) := G(u_H) + \left[F(z_H) - b(u_H, z_H)\right]$ allows to control the goal error by continuity of $b(\cdot,\cdot)$

$$|G(u^\star) - G_H(u_H, z_H)| \leq |b(u^\star - u_H, z^\star - z_H)| \leq L\,|||u^\star - u_H|||\,|||z^\star - z_H|||. \tag{5.12}$$

We emphasize that (5.12) holds for any u_H, z_H and, in particular, for those stemming from an iterative solution step. Moreover, if $u_H = u_H^\star$, then $G(u_H, z_H) = G(u_H^\star)$ as expected.

5.2.3 Zarantonello iteration

The discrete formulations (5.10) lead to positive definite, but *nonsymmetric* linear systems of equations. To reduce the formulation to symmetric and positive definite (SPD) problems, we follow previous own work [BIM+24a] for the primal problem and employ the Zarantonello iteration [Zar60]. Typically, the latter is used in the up-to-date proof of the Lax–Milgram lemma and also defines a linearization scheme for the treatment of a certain class of nonlinear elliptic PDEs (see, e.g., [CW17; GHPS18; HPSV21; BFM+23]). In its core, it is a fixed-point method, thus also applicable in the nonsymmetric setting at hand. For a damping parameter $\delta > 0$ and given $u_H, z_H \in \mathcal{X}_H$, the Zarantonello iterations $\Phi^u_H, \Phi^z_H \colon (0,\infty) \times \mathcal{X}_H \to \mathcal{X}_H$ compute the unique solutions $\Phi^u_H(\delta; u_H)$, $\Phi^z_H(\delta; z_H) \in \mathcal{X}_H$ to the symmetric variational formulations

$$a(\Phi^u_H(\delta; u_H), v_H) = a(u_H, v_H) + \delta\,\big[F(v_H) - b(u_H, v_H)\big] \quad \text{for all} \quad v_H \in \mathcal{X}_H, \tag{5.13a}$$

$$a(v_H, \Phi^z_H(\delta; z_H)) = a(v_H, z_H) + \delta\,\big[G(v_H) - b(v_H, z_H)\big] \quad \text{for all} \quad v_H \in \mathcal{X}_H. \tag{5.13b}$$

The Riesz–Fischer theorem (and also the Lax–Milgram lemma) guarantees existence and uniqueness of $\Phi^u_H(\delta; u_H), \Phi^z_H(\delta; z_H) \in \mathcal{X}_H$, i.e., the Zarantonello operators $\Phi^u_H(\delta; \cdot)$ and $\Phi^z_H(\delta; \cdot)$ are well-defined. In particular, the exact discrete solutions $u^\star_H = \Phi^u_H(\delta; u^\star_H)$ and $z^\star_H = \Phi^z_H(\delta; z^\star_H)$ are the unique fixed points for all $\delta > 0$. Moreover, for a sufficiently small damping parameter δ, i.e., $0 < \delta < \delta^\star := 2\alpha/L^2$, the Banach fixed-point theorem [Zei90a, Section 25.4] guarantees that $\Phi^u_H(\delta, \cdot)$ and $\Phi^z_H(\delta, \cdot)$ are contractive with constant $0 < q_{\mathrm{sym}} := \big[1 - \delta\,(2\alpha - \delta L^2)\big]^{1/2} < 1$, i.e., for all functions $v_H, w_H \in \mathcal{X}_H$, it holds that

$$\max\big\{|||\Phi^u_H(\delta; v_H) - \Phi^u_H(\delta; w_H)|||, |||\Phi^z_H(\delta; v_H) - \Phi^z_H(\delta; w_H)|||\big\} \le q_{\mathrm{sym}}\, |||v_H - w_H|||. \tag{5.14}$$

The optimal value $\delta_{\mathrm{opt}} = \alpha/L^2$ yields the minimal contraction value $q_{\mathrm{sym}} = 1 - \alpha^2/L^2$.

5.2.4 Algebraic solver

A canonical candidate for solving (5.10) directly is a generalized minimal residual method [Saa03; SS86] with optimal preconditioner for the symmetric part. While this guarantees uniform contraction of the algebraic residuals in a *discrete vector norm*, the link between the algebraic residuals and the functional setting is still open [BIM+24a]. Instead, after a symmetrization with the Zarantonello iteration, it remains to solve the SPD systems (5.13). Since large SPD problems are still computationally expensive and the exact solution cannot be computed in linear computational complexity, we employ an iterative algebraic solver whose iteration is expressed by the operator $\Psi_H \colon \mathcal{X}' \times \mathcal{X}_H \to \mathcal{X}_H$. More precisely, given a bounded linear functional $\psi \in \mathcal{X}'$ and an approximation $w_H \in \mathcal{X}_H$ of the exact solution $w^\star_H \in \mathcal{X}_H$ to $a(w^\star_H, v_H) = \psi(v_H)$ for all $v_H \in \mathcal{X}_H$, the algebraic solver returns an improved approximation $\Psi_H(\psi; w_H) \in \mathcal{X}_H$ in the sense that there exists $0 < q_{\mathrm{ctr}} < 1$ independent of ψ and $\mathcal{X}_H$ such that

$$|||w^\star_H - \Psi_H(\psi; w_H)||| \le q_{\mathrm{ctr}}\, |||w^\star_H - w_H||| \quad \text{for all} \quad w_H \in \mathcal{X}_H. \tag{5.15}$$

To simplify notation, we shall identify ψ with its Riesz representative $w^\star_H \in \mathcal{X}_H$ and write $\Psi_H(w^\star_H; \cdot)$ instead of $\Psi_H(\psi; \cdot)$, even though $w^\star_H$ is unknown in practice and will only be approximated by an optimal algebraic solver, e.g., [CNX12; WZ17; IMPS24]. In the following, we use the hp-robust multigrid method from [IMPS24] with localized lowest-order smoothing on intermediate levels and patchwise higher-order smoothing on the finest mesh as an innermost algebraic solver loop.

5.2.5 Mesh refinement

The mesh refinement employs newest-vertex bisection (NVB). We refer to [Ste08] for NVB with admissible initial triangulation $\mathcal{T}_0$ and $d \geq 2$, to [AFF+15; KPP13] for NVB with general $\mathcal{T}_0$ for $d \in \{1, 2\}$, and to the recent work [DGS23] for NVB with general $\mathcal{T}_0$ in any dimension $d \geq 2$. For each triangulation $\mathcal{T}_H$ and marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, let $\mathcal{T}_h := \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ be the coarsest conforming refinement of $\mathcal{T}_H$ such that at least all $T \in \mathcal{M}_H$ have been refined, i.e., $\mathcal{M}_H \subseteq \mathcal{T}_H \setminus \mathcal{T}_h$. We write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ can be obtained from $\mathcal{T}_H$ by finitely many steps of NVB, and $\mathcal{T}_h \in \mathbb{T}_N(\mathcal{T}_H)$ if $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with $\#\mathcal{T}_h - \#\mathcal{T}_H \leq N$ with the number of additional elements $N \in \mathbb{N}_0$. To simplify notation, we write $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$ and $\mathbb{T}_N := \mathbb{T}_N(\mathcal{T}_0)$. We note that the nestedness of meshes $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ implies nestedness of the corresponding finite element spaces $\mathcal{X}_H \subseteq \mathcal{X}_h \subset \mathcal{X}$ from (5.9).

5.2.6 *A posteriori* error estimation

For a triangle $T \in \mathcal{T}_H \in \mathbb{T}$ and $v_H \in \mathcal{X}_H$, let $\boldsymbol{n}$ denote the outer unit normal vector and $[\![\cdot]\!]$ the jump along inner edges of $\mathcal{T}_H$. We define the refinement indicators $\eta_H(T; v_H) \geq 0$ and $\zeta_H(T; v_H) \geq 0$ for the primal and dual problem from (5.10), respectively, by

$$\begin{aligned}
\eta_H(T; v_H)^2 &:= |T|^{2/d} \, \| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) + \boldsymbol{b} \cdot \nabla v_H + c\, v_H - f \|^2_{L^2(T)} \\
&\quad + |T|^{1/d} \, \| [\![(\boldsymbol{A}\nabla v_H - \boldsymbol{f}) \cdot \boldsymbol{n}]\!] \|^2_{L^2(\partial T \cap \Omega)}, \\
\zeta_H(T; v_H)^2 &:= |T|^{2/d} \, \| -\operatorname{div}(\boldsymbol{A}\nabla v_H - \boldsymbol{g}) - \boldsymbol{b} \cdot \nabla v_H + \big(c - \operatorname{div}(\boldsymbol{b})\big)\, v_H - g \|^2_{L^2(T)} \\
&\quad + |T|^{1/d} \, \| [\![(\boldsymbol{A}\nabla v_H - \boldsymbol{g}) \cdot \boldsymbol{n}]\!] \|^2_{L^2(\partial T \cap \Omega)}.
\end{aligned} \tag{5.16a}$$

For any subset $\mathcal{U}_H \subseteq \mathcal{T}_H$, we abbreviate

$$\eta_H(\mathcal{U}_H; v_H)^2 := \sum_{T \in \mathcal{U}_H} \eta_H(T; v_H)^2 \quad \text{and} \quad \zeta_H(\mathcal{U}_H; v_H)^2 := \sum_{T \in \mathcal{U}_H} \zeta_H(T; v_H)^2 \tag{5.16b}$$

as well as $\eta_H(v_H) := \eta_H(\mathcal{T}_H; v_H)$ and $\zeta_H(v_H) := \zeta_H(\mathcal{T}_H; v_H)$ for all $v_H \in \mathcal{X}_H$. For details on residual-based error estimators, we refer to [AO00; Ver94]. Throughout the paper, the index of the estimators refer to the underlying mesh, e.g., η_h and ζ_h on the refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ or η_ℓ and ζ_ℓ on a sequence of meshes $\mathcal{T}_\ell$ with $\ell \in \mathbb{N}_0$. It is well-known that η_H, ζ_H satisfy the following *axioms of adaptivity*.

Lemma 5.1 ([CFPP14, Section 6.1]). *The error estimators η_H, ζ_H from (5.16) satisfy the following properties with constants $C_{\mathrm{stab}}, C_{\mathrm{rel}}, C_{\mathrm{drel}}, C_{\mathrm{mon}} > 0$ and $0 < q_{\mathrm{red}} < 1$ for any triangulation $\mathcal{T}_H \in \mathbb{T}$ and any conforming refinement $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ with the corresponding Galerkin solutions $u_H^\star, z_H^\star \in \mathcal{X}_H$, $u_h^\star, z_h^\star \in \mathcal{X}_h$ to (5.10), any subset $\mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h$, and arbitrary $v_H \in \mathcal{X}_H$, $v_h \in \mathcal{X}_h$.*

(A1) ***stability:*** $|\eta_h(\mathcal{U}_H; v_h) - \eta_H(\mathcal{U}_H; v_H)| + |\zeta_h(\mathcal{U}_H; v_h) - \zeta_H(\mathcal{U}_H; v_H)| \leq C_{\mathrm{stab}} \, |\!|\!| v_h - v_H |\!|\!|$.

(A2) ***reduction:*** $\eta_h(\mathcal{T}_h \setminus \mathcal{T}_H; v_H) \leq q_{\mathrm{red}} \, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h; v_H)$ *and* $\zeta_h(\mathcal{T}_h \setminus \mathcal{T}_H; v_H) \leq q_{\mathrm{red}} \zeta_H(\mathcal{T}_H \setminus \mathcal{T}_h; v_H)$.

(A3) ***reliability:*** $|\!|\!| u^\star - u_H^\star |\!|\!| \leq C_{\mathrm{rel}} \, \eta_H(u_H^\star)$ *and* $|\!|\!| z^\star - z_H^\star |\!|\!| \leq C_{\mathrm{rel}} \, \zeta_H(z_H^\star)$.

(A3+) ***discrete reliability:*** $|\!|\!| u_h^\star - u_H^\star |\!|\!| \leq C_{\mathrm{drel}} \, \eta_H(\mathcal{T}_H \setminus \mathcal{T}_h, u_H^\star)$ *and* $|\!|\!| z_h^\star - z_H^\star |\!|\!| \leq C_{\mathrm{drel}} \, \zeta_H(\mathcal{T}_H \setminus \mathcal{T}_h, z_H^\star)$.

(QM) ***quasi-monotonicity:*** $\eta_h(u_h^\star) \le C_{\rm mon}\, \eta_H(u_H^\star)$ *and* $\zeta_h(z_h^\star) \le C_{\rm mon}\, \zeta_H(z_H^\star)$.

The constant $C_{\rm rel}$ *depends only on the uniform* γ*-shape regularity of all* $\mathcal{T}_H \in \mathbb{T}$ *and on the space dimension* d*, while* $C_{\rm stab}$ *and* $C_{\rm drel}$ *additionally depend on the polynomial degree* p*. For NVB, reduction (A2) holds with* $q_{\rm red} := 2^{-1/(2d)}$*. Moreover, the constant in quasi-monotonicity (QM) satisfies* $C_{\rm mon} \le \min\{1 + C_{\rm stab}(1 + C_{\rm Céa})C_{\rm rel}\,,\ 1 + C_{\rm stab}\, C_{\rm drel}\}$. □

Reliability (A3) and stability (A1) verify

$$\begin{aligned} |||u^\star - u_H||| &\le \max\{C_{\rm rel}, 1 + C_{\rm stab}\, C_{\rm rel}\}\, \big[\eta_H(u_H) + |||u_H^\star - u_H|||\big],\\ |||z^\star - z_H||| &\le \max\{C_{\rm rel}, 1 + C_{\rm stab}\, C_{\rm rel}\}\, \big[\zeta_H(z_H) + |||z_H^\star - z_H|||\big]. \end{aligned}$$

In combination with the estimate (5.12), we finally conclude for $C_{\rm goal} := L \max\{C_{\rm rel}, 1 + C_{\rm stab}\, C_{\rm rel}\}^2$ the reliable *goal-error estimate*

$$|G(u^\star) - G_H(u_H, z_H)| \le C_{\rm goal}\, \big[\eta_H(u_H) + |||u_H^\star - u_H|||\big]\, \big[\zeta_H(z_H) + |||z_H^\star - z_H|||\big], \tag{5.17}$$

which provides the core estimate of the proposed adaptive algorithm in Section 5.3 below.

The ellipticity of $b(\cdot,\cdot)$ from (5.7) ensures inf-sup stability of the elliptic problem at hand. Recall from [Fei22] that inf-sup stability implies the generalized quasi-orthogonality, which will be an important tool in the subsequent analysis.

Proposition 5.2 (validity of quasi-orthogonality [Fei22, Equation (8)])**.** *For any sequence* $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset \mathcal{X}$ *of nested discrete subspaces with* $\ell \ge 0$*, there holds*

(A4) ***quasi-orthogonality:*** *There exist constants* $C_{\rm orth} > 0$ *and* $0 < \delta < 1$ *such that the corresponding Galerkin solutions* $u_\ell^\star, z_\ell^\star \in \mathcal{X}_\ell$ *to (5.10) satisfy, for all* $\ell, M \in \mathbb{N}_0$,

$$\sum_{\ell'=\ell}^{\ell+M} |||u_{\ell'+1}^\star - u_{\ell'}^\star|||^2 \le C_{\rm orth}\,(M+1)^{1-\delta}\, |||u^\star - u_\ell^\star|||^2, \tag{5.18a}$$

$$\sum_{\ell'=\ell}^{\ell+M} |||z_{\ell'+1}^\star - z_{\ell'}^\star|||^2 \le C_{\rm orth}\,(M+1)^{1-\delta}\, |||z^\star - z_\ell^\star|||^2. \tag{5.18b}$$

The constants $C_{\rm orth}$ *and* δ *depend only on the dimension* d*, the elliptic bilinear form* $b(\,\cdot\,,\,\cdot\,)$*, and the chosen norm* $|||\cdot|||$*, but are independent of the spaces* $\mathcal{X}_\ell$. □

5.3 Adaptive algorithm

In this section, we introduce our goal-oriented adaptive iteratively symmetrized algorithm. It utilizes specific stopping indices denoted by an underline, e.g., $\underline{\ell}, \underline{m}[\ell], \underline{n}[\ell,k] \in \mathbb{N}_0$. For an overview, see Table 5.1 above. However, we may omit the dependence whenever it is apparent from the context, such as in the abbreviation $\underline{n} := \underline{n}[\ell,m]$ for $u_\ell^{m,\underline{n}}$.

