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# Some optimisation problems in insurance with a terminal distribution constraint

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#### ABSTRACT

In this paper, we study two optimisation settings for an insurance company, under the constraint that the terminal surplus at a deterministic and finite time T follows a normal distribution with a given mean and a given variance. In both cases, the surplus of the insurance company is assumed to follow a Brownian motion with drift. First, we allow the insurance company to pay dividends and seek to maximise the expected discounted dividend payments or to minimise the ruin probability under the terminal distribution constraint. Here, we find explicit expressions for the optimal strategies in both cases, when the dividend strategy is updated at discrete points in time and continuously in time. Second, we let the insurance company buy a reinsurance contract for a pool of insured or a branch of business. We set the initial capital to zero in order to verify whether the premia are sufficient to buy reinsurance and to manage the risk of incoming claims in such a way that the desired risk characteristics are achieved at some terminal time without external help (represented, for instance, by a positive initial capital). We only allow for piecewise constant reinsurance strategies producing a normally distributed terminal surplus, whose mean and variance lead to a given Value at Risk or Expected Shortfall at some confidence level  $\alpha$ . We investigate the question which admissible reinsurance strategy produces a smaller ruin probability, if the ruin-checks are due at discrete deterministic points in time.

# 1. Introduction

This paper investigates the problem of dividend maximisation and the problem of ruin minimisation for an insurance company who aims to achieve a certain surplus distribution at a particular future date. Indeed, the knowledge of the surplus distribution, for instance, at regulatory check-times is an important information for the calculation of the necessary capital reserves, in view of the Solvency regulations. Measuring the solvency of a collective of risks remains a key task in insurance mathematics. In 1903 Filip Lundberg proposes a model (nowadays called the classical risk model or Cramer–Lundberg model) for the surplus where the total claim amount of an insurance company or of a collective of insured is represented by a compound Poisson process, whilst the premia are assumed to arrive at a constant positive rate. The surplus process which is made of incomes, the initial capital and premia, and payments, the claims, is then described by a jump process with drift that may eventually take negative values if for instance claims exceed premia. One way to assess the solvency

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is to look at ruin probability, that is the probability that the surplus becomes negative in finite time. This is the central object in the classical risk model and has been studied in diverse settings since Harald Cramer republished the results of Lundberg in the 1930s. However, due to the presence of jumps (claim sizes) in the expression of the surplus under the Cramer–Lundberg model, derivations are quite complex. Explicit form representations of the ruin probability can be found just for a few claim size distributions, exponential distribution above all. For this reason, one often considers an approximation of a compound Poisson process by a diffusion, see, e.g. Schmidli (2008, p. 226) for more details. It is clear that if the risk of having a claim cannot be entirely eliminated (for instance, by buying a full reinsurance contract), the distribution of the surplus in the classical risk model, or in it's diffusion approximation, should allow for negative values. Thus, choosing a specific target distribution for the future surplus one should keep in mind that this distribution cannot be concentrated on the positive real line.

Over the years, several risk measures have been proposed and their properties have been investigated, by adding risk constraints like, for example, value at risk.

One of the most popular risk measures is the value of expected discounted dividends. Here, one searches for the 'optimal' dividend strategy, i.e. a strategy maximising the value of expected discounted dividends up to the time when the surplus becomes negative. By considering the optimal strategy, the focus is deliberately placed on the surplus' evolution characteristics rather than on the company's managerial skills. Some results on dividend maximisation problems can be found, for instance, in Asmussen & Taksar (1997) and Shreve et al. (1984). We further refer to Albrecher & Thonhauser (2009), Avanzi (2009), Hipp (2020) and references therein for an overview of the existing results.

The optimal dividend payout strategy in the most 'unconstrained' settings turns out to be of a barrier or a band type, meaning that the strategy can change from 'paying the maximal possible amount' to 'paying nothing' in dependence on the current surplus value. This setup cannot be considered realistic or doable for an insurance company. Moreover, solvency requirements imposed by regulators may not allow paying dividends according to the optimal, possibly bang-bang, strategy. To make the models more realistic, one needs to impose restrictions. Paulsen (2003) studies the optimal dividend problem with a no-bankruptcy constraint – dividends will not be paid if the surplus is below a certain barrier. An extended setting with transaction costs is analysed in Bai et al. (2012). Hipp (2003), considers optimal dividend payment strategies under the constraint that the ruin probability stays under a given boundary. Thonhauser & Albrecher (2011) maximise the total discounted utility of dividend payments under strictly positive transaction costs.

The setting considered in the first part of this paper is novel in the following way. The surplus of an insurance company in a finite time interval is modelled by a diffusion process. We concentrate on the dividend payments – described by dividend rates – with two different objective functions: expected discounted dividend payments and ruin probability. In the first case, one faces a maximisation problem, whereas in the second case the ruin probability should be minimised. The surplus can only be controlled at discrete equidistant time points. We introduce a constraint on the set of admissible strategies by requiring that the ex-dividend terminal wealth should be normally distributed with fixed exogenously given mean and variance. To the best of our knowledge, such a constraint has not been considered in an insurance optimisation problems before. As the surplus is modelled by a diffusion, the choice of a normal distribution as a target distribution seems to be natural. We prove that the optimal strategy in both cases should be deterministic, i.e. it is decided at time zero. As it is intuitively clear, the strategy leading to the maximal discounted dividend value starts with high payments in the very beginning and decreases approaching the time horizon; the strategy minimising the ruin probability behaves in an opposite way.

The results obtained in this first part of the paper heavily rely on the very nature of dividend payments. The control is acting solely on the drift, letting the volatility unchanged. This allows comparing different strategies by comparing their paths. However, choosing a control problem with an impact on the volatility of the surplus process will not permit the use of a path-comparison method and will require different, more complex techniques. A well-established, well-investigated and still quite popular risk measure is, as mentioned above, the ruin probability – the probability that a company, a strain of business or a pool of insured risks goes bankrupt in finite time, i.e. writes red numbers – the aggregate claims exceed the collected premia. The surplus, with continuous paths or having jumps, can be controlled, for instance, by a reinsurance, dividend payments, possible surplus investments into dependent or independent markets.

A *technical ruin*, that is, the event in which the surplus becomes negative or touches zero, does not compulsory mean that the company has to entirely stop operating. The time and the severity of ruin are completely neglected by looking solely at the ruin probability. Also, due to Solvency II requirements, companies have enough reserves to bridge a certain period of unfavourable business development. For these reasons, the ruin probability might not be a desired risk measure to assess a company's performance. However, paired with some additional constraints it can help choosing a strategy which is, for instance, more risk averse in the eyes of the insurance company.

In the second part of this manuscript, we are looking at the surplus of an insurance company who buys proportional reinsurance contracts of a specific type. To control the risk exposure and to be able to meet regulatory requirements, insurance companies need to pay attention to various constraints. For instance, Bernard & Tian (2009), Lo (2017a, 2017b) and Huang & Yin (2019) search for the optimal reinsurance strategy under a constraint (a strictly positive surplus or a fixed risk measure to stay under some prespecified boundary) on the loss at the terminal time. Optimal investment and reinsurance have been considered with constraints on the budget, see Bi et al. (2014), or on Value at Risk, see, e.g. Choulli et al. (2001), Bi & Cai (2019) and Wang & Siu (2020). The problem of choosing a reinsurance strategy to minimise the ruin probability with a Value at Risk (or a Conditional Value at Risk) constraint is considered for instance in Zhang et al. (2016) and Chen et al. (2010) under finite and infinite time horizons.

In this paper, we seek to find a proportional reinsurance strategy that minimises the ruin probability under a constraint imposed on the distribution of the terminal wealth. Adding a constraint of this type on the terminal surplus has several advantages. For instance, one will be able to calculate any risk measure acting on the terminal wealth: the Value at Risk, the Expected Shortfall or the expected terminal utility, and hence address many regulatory requirements all at once.

We consider a finite time interval [0, T], and the direct insurer can choose the deductible only twice – in the beginning and in the middle of the interval. The target is twofold: the terminal postreinsurance surplus at *T* should be normally distributed with given mean and variance, and the chosen admissible strategy should lead to the smallest possible ruin probability. We show that the optimal strategy is deterministic, i.e. it is chosen at time 0, and one is always acting in a risk averse way. That is, the insurer buys less reinsurance in the beginning, in order to let the drift push the surplus upwards, and buys more reinsurance in the second half, reducing the risk of ruin shortly before the regulator's check. We briefly discuss the case where the insurer can update the reinsurance strategy three times, which provides some intuition on how to deal with more than two updates. To the best of our knowledge, the presented approach is new in many aspects. Moreover, the discrete nature of the optimal strategies, make this setting easily applicable from a practical point of view.

The paper is organised as follows. In Section 2, we introduce and solve the dividend maximisation problem. We address the ruin minimising problem in Section 3. The reinsurance optimisation problem is discussed in Section 4. We conclude in Section 5.

#### 2. Maximising dividends under a terminal distribution constraint

In this section, we consider an insurance company who is allowed to pay dividends. The dividend rate has to be chosen in such a way that the surplus at some future deterministic time *T* achieves a given distribution. At the same time, the value of expected discounted dividends should be maximised.

We consider a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , a finite time horizon T > 0 and a Brownian motion  $W = (W_t)_{t \in [0,T]}$ . We denoted by  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$  the natural complete and right continuous filtration

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of *W*, and set  $\mathcal{F}_T = \mathcal{F}$ . The surplus of the insurance company in the interval [0, *T*] is modelled by a Brownian motion with drift as

$$X_t^{\mathbf{0}} = x + \bar{\mu}t + \bar{\sigma}W_t, \quad t \in [0, T],$$

where  $x \ge 0$  represents the initial capital and  $\bar{\mu}, \bar{\sigma} > 0$ .

The company is allowed to pay dividends in form of dividend rates  $0 \le c \le \xi$  for some given  $\xi > 0$ . It means that the post-dividend process under a dividend strategy  $\mathbf{c} = (c_s)_{s \in [0,T]}$  is given by

$$X_{t}^{c} = x + \bar{\mu}t - \int_{0}^{t} c_{s} \, \mathrm{d}s + \bar{\sigma} \, W_{t}, \quad t \in [0, T].$$
<sup>(1)</sup>

Our objective is to determine the strategies that maximise the expected discounted dividends and simultaneously lead to a normally distributed post-dividend terminal surplus  $X_T^c$ . We assume that the target distribution is Gaussian with the mean x + MT, and the variance  $\delta^2 T$ , for some  $M \in \mathbb{R}$  and  $\delta > 0$ .

At first, the company is only allowed to update a dividend strategy at  $n \in \mathbb{N}$  equidistant time points  $kT/n, k \in \{0, ..., n-1\}$  in the period [0, T]. An admissible strategy is a sequence  $\mathbf{c} = (c_0, ..., c_{n-1})$  of dividend rates such that for all k = 0, 1, ..., n-1,  $c_k \in [0, \xi]$  is an  $\mathcal{F}_{\frac{kT}{n}}$ -measurable random variable and the total surplus at time T satisfies  $X_T^{\mathbf{c}} \sim N(x + MT, \delta^2 T)$ . We denote the set of admissible strategies by  $\mathcal{A}_{(n)}$ , where the subscript (n) indicates the number of the allowed change points. The accumulated dividends up to time t are then given by

$$\sum_{k=0}^{n-1} c_k \left( \frac{(k+1)T}{n} \wedge t - \frac{kT}{n} \wedge t \right).$$

It is worth mentioning that differently than in the classical dividend problems, see for instance (Asmussen & Taksar 1997), dividends can be paid (up to time T) even if the surplus is negative. This feature of our model alleviates, to some extent, the drawback of models stopping at the ruin time. A technical, in fact, ruin does not mean that the company stops operating. In reality, some insurance companies proceed with dividend payments even during protracted crisis times. A famous example provided by Munich Re, known for not reducing its dividends since at least 2006, see Munich Re (2022).

