Szegő's theorem on a Jordan arc

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Preliminaries

Let Γ be a Jordan arc of class C^{2+}. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We define

$$\Phi : \mathbb{C}_\infty \backslash \Gamma \to \mathbb{C}_\infty \backslash \overline{\mathbb{D}}$$

to be the conformal mapping satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.
Since Γ is an arc, Φ extends to the two sides as Φ_+ and Φ_-.
Let ω_{z_0} be the **harmonic measure** of $\Omega = \mathbb{C}_\infty \backslash \Gamma$ and the point $z_0 \in \Omega$.
All harmonic measures are absolutely continuous with respect to the arclength dz. Let ρ_{z_0} be the corresponding density, i.e.

$$\omega_{z_0} = \rho_{z_0} dz.$$

Szegő-function

We say that a positive and finite measure $\mu = f d\omega_{z_0} + \mu_s$ where μ_s is singular w.r.t. dz satisfies the **Szegő-condition** if

$$\int_\Gamma \log f \, d\omega_{z_0} > -\infty.$$

In this case we can introduce the Szegő-function for μ. Let

$$R_\mu(z) = \exp\left(\frac{1}{2}\int_\Gamma \log(f\rho_{z_0})\, d\omega_z + \frac{1}{2} i * \int_\Gamma \log(f\rho_{z_0})\, d\omega_z\right)$$

where $i * \int \log(f\rho_{z_0}) d\omega_z$ is the harmonic conjugate such that $R_\mu(z_0) > 0$.
Then R_μ is nonzero, analytic in Ω and $|R_\mu(z)|^2 = f(z)\rho_{z_0}(z)$ for $z \in \Gamma$ a.e.
Similarly, we define R_f as nonzero, analytic in Ω and $|R_f|^2 = f$ on Γ a.e.

The Hardy-space $H^2(\Omega, \mu)$

Let Ψ be the inverse of Φ. If μ satisfies the Szegő-condition then the corresponding Hardy space is defined by

$$H^2(\Omega,\mu) := \left\{ F \in \mathrm{Hol}(\Omega) : F(\Psi)\frac{R_\mu(\Psi)}{\sqrt{\Phi'(\Psi)}} \in H^2(\mathbb{C}_\infty \backslash \overline{\mathbb{D}}) \right\}.$$

with the norm

$$\|F\|_{H^2(\Omega,\mu)} := \left(\oint_\Gamma |F|^2 f d\omega_{z_0}\right)^{1/2} = \left(\int_\Gamma (|F_+|^2 + |F_-|^2) f d\omega_{z_0}\right)^{1/2}.$$

$H^2(\Omega,\mu)$ is a **reproducing kernel Hilbert space** with the kernel

$$K_\mu(z,w) = \frac{1}{2\pi} \cdot \frac{\sqrt{\Phi'(z)}}{R_\mu(z)} \cdot \frac{\overline{\sqrt{\Phi'(w)}}}{\overline{R_\mu(w)}} \cdot \frac{1}{1 - 1/(\Phi(z)\overline{\Phi(w)})}.$$

The $H^2(\Omega, \mu)$ problem

If μ satisfies the Szegő-condition and $z_0 \in \mathbb{C}_\infty \backslash \Gamma$ we define

$$\nu(\mu, z_0) := \inf\left\{ \oint_\Gamma |F|^2 f d\omega_{z_0} : F \in H^2(\Omega,\mu), F(z_0) = 1 \right\}.$$

where the infinum is uniquely attained by $F_\mu = \frac{K_\mu(\cdot, z_0)}{K_\mu(z_0,z_0)}$. Then

$$\nu(\mu,z_0) = 2\pi(1 - |\Phi(z_0)|^{-2})\frac{R_\mu(z_0)^2}{|\Phi'(z_0)|} = \frac{R_f(z_0)^2}{K_{\omega_{z_0}}(z_0,z_0)},$$

and

$$F_\mu(z) = \frac{1 - |\Phi(z_0)|^{-2}}{1 - \left(\Phi(z)\overline{\Phi(z_0)}\right)^{-1}} \sqrt{\frac{\Phi'(z)}{\Phi'(z_0)}} \frac{R_\mu(z_0)}{R_\mu(z)}.$$

The Christoffel-function

Let $\Gamma \in C^{2+}$ be a Jordan arc, μ be a positive and finite measure with $\mathrm{supp}(\mu) \subseteq \Gamma$ infinite and $z_0 \in \mathbb{C} \backslash \Gamma$. Then we define

$$\lambda_n(\mu, z_0) = \inf\left\{ \int_\Gamma |p|^2 d\mu : p \text{ is a polynomial of degree} \le n, p(z_0) = 1 \right\}.$$

The unique minimizer P_{n,z_0} is called the n-th **minimizing polynomial** of the measure μ.
For the point ∞ we define

$$\lambda_n(\mu, \infty) = \inf\left\{ \int_\Gamma |p|^2 d\mu : p \text{ is a monic polynomial of degree } n \right\}.$$

The unique minimizer $P_{n,\infty}$ is the n-th monic **orthogonal polynomial** of the measure μ.
For $z_0 \in \mathbb{C}_\infty \backslash \Gamma$ we define

$$C(\Gamma, z_0) := \begin{cases} 1/\Phi(z_0) & , z_0 \neq \infty \\ 1/\Phi'(\infty) & , z_0 = \infty. \end{cases}$$

Theorem (Szegő-Widom asymptotics)

Let $\Gamma \in C^{2+}$ be a Jordan arc, $\mu = f d\omega_{z_0} + \mu_s$ a positive and finite measure supported on Γ with singular part μ_s and $z_0 \in \mathbb{C}_\infty \backslash \Gamma$. Then the limit $\lambda_\infty(\mu, z_0) := \lim_{n\to\infty} |C(\Gamma, z_0)|^{-2n} \lambda_n(\mu, z_0)$ exists and

$$\lambda_\infty(\mu, z_0) \neq 0 \iff \int_\Gamma \log f \, d\omega_{z_0} > -\infty.$$

If μ satisfies the Szegő-condition, then

$$|C(\Gamma, z_0)|^{-2n}\lambda_n(\mu, z_0) \xrightarrow{n\to\infty} \nu(\mu, z_0),$$

$$\int_\Gamma |C(\Gamma, z_0)^{-n} P_{n,z_0} - H_n|^2 f d\omega_{z_0} \xrightarrow{n\to\infty} 0,$$

$$\frac{C(\Gamma, z_0)^{-n}}{\Phi(z)^n} P_{n,z_0}(z) \xrightarrow{n\to\infty} F_\mu(z)$$

locally uniform in Ω, where

$$H_n(z) = \Phi_+(z)^n F_\mu^+(z) + \Phi_-(z_0)^n F_\mu^-(z).$$

Corollary

The harmonic measure ω_{z_0} is the unique maximizer of $\lambda_\infty(\mu, z_0)$ over all probability measures μ.

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