Algorithm 5A: GOAISFEM

Input: Initial mesh $\mathcal{T}_0$, polynomial degree $p \in \mathbb{N}$, marking parameters $0 < \theta \le 1$, $C_{\rm mark} \ge 1$, solver parameters $\lambda_{\rm sym} > 0$, $\lambda_{\rm alg} > 0$, Zarantonello damping parameter $\delta > 0$, and initial guesses $u_0^{0,0} = u_0^{0,\underline{n}}$, $z_0^{0,0} = z_0^{0,\underline{\nu}} \in \mathcal{X}_0$.

Adaptive loop: For all $\ell = 0, 1, 2, \dots$, repeat the following steps (I)–(IV):

(I) `SOLVE & ESTIMATE (PRIMAL)`. For all $m = 1, 2, 3, \dots$, repeat (a)–(c):

(a) Set $u_\ell^{m,0} := u_\ell^{m-1,\underline{n}}$ and define for theoretical reasons $u_\ell^{m,\star} := \Phi_\ell^u(\delta; u_\ell^{m-1,\underline{n}})$.

(b) For all $n = 1, 2, 3, \dots$, repeat the following steps (i)–(ii):

(i) Compute $u_\ell^{m,n} := \Psi_\ell(u_\ell^{m,\star}; u_\ell^{m,n-1})$ and corresponding refinement indicators $\eta_\ell(T; u_\ell^{m,n})$ for all $T \in \mathcal{T}_\ell$.

(ii) Terminate n-loop and define $\underline{n}[\ell, m] := n$ if

$$|||u_\ell^{m,n} - u_\ell^{m,n-1}||| \le \lambda_{\rm alg} \big[\lambda_{\rm sym}\, \eta_\ell(u_\ell^{m,n}) + |||u_\ell^{m,n} - u_\ell^{m,0}|||\big]. \tag{5.19}$$

(c) Terminate m-loop and define $\underline{m}[\ell] := m$ if

$$|||u_\ell^{m,\underline{n}} - u_\ell^{m,0}||| \le \lambda_{\rm sym}\, \eta_\ell(u_\ell^{m,\underline{n}}). \tag{5.20}$$

(II) `SOLVE & ESTIMATE (DUAL)`. For all $\mu = 1, 2, 3, \dots$, repeat (a)–(c):

(a) Set $z_\ell^{\mu,0} := z_\ell^{\mu-1,\underline{\nu}}$ and define for theoretical reasons $z_\ell^{\mu,\star} := \Phi_\ell^z(\delta; z_\ell^{\mu-1,\underline{\nu}})$.

(b) For all $\nu = 1, 2, 3, \dots$, repeat the following steps (i)–(ii):

(i) Compute $z_\ell^{\mu,\nu} := \Psi_\ell(z_\ell^{\mu,\star}; z_\ell^{\mu,\nu-1})$ and corresponding refinement indicators $\zeta_\ell(T; z_\ell^{\mu,\nu})$ for all $T \in \mathcal{T}_\ell$.

(ii) Terminate ν-loop and define $\underline{\nu}[\ell, \mu] := \nu$ if

$$|||z_\ell^{\mu,\nu} - z_\ell^{\mu,\nu-1}||| \le \lambda_{\rm alg} \big[\lambda_{\rm sym}\, \zeta_\ell(z_\ell^{\mu,\nu}) + |||z_\ell^{\mu,\nu} - z_\ell^{\mu,0}|||\big]. \tag{5.21}$$

(c) Terminate μ-loop and define $\underline{\mu}[\ell] := \mu$ if

$$|||z_\ell^{\mu,\underline{\nu}} - z_\ell^{\mu,0}||| \le \lambda_{\rm sym}\, \zeta_\ell(z_\ell^{\mu,\underline{\nu}}). \tag{5.22}$$

(III) `MARK`. Determine sets

$$\begin{aligned} \overline{\mathcal{M}}_\ell^u &\in \mathbb{M}_\ell^u[\theta, u_\ell^{\underline{m},\underline{n}}] := \{\mathcal{U}_\ell \subseteq \mathcal{T}_\ell : \theta\, \eta_\ell(u_\ell^{\underline{m},\underline{n}})^2 \le \eta_\ell(\mathcal{U}_\ell, u_\ell^{\underline{m},\underline{n}})^2\},\\ \overline{\mathcal{M}}_\ell^z &\in \mathbb{M}_\ell^z[\theta, z_\ell^{\underline{\mu},\underline{\nu}}] := \{\mathcal{U}_\ell \subseteq \mathcal{T}_\ell : \theta\, \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})^2 \le \zeta_\ell(\mathcal{U}_\ell, z_\ell^{\underline{\mu},\underline{\nu}})^2\} \end{aligned}$$

satisfying the following Dörfler criterion [Dör96] with quasi-minimal cardinality

$$\#\overline{\mathcal{M}}_\ell^u \le C_{\rm mark} \min_{\mathcal{U}_\ell^\star \in \mathbb{M}_\ell^u[\theta, u_\ell^{\underline{m},\underline{n}}]} \mathcal{U}_\ell^\star \quad \text{and} \quad \#\overline{\mathcal{M}}_\ell^z \le C_{\rm mark} \min_{\mathcal{U}_\ell^\star \in \mathbb{M}_\ell^z[\theta, z_\ell^{\underline{\mu},\underline{\nu}}]} \mathcal{U}_\ell^\star. \tag{5.23}$$

As in [FPZ16], define the set of marked elements $\mathcal{M}_\ell := \mathcal{M}_\ell^u \cup \mathcal{M}_\ell^z$, where $\mathcal{M}_\ell^u \subseteq \overline{\mathcal{M}}_\ell^u$ and $\mathcal{M}_\ell^z \subseteq \overline{\mathcal{M}}_\ell^z$ satisfy $\#\mathcal{M}_\ell^u = \#\mathcal{M}_\ell^z = \min\{\#\overline{\mathcal{M}}_\ell^u, \#\overline{\mathcal{M}}_\ell^z\}$.

(IV) `REFINE`. Generate the new mesh $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{M}_\ell, \mathcal{T}_\ell)$ by NVB and define $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{n}} := u_{\ell+1}^{0,\star} := u_\ell^{\underline{m},\underline{n}}$ and $z_{\ell+1}^{0,0} := z_{\ell+1}^{0,\underline{\nu}} := z_{\ell+1}^{0,\star} := z_\ell^{\underline{\mu},\underline{\nu}}$ (nested iteration).

Output: Sequences of successively refined triangulations $\mathcal{T}_\ell$, successive discrete approximations $u_\ell^{m,n}$, $z_\ell^{\mu,\nu}$, and corresponding error estimators $\eta_\ell(u_\ell^{m,n})$, $\zeta(z_\ell^{\mu,\nu})$.

Remark 5.3. (i) *Although the primal loop (I) and dual loop (II) in Algorithm 5A are displayed sequentially, they are independent of each other. Therefore, a practical implementation will realize these iterations simultaneously since the system matrix is the same (thanks to the symmetrization step).*

(ii) *In order to investigate the asymptotic behavior, it is reasonable to analyze Algorithm 5A in the present formulation with infinitely many steps. We note that a practical implementation will terminate with $\underline{\ell} := \ell$ provided that the estimator product is smaller than a user-specified tolerance.*

For the analysis of Algorithm 5A, we define the index set $\mathcal{Q} := \mathcal{Q}^u \cup \mathcal{Q}^z$ with

$$\mathcal{Q}^u := \{(\ell, m, n) \in \mathbb{N}_0^3 \colon u_\ell^{m,n} \text{ is used in Algorithm 5A}\}, \tag{5.24a}$$
$$\mathcal{Q}^z := \{(\ell, \mu, \nu) \in \mathbb{N}_0^3 \colon z_\ell^{\mu,\nu} \text{ is used in Algorithm 5A}\}. \tag{5.24b}$$

Furthermore, we require the following final indices and notice that these are consistent with those defined in Algorithm 5A:

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 \colon (\ell, 0, 0) \in \mathcal{Q}^u \text{ or } (\ell, 0, 0) \in \mathcal{Q}^z\} \in \mathbb{N}_0 \cup \{\infty\}, \tag{5.25a}$$
$$\underline{m}[\ell] := \sup\{m \in \mathbb{N} \colon (\ell, m, 0) \in \mathcal{Q}^u\}, \qquad \underline{\mu}[\ell] := \sup\{\mu \in \mathbb{N} \colon (\ell, \mu, 0) \in \mathcal{Q}^z\}, \tag{5.25b}$$
$$\underline{n}[\ell, m] := \sup\{n \in \mathbb{N} \colon (\ell, m, n) \in \mathcal{Q}^u\}, \qquad \underline{\nu}[\ell, \mu] := \sup\{\nu \in \mathbb{N} \colon (\ell, \mu, \nu) \in \mathcal{Q}^z\}. \tag{5.25c}$$

In addition, we set $\underline{k}[\ell] := \max\{\underline{m}[\ell], \underline{\mu}[\ell]\}$ as well as $\underline{j}[\ell, k] := \max\{\underline{n}[\ell, k], \underline{\nu}[\ell, k]\}$.

Finally, we introduce the total step counter $|\cdot,\cdot,\cdot|$ defined for all $(\ell, k, j) \in \mathcal{Q}$ by

$$|\ell, k, j| = \sum_{\ell'=0}^{\ell-1} \sum_{k'=0}^{\underline{k}[\ell']} \sum_{j'=0}^{\underline{j}[\ell',k']} 1 + \sum_{k'=0}^{k-1} \sum_{j'=0}^{\underline{j}[\ell,k']} 1 + \sum_{j'=0}^{j-1} 1.$$

This definition indeed provides a lexicographic ordering on $\mathcal{Q}$, if the solver steps 5A(I) for $u_\ell^{m,n}$ and 5A(II) for $z_\ell^{\mu,\nu}$ are done in *parallel*. We note that one solver step of an optimal geometric multigrid method on graded meshes can be performed in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations; see, e.g., [WZ17; IMPS24]. For given $u_\ell^{m,n}, z_\ell^{\mu,\nu} \in \mathcal{X}_\ell$, the simultaneous computation of the refinement indicators $\eta_\ell(T, u_\ell^{m,n})$ and $\zeta_\ell(T, z_\ell^{\mu,\nu})$ requires $\mathcal{O}(\#\mathcal{T}_\ell)$ operations, hence the steps 5A(I)–(II) require $\mathcal{O}(\#\mathcal{T}_\ell)$ operations as well. Furthermore, Dörfler marking can be performed in $\mathcal{O}(\#\mathcal{T}_\ell)$ operations; see, e.g., [Ste07; PP20]. Therefore, the total work to compute $u_\ell^{m,n}$ and $z_\ell^{\mu,\nu}$ is (up to a constant) given by

$$\texttt{cost}(\ell, k, j) := \sum_{\substack{(\ell',m',n') \in \mathcal{Q}^u \\ |\ell',m',n'| \le |\ell,k,j|}} \#\mathcal{T}_{\ell'} + \sum_{\substack{(\ell',\mu',\nu') \in \mathcal{Q}^z \\ |\ell',\mu',\nu'| \le |\ell,k,j|}} \#\mathcal{T}_{\ell'} \simeq \sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \#\mathcal{T}_{\ell'}. \tag{5.26}$$

Since $\#\mathcal{Q} = \infty$, we have either $\underline{\ell} = \infty$, or $\underline{k}[\underline{\ell}] = \infty$, or $\underline{j}[\underline{\ell}, \underline{k}] = \infty$. A further observation about Algorithm 5A is that the nested algebraic solver loop within the Zarantonello loop is guaranteed to terminate, and the latter case $\underline{j}[\underline{\ell}, \underline{k}] = \infty$ is therefore excluded.

Lemma 5.4 (finite termination of algebraic solver [BIM+24a, Lemma 3.2]). *Independently of the algorithmic parameters δ, θ, λ_{sym}, and λ_{alg}, the innermost n- and ν-loops of Algorithm 5A always terminate. In particular, $\underline{j}[\ell,k] < \infty$ for all $(\ell,k,0) \in \mathcal{Q}$.* □

5.4 *A posteriori* error analysis

Algorithm 5A does not provide the exact algebraic solutions $u_\ell^{m,\star}$ and $z_\ell^{\mu,\star}$ to (5.13) but instead uses an inexact algebraic solver. However, the following result from [BIM+24a] applies to the primal and the dual problem alike and shows that these inexact Zarantonello iterations remain contractions except for the final iterate on each mesh (see also [BIM+24b] for an extended version).

Lemma 5.5 (contraction of inexact Zarantonello iteration [BIM+24a, Lemma 5.1]). *Choose any damping parameter $0 < \delta < \delta^\star = 2\alpha/L^2$ to ensure the contraction (5.14) of the Zarantonello iteration and*

$$0 < \lambda_{\mathrm{alg}}^\star < \frac{(1-q_{\mathrm{sym}})(1-q_{\mathrm{ctr}})}{4q_{\mathrm{ctr}}} \quad \text{such that} \quad 0 < \overline{q}_{\mathrm{sym}} := \frac{q_{\mathrm{sym}} + 2\,\frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}^\star}{1 - 2\,\frac{q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}^\star} < 1. \tag{5.27}$$

Then, for arbitrary $\lambda_{\mathrm{sym}} > 0$ and any $0 < \lambda_{\mathrm{alg}} \le \lambda_{\mathrm{alg}}^\star$, we have for all $(\ell, m, \underline{n}) \in \mathcal{Q}^u$ with $1 \le m < \underline{m}[\ell]$ and all $(\ell,\mu,\underline{\nu}) \in \mathcal{Q}^z$ with $1 \le \mu < \underline{\mu}[\ell]$ that

$$|||u_\ell^\star - u_\ell^{m,\underline{n}}||| \le \overline{q}_{\mathrm{sym}}\,|||u_\ell^\star - u_\ell^{m-1,\underline{n}}||| \quad \text{and} \quad |||z_\ell^\star - z_\ell^{\mu,\underline{\nu}}||| \le \overline{q}_{\mathrm{sym}}\,|||z_\ell^\star - z_\ell^{\mu-1,\underline{\nu}}|||. \tag{5.28}$$

Moreover, for $m = \underline{m}[\ell]$ resp. $\mu = \underline{\mu}[\ell]$, it holds that

$$\begin{aligned} |||u_\ell^\star - u_\ell^{m,\underline{n}}||| &\le q_{\mathrm{sym}}\,|||u_\ell^\star - u_\ell^{m-1,\underline{n}}||| + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}\,\lambda_{\mathrm{sym}}\,\eta_\ell(u_\ell^{m,\underline{n}}), \\ |||z_\ell^\star - z_\ell^{\mu,\underline{\nu}}||| &\le q_{\mathrm{sym}}\,|||z_\ell^\star - z_\ell^{\mu-1,\underline{\nu}}||| + \frac{2\,q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\,\lambda_{\mathrm{alg}}\,\lambda_{\mathrm{sym}}\,\zeta_\ell(z_\ell^{\mu,\underline{\nu}}). \end{aligned} \tag{5.29}$$ □

The subsequent lemma gathers *a posteriori* error estimates following directly from the corresponding contraction of the symmetrization, algebraic solver, and the inexact Zarantonello iteration. Further details of the elementary proof are omitted.

Lemma 5.6 (stability and *a posteriori* error control). *For all $(\ell, m, 0) \in \mathcal{Q}^u$, contraction (5.14) shows*

$$\frac{1-q_{\mathrm{sym}}}{q_{\mathrm{sym}}}\,|||u_\ell^\star - u_\ell^{m,\star}||| \le |||u_\ell^{m,\star} - u_\ell^{m-1,\underline{n}}||| \le (1+q_{\mathrm{sym}})\,|||u_\ell^\star - u_\ell^{m-1,\underline{n}}|||. \tag{5.30}$$

Analogously, for all $(\ell, m, n) \in \mathcal{Q}^u$ the contraction (5.15) ensures

$$\frac{1-q_{\mathrm{ctr}}}{q_{\mathrm{ctr}}}\,|||u_\ell^{m,\star} - u_\ell^{m,n}||| \le |||u_\ell^{m,n} - u_\ell^{m,n-1}||| \le (1+q_{\mathrm{ctr}})\,|||u_\ell^{m,\star} - u_\ell^{m,n-1}|||. \tag{5.31}$$

For all $(\ell, m, \underline{n}) \in Q^u$ with $m < \underline{m}[\ell]$, the contraction (5.28) leads to

$$\frac{1-\overline{q}_{\rm sym}}{\overline{q}_{\rm sym}} \, |||u_\ell^\star - u_\ell^{m,\underline{n}}||| \le |||u_\ell^{m,\underline{n}} - u_\ell^{m-1,\underline{n}}||| \le (1+\overline{q}_{\rm sym}) \, |||u_\ell^\star - u_\ell^{m-1,\underline{n}}|||. \tag{5.32}$$

The analogous estimates are also valid for the dual variable. □

Finally, the following lemma shows that in the case of finitely many mesh-refinement steps, the Zarantonello iteration does not terminate and one of the two exact continuous solutions is already the discrete solution to (5.10).

