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ABSTRACT

Modeling and analysis of multicomponent systems for gas mixtures

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The aim of this thesis is to understand and analyze diffussive and thermal effects in multicomponent systems for gas mixtures through the perspective of partial differential equations. Starting from Class–II models of thermodynamics, diffusion equations are derived formally by a Chapman–Enskog expansion and the expansion is justified as a relaxation limit by means of the relative entropy method. The resulting model is analyzed and compared with related models from the literature and a comparison among thermomechanical theories is presented. In the case of zero mean flow, the system reduces to a cross–diffusion system of Maxwell–Stefan type. In the isothermal case, renormalized solutions are employed to prove the uniqueness of weak solutions and an energy study is performed, showing the absence of anomalous dissipation. In the nonisothermal case, the global–in–time existence of weak solutions is studied, using the boundedness–by–entropy method and it is shown that strong solutions are unique in the class of weak solutions, a property known as weak–strong uniqueness.

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السلام عليكم

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Chapter 1

Introduction

Multicomponent fluid systems are prevalent in both natural and industrial settings. Examples include Earth's atmosphere with its diverse composition of nitrogen, oxygen, argon, carbon dioxide, and other gases, as well as natural gas composed of hydrocarbons like methane, ethane, and propane. The ubiquity of such multicomponent fluids underscores the importance of comprehensively modeling and predicting their behavior.

The complexity of models for these systems can vary based on the specific phenomenon and application, with a trade–off between detailed modeling and practical limitations in obtaining experimental data. Finding the right balance is crucial; a theory must be detailed enough to accurately describe a phenomenon yet not overly intricate to hinder comprehension. The modeling of multicomponent fluids has been extensively explored, and interested readers can refer to [1, 2, 3, 4, 5, 6] for diverse approaches.

Analytical results on multicomponent flows extend in various directions. We refer to [7, 8, 9, 10, 11, 12] for existence and uniqueness studies of strong and/or weak solutions for Maxwell–Stefan systems of mass diffusion, situations that involve no mean flow and pertain to the general area of parabolic systems. By contrast, situations that involve mean flow lead to questions in the realm of hyperbolic or hyperbolic–parabolic systems. There are available analyses for isothermal viscous flows of multicomponent systems [13] and even for multicomponent compressible Euler flows [14, 15, 16, 17]. For analyses of nonisothermal multicomponent systems that include effects of heat–conduction we refer to [18, 19, 20, 21, 22, 11]. The above works concern the mathematical structure of multicomponent systems and existence of solutions for steady and dynamic problems. There has been recent interest in the convergence from compressible multicomponent Euler equations in the high–friction limit to Maxwell–Stefan systems, a problem pertaining to the subject of relaxation approximations. A number of studies have appeared regarding isothermal flows [23, 24, 25, 15] achieving in the limit the classical Maxwell–Stefan system [23], or (for more general chemical potentials) porous media variants or even fourth order diffusions [25, 15]. For nonisothermal flows, the results are limited [26] and achieve in the limit the nonisothermal analogue of the Maxwell–Stefan system, known as Maxwell–Stefan–Fourier.

This thesis focuses on presenting modeling and analysis results for multicomponent systems. The approach involves developing general theories (Class–II models) consistent with thermodynamic laws. Asymptotic techniques are then employed to transition to approximate (Class–I) models, balancing accuracy and analytical tractability. The transition is achieved through asymptotic expansions (Chapman–Enskog expansion), which are formal, but can be made rigorous if we see them as relaxation processes and exploit the dissipative structure of the models (Relative entropy method). The resulting (Class–I) model fits into a specific class of problems (hyperbolic–parabolic type), which provides a well–posed theory. This makes possible the development of a hierarchy of models among thermomechanical theories. The thesis further delves into the parabolic part of the model, addressing issues such as the existence and uniqueness of weak solutions and the absence of anomalous dissipation.

The thesis structure is as follows: Chapters 2 to 4 introduce the modeling of multicomponent systems. Chapter 2 focuses on the thermodynamics of Class–II models (Section 2.1) and their derivation (Section 2.2). In chapter 3 we perform some formal asymptotics to derive a Class–I model (section 3.1) and we justify the process (Section 3.2) by means of the relative entropy method. Chapter 4 analyzes the resulting Class–I model, demonstrating its fit into the theory of hyperbolic–parabolic composite type systems (Section 4.1), and presenting a hierarchy of thermomechanical theories for strong solutions (Section 4.2). Chapters 5 and 6 delve into the analysis of parabolic systems of gas mixtures, focusing on the Maxwell–Stefan cross–diffusion system. Chapter 5 studies the isothermal Maxwell–Stefan system, discussing uniqueness issues (Section 5.1) and the absence of turbulence (Section 5.2). Finally, Chapter 6 explores the non–isothermal analogue, known as Maxwell–Stefan–Fourier, showing the global–in–time existence of weak solutions (Section 6.2) and their weak–strong uniqueness (Section 6.3).

Chapter 2

Modeling of multicomponent systems of fluids

This chapter is an extended version of [22, App. A and B]. It presents the natural modeling framework of multicomponent models and outline its thermodynamic structure. In section 2.1, following [27], we describe the fundamentals of the thermodynamic theory in equilibrium, known as thermostatics. In section 2.2 we present the results of [6] through the scope of thermodynamic reduction, which allows for the derivation of closures that are consistent with the second law of thermodynamics in the form of the Clausius–Duhem inequality.

2.1 Equilibrium thermodynamics

Thermostatics seeks to describe the equilibrium states to which systems eventually evolve. We assume that these states exist and are characterized completely (at least macroscopically) by the internal energy of the system E, the volume V and the mole numbers $M_1, \ldots, M_n$ of the components of the system. These quantities are called extensive parameters.

As in many physical theories, the criterion for determining the equilibrium state is an extremum principle. More precisely, we postulate that there exists a function H of the extensive parameters, called entropy, defined for all equilibrium states, so that the values assumed by the extensive parameters in the absence of internal constraints, are those that maximize the entropy over the manifold of constrained equilibrium states. Note that the existence of the entropy is postulated only for equilibrium states. For nonequilibrium thermodynamics, we do not have information regarding what happens exactly, so we need to place some further assumptions. For the moment we deal with equilibrium, but we will come back to this in section 2.2.

Based on our setting above, any problem in thermodynamics can be completely solved by the extremum principle provided that the entropy of the system is known as a function of the extensive parameters. The relation that gives the entropy as a function of the extensive parameters is known as fundamental relation

$$H = H(E, V, M_1, \dots, M_n) \tag{2.1}$$

and contains all thermodynamic information about the system.

The next postulate, as we are expecting, is that the entropy of a composite system is additive over the constituent subsystems. Moreover, it is differentiable and increasing as a function of the internal energy. Interestingly enough, the additivity property implies that the entropy is homogeneous of first order, i.e.

$$H(\lambda E, \lambda V, \lambda M_1, \dots, \lambda M_n) = \lambda H(E, V, M_1, \dots, M_n) \quad \forall \lambda \in \mathbb{R}. \tag{2.2}$$

The differentiability of the entropy, along with the fact that it is an increasing function of the internal energy suggest that (2.1) can be inverted with respect to the energy. Thus, the energy is a differentiable function of $H, V, M_1, \dots, M_n$ and an alternative form of the fundamental relation is

$$E = E(H, V, M_1, \dots, M_n), \tag{2.3}$$

for which the homogeneity of first order applies as well.

The last postulate is that the entropy of any system vanishes in the state for which

$$\frac{\partial E}{\partial H} = 0, \tag{2.4}$$

that is, as we will see in a few lines, at zero temperature (third law of thermodynamics).

With the aforementioned postulates we are ready to proceed to the thermodynamic analysis. Since the energy is a quantity that is easier to understand intuitively and also to measure in experiments, we prefer to use the energy representation (2.3) of the fundamental relation, instead of (2.1). In this case, the extremum principle is no longer the maximization of the entropy in equilibrium, but rather the minimization of the energy. The condition that guarantees the equivalence between the two principles is the fact that the entropy is an increasing function of the energy.

2.1.1 Equations of state

Using (2.3) we write down the differential of the internal energy E:

$$dE = \frac{\partial E}{\partial H} dH + \frac{\partial E}{\partial V} dV + \sum_{j=1}^{n} \frac{\partial E}{\partial M_j} dM_j. \tag{2.5}$$

The partial derivatives in (2.5) are called intensive parameters and occur so frequently that we introduce special symbols for them:

$$\frac{\partial E}{\partial H} = \theta \quad \text{the temperature,} \tag{2.6}$$

$$\frac{\partial E}{\partial V} = -p \quad \text{the negative (total) pressure,} \tag{2.7}$$

$$\frac{\partial E}{\partial M_i} = \mu_i \quad \text{the chemical potential of the } i\text{–th component.} \tag{2.8}$$

We briefly mention here that in the (equivalent) entropy representation (2.1) the intensive parameters are

$$\frac{\partial H}{\partial E} = \frac{1}{\theta}, \qquad \frac{\partial H}{\partial V} = \frac{p}{\theta}, \qquad \frac{\partial H}{\partial M_i} = -\frac{\mu_i}{\theta} \tag{2.9}$$

as in [6].

With the notation of (2.6)–(2.8), (2.5) reads:

$$dE = \theta dH - p dV + \sum_j \mu_j dM_j. \tag{2.10}$$

In the special case of constant mole numbers (2.10) reduces to

$$dE = \theta dH - pdV, \tag{2.11}$$

which compared to the conservation of energy

$$dQ = dE + pdV, \tag{2.12}$$

where Q is the heat flux, gives

$$dQ = \theta dH, \tag{2.13}$$

which says that flux of heat into a system is associated with an increase of entropy. (2.13) is another relation that holds only in equilibrium and as we will see in section 2.2, the entropy flux far from equilibrium contains extra terms related to diffusion of matter.

Relations (2.6)–(2.8) define a set of functional relations known as equations of state

$$\theta = \theta(H, V, M_1, \ldots, M_n), \tag{2.14}$$

$$p = p(H, V, M_1, \ldots, M_n), \tag{2.15}$$

$$\mu_i = \mu_i(H, V, M_1, \ldots, M_n). \tag{2.16}$$

Knowledge of all the equations of state is equivalent to knowing the fundamental equation. Furthermore, the homogeneity of first order of the fundamental equation implies that the equations of state are homogeneous of order zero.

By exploiting the extremum principle in either of its form, we observe that the intensive parameters can be seen as potentials for fluxes: the temperature can be looked upon as a potential for heat flux, the pressure for volume changes and the chemical potentials for matter flow (diffusion). In other words, a difference in temperature provides a generalized force for heat flow, a difference in pressure provides a generalized force for change of volume and a difference in the chemical potential provides a generalized force for matter flow, phase transition and chemical reactions. In fact, further analysis, shows that the direction of the flows agrees with our intuitive concept: heat flows from regions of high temperature to regions of low temperature. Likewise, matter tends to flow from regions of high chemical potential to regions of low chemical potential.

2.1.2 Gibbs–Duhem relation

Further exploitation of the first–order homogeneity of the fundamental equation

$$E(\lambda H, \lambda V, \lambda M_1, \dots, \lambda M_n) = \lambda E(H, V, M_1, \dots, M_n) \quad \forall \lambda \in \mathbb{R} \tag{2.17}$$

provides an essential relation that will be used later in section 2.2. Differentiation of (2.17) with respect to λ gives

$$\frac{\partial E}{\partial(\lambda H)} H + \frac{\partial E}{\partial(\lambda V)} V + \sum_j \frac{\partial E}{\partial(\lambda M_j)} M_j = E. \tag{2.18}$$

By choosing $\lambda = 1$ and using (2.6)–(2.8), we obtain

$$E = \theta H - pV + \sum_j \mu_j M_j. \tag{2.19}$$

Equation (2.19) is known as Euler equation, even though many authors refer to it as Gibbs–Duhem relation and throughout this text we keep this name so as to not have any misunderstandings with other texts. Taking the infinitesimal variation of (2.19) we find

$$dE = \theta dH + H d\theta - pdV - V dp + \sum_j \mu_j dM_j + \sum_j M_j d\mu_j, \tag{2.20}$$

which after comparison with (2.5) yields (the real) Gibbs–Duhem relation

$$H d\theta - V dp + \sum_j M_j d\mu_j = 0. \tag{2.21}$$

The (real) Gibbs–Duhem relation provides the relations among the intensive parameters in differential form. Integration of (2.20) gives the relation in explicit form, provided one knows the equations of state. If two equations of state are given, one can integrate the Gibbs–Duhem relation (2.20) to obtain the third one up to an undetermined constant of integration. In other words, two equations of state are sufficient to determine the fundamental equation, except for an undetermined constant.

Back to the homogeneity of (2.17), we can choose $\lambda = \frac{1}{V}$ and by introducing the partial mass densities $\rho_i = \frac{M_i}{V}$ and thus the total mass density $\rho = \frac{1}{V}$ (normalizing the total mass $\sum_i M_i = 1$), (2.3) reads

$$\rho e(\rho\eta, \rho_1, \dots, \rho_n) = \frac{1}{V} E(H, V, M_1, \dots, M_n), \tag{2.22}$$

where ρe is the specific internal energy and $\rho\eta$ the specific entropy. Then

$$\theta = \frac{\partial E}{\partial H} = V \frac{\partial(\rho e)}{\partial(\rho\eta)} \frac{\partial(\rho\eta)}{\partial H} = \frac{\partial(\rho e)}{\partial(\rho\eta)} \tag{2.23}$$

and

$$\mu_i = \frac{\partial E}{\partial M_i} = V \frac{\partial(\rho e)}{\partial \rho_i} \frac{\partial \rho_i}{\partial M_i} = \frac{\partial(\rho e)}{\partial \rho_i}. \tag{2.24}$$

Using this notation, equations (2.19) and (2.21) read

$$\rho e = \rho\eta\theta - p + \sum_j \rho_j \mu_j \tag{2.25}$$

and

$$\rho\eta d\theta - dp + \sum_j \rho_j d\mu_j = 0 \tag{2.26}$$

respectively.

2.1.3 Thermodynamic potentials

In thermodynamics, we often need to pass from a set of variables that contains only extensive parameters to another that contains some intensive parameters as well. For instance, in the analysis of later sections we assume that the temperature is one of the prime variables instead of the entropy that we have considered so far (mainly for reasons related to the measurability of the prime variables). To do so, we use partial Legendre transforms of the internal energy that replace some extensive variables by the corresponding intensive ones. These partial Legendre transforms are called thermodynamic potentials. A well–known example is the enthalpy: the enthalpy is the partial Legendre transform of the internal energy E that replaces the volume V by the pressure p as an independent variable. Thus, the enthalpy h is a function of $H, p, M_1, \ldots, M_n$, defined by

$$h = E + pV, \tag{2.27}$$

where the replacement is achieved by first solving the relation

$$\frac{\partial E}{\partial V} = -p \tag{2.28}$$

with respect to V and then eliminating V from (2.27).

Another famous example is the Gibbs free energy G, which is the partial Legendre transform of the internal energy E that simultaneously replaces the entropy H by the temperature θ and the volume V by the pressure p as independent variables. The Gibbs free energy is given by

$$G = E - \theta H + pV \tag{2.29}$$

and depends on the variables $\theta, p, M_1, \ldots, M_n$. Note that the sign in front of the product pV is plus, because the intensive parameter is $-p$! As before, the replacement of the extensive parameters is done by first solving the relations $\frac{\partial E}{\partial H} = \theta$ with respect to H and $\frac{\partial E}{\partial V} = -p$ with respect to V and eliminating H and V from (2.29).

In our case, we are interested in passing from the set $\{H, V, M_1, \dots, M_n\}$ to $\{\theta, V, M_1, \dots, M_n\}$, i.e. to replace the entropy by the temperature. The thermodynamic potential that allows us to do so is the Helmholtz free energy defined by

$$\Psi = E - H\theta. \tag{2.30}$$

The replacement is achieved by solving

$$\frac{\partial E}{\partial H} = \theta \tag{2.31}$$

with respect to H and eliminating H from (2.30). Using the homogeneity of the fundamental equation as in (2.22) we obtain the specific Helmholtz free energy $\rho\psi = \rho\psi(\theta, \rho_1, \dots, \rho_n)$ with $\rho\psi = \rho e - \rho\eta\theta$, where $e(\theta, \rho_1, \dots, \rho_n) = -\theta^2 \left(\frac{\psi}{\theta}\right)_\theta$ and (2.25) reads

$$\rho\psi + p = \sum_j \rho_j \mu_j. \tag{2.32}$$

2.1.4 Thermodynamic stability

Our first postulate stated that the entropy attains its maximum in equilibrium. This is translated as follows: the first variation must vanish and the second one must be negative. The second condition in particular, determines the stability of the predicted equilibrium states and suggests that the entropy is a concave function. More precisely, the concavity of the entropy reduces to the following relations:

$$\frac{\partial^2 H}{\partial E^2} \leq 0, \quad \frac{\partial^2 H}{\partial V^2} \leq 0, \quad \frac{\partial^2 H}{\partial E^2}\frac{\partial^2 H}{\partial V^2} - \left(\frac{\partial^2 H}{\partial E \partial V}\right)^2 \geq 0. \tag{2.33}$$

A byproduct of the first inequality of (2.33) is that the specific heat at constant volume $c_v := e_\theta(\theta, \rho_1, \ldots, \rho_2)$ is positive, since

$$\frac{\partial^2 H}{\partial E^2} = -\frac{1}{T}\frac{\partial \theta}{\partial E} = -\frac{1}{NT^2 c_v} \leq 0,$$

while the remaining stability conditions place analogous restrictions on other physical quantities.

Should we reformulate the stability criteria in the energy representation, we see that the energy attains its minimum in equilibrium, hence it is a convex function, i.e.

$$\frac{\partial^2 E}{\partial H^2} \geq 0 \quad , \quad \frac{\partial^2 E}{\partial V^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial H^2}\frac{\partial^2 E}{\partial V^2} - \left(\frac{\partial^2 E}{\partial H \partial V}\right)^2 \geq 0. \tag{2.34}$$

These conditions can be extended to the thermodynamic potentials. In particular, the Helmholtz free energy is a concave function of the temperature and a convex function of the volume, with the relations

$$\frac{\partial^2 \Psi}{\partial \theta^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial V^2} \geq 0. \tag{2.35}$$

2.2 Derivation of multicomponent models

We are interested in modeling multicomponent mixtures of fluids, i.e. systems of gases or liquids that are composed of more than one constituents. Our framework is the one of continuum mechanics, in which bodies are modeled as continuous media and their behavior is governed by (i) balance laws, in the form of partial differential equations that describe the behavior of the basic variables that characterize our physical system, and (ii) constitutive relations, that complement our system of balance laws, by connecting the various quantities and characterizing the material response. For more details we refer to [28] which presents the topic from the point of view of elasticity theory and [3] and [4] from the perspective of fluid mechanics.

The primitive variables in consideration are the mass densities, the velocities and the temperatures of the constituents. Depending on the level of precision we would like to have, we distinguish among three types of models: in Class–I models, each of the constituents is described by its own mass density ρ_i, a common velocity v, which is the barycentric velocity of the mixture and a common temperature θ, which is the temperature of the mixture. In Class–II models, each of the constituents is described by its own mass density ρ_i, its own velocity v_i and a common temperature θ. Finally, in Class–III models, each of the constituents has its own mass density ρ_i, its own velocity v_i and its own temperature θ_i. For a short discussion on motivations and reasons to employ each model we refer to the introduction of [6].

To this extend, we consider a Class–II model. We employ the field equations of the basic extensive quantities, i.e. partial mass, partial momentum and partial energy balances respectively:

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = m_i, \tag{2.36}$$

$$\partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = \operatorname{div}(S_i) + \rho_i b_i + f_i, \tag{2.37}$$

$$\begin{aligned} \partial_t(\rho_i e_i + \frac{1}{2}\rho_i v_i^2) + \operatorname{div}((\rho_i e_i + \frac{1}{2}\rho_i v_i^2)v_i) &= \operatorname{div}(v_i \cdot S_i) + \operatorname{div}(q_i) \\ &\quad + \rho_i b_i \cdot v_i + \rho_i r_i + \ell_i, \end{aligned} \tag{2.38}$$

where ρ_i is the mass density of the i–th component, m_i the production of mass, v_i the velocity, S_i the partial stress tensor, b_i the body forces applied to the i–th component, f_i the momentum production, e_i the specific internal energy, q_i the heat flux, r_i the heat supply in the form of radiation and ℓ_i the internal energy production. Moreover, we define the total mass $\rho := \sum_i \rho_i$ and the barycentric velocity of the mixture v such that $\rho v = \sum_i \rho_i v_i$.

Here are some remarks on the model:

(a) The unknown quantities are the primitive variables and are functions of space and time (x,t), with the temperature being a positive function, since it represents the absolute temperature. The rest of the quantities of the model depend on the primitive variables and their derivatives, in a way that is compatible with the principle of material frame indifference and Galilean invariance.

(b) The production of mass is often due to chemical reactions, but in this paper we consider non–reactive fluids, i.e. we take $m_i = 0$. For a treatment of the same model but with chemical reactions we refer to [6].

(c) The momentum production f_i can be present because of mass production, but even more important due to the binary interactions between the species, e.g. friction. Likewise, we also consider energy production, which however is not important, because as we will see in a moment we eventually consider only the total energy balance.

(d) Even though we allow for mass, momentum and energy production, we require that the total mass, total momentum and total energy be conserved, thus having the constraints

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \ell_i = 0, \tag{2.39}$$

where $n \in \mathbb{N}$ is the number of constituents.

(e) The consideration of the terms b_i and r_i is not essential, since both are external factors and can be modified accordingly (e.g. they can be taken equal to zero), however their presence allows for a better understanding of the mechanics of the model, as well as for measurements and experiments.

(f) The nature of a Class–II model implies that we are interested only in a single energy equation, which is the balance of total energy, since we consider one temperature, the temperature of the whole mixture. For this reason, we sum up equations (2.38) over $i \in \{1, \ldots, n\}$ to obtain the total energy balance

$$\partial_t(\rho e + \sum_i \frac{1}{2}\rho_i v_i^2) + \operatorname{div}\left((\rho e + \sum_i \frac{1}{2}\rho_i v_i^2)v\right)$$
$$= \operatorname{div}\left(q - \sum_i(\rho_i e_i + p_i + \frac{1}{2}\rho_i v_i^2)u_i\right) - \operatorname{div}(pv) \quad (2.40)$$
$$+ \operatorname{div}(v_i \cdot \sigma_i) + \rho b \cdot v + \rho r + \sum_i \rho_i b_i \cdot u_i,$$

in which we have introduced the diffusional velocities $u_i := v_i - v$ and we denote $\rho e := \sum_i \rho_i e_i$, $p := \sum_i p_i$, $q := \sum_i q_i$, $\rho b := \sum_i \rho_i b_i$ and $\rho r := \sum_i \rho_i r_i$. Notice that $\sum_i \rho_i u_i = 0$.

(g) The partial stresses are decomposed as $S_i = -p_i\mathbb{I} + \sigma_i$, where $-p_i\mathbb{I}$ is the elastic part of the stress tensor, with p_i the partial pressures and $\mathbb{I}$ the identity tensor and σ_i the viscous part. Here, for simplicity, we take $\sigma_i = 0$ and thus $\operatorname{div}(S_i)$ reduces to $-\nabla p_i$. For the treatment of the theory with viscosity we refer to [6].

2.2.1 Consistency with Clausius–Duhem inequality

In order for our system to be thermodynamically complete, we add the imbalance of entropy in the form of the Clausius–Duhem inequality:

$$\partial_t(\rho\eta) + \operatorname{div}(\rho\eta v) \geq \operatorname{div}\Phi + \frac{\rho r}{\theta}, \quad (2.41)$$

where η stands for the specific entropy, Φ for the entropy flux and θ for the temperature. The fact that we have an inequality signifies that there is an additional entropy production ζ, which is not specified a priori in the context of this theory. The only information available is that according to the second law of thermodynamics, the entropy production must be nonnegative. We note here that in the case of one species the entropy flux is the heat flux divided by the temperature, however there is no available information for the multicomponent case, hence we write Φ and it needs to be determined along with the entropy production. As opposed to the balance laws of mass, momentum and energy, which determine the thermodynamic process from assigned body force b, heat supply r, boundary and initial conditions, the Clausius–Duhem inequality plays the role of an admissibility criterion for thermodynamic processes that already comply with the balance laws [29].

Having said these, we write down our model:

$$\partial_t(\rho_i) + \mathrm{div}(\rho_i v_i) = 0, \tag{2.42}$$

$$\partial_t(\rho_i v_i) + \mathrm{div}(\rho_i v_i \otimes v_i) = -\nabla p_i + \rho_i b_i + f_i, \tag{2.43}$$

$$\partial_t\left(\rho e + \sum_i \frac{1}{2}\rho_i v_i^2\right) + \mathrm{div}\left((\rho e + \sum_i \frac{1}{2}\rho_i v_i^2)v\right) = -\mathrm{div}(pv) + \rho b\cdot v \tag{2.44}$$
$$+\mathrm{div}\left((q - \sum_i(\rho_i e_i + p_i + \frac{1}{2}\rho_i v_i^2)u_i\right) + \rho r + \sum_i \rho_i b_i \cdot u_i,$$

$$\partial_t(\rho\eta) + \mathrm{div}(\rho\eta v) = \mathrm{div}\Phi + \frac{\rho r}{\theta} + \zeta. \tag{2.45}$$

The next step is the so–called thermodynamic reduction, which aims to determine the rest of the quantities in our model, apart from the unknown quantities $(\rho_i)_i, (v_i)_i, \theta$ and the external fields $(b_i)_i, (r_i)_i$. To do so, we first introduce the Helmholtz free energy $\psi := e - \eta\theta$, which we postulate to be a function of the mass densities, the temperature and the temperature gradient. In general, ψ cannot depend on the velocities v_i, otherwise it is incompatible with Galilean invariance, but it can depend on the differences $v_i - v_j$ and the velocity gradient ∇v_i. It can also depend on the gradients of the mass densities, as well as higher derivatives of them, which however make our model extremely complicated. For an extensive discussion on the dependence of Helmholtz free energy on further quantities and its implications we refer to [1].

After plugging into (2.45) the free energy, carrying out the differentiations and using the balance laws (2.42)–(2.44) we obtain:

$$
\begin{aligned}
&(\rho\psi)^{\cdot} + \rho\eta\dot{\theta} + (\rho\psi + p)\mathrm{div}v + \mathrm{div}\left(\theta\Phi - q + \sum_i (\rho_i e_i + p_i)u_i\right) \\
&\quad + \sum_i u_i \cdot (f_i - \nabla p_i) - \Phi \cdot \nabla\theta \leq 0,
\end{aligned}
\tag{2.46}
$$

where the dot stands for the material derivative, i.e. $\dot{\theta} = \theta_t + v \cdot \nabla\theta$. At this point we use an information that comes from thermostatics in the form of the Gibbs–Duhem equation (2.32), according to which in thermodynamic equilibrium there holds:

$$
\rho\psi + p = \sum_i \rho_i \mu_i, \tag{2.47}
$$

where μ_i are the (electro)chemical potentials. Then, after introducing the chemical potentials and performing some further calculations, we obtain the so–called dissipation inequality:

$$
\begin{aligned}
&- [(\rho\psi)_\theta + \rho\eta]\dot{\theta} - (\rho\psi)_{\nabla\theta}(\nabla\theta)^{\cdot} - \sum_i [(\rho\psi)_{\rho_i} - \mu_i]\dot{\rho}_i \\
&- \theta\mathrm{div}\left(\Phi - \frac{q}{\theta} + \frac{1}{\theta}\sum_i (\rho_i e_i + p_i - \rho_i\mu_i)u_i\right) + \frac{1}{\theta} q \cdot \nabla\theta \\
&- \sum_i u_i \cdot \left(f_i - \nabla p_i + \rho_i \nabla\mu_i + \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i\mu_i)\nabla\theta\right) \geq 0.
\end{aligned}
\tag{2.48}
$$

It is now clear from what we noted above, that by controlling the body forces b_i and heat supplies r_i we can construct smooth processes that satisfy the mass, momentum and energy balances and attain at some point (x, t) arbitrarily prescribed values of $\rho_i, \theta, \dot{\rho}_i, \dot{\theta}$ [29],[3]. Hence, (2.48) is violated unless the following relations hold:

$$
\psi = \psi(\rho_1, \ldots, \rho_n, \theta), \tag{2.49}
$$

$$
\eta = -\psi_\theta, \tag{2.50}
$$

$$
\mu_i = (\rho\psi)_{\rho_i}, \tag{2.51}
$$

$$
\Phi = \frac{q}{\theta} - \frac{1}{\theta}\sum_i (\rho_i e_i + p_i - \rho_i\mu_i)u_i, \tag{2.52}
$$

and moreover

$$\frac{1}{\theta}q \cdot \nabla\theta - \sum_i u_i \cdot \left(f_i - \nabla p_i + \rho_i \nabla \mu_i + \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta \right) \geq 0. \tag{2.53}$$

The last inequality is essentially the requirement $\theta\zeta \geq 0$ which indicates that

$$\zeta = \frac{1}{\theta^2}q \cdot \nabla\theta - \frac{1}{\theta}\sum_i u_i \cdot \left(f_i - \nabla p_i + \rho_i \nabla \mu_i + \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta \right), \tag{2.54}$$

i.e. the entropy production contains a term due to heat conduction and another one due to diffusion, which verifies that energy gets dissipated because of heat conduction and diffusion and this dissipation, in turn, results in production of entropy. In case we considered chemical reactions and viscosity, we would also have their contributions to the entropy production (as well as an extra contribution to the diffusion term due to chemical reactions).

The problem with (2.53) is that it is too complicated to allow for investigating necessary and sufficient conditions so that it holds everywhere. Therefore, a technique that is usually employed is to ask for each one of the terms to be nonnegative. According to this, a sufficient condition for the heat conduction term to be nonnegative is the ansatz:

$$q = \kappa \nabla\theta, \tag{2.55}$$

with $\kappa = \kappa(\rho_1, \ldots, \rho_n, \theta) \geq 0$, which reminds us of the standard Fourier's law. Likewise, but following a much more sophisticated process employing the constraint (2.39), the Gibbs–Duhem relation (2.32) and certain properties of the second term of (2.53), we obtain the ansatz

$$f_i = \nabla p_i - \rho_i \nabla \mu_i - \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta - \theta \sum_{j \neq i} b_{ij} \rho_i \rho_j (v_i - v_j), \tag{2.56}$$

where $b_{ij} = b_{ij}(\rho_i, \rho_j, \theta)$ are symmetric, nonnegative binary–type interactions.

For the full derivation we refer to [6]. After plugging the two closures (2.55) and (2.56), (2.54) reads due to the symmetry of b_{ij}:

$$\zeta = \frac{1}{\theta^2}\kappa|\nabla\theta|^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} b_{ij}\rho_i\rho_j|u_i - u_j|^2. \tag{2.57}$$

Another interesting comment is that even after determining the constitutive relations (2.49)–(2.56) via the thermodynamic reduction, our model is not fully closed. Indeed, even though we can compute the thermal energy from the formula

$$e = -\theta^2\left(\frac{\psi}{\theta}\right)_\theta, \tag{2.58}$$

and the total pressure from the Gibbs–Duhem equation

$$p = -\rho\psi + \sum_i \rho_i\mu_i, \tag{2.59}$$

it turns out that if we are just given the free energy density ψ there is no way to determine the partial pressures p_i and partial internal energies e_i. In other words, for a Class–II model it is not sufficient to be given solely the Helmholtz free energy, as it is for a Class–I model [6], or in elasticity theory [29]. Therefore, we require some more information: a first approach is to ask for the constitutive function of $(e_i)_i$ and $(p_i)_i$. In fact, since we can compute the total pressure and thermal energy by (2.59) and (2.58) anyway, the fact that they are defined as the sum of the partial pressures and internal energies respectively, allows us to ask for $n-1$ of them, as the last one can be computed by a simple subtraction.

Another approach that is often used (and will be employed in the next sections) is the so–called simple mixture: this is a simplification of the thermodynamics of the model, in which we assume that each component has its own thermodynamics, i.e. behaves as if it were on its own. This is denoted by assuming that each component is described by its own free energy density $\psi_i = \psi_i(\rho_i, \theta)$, thus obtaining the following relations:

$$e_i = e_i(\rho_i, \theta) = -\theta^2 \left(\frac{\psi_i}{\theta} \right)_\theta, \tag{2.60}$$

$$\mu_i = \mu_i(\rho_i, \theta) = (\rho_i \psi_i)_{\rho_i}, \tag{2.61}$$

$$p_i = p_i(\rho_i, \theta) = -\rho_i \psi_i + \rho_i \mu_i. \tag{2.62}$$

This also has the implication that $\rho\psi = \sum_i \rho_i \psi_i$. For an extensive discussion on simple mixtures we refer to [1], while for special cases of simple mixtures to [6].

Chapter 3

High–friction limit

This chapter consists of the work from [22] and [26]. As mentioned in chapter 2, in a multicomponent theory with primitive variables mass density, velocity and temperature, one distinguishes among three classes of models: In a Class–I model, each component is described by its own mass density, but the components move with a common velocity and have a common temperature. In Class–II models, each component is described by its own mass density and velocity, but the components have a common temperature. Class–III models (which are not considered here) are described via the individual densities, velocities and temperatures of each component. For information regarding Class–III models we refer to [5], while for a short discussion on motivations and reasons to employ each model we refer to [6].

It is useful to have systematic ways of passing from a detailed theory to a less detailed one, which is mathematically easier to handle and experimentally easier to measure. Previous works trying to pass from a Class–II to a Class–I model are already known, for example in [6] in which the authors investigate the reduction of a non–isothermal model using an entropy invariant method, or in [25], in which the reduction is done using asymptotic methods, but for the isothermal case and for a simple mixture.

The objective of this chapter is to derive a Class–I model using the asymptotic method of [25], but for the non–isothermal, non–simple mixture model treated in [6]. The model consists of equations (2.42)–(2.45), and the constitutive relations (2.49)–(2.52), (2.55)–(2.57), i.e.

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \tag{3.1}$$

$$\begin{aligned} \partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = \rho_i b_i - \rho_i \nabla \mu_i - \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta \\ - \frac{\theta}{\epsilon}\sum_{j\neq i} b_{ij}\rho_i\rho_j(u_i - u_j), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \partial_t \left(\rho e + \sum_i \frac{1}{2}\rho_i v_i^2\right) + \operatorname{div}\left(\left(\rho e + \sum_i \frac{1}{2}\rho_i v_i^2\right) v\right) + \operatorname{div}(pv) \\ = \operatorname{div}\left(\kappa\nabla\theta - \sum_i (\rho_i e_i + p_i + \frac{1}{2}\rho_i v_i^2)u_i\right) + \rho b \cdot v + \rho r + \sum_i \rho_i b_i \cdot u_i. \end{aligned} \tag{3.3}$$

Notice, that the last term of (3.2), which corresponds to friction, has been scaled by a small positive parameter ϵ, which in the kinetic level represents the time between two consecutive collisions. As $\epsilon \to 0$, friction becomes more dominant and the partial velocities tend to average and align to a common velocity. This idea, which gives the intuition behind this chapter, is known as the high–friction limit.

The resulting system reads

$$\partial_t \rho_i + \operatorname{div}(\rho_i v) = -\operatorname{div}(\rho_i u_i), \tag{3.4}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \rho b - \nabla p, \tag{3.5}$$

$$\begin{aligned} \partial_t \left(\rho e + \frac{1}{2}\rho v^2\right) + \operatorname{div}\left((\rho e + \frac{1}{2}\rho v^2)v\right) = \operatorname{div}\left(\kappa\nabla\theta - \sum_i (\rho_i e_i + p_i)u_i\right) \\ - \operatorname{div}(pv) + \rho r + \rho b \cdot v + \sum_i \rho_i b_i \cdot u_i, \end{aligned} \tag{3.6}$$

where u_i is determined by solving the constrained linear system

$$\begin{aligned} -\sum_{j\neq i} b_{ij}\theta\rho_i\rho_j(u_i - u_j) = \epsilon\left(\frac{\rho_i}{\rho}(\rho b - \nabla p) + \rho_i\theta\nabla\frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i)\nabla\frac{1}{\theta}\right) \\ \sum_i \rho_i u_i = 0. \end{aligned} \tag{3.7}$$

System (3.4)–(3.7) contains a common velocity v, instead of n partial velocities v_i, thus the name Class–I. Nevertheless, the diffusional velocities u_i survive the asymptotic process as $\mathcal{O}(\epsilon)$ terms and are determined by solving (3.7); it is due to them, that the limiting system describes diffusive phenomena, despite it containing no partial velocities.

3.1 The Chapman–Enskog expansion

The goal of the asymptotic procedure is to obtain from (3.1)–(3.3) a Class–I model via a Chapman–Enskog expansion that will provide an approximation of the initial system of order ϵ^2. The resulting system will consist of n partial momentum balances, a single (total) momentum balance and one total energy balance. It will also be complemented by the constraint $\sum_i \rho_i u_i = 0$ (at least up to order $\mathcal{O}(\epsilon^2)$) and a linear system for determining the diffusional velocities u_i.

The resulting system should contain no partial velocities. Using the definition of the diffusional velocities $u_i = v_i - v$ we perform the change of variables $(v_1, \ldots, v_n) \mapsto (v, u_1, \ldots, u_n)$. Then system (3.1)–(3.3) reads:

$$\partial_t \rho_i + \operatorname{div}(\rho_i u_i + \rho_i v) = 0, \tag{3.8}$$

$$\begin{aligned}\partial_t \left(\rho_i(u_i + v)\right) + \operatorname{div}(\rho_i(u_i + v) \otimes (u_i + v)) &= \rho_i b_i - \rho_i \nabla \mu_i \\ -\frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta &- \frac{\theta}{\epsilon}\sum_{j\neq i} b_{ij}\rho_i\rho_j(u_i - u_j),\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\partial_t\Big(\rho e + \sum_i \frac{1}{2}\rho_i(u_i+v)^2\Big) + \operatorname{div}\left(\Big(\rho e + p + \sum_i \frac{1}{2}\rho_i(u_i+v)^2\Big)v\right) - \rho r \\ &= \operatorname{div}\left(\kappa\nabla\theta - \sum_i(\rho_i e_i + p_i + \frac{1}{2}\rho_i(u_i+v)^2)u_i\right) + \sum_i \rho_i b_i \cdot (u_i + v),\end{aligned} \tag{3.10}$$

subject to the constraint

$$\sum_i \rho_i u_i = 0. \tag{3.11}$$

3.1.1 Hilbert Expansion

Observe that, as $\epsilon \to 0$, we formally obtain the equation

$$-\theta \sum_{j \neq i} b_{ij} \rho_i \rho_j (u_i - u_j) = 0 \,.$$

In [25] it was proved that the system

$$-\sum_{j \neq i} B_{ij}(u_i - u_j) = d_i, \quad i = 1, \ldots, n \;\text{ subject to }\; \sum_i \rho_i u_i = 0, \tag{3.12}$$

under the hypotheses:

(i) $d_1, \ldots, d_n \in \mathbb{R}^3$ satisfy $\sum_i d_i = 0$,

(ii) $\rho_1, \ldots, \rho_n > 0$,

(iii) $(B_{ij}) \in \mathbb{R}^{n \times n}$ is a symmetric matrix with $B_{ij} \geq 0$ for $i, j = 1, \ldots, n$ and

(iv) all solutions of the homogeneous system $\sum_{j \neq i} B_{ij}(u_i - u_j) = 0$, $i = 1, \ldots, n$ lie in the space spanned by $(1, \ldots, 1) \in \mathbb{R}^n$,

has the unique solution

$$\rho_i u_i = -\sum_{j,k=1}^{n-1} (\delta_{ij}\rho_i - \frac{\rho_i \rho_j}{\rho}) \tau_{jk}^{-1} d_k, \quad i = 1, \ldots, n-1, \tag{3.13}$$
$$\rho_n u_n = -\sum_{j=1}^{n-1} \rho_j u_j,$$

where $(\tau_{ij}^{-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is the inverse of a regular submatrix, obtained by reordering the matrix $(\tau_{ij}) \in \mathbb{R}^{n \times n}$ of rank $n-1$ with coefficients

$$\tau_{ij} = \delta_{ij} \sum_{k=1}^{n} B_{ik} - B_{ij} \,.$$

Our case is simpler since the coefficients b_{ij} are symmetric and strictly positive. We also require that $\theta > 0$, which means that hypotheses (iii) and (iv) are automatically satisfied for $\rho_i > 0$, $i = 1, \ldots, n$.

Next, introduce the Hilbert expansion

$$\rho_i = \rho_i^0 + \epsilon\rho_i^1 + \epsilon^2\rho_i^2 + \mathcal{O}(\epsilon^3), \tag{3.14}$$

$$u_i = u_i^0 + \epsilon u_i^1 + \epsilon^2 u_i^2 + \mathcal{O}(\epsilon^3), \tag{3.15}$$

$$v = v^0 + \epsilon v^1 + \mathcal{O}(\epsilon^2), \tag{3.16}$$

$$\theta = \theta^0 + \epsilon\theta^1 + \epsilon^2\theta^2 + \mathcal{O}(\epsilon^3). \tag{3.17}$$

Inserting this into $\rho = \sum_i \rho_i$ and the constraint (3.11) we obtain

$$\rho = \rho^0 + \epsilon\rho^1 + \mathcal{O}(\epsilon^2), \tag{3.18}$$

where we set $\rho^0 := \sum_i \rho_i^0$ and $\rho^1 := \sum_i \rho_i^1$ and

$$\sum_i \rho_i^0 u_i^0 + \epsilon \sum_i (\rho_i^1 u_i^0 + \rho_i^0 u_i^1) + \mathcal{O}(\epsilon^2) = 0. \tag{3.19}$$

Equating terms of the same order gives

$$\sum_i \rho_i^0 u_i^0 = 0 \quad \text{and} \quad \sum_i (\rho_i^1 u_i^0 + \rho_i^0 u_i^1) = 0. \tag{3.20}$$

Next we insert the expansion (3.14)–(3.17) into system (3.8)–(3.10) and identify terms of the same order:

(i) The terms at the order $\mathcal{O}(1/\epsilon)$ give:

$$-\theta^0 \sum_{j\neq i} b_{ij}\rho_i^0\rho_j^0(u_i^0 - u_j^0) = 0. \tag{3.21}$$

For the expansion we assume $\rho_i^0 > 0$ and note that (3.21) is of the form (3.12) with $d_i = 0$ and B_{ij} symmetric and strictly positive, which implies that the null–

space of the homogeneous system is span$\{(1,\dots,1)\}$. This, along with (3.13), gives $u_i^0 = 0$ for $i \in \{1,\dots,n\}$. This is incorporated in the remaining expansions.

For the expansion of $\frac{1}{\theta}$ we use the Taylor series $\frac{1}{1+x} = 1 - x + x^2 + \mathcal{O}(x^3)$ and thus

$$\frac{1}{\theta} = \frac{1}{\theta^0(1+\epsilon\frac{\theta^1}{\theta^0}+\epsilon^2\frac{\theta^2}{\theta^0}+\cdots)} = \frac{1}{\theta^0} - \epsilon\frac{\theta^1}{(\theta^0)^2} + \epsilon^2\frac{(\theta^1)^2-\theta^2\theta^0}{(\theta^0)^3} + \mathcal{O}(\epsilon^3). \tag{3.22}$$

For the expansions of the thermodynamic functions we use

$$\begin{aligned} e_i &= \hat{e}_i(\rho_1^0+\epsilon\rho_1^1+\mathcal{O}(\epsilon^2),\dots,\rho_n^0+\epsilon\rho_n^1+\mathcal{O}(\epsilon^2),\theta^0+\epsilon\theta^1+\mathcal{O}(\epsilon^2)) \\ &= e_i^0+\epsilon e_i^1+\epsilon^2 e_i^2+\mathcal{O}(\epsilon^3), \end{aligned} \tag{3.23}$$

where e_i^0, e_i^1, $e_i^2, \dots$ are computed by using the Taylor theorem, for instance

$$e_i^0 = \hat{e}_i(\rho_1^0,\dots,\rho_n^0,\theta^0)\,, \quad e_i^1 = \sum_{j=1}^n \frac{\partial\hat{e}_i}{\partial\rho_j}(\rho_1^0,\dots,\rho_n^0,\theta^0)\,\rho_j^1 + \frac{\partial\hat{e}_i}{\partial\theta}(\rho_1^0,\dots,\rho_n^0,\theta^0)\,\theta^1\,,$$

$$\begin{aligned} e_i^2 = \sum_{j=1}^n \frac{\partial\hat{e}_i}{\partial\rho_j}(\rho^0,\theta^0)\,\rho_j^2 + \frac{\partial\hat{e}_i}{\partial\theta}(\rho^0,\theta^0)\,\theta^2 + \sum_{k,l=1}^n \frac{\partial^2\hat{e}_i}{\partial\rho_k\partial\rho_l}(\rho^0,\theta^0)\,\rho_k^1\rho_l^1 \\ + \sum_{k=1}^n \frac{\partial^2\hat{e}_i}{\partial\rho_k\partial\theta}(\rho^0,\theta^0)\,\rho_k^1\theta^1 + \frac{\partial^2\hat{e}_i}{\partial\theta^2}(\rho^0,\theta^0)\,(\theta^1)^2 \end{aligned}$$

and so on. A similar expansion is used for the functions e_i, p_i and κ all of which only depend on $(\rho_1,\dots,\rho_n)$ and θ. We then obtain:

(ii) Terms of order $\mathcal{O}(1)$:

$$\partial_t\rho_i^0 + \operatorname{div}(\rho_i^0 v^0) = 0, \tag{3.24}$$

$$\begin{aligned} \partial_t(\rho_i^0 v^0) + \operatorname{div}(\rho_i^0 v^0\otimes v^0) = \rho_i^0 b_i^0 - \rho_i^0\nabla\mu_i^0 - \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0 \\ - \theta^0\sum_{j\neq i} b_{ij}\rho_i^0\rho_j^0(u_i^1-u_j^1), \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&\partial_t\Big(\sum_i \rho_i^0 e_i^0 + \sum_i \frac{1}{2}\rho_i^0 (v^0)^2\Big) + \operatorname{div}\left(\left(\sum_i \rho_i^0 e_i^0 + \sum_i \frac{1}{2}\rho_i^0 (v^0)^2\right) v^0\right) \\
&\qquad = -\operatorname{div}(p^0 v^0) + \sum_i \rho_i^0 r_i^0 + \sum_i \rho_i^0 b_i^0 \cdot v^0 + \operatorname{div}\left(\kappa^0 \nabla \theta^0\right).
\end{aligned} \tag{3.26}
$$

(iii) Terms of order $\mathcal{O}(\epsilon)$:

$$
\partial_t \rho_i^1 + \operatorname{div}(\rho_i^0 (u_i^1 + v^1) + \rho_i^1 v^0) = 0, \tag{3.27}
$$

$$
\begin{aligned}
&\partial_t(\rho_i^1 v^0 + \rho_i^0 (u_i^1 + v^1)) + \operatorname{div}\Big(\rho_i^1 v^0 \otimes v^0 + \rho_i^0 (u_i^1 + v^1) \otimes v^0 \\
&+ \rho_i^0 v^0 \otimes (u_i^1 + v^1)\Big) = \rho_i^1 b_i^0 + \rho_i^0 b_i^1 - \rho_i^1 \nabla \mu_i^0 - \rho_i^0 \nabla \mu_i^1 \\
&+ \frac{\theta^1}{(\theta^0)^2}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0 \mu_i^0)\nabla\theta^0 - \frac{1}{\theta^0}(\rho_i^1 e_i^0 + \rho_i^0 e_i^1 + p_i^1 - \rho_i^1 \mu_i^0 - \rho_i^0 \mu_i^1)\nabla\theta^0 \\
&- \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0 \mu_i^0)\nabla\theta^1 - \theta^0 \sum_{j\neq i} b_{ij}\Big\{(\rho_i^1 \rho_j^0 + \rho_i^0 \rho_j^1)(u_i^1 - u_j^1) \\
&+ \rho_i^0 \rho_j^0 (u_i^2 - u_j^2)\Big\} - \theta^1 \sum_{j\neq i} b_{ij} \rho_i^0 \rho_j^0 (u_i^1 - u_j^1),
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
&\partial_t\Big(\sum_i(\rho_i^0 e_i^1 + \rho_i^1 e_i^0) + \sum_i \frac{1}{2}\rho_i^1 (v^0)^2 + \sum_i \frac{1}{2}\rho_i^0 2 v^0 (u_i^1 + v^1)\Big) \\
&+ \operatorname{div}\Big(\Big(\sum_i \rho_i^0 e_i^0 + \sum_i \frac{1}{2}\rho_i^0 (v^0)^2\Big) v^1 + \Big(\sum_i (\rho_i^0 e_i^1 + \rho_i^1 e_i^0) \\
&+ \sum_i \frac{1}{2}\rho_i^0 2 v^0 (u_i^1 + v^1) + \sum_i \frac{1}{2}\rho_i^1 (v^0)^2\Big) v^0\Big) = -\operatorname{div}(p^0 v^1 + p^1 v^0) \\
&+ \sum_i (\rho_i^0 b_i^0 \cdot v^1 + (\rho_i^1 b_i^0 + \rho_i^0 b_i^1) \cdot v^0) + \sum_i \rho_i^0 b_i^0 \cdot u_i^1 + \sum_i (\rho_i^0 r_i^1 + \rho_i^1 r_i^0) \\
&+ \operatorname{div}\left(\kappa^0 \nabla\theta^1 + \kappa^1 \nabla\theta^0 - \sum_i \left\{(\rho_i^0 e_i^0 + p_i^0) + \frac{1}{2}\rho_i^0 (v^0)^2\right\} u_i^1\right).
\end{aligned} \tag{3.29}
$$

Summing (3.25) over i and using the symmetry of b_{ij} we obtain

$$
\partial_t(\sum_i \rho_i^0 v^0) + \operatorname{div}(\sum_i \rho_i^0 v^0 \otimes v^0) = \sum_i \rho_i^0 b_i^0 - \nabla p^0. \tag{3.30}
$$

The reason behind the simplification of the right–hand side comes from the thermodynamic relations of the problem. More precisely, we have:

$$
\begin{aligned}
&\sum_i \Big(\rho_i b_i - \rho_i \nabla \mu_i - \frac{1}{\theta}(\rho_i e_i + p_i - \rho_i \mu_i)\nabla\theta\Big) \\
&\qquad = \rho b - \sum_i \rho_i \nabla\mu_i - \frac{1}{\theta}(\rho e + p - \sum_i \rho_i\mu_i)\nabla\theta \\
&\qquad \overset{(2.32)}{=} \rho b - \sum_i \rho_i \nabla \mu_i - \rho\eta\nabla\theta \\
&\qquad = \rho b + \sum_i \mu_i \nabla\rho_i - \nabla\left(\sum_i \rho_i\mu_i\right) - \rho\eta\nabla\theta \\
&\qquad \overset{(2.51)-(2.51)}{=} \rho b - \nabla p.
\end{aligned}
$$

Expanding both sides, we get:

$$
\sum_i \left(\rho_i^0 b_i^0 - \rho_i^0 \nabla\mu_i^0 - \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0\right) = \sum_i \rho_i^0 b_i^0 - \nabla p^0 \tag{3.31}
$$

and

$$
\begin{aligned}
&\sum_i \Big(\rho_i^1 b_i^0 + \rho_i^0 b_i^1 - \rho_i^1\nabla\mu_i^0 - \rho_i^0\nabla\mu_i^1 + \frac{\theta^1}{(\theta^0)^2}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0 \\
&- \frac{1}{\theta^0}(\rho_i^1 e_i^0 + \rho_i^0 e_i^1 + p_i^1 - \rho_i^1\mu_i^0 - \rho_i^0\mu_i^1)\nabla\theta^0 - \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^1\Big) \\
&= \sum_i(\rho_i^1 b_i^0 + \rho_i^0 b_i^1) - \nabla p^1,
\end{aligned} \tag{3.32}
$$

where $p^0 = \sum_i p_i^0$ and $p^1 = \sum_i p_i^1$.