The following lemma indicates the range of achievable target expectations x + MT, by a postdividend Brownian surplus, see Equation (1), at time *T*.

**Lemma 2.1:** The parameter M in the target distribution of the surplus at time T has to fulfil  $\bar{\mu} - \xi \le M \le \bar{\mu}$ .

**Proof:** For any admissible dividend strategy  $\mathbf{c} = (c_0, \dots, c_{n-1}) \in \mathcal{A}_{(n)}$ , the distribution of the surplus in Equation (1) at time *T* is Gaussian with mean

$$x + \left(\bar{\mu} - \sum_{k=0}^{n-1} \frac{\mathbb{E}[c_k]}{n}\right)T = x + MT.$$

Using the fact that  $0 \le c_k \le \xi$ , for every  $k \in \{0, \dots, n-1\}$  we get that

$$x + (\bar{\mu} - \xi) T \le x + \left(\bar{\mu} - \frac{\sum_{k=0}^{n-1} \mathbb{E}[c_k]}{n}\right) T \le x + \bar{\mu}T,$$

which proves the statement.

Note that, for large values of  $\xi$ , the range of achievable means may include negative values. Although this is mathematically feasible, an insurance company would not pursue a strategy to achieve a negative expected surplus, but it would rather choose  $M \in [0, \bar{\mu}]$ , so to obtain a expected net profit at time T, even if small. Next we better identify the characteristics of admissible strategies.

**Proposition 2.2:** Any admissible strategy  $\mathbf{c} = (c_0, \ldots, c_{n-1}) \in \mathcal{A}_{(n)}$  is such that  $\sum_{i=1}^{n-1} c_i$  is  $\mathcal{F}_0$ measurable, i.e. deterministic.

**Proof:** Let  $\mathbf{c} = (c_0, \dots, c_{n-1})$  be an arbitrary admissible dividend strategy. The corresponding surplus at time T is then given by

$$X_T^{\mathbf{c}} = x + \bar{\mu}T - c_0 \frac{T}{n} - \frac{T}{n} \sum_{k=1}^{n-1} c_k + \bar{\sigma} W_T.$$
 (2)

We now identify the set of dividend strategies that lead a normal distribution with mean x + MT and variance  $\delta^2 T$ . Let Y be a generic random variable with  $Y \sim N(x + MT, \delta^2 T)$ . Then, for  $\zeta \in \mathbb{R}$  it holds that  $\mathbb{E}[e^{\zeta Y}] = e^{\zeta(x+MT) + \frac{\delta^2}{2}\zeta^2 T}$ . Now we consider the surplus at time T,  $X_T^{\mathbf{c}}$ . From Equation (2) and the fact that **c** is an admissible strategy we get that  $X_T^{\mathbf{c}} \sim N(x + MT, \delta^2 T)$  and it holds that

$$e^{\zeta(x+MT)+\frac{\delta^2}{2}\zeta^2 T} = \mathbb{E}[e^{\zeta X_T^C}] = e^{\zeta(x+\bar{\mu}T-c_0T/n)} \mathbb{E}[e^{\zeta\bar{\sigma}W_T-\zeta\sum_{k=1}^{n-1}c_kT/n}].$$
(3)

Let **Q** be a probability measure on  $(\Omega, \mathcal{F}_T)$  equivalent to **P**, with the Radon–Nikodym derivative  $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{O}}|_{\mathcal{F}_T} = e^{-\zeta \bar{\sigma} W_T + \frac{\zeta^2 \bar{\sigma}^2 T}{2}}.$  By applying change of measure techniques in Equation (3) we obtain

$$\mathbb{E}[e^{\zeta \,\tilde{\sigma} \,W_T - \zeta \sum_{k=1}^{n-1} c_k T/n}] = e^{\frac{\zeta^2 \,\tilde{\sigma}^2 T}{2}} \mathbb{E}_{\mathbf{Q}}[e^{-\zeta \sum_{k=1}^{n-1} c_k T/n}].$$

Together with Equation (3), one gets for all  $\zeta \in \mathbb{R}$ 

$$e^{\zeta(x+MT)+\frac{\delta^2}{2}\zeta^2 T} = e^{\zeta(x+\bar{\mu}T-c_0T/n)+\frac{\bar{\sigma}^2T\zeta^2}{2}} \mathbb{E}_{\mathbf{Q}}[e^{-\zeta\sum_{k=1}^{n-1}c_kT/n}],$$

leading to

$$\mathbb{E}_{\mathbf{Q}}[e^{-\zeta \sum_{k=1}^{n-1} c_k T/n}] = e^{\zeta (M - \bar{\mu} + c_0/n)T + \frac{\delta^2 - \bar{\sigma}^2}{2} \zeta^2 T}.$$

If  $\delta^2 - \bar{\sigma}^2 > 0$ , by uniqueness of the moment generating functions the variable  $\sum_{k=1}^{n-1} c_k \frac{T}{n}$  is normally distributed with mean  $(M - \bar{\mu} + c_0/n)T$  and variance  $(\delta^2 - \bar{\sigma}^2)T$ . Hence, it has positive Q-probability to attain negative values, which contradicts the equivalence of Q and P, since  $\sum_{k=1}^{n-1} c_k T/n \ge 0$ **P**-a.s. If, instead,  $\delta^2 - \bar{\sigma}^2 < 0$ , there is no random variable with such a moment generating function.

Finally, if  $\delta = \bar{\sigma}$ , the variable  $\sum_{k=1}^{n-1} c_k T/n$  must be a constant, i.e. deterministic.

For the special case n = 2 we obtain the following corollary.

**Corollary 2.3:** The set of admissible strategies  $A_{(2)}$  only consists of deterministic pairs  $(c_0, c_1)$ , i.e.  $c_1$  is  $\mathcal{F}_0$ -measurable.

Note that the dividend strategies act solely on the drift and do not affect the volatility. This fact allows comparing different strategies by looking at the surplus 'path by path'. Another implication is that, in the case n = 2, the optimal dividend strategy is completely decided at time t = 0; meaning that once the dividend rate  $c_0$ , to be valid in [0, T/2], is chosen, then  $c_1$  is also uniquely determined

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at time t = 0 so that the final distribution can be achieved. We will see in the reminder of the section that the optimal strategy is deterministic also for n > 2.

Let now r > 0 be the preference rate of the insurer. We call *return function* corresponding to a strategy  $\mathbf{c} = (c_0, \ldots, c_{n-1}) \in \mathcal{A}_{(n)}$  the expected total present value of the dividends paid up to time *T*, that is

$$V^{\mathbf{c}}(x) := \mathbb{E}_{x}\left[\sum_{k=0}^{n-1} \frac{c_{k}}{r} e^{-r\frac{kT}{n}} \left(1 - e^{-r\frac{T}{n}}\right)\right].$$

The function  $V^{\mathbf{c}}(x)$  can be derived from the usual definition of return functions corresponding to continuous time dividend controls – as an integral over discounted dividend rates. Let  $\zeta = (\zeta_t)_{t \in [0,T]}$  with  $\zeta_t = c_k \mathbf{I}_{[t \in [kT/n, (k+1)T/n)]}$  for  $k \in \{0, ..., n-1\}$ . That is,

$$V^{\zeta}(x) = \mathbb{E}_{x} \left[ \int_{0}^{T} e^{-rs} \zeta_{s} \, \mathrm{d}s \right] = \mathbb{E}_{x} \left[ \sum_{k=0}^{n-1} \int_{kT/n}^{(k+1)T/n} e^{-rs} c_{k} \, \mathrm{d}s \right]$$
$$= \mathbb{E}_{x} \left[ \sum_{k=0}^{n-1} \frac{c_{k}}{r} e^{-r\frac{kT}{n}} \left( 1 - e^{-r\frac{T}{n}} \right) \right]$$
$$= V^{\mathbf{c}}(x).$$

Note that the dependence on the initial capital x is in this setting purely nominal. As stressed before, we do not stop our considerations at the time of ruin. The strategy will depend solely on the parameters of the surplus process and the target distribution.

The target of the insurance company is to find a strategy  $\mathbf{c}^* = (c_0^*, \dots, c_{n-1}^*) \in \mathcal{A}_{(n)}$  leading to

$$V^{\mathbf{c}^{*}}(x) = \max_{\mathbf{c}\in\mathcal{A}_{(n)}} \mathbb{E}_{x} \left[ \sum_{k=0}^{n-1} \frac{c_{k}}{r} e^{-r\frac{kT}{n}} \left( 1 - e^{-r\frac{T}{n}} \right) \right].$$
(4)

To analyse Problem (4), we start with the case of two periods, i.e. n = 2.

#### 2.1. A 2-period model

Suppose that the insurance company is allowed to update its dividend strategy only once, at time T/2. Due to Corollary 2.3, we get that the set of admissible dividend strategies  $\mathcal{A}_{(2)}$  consists of all deterministic pairs  $\mathbf{c} = (c_0, c_1)$  with  $c_0, c_1 \in [0, \xi]$ , and such that

$$\bar{\mu} - (c_0 + c_1)/2 = M.$$

As a direct consequence of the fact that  $c_0, c_1 \in \mathcal{F}_0$ , it must also hold that  $\bar{\sigma}^2 = \delta^2$ , otherwise the target distribution would not be reachable. In the next step, we investigate how to determine the optimal strategy.

**Proposition 2.4:** The optimal strategy  $\mathbf{c}^* = (c_0^*, c_1^*)$  is given by

$$c_0^* = \xi \wedge 2(\bar{\mu} - M),$$
  
$$c_1^* = \begin{cases} 0 & \text{if } 2\bar{\mu} - 2M \le \xi \\ 2\bar{\mu} - 2M - \xi & \text{if } 2\bar{\mu} - 2M > \xi \end{cases}$$

*Proof:* We consider the problem

$$\max_{(c_0,c_1)\in\mathcal{A}_{(2)}} \frac{c_0 T}{r} \left(1 - e^{-rT/2}\right) + \frac{c_1 T}{r} e^{-rT/2} \left(1 - e^{-rT/2}\right).$$
(5)

It is easy to see that, for r > 0, the discounting coefficient in the first period,  $1 - e^{-rT/2}$ , is larger than in the second period,  $e^{-rT/2}(1 - e^{-rT/2})$ . Therefore, to maximise the discounted dividends,  $c_0$  must be chosen as big as possible. Taking into account that  $0 \le c_0 \le \xi$  and that  $c_0 + c_1 = 2(\bar{\mu} - M)$ , we get that  $c_0 = \min(\xi, 2(\bar{\mu} - M))$ , and consequently,  $c_1 = 2(\bar{\mu} - M) - \xi$  if  $c_0 = \xi$  and  $c_1 = 0$  if  $c_0 = 2(\bar{\mu} - M)$ .

To summarise the result, in a two-period setting, the optimal dividend strategy pays dividends at the maximum rate in the first period, and then adjusts the strategy to achieve the target distribution in the second period. Such behaviour is justified by the effect of discounting which has a bigger impact in the time interval [T/2, T].