Lemma 5.7 (case of finite mesh-refinement steps)**.** *Suppose that the inexact Zarantonello iteration satisfies contraction (5.28) and that η and ζ satisfy (A1)–(A3). If $\underline{\ell} < \infty$, then $\underline{k}[\underline{\ell}] = \infty$ and $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$ (so that $u^\star = u_{\underline{\ell}}^\star$) or $\zeta_{\underline{\ell}}(z_{\underline{\ell}}^\star) = 0$ (so that $z^\star = z_{\underline{\ell}}^\star$).*

Proof. By Lemma 5.4, we have $\underline{j}[\ell, k] < \infty$. If $\underline{\ell} < \infty$, then $\underline{k}[\underline{\ell}] = \infty$ and, hence,

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^{m,\underline{n}}) \overset{(5.20)}{<} \lambda_{\rm sym}^{-1} \, |||u_{\underline{\ell}}^{m,\underline{n}} - u_{\underline{\ell}}^{m-1,\underline{n}}||| \quad \text{for all } m \in \mathbb{N} \tag{5.33}$$

or

$$\zeta_{\underline{\ell}}(z_{\underline{\ell}}^{\mu,\underline{\nu}}) \overset{(5.22)}{<} \lambda_{\rm sym}^{-1} \, |||z_{\underline{\ell}}^{\mu,\underline{\nu}} - z_{\underline{\ell}}^{\mu-1,\underline{\nu}}||| \quad \text{for all } \mu \in \mathbb{N}. \tag{5.34}$$

If (5.33) holds, then the inexact Zarantonello iterates $u_{\underline{\ell}}^{m,\underline{n}}$ are convergent with limit $u_{\underline{\ell}}^\star$ and we obtain by stability (A1) that

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) \overset{(A1)}{\le} \eta_{\underline{\ell}}(u_{\underline{\ell}}^{m,\underline{n}}) + C_{\rm stab} \, |||u_{\underline{\ell}}^\star - u_{\underline{\ell}}^{m,\underline{n}}||| \overset{(5.33)}{\lesssim} |||u_{\underline{\ell}}^{m,\underline{n}} - u_{\underline{\ell}}^{m-1,\underline{n}}||| \xrightarrow{m\to\infty} 0.$$

This proves that $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$, and we infer from reliability (A3) that $u_{\underline{\ell}}^\star = u^\star$. The same arguments apply to $z_{\underline{\ell}}^\star$ in the case of (5.34). □

Due to the contraction of the inexact Zarantonello iteration (5.28), we have the following a posteriori error estimates for the final iterates.

Lemma 5.8 (stability of final iterates)**.** *Suppose that the inexact Zarantonello iteration satisfies (5.28). Then, for all $(\ell+1, \underline{m}, \underline{n}) \in Q^u$ and $(\ell+1, \underline{\mu}, \underline{\nu}) \in Q^z$, there holds*

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{m}-1,\underline{n}}||| \le |||u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}}|||, \quad |||z_{\ell+1}^\star - z_{\ell+1}^{\underline{\mu}-1,\underline{\nu}}||| \le |||z_{\ell+1}^\star - z_\ell^{\underline{\mu},\underline{\nu}}|||, \tag{5.35}$$

$$|||u_{\ell+1}^{\underline{m},\underline{n}} - u_\ell^{\underline{m},\underline{n}}||| \le 4\, |||u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}}|||, \quad |||z_{\ell+1}^{\underline{\mu},\underline{\nu}} - z_\ell^{\underline{\mu},\underline{\nu}}||| \le 4\, |||z_{\ell+1}^\star - z_\ell^{\underline{\mu},\underline{\nu}}|||, \tag{5.36}$$

$$|||u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m}-1,\underline{n}}||| \le 4\, |||u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}}|||, \quad |||z_\ell^{\underline{\mu},\underline{\nu}} - z_\ell^{\underline{\mu}-1,\underline{\nu}}||| \le 4\, |||z_\ell^\star - z_\ell^{\underline{\mu}-1,\underline{\nu}}|||. \tag{5.37}$$

Proof. For $(\ell+1, \underline{m}, \underline{n}) \in Q^u$, nested iteration $u_{\ell+1}^{0,\underline{n}} = u_\ell^{\underline{m},\underline{n}}$ together with the contraction of the inexact Zarantonello iteration (5.28) and $\underline{m}[\ell+1] \ge 1$ prove (5.35) by

$$|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{m}-1,\underline{n}}||| \overset{(5.28)}{\le} \overline{q}_{\rm sym}^{\,\underline{m}[\ell+1]-1} \, |||u_{\ell+1}^\star - u_{\ell+1}^{0,\underline{n}}||| \le |||u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}}|||.$$

Let $(\ell, \underline{m}, \underline{n}) \in \mathcal{Q}^u$. Contraction of the algebraic solver (5.15), the fact $\underline{n}[\ell, \underline{m}] \geq 1$, and nested iteration $u_\ell^{\underline{m},0} = u_\ell^{\underline{m}-1,\underline{n}}$ show that

$$\|\!|\!| u_\ell^{\underline{m},\star} - u_\ell^{\underline{m},\underline{n}} |\!|\!\| \overset{(5.15)}{\leq} q_{\mathrm{ctr}}^{\underline{n}[\ell,\underline{m}]} \, |\!|\!| u_\ell^{\underline{m},\star} - u_\ell^{\underline{m},0} |\!|\!| \leq q_{\mathrm{ctr}} \, |\!|\!| u_\ell^{\underline{m},\star} - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!|. \tag{5.38}$$

This and with the contraction of the exact Zarantonello iteration (5.14) result in

$$\begin{aligned} |\!|\!| u_\ell^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| &\leq |\!|\!| u_\ell^\star - u_\ell^{\underline{m},\star} |\!|\!| + |\!|\!| u_\ell^{\underline{m},\star} - u_\ell^{\underline{m},\underline{n}} |\!|\!| \\ &\overset{(5.38)}{\leq} (1 + q_{\mathrm{ctr}}) \, |\!|\!| u_\ell^\star - u_\ell^{\underline{m},\star} |\!|\!| + q_{\mathrm{ctr}} \, |\!|\!| u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!| \\ &\overset{(5.14)}{\leq} \big[(1 + q_{\mathrm{ctr}}) q_{\mathrm{sym}} + q_{\mathrm{ctr}}\big] \, |\!|\!| u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!| \leq 3 \, |\!|\!| u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!|. \end{aligned} \tag{5.39}$$

Consequently, the combination of (5.39) and (5.35) validates (5.36) via

$$\begin{aligned} |\!|\!| u_{\ell+1}^{\underline{m},\underline{n}} - u_\ell^{\underline{m},\underline{n}} |\!|\!| &\leq |\!|\!| u_{\ell+1}^\star - u_{\ell+1}^{\underline{m},\underline{n}} |\!|\!| + |\!|\!| u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| \\ &\overset{(5.39)}{\leq} 3 \, |\!|\!| u_{\ell+1}^\star - u_{\ell+1}^{\underline{m}-1,\underline{n}} |\!|\!| + |\!|\!| u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| \overset{(5.35)}{\leq} 4 \, |\!|\!| u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!|. \end{aligned}$$

The estimate (5.39) also implies (5.37), because

$$|\!|\!| u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!| \leq |\!|\!| u_\ell^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| + |\!|\!| u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!| \overset{(5.39)}{\leq} 4 \, |\!|\!| u_\ell^\star - u_\ell^{\underline{m}-1,\underline{n}} |\!|\!|.$$

The same arguments prove the estimates for the dual variable and conclude the proof. □

The subsequent lemma states the estimator reduction for only one of the two error estimators. This poses a significant challenge in the proof of full linear convergence due to the required contraction of the nonlinear quasi-error product in Lemma 5.11 below.

Lemma 5.9 (estimator reduction and stability)**.** *Define the constant* $0 < q(\theta) := \big[1 - (1 - q_{\mathrm{red}}^2)\,\theta\big]^{1/2} < 1$ *and suppose that the estimators* η *and* ζ *satisfy* (A1)–(A2)*. If the primal error estimator satisfies the Dörfler criterion, i.e.,* $\mathcal{M}_\ell^u = \overline{\mathcal{M}}_\ell^u \subseteq \mathcal{M}_\ell$ *in Algorithm* 5A(III)*, then*

$$\begin{aligned} \eta_{\ell+1}(u_{\ell+1}^{\underline{m},\underline{n}}) &\leq q(\theta)\, \eta_\ell(u_\ell^{\underline{m},\underline{n}}) + 4\, C_{\mathrm{stab}} \, |\!|\!| u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| \quad && \text{for all} \quad (\ell+1, \underline{m}, \underline{n}) \in \mathcal{Q}^u, \\ \zeta_{\ell+1}(z_{\ell+1}^{\underline{\mu},\underline{\nu}}) &\leq \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}}) + 4\, C_{\mathrm{stab}} \, |\!|\!| z_{\ell+1}^\star - z_\ell^{\underline{\mu},\underline{\nu}} |\!|\!| \quad && \text{for all} \quad (\ell+1, \underline{\mu}, \underline{\nu}) \in \mathcal{Q}^z. \end{aligned} \tag{5.40}$$

If the dual error estimator satisfies the Dörfler criterion, i.e., $\mathcal{M}_\ell^z = \overline{\mathcal{M}}_\ell^z \subseteq \mathcal{M}_\ell$ *in Algorithm* 5A(III)*, then*

$$\begin{aligned} \eta_{\ell+1}(u_{\ell+1}^{\underline{m},\underline{n}}) &\leq \eta_\ell(u_\ell^{\underline{m},\underline{n}}) + 4\, C_{\mathrm{stab}} \, |\!|\!| u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}} |\!|\!| \quad && \text{for all} \quad (\ell+1, \underline{m}, \underline{n}) \in \mathcal{Q}^u, \\ \zeta_{\ell+1}(z_{\ell+1}^{\underline{\mu},\underline{\nu}}) &\leq q(\theta)\, \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}}) + 4\, C_{\mathrm{stab}} \, |\!|\!| z_{\ell+1}^\star - z_\ell^{\underline{\mu},\underline{\nu}} |\!|\!| \quad && \text{for all} \quad (\ell+1, \underline{\mu}, \underline{\nu}) \in \mathcal{Q}^z. \end{aligned} \tag{5.41}$$

Proof. For $(\ell+1, 0, 0) \in \mathcal{Q}^u$, stability (A1) and reduction (A2) yield that

$$\begin{aligned} \eta_{\ell+1}(u_\ell^{\underline{m},\underline{n}})^2 &= \eta_{\ell+1}(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell; u_\ell^{\underline{m},\underline{n}})^2 + \eta_{\ell+1}(\mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell; u_\ell^{\underline{m},\underline{n}})^2 \\ &\leq \eta_\ell(\mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell; u_\ell^{\underline{m},\underline{n}})^2 + q_{\mathrm{red}}^2\, \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}; u_\ell^{\underline{m},\underline{n}})^2 \\ &= \eta_\ell(u_\ell^{\underline{m},\underline{n}})^2 - (1 - q_{\mathrm{red}}^2)\, \eta_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}; u_\ell^{\underline{m},\underline{n}})^2. \end{aligned} \tag{5.42}$$

The Dörfler marking in Algorithm 5A(III) for the primal error estimator η and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ prove the contraction in (5.40)

$$\eta_{\ell+1}(u_\ell^{\underline{m},\underline{n}})^2 \le \eta_\ell(u_\ell^{\underline{m},\underline{n}})^2 - (1-q_{\mathrm{red}}^2)\,\eta_\ell(\mathcal{M}_\ell; u_\ell^{\underline{m},\underline{n}})^2 \le q(\theta)^2\,\eta_\ell(u_\ell^{\underline{m},\underline{n}})^2. \tag{5.43}$$

For $(\ell+1, \underline{m}, \underline{n}) \in \mathcal{Q}^u$, this and (5.36) lead to

$$\begin{aligned}\eta_{\ell+1}(u_{\ell+1}^{\underline{m},\underline{n}}) &\overset{\text{(A1)}}{\le} \eta_{\ell+1}(u_\ell^{\underline{m},\underline{n}}) + C_{\mathrm{stab}}\,|||u_{\ell+1}^{\underline{m},\underline{n}} - u_\ell^{\underline{m},\underline{n}}||| \\ &\overset{(5.43)}{\le} q(\theta)\,\eta_\ell(u_\ell^{\underline{m},\underline{n}}) + C_{\mathrm{stab}}\,|||u_{\ell+1}^{\underline{m},\underline{n}} - u_\ell^{\underline{m},\underline{n}}||| \\ &\overset{(5.36)}{\le} q(\theta)\,\eta_\ell(u_\ell^{\underline{m},\underline{n}}) + 4\,C_{\mathrm{stab}}\,|||u_{\ell+1}^\star - u_\ell^{\underline{m},\underline{n}}|||.\end{aligned}$$

For $(\ell+1, \underline{\mu}, \underline{\nu}) \in \mathcal{Q}^z$, we argue analogously to (5.42) in order to obtain that $\zeta_{\ell+1}(z_\ell^{\underline{\mu},\underline{\nu}}) \le \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})$. Together with (5.36), it follows that

$$\zeta_{\ell+1}(z_{\ell+1}^{\underline{\mu},\underline{\nu}}) \overset{\text{(A1)}}{\le} \zeta_{\ell+1}(z_\ell^{\underline{\mu},\underline{\nu}}) + C_{\mathrm{stab}}\,|||z_{\ell+1}^{\underline{\mu},\underline{\nu}} - z_\ell^{\underline{\mu},\underline{\nu}}||| \overset{(5.36)}{\le} \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}}) + 4\,C_{\mathrm{stab}}\,|||z_{\ell+1}^\star - z_\ell^{\underline{\mu},\underline{\nu}}|||.$$

The proof holds verbatim in the case of Dörfler marking for the dual error estimator, albeit with reversed roles. This concludes the proof. □

5.5 Full linear convergence

This section presents full linear convergence of Algorithm 5A as the first main result of this work. Recall the goal-error estimate from (5.17) motivating the product structure of the respective primal and dual error components. Thus, we define the quasi-errors

$$\mathsf{H}_\ell^{m,n} := |||u_\ell^\star - u_\ell^{m,n}||| + |||u_\ell^{m,\star} - u_\ell^{m,n}||| + \eta_\ell(u_\ell^{m,n}) \quad \text{for all} \quad (\ell, m, n) \in \mathcal{Q}^u, \tag{5.44a}$$

$$\mathsf{Z}_\ell^{\mu,\nu} := |||z_\ell^\star - z_\ell^{\mu,\nu}||| + |||z_\ell^{\mu,\star} - z_\ell^{\mu,\nu}||| + \zeta_\ell(z_\ell^{\mu,\nu}) \quad \text{for all} \quad (\ell, \mu, \nu) \in \mathcal{Q}^z. \tag{5.44b}$$

The quasi-errors naturally extend to the full index set $(\ell, k, j) \in \mathcal{Q}$ by

$$\begin{aligned}\mathsf{H}_\ell^{k,j} &:= \begin{cases}\mathsf{H}_\ell^{k,\underline{n}} & \text{if } (\ell,k,0) \in \mathcal{Q}^u \text{ but } (\ell,k,j) \notin \mathcal{Q}^u,\\ \mathsf{H}_\ell^{\underline{m},\underline{n}} & \text{if } (\ell,k,0) \notin \mathcal{Q}^u,\end{cases}\\ \mathsf{Z}_\ell^{k,j} &:= \begin{cases}\mathsf{Z}_\ell^{k,\underline{\nu}} & \text{if } (\ell,k,0) \in \mathcal{Q}^z \text{ but } (\ell,k,j) \notin \mathcal{Q}^z,\\ \mathsf{Z}_\ell^{\underline{\mu},\underline{\nu}} & \text{if } (\ell,k,0) \notin \mathcal{Q}^z.\end{cases}\end{aligned} \tag{5.45}$$

The following theorem asserts full linear convergence of the quasi-error product.