Equation (3.30) along with (3.24) and (3.26) provide a closed system for determining $(\rho_1^0, \ldots, \rho_n^0, v^0, \theta^0)$. Now, by (3.25) follows that $(u_1^1, \ldots, u_n^1)$ satisfies the linear system

$$
-\sum_{j\neq i} b_{ij}\theta^0\rho_i^0\rho_j^0(u_i^1 - u_j^1) = d_i^0, \tag{3.33}
$$

where $d_i^0 = \partial_t(\rho_i^0 v^0) + \operatorname{div}(\rho_i^0 v^0 \otimes v^0) - \rho_i^0 b_i^0 + \rho_i^0\nabla\mu_i^0 + \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0$. Since $u_i^0 = 0$ the first constraint of (3.20) is satisfied trivially and the second one becomes

$$
\sum_i \rho_i^0 u_i^1 = 0. \tag{3.34}
$$

Moreover, due to (3.30) and (3.31) we see that $\sum_i d_i^0 = 0$. This guarantees the solvability of system (3.33), since assumptions (i)–(iv) are satisfied according to the analysis of system (3.12). Hence, there exists a unique solution $(u_1^1, \dots, u_n^1)$ to system (3.33).

Similarly, summing (3.28) over i, using the symmetry of b_{ij}, the identity (3.32) and the constraint (3.34) we get

$$\begin{aligned}\partial_t(\sum_i \rho_i^1 v^0 + \rho^0 v^1) + \mathrm{div}(\sum_i \rho_i^1 v^0 \otimes v^0 + \rho^0 v^1 \otimes v^0 + \rho^0 v^0 \otimes v^1) \\ = \sum_i (\rho_i^1 b_i^0 + \rho_i^0 b_i^1) - \nabla p^1,\end{aligned} \tag{3.35}$$

which along with (3.27) and (3.29) provide a closed system for determining $(\{\rho_i^1\}_{i=1}^n, v^1, \theta^1)$. Note that after using the constraint (3.34), (3.29) reads

$$\begin{aligned}&\partial_t\Big(\sum_i(\rho_i^0 e_i^1 + \rho_i^1 e_i^0) + \sum_i \frac{1}{2}\rho_i^1 (v^0)^2 + \rho^0 v^0 v^1\Big) + \mathrm{div}\Big(\Big(\sum_i \rho_i^0 e_i^0 \\ &+ \sum_i \frac{1}{2}\rho_i^0 (v^0)^2\Big) v^1 + \Big(\sum_i(\rho_i^0 e_i^1 + \rho_i^1 e_i^0) + \rho^0 v^0 v^1 + \sum_i \frac{1}{2}\rho_i^1 (v^0)^2\Big) v^0\Big) \\ &= \sum_i(\rho_i^0 b_i^0 \cdot v^1 + (\rho_i^1 b_i^0 + \rho_i^0 b_i^1)\cdot v^0) + \sum_i \rho_i^0 b_i^0 \cdot u_i^1 + \sum_i(\rho_i^0 r_i^1 + \rho_i^1 r_i^0) \\ &+ \mathrm{div}\left(\kappa^0 \nabla\theta^1 + \kappa^1 \nabla \theta^0 - \sum_i (\rho_i^0 e_i^0 + p_i^0) u_i^1\right) - \mathrm{div}(p^0 v^1 + p^1 v^0).\end{aligned}$$

3.1.2 Reconstruction of the effective equations

Next, we reconstruct the effective equations that are valid asymptotically up to order $\mathcal{O}(\epsilon^2)$. We add back (3.24) plus ϵ times (3.27), (3.30) plus ϵ times (3.35) and (3.26) plus ϵ times (3.29) to obtain

$$\partial_t(\rho_i^0 + \epsilon\rho_i^1) + \mathrm{div}(\rho_i^0 v^0 + \epsilon(\rho_i^0 v^1 + \rho_i^1 v^0)) = -\epsilon \mathrm{div}(\rho_i^0 u_i^1), \tag{3.36}$$

$$\begin{aligned}\partial_t\Big(\rho^0 v^0 + \epsilon(\rho^1 v^0 + \rho^0 v^1)\Big) + \mathrm{div}\Big(\rho^0 v^0 + \epsilon(\rho^1 v^0 \otimes v^0 + \rho^0 v^1 \otimes v^0 \\ + \rho^0 v^0 \otimes v^1)\Big) = \sum_i \{\rho_i^0 b_i^0 + \epsilon(\rho_i^1 b_i^0 + \rho_i^0 b_i^1)\} - (\nabla p^0 + \epsilon \nabla p^1),\end{aligned} \tag{3.37}$$

$$
\begin{aligned}
&\partial_t\Big(\sum_i \rho_i^0 e_i^0 + \epsilon\sum_i(\rho_i^0 e_i^1 + \rho_i^1 e_i^0) + \sum_i \frac{1}{2}(\rho_i^0 + \epsilon\rho_i^1)(v^0)^2) \\
&+ \epsilon\sum_i \frac{1}{2}\rho_i^0 2v^0 v^1\Big) + \operatorname{div}\Big(\sum_i \rho_i^0 e_i^0(v^0 + \epsilon v^1) + \epsilon\sum_i(\rho_i^1 e_i^0 + \rho_i^0 e_i^1)v^0 \\
&+ \sum_i \frac{1}{2}\rho_i^0(v^0)^2(v^0 + \epsilon v^1) + \epsilon\sum_i \frac{1}{2}\rho_i^1(v^0)^2 v^0) + \epsilon\sum_i \frac{1}{2}\rho_i^0 2v^0(u_i^1 \\
&+ v^1)v^0\Big) = -\operatorname{div}(p^0 v^0 + \epsilon(p^1 v^0 + p^0 v^1) + \sum_i(\rho_i^0 r_i^0 + \epsilon(\rho_i^0 r_i^1 + \rho_i^1 r_i^0)) \\
&+ \sum_i\Big(\rho_i^0 b_i^0\cdot v^0 + \epsilon(\rho_i^0 b_i^0\cdot v^1 + (\rho_i^0 b_i^1 + \rho_i^1 b_i^0)\cdot v^0)\Big) + \epsilon\sum_i \rho_i^0 b_i^0\cdot u_i^1 \\
&+ \operatorname{div}\Big((\kappa^0\nabla\theta^0) + \epsilon(\kappa^1\nabla\theta^0 + \kappa^1\nabla\theta^0) - \epsilon\sum_i(\rho_i^0 e_i^0 + p_i^0)u_i^1\Big).
\end{aligned} \tag{3.38}
$$

Now set

$$\rho_i^\epsilon = \rho_i^0 + \epsilon\rho_i^1 + \mathcal{O}(\epsilon^2)\,, \qquad \rho^\epsilon = \sum_i \rho_i^\epsilon, \tag{3.39}$$

$$u_i^\epsilon = u_i^0 + \epsilon u_i^1 + \mathcal{O}(\epsilon^2), \tag{3.40}$$

$$v^\epsilon = v^0 + \epsilon v^1 + \mathcal{O}(\epsilon^2), \tag{3.41}$$

$$\theta^\epsilon = \theta^0 + \epsilon\theta^1 + \mathcal{O}(\epsilon^2). \tag{3.42}$$

We also have the expansions obtained via (3.23) and reading

$$
\begin{aligned}
e_i^\epsilon &= e_i^0 + \epsilon e_i^1 + \epsilon^2 e_i^2 + \mathcal{O}(\epsilon^3), \\
p_i^\epsilon &= p_i^0 + \epsilon p_i^1 + \epsilon^2 p_i^2 + \mathcal{O}(\epsilon^3), \\
\kappa^\epsilon &= \kappa^0 + \epsilon\kappa^1 + \epsilon^2\kappa^2 + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{3.43}
$$

Then equations (3.36)–(3.38) read

$$\partial_t\rho_i^\epsilon + \operatorname{div}(\rho_i^\epsilon v^\epsilon) = -\operatorname{div}(\rho_i^\epsilon u_i^\epsilon) + \mathcal{O}(\epsilon^2), \tag{3.44}$$

$$\partial_t(\rho^\epsilon v^\epsilon) + \operatorname{div}(\rho^\epsilon v^\epsilon \otimes v^\epsilon) = \rho^\epsilon b^\epsilon - \nabla p^\epsilon + \mathcal{O}(\epsilon^2), \tag{3.45}$$

$$\partial_t\left(\rho^\epsilon e^\epsilon + \frac{1}{2}\rho^\epsilon(v^\epsilon)^2\right) + \operatorname{div}\left(\left(\rho^\epsilon e^\epsilon + p^\epsilon + \frac{1}{2}\rho^\epsilon(v^\epsilon)^2\right)v^\epsilon\right) = \rho^\epsilon r^\epsilon$$
$$+\sum_i \rho_i^\epsilon b_i^\epsilon\cdot(u_i^\epsilon + v) + \operatorname{div}\left(\kappa^\epsilon\nabla\theta^\epsilon - \sum_i(\rho_i^\epsilon e_i^\epsilon + p_i^\epsilon)u_i^\epsilon\right) + \mathcal{O}(\epsilon^2). \tag{3.46}$$

Finally, we need to construct the formulas determining $(u_i^\epsilon)_i$. Using (3.33) we deduce that

$$-\sum_{j\neq i} b_{ij}\theta^\epsilon\rho_i^\epsilon\rho_j^\epsilon(u_i^\epsilon - u_j^\epsilon) = -\epsilon\sum_{j\neq i} b_{ij}\theta^0\rho_i^0\rho_j^0(u_i^1 - u_j^1) + \mathcal{O}(\epsilon^2)$$
$$= \epsilon d_i^0 + \mathcal{O}(\epsilon^2), \tag{3.47}$$

where from (3.24), summing (3.24) over i and (3.30):

$$\begin{aligned}
d_i^0 &= \partial_t(\rho_i^0 v^0) + \operatorname{div}(\rho_i^0 v^0\otimes v^0) - \rho_i^0 b_i^0 + \rho_i^0\nabla\mu_i^0 + \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0\\
&= \rho_i^0\left(\partial_t v^0 + v^0\cdot\nabla v^0\right) - \rho_i^0 b_i^0 + \rho_i^0\nabla\mu_i^0 + \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0\\
&= \frac{\rho_i^0}{\rho^0}\left(\partial_t(\rho^0 v^0) + \operatorname{div}(\rho^0 v^0\otimes v^0)\right) - \rho_i^0 b_i^0 + \rho_i^0\nabla\mu_i^0 + \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0\\
&= \frac{\rho_i^0}{\rho^0}\sum_j(\rho_j^0 b_j^0 - \nabla p_j^0) - \rho_i^0 b_i^0 + \rho_i^0\nabla\mu_i^0 + \frac{1}{\theta^0}(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)\nabla\theta^0.
\end{aligned}$$

This motivates us to define

$$d_i^\epsilon := \frac{\rho_i^\epsilon}{\rho^\epsilon}\sum_j(\rho_j^\epsilon b_j^\epsilon - \nabla p_j^\epsilon) - \rho_i^\epsilon b_i^\epsilon + \rho_i^\epsilon\nabla\mu_i^\epsilon + \frac{1}{\theta^\epsilon}(\rho_i^\epsilon e_i^\epsilon + p_i^\epsilon - \rho_i^\epsilon\mu_i^\epsilon)\nabla\theta^\epsilon, \tag{3.48}$$

which by (3.31)–(3.32) sums up to zero and thus the linear system for determining $\{u_i^\epsilon\}_{i=1}^n$ is

$$-\sum_{j\neq i} b_{ij}\theta^\epsilon\rho_i^\epsilon\rho_j^\epsilon(u_i^\epsilon - u_j^\epsilon) = \epsilon d_i^\epsilon + \mathcal{O}(\epsilon^2) \tag{3.49}$$

and is solvable according to the analysis of (3.12). Moreover, the constraint becomes

$$\sum_i \rho_i^\epsilon u_i^\epsilon = \mathcal{O}(\epsilon^2), \tag{3.50}$$

which means that since the variables ρ_i and u_i are defined up to order $\mathcal{O}(\epsilon^2)$ we can set $\sum_i \rho_i^\epsilon u_i^\epsilon = 0$ up to that order.

Therefore, omitting the ϵ–notation and the higher-order terms, we conclude that system (3.1)–(3.3) is approximated within $\mathcal{O}(\epsilon^2)$ by the system:

$$\partial_t \rho_i + \mathrm{div}(\rho_i v) = -\mathrm{div}(\rho_i u_i), \tag{3.51}$$

$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = \rho b - \nabla p, \tag{3.52}$$

$$\begin{aligned} \partial_t \left(\rho e + \frac{1}{2}\rho v^2\right) + \mathrm{div}\Big((\rho e + p + \frac{1}{2}\rho v^2)v\Big) &= \rho r + \rho b \cdot v + \sum_i \rho_i b_i \cdot u_i \\ &+ \mathrm{div}\left(\kappa \nabla \theta - \sum_i (\rho_i e_i + p_i) u_i\right), \end{aligned} \tag{3.53}$$

where u_i are determined by solving the constrained linear system:

$$\begin{aligned} -\sum_{j \neq i} b_{ij} \theta \rho_i \rho_j (u_i - u_j) &= \epsilon d_i, \\ \sum_i \rho_i u_i &= 0 \end{aligned} \tag{3.54}$$

and

$$d_i = \frac{\rho_i}{\rho}(-\nabla p + \rho b) + \rho_i \theta \nabla \frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i) \nabla \frac{1}{\theta} - \rho_i b_i.$$

Our system (3.51)–(3.54) agrees with the Class–I model obtained in [6] using the entropy invariant method, in the case of an inviscid, non–reactive mixture with zero thermal diffusivities [6, Section 8].

3.1.3 Asymptotic expansion of the entropy equation

Given the Class–II system (3.1)–(3.3), we have derived in section 2.2 the entropy equation:

$$\begin{aligned} \partial_t(\rho \eta) + \mathrm{div}(\rho \eta v) &= \mathrm{div}\left(\frac{\kappa \nabla \theta - \sum_i (\rho_i e_i + p_i - \rho_i \mu_i) u_i}{\theta}\right) + \frac{1}{\theta^2} \kappa |\nabla \theta|^2 \\ &+ \frac{1}{2\epsilon} \sum_i \sum_j b_{ij} \rho_i \rho_j (u_i - u_j)^2 + \frac{\rho r}{\theta} \end{aligned} \tag{3.55}$$

and the frictional dissipation has been appropriately scaled by $\epsilon > 0$. There are two approaches for deriving the entropy equation of the asymptotic limit system (3.51)–(3.53): (i) to derive it by using the system and the constitutive relations as was done in section 2.2 to obtain (3.55) and (ii) to expand the entropy equation of the Class–II model and obtain its ϵ^2–approximation the same way we obtained system (3.51)–(3.53). Here we present the latter way, but it is easy to verify that the two results coincide.

As in subsection 3.1.1, we introduce the same Hilbert expansion, insert it into (3.55) and identify terms of the same order:

(i) Terms of order $\mathcal{O}(1/\epsilon)$:

$$\frac{1}{2}\sum_{i,j} b_{ij}\rho_i^0\rho_j^0(u_i^0 - u_j^0)^2 = 0, \tag{3.56}$$

which is consistent with $u_i^0 = 0$ in subsection 3.1.1.

Next, using $u_i^0 = 0$, the expansion (3.22) for $\frac{1}{\theta}$ and

$$\frac{1}{\theta^2} = \frac{1}{(\theta^0)^2} - \epsilon\frac{2\theta^1}{(\theta^0)^3} + \epsilon^2\frac{3(\theta^1)^2 - \theta^2\theta^0}{(\theta^0)^4} + \mathcal{O}(\epsilon^3)\,,$$

as well as the expansions (3.43) and

$$\begin{aligned}\eta &= \hat{\eta}\left(\rho_1^0 + \epsilon\rho_1^1 + \mathcal{O}(\epsilon^2), \dots, \rho_n^0 + \epsilon\rho_n^1 + \mathcal{O}(\epsilon^2), \theta^0 + \epsilon\theta^1 + \mathcal{O}(\epsilon^2)\right)\\ &= \eta^0 + \epsilon\eta^1 + \epsilon^2\eta^2 + \mathcal{O}(\epsilon^3),\end{aligned}$$

we obtain:

(ii) at order $\mathcal{O}(1)$:

$$\partial_t(\rho^0\eta^0) + \operatorname{div}(\rho^0\eta^0 v^0) = \operatorname{div}\left(\frac{1}{\theta^0}\kappa^0\nabla\theta^0\right) + \frac{1}{(\theta^0)^2}\kappa^0(\nabla\theta^0)^2 + \frac{\rho^0 r^0}{\theta^0}, \tag{3.57}$$

(iii) at order $\mathcal{O}(\epsilon)$:

$$\partial_t(\rho^1\eta^0 + \rho^0\eta^1) + \operatorname{div}((\rho^1\eta^0 + \rho^0\eta^1)v^0 + \rho^0\eta^0 v^1) = \operatorname{div}\Big(\frac{1}{\theta^0}(\kappa^1\nabla\theta^0 + \kappa^0\nabla\theta^1) - \frac{\theta^1}{(\theta^0)^2}\kappa^0\nabla\theta^0 - \frac{1}{\theta^0}\sum_i(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)u_i^1\Big) - \frac{2\theta^1}{(\theta^0)^3}\kappa^0(\nabla\theta^0)^2 + \frac{1}{(\theta^0)^2}\kappa^1(\nabla\theta^0)^2 + \frac{1}{(\theta^0)^2}\kappa^0 2\nabla\theta^0\nabla\theta^1 + \frac{1}{2}\sum_{i,j} b_{ij}\rho_i^0\rho_j^0(u_i^1 - u_j^1)^2 + \frac{\rho^1 r^0 + \rho^0 r^1}{\theta^0} - \frac{\rho^0 r^0\theta^1}{(\theta^0)^2}. \tag{3.58}$$

Moreover, using (3.33) in the entropy expansion, the third last term of (3.58) reads

$$\frac{1}{2}\sum_{i,j} b_{ij}\rho_i^0\rho_j^0(u_i^1 - u_j^1)^2 = \frac{1}{2}\sum_i u_i^1\sum_j b_{ij}\rho_i^0\rho_j^0(u_i^1 - u_j^1) - \frac{1}{2}\sum_j u_j^1\sum_i b_{ij}\rho_i^0\rho_j^0(u_i^1 - u_j^1) = -\frac{1}{2\theta^0}\sum_i u_i^1 d_i^0 - \frac{1}{2\theta^0}\sum_j u_j^1 d_j^0 = -\frac{1}{\theta^0}\sum_i u_i^1\cdot d_i^0.$$

Thus, in the reconstruction, we add (3.57) plus ϵ times (3.58), to obtain

$$\partial_t(\rho^0\eta^0 + \epsilon(\rho^1\eta^0 + \rho^0\eta^1)) + \operatorname{div}(\rho^0\eta^0 v^0 + \epsilon((\rho^1\eta^0 + \rho^0\eta^1)v^0 + \rho^0\eta^0 v^1)) = \operatorname{div}\Big[\frac{1}{\theta^0}\kappa^0\nabla\theta^0 + \epsilon\Big(\frac{1}{\theta^0}(\kappa^1\nabla\theta^0 + \kappa^0\nabla\theta^1) - \frac{\theta^1}{(\theta^0)^2}\kappa^0\nabla\theta^0 - \frac{1}{\theta^0}\sum_i(\rho_i^0 e_i^0 + p_i^0 - \rho_i^0\mu_i^0)u_i^1\Big)\Big] + \frac{1}{(\theta^0)^2}\kappa^0(\nabla\theta^0)^2 - \epsilon\Big(\frac{2\theta^1}{(\theta^0)^3}\kappa^0(\nabla\theta^0)^2 - \frac{1}{(\theta^0)^2}\kappa^1(\nabla\theta^0)^2 - \frac{1}{(\theta^0)^2}\kappa^0 2\nabla\theta^0\nabla\theta^1\Big) - \frac{\epsilon}{\theta^0}\sum_i u_i^1\cdot d_i^0 + \frac{\rho^0 r^0}{\theta^0} + \epsilon\Big(\frac{\rho^1 r^0 + \rho^0 r^1}{\theta^0} - \frac{\rho^0 r^0\theta^1}{(\theta^0)^2}\Big), \tag{3.59}$$

which in turn gives

$$\partial_t(\rho^\epsilon\eta^\epsilon) + \operatorname{div}(\rho^\epsilon\eta^\epsilon v^\epsilon) = \operatorname{div}\left(\kappa^\epsilon\frac{\nabla\theta^\epsilon}{\theta^\epsilon} - \sum_i\frac{\rho_i^\epsilon e_i^\epsilon + p_i^\epsilon - \rho_i^\epsilon\mu_i^\epsilon}{\theta^\epsilon}u_i^\epsilon\right) + \kappa^\epsilon\frac{(\nabla\theta^\epsilon)^2}{(\theta^\epsilon)^2} + \frac{\rho^\epsilon r^\epsilon}{\theta^\epsilon} - \frac{1}{\theta^\epsilon}\sum_i u_i^\epsilon\cdot d_i^\epsilon + \mathcal{O}(\epsilon^2). \tag{3.60}$$

Omitting the $O(\epsilon^2)$ correction, the entropy equation for the Class–I system (3.51)–

(3.53) reads

$$\partial_t(\rho\eta) + \operatorname{div}(\rho\eta v) = \operatorname{div}\left(\frac{1}{\theta}\kappa\nabla\theta - \frac{1}{\theta}\sum_i(\rho_i e_i + p_i - \rho_i\mu_i)u_i\right) + \frac{\rho r}{\theta} + \frac{1}{\theta^2}\kappa|\nabla\theta|^2 - \frac{1}{\theta}\sum_i u_i \cdot d_i\,.$$

A direct computation using (3.54) and the symmetry of b_{ij}, shows that

$$-\frac{1}{\theta}\sum_i u_i \cdot d_i = \sum_i\sum_{j\neq i} b_{ij}\rho_i\rho_j u_i \cdot (u_i - u_j) = \frac{1}{2\epsilon}\sum_i\sum_{j\neq i} b_{ij}\rho_i\rho_j(u_i - u_j)^2$$

and as a result we conclude that the order $O(\epsilon^2)$ expansion of the entropy dissipation structure (3.55) for the Class–II model yields the entropy dissipation structure

$$\partial_t(\rho\eta) + \operatorname{div}(\rho\eta v) = \operatorname{div}\left(\frac{1}{\theta}\kappa\nabla\theta - \frac{1}{\theta}\sum_i(\rho_i e_i + p_i - \rho_i\mu_i)u_i\right) + \frac{\rho r}{\theta} + \frac{1}{\theta^2}\kappa|\nabla\theta|^2 - \frac{1}{\theta}\sum_i u_i \cdot d_i \tag{3.61}$$

for the emerging Class–I model.

3.2 Validation of the expansion

Having obtained formally the Class–I model (3.51)–(3.54), we proceed to justify the expansion rigorously, following the work in [26]. To this direction, we develop a relative entropy identity for Class–II models and use it to validate the high–friction limit in the weak–strong solution context, i.e. we compare a weak solution of the Class–II model with a strong solution of the Class–I model. We show that, as $\epsilon \to 0$, the weak solution converges to the strong one, in the relative entropy sense. Such a result assumes that a strong solution to the Class–I model truly exists, which has been established near equilibrium, [4]. More precisely, there exists a unique strong solution to Class–I models, which is local–in–time for general initial data and can be extended for all positive times for initial data close to an equilibrium state (see [4, Chapter 8] for more details).

For ease of notation, we focus on the case of a simple mixture (cf. section 2.2). Then, the Class–II model reads

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \tag{3.62}$$

$$\partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = \rho_i b_i - \nabla p_i - \frac{1}{\epsilon}\theta \sum_{j\neq i} b_{ij}\rho_i\rho_j(v_i - v_j), \tag{3.63}$$

$$\begin{aligned} \partial_t\Big(\rho e + \sum_{j=1}^n \frac{1}{2}\rho_j v_j^2\Big) + \operatorname{div}\Big(\sum_{j=1}^n \Big(\rho_j e_j + p_j + \frac{1}{2}\rho_j v_j^2\Big) v_j\Big) \\ = \operatorname{div}(\kappa\nabla\theta) + \sum_{j=1}^n \rho_j b_j \cdot v_j + \rho r, \end{aligned} \tag{3.64}$$

with the entropy equation

$$\begin{aligned} \partial_t(\rho\eta) + \operatorname{div}\left(\sum_{j=1}^n \rho_j \eta_j v_j\right) = \operatorname{div}\left(\frac{1}{\theta}\kappa\nabla\theta\right) + \frac{1}{\theta^2}\kappa|\nabla\theta|^2 \\ + \frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\rho_i\rho_j|v_i - v_j|^2 + \frac{\rho r}{\theta} \end{aligned} \tag{3.65}$$

and the Class–I model reduces to

$$\partial_t \bar\rho_i + \operatorname{div}(\bar\rho_i \bar v) = -\operatorname{div}(\bar\rho_i \bar u_i), \tag{3.66}$$

$$\partial_t(\bar\rho\bar v) + \operatorname{div}(\bar\rho\bar v \otimes \bar v) = \bar\rho\bar b - \nabla\bar p, \tag{3.67}$$

$$\begin{aligned} \partial_t\Big(\bar\rho\bar e + \frac{1}{2}\bar\rho\bar v^2\Big) + \operatorname{div}\Big((\bar\rho\bar e + \bar p + \frac{1}{2}\bar\rho\bar v^2)\bar v\Big) = \bar\rho\bar r + \bar\rho\bar b\cdot\bar v \\ + \operatorname{div}\Big(\bar\kappa\nabla\bar\theta - \sum_{j=1}^n(\bar\rho_j\bar e_j + \bar p_j)\bar u_j\Big) + \sum_{j=1}^n \bar\rho_j\bar b_j\cdot\bar u_j, \end{aligned} \tag{3.68}$$

where $\bar u_i$ is determined by solving the Maxwell–Stefan system:

$$-\sum_{j\neq i} b_{ij}\bar\theta\bar\rho_i\bar\rho_j(\bar u_i - \bar u_j) = \epsilon\left(\frac{\bar\rho_i}{\bar\rho}(\bar\rho\bar b - \nabla\bar p) - \bar\rho_i\bar b_i + \nabla\bar p_i\right), \tag{3.69}$$

subject to the constraint

$$\sum_{i=1}^{n} \bar{\rho}_i \bar{u}_i = 0, \tag{3.70}$$

and the entropy balance now takes the form

$$\begin{aligned} \partial_t(\bar{\rho}\bar{\eta}) + \operatorname{div}(\bar{\rho}\bar{\eta}\bar{v}) = \operatorname{div}\left(\frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta} - \sum_{j=1}^{n} \bar{\rho}_j \bar{\eta}_j \bar{u}_j\right) + \frac{1}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2 \\ + \frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{u}_i - \bar{u}_j|^2 + \frac{\bar{\rho}\bar{r}}{\bar{\theta}}. \end{aligned} \tag{3.71}$$

We consider the problem in $\mathbb{T}^3 \times (0, \infty)$, where $\mathbb{T}^3$ denotes the three dimensional torus, with space–periodic boundary conditions, but we note that the same analysis can be performed in a bounded domain Ω, with no–flux boundary conditions, i.e.

$$\rho_i v_i \cdot \nu = 0, \quad (\rho_i v_i \otimes v_i + p_i \mathbb{I})\nu = 0, \quad \left(\sum_{j=1}^{n}\left(\rho_j e_j + p_j + \frac{1}{2}\rho_j v_j^2\right)v_j - \kappa\nabla\theta\right)\cdot\nu = 0,$$

for all $i = 1, \ldots, n$, on the parabolic boundary $\partial\Omega \times [0, \infty)$, where ν denotes the outward vector that is normal to the boundary $\partial\Omega$.

3.2.1 The relative entropy inequality for Class–II systems

Let $\omega = (\rho_1, \ldots, \rho_n, v_1, \ldots, v_n, \theta)$ and $\bar{\omega} = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{v}_1, \ldots, \bar{v}_n, \bar{\theta})$ be two solutions of the Class–II system. Motivated by [30] and [22], we define the relative entropy of ω and $\bar{\omega}$ as follows:

$$\mathcal{H}(\omega|\bar{\omega})(t) = \int_{\mathbb{T}^3}\left(\frac{1}{2}\sum_{i=1}^{n}\rho_i|v_i - \bar{v}_i|^2 + \sum_{i=1}^{n}(\rho_i\psi_i)(\omega|\bar{\omega}) + (\rho\eta - \bar{\rho}\bar{\eta})(\theta - \bar{\theta})\right)\mathrm{d}x, \tag{3.72}$$

where

$$(\rho_i\psi_i)(\omega|\bar{\omega}) = \rho_i\psi_i - \bar{\rho}_i\bar{\psi}_i - (\bar{\rho}_i\bar{\psi}_i)_{\rho_i}(\rho_i - \bar{\rho}_i) - (\bar{\rho}_i\bar{\psi}_i)_\theta(\theta - \bar{\theta}).$$

Throughout the thesis, we use the convention

$$\bar{f} = f(\bar{\omega})$$

and therefore when we write $\bar{\rho}_i\bar{\psi}_i$, we mean $(\rho_i\psi_i)(\bar{\omega})$, while

$$(\bar{\rho}_i\bar{\psi}_i)_{\rho_i} = \frac{\partial(\rho_i\psi_i)}{\partial\rho_i}\Big|_{\omega=\bar{\omega}}$$

and similarly

$$(\bar{\rho}_i\bar{\psi}_i)_{\theta} = \frac{\partial(\rho_i\psi_i)}{\partial\theta}\Big|_{\omega=\bar{\omega}}.$$

If $\rho_i\psi_i$ are C^3 on the set

$$U = \{0 \leq \rho_i \leq M, \quad 0 < \gamma \leq \rho \leq M, \quad 0 < \gamma \leq \theta \leq M, \quad \text{for some } \gamma, M > 0\},$$

such that

$$(\rho_i\psi_i)_{\rho_i\rho_i} > 0 \quad \text{and} \quad (\rho_i\psi_i)_{\theta\theta} < 0, \tag{3.73}$$

then according to [22], there exists $C > 0$ such that

$$\sum_{i=1}^{n}(\rho_i\psi_i)(\omega|\bar{\omega}) + (\rho\eta - \bar{\rho}\bar{\eta})(\theta - \bar{\theta}) \geq C\left(\sum_{i=1}^{n}|\rho_i - \bar{\rho}_i|^2 + |\theta - \bar{\theta}|^2\right). \tag{3.74}$$

Therefore, the relative entropy defined in (3.72) can serve as a measure of the distance between ω and $\bar{\omega}$. We note, that the conditions in (3.73) (known as Gibbs thermodynamic stability conditions) are natural in thermodynamics, as they follow from the basic assumptions that the temperature is a strictly positive quantity and that the energy is a convex function of the entropy, satisfied for example for the ideal gas (for more details see [22, Appendix A]).

Let

$$H(\omega|\bar{\omega}) = \sum_{i=1}^{n}\left[\frac{1}{2}\rho_i|v_i - \bar{v}_i|^2 + (\rho_i\psi_i)(\omega|\bar{\omega}) + (\rho_i\eta_i - \bar{\rho}_i\bar{\eta}_i)(\theta - \bar{\theta})\right]$$

and

$$Q(\omega|\bar{\omega}) = \sum_{i=1}^{n} \Big[\frac{1}{2}\rho_i v_i |v_i - \bar{v}_i|^2 + (\rho_i\psi_i)(\omega|\bar{\omega})v_i$$
$$+ (\rho_i\eta_i - \bar{\rho}_i\bar{\eta}_i)(\theta - \bar{\theta})v_i + (p_i - \bar{p}_i)(v_i - \bar{v}_i)\Big].$$

We want to obtain an identity of the form:

$$\partial_t H(\omega|\bar{\omega}) + \operatorname{div} Q(\omega|\bar{\omega}) = \mathrm{RHS}. \tag{3.75}$$

Carrying out the differentiations and using the equations (3.76), (3.80), (3.82) and (3.79), along with the thermodynamic relations from chapter 2, we find that

$$\begin{aligned}
\mathrm{RHS} = &-\frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\theta} b_{ij}\rho_i\rho_j |(v_i - v_j) - (\bar{v}_i - \bar{v}_j)|^2 - \bar{\theta}\kappa |\nabla \log\theta - \nabla \log\bar{\theta}|^2 \\
&- \sum_{i=1}^{n} p_i(\omega|\bar{\omega}) \operatorname{div} \bar{v}_i - \sum_{i=1}^{n} \rho_i (v_i - \bar{v}_i)\cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \\
&+ \sum_{i=1}^{n} \rho_i (b_i - \bar{b}_i)\cdot (v_i - \bar{v}_i) - \sum_{i=1}^{n} (\rho_i\eta_i)(\omega|\bar{\omega})(\partial_s\bar{\theta} + \bar{v}_i \cdot \nabla\bar{\theta}) \\
&- \sum_{i=1}^{n} \rho_i (v_i - \bar{v}_i)\cdot \nabla\bar{\theta}(\eta_i - \bar{\eta}_i) + \left(\frac{\rho r}{\theta} - \frac{\bar{\rho}\bar{r}}{\bar{\theta}}\right)(\theta - \bar{\theta}) \\
&+ (\nabla \log\theta - \nabla\log\bar{\theta})\cdot \nabla \log\bar{\theta}(\theta\bar{\kappa} - \bar{\theta}\kappa) \\
&+ \operatorname{div}\left((\theta - \bar{\theta})\left(\kappa\nabla\log\theta - \bar{\kappa}\nabla\log\bar{\theta}\right)\right) \\
&- \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\theta} b_{ij}\rho_i (v_i - \bar{v}_i)\cdot(\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j) \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}(\theta - \bar{\theta}) b_{ij}\rho_i\rho_j (v_i - \bar{v}_i)\cdot(\bar{v}_i - \bar{v}_j) \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}(\theta - \bar{\theta}) b_{ij}(\rho_i - \bar{\rho}_i)\bar{\rho}_j(\bar{v}_i - \bar{v}_j)\cdot \bar{v}_i \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}(\theta - \bar{\theta}) b_{ij}\rho_i(\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j)\cdot \bar{v}_i,
\end{aligned}$$

where the relative quantities are given by

$$p_i(\omega|\bar{\omega}) = p_i - \bar{p}_i - (\bar{p}_i)_{\rho_i}(\rho_i - \bar{\rho}_i) - (\bar{p}_i)_\theta(\theta - \bar{\theta})$$

$$(\rho_i\eta_i)(\omega|\bar{\omega}) = \rho_i\eta_i - \bar{\rho}_i\bar{\eta}_i - (\bar{\rho}_i\bar{\eta}_i)_{\rho_i}(\rho_i - \bar{\rho}_i) - (\bar{\rho}_i\bar{\eta}_i)_\theta(\theta - \bar{\theta}).$$

We proceed as follows: (i) First, we interpret a Class–I model as a Class–II system with error terms. (ii) Using the relative entropy formula we compare an exact solution to an approximate solution of a Class–II system. (iii) This needs to be done at some prescribed level of solutions; this is made precise in section 3.2.3. (iv) Finally, the derivation of the convergence result is done in sections 3.2.4 and 3.2.5.

3.2.2 Reformulation of the Class–I model

First, we embed a solution of a Class–I model into an approximate solution of a Class–II model. The equations of the Class–II model contain the partial velocities v_i, while the equations of the Class–I model contain the barycentric velocity v and the diffusional velocities u_i.

Let $(\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{v}, \bar{\theta})$ be a solution of (3.66)–(3.68). Then we set

$$\bar{v}_i = \bar{v} + \bar{u}_i$$

and (3.66)–(3.68) and (3.71) read:

$$\partial_t\bar{\rho}_i + \operatorname{div}(\bar{\rho}_i\bar{v}_i) = 0, \tag{3.76}$$

$$\partial_t(\bar{\rho}\bar{v}) + \operatorname{div}(\bar{\rho}\bar{v} \otimes \bar{v}) = \bar{\rho}\bar{b} - \nabla\bar{p}, \tag{3.77}$$

$$\begin{aligned}\partial_t\left(\bar{\rho}\bar{e} + \frac{1}{2}\bar{\rho}\bar{v}^2\right) + \operatorname{div}\left(\sum_{j=1}^n(\bar{\rho}_j\bar{e}_j + \bar{p}_j)\bar{v}_j + \frac{1}{2}\bar{\rho}\bar{v}^2\bar{v}\right)& \\ = \operatorname{div}(\bar{\kappa}\nabla\bar{\theta}) + \sum_{j=1}^n \bar{\rho}_j\bar{b}_j \cdot \bar{v}_j + \bar{\rho}\bar{r},&\end{aligned} \tag{3.78}$$

$$\partial_t(\bar\rho\bar\eta) + \operatorname{div}\left(\sum_{j=1}^n \bar\rho_j\bar\eta_j\bar v_j\right) = \operatorname{div}\left(\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right) + \frac{1}{\bar\theta^2}\bar\kappa|\nabla\bar\theta|^2 + \frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\bar\rho_i\bar\rho_j|\bar v_i - \bar v_j|^2 + \frac{\bar\rho\bar r}{\bar\theta}. \tag{3.79}$$

Next, we rewrite (3.77) and (3.78) in a form that resembles the equations of the Class–II model. We reformulate (3.77) as:

$$\partial_t(\bar\rho_i\bar v_i) + \operatorname{div}(\bar\rho_i\bar v_i\otimes\bar v_i) = \bar\rho_i\bar b_i - \nabla\bar p_i - \frac{\bar\theta}{\epsilon}\sum_{j\neq i} b_{ij}\bar\rho_i\bar\rho_j(\bar v_i - \bar v_j) + \bar R_i, \tag{3.80}$$

where

$$\bar R_i = \partial_t(\bar\rho_i\bar v) + \partial_t(\bar\rho_i\bar u_i) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar v) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar u_i) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar v) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar u_i) - \bar\rho_i\bar b_i + \nabla\bar p_i + \frac{\bar\theta}{\epsilon}\sum_{j\neq i} b_{ij}\bar\rho_i\bar\rho_j(\bar v_i - \bar v_j).$$

Using (3.69), we obtain

$$\bar R_i = \partial_t(\bar\rho_i\bar v) + \partial_t(\bar\rho_i\bar u_i) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar v) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar u_i) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar v) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar u_i) - \frac{\bar\rho_i}{\bar\rho}(\bar\rho\bar b - \nabla\bar p)$$

and by virtue of $\partial_t\bar\rho + \operatorname{div}(\bar\rho\bar v) = 0$, we see that

$$\begin{aligned}\partial_t(\bar\rho_i\bar v) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar v) &= (\partial_t\bar\rho_i + \operatorname{div}(\bar\rho_i\bar v))\bar v + \bar\rho_i(\partial_t\bar v + \nabla\bar v\cdot\bar v)\\ &= -\operatorname{div}(\bar\rho_i\bar u_i)\bar v + \frac{\bar\rho_i}{\bar\rho}(\partial_t(\bar\rho\bar v) + \operatorname{div}(\bar\rho\bar v\otimes\bar v))\\ &= -\operatorname{div}(\bar\rho_i\bar u_i)\bar v + \frac{\bar\rho_i}{\bar\rho}(\bar\rho\bar b - \nabla\bar p)\end{aligned}$$

and thus

$$\bar R_i = -\operatorname{div}(\bar\rho_i\bar u_i)\bar v + \partial_t(\bar\rho_i\bar u_i) + \operatorname{div}(\bar\rho_i\bar v\otimes\bar u_i) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar v) + \operatorname{div}(\bar\rho_i\bar u_i\otimes\bar u_i). \tag{3.81}$$

Similarly, we reformulate (3.78) as:

$$\begin{aligned}\partial_t\left(\bar\rho\bar e+\sum_{j=1}^n\frac{1}{2}\bar\rho_j\bar v_j^2\right)+\operatorname{div}\left(\sum_{j=1}^n\left(\bar\rho_j\bar e_j+\bar p_j+\frac{1}{2}\bar\rho_j\bar v_j^2\right)\bar v_j\right)&\\ =\operatorname{div}\left(\bar\kappa\nabla\bar\theta\right)+\sum_{j=1}^n\bar\rho_j\bar b_j\cdot\bar v_j+\bar\rho\bar r+\bar Q,&\end{aligned}\tag{3.82}$$

where

$$\bar Q=\partial_t\left(\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2\right)+\operatorname{div}\left(\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2\bar u_j\right)+\operatorname{div}\left(\frac{3}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2\bar v\right),\tag{3.83}$$

because due to (3.70)

$$\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar v_j^2-\frac{1}{2}\bar\rho\bar v^2=\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2$$

and

$$\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar v_j^2\bar v_j-\frac{1}{2}\bar\rho\bar v^2\bar v=\frac{1}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2\bar u_j+\frac{3}{2}\sum_{j=1}^n\bar\rho_j\bar u_j^2\bar v.$$

The equations of the Class–I model are thus reformulated as equations of a Class–II model (namely equations (3.76), (3.80), (3.82)), with the terms $\bar R_i$ and $\bar Q$ given by (3.81) and (3.83), respectively. The latter are viewed as error terms. The Maxwell–Stefan system

$$\begin{aligned}-\sum_{j\neq i}b_{ij}\bar\theta\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j)&=\epsilon\left(\frac{\bar\rho_i}{\bar\rho}(\bar\rho\bar b-\nabla\bar p)-\bar\rho_i\bar b_i+\nabla\bar p_i\right),\\ \sum_{j=1}^n\bar\rho_j\bar u_j&=0,\end{aligned}\tag{3.84}$$

is uniquely solvable [31, 25], which implies $\bar u_i=\mathcal{O}(\epsilon)$ and thus for smooth solutions $\bar R_i$ and $\bar Q$ are of order $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ respectively.

3.2.3 Notions of solutions

In the following, we give the definitions of solutions that will be used. We use the notation $\omega = ((\rho_1, \dots, \rho_n, v_1, \dots, v_n, \theta)$.

Definition 1. *A function* $(\rho_1, \dots, \rho_n, v_1, \dots, v_n, \theta)$ *is called a weak solution of the Class–II model* (3.62)–(3.64)*, if for all* $i \in \{1, \dots, n\}$*:*

$$0 \le \rho_i \in C^0([0,\infty); L^1(\mathbb{T}^3)), \quad \rho_i v_i \in C^0([0,\infty); L^1(\mathbb{T}^3;\mathbb{R}^3)),$$

$$\rho_i v_i \otimes v_i \in L^1_{\text{loc}}(\mathbb{T}^3 \times [0,\infty); \mathbb{R}^3 \times \mathbb{R}^3), \quad p_i \in L^1_{\text{loc}}(\mathbb{T}^3 \times [0,\infty)),$$

$$\rho_i b_i \in L^1_{\text{loc}}(\mathbb{T}^3 \times [0,\infty); \mathbb{R}^3),$$

$$0 < \theta \in C^0([0,\infty); L^1(\mathbb{R}^3)), \quad (\rho_i e_i + \frac{1}{2}\rho_i v_i^2) \in C^0([0,\infty); L^1(\mathbb{R}^3)),$$

$$(\rho_i e_i + p_i + \frac{1}{2}\rho_i v_i^2) v_i \in L^1_{\text{loc}}(\mathbb{R}^3 \times [0,\infty); \mathbb{R}^3), \quad \kappa \nabla\theta \in L^1_{\text{loc}}(\mathbb{R}^3 \times [0,\infty); \mathbb{R}^3),$$

$$(\rho_i b_i \cdot v_i + \rho r) \in L^1_{\text{loc}}(\mathbb{R}^3 \times [0,\infty))$$

and $(\rho_1, \dots, \rho_n, v_1, \dots, v_n, \theta)$ *solves for all* $\psi_i, \xi \in C_c^\infty([0,\infty); C^\infty(\mathbb{T}^3))$ *and* $\phi_i \in C_c^\infty([0,\infty); C^\infty(\mathbb{T}^3;\mathbb{R}^3))$*:*

$$\begin{aligned} -\int_{\mathbb{T}^3} \rho_i(x,0)\psi_i(x,0)\mathrm{d}x - \int_0^\infty \int_{\mathbb{T}^3} \rho_i \partial_t \psi_i \mathrm{d}x\mathrm{d}t \\ - \int_0^\infty \int_{\mathbb{T}^3} \rho_i v_i \cdot \nabla\psi_i \mathrm{d}x\mathrm{d}t = 0, \end{aligned} \tag{3.85}$$

$$\begin{aligned} &-\int_{\mathbb{T}^3} (\rho_i v_i)(x,0)\phi_i(x,0)\mathrm{d}x - \int_0^\infty \int_{\mathbb{T}^3} \rho_i v_i \cdot \partial_t \phi_i \mathrm{d}x\mathrm{d}t \\ &- \int_0^\infty \int_{\mathbb{T}^3} (\rho_i v_i \otimes v_i + p_i \mathbb{I}) : \nabla\phi_i \mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\mathbb{T}^3} \rho_i b_i \phi_i \mathrm{d}x\mathrm{d}t \\ &- \frac{1}{\epsilon} \int_0^\infty \int_{\mathbb{T}^3} \theta \sum_{j=1}^n b_{ij}\rho_i\rho_j(v_i - v_j)\phi_i \mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.86}$$

and

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{T}^3} (\rho e + \frac{1}{2}\sum_{j=1}^n \rho_j v_j^2)(x,0)\xi(x,0)\mathrm{d}x\mathrm{d}t \\
&-\int_0^\infty \int_{\mathbb{T}^3} (\rho e + \frac{1}{2}\sum_{j=1}^n \rho_j v_j^2)\partial_t \xi \mathrm{d}x\mathrm{d}t \\
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{j=1}^n (\rho_j e_j + p_j + \frac{1}{2}\rho_j v_j^2) v_j \cdot \nabla \xi \mathrm{d}x\mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{T}^3} \kappa \nabla\theta \cdot \nabla \xi \mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} (\sum_{j=1}^n \rho_j b_j \cdot v_j + \rho r)\xi \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.87}
$$

Definition 2. *A function* $(\rho_1, \dots, \rho_n, v_1, \dots, v_n, \theta)$ *is called a dissipative weak solution of the Class–II model* (3.62)–(3.64)*, if it is a weak solution and moreover:*

$$
\rho\eta \in C^0([0,\infty); L^1(\mathbb{T}^3)), \quad \rho_i \eta_i v_i \in L^1_{\mathrm{loc}}(\mathbb{T}^3 \times [0,\infty); \mathbb{R}^3),
$$

$$
\kappa \nabla \log\theta \in L^1_{\mathrm{loc}}(\mathbb{T}^3 \times [0,\infty); \mathbb{R}^3), \quad \kappa |\nabla \log\theta|^2 \in L^1_{\mathrm{loc}}(\mathbb{T}^3 \times [0,\infty)),
$$

$$
\frac{\rho r}{\theta} \in L^1_{\mathrm{loc}}(\mathbb{T}^3 \times [0,\infty)),
$$

for all $i \in \{1, \dots, n\}$ *and the integrated entropy inequality*

$$
\begin{aligned}
&-\int_{\mathbb{T}^3} (\rho\eta)(x,0)\chi(x,0)\mathrm{d}x - \int_0^\infty \int_{\mathbb{T}^3} \rho\eta \partial_t \chi \mathrm{d}x\mathrm{d}t \\
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{j=1}^n \rho_j \eta_j v_j \cdot \nabla\chi \mathrm{d}x\mathrm{d}t \geq -\int_0^\infty \int_{\mathbb{T}^3} \frac{1}{\theta}\kappa \nabla\theta \cdot \nabla\chi \mathrm{d}x\mathrm{d}t \\
&+\int_0^\infty \int_{\mathbb{T}^3} \frac{1}{\theta^2}\kappa |\nabla\theta|^2 \chi \mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} \frac{\rho r}{\theta}\chi \mathrm{d}x\mathrm{d}t \\
&+\frac{1}{2\epsilon}\sum_{i=1}^n \sum_{j=1}^n \int_0^\infty \int_{\mathbb{T}^3} b_{ij}\rho_i \rho_j |v_i - v_j|^2 \chi \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.88}
$$

holds for all test functions $\chi \in C_c^\infty([0,\infty); C^\infty(\mathbb{T}^3))$.

Definition 3. *A function* $(\bar\rho_1, \dots, \bar\rho_n, \bar v_1, \dots, \bar v_n, \bar\theta)$ *is called a strong solution of the Class–I model* (3.66)–(3.70)*, if* (3.66)–(3.71) *hold almost everywhere in* $\mathbb{T}^3$ *and for all* $t > 0$.