#### 2.2. An n-period model

We now extend our analysis to an *n*-period framework. That is, the dividend strategy can be adjusted n times in the interval [0, T]. Recall that, according to Proposition 2.2, strategies are not necessarily deterministic, but the sum of dividend rates is.

To better explain the mechanism for the computation of the optimal dividend strategy, we consider an example with n = 3.

**Example 2.5:** Let n = 3 and let  $\mathbf{c} = (c_0, c_1, c_2)$  be an admissible strategy. The expected discounted total dividends are given by

$$\frac{c_0}{r}(1-e^{-rT/3})+\frac{e^{-rT/3}(1-e^{-rT/3})}{r}\mathbb{E}\left[c_1+c_2e^{-rT/3}\right].$$

We easily see that, that due to discounting (r > 0), the strategy  $c_0$  to be applied in the first period has a larger weight than the others, hence, as in the two period model, it would be optimal to choose it the largest possible. Taking into account that  $x + MT = x + \bar{\mu}T - \frac{(c_0 + \mathbb{E}[c_1 + c_2])T}{3}$ , and that  $c_k \in [0, \xi]$  for k = 0, ..., 2, we have that

$$c_0 = \begin{cases} 3(\bar{\mu} - M) & \text{if } 3(\bar{\mu} - M) \le \xi \\ \xi & \text{if } 3(\bar{\mu} - M) > \xi \end{cases}$$

equivalently,  $c_0 = \min(3\bar{\mu} - 3M, \xi)$ . Now we move to the choice of  $c_1, c_2$ . After choosing  $c_0$  we get that  $\mathbb{E}[c_1 + c_2] = c_1 + c_2 = \max(0, 3(\bar{\mu} - M) - \xi)$ , according to Proposition 2.2. If  $c_0 = 3\bar{\mu} - 3M$ , since  $c_1$  and  $c_2$  are nonnegative, it holds that  $c_1 = c_2 = 0$ . If instead,  $c_0 = \xi$ , using the same argument like for  $c_0$ , we choose  $c_1$  and  $c_2$  so that  $c_1$  is the largest possible value according to the constraints, i.e.  $c_1 = \min(3(\bar{\mu} - M) - \xi, \xi)$ , and  $c_2 = \max(3(\bar{\mu} - M) - 2\xi, 0)$ . Put in other words, if  $2\xi \leq 3\bar{\mu} - 3M < 3\xi$ , then  $c_0 = c_1 = \xi$  and  $c_2 = 3\bar{\mu} - 3M - 2\xi$ . If  $\xi < 3\bar{\mu} - 3M < 2\xi$ , at time *T*/3 we determine both  $c_1$  and  $c_2$ , depending on the current surplus so that

$$\bar{\mu}T - (c_1 + c_2)T/3 = MT + \xi T/3.$$

We stress that because  $c_1 + c_2$  must be deterministic, we immediately get that  $c_2$  is  $\mathcal{F}_{T/3}$  measurable. That means, once  $c_1$  is found, then  $c_2$  is also determined, so that the constraint on the distribution is satisfied. Moreover, the value  $3(\bar{\mu} - M) - \xi$  is the biggest possible choice for  $c_1$ . The deterministic strategy  $\mathbf{c}^* = (c_0, c_1, c_2)$  where

$$\begin{cases} c_0^* = \min(\xi, 3(\bar{\mu} - M)), \\ c_1^* = \max(\min(\xi, 3(\bar{\mu} - M) - \xi), 0), \\ c_2^* = \max(3(\bar{\mu} - M) - 2\xi, 0), \end{cases}$$
(6)

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fulfils all necessary conditions.

Next, we show that we cannot find a different, possibly stochastic, strategy with a higher expected discounted dividends value, meaning that the optimal strategy is indeed deterministic.

Let  $\mathbf{c}^* = (c_0^*, c_1^*, c_2^*)$  be the strategy in Equation (6) and let  $\tilde{\mathbf{c}} = (\tilde{c}_0, \tilde{c}_1, \tilde{c}_2) \in \mathcal{A}_{(3)}$  be an arbitrary admissible strategy, i.e. such that  $\tilde{c}_m \in [0, \xi]$  for  $m = 0, 1, 2, \text{ and } X_T^{\tilde{\mathbf{c}}} \sim N(x + MT, \delta^2 T)$ .

Then, there exist two random variables  $d_1, d_2$  such that  $\mathbb{E}[d_1 + d_2] = c_0^* - \tilde{c}_0 \ge 0$ , because  $c_0^*$  is the largest possible dividend rate, and  $\tilde{c}_1 = c_1^* + d_1$ ,  $\tilde{c}_2 = c_2^* + d_2$ . It holds that

$$\begin{split} V^{\tilde{\mathbf{c}}}(x) &= \tilde{c}_0(1 - e^{-\frac{rT}{3}}) + (1 - e^{-\frac{rT}{3}})e^{-\frac{rT}{3}}\mathbb{E}[\tilde{c}_1 + \tilde{c}_2 e^{-\frac{rT}{3}}] \\ &= V^{\mathbf{c}^*}(x) - (c_0^* - \tilde{c}_0)(1 - e^{-\frac{rT}{3}}) + (1 - e^{-\frac{rT}{3}})e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}] \\ &= V^{\mathbf{c}^*}(x) - (1 - e^{-\frac{rT}{3}})\left(\mathbb{E}[d_1 + d_2] - e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}]\right), \end{split}$$

where in the last equality we have used the fact that  $(c_0^* - \tilde{c}_0) = \mathbb{E}[d_1 + d_2]$ .

If  $\mathbb{E}[d_2] < 0$ , then,  $\mathbb{E}[c_2^*] > \mathbb{E}[\tilde{c}_2] \ge 0$ ; hence, necessarily  $c_2^* = 3(\bar{\mu} - M) - 2\xi > 0$  and  $c_1^* = c_0^* = \xi$ . Since  $c_0^* + c_1^* + c_2^* = \tilde{c}_0 + \mathbb{E}[\tilde{c}_1] + \mathbb{E}[\tilde{c}_2]$  we get that  $\tilde{c}_0 + \mathbb{E}[\tilde{c}_1] > c_0^* + \mathbb{E}[c_1^*] = 2\xi$  leading to a contradiction. Then, it must hold that  $\mathbb{E}[d_2] \ge 0$ . Now we have two cases:

- (i) If  $\mathbb{E}[d_1] < -e^{-rT/3}\mathbb{E}[d_2]$ , then it is immediate that  $\mathbb{E}[d_1 + d_2] e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2e^{-\frac{rT}{3}}] > 0$ and then  $V^{\tilde{\mathbf{c}}} \leq V^{\mathbf{c}^*}$ ;
- (ii) If  $\mathbb{E}[d_1] \ge -e^{-rT/3}\mathbb{E}[d_2]$ , we get that

$$c_0^* - \tilde{c}_0 = \mathbb{E}[d_1 + d_2] \ge \mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}] \ge e^{-\frac{rT}{3}} \mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}]$$

which implies that  $V^{\tilde{c}} \leq V^{c^*}$ .

To conclude we observe that if  $\mathbb{E}[d_2] > 0$  then the inequality is strict and the strategy  $(c_0^*, c_1^*, c_2^*)$ , is optimal. If  $\mathbb{E}[d_2] = 0$  we get that either  $\mathbb{E}[d_1] > 0$  in which case the inequality is strict again, or  $\mathbb{E}[d_1] = 0$  which corresponds to the case where  $\tilde{\mathbf{c}} = \mathbf{c}$ .

The above example provides the argument for computing the optimal dividend strategy in an *n*-period framework.

**Proposition 2.6:** Let n be the number of sub-periods in the interval [0, T] and let

$$\kappa := \min\{m \ge 0 : n(\bar{\mu} - M) < (m+1)\xi\}.$$
(7)

Then, an optimal strategy  $\mathbf{c}^* = (c_0^*, c_1^*, \dots, c_{n-1}^*)$  is given by

$$\begin{cases} c_0^* = \dots = c_{\kappa-1}^* = \xi, \\ c_{\kappa}^* = n(\bar{\mu} - M) - \kappa \xi, \\ c_{\kappa+1}^* = \dots c_{n-1}^* = 0. \end{cases}$$
(8)

**Proof:** Assume first  $\kappa = n - 1$ , then obviously the optimal strategy is  $(\xi, \ldots, \xi)$ .

Let now  $\kappa < n-1$  and let  $\tilde{\mathbf{c}} = (\tilde{c}_0, \dots, \tilde{c}_{n-1})$  be an admissible strategy. Like in Example 2.5, there exist  $d_1, \dots, d_{n-1}$  such that  $\tilde{c}_m = c_m^* + d_m$  for  $m \in \{1, \dots, n-1\}$  and  $\sum_{m=1}^{n-1} \mathbb{E}[d_m] = c_0^* - \tilde{c}_0 \ge 0$ . Then we have that

$$V^{\tilde{\mathbf{c}}}(x) = V^{\mathbf{c}^*}(x) - (c_0^* - \tilde{c}_0)(1 - e^{-rT/n}) + (1 - e^{-rT/n})\sum_{m=1}^{n-1} e^{-rTm/n} \mathbb{E}[d_m].$$

Note that since  $c_m^* = \xi$  for all  $m \le \kappa - 1$ , and  $c_m^* = 0$  for all  $m > \kappa + 1$ , it must hold  $d_m \le 0$  for  $m \le \kappa - 1$  and  $d_m \ge 0$  for  $m \ge \kappa + 1$ .

Now we observe that, the function  $t \mapsto \sum_{m=1}^{n-1} e^{rt \frac{\kappa-m}{n}} \mathbb{E}[d_m]$  is decreasing, and hence it attains its maximum at t = 0, i.e.  $\sum_{m=1}^{n-1} \mathbb{E}[d_m] \ge \sum_{m=1}^{n-1} e^{rT \frac{\kappa-m}{n}} \mathbb{E}[d_m]$ . Therefore, we conclude that

$$c_0^* - \tilde{c}_0 = \sum_{m=1}^{n-1} \mathbb{E}[d_m] \ge e^{-rT\frac{\kappa}{n}} \sum_{m=1}^{n-1} \mathbb{E}[d_m] \ge e^{-rT\frac{\kappa}{n}} \sum_{m=1}^{n-1} e^{rT\frac{\kappa-m}{n}} \mathbb{E}[d_m]$$

The strict inequality holds true if there is at least one *m* with  $\mathbb{E}[d_m] \neq 0$ . If instead  $\mathbb{E}[d_m] = 0$  for all m = 1, ..., n - 1, then strategies  $\tilde{c}$  and  $c^*$  coincide, i.e. in particular  $d_m = 0$  almost surely for all  $m = 0, \ldots, n-1$ . This leads to  $V^{\tilde{c}} < V^{c^*}$  if  $\tilde{c} \neq c^*$ .

Remark 2.7 (Continuous time): This procedure allows extending the setting to continuous time.

We denote by  $\mathcal{A}_{(\infty)}$  the set of admissible strategies, consisting of the  $\mathbb{F}$ -adapted processes  $\mathbf{c} =$  $(c_s)_{s \in [0,T]}$  with  $0 \le c_s \le \xi$  and  $X_T^{\mathbf{c}}$  from Equation (1) normally distributed with mean x + MT and variance  $\delta^2 T$ . Letting  $n \to \infty$  in the *n*-period models, the optimal strategy as given in Equation (8) converges to a deterministic strategy in continuous time:

$$c_s^* = \begin{cases} \xi & : t \le T \land t^*, \\ 0 & : t > T \land t^*, \end{cases}$$

where  $t^* = (\bar{\mu} - M)T/\xi$ . We assume  $t^* < T$ .

