

# Szegő's theorem on a Jordan arc

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## Preliminaries

Let  $\Gamma$  be a Jordan arc of class  $C^{2+}$ . Let  $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ . We define

$$\Phi : \mathbb{C}_\infty \setminus \Gamma \rightarrow \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$$

to be the conformal mapping satisfying  $\Phi(\infty) = \infty$  and  $\Phi'(\infty) > 0$ . Since  $\Gamma$  is an arc,  $\Phi$  extends to the two sides as  $\Phi_+$  and  $\Phi_-$ .

Let  $\omega_{z_0}$  be the **harmonic measure** of  $\Omega = \mathbb{C}_\infty \setminus \Gamma$  and the point  $z_0 \in \Omega$ .

All harmonic measures are absolutely continuous with respect to the arclength  $dz$ . Let  $\rho_{z_0}$  be the corresponding density, i.e.

$$\omega_{z_0} = \rho_{z_0} dz.$$

## Szegő-function

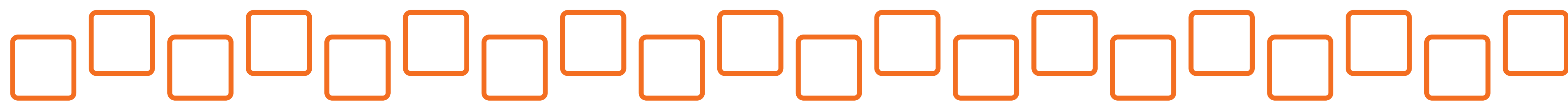
We say that a positive and finite measure  $\mu = f d\omega_{z_0} + \mu_s$  where  $\mu_s$  is singular w.r.t.  $dz$  satisfies the **Szegő-condition** if

$$\int_{\Gamma} \log f d\omega_{z_0} > -\infty.$$

In this case we can introduce the Szegő-function for  $\mu$ . Let

$$R_\mu(z) = \exp \left( \frac{1}{2} \int_{\Gamma} \log(f \rho_{z_0}) d\omega_z + \frac{1}{2} i * \int_{\Gamma} \log(f \rho_{z_0}) d\omega_z \right)$$

where  $i * \int \log(f \rho_{z_0}) d\omega_z$  is the harmonic conjugate such that  $R_\mu(z_0) > 0$ . Then  $R_\mu$  is nonzero, analytic in  $\Omega$  and  $|R_\mu(z)|^2 = f(z) \rho_{z_0}(z)$  for  $z \in \Gamma$  a.e. Similarly, we define  $R_f$  as nonzero, analytic in  $\Omega$  and  $|R_f|^2 = f$  on  $\Gamma$  a.e.



## The Hardy-space $H^2(\Omega, \mu)$

Let  $\Psi$  be the inverse of  $\Phi$ . If  $\mu$  satisfies the Szegő-condition then the corresponding Hardy space is defined by

$$H^2(\Omega, \mu) := \left\{ F \in \text{Hol}(\Omega) : F(\Psi) \frac{R_\mu(\Psi)}{\sqrt{\Phi'(\Psi)}} \in H^2(\mathbb{C}_\infty \setminus \overline{\mathbb{D}}) \right\}.$$

with the norm

$$\|F\|_{H^2(\Omega, \mu)} := \left( \int_{\Gamma} |F|^2 f d\omega_{z_0} \right)^{1/2} = \left( \int_{\Gamma} (|F_+|^2 + |F_-|^2) f d\omega_{z_0} \right)^{1/2}.$$

$H^2(\Omega, \mu)$  is a **reproducing kernel Hilbert space** with the kernel

$$K_\mu(z, w) = \frac{1}{2\pi} \cdot \frac{\sqrt{\Phi'(z)}}{R_\mu(z)} \cdot \frac{\overline{\sqrt{\Phi'(w)}}}{\overline{R_\mu(w)}} \cdot \frac{1}{1 - 1/(\Phi(z)\overline{\Phi(w)})}.$$

## The $H^2(\Omega, \mu)$ problem

If  $\mu$  satisfies the Szegő-condition and  $z_0 \in \mathbb{C}_\infty \setminus \Gamma$  we define