3.2.4 Derivation of the relative entropy inequality

We, now, make the derivation of the relative entropy inequality rigorous:

Proposition 4. *Let* ω *be a dissipative weak solution of the Class–II model* (3.62)–(3.64) *and* $\bar{\omega}$ *a strong solution of the Class–I model* (3.66)–(3.71). *Then, the following relative entropy inequality holds for every* $t > 0$*:*

$$
\begin{aligned}
&\mathcal{H}(\omega|\bar{\omega})(t) + \frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar{\theta}b_{ij}\rho_i\rho_j|(v_i - v_j) - (\bar{v}_i - \bar{v}_j)|^2 \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t\int_{\mathbb{T}^3}\bar{\theta}\kappa|\nabla\log\theta - \nabla\log\bar{\theta}|^2\mathrm{d}x\mathrm{d}s \le \mathcal{H}(\omega|\bar{\omega})(0) \\
&- \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}p_i(\omega|\bar{\omega})\operatorname{div}\bar{v}_i\mathrm{d}x\mathrm{d}s + \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\rho_i(b_i - \bar{b}_i)\cdot(v_i - \bar{v}_i)\mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\rho_i(v_i - \bar{v}_i)\cdot\nabla\bar{v}_i(v_i - \bar{v}_i)\mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i\eta_i)(\omega|\bar{\omega})(\partial_s\bar{\theta} + \bar{v}_i\cdot\nabla\bar{\theta})\mathrm{d}x\mathrm{d}s + \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar{R}_i\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\rho_i(v_i - \bar{v}_i)\cdot\nabla\bar{\theta}(\eta_i - \bar{\eta}_i)\mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\frac{\rho_i}{\bar{\rho}_i}(v_i - \bar{v}_i)\cdot\bar{R}_i\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathbb{T}^3}\bar{Q}\mathrm{d}x\mathrm{d}s \qquad (3.89)\\
&+ \int_0^t\int_{\mathbb{T}^3}(\nabla\log\theta - \nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}(\theta\bar{\kappa} - \bar{\theta}\kappa)\mathrm{d}x\mathrm{d}s \\
&- \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar{\theta}b_{ij}\rho_i(v_i - \bar{v}_i)\cdot(\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j)\mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\theta - \bar{\theta})b_{ij}\rho_i\rho_j(v_i - \bar{v}_i)\cdot(\bar{v}_i - \bar{v}_j)\mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\theta - \bar{\theta})b_{ij}(\rho_i - \bar{\rho}_i)\bar{\rho}_j(\bar{v}_i - \bar{v}_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\theta - \bar{\theta})b_{ij}\rho_i(\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t\int_{\mathbb{T}^3}\left(\frac{\rho r}{\theta} - \frac{\bar{\rho}\bar{r}}{\bar{\theta}}\right)(\theta - \bar{\theta})dxds.
\end{aligned}
$$

Proof. Multiply (3.76),(3.80),(3.82) and (3.79) by the test functions $\psi_i, \phi_i, \xi, \chi$ respectively, as in the weak formulation of the equations of the Class–II model, integrate them over $\mathbb{T}^3 \times (0,\infty)$ and subtract them from (3.85)–(3.88), in order to obtain:

$$
\begin{aligned}
-\int_{\mathbb{T}^3}(\rho_i-\bar\rho_i)(x,0)\psi_i(x,0)\mathrm{d}x-\int_0^\infty\int_{\mathbb{T}^3}(\rho_i-\bar\rho_i)\partial_t\psi_i\mathrm{d}x\mathrm{d}t& \\
-\int_0^\infty\int_{\mathbb{T}^3}(\rho_iv_i-\bar\rho_i\bar v_i)\cdot\nabla\psi_i\mathrm{d}x\mathrm{d}t&=0,
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}(\rho_iv_i-\bar\rho_i\bar v_i)(x,0)\phi_i(x,0)\mathrm{d}x-\int_0^\infty\int_{\mathbb{T}^3}(\rho_iv_i-\bar\rho_i\bar v_i)\partial_t\phi_i\mathrm{d}x\mathrm{d}t\\
&-\int_0^\infty\int_{\mathbb{T}^3}(\rho_iv_i\otimes v_i-\bar\rho_i\bar v_i\otimes\bar v_i):\nabla\phi_i\mathrm{d}x\mathrm{d}t\\
&=\int_0^\infty\int_{\mathbb{T}^3}(\rho_ib_i-\nabla p_i-\bar\rho_i\bar b_i+\nabla\bar p_i)\cdot\phi_i\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{T}^3}\bar R_i\cdot\phi_i\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{\epsilon}\int_0^\infty\int_{\mathbb{T}^3}\left(\theta\sum_{j\neq i}b_{ij}\rho_i\rho_j(v_i-v_j)-\bar\theta\sum_{j\neq i}b_{ij}\bar\rho_i\bar\rho_j(\bar v_i-\bar v_j)\right)\cdot\phi_i\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}\left(\rho e+\frac12\sum_{j=1}^n\rho_jv_j^2-\bar\rho\bar e-\frac12\sum_{j=1}^n\bar\rho_j\bar v_j^2\right)(x,0)\xi(x,0)\mathrm{d}x\\
&-\int_0^\infty\int_{\mathbb{T}^3}\left(\rho e+\frac12\sum_{j=1}^n\rho_jv_j^2-\bar\rho\bar e-\frac12\sum_{j=1}^n\bar\rho_j\bar v_j^2\right)\partial_t\xi\mathrm{d}x\mathrm{d}t\\
&+\int_0^\infty\int_{\mathbb{T}^3}(\kappa\nabla\theta-\bar\kappa\nabla\bar\theta)\cdot\nabla\xi\mathrm{d}x\mathrm{d}t+\int_0^\infty\int_{\mathbb{T}^3}\bar Q\xi\mathrm{d}x\mathrm{d}t\\
&-\int_0^\infty\int_{\mathbb{T}^3}\sum_{j=1}^n\left((\rho_je_j+p_j+\frac12\rho_jv_j^2)v_j-(\bar\rho_j\bar e_j+\bar p_j+\frac12\bar\rho_j\bar v_j^2)\bar v_j\right)\cdot\nabla\xi\mathrm{d}x\mathrm{d}t\\
&=\int_0^\infty\int_{\mathbb{T}^3}\left(\rho r+\sum_{j=1}^n\rho_jb_j\cdot v_j-\bar\rho\bar r-\sum_{j=1}^n\bar\rho_j\bar b_j\cdot\bar v_j\right)\xi\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and

$$-\int_{\mathbb{T}^3}(\rho\eta-\bar{\rho}\bar{\eta})(x,0)\chi(x,0)\mathrm{d}x-\int_0^\infty\int_{\mathbb{T}^3}(\rho\eta-\bar{\rho}\bar{\eta})\partial_t\chi\mathrm{d}x\mathrm{d}t$$
$$-\int_0^\infty\int_{\mathbb{T}^3}\sum_{j=1}^n(\rho_j\eta_jv_j-\bar{\rho}_j\bar{\eta}_j\bar{v}_j)\cdot\nabla\chi\mathrm{d}x\mathrm{d}t$$
$$\geq-\int_0^\infty\int_{\mathbb{T}^3}\left(\frac{1}{\theta}\kappa\nabla\theta-\frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\right)\cdot\nabla\chi\mathrm{d}x\mathrm{d}t+\int_0^\infty\int_{\mathbb{T}^3}\left(\frac{\rho r}{\theta}-\frac{\bar{\rho}\bar{r}}{\bar{\theta}}\right)\chi\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\infty\int_{\mathbb{T}^3}\left(\frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\rho_i\rho_j|v_i-v_j|^2-\frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2\right)\chi\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\infty\int_{\mathbb{T}^3}\left(\frac{1}{\theta^2}\kappa|\nabla\theta|^2-\frac{1}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2\right)\chi\mathrm{d}x\mathrm{d}t.$$

We choose the test functions $\psi_i=\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)\zeta,\ \ \psi_i=\bar{v}_i\zeta,\ \ \xi=-\zeta$ and $\chi=\bar{\theta}\zeta$ where

$$\zeta(s)=\begin{cases}1 & 0\leq s<t\\ \frac{t-s}{\delta}+1 & t\leq s<t+\delta\ ,\\ 0 & s\geq t+\delta\end{cases}$$

and let $\delta\to 0$, to obtain:

$$-\int_{\mathbb{T}^3}(\rho_i-\bar{\rho}_i)(x,0)\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)(x,0)\mathrm{d}x-\int_0^t\int_{\mathbb{T}^3}(\rho_i-\bar{\rho}_i)\partial_s\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)\mathrm{d}x\mathrm{d}s$$
$$+\int_{\mathbb{T}^3}(\rho_i-\bar{\rho}_i)\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}(\rho_iv_i-\bar{\rho}_i\bar{v}_i)\cdot\nabla\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)\mathrm{d}x=0,$$

$$-\int_{\mathbb{T}^3}(\rho_iv_i-\bar{\rho}_i\bar{v}_i)(x,0)\bar{v}_i(x,0)\mathrm{d}x-\int_0^t\int_{\mathbb{T}^3}(\rho_iv_i-\bar{\rho}_i\bar{v}_i)\partial_s\bar{v}_i\mathrm{d}x\mathrm{d}s$$
$$+\int_{\mathbb{T}^3}(\rho_iv_i-\bar{\rho}_i\bar{v}_i)\bar{v}_i\mathrm{d}x-\int_0^t\int_{\mathbb{T}^3}(\rho_iv_i\otimes v_i-\bar{\rho}_i\bar{v}_i\otimes\bar{v}_i):\nabla\bar{v}_i\mathrm{d}x\mathrm{d}s$$
$$=\int_0^t\int_{\mathbb{T}^3}(\rho_ib_i-\nabla p_i-\bar{\rho}_i\bar{b}_i+\nabla\bar{p}_i)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\bar{R}_i\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\left(\theta\sum_{j\neq i}b_{ij}\rho_i\rho_j(v_i-v_j)-\bar{\theta}\sum_{j\neq i}b_{ij}\bar{\rho}_i\bar{\rho}_j(\bar{v}_i-\bar{v}_j\right)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s,$$

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\left(\rho e+\frac{1}{2}\sum_{j=1}^n\rho_j v_j^2-\bar{\rho}\bar{e}-\frac{1}{2}\sum_{j=1}^n\bar{\rho}_j\bar{v}_j^2\right)(x,0)\mathrm{d}x\\
&-\int_{\mathbb{T}^3}\left(\rho e+\frac{1}{2}\sum_{j=1}^n\rho_j v_j^2-\bar{\rho}\bar{e}-\frac{1}{2}\sum_{j=1}^n\bar{\rho}_j\bar{v}_j^2\right)\mathrm{d}x\\
&=-\int_0^t\int_{\mathbb{T}^3}\left(\rho r+\sum_{j=1}^n\rho_j b_j\cdot v_j-\bar{\rho}\bar{r}-\sum_{j=1}^n\bar{\rho}_j\bar{b}_j\cdot\bar{v}_j\right)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\bar{Q}\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}(\rho\eta-\bar{\rho}\bar{\eta})(x,0)\bar{\theta}(x,0)\mathrm{d}x-\int_0^t\int_{\mathbb{T}^3}(\rho\eta-\bar{\rho}\bar{\eta})\partial_s\bar{\theta}\mathrm{d}x\mathrm{d}s+\int_{\mathbb{T}^3}(\rho\eta-\bar{\rho}\bar{\eta})\bar{\theta}\mathrm{d}x\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{j=1}^n(\rho_j\eta_j v_j-\bar{\rho}_j\bar{\eta}_j\bar{v}_j)\cdot\nabla\bar{\theta}\mathrm{d}x\mathrm{d}s\geq-\int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{\theta}\kappa\nabla\theta-\frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\right)\cdot\nabla\bar{\theta}\mathrm{d}x\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\rho_i\rho_j|v_i-v_j|^2-\frac{1}{2\epsilon}\sum_{i=1}^n\sum_{j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2\right)\bar{\theta}\mathrm{d}x\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{\theta^2}\kappa|\nabla\theta|^2-\frac{1}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2\right)\bar{\theta}\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\left(\frac{\rho r}{\theta}-\frac{\bar{\rho}\bar{r}}{\bar{\theta}}\right)\bar{\theta}\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Then, summing everything up and by virtue of the computation

$$
\begin{aligned}
&-\sum_{i=1}^n(\rho_i-\bar{\rho}_i)\left(\bar{\mu}_i-\frac{1}{2}\bar{v}_i^2\right)-\sum_{i=1}^n(\rho_i v_i-\bar{\rho}_i\bar{v}_i)\bar{v}_i\\
&+\left(\rho e+\frac{1}{2}\sum_{i=1}^n\rho_i v_i^2-\bar{\rho}\bar{e}-\frac{1}{2}\sum_{i=1}^n\bar{\rho}_i\bar{v}_i^2\right)-(\rho\eta-\bar{\rho}\bar{\eta})\bar{\theta}\\
&=-\sum_{i=1}^n(\rho_i-\bar{\rho}_i)\bar{\mu}_i+\frac{1}{2}\sum_{i=1}^n(\rho_i\bar{v}_i^2-2\rho_i v_i\cdot\bar{v}_i+\rho_i v_i^2)+(\rho e-\bar{\rho}\bar{e})\\
&-(\rho\eta-\bar{\rho}\bar{\eta})\bar{\theta}=\frac{1}{2}\sum_{i=1}^n\rho_i|v_i-\bar{v}_i|^2-\sum_{i=1}^n(\rho_i\psi_i)_{\rho_i}(\rho_i-\bar{\rho}_i)+\rho e-\bar{\rho}\bar{e}-\rho\eta\bar{\theta}+\bar{\rho}\bar{\eta}\bar{\theta}\\
&=\frac{1}{2}\sum_{i=1}^n\rho_i|v_i-\bar{v}_i|^2+\sum_{i=1}^n(\rho_i\psi_i)(\omega|\bar{\omega})+(\rho\eta-\bar{\rho}\bar{\eta})(\theta-\bar{\theta})=\mathcal{H}(\omega|\bar{\omega}),
\end{aligned}
$$

one gets the inequality:

$$
\mathcal{H}(\omega|\bar{\omega})(t)\leq\mathcal{H}(\omega|\bar{\omega})(0)+I_1+I_2+I_3+I_4+I_5+I_6+I_7,
$$

where

$$I_1 = -\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i-\bar\rho_i)\cdot\partial_s\left(\bar\mu_i-\frac{1}{2}\bar v_i^2\right)\mathrm{d}x\mathrm{d}s$$
$$-\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i v_i-\bar\rho_i\bar v_i)\partial_s\bar v_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}(\rho\eta-\bar\rho\bar\eta)\partial_s\bar\theta\mathrm{d}x\mathrm{d}s,$$

$$I_2 = -\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i v_i-\bar\rho_i\bar v_i)\cdot\nabla\left(\bar\mu_i-\frac{1}{2}\bar v_i^2\right)\mathrm{d}x\mathrm{d}s$$
$$-\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i v_i\otimes v_i-\bar\rho_i\bar v_i\otimes\bar v_i):\nabla\bar v_i\mathrm{d}x\mathrm{d}s$$
$$-\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i v_i\eta_i-\bar\rho_i\bar v_i\bar\eta_i)\cdot\nabla\bar\theta\mathrm{d}x\mathrm{d}s,$$

$$I_3 = \frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\theta b_{ij}\rho_i\rho_j(v_i-v_j)\cdot\bar v_i\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar\theta b_{ij}\bar\rho_i\bar\rho_j(\bar v_i-\bar v_j)\cdot\bar v_i\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar\theta b_{ij}\rho_i\rho_j|v_i-v_j|^2\mathrm{d}x\mathrm{d}s$$
$$+\frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar\theta b_{ij}\bar\rho_i\bar\rho_j|\bar v_i-\bar v_j|^2\mathrm{d}x\mathrm{d}s,$$

$$I_4 = \int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{\theta}\kappa\nabla\theta-\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right)\cdot\nabla\bar\theta\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{\theta^2}\kappa|\nabla\theta|^2-\frac{1}{\bar\theta^2}\bar\kappa|\nabla\bar\theta|^2\right)\bar\theta\mathrm{d}x\mathrm{d}s,$$

$$I_5 = -\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(\rho_i b_i-\bar\rho_i\bar b_i)\cdot\bar v_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\left(\frac{\rho r}{\theta}-\frac{\bar\rho\bar r}{\bar\theta}\right)\bar\theta\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_{\mathbb{T}^3}\left(\rho r+\sum_{i=1}^{n}\rho_i b_i\cdot v_i-\bar\rho\bar r-\sum_{i=1}^{n}\bar\rho_i\bar b_i\cdot\bar v_i\right)\mathrm{d}x\mathrm{d}s,$$

$$I_6 = -\sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}(p_i-\bar p_i)\operatorname{div}\bar v_i\mathrm{d}x\mathrm{d}s$$

$$I_7 = \sum_{i=1}^{n}\int_0^t\int_{\mathbb{T}^3}\bar R_i\cdot\bar v_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\bar Q\mathrm{d}x\mathrm{d}s.$$

We, now, want to rearrange the above terms. We start with I_1 and carry out the following calculation:

$$
\begin{aligned}
I_1 = &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i) \left((\bar{\mu}_i)_{\rho_i} \partial_s \bar{\rho}_i + (\bar{\mu}_i)_\theta \partial_s \bar{\theta} \right) \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \partial_s \bar{v}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i - \bar{\rho}_i \bar{\eta}_i) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s \\
= &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i) \left((\bar{\mu}_i)_{\rho_i} \partial_s \bar{\rho}_i + (\bar{\mu}_i)_\theta \partial_s \bar{\theta} \right) \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \partial_s \bar{v}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i)(\omega|\bar{\omega}) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i \bar{\eta}_i)_{\rho_i} (\rho_i - \bar{\rho}_i) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i \bar{\eta}_i)_\theta (\theta - \bar{\theta}) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and since

$$
(\rho_i \eta_i)_{\rho_i} = (-(\rho_i \psi_i)_\theta)_{\rho_i} = -((\rho_i \psi_i)_{\rho_i})_\theta = -(\mu_i)_\theta, \tag{3.90}
$$

we see that

$$
\begin{aligned}
I_1 = &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \partial_s \bar{v}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i)(\omega|\bar{\omega}) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i \bar{\eta}_i)_\theta (\theta - \bar{\theta}) \partial_s \bar{\theta} \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)(\bar{\mu}_i)_{\rho_i} \partial_s \bar{\rho}_i \mathrm{d}x\mathrm{d}s \\
=: &\, I_{11} + \cdots + I_{14},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{13} &= -\int_0^t \int_{\mathbb{T}^3} \partial_s(\bar{\rho}\bar{\eta})(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i\bar{\eta}_i)_{\rho_i}(\theta-\bar{\theta})\partial_s\bar{\rho}_i \mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{T}^3} \operatorname{div}\left(\sum_{i=1}^n \bar{\rho}_i\bar{\eta}_i\bar{v}_i\right)(\theta-\bar{\theta}) \\
&\quad + \int_0^t \int_{\mathbb{T}^3} \frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\cdot\nabla(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \frac{1}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
&\quad - \frac{1}{2\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n\sum_{j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \frac{\bar{\rho}\bar{r}}{\bar{\theta}}(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i\bar{\eta}_i)_{\rho_i}(\theta-\bar{\theta})\nabla\bar{\rho}_i\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{\rho}_i\bar{\eta}_i)_{\rho_i}(\theta-\bar{\theta})\bar{\rho}_i \operatorname{div}\bar{v}_i\mathrm{d}x\mathrm{d}s \\
&=: I_{131} + \cdots + I_{137}
\end{aligned}
$$

and we have used (3.76) and (3.79) and an integration by parts in the term I_{132}.

Moreover,

$$
\begin{aligned}
I_{131} = \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla(\bar{\rho}_i\bar{\eta}_i)\cdot\bar{v}_i(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
+ \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{\rho}_i\bar{\eta}_i \operatorname{div}\bar{v}_i(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s =: I_{1311} + I_{1312}.
\end{aligned}
$$

Again using (3.76),

$$
\begin{aligned}
I_{14} &= \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)(\bar{\mu}_i)_{\rho_i} \nabla \bar{\rho}_i \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)(\bar{\mu}_i)_{\rho_i} \bar{\rho}_i \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i) \nabla \bar{\mu}_i \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)(\bar{\mu}_i)_{\theta} \nabla \bar{\theta} \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)(\bar{\mu}_i)_{\rho_i} \bar{\rho}_i \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&=: I_{141} + \cdots + I_{143}.
\end{aligned}
$$

We now write I_2 as

$$
\begin{aligned}
I_2 &= - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i v_i - \bar{\rho}_i \bar{v}_i) \cdot \nabla \bar{\mu}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i v_i \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i v_i - \bar{\rho}_i \bar{\eta}_i \bar{v}_i) \cdot \nabla \bar{\theta} \mathrm{d}x\mathrm{d}s \\
&= - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i v_i - \bar{\rho}_i \bar{v}_i) \cdot \nabla \bar{\mu}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i \bar{v}_i \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i v_i - \bar{\rho}_i \bar{\eta}_i \bar{v}_i) \cdot \nabla \bar{\theta} \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and if we add and subtract the term with the relative pressure:

$$
\begin{aligned}
I_2 &= - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i v_i - \bar{\rho}_i \bar{v}_i) \cdot \nabla \bar{\mu}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i \bar{v}_i \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n p_i(\omega | \bar{\omega}) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{p}_i)_{\rho_i} (\rho_i - \bar{\rho}_i) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar{p}_i)_{\theta} (\theta - \bar{\theta}) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i v_i - \bar{\rho}_i \bar{\eta}_i \bar{v}_i) \cdot \nabla \bar{\theta} \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (p_i - \bar{p}_i) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&=: I_{21} + \cdots + I_{28}.
\end{aligned}
$$

Now, I_{28} cancels out with I_6, while

$$\begin{aligned}
I_{27} = &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i\eta_i - \bar\rho_i\bar\eta_i)\bar v_i \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i\eta_i(v_i - \bar v_i) \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
= &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i\eta_i)(\omega|\bar\omega)\bar v_i \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar\rho_i\bar\eta_i)_{\rho_i}(\rho_i - \bar\rho_i)\bar v_i \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\bar\rho_i\bar\eta_i)_{\theta}(\theta - \bar\theta)\bar v_i \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i\eta_i(v_i - \bar v_i) \cdot \nabla\bar\theta \mathrm{d}x\mathrm{d}s \\
=: &\, I_{271} + \cdots + I_{274}
\end{aligned}$$

and thus I_{272} cancels out with I_{142} and I_{1311} cancels out with I_{136} and I_{273}. Furthermore, due to

$$\nabla p_i = \rho_i \nabla\mu_i + \rho_i\eta_i\nabla\theta, \tag{3.91}$$

which can be obtained by applying the gradient operator to (2.32) and using (2.50) and (2.51), we have

$$\begin{aligned}
I_{21} = &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla\bar\mu_i \cdot (\rho_i - \bar\rho_i)v_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla\bar\mu_i \cdot \bar\rho_i(v_i - \bar v_i)\mathrm{d}x\mathrm{d}s \\
= &-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla\bar\mu_i \cdot (\rho_i - \bar\rho_i)(v_i - \bar v_i)\mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla\bar\mu_i \cdot (\rho_i - \bar\rho_i)\bar v_i \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar\rho_i\bar\eta_i\nabla\bar\theta \cdot (v_i - \bar v_i)\mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \nabla\bar p_i \cdot (v_i - \bar v_i)\mathrm{d}x\mathrm{d}s \\
=: &\, I_{211} + \cdots + I_{214},
\end{aligned}$$

where I_{212} cancels out with I_{141} and

$$I_{213} + I_{274} = -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i \eta_i - \bar{\rho}\bar{\eta}_i)(v_i - \bar{v}_i) \cdot \nabla \bar{\theta} \mathrm{d}x \mathrm{d}s.$$

Regarding I_{11}, using (3.80) we get

$$\begin{aligned}
I_{11} &= \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i \bar{v}_i \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \mathrm{d}x \mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \cdot \bar{b}_i \mathrm{d}x \mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i} (v_i - \bar{v}_i) \cdot \nabla \bar{p}_i \mathrm{d}x \mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i} (v_i - \bar{v}_i) \cdot \bar{R}_i \mathrm{d}x \mathrm{d}s \\
&+ \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \rho_i (v_i - \bar{v}_i) \bar{\theta} b_{ij} \bar{\rho}_j (\bar{v}_i - \bar{v}_j) \mathrm{d}x \mathrm{d}s =: I_{111} + \cdots + I_{115}.
\end{aligned}$$

Notice that I_{111} cancels out with I_{23} and

$$I_{214} + I_{113} = \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{1}{\bar{\rho}_i} \nabla \bar{p}_i \cdot (v_i - \bar{v}_i)(\rho_i - \bar{\rho}_i),$$

which combined with I_{211} gives, due to (3.91),

$$I_{214} + I_{113} + I_{211} = \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{\eta}_i (\rho_i - \bar{\rho}_i)(v_i - \bar{v}_i) \cdot \nabla \bar{\theta} \mathrm{d}x \mathrm{d}s$$

and hence

$$I_{214} + I_{113} + I_{211} + I_{213} + I_{274} = -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i (v_i - \bar{v}_i) \cdot \nabla \bar{\theta} (\eta_i - \bar{\eta}_i) \mathrm{d}x \mathrm{d}s.$$

Finally, (3.91) implies

$$(p_i)_{\rho_i} = \rho_i (\mu_i)_{\rho_i}, \tag{3.92}$$

$$(p_i)_\theta = \rho_i \eta_i + \rho_i (\mu_i)_\theta, \tag{3.93}$$

and due to (3.93),

$$I_{1312} + I_{26} = -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{\rho}_i (\bar{\mu}_i)_\theta (\theta - \bar{\theta}) \operatorname{div} \bar{v}_i \mathrm{d}x \mathrm{d}s,$$

which cancels out with I_{137}, because of (3.90), while due to (3.92), I_{25} cancels out with I_{143}.

Putting together I_1, I_2 and I_6, we get

$$
\begin{aligned}
I_1 + I_2 + I_6 = &- \sum_{i=1}^{n} \int_0^t \int_{\mathbb{T}^3} p_i(\omega|\bar{\omega}) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n} \int_0^t \int_{\mathbb{T}^3} \frac{\rho_i}{\bar{\rho}_i}(v_i - \bar{v}_i) \cdot \bar{R}_i \mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n} \int_0^t \int_{\mathbb{T}^3} \rho_i(v_i - \bar{v}_i) \cdot \nabla \bar{v}_i (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \rho_i(v_i - \bar{v}_i) \cdot \bar{b}_i \mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n} \int_0^t \int_{\mathbb{T}^3} (\rho_i \eta_i)(\omega|\bar{\omega})(\partial_s \bar{\theta} + \bar{v}_i \cdot \nabla \bar{\theta}) \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{T}^3} \frac{1}{\bar{\theta}} \bar{\kappa} \nabla \bar{\theta} \cdot \nabla(\theta - \bar{\theta}) \mathrm{d}x\mathrm{d}s \\
&- \sum_{i=1}^{n} \int_0^t \int_{\mathbb{T}^3} \rho_i(v_i - \bar{v}_i) \cdot \nabla \bar{\theta} (\eta_i - \bar{\eta}_i) \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} \frac{1}{\bar{\theta}^2} \bar{\kappa} |\nabla \bar{\theta}|^2 (\theta - \bar{\theta}) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \frac{\bar{\rho}\bar{r}}{\bar{\theta}} (\theta - \bar{\theta}) \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_i (v_i - \bar{v}_i) \bar{\theta} b_{ij} \bar{\rho}_j (\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \\
&- \frac{1}{2\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \bar{\rho}_i \bar{\rho}_j |\bar{v}_i - \bar{v}_j|^2 (\theta - \bar{\theta}) \mathrm{d}x\mathrm{d}s.
\end{aligned}
\tag{3.94}
$$

The plan for the rest of the proof is to combine I_3 and I_4 with the friction and heat–conduction terms in (3.94) respectively, I_5 with the terms containing forces and heat supply and I_7 with the term containing the remainder $\bar{R}_i$.

We start by noticing that

$$
\begin{aligned}
I_3 = &\frac{1}{\epsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \int_{\mathbb{T}^3} \theta b_{ij} \rho_i \rho_j (v_i - v_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
&- \frac{1}{2\epsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \int_{\mathbb{T}^3} \bar{\theta} b_{ij} \rho_i \rho_j |v_i - v_j|^2 \mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

The reason is that due to the symmetry of b_{ij}

$$-\frac{1}{\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\theta}b_{ij}\bar{\rho}_i\bar{\rho}_j(\bar{v}_i-\bar{v}_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s=-\frac{1}{2\epsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\theta}b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2\mathrm{d}x\mathrm{d}s$$

and thus the second and fourth terms of I_3 cancel out with each other. Therefore, for the friction terms, we have:

$$\begin{aligned}
F:=I_3&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_i(v_i-\bar{v}_i)\bar{\theta}b_{ij}\bar{\rho}_j(\bar{v}_i-\bar{v}_j)\mathrm{d}x\mathrm{d}s\\
&\qquad-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s\\
&=\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta b_{ij}\rho_i\rho_j(v_i-v_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s\\
&-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\theta}b_{ij}\rho_i\rho_j|v_i-v_j|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_i(v_i-\bar{v}_i)\bar{\theta}b_{ij}\bar{\rho}_j(\bar{v}_i-\bar{v}_j)\mathrm{d}x\mathrm{d}s\\
&-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}\theta\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}\bar{\theta}\bar{\rho}_i\bar{\rho}_j|\bar{v}_i-\bar{v}_j|^2\mathrm{d}x\mathrm{d}s\\
&=:F_1+\cdots+F_5
\end{aligned}$$

and we start by collecting only the terms that are multiplied by $\bar{\theta}$:

$$
\begin{aligned}
F_2 + F_3 + F_5 = &- \frac{1}{2\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i \rho_j |v_i - v_j|^2 \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{2\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \bar{\rho}_i \bar{\rho}_j |\bar{v}_i - \bar{v}_j|^2 \mathrm{d}x\mathrm{d}s \\
&- \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i \rho_j (v_i - v_j) \cdot (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i \rho_j (v_i - v_j) \cdot (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i (v_i - \bar{v}_i) \cdot \bar{\rho}_j (\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \\
=&: f_1 + \cdots + f_5.
\end{aligned}
$$

The last term can be written as

$$
\begin{aligned}
f_5 = &-\frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i (v_i - \bar{v}_i) \cdot (\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i (v_i - \bar{v}_i) \cdot \rho_j (\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \\
=&: f_{51} + f_{52}
\end{aligned}
\tag{3.95}
$$

and f_{52} can be combined with f_3, in order to give

$$
f_{52} + f_3 = -\frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i \rho_j (v_i - \bar{v}_i) \cdot ((v_i - v_j) - (\bar{v}_i - \bar{v}_j)) \mathrm{d}x\mathrm{d}s,
$$

and because of the symmetry in the indices i, j

$$
f_{52} + f_3 = -\frac{1}{2\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i \rho_j |(v_i - v_j) - (\bar{v}_i - \bar{v}_j)|^2 \mathrm{d}x\mathrm{d}s.
$$

Now, we collect F_1 and F_4 with f_1, f_2 and f_4 to obtain

$$
\begin{aligned}
F_1 + F_4 + f_1 + f_2 + f_4 = &-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_j \cdot (v_i - \bar{v}_i)\mathrm{d}x\mathrm{d}s \\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_i \cdot \bar{v}_i\mathrm{d}x\mathrm{d}s \\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_j \cdot \bar{v}_i\mathrm{d}x\mathrm{d}s \\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \theta b_{ij}\bar{\rho}_i\bar{\rho}_j(\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i\mathrm{d}x\mathrm{d}s \\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j(v_i - v_j) \cdot v_i\mathrm{d}x\mathrm{d}s \\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \bar{\theta} b_{ij}\bar{\rho}_i\bar{\rho}_j(\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i\mathrm{d}x\mathrm{d}s \\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_i \cdot (v_i - \bar{v}_i)\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and, due to symmetry,

$$
-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_j \cdot \bar{v}_i\mathrm{d}x\mathrm{d}s = -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_i \cdot \bar{v}_j\mathrm{d}x\mathrm{d}s,
$$

which implies that

$$
\begin{aligned}
F_1 + F_4 + f_1 + f_2 + f_4 = \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_i \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \theta b_{ij}\rho_i\rho_j v_i \cdot \bar{v}_j \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \theta b_{ij}\bar{\rho}_i\bar{\rho}_j (\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_i \cdot v_i \mathrm{d}x\mathrm{d}s \\
+ \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_i \cdot v_j \mathrm{d}x\mathrm{d}s \\
+ \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij}\bar{\rho}_i\bar{\rho}_j (\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
+ \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_i \cdot (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij}\rho_i\rho_j v_j \cdot (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}s \\
= \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\rho_i\rho_j v_i \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\rho_i\rho_j \bar{v}_i \cdot \bar{v}_j \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\rho_i\rho_j (v_i - \bar{v}_i) \cdot \bar{v}_j \mathrm{d}x\mathrm{d}s \\
- \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\bar{\rho}_i\bar{\rho}_j (\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\rho_i\rho_j (v_i - \bar{v}_i) \cdot (\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \\
+ \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}(\rho_i - \bar{\rho}_i)\bar{\rho}_j (\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s \\
+ \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n (\theta - \bar{\theta}) b_{ij}\rho_i(\rho_j - \bar{\rho}_j) (\bar{v}_i - \bar{v}_j) \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

so that

$$
\begin{aligned}
F = & -\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n\bar{\theta}b_{ij}\rho_i\rho_j|(v_i-v_j)-(\bar{v}_i-\bar{v}_j)|^2\mathrm{d}x\mathrm{d}s \\
& -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n\bar{\theta}b_{ij}\rho_i(v_i-\bar{v}_i)\cdot(\rho_j-\bar{\rho}_j)(\bar{v}_i-\bar{v}_j)\mathrm{d}x\mathrm{d}s \\
& +\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n(\theta-\bar{\theta})b_{ij}\rho_i\rho_j(v_i-\bar{v}_i)\cdot(\bar{v}_i-\bar{v}_j)\mathrm{d}x\mathrm{d}s \\
& +\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n(\theta-\bar{\theta})b_{ij}(\rho_i-\bar{\rho}_i)\bar{\rho}_j(\bar{v}_i-\bar{v}_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s \\
& +\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\sum_{j=1}^n(\theta-\bar{\theta})b_{ij}\rho_i(\rho_j-\bar{\rho}_j)(\bar{v}_i-\bar{v}_j)\cdot\bar{v}_i\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Regarding the heat conduction terms,

$$
\begin{aligned}
& I_4+\int_0^t\int_{\mathbb{T}^3}\frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\cdot\nabla(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\frac{1}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
& =\int_0^t\int_{\mathbb{T}^3}\left(\kappa\nabla\log\theta-\bar{\kappa}\nabla\log\bar{\theta}\right)\cdot\nabla\bar{\theta}\mathrm{d}x\mathrm{d}s \\
& -\int_0^t\int_{\mathbb{T}^3}(\kappa|\nabla\log\theta|^2-\bar{\kappa}|\nabla\log\bar{\theta}|^2)\bar{\theta}\mathrm{d}x\mathrm{d}s \\
& \quad+\int_0^t\int_{\mathbb{T}^3}\bar{\kappa}\nabla\log\bar{\theta}\cdot(\nabla\theta-\nabla\bar{\theta})\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\bar{\kappa}|\nabla\log\bar{\theta}|^2(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
& =-\int_0^t\int_{\mathbb{T}^3}\bar{\theta}\kappa|\nabla\log\theta-\nabla\log\bar{\theta}|^2\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\bar{\theta}\kappa\mathrm{d}x\mathrm{d}s \\
& \quad+\int_0^t\int_{\mathbb{T}^3}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}\theta\bar{\kappa}\mathrm{d}x\mathrm{d}s \\
& =-\int_0^t\int_{\mathbb{T}^3}\bar{\theta}\kappa|\nabla\log\theta-\nabla\log\bar{\theta}|^2\mathrm{d}x\mathrm{d}s \\
& -\int_0^t\int_{\mathbb{T}^3}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}(\theta\bar{\kappa}-\bar{\theta}\kappa)\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We, now, pass to the terms containing forces and heat supplies:

$$
\begin{aligned}
& I_5-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(v_i-\bar{v}_i\cdot\bar{b}_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\frac{\bar{\rho}\bar{r}}{\bar{\theta}}(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s \\
& =\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(b_i-\bar{b}_i)\cdot(v_i-\bar{v}_i)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\left(\frac{\rho r}{\theta}-\frac{\bar{\rho}\bar{r}}{\bar{\theta}}\right)(\theta-\bar{\theta})\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and finally,

$$I_7 - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i}(v_i - \bar{v}_i) \cdot \bar{R}_i \mathrm{d}x\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{R}_i \cdot \bar{v}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \bar{Q}\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i}(v_i - \bar{v}_i) \cdot \bar{R}_i \mathrm{d}x\mathrm{d}s.$$

Putting everything together, we arrive at (3.89).

□

3.2.5 Validation of the high–friction limit

A careful estimation of the terms on the right-hand side of (3.89) implies the following theorem:

Theorem 5. *Let ω be a dissipative weak solution of the Class–II model* (3.62)–(3.64) *and $\bar{\omega}$ a strong solution of the Class–I model* (3.66)–(3.70). *We assume that the weak solution satisfies*

$$0 \leq \rho_1, \ldots, \rho_n \leq M, \quad 0 < \gamma \leq \rho \leq M, \quad 0 < \gamma \leq \theta \leq M$$

and the strong solution satisfies

$$0 < \gamma \leq \bar{\rho}_1, \ldots, \bar{\rho}_n \leq M, \quad |\bar{v}_1|, \ldots, |\bar{v}_n| \leq M, \quad 0 < \gamma \leq \bar{\theta} \leq M,$$

$$|\nabla \bar{v}_1|, \ldots, |\nabla \bar{v}_n| \leq M, \quad |\partial_t \bar{\theta}| \leq M, \quad |\nabla \bar{\theta}| \leq M,$$

for some $\gamma, M > 0$. Moreover, assume that κ and $\frac{\rho r}{\theta}$ are Lipschitz functions of $(\rho_1, \ldots, \rho_n, \theta)$, with κ bounded away from zero, b_i are Lipschitz functions of $(\rho_1, \ldots, \rho_n, v_1, \ldots, v_n, \theta)$, for all $i \in \{1, \ldots, n\}$ and the free energy functions $\rho_i \psi_i \in C^3(U)$ satisfy (3.73), *for all $i \in \{1, \ldots, n\}$. Then, if the initial data are such that $\mathcal{H}(\omega|\bar{\omega})(0) \to 0$, as $\epsilon \to 0$, we have that $\mathcal{H}(\omega|\bar{\omega})(t) \to 0$, for all $t > 0$, as $\epsilon \to 0$.*

Remark 6. *Notice that the assumption $\rho_i \psi_i \in C^3(U)$ does not in general hold* *for all free energy functions. In the case of the ideal gas, for instance, $\psi_i = R_i\theta \log \rho_i - c_i\theta \log \theta$, where R_i, c_i are constants, which fails to be C^3, unless both ρ_i and θ are bounded away from zero. In this case, Theorem 5 is still valid under the additional assumption $0 < \gamma \leq \rho_1, \ldots, \rho_n \leq M$ (see [22, Section 5]).*

Proof. Having obtained the relative entropy inequality (3.89), Theorem 5 is a direct application of Young's inequality and Grönwall's Lemma. In particular, we estimate each term on the right-hand side of (3.89), as follows:

We start by noticing that, according to [22, Lemma 4.1], due to the smoothness of the free energy and the bounds on the strong solution, we have the following bounds:

$$|p_i(\omega|\bar{\omega})| \leq C \left(|\rho_i - \bar{\rho}_i|^2 + |\theta - \bar{\theta}|^2\right)$$

and

$$|(\rho_i\eta_i)(\omega|\bar{\omega})| \leq C \left(|\rho_i - \bar{\rho}_i|^2 + |\theta - \bar{\theta}|^2\right),$$

which imply that

$$\begin{aligned} &\left| -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1^n} (\rho_i\eta_i)(\omega|\bar{\omega})(\partial_s\bar{\theta} + \bar{v}_i \cdot \nabla\bar{\theta}) \mathrm{d}x\mathrm{d}s \right| \\ &\qquad \leq C \int_0^t \int_{\mathbb{T}^3} \left(\sum_{i=1}^n |\rho_i - \bar{\rho}_i|^2 + |\theta - \bar{\theta}|^2 \right) \mathrm{d}x\mathrm{d}s \end{aligned}$$

and

$$\left| -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1^n} p_i(\omega|\bar{\omega}) \operatorname{div} \bar{v}_i \mathrm{d}x\mathrm{d}s \right| \leq C \int_0^t \int_{\mathbb{T}^3} \left(\sum_{i=1}^n |\rho_i - \bar{\rho}_i|^2 + |\theta - \bar{\theta}|^2 \right) \mathrm{d}x\mathrm{d}s.$$

Again by the smoothness of the free energy, and thus the entropy, we obtain

$$|\eta_i - \bar{\eta}_i| \leq C(|\rho_i - \bar{\rho}_i| + |\theta - \bar{\theta}|)$$

and thus by Young's inequality,

$$\begin{aligned}&\left|-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(v_i-\bar{v}_i)\cdot\nabla\bar{\theta}(\eta_i-\bar{\eta}_i)\mathrm{d}x\mathrm{d}s\right|\\&\quad\le C\int_0^t\int_{\mathbb{T}^3}\left(\sum_{i=1}^n|\rho_i-\bar{\rho}_i|^2+\sum_{i=1}^n\rho_i|v_i-\bar{v}_i|^2+|\theta-\bar{\theta}|^2\right)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Moreover, by Young's inequality and the Lipschitz continuity of b_i,

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(b_i-\bar{b}_i)\cdot(v_i-\bar{v}_i)\mathrm{d}x\mathrm{d}s\right|\\&\qquad\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\rho_i|v_i-\bar{v}_i|^2+\rho_i|b_i-\bar{b}_i|^2\right)\mathrm{d}x\mathrm{d}s\\&\qquad\le C\int_0^t\int_{\mathbb{T}^3}\left(\sum_{i=1}^n|\rho_i-\bar{\rho}_i|^2+\sum_{i=1}^n\rho_i|v_i-\bar{v}_i|^2+|\theta-\bar{\theta}|^2\right)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left|-\int_0^t\int_{\mathbb{T}^3}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}(\theta\bar{\kappa}-\bar{\theta}\kappa)\mathrm{d}x\mathrm{d}s\right|\\&\quad\le\int_0^t\int_{\mathbb{T}^3}|\sqrt{\bar{\theta}}\sqrt{\kappa}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}(\kappa-\bar{\kappa})\frac{\theta}{\sqrt{\bar{\theta}\kappa}}|\mathrm{d}x\mathrm{d}s\\&\qquad+\int_0^t\int_{\mathbb{T}^3}|\sqrt{\bar{\theta}}\sqrt{\kappa}(\nabla\log\theta-\nabla\log\bar{\theta})\cdot\nabla\log\bar{\theta}(\theta-\bar{\theta})\frac{\sqrt{\kappa}}{\sqrt{\bar{\theta}}}|\mathrm{d}x\mathrm{d}s\\&\quad\le\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\bar{\theta}\kappa|\nabla\log\theta-\nabla\log\bar{\theta}|^2\mathrm{d}x\mathrm{d}s\\&\qquad+C\int_0^t\int_{\mathbb{T}^3}\left(\sum_{i=1}^n|\rho_i-\bar{\rho}_i|^2+|\theta-\bar{\theta}|^2\right)\mathrm{d}x\mathrm{d}s\end{aligned}$$

by Young's inequality, the lower bounds of $\bar{\theta}$ and κ and the Lipschitz continuity of κ.

Also,

$$\left|\int_0^t\int_{\mathbb{T}^3}\left(\sum_{i=1}^n\bar{R}_i\cdot\bar{v}_i-\bar{Q}\right)\mathrm{d}x\mathrm{d}s\right|\le\mathcal{O}(\epsilon)$$

and

$$\left| -\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i}(v_i - \bar{v}_i) \cdot \bar{R}_i \mathrm{d}x\mathrm{d}s \right|$$
$$\leq C \left(\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i |v_i - \bar{v}_i|^2 \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i \frac{\bar{R}_i^2}{\bar{\rho}_i^2} \mathrm{d}x\mathrm{d}s \right)$$
$$\leq C \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i |v_i - \bar{v}_i|^2 \mathrm{d}x\mathrm{d}s + \mathcal{O}(\epsilon).$$

Finally,

$$\left| -\frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \sum_{j=1}^n \bar{\theta} b_{ij} \rho_i (v_i - \bar{v}_i) \cdot (\rho_j - \bar{\rho}_j)(\bar{v}_i - \bar{v}_j) \mathrm{d}x\mathrm{d}s \right|$$
$$\leq C \int_0^t \int_{\mathbb{T}^3} \left(\sum_{i=1}^n |\rho_i - \bar{\rho}_i|^2 + \sum_{i=1}^n \rho_i |v_i - \bar{v}_i|^2 \right) \mathrm{d}x\mathrm{d}s$$

and C does not depend on ϵ, because $\frac{1}{\epsilon}(\bar{v}_i - \bar{v}_j) = \mathcal{O}(1)$ and the remaining terms are treated in a similar fashion.

Putting everything together, we obtain

$$\begin{aligned} &\mathcal{H}(\omega|\bar{\omega})(t) + \frac{1}{2\epsilon} \sum_{i=1}^n \sum_{j=1}^n \int_0^t \int_{\mathbb{T}^3} \bar{\theta} b_{ij} \rho_i \rho_j |(v_i - v_j) - (\bar{v}_i - \bar{v}_j)|^2 \mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} \bar{\theta} \kappa |\nabla \log \theta - \nabla \log \bar{\theta}|^2 \mathrm{d}x\mathrm{d}s \leq \mathcal{H}(\omega|\bar{\omega})(0) \\ &+ C \int_0^t \int_{\mathbb{T}^3} \left(\sum_{i=1}^n |\rho_i - \bar{\rho}_i|^2 + \sum_{i=1}^n \rho_i |v_i - \bar{v}_i|^2 + |\theta - \bar{\theta}|^2 \right) \mathrm{d}x\mathrm{d}s + \mathcal{O}(\epsilon), \end{aligned} \tag{3.96}$$

where by virtue of (3.72) and (3.74),

$$\int_0^t \int_{\mathbb{T}^3} \left(\sum_{i=1}^n |\rho_i - \bar{\rho}_i|^2 + \sum_{i=1}^n \rho_i |v_i - \bar{v}_i|^2 + |\theta - \bar{\theta}|^2 \right) \mathrm{d}x\mathrm{d}s \leq C \int_0^t \mathcal{H}(\omega|\bar{\omega})(s) \mathrm{d}s.$$

The dissipation terms on the left-hand side of (3.96) are non-negative and thus can be neglected, yielding

$$\mathcal{H}(\omega|\bar{\omega})(t) \leq [\mathcal{H}(\omega|\bar{\omega})(0) + \mathcal{O}(\epsilon)] + C \int_0^t \mathcal{H}(\omega|\bar{\omega})(s) ds,$$

where $C > 0$ is independent of ϵ.

By Grönwall's Lemma

$$\mathcal{H}(\omega|\bar{\omega})(t) \leq [\mathcal{H}(\omega|\bar{\omega})(0) + \mathcal{O}(\epsilon)]e^{Ct},$$

where $C > 0$ does not depend on ϵ. Letting $\epsilon \to 0$, $\mathcal{H}(\omega|\bar{\omega})(0) \to 0$ and thus $\mathcal{H}(\omega|\bar{\omega})(t) \to 0$, for all $t > 0$ and the proof is completed.

□

Chapter 4

Class–I systems

In this chapter, we first study the structure of the emerging Class–I model. It turns out that it falls into the general theory of hyperbolic–parabolic composite type systems, for which are locally well–posed. This allows us to study the zero–diffusion limit for smooth solutions and develop a theory of convergence among thermomechanical theories as the parameters ϵ and κ tend to zero. This theory is based on a formal relative entropy computation that resembles the one from section 3.2.4 and can be made rigorous following the same steps. At the end of the chapter, we also present the case of a simple mixture of ideal gases.

4.1 Dissipative structure

For simplicity we select the external fields r_i and b_i to be zero and proceed to study the mathematical structure of the Class–I model, which can be written as follows:

$$\partial_t \rho_i + \operatorname{div}(\rho_i v) = \operatorname{div}(-\rho_i u_i) \tag{4.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p\mathbb{I}) = 0 \tag{4.2}$$

$$\partial_t \left(\rho e + \frac{1}{2}\rho v^2\right) + \operatorname{div}\left((\rho e + \frac{1}{2}\rho v^2)v + pv\right) = \operatorname{div}\left(\kappa \nabla \theta - \sum_i (\rho_i e_i + p_i) u_i\right) \tag{4.3}$$

where u_i are determined by inverting the constrained linear system

$$-\sum_{j\neq i} b_{ij}\theta\rho_i\rho_j(u_i-u_j)=\epsilon\left(-\frac{\rho_i}{\rho}\nabla p+\rho_i\theta\nabla\frac{\mu_i}{\theta}-\theta(\rho_i e_i+p_i)\nabla\frac{1}{\theta}\right) \tag{4.4}$$

$$\sum_i \rho_i u_i = 0\,. \tag{4.5}$$

Given the constitutive choices outlined in chapter 2 smooth solutions of (4.1)–(4.5) satisfy the entropy identity

$$\begin{aligned}\partial_t(-\rho\eta)+\operatorname{div}(-\rho\eta v)=\operatorname{div}&\left(\frac{1}{\theta}\sum_i(\rho_i e_i+p_i-\rho_i\mu_i)u_i-\frac{1}{\theta}\kappa\nabla\theta\right)\\ &-\frac{1}{\theta^2}\kappa|\nabla\theta|^2+\frac{1}{\theta}\sum_i u_i\cdot d_i.\end{aligned} \tag{4.6}$$

Notice we multiplied the entropy identity by minus one, because we want to study the mathematical entropy, which is defined as the negative thermodynamic entropy.