Let  $\tilde{\mathbf{c}} = (\tilde{c}_s)_{s \in [0,T]}$  be an admissible strategy and define  $d_s := \tilde{c}_s - c_s^*$ . Since we would like to achieve the same final distribution with strategies  $\mathbf{c}^*$  and  $\tilde{\mathbf{c}}$ , it must hold that  $\mathbb{E}[\int_0^T d_s] = 0$ . Moreover, it is clear that  $d_s \leq 0$  for  $s \leq t^*$ ,  $d_s \geq 0$  for  $s > t^*$ . As for the *n*-period models we get

$$V^{\tilde{\mathbf{c}}}(x) = \mathbb{E}_{x} \left[ \int_{0}^{T} e^{-rs} \tilde{c}_{s} \, \mathrm{d}s \right] = V^{\mathbf{c}^{*}}(x) + \mathbb{E}_{x} \left[ \int_{0}^{T} e^{-rs} d_{s} \, \mathrm{d}s \right]$$
$$= V^{\mathbf{c}^{*}}(x) + e^{-rt^{*}} \mathbb{E}_{x} \left[ \int_{0}^{T} e^{-r(s-t^{*})} d_{s} \, \mathrm{d}s \right]$$
$$\leq V^{\mathbf{c}^{*}}(x) + e^{-rt^{*}} \mathbb{E}_{x} \left[ \int_{0}^{T} d_{s} \, \mathrm{d}s \right] = V^{\mathbf{c}^{*}}(x).$$

A strict inequality holds true if  $\mathbb{E}[d_s] \neq 0$  for all  $s \in \mathcal{T}$  where  $\mathcal{T} \subseteq [0, T]$  is a Lebesgue measurable non-zero set.

Therefore, in continuous time it is optimal to pay on the maximal rate as long as possible, and to pay nothing afterwards.

Note, that we have only considered the case of dividend rates. However, it is also possible to allow for lump sum payments. Then, because r > 0 it is clear that one should pay the amount  $(\bar{\mu} - M)T$ directly at time zero.

Considering a setting with dividend rates may be more preferable for reputational reasons. Indeed, distributing the dividends over the whole period [0, T] rather than paying a lump sum at the beginning of the period, may give a better impression to shareholders. This is guaranteed in our model by the upper bound  $\xi$  on the admissible dividend rates. The value of  $\xi$  is a management decision: in our setting it has to be small enough to distribute dividends over the whole period, and large enough to achieve the target distribution.

# 3. Dividend Minimising the Ruin Probability

At first glance, the title of this sections sounds controversial. Indeed, paying dividends increases the probability of ruin, and in many settings the optimal dividend strategy even leads to a certain ruin.

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However, the constraint put on the terminal distribution allows finding a non-zero dividend strategy that minimises the ruin probability over the set of admissible strategies.

We consider again an insurance company who pays dividends and aims to achieve a target distribution of the terminal surplus at time *T*. However, we now assume that the objective of the insurer is to minimise the ruin probability.

We consider the same setting like in Section 2 with a surplus, after dividends, described by Equation (1). We recall that the set of achievable target means is given by  $\bar{\mu} - \xi < M < \bar{\mu}$  (see Lemma 2.1) and that the set of admissible strategies  $\mathcal{A}_{(n)}$  is the set of all strategies  $\mathbf{c} = (c_0, \ldots, c_{n-1})$ , where  $\sum_{k=1}^{n-1} c_k$  is  $\mathcal{F}_0$ -measurable (see Proposition 2.2), and  $c_0 + \sum_{k=1}^{n-1} c_k = n(\bar{\mu} - M)$ .

The goal of the insurance company is to minimise the ruin probability, which is given by

$$\min \mathbf{P}\left[\inf_{0 \le t \le T} X_t^{\mathbf{c}} < 0\right] \tag{9}$$

over all admissible dividend strategies  $\mathbf{c} \in \mathcal{A}_{(n)}$ .

Like in Section 2, we begin by addressing Problem (9) in a two-period framework.

#### 3.1. A 2-period model

The set of admissible strategies is denoted by  $A_{(2)}$ , given in Section 2.1. That is, all admissible strategies are of the form  $\mathbf{c} = (c_0, c_1)$  with  $c_0, c_1 \in [0, \xi]$  deterministic (see Corollary 2.3) and  $c_0 + c_1 = 2(\bar{\mu} - M)$ . We target to minimise the ruin probability in the time interval [0, T], i.e.

$$p(\mathbf{c}, x) := \mathbf{P}\left[\inf_{0 \le t \le T} X_t^{\mathbf{c}} < 0\right],$$

over all  $\mathbf{c} \in \mathcal{A}_{(2)}$ . Note that differently than in Section 2 the dependence on the initial capital *x* is crucial in this setting.

**Proposition 3.1:** Let  $\mathbf{c} = (c_0, c_1)$  and  $\tilde{\mathbf{c}} = (\tilde{c}_0, \tilde{c}_1)$  be two admissible strategies, i.e.  $X^{\mathbf{c}}, X^{\mathbf{c}} \sim N(x + MT, \delta^2 T)$ . We assume that  $c_0 > \tilde{c}_0$ . Then,  $\tilde{\mathbf{c}}$  is better than  $\mathbf{c}$ , in the sense that

$$p(\tilde{\mathbf{c}}, x) < p(\mathbf{c}, x)$$

**Proof:** We first observe that at time *T*, both strategies **c** and  $\tilde{\mathbf{c}}$  lead to the same distribution of the final surplus, i.e.  $X_T^{\mathbf{c}}, X_T^{\mathbf{c}} \sim N(x + MT, \delta^2 T)$ . Then, we have

$$\begin{split} \inf_{0 \le t \le T} X_t^{\mathbf{c}} &= \inf_{0 \le s \le T} \begin{cases} x + (\bar{\mu} - c_0)s + \bar{\sigma} W_s &: \text{ if } s \le \frac{T}{2} \\ x + (\bar{\mu} - c_0)\frac{T}{2} + \bar{\sigma} W_s + (\bar{\mu} - c_1)(s - \frac{T}{2}) &: \text{ if } s \in (\frac{T}{2}, T] \end{cases} \\ &= \inf_{0 \le s \le T} \begin{cases} x + (\bar{\mu} - c_0)s + \bar{\sigma} W_s &: \text{ if } s \le \frac{T}{2} \\ x + \bar{\mu}s + c_0(s - T) + \bar{\sigma} W_s - 2(\bar{\mu} - M)(s - \frac{T}{2}) &: \text{ if } s \in (\frac{T}{2}, T] \end{cases} \\ &= \inf_{0 \le s \le T} \begin{cases} X_s^{\tilde{\mathbf{c}}} + (\tilde{c}_0 - c_0)s &: \text{ if } s \le \frac{T}{2} \\ X_s^{\tilde{\mathbf{c}}} + (c_0 - \tilde{c}_0)(s - T) &: \text{ if } s \in (\frac{T}{2}, T] \end{cases} \\ &< \inf_{0 \le t \le T} X_s^{\tilde{\mathbf{c}}}. \end{split}$$

Therefore, for all x > 0 we get that  $p(\mathbf{c}, x) > p(\tilde{\mathbf{c}}, x)$ .

As a consequence of Proposition 3.1, we get that  $c_0$  should be chosen as the smallest possible value. This leads to the following result.

**Corollary 3.2:** In a two-period framework, the ruin minimising dividend strategy is  $\mathbf{c}^* = (c_0^*, c_1^*)$  where

$$\begin{cases} c_0^* = \max(2(\bar{\mu} - M) - \xi, 0), \\ c_1^* = \min(\xi, 2(\bar{\mu} - M)). \end{cases}$$

#### 3.2. An n-period model

The extension to n-periods is obtained by replicating the reasoning of Proposition 3.1 and Corollary 3.2.

**Proposition 3.3:** Let  $k := \min\{m \ge 0 : n(\bar{\mu} - M) < (m+1)\xi\}$ . Then, the ruin minimising dividend strategy  $\mathbf{c} = (c_0, \ldots, c_{n-1})$  fulfils

$$\begin{cases} c_{n-1} = \dots = c_{n-k} = \xi, \\ c_{n-k-1} = n(\bar{\mu} - M) - k\xi, \\ c_0 = \dots = c_{n-k-2} = 0. \end{cases}$$

**Remark 3.4 (Continuous time):** Letting  $n \to \infty$  will produce the following optimal strategy: we define  $t^*$  as the time that realises  $(\bar{\mu} - M)T = \xi t^*$ . Then, the optimal dividend rate is  $c_t = 0$  for all  $0 < t < t^*$  and  $c_t = \xi$  for  $t \ge t^*$ .

Like in the dividend maximisation problem in Section 2, the ruin minimising strategy is deterministic. An intuitive consequence of the result above (Proposition 3.3) is that the optimal strategy that minimises ruin probability is also the strategy that minimises the value of expected discounted dividends.

#### 4. Reinsurance With a Target Terminal Distribution

In this section, we change the setting of Sections 2 and 3, and instead we consider an insurance company who buys reinsurance for a certain branch of their business or a pool of insured claims.

We let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space and T a fixed time horizon. We also let Z be a random variable representing a claim size having positive finite first and second moments denoted by  $\mathbb{E}[Z] = \mu$  and  $\mathbb{E}[Z^2] = \mu_2$ , respectively. We assume that the surplus of the insurance company is described by a Brownian motion with drift, approximating a Cramer–Lundberg model like, e.g. in Schmidli (2008, p. 226),

$$X_t = x + \lambda \eta \mu t + \sqrt{\lambda \mu_2 W_t}, \quad t \in [0, T],$$

where  $\lambda, \eta > 0$  and  $W = (W_t)_{t \in [0,T]}$  is a Brownian motion. We also define by  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$  natural filtration of the Brownian motion, under the usual hypotheses. The insurance company is allowed to buy proportional reinsurance with retention  $b \in [0, 1]$  to mitigate the losses. We assume that the reinsurance premium is calculated via the expected value principle, that is the reinsurance premium rate is  $(1 + \theta)(\lambda \mu - \lambda \mathbb{E}[r(Z, b)])$ , where r(Z, b) = bZ. Then, the premium rate that remains to the insurer (i.e. the difference between the insurance premium and the reinsurance premium) is

$$c(b) = \lambda \theta \mathbb{E}[r(Z, b)] - \lambda \mu(\theta - \eta),$$

with c(0) < 0, see, e.g. Schmidli (2008, Ch. 2.2) for more details.

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Under a reinsurance strategy  $\mathbf{b} = (b_s)_{s \in [0,T]}$ , the surplus is given by

$$X_t^{\mathbf{b}} = x + \lambda \mu \int_0^t (\theta b_s - (\theta - \eta)) \, \mathrm{d}s + \sqrt{\lambda \mu_2} \int_0^t b_s \, \mathrm{d}W_s, \quad t \in [0, T].$$

We denote by

$$\bar{X}_t^{\mathbf{b}} := X_t^{\mathbf{b}} - x, \quad t \in [0, T],$$

the net value of collective, i.e. the part of the surplus that only accounts for insurance/reinsurance premia and claims.