Theorem 5.10: full linear convergence

Suppose that the estimators η and ζ satisfy (A1)–(A3) and (QM) and suppose (A4). Recall $\lambda_{\mathrm{alg}}^\star$ and $\overline{q}_{\mathrm{sym}}$ from Lemma 5.5. With the constant $q(\theta)$ from Lemma 5.9 and $\overline{q} := \max\{q(\theta)^{1/2}, (1+q_{\mathrm{sym}})/2\} < 1$, let

$$0 < \lambda^\star := \frac{(1-q_{\mathrm{ctr}})\,(\overline{q}-q_{\mathrm{sym}})\,(1-\overline{q})}{8\,q_{\mathrm{ctr}}\,C_{\mathrm{stab}}}. \tag{5.46}$$

Then, for arbitrary marking parameter $0 < \theta \le 1$ and any solver parameters $\lambda_{\mathrm{sym}} > 0$ and

$0 < \lambda_{\mathrm{alg}} \le \lambda^\star_{\mathrm{alg}}$ *with* $\lambda_{\mathrm{sym}}\lambda_{\mathrm{alg}} \le \lambda^\star$, *Algorithm 5A guarantees full linear convergence: There exist constants* $C_{\mathrm{lin}} \ge 1$ *and* $0 < q_{\mathrm{lin}} < 1$ *such that the quasi-error product satisfies, for all* $(\ell, k, j), (\ell', k', j') \in \mathcal{Q}$ *with* $|\ell', k', j'| \le |\ell, k, j|$

$$\mathsf{H}_\ell^{k,j}\,\mathsf{Z}_\ell^{k,j} \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\,\mathsf{H}_{\ell'}^{k',j'}\,\mathsf{Z}_{\ell'}^{k',j'}. \tag{5.47}$$

The constants C_{lin} *and* q_{lin} *depend only on* C_{stab}, C_{rel}, C_{mon}, C_{orth}, $C_{\mathrm{Céa}}$, θ, q_{red}, $\overline{q}_{\mathrm{sym}}$, q_{sym}, q_{ctr}, λ_{sym}, *and* λ_{alg}.

Three lemmas are required to prove Theorem 5.10. The characterization of R-linear convergence from [BFM+23, Lemma 5 and 10] is the primary tool for the proof of Theorem 5.10; see (5.67) below. The proof of Theorem 5.10 departs with the contraction of the quasi-error for the final iterates of the inexact Zarantonello loop up to a remainder on the mesh level ℓ. To this end, we define the simplified weighted quasi-error

$$\mathsf{H}_\ell := \big[|||u_\ell^\star - u_\ell^{\underline{m},\underline{n}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{m},\underline{n}})\big],\ \mathsf{Z}_\ell := \big[|||z_\ell^\star - z_\ell^{\underline{\mu},\underline{\nu}}||| + \gamma\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})\big] \ \text{ for all } (\ell, \underline{k}, \underline{j}) \in \mathcal{Q}, \tag{5.48}$$

where $\gamma > 0$ is a free parameter chosen in (5.51) below. This quasi-error quantity satisfies contraction up to a tail-summable remainder due to estimator reduction (5.40)–(5.41).

Lemma 5.11 (contraction in mesh level up to tail-summable remainder)**.** *Under the assumptions of Theorem 5.10, there exists* $0 < q < 1$ *such that the quasi-error product* $\mathsf{H}_\ell\,\mathsf{Z}_\ell$ *from (5.48) satisfies contraction up to a remainder* $R_\ell \ge 0$,

$$\mathsf{H}_{\ell+1}\,\mathsf{Z}_{\ell+1} \le q\,\mathsf{H}_\ell\,\mathsf{Z}_\ell + q\,R_\ell \quad \textit{for all } (\ell+1, \underline{k}, \underline{j}) \in \mathcal{Q}. \tag{5.49}$$

The remainder R_ℓ *satisfies*

$$R_{\ell+M} \lesssim \mathsf{H}_\ell\,\mathsf{Z}_\ell \ \textit{ and } \ \sum_{\ell'=\ell}^{\ell+M} R_{\ell'}^2 \lesssim (M+1)^{1-\delta}\,\mathsf{H}_\ell^2\,\mathsf{Z}_\ell^2 \ \textit{ for all } \ell, M \in \mathbb{N}_0 \textit{ with } \ell + M < \underline{\ell}. \tag{5.50}$$

Proof. The proof consists of four steps.

Step 1 (choice of constants). Recall the constants $0 < q(\theta) < 1$ from Lemma 5.9 and $\lambda^\star > 0$ and $0 < \overline{q} < 1$ defined in the statement of Theorem 5.10 and define the constants

$$C(\gamma, \lambda) := 1 + \frac{2\,q_{\mathrm{ctr}}}{1 - q_{\mathrm{ctr}}}\,\frac{\lambda}{\gamma} > 1 \text{ and } 0 < q_{\mathrm{ctr}} := \max\big\{q_{\mathrm{sym}} + 4C_{\mathrm{stab}}\,C(\gamma,\lambda)\,\gamma,\ q(\theta)C(\gamma,\lambda)\big\}.$$

Elementary calculations show that the choice of

$$\gamma := \frac{\overline{q}\,(\overline{q} - q_{\mathrm{sym}})}{4\,C_{\mathrm{stab}}} < 1 \tag{5.51}$$

ensures $q_{\mathrm{sym}}\,C(\gamma,\lambda) + 4\,C_{\mathrm{stab}}\,\gamma\,C(\gamma,\lambda)^2 < 1$ as well as, for all $0 < \lambda < \lambda^\star$,

$$C(\gamma, \lambda) = 1 + \frac{2\,q_{\mathrm{ctr}}}{1 - q_{\mathrm{ctr}}}\,\frac{\lambda}{\gamma} < 1 + \frac{1 - \overline{q}}{\overline{q}} = \frac{1}{\overline{q}} \le \frac{1}{q(\theta)^{1/2}}. \tag{5.52}$$

Consequently, we have $q(\theta)\,C(\gamma,\lambda)^2 < 1$ and thus $0 < q'_{\mathrm{ctr}} := C(\gamma,\lambda)\,q_{\mathrm{ctr}} < 1$ and $q_{\mathrm{ctr}} < 1$.

Step 2 (contraction of H_ℓ and Z_ℓ). Abbreviate $\lambda \coloneqq \lambda_{\rm alg}\,\lambda_{\rm sym}$. Recall that marking in Algorithm 5A(III) ensures that the estimate (5.40) or (5.41) hold. If (5.40) is satisfied, the quasi-contraction of the inexact Zarantonello iteration (5.29) for the final iterate, the stability estimate (5.35), and the estimator reduction (5.40) lead, for all $(\ell+1,\underline{k},\underline{j}) \in \mathcal{Q}^u$, to

$$
\begin{aligned}
\mathsf{H}_{\ell+1} &\overset{(5.29)}{\le} q_{\rm sym}\,|||u^\star_{\ell+1} - u^{\underline{m}-1,\underline{n}}_{\ell+1}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u^{\underline{m},\underline{n}}_{\ell+1}) \\
&\overset{(5.35)}{\le} q_{\rm sym}\,|||u^\star_{\ell+1} - u^{\underline{m},\underline{n}}_{\ell}||| + C(\gamma,\lambda)\,\gamma\,\eta_{\ell+1}(u^{\underline{m},\underline{n}}_{\ell+1}) \\
&\overset{(5.40)}{\le} \big(q_{\rm sym} + 4\,C_{\rm stab}\,C(\gamma,\lambda)\,\gamma\big)\,|||u^\star_{\ell+1} - u^{\underline{m},\underline{n}}_{\ell}||| + q(\theta)\,C(\gamma,\lambda)\,\gamma\,\eta_\ell(u^{\underline{m},\underline{n}}_{\ell}) \\
&\le q_{\rm ctr}\,\big[|||u^\star_{\ell+1} - u^{\underline{m},\underline{n}}_{\ell}||| + \gamma\,\eta_\ell(u^{\underline{m},\underline{n}}_{\ell})\big].
\end{aligned}
\tag{5.53}
$$

The same arguments yield, for all $(\ell+1,\underline{\mu},\underline{\nu}) \in \mathcal{Q}^z$,

$$
\begin{aligned}
\mathsf{Z}_{\ell+1} &\overset{(5.29)}{\le} q_{\rm sym}\,|||z^\star_{\ell+1} - z^{\underline{\mu}-1,\underline{\nu}}_{\ell+1}||| + C(\gamma,\lambda)\,\gamma\,\zeta_{\ell+1}(z^{\underline{\mu},\underline{\nu}}_{\ell+1}) \\
&\overset{(5.35)}{\le} q_{\rm sym}|||z^\star_{\ell+1} - z^{\underline{\mu},\underline{\nu}}_{\ell}||| + C(\gamma,\lambda)\gamma\,\zeta_{\ell+1}(z^{\underline{\mu},\underline{\nu}}_{\ell+1}) \\
&\overset{(5.40)}{\le} C(\gamma,\lambda)\,\big[q_{\rm ctr}\,|||z^\star_{\ell+1} - z^{\underline{\mu},\underline{\nu}}_{\ell}||| + \gamma\,\zeta_\ell(z^{\underline{\mu},\underline{\nu}}_{\ell})\big].
\end{aligned}
\tag{5.54}
$$

For $0 < q_{\rm ctr} < q'_{\rm ctr} = C(\gamma,\lambda)\,q_{\rm ctr} < 1$, the product of (5.53) and (5.54) reads

$$
\begin{aligned}
\mathsf{H}_{\ell+1}\,\mathsf{Z}_{\ell+1} &\le C(\gamma,\lambda)\,q_{\rm ctr}\,\big[|||u^\star_{\ell+1} - u^{\underline{m},\underline{n}}_{\ell}||| + \gamma\,\eta_\ell(u^{\underline{m},\underline{n}}_{\ell})\big]\big[|||z^\star_{\ell+1} - z^{\underline{\mu},\underline{\nu}}_{\ell}||| + \gamma\,\zeta_\ell(z^{\underline{\mu},\underline{\nu}}_{\ell})\big] \\
&= q'_{\rm ctr}\big[|||u^\star_{\ell+1} - u^{\underline{m},\underline{n}}_{\ell}||| + \gamma\,\eta_\ell(u^{\underline{m},\underline{n}}_{\ell})\big]\big[|||z^\star_{\ell+1} - z^{\underline{\mu},\underline{\nu}}_{\ell}||| + \gamma\,\zeta_\ell(z^{\underline{\mu},\underline{\nu}}_{\ell})\big].
\end{aligned}
\tag{5.55}
$$

If (5.41) is satisfied, we obtain the same estimate with reversed roles in the derivation.

Step 3 (quasi-monotonicity of H_ℓ and Z_ℓ). The Céa estimate (5.11), nestedness of the discrete spaces, reliability (A3), quasi-monotonicity (QM), stability (A1), and the definition (5.48) prove, for all $\ell \le \ell' \le \ell'' \le \underline{\ell}$ with $(\ell,\underline{m},\underline{n}) \in \mathcal{Q}^u$ and $(\ell,\underline{\mu},\underline{\nu}) \in \mathcal{Q}^z$, that

$$
|||u^\star_{\ell''} - u^\star_{\ell'}||| \overset{(5.11)}{\lesssim} |||u^\star - u^\star_{\ell'}||| \overset{(A3)}{\lesssim} \eta_{\ell'}(u^\star_{\ell'}) \overset{(QM)}{\lesssim} \eta_\ell(u^\star_\ell) \overset{(A1)}{\lesssim} \eta_\ell(u^{\underline{m},\underline{n}}_\ell) + |||u^\star_\ell - u^{\underline{m},\underline{n}}_\ell||| \overset{(5.48)}{\simeq} \mathsf{H}_\ell, \tag{5.56a}
$$

$$
|||z^\star_{\ell''} - z^\star_{\ell'}||| \overset{(5.11)}{\lesssim} |||z^\star - z^\star_{\ell'}||| \overset{(A3)}{\lesssim} \zeta_{\ell'}(z^\star_{\ell'}) \overset{(QM)}{\lesssim} \zeta_\ell(z^\star_\ell) \overset{(A1)}{\lesssim} \zeta_\ell(z^{\underline{\mu},\underline{\nu}}_\ell) + |||z^\star_\ell - z^{\underline{\mu},\underline{\nu}}_\ell||| \overset{(5.48)}{\simeq} \mathsf{Z}_\ell, \tag{5.56b}
$$

where the hidden constants depend only on γ^{-1}, $C_{\rm Céa}$, $C_{\rm stab}$, $C_{\rm rel}$, and $C_{\rm mon}$. A successive application of (5.53), the quasi-monotonicity (QM), the geometric series, and (5.56a) show

$$
\begin{aligned}
\mathsf{H}_{\ell+M} &\overset{(5.53)}{\le} q_{\rm ctr}\,\mathsf{H}_{\ell+M-1} + q_{\rm ctr}\,|||u^\star_{\ell+M} - u^\star_{\ell+M-1}||| \le q_{\rm ctr}^M\,\mathsf{H}_\ell + \sum_{j=0}^{M-1}\Big(q^{M-j}|||u^\star_{\ell+j+1} - u^\star_{\ell+j}|||\Big) \\
&\overset{(5.56a)}{\simeq} \mathsf{H}_\ell \quad \text{for all } \ell, M \in \mathbb{N}_0 \text{ with } \ell + M < \underline{\ell}.
\end{aligned}
\tag{5.57a}
$$

Thus, we obtain from (5.54) that

$$
\begin{aligned}
\mathsf{Z}_{\ell+M} &\overset{(5.54)}{\le} C(\gamma,\lambda)\big[q_{\rm ctr}\,|||z^\star_{\ell+M} - z^{\underline{\mu},\underline{\nu}}_{\ell+M-1}||| + \gamma\,\zeta_{\ell+M-1}(z^{\underline{\mu},\underline{\nu}}_{\ell+M-1})\big] \\
&\overset{(5.52)}{\le} \max\Big\{q'_{\rm ctr}, \frac{\gamma}{q(\theta)}\Big\}\big[\mathsf{Z}_{\ell+M-1} + |||z^\star_{\ell+M} - z^\star_{\ell+M-1}|||\big]
\end{aligned}
$$

and the analogous induction argument to (5.57a) shows quasi-monotonicity

$$\mathsf{Z}_{\ell+M} \lesssim \mathsf{Z}_\ell \quad \text{for all } M \in \mathbb{N}_0 \text{ with } \ell + M < \underline{\ell}. \tag{5.57b}$$

Step 4 (contraction of $\mathsf{H}_\ell\,\mathsf{Z}_\ell$ up to tail-summable remainder). Define

$$\begin{aligned} R_\ell &:= |||u^\star_{\ell+1} - u^\star_\ell|||\,\big[|||z^\star_\ell - z_\ell^{\underline{\mu},\underline{\nu}}||| + |||z^\star_{\ell+1} - z^\star_\ell||| + \gamma\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})\big] \\ &\quad + |||z^\star_{\ell+1} - z^\star_\ell|||\big[|||u^\star_\ell - u_\ell^{\underline{m},\underline{n}}||| + |||u^\star_{\ell+1} - u^\star_\ell||| + \gamma\,\eta_\ell(u_\ell^{\underline{m},\underline{n}})\big]. \end{aligned}$$

The contraction (5.55) proves the quasi-contraction (5.49) via

$$\begin{aligned} \mathsf{H}_{\ell+1}\,\mathsf{Z}_{\ell+1} &\overset{(5.55)}{\le} q'_{\mathrm{ctr}}\big[|||u^\star_{\ell+1} - u_\ell^{\underline{m},\underline{n}}||| + \gamma\,\eta_\ell(u_\ell^{\underline{m},\underline{n}})\big]\big[|||z^\star_{\ell+1} - z_\ell^{\underline{\mu},\underline{\nu}}||| + \gamma\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})\big] \\ &\le q'_{\mathrm{ctr}}\,\big[|||u^\star_\ell - u_\ell^{\underline{m},\underline{n}}||| + |||u^\star_{\ell+1} - u^\star_\ell||| + \gamma\,\eta_\ell(u_\ell^{\underline{m},\underline{n}})\big] \\ &\qquad \times \big[|||z^\star_\ell - z_\ell^{\underline{\mu},\underline{\nu}}||| + |||z^\star_{\ell+1} - z^\star_\ell||| + \gamma\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})\big] \\ &\le q'_{\mathrm{ctr}}\,\mathsf{H}_\ell\,\mathsf{Z}_\ell + q'_{\mathrm{ctr}}\,R_\ell. \end{aligned}$$