Next, setting $U=[\rho_1,\ldots,\rho_n,v,\theta]^\top$ with $v=(v_1,v_2,v_3)$, system (4.1)–(4.3) can be written in the form:

$$\partial_t(A(U))+\operatorname{div}(F(U))=\operatorname{div}(\varepsilon B(U)\nabla U) \tag{4.7}$$

where

$$A(U)=\begin{bmatrix}\rho_1\\ \vdots\\ \rho_n\\ \rho v_1\\ \rho v_2\\ \rho v_3\\ \rho e+\frac{1}{2}\rho v^2\end{bmatrix} \tag{4.8}$$

$$F(U)=\begin{bmatrix} \rho_1 v_1 & \rho_1 v_2 & \rho_1 v_3 \\ \vdots & \vdots & \vdots \\ \rho_n v_1 & \rho_n v_2 & \rho_n v_3 \\ \rho v_1^2+p & \rho v_1 v_2 & \rho v_1 v_3 \\ \rho v_1 v_2 & \rho v_2^2+p & \rho v_2 v_3 \\ \rho v_1 v_3 & \rho v_2 v_3 & \rho v_3^2+p \\ (\rho e+\frac{1}{2}\rho v^2+p)v_1 & (\rho e+\frac{1}{2}\rho v^2+p)v_2 & (\rho e+\frac{1}{2}\rho v^2+p)v_3 \end{bmatrix} \tag{4.9}$$

$$B^\kappa(U)=\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \nabla U=\begin{bmatrix} \nabla\rho_1 \\ \vdots \\ \nabla\rho_n \\ \nabla v_1 \\ \nabla v_2 \\ \nabla v_3 \\ \nabla\theta \end{bmatrix}$$

$$B^\epsilon(U)=\begin{bmatrix} A_{11} & \cdots & A_{1n} & 0 & 0 & 0 & A_{1,n+4} \\ \vdots & \ddots & \vdots & 0 & 0 & 0 & \vdots \\ A_{n1} & \cdots & A_{nn} & 0 & 0 & 0 & A_{n,n+4} \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \sum_{i=1}^n \frac{h_i}{\rho_i}A_{i1} & \cdots & \sum_{i=1}^n \frac{h_i}{\rho_i}A_{in} & 0 & 0 & 0 & \sum_{i=1}^n \frac{h_i}{\rho_i}A_{i,n+4} \end{bmatrix}$$

where we denote $h_i := \rho_i e_i + p_i$ and write

$$-\rho_i u_i = \sum_{j=1}^n A_{ij}\nabla\rho_j + A_{i,n+4}\nabla\theta, \text{ for } i=1,\dots,n$$

and in (4.7) we set $\varepsilon = (\kappa, \epsilon)$ and $B(U) = (B^\kappa(U), B^\epsilon(U))$. We note that the coefficients A_{ij} and $A_{i,n+4}$ are determined by solving system (4.4)–(4.5), as explained later in this section.

According to [30] system (4.7) fits into the class of hyperbolic–parabolic systems (see also [32]) provided the following conditions hold:

(i) A is a C^2 and bijective map from its domain onto its range, with $\nabla A(U)$ non-singular for any U in the domain of A

(ii) there is an entropy–entropy flux pair $(H(U), Q_i(U))$, $i = 1, 2, 3$, generated by a smooth (vector valued) multiplier $G(U)$, such that

$$\nabla H = G \cdot \nabla A \quad \text{and} \quad \nabla Q_i = G \cdot \nabla F_i \quad \text{for } i = 1, 2, 3\,.$$

(iii) the matrix $\nabla^2 H - G \cdot \nabla^2 A$ is symmetric and positive definite

(iv) smooth solutions of (4.7) satisfy an additional conservation law (the so–called entropy identity)

$$\partial_t(H(U)) + \operatorname{div}(Q(U)) = \operatorname{div}(G(U)\varepsilon B(U)\nabla U) - \nabla U^\top \nabla G(U)^\top \varepsilon B(U)\nabla U \tag{4.10}$$

with the matrices $\nabla G^\top B$ inducing entropy dissipation, i.e.

$$\nabla U^\top \nabla G(U)^\top \varepsilon B(U) \nabla U \geq 0\,.$$

Let us check whether these conditions are satisfied in the case of system (4.1)–(4.5). Indeed, setting

$$H(U) := -\rho\eta(\rho_1, \ldots, \rho_n, \theta)\,, \quad Q_i(U) := -\rho\eta(\rho_1, \ldots, \rho_n, \theta)v_i\,, \tag{4.11}$$

we see that (4.10) is precisely equation (4.6).

Concerning (i), if $U = [\rho_1, \dots, \rho_n, v, \theta]^\top$ and $\tilde{U} = [\tilde{\rho}_1, \dots, \tilde{\rho}_n, \tilde{v}, \tilde{\theta}]^\top$ we have

$$A(U) = A(\tilde{U}) \Leftrightarrow \begin{cases} \rho_i = \tilde{\rho}_i \\ v = \tilde{v} \\ e(\rho_1, \dots, \rho_n, \theta) = e(\tilde{\rho}_1, \dots, \tilde{\rho}_n, \tilde{\theta}) \end{cases}$$

where the last equality implies that $\theta = \tilde{\theta}$, provided the internal energy is an increasing function of the temperature thus one-to-one. This follows from assuming that $e_\theta \equiv c_v > 0$, a property connected to the stability of equilibrium states (see 2.1.4 for more). Hence, if $\psi \in C^3$ and $\rho > 0$, A is C^2 and bijective in its domain, with

$$\det(\nabla A) = \det \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \\ v_1 & v_1 & \dots & v_1 & \rho & 0 & 0 & 0 \\ v_2 & v_2 & \dots & v_2 & 0 & \rho & 0 & 0 \\ v_3 & v_3 & \dots & v_3 & 0 & 0 & \rho & 0 \\ (\rho e)_{\rho_1} + \frac{1}{2}v^2 & (\rho e)_{\rho_2} + \frac{1}{2}v^2 & \dots & (\rho e)_{\rho_n} + \frac{1}{2}v^2 & \rho v_1 & \rho v_2 & \rho v_3 & \rho c_v \end{bmatrix}$$

$$= \rho^4 c_v > 0$$

For (ii) using the notation $\rho\hat{\eta} = \rho\eta(\rho_1, \dots, \rho_n, \theta)$ and $\rho\tilde{\eta} = \rho\eta(\rho_1, \dots, \rho_n, \rho e)$ and the thermodynamic relations from 2 we compute the partial derivatives of the entropy

$$(\rho\hat{\eta})_{\rho_i} = (\rho\tilde{\eta})_{\rho e}(\rho\hat{e})_{\rho_i} + (\rho\tilde{\eta})_{\rho_i} = \frac{(\rho e)_{\rho_i} - \mu_i}{\theta}$$

$$(\rho\hat{\eta})_\theta = \left(\frac{\rho e - \rho\psi}{\theta}\right)_\theta = \frac{[(\rho e)_\theta - (\rho\psi)_\theta]\theta - (\rho e - \rho\psi)}{\theta^2} = \frac{\rho e_\theta}{\theta} = \frac{\rho c_v}{\theta}.$$

By virtue of the relation $\nabla(-\rho\eta) = G \cdot \nabla A$ this determines the multiplier G to be

$$G(U) = \Big(G_j(U)\Big)_{j=1}^{n+4} = \frac{1}{\theta}\begin{bmatrix} \mu_1 - \frac{1}{2}v^2 \\ \vdots \\ \mu_n - \frac{1}{2}v^2 \\ v_1 \\ v_2 \\ v_3 \\ -1 \end{bmatrix} \tag{4.12}$$

while relations $\nabla(-\rho\eta v_i) = G \cdot \nabla F_i$ serve as a way of verifying that our calculations are correct; indeed, they hold for the G we found, using the properties

$$(\mu_i)_\theta = ((\rho\psi)_{\rho_i})_\theta = ((\rho\psi)_\theta)_{\rho_i} = (-\rho\hat{\eta})_{\rho_i} = \frac{\mu_i - (\rho e)_{\rho_i}}{\theta} \tag{4.13}$$

Now, for (iii), note that $G \cdot \nabla^2 A := \sum_{j=1}^{n+4} G_j(U)\nabla^2 A_j(U)$. One easily sees that $\nabla^2 A_j = 0$ for $j = 1, \ldots, n$. Then a tedious but straightforward calculation, using the thermodynamic relations (2.50)-(2.51) and

$$\begin{aligned} \frac{1}{\theta}(\rho e)_\theta - (\rho\eta)_\theta &= \frac{1}{\theta}\big((\rho\psi)_\theta + \rho\eta\big) = 0 \\ \frac{1}{\theta}(\rho e)_{\theta\theta} - (\rho\eta)_{\theta\theta} &= \frac{1}{\theta}(\rho\psi)_{\theta\theta} + \frac{2}{\theta}(\rho\eta)_\theta = \frac{\rho}{\theta}\eta_\theta = \frac{\rho}{\theta^2}e_\theta\,, \end{aligned}$$

implies

$$\nabla^2(-\rho\eta) - G \cdot \nabla^2 A = \frac{1}{\theta}\begin{bmatrix} (\rho\psi)_{\rho_1\rho_1} & \cdots & (\rho\psi)_{\rho_1\rho_n} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\rho\psi)_{\rho_n\rho_1} & \cdots & (\rho\psi)_{\rho_n\rho_n} & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \rho & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \rho & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \rho & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \frac{1}{\theta}\rho c_v \end{bmatrix} \tag{4.14}$$

Clearly, for $\psi \in C^3$, $\rho, \theta > 0$ and $c_v > 0$, the latter will be positive definite provided the matrix $(\rho\psi)_{\rho_i\rho_j}$ is positive definite. Indeed, for any $\xi \in \mathbb{R}^{n+4} \setminus \{(0, \dots, 0)\}$ we see that

$$\xi^\top(\nabla^2(-\rho\eta) - G\cdot\nabla^2 A)\xi = \frac{1}{\theta}\sum_{i=1}^{n}\sum_{j=1}^{n}(\rho\psi)_{\rho_i\rho_j}\xi_i\xi_j + \frac{\rho}{\theta}(\xi_{n+1}^2 + \xi_{n+2}^2 + \xi_{n+3}^2) + \frac{\rho c_v}{\theta^2}\xi_{n+4}^2 > 0$$

The assumption that $(\rho\psi)_{\rho_i\rho_j}$ is positive definite is a natural assumption in thermodynamics that is related to the convexity of the entropy (again we refer to [27] regarding the stability of equilibrium states, or to [6]).

For the last condition, we need to show that $\nabla U^\top \nabla G(U)^\top \varepsilon B(U)\nabla U \geq 0$, in other words, the entropy production is non-negative:

$$\zeta = \frac{1}{\theta^2}\kappa|\nabla\theta|^2 - \frac{1}{\theta}\sum_i u_i \cdot d_i \geq 0. \tag{4.15}$$

Clearly the first term is non-negative, since $\kappa \geq 0$. For the second term we need to invert system (4.4), in order to determine $\rho_i u_i$.

Various methods for the inversion of the Maxwell-Stefan system are available in the literature, see for example [6, 25] and [4, Sec. 7.7]. In the present work, we invert (4.4) using the Bott–Duffin inverse, following the analysis of [10]. The advantage of this method is that it provides an explicit formula for the solution of the linear system (4.4), (4.5), which is helpful to estimate the dissipation.

The need to introduce the Bott–Duffin inverse arises from the fact that the desired inversion has to respect the constraint $\sum_i \rho_i u_i = 0$, i.e. we invert within the appropriate subspace. More precisely, we consider the solution of the generic system

$$Mx + y = w, \quad x \in L, \; y \in L^\perp \tag{4.16}$$

where $M \in \mathbb{R}^{m\times m}$, for some $m \in \mathbb{N}$, is a matrix and $L \subset \mathbb{R}^m$ a subspace. Let $\mathbb{P}_L$ and $\mathbb{P}_{L^\perp}$ be the projection operators onto the subspaces L and $L^\perp$, respectively. Then, the set of solutions of system (4.16) is the same as the set of solutions of the system

$$(M\mathbb{P}_L + \mathbb{P}_{L^\perp})z = w \tag{4.17}$$

and $[x, y]^\top$ solves (4.16) if and only if $x = \mathbb{P}_L z$ and $y = \mathbb{P}_{L^\perp} z = w - M\mathbb{P}_L z$. Now, if the matrix $M\mathbb{P}_L + \mathbb{P}_{L^\perp}$ is invertible, we define the Bott–Duffin inverse of M with respect to L by

$$M^{BD} = \mathbb{P}_L(M\mathbb{P}_L + \mathbb{P}_{L^\perp})^{-1} \tag{4.18}$$

so that the solution of (4.16) is given by

$$x = M^{BD}d, \quad y = w - Mx \tag{4.19}$$

In our context, if we introduce the molar fractions

$$c_i = \frac{\rho_i}{\rho} \tag{4.20}$$

the left–hand side of (4.4) reads:

$$\begin{aligned}
&-\sum_{j\neq i} b_{ij}\theta\rho_i\rho_j(u_i - u_j)\\
&\qquad = -\rho\theta\left(\sqrt{\rho_i}\sum_{j\neq i} b_{ij}c_j(\sqrt{\rho_i}u_i) - \sqrt{\rho_i}\sum_{j\neq i} b_{ij}\sqrt{c_i}\sqrt{c_j}(\sqrt{\rho_j}u_j)\right)\\
&\qquad = -\rho\theta\sqrt{\rho_i}\sum_{j=1}^{n}\left(\sum_{k\neq i} c_k b_{ik}\delta_{ij} - \sqrt{c_i c_j}b_{ij}\right)\sqrt{\rho_j}u_j\\
&\qquad = -\rho\theta\sqrt{\rho_i}\sum_{j=1}^{n} M_{ij}\sqrt{\rho_j}u_j
\end{aligned}$$

where we introduce the matrix $M = (M_{ij})$ given by

$$M_{ij} = \begin{cases} \sum_{k\neq i} c_k b_{ik} & i = j \\ -\sqrt{c_i c_j}b_{ij} & i \neq j \end{cases} \tag{4.21}$$

and we are interested in the constrained inversion $Mx = w$, $x \in L$, where $w_i =$

$-\frac{\epsilon d_i}{\rho\theta\sqrt{\rho_i}}$, using d_i from the end of section 3.1.2, $x_i = \sqrt{\rho_i}u_i$ and $L = \{(y_1, \dots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n \sqrt{\rho_i}y_i = 0\}$. Moreover, the projection matrix $\mathbb{P}_L$ on L is given by

$$(\mathbb{P}_L)_{ij} = \delta_{ij} - \frac{\sqrt{\rho_i\rho_j}}{\rho}. \tag{4.22}$$

In [10] it was proven that the matrix M from (4.21) satisfies the relation

$$z^\top M z \geq \mu|\mathbb{P}_L z|^2 \quad \forall z \in \mathbb{R}^n \quad \text{where } \mu = \min_{i\neq j} b_{ij} \tag{4.23}$$

which in turn implies that the Bott-Duffin inverse of M, namely M^{BD}, is well–defined, symmetric and satisfies

$$z^\top M^{BD} z \geq \lambda|\mathbb{P}_L z|^2 \quad \forall z \in \mathbb{R}^n \quad \text{where } \lambda = \left(2\sum_{i\neq j}(b_{ij}+1)\right)^{-1} \tag{4.24}$$

Therefore, system (4.4) is written

$$\sum_{j=1}^n M_{ij}\sqrt{\rho_j}u_j = -\frac{\epsilon d_i}{\rho\theta\sqrt{\rho_i}} \tag{4.25}$$

and can be inverted

$$\sqrt{\rho_i}u_i = -\sum_{j=1}^n M_{ij}^{BD}\frac{\epsilon d_j}{\rho\theta\sqrt{\rho_j}} \tag{4.26}$$

Plugging (4.26) into the last term of (4.15) we get

$$\begin{aligned} -\frac{1}{\theta}\sum_i u_i \cdot d_i &= -\frac{1}{\theta}\sum_i \sqrt{\rho_i}u_i \cdot \frac{d_i}{\sqrt{\rho_i}} \\ &= \frac{\epsilon}{\rho\theta^2}\sum_i\sum_j M_{ij}^{BD}\frac{d_i}{\sqrt{\rho_i}}\frac{d_j}{\sqrt{\rho_j}} \geq 0 \end{aligned}$$

by (4.24). Hence, (4.15) is satisfied, which means that the matrices $\nabla G^\top B$ induce entropy dissipation as we wanted, concluding that system (4.1)–(4.3) along with the linear system (4.4) subject to the constraint (4.5) is of hyperbolic–parabolic type. In fact, condition (4.15) is the minimum framework inducing entropy dissipation along the evolution and at the same time allowing for degenerate diffusion matrices (see [30]). In the next section we shall show that our problem enjoys a stronger dissipative structure, which allows us to establish some convergence results.

A particular case of system (4.1)–(4.5) is the system without mass–diffusion and heat–conduction, obtained by setting $\kappa = \epsilon = 0$:

$$\partial_t \rho_i + \operatorname{div}(\rho_i v) = 0 \tag{4.27}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p\mathbb{I}) = 0 \tag{4.28}$$

$$\partial_t \left(\rho e + \frac{1}{2}\rho v^2\right) + \operatorname{div}\left((\rho e + \frac{1}{2}\rho v^2)v + pv\right) = 0 \tag{4.29}$$

equipped with the entropy identity

$$\partial_t(-\rho\eta) + \operatorname{div}(-\rho\eta v) = 0 \tag{4.30}$$

We would like to show that (4.27)–(4.29) is hyperbolic under the assumptions $(\rho\psi)_{\rho_i\rho_j}$ is positive definite and $e_\theta > 0$, as long as the total mass ρ remains away from zero. To do so, one needs to rewrite system (4.27)–(4.29) in the form

$$\nabla A(U)\partial_t U + \sum_{\alpha=1}^{3} \nabla F_\alpha(U)\partial_{x_\alpha} U = 0$$

where A, F_α, U are as in (4.7). Since ∇A is non-singular, we proceed to find the characteristic speeds of the system by solving, for any $N = (N_1, N_2, N_3)$ on the sphere, the eigenvalue problem

$$\left[\sum_{\alpha=1}^{3} \nabla F_\alpha(U)N_\alpha - \lambda(U,N)\nabla A(U)\right] r(U,N) = 0$$

where λ, r are the eigenvalues and eigenvectors respectively and

$$
\sum_{\alpha=1}^{3} \nabla F_\alpha(U) N_\alpha - \lambda(U,N)\nabla A(U)
= \begin{bmatrix}
v\cdot N-\lambda & \cdots & 0 & \rho_1 N_1 & \rho_1 N_2 & \rho_1 N_3 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & v\cdot N-\lambda & \rho_n N_1 & \rho_n N_2 & \rho_n N_3 & 0 \\
p_{\rho_1} N_1 & \cdots & p_{\rho_n} N_1 & \rho(v\cdot N-\lambda) & 0 & 0 & N_1 p_\theta \\
p_{\rho_1} N_2 & \cdots & p_{\rho_n} N_2 & 0 & \rho(v\cdot N-\lambda) & 0 & N_2 p_\theta \\
p_{\rho_1} N_3 & \cdots & p_{\rho_n} N_3 & 0 & 0 & \rho(v\cdot N-\lambda) & N_3 p_\theta \\
0 & \cdots & 0 & N_1 \theta p_\theta & N_2 \theta p_\theta & N_3 \theta p_\theta & \rho c_v (v\cdot N-\lambda)
\end{bmatrix}
$$

By using the property

$$
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B)
$$

from [33, Sec. 5], for any block of matrices A, B, C, D, we determine the characteristic equation:

$$
\rho^4 c_v (v\cdot N-\lambda)^{n+2} \left((v\cdot N-\lambda)^2 - \frac{1}{\rho}\sum_i \rho_i p_{\rho_i} - \frac{\theta p_\theta^2}{c_v \rho^2} \right) = 0
$$

which yields the wave speeds

$$
\lambda_1 = \cdots = \lambda_{n+2} = v\cdot N, \ \ \lambda_{n+3,n+4} = v\cdot N \pm \sqrt{\frac{1}{\rho}\sum_i \rho_i p_{\rho_i} + \frac{\theta p_\theta^2}{c_v\rho^2}}
$$

Therefore, system (4.27)–(4.29) is hyperbolic if all eigenvalues are real, which holds under the hypotheses $\rho, \theta > 0$, $c_v > 0$ and the matrix $(\rho\psi)_{\rho_i\rho_j}$ is positive semi–definite. The last hypothesis ensures that the term $\sum_i \rho_i p_{\rho_i}$ is non–negative, since

$$
\sum_i \rho_i p_{\rho_i} = \sum_{i,j} \rho_i \rho_j (\mu_i)_{\rho_j} \geq 0.
$$

In fact, the eigenvectors corresponding to the repeated eigenvalues are given by the formula

$$\xi = (\xi_1, \xi_2, \dots, \xi_n, \xi_{n+1}, \xi_{n+2}, -\frac{N_1}{N_3}\xi_{n+1} - \frac{N_2}{N_3}\xi_{n+2}, -\frac{1}{p_\theta}\sum_{i=1}^{n} \xi_i p_{\rho_i})$$

for $(\xi_1, \dots, \xi_{n+2}) \in \mathbb{R}^{n+2}$ and thus the dimension of the eigenspace is $n+2$.

The reader should notice that hyperbolicity is not valid at $\rho = 0$ (as $\det A$ vanishes) but strict hyperbolicity still holds when some of the ρ_i's vanish provided $\rho \neq 0$.

4.2 The zero–diffusion limit

As in the previous section, we consider the hyperbolic–parabolic system

$$\partial_t \rho_i + \operatorname{div}(\rho_i v) = \operatorname{div}(-\rho_i u_i) \tag{4.31}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p\mathbb{I}) = 0 \tag{4.32}$$

$$\begin{aligned} \partial_t \left(\rho e + \frac{1}{2}\rho v^2\right) &+ \operatorname{div}\Big((\rho e + \frac{1}{2}\rho v^2)v + pv\Big) \\ &= \operatorname{div}\left(\kappa \nabla\theta - \sum_i (\rho_i e_i + p_i) u_i\right), \end{aligned} \tag{4.33}$$

where u_i are determined by solving

$$-\sum_{j\neq i} b_{ij}\theta\rho_i\rho_j(u_i - u_j) = \epsilon\left(-\frac{\rho_i}{\rho}\nabla p + \rho_i\theta\nabla\frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i)\nabla\frac{1}{\theta}\right) \tag{4.34}$$

$$\sum_i \rho_i u_i = 0\,, \tag{4.35}$$

which is endowed with the dissipative structure

$$\partial_t(-\rho\eta) + \mathrm{div}(-\rho\eta v) + \frac{1}{\theta^2}\kappa|\nabla\theta|^2 - \frac{1}{\theta}\sum_i u_i \cdot d_i \tag{4.36}$$
$$= \mathrm{div}\left(\frac{1}{\theta}\sum_i(\rho_i e_i + p_i - \rho_i\mu_i)u_i - \frac{1}{\theta}\kappa\nabla\theta\right).$$

Moreover, consider the system obtained when neglecting the mass diffusive effects ($\epsilon = 0$) but including heat conduction:

$$\partial_t\rho_i + \mathrm{div}(\rho_i v) = 0 \tag{4.37}$$

$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v + p\mathbb{I}) = 0 \tag{4.38}$$

$$\partial_t\left(\rho e + \frac{1}{2}\rho v^2\right) + \mathrm{div}\left((\rho e + \frac{1}{2}\rho v^2)v + pv\right) = \mathrm{div}\left(\kappa\nabla\theta\right) \tag{4.39}$$

endowed with the limiting dissipation structure

$$\partial_t(-\rho\eta) + \mathrm{div}(-\rho\eta v) = \mathrm{div}\left(-\frac{1}{\theta}\kappa\nabla\theta\right) - \frac{1}{\theta^2}\kappa|\nabla\theta|^2. \tag{4.40}$$

Local existence results for smooth solutions are available for multicomponent systems (see [34] and [4, Ch. 8]). For hyperbolic–parabolic systems like the ones above with general initial data one can expect existence of a unique smooth solution, which however can break down at finite time and the time of existence in general depends on the diffusion constants ϵ and κ. More precisely, under sufficient conditions on the initial data, the diffusion coefficients and the free energy function, if $Q_T = \Omega \times (0,T)$, where Ω is bounded and $T > 0$, there exists a $\tau \in (0,T]$ such that the above problem possesses a unique solution

$$(\rho_1,\ldots,\rho_n) \in W_p^1(Q_\tau;\mathbb{R}^n_+),\ v \in W_p^{2,1}(Q_\tau;\mathbb{R}^3),\ \theta \in W^{2,1}(Q_\tau;\mathbb{R}_+)$$

where the spaces above are defined as follows:

$$W_p^{2,1}(Q_T) = \{u \in L^p(Q_T) : \partial_t^\beta \partial_x^\alpha u \in L^p(Q_T) \text{ for all } 0 < 2\beta + |\alpha| \leq 2\}$$

$$W_p^{1,0}(Q_T) = \{u \in L^p(Q_T) : \partial_x^\alpha u \in L^p(Q_T) \text{ for all } |\alpha| = 1\}.$$

For more details we refer to [35].

In this section, we show that smooth solutions of (4.31)–(4.35) converge to solutions of (4.37)–(4.40) as ϵ tends to zero, so long as the solutions of the latter remain in the smooth regime and the theory developed should be understood as indicating conditions for convergence of thermomechanical theories in the smooth regime.

In order to show the convergence from (4.7) as $\varepsilon \to 0$ we use the following ingredients:

(a) The method of relative entropy introduced in [28], here employed in the form proposed in [30] :

$$H(U|\bar{U}) = H(U) - H(\bar{U}) - G(\bar{U}) \cdot (A(U) - A(\bar{U}))\,. \tag{4.41}$$

By [30, Appendix A], whenever conditions (i)–(iv) for (4.10) hold, the relative entropy can be written as $H(U|\bar{U}) = \hat{H}(A(U)|A(\bar{U}))$, where $\hat{H}(V)$ is a strictly convex function. Hence, $H(U|\bar{U})$ vanishes if and only if $A(U) = A(\bar{U})$ (and by (i), $U = \bar{U}$) and it can serve to measure the distance between two solutions.

(b) A second ingredient is the control of diffusion by dissipation (see [30],[29, Sec 4.6]), that is a hypothesis that there exist constants $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$\sum_{\alpha,\beta} \nabla G(U) \partial_\alpha U B^\kappa_{\alpha\beta}(U) \partial_\beta U \geq \nu_1 \sum_\alpha \left| \sum_\beta B^\kappa_{\alpha\beta}(U) \partial_\beta U \right|^2 \tag{4.42}$$

$$\sum_{\alpha,\beta} \nabla G(U) \partial_\alpha U B^\epsilon_{\alpha\beta}(U) \partial_\beta U \geq \nu_2 \sum_\alpha \left| \sum_\beta B^\epsilon_{\alpha\beta}(U) \partial_\beta U \right|^2 . \tag{4.43}$$

Next, we list hypotheses used on the thermodynamic functions. For the internal energy, when expressed in the form $\rho e = \rho\tilde{e}(\rho_1, \ldots, \rho_n, \rho\eta)$, we require

$$\frac{\partial(\rho\tilde{e})}{\partial(\rho\eta)} > 0\,, \quad \frac{\partial^2(\rho\tilde{e})}{\partial(\rho\eta)^2} > 0 \tag{H$_1$}$$

$$\nabla^2_{(\rho_1,\ldots,\rho_n,\rho\eta)}(\rho\tilde{e}) > 0 \tag{H$_2$}$$

Hypothesis (H$_1$) is natural in thermodynamics stating that the temperature $\theta > 0$ and ensuring convexity of the energy as a function of entropy. Hypothesis (H$_2$) implies that the system (4.27)–(4.29) is hyperbolic and excludes various interesting models related to pressure laws of Van–der–Waals type.

Using (H$_1$) one may invert the equation $\theta = \frac{\partial(\rho\tilde{e})}{\partial(\rho\eta)}$ and define the inverse function $\rho\eta = (\rho\eta)^*(\rho_1, \ldots, \rho_n, \theta)$. This yields the Legendre transform [27, Sec 5]

$$\rho\psi = \rho\tilde{e}(\rho_1, \ldots, \rho_n, (\rho\eta)^*) - \theta(\rho\eta)^* \quad \text{where} \quad \frac{\partial(\rho\tilde{e})}{\partial(\rho\eta)}(\rho_1, \ldots, \rho_n, (\rho\eta)^*) = \theta. \tag{4.44}$$

A computation shows that (H$_1$), (H$_2$) imply

$$\nabla^2_{(\rho_1,\ldots,\rho_n)}(\rho\psi) > 0\,, \quad (\rho\psi)_{\theta\theta} < 0\,. \tag{4.45}$$

The latter should be compared to (4.14) and property (iii).

An alternative is to define the Legendre transform through the direct formula

$$(\rho\psi)(\rho_1, \ldots, \rho_n, \theta) = \inf_{0<\rho\eta<\infty} \big\{\rho\tilde{e}(\rho_1, \ldots, \rho_n, \rho\eta) - \theta\rho\eta\big\} \tag{4.46}$$

Under (H$_1$) and

$$\lim_{\rho\eta\to 0} \frac{\partial(\rho\tilde{e})}{\partial(\rho\eta)} = 0\,, \quad \lim_{\rho\eta\to\infty} \frac{\rho\tilde{e}}{\rho\eta} = \infty \tag{4.47}$$

this problem has at most one solution computed via (4.46). An advantage of this approach is that the convexity conditions (4.45) follow directly from the minimization formula (4.46). On the other hand to solve (4.46) requires the assumption (4.47). This formulation and (4.47) is consistent with the third law of thermodynamics (that the entropy vanishes at the state of zero–temperature, see [27, Sec 1.10]) but the popular model of the ideal gas (with constant heat capacity) violates the third law and presents negative entropies. Nevertheless, the relation (4.44) between internal energy ρe and Helmholtz free energy $\rho\psi$ is still valid.

The adaptation of the general framework to the system (4.31)–(4.35) requires some computations, and it is remarkable that such a complicated system hides a simple structure. Let $U = (\rho_1, \dots, \rho_n, v, \theta)^\top$ be a solution of (4.31)–(4.36) and $\bar{U} = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{v}, \bar{\theta})^\top$ of (4.37)–(4.40). Using (4.11), (4.12) and (2.50)–(2.57), (2.32) we arrive at

$$
\begin{aligned}
H(U|\bar{U}) &= -\rho\eta + \bar{\rho}\bar{\eta} - \frac{1}{\bar{\theta}} \sum_{j=1}^{n} (\bar{\mu}_j - \tfrac{1}{2}\bar{v}^2)(\rho_j - \bar{\rho}_j) - \frac{\bar{v}}{\bar{\theta}} \cdot (\rho v - \bar{\rho}\bar{v}) \\
&\quad + \frac{1}{\bar{\theta}} \Big(\rho e + \tfrac{1}{2}\rho v^2 - \bar{\rho}\bar{e} - \tfrac{1}{2}\bar{\rho}\bar{v}^2 \Big) \\
&= \frac{1}{2\bar{\theta}} \rho |v - \bar{v}|^2 + \frac{1}{\bar{\theta}} \Big[\Big(\rho\psi - \bar{\rho}\bar{\psi} - \sum_{j=1}^{n} \bar{\mu}_j (\rho_j - \bar{\rho}_j) + \bar{\rho}\bar{\eta}(\theta - \bar{\theta}) \Big) \\
&\quad + (\theta - \bar{\theta})(\rho\eta - \bar{\rho}\bar{\eta}) \Big] \\
&= \frac{1}{\bar{\theta}} \left(\frac{1}{2} \rho |v - \bar{v}|^2 + J(\rho_1, \dots, \rho_n, \theta | \bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\theta}) \right)
\end{aligned}
\tag{4.48}
$$

where we set $\omega = (\rho_1, \dots, \rho_n, \theta)$, $\bar{\omega} = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\theta})$ and use (2.50) to write

$$
\begin{aligned}
J(\omega|\bar{\omega}) &:= \rho\psi - \bar{\rho}\bar{\psi} - \sum_i \bar{\mu}_i (\rho_i - \bar{\rho}_i) + \bar{\rho}\bar{\eta}(\theta - \bar{\theta}) + (\theta - \bar{\theta})(\rho\eta - \bar{\rho}\bar{\eta}) \qquad (4.49) \\
&= \rho e - \bar{\rho}\bar{e} - \sum_i \bar{\mu}_i (\rho_i - \bar{\rho}_i) - \bar{\theta}(\rho\eta - \bar{\rho}\bar{\eta}) \\
I(U|\bar{U}) &:= \bar{\theta} H(U|\bar{U}) = \frac{1}{2} \rho |v - \bar{v}|^2 + J(\omega|\bar{\omega}) \,. \qquad (4.50)
\end{aligned}
$$

Due to hypothesis (H_2), the quantity $J(\omega|\bar{\omega})$ will serve as a measure of the distance between the states ω and $\bar{\omega}$, in analogy to the situation in single component fluids [28]. This suggests to calculate the evolution of the quantity (4.50).

Subtracting the entropy identities (4.40) from (4.36) and multiplying by $\bar{\theta}$, we obtain

$$
\begin{aligned}
&\partial_t(-\bar\theta\rho\eta+\bar\theta\bar\rho\bar\eta)+\operatorname{div}(-\rho\eta v\bar\theta+\bar\rho\bar\eta\bar v\bar\theta)\\
&=\operatorname{div}\left(\frac{\bar\theta}{\theta}\sum_j(h_j-\rho_j\mu_j)u_j-\frac{\bar\theta}{\theta}\kappa\nabla\theta\right)-\partial_t\bar\theta(\rho\eta-\bar\rho\bar\eta)\\
&+\nabla\bar\theta\cdot(-\rho\eta v+\bar\rho\bar\eta\bar v)-\frac{1}{\theta}\sum_j(h_j-\rho_j\mu_j)u_j\cdot\nabla\bar\theta+\frac{1}{\theta}\kappa\nabla\theta\cdot\nabla\bar\theta\\
&-\frac{\bar\theta}{\theta^2}\kappa|\nabla\theta|^2+\frac{\bar\theta}{\theta}\sum_j u_j\cdot d_j+\operatorname{div}\left(\frac{\theta}{\bar\theta}\bar\kappa\nabla\bar\theta\right)-\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\cdot\nabla\bar\theta+\frac{\theta}{\bar\theta^2}\bar\kappa|\nabla\bar\theta|^2
\end{aligned}
\tag{4.51}
$$

Likewise, subtracting system (4.37)–(4.39) from system (4.31)–(4.33) and multiplying the result by $-\bar\theta G(\bar U)$, where G is the multiplier from (4.12), we obtain:

$$
\begin{aligned}
&\partial_t\Big(\sum_i\left(\frac12\bar v^2-\bar\mu_i\right)(\rho_i-\bar\rho_i)-\bar v(\rho v-\bar\rho\bar v)+\Big(\rho e+\frac12\rho v^2-\bar\rho\bar e\\
&-\frac12\bar\rho\bar v^2\Big)\Big)+\operatorname{div}\Big(\sum_i\left(\frac12\bar v^2-\bar\mu_i\right)(\rho_i v-\bar\rho_i\bar v)-\bar v(\rho v\otimes v-\bar\rho\bar v\otimes\bar v\\
&+(p-\bar p)\mathbb{I})+\left(\rho e+\frac12\rho v^2+p\right)v-\left(\bar\rho\bar e+\frac12\bar\rho\bar v^2+\bar p\right)\bar v\Big)\\
&=\sum_i\partial_t\left(\frac12\bar v^2-\bar\mu_i\right)(\rho_i-\bar\rho_i)+\sum_i\nabla\left(\frac12\bar v^2-\bar\mu_i\right)\cdot(\rho_i v-\bar\rho_i\bar v)\\
&-\sum_i\left(\frac12\bar v^2-\bar\mu_i\right)\operatorname{div}(\rho_i u_i)-\partial_t\bar v\cdot(\rho v-\bar\rho\bar v)-\rho v\nabla\bar v\cdot v+\bar\rho\bar v\nabla\bar v\cdot\bar v\\
&-(p-\bar p)\operatorname{div}\bar v+\operatorname{div}\left(\kappa\nabla\theta-\sum_j h_j u_j\right)-\operatorname{div}(\bar\kappa\nabla\bar\theta)
\end{aligned}
\tag{4.52}
$$

where we used the abbreviation $h_j:=\rho_j e_j+p_j$.

Next, we add equations (4.51) and (4.52) and use the formula (4.48) and

$$
\begin{aligned}
Q(U|\bar U)&=-\rho\eta v\bar\theta+\bar\rho\bar\eta\bar v\bar\theta+\sum_i\left(\frac12\bar v^2-\bar\mu_i\right)(\rho_i v-\bar\rho_i\bar v)-\bar v\rho v^2\\
&+\bar v\bar\rho\bar v^2-\bar v(p-\bar p)+\left(\rho e+\frac12\rho v^2+p\right)v-\left(\bar\rho\bar e+\frac12\bar\rho\bar v^2+\bar p\right)\bar v\\
&=vI(U|\bar U)+(p-\bar p)(v-\bar v)
\end{aligned}
\tag{4.53}
$$

to arrive at

$$
\partial_t I(U|\bar U)+\operatorname{div}Q(U|\bar U)=T_1+T_2+T_3+T_4 \tag{4.54}
$$

where

$$T_1 := \sum_i \partial_t \left(\frac{1}{2}\bar{v}^2 - \bar{\mu}_i\right)(\rho_i - \bar{\rho}_i) - \partial_t \bar{v} \cdot (\rho v - \bar{\rho}\bar{v}) - \partial_t \bar{\theta}(\rho\eta - \bar{\rho}\bar{\eta})$$

$$\begin{aligned} T_2 := &\sum_i \nabla \left(\frac{1}{2}\bar{v}^2 - \bar{\mu}_i\right) \cdot (\rho_i v - \bar{\rho}_i \bar{v}) - \rho v \nabla \bar{v} \cdot v + \bar{\rho}\bar{v}\nabla\bar{v}\cdot\bar{v} \\ &- (p - \bar{p})\mathrm{div}\bar{v} + \nabla\bar{\theta}\cdot(-\rho\eta v + \bar{\rho}\bar{\eta}\bar{v}) \end{aligned}$$

$$\begin{aligned} T_3 := &\mathrm{div}\left(-\frac{\bar{\theta}}{\theta}\kappa\nabla\theta\right) + \frac{1}{\theta}\kappa\nabla\theta\cdot\nabla\bar{\theta} - \frac{\bar{\theta}}{\theta^2}\kappa|\nabla\theta|^2 + \mathrm{div}\left(\frac{\theta}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\right) \\ &- \frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\cdot\nabla\theta + \frac{\theta}{\bar{\theta}^2}\bar{\kappa}|\nabla\bar{\theta}|^2 + \mathrm{div}\left(\kappa\nabla\theta\right) - \mathrm{div}(\bar{\kappa}\nabla\bar{\theta}) \end{aligned}$$

$$\begin{aligned} T_4 := &\mathrm{div}\left(\frac{\bar{\theta}}{\theta}\sum_j (h_j - \rho_j\mu_j)u_j\right) - \frac{1}{\theta}\sum_j (h_j - \rho_j\mu_j)u_j\cdot\nabla\bar{\theta} + \frac{\bar{\theta}}{\theta}\sum_j u_j\cdot d_j \\ &- \sum_i \left(\frac{1}{2}\bar{v}^2 - \bar{\mu}_i\right)\mathrm{div}(\rho_i u_i) + \mathrm{div}\left(-\sum_j h_j u_j\right). \end{aligned}$$

Now

$$\begin{aligned} T_1 = &-\rho(v - \bar{v})\partial_t\bar{v} + \partial_t\bar{\theta}(-\rho\eta)(U|\bar{U}) - (\bar{\rho}\bar{\eta})_\theta(\theta - \bar{\theta})\partial_t\bar{\theta} \\ &- \sum_j\sum_i (\bar{\mu}_j)_{\rho_i}(\rho_i - \bar{\rho}_i)\partial_t\bar{\rho}_j =: T_{11} + T_{12} + T_{13} + T_{14} \end{aligned}$$

where we have used the fact that $(\mu_i)_\theta = -(\rho\eta)_{\rho_i}$ and introduced the relative quantity

$$(-\rho\eta)(U|\bar{U}) = -\rho\eta + \bar{\rho}\bar{\eta} + \sum_j (\bar{\rho}\bar{\eta})_{\rho_j}(\rho_j - \bar{\rho}_j) + (\bar{\rho}\bar{\eta})_\theta(\theta - \bar{\theta}).$$

Using the entropy balance (4.40) we get

$$
\begin{aligned}
T_{13} &= \partial_t(-\bar\rho\bar\eta)(\theta-\bar\theta) + \sum_j (\bar\rho\bar\eta)_{\rho_j}\partial_t\bar\rho_j(\theta-\bar\theta) \\
&= \operatorname{div}(\bar\rho\bar\eta\bar v)(\theta-\bar\theta) + \operatorname{div}\left(-\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right)(\theta-\bar\theta) - \frac{1}{\bar\theta^2}\bar\kappa|\nabla\bar\theta|^2(\theta-\bar\theta) \\
&- \sum_j (\bar\rho\bar\eta)_{\rho_j}\nabla\bar\rho_j\cdot\bar v(\theta-\bar\theta) - \sum_j (\bar\rho\bar\eta)_{\rho_j}\bar\rho_j\operatorname{div}\bar v(\theta-\bar\theta) \\
&=: T_{131} + T_{132} + T_{133} + T_{134} + T_{135}
\end{aligned}
$$

where

$$
\begin{aligned}
T_{131} &= \nabla(\bar\rho\bar\eta)\cdot\bar v(\theta-\bar\theta) + \bar\rho\bar\eta\operatorname{div}\bar v(\theta-\bar\theta) \\
&=: T_{1311} + T_{1312}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
T_{14} &= \sum_j\sum_i (\bar\mu_i)_{\rho_j}(\rho_i-\bar\rho_i)\nabla\bar\rho_j\cdot\bar v + \sum_j\sum_i (\bar\mu_i)_{\rho_j}(\rho_i-\bar\rho_i)\bar\rho_j\operatorname{div}\bar v \\
&= \sum_i \nabla\bar\mu_i\cdot\bar v(\rho_i-\bar\rho_i) - \sum_i (\bar\mu_i)_\theta\nabla\bar\theta\cdot\bar v(\rho_i-\bar\rho_i) \\
&\quad + \sum_j\sum_i (\bar\mu_i)_{\rho_j}(\rho_i-\bar\rho_i)\bar\rho_j\operatorname{div}\bar v =: T_{141} + T_{142} + T_{143}.
\end{aligned}
$$

Introducing the relative pressure

$$
p(U|\bar U) = p - \bar p - \sum_j \bar p_{\rho_j}(\rho_j-\bar\rho_j) - \bar p_\theta(\theta-\bar\theta)
$$

we get

$$
\begin{aligned}
T_2 &= -\rho(v-\bar v)\nabla\bar v\cdot(v-\bar v) - \sum_i \nabla\bar\mu_i\cdot(\rho_i v - \bar\rho_i\bar v) - p(U|\bar U)\operatorname{div}\bar v \\
&- \sum_j \bar p_{\rho_j}(\rho_j-\bar\rho_j)\operatorname{div}\bar v - \bar p_\theta(\theta-\bar\theta)\operatorname{div}\bar v + \bar v\cdot\nabla\bar\theta(-\rho\eta)(U|\bar U) \\
&- \bar v\cdot\nabla\bar\theta\sum_j (\bar\rho\bar\eta)_{\rho_j}(\rho_j-\bar\rho_j) - \bar v\cdot\nabla\bar\theta(\bar\rho\bar\eta)_\theta(\theta-\bar\theta) + \nabla\bar\theta(-\rho\eta)\cdot(v-\bar v) \\
&- \rho\bar v\nabla\bar v\cdot(v-\bar v) =: T_{21} + \cdots + T_{210}
\end{aligned}
$$

where

$$T_{22} = -\sum_i \nabla\bar{\mu}_i \cdot (\rho_i - \bar{\rho}_i)v - \sum_i \nabla\bar{\mu}_i \cdot \bar{\rho}_i(v - \bar{v})$$
$$= -\sum_i \nabla\bar{\mu}_i \cdot (\rho_i - \bar{\rho}_i)(v - \bar{v}) - \sum_i \nabla\bar{\mu}_i \cdot (\rho_i - \bar{\rho}_i)\bar{v} + (v - \bar{v})\bar{\rho}\bar{\eta} \cdot \nabla\bar{\theta}$$
$$- (v - \bar{v}) \cdot \nabla\bar{p} =: T_{221} + T_{222} + T_{223} + T_{224}.$$

Now, T_{28} and T_{134} cancel with T_{1311} and the same holds for T_{141} with T_{222} and T_{142} with T_{27}. Moreover

$$T_{29} + T_{223} = -(\rho\eta - \bar{\rho}\bar{\eta})(v - \bar{v}) \cdot \nabla\bar{\theta}$$

and by the momentum balance (4.38)

$$T_{224} = \frac{1}{\bar{\rho}}(\rho - \bar{\rho})(v - \bar{v}) \cdot \nabla\bar{p} - \frac{\rho}{\bar{\rho}}(v - \bar{v}) \cdot \nabla\bar{p}$$
$$= \frac{1}{\bar{\rho}}(\rho - \bar{\rho})(v - \bar{v}) \cdot \nabla\bar{p} - \rho(v - \bar{v}) \cdot \partial_t\bar{v} + \rho\bar{v}\nabla\bar{v} \cdot (v - \bar{v})$$
$$=: T_{2241} + T_{2242} + T_{2243}$$

where T_{2242} cancels with T_{11} and T_{2243} with T_{210}.

Differentiation of the Gibbs–Duhem relation (2.32) with respect to mass density and temperature, respectively, gives

$$p_{\rho_i} = \sum_j \rho_j(\mu_j)_{\rho_i}\,, \qquad p_\theta = \rho\eta + \sum_j \rho_j(\mu_j)_\theta\,. \tag{4.55}$$

Hence T_{25}, T_{1312} cancel with T_{135} and T_{24} with T_{143}. Finally, taking the gradient of the Gibbs–Duhem relation and using (4.55), we obtain the following identity

$$\nabla p = \rho\eta\nabla\theta + \sum_j \rho_j \nabla\mu_j$$

which allows us to write

$$
\begin{aligned}
&-(\rho\eta-\bar\rho\bar\eta)(v-\bar v)\cdot\nabla\bar\theta-\sum_j\nabla\bar\mu_j\cdot(\rho_j-\bar\rho_j)\,(v-\bar v)+\frac{1}{\bar\rho}(\rho-\bar\rho)(v-\bar v)\cdot\nabla\bar p\\
&=-(\eta-\bar\eta)\rho(v-\bar v)\cdot\nabla\bar\theta-\sum_j\nabla\bar\mu_j\left(\frac{\rho_j}{\rho}-\frac{\bar\rho_j}{\bar\rho}\right)\cdot\rho(v-\bar v)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T_1+T_2&=(\partial_t\bar\theta+\bar v\cdot\nabla\bar\theta)(-\rho\eta)(\omega|\bar\omega)-p(\omega|\bar\omega)\mathrm{div}\bar v\\
&\quad-(\eta-\bar\eta)\rho(v-\bar v)\cdot\nabla\bar\theta-\sum_j\nabla\bar\mu_j\left(\frac{\rho_j}{\rho}-\frac{\bar\rho_j}{\bar\rho}\right)\cdot\rho(v-\bar v)\\
&\quad-\rho(v-\bar v)\nabla\bar v\cdot(v-\bar v)+T_{132}+T_{133}.
\end{aligned}
\tag{4.56}
$$

Regarding T_3, we have

$$
\begin{aligned}
T_3=&-\mathrm{div}\left(\bar\theta\frac{1}{\theta}\kappa\nabla\theta-\bar\theta\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right)+\mathrm{div}\left(\kappa\nabla\theta-\bar\kappa\nabla\bar\theta\right)\\
&+\nabla\bar\theta\left(\frac{1}{\theta}\kappa\nabla\theta-\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right)-\bar\theta\left(\frac{1}{\theta^2}\kappa|\nabla\theta|^2-\frac{1}{\bar\theta^2}\bar\kappa|\nabla\bar\theta|^2\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
T_3+T_{132}+T_{133}=\mathrm{div}\left[(\theta-\bar\theta)\left(\frac{1}{\theta}\kappa\nabla\theta-\frac{1}{\bar\theta}\bar\kappa\nabla\bar\theta\right)\right]-\bar\theta\kappa\left(\frac{\nabla\theta}{\theta}-\frac{\nabla\bar\theta}{\bar\theta}\right)^2\\
-\left(\frac{\nabla\theta}{\theta}-\frac{\nabla\bar\theta}{\bar\theta}\right)\frac{\nabla\bar\theta}{\bar\theta}(\bar\theta\kappa-\theta\bar\kappa)
\end{aligned}
\tag{4.57}
$$

As for T_4, we have

$$
\begin{aligned}
T_4=&-\mathrm{div}\Big[\sum_j\rho_ju_j(\mu_j-\bar\mu_j)+\frac{1}{\theta}(\theta-\bar\theta)\sum_j(h_j-\rho_j\mu_j)u_j\Big]\\
&+\frac{\bar\theta}{\theta}\sum_j u_j\cdot d_j-\sum_j\nabla\bar\mu_j\cdot\rho_ju_j-\frac{1}{\bar\theta}\nabla\bar\theta\sum_j(h_j-\rho_j\mu_j)u_j.
\end{aligned}
\tag{4.58}
$$

Then, by (4.53)–(4.54) and (4.56)–(4.58) we obtain the relative entropy identity (4.59).

The resulting relative entropy identity, then, reads:

$$
\begin{aligned}
&\partial_t I(U|\bar{U}) + \operatorname{div}\Big[vI(U|\bar{U}) + (p-\bar{p})(v-\bar{v}) + \sum_j \rho_j u_j(\mu_j - \bar{\mu}_j) \\
&- (\theta-\bar{\theta})\left(\frac{1}{\theta}\kappa\nabla\theta - \frac{1}{\bar{\theta}}\bar{\kappa}\nabla\bar{\theta}\right) + \frac{1}{\theta}(\theta-\bar{\theta})\sum_j (h_j - \rho_j\mu_j)u_j \Big] \\
&+ \bar{\theta}\kappa\left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}}\right)^2 - \frac{\bar{\theta}}{\theta}\sum_j u_j\cdot d_j = (\partial_t\bar{\theta} + \bar{v}\cdot\nabla\bar{\theta})(-\rho\eta)(\omega|\bar{\omega}) \\
&- p(\omega|\bar{\omega})\operatorname{div}\bar{v} - (\eta-\bar{\eta})\rho(v-\bar{v})\cdot\nabla\bar{\theta} - \sum_j \nabla\bar{\mu}_j\left(\frac{\rho_j}{\rho} - \frac{\bar{\rho}_j}{\bar{\rho}}\right)\rho(v-\bar{v}) \\
&- \rho(v-\bar{v})\nabla\bar{v}\cdot(v-\bar{v}) - \sum_j \nabla\bar{\mu}_j\cdot\rho_j u_j - \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}}\right)\frac{\nabla\bar{\theta}}{\bar{\theta}}(\bar{\theta}\kappa - \theta\bar{\kappa}) \\
&- \frac{1}{\bar{\theta}}\nabla\bar{\theta}\sum_j (h_j - \rho_j\mu_j)u_j
\end{aligned}
\tag{4.59}
$$

where

$$
\begin{aligned}
p(\omega|\bar{\omega}) &= p - \bar{p} - \sum_j \bar{p}_{\rho_j}(\rho_j - \bar{\rho}_j) - \bar{p}_\theta(\theta - \bar{\theta}) \\
(-\rho\eta)(\omega|\bar{\omega}) &= -\rho\eta + \bar{\rho}\bar{\eta} + \sum_j (\bar{\rho}\bar{\eta})_{\rho_j}(\rho_j - \bar{\rho}_j) + (\bar{\rho}\bar{\eta})_\theta(\theta - \bar{\theta}).
\end{aligned}
$$

In the sequel, let $\mathcal{U} \subset (\mathbb{R}^+)^{n+1}$ be a set in the positive cone $(\mathbb{R}^+)^{n+1}$ with $\overline{\mathcal{U}}$ compact, and suppose that the states $\omega, \bar{\omega} \in \mathcal{U}$ satisfy

$$0 < \rho_j, \bar{\rho}_j \leq M \tag{4.60}$$

$$0 < \delta \leq \rho, \bar{\rho} \leq M \tag{4.61}$$

$$0 < \delta \leq \theta, \bar{\theta} \leq M \tag{4.62}$$

for some $\delta, M > 0$.