The insurance company wants to make sure that the collected premia are sufficient (in a certain sense) to buy reinsurance, if necessary, and to pay the occurring claims. To achieve such level of *sustainability* the target of the insurance is to choose a reinsurance strategy such that at time T the distribution of the net collective is normal with mean MT, for some small M > 0 and variance  $\delta^2 T$ . To gain some intuition on the choice of M and  $\delta$ , we may interpret M as a (small) positive target gain, and  $\delta$  is fixed to fulfil  $\mathbf{P}[-\bar{X}_T^B > \ell] \le 1 - \alpha$  for some given  $\ell > 0$  and  $\alpha \in (0, 1)$ . The latter is a condition on the Value at Risk (VaR) at the confidence level  $\alpha$  (for instance  $\alpha = 99.5\%$ ). In particular,  $\ell$  represents the loss that the insurer can bear with at most probability  $1 - \alpha$ . This can be interpreted as the required capital ensuring the system's solvency. Aiming at  $\bar{X}_T^B \sim N(MT, \delta^2 T)$  as a target distribution is justified, for instance, by the existence of the closed form formulas for the VaR or Expected Shortfall (ES) for Gaussian random variables, which can be easily calculated.<sup>1</sup>

To reach the target distribution, the insurance company follows a *sustainable* strategy, that is reinsurance is only financed through premia, and we additionally require that reinsurance strategies do not produce a negative premium rate for the insurer. This condition is, in spirit, similar to the selffinancing condition which is often assumed in finance. Indeed, a branch of business or a pool of insured is considered as a closed system, where the insurance company does not intervene by injecting or withdrawing additional capital.

Our next step is to define the set of possible controls leading to the desired distribution. We let  $\mathcal{B}$  denote the set of strategies  $\mathbf{b} = (b_t)_{t \in [0,T]}$  with  $b_t \in [0,1]$  for all  $t \in [0,T]$ , that are adapted to  $\mathbb{F}$  and such that  $\bar{X}_T^{\mathbf{b}} \sim N(MT, \delta^2 T)$ .

Note that, in particular, deterministic controls make the terminal distribution of the net collective surplus Gaussian, see Example 4.1.

**Example 4.1 (Deterministic controls):** Let  $\mathbf{b} = (b(t))_{t \in [0,T]}$  be a continuous deterministic reinsurance strategy, with  $b(t) \in [0, 1]$  for all  $t \in [0, T]$ .<sup>2</sup> Then **b** is an admissible control if the following two conditions hold:

$$\begin{cases} \lambda \mu \int_0^T (\theta b(s) - (\theta - \eta)) \, \mathrm{d}s = MT, \\ \lambda \mu_2 \int_0^T b(s)^2 \, \mathrm{d}s = \delta^2 T. \end{cases}$$
(10)

<sup>1</sup> Denoting by  $L_T$  the terminal loss at time T, we immediately get that

$$\operatorname{VaR}_{\alpha}(L_T) = -MT + \delta\sqrt{T}\Phi^{-1}(\alpha),$$

and

$$ES_{\alpha}(L_{T}) = -MT + \delta\sqrt{T}\frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}$$

for  $\alpha \in (0, 1)$ , where  $\varphi$  and  $\Phi$  denote the density and the cumulative distribution function of the standard normal, respectively. <sup>2</sup> In this example we use the notation b(t) in place of  $b_t$  to emphasise the deterministic nature of the strategy.



**Figure 1.** The strategy  $b(t) = \frac{A}{A+Ct}$  with A = 3.61, C = 6.58.

To make an example,  $b(t) = \frac{A}{A+Ct}$  is an admissible control for constants A, C which satisfy

$$\int_0^T \frac{A}{A+Cs} \, \mathrm{d}s = \frac{A^2}{C} \ln\left(\frac{A+CT}{A}\right) = \frac{M+\lambda\mu(\theta-\eta)T}{\lambda\theta\mu},$$
$$\int_0^T \frac{A^2}{(A+Cs)^2} \, \mathrm{d}s = \frac{A}{C} \left(1-\frac{A}{A+CT}\right) = \frac{\delta^2}{\lambda\mu_2}.$$

For the parameter set given by  $\mu = 0.05$ ,  $\lambda = 1$ ,  $\eta = 0.3$ ,  $\theta = 0.5$ ,  $\mu_2 = 0.05$ , M = 0.06,  $\delta = 0.15$ , T = 2.5, we get that A = 3.613, C = 6.5837. The strategy is illustrated in Figure 1.

In the sequel we restrict to the case where reinsurance strategies can be updated only at deterministic time points, which represent some apriori fixed checking dates, for instance all the four months or all the six months. In fact, we concentrate on the case n = 2 and we refer to this case as the *two period model*. The technical reason is that, differently than in the dividend case, reinsurance controls affect both the drift and the volatility. Therefore, in this case, a pathwise comparison is not possible anymore, and the problem must be addressed with different techniques. In the case n = 2, we are still able to obtain an explicit solution with probabilistic methods. However, the problem becomes immediately more complicated when we increase the number of periods (see Section 4.4), even if we restrict to deterministic strategies.

#### 4.1. Admissible strategies in a 2-period model

We denote the set of admissible strategies by  $\mathcal{B}_{(2)}$ , where, like before, the subscript indicates the number of strategy updates up to time *T*. An admissible strategy is a pair  $\mathbf{b} = (b_0, b_1)$ , where  $b_0$  is  $\mathcal{F}_0$ -measurable and  $b_1$  is  $\mathcal{F}_{T/2}$  measurable. In this setting the retention level is updated only once, at time *T*/2. Hence, at time *T* the net surplus satisfies

$$\bar{X}_{T}^{\mathbf{b}} = \frac{\lambda\mu\theta T}{2}(b_{0} + b_{1}) - \lambda\mu(\theta - \eta)T + \sqrt{\lambda\mu_{2}}b_{0}W_{T/2} + \sqrt{\lambda\mu_{2}}b_{1}(W_{T} - W_{T/2})$$

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$$= \bar{X}_{T/2}^{b_0} + b_1 \frac{T}{2} \lambda \mu \theta - \lambda \mu (\theta - \eta) \frac{T}{2} + b_1 \sqrt{\lambda \mu_2} (W_T - W_{T/2}),$$

where  $\bar{X}_{T/2}^{b_0} = \lambda \mu (\theta b_0 - \theta + \eta) \frac{T}{2} + \sqrt{\lambda \mu_2} b_0 W_{T/2}$ .

The set of admissible strategies is characterised in the lemma below. In particular, we show that admissible strategies are deterministic.

**Lemma 4.2:** The set  $\mathcal{B}_{(2)}$  consists of all strategies  $\mathbf{b} = (b_0, b_1)$  where  $b_0, b_1$  are both  $\mathcal{F}_0$ -measurable, taking values in [0, 1], and satisfying the following two conditions:

$$b_1 = 2 \frac{M + \lambda \mu (\theta - \eta)}{\lambda \theta \mu} - b_0,$$
  

$$b_1^2 = \frac{2\delta^2}{\lambda \mu_2} - b_0^2.$$
(11)

**Proof:** Recall that for any normally distributed random variable Y with mean MT and variance  $\delta^2 T$ , the moment generating function is given by  $\mathbb{E}[e^{\zeta Y}] = e^{\zeta MT + \frac{1}{2}\zeta^2 \delta^2 T}$ , for all  $\zeta \in \mathbb{R}$ . Let  $\tilde{W}_{T/2} = W_T - W_{T/2}$ ; then  $W_{T/2}$  and  $\tilde{W}_{T/2}$  are independent. Since **b** is chosen so that  $X_T^{\mathbf{b}} \sim N(MT, \delta^2 T)$ , it holds that

$$\mathbb{E}[e^{\zeta X_T^B}] = e^{\zeta MT + \frac{1}{2}\zeta^2 \delta^2 T}$$

for all  $\zeta \in \mathbb{R}$ . Now, we let **Q** be a probability measure equivalent to **P**, with the Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_T} = e^{r\sqrt{\lambda\mu_2}b_1\widetilde{W}_{T/2} - \frac{r^2}{2}\lambda\mu_2b_1^2\frac{T}{2}}$$

Using the independence of  $\tilde{W}_{T/2}$  and  $W_{T/2}$  and the change of measure we get that

$$e^{\zeta MT + \frac{1}{2}\zeta^2 \delta^2 T} = \mathbb{E}[e^{\zeta \bar{X}_T^B}] = \mathbb{E}\left[e^{\zeta \bar{X}_{T/2}^{b_0} + r\lambda\mu(\theta b_1 - \theta + \eta)\frac{T}{2} + \zeta\sqrt{\lambda\mu_2}b_1\tilde{W}_{T/2}}\right]$$
$$= e^{\zeta\lambda\mu(b_0\theta - \theta + \eta)T/2 + \frac{\zeta^2\lambda\mu_2b_0^2T}{4}} \mathbb{E}\left[e^{\zeta\lambda\mu(\theta b_1 - \theta + \eta)\frac{T}{2} + \zeta\sqrt{\lambda\mu_2}b_1\tilde{W}_{T/2}}\right]$$
$$= e^{\zeta\lambda\mu(b_0\theta - \theta + \eta)T/2 + \frac{\zeta^2\lambda\mu_2b_0^2T}{4}} \mathbb{E}_{\mathbf{Q}}\left[e^{\zeta\lambda\mu(\theta b_1 - \theta + \eta)\frac{T}{2} + \frac{\zeta^2}{2}\lambda\mu_2b_1^2\frac{T}{2}}\right],$$

for all  $\zeta \in \mathbb{R}$ . This can be simplified to

$$\mathbb{E}_{\mathbf{Q}}[e^{\zeta\lambda\mu(\theta b_{1}-\theta+\eta)\frac{T}{2}+\frac{\zeta^{2}}{2}\lambda\mu_{2}b_{1}^{2}\frac{T}{2}}] = e^{\zeta MT-\zeta\lambda\mu(b_{0}\theta-\theta+\eta)\frac{T}{2}+\frac{1}{2}\zeta^{2}\delta^{2}T-\frac{\zeta^{2}\lambda\mu_{2}b_{0}^{2}T}{4}}.$$

Deriving the above expression with respect to  $\zeta$  and letting  $\zeta = 0$  we see that all moments of  $b_1$  correspond to the moments of a normal distribution, meaning that the moment generating function of  $b_1$  (written as a power series with the moments as coefficients) corresponds to that of a normal distribution. Therefore, we conclude

$$b_1 \sim N\left(2\frac{M+\lambda\mu(\theta-\eta)}{\lambda\mu\theta}-b_0,\frac{2\delta^2}{\lambda\mu_2}-b_0^2\right)$$

However, this is impossible because  $b_1$  can attain values only in [0, 1] **P**-a.s. (hence also **Q**-a.s.), which means that  $b_1$  must be constant.

Because  $b_0$ ,  $b_1$  can take values only in [0, 1], it is clear that not all arbitrary values of M and  $\delta$  are reachable. In the next lemma we specify the ranges of M and  $\delta$ .