The remainder term R_ℓ can be estimated by (5.56) and the Young inequality to show

$$R_\ell^2 \overset{(5.56)}{\lesssim} \big(|||u^\star_{\ell+1} - u^\star_\ell|||\,\mathsf{Z}_\ell + |||z^\star_{\ell+1} - z^\star_\ell|||\,\mathsf{H}_\ell\big)^2 \lesssim |||u^\star_{\ell+1} - u^\star_\ell|||^2\,\mathsf{Z}_\ell^2 + |||z^\star_{\ell+1} - z^\star_\ell|||^2\,\mathsf{H}_\ell^2. \tag{5.58}$$

Thus, the quasi-monotonicity (5.57) verifies

$$R_{\ell+M} \lesssim \mathsf{H}_{\ell+M}\,\mathsf{Z}_{\ell+M} \overset{(5.57)}{\lesssim} \mathsf{H}_\ell\,\mathsf{Z}_\ell \quad \text{for all } \ell, M \in \mathbb{N} \text{ with } \ell + M < \underline{\ell}.$$

Quasi-orthogonality (A4), reliability (A3), and the estimates (5.56) imply, for all $\ell, M \in \mathbb{N}_0$ with $\ell + M < \underline{\ell}$,

$$\begin{aligned} \sum_{\ell'=\ell}^{\ell+M} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 &\overset{(A4)}{\lesssim} (M+1)^{1-\delta}\,|||u^\star - u^\star_\ell|||^2 \overset{(A3)}{\lesssim} (M+1)^{1-\delta}\,\eta_\ell(u^\star_\ell)^2 \overset{(5.56a)}{\lesssim} (M+1)^{1-\delta}\,\mathsf{H}_\ell^2, \\ \sum_{\ell'=\ell}^{\ell+M} |||z^\star_{\ell'+1} - z^\star_{\ell'}|||^2 &\overset{(A4)}{\lesssim} (M+1)^{1-\delta}\,|||z^\star - z^\star_\ell|||^2 \overset{(A3)}{\lesssim} (M+1)^{1-\delta}\,\zeta_\ell(z^\star_\ell)^2 \overset{(5.56b)}{\lesssim} (M+1)^{1-\delta}\,\mathsf{Z}_\ell^2. \end{aligned} \tag{5.59}$$

Using (5.58), the quasi-monotonicity (5.57), and (5.59), we conclude the proof of (5.50), for all $\ell, M \in \mathbb{N}_0$ with $\ell + M < \underline{\ell}$,

$$\begin{aligned} \sum_{\ell'=\ell}^{\ell+M} R_{\ell'}^2 &\overset{(5.58)}{\lesssim} \sum_{\ell'=\ell}^{\ell+M} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2\,\mathsf{Z}_{\ell'}^2 + \sum_{\ell'=\ell}^{\ell+M} |||z^\star_{\ell'+1} - z^\star_{\ell'}|||^2\,\mathsf{H}_{\ell'}^2 \\ &\overset{(5.57)}{\lesssim} \mathsf{Z}_\ell^2 \sum_{\ell'=\ell}^{\ell+M} |||u^\star_{\ell'+1} - u^\star_{\ell'}|||^2 + \mathsf{H}_\ell^2 \sum_{\ell'=\ell}^{\ell+M} |||z^\star_{\ell'+1} - z^\star_{\ell'}|||^2 \overset{(5.59)}{\lesssim} (M+1)^{1-\delta}\,\mathsf{H}_\ell^2\,\mathsf{Z}_\ell^2. \end{aligned}$$

□

The tail-summability in ℓ provides the basis for the proof of tail-summability on the mesh level ℓ together with the Zarantonello symmetrization index k for the final iterates of the algebraic solver. The main ingredients in the proof of tail-summability in (ℓ, k) are Lemma 5.11 and the following quasi-contraction in the symmetrization index k.

Lemma 5.12 (quasi-contraction of inexact Zarantonello symmetrization)**.** *There holds*

$$\mathsf{H}_\ell^{k',\underline{j}}\,\mathsf{Z}_\ell^{k',\underline{j}} \lesssim \overline{q}_{\mathrm{sym}}^{k'-k}\,\mathsf{H}_\ell^{k,\underline{j}}\,\mathsf{Z}_\ell^{k,\underline{j}} \qquad \textit{for all } (\ell,k',\underline{j}) \in \mathcal{Q} \textit{ with } 0 \le k \le k' \le \underline{k}[\ell], \tag{5.60}$$

$$\mathsf{H}_\ell^{0,\underline{j}}\,\mathsf{Z}_\ell^{0,\underline{j}} \lesssim \mathsf{H}_{\ell-1}\,\mathsf{Z}_{\ell-1} \qquad \textit{for all } (\ell,0,0) \in \mathcal{Q} \textit{ with } \ell \ge 1. \tag{5.61}$$

Proof. First, we note that the *a posteriori* error control (5.31) and the stopping criteria of the algebraic solver (5.19) and of the symmetrization (5.20) lead, for $(\ell,\underline{m},\underline{n}) \in \mathcal{Q}^u$, to

$$|||u_\ell^{\underline{m},\star} - u_\ell^{\underline{m},\underline{n}}||| \overset{(5.31)}{\lesssim} |||u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m},\underline{n}-1}||| \overset{(5.19)}{\lesssim} \eta_\ell(u_\ell^{\underline{m},\underline{n}}) + |||u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m},0}||| \overset{(5.20)}{\lesssim} \eta_\ell(u_\ell^{\underline{m},\underline{n}}) \lesssim \mathsf{H}_\ell.$$

Since the two notions of quasi-errors H_ℓ and $\mathsf{H}_\ell^{\underline{k},\underline{j}}$ only differ by the middle term $|||u_\ell^{\underline{m},\star} - u_\ell^{\underline{m},\underline{n}}|||$ and the fixed constant factor $0 < \gamma < 1$, this and the analogous estimate for the dual variable show

$$\mathsf{H}_\ell \le \mathsf{H}_\ell^{\underline{k},\underline{j}} \lesssim \mathsf{H}_\ell \quad \text{and} \quad \mathsf{Z}_\ell \le \mathsf{Z}_\ell^{\underline{k},\underline{j}} \lesssim \mathsf{Z}_\ell \quad \text{for all } (\ell,\underline{k},\underline{j}) \in \mathcal{Q}. \tag{5.62}$$

For $0 \le k < k' < \underline{m}[\ell] < \underline{k}[\ell]$ (i.e., the primal iteration stops earlier than the dual iteration), the validity of the stopping criterion (5.19) for the algebraic solver and the failure of criterion (5.20) for the inexact Zarantonello symmetrization prove that

$$\begin{aligned}
\mathsf{H}_\ell^{k',\underline{n}} &\overset{(5.31)}{\lesssim} |||u_\ell^\star - u_\ell^{k',\underline{n}}||| + |||u_\ell^{k',\underline{n}} - u_\ell^{k',\underline{n}-1}||| + \eta_\ell(u_\ell^{k',\underline{n}}) \\
&\overset{(5.19)}{\lesssim} |||u_\ell^\star - u_\ell^{k'-1,\underline{n}}||| + |||u_\ell^{k',\underline{n}} - u_\ell^{k'-1,\underline{n}}||| + \eta_\ell(u_\ell^{k',\underline{n}}) \\
&\overset{(5.20)}{\lesssim} |||u_\ell^\star - u_\ell^{k',\underline{n}}||| + |||u_\ell^{k',\underline{n}} - u_\ell^{k'-1,\underline{n}}||| \\
&\overset{(5.32)}{\le} |||u_\ell^\star - u_\ell^{k'-1,\underline{n}}||| \overset{(5.28)}{\lesssim} \overline{q}_{\mathrm{sym}}^{k'-k}\,|||u_\ell^\star - u_\ell^{k,\underline{n}}||| \lesssim \overline{q}_{\mathrm{sym}}^{k'-k}\,\mathsf{H}_\ell^{k,\underline{n}}.
\end{aligned} \tag{5.63}$$

Moreover, for $0 \le k < k' = \underline{m}[\ell]$, stability (A1) and the estimate (5.37) verify

$$\begin{aligned}
\mathsf{H}_\ell^{\underline{m},\underline{n}} &\overset{(5.62)}{\simeq} |||u_\ell^\star - u_\ell^{\underline{m},\underline{n}}||| + \eta_\ell(u_\ell^{\underline{m},\underline{n}}) \overset{(A1)}{\lesssim} |||u_\ell^\star - u_\ell^{\underline{m},\underline{n}}||| + |||u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m}-1,\underline{n}}||| + \eta_\ell(u_\ell^{\underline{m}-1,\underline{n}}) \\
&\lesssim \mathsf{H}_\ell^{\underline{m}-1,\underline{n}} + |||u_\ell^{\underline{m},\underline{n}} - u_\ell^{\underline{m}-1,\underline{n}}||| \overset{(5.37)}{\lesssim} \mathsf{H}_\ell^{\underline{m}-1,\underline{n}} \overset{(5.63)}{\lesssim} \overline{q}_{\mathrm{sym}}^{\underline{m}[\ell]-1-k}\,\mathsf{H}_\ell^{k,\underline{n}} \simeq \overline{q}_{\mathrm{sym}}^{\underline{m}[\ell]-k}\,\mathsf{H}_\ell^{k,\underline{n}}.
\end{aligned}$$

For $0 \le k \le \underline{m}[\ell] < k' \le \underline{k}[\ell]$, it follows $\mathsf{H}_\ell^{k',\underline{n}} = \mathsf{H}_\ell^{\underline{m},\underline{n}} \lesssim \overline{q}_{\mathrm{sym}}^{\underline{m}[\ell]-k}\,\mathsf{H}_\ell^{k,\underline{n}}$. Finally, for $\underline{m}[\ell] \le k < k' \le \underline{k}[\ell]$, we have $\mathsf{H}_\ell^{k',\underline{n}} = \mathsf{H}_\ell^{\underline{m}[\ell],\underline{n}} = \mathsf{H}_\ell^{k,\underline{n}}$. Notice that the same argumentation holds for the dual quasi-error $\mathsf{Z}_\ell^{k,\underline{\nu}}$ in the remaining cases with $\underline{\mu}[\ell] < \underline{k}[\ell]$ (i.e., the dual iteration stops earlier than the primal iteration).

Since $\underline{k}[\ell] = \underline{m}[\ell]$ or $\underline{k}[\ell] = \underline{\mu}[\ell]$ by definition, we obtain, for all $(\ell,k',\underline{j}) \in \mathcal{Q}$ with $0 \le k \le k' \le \underline{k}[\ell]$,

$$\mathsf{H}_\ell^{k',\underline{j}} \lesssim \overline{q}_{\mathrm{sym}}^{k'-k}\,\mathsf{H}_\ell^{k,\underline{j}} \quad \text{if } \underline{k}[\ell] = \underline{m}[\ell] \quad \text{or} \quad \mathsf{Z}_\ell^{k',\underline{j}} \lesssim \overline{q}_{\mathrm{sym}}^{k'-k}\,\mathsf{Z}_\ell^{k,\underline{j}} \quad \text{if } \underline{k}[\ell] = \underline{\mu}[\ell].$$

Furthermore, there holds $\mathsf{H}_\ell^{k',\underline{j}} \lesssim \mathsf{H}_\ell^{k,\underline{j}}$ and $\mathsf{Z}_\ell^{k',\underline{j}} \lesssim \mathsf{Z}_\ell^{k,\underline{j}}$ in any case. This yields (5.60) via

$$\mathsf{H}_\ell^{k',\underline{j}}\,\mathsf{Z}_\ell^{k',\underline{j}} \lesssim \overline{q}_{\mathrm{sym}}^{k'-k}\,\mathsf{H}_\ell^{k,\underline{j}}\,\mathsf{Z}_\ell^{k,\underline{j}} \quad \text{for all } (\ell,k',\underline{j}) \in \mathcal{Q} \text{ with } 0 \le k \le k' \le \underline{k}[\ell],$$

where the hidden constant depends only on C_{stab}, λ_{sym}, and $\overline{q}_{\mathrm{sym}}$.

Nested iteration $u_{\ell-1}^{m,\underline{n}} = u_\ell^{0,\underline{n}}$ and $z_{\ell-1}^{\mu,\underline{\nu}} = z_\ell^{0,\underline{\nu}}$ and the estimates (5.56) yield, for all $(\ell,0,0) \in \mathcal{Q}$ with $\ell > 0$,

$$\mathsf{H}_\ell^{0,\underline{j}} \overset{(5.62)}{\simeq} |||u_\ell^\star - u_{\ell-1}^{m,\underline{n}}||| + \eta_\ell(u_{\ell-1}^{m,\underline{n}}) \le |||u_\ell^\star - u_{\ell-1}^\star||| + \mathsf{H}_{\ell-1}^{\underline{k},\underline{j}} \overset{(5.56)}{\lesssim} \mathsf{H}_{\ell-1} + \mathsf{H}_{\ell-1}^{\underline{k},\underline{j}} \overset{(5.62)}{\simeq} \mathsf{H}_{\ell-1},$$
$$\mathsf{Z}_\ell^{0,\underline{j}} \overset{(5.62)}{\simeq} |||z_\ell^\star - z_{\ell-1}^{\mu,\underline{\nu}}||| + \zeta_\ell(z_{\ell-1}^{\mu,\underline{\nu}}) \le |||z_\ell^\star - z_{\ell-1}^\star||| + \mathsf{Z}_{\ell-1}^{\underline{k},\underline{j}} \overset{(5.56)}{\lesssim} \mathsf{Z}_{\ell-1} + \mathsf{Z}_{\ell-1}^{\underline{k},\underline{j}} \overset{(5.62)}{\simeq} \mathsf{Z}_{\ell-1}.$$

A multiplication of the two previous estimates proves (5.61). □

Finally, the quasi-contraction in (ℓ,k) from Lemma 5.12 together with a quasi-contraction in the algebraic solver index j leads to tail-summability in (ℓ,k,j).

Lemma 5.13 (quasi-contraction and stability by algebraic solver)**.** *There holds*

$$\mathsf{H}_\ell^{k,j'}\,\mathsf{Z}_\ell^{k,j'} \lesssim q_{\mathrm{ctr}}^{j'-j}\,\mathsf{H}_\ell^{k,j}\,\mathsf{Z}_\ell^{k,j} \quad \text{for all } (\ell,k,j') \in \mathcal{Q} \text{ with } 0 \le j \le j' \le \underline{j}[\ell,k] \tag{5.64}$$

and, with the abbreviation $(m-1)_+ := \max\{m-1,0\}$,

$$\mathsf{H}_\ell^{m,0} \le 3\,\mathsf{H}_\ell^{(m-1)_+,\underline{n}} \text{ and } \mathsf{Z}_\ell^{\mu,0} \le 3\,\mathsf{Z}_\ell^{(\mu-1)_+,\underline{\nu}} \text{ for all } (\ell,m,0) \in \mathcal{Q}^u, (\ell,\mu,0) \in \mathcal{Q}^z. \tag{5.65}$$

Proof. We recall that $u_\ell^{0,0} = u_\ell^{0,\underline{n}} = u_\ell^{0,\star}$ by definition and, hence, $\mathsf{H}_\ell^{0,0} = \mathsf{H}_\ell^{0,\underline{n}} = \mathsf{H}_\ell^{0,\underline{j}}$. Nested iteration $u_\ell^{m,0} = u_\ell^{m-1,\underline{n}}$ implies that

$$|||u_\ell^{m,\star} - u_\ell^{m,0}||| \overset{(5.30)}{\le} (q_{\mathrm{sym}}+1)\,|||u_\ell^\star - u_\ell^{m-1,\underline{n}}||| \le 2\,\mathsf{H}_\ell^{m-1,\underline{j}} \quad \text{for all } (\ell,m,0) \in \mathcal{Q}^u.$$

Therewith, we derive (5.65).