Lemma 7. *Let $\omega, \bar{\omega} \in \mathcal{U}$ satisfy* (4.60)–(4.62) *and suppose that $\psi(\rho_1, \dots, \rho_n, \theta) \in C^3(\overline{\mathcal{U}})$ satisfies the thermodynamic relations* (2.49)–(2.51), (2.32) *and the hypotheses* (H_1),(H_2) *(and thus* (4.45)*). There exist constants $c_1, c_2, c_3 > 0$ depend-*

ing on δ, M, such that for $\omega, \bar{\omega} \in \mathcal{U}$ we have

$$c_1|\omega - \bar{\omega}|^2 \leq J(\omega|\bar{\omega}) \tag{4.63}$$

$$|p(\omega|\bar{\omega})| \leq c_2 J(\omega|\bar{\omega}) \tag{4.64}$$

$$|(-\rho\eta)(\omega|\bar{\omega})| \leq c_3 J(\omega|\bar{\omega})\,. \tag{4.65}$$

Proof. Consider the form (4.49) and use $(\rho e)_{\rho\eta} = \theta$ and the convexity of $\rho e = \rho\tilde{e}(\rho_1, \dots, \rho_n, \rho\eta)$ in the variables $(\rho_1, \dots, \rho_n, \rho\eta)$ to obtain

$$\begin{aligned} J(\omega|\bar{\omega}) &= \rho\tilde{e} - \overline{\rho\tilde{e}} - \sum_i \overline{\left(\frac{\partial \rho\tilde{e}}{\partial \rho_i}\right)}(\rho_i - \bar{\rho}_i) - \overline{\left(\frac{\partial \rho\tilde{e}}{\partial \rho\eta}\right)}(\rho\eta - \bar{\rho}\bar{\eta}) \\ &\geq c\left(\sum_i |\rho_i - \bar{\rho}_i|^2 + |\rho\eta - \bar{\rho}\bar{\eta}|^2\right) \end{aligned} \tag{4.66}$$

where $c = \inf_{(\rho_1,\dots,\rho_n,\rho\eta)\in\overline{\mathcal{U}}} \nabla^2_{(\rho_1,\dots,\rho_n,\rho\eta)}\rho\tilde{e} > 0$, for $0 < \delta \leq \rho < M$. Next, the map $(\rho_1, \dots, \rho_n, \theta) \mapsto (\rho_1, \dots, \rho_n, \rho\eta)$ defined by $\eta(\rho_1, \dots, \rho_n, \theta) = -\psi_\theta$ can be inverted on the set $\overline{\mathcal{U}}$ and since $\frac{\partial(\rho\eta)}{\partial\theta} = \frac{1}{\theta}\rho c_v > 0$, for $\rho, \theta > 0$, the inverse map is Lipschitz and

$$|\theta - \bar{\theta}|^2 + \sum_i |\rho_i - \bar{\rho}_i|^2 \leq C\left(|\rho\eta - \bar{\rho}\bar{\eta}|^2 + \sum_i |\rho_i - \bar{\rho}_i|^2\right) \tag{4.67}$$

where C depends on $\min_{\overline{\mathcal{U}}} \frac{\partial(\rho\eta)}{\partial\theta}$. Combining (4.66) with (4.67) gives (4.63).

The bounds (4.64)–(4.65) follow from the Taylor theorem, which provides

$$p(\omega|\bar{\omega}) = (\omega - \bar{\omega})^\top \left(\int_0^1 \int_0^s \nabla^2 p(\tau\omega + (1-\tau)\bar{\omega})d\tau ds\right)(\omega - \bar{\omega}) \tag{4.68}$$

$$(-\rho\eta)(\omega|\bar{\omega}) = (\omega - \bar{\omega})^\top \left(\int_0^1 \int_0^s \nabla^2(-\rho\eta)(\tau\omega + (1-\tau)\bar{\omega})d\tau ds\right)(\omega - \bar{\omega}) \tag{4.69}$$

and the regularity of ψ implies that $p, \rho\eta \in C^2(\overline{\mathcal{U}})$. ☐

Let $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ be the three–dimensional torus. We have the following convergence result:

Theorem 8. *Let $\bar{U}^\kappa$ be a classical solution of* (4.37)–(4.39) *defined on a maximal interval of existence $\mathbb{T} \times [0, T^*)$ and let $U^{\epsilon,\kappa}$ be a family of classical solutions of* (4.31)–(4.33) *defined on $\mathbb{T}^3 \times [0, T]$ for some $T < T^*$, which emanate from smooth data $\bar{U}_0^\kappa$, $U_0^{\epsilon,\kappa}$, respectively, and satisfy the uniform bounds* (4.60)–(4.62) *for $\delta, M > 0$. Moreover, assume that $\psi \in C^3(\bar{\mathcal{U}})$ satisfies* (H$_1$), (H$_2$), *that*

$$\sum_j \left| \frac{h_j - \rho_j \mu_j}{\sqrt{\rho_j}} \right|^2 \le \alpha(\rho e + 1) \tag{4.70}$$

for some $\alpha > 0$ and that $0 \le \kappa(\rho_1, \dots, \rho_n, \theta) \le M$. Then, there exist constants $c, C > 0$ depending on δ, M, α but otherwise independent of ϵ such that

$$\int_{\mathbb{T}^3} I(U^{\epsilon,\kappa}|\bar{U}^\kappa)dx \le c \int_{\mathbb{T}^3} I(U_0^{\epsilon,\kappa}|\bar{U}_0^\kappa)dx + \epsilon C \tag{4.71}$$

In particular, if $\int_{\mathbb{T}^3} I(U_0^{\epsilon,\kappa}|\bar{U}_0^\kappa)dx \to 0$ as $\epsilon \to 0$, then

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^3} I(U^{\epsilon,\kappa}|\bar{U}^\kappa)dx \to 0 \quad \text{as } \epsilon \to 0. \tag{4.72}$$

Proof. We first integrate the relative entropy identity (4.59) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3} I(U^\varepsilon|\bar{U})dx + \int_{\mathbb{T}^3} \bar{\theta}\kappa \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}}\right)^2 dx - \int_{\mathbb{T}^3} \frac{\bar{\theta}}{\theta} \sum_j u_j \cdot d_j dx \\
&= \int_{\mathbb{T}^3} (\partial_t\bar{\theta} + \bar{v}\cdot\nabla\bar{\theta})(-\rho\eta)(U^\varepsilon|\bar{U})dx - \int_{\mathbb{T}^3} p(U^\varepsilon|\bar{U})\mathrm{div}\bar{v}dx \\
&+ \int_{\mathbb{T}^3} (\eta - \bar{\eta})\rho(v - \bar{v})\cdot\nabla\bar{\theta}dx - \int_{\mathbb{T}^3} \sum_j \nabla\bar{\mu}_j \left(\frac{\rho_j}{\rho} - \frac{\bar{\rho}_j}{\bar{\rho}}\right)\rho(v - \bar{v})dx \\
&- \int_{\mathbb{T}^3} \rho(v - \bar{v})\nabla\bar{v}\cdot(v - \bar{v})dx - \int_{\mathbb{T}^3} \sum_j \nabla\bar{\mu}_j \cdot \rho_j u_j dx \\
&- \int_{\mathbb{T}^3} \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}}\right)\frac{\nabla\bar{\theta}}{\bar{\theta}}\kappa(\bar{\theta} - \theta)dx - \int_{\mathbb{T}^3} \frac{1}{\bar{\theta}}\nabla\bar{\theta}\sum_j (h_j - \rho_j\mu_j)u_j dx \\
&=: I_1 + \dots + I_8 \,.
\end{aligned} \tag{4.73}$$

Our strategy is to control the terms I_1 to I_5 by the integral of $I(U|\bar{U})$ and terms I_6 to I_8 by the dissipation on the left–hand side. In particular, to control I_1 and I_2, we use (4.65) and (4.64) respectively. For I_3, using the regularity of ψ, and thus η, we have

$$\begin{aligned}(\eta-\bar{\eta})\rho(v-\bar{v})\cdot\nabla\bar{\theta} &\leq c\left(\rho|\eta-\bar{\eta}|^2+\rho|v-\bar{v}|^2\right)\\ &\leq c\left(\sum_j|\rho_j-\bar{\rho}_j|^2+|\theta-\bar{\theta}|^2+\rho|v-\bar{v}|^2\right)\end{aligned}$$

Regarding I_4, we have

$$\begin{aligned}\sum_j\nabla\bar{\mu}_j\left(\frac{\rho_j}{\rho}-\frac{\bar{\rho}_j}{\bar{\rho}}\right)\rho(v-\bar{v}) &\leq c\left(\rho\sum_j\left(\frac{\rho_j}{\rho}-\frac{\bar{\rho}_j}{\bar{\rho}}\right)^2+\rho(v-\bar{v})^2\right)\\ &\leq c\left(\sum_j(\rho_j-\bar{\rho}_j)^2+\rho(v-\bar{v})^2\right)\end{aligned}$$

since the map $f_j:(\rho_1,\dots,\rho_n)\mapsto\frac{\rho_j}{\rho_1+\dots+\rho_n}$, for $j\in\{1,\dots,n\}$ is Lipschitz under the assumption $0<\delta\leq\rho$. Indeed

$$\frac{\partial f_j}{\partial\rho_i}=\begin{cases}-\frac{\rho_j}{\rho^2}, & i\neq j\\ \frac{\rho-\rho_j}{\rho^2}, & i=j\end{cases}$$

hence

$$\left|\frac{\partial f_j}{\partial\rho_i}\right|\leq\frac{c}{\rho}\leq\frac{c}{\delta}\,.$$

As for I_5

$$\rho(v-\bar{v})\nabla\bar{v}\cdot(v-\bar{v})\leq c\rho|v-\bar{v}|^2$$

Now, since $u_i=\mathcal{O}(\epsilon)$, we can set $\rho_j u_j=\epsilon\rho_j\tilde{u}_j$ and we have, by Young's inequality and for $\mu=\min_{i\neq j}b_{ij}>0$:

$$\begin{aligned}I_6 &\leq\frac{\epsilon}{\mu\delta}\int_{\mathbb{T}^3}\sum_j\frac{\rho_j}{\rho}|\nabla\bar{\mu}_j|^2dx+\frac{\epsilon\mu\delta}{4}\int_{\mathbb{T}^3}\rho\sum_j|\sqrt{\rho_j}\tilde{u}_j|^2dx\\ &\leq\epsilon C+\frac{\epsilon\mu\delta}{4}\int_{\mathbb{T}^3}\rho\sum_j|\sqrt{\rho_j}\tilde{u}_j|^2dx=:I_{61}+I_{62}\end{aligned}$$

Likewise, using assumption (4.70), we have that

$$\begin{aligned} I_8 &\le \frac{\epsilon}{\mu\delta}\int_{\mathbb{T}^3}\frac{1}{\rho}\sum_j\left|\frac{h_j-\rho_j\mu_j}{\sqrt{\rho_j}}\right|^2\left|\frac{\nabla\bar\theta}{\bar\theta}\right|^2dx+\frac{\epsilon\mu\delta}{4}\int_{\mathbb{T}^3}\rho\sum_j|\sqrt{\rho_j}\tilde u_j|^2dx \\ &\le \epsilon C+\frac{\epsilon\mu\delta}{4}\int_{\mathbb{T}^3}\rho\sum_j|\sqrt{\rho_j}\tilde u_j|^2dx =: I_{81}+I_{82} \end{aligned}$$

Finally

$$\begin{aligned} I_7 &\le \frac{1}{2}\int_{\mathbb{T}^3}\bar\theta\kappa\left(\frac{\nabla\theta}{\theta}-\frac{\nabla\bar\theta}{\bar\theta}\right)^2dx+\frac{1}{2}\int_{\mathbb{T}^3}\kappa\frac{|\nabla\bar\theta|^2}{\bar\theta^3}(\theta-\bar\theta)^2dx \\ &\le \frac{1}{2}\int_{\mathbb{T}^3}\bar\theta\kappa\left(\frac{\nabla\theta}{\theta}-\frac{\nabla\bar\theta}{\bar\theta}\right)^2dx+c\int_{\mathbb{T}^3}(\theta-\bar\theta)^2dx =: I_{71}+I_{72} \end{aligned}$$

Using (4.63) we conclude that

$$I_1+\cdots+I_5+I_{72}\le c\int_{\mathbb{T}^3}I(U^\varepsilon|\bar U)dx\,.$$

The error terms I_{62}, I_{71} and I_{82} are controlled by the dissipation on the left–hand side of (4.73). This is due to the assumption that $\delta\le\bar\theta$ and the following estimate, in which we use (4.25) and (4.23):

$$\begin{aligned} -\frac{1}{\theta}\sum_j u_j\cdot d_j &= \epsilon\rho\sum_{i,j}M_{ij}\sqrt{\rho_i}\tilde u_i\cdot\sqrt{\rho_j}\tilde u_j \\ &\ge \epsilon\rho\mu|\mathbb{P}_L\sqrt{\rho}\tilde u|^2 \end{aligned}$$

where $\sqrt{\rho}\tilde u$ is the vector with components $\sqrt{\rho_i}\tilde u_i$, whose projection is computed as

$$\begin{aligned} \mathbb{P}_L\sqrt{\rho}\tilde u &= \sum_k(\mathbb{P}_L)_{ik}\sqrt{\rho_k}\tilde u_k \\ &= \sqrt{\rho_i}\tilde u_i-\frac{\sqrt{\rho_i}}{\rho}\sum_k\rho_k\tilde u_k \\ &= \sqrt{\rho_i}\tilde u_i \end{aligned}$$

Putting everything together we obtain:

$$\frac{d}{dt}\int_{\mathbb{T}^3} I(U^\varepsilon|\bar{U})dx + \frac{\delta}{2}\int_{\mathbb{T}^3}\kappa\left(\frac{\nabla\theta}{\theta}-\frac{\nabla\bar{\theta}}{\bar{\theta}}\right)^2 dx + \frac{\delta\epsilon\mu}{2}\int_{\mathbb{T}^3}\rho\sum_j|\sqrt{\rho_j}\tilde{u}_j|^2 dx$$
$$\leq c\int_{\mathbb{T}^3} I(U^\varepsilon|\bar{U})dx + \epsilon C \tag{4.74}$$

for appropriate constants $c, C > 0$ independent of ϵ. The dissipation terms are neglected and we obtain the differential inequality

$$\frac{d\varphi^\varepsilon(t)}{dt} \leq c\varphi^\varepsilon(t) + \epsilon C \tag{4.75}$$

for

$$\varphi^\varepsilon(t) = \int_{\mathbb{T}^3} I(U^\varepsilon(x,t)|\bar{U}(x,t))dx$$

Then, (4.71) follows by Grönwall's Lemma. □

A particular case of the above analysis is the convergence to the adiabatic theory, i.e. when also $\kappa = 0$. Consider the hyperbolic-parabolic system (4.31)–(4.36) with the linear system (4.34) and the constraint (4.35) and its hyperbolic counterpart (4.27)–(4.30). Then, following the same process as before, we can obtain that the hyperbolic-parabolic system (4.31)–(4.36) converges as $\epsilon, \kappa \to 0$ to the hyperbolic system without diffusion and heat conduction (4.27)–(4.30):

Theorem 9. *Let $\bar{U}$ be a classical solution of (4.27)–(4.29) defined on a maximal interval of existence $\mathbb{T}^3 \times [0, T^*)$ and let $U^{\epsilon,\kappa}$ be a family of classical solutions of (4.31)–(4.33) defined on $\mathbb{T}^3 \times [0, T]$, for $T < T^*$, emanating from smooth data $\bar{U}_0, U_0^{\epsilon,\kappa}$ respectively. Under the hypotheses of theorem 8, there exist constants $c, C_1, C_2 > 0$ independent of ϵ and κ such that*

$$\int_{\mathbb{T}^3} I(U^{\epsilon,\kappa}|\bar{U})dx \leq c\int_{\mathbb{T}^3} I(U_0^{\epsilon,\kappa}|\bar{U}_0)dx + \|\kappa\|_\infty C_1 + \epsilon C_2 \tag{4.76}$$

In particular, if $\int_{\mathbb{T}^3} I(U_0^{\epsilon,\kappa}|\bar{U}_0)dx \to 0$ *as* $\epsilon, \kappa \to 0$, *then*

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^3} I(U^{\epsilon,\kappa}|\bar{U})dx \to 0 \quad \text{as } \epsilon, \kappa \to 0. \tag{4.77}$$

Proof. The proof is identical to the proof of Theorem 8, with the only difference that $\bar{\kappa} = 0$ (whereas in the previous case $\kappa = \bar{\kappa}$) and thus I_7 is controlled by:

$$\begin{aligned} I_7 &= -\int_{\mathbb{T}^3} \kappa \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}} \right) \cdot \nabla\bar{\theta} \\ &\leq \frac{1}{2} \int_{\mathbb{T}^3} \bar{\theta}\kappa \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}} \right)^2 dx + \frac{1}{2} \int_{\mathbb{T}^3} \kappa \frac{|\nabla\bar{\theta}|^2}{\bar{\theta}} dx \\ &\leq \frac{1}{2} \int_{\mathbb{T}^3} \bar{\theta}\kappa \left(\frac{\nabla\theta}{\theta} - \frac{\nabla\bar{\theta}}{\bar{\theta}} \right)^2 dx + \|\kappa\|_\infty C_1. \end{aligned}$$

□

Next, we present a commentary on the hypotheses for Theorems 8 and 9. The goal here is to discuss aspects of the theory of smooth solutions; the situation for weak solutions presents serious challenges. The hypotheses on the bounds (4.61), (4.62) reflect the loss of strict hyperbolicity (and even hyperbolicity) of the model at $\rho = 0$. Also, we would expect that such continuum models are not valid for very large temperatures or temperature near zero.

The hypothesis $\psi \in C^3(\bar{U})$ is a drawback as it does not hold at $\rho_i = 0$ for realistic models and at the same time one would insist on the range (4.60) that guarantees some of the components may disappear as an outcome of interactions. The main problematic term is I_3. Realistic models are discussed in section 4.2.1 dealing with the multicomponent ideal gas. The reader will notice that $\psi \in C^3(\bar{U})$ holds on the restricted range

$$0 < \delta \leq \rho_j, \bar{\rho}_j \leq M \tag{4.78}$$

and gives convergence for solutions taking values in that range (4.78), (4.61), (4.62).

The need concerning (4.70) originates from a deficiency of the models discussed here. As mentioned at the end of section 2.2, for the general model, the partial energies e_i and pressures p_i and thus the enthalpies h_i are not determined from the total free energy $\rho\psi$, yet they enter the balance equations (3.4)–(3.7) and thus extra constitutive relations have to be supplied. Hypothesis (4.70) concerns these extra constitutive relations. For the case of simple mixtures, $\rho\psi = \sum_i \rho_i \psi_i(\rho_i, \theta)$, the need for extra constitutive relations does not arise; one computes using the constitutive relations that $\frac{1}{\sqrt{\rho_j}}(h_j - \rho_j \mu_j) = -\theta \sqrt{\rho_j} \frac{\partial \psi_j}{\partial \theta}$ and the terms are bounded from (4.60)–(4.62).

4.2.1 Multicomponent ideal gases

We present the constitutive model of an ideal multicomponent gas and compute the relative constitutive functions that appear in the relative entropy formula (4.59). We refer to Callen [27, Sec 13] and Giovangigli [4, Sec 6] for details on the multicomponent ideal gas laws. Here, we outline a constitutive model defined in terms of densities ρ_i and temperature θ. The model is a simple mixture of ideal gases where the free energy of each component is given by $\rho_i \psi_i = R_i \theta \rho_i \log \rho_i - c_i \rho_i \theta \log \theta$, where $R_i > 0$ is the engineering gas constant and $c_i > 0$ the constant heat capacity of the i-th component, and the mixture free energy is

$$\rho\psi = \sum_i \rho_i \psi_i = \sum_i R_i \theta \rho_i \log \rho_i - \sum_i c_i \rho_i \theta \log \theta \,. \tag{4.79}$$

We then have

$$\begin{aligned} \mu_j &= \frac{\partial(\rho\psi)}{\partial \rho_j} = R_j (1 + \log \rho_j)\theta - c_j \theta \log \theta \\ \rho\eta &= -\frac{\partial(\rho\psi)}{\partial \theta} = -\sum_i R_i \rho_i \log \rho_i + \sum_i c_i \rho_i (1 + \log \theta) \\ \rho e &= \sum_i c_i \rho_i \theta \\ p &= -\rho\psi + \sum_j \rho_j \mu_j = \sum_j R_j \rho_j \theta. \end{aligned} \tag{4.80}$$

The relative quantities are computed as follows: Using (4.49), we have

$$\begin{aligned} J(\omega|\bar{\omega}) &= \bar{\theta}\sum_i R_i\Big(\rho_i\log\rho_i - \bar{\rho}_i\log\bar{\rho}_i - (1+\log\bar{\rho}_i)(\rho_i-\bar{\rho}_i)\Big) \\ &\qquad + \bar{\theta}\sum_i c_i\rho_i\Big(-\log\theta + \log\bar{\theta} + \frac{1}{\bar{\theta}}(\theta-\bar{\theta})\Big) \\ &= \bar{\theta}\sum_i R_i(x\log x)\big(\rho_i|\bar{\rho}_i\big) + \sum_i c_i\rho_i\bar{\theta}(-\log y)(\theta|\bar{\theta}) \end{aligned} \tag{4.81}$$

where $(x\log x)\big(\rho_i|\bar{\rho}_i\big)$ is the quadratic part of the Taylor expansion of $x\log x$ and ditto for $(-\log y)(\theta|\bar{\theta})$. Due to the convexity of $(x\log x)$ and $(-\log y)$ both terms are positive.

Similarly, we compute

$$p(\omega|\bar{\omega}) = \sum_j R_j(\rho_j-\bar{\rho}_j)(\theta-\bar{\theta}) \tag{4.82}$$

and

$$\begin{aligned} (-\rho\eta)(\omega|\bar{\omega}) &= -\rho\eta + \bar{\rho}\bar{\eta} + \sum_j \overline{\frac{\partial(\rho\eta)}{\partial\rho_j}}(\rho_j-\bar{\rho}_j) + \overline{\frac{\partial(\rho\eta)}{\partial\theta}}(\theta-\bar{\theta}) \\ &= \sum_i R_i\big(x\log x\big)\big(\rho_i|\bar{\rho}_i\big) + \sum_i c_i\rho_i(-\log y)(\theta|\bar{\theta}) \\ &\qquad - \sum_i \frac{c_i}{\bar{\theta}}(\rho_i-\bar{\rho}_i)(\theta-\bar{\theta}). \end{aligned} \tag{4.83}$$

Finally observe that the relative entropy (4.50) takes the form

$$I(U|\bar{U}) = \tfrac{1}{2}\rho|v-\bar{v}|^2 + \bar{\theta}\sum_i R_i\big(x\log x\big)\big(\rho_i|\bar{\rho}_i\big) + \bar{\theta}\sum_i c_i\rho_i\big(-\log y\big)\big(\theta|\bar{\theta}\big) \tag{4.84}$$

Using the explicit formulas (4.81), (4.82) and (4.83) the facts $(x\log x)'' = \frac{1}{x}$ and $(-\log x)'' = \frac{1}{x^2}$ we obtain an analog of Lemma 7 for the ideal gas:

Lemma 10. *Let $\omega, \bar{\omega} \in \mathcal{U}$ satisfy* (4.60)–(4.62) *and let $\rho\psi$ be given in* (4.79).

There exist constants $c_1, c_2 > 0$ depending on δ, M, such that

$$c_1|\omega - \bar{\omega}|^2 \leq J(\omega|\bar{\omega}) \tag{4.85}$$

$$|p(\omega|\bar{\omega})| + |(-\rho\eta)(\omega|\bar{\omega})| \leq c_2 J(\omega|\bar{\omega})\,. \tag{4.86}$$

The missing element to conclude the proof of an analog of Theorem 8 for the multicomponent ideal gas is an estimate of the type

$$|(\eta - \bar{\eta})\rho(v - \bar{v}) \cdot \nabla\bar{\theta}| \leq C\Big(\frac{1}{2}\rho|v - \bar{v}|^2 + J(\omega|\bar{\omega})\Big)$$

However, such an estimate is not valid for solutions that take value on the range (4.60)–(4.62).

Chapter 5

The Maxwell–Stefan system

The Maxwell–Stefan system describes diffusive phenomena in multicomponent systems of gases. It appears in applications from various domains, for instance in dialysis, ion exchange, sedimentation and electrolysis. It describes the evolution of the vector function $\mathbf{c} = (c_1, \ldots, c_n)$ where $c_i(x,t)$ is the concentration of the i–th component, $i = 1, \ldots, n$. Their dynamics is described by n equations for the conservation of mass,

$$\partial_t c_i + \nabla \cdot J_i = 0\,, \tag{5.1}$$

with $J_i = c_i u_i$, subject to a linear system for determining the molar fluxes $J_i(x,t)$,

$$\begin{cases} -\displaystyle\sum_{j=1, j\neq i}^{n} \frac{c_j J_i - c_i J_j}{D_{ij}} = \nabla c_i \qquad i = 1, \ldots, n\,, & (5.2) \\ \displaystyle\sum_{i=1}^{n} J_i = 0, & (5.3) \end{cases}$$

where $D_{ij} = D_{ji} > 0$, $i \neq j$, are the diffusion coefficients modeling binary interactions between the components.

The Maxwell–Stefan system can be seen as the isothermal analogue of the Class–I system that has been studied in chapter 4, under the hypothesis of zero mean flow, i.e. system (4.31)–(4.35) for which θ is constant, $v = 0$ and we set $D_{ij} = \frac{1}{b_{ij}}$ and $c_i = \frac{\rho_i}{\rho}$. The reader will notice that system (4.31)–(4.35) is written in terms of the mass densities ρ_i, while system (5.1)–(5.3) in terms of the concentrations c_i and one does not imply the other since the total mass density ρ is not necessarily constant. Nevertheless, we ignore this inconsistency in order to be consistent with the references that we follow.

The first mathematically rigorous work studying the Maxwell–Stefan system was probably in [34], where the authors showed local well–posedness of the problem in the whole space $\mathbb{R}^3$ for strong solutions. In particular, they proved that for generic initial data, solutions can break down at finite time, unlike the case of initial data close to an equilibrium state, in which the solution can be extended globally in time. The problem in a bounded domain with no–flux boundary conditions was later studied in [7], where local well–posedness of strong solutions was shown. The key ingredient in the analysis of [7] was the characterization of the spectrum of the linear system (5.2), which paved the way for existence of global–in–time weak solutions, as was shown in [8]. For more details on the inversion of the constrained linear system (5.2)–(5.3) we refer to [31] and [4, Chapter 7].

5.1 Uniqueness of weak solutions

Uniqueness of weak solutions to cross–diffusion systems, i.e. systems with non-trivial interactions among the species, is in general a hard task. For this reason, the best one can hope for is a weak–strong uniqueness property, meaning that for any pair of weak and strong solutions emanating from the same initial data, the weak solution will coincide with the strong one, as long as the latter exists. In the case of the Maxwell–Stefan system strong solutions indeed exist (and as we mentioned are unique), however they are, in general, only locally defined in time and thus uniqueness of weak solutions can only be shown for small times, until the strong solution collapses. The weak–strong uniqueness result for the Maxwell–Stefan system was proved in [10].

In our work [12], we develop a theory based on the relative entropy method that allows us to prove uniqueness of weak solutions in the three dimensional torus $\mathbb{T}^3$.

A commonly used method for stability proceeds by exploiting the entropy structure of the system and computing the evolution of the relative entropy; the method is usually applicable when one of the solutions is a strong solution. It has been used in [10] to provide a weak–strong uniqueness result under the assumptions that the strong solution is sufficiently smooth and presents no anomalous diffusion (concentrations in the entropy identity). The required smoothness of the strong solution is considerably more than what is provided by the global existence result in [8] and regularity results for weak solutions (even if expected) are currently not available.

The Maxwell-Stefan system is endowed with an entropy functional, which for simple gases takes the form

$$H(\mathbf{c}) = \int_{\mathbb{T}^3} \sum_{i=1}^{n} c_i(\ln c_i - 1)\mathrm{d}x\,. \tag{5.4}$$

To prove uniqueness under reduced regularity assumptions two new tools are introduced: (i) the relative entropy is replaced by the symmetrized relative entropy

$$H^{sym}(\mathbf{c}, \bar{\mathbf{c}}) := \int_{\mathbb{T}^3} \sum_{i=1}^{n} (\ln c_i - \ln \bar{c}_i)(c_i - \bar{c}_i)\ \mathrm{d}x \tag{5.5}$$

which satisfies a nice formal identity (see (5.26)). (ii) For the rigorous derivation of the relative entropy identity we introduce renormalized solutions.

Renormalized solutions were introduced in the study of the Boltzmann equation [36] and extended to transport equations [37], nonlinear elliptic problems [38], equations of fluid mechanics [39], nonlinear parabolic problems [40]. Closest to the use here are the references [39, pp. 227], [41, Sec 4.1.5] concerning the compressible Navier–Stokes system. To motivate the definition of a renormalized solution, compute the evolution of a smooth function $\beta(c_i)$. Formal multiplication of (5.1) by $\beta'(c_i)$ gives:

$$\partial_t \beta(c_i) + \nabla \cdot (\beta'(c_i) c_i u_i) - 2c_i \beta''(c_i) \nabla\sqrt{c_i} \cdot \sqrt{c_i} u_i = 0 \tag{5.6}$$

From the existence theory of weak solutions [8], we know that $\nabla\sqrt{c_i} \cdot \sqrt{c_i} u_i \in L^1$ and use this information to give meaning to (5.6).

The formal identity (5.26) leads after integration to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\Big(\sum_i(\ln c_i-\ln\bar c_i)(c_i-\bar c_i)\Big)dx$$
$$+\int_{\mathbb{T}^3}\underbrace{\sum_{i,j}\frac{1}{2D_{ij}}(c_ic_j+\bar c_i\bar c_j)\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2}_{Q}dx=E_1+E_2 \tag{5.7}$$

where E_1, E_2 are error terms. Renormalized solutions are used to give a precise meaning to (5.7). The last step combines information on the spectral gap of an operator associated to the frictional relative work Q between two solutions from [10] with a delicate estimation of the error terms E_1 and E_2 to circumvent the logarithmic singularities of the relative entropy. It leads to a uniqueness result for renormalized solutions of the Maxwell–Stefan system, see Theorem 15.

We work on the torus $\mathbb{T}^3$ with initial data that satisfy

$$0\le c_i^0\in L^\infty(\mathbb{T}^3)\,,\quad i=1,\dots,n\,;\qquad \sum_{i=1}^n c_i^0=1\,;\qquad H(\mathbf{c}^0)<+\infty\,. \tag{A}$$

The solution propagates from the data the constraint $\sum_i c_i=1$. We recall the notion of weak solution.

Definition 11. *A function $\mathbf{c}=(c_1,\dots,c_n)$ is a weak solution of* (5.1)–(5.3)*, if for $i\in\{1,\dots,n\}$:*

(i) $0\le c_i\le 1$ *and* $\sum_{i=1}^n c_i=1$,

(ii) $\nabla\sqrt{c_i}\in L^2_{\mathrm{loc}}(\mathbb{T}^3\times(0,\infty))\,,\ \sqrt{c_i}u_i\in L^2_{\mathrm{loc}}(\mathbb{T}^3\times(0,\infty)))$,

(iii) for any test function $\phi_i\in C_c^\infty(\mathbb{T}^3\times[0,\infty))$, we have

$$\int_{\mathbb{T}^3}c_i^0(x)\phi_i(x,0)\mathrm{d}x+\int_0^\infty\int_{\mathbb{T}^3}c_i\partial_t\phi_i\mathrm{d}x\mathrm{d}t+\int_0^\infty\int_{\mathbb{T}^3}c_iu_i\cdot\nabla\phi_i\mathrm{d}x\mathrm{d}t=0 \tag{5.8}$$

and c_iu_i satisfies (5.2)–(5.3).

(iv) $\lim_{t\to 0} \int_{\mathbb{T}^3} |c_i(x,t) - c_i^0(x)|dx \to 0.$

Using [8] and spectral properties for the Maxwell–Stefan system (5.2)–(5.3) from [10], property (ii) implies the regularity

$$c_i \in C^0_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3)), \quad \sqrt{c_i}u_i \in L^2_{\text{loc}}(\mathbb{T}^3 \times (0,\infty)) \tag{5.9}$$

and gives meaning to the terms of (5.8).

The definition of renormalized solutions is motivated by the formal identity (5.6) and the regularity information provided by the existence theory, notably that $\nabla\sqrt{c_i}\cdot\sqrt{c_i}u_i \in L^1_{\text{loc}}(\mathbb{T}^3\times(0,\infty))$. For test functions $\beta(s)$ of class $C^2([0,\infty))$ we introduce the definition:

Definition 12. *A function* $\mathbf{c} = (c_1,\dots,c_n)$ *is a renormalized solution of* (5.1)-(5.3)*, if for* $i \in \{1,\dots,n\}$ *it satisfies:*

(i) $0 \le c_i \le 1$ *and* $\sum_{i=1}^n c_i = 1,$

(ii) $\nabla\sqrt{c_i} \in L^2_{\text{loc}}(\mathbb{T}^3\times(0,\infty))\,,\ \sqrt{c_i}u_i \in L^2_{\text{loc}}(\mathbb{T}^3\times(0,\infty))),$

(iii) for any test function $\phi_i \in C_c^\infty(\mathbb{T}^3\times[0,\infty))$ *and for* $\beta \in C^2([0,\infty))$

$$\begin{aligned} &\int_{\mathbb{T}^3} \beta(c_i^0)\phi_i(x,0)\mathrm{d}x + \int_0^\infty\int_{\mathbb{T}^3} \beta(c_i)\partial_t\phi_i \mathrm{d}x\mathrm{d}t \\ &+ \int_0^\infty\int_{\mathbb{T}^3} \sqrt{c_i}\beta'(c_i)\sqrt{c_i}u_i\cdot\nabla\phi_i \mathrm{d}x\mathrm{d}t \\ &+ 2\int_0^\infty\int_{\mathbb{T}^3} c_i\beta''(c_i)\nabla\sqrt{c_i}\cdot\sqrt{c_i}u_i\phi_i \mathrm{d}x\mathrm{d}t = 0 \end{aligned} \tag{5.10}$$

and c_iu_i *satisfies* (5.2)–(5.3)*.*

(iv) $\lim_{t\to 0}\int_{\mathbb{T}^3} |\beta(c_i(x,t)) - \beta(c_i^0(x))|\mathrm{d}x \to 0,$ *for* $\beta \in C([0,\infty))$*.*

Renormalized solutions are often introduced to handle the behavior near infinity. Here, the solution is bounded and the goal is to handle the behavior at $c_i = 0$. Within the regularity class of weak solutions and for $\beta \in C^2([0,\infty))$ all terms in (5.10) are well defined.

The objective is to derive an identity for the symmetrized relative entropy (5.5). A formal derivation is presented in section 5.1.3 and leads to the formal identity (5.26). At the same time (5.5) only allows for non-vanishing solutions because of the logarithmic term $\ln c_i$. Thus, we introduce a symmetrized relative entropy, which is regularized so as to also allow for zero concentrations:

$$F(\mathbf{c}, \bar{\mathbf{c}}) := \int_{\mathbb{T}^3} \sum_{i=1}^{n} (\ln(c_i + \delta) - \ln(\bar{c}_i + \delta))(c_i - \bar{c}_i)\,\mathrm{d}x \tag{5.11}$$

where $\delta > 0$ is a constant to be selected. The strategy will be to calculate a Grönwall inequality for the functional $F(\mathbf{c}, \bar{\mathbf{c}}) > 0$.

The first step is to derive an identity for the symmetrized relative entropy between two renormalized solutions, using the method of doubling the variables. This method, due to Kruzhkov [42], was further developed in [43], and allows to treat one solution as a constant with respect to the evolution of the other, in order to derive a relative entropy identity, overcoming the lack of regularity. Compared to the use of doubling of variables in hyperbolic problems, we here employ also the regularity properties emerging from the existence theory.

The work proceeds as follows: First we establish a symmetrized relative entropy inequality between two renormalized solutions:

Theorem 13. *Let* $\mathbf{c}$ *and* $\bar{\mathbf{c}}$ *be renormalized solutions of* (5.1)–(5.2) *emanating from the same initial data. Then for any* $T > 0$ *and* $\beta \in C^2([0, \infty))$*, the following identity holds:*

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\mathbb{T}^3} \big(\beta(c_i) - \beta(\bar{c}_i)\big)(c_i - \bar{c}_i)\,\mathrm{d}x\Big|_{t=T} \\ &= \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} \nabla\left(\beta(c_i) - \beta(\bar{c}_i)\right) \cdot (c_i u_i - \bar{c}_i \bar{u}_i)\mathrm{d}x\mathrm{d}t \\ &+ \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} \Big(\nabla\big[(c_i - \bar{c}_i)\beta'(c_i)\big] \cdot c_i u_i - \nabla\big[(c_i - \bar{c}_i)\beta'(\bar{c}_i)\big] \cdot \bar{c}_i \bar{u}_i\Big)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.12}$$

Now, the choice of $\beta(s) = \ln(s + \delta)$, which is indeed in $C^2([0, \infty))$, in (5.12) implies the following result:

Corollary 14. *Let* $\mathbf{c}$ *and* $\bar{\mathbf{c}}$ *be renormalized solutions of* (5.1)–(5.2)*. Then, for any* $T > 0$ *and* $\delta > 0$*:*

$$\begin{aligned} &F(\mathbf{c}, \bar{\mathbf{c}})(T) \\ &= \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} (c_i + \bar{c}_i + 2\delta)\ \nabla\big(\ln(c_i+\delta) - \ln(\bar{c}_i+\delta)\big) \cdot \left(\frac{c_i u_i}{c_i+\delta} - \frac{\bar{c}_i \bar{u}_i}{\bar{c}_i+\delta} \right). \end{aligned} \tag{5.13}$$

We remark that up to Corollary 14 only the transport structure (5.1) was used. The question becomes how the formal identity (5.25) is affected by the introduction of the corrections in the logarithmic terms. To achieve this, a change of variables is quite helpful, namely, the functions $d_i = c_i + \delta$ and $v_i = \frac{c_i u_i}{c_i+\delta}$ satisfy an approximate Maxwell–Stefan system in the form (5.33). This motivates to express the error estimate for the new system. A careful estimation of the terms involved in (5.13) yields the uniqueness of nonnegative renormalized solutions:

Theorem 15. *Let* $\mathbf{c}$ *and* $\bar{\mathbf{c}}$ *be renormalized solutions of* (5.1)–(5.2) *emanating from the same initial data. Then, if* $c_i u_i$ *and* $\bar{c}_i \bar{u}_i$ *are in* $L^\infty(\mathbb{T}^3 \times (0,\infty))$*, there exists at most one nonnegative renormalized solution to the classical Maxwell–Stefan system* (5.1)–(5.2)*.*

Remark 16. *The regularity assumption* $c_i u_i \in L^\infty(\mathbb{T}^3 \times (0,\infty))$ *implies, due to the force-flux relation* (5.2)*, that* $\nabla c_i \in L^\infty(\mathbb{T}^3 \times (0,\infty))$*. We recall the regularity offered by the existence theory of [8] is* $\nabla\sqrt{c_i} \in L^2_{\mathrm{loc}}(\mathbb{T}^3 \times (0,\infty))$ *and slightly less than that required for uniqueness.*

We, then, study the relation between weak and renormalized solutions and show that our definition of renormalized solutions is implied by the definition of weak solutions and, as a result, Theorem 15 holds for weak solutions as well.

Theorem 17. *Let* $\mathbf{c}$ *be a weak solution of* (5.1)–(5.2) *in the sense of Definition 11 of regularity* (5.9)*. Then* $\mathbf{c}$ *is a renormalized solution of* (5.1)–(5.2)*.*

As a consequence, the uniqueness of weak solutions follows:

Corollary 18. *Let* $\mathbf{c}$ *and* $\bar{\mathbf{c}}$ *be weak solutions of* (5.1)–(5.2) *emanating from the same initial data and suppose that* $c_i u_i$ *and* $\bar{c}_i \bar{u}_i$ *are in* $L^\infty(\mathbb{T}^3 \times (0, \infty))$. *Then,* $\mathbf{c} = \bar{\mathbf{c}}$, *i.e. weak solutions are unique.*

5.1.1 Formal comparison of solutions of transport equations

The entropy (5.4) of the Maxwell-Stefan system plays an important role in the existence theory of weak solutions [8], and the relative entropy

$$H(\mathbf{c}|\bar{\mathbf{c}}) = \int_{\mathbb{T}^3} \sum_{i=1}^n \Big(c_i \ln \frac{c_i}{\bar{c}_i} - (c_i - \bar{c}_i) \Big) \mathrm{d}x \tag{5.14}$$

was used in [10] to estimate the distance between two solutions $\mathbf{c}$ and $\bar{\mathbf{c}}$. Here, we will employ instead a symmetrized version of the relative entropy

$$H^{sym}(\mathbf{c}, \bar{\mathbf{c}}) := H(\mathbf{c}|\bar{\mathbf{c}}) + H(\bar{\mathbf{c}}|\mathbf{c}) = \int_{\mathbb{T}^3} \sum_{i=1}^n (\ln c_i - \ln \bar{c}_i)(c_i - \bar{c}_i) \mathrm{d}x \tag{5.15}$$

The quantity $(\ln c_i - \ln \bar{c}_i)(c_i - \bar{c}_i)$ is positive for $0 < c_i, \bar{c}_i < 1$ when $c_i \neq \bar{c}_i$. It takes the value ∞ when one out of c_i or $\bar{c}_i$ vanishes, and it can be defined to be $+\infty$ when both vanish. Hence, it can serve to measure the distance between two solutions.

For a renormalized solution corresponding to (5.6), an entropy functional is defined by

$$H_B(\mathbf{c}) = \int_{\mathbb{T}^3} \sum_{i=1}^n \int_0^{c_i} \beta(s) \mathrm{d}s \, \mathrm{d}x \,. \tag{5.16}$$

Setting $B(s) := \int_0^s \beta(\tau) \mathrm{d}\tau$, the symmetrized relative entropy of two renormalized solutions $\mathbf{c}$ and $\bar{\mathbf{c}}$ is given by

$$H_B^{sym}(\mathbf{c}, \bar{\mathbf{c}}) = \int_{\mathbb{T}^3} \sum_{i=1}^n (\beta(c_i) - \beta(\bar{c}_i))(c_i - \bar{c}_i) \, \mathrm{d}x \,. \tag{5.17}$$

When β is a monotone map, the symmetrized relative entropy is nonnegative and offers a suitable tool to measure the distance between $\mathbf{c}$ and $\bar{\mathbf{c}}$.

5.1.2 Uniqueness for the heat equation

In this section we present a uniqueness/stability estimate for the heat equation. The heat equation is first written in the form suggested by the so-called Otto calculus [44]

$$\partial_t \rho + \operatorname{div} \rho u = 0 \qquad u = -\nabla \ln \rho. \tag{5.18}$$

We present an identity based only on the transport equation, $\partial_t \rho + \operatorname{div} \rho u = 0$, which yields a relative entropy comparison between two solutions of (5.18). Let (ρ, u) and $(\bar{\rho}, \bar{u})$ be two solutions of the transport equation and we write down the identities

$$\partial_t \rho + \operatorname{div} \rho u = 0 \tag{5.19}$$

$$\partial_t \ln \bar{\rho} + \operatorname{div} \bar{u} + \nabla \ln \bar{\rho} \cdot \bar{u} = 0. \tag{5.20}$$

We multiply (5.19) by $\ln \bar{\rho}$ and (5.20) by ρ and add to obtain

$$\partial_t(\rho \ln \bar{\rho}) + \operatorname{div}(\rho \ln \bar{\rho}\, u + \rho \bar{u}) + \nabla \ln \frac{\bar{\rho}}{\rho} \cdot \rho \bar{u} - \nabla \ln \bar{\rho} \cdot \rho u = 0. \tag{5.21}$$

In particular, we have

$$\partial_t(\rho \ln \rho) + \operatorname{div}(\rho \ln \rho\, u + \rho u) - \nabla \ln \rho \cdot \rho u = 0. \tag{5.22}$$

Next, we add (5.22) with the corresponding equation for $(\bar{\rho}, \bar{u})$, and then from the result we subtract (5.21) and the analog of (5.21) with (ρ, u) and $(\bar{\rho}, \bar{u})$ interchanged. After re-arranging the terms we obtain

$$\begin{aligned} \partial_t\big((\ln \rho - \ln \bar{\rho})(\rho - \bar{\rho})\big) + \operatorname{div}\big((\ln \rho - \ln \bar{\rho})(\rho u - \bar{\rho}\bar{u}) + (\rho - \bar{\rho})(u - \bar{u})\big) \\ - (\rho + \bar{\rho})\nabla(\ln \rho - \ln \bar{\rho}) \cdot (u - \bar{u}) = 0. \end{aligned} \tag{5.23}$$

We emphasize that up to here we only used the transport equation (5.19).

Now, consider the heat equation (5.18) and introduce the formula $u = -\nabla \ln \rho$ to (5.23). After an integration, we obtain the identity

$$\frac{d}{dt}\int_\Omega \big((\ln\rho - \ln\bar\rho)(\rho-\bar\rho)\big)\mathrm{d}x + \int_\Omega (\rho+\bar\rho)|\nabla(\ln\rho - \ln\bar\rho)|^2\mathrm{d}x = 0 \tag{5.24}$$

which provides an alternative approach to show uniqueness and stability for the heat equation, based on its entropy structure.

5.1.3 The symmetrized relative entropy identity for the Maxwell–Stefan system

Consider next the Maxwell–Stefan system and return to the notation c_i for the densities and $J_i = c_i u_i$ for the fluxes. Using the transport equations (5.1) for the individual components and following the derivation of section 5.1.2 we obtain the identity

$$\begin{aligned}
&\partial_t\Big(\sum_i (\ln c_i - \ln\bar c_i)(c_i - \bar c_i)\Big)\\
&\quad + \operatorname{div}\Big(\sum_i (\ln c_i - \ln\bar c_i)(c_i u_i - \bar c_i \bar u_i) + (c_i - \bar c_i)(u_i - \bar u_i)\Big)\\
&\qquad - \sum_i \Big(\nabla(\ln c_i - \ln\bar c_i)\cdot(c_i+\bar c_i)(u_i - \bar u_i)\Big) = 0.
\end{aligned}\tag{5.25}$$

Next, we use the algebraic system (5.2),

$$-\sum_{j=1}^n \frac{1}{D_{ij}} c_j(u_i - u_j) = \nabla \ln c_i$$

and the symmetry of D_{ij} to express

$$\begin{aligned}
I = &-\sum_i (c_i+\bar c_i)(u_i-\bar u_i)\cdot\nabla(\ln c_i - \ln\bar c_i)\cdot(c_i+\bar c_i)(u_i-\bar u_i)\\
= &\quad \sum_{i,j}\tfrac12 c_i c_j \frac{1}{D_{ij}}\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2 + \sum_{i,j}(c_j-\bar c_j)(u_i-\bar u_i)\cdot\frac{c_i}{D_{ij}}(\bar u_i - \bar u_j)\\
&+\sum_{i,j}\tfrac12 \bar c_i\bar c_j\frac{1}{D_{ij}}\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2 + \sum_{i,j}(c_j-\bar c_j)(u_i-\bar u_i)\cdot\frac{\bar c_i}{D_{ij}}(u_i-u_j).
\end{aligned}$$

Combining with (5.25) we arrive at the (formal) identity

$$
\begin{aligned}
&\partial_t\Big(\sum_i(\ln c_i-\ln\bar c_i)(c_i-\bar c_i)\Big)\\
&\qquad+\operatorname{div}\Big(\sum_i(\ln c_i-\ln\bar c_i)(c_iu_i-\bar c_i\bar u_i)+(c_i-\bar c_i)(u_i-\bar u_i)\Big)\\
&\qquad+\sum_{i,j}\frac{1}{2D_{ij}}(c_ic_j+\bar c_i\bar c_j)\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2\\
&\quad=-\sum_{i,j}(c_j-\bar c_j)(u_i-\bar u_i)\cdot\frac{1}{D_{ij}}\Big(c_i(\bar u_i-\bar u_j)+\bar c_i(u_i-u_j)\Big).
\end{aligned}
\tag{5.26}
$$

The identity (5.26) will guide the analysis of uniqueness in the following sections. It bears analogies to the identity used for weak–strong uniqueness in [10], but here the evolution is computed for the symmetrized relative entropy and is complemented with spectral properties of the matrix in (5.2) that will quantify the effect of friction. It should be compared to Corollary 14 for $\delta=0$.