**Lemma 4.3:** If there exist  $b_0, b_1 \in [0, 1]$  such that Condition (11) holds, then the target mean M and the variance  $\delta > 0$  satisfy:

$$0 \le M \le \lambda \mu \eta,$$

$$\left(\frac{M + \lambda \mu (\theta - \eta)}{\lambda \mu \theta}\right)^2 \frac{M}{\lambda \mu \theta} \le \frac{\delta^2}{\lambda \mu_2} \le \min\left\{2\left(\frac{M + \lambda \mu (\theta - \eta)}{\lambda \mu \theta}\right)^2, 1\right\}.$$
(12)

**Proof:** From Condition (11), and the fact that  $b_0, b_1$  take values in [0, 1], we get that  $0 \le M \le \lambda \mu \eta$  and that  $\frac{\delta^2}{\lambda \mu_2} \le 1$ . Using again the Conditions (11) and substituting the value of  $b_1$  into the second equation we get that  $b_0$  must solve

$$2b_0^2 - 4b_0 \frac{M + \lambda\mu(\theta - \eta)}{\lambda\mu\theta} + 4\left(\frac{M + \lambda\mu(\theta - \eta)}{\lambda\mu\theta}\right)^2 - \frac{2\delta^2}{\lambda\mu} = 0.$$

Imposing the existence of a real solution leads to

$$\frac{\delta^2}{\lambda\mu_2} \ge \left(\frac{M + \lambda\mu(\theta - \eta)}{\lambda\mu\theta}\right)^2.$$

Then, using the fact that  $b_0$  must take non-negative values leads to the bound:

$$\frac{\delta^2}{\lambda\mu_2} \le \min\left\{2\left(\frac{M+\lambda\mu(\theta-\eta)}{\lambda\mu\theta}\right)^2, 1\right\}.$$

Notice that because we do not allow for arbitrage and require  $\eta < \theta$ , to ensure the existence of a solution at least for the case  $\eta = \theta$ , we must have that  $\frac{\delta^2}{\lambda\mu_2} \ge (\frac{M}{\lambda\mu\theta})^2$ , which is guaranteed by Equations (12).

There is a clear trade-off between increasing profits and reducing risks. This is due to the fact that a reinsurance strategy controls both the mean and the volatility. Under a reinsurance strategy the mean and the volatility move into the same direction: increasing the retention level makes the mean larger, but also the volatility. This observation has important consequences for the ruin probability. Indeed, a bigger retention level would make the drift of the net collective larger, meaning that it potentially can push the surplus away from zero; however, at the same time, it increases the riskiness by making the volatility larger. For instance, considering the parameters  $\mu = 0.22$ ;  $\mu_2 = 0.05$ ;  $\eta = 0.3$ ;  $\theta = 0.35$ ;  $\lambda = 2$ , M = 0.08 < 0.132, we get that the admissible values of  $\delta$  vary in the range [0.2094, 0.2962]. If an insurance company aims at getting an expected gain of 8% at the end of the observation period, it has to account for a relatively large risk of at least 21%.

We can write the range for  $\delta$  as

$$\left(1+\frac{M-\lambda\mu\eta}{\lambda\mu\theta}\right)^2 \leq \frac{\delta^2}{\lambda\mu_2} \leq \min\left\{2\left(1+\frac{M-\lambda\mu\eta}{\lambda\mu\theta}\right)^2, 1\right\}.$$

From this expression it is clear that if the target return is close to  $\lambda \mu \eta$ , the variance  $\delta^2$  is approximately  $\lambda \mu_2$ , which corresponds to the case where no reinsurance is bought.

η	$b_0$	$b_1$	$p(b_0, b_1)$	<b>p</b> ( <i>b</i> <sub>1</sub> , <i>b</i> <sub>0</sub> )
0.25	0.4448	0.7760	0.4088	0.5117
0.26	0.3339	0.8298	0.3772	0.5372
0.27	0.2468	0.8597	0.3485	0.5561
0.28	0.1715	0.8778	0.3154	0.5720
0.29	0.1038	0.8884	0.2637	0.5857
0.3	0.0416	0.8935	0.1254	0.5967

**Table 1.** Admissible strategies and survival probabilities for different values of  $\eta < \theta$ .

#### 4.2. Ruin probabilities in a 2-period model

For the case n = 2, the pairs of strategies that satisfy Conditions (11) are of the type  $(b_0, b_1)$  and  $(b_1, b_0)$ .

We assume that  $\frac{T}{2}$  and T are the regulatory authorities' inspection dates. A reinsurance strategy is chosen so that the probability of having a positive surplus at both dates is maximised.

We now give a definition of ruin within this setting. We say that *the ruin occurs if the insurance company showcases a negative surplus at any of the time points* T/2 *or* T. Then, an equivalent formulation of the problem is:

Find a reinsurance strategy that minimises the ruin probability.

In mathematical terms, the problem is formulated as follows. Let  $\mathbf{b} = (b_0, b_1)$  and  $\mathbf{b} = (b_1, b_0)$  be the two admissible strategies. Without loss of generality, we assume that  $b_0 \le b_1$ . For each strategy we define the corresponding *survival probabilities*:

$$p(\mathbf{b}) = \mathbf{P} \left[ \bar{X}_{T/2}^{\mathbf{b}} > 0, \bar{X}_{T}^{\mathbf{b}} > 0 \right],$$
$$p(\tilde{\mathbf{b}}) = \mathbf{P} \left[ \bar{X}_{T/2}^{\tilde{\mathbf{b}}} > 0, \bar{X}_{T}^{\tilde{\mathbf{b}}} > 0 \right].$$

Our objective is to decide which of these two probabilities,  $p(\mathbf{b})$  or  $p(\tilde{\mathbf{b}})$ , is the largest.

Table 1 below illustrates survival probability for different values of  $\eta < \theta$  so that M and  $\delta$  are achievable for  $T = 1, \lambda = 2, \mu = 0.22, \mu_2 = 0.05, \theta = 0.35, M = 0.05, \delta = 0.2$ . The last two columns suggest that  $p(\mathbf{b}) < p(\tilde{\mathbf{b}})$ . This result is proved in Proposition 4.4 below.

**Proposition 4.4:** Let  $b_0 < b_1$ . Then the strategy  $(b_1, b_0)$  is better than the strategy  $(b_0, b_1)$ , *i.e.*  $p(\tilde{\mathbf{b}}) > p(\mathbf{b})$ .

**Proof:** Let  $(W_t)_{t>0}$  and  $(\hat{W}_t)_{t>0}$  be two independent Brownian motions and denote

$$\begin{split} \bar{X}^b_{T/2} &= \lambda \mu \left( \theta (1-b) - \eta \right) T/2 + \sqrt{\lambda \mu_2} b W_{T/2}, \\ \hat{X}^b_{T/2} &= \lambda \mu \left( \theta (1-b) - \eta \right) T/2 + \sqrt{\lambda \mu_2} b \hat{W}_{T/2}. \end{split}$$

Then, the survival probabilities can be rewritten as

$$p(\mathbf{b}) = \mathbf{P} \left[ \bar{X}_{T/2}^{b_0} > 0, \bar{X}_{T/2}^{b_0} + \hat{X}_{T/2}^{b_1} > 0 \right],$$
  

$$p(\tilde{\mathbf{b}}) = \mathbf{P} \left[ \bar{X}_{T/2}^{b_1} > 0, \bar{X}_{T/2}^{b_1} + \hat{X}_{T/2}^{b_0} > 0 \right],$$
(13)

having set  $\hat{W}_{T/2} = W_T - W_{T/2}$ . The advantage of this representation stands in the fact that for every  $b \in [0, 1]$ ,  $\bar{X}^b_{T/2}$  and  $\hat{X}^b_{T/2}$  are independent. We observe that there exist standard Brownian motions

 $W^0$  and  $W^1$  such that

$$\sqrt{\lambda\mu_2}b_0W_t + \sqrt{\lambda\mu_2}b_1\hat{W}_t = \sqrt{\lambda\mu_2(b_0^2 + b_1^2)}W_t^0,$$

$$\sqrt{\lambda\mu_2}b_1W_t + \sqrt{\lambda\mu_2}b_0\hat{W}_t = \sqrt{\lambda\mu_2(b_0^2 + b_1^2)}W_t^1,$$
(14)

for all  $t \in [0, T]$ . We now let

$$Y_t^0 := \lambda \mu \theta (b_0 + b_1) t - 2\lambda \mu (\theta - \eta) t + \sqrt{\lambda \mu_2 (b_0^2 + b_1^2)} W_t^0,$$
  
$$Y_t^1 := \lambda \mu \theta (b_0 + b_1) - 2\lambda \mu (\theta - \eta) t + \sqrt{\lambda \mu_2 (b_0^2 + b_1^2)} W_t^1,$$

for all  $t \in [0, T]$ . Due to Equations (14) we get that for all  $t \in [0, T]$ ,  $Y_t^0 = Y_t^1 = 2Mt + \sqrt{2\delta^2} \hat{W}_t$ , hence they are identically distributed.

Next, we write  $\bar{X}_{T/2}^{b_0}$  and  $\bar{X}_{T/2}^{b_1}$  in terms of  $Y_{T/2}^0$  and  $Y_{T/2}^1$ . Since  $\bar{X}_{T/2}^{b_0}$  and  $\bar{X}_{T/2}^{b_1}$  are normally distributed, we have that

$$\begin{split} \bar{X}_{T/2}^{b_0} &= \rho Y_{T/2}^0 + Z^0, \quad \rho := \frac{Cov(\bar{X}_{T/2}^{b_0}, Y_{T/2}^0)}{Var(Y_{T/2}^0)} = \frac{Var[\bar{X}_{T/2}^{b_0}]}{Var[Y_{T/2}^0]} = \frac{\lambda \mu_2 b_0^2}{2\delta^2}, \\ \bar{X}_{T/2}^{b_1} &= \gamma Y_{T/2}^1 + Z^1, \quad \gamma := \frac{Cov(\bar{X}_{T/2}^{b_1}, Y_{T/2}^1)}{Var[Y_{T/2}^1]} = \frac{Var[\bar{X}_{T/2}^{b_1}]}{Var[Y_{T/2}^1]} = \frac{\lambda \mu_2 b_1^2}{2\delta^2} = 1 - \rho, \end{split}$$

where  $Y_{T/2}^0$  and  $Z^0$ ,  $Y_{T/2}^1$  and  $Z^1$  are independent, since they are all normally distributed and  $Cov(Y_{T/2}^0, Z^0) = Cov(Y_{T/2}^1, Z^1) = 0.$ 

Expectations and variances of  $Z^0$  and  $Z^1$  are given by

$$\begin{cases} \mathbb{E}[Z^{0}] = \mathbb{E}[\bar{X}_{T/2}^{b_{0}} - \rho Y_{T/2}^{0}] = \lambda \mu (\theta b_{0} - \theta + \eta) T/2 - 2\rho MT/2, \\ \mathbb{E}[Z^{1}] = \mathbb{E}[\bar{X}_{T/2}^{b_{1}} - \gamma Y_{T/2}^{1}] = \lambda \mu (\theta b_{1} - \theta + \eta) T/2 - 2\gamma MT/2 = -\mathbb{E}[Z^{0}], \\ Var[Z^{0}] = \lambda \mu_{2} b_{0}^{2} T/2 - 2\rho^{2} \delta^{2} T/2 = 2\delta^{2} \rho \gamma T/2, \\ Var[Z^{1}] = \lambda \mu_{2} b_{1}^{2} T/2 - 2\gamma^{2} \delta^{2} T/2 = 2\delta^{2} \gamma \rho T/2. \end{cases}$$

$$(15)$$

Using Fubini's theorem, we get

$$\begin{split} p(\mathbf{b}) &= \mathbf{P} \left[ Y_{T/2}^{0} + \frac{Z^{0}}{\rho} > 0, \ Y_{T/2}^{0} > 0 \right] = \mathbf{P} \left[ \frac{Z^{0}}{\rho} > -Y_{T/2}^{0}, \ Y_{T/2}^{0} > 0 \right] \\ &= \int_{0}^{\infty} \left( 1 - \Phi \left( \frac{-y - \frac{\mathbb{E}[Z^{0}]}{\rho}}{\sqrt{\delta^{2} \frac{y}{\rho} T}} \right) \right) f_{Y_{T/2}^{0}}(y) \ dy, \\ p(\tilde{\mathbf{b}}) &= \mathbf{P} \left[ Y_{T/2}^{1} + \frac{Z^{1}}{\gamma} > 0, \ Y_{T/2}^{1} > 0 \right] = \mathbf{P} \left[ \frac{Z^{1}}{\gamma} > -Y_{T/2}^{1}, \ Y_{T/2}^{1} > 0 \right] \\ &= \int_{0}^{\infty} \left( 1 - \Phi \left( \frac{-y + \frac{\mathbb{E}[Z^{0}]}{\gamma}}{\sqrt{\delta^{2} \frac{\rho}{\gamma} T}} \right) \right) f_{Y_{T/2}^{1}}(y) \ dy, \end{split}$$

where  $\Phi$  is the standard normal distribution,  $f_{Y_{T/2}^0}(y) = f_{Y_{T/2}^1}(y)$  are the densities of the random variables  $Y_{T/2}^0$  and  $Y_T^1/2$ , respectively, and we have used that  $\mathbb{E}[Z^1] = -\mathbb{E}[Z^0]$ .