The combination of *a posteriori* error control (5.30) for the exact Zarantonello iteration, for the algebraic solver (5.31), and the failure of the stopping criterion (5.19) in Algorithm 5A(I.b.ii) for the algebraic solver proves, for $0 \le j < j' < \underline{n}[\ell,m] < \underline{j}[\ell,m]$,

$$\begin{aligned}
\mathsf{H}_\ell^{m,j'} &\le |||u_\ell^\star - u_\ell^{m,\star}||| + 2\,|||u_\ell^{m,\star} - u_\ell^{m,j'}||| + \eta_\ell(u_\ell^{m,j'})\\
&\overset{(5.30)}{\le} \frac{q_{\mathrm{sym}}}{1-q_{\mathrm{sym}}}\,|||u_\ell^{m,j'} - u_\ell^{m-1,\underline{j}}||| + \Big(2 + \frac{q_{\mathrm{sym}}}{1-q_{\mathrm{sym}}}\Big)\,|||u_\ell^{m,\star} - u_\ell^{m,j'}||| + \eta_\ell(u_\ell^{m,j'})\\
&\overset{(5.31)}{\lesssim} |||u_\ell^{m,j'} - u_\ell^{m-1,\underline{j}}||| + |||u_\ell^{m,j'} - u_\ell^{m,j'-1}||| + \eta_\ell(u_\ell^{m,j'}) \overset{(5.19)}{\lesssim} |||u_\ell^{m,j'} - u_\ell^{m,j'-1}|||\\
&\overset{(5.31)}{\lesssim} |||u_\ell^{m,\star} - u_\ell^{m,j'-1}||| \overset{(5.15)}{\le} q_{\mathrm{ctr}}^{(j'-1)-j}\,|||u_\ell^{m,\star} - u_\ell^{m,j}||| \lesssim q_{\mathrm{ctr}}^{j'-j}\,\mathsf{H}_\ell^{m,j}.
\end{aligned} \tag{5.66}$$

For $0 \le j < \underline{n}[\ell,m] \le j' \le \underline{j}[\ell,m]$, stability (A1) and contraction of the algebraic solver (5.15) verify that

$$\begin{aligned}
\mathsf{H}_\ell^{m,j'} = \mathsf{H}_\ell^{m,\underline{n}} &\overset{(5.15)}{\le} |||u_\ell^\star - u_\ell^{m,\underline{n}-1}||| + |||u_\ell^{m,\underline{n}} - u_\ell^{m,\underline{n}-1}||| + q_{\mathrm{ctr}}\,|||u_\ell^{m,\star} - u_\ell^{m,\underline{n}-1}||| + \eta_\ell(u_\ell^{m,\underline{n}})\\
&\overset{(A1)}{\le} \mathsf{H}_\ell^{m,\underline{n}-1} + (2 + C_{\mathrm{stab}})\,|||u_\ell^{m,\underline{n}} - u_\ell^{m,\underline{n}-1}|||\\
&\overset{(5.31)}{\lesssim} \mathsf{H}_\ell^{m,\underline{n}-1} + |||u_\ell^{m,\star} - u_\ell^{m,\underline{n}-1}||| \lesssim \mathsf{H}_\ell^{m,\underline{n}-1} \overset{(5.66)}{\lesssim} q_{\mathrm{ctr}}^{\underline{n}[\ell]-j}\,\mathsf{H}_\ell^{m,j}.
\end{aligned}$$

For $\underline{n}[\ell, m] \le j < j' \le \underline{j}[\ell, m]$, it holds that $\mathsf{H}_\ell^{m,j} = \mathsf{H}_\ell^{m,\underline{n}} = \mathsf{H}_\ell^{m,j'}$. Since $\underline{j}[\ell,k] = \underline{n}[\ell,k]$ or $\underline{j}[\ell,k] = \underline{\nu}[\ell,k]$, we have, for all $(\ell,k,j') \in \mathcal{Q}$ with $0 \le j \le j' \le \underline{j}[\ell,k]$,

$$\mathsf{H}_\ell^{k,j} \lesssim q_{\mathrm{ctr}}^{j-j'}\, \mathsf{H}_\ell^{k,j'} \quad \text{if } \underline{j}[\ell,k] = \underline{n}[\ell,k] \quad \text{or} \quad \mathsf{Z}_\ell^{k,j} \lesssim q_{\mathrm{ctr}}^{j-j'}\, \mathsf{Z}_\ell^{k,j'} \quad \text{if } \underline{j}[\ell,k] = \underline{\nu}[\ell,k].$$

Furthermore, we have $\mathsf{H}_\ell^{k,j} \lesssim \mathsf{H}_\ell^{k,j'}$ and $\mathsf{Z}_\ell^{k,j} \lesssim \mathsf{Z}_\ell^{k,j'}$ in any case. Hence, we obtain

$$\mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \lesssim q_{\mathrm{ctr}}^{j-j'}\, \mathsf{H}_\ell^{k,j'}\, \mathsf{Z}_\ell^{k,j'} \qquad \text{for all } (\ell,k,j) \in \mathcal{Q} \text{ with } 0 \le j' \le j \le \underline{j}[\ell,k],$$

where the hidden constant depends only on q_{sym}, λ_{sym}, q_{ctr}, λ_{alg}, and C_{stab}. □

Ultimately, synthesizing the preceding lemmas yields tail-summability of the quasi-error product and thus leads to the following proof of Theorem 5.10.

Proof of Theorem 5.10. The proof consists of four steps.

Step 1 (tail-summability in mesh level $\boldsymbol{\ell}$). We apply the tail-summability criterion from [BFM+23, Lemma 5] to the sequences $a_\ell := \mathsf{H}_\ell\, \mathsf{Z}_\ell$ and $b_\ell := q'_{\mathrm{ctr}}\, R_\ell$. Therein, it is shown that R-linear convergence is equivalent to tail-summability and that, for tail-summability, it is sufficient to guarantee

$$a_{\ell+1} \le q a_\ell + b_\ell, \quad b_{\ell+M} \le C_1\, a_\ell, \text{ and } \sum_{\ell'=\ell}^{\ell+M} b_{\ell'}^2 \le C_2\, (M+1)^{1-\delta}\, a_\ell^2 \text{ for all } \ell, M \in \mathbb{N}_0. \tag{5.67}$$

Indeed, contraction up to a remainder from (5.49), the estimate of the remainder from (5.50), and the quasi-monotonicity of H_ℓ and Z_ℓ from (5.57) validate the assumptions of the tail-summability criterion (5.67) and lead to tail-summability

$$\sum_{\ell'=\ell+1}^{\underline{\ell}-1} \mathsf{H}_{\ell'}\, \mathsf{Z}_{\ell'} \lesssim \mathsf{H}_\ell\, \mathsf{Z}_\ell \quad \text{for all } (\ell, \underline{k}, \underline{j}) \in \mathcal{Q}. \tag{5.68}$$

Step 2 (tail-summability in $(\boldsymbol{\ell}, \boldsymbol{k})$). For $(\ell,k,\underline{j}) \in \mathcal{Q}$, the estimates (5.60)–(5.61) and the geometric series prove tail-summability

$$\begin{aligned}
\sum_{\substack{(\ell',k',\underline{j}) \in \mathcal{Q} \\ |\ell',k',\underline{j}| > |\ell,k,\underline{j}|}} \mathsf{H}_\ell^{k',\underline{j}}\, \mathsf{Z}_\ell^{k',\underline{j}} &= \sum_{k'=k+1}^{\underline{k}[\ell]} \mathsf{H}_\ell^{k',\underline{j}}\, \mathsf{Z}_\ell^{k',\underline{j}} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathsf{H}_{\ell'}^{k',\underline{j}}\, \mathsf{Z}_{\ell'}^{k',\underline{j}} \\
&\overset{(5.60)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}}\, \mathsf{Z}_\ell^{k,\underline{j}} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \mathsf{H}_{\ell'}^{0,\underline{j}}\, \mathsf{Z}_{\ell'}^{0,\underline{j}} \overset{(5.61)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}}\, \mathsf{Z}_\ell^{k,\underline{j}} + \sum_{\ell'=\ell}^{\underline{\ell}-1} \mathsf{H}_{\ell'}\, \mathsf{Z}_{\ell'} \\
&\overset{(5.68)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}}\, \mathsf{Z}_\ell^{k,\underline{j}} + \mathsf{H}_\ell\, \mathsf{Z}_\ell \overset{(5.62)}{\lesssim} \mathsf{H}_\ell^{k,\underline{j}}\, \mathsf{Z}_\ell^{k,\underline{j}}.
\end{aligned} \tag{5.69}$$

Step 3 (tail-summability in (ℓ, k, j)). Finally, for all $(\ell, k, j) \in Q$, we observe that

$$\sum_{\substack{(\ell',k',j')\in Q \\ |\ell',k',j'|>|\ell,k,j|}} \mathsf{H}_{\ell'}^{k',j'}\,\mathsf{Z}_{\ell'}^{k',j'} = \sum_{j'=j+1}^{\underline{j}[\ell,k]} \mathsf{H}_{\ell}^{k,j'}\,\mathsf{Z}_{\ell}^{k,j'} + \sum_{k'=k+1}^{\underline{k}[\ell]} \sum_{j'=0}^{\underline{j}[\ell,k']} \mathsf{H}_{\ell}^{k',j'}\,\mathsf{Z}_{\ell}^{k',j'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \sum_{j'=0}^{\underline{j}[\ell',k']} \mathsf{H}_{\ell'}^{k',j'}\,\mathsf{Z}_{\ell'}^{k',j'}$$

$$\overset{(5.64)}{\lesssim} \mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j} + \sum_{k'=k+1}^{\underline{k}[\ell]} \mathsf{H}_{\ell}^{k',0}\,\mathsf{Z}_{\ell}^{k',0} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \mathsf{H}_{\ell'}^{k',0}\,\mathsf{Z}_{\ell'}^{k',0}$$

$$\overset{(5.65)}{\lesssim} \mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j} + \sum_{\substack{(\ell',k',\underline{j})\in Q \\ |\ell',k',\underline{j}|>|\ell,k,\underline{j}|}} \mathsf{H}_{\ell'}^{k',\underline{j}}\,\mathsf{Z}_{\ell'}^{k',\underline{j}} \overset{(5.69)}{\lesssim} \mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j} + \mathsf{H}_{\ell}^{k,\underline{j}}\,\mathsf{Z}_{\ell}^{k,\underline{j}} \overset{(5.64)}{\lesssim} \mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j}.$$

Step 4. Since the index set Q is linearly ordered with respect to the total step counter $|\cdot,\cdot,\cdot|$, tail-summability in Step 3 and the equivalence of tail-summability and R-linear convergence from [BFM+23, Lemma 10] conclude the proof of (5.47) in Theorem 5.10. □

5.6 Optimal complexity of Algorithm 5A

Full linear convergence (5.47) has a simple but crucial consequence. Using a geometric series argument, one can prove that the cumulative computational cost up to a given level is bounded by the cost of the said level; see [BFM+23, Corollary 14], where only the primal quasi-error $\mathsf{H}_{\ell}^{k,j}$ has to be replaced by the quasi-error product $\mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j}$. As a consequence, the convergence rates with respect to the number of degrees of freedom (defined as $M(r)$ in (5.70) below) and the rates with respect to the overall computational cost (cf. (5.26) and the discussion following the statement of Algorithm 5A) coincide.

Corollary 5.14 (rates = complexity [BFM+23, Corollary 14])**.** *Suppose the assumptions of Theorem 5.10. For all $r > 0$, the output $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ of Algorithm 5A satisfies*

$$M(r) := \sup_{(\ell,k,j)\in Q} (\#\mathcal{T}_\ell)^r\,\mathsf{H}_{\ell}^{k,j}\,\mathsf{Z}_{\ell}^{k,j} \le \sup_{(\ell,k,j)\in Q} \Big(\sum_{\substack{(\ell',k',j')\in Q \\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^r \mathsf{H}_{\ell}^{k,j}\mathsf{Z}_{\ell}^{k,j} \le C_{\mathrm{cost}}(r) M(r), \tag{5.70}$$

with the constant $C_{\mathrm{cost}}(r) := C_{\mathrm{lin}}/(1-q_{\mathrm{lin}}^{1/r})^r > 0$. □

While Theorem 5.10 only concerns R-linear convergence, a sufficiently small choice of the adaptivity parameters θ, λ_{sym}, and λ_{alg} even guarantees the optimal convergence rate $r = s + t$ with respect to computational cost, i.e., the overall computational time. Here, we suppose that the primal solution $u^\star$ to (5.5) can be approximated at rate s and the dual solution $z^\star$ to (5.8) can be approximated at rate t. To formalize this idea, we introduce the notion of approximation classes [BDD04; Ste07; CKNS08; CFPP14]. For $s, t > 0$, define

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_{\mathrm{opt}}\in\mathbb{T}_N} \eta_{\mathrm{opt}}(u_{\mathrm{opt}}^\star) \Big), \quad \|z^\star\|_{\mathbb{A}_t} := \sup_{N\in\mathbb{N}_0} \Big((N+1)^t \min_{\mathcal{T}_{\mathrm{opt}}\in\mathbb{T}_N} \zeta_{\mathrm{opt}}(z_{\mathrm{opt}}^\star) \Big),$$

where $\eta_{\mathrm{opt}}(\cdot)$ and $\zeta_{\mathrm{opt}}(\cdot)$ denote the estimator values for the exact discrete solutions $u_{\mathrm{opt}}^\star$ and $z_{\mathrm{opt}}^\star$ on the unavailable optimal triangulations $\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T})$. We stress that $\|u^\star\|_{\mathbb{A}_s}$ and $\|z^\star\|_{\mathbb{A}_t}$ can equivalently be defined by energy error plus data oscillations [FFP14; CFPP14].

Theorem 5.15: optimal complexity

Suppose that the estimators η and ζ satisfy (A1)–(A3$^+$) and (QM) and suppose quasi-orthogonality (A4). Recall $\lambda^\star_{\mathrm{alg}}$ from Lemma 5.5 and $\lambda^\star$ from (5.46) in Theorem 5.10. Define the constants

$$\begin{aligned} \lambda^\star_{\mathrm{sym}} &:= \min\{1, C_{\mathrm{stab}}^{-1} C_{\mathrm{alg}}^{-1}\} \le 1 \quad \textit{with} \quad C_{\mathrm{alg}} := \frac{1}{1-q_{\mathrm{sym}}} \Big(\frac{2\, q_{\mathrm{ctr}}}{1-q_{\mathrm{ctr}}}\, \lambda^\star_{\mathrm{alg}} + q_{\mathrm{sym}} \Big), \\ \theta^\star &:= (1 + C_{\mathrm{stab}}^2\, C_{\mathrm{rel}}^2)^{-1} < 1. \end{aligned} \tag{5.71}$$

Suppose that θ, λ_{sym}, and λ_{alg} are sufficiently small in the sense of

$$\begin{aligned} &0 < \lambda_{\mathrm{alg}} \le \lambda^\star_{\mathrm{alg}}, \quad 0 < \lambda_{\mathrm{sym}} < \lambda^\star_{\mathrm{sym}}, \quad \textit{and} \quad \lambda_{\mathrm{alg}}\, \lambda_{\mathrm{sym}} < \lambda^\star, \\ &0 < \theta_{\mathrm{mark}} := \frac{(\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}})^2}{(1-\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}})^2} < \theta^\star < 1. \end{aligned} \tag{5.72}$$

Then, Algorithm 5A guarantees, for all $s, t > 0$, that

$$\sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^{s+t} \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \le C_{\mathrm{opt}} \max\{\|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t}, \mathsf{H}_0^{0,0}\, \mathsf{Z}_0^{0,0}\}. \tag{5.73}$$

The constant C_{opt} depends only on C_{stab}, C_{rel}, C_{drel}, C_{mark}, C_{mesh}, C_{lin}, q_{lin}, $\#\mathcal{T}_0$, and $s+t$. In particular, there holds optimal complexity of Algorithm 5A.

The proof of Theorem 5.15 employs the following result from [BIM+24b] providing estimator equivalence between the (unavailable) estimators for the exact discrete solutions $u^\star_\ell, z^\star_\ell$ and the estimators at the computed approximations $u_\ell^{\underline{m},\underline{n}}, z_\ell^{\underline{\mu},\underline{\nu}}$.