5.1.4 Derivation of the relative entropy identity

In this section we give a proof of Theorem 13. Let $c_i,\bar c_i$ be two renormalized solutions and we proceed to establish Theorem 13. By the weak formulation for the difference $A(x,t):=c_i(x,t)-\bar c_i(x,t)$, corresponding to $\beta=\mathrm{id}$, we have

$$
\begin{aligned}
&\int_{\mathbb{T}^3}A(x,0)\phi(x,0)\mathrm{d}x\\
&\qquad+\int_0^\infty\int_{\mathbb{T}^3}[A(x,t)\partial_t\phi(x,t)+C(x,t)\cdot\nabla_x\phi(x,t)]\mathrm{d}x\mathrm{d}t=0,
\end{aligned}
\tag{5.27}
$$

where $C(x,t):=c_iu_i(x,t)-\bar c_i\bar u_i(x,t)$ and $\phi(x,t)\in C^1_c(Q_\infty)$ a test function compactly supported in $Q_\infty=\mathbb{T}^3\times[0,\infty)$. Likewise, write the weak formulation for the difference $B(y,\tau):=\beta(c_i(y,\tau))-\beta(\bar c_i(y,\tau))$, for a generic β, namely

$$
\begin{aligned}
&\int_{\mathbb{T}^3}B(y,0)\phi(y,0)\mathrm{d}y\\
&\qquad+\int_0^\infty\int_{\mathbb{T}^3}B(y,\tau)\partial_\tau\phi(y,\tau)+D(y,\tau)\cdot\nabla_y\phi(y,\tau)]\mathrm{d}y\mathrm{d}\tau\\
&\qquad+\int_0^\infty\int_{\mathbb{T}^3}E(y,\tau)\phi(y,\tau)\mathrm{d}y\mathrm{d}\tau=0,
\end{aligned}
\tag{5.28}
$$

where $\phi(y,\tau) \in C_c^1(Q_\infty)$,

$$D(y,\tau) = \beta'(c_i(y,\tau))c_i u_i(y,\tau) - \beta'(\bar{c}_i(y,\tau))\bar{c}_i\bar{u}_i(y,\tau)$$

$$E(y,\tau) = 2\sqrt{c_i}\beta''(c_i)\nabla\sqrt{c_i}\cdot c_i u_i - 2\sqrt{\bar{c}_i}\beta''(\bar{c}_i)\nabla\sqrt{\bar{c}_i}\cdot \bar{c}_i\bar{u}_i.$$

Let $\varphi(z,s) \in C_c^1(Q_\infty)$, that is ϕ is periodic in space and compactly supported on $\mathbb{T}^3 \times [0,\infty)$. Let $\rho(\sigma) \geq 0$ be a symmetric mollifier, supported in the ball centered at 0 with radius 1, with $\int \rho(\sigma)d\sigma = 1$. Define

$$\begin{aligned}\Phi(x,t;y,\tau) &= \varphi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\,\frac{1}{\epsilon}\rho\left(\frac{t-\tau}{\epsilon}\right)\prod_{\alpha=1}^{3}\frac{1}{\epsilon}\rho\left(\frac{x_\alpha - y_\alpha}{\epsilon}\right) \\ &=: \varphi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\,\phi^\epsilon(x,y,t,\tau)\end{aligned} \tag{5.29}$$

and note that $\Phi \in C_c^1(Q_\infty \times Q_\infty)$ and if z and s denote the first and second argument of φ respectively, we have

$$(\nabla_x + \nabla_y)\Phi = (\nabla_z\varphi)\Big(\frac{x+y}{2}, \frac{t+\tau}{2}\Big)\,\phi^\epsilon \tag{5.30}$$

$$(\partial_t + \partial_\tau)\Phi = (\partial_s\varphi)\Big(\frac{x+y}{2}, \frac{t+\tau}{2}\Big)\,\phi^\epsilon. \tag{5.31}$$

Φ is jointly periodic in space: if $p \in \mathbb{R}^3$ is a period then $\Phi(x+p,t;y+p,\tau) = \Phi(x,t;y,\tau)$.

Now, in (5.27) we employ the test function

$$\phi_B(x,t) = \int_0^\infty\int_{\mathbb{T}^3} B(y,\tau)\Phi(x,t;y,\tau)\mathrm{d}y\mathrm{d}\tau$$

which is in $C_c^1(Q_\infty)$, since

$$\begin{aligned}
\phi_B(x+p,t) &= \int_0^\infty \int_{\mathbb{T}^3} B(y,\tau)\Phi(x+p,t;y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&= \int_0^\infty \int_{\mathbb{T}^3-p} B(y+p,\tau)\Phi(x+p,t;y+p,\tau)\mathrm{d}y\mathrm{d}\tau \\
&= \int_0^\infty \int_{\mathbb{T}^3} B(y,\tau)\Phi(x,t;y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&= \phi_B(x,t).
\end{aligned}$$

The resulting identity is integrated over $\mathbb{T}^3 \times [0,\infty)$ and gives, using (5.30),

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^3} \int_0^\infty \int_{\mathbb{T}^3} A(x,t)B(y,\tau)(\partial_t\Phi)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad + \int_0^\infty \int_{\mathbb{T}^3} \int_0^\infty \int_{\mathbb{T}^3} C(x,t)B(y,\tau)\cdot\Big(\nabla_z\varphi\big(\tfrac{x+y}{2},\tfrac{t+\tau}{2}\big)\phi^\epsilon - \nabla_y\Phi\Big)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad = -\int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} A(x,0)B(y,\tau)\Phi(x,0;y,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\,.
\end{aligned}$$

Since $B(y,\tau) = \beta(c_i(y,\tau)) - \beta(\bar{c}_i(y,\tau)) \in H^1(\mathbb{T}^3)$ for τ fixed, we can integrate by parts in y and use the periodic boundary conditions to get

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} A(x,0)B(y,\tau)\varphi\big(\tfrac{x+y}{2},\tfrac{\tau}{2}\big)\phi^\epsilon(x,y,0,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\
&= -\int_0^\infty \int_{\mathbb{T}^3} \int_0^\infty \int_{\mathbb{T}^3} A(x,t)B(y,\tau)\partial_t\big(\Phi(x,t;y,\tau)\big)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&- \int_0^\infty \int_{\mathbb{T}^3} \int_0^\infty \int_{\mathbb{T}^3} \Big[C(x,t)B(y,\tau)\cdot(\nabla_z\varphi)\big(\tfrac{x+y}{2},\tfrac{t+\tau}{2}\big)\phi^\epsilon \\
&\qquad + C(x,t)\cdot\nabla_y B(y,\tau)\Phi\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

Similarly, we choose the test function

$$\phi_A(y,\tau) = \int_0^\infty \int_{\mathbb{T}^3} A(x,t)\Phi(x,t;y,\tau)\mathrm{d}x\mathrm{d}t$$

in (5.28), which is $C_c^1(Q_\infty)$ and follow similar steps to arrive at

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} A(x,t)B(y,0)\varphi\big(\tfrac{x+y}{2},\tfrac{\tau}{2}\big)\phi^\epsilon(x,y,t,0)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} A(x,t)B(y,\tau)\partial_\tau\big(\Phi(x,t;y,\tau)\big) + A(x,t)E(y,\tau)\Phi \,\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad -\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} A(x,t)D(y,\tau)\cdot(\nabla_z\varphi)\big(\tfrac{x+y}{2},\tfrac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad +\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} D(y,\tau)\cdot\nabla_x A(x,t)\Phi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

Adding the two equations and using (5.31), we get

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} A(x,0)B(y,\tau)\varphi\big(\tfrac{x+y}{2},\tfrac{\tau}{2}\big)\phi^\epsilon(x,y,0,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\
&+\int_0^\infty \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} A(x,t)B(y,0)\varphi\big(\tfrac{x+y}{2},\tfrac{\tau}{2}\big)\phi^\epsilon(x,y,t,0)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} A(x,t)B(y,\tau)\varphi_s\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&-\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} C(x,t)B(y,\tau)\cdot\nabla_z\varphi\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&-\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} A(x,t)D(y,\tau)\cdot\nabla_z\varphi\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&-\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} C(x,t)\cdot\nabla_y B(y,\tau)\varphi\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&-\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} D(y,\tau)\cdot\nabla_x A(x,t)\varphi\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&-\int_0^\infty \int_{\mathbb{T}^3}\int_0^\infty \int_{\mathbb{T}^3} A(x,t)E(y,\tau)\varphi\big(\frac{x+y}{2},\frac{t+\tau}{2}\big)\phi^\epsilon\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

We, turn to passing to the limit $\epsilon \to 0$. This is effected by standard approximation results that are listed for the reader's convenience. Lemma 20 indicates the subtle role played by the initial trace.

Lemma 19. *Let $f(x,t) \in L^1(\mathbb{R}^d \times (0,\infty))$. Then*

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d}\int_0^\infty \int_{\mathbb{R}^d} f(y,\tau)\varphi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\frac{1}{\epsilon}\rho\left(\frac{t-\tau}{\epsilon}\right)\frac{1}{\epsilon^d}\rho\left(\frac{x-y}{\epsilon}\right)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\mathrm{d}t \\
&\qquad \to \int_0^\infty \int_{\mathbb{R}^d} f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \forall \varphi \in C_c(\mathbb{R}^d \times [0,\infty)),
\end{aligned}$$

where we denote

$$\rho\left(\frac{x-y}{\epsilon}\right)=\prod_{\alpha=1}^{d}\rho\left(\frac{x_\alpha-y_\alpha}{\epsilon}\right).$$

Proof. Extend f by setting

$$\bar f(x,t)=\begin{cases} f(x,t) & t>0\\ 0 & \text{else}\end{cases}$$

and set $X=(x,t)$ and $Y=(y,\tau)$. Then, it is enough to show that for all $\varphi\in C_c(\mathbb{R}^d\times[0,\infty))$

$$\int\int \bar f(Y)\varphi\left(\frac{X+Y}{2}\right)\frac{1}{\epsilon^{d+1}}\rho\left(\frac{X-Y}{\epsilon}\right)\mathrm{d}Y\mathrm{d}X\to\int\bar f(X)\varphi(X)\mathrm{d}X.$$

Indeed, using the change of variables $Y=X+\epsilon Z$,

$$\begin{aligned}
I&=\left|\int\int \bar f(Y)\varphi\left(\frac{X+Y}{2}\right)\frac{1}{\epsilon^{d+1}}\rho\left(\frac{X-Y}{\epsilon}\right)\mathrm{d}Y\mathrm{d}X-\int\bar f(X)\varphi(X)\mathrm{d}X\right|\\
&=\left|\int\int\left(\bar f(X+\epsilon Z)\varphi\left(X+\frac{1}{2}\epsilon Y\right)-\bar f(X)\varphi(X)\right)\rho\left(Z\right)\mathrm{d}Z\mathrm{d}X\right|\\
&\le\int\int|\bar f(X+\epsilon Z)-\bar f(X)|\left|\varphi\left(X+\frac{1}{2}\epsilon Z\right)\right|\rho(Z)\mathrm{d}Z\mathrm{d}X\\
&\qquad+\int\int\left|\varphi\left(X+\frac{1}{2}\epsilon Z\right)-\varphi(X)\right||\bar f(X)|\rho(Z)\mathrm{d}Z\mathrm{d}X,
\end{aligned}$$

where both terms tend to zero by the dominated convergence theorem since $f\in L^1$. □

Lemma 20. *Let $f(\cdot,t)\in L^1(\mathbb{R}^d)$ for all $t>0$ and*

$$\lim_{t\to0}\int_{\mathbb{R}^d}|f(x,t)-f(x,0)|\mathrm{d}x=0.$$

Then

$$\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d} f(y,\tau)\varphi\left(\frac{x+y}{2},\frac{\tau}{2}\right)\frac{1}{\epsilon}\rho\left(-\frac{\tau}{\epsilon}\right)\frac{1}{\epsilon^d}\rho\left(\frac{x-y}{\epsilon}\right)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\ \to \frac{1}{2}\int_{\mathbb{R}^d} f(x,0)\varphi(x,0)\mathrm{d}x \quad \forall\varphi\in C_c(\mathbb{R}^d\times[0,\infty)).$$

Proof. We compare the difference between K^ϵ and K

$$K^\epsilon := \int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d} f(y,\tau)\varphi\left(\frac{x+y}{2},\frac{\tau}{2}\right)\frac{1}{\epsilon}\rho\left(-\frac{\tau}{\epsilon}\right)\frac{1}{\epsilon^d}\rho\left(\frac{x-y}{\epsilon}\right)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x$$
$$K := \frac{1}{2}\int_{\mathbb{R}^d} f(x,0)\varphi(x,0)\mathrm{d}x$$

By the change of variables $\frac{y-x}{\epsilon}=z$ and $\frac{\tau}{\epsilon}=\sigma$, the symmetry of ρ, and

$$\int_0^1 \rho(\sigma)d\sigma = 1/2$$

we have

$$\begin{aligned}
&|K^\epsilon - K| \\
&\le \int_{\mathbb{R}^d}\int_0^1\int_{|z|\le 1}\left|f(x+\epsilon z,\epsilon\sigma)\varphi\left(x+\frac{1}{2}\epsilon z,\frac{1}{2}\epsilon\sigma\right)-f(x,0)\varphi(x,0)\right|\rho(\sigma)\rho(z)\mathrm{d}z\mathrm{d}\sigma\mathrm{d}x \\
&\le \int_{\mathbb{R}^d}\int_0^1\int_{|z|\le 1}|f(x+\epsilon z,\epsilon\sigma)|\left|\varphi\left(x+\frac{1}{2}\epsilon z,\frac{1}{2}\epsilon\sigma\right)-\varphi(x,0)\right|\rho(\sigma)\rho(z)\mathrm{d}z\mathrm{d}\sigma\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d}\int_0^1\int_{|z|\le 1}|f(x+\epsilon z,\epsilon\sigma)-f(x,0)|\,|\varphi(x,0)|\,\rho(\sigma)\rho(z)\mathrm{d}z\mathrm{d}\sigma\mathrm{d}x \\
&=: J_1 + J_2,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^1\rho(\sigma)\int_{|z|\le 1}\rho(z)\int_{\mathbb{R}^d}|f(x+\epsilon z,\epsilon\sigma)|\left|\varphi\left(x+\frac{1}{2}\epsilon z,\frac{1}{2}\epsilon\sigma\right)-\varphi(x,0)\right|\mathrm{d}x\mathrm{d}z\mathrm{d}\sigma \\
&\le \mathcal{O}(\epsilon)\sup_{(z,s)}|\nabla_{(z,s)}\varphi(z,s)|\sup_{|\tau|\le\epsilon}\int_{\mathbb{R}^d}|f(y,\tau)|\,\mathrm{d}y
\end{aligned}$$

which goes to zero as $\epsilon\to 0$, provided $\limsup_{\tau\to 0}\|f(\cdot,\tau)\|_{L^1(\mathbb{R}^d)}\le C$, where C is a constant independent of ϵ.

For the second term, we have

$$
\begin{aligned}
J_2 &= \int_0^1 \rho(\sigma) \int_{|z|\le 1} \rho(z) \int_{\mathbb{R}^d} |f(x+\epsilon z, \epsilon\sigma) - f(x,0)|\, |\varphi(x,0)|\, \mathrm{d}x\mathrm{d}z\mathrm{d}\sigma \\
&\le \int_0^1 \rho(\sigma) \int_{|z|\le 1} \rho(z) \int_{\mathbb{R}^d} |f(x+\epsilon z, \epsilon\sigma) - f(x+\epsilon z,0)|\, |\varphi(x,0)|\, \mathrm{d}x\mathrm{d}z\mathrm{d}\sigma \\
&\quad + \int_0^1 \rho(\sigma) \int_{|z|\le 1} \rho(z) \int_{\mathbb{R}^d} |f(x+\epsilon z, 0) - f(x,0)|\, |\varphi(x,0)|\, \mathrm{d}x\mathrm{d}z\mathrm{d}\sigma \\
&\le \sup_x |\varphi(x,0)| \sup_{0\le\sigma\le 1} \int_{\mathbb{R}^d} |f(y,\epsilon\sigma) - f(y,0)|\, \mathrm{d}y\mathrm{d}\sigma \\
&\quad + \frac{1}{2} \sup_x |\varphi(x,0)| \|f(\cdot+\epsilon z, 0) - f(\cdot,0)\|_{L^1(\mathbb{R}^d)}
\end{aligned}
$$

Since $\lim_{\tau\to 0} \int_{\mathbb{R}^d} |f(y,\tau) - f(y,0)| \mathrm{d}y = 0$ and $\lim_{z\to 0} \|f(\cdot+z,0) - f(\cdot,0)\|_{L^1(\mathbb{R}^d)} = 0$ the last term goes to zero as $\epsilon \to 0$. □

Using the lemmas we take $\epsilon \to 0$ to obtain

$$
\begin{aligned}
&-\int_{\mathbb{T}^3} A(x,0)B(x,0)\varphi(x,0)\mathrm{d}x = \int_0^\infty \int_{\mathbb{T}^3} A(x,t)B(x,t)\varphi_t(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\infty \int_{\mathbb{T}^3} [C(x,t)B(x,t) + A(x,t)D(x,t)] \cdot \nabla_x \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\infty \int_{\mathbb{T}^3} [C(x,t)\cdot\nabla_x B(x,t) + D(x,t)\cdot\nabla_x A(x,t)]\, \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\infty \int_{\mathbb{T}^3} A(x,t)E(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Now, we choose

$$
\varphi(x,t) = \begin{cases} 1 & t \in [0,T] \\ \frac{T-t}{\sigma} + 1 & T < t \le T+\sigma \\ 0 & t > T+\sigma \end{cases} \tag{5.32}
$$

for some $\sigma > 0$ and let $\sigma \to 0$:

$$
\begin{aligned}
&\int_{\mathbb{T}^3} A(x,T)B(x,T)\mathrm{d}x - \int_{\mathbb{T}^3} A(x,0)B(x,0)\mathrm{d}x \\
&= \int_0^T \int_{\mathbb{T}^3} [C(x,t)\cdot\nabla_x B(x,t) + D(x,t)\cdot\nabla_x A(x,t) + A(x,t)E(x,t)]\, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Finally, substituting back A, B, C, D and E, summing the resulting equation over $i \in \{1, \ldots, n\}$, and using the same initial conditions for the two solutions, we arrive at (5.12).

We, now, proceed with the proof of Corollary 14. Let $\beta(s) = \ln(s + \delta)$, for some fixed $\delta \in (0, 1)$. Note that $\beta \in C^2([0, \infty))$, since for all $s \in [0, 1]$:

$$|\beta(s)| = |\ln(s + \delta)| \leq \max\{|\ln \delta|, \ln(1 + \delta)\}$$

$$|\beta'(s)| = \left|\frac{1}{s + \delta}\right| \leq \frac{1}{\delta},$$

$$|\sqrt{s}\beta''(s)| = \left|\frac{\sqrt{s}}{(s + \delta)^2}\right| \leq \frac{1}{\delta^2}.$$

Hence, we can choose it in (5.12) and carry out the following calculation:

$$\begin{aligned}
& \nabla\big[(c_i - \bar{c}_i)\beta'(c_i)\big] \cdot c_i u_i - \nabla\big[(c_i - \bar{c}_i)\beta'(\bar{c}_i)\big] \cdot \bar{c}_i \bar{u}_i \\
= \;& \nabla\left(1 - \frac{\bar{c}_i + \delta}{c_i + \delta}\right) \cdot c_i u_i - \nabla\left(\frac{c_i + \delta}{\bar{c}_i + \delta} - 1\right) \cdot \bar{c}_i \bar{u}_i \\
= \;& -\nabla\left(\frac{\bar{c}_i + \delta}{c_i + \delta}\right) \cdot c_i u_i - \nabla\left(\frac{c_i + \delta}{\bar{c}_i + \delta}\right) \cdot \bar{c}_i \bar{u}_i \\
= \;& -\nabla \ln\left(\frac{\bar{c}_i + \delta}{c_i + \delta}\right) \cdot c_i u_i \frac{\bar{c}_i + \delta}{c_i + \delta} - \nabla \ln\left(\frac{c_i + \delta}{\bar{c}_i + \delta}\right) \cdot \bar{c}_i \bar{u}_i \frac{c_i + \delta}{\bar{c}_i + \delta} \\
= \;& \nabla\big(\ln(c_i + \delta) - \ln(\bar{c}_i + \delta)\big) \cdot \left(c_i u_i \frac{\bar{c}_i + \delta}{c_i + \delta} - \bar{c}_i \bar{u}_i \frac{c_i + \delta}{\bar{c}_i + \delta}\right)
\end{aligned}$$

Then,

$$\begin{aligned}
& \nabla\big(\ln(c_i + \delta) - \ln(\bar{c}_i + \delta)\big) \cdot \left(c_i u_i \frac{\bar{c}_i + \delta}{c_i + \delta} - \bar{c}_i \bar{u}_i \frac{c_i + \delta}{\bar{c}_i + \delta}\right) \\
& + \nabla\big(\ln(c_i + \delta) - \ln(\bar{c}_i + \delta)\big) \cdot (c_i u_i - \bar{c}_i \bar{u}_i) \\
= \;& \nabla\big(\ln(c_i + \delta) - \ln(\bar{c}_i + \delta)\big) \cdot \left(c_i u_i \left(1 + \frac{\bar{c}_i + \delta}{c_i + \delta}\right) - \bar{c}_i \bar{u}_i \left(1 + \frac{c_i + \delta}{\bar{c}_i + \delta}\right)\right) \\
= \;& (c_i + \delta + \bar{c}_i + \delta)\; \nabla\big(\ln(c_i + \delta) - \ln(\bar{c}_i + \delta)\big) \cdot \left(\frac{c_i u_i}{c_i + \delta} - \frac{\bar{c}_i \bar{u}_i}{\bar{c}_i + \delta}\right)
\end{aligned}$$

Since the above calculation is purely algebraic, one would expect that it holds rigorously provided all the terms are well-defined. Indeed,

$$c_i \in L^\infty(\mathbb{T}^3 \times (0,T)),$$

$$\nabla \ln(c_i + \delta) = \frac{2\sqrt{c_i}}{c_i + \delta} \nabla \sqrt{c_i} \in L^2(\mathbb{T}^3 \times (0,T)),$$

$$\frac{c_i u_i}{c_i + \delta} = \frac{\sqrt{c_i}}{c_i + \delta} \sqrt{c_i} u_i \in L^2(\mathbb{T}^3 \times (0,T))$$

and thus (5.13) is justified.

We define, now, a new set of variables $d_i(x,t)$ and $v_i(x,t)$ as follows:

$$\begin{aligned} d_i(x,t) &= c_i(x,t) + \delta \\ v_i(x,t) &= \frac{c_i u_i}{c_i + \delta} \end{aligned}$$

for some $\delta \in (0,1)$. Note that the flux is preserved in the sense that $c_i u_i = d_i v_i$, i.e. the new flux also satisfies the constraint $\sum_{i=1}^n d_i v_i = 0$ and if $c_i u_i \in L^\infty(\mathbb{T}^3 \times (0,T))$, then $d_i v_i \in L^\infty(\mathbb{T}^3 \times (0,T))$ as well. Therefore, (5.1)–(5.2) reads:

$$\partial_t d_i + \nabla \cdot (d_i v_i) = 0, \tag{5.33}$$

$$\nabla d_i = -\sum_{j=1}^n \frac{d_i d_j}{D_{ij}} (v_i - v_j) + \delta \sum_{j=1}^n \frac{1}{D_{ij}} (d_i v_i - d_j v_j) \tag{5.34}$$

for all $i = 1, \dots, n$ and the initial conditions become:

$$d_i^0 = c_i^0 + \delta \text{ on } \mathbb{T}^3, \ i = 1, \dots, n. \tag{5.35}$$

The new variables enjoy the properties:

$$\delta \le d_i \le 1 + \delta \le 2, \tag{5.36}$$

$$\sum_{i=1}^n d_i = 1 + n\delta \tag{5.37}$$

and

$$\|v_i\|_{L^\infty(\mathbb{T}^3\times(0,\infty))} = \left\| \frac{c_i u_i}{c_i+\delta} \right\|_{L^\infty} \le \frac{1}{\delta}\|cu\|_{L^\infty}, \tag{5.38}$$

where $\|cu\|_{L^\infty} = \max_{1\le i,j\le n}\{\|c_i u_i\|_{L^\infty}, \|\bar{c}_j \bar{u}_j\|_{L^\infty}\}$.

Now, we introduce the two matrices

$$A_{ij}(\mathbf{d}) = \begin{cases} \sum_{k\neq i} \frac{d_k}{D_{ik}}, & i=j, \\ -\frac{\sqrt{d_i d_j}}{D_{ij}}, & i\neq j, \end{cases} \tag{5.39}$$

$$B_{ij}(\mathbf{d}) = \begin{cases} -\sum_{k\neq i} \frac{1}{D_{ik}}, & i=j, \\ \frac{\sqrt{d_j}}{D_{ij}\sqrt{d_i}}, & i\neq j. \end{cases} \tag{5.40}$$

Note that $A_{ij}(d)$ is symmetric and its spectrum plays a significant role later, while $B_{ij}(d)$ is not symmetric. The linear system in (5.33) can be expressed as

$$2\nabla\sqrt{d_i} = -\sum_{j=1}^{n}\Big(A_{ij}(\mathbf{d}) + \delta B_{ij}(\mathbf{d})\Big)\sqrt{d_j}v_j \tag{5.41}$$

for all $i = 1,\ldots,n$. It is a perturbation of (5.2) and when $\delta = 0$ we recover (5.2).

Let $\boldsymbol{z}\in\mathbb{R}^n$. The computation

$$(A(\mathbf{d})z)_i = \sum_{j=1}^{n} A_{ij}(\mathbf{d})z_j = \sum_{j\neq i}\frac{d_j}{D_{ij}}z_i - \sum_{j\neq i}\frac{\sqrt{d_i d_j}}{D_{ij}}z_j = \sum_{j\neq i}\frac{\sqrt{d_j}}{D_{ij}}(\sqrt{d_j}z_i - \sqrt{d_i}z_j)$$

shows that

$$\begin{aligned} \operatorname{ran} A(\mathbf{d}) &= \left\{x\in\mathbb{R}^n : \sqrt{\mathbf{d}}\cdot x = 0\right\} =: L(\mathbf{d}) \\ \ker A(\mathbf{d}) &= \operatorname{span}\left\{\sqrt{\mathbf{d}}\right\} =: L(\mathbf{d})^\perp \end{aligned}$$

where by $\sqrt{\mathbf{d}}$ we denote the vector $(\sqrt{d_1},\ldots,\sqrt{d_n})$ and the projection matrices onto L and $L^\perp$ are given by

$$(\mathbb{P}_{L(\mathbf{d})})_{ij} = \delta_{ij} - \frac{\sqrt{d_i d_j}}{1+n\delta} \quad\text{and}\quad (\mathbb{P}_{L(\mathbf{d})^\perp})_{ij} = \frac{\sqrt{d_i d_j}}{1+n\delta} \tag{5.42}$$

respectively.

Due to (5.37),

$$\sum_{i=1}^{n} \frac{d_i}{1+n\delta} = 1$$

and using the scaling property

$$A(\mathbf{d}) = (1+n\delta) A\left(\frac{\mathbf{d}}{1+n\delta}\right)$$

we can apply the Perron-Frobenius Theorem as in [10], in order to obtain the estimate:

$$\boldsymbol{z}^\top A(\mathbf{d})\boldsymbol{z} = (1+n\delta)\ \boldsymbol{z}^\top A\left(\frac{\mathbf{d}}{1+n\delta}\right)\boldsymbol{z} \geq (1+n\delta)\mu|\mathbb{P}_{L(\mathbf{d})}\boldsymbol{z}|^2 \tag{5.43}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$, where $\mu = \min\limits_{i\neq j} \frac{1}{D_{ij}}$.

Using the variables d_i and v_i, the relative entropy identity of Corollary 14 reads:

$$F(\mathbf{d}, \bar{\mathbf{d}})\Big|_{t=T} = \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} (d_i + \bar{d}_i)\ \nabla(\ln d_i - \ln \bar{d}_i) \cdot (v_i - \bar{v}_i) \mathrm{d}x\mathrm{d}t. \tag{5.44}$$

Using the linear system in (5.33), we can rewrite the integrand on the right–hand side of (5.44) as follows:

$$\begin{aligned}
&\sum_{i=1}^{n}(d_i+\bar{d}_i)(v_i-\bar{v}_i)\cdot\nabla(\ln d_i - \ln \bar{d}_i) \\
&= -\sum_{i=1}^{n}(d_i+\bar{d}_i)(v_i-\bar{v}_i)\cdot\sum_{j\neq i}\frac{1}{D_{ij}}\left(d_j(v_i-v_j) - \bar{d}_j(\bar{v}_i-\bar{v}_j)\right) \\
&+\ \delta\sum_{i=1}^{n}(d_i+\bar{d}_i)(v_i-\bar{v}_i)\cdot\sum_{j\neq i}\frac{1}{D_{ij}}\left(v_i - \frac{d_j}{d_i}v_j - \bar{v}_i + \frac{\bar{d}_j}{\bar{d}_i}\bar{v}_j\right) \\
&:= I_1 + I_2.
\end{aligned}$$

We start with I_1:

$$I_1 = -\sum_{i=1}^n d_i(v_i - \bar{v}_i) \cdot \sum_{j\neq i} \frac{1}{D_{ij}} \left(d_j(v_i - v_j) - \bar{d}_j(\bar{v}_i - \bar{v}_j)\right)$$
$$- \sum_{i=1}^n \bar{d}_i(v_i - \bar{v}_i) \cdot \sum_{j\neq i} \frac{1}{D_{ij}} \left(d_j(v_i - v_j) - \bar{d}_j(\bar{v}_i - \bar{v}_j)\right)$$

and using the symmetry of D_{ij}:

$$\begin{aligned} I_1 \quad = \quad & -\sum_{i=1}^n \sum_{j\neq i} \frac{d_i d_j}{2D_{ij}} |(v_i - \bar{v}_i) - (v_j - \bar{v}_j)|^2 - \sum_{i=1}^n \sum_{j\neq i} \frac{d_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i) \cdot (\bar{v}_i - \bar{v}_j) \\ & -\sum_{i=1}^n \sum_{j\neq i} \frac{\bar{d}_i \bar{d}_j}{2D_{ij}} |(v_i - \bar{v}_i) - (v_j - \bar{v}_j)|^2 - \sum_{i=1}^n \sum_{j\neq i} \frac{\bar{d}_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i) \cdot (v_i - v_j). \end{aligned}$$

Regarding I_2, we can split it into two parts:

$$I_2 = \delta \sum_{i=1}^n \Big(\sum_{j\neq i} \frac{1}{D_{ij}} \Big) (d_i + \bar{d}_i) |v_i - \bar{v}_i|^2$$
$$- \delta \sum_{i=1}^n \sum_{j\neq i} \frac{1}{D_{ij}} (d_i + \bar{d}_i)(v_i - \bar{v}_i) \cdot \left(\frac{d_j}{d_i} v_j - \frac{\bar{d}_j}{\bar{d}_i} \bar{v}_j \right).$$

This shows that

$$\begin{aligned} & F(\mathbf{d}, \bar{\mathbf{d}})\Big|_{t=T} + \sum_{i=1}^n \sum_{j\neq i} \int_0^T \int_{\mathbb{T}^3} \frac{1}{2D_{ij}} (d_i d_j + \bar{d}_i \bar{d}_j) |(v_i - \bar{v}_i) - (v_j - \bar{v}_j)|^2 \\ & = -\sum_{i=1}^n \sum_{j\neq i} \int_0^T \int_{\mathbb{T}^3} \frac{d_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i) \cdot (\bar{v}_i - \bar{v}_j) \, \mathrm{d}x \mathrm{d}t \\ & \quad - \sum_{i=1}^n \sum_{j\neq i} \int_0^T \int_{\mathbb{T}^3} \frac{\bar{d}_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i) \cdot (v_i - v_j) \, \mathrm{d}x \mathrm{d}t \\ & \quad + \delta \sum_{i=1}^n \Big(\sum_{j\neq i} \frac{1}{D_{ij}} \Big) \int_0^T \int_{\mathbb{T}^3} (d_i + \bar{d}_i) |v_i - \bar{v}_i|^2 \, \mathrm{d}x \mathrm{d}t \\ & \quad - \delta \sum_{i=1}^n \sum_{j\neq i} \int_0^T \int_{\mathbb{T}^3} \frac{1}{D_{ij}} (d_i + \bar{d}_i)(v_i - \bar{v}_i) \cdot \left(\frac{d_j}{d_i} v_j - \frac{\bar{d}_j}{\bar{d}_i} \bar{v}_j \right) \, \mathrm{d}x \mathrm{d}t \end{aligned} \tag{5.45}$$

and we aim to control the terms on the right–hand side by the relative entropy and the dissipation on the left–hand side.

5.1.5 Estimation of the dissipation

Following the ideas of [10], we use the symmetry of D_{ij} to rewrite the dissipation terms as follows:

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\neq i}\frac{d_i d_j}{2D_{ij}}|(v_i-\bar{v}_i)-(v_j-\bar{v}_j)|^2 \\
&\quad=\sum_{i=1}^{n}\sum_{j\neq i}\frac{d_i d_j}{D_{ij}}((v_i-\bar{v}_i)-(v_j-\bar{v}_j))\cdot(v_i-\bar{v}_i) \\
&\quad=\sum_{i=1}^{n}\left[\sum_{j\neq i}\frac{d_i d_j}{D_{ij}}|v_i-\bar{v}_i|^2-\sum_{j\neq i}\frac{d_i d_j}{D_{ij}}(v_j-\bar{v}_j)\cdot(v_i-\bar{v}_i)\right] \\
&\quad=\sum_{i=1}^{n}\Big[\sum_{j\neq i}\frac{d_j}{D_{ij}}\sqrt{d_i}(v_i-\bar{v}_i)\cdot\sqrt{d_i}(v_i-\bar{v}_i) \\
&\qquad-\sum_{j\neq i}\frac{\sqrt{d_i d_j}}{D_{ij}}\sqrt{d_i}(v_i-\bar{v}_i)\cdot\sqrt{d_j}(v_j-\bar{v}_j)\Big] \\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\mathbf{d})\sqrt{d_i}(v_i-\bar{v}_i)\cdot\sqrt{d_j}(v_j-\bar{v}_j).
\end{aligned}$$

where $A_{ij}(d)$ is the matrix in (5.39). Let $\boldsymbol{Y}=(Y_1,\ldots,Y_n)$, where

$$Y_i=\sqrt{d_i}(v_i-\bar{v}_i).$$

By (5.43),

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\mathbf{d})\sqrt{d_i}(v_i-\bar{v}_i)\cdot\sqrt{d_j}(v_j-\bar{v}_j)&\geq(1+n\delta)\mu|\mathbb{P}_L(\mathbf{d})\boldsymbol{Y}|^2 \\
&\geq\mu|\mathbb{P}_L(\mathbf{d})\boldsymbol{Y}|^2,
\end{aligned}$$

where

$$|\mathbb{P}_L(\mathbf{d})\boldsymbol{Y}|^2=|\boldsymbol{Y}|^2-|\mathbb{P}_{L^\perp}(\mathbf{d})\boldsymbol{Y}|^2$$

with

$$|\boldsymbol{Y}|^2=\sum_{i=1}^{n}d_i|v_i-\bar{v}_i|^2$$

and

$$
\begin{aligned}
(\mathbb{P}_{L^\perp}(\mathbf{d})\boldsymbol{Y})_i &= \sum_{j=1}^{n} \frac{\sqrt{d_i d_j}}{1+n\delta}\sqrt{d_j}(v_j - \bar{v}_j) \\
&= \frac{\sqrt{d_i}}{1+n\delta}\sum_{j=1}^{n}(d_j v_j - d_j \bar{v}_j) \\
&= \frac{\sqrt{d_i}}{1+n\delta}\sum_{j=1}^{n}(\bar{d}_j \bar{v}_j - d_j \bar{v}_j) \\
&= \frac{\sqrt{d_i}}{1+n\delta}\sum_{j=1}^{n}(\bar{d}_j - d_j)\bar{v}_j
\end{aligned}
$$

where we have used the fact that $\sum_{i=1}^{n} d_i v_i = \sum_{i=1}^{n} \bar{d}_i \bar{v}_i = 0$.

Then, by (5.38),

$$
\begin{aligned}
|\mathbb{P}_{L^\perp}(\mathbf{d})Y|^2 &= \frac{1}{1+n\delta}\left(\sum_{j=1}^{n}(\bar{d}_j - d_j)\bar{v}_j\right)^2 \\
&\leq \frac{n}{1+n\delta}\sum_{j=1}^{n}|d_j - \bar{d}_j|^2 |\bar{v}_j|^2 \\
&\leq n\|\bar{v}_j\|_{L^\infty}^2 \sum_{j=1}^{n}|d_j - \bar{d}_j|^2 \\
&\leq \frac{n}{\delta^2}\|cu\|_{L^\infty}^2 \sum_{j=1}^{n}|d_j - \bar{d}_j|^2.
\end{aligned}
$$

Therefore

$$
\sum_{i=1}^{n}\sum_{j\neq i}\frac{d_i d_j}{2D_{ij}}|(v_i - \bar{v}_i) - (v_j - \bar{v}_j)|^2 \geq \mu\sum_{i=1}^{n} d_i |v_i - \bar{v}_i|^2 - \frac{\mu n}{\delta^2}\|cu\|_{L^\infty}^2 \sum_{j=1}^{n}|d_j - \bar{d}_j|^2
$$

and similarly

$$
\sum_{i=1}^{n}\sum_{j\neq i}\frac{\bar{d}_i \bar{d}_j}{2D_{ij}}|(v_i - \bar{v}_i) - (v_j - \bar{v}_j)|^2 \geq \mu\sum_{i=1}^{n} \bar{d}_i |v_i - \bar{v}_i|^2 - \frac{\mu n}{\delta^2}\|cu\|_{L^\infty}^2 \sum_{j=1}^{n}|d_j - \bar{d}_j|^2.
$$

Putting everything together, we find:

$$
\begin{aligned}
&F(\mathbf{d},\bar{\mathbf{d}})\Big|_{t=T} + \mu \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 \, \mathrm{d}x\mathrm{d}t \\
&\leq \frac{2n\mu}{\delta^2} \|cu\|_{L^\infty}^2 \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{T}^3} \underbrace{-\sum_{i=1}^{n} \sum_{j\neq i} \frac{d_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i)\cdot(\bar{v}_i - \bar{v}_j)}_{J_1} \, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{T}^3} \underbrace{-\sum_{i=1}^{n} \sum_{j\neq i} \frac{\bar{d}_i}{D_{ij}} (d_j - \bar{d}_j)(v_i - \bar{v}_i)\cdot(v_i - v_j)}_{J_2} \, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{T}^3} \underbrace{\delta \sum_{i=1}^{n} \Big(\sum_{j\neq i} \frac{1}{D_{ij}}\Big)(d_i + \bar{d}_i)|v_i - \bar{v}_i|^2}_{J_3} \, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{T}^3} \underbrace{-\delta \sum_{i=1}^{n} \sum_{j\neq i} \frac{1}{D_{ij}} (d_i + \bar{d}_i)(v_i - \bar{v}_i)\cdot\left(\frac{d_j}{d_i} v_j - \frac{\bar{d}_j}{\bar{d}_i}\bar{v}_j\right)}_{J_4} \, \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{5.46}
$$

and we need to estimate the error terms J_1, J_2, J_3 and J_4.

We start with J_3. Setting $M = \max_{i\neq j} \frac{1}{D_{ij}}$, we get

$$
J_3 \leq n\delta M \sum_{i=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2. \tag{5.47}
$$

Regarding the terms J_1 and J_2, we have:

$$
\begin{aligned}
J_1 &\leq \sum_{i=1}^{n} \sum_{j\neq i} \frac{d_i}{D_{ij}} |d_j - \bar{d}_j||v_i - \bar{v}_i||\bar{v}_i - \bar{v}_j| \\
&= \sum_{i=1}^{n} \sqrt{d_i}|v_i - \bar{v}_i| \sum_{j\neq i} \frac{\sqrt{d_i}}{D_{ij}} |d_j - \bar{d}_j||\bar{v}_i - \bar{v}_j|
\end{aligned}
$$

and using Young's and Jensen's inequality

$$
\begin{aligned}
J_1 &\le \frac{\mu}{4}\sum_{i=1}^n d_i|v_i-\bar v_i|^2 + \frac{1}{\mu}\sum_{i=1}^n\left(\sum_{j\ne i}\frac{\sqrt{d_i}}{D_{ij}}|d_j-\bar d_j||\bar v_i-\bar v_j|\right)^2\\
&\le \frac{\mu}{4}\sum_{i=1}^n d_i|v_i-\bar v_i|^2 + \frac{8n^2M^2}{\mu}\|v\|_{L^\infty}^2\sum_{j=1}^n|d_j-\bar d_j|^2,
\end{aligned}
$$

where $\|v\|_{L^\infty} := \max_{1\le i\le n}\|v_i\|_{L^\infty}$. Finally, by (5.38):

$$
J_1 \le \frac{\mu}{4}\sum_{i=1}^n d_i|v_i-\bar v_i|^2 + \frac{8n^2M^2}{\mu\delta^2}\|cu\|_{L^\infty}^2\sum_{j=1}^n|d_j-\bar d_j|^2
$$

and similarly,

$$
J_2 \le \frac{\mu}{4}\sum_{i=1}^n \bar d_i|v_i-\bar v_i|^2 + \frac{8n^2M^2}{\mu\delta^2}\|cu\|_{L^\infty}^2\sum_{j=1}^n|d_j-\bar d_j|^2.
$$

Hence, we see that

$$
J_1+J_2 \le \frac{\mu}{4}\sum_{i=1}^n(d_i+\bar d_i)|v_i-\bar v_i|^2 + \frac{C_1}{\delta^2}\sum_{j=1}^n|d_j-\bar d_j|^2 \tag{5.48}
$$

and $C_1>0$ depends on n, μ, M, and $\|cu\|_{L^\infty}$, but not on δ.

For J_4, we first split it into two parts

$$
\begin{aligned}
J_4 &\le \delta M\sum_{i=1}^n\sum_{j\ne i}(d_i+\bar d_i)|v_i-\bar v_i|\left|\frac{d_j}{d_i}v_j-\frac{\bar d_j}{\bar d_i}\bar v_j\right|\\
&\le \delta M\sum_{i=1}^n\sum_{j\ne i}d_i|v_i-\bar v_i|\Big(\left|\frac{d_j}{d_i}v_j-\frac{d_j}{d_i}\bar v_j\right|+\left|\frac{d_j}{d_i}\bar v_j-\frac{\bar d_j}{\bar d_i}\bar v_j\right|\Big)\\
&\quad+\delta M\sum_{i=1}^n\sum_{j\ne i}\bar d_i|v_i-\bar v_i|\Big(\left|\frac{d_j}{d_i}v_j-\frac{\bar d_j}{\bar d_i}v_j\right|+\left|\frac{\bar d_j}{\bar d_i}v_j-\frac{\bar d_j}{\bar d_i}\bar v_j\right|\Big)\\
&=\delta M\sum_{i=1}^n\sum_{j\ne i}d_j|v_i-\bar v_i|\,|v_j-\bar v_j|+\delta M\sum_{i=1}^n\sum_{j\ne i}d_i|v_i-\bar v_i||\bar v_j|\left|\frac{d_j}{d_i}-\frac{\bar d_j}{\bar d_i}\right|\\
&\quad+\delta M\sum_{i=1}^n\sum_{j\ne i}\bar d_i|v_i-\bar v_i||v_j|\left|\frac{d_j}{d_i}-\frac{\bar d_j}{\bar d_i}\right|+\delta M\sum_{i=1}^n\sum_{j\ne i}\bar d_j|v_i-\bar v_i|\,|v_j-\bar v_j|
\end{aligned}
$$

and thus

$$
\begin{aligned}
J_4 &= \delta M \sum_{i=1}^{n} \sum_{j\neq i} (d_j + \bar{d}_j)|v_i - \bar{v}_i|\,|v_j - \bar{v}_j| \\
&\quad + \delta M \sum_{i=1}^{n} \sum_{j\neq i} (d_i|\bar{v}_j| + \bar{d}_i|v_j|)|v_i - \bar{v}_i| \left| \frac{d_j}{d_i} - \frac{\bar{d}_j}{\bar{d}_i} \right| \\
&=: J_4^1 + J_4^2.
\end{aligned}
$$

The first term is handled by Young's inequality:

$$
\begin{aligned}
J_4^1 &= \sum_{j=1}^{n} \sqrt{d_j + \bar{d}_j}|v_j - \bar{v}_j| \left(\delta M \sqrt{d_j + \bar{d}_j} \sum_{i=1}^{n} |v_i - \bar{v}_i| \right) \\
&\leq \frac{\mu}{4} \sum_{j=1}^{n} (d_j + \bar{d}_j)\,|v_j - \bar{v}_j|^2 + \frac{4n\delta^2 M^2}{\mu} \left(\sum_{i=1}^{n} |v_i - \bar{v}_i| \right)^2
\end{aligned}
$$

then Jensen's inequality in the second term:

$$
J_4^1 \leq \frac{\mu}{4} \sum_{j=1}^{n} (d_j + \bar{d}_j)\,|v_j - \bar{v}_j|^2 + \frac{4n^2\delta^2 M^2}{\mu} \sum_{i=1}^{n} |v_i - \bar{v}_i|^2
$$

and finally we multiply and divide by $d_i + \bar{d}_i$:

$$
\begin{aligned}
J_4^1 &\leq \frac{\mu}{4} \sum_{j=1}^{n} (d_j + \bar{d}_j)\,|v_j - \bar{v}_j|^2 + \frac{4n^2\delta^2 M^2}{\mu} \sum_{i=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 \frac{1}{d_i + \bar{d}_i} \\
&\leq \frac{\mu}{4} \sum_{j=1}^{n} (d_j + \bar{d}_j)\,|v_j - \bar{v}_j|^2 + \frac{2n^2\delta M^2}{\mu} \sum_{i=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2
\end{aligned}
$$

where in the last inequality we used that $d_i + \bar{d}_i \geq 2\delta$.

The second term is estimated in a similar way:

$$
\begin{aligned}
J_4^2 &\leq \delta M \max\{\|\bar{v}_j\|_{L^\infty}, \|v_j\|_{L^\infty}\} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left| \frac{d_j}{d_i} - \frac{\bar{d}_j}{\bar{d}_i} \right| \\
&\overset{(5.38)}{\leq} M\|cu\|_{L^\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left| \frac{d_j}{d_i} - \frac{\bar{d}_j}{\bar{d}_i} \right|
\end{aligned}
$$

and using triangle inequality

$$
\begin{aligned}
J_4^2 &\leq M\|cu\|_{L^\infty} \sum_{i=1}^n \sum_{j=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left(\left| \frac{d_j}{d_i} - \frac{d_j}{\bar{d}_i} \right| + \left| \frac{d_j}{\bar{d}_i} - \frac{\bar{d}_j}{\bar{d}_i} \right| \right) \\
&= M\|cu\|_{L^\infty} \sum_{i=1}^n \sum_{j=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left(\frac{d_j}{d_i \bar{d}_i} \left|d_i - \bar{d}_i\right| + \frac{1}{\bar{d}_i} \left|d_j - \bar{d}_j\right| \right) \\
&\leq M\|cu\|_{L^\infty} \sum_{i=1}^n \sum_{j=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left(\frac{2}{\delta^2} \left|d_i - \bar{d}_i\right| + \frac{1}{\delta} \left|d_j - \bar{d}_j\right| \right)
\end{aligned}
$$

because $\delta \leq d_i, \bar{d}_i \leq 2$ and thus $\frac{1}{d_i \bar{d}_i} = \frac{1}{(c_i+\delta)(\bar{c}_i+\delta)} \leq \frac{1}{\delta^2}$. Then,

$$
\begin{aligned}
J_4^2 &= \frac{2M\|cu\|_{L^\infty}}{\delta^2} \sum_{i=1}^n \sum_{j=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left|d_i - \bar{d}_i\right| \\
&\quad + \frac{M\|cu\|_{L^\infty}}{\delta} \sum_{i=1}^n \sum_{j=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left|d_j - \bar{d}_j\right| =: J_4^{2,1} + J_4^{2,2}\,.
\end{aligned}
$$

Using Young's inequality the first term gives

$$
\begin{aligned}
J_4^{2,1} &= \frac{2nM\|cu\|_{L^\infty}}{\delta^2} \sum_{i=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i| \left|d_i - \bar{d}_i\right| \\
&= \sum_{i=1}^n \sqrt{d_i + \bar{d}_i}|v_i - \bar{v}_i| \frac{2nM\|cu\|_{L^\infty}}{\delta^2} \sqrt{d_i + \bar{d}_i}|d_i - \bar{d}_i| \\
&\leq \frac{\mu}{8} \sum_{i=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 + \frac{16n^2M^2\|cu\|_{L^\infty}^2}{\mu\delta^4} \sum_{i=1}^n |d_i - \bar{d}_i|^2\,.
\end{aligned}
$$

Similarly, Young's and Jensen's inequalities give

$$
\begin{aligned}
J_4^{2,2} &= \sum_{i=1}^n \sqrt{d_i + \bar{d}_i}|v_i - \bar{v}_i| \frac{M\|cu\|_{L^\infty}}{\delta} \sqrt{d_i + \bar{d}_i} \sum_{j=1}^n |d_j - \bar{d}_j| \\
&\leq \frac{\mu}{8} \sum_{i=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 + \frac{2M^2\|cu\|_{L^\infty}^2}{\mu\delta^2} \sum_{i=1}^n (d_i + \bar{d}_i) \left(\sum_{j=1}^n |d_j - \bar{d}_j| \right)^2 \\
&\leq \frac{\mu}{8} \sum_{i=1}^n (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 + \frac{8M^2n^2\|cu\|_{L^\infty}^2}{\mu\delta^2} \sum_{j=1}^n |d_j - \bar{d}_j|^2.
\end{aligned}
$$

Summarizing,

$$J_4 \le \left(\frac{\mu}{2} + C_2\delta\right) \sum_{i=1}^{n} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 + \frac{C_3}{\delta^4} \sum_{j=1}^{n} |d_j - \bar{d}_j|^2 \qquad (5.49)$$

where $C_2 > 0$ depends only on n, μ and M and $C_3 > 0$ on n, μ, M and $\|cu\|_{L^\infty}^2$ and neither of them depends on δ.

Finally, putting together (5.46), (5.47), (5.48) and (5.49), we obtain the differential inequality

$$F(\mathbf{d}, \bar{\mathbf{d}})\Big|_{t=T} + \left(\frac{\mu}{4} - C_4\delta\right) \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} (d_i + \bar{d}_i)|v_i - \bar{v}_i|^2 \, \mathrm{d}x\mathrm{d}t \le \frac{C_5}{\delta^4} \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \, \mathrm{d}x\mathrm{d}t$$

where C_4 and C_5 are positive constants independent of δ.