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Since  $\Phi$  is increasing, we consider the crucial quantities

$$z_0 := \frac{-y - \frac{\mathbb{E}[Z^0]}{\rho}}{\sqrt{\delta^2 \frac{\gamma}{\rho} \frac{T}{2}}} \quad \text{and} \quad z_1 := \frac{-y + \frac{\mathbb{E}[Z^0]}{\gamma}}{\sqrt{\delta^2 \frac{\rho}{\gamma} \frac{T}{2}}}.$$
 (16)

We have the following two cases:

- (i) Assume first that  $\mathbb{E}[Z^0] \leq 0$ , with  $\mathbb{E}[Z^0]$  from (15). Since  $1 2\rho = \frac{\lambda \mu_2}{\delta^2} (\frac{\delta^2}{\lambda \mu_2} b_0^2) = \frac{\lambda \mu_2}{\delta^2} \frac{b_1^2 b_0^2}{2} > 0$ , it holds that  $z_0 > z_1$  for all y > 0. Then, we can immediately conclude, that  $p(\mathbf{b}) < p(\tilde{\mathbf{b}})$ , and hence the strategy  $\tilde{\mathbf{b}} = (b_1, b_0)$  is better than the strategy  $\mathbf{b} = (b_0, b_1)$ .
- (ii) Assume next that E[Z<sup>0</sup>] > 0, with E[Z<sup>0</sup>] from (15). Note that since b<sub>0</sub> ≤ b<sub>1</sub> and b<sub>0</sub>, b<sub>1</sub> ∈ [0, 1], then either b<sub>0</sub> = b<sub>1</sub> = 1, in which case there is nothing to prove since **b** and **b** are equal, or it cannot hold that b<sub>0</sub> = 1. The latter implies that there always exists an y<sup>\*</sup> ∈ (0, +∞) such that for y < y<sup>\*</sup>, it holds that z<sub>0</sub> < z<sub>1</sub>, and the opposite holds true for y > y<sup>\*</sup>, see, e.g. the right panel in Figure 2.Now, we consider the functions η → b<sub>0</sub>(η) and η → b<sub>1</sub>(η) for η ∈ [0, θ]. We know that

$$b_0(\eta) + b_1(\eta) = 2 \frac{M - \lambda \mu(\theta - \eta)}{\lambda \mu \theta}$$
 and  $b_0(\eta)^2 + b_1(\eta)^2 = 2 \frac{\delta^2}{\lambda \mu_2}$ 

meaning that  $b'_0(\eta) + b'_1(\eta) = \frac{2}{\theta} > 0$  and  $b'_0(\eta)b_0(\eta) + b'_1(\eta)b_1(\eta) = 0$ . Consequently, since  $b_0 < b_1$  we get that  $b'_1(\eta) > 0$  and  $b'_0(\eta) < 0$ , that is to say,  $b_0$  is decreasing with respect to  $\eta$  and  $b_1$  is increasing. We let  $\eta = \theta$  and note that, in this case for any  $a \in [0, 1]$  it holds that

$$\bar{X}_t^b = b\left\{\lambda\mu\theta + \sqrt{\lambda\mu_2}W_t\right\} = \frac{b}{a}\bar{X}_t^a, \quad t \in [0, T].$$

Using this fact, Equation (13) and the fact that  $b_0 < b_1$ , we get

$$\begin{split} p(\mathbf{b}) &= \mathbf{P} \left[ \bar{X}_{T/2}^{b_0} > 0, \bar{X}_{T/2}^{b_0} + \hat{X}_{T/2}^{b_1} \right] = \mathbf{P} \left[ \bar{X}_{T/2}^{b_1} > 0, \frac{b_0}{b_1} \bar{X}_{T/2}^{b_0} + \hat{X}_{T/2}^{b_0} \right] \\ &= \mathbf{P} \left[ \bar{X}_{T/2}^{b_1} > 0, \frac{b_0^2}{b_1^2} \bar{X}_{T/2}^{b_1} + \hat{X}_{T/2}^{b_0} \right] \\ &< \mathbf{P} \left[ \bar{X}_{T/2}^{b_1} > 0, \bar{X}_{T/2}^{b_1} + \hat{X}_{T/2}^{b_0} \right] \\ &= p(\tilde{\mathbf{b}}), \end{split}$$

which proves the statement in case  $\eta = \theta$ . Now, we let  $\mathbf{b}(\eta)$  and  $\mathbf{\tilde{b}}(\eta)$  be, respectively, the strategies  $(b_0, b_1)$  and  $(b_1, b_0)$  corresponding to  $\eta \in (0, \theta)$ . Assume there is an  $\bar{\eta} \in (0, \theta)$  such that  $p(\mathbf{b}(\bar{\eta})) > p(\mathbf{\tilde{b}}(\bar{\eta}))$ . Then, by the intermediate value theorem there exists an  $\eta^* \in (\bar{\eta}, \theta)$  such that  $p(\mathbf{b}(\eta^*)) = p(\mathbf{\tilde{b}}(\eta^*))$ . Assume that  $\mathbf{b}(\eta^*) \neq \mathbf{\tilde{b}}(\eta^*)$ . Let  $\bar{X}$  be a random variable, independent of  $Z^1$  and  $Z^0$  with  $\bar{X} \sim N(MT, \delta^2 T)$ .

$$0 = p(\mathbf{b}(\eta^*)) - p(\mathbf{b}(\eta^*))$$
  
=  $\mathbf{P}\left[\gamma \bar{X} + Z^1 > 0, \bar{X} > 0\right] - \mathbf{P}\left[\eta \bar{X} + Z^0 > 0, \bar{X} > 0\right]$   
=  $\mathbf{P}\left[\max\left(-\frac{Z^1}{\gamma}, 0\right) < \bar{X} < \max\left(-\frac{Z^0}{\eta}, 0\right)\right] > 0.$ 



**Figure 2.** Survival density under the case  $\mathbb{E}[Z^0] \le 0$  (left panel), and under  $\mathbb{E}[Z^0] > 0$  (right panel).

The last inequality follows from the fact that  $\bar{X}$ ,  $Z^1$  and  $Z^0$  are normally distributed. Hence, this contradiction yields that  $\mathbf{b}(\eta^*) = \tilde{\mathbf{b}}(\eta^*)$ . However, since it holds that  $b'_1 > 0$  and  $b'_0 < 0$  for  $b_0 < b_1$ , that means  $b_0(\bar{\eta}) > b_0(\eta^*)$  and  $b_1(\bar{\eta}) < b_1(\eta^*)$ , contradicting  $\mathbf{b}(\eta^*) = \tilde{\mathbf{b}}(\eta^*)$ .

That concludes the fact that  $\tilde{\mathbf{b}} = (b_1, b_0)$  is always better than  $\mathbf{b} = (b_0, b_1)$ .

In the following, we discuss the situations where  $\mathbb{E}[Z^0] \leq 0$  and derive sufficient conditions for this to hold.

Lemma 4.5: If

$$\frac{\mu\eta}{\mu_2} \le \frac{M}{\delta^2} \quad and \quad \frac{\mu\theta}{\mu_2} \ge \frac{2M}{\delta^2},\tag{17}$$

then  $\mathbb{E}[Z^0] \leq 0$  with  $\mathbb{E}[Z^0]$  given in (15).

**Proof:** We observe that  $\mathbb{E}[Z_0] \leq 0$  is equivalent to

$$-\frac{\lambda\mu_2 b_0^2}{\delta^2} MT + \lambda\mu(\theta b_0 - \theta + \eta)T \le 0,$$
(18)

for all  $b_0 \in [0, 1]$ , where we have substituted  $\rho = \frac{\lambda \mu_2 b_0^2}{2\delta^2}$ . To show that (18) holds for all  $b_0 \in [0, 1]$ , we consider the function

$$F(b) = -\frac{\mu_2 b^2}{\delta^2} M + \mu(\theta b - \theta + \eta).$$

This function is concave and has a maximum at  $b^* = \frac{\mu \theta \delta^2}{2\mu_2 M} > 0$ . We observe that F(0) < 0 and that  $F(1) = -\frac{\mu_2}{\delta^2}M + \mu\eta$ , which is negative if the first of condition (17) holds true. Moreover, under the second condition in (17) we also get that  $b^* \ge 1$ , which guarantees that  $\mathbb{E}[Z] \le 0$ .

Condition (17) is meaningful in terms of insurance and reinsurance premia. Indeed, it tells us that the reinsurance is expensive and the income from the direct insurance premia is low. In this case the result of Proposition 4.4 is intuitively clear: since reinsurance is costly, choosing a bigger retention level (less reinsurance) in the first period has the advantage that a larger drift can drive the surplus

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away from zero, and hence it is better in terms of survival probability (i.e. minimises the ruin at times T/2 and T).

To better understand the different cases (i.e.  $\mathbb{E}[Z_0] \le 0$  and  $\mathbb{E}[Z_0] > 0$ ) we let

$$G_0(y) = \left(1 - \phi\left(\frac{-y - \frac{\mathbb{E}[Z^0]}{\rho}}{\sqrt{\delta^2 \frac{1-\rho}{\rho}T}}\right)\right) f_{Y_{T/2}^0}(y),$$
  
$$G_1(y) = \left(1 - \phi\left(\frac{-y + \frac{\mathbb{E}[Z^0]}{\gamma}}{\sqrt{\delta^2 \frac{1-\gamma}{\gamma}T}}\right)\right) f_{Y_{T/2}^1}(y),$$

for all y > 0, so that

$$p(\mathbf{b}) = \int_0^\infty G_0(y) \mathrm{d}y, \quad p(\tilde{\mathbf{b}}) = \int_0^\infty G_1(y) \mathrm{d}y.$$

Figure 2 represents the densities of the survival probability (i.e.  $G_0(y)$  and  $G_1(y)$ ) relative to the strategy  $\mathbf{b} = (b_0, b_1)$  (dashed line) and the strategy  $\tilde{\mathbf{b}}(b_1, b_0)$  (solid line), under given parameters.