Lemma 5.16 (estimator equivalence [BIM+24b, Lemma 15])**.** *Recall the constants $\lambda^\star_{\mathrm{sym}}$, $C_{\mathrm{alg}} > 0$ from (5.71) and $\lambda^\star_{\mathrm{alg}} > 0$ from Lemma 5.5. Then, for all $0 < \theta \le 1$, $0 < \lambda_{\mathrm{alg}} \le \lambda^\star_{\mathrm{alg}}$, $0 < \lambda_{\mathrm{sym}} < \lambda^\star_{\mathrm{sym}}$, it holds that*

$$\begin{aligned} \big(1-\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}\big)\, \eta_\ell(u_\ell^{\underline{m},\underline{n}}) \le \eta_\ell(u^\star_\ell) \le \big(1+\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}\big)\, \eta_\ell(u_\ell^{\underline{m},\underline{n}}) \quad &\textit{for all } (\ell,\underline{m},\underline{n}) \in \mathcal{Q}^u, \\ \big(1-\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}\big)\, \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}}) \le \zeta_\ell(z^\star_\ell) \le \big(1+\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}}\big)\, \zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}}) \quad &\textit{for all } (\ell,\underline{\mu},\underline{\nu}) \in \mathcal{Q}^z. \ \square \end{aligned} \tag{5.74}$$

Proof of Theorem 5.15. By Corollary 5.14, it suffices to prove that, for any $s, t > 0$,

$$\sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^{s+t}\, \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \lesssim \max\{\|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t}, \mathsf{H}_0^{0,0}\, \mathsf{Z}_0^{0,0}\}. \tag{5.75}$$

Since the inequality becomes trivial if either $\|u^\star\|_{\mathbb{A}_s} = \infty$ or $\|z^\star\|_{\mathbb{A}_t} = \infty$, we may assume $\|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t} < \infty$. The proof consists of three steps.

Step 1. With $0 < \theta_{\mathrm{mark}} := (\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}})^2\, (1-\lambda_{\mathrm{sym}}/\lambda^\star_{\mathrm{sym}})^{-2} < \theta^\star$, the validity of (A3$^+$) for both estimators and [FGH+16, Lemma 14] guarantee the existence of sets $\mathcal{R}_{\ell'} \subseteq \mathcal{T}_{\ell'}$ with $0 \le \ell' < \underline{\ell}$

such that

$$\#\mathcal{R}_{\ell'} \lesssim \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)}\,\big[\eta_{\ell'}(u^\star_{\ell'})\,\zeta_{\ell'}(z^\star_{\ell'})\big]^{-1/(s+t)}, \tag{5.76a}$$

$$\theta_{\rm mark}\,\eta_{\ell'}(u^\star_{\ell'}) \le \eta_{\ell'}(\mathcal{R}_{\ell'}, u^\star_{\ell'}) \quad\text{or}\quad \theta_{\rm mark}\,\zeta_{\ell'}(z^\star_{\ell'}) \le \zeta_{\ell'}(\mathcal{R}_{\ell'}, z^\star_{\ell'}). \tag{5.76b}$$

For $0 \le \ell' < \underline{\ell}$, the estimator equivalence (5.74) in Lemma 5.16 leads to

$$\big(1-\lambda_{\rm sym}/\lambda^\star_{\rm sym}\big)\,\eta_{\ell'}(u^{\underline{m},\underline{n}}_{\ell'}) \le \eta_{\ell'}(u^\star_{\ell'}) \quad\text{and}\quad \big(1-\lambda_{\rm sym}/\lambda^\star_{\rm sym}\big)\,\zeta_{\ell'}(z^{\underline{\mu},\underline{\nu}}_{\ell'}) \le \zeta_{\ell'}(z^\star_{\ell'})$$

and consequently with (5.76a) to

$$\#\mathcal{R}_{\ell'} \lesssim \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)}\,\big[\eta_{\ell'}(u^{\underline{m},\underline{n}}_{\ell'})\,\zeta_{\ell'}(z^{\underline{\mu},\underline{\nu}}_{\ell'})\big]^{-1/(s+t)}. \tag{5.77}$$

Note that the stopping criteria (5.20) and (5.22) lead to

$$\mathsf{H}_{\ell'} \simeq |||u^\star_{\ell'} - u^{\underline{m},\underline{n}}_{\ell'}||| + \eta_{\ell'}(u^{\underline{m},\underline{n}}_{\ell'}) \overset{(5.20)}{\lesssim} \eta_{\ell'}(u^{\underline{m},\underline{n}}_{\ell'}) \quad\text{and}\quad \mathsf{Z}_{\ell'} \simeq |||z^\star_{\ell'} - z^{\underline{\mu},\underline{\nu}}_{\ell'}||| + \zeta_{\ell'}(z^{\underline{\mu},\underline{\nu}}_{\ell'}) \overset{(5.22)}{\lesssim} \zeta_{\ell'}(z^{\underline{\mu},\underline{\nu}}_{\ell'})$$

and with (5.61) to

$$\mathsf{H}^{0,\underline{j}}_{\ell'+1}\,\mathsf{Z}^{0,\underline{j}}_{\ell'+1} \overset{(5.61)}{\lesssim} \mathsf{H}_{\ell'}\,\mathsf{Z}_{\ell'} \lesssim \eta_{\ell'}(u^{\underline{m},\underline{n}}_{\ell'})\,\zeta_{\ell'}(z^{\underline{\mu},\underline{\nu}}_{\ell'}). \tag{5.78}$$

Hence, the combination of (5.77) and (5.78) reads

$$\#\mathcal{R}_{\ell'} \lesssim \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)}\,\big[\mathsf{H}^{0,\underline{j}}_{\ell'+1}\,\mathsf{Z}^{0,\underline{j}}_{\ell'+1}\big]^{-1/(s+t)}. \tag{5.79}$$

Step 2. Recall from [BGIP23, Theorem 8] that the set $\mathcal{R}_{\ell'}$ satisfies the Dörfler criterion from Algorithm 5A(III) with the same parameter θ. The quasi-minimality of $\mathcal{M}_{\ell'}$ implies

$$\#\mathcal{M}_{\ell'} \le C_{\rm mark}\,\#\mathcal{R}_{\ell'} \quad\text{for all } 0 \le \ell' < \underline{\ell} \tag{5.80}$$

with the constant $C_{\rm mark} \ge 1$ from Algorithm 5A.

Step 3. Let $(\ell, k, j) \in \mathcal{Q}$. Full linear convergence (5.47) from Theorem 5.10 yields that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|}} \big(\mathsf{H}^{k',j'}_{\ell'}\,\mathsf{Z}^{k',j'}_{\ell'}\big)^{-1/s} \overset{(5.47)}{\lesssim} \big(\mathsf{H}^{k,j}_{\ell}\,\mathsf{Z}^{k,j}_{\ell}\big)^{-1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|}} \big(q_{\rm lin}^{1/s}\big)^{|\ell,k,j|-|\ell',k',j'|} \lesssim \big(\mathsf{H}^{k,j}_{\ell}\,\mathsf{Z}^{k,j}_{\ell}\big)^{-1/s}. \tag{5.81}$$

NVB refinement satisfies the mesh-closure estimate [CFPP14, Eqn. (2.9)] reading,

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm mesh} \sum_{\ell'=0}^{\ell-1} \#\mathcal{M}_{\ell'} \quad\text{for all } \ell \ge 0, \tag{5.82}$$

where $C_{\rm mesh} > 1$ depends only on $\mathcal{T}_0$. Thus, for $(\ell, k, j) \in \mathcal{Q}$, we have by the mesh-closure estimate (5.82), quasi-optimality of Dörfler marking (5.80), and the result (5.81) that

$$\begin{aligned}
\#\mathcal{T}_\ell - \#\mathcal{T}_0 &\overset{(5.82)}{\lesssim} \sum_{\ell'=0}^{\ell-1} \#\mathcal{M}_{\ell'} \overset{(5.80)}{\lesssim} \sum_{\ell'=0}^{\ell-1} \#\mathcal{R}_{\ell'} \overset{(5.79)}{\lesssim} \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)} \sum_{\ell'=0}^{\ell-1} \big(\mathsf{H}^{0,\underline{j}}_{\ell'+1}\,\mathsf{Z}^{0,\underline{j}}_{\ell'+1}\big)^{-1/(s+t)} \\
&\le \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)} \sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|}} \big(\mathsf{H}^{k',j'}_{\ell'}\,\mathsf{Z}^{k',j'}_{\ell'}\big)^{-1/(s+t)} \\
&\overset{(5.81)}{\lesssim} \big(\|u^\star\|_{\mathbb{A}_s}\,\|z^\star\|_{\mathbb{A}_t}\big)^{1/(s+t)}\,\big(\mathsf{H}^{k,j}_{\ell}\,\mathsf{Z}^{k,j}_{\ell}\big)^{-1/(s+t)}.
\end{aligned}$$

Rearranging the terms and noting that $1 \leq \#\mathcal{T}_\ell - \#\mathcal{T}_0$ implies $\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \leq 2\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)$, we obtain, for $\ell > 0$, that

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^{s+t}\, \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \lesssim \|u^\star\|_{\mathbb{A}_s}\, \|z^\star\|_{\mathbb{A}_t}. \tag{5.83a}$$

Moreover, full linear convergence (5.47) proves that

$$(\#\mathcal{T}_0 - \#\mathcal{T}_0 + 1)^{s+t}\, \mathsf{H}_0^{k,j}\, \mathsf{Z}_0^{k,j} = \mathsf{H}_0^{k,j}\, \mathsf{Z}_0^{k,j} \lesssim \mathsf{H}_0^{0,0}\, \mathsf{Z}_0^{0,0}. \tag{5.83b}$$

We recall from [BHP17, Lemma 22] that, for all $\mathcal{T}_\ell \in \mathbb{T}$, it holds

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \leq \#\mathcal{T}_\ell \leq \#\mathcal{T}_0\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1). \tag{5.84}$$

This shows, for all $(\ell, k, j) \in \mathcal{Q}$,

$$(\#\mathcal{T}_\ell)^{s+t}\, \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \overset{(5.84)}{\lesssim} (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^{s+t}\, \mathsf{H}_\ell^{k,j}\, \mathsf{Z}_\ell^{k,j} \overset{(5.83)}{\lesssim} \max\{\|u^\star\|_{\mathbb{A}_s}\, \|z^\star\|_{\mathbb{A}_t}, \mathsf{H}_0^{0,0}\, \mathsf{Z}_0^{0,0}\}$$

and concludes the proof of (5.75). □

5.7 Numerical examples

In this section, we present numerical experiments using the open source software package MooAFEM [IP23][1]. In the following, Step (I) and (II) of Algorithm 5A employ the optimal hp-robust local multigrid method from [IMPS24] as an algebraic solver. If not explicitly stated otherwise, we choose the parameters $\theta = 0.5$, $\delta = 0.5$, $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.7$ in Algorithm 5A throughout the numerical experiments.

5.7.1 Singularity in the goal functional

The first model problem is a nonsymmetric variant of the benchmark problem from [BGIP23, Section 4.1] with a singularity only in the goal functional. On the unit square $\Omega = (0,1)^2 \subset \mathbb{R}^2$, we consider

$$-\Delta u^\star + x \cdot \nabla u^\star + u^\star = f \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega, \tag{5.85}$$

where the right-hand side is chosen such that the exact solution $u^\star$ reads

$$u^\star(x) = x_1\, x_2\, (1 - x_1)\, (1 - x_2).$$

Consider $g = 0$ and $\boldsymbol{g} = \chi_K\, (1,0)^\top$ in the quantity of interest

$$G(u^\star) \coloneqq \int_K \partial_{x_1} u^\star \,\mathrm{d}x = 11/960 \quad \text{with } K \coloneqq \operatorname{conv}\{(1/2, 1), (1, 1/2), (1,1)\}.$$

Figure 5.2 (left) displays a mesh generated by Algorithm 5A and the support K of $\boldsymbol{g}$. The error estimator captures and resolves the two point singularities induced by G.

[1] All experiments presented in this paper are reproducible with the openly available software package under https://www.tuwien.at/mg/asc/praetorius/software/mooafem.

5.7.2 Geometric singularity and strong convection

The second benchmark problem investigates $\Omega = (-1,1)^2 \setminus \mathrm{conv}\{(0,0),(-1,0),(-1,-1)\} \subset \mathbb{R}^2$ with the Dirichlet boundary $\Gamma_D = \mathrm{conv}\{(-1,0),(0,0)\} \cup \mathrm{conv}\{(0,0),(-1,-1)\}$ and Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$; see Figure 5.2 (right) for a visualization of the geometry. We consider

$$-\Delta u^\star + (5,5)^\top \cdot \nabla u^\star = 1 \text{ in } \Omega \quad \text{subject to} \quad u^\star = 0 \text{ on } \Gamma_D \text{ and } \nabla u^\star \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_N. \tag{5.86}$$

Consider $g = 0$ and $\boldsymbol{g} = \chi_S\,(1,1)^\top$ in the quantity of interest

$$G(u^\star) = \int_S \partial_{x_1} u^\star + \partial_{x_2} u^\star \,\mathrm{d}x \quad \text{with } S := (-1/2, 1/2)^2 \cap \Omega.$$

The exact solution $u^\star$ is not known analytically in this case so that we do not have access to the exact goal error $|G(u^\star) - G_\ell(u_\ell^{\underline{m},\underline{n}}, z_\ell^{\underline{\mu},\underline{\nu}})|$. Figure 5.2 (right) shows a mesh generated by Algorithm 5A as well as the configuration, i.e., the support S of $\boldsymbol{g}$ in blue, the Dirichlet boundary in red solid lines, and the Neumann boundary in green dashed lines.

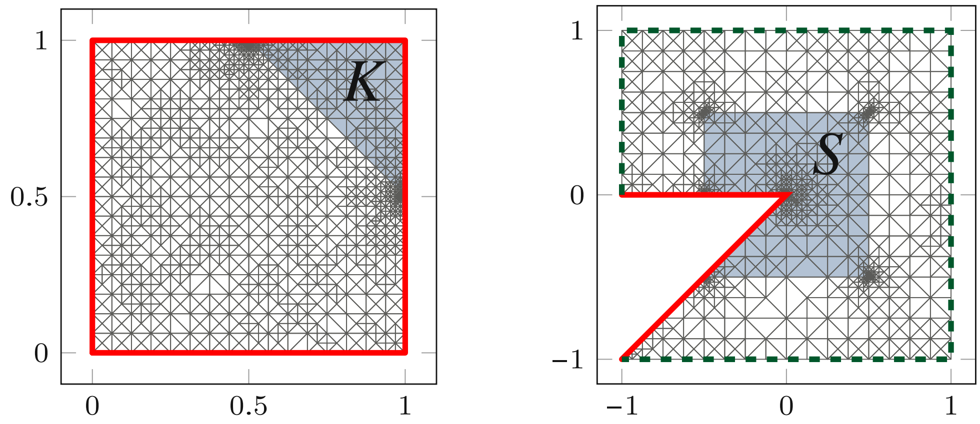

Figure 5.2: Left: Mesh $\mathcal{T}_{15}$ for the problem (5.85) generated by Algorithm 5A with $\#\mathcal{T}_{15} = 2315$. Right: Mesh $\mathcal{T}_{18}$ for the problem (5.86) with $\#\mathcal{T}_{18} = 2130$, where the Dirichlet boundary part Γ_D is marked by red solid lines and the Neumann boundary part Γ_N by green dashed lines.