Next, by selecting $0 < \delta < \min\{1, \frac{\mu}{4C_4}\}$, we obtain

$$F(\mathbf{d}, \bar{\mathbf{d}})\Big|_{t=T} \le \frac{C_5}{\delta^4} \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \, \mathrm{d}x\mathrm{d}t, \qquad (5.50)$$

where

$$F(\mathbf{d}, \bar{\mathbf{d}}) = \sum_{i=1}^{n} \int_{\mathbb{T}^3} (\ln d_i - \ln \bar{d}_i)(d_i - \bar{d}_i)\mathrm{d}x.$$

By Taylor's theorem,

$$(\ln d_i - \ln \bar{d}_i)(d_i - \bar{d}_i) = \frac{1}{\xi}|d_i - \bar{d}_i|^2,$$

for some $\xi \in (\min\{d_i, \bar{d}_i\}, \max\{d_i, \bar{d}_i\}) \subset (\delta, 1+\delta)$, i.e.

$$(\ln d_i - \ln \bar{d}_i)(d_i - \bar{d}_i) \ge \frac{1}{1+\delta}|d_i - \bar{d}_i|^2. \qquad (5.51)$$

Therefore, (5.50) and (5.51) imply that

$$\sum_{i=1}^{n} \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \,\mathrm{d}x\Big|_{t=T} \le C_5 \frac{1+\delta}{\delta^4} \sum_{i=1}^{n} \int_0^T \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \,\mathrm{d}x\mathrm{d}t$$

and a direct application of Grönwall's Lemma gives

$$\sum_{i=1}^{n} \int_{\mathbb{T}^3} |d_i - \bar{d}_i|^2 \,\mathrm{d}x\Big|_{t=T} \le 0$$

i.e. $d_i = \bar{d}_i$ and thus $c_i = \bar{c}_i$, for all $i = 1, \dots, n$. This completes the proof.

5.1.6 Relation between weak and renormalized equations

It is evident from Definition 12 that weak solutions are a special case of renormalized solutions. Indeed, choosing $\beta(s) = s$ (which is admissible) Definition 12 reduces to Definition 11. In this section, we investigate the other direction (weak $\Rightarrow$ renormalized) and prove Theorem 17.

We, first, prove a result for generic transport equations, which is inspired by [45, Lemmas 4 and 5] and which is used in the proof of Theorem 17.

Lemma 21. *Let $u \in C^0([0,T]; L^2(\mathbb{T}^3))$, with $u \in L^2(0,T; H^1(\mathbb{T}^3))$ and $u^0 \in L^2(\mathbb{T}^3)$, and $z \in L^2(0,T; (L^2(\mathbb{T}^3))^3)$, be such that for all $\psi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$*

$$-\int_0^T \int_{\mathbb{T}^3} u \partial_t \psi \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} u^0 \psi(x,0) \mathrm{d}x = \int_0^T \int_{\mathbb{T}^3} z \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t. \tag{5.52}$$

Let $\mathcal{B} = \{\beta \in C^2(\mathbb{R};\mathbb{R})$ s.t. $|\beta(w)| \le C|w|^2,\ |\beta'(w)| \le C|w|,\ |\beta''(w)| \le C\}$. Then, for $\beta \in \mathcal{B}$ and $\psi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$, there holds

$$\begin{aligned} -\int_0^T \int_{\mathbb{T}^3} \beta(u) \partial_t \psi \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \beta(u^0) \psi(x,0) \mathrm{d}x \\ = \int_0^T \int_{\mathbb{T}^3} \beta'(u) z \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \psi \beta''(u) \nabla u \cdot z \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Proof. Let ρ_σ denote a standard symmetric mollifier with respect to space. We choose the test function $\rho_\sigma * \psi$ in (5.52) and integrate by parts (in space) on the right-hand side, to obtain for all $\psi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$

$$-\int_0^T\int_{\mathbb{T}^3}(\rho_\sigma * u)\partial_t\psi\,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3}(\rho_\sigma * u^0)\psi(x,0)\mathrm{d}x = -\int_0^T\int_{\mathbb{T}^3}\mathrm{div}(\rho_\sigma * z)\psi\,\mathrm{d}x\mathrm{d}t. \tag{5.53}$$

Define the functions $f^\sigma, g^\sigma : \mathbb{T}^3 \times [-T,T] \to \mathbb{R}$ by:

$$f^\sigma(x,t) = \begin{cases} (\rho_\sigma * u)(x,t), & t > 0 \\ (\rho_\sigma * u^0)(x), & t < 0 \end{cases} \tag{5.54}$$

and

$$g^\sigma(x,t) = \begin{cases} \mathrm{div}(\rho_\sigma * z)(x,t), & t > 0 \\ 0, & t < 0 \end{cases} \tag{5.55}$$

and notice that $f^\sigma, g^\sigma \in L^2(-T,T;L^2(\mathbb{T}^3))$. Then, for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (-T,T))$, f^σ and g^σ satisfy, due to (5.53),

$$-\int_{-T}^T\int_{\mathbb{T}^3} f^\sigma(x,t)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = -\int_{-T}^T\int_{\mathbb{T}^3} g^\sigma(x,t)\varphi\,\mathrm{d}x\mathrm{d}t. \tag{5.56}$$

Let, now, ρ_τ be a standard symmetric mollifier with respect to time. Using $\rho_\tau * \varphi$ as a test function in (5.56), we get for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (-T+2\tau, T+2\tau))$

$$-\int_{-T}^T\int_{\mathbb{T}^3}(\rho_\tau * f^\sigma)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = -\int_{-T}^T\int_{\mathbb{T}^3}(\rho_\tau * g^\sigma)\varphi\,\mathrm{d}x\mathrm{d}t,$$

which implies that $\rho_\tau * f^\sigma \in C^1((-T+\tau, T-\tau); L^1(\mathbb{T}^3))$, with the representation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_\tau * f^\sigma) = -\rho_\tau * g^\sigma, \quad \text{in the } L^1(\mathbb{T}^3) \text{ sense.}$$

Now, for $\beta \in \mathcal{B}$, we have by the chain rule

$$-\int_{-T}^{T}\int_{\mathbb{T}^3}\beta(\rho_\tau * f^\sigma)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = -\int_{-T}^{T}\int_{\mathbb{T}^3}\beta'(\rho_\tau * f^\sigma)(\rho_\tau * g^\sigma)\varphi\,\mathrm{d}x\mathrm{d}t \qquad (5.57)$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (-T+2\tau, T-2\tau))$.

Keeping $\sigma > 0$ fixed, we let $\tau \to 0$. For the term on the left–hand side, β is continuous and $\rho_\tau * f^\sigma \to f^\sigma$ a.e., hence $\beta(\rho_\tau * f^\sigma) \to \beta(f^\sigma)$ a.e. and due to f^σ being in $L^2(-T,T;L^2(\mathbb{T}^3))$ and the growth of β, we have

$$|\beta(\rho_\tau * f^\sigma)\partial_t\varphi| \le C|\rho_\tau * f^\sigma|^2\|\partial_t\varphi\|_\infty \quad \text{and}$$
$$\rho_\tau * f^\sigma \to f^\sigma, \text{ in } L^2(K) \text{ for any compact } K \subset \mathbb{T}^3 \times (-T,T).$$

Similarly, for the right–hand side, since $\rho_\tau * g^\sigma \to g^\sigma$ a.e. and $\rho_\tau * f^\sigma \to f^\sigma$ a.e. and β' is continuous, we have $\beta'(\rho_\tau * f^\sigma)(\rho_\tau * g^\sigma) \to \beta'(f^\sigma)g^\sigma$ a.e. and due to the growth of β' and f^σ, g^σ being in $L^2(-T,T;L^2(\mathbb{T}^3))$, we have

$$|\beta'(\rho_\tau * f^\sigma)(\rho_\tau * g^\sigma)\varphi| \le C|\rho_\tau * f^\sigma||\rho_\tau * g^\sigma|\|\varphi\|_\infty \quad \text{and}$$
$$\rho_\tau * f^\sigma \to f^\sigma, \text{ in } L^2(K) \text{ for any compact } K \subset \mathbb{T}^3 \times (-T,T),$$
$$\rho_\tau * g^\sigma \to g^\sigma, \text{ in } L^2(K) \text{ for any compact } K \subset \mathbb{T}^3 \times (-T,T).$$

Letting $\tau \to 0$ we can apply the dominated convergence theorem to deduce that

$$-\int_{-T}^{T}\int_{\mathbb{T}^3}\beta(f^\sigma)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = -\int_{-T}^{T}\int_{\mathbb{T}^3}\beta'(f^\sigma)(g^\sigma)\varphi\,\mathrm{d}x\mathrm{d}t,$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (-T,T))$ and because of (5.54)-(5.55) we recover

$$-\int_0^T\int_{\mathbb{T}^3}\beta(\rho_\sigma * u)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3}\beta(\rho_\sigma * u^0)\varphi(x,0)\,\mathrm{d}x$$
$$= -\int_0^T\int_{\mathbb{T}^3}\beta'(\rho_\sigma * u)\,\mathrm{div}(\rho_\sigma * z)\varphi\,\mathrm{d}x\mathrm{d}t.$$

We, now, integrate by parts on the right–hand side

$$
\begin{aligned}
-\int_0^T\int_{\mathbb{T}^3}\beta(\rho_\sigma * u)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3}\beta(\rho_\sigma * u^0)\varphi(x,0)\,\mathrm{d}x \\
= \int_0^T\int_{\mathbb{T}^3}\beta''(\rho_\sigma * u)\nabla(\rho_\sigma * u)\cdot(\rho_\sigma * z)\varphi\,\mathrm{d}x\mathrm{d}t \\
+ \int_0^T\int_{\mathbb{T}^3}\beta'(\rho_\sigma * u)(\rho_\sigma * z)\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and let $\sigma \to 0$. For the terms on the left–hand side, we have $\rho_\sigma * u \to u$ a.e. and the continuity of β implies that $\beta(\rho_\sigma * u) \to \beta(u)$ a.e.. Moreover, by $u \in L^2(0,T;L^2(\mathbb{T}^3))$ and the growth condition on β, we see that

$$
\begin{aligned}
&|\beta(\rho_\sigma * u)\partial_t\varphi| \le C|\rho_\sigma * u|^2\|\partial_t\varphi\|_\infty, \text{ where} \\
&\rho_\sigma * u \to u \text{ in } L^2(K), \text{ for any compact } K \subset \mathbb{T}^3 \times (0,T).
\end{aligned}
$$

For the first term on the right–hand side, we have $\beta''(\rho_\sigma * u)\nabla(\rho_\sigma * u)\cdot(\rho_\sigma * z) \to \beta''(u)\nabla u \cdot z$ a.e. and use the growth condition $|\beta''(w)| \le C$ and the fact that $z, \nabla u \in L^2(0,T;(L^2(\mathbb{T}^3))^3)$ to get

$$
\begin{aligned}
&|\beta''(\rho_\sigma * u)\nabla(\rho_\sigma * u)\cdot(\rho_\sigma * z)\varphi| \le C|\nabla(\rho_\sigma * u)||\rho_\sigma * z|\|\varphi\|_\infty, \text{ where} \\
&\nabla(\rho_\sigma * u) \to \nabla u \text{ and } \rho_\sigma * z \to z \text{ in } L^2(K) \text{ for any compact } K \subset \mathbb{T}^3 \times (0,T).
\end{aligned}
$$

Finally, for the last term on the right–hand side $\beta'(\rho_\sigma * u)(\rho_\sigma * z) \to \beta'(u)z$ a.e. and since $|\beta'(w)| \le C|w|$, $u \in L^2(0,T;L^2(\mathbb{T}^3))$ and $z \in L^2(0,T;(L^2(\mathbb{T}^3))^3)$ we obtain

$$
\begin{aligned}
&|\beta'(\rho_\sigma * u)(\rho_\sigma * z)\cdot\nabla\varphi| \le C|\rho_\sigma * u||\rho_\sigma * z|\|\nabla\varphi\|_\infty, \text{ where} \\
&\rho_\sigma * u \to u \text{ and } \rho_\sigma * z \to z \text{ in } L^2(K) \text{ for any compact } K \subset \mathbb{T}^3 \times (0,T).
\end{aligned}
$$

By the dominated convergence theorem, taking the limit $\sigma \to 0$, we conclude

$$
\begin{aligned}
-\int_0^T \int_{\mathbb{T}^3} \beta(u)\partial_t \varphi \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \beta(u^0)\varphi(x,0)\,\mathrm{d}x \\
= \int_0^T \int_{\mathbb{T}^3} \beta''(u)\nabla u \cdot z\varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \beta'(u) z \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

□

If the solution satisfies the bounds $0 \le u \le M$, for some $M > 0$, the growth conditions on β and its derivatives are not needed:

Lemma 22. *Let $u \in C([0,T]; L^2(\mathbb{T}^3)) \cap L^2(0,T; H^1(\mathbb{T}^3))$, with $0 \le u(x,t) \le M$, for some $M > 0$, $u^0 \in L^2(\mathbb{T}^3)$ and $z \in L^2(0,T; (L^2(\mathbb{T}^3))^3)$ satisfy for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$*

$$
-\int_0^T \int_{\mathbb{T}^3} u\partial_t \varphi \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} u^0 \varphi(x,0)\mathrm{d}x = \int_0^T \int_{\mathbb{T}^3} z \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t.
$$

Let $\beta \in C^2([0,\infty))$. Then, for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,T))$, there holds

$$
\begin{aligned}
-\int_0^T \int_{\mathbb{T}^3} \beta(u)\partial_t \varphi \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \beta(u^0)\varphi(x,0)\mathrm{d}x \\
= \int_0^T \int_{\mathbb{T}^3} \beta'(u) z \cdot \nabla\varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \beta''(u)\nabla u \cdot z\varphi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Proof. As in the proof of Lemma 21, we derive (5.57) and we pass to the limit $\tau \to 0$. The limit is done via a similar (but simpler) argument using the bound $0 \le u \le M$ and the continuity of β, β' and β''. In particular, since u and u^0 take values between 0 and M, the same holds for $\rho_\tau * f^\sigma$, and the continuity of β and β' implies that $\beta(\rho_\tau * f^\sigma)$ and $\beta'(\rho_\tau * f^\sigma)$ are bounded. This is enough to apply the dominated convergence theorem to (5.57) and pass to the limit $\tau \to 0$. We, then, proceed as in the proof of Lemma 21 and use the same idea for the limit $\sigma \to 0$. □

Now, we are ready to prove Theorem 17:

Proof. To this end, let $\mathbf{c}$ be a weak solution. Then it satisfies (i)–(iv) of Definition 11. In order to show that $\mathbf{c}$ is a renormalized solution, we have to prove that it also satisfies (i)–(iv) of Definition 12. Conditions (i) and (ii) are automatically satisfied and they imply that $c_i \in L^2(0,T;H^1(\mathbb{T}^3))$ and $c_i u_i \in L^2(0,T;(L^2(\mathbb{T}^3))^3)$. Moreover, $c_i \in C([0,T];L^2(\mathbb{T}^3))$ (see (5.9)). This allows for a direct application of Lemma 22 with $u = c_i$ and $z = c_i u_i$, which implies that $\mathbf{c}$ satisfies (5.10) (notice that $2c_i \nabla\sqrt{c_i} \cdot \sqrt{c_i} u_i = \nabla c_i \cdot c_i u_i$). Finally, since $\beta \in C^2$, we have β is Lipschitz continuous, hence (iv) of Definition 12 follows due to condition (iv) of Definition 11. Hence, (i)–(iv) of Definition 12 are satisfied and $\mathbf{c}$ is a renormalized solution. □

5.2 Absence of anomalous dissipation

Previous works [8, 10] have shown that weak solutions automatically satisfy the entropy inequality:

$$H(\mathbf{c}(T)) + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{c_i c_j}{D_{ij}}|u_i - u_j|^2 \mathrm{d}x\mathrm{d}t \le H(\mathbf{c}^0) \tag{5.58}$$

where the dissipation term is understood as

$$\frac{1}{2}\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{1}{D_{ij}}|\sqrt{c_i}(\sqrt{c_i}u_i) - \sqrt{c_i}(\sqrt{c_j}u_j)|^2 \mathrm{d}x\mathrm{d}t$$

and thus is well–defined, since $c_i \in L^\infty(0,T;L^\infty(\Omega))$ and $\sqrt{c_i}u_i \in L^2(0,T;L^2(\Omega))$. In [46] we prove that, in fact, weak solutions satisfy (5.58) as an equality, implying that there is no anomalous dissipation, and thus the flow is not turbulent.

Theorem 23. *Let* $\mathbf{c}$ *be a weak solution to the Maxwell–Stefan system with no–flux boundary conditions and let the initial data be such that* $c_i^0(x) \ge 0$ *and* $\sum_{i=1}^n c_i^0(x) = 1$ *in* Ω*. Then* $\mathbf{c}$ *satisfies the entropy equality*

$$H(\mathbf{c}(T)) + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{c_i c_j}{D_{ij}}|u_i - u_j|^2 \mathrm{d}x\mathrm{d}t = H(\mathbf{c}^0). \tag{5.59}$$

Proof. We first notice that, a–posteriori, the solution enjoys better regularity:

Lemma 24. *Let* **c** *be a weak solution. Then*

$$\partial_t c_i \in L^2(0,T;H^1(\Omega)^*) \tag{5.60}$$

and thus

$$c_i \in C([0,T];L^2(\Omega)). \tag{5.61}$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the duality bracket between the spaces $L^2(0,T;H^1(\Omega)^*)$ and $L^2(0,T;H^1(\Omega))$. Then, the computation

$$\begin{aligned}\int_0^T |\langle\partial_t c_i,\varphi\rangle|\mathrm{d}t &= \int_0^T |\langle -\operatorname{div}(c_i u_i),\varphi\rangle|\mathrm{d}t\\ &= \int_0^T |\langle c_i u_i,\nabla\varphi\rangle|\mathrm{d}t\\ &\le \|\sqrt{c_i}\|_{L^\infty(\Omega_T)}\|\sqrt{c_i}u_i\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}\end{aligned}$$

shows that due to $\sqrt{c_i}u_i$ being in $L^2(0,T;L^2(\Omega))$,

$$\partial_t c_i \in L^2(0,T;(H^1(\Omega)^*).$$

Moreover, since

$$c_i \in L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \partial_t c_i \in L^2(0,T;H^1(\Omega)^*)$$

and

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)^*$$

is a Gelfand triple, it follows that $c_i \in H^1(\Omega\times(0,T))$, which is continuously embedded in $C([0,T];L^2(\Omega))$ and this proves (5.61). □

We divide the proof of Theorem 23 in three steps:

Step 1: We start by writing the weak formulation

$$\int_\Omega c_i(t)\varphi_i(t)\mathrm{d}x - \int_\Omega c_i(s)\varphi(s)\mathrm{d}x - \int_s^t\int_\Omega c_i\partial_t\varphi_i\mathrm{d}x\mathrm{d}\tau - \int_s^t\int_\Omega c_iu_i\cdot\nabla\varphi_i\mathrm{d}x\mathrm{d}\tau = 0$$

for $0 < s < t < T$, choose $\varphi_i = (\ln(c_i^\epsilon + \delta))^\epsilon$, for some $\delta > 0$, where v^ϵ denotes mollification of v in time and let $\epsilon \to 0$. We note that admissible test functions belong to the space $C^1([0,T];C^1(\bar\Omega))$, but by a density argument, it is enough to be in $L^2(0,T;H^1(\Omega))$. This yields for the first two terms:

$$\int_\Omega c_i(t)\varphi_i(t)\mathrm{d}x = \int_\Omega c_i^\epsilon(t)\ln(c_i^\epsilon(t)+\delta)\mathrm{d}x \to \int_\Omega c_i(t)\ln(c_i(t)+\delta)\mathrm{d}x$$

and

$$\int_\Omega c_i(s)\varphi(s)\mathrm{d}x = \int_\Omega c_i^\epsilon(s)\ln(c_i^\epsilon(s)+\delta)\mathrm{d}x \to \int_\Omega c_i(s)\ln(c_i(s)+\delta)\mathrm{d}x$$

as $\epsilon \to 0$ by the Dominated Convergence Theorem, since $c_i^\epsilon\ln(c_i^\epsilon+\delta) \to c_i\ln(c_i+\delta)$ almost everywhere as $\epsilon \to 0$ and $|c_i^\epsilon\ln(c_i^\epsilon+\delta)| \leq \max\{|\ln\delta|, |\ln(1+\delta)|\}$ uniformly in ϵ and $\max\{|\ln\delta|, |\ln(1+\delta)|\} \in L^1(\Omega)$, due to Ω being bounded.

The third term gives

$$\begin{aligned}\int_s^t\int_\Omega c_i\partial_t\varphi_i\mathrm{d}x\mathrm{d}\tau &= \int_s^t\int_\Omega c_i^\epsilon\partial_t\ln(c_i^\epsilon+\delta)\mathrm{d}x\mathrm{d}\tau\\ &= \int_s^t\int_\Omega \partial_t c_i^\epsilon\mathrm{d}x\mathrm{d}\tau - \delta\int_s^t\int_\Omega\partial_t\ln(c_i^\epsilon+\delta)\mathrm{d}x\mathrm{d}\tau\\ &= \int_\Omega c_i^\epsilon(t)\mathrm{d}x - \int_\Omega c_i^\epsilon(s)\mathrm{d}x - \delta\int_\Omega\ln(c_i^\epsilon(t)+\delta)\mathrm{d}x\\ &\quad + \delta\int_\Omega\ln(c_i^\epsilon(s)+\delta)\mathrm{d}x\end{aligned}$$

and thus

$$\begin{aligned}\int_s^t\int_\Omega c_i\partial_t\varphi_i\mathrm{d}x\mathrm{d}\tau \to \int_\Omega c_i(t)\mathrm{d}x - \int_\Omega c_i(s)\mathrm{d}x - \delta\int_\Omega\ln(c_i(t)+\delta)\mathrm{d}x\\ + \delta\int_\Omega\ln(c_i(s)+\delta)\mathrm{d}x,\end{aligned}$$

as $\epsilon \to 0$ by the Dominated Convergence Theorem, since for the first two terms $c_i^\epsilon \to c_i$ almost everywhere as $\epsilon \to 0$ and $|c_i^\epsilon| \leq 1$ uniformly in ϵ and $1 \in L^1(\Omega)$, while for the last two terms $\ln(c_i^\epsilon+\delta) \to \ln(c_i+\delta)$ almost everywhere and $|\ln(c_i^\epsilon+\delta)| \leq \max\{|\ln\delta|, |\ln(1+\delta)|\}$ uniformly and $\max\{|\ln\delta|, |\ln(1+\delta)| \in L^1(\Omega)$.

Finally, for the last term

$$\begin{aligned}\int_s^t \int_\Omega c_i u_i \cdot \nabla\varphi_i \mathrm{d}x\mathrm{d}\tau &= \int_s^t \int_\Omega (c_i u_i)^\epsilon \cdot \nabla \ln(c_i^\epsilon + \delta)\mathrm{d}x\mathrm{d}\tau \\ &= \int_s^t \int_\Omega (c_i u_i)^\epsilon \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon + \delta}\mathrm{d}x\mathrm{d}\tau \\ &= \int_s^t \int_\Omega ((c_i u_i)^\epsilon - c_i u_i) \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon + \delta}\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_s^t \int_\Omega c_i u_i \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon + \delta}\mathrm{d}x\mathrm{d}\tau \\ &\to \int_s^t \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta}\mathrm{d}x\mathrm{d}\tau\end{aligned}$$

as $\epsilon \to 0$ because the first term gives

$$\begin{aligned}\left|\int_s^t \int_\Omega ((c_i u_i)^\epsilon - c_i u_i) \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon + \delta}\mathrm{d}x\mathrm{d}\tau\right| &\leq \int_s^t \int_\Omega |(c_i u_i)^\epsilon - c_i u_i| \frac{|\nabla c_i^\epsilon|}{c_i^\epsilon + \delta}\mathrm{d}x\mathrm{d}\tau \\ &\leq \frac{1}{\delta}\|(c_i u_i)^\epsilon - c_i u_i\|_{L^2(\Omega_T)} \|\nabla c_i^\epsilon\|_{L^2(\Omega_T)} \\ &\leq \frac{1}{\delta}\|(c_i u_i)^\epsilon - c_i u_i\|_{L^2(\Omega_T)} \|\nabla c_i\|_{L^2(\Omega_T)} \to 0\end{aligned}$$

and the last term converges by the Dominated Convergence Theorem, since $c_i u_i \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon+\delta} \to c_i u_i \cdot \frac{\nabla c_i}{c_i+\delta}$ almost everywhere as $\epsilon \to 0$ and $|c_i u_i \cdot \frac{\nabla c_i^\epsilon}{c_i^\epsilon+\delta}| \leq \frac{1}{\delta}|c_i u_i| \|\nabla c_i\|_{L^2(0,T)}$ uniformly in ϵ, with $\frac{1}{\delta}|c_i u_i| \|\nabla c_i\|_{L^2(0,T)} \in L^1(\Omega_T)$.

Putting everything together, we obtain:

$$\begin{aligned}&\int_\Omega c_i(t)\ln(c_i(t)+\delta)\mathrm{d}x - \int_\Omega c_i(s)\ln(c_i(s)+\delta)\mathrm{d}x - \int_\Omega c_i(t)\mathrm{d}x \\ &+ \int_\Omega c_i(s)\mathrm{d}x + \delta\int_\Omega \ln(c_i(t)+\delta)\mathrm{d}x - \delta\int_\Omega \ln(c_i(s)+\delta)\mathrm{d}x \\ &- \int_s^t \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i+\delta}\mathrm{d}x\mathrm{d}\tau = 0.\end{aligned} \tag{5.62}$$

Step 2: Now let $t \to T$ and $s \to 0$. The first two terms are handled as follows:

$$\begin{aligned}
\int_\Omega c_i(t)\ln(c_i(t)+\delta)\mathrm{d}x &= \int_\Omega (c_i(t)-c_i(T))\ln(c_i(t)+\delta)\mathrm{d}x \\
&\quad + \int_\Omega c_i(T)\ln(c_i(t)+\delta)\mathrm{d}x \\
&= \int_\Omega (c_i(t)-c_i(T))(\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta))\mathrm{d}x \\
&\quad + \int_\Omega (c_i(t)-c_i(T))\ln(c_i(T)+\delta)\mathrm{d}x \\
&\quad + \int_\Omega c_i(T)\ln(c_i(t)+\delta)\mathrm{d}x \\
&= \int_\Omega (c_i(t)-c_i(T))(\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta))\mathrm{d}x \\
&\quad + \int_\Omega (c_i(t)-c_i(T))\ln(c_i(T)+\delta)\mathrm{d}x \\
&\quad + \int_\Omega c_i(T)\ln(c_i(T)+\delta)\mathrm{d}x \\
&\quad + \int_\Omega c_i(T)(\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta))\mathrm{d}x.
\end{aligned}$$

By the Taylor theorem,

$$\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta) = \frac{1}{\xi}(c_i(t)-c_i(T))$$

for some ξ between $c_i(t)+\delta$ and $c_i(T)+\delta$ and thus $\frac{1}{\xi} \le \frac{1}{\delta}$. Then

$$\begin{aligned}
\left|\int_\Omega (c_i(t)-c_i(T))(\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta))\mathrm{d}x\right| &= \left|\int_\Omega \frac{1}{\xi}|c_i(t)-c_i(T)|^2\mathrm{d}x\right| \\
&\le \frac{1}{\delta}\|c_i(t)-c_i(T)\|^2_{L^2(\Omega)} \\
&\to 0 \quad \text{as } t\to T
\end{aligned}$$

because of (5.61). Moreover

$$\begin{aligned}
\left|\int_\Omega (c_i(t)-c_i(T))\ln(c_i(T)+\delta)\mathrm{d}x\right| &\le \|\ln(c_i(T)+\delta)\|_{L^2(\Omega)}\|c_i(t)-c_i(T)\|_{L^2(\Omega)} \\
&\le |\Omega|\max\{|\ln\delta|,\ln(1+\delta)\}\|c_i(t)-c_i(T)\|_{L^2(\Omega)} \\
&\to 0 \quad \text{as } t\to T
\end{aligned}$$

because of (5.61). Finally

$$\left|\int_\Omega c_i(T)(\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta)\mathrm{d}x\right| = \left|\int_\Omega \frac{1}{\xi}c_i(T)(c_i(t)-c_i(T))\mathrm{d}x\right|$$
$$\leq \frac{1}{\delta}\|c_i(T)\|_{L^2(\Omega)}\|c_i(t)-c_i(T)\|_{L^2(\Omega)}$$
$$\leq \frac{1}{\delta}|\Omega|\|c_i(t)-c_i(T)\|_{L^2(\Omega)}$$
$$\to 0 \quad \text{as } t\to T$$

again due to (5.61). Hence

$$\int_\Omega c_i(t)\ln(c_i(t)+\delta)\mathrm{d}x \to \int_\Omega c_i(T)\ln(c_i(T)+\delta)\mathrm{d}x$$

and

$$\int_\Omega c_i(s)\ln(c_i(s)+\delta)\mathrm{d}x \to \int_\Omega c_i^0\ln(c_i^0+\delta)\mathrm{d}x.$$

The next four terms are treated in a similar way, since

$$\int_\Omega c_i(t)\mathrm{d}x = \int_\Omega (c_i(t)-c_i(T))\mathrm{d}x + \int_\Omega c_i(T)\mathrm{d}x \to \int_\Omega c_i(T)\mathrm{d}x$$

and

$$\int_\Omega \ln(c_i(t)+\delta)\mathrm{d}x = \int_\Omega (\ln(c_i(t)+\delta)-\ln(c_i(T)+\delta))\mathrm{d}x + \int_\Omega \ln(c_i(T)+\delta)\mathrm{d}x$$
$$\to \int_\Omega \ln(c_i(T)+\delta)\mathrm{d}x$$

as above.

The last term converges after rewritting is as

$$\int_s^t\int_\Omega c_iu_i\cdot\frac{\nabla c_i}{c_i+\delta}\mathrm{d}x\mathrm{d}\tau = \int_0^T\int_\Omega c_iu_i\cdot\frac{\nabla c_i}{c_i+\delta}\chi_{[s,t]}(\tau)\mathrm{d}x\mathrm{d}\tau$$

where χ denotes the characteristic function. Then $c_iu_i\cdot\frac{\nabla c_i}{c_i+\delta}\chi_{[s,t]} \to c_iu_i\cdot\frac{\nabla c_i}{c_i+\delta}\chi_{[0,T]}$

almost everywhere as $t \to T$ and $s \to 0$ and

$$\left| c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \chi_{[s,t]} \right| \leq |c_i u_i| \frac{|\nabla c_i|}{c_i + \delta} \in L^1(0, T; L^1(\Omega))$$

since for fixed $\delta > 0$, $\frac{1}{c_i+\delta}$ is bounded above and $c_i u_i, \nabla c_i \in L^2(0, T; L^2(\Omega))$, where the bound is uniform in s and t. By the Dominated Convergence Theorem we obtain the convergence

$$\int_s^t \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \mathrm{d}x\mathrm{d}\tau \to \int_0^T \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \mathrm{d}x\mathrm{d}\tau \quad \text{as } t \to T \text{ and } s \to 0$$

and we conclude

$$\begin{aligned} &\int_\Omega c_i(T) \ln(c_i(T) + \delta)\mathrm{d}x - \int_\Omega c_i^0 \ln(c_i^0 + \delta)\mathrm{d}x - \int_\Omega c_i(T)\mathrm{d}x \\ &+ \int_\Omega c_i^0 \mathrm{d}x + \delta \int_\Omega \ln(c_i(T) + \delta)\mathrm{d}x - \delta \int_\Omega \ln(c_i^0 + \delta)\mathrm{d}x \\ &- \int_0^T \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \mathrm{d}x\mathrm{d}\tau = 0. \end{aligned} \tag{5.63}$$

Step 3: Finally, we let $\delta \to 0$. The first term gives

$$\int_\Omega c_i(T) \ln(c_i(T) + \delta)\mathrm{d}x \to \int_\Omega c_i(T) \ln c_i(T)\mathrm{d}x$$

by the Dominated Convergence Theorem, since $c_i(T) \ln(c_i(T)+\delta) \to c_i(T) \ln c_i(T)$ almost everywhere as $\delta \to 0$ and $|c_i(T) \ln(c_i(T) + \delta)| \leq 1$ uniformly in δ and $1 \in L^1(\Omega)$.

The second term is treated as the first one, yielding

$$\int_\Omega c_i^0 \ln(c_i^0 + \delta)\mathrm{d}x \to \int_\Omega c_i^0 \ln c_i^0 \mathrm{d}x$$

and the next two are independent of δ. Furthermore, $\delta \ln(c_i(T) + \delta) \to 0$ almost everywhere as $\delta \to 0$ and $|\delta \ln(c_i(T) + \delta)| \leq 1$ uniformly in δ, with $1 \in L^1(\Omega)$, showing that

$$\delta \int_\Omega \ln(c_i(T) + \delta)\mathrm{d}x \to 0$$

and

$$\delta \int_\Omega \ln(c_i^0 + \delta)\mathrm{d}x \to 0$$

as $\delta \to 0$, by the Dominated Convergence Theorem. We note at this point, that as $\delta \to 0$, the logarithm might blow-up for small values of $c_i(T)$ and c_i^0, however since δ goes to zero faster that the logarithm blows up, the product converges to zero without any problem.

Finally, for the last term we obtain

$$\int_0^T \int_\Omega c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \mathrm{d}x\mathrm{d}\tau \to 2\int_0^T \int_\Omega \sqrt{c_i} u_i \cdot \nabla\sqrt{c_i} \mathrm{d}x\mathrm{d}\tau$$

by the Dominated Convergence Theorem, since

$$c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \to 2\sqrt{c_i} u_i \cdot \nabla\sqrt{c_i} \quad \text{almost everywhere as } \delta \to 0$$

and

$$\begin{aligned} \left| c_i u_i \cdot \frac{\nabla c_i}{c_i + \delta} \right| &= |\sqrt{c_i} u_i||2\nabla\sqrt{c_i}| \frac{c_i}{c_i + \delta} \\ &\leq |\sqrt{c_i} u_i||2\nabla\sqrt{c_i}| \in L^1(0, T; L^1(\Omega)) \end{aligned}$$

since by the regularity of weak solutions $\sqrt{c_i} u_i$ and $\nabla\sqrt{c_i}$ are in $L^2(0, T; L^2(\Omega))$ and the bound is uniform in δ.

Therefore, letting $\delta \to 0$ we obtain the entropy equality

$$\int_\Omega c_i(T)(\ln c_i(T) - 1)\mathrm{d}x - 2\int_0^T \int_\Omega \sqrt{c_i} u_i \cdot \nabla\sqrt{c_i} \mathrm{d}x\mathrm{d}t = \int_\Omega c_i^0(\ln c_i^0 - 1)\mathrm{d}x. \tag{5.64}$$

Finally, due to the linear system and the symmetry of D_{ij}, we have

$$\begin{aligned}
-\sum_{i=1}^{n} 2\sqrt{c_i}u_i \cdot \nabla\sqrt{c_i} &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\sqrt{c_j}u_j\sqrt{c_i}u_i \\
&= -\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{D_{ij}}c_iu_i \cdot c_ju_j + \sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{D_{ij}}c_jc_iu_i^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{c_ic_j}{D_{ij}}|u_i - u_j|^2.
\end{aligned}$$

Summing (5.64) over all $i \in \{1, \ldots, n\}$ and using the definition of the entropy functional, we arrive at (5.59) and the proof is complete.

□

Chapter 6

The Maxwell–Stefan–Fourier system

The dynamics of multicomponent gaseous mixtures with vanishing barycentric velocity and constant temperature can be described by the Maxwell–Stefan equations that were studied in chapter 5. In this chapter we deal with the nonisothermal analogue of the Maxwell–Stefan system, known as Maxwell–Stefan–Fourier. The existence of local–in–time solutions was shown in [47], while [9] investigated a special nonisothermal case. Here, we present our work [11] where we prove the existence of global–in–time weak solutions and the weak–strong uniqueness property for a rather general nonisothermal Maxwell–Stefan system. The novelty of our approach is the consistent thermodynamic modeling.

The evolution of the mass densities $\rho_i(x,t)$ of the ith gas component and the temperature $\theta(x,t)$ of the mixture is described by the mass and energy balances

$$\partial_t\rho_i + \operatorname{div} J_i = 0, \quad \partial_t(\rho e) + \operatorname{div} J_e = 0, \quad i = 1, \dots, n, \tag{6.1}$$

$$J_i = \rho_i u_i, \quad J_e = -\kappa(\theta)\nabla\theta + \sum_{j=1}^n (\rho_j e_j + p_j)u_j \quad \text{in } \Omega,\ t > 0, \tag{6.2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, J_i and J_e are the diffusion and energy fluxes, respectively, u_i are the diffusional velocities, $\rho = \sum_{i=1}^n \rho_i$ is the total mass density, p_i the partial pressure with the total pressure $p = \sum_{i=1}^n p_i$, $\rho_i e_i$ the partial internal energy $\rho_i e_i$ with the total energy $\rho e = \sum_{i=1}^n \rho_i e_i$, and $\kappa(\theta)$ is the heat conductivity. Equations (6.1)–(6.2) are supplemented with the boundary and initial conditions

$$J_i \cdot \nu = 0, \quad J_e \cdot \nu = \lambda(\theta - \theta_0) \quad \text{on } \partial\Omega,\ t > 0, \tag{6.3}$$

$$\rho_i(0) = \rho_i^0, \quad \theta(0) = \theta^0 \quad \text{in } \Omega,\ i = 1, \ldots, n, \tag{6.4}$$

where ν is the exterior unit normal vector to $\partial\Omega$, $\theta_0 > 0$ is the given background temperature, and $\lambda > 0$ is a relaxation constant. The boundary conditions mean that the gas components cannot leave the domain, while heat exchange through the boundary is possible and proportional to the difference between the gas and background temperatures. To close the model, we need to determine u_i, $\rho_i e_i$, and p_i.

The velocities u_i are computed from the constrained algebraic Maxwell–Stefan system

$$-\theta \sum_{j=1}^{n} b_{ij}\rho_i\rho_j(u_i - u_j) = d_i \quad \text{for } i = 1, \ldots, n, \quad \sum_{i=1}^{n} \rho_i u_i = 0, \tag{6.5}$$

where the constant coefficients $b_{ij} = b_{ji} > 0$ model the interaction between the ith and jth components. The driving force d_i is given by

$$d_i = \rho_i \theta \nabla \frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i)\nabla \frac{1}{\theta}, \quad i = 1, \ldots, n, \tag{6.6}$$

where μ_i is the chemical potential. The constraint

$$\nabla p = 0 \quad \text{in } \Omega,\ t > 0, \tag{6.7}$$

is needed in order for our system to be thermodynamically consistent. We refer to Section 6.1 for details.

The internal energies $\rho_i e_i$ and chemical potentials μ_i are determined from the Helmholtz free energy (see (6.17)), and the pressure is computed from the Gibbs–Duhem relation. As shown in Section 6.1, these quantities are explicitly given by

$$\begin{aligned} \mu_i &= \frac{\theta}{m_i}\log\frac{\rho_i}{m_i} - c_w\theta(\log\theta - 1), & \rho_i e_i &= c_w\rho_i\theta, \\ \rho_i\eta_i &= -\frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) + c_w\rho_i\log\theta, & p_i &= \frac{\rho_i\theta}{m_i}, \quad i = 1,\ldots,n, \end{aligned} \tag{6.8}$$

where $\rho_i\eta_i$ is the entropy density of the ith component and $c_w > 0$ is the heat capacity. Then the driving force d_i and energy flux J_e simplify to

$$d_i = \frac{\nabla(\rho_i\theta)}{m_i}, \quad J_e = -\kappa\nabla\theta + \theta\sum_{i=1}^n \frac{\rho_i u_i}{m_i}. \tag{6.9}$$

The matrix M associated to the algebraic system (6.5) is singular (since $\sum_{i=1}^n d_i = 0$) and thus not positive definite. However, we recall in Section 6.1.1 that it is positive definite on the subspace $L = \{\boldsymbol{y} = (y_1,\ldots,y_n) \in \mathbb{R}^n : \sqrt{\boldsymbol{\rho}}\cdot\boldsymbol{y} = 0\}$ (here, $\sqrt{\boldsymbol{\rho}}$ is the vector with components $\sqrt{\rho_i}$). Therefore, the Bott–Duffin inverse of M, denoted by $M^{BD} = M^{BD}(\boldsymbol{\rho})$, exists and is symmetric and positive definite on L. Moreover, we show in Section 6.1.3 below that the fluxes can be expressed as a linear combination of the entropy variables (or thermochemical potentials) $\boldsymbol{\mu}/\theta = (\mu_1/\theta,\ldots,\mu_n/\theta)$ and $-1/\theta$,

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -Q(\boldsymbol{\rho},\theta)\begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}, \quad \text{where } Q(\boldsymbol{\rho},\theta) = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix}, \tag{6.10}$$

and $A = (A_{ij}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} = (B_i) \in \mathbb{R}^n$, $a > 0$ are given by

$$A_{ij}(\boldsymbol{\rho}) = M_{ij}^{BD}\sqrt{\rho_i\rho_j}, \quad B_i(\boldsymbol{\rho},\theta) = \theta\sum_{j=1}^n \frac{A_{ij}}{m_j}, \quad a(\boldsymbol{\rho},\theta) = \theta^2\left(\kappa + \sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j}\right). \tag{6.11}$$

Here, variables in bold font are n–dimensional vectors. The Onsager matrix Q turns out to be positive semidefinite (see (6.34)), which reveals the parabolic structure of equations (6.1)–(6.2).

The isothermal Maxwell–Stefan equations can be derived from the multi-species Boltzmann equations in the diffusive approximation [48]. The high–friction limit in Euler (–Korteweg) equations reveals a formal gradient–flow form of the Maxwell–Stefan equations [25], leading to Fick–Onsager diffusion fluxes instead of (6.5). In fact, it is shown in [49] that the Fick–Onsager and generalized Maxwell–Stefan approaches are equivalent. A formal Chapman–Enskog expansion of the stationary nonisothermal model was given in [50]. Another nonisothermal Maxwell–Stefan system was derived in [51], but with a different energy flux than ours.

Maxwell–Stefan systems with nonvanishing barycentric velocities can be formulated in the framework of hyperbolic–parabolic systems, which allows one to perform a local–in–time existence analysis [34]. Global–in–time regular solutions around the constant equilibrium state were found to exist in [52]. An existence analysis for Maxwell–Stefan systems coupled to the Navier–Stokes equations for the barycentric velocity can be found in [53] for the incompressible case and in [13] for the compressible situation. For steady–state problems, we refer to, e.g., [21, 18].

Concerning the nonisothermal equations, we refer to [9], where an existence analysis for global–in–time weak solutions was presented. However, this model has some modeling deficiencies explained below. Therefore, our aim is to prove (i) global existence for a thermodynamically consistent nonisothermal model and (ii) that strong solutions are unique in the class of weak solutions, which is known as the weak–strong uniqueness property.

Let us detail the main differences of our work compared to [9]:

(i) The most important difference is the lack of validity of the Onsager reciprocity relations in the model of [9]. The relations imply the symmetry of the coefficients of the Onsager matrix; see (6.10). The choice in [9] leads to a cancelation in the entropy inequality, thus simplifying the estimation. Our results do not rely on this simplification; see Remark 30 for further details.

(ii) The constraint (6.7) on the pressure is not taken into account in [9]. This condition is not necessary mathematically, but its lack creates an inconsistency with the assumption of vanishing barycentric velocity. Indeed, a difference in pressure induces a force difference, which can result in an acceleration according to Newton's second law, if there is no additional force to balance it.

(iii) According to Onsager's reciprocity relations, the Onsager matrix Q in (6.10) has to be positive semidefinite. We show that Q is in fact positive definite on the subspace $L = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} \cdot \sqrt{\boldsymbol{\rho}} = 0\}$. In [9], is is *assumed* that this subspace equals $\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} \cdot \mathbf{1} = 0\}$. This is not consistent with the thermodynamic modeling.

(iv) We consider different molar masses m_i, while they are assumed to be the same in [9]. When we assume equal molar masses, the cross–terms cancel, and we end up with the simple heat flux $J_e = -\kappa\nabla\theta$ (see (6.9) and the constraint in (6.5)), thus decoupling the equations.

We impose the following assumptions:

(A1) Domain: $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary, and $T > 0$. We set $\Omega_T = \Omega \times (0, T)$ and $\mathbb{R}_+ = [0, \infty)$.

(A2) Data: $\rho_i^0 \in L^\infty(\Omega)$ satisfies $\rho_i^0 \geq 0$ in Ω and $0 < \rho_* \leq \sum_{i=1}^n \rho_i^0 \leq \rho^*$ in Ω for some $\rho_*, \rho^* > 0$ and for all $i = 1, \ldots, n$; $\theta^0 \in L^\infty(\Omega)$ satisfies $\inf_\Omega \theta^0 > 0$.

(A3) Coefficients: $b_{ij} = b_{ji} > 0$ for all $i, j = 1, \ldots, n$.

(A4) Heat conductivity: $\kappa \in C^0(\mathbb{R}_+^n \times \mathbb{R}_+)$ satisfies $c_\kappa(1+\theta^2) \leq \kappa(\theta) \leq C_\kappa(1+\theta^2)$ for some $c_\kappa, C_\kappa > 0$ and all $(\boldsymbol{\rho}, \theta) \in \mathbb{R}_+^n \times \mathbb{R}_+$.

The lower bound for the total mass density ρ is needed to derive uniform estimates for the temperature. The proof of Lemma 10 in [10] shows that $M_{ij}^{BD}(\boldsymbol{\rho})$ is bounded for all $\boldsymbol{\rho} \in \mathbb{R}_+^n$. The growth condition for the heat conductivity is used to derive higher integrability bounds for the temperature, which are needed to derive a uniform estimate for the discrete time derivative of the temperature. We may also assume reaction terms R_i in (6.1) with the properties that the total reaction rate $\sum_{i=1}^n R_i$ vanishes and the vector of reaction rates R_i is derived from a convex, nonnegative potential [54, Section 2.2].

The first result we present is the existence of weak solutions.

Theorem 25. *Let Assumptions (A1)–(A4) hold. Then there exists a weak solution to* (6.1)–(6.8) *satisfying* $\rho_i > 0$, $\theta > 0$ *a.e. in* $\Omega_T = \Omega \times (0,T)$ *and*

$$\sqrt{\rho_i} \in L^\infty(\Omega_T) \cap C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \quad \partial_t \rho_i \in L^2(0,T; H^1(\Omega)^*),$$
$$\theta \in C^0_w([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \quad \partial_t(\rho\theta) \in L^{16/11}(0,T; W^{1,16/11}(\Omega)^*),$$
$$\theta^2,\ \log\theta \in L^2(0,T; H^1(\Omega)), \quad i = 1, \dots, n,$$

the weak formulation

$$\int_0^T \langle \partial_t \rho_i, \phi_i \rangle_{H^1(\Omega)^*} \mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=}^n M_{ij}^{BD} \big(2\nabla\sqrt{\rho_j} + \rho_j \nabla \log\theta\big) \mathrm{d}x\mathrm{d}t = 0,$$
$$\int_0^T \int_\Omega \langle \partial_t(\rho\theta), \phi_0 \rangle_{W^{1,16/5}(\Omega)^*} \mathrm{d}t + \int_0^T \int_\Omega \kappa \nabla\theta \cdot \nabla\phi_0 \mathrm{d}x\mathrm{d}t - \lambda \int_0^T \int_{\partial\Omega} (\theta_0 - \theta)\phi_0 \mathrm{d}s\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \sum_{i,j=1}^n \frac{\theta M_{ij}^{BD}}{m_i m_j} \sqrt{\rho_i} \big(2\nabla\sqrt{\rho_j} + \sqrt{\rho_j}\nabla \log\theta\big) \cdot \nabla\phi_0 \mathrm{d}x\mathrm{d}t = 0$$

holds for all $\phi_1, \dots, \phi_n \in L^2(0,T; H^1(\Omega))$ *and* $\phi_0 \in L^{16/5}(0,T; W^{1,16/5}(\Omega)^*)$, *and the initial conditions* (6.4) *are satisfied in the sense* $\rho_i(0) = \rho_i^0$ *in* $L^2(\Omega)$ *and* $\theta(0) = \theta^0$ *weakly in* $L^2(\Omega)$.

The idea of the proof is to apply the boundedness–by–entropy method, which automatically yields $L^\infty(\Omega_T)$ bounds [55]. More precisely, we formulate system (6.1)–(6.2) in terms of the relative entropy variables $(\mu_i - \mu_n)/\theta$ for $i = 1, \dots, n-1$ and $\log\theta$. We show in Lemma 27 that this defines the mass densities and temperature uniquely as a function of $(w_1, \dots, w_{n-1}, w)$. We introduce the mathematical entropy density

$$h(\boldsymbol{\rho}', \theta) = \sum_{i=1}^{n} \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w\rho\log\theta,$$

where the nth partial mass density is computed from $\rho_n = \rho - \sum_{i=1}^{n-1}\rho_i$, i.e., h depends on $\boldsymbol{\rho}' = (\rho_1, \ldots, \rho_{n-1})$ and θ. Gradient estimates for $(\boldsymbol{\rho}, \theta)$ are first derived from the entropy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h(\boldsymbol{\rho}', \theta)\mathrm{d}x + \int_\Omega \frac{\kappa}{\theta^2}|\nabla\theta|^2\mathrm{d}x + \sum_{i,j=1}^{n}\int_\Omega M_{ij}^{BD}\frac{d_i}{\theta\sqrt{\rho_i}}\frac{d_j}{\theta\sqrt{\rho_j}}\mathrm{d}x = 0,$$

which becomes an inequality for weak solutions. Second, as in [9], the energy balance equation (6.2) yields a bound for θ^2 in $L^2(0,T;H^1(\Omega))$. As mentioned before, the derivation of the entropy inequality differs from that one in [9], because the cross–term

$$I_5 = 2\int_\Omega \sum_{i=1}^{n-1}\frac{B_i}{\theta}\nabla\frac{\mu_i - \mu_n}{\theta}\cdot\nabla\log\theta\mathrm{d}x,$$

which cancels out in [9], needs to be controlled. (We recall definition (6.11) of B_i.) This is done by observing that the sum $I_4 + I_5 + I_8$ (see (6.41)) is nonnegative,

$$I_4 + I_5 + I_8 = \int_\Omega \sum_{i,j=1}^{n} A_{ij}\nabla\left(\frac{\mu_i}{\theta} + \frac{1}{m_i}\log\theta\right)\cdot\nabla\left(\frac{\mu_j}{\theta} + \frac{1}{m_j}\log\theta\right)\mathrm{d}x \geq 0,$$

as (A_{ij}) is positive semidefinite due to (6.34).