The left panel corresponds to the case where condition (17) holds, i.e. insurance is cheap and reinsurance is expensive. Here the survival probability of the strategy  $\tilde{\mathbf{b}}$  dominates that of the strategy **b** for all values of *y*. In the right panel there exist a level  $y^* > 0$ , small, at which these two curves switch. However the area under the curve  $G_1$  in the set  $\{y > y^*\}$  largely compensates that in the set  $\{y < y^*\}$ . Such point  $y^*$  corresponds to  $y^* = \frac{2\mathbb{E}[Z^0]}{1-2\rho}$  (and it only exists in case  $\mathbb{E}[Z^0] > 0$ ). Note that, natural bounds on the value of  $\eta$ , i.e.  $0 \le \eta \le \theta$ , guarantee that such compensation of areas always applies and hence  $p(\tilde{\mathbf{b}}) > p(\mathbf{b})$ , (see Proposition 4.4).

#### 4.3. The penalisation problem

Suppose now, we have the following situation: the insurer may decide (at time zero) to have or not to have an update in the reinsurance contract at time T/2. If she updates the contract, she will pay a penalty amounting to PT at time T/2. In case of no changes, no penalty will be applied. The strategies corresponding to these two different scenarios are chosen to achieve a Gaussian distribution at time T with the same target variance  $\delta^2 T$ . If the insurer does not change the strategy at time T/2 then the mean of the net collective is M'T, uniquely determined by the condition on the target variance. In case the strategy is changed at time T/2, the final expected wealth will be M < M'. Next, we show that, due to the insurer's objective to minimise the ruin probability, changing the strategy at time T/2 is more preferable, even with a smaller expected mean.

We assume that M = M' - P. Let  $\hat{\mathbf{b}} = (\hat{b}, \hat{b})$  be the strategy where the insurer decides to make no changes at time T/2 and let  $\mathbf{b} = (b_0, b_1)$  and  $\tilde{\mathbf{b}} = (b_1, b_0)$  be the admissible strategy where the insurer switches, with  $b_0 < b_1$ . We already know, by Proposition 4.4, that strategy  $\tilde{\mathbf{b}}$  is better than **b**. The survival probability of strategy  $\hat{\mathbf{b}}$  is given by

$$p(\hat{\mathbf{b}}) = \mathbf{P}\left[\bar{X}_{T/2}^{\hat{b}} > 0, \ \bar{X}_{T}^{\hat{b}} > 0\right].$$

We let  $\hat{Y} = \bar{X}_T^{\hat{b}}$ . Then we get that  $Y \sim N(M'T, \delta^2 T)$ , and we observe that

$$\bar{X}_{T/2}^{\hat{b}} = \frac{1}{2}\hat{Y} + \hat{Z}_{T/2}$$

where

$$\hat{Z} \sim N\left(\lambda\mu(\theta\hat{b}-\theta+\eta)\frac{T}{2}-M'\frac{T}{2},\frac{1}{4}\delta^2T\right).$$



**Figure 3.** Survival probabilities under penalisation. Black line corresponds to the survival probability of the strategy  $\hat{\mathbf{b}}$ , grey line to that of  $\mathbf{b}$  and the dashed line to that of the constant strategy  $\hat{\mathbf{b}}$ .

Random variables  $\hat{Y}$  and  $\hat{Z}$  are independent. This implies that

$$p(\hat{\mathbf{b}}) = \left[2\hat{Z} > -\hat{Y}, \hat{Y} > 0\right] = \int_0^\infty \left(1 - \phi\left(\frac{-y - 2\mathbb{E}[\hat{Z}]}{\sqrt{\delta^2 T}}\right)\right) f_{\hat{Y}}(y) \, \mathrm{d}y.$$

Next, for the strategy  $\tilde{\mathbf{b}}$  the survival probability is given by:

$$p(\tilde{\mathbf{b}}) = \int_{PT}^{\infty} \left( 1 - \phi \left( \frac{-y + PT - \frac{\mathbb{E}[Z^1]}{\gamma}}{\sqrt{\delta^2 \frac{\rho}{\gamma} T}} \right) \right) f_{\hat{Y}}(y) \, \mathrm{d}y,$$

where  $Z^1 = N(\lambda \mu(\theta b_1 - \theta + \eta)T/2 - 2\gamma(M' - P)T/2, \gamma(1 - \gamma)\delta^2 T)$ , like in the proof of Proposition 4.4.

It is clear that for P = 0, there is a unique strategy,  $\hat{\mathbf{b}}$ , that leads to the desired distribution for the net collective. For P > 0, however the strategy  $\hat{\mathbf{b}}$  has a survival probability that is always smaller than the survival probability of the optimal strategy  $\mathbf{b}$  and larger than that of  $\tilde{\mathbf{b}}$ . This difference is illustrated in Figure 3.

## 4.4. A 3 period model

To explain the complexity of the problem for n > 2, we consider the case n = 3. Here, the form of the survival probability does not allow deriving conditions that ensure a clear dominance of one strategy. In addition, the computational time increases with the number of periods.

To give some intuition on how to deal with this case, we restrict to deterministic strategies  $\mathbf{b} = (b_0, b_1, b_2)$ . These strategies satisfy

$$b_{0} + b_{1} + b_{2} = 3 \frac{M + \lambda \mu (\theta - \eta)}{\lambda \mu \theta}$$

$$b_{0}^{2} + b_{1}^{2} + b_{2}^{2} = \frac{3\delta^{2}}{\lambda \mu_{2}},$$

$$b_{0}, b_{1}, b_{2} \in [0, 1],$$



**Figure 4.** Admissible deterministic strategies for n = 3 and parameters T = 1;  $\mu = 0.15$ ;  $\mu_2 = 0.06$ ;  $\lambda = 1$ ;  $\theta = 0.35$ ;  $\eta = 0.2$ ; M = 0.02;  $\delta = 0.2$ ;

which means that there are infinitely many combinations of  $(b_0, b_1, b_2)$  that lead to the target distribution. In particular, admissible triplets build (a part of) a circle as shown in Figure 4. To choose the ruin-minimising strategy we look at survival probability

$$p(\mathbf{b}) = \mathbf{P}\left[\bar{X}_{\frac{T}{3}}^{b_0} > 0, \bar{X}_{\frac{2T}{3}}^{b_0, b_1} > 0, \bar{X}_T^{\mathbf{b}} > 0\right].$$

We define auxiliary random variables  $\zeta^0$ ,  $\zeta^1$  such that

$$\begin{split} \zeta^0 &\sim N\left(\lambda\mu(\theta b_0 - \theta + \eta)\frac{T}{3} - M\rho_0 T, \ \rho_0(1 - \rho_0)\delta^2 T\right),\\ \rho_2 \zeta^0 &+ \zeta^1 &\sim N\left(-\lambda\mu\theta b_2\frac{T}{3} + \lambda\mu(\theta - \eta)T + (1 - \rho_1)MT, \ \rho_1(1 - \rho_1)\delta^2 T\right), \end{split}$$

which are correlated. Then, we have that

$$p(\mathbf{b}) = \mathbf{P}\left[\frac{\zeta^{0}}{\rho_{0}} > -\bar{X}_{T}^{\mathbf{b}}, \frac{\rho_{2}\zeta^{0} + \zeta^{1}}{\rho_{1}} > -\bar{X}_{T}^{\mathbf{b}}, \bar{X}_{T}^{\mathbf{b}} > 0\right]$$
$$= \int_{0}^{\infty} \mathbf{P}\left[\frac{\zeta^{0}}{\rho_{0}} > y, \frac{\rho_{2}\zeta^{0} + \zeta^{1}}{\rho_{1}} > y\right] f_{Y}(y) \, \mathrm{d}y.$$

where  $Y \sim N(MT, \delta^2 T)$ , and  $f_Y(y)$  is the corresponding density.

Figure 5 shows the survival probability with respect to the first component  $b_0$ . It is clear that, once  $b_0$  is chosen, there are only two possible choices for  $b_1$  and  $b_2$ . Suppose that, for instance  $b_1 > b_2$ , then for a fixed  $b_0$ , the possible strategies are  $(b_0, b_1, b_2)$  and  $(b_0, b_2, b_1)$ . We see that the survival probability is maximised by the largest  $b_0$  and the combination that leads to the higher survival probability is the sorted one, i.e.  $(b_0, b_1, b_2)$  with  $b_0 > b_1 > b_2$ .

We conclude the section by showing that the sorted sequence  $\mathbf{b} = (b_0, b_1, b_2)$  leads to a bigger survival probability than the 'unsorted' sequence  $\tilde{\mathbf{b}} = (b_0, b_2, b_1)$ . This means, in particular, that as shown in Figure 5 any unsorted sequence will be overperformed by a sorted one. To prove this, we denote by  $p^x(\cdot)$  the survival probability of a strategy where *x* is the initial capital. Then,

$$p(\mathbf{b}) = \mathbf{P}\left[\bar{X}_{T/3}^{b_0} > 0, \bar{X}_{2/3T}^{(b_0,b_1)} > 0, \bar{X}_T^{(b_0,b_1,b_2)} > 0\right] = \mathbb{E}\left[\mathbf{I}_{\bar{X}_{T/3}^{b_0} > 0} p^{\bar{X}_{T/3}^{b_0}}((b_1,b_2))\right]$$



**Figure 5.** Survival probabilities as functions of the first component  $b_0$  of the strategy  $(b_0, b_1, b_2)$ .

> 
$$\mathbb{E}\left[\mathbf{I}_{\tilde{X}_{T/3}^{b_0}>0} p^{\tilde{X}_{T/3}^{b_0}}((b_2, b_1))\right] = p(\tilde{\mathbf{b}}),$$

where the inequality follows from the case n = 2.

#### 5. Conclusions

In this paper, we consider an insurance company whose objective is to choose a dividend payment or a reinsurance strategy leading to a certain surplus distribution. The question which strategy to prefer depends on the underlying target functional – the value of expected discounted dividends (to be maximised) or the ruin probability (to be minimised). Such a problem is motivated by the necessity of being able to compute risk measures, typically based on the distribution of a future loss at some fixed date. Fixing a terminal wealth distribution would allow computing several risk measures at once, instead of choosing a specific constraint in the beginning of an optimisation task. In this line, the present paper represents the first step towards a more detailed and more realistic analysis of the problems faced by practitioners on the almost daily basis.

In addition, we would like to stress that the dividend related problems can be easily generalised to a continuous time framework. In the reinsurance setting we are able to fully analyse the 2-period problem. Since, reinsurance contracts are usually difficult or even impossible to update before the maturity date, this setting seems to be the most realistic one from a practical point of view.

Using a pool of possible distributions and mean/variance combinations instead of only one specific distribution is a possible extension direction. However, in this paper our main target is to introduce an idea and to illustrate with two clear settings how this idea can be implemented. For instance, in the reinsurance setting, the discrete nature of the problem does not allow using the differential equation approach. Any return function would depend on the initial surplus and on time. Changing the length of one interval would completely change the optimal strategy, as more weight will be put on the remaining intervals. Therefore, we are using purely probabilistic methods to prove our claims for the 2-period case. In the general n-period setting, the admissible strategies are not necessarily deterministic, and the optimal strategy may even not exist.

Our future research will concentrate on the extensions of the presented models. We plan to work on the n-period model for the reinsurance setting. We will also consider problems with non-normal target distributions and allow for continuous time ruin-checks. 678 👄 K. COLANERI ET AL.

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