Optimality of Algorithm 5A. Figure 5.3 displays the estimator product $\eta_\ell(u_\ell^{\underline{m},\underline{n}})\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})$ and the goal error $|G(u^\star) - G_\ell(u_\ell^{\underline{m},\underline{n}}, z_\ell^{\underline{\mu},\underline{\nu}})|$ from (5.17) for the problem (5.85), due to higher-order approximations, we only show results prior to machine precision. For all investigated polynomial degrees p, the goal error and the estimator product are indeed equivalent. Algorithm 5A achieves the optimal rate $-p$ with respect to the cumulative computational work and with respect to the cumulative computational time in Figure 5.3 for problem (5.85) and Figure 5.4 for problem (5.86). Figure 5.5 shows that the proposed algorithm indeed achieves linear complexity and is substantially faster than the MATLAB built-in direct solver as the slightly larger slope of the latter indicates super-linear complexity. Table 5.2 displays the weighted costs

$$\eta_\ell(u_\ell^{\underline{m},\underline{n}})\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})\Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\le|\ell,\underline{k},\underline{j}|}} \mathtt{time}(\ell',k',j')\Big)^p \tag{5.87}$$

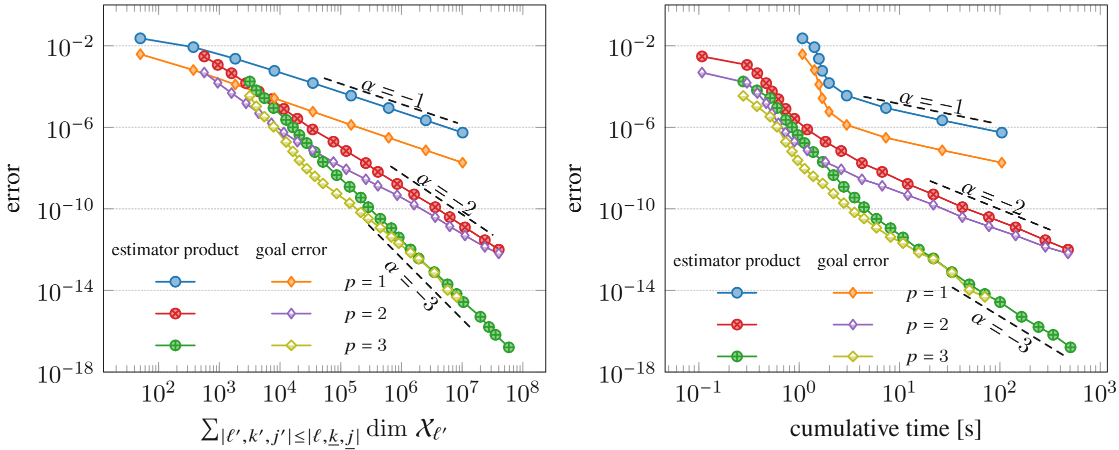

Figure 5.3: Convergence history plot of estimator product $\eta_\ell(u_\ell^{\underline{m},\underline{n}})\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})$ indicated by bullets and goal error from (5.17) indicated by diamonds with respect to the cumulative computational work (left) and with respect to the cumulative computational time (right) for the benchmark problem (5.85).

of Algorithm 5A for polynomial degree $p = 2$ with $\texttt{time}(\ell', k', j')$ in seconds and highlights the corresponding optimal choices of the parameters. This justifies the selection of $\theta = 0.5$ together with larger symmetrization parameter $\lambda_{\mathrm{sym}} = 0.7$, and algebraic solver parameter $\lambda_{\mathrm{alg}} = 0.7$. The table for the second benchmark problem from (5.86) leads to similar results and is therefore omitted. While the choice of the damping parameter $0 < \delta < 2\alpha/L^2$ in (5.13) is crucial in the case of large convection to guarantee the contraction property (5.14), the adaptivity parameters appear more robust with respect to other coefficients in (5.4).

Finally, in Figure 5.6, we display the number of total solver steps $|\ell, \underline{m}, \underline{n}| - |\ell, 0, 0|$ resp. $|\ell, \underline{\mu}, \underline{\nu}| - |\ell, 0, 0|$ on each mesh level for both benchmark problems (5.85) and (5.86). The plots show that the two iterations often stop after the same number of steps.

5.8 Summary

In this work, we developed a cost-optimal goal-oriented adaptive finite element method for the efficient computation of the quantity of interest $G(u^\star)$ with solution $u^\star$ to the general second-order linear elliptic partial differential equation (5.4). Since the current analysis of iterative algebraic solvers for nonsymmetric systems with optimal preconditioner only leads to contraction of the residual in a vector norm, we proposed a nested iterative solver for the primal and dual problem in parallel. The strategy consists of the Zarantonello iteration (5.13) as an outer solver loop and an optimal multigrid solver for the arising SPD system as an innermost solver loop. In recent own work [BFM+23], we have shown that the link between convergence rates with respect to the degrees of freedom and the total computational cost is full linear convergence of the quasi-error $\mathsf{H}_\ell^{k,j}\,\mathsf{Z}_\ell^{k,j}$. To this end, Theorem 5.10 shows that the proposed adaptive algorithm contracts (up to a multiplicative constant) the quasi-error product $\mathsf{H}_\ell^{k,j}\,\mathsf{Z}_\ell^{k,j}$ in every step, independently of the algorithmic decision to employ mesh refinement, symmetrization, or the algebraic solver. A particular problem in the analysis is that the nested iterative solver procedure only guarantees contraction as long as $1 \le k < \underline{k}[\ell]$,

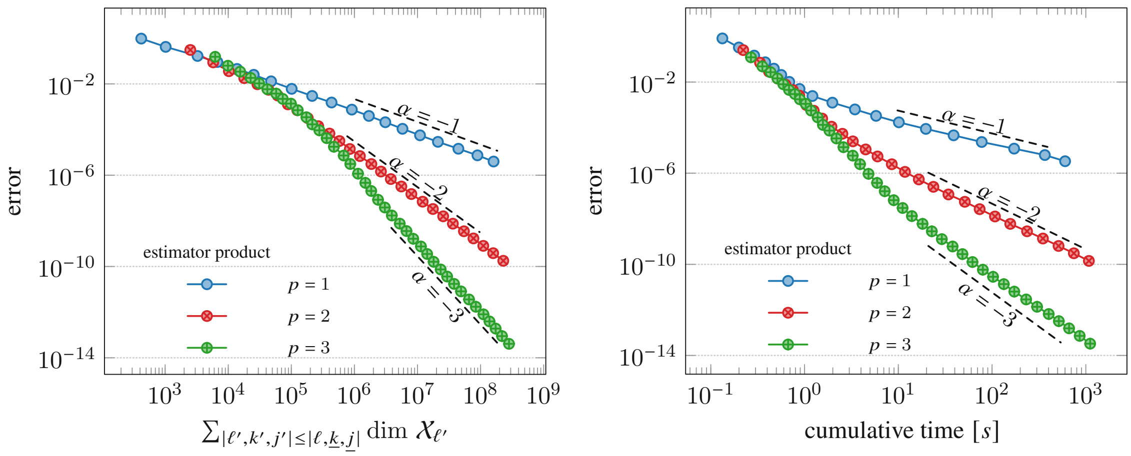

Figure 5.4: Convergence history plot of estimator product $\eta_\ell(u_\ell^{\underline{m},\underline{n}})\,\zeta_\ell(z_\ell^{\underline{\mu},\underline{\nu}})$ with respect to the cumulative computational cost (left) and the cumulative computational time (right) for the benchmark problem (5.86).

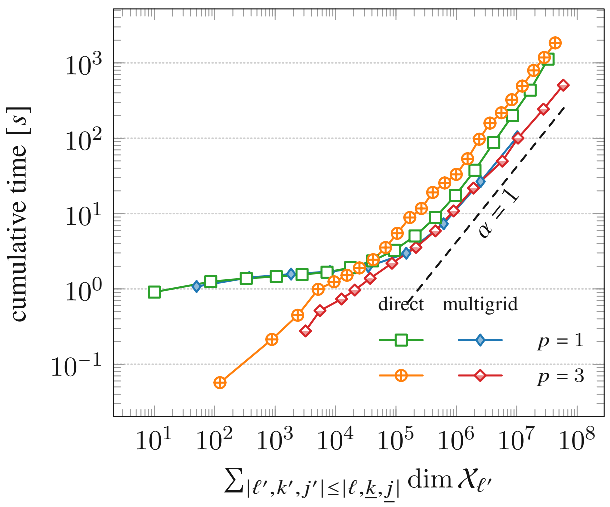

Figure 5.5: Comparison of cumulative time of the local multigrid solver with the MATLAB built-in direct solver `mldivide` with respect to the cumulative computational cost for the benchmark problem (5.86).

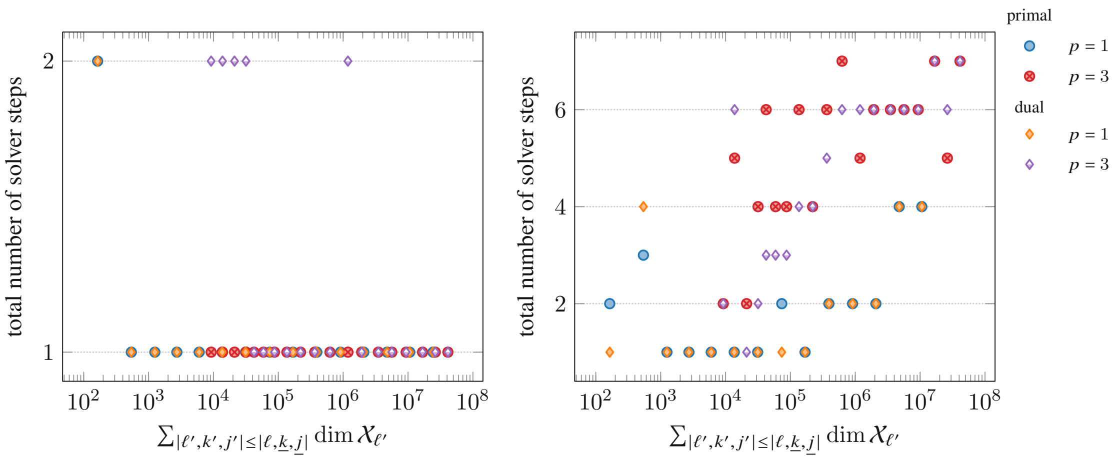

Figure 5.6: Number of total solver steps $|\ell, \underline{m}, \underline{n}| - |\ell, 0, 0|$ resp. $|\ell, \underline{\mu}, \underline{\nu}| - |\ell, 0, 0|$ on each mesh level for the benchmark problems (5.85) (left) and (5.86) (right).

$\cdot 10^{-7}$	$\theta = 0.1$					$\theta = 0.3$					$\theta = 0.5$				
λ_{alg} \ λ_{sym}	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
0.1	38.7	33.4	29.6	22.1	24.4	10.2	5.12	4.90	4.83	4.74	6.18	4.48	4.66	4.89	5.25
0.3	36.2	24.7	24.5	21.8	23.1	7.28	4.98	3.53	3.27	3.26	4.18	4.54	4.79	5.01	5.13
0.5	24.3	24.7	24.7	23.4	23.6	5.84	3.64	3.39	3.27	3.37	3.41	2.71	2.52	2.49	2.68
0.7	24.1	24.8	23.8	22.2	24.0	4.95	3.59	3.30	3.25	3.42	2.74	2.35	2.41	2.24	2.46
0.9	23.5	24.6	22.3	24.4	23.8	4.90	3.58	3.29	3.26	3.41	2.81	2.30	2.43	2.27	2.41
	$\theta = 0.7$					$\theta = 0.8$					$\theta = 0.9$				
0.1	5.82	5.18	5.43	5.40	5.93	8.53	6.10	7.31	6.67	7.77	11.6	8.86	9.12	9.87	9.97
0.3	4.65	4.86	5.35	5.98	6.67	6.27	5.92	7.20	7.46	7.57	8.62	8.40	9.27	10.6	11.5
0.5	3.69	2.89	2.88	2.95	3.13	5.09	3.61	3.66	3.63	3.66	7.27	5.32	4.84	4.93	5.12
0.7	2.99	2.56	2.64	2.62	2.89	3.75	3.12	3.23	3.03	3.11	4.58	3.95	4.04	4.43	4.79
0.9	2.89	2.49	2.65	2.66	2.89	3.79	3.11	3.19	3.13	3.27	4.67	4.06	4.16	4.35	4.61

Table 5.2: Optimal selection of parameters with respect to the cumulative computational costs (overall computation time in seconds) for the experiment (5.85) with fixed polynomial degree $p = 2$ and $\delta = 0.5$. For comparison, the table displays the value of the weighted costs from (5.87) (in 10^{-7}) with overall stopping criterion $\eta_\ell(u_\ell^{\underline{m},\underline{n}})\,\zeta_\ell(u_\ell^{\underline{\mu},\underline{\nu}}) < 5 \cdot 10^{-10}$ for various choices of λ_{sym}, λ_{alg}, and θ. For each θ-block, we mark the row-wise optimal values in blue, the column-wise optimal values in yellow, and in green if both optimal values coincide.

whereas contraction for the final iterate is only guaranteed up to an estimator term (cf. (5.29)). Another difficulty arises from the nonsymmetric setting with a quasi-Pythagorean estimate (5.18) replacing the usual Pythagorean estimate. Therefore, the proof of Theorem 5.10 employs the equivalence of R-linear convergence and tail-summability of the quasi-error product $\mathsf{H}_\ell^{k,j}\,\mathsf{Z}_\ell^{k,j}$ and leads to mild restriction on the product $\lambda_{\mathrm{sym}}\,\lambda_{\mathrm{alg}}$ of the involved solver stopping parameters. The key ingredients to cost-optimality are an adaptive mesh-refinement algorithm with optimal convergence rate with respect to the number of degrees of freedom (under the assumption of exact solution) and an algebraic solver for the linear system of equations that is contractive with respect to the underlying Sobolev norm. In this regard, the analysis in this paper may guide the generalization to conforming discretizations of vector-valued elliptic problems. Finally, the numerical experiments in Section 5.7 suggest that the proposed strategy allows for large stopping parameter in practice and that a larger choice is beneficial in terms of total runtime. Admittedly, the development of an optimal solver for the nonsymmetric problem (5.10) would allow to prove full linear convergence with an arbitrary selection of the stopping parameter.

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Curriculum Vitae

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Education

02/2022–06/2024 (expected)	**PhD studies in Technical Mathematics**, TU Wien, Austria. Supervisor: Prof. Dirk Praetorius.
10/2019–12/2021	**Master's degree in Mathematics**, HU Berlin, Germany. Passed with distinction, Master thesis: *Discrete stability of dGFEM for biharmonic plates*, supervisor: Carsten Carstensen.
10/2016–09/2019	**Bachelor's degree in Mathematics**, PLUS Salzburg, Austria. Passed with distinction, Bachelor thesis: *Lowest-order mixed FEM for the Poisson problem*, supervisor: Andreas Schröder.
10/2015–11/2018	**Bachelor's degree in the teaching programme Mathematics / History and political education**, PLUS Salzburg, Austria. Bachelor theses: *A diophantine equation for sums of consecutive like powers*, supervisor: Clemens Fuchs & *Schuschnigg und die Dollfuß-Straße, Eine Analyse des Personenkults im Austrofaschismus in den Jahren 1934–1938*, supervisor: Matthias Marschik.

Research experience

since 02/2022	**University assistent**, TU Wien, Austria. Institute of Analysis and Scientific Computing, work group on *Numerics of PDEs*.
2018–2019	**Student assistent**, PLUS Salzburg, Austria. Departement of Mathematics, work group on *Technical Mathematics* and work group on *Statistics*.

Teaching experience

2022–2024 **Teaching assistant**, TU Wien, Austria.
Scientific Programming for Interdisciplinary Mathematics, VU

2023 **Teaching assistant**, TU Wien, Austria.
Numerics of partial differential equations: stationary problems, exercise class.

2022 **Teaching assistant**, TU Wien, Austria.
Numerische Mathematik, exercise class.

2022 **Teaching assistant**, TU Wien, Austria.
Seminar with seminar thesis in numerics: fixed-point theorems, seminar.

Scholarships and Awards

2020 **Leistungsstipendium für hervorragende Studienleistungen** der Universität Salzburg.

2018 **Early Student Award** of the Austrian Mathematical Society.

Scientific talks

2023 **European Conference on Numerical Mathematics and Advanced Applications (ENUMATH)**, Lisbon, Portugal. *Cost-optimal goal-oriented adaptive FEM for linear elliptic PDEs*.

2023 **Austrian Numerical Analysis Day (ANADay)**, Wien, Austria. *Adaptive FEM for linear elliptic PDEs: optimal complexity*.

2023 **2nd SFB International Workshop 2023 "Taming Complexity in Partial Differential Systems"**, Wien, Austria. *Adaptive FEM for linear elliptic PDEs: optimal complexity*.

Own scientific publications

2024 M. Brunner, D. Praetorius, and J. Streitberger. Cost-optimal adaptive FEM with linearization and algebraic solver for semilinear elliptic PDEs, 2024. arXiv: 2401.06486

2024 A. Miraçi, D. Praetorius, and J. Streitberger. Parameter-robust full linear convergence and optimal complexity of adaptive iteratively linearized FEM for strongly monotone nonlinear PDEs, 2024. arXiv: 2401.17778

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