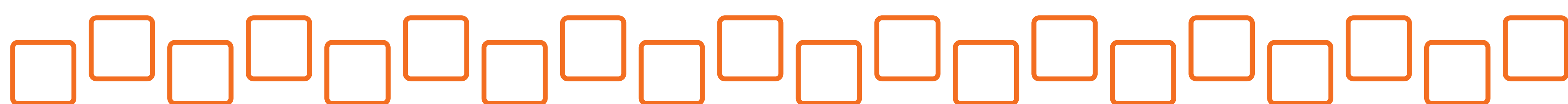
$$\nu(\mu, z_0) := \inf \left\{ \int_{\Gamma} |F|^2 f d\omega_{z_0} : F \in H^2(\Omega, \mu), F(z_0) = 1 \right\}.$$

where the infimum is uniquely attained by  $F_\mu = \frac{K_\mu(\cdot, z_0)}{K_\mu(z_0, z_0)}$ . Then

$$\nu(\mu, z_0) = 2\pi(1 - |\Phi(z_0)|^{-2}) \frac{R_\mu(z_0)^2}{|\Phi'(z_0)|} = \frac{R_f(z_0)^2}{K_{\omega_{z_0}}(z_0, z_0)},$$

and

$$F_\mu(z) = \frac{1 - |\Phi(z_0)|^{-2}}{1 - (\Phi(z)\overline{\Phi(z_0)})^{-1}} \sqrt{\frac{\Phi'(z) R_\mu(z_0)}{\Phi'(z_0) R_\mu(z)}}.$$



## The Christoffel-function

Let  $\Gamma \in C^{2+}$  be a Jordan arc,  $\mu$  be a positive and finite measure with  $\text{supp}(\mu) \subseteq \Gamma$  infinite and  $z_0 \in \mathbb{C} \setminus \Gamma$ . Then we define

$$\lambda_n(\mu, z_0) = \inf \left\{ \int_{\Gamma} |p|^2 d\mu : p \text{ is a polynomial of degree } \leq n, p(z_0) = 1 \right\}.$$

The unique minimizer  $P_{n, z_0}$  is called the  $n$ -th **minimizing polynomial** of the measure  $\mu$ .

For the point  $\infty$  we define

$$\lambda_n(\mu, \infty) = \inf \left\{ \int_{\Gamma} |p|^2 d\mu : p \text{ is a monic polynomial of degree } n \right\}.$$

The unique minimizer  $P_{n, \infty}$  is the  $n$ -th monic **orthogonal polynomial** of the measure  $\mu$ .

For  $z_0 \in \mathbb{C}_\infty \setminus \Gamma$  we define

$$C(\Gamma, z_0) := \begin{cases} 1/\Phi(z_0) & , z_0 \neq \infty \\ 1/\Phi'(\infty) & , z_0 = \infty. \end{cases}$$

## Theorem (Szegő-Widom asymptotics)

Let  $\Gamma \in C^{2+}$  be a Jordan arc,  $\mu = f d\omega_{z_0} + \mu_s$  a positive and finite measure supported on  $\Gamma$  with singular part  $\mu_s$  and  $z_0 \in \mathbb{C}_\infty \setminus \Gamma$ . Then the limit  $\lambda_\infty(\mu, z_0) := \lim_{n \rightarrow \infty} |C(\Gamma, z_0)|^{-2n} \lambda_n(\mu, z_0)$  exists and

$$\lambda_\infty(\mu, z_0) \neq 0 \iff \int_{\Gamma} \log f d\omega_{z_0} > -\infty.$$

If  $\mu$  satisfies the Szegő-condition, then

$$|C(\Gamma, z_0)|^{-2n} \lambda_n(\mu, z_0) \xrightarrow{n \rightarrow \infty} \nu(\mu, z_0),$$

$$\int_{\Gamma} |C(\Gamma, z_0)^{-n} P_{n, z_0} - H_n|^2 f d\omega_{z_0} \xrightarrow{n \rightarrow \infty} 0,$$

$$\frac{C(\Gamma, z_0)^{-n}}{\Phi(z)^n} P_{n, z_0}(z) \xrightarrow{n \rightarrow \infty} F_\mu(z)$$

locally uniform in  $\Omega$ , where

$$H_n(z) = \Phi_+(z)^n F_\mu^+(z) + \Phi_-(z_0)^n F_\mu^-(z).$$

## Corollary

The harmonic measure  $\omega_{z_0}$  is the unique maximizer of  $\lambda_\infty(\mu, z_0)$  over all probability measures  $\mu$ .