From a technical viewpoint, we approximate equations (6.1)–(6.2) by replacing the time derivative by the implicit Euler discretization to avoid issues with the time regularity and by adding a higher–order regularization to achieve $H^2(\Omega)$ and hence $L^\infty(\Omega)$ regularity for the entropy variables. The approximation is chosen in such a way that a discrete entropy inequality can be derived, yielding uniform estimates for both the compactness of the fixed-point operator (to obtain a solution to the approximate problem) and the de–regularization limit (to obtain a solution to the original problem).

Our second result concerns the weak–strong uniqueness property.

Theorem 26. *Let the assumptions of Theorem 25 hold, let* $\lambda = 0$ *in* (6.3)*, let* $(\boldsymbol{\rho}, \theta)$ *be a weak solution and* $(\bar{\boldsymbol{\rho}}, \bar{\theta})$ *be a strong solution to* (6.1)*–*(6.8)*. We assume that there exist* $m, M > 0$ *such that*

$$0 < \rho_i \le \rho^*, \quad 0 < \theta \le M, \quad 0 < \bar{\rho}_i \le \rho^*, \quad 0 < m \le \bar{\theta} \le M \quad \text{in } \Omega_T.$$

Furthermore, we suppose that $\bar{u}_i$, $|\nabla \log \bar{\theta}| \in L^\infty(\Omega_T)$ *for* $i = 1, \ldots, n$ *and that the thermal conductivity* κ *is Lipschitz continuous. If the initial data of* $(\boldsymbol{\rho}, \theta)$ *and* $(\bar{\boldsymbol{\rho}}, \bar{\theta})$ *coincide then* $\boldsymbol{\rho}(x,t) = \bar{\boldsymbol{\rho}}(x,t)$ *and* $\theta(x,t) = \bar{\theta}(x,t)$ *for a.e.* $x \in \Omega$ *and all* $t > 0$.

By a strong solution, we understand a solution that has sufficient regularity to satisfy the entropy equality stated in Lemma 38; see Section 6.3. Observe that we require the boundedness of the temperature θ, which is not proved in Theorem 25. The proof of Theorem 26 is based on the relative entropy, defined by

$$\begin{aligned} H(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta}) &= \int_\Omega h(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta}) \mathrm{d}x \\ &= \int_\Omega \bigg\{ \sum_{i=1}^n \frac{1}{m_i} \Big(\rho_i \log \frac{\rho_i}{\bar{\rho}_i} - (\rho_i - \bar{\rho}_i) \Big) - c_w \rho \Big(\log \frac{\theta}{\bar{\theta}} - (\theta - \bar{\theta}) \Big) \bigg\} \mathrm{d}x, \end{aligned} \tag{6.12}$$

where

$$h(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta}) = h(\boldsymbol{\rho}, \theta) - h(\bar{\boldsymbol{\rho}}, \bar{\theta}) - \sum_{i=1}^n \frac{\partial h}{\partial \rho_i}(\bar{\boldsymbol{\rho}}, \bar{\theta})(\rho_i - \bar{\rho}_i) - \frac{\partial h}{\partial E}(\bar{\boldsymbol{\rho}}, \bar{\theta})(E - \bar{E})$$

and $E = c_w \rho \theta$ and $\bar{E} = c_w \rho \bar{\theta}$ are the internal energy densities. The idea is to compute the time derivative:

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t}(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta}) + c \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x + c \int_\Omega |\nabla(\log \theta - \log \bar{\theta})|^2 \mathrm{d}x \\ \le C \int_\Omega \Big(\sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 + (\theta - \bar{\theta})^2 \Big) \mathrm{d}x, \end{aligned}$$

where $c > 0$ is some constant and $C > 0$ depends on the $L^\infty(\Omega_T)$ norms of θ, $\bar{u}_i$, and $\nabla \log \bar{\theta}$, $i = 1, \dots, n$. The difficulty is to estimate the expressions arising from the time derivative of the relative entropy in such a way that only $\bar{u}_i$ and $\bar{\theta}$ need to be bounded. Thanks to the positive lower bound for $\bar{\theta}$, we can bound the right–hand side in terms of the relative entropy,

$$\int_\Omega \left(\sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 + (\theta - \bar{\theta})^2 \right) \mathrm{d}x \le \int_\Omega H(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta}) \mathrm{d}x.$$

Then Gronwall's lemma shows that $H((\boldsymbol{\rho}, \theta)(t) | (\bar{\boldsymbol{\rho}}, \bar{\theta})(t)) = 0$ for $t > 0$ and hence $(\boldsymbol{\rho}, \theta)(t) = (\bar{\boldsymbol{\rho}}, \bar{\theta})(t)$. Compared to [10], we include the temperature terms and combine them with the entropy variables w_i in such a way that the positive semidefiniteness of M^{BD} can be exploited.

6.1 Thermodynamic structure

We consider the following system of equations modeling the dynamics of a non-isothermal gas mixture of n components with mass diffusion and heat conduction:

$$\partial_t \rho_i + \operatorname{div}(\rho_i(v + u_i)) = 0, \quad i = 1, \dots, n, \tag{6.13}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \rho b - \nabla p, \tag{6.14}$$

$$\begin{aligned} \partial_t \left(\rho e + \frac{1}{2} \rho |v|^2 \right) + \operatorname{div} \left(\left(\rho e + \frac{1}{2} \rho |v|^2 \right) v \right) &= \operatorname{div}(\kappa \nabla \theta) \\ - \operatorname{div} \sum_{j=1}^n (\rho_i e_i + p_i) u_i - \operatorname{div}(pv) + \rho r + \rho b \cdot v &+ \sum_{i=1}^n \rho_i b_i \cdot u_i. \end{aligned} \tag{6.15}$$

Besides of the variables introduced in the introduction, v denotes the barycentric velocity of the mixture. The quantities $\rho_i b_i$ are the body forces, where $\rho b = \sum_{i=1}^n \rho_i b_i$ is the total force exerted on the mixture, and ρr is the total heat supply due to radiation. The diffusional velocities u_i are determined by solving the linear system (6.5) with driving forces given by (6.16), while the partial internal energy densities $\rho_i e_i$, and the partial pressures p_i are determined from the free energy; see below.

Equations (6.13)–(6.15) correspond to a so-called class–I model. They can be derived either via an entropy invariant model reduction [6] or in the high-friction limit [22] from a class–II model, in which each component has its own velocity v_i. Equations (6.13) are the partial mass balances, (6.14) is the momentum balance, and (6.15) the energy balance. As proved in [22], system (6.13)–(6.15) and (6.5) fits into the general theory of hyperbolic–parabolic composite-type systems introduced in [56] and further explored in [57].

As mentioned in the introduction, system (6.1)–(6.2) and (6.7) is supplemented by the constrained Maxwell–Stefan system (6.5) for the velocities u_i. These equations can be derived from a class-II model in the diffusion approximation [6, Section 14, (210)] or in the high-friction limit [22, Section 2, (2.50)] with the driving forces

$$d_i = -\frac{\rho_i}{\rho}\nabla p + \rho_i\theta\nabla\frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i)\nabla\frac{1}{\theta} + \rho_i(b - b_i), \tag{6.16}$$

where μ_i is the chemical potential of the ith component. Since the pressure is uniform in space, $\nabla p = 0$, and we have neglected external forces, the driving force becomes (6.6). Then equations (6.1)–(6.2) and (6.7) are obtained by setting $v = 0$ and $r = b_i = 0$.

The internal energy densities $\rho_i e_i$, partial pressures p_i, and the chemical potential μ_i are determined from the Helmholtz free energy. We assume that the gas is a simple mixture, which implies that these quantities can be calculated from the partial free energy densities $\psi_i(\rho_i, \theta)$, $i = 1, \ldots, n$. We have

$$\mu_i = \frac{\partial\psi_i}{\partial\rho_i}, \quad \rho_i\eta_i = -\frac{\partial\psi_i}{\partial\theta}, \quad \rho_i e_i = \psi_i + \theta\rho_i\eta_i, \quad p_i = \rho_i\mu_i - \psi_i,$$

where $\rho_i\eta_i$ is the entropy density of the ith component and the equation for p_i is called the Gibbs–Duhem relation. Defining the partial Helmholtz free energy as

$$\psi_i = \theta\frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w\rho\theta(\log\theta - 1), \quad i = 1, \ldots, n, \tag{6.17}$$

the thermodynamic quantities are given by (6.8). Moreover, the driving force d_i and enthalpy $h_i := \rho_i e_i + p_i$ read as

$$d_i = \frac{\nabla(\rho_i\theta)}{m_i}, \quad h_i = \left(c_w + \frac{1}{m_i}\right)\rho_i\theta, \quad i = 1, \dots, n. \tag{6.18}$$

This corresponds to equations (6.9).

6.1.1 Inversion of the Maxwell–Stefan system

We discuss the inversion of the Maxwell–Stefan system (6.5) following [22] and [10, Section 2]. We write (6.5) equivalently as

$$-\theta\sqrt{\rho_i}\sum_{j=1}^{n} M_{ij}\sqrt{\rho_j}u_j = d_i, \quad i = 1, \dots, n, \tag{6.19}$$

where the matrix $M(\boldsymbol{\rho}) = (M_{ij}) \in \mathbb{R}^{n\times n}$ is given by

$$M_{ij} = \begin{cases} \sum_{k=1,\,k\neq i}^{n} b_{ik}\rho_k & \text{if } i = j, \\ -b_{ij}\sqrt{\rho_i\rho_j} & \text{if } i \neq j. \end{cases} \tag{6.20}$$

We wish to invert $M\boldsymbol{v} = \boldsymbol{w}$, where $v_i = \sqrt{\rho_i}u_i$ and $w_i = -d_i/(\theta\sqrt{\rho_i})$. Since (b_{ij}) is symmetric, $0 = (M\boldsymbol{v})_i = \sum_{i\neq j} b_{ij}\sqrt{\rho_j}(\sqrt{\rho_j}v_i - \sqrt{\rho_i}v_j)$ shows that the kernel of M consists of span$\{\sqrt{\boldsymbol{\rho}}\}$. Thus, we can invert M only on the subspace $L = \{\boldsymbol{y} \in \mathbb{R}^n : \sqrt{\boldsymbol{\rho}}\cdot\boldsymbol{y} = 0\}$. We define the projections P_L on L and $P_{L^\perp}$ on $L^\perp$ by

$$(P_L)_{ij} = \delta_{ij} - \rho^{-1}\sqrt{\rho_i\rho_j}, \quad (P_{L^\perp})_{ij} = \rho^{-1}\sqrt{\rho_i\rho_j} \quad \text{for } i, j = 1, \dots, n,$$

where δ_{ij} is the Kronecker symbol. The matrix $M = (M_{ij})$ is positive definite on L [10, Lemma 4]:

$$\boldsymbol{z}^T M\boldsymbol{z} \geq \mu_M|P_L\boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{6.21}$$

where $\mu_M = \min_{i\neq j} b_{ij} > 0$. Since the matrix $MP_L + P_{L^\perp}$ is invertible [10, Lemma 4], we can define the Bott–Duffin inverse of M with respect to L as

$M^{BD} = P_L(MP_L + P_{L^\perp})^{-1}$. Hence, we can invert (6.19) by

$$\sqrt{\rho_i}u_i = -\sum_{j=1}^n M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}, \quad i = 1, \ldots, n. \tag{6.22}$$

The matrix $M^{BD} = M^{BD}(\boldsymbol{\rho})$ is symmetric and positive definite on L [10, Lemma 4],

$$\boldsymbol{z}^T M^{BD}\boldsymbol{z} \geq \mu|P_L\boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{6.23}$$

where $\mu = (2\sum_{i\neq j}(b_{ij} + 1))^{-1}$.

6.1.2 Entropy variables

The mathematical analysis becomes easier when formulating the system in terms of the so-called entropy variables. To this end, we introduce the mathematical entropy density

$$h = -\sum_{i=1}^n \rho_i\eta_i = \sum_{i=1}^n \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w\rho\log\theta, \tag{6.24}$$

which is the negative of the physical (total) entropy density (6.8). Summing the mass balances (6.1) over $i = 1, \ldots, n$ and using the constraint $\sum_{i=1}^n \rho_i u_i = 0$ from (6.5), we obtain $\partial_t\rho = 0$. Thus, the total density is determined by the initial total density, $\rho(x, t) = \sum_{i=1}^n \rho_i^0(x)$ for $x \in \Omega$, and is independent of time. This suggests to compute only the first $n-1$ mass densities, since the last one can be determined by $\rho_n = \rho - \sum_{i=1}^{n-1}\rho_i$. Then we interpret the entropy density h as a function of $(\boldsymbol{\rho}', \theta) := (\rho_1, \ldots, \rho_{n-1}, \theta)$:

$$h(\boldsymbol{\rho}', \theta) = \sum_{i=1}^{n-1}\frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) + \frac{\rho_n}{m_n}\left(\log\frac{\rho_n}{m_n} - 1\right) - c_w\rho\log\theta$$

with the partial derivatives

$$\frac{\partial h}{\partial\rho_i} = \frac{1}{m_i}\log\frac{\rho_i}{m_i} - \frac{1}{m_n}\log\frac{\rho_n}{m_n}, \quad i = 1, \ldots, n-1, \quad \frac{\partial h}{\partial\theta} = -c_w\frac{\rho}{\theta}.$$

The Hessian matrix

$$\mathrm{D}^2 h = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0}^T & c_w\rho/\theta^2 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \text{where } R_{ij} = \frac{\delta_{ij}}{m_i\rho_i} + \frac{1}{m_n\rho_n},$$

is positive definite, showing that the entropy is convex.

According to thermodynamics [6], there exists another set of variables, known as entropy variables and are $(\mu_1/\theta, \ldots, \mu_n/\theta, -1/\theta)$. We set

$$q_i = \frac{\mu_i}{\theta} = \frac{1}{m_i}\log\frac{\rho_i}{m_i} - c_w(\log\theta - 1) \quad \text{for } i = 1, \ldots, n. \tag{6.25}$$

Since the nth partial density is determined by the densities $\rho_1, \ldots, \rho_{n-1}$, we prefer to work with the relative entropy variables

$$w_i = q_i - q_n = \frac{\mu_i - \mu_n}{\theta} = \frac{\partial h}{\partial \rho_i}, \quad i = 1, \ldots, n-1. \tag{6.26}$$

Setting additionally $w = \log\theta$, our new set of variables is $(w_1, \ldots, w_{n-1}, w)$. The following lemma states that the mapping $(\rho_1, \ldots, \rho_n, \theta) \mapsto (w_1, \ldots, w_{n-1}, w)$ is invertible.

Lemma 27. *Let $(w_1, \ldots, w_{n-1}, w) \in \mathbb{R}^n$ and $\rho > 0$ be given. Then there there exists a unique $(\rho_1, \ldots, \rho_n, \theta) \in \mathbb{R}_+^{n+1}$ with $\rho_i > 0$ for $i = 1, \ldots, n$ satisfying $\sum_{i=1}^n \rho_i = \rho$, $w_i = \partial h/\partial\rho_i$ for $i = 1, \ldots, n-1$, and $w = \log\theta$.*

Proof. The proof is similar to [53, Lemma 6] with some small changes. Given $w \in \mathbb{R}$, the temperature equals $\theta = \exp(w) > 0$. The function

$$f(s) = \sum_{i=1}^{n-1} m_i e^{m_i w_i}\left(\frac{\rho - s}{m_n}\right)^{m_i/m_n} \quad \text{for } s \in [0, \rho],$$

is strictly decreasing and $0 = f(\rho) < f(s) < f(0)$ for $s \in (0, \rho)$. By continuity, there exists a unique fixed point $s_0 \in (0, \rho)$. Then $\rho_i := m_i \exp(m_i w_i)((\rho - s_0)/m_n)^{m_i/m_n}$ for $i = 1, \ldots, n$ satisfies $\rho_i > 0$ and $\sum_{i=1}^{n-1} \rho_i = f(s_0) = s_0 < \rho$. Consequently, $\rho_n := \rho - \sum_{i=1}^{n-1} \rho_i = \rho - s_0 > 0$ and $\rho_i/m_i = \exp(m_i w_i)(\rho_n/m_n)^{m_i/m_n}$ is equivalent to

$$w_i = \frac{1}{m_i} \log \frac{\rho_i}{m_i} - \frac{1}{m_n} \log \frac{\rho_n}{m_n} = \frac{\partial h}{\partial \rho_i}$$

for $i = 1, \dots, n-1$, which finishes the proof. □

6.1.3 Formulation of the fluxes and parabolicity

We can compute the fluxes as a linear combination of $\nabla(w_1, \dots, w_{n-1}, w)$ or $\nabla(q_1, \dots, q_n, -1/\theta)$.

Lemma 28. *It holds for $i = 1, \dots, n$ that*

$$J_i = -\sum_{j=1}^{n-1} A_{ij} \nabla w_j - \frac{B_i}{\theta} \nabla w = -\sum_{j=1}^{n} A_{ij} \nabla q_j - B_i \nabla\left(-\frac{1}{\theta}\right), \tag{6.27}$$

$$\begin{aligned} J_e &= -\kappa\theta\nabla w - \sum_{j=1}^{n-1} B_j \nabla w_j - \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \nabla w \\ &= -\sum_{j=1}^{n} B_j \nabla q_j - \theta^2 \left(\kappa + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j}\right) \nabla\left(-\frac{1}{\theta}\right), \end{aligned} \tag{6.28}$$

where the coefficients

$$A_{ij} = M_{ij}^{BD} \sqrt{\rho_i \rho_j}, \quad B_i = \theta \sum_{j=1}^{n} A_{ij} \left(c_w + \frac{1}{m_j}\right) = \theta \sum_{j=1}^{n} \frac{A_{ij}}{m_j} \tag{6.29}$$

for $i, j = 1, \dots, n$ depend on $(\boldsymbol{\rho}, \theta)$ and satisfy the relations

$$\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = \sum_{i=1}^{n} B_i = 0. \tag{6.30}$$

Proof. We wish to express the driving force $d_j = \nabla(\rho_j \theta)/m_j$ from (6.9) in terms of $\nabla q_j = \nabla \log \rho_j/m_j - c_w \nabla \log \theta$. A computation, using $w = \log \theta$, yields

$$d_j = \rho_j \theta \nabla q_j + \rho_j \theta \left(c_w + \frac{1}{m_j}\right) \nabla w. \tag{6.31}$$

Therefore, by (6.22), for $i = 1, \dots, n$,

$$J_i = \rho_i u_i = -\sqrt{\rho_i} \sum_{j=}^{n} M_{ij}^{BD} \frac{d_j}{\theta\sqrt{\rho_j}} = -\sum_{j=1}^{n} M_{ij}^{BD} \sqrt{\rho_i\rho_j} \left\{ \nabla q_j + \left(c_w + \frac{1}{m_j} \right) \nabla w \right\}$$
$$= -\sum_{j=1}^{n} A_{ij} \nabla q_j - \sum_{j=1}^{n} A_{ij} \left(c_w + \frac{1}{m_j} \right) \nabla \log\theta = -\sum_{j=1}^{n} A_{ij} \nabla q_j - \frac{B_i}{\theta} \nabla \log\theta.$$

This shows the second relation in (6.27). The first relation then follows from (6.30) (which is proved below), since, using $q_j = w_j + q_n$ for $j = 1, \dots, n-1$ (see (6.26)),

$$\sum_{j=1}^{n} A_{ij} \nabla q_j = \sum_{j=1}^{n-1} A_{ij} (\nabla w_j + \nabla q_n) + A_{in} \nabla q_n = \sum_{j=1}^{n-1} A_{ij} \nabla w_j. \tag{6.32}$$

Next, we compute the energy flux defined in (6.2). We use (6.18), (6.22), and (6.31):

$$\begin{aligned} J_e &= -\kappa\theta\nabla w + \sum_{i=1}^{n} \sqrt{\rho_i}\theta \left(c_w + \frac{1}{m_i} \right) \sqrt{\rho_i} u_i \\ &= -\kappa\theta\nabla w - \theta \sum_{i,j=1}^{n} \sqrt{\rho_i} \left(c_w + \frac{1}{m_i} \right) M_{ij}^{BD} \frac{d_j}{\theta\sqrt{\rho_j}} \\ &= -\kappa\theta\nabla w - \theta \sum_{i,j=1}^{n} \left(c_w + \frac{1}{m_i} \right) M_{ij}^{BD} \sqrt{\rho_i\rho_j} \left\{ \nabla q_j + \left(c_w + \frac{1}{m_j} \right) \nabla w \right\} \\ &= -\kappa\theta\nabla w - \sum_{j=1}^{n} B_j \nabla q_j - \theta \sum_{i,j=1}^{n} A_{ij} \left(c_w + \frac{1}{m_i} \right) \left(c_w + \frac{1}{m_j} \right) \nabla w \\ &= -\kappa\theta\nabla w - \sum_{j=1}^{n} B_j \nabla q_j - \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \nabla w, \end{aligned}$$

where the last equation follows from (6.30). Moreover, because of

$$\sum_{j=1}^{n} B_j \nabla q_j = \sum_{j=1}^{n-1} B_j \nabla (w_j + q_n) + B_n \nabla q_n = \sum_{j=1}^{n-1} B_j \nabla w_j, \tag{6.33}$$

we have proved (6.28).

It remains to verify (6.30). We recall the property $P_L(MP_L + P_{L^\perp})^{-1} P_{L^\perp} = 0$ from [58, Lemma 2], which implies that $M^{BD} P_{L^\perp} = 0$. Hence, $L^\perp \subset \ker M^{BD}$

and since $L^\perp = \text{span}\{\sqrt{\boldsymbol{\rho}}\}$, we conclude that $\sum_{j=1}^n M_{ij}^{BD}\sqrt{\rho_j} = 0$. This shows that, by the definition of A_{ij},

$$\sum_{j=1}^n A_{ij} = \sqrt{\rho_i}\sum_{j=1}^n M_{ij}^{BD}\sqrt{\rho_j} = 0.$$

The symmetry of (A_{ij}) immediately gives $\sum_{i=1}^n A_{ij} = 0$. Finally, by the definition of B_i,

$$\sum_{i=1}^n B_i = \theta\sum_{i,j=1}^n M_{ij}^{BD}\sqrt{\rho_i\rho_j}\left(c_w + \frac{1}{m_j}\right) = \theta\sum_{j=1}^n\left(c_w + \frac{1}{m_j}\right)\sum_{i=1}^n A_{ij} = 0.$$

This finishes the proof. □

The previous proof shows that we can formulate the diffusion fluxes in different ways.

Corollary 29. *It holds for $i = 1, \ldots, n$ that*

$$J_i = \rho_i u_i = -\sum_{j=1}^n A_{ij}\nabla\left(q_j + \frac{w}{m_j}\right) = -\sqrt{\rho_i}\sum_{j=1}^n M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}.$$

We claim that the Onsager matrix $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ in (6.10) is positive semidefinite. Let $a = \theta(\kappa + \sum_{i,j=1}^n A_{ij}/(m_i m_j))$. We compute for $\xi \in \mathbb{R}^{n+1}$:

$$\begin{aligned} \xi^T Q\xi &= \sum_{i,j=1}^n A_{ij}\xi_i\xi_j + 2\sum_{i=1}^n B_i\xi_i\xi_{n+1} + a\xi_{n+1}^n \qquad (6.34)\\ &= \sum_{i,j=1}^n A_{ij}\xi_i\xi_j + 2\theta\sum_{i,j=1}^n \frac{A_{ij}}{m_j}\xi_i\xi_{n+1} + \theta^2\left(\kappa + \sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j}\right)\xi_{n+1}^2 \\ &= \sum_{i,j=1}^n A_{ij}\left(\xi_i + \frac{\theta\xi_{n+1}}{m_i}\right)\left(\xi_j + \frac{\theta\xi_{n+1}}{m_j}\right) + \kappa\theta^2\xi_{n+1}^2 \geq 0, \end{aligned}$$

where the nonnegativity follows from the positive semidefiniteness (6.23) of M^{BD}. This reveals the parabolicity of our system in terms of the entropy variables.

6.2 Existence of weak solutions

In this section we prove Theorem 25. The previous subsection shows that we can write our system as the mass and energy balances (6.1)–(6.2) with the fluxes (6.27)–(6.28). The weak formulation in the relative entropy variables (6.26) reads as

$$\int_0^T \langle \partial_t \rho_i, \phi_i \rangle \mathrm{d}t + \int_0^T \int_\Omega \left(\sum_{j=1}^{n-1} A_{ij} \nabla w_j + e^{-w} B_i \nabla w \right) \cdot \nabla \phi_i \mathrm{d}x\mathrm{d}t = 0, \tag{6.35}$$

$$\int_0^T \langle \partial_t E, \phi_0 \rangle \mathrm{d}t + \int_0^T \int_\Omega e^w \left(\kappa + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \right) \nabla w \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t \tag{6.36}$$

$$+ \int_0^T \int_\Omega \sum_{j=1}^{n-1} B_j \nabla w_j \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t = \lambda \int_0^T \int_{\partial\Omega} (\theta_0 - \theta) \phi_0 \mathrm{d}s\mathrm{d}t$$

for test functions $\phi_1, \ldots, \phi_n \in L^2(0, T; H^1(\Omega))$ and $\phi_0 \in L^\infty(0, T; W^{1,\infty}(\Omega))$. According to (6.8), the energy is given by $E = c_w \rho\theta$. Moreover, ρ_i, A_{ij}, B_i, and E are interpreted as functions of $(w_1, \ldots, w_{n-1}, w)$.

The proof follows the lines of [9, Section 3], which is based on the boundedness–by–entropy method [55], but some details are different. We approximate equations (6.35)–(6.36) by replacing the time derivative by the implicit Euler scheme and adding a higher-order regularization in w_i. The existence of solutions to the approximate system is shown by means of the Leray–Schauder fixed–point theorem, where the compactness of the fixed–point operator is obtained by the approximate entropy inequality. This inequality yields estimates uniform in the regularization parameters, allowing for the de–regularization limit via the Aubin–Lions compactness lemma.

Let $\epsilon \in (0, 1)$, $N \in \mathbb{N}$, and $\tau = T/N$. We set $w_0 = \log \theta_0$ and $\boldsymbol{w} = (w_1, \ldots, w_{n-1}, w)$. Let $\bar{\boldsymbol{w}} = (\bar{w}_1, \ldots, \bar{w}_{n-1}, \bar{w}) \in L^\infty(\Omega; \mathbb{R}^n)$ be given. We define for test functions $\phi_i \in H^2(\Omega)$, $i = 0, \ldots, n-1$, the approximate scheme

$$
\begin{aligned}
0 &= \frac{1}{\tau}\int_\Omega \big(\rho_i(\boldsymbol{w}) - \rho_i(\bar{\boldsymbol{w}})\big)\phi_i \mathrm{d}x + \epsilon\int_\Omega \big(\mathrm{D}^2 w_i : \mathrm{D}^2\phi_i + w_i\phi_i\big)\mathrm{d}x \\
&\quad + \int_\Omega \bigg(\sum_{j=1}^{n-1} A_{ij}\nabla w_j + e^{-w}B_i\nabla w\bigg)\cdot\nabla\phi_i \mathrm{d}x
\end{aligned} \tag{6.37}
$$

$$
\begin{aligned}
0 &= \frac{1}{\tau}\int_\Omega \big(E(\boldsymbol{w}) - E(\bar{\boldsymbol{w}})\big)\phi_0 \mathrm{d}x + \int_\Omega \sum_{i=1}^{n-1} B_i\nabla w_i\cdot\nabla\phi_0 \mathrm{d}x \\
&\quad + \int_\Omega e^w\bigg(\kappa(e^w) + \sum_{i,j=1}^{n}\frac{A_{ij}}{m_i m_j}\bigg)\nabla w\cdot\nabla\phi_0 \mathrm{d}x - \lambda\int_{\partial\Omega}(e^{w_0} - e^w)\phi_0 \mathrm{d}s \\
&\quad + \epsilon\int_\Omega e^w\big(\mathrm{D}^2 w : \mathrm{D}^2\phi_0 + |\nabla w|^2\nabla w\cdot\nabla\phi_0\big)\mathrm{d}x \\
&\quad + \epsilon\int_\Omega (e^{w_0} + e^w)(w - w_0)\phi_0 \mathrm{d}x,
\end{aligned} \tag{6.38}
$$

where $\mathrm{D}^2 w_i$ is the Hesse matrix of w_i, the double point ":" denotes the Frobenius matrix product, we recall that $E(\boldsymbol{w}) = c_w\rho\theta$, and A_{ij} and B_i are interpreted as functions of $\boldsymbol{w}$. The higher-order regularization yields solutions $w_i, w \in H^2(\Omega)$, and the $W^{1,4}(\Omega)$ regularization allows us to estimate the higher-order terms when using the test function $e^{-w_0} - e^{-w}$ (see the estimate of I_{11} below). The lower-order regularization $(e^{w_0} - e^w)(w - w_0)$ provides an ϵ-dependent $L^2(\Omega)$ bound for w.

6.2.1 Solution of the linearized approximate problem

Let $\boldsymbol{w}^* \in W^{1,4}(\Omega;\mathbb{R}^n)$ and $\sigma \in [0,1]$. We want to find a solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to the linear problem

$$
a(\boldsymbol{w},\boldsymbol{\phi}) = \sigma F(\boldsymbol{\phi}) \quad \text{for } \boldsymbol{\phi} = (\phi_1,\ldots,\phi_{n-1},\phi_0) \in H^2(\Omega;\mathbb{R}^n), \tag{6.39}
$$

where

$$
\begin{aligned}
a(\boldsymbol{w},\boldsymbol{\phi}) &= \int_\Omega \kappa(e^{w^*})e^{w^*}\nabla w\cdot\nabla\phi_0 \mathrm{d}x + \epsilon\int_\Omega\sum_{i=1}^{n-1}\big(\mathrm{D}^2 w_i : \mathrm{D}^2\phi_i + w_i\phi_i\big)\mathrm{d}x \\
&\quad + \epsilon\int_\Omega (e^{w_0} + e^{w^*})w\phi_0 \mathrm{d}x + \epsilon\int_\Omega e^{w^*}\big(\mathrm{D}^2 w : \mathrm{D}^2\phi_0 + |\nabla w^*|^2\nabla w\cdot\nabla\phi_0\big)\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
F(\boldsymbol{\phi}) = &- \int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w}^*)\nabla w_j^* \cdot \nabla\phi_i \mathrm{d}x - \int_\Omega e^{w^*} \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w}^*)}{m_i m_j}\nabla w^* \cdot \nabla\phi_0 \mathrm{d}x \\
&- \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*) e^{-w^*}\nabla w^* \cdot \nabla\phi_i \mathrm{d}x - \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*)\nabla w_i^* \cdot \nabla\phi_0 \mathrm{d}x \\
&- \frac{1}{\tau}\int_\Omega \sum_{i=1}^{n-1}(\rho_i^* - \bar{\rho}_i)\phi_i \mathrm{d}x - \frac{1}{\tau}\int_\Omega (E^* - \bar{E})\phi_0 \mathrm{d}x + \lambda \int_{\partial\Omega}(e^{w_0} - e^{w^*})\phi_0 \mathrm{d}s \\
&+ \epsilon \int_\Omega (e^{w_0} + e^{w^*}) w_0 \phi_0 \mathrm{d}x,
\end{aligned}
$$

where we abbreviated $\rho_i^* = \rho_i(\boldsymbol{w}^*)$, $\bar{\rho}_i = \rho_i(\bar{\boldsymbol{w}})$, $E^* = c_w \rho e^{w^*}$, and $\bar{E} = c_w \rho e^{\bar{w}}$. The bilinear form a is clearly coercive on $H^2(\Omega;\mathbb{R}^n)$, and both a and F are continuous on this space. By the Lax–Milgram lemma, there exists a unique solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to (6.39).

6.2.2 Solution of the approximate problem

The solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to (6.39) defines the fixed-point operator

$$
S : W^{1,4}(\Omega;\mathbb{R}^n) \times [0,1] \to W^{1,4}(\Omega;\mathbb{R}^n), \quad S(\boldsymbol{w}^*, \sigma) = \boldsymbol{w}.
$$

Because of the compact embedding $H^2(\Omega;\mathbb{R}^n) \hookrightarrow W^{1,4}(\Omega;\mathbb{R}^n)$, the operator is continuous, compact, and it satisfies $S(\boldsymbol{w}^*, 0) = 0$ for all $\boldsymbol{w}^* \in W^{1,4}(\Omega;\mathbb{R}^n)$. It remains to find a uniform bound for all fixed points of $S(\cdot, \sigma)$. Let $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ be such a fixed point. Then $\boldsymbol{w}$ solves (6.39) with $\boldsymbol{w}^* = \boldsymbol{w}$. We choose the test functions $\phi_i = w_i$ for $i = 1, \ldots, n-1$ and $\phi_0 = e^{-w_0} - e^{-w}$ in (6.39):

$$\begin{aligned}
0 &= \frac{\sigma}{\tau}\int_\Omega \sum_{i=1}^{n-1}(\rho_i - \bar{\rho}_i)w_i \mathrm{d}x + \frac{\sigma}{\tau}\int_\Omega (E-\bar{E})(-e^{-w})\mathrm{d}x \\
&+ \frac{\sigma}{\tau}\int_\Omega (E-\bar{E})e^{-w_0}\mathrm{d}x + \sigma\int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w})\nabla w_i \cdot \nabla w_j \mathrm{d}x \\
&+ 2\sigma \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w})e^{-w}\nabla w_i \cdot \nabla w \mathrm{d}x + \int_\Omega \kappa(e^w)|\nabla w|^2 \mathrm{d}x \\
&+ \epsilon \int_\Omega \sum_{i=1}^{n-1}\big(|\mathrm{D}^2 w_i|^2 + w_i^2\big)\mathrm{d}x + \sigma\int_\Omega \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w})}{m_i m_j}|\nabla w|^2 \mathrm{d}x \qquad (6.40)\\
&- \sigma\lambda \int_{\partial\Omega} (e^{w_0} - e^w)(e^{-w_0} - e^{-w})\mathrm{d}s \\
&+ \epsilon \int_\Omega (e^{w_0} + e^w)(e^{-w_0} - e^{-w})(w - \sigma w_0)\mathrm{d}x \\
&+ \epsilon \int_\Omega \big(|\mathrm{D}^2 w|^2 - \mathrm{D}w : (\nabla w \otimes \nabla w) + |\nabla w|^4\big)\mathrm{d}x =: I_1 + \cdots + I_{11}.
\end{aligned}$$

We estimate the terms $I_1, \ldots, I_{11}$ step by step. First, by the convexity of the entropy and arguing similarly as in [9, Section 3, Step 2],

$$\begin{aligned}
I_1 + I_2 &= \frac{\sigma}{\tau}\int_\Omega \sum_{i=1}^{n-1}\bigg((\rho_i - \bar{\rho}_i)\frac{\partial h}{\partial \rho_i} + (\theta - \bar{\theta})\frac{\partial h}{\partial \theta}\bigg)\mathrm{d}x \\
&\geq \frac{\sigma}{\tau}\int_\Omega \big(h(\rho_1, \ldots, \rho_{n-1}, \theta) - h(\bar{\rho}_1, \ldots, \bar{\rho}_{n-1}, \bar{\theta})\big)\mathrm{d}x,
\end{aligned}$$

where we have set $\theta = e^w$ and $\bar{\theta} = e^{\bar{w}}$. Definition (6.26) of w_i, definition (6.29) of B_i, and the relations

$$\sum_{j=1}^{n-1} A_{ij}(\boldsymbol{w})\nabla w_j = \sum_{j=1}^{n} A_{ij}(\boldsymbol{w})\nabla q_j, \quad \sum_{i=1}^{n-1} B_i(\boldsymbol{w})\nabla w_i = \sum_{j=1}^{n} B_i(\boldsymbol{w})\nabla q_i$$

from (6.32)–(6.33) allow us to rewrite the sum $I_4 + I_5 + I_8$ as

$$I_4 + I_5 + I_8 = \sigma \int_\Omega \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{w})\nabla\Big(q_i + \frac{w}{m_i}\Big) \cdot \nabla\Big(q_j + \frac{w}{m_j}\Big)\mathrm{d}x. \qquad (6.41)$$

This expression is nonnegative because of the positive semidefiniteness of $A_{ij} = M_{ij}^{BD}\sqrt{\rho_i\rho_j}$; see (6.23). Furthermore, since $\sinh(z)/z \geq 1$ for $z \in \mathbb{R}$, $z \neq 0$,

$$
\begin{aligned}
I_9 &= \sigma\lambda \int_{\partial\Omega} e^{-w-w_0}(e^w - e^{w_0})^2 \mathrm{d}x \geq 0, \\
I_{10} &= 2\epsilon \int_\Omega \sinh(w-w_0)(w-\sigma w_0)\mathrm{d}x = 2\epsilon \int_\Omega (w-w_0)(w-\sigma w_0)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x \\
&= \epsilon \int_\Omega w^2 \frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x + \epsilon \int_\Omega \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x \\
&\geq \epsilon \int_\Omega w^2 \mathrm{d}x + \epsilon \int_\Omega \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x.
\end{aligned}
$$

We claim that there exists $m = m(w_0, \sigma) > 0$ such that for all $w \in \mathbb{R}$,

$$
g(w) = \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0} \geq -m,
$$

where $w_0 \in \mathbb{R}$ and $\sigma \in (0,1]$ are given. Indeed, this follows from $g(w) \to \infty$ as $|w| \to \infty$ and $g((1+\sigma)w_0) < 0$ (unless $w_0 = 0$). We conclude that

$$
I_{10} \geq \epsilon \int_\Omega w^2 \mathrm{d}x - \epsilon m.
$$

Finally, we can estimate

$$
I_{11} = \frac{\epsilon}{2}\int_\Omega \big(|\mathrm{D}^2 w|^2 + |\mathrm{D}^2 w - \nabla w \otimes \nabla w|^2 + |\nabla w|^4\big)\mathrm{d}x \geq \frac{\epsilon}{2}\int_\Omega \big(|\mathrm{D}^2 w|^2 + |\nabla w|^4\big)\mathrm{d}x.
$$

Summarizing these estimates, we find that

$$
\begin{aligned}
&\frac{\sigma}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta) + Ee^{-w_0}\big)\mathrm{d}x + \epsilon C\big(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\big) \\
&\quad + \int_\Omega \kappa(e^w)|\nabla w|^2 \mathrm{d}x \leq \frac{\sigma}{\tau}\int_\Omega \big(h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) + \bar{E}e^{-w_0}\big)\mathrm{d}x + \epsilon m.
\end{aligned} \tag{6.42}
$$

The right–hand side is bounded since $\bar{\boldsymbol{w}} \in L^\infty(\Omega;\mathbb{R}^n)$ by assumption, implying that $(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) \in L^\infty(\Omega;\mathbb{R}^n)$. The first term on the left–hand side is bounded from below since, by definition (6.24) of h and $Ee^{-w_0} = c_w\rho\theta/\theta_0$,

$$h(\rho_1, \ldots, \rho_{n-1}, \theta) + Ee^{-w_0} = \sum_{i=1}^{n} \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w\rho\left(\log\theta - \frac{\theta}{\theta_0}\right).$$

Thus, we obtain a uniform bound for $\boldsymbol{w}$ in $H^2(\Omega;\mathbb{R}^n)$ and consequently also in $W^{1,4}(\Omega;\mathbb{R}^n)$. We can apply the Leray–Schauder fixed–point theorem to conclude the existence of a fixed point of $S(\cdot,1)$. This, in turn, shows that $\boldsymbol{w}$ is a weak solution to the approximate problem (6.37)–(6.38).

Remark 30 (Treatment of the cross–terms)**.** In the paper [9], the fluxes are given by

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} M & -\boldsymbol{G} \\ \boldsymbol{G}^T & \kappa\theta^2 \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},$$

where $M = M(\boldsymbol{\rho},\theta) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{\rho},\theta) \in \mathbb{R}^n$. A multiplication of this equation by $\nabla(\boldsymbol{\mu}/\theta, -1/\theta)$ shows that the cross-terms cancel out,

$$-\nabla\begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^{n} M_{ij}\nabla\frac{\mu_i}{\theta}\cdot\nabla\frac{\mu_j}{\theta} + \kappa|\nabla\log\theta|^2 \geq 0,$$

since M is assumed to be positive semidefinite in [9]. In the present work, we have

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},$$

and the cross–terms do not cancel.

This is compensated by the sum $\sum_{i,j=1}^n A_{ij}/(m_i m_j)$. Indeed, a computation shows that (also see (6.41))

$$-\nabla\begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^{n} A_{ij}\nabla\left(q_i + \frac{w}{m_i}\right)\cdot\nabla\left(q_j + \frac{w}{m_j}\right) + \kappa|\nabla\log\theta|^2 \geq 0,$$

since A is positive semidefinite because of (6.34). □

6.2.3 Discrete entropy inequality

We derive some estimates from (6.40) with $\sigma = 1$, which are uniform in (ϵ, τ), by exploiting the sum $I_4 + I_5 + I_8$, which we have neglected in (6.42). Taking into account that the estimate of I_{10} becomes for $\sigma = 1$

$$I_{10} = 2\epsilon \int_\Omega \sinh(w - w_0)(w - w_0)\mathrm{d}x \geq 2\epsilon \int_\Omega (w - w_0)^2 \mathrm{d}x \geq 0,$$

we obtain the discrete entropy inequality

$$\begin{aligned} &\frac{\sigma}{\tau} \int_\Omega \big(h(\rho_1, \ldots, \rho_{n-1}, \theta) + Ee^{-w_0}\big)\mathrm{d}x + \epsilon C\big(\|\boldsymbol{w}\|^2_{H^2(\Omega)} + \|\nabla w\|^4_{L^4(\Omega)}\big) \\ &\quad + \int_\Omega \kappa(e^w)|\nabla w|^2 \mathrm{d}x + \int_\Omega \sum_{i,j=1}^n A_{ij} \nabla\Big(q_i + \frac{w}{m_i}\Big) \cdot \nabla\Big(q_j + \frac{w}{m_j}\Big)\mathrm{d}x \\ &\leq \frac{\sigma}{\tau} \int_\Omega \big(h(\bar\rho_1, \ldots, \bar\rho_{n-1}, \bar\theta) + \bar{E}e^{-w_0}\big)\mathrm{d}x. \end{aligned} \tag{6.43}$$

Lemma 31. *It holds that*

$$\int_\Omega \sum_{i,j=1}^n A_{ij} \nabla\Big(q_i + \frac{w}{m_i}\Big) \cdot \nabla\Big(q_j + \frac{w}{m_j}\Big)\mathrm{d}x \geq \int_\Omega \sum_{i=1}^n \frac{\mu}{m_i^2} |2\nabla\sqrt{\rho_i} + \sqrt{\rho_i}\nabla w|^2 \mathrm{d}x, \tag{6.44}$$

where $\mu > 0$ *is defined in* (6.23).

We deduce from Assumption (A4) that $\kappa(e^w)|\nabla w|^2 \geq c_\kappa |\nabla w|^2$, and in view of (6.43), this quantity is bounded in $L^2(\Omega)$. Therefore, Lemma 31 yields a gradient bound for $\sqrt{\rho_i}$ in $L^2(\Omega)$, since

$$4|\nabla\sqrt{\rho_i}|^2 \leq |2\nabla\sqrt{\rho_i} + \sqrt{\rho_i}\nabla w|^2 + \rho_i |\nabla w|^2.$$

Proof of Lemma 31. It follows from (6.25) and (6.30) that

$$\sum_{i,j=1}^n A_{ij} \nabla q_i = \sum_{i,j=1}^n A_{ij} \frac{\nabla \log \rho_i}{m_i} - c_w \sum_{i,j=1}^n A_{ij} \nabla w = \sum_{i,j=1}^n A_{ij} \frac{\nabla \rho_i}{m_i \rho_i}$$

and therefore, in view of the definition $A_{ij} = M_{ij}^{BD}\sqrt{\rho_i\rho_j}$ and the positive definiteness (6.23) on the subspace L,

$$\begin{aligned}\sum_{i,j=1}^n A_{ij}\nabla\left(q_i + \frac{w}{m_i}\right)\cdot\nabla\left(q_j + \frac{w}{m_j}\right) &= \sum_{i,j=1}^n A_{ij}\left(\frac{\nabla\rho_i}{m_i\rho_i} + \frac{\nabla w}{m_i}\right)\cdot\left(\frac{\nabla\rho_j}{m_j\rho_j} + \frac{\nabla w}{m_j}\right) \\ &= \sum_{i,j=1}^n M_{ij}^{BD}\frac{1}{m_i}\left(\frac{\nabla\rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i}\nabla w\right)\cdot\frac{1}{m_j}\left(\frac{\nabla\rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j}\nabla w\right) \\ &\geq \mu\left|P_L\left(\frac{1}{m_i}\left(\frac{\nabla\rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i}\nabla w\right)\right)_{i=1}^n\right|^2.\end{aligned}$$

We insert the definition of the projection matrix P_L:

$$\begin{aligned}\left[P_L\left(\frac{1}{m_j}\left(\frac{\nabla\rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j}\nabla w\right)\right)_{j=1}^n\right]_i &= \sum_{j=1}^n\left(\delta_{ij} - \frac{\sqrt{\rho_i\rho_j}}{\rho}\right)\frac{1}{m_j}\left(\frac{\nabla\rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j}\nabla w\right) \\ = \frac{1}{m_i}\left(\frac{\nabla\rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i}\nabla w\right) - \frac{\sqrt{\rho_i}}{\rho}\sum_{j=1}^n\frac{1}{m_j}(\nabla\rho_j + \rho_j\nabla w) &= \frac{1}{m_i}\left(\frac{\nabla\rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i}\nabla w\right).\end{aligned}$$

The last step follows from the pressure constraint (6.7). Indeed, by (6.8),

$$\sum_{j=1}^n\frac{1}{m_j}(\nabla\rho_j + \rho_j\nabla w) = \frac{1}{\theta}\sum_{j=1}^n\frac{\nabla(\rho_j\theta)}{m_j} = \frac{1}{\theta}\nabla p = 0. \tag{6.45}$$

We have shown that

$$\sum_{i,j=1}^n A_{ij}\nabla\left(q_i + \frac{w}{m_i}\right)\cdot\nabla\left(q_j + \frac{w}{m_j}\right) \geq \sum_{i=1}^n\frac{\mu}{m_i^2}\left|2\nabla\sqrt{\rho_i} + \sqrt{\rho_i}\nabla w\right|^2,$$

which equals (6.44) after integration over Ω. □

Remark 32. We observe that the sum (6.45) vanishes even without requiring the constraint (6.7). Indeed, by (6.18),

$$\sum_{j=1}^n\frac{1}{m_j}(\nabla\rho_j + \rho_j\nabla w) = \frac{1}{\theta}\sum_{j=1}^n\frac{1}{m_j}\nabla(\rho_j\theta) = \frac{1}{\theta}\sum_{j=1}^n d_j = 0.$$

The fact that $\sum_{j=1}^n d_j$ vanishes is a necessary condition for the invertibility of the linear system (6.19). □

In view of Lemma 31 and the lower bound $\kappa \geq c_\kappa(1+\theta^2)$, we conclude from (6.43) the following discrete entropy inequality.

Lemma 33 (Discrete entropy inequality)**.** *It holds that*

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta)+Ee^{-w_0}\big)\mathrm{d}x + \epsilon C\big(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\big)\\
&\qquad + \int_\Omega \big(|\nabla w|^2 + |\nabla\theta|^2\big)\mathrm{d}x + \int_\Omega \sum_{i=1}^n \frac{\mu}{m_i^2}\big|2\nabla\sqrt{\rho_i}+\sqrt{\rho_i}\nabla w\big|^2\mathrm{d}x\\
&\quad \leq \frac{1}{\tau}\int_\Omega \big(h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta)+\bar E e^{-w_0}\big)\mathrm{d}x.
\end{aligned}$$

Finally, we derive an estimate for the temperature.

Lemma 34. *There exists a constant $C>0$, only depending on λ, Ω, $\partial\Omega$, and θ^0 such that*

$$\frac{c_w}{2\tau}\int_\Omega \rho\theta^2\mathrm{d}x + \frac{c_\kappa}{2}\int_\Omega (1+\theta^2)|\nabla\theta|^2\mathrm{d}x \leq C + C\int_\Omega\sum_{i=1}^n|\nabla\sqrt{\rho_i}|^2\mathrm{d}x + \frac{c_w}{2\tau}\int_\Omega \rho\bar\theta^2\mathrm{d}x.$$

Proof. We use θ as a test function in the approximate energy equation (6.38). Observing that $\nabla w_i = \nabla\rho_i/(m_i\rho_i) - \nabla\rho_n/(m_n\rho_n)$ by (6.26) and $\sum_{i=1}^n B_i\nabla w_i = \sum_{i=1}^n B_i(m_i\rho_i)^{-1}\nabla\rho_i$ by (6.30), we find that

$$\begin{aligned}
0 &= \frac{c_w}{\tau}\int_\Omega \rho(\theta-\bar\theta)\mathrm{d}x + \int_\Omega \kappa(\theta)|\nabla\theta|^2\mathrm{d}x + \int_\Omega\sum_{i,j=1}^n \frac{A_{ij}}{m_im_j}|\nabla\theta|^2\mathrm{d}x\\
&\int_\Omega\sum_{i=1}^n\frac{B_i}{m_i\rho_i}\nabla\rho_i\cdot\nabla\theta\mathrm{d}x - \lambda\int_{\partial\Omega}(\theta_0-\theta)\theta\mathrm{d}s + \epsilon\int_\Omega(\theta_0+\theta)(\log\theta-\log\theta_0)\theta\mathrm{d}x\\
&+\epsilon\int_\Omega\left(|\mathrm{D}^2\theta|^2 - \frac{1}{\theta}\mathrm{D}^2\theta:(\nabla\theta\otimes\nabla\theta) + \frac{|\nabla\theta|^4}{\theta^2}\right)\mathrm{d}x = J_1+\cdots+J_7.
\end{aligned}$$

We deduce from Young's inequality and Assumption (A4) on κ that

$$J_1 \geq \frac{c_w}{2\tau}\int_\Omega \rho(\theta^2-\bar\theta^2)\mathrm{d}x, \quad J_2 \geq c_\kappa\int_\Omega(1+\theta^2)|\nabla\theta|^2\mathrm{d}x.$$

Furthermore, $J_3\geq 0$. Definition (6.29) of B_i and A_{ij} as well as the bound $\rho_j \leq \rho^*$

show that

$$J_4 = \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j \rho_i} \nabla\rho_i \cdot \nabla\theta \mathrm{d}x = \theta \sum_{i,j=1}^{n} \frac{M_{ij}^{BD}}{m_i m_j} \frac{\sqrt{\rho_j}}{\sqrt{\rho_i}} \nabla\rho_i \cdot \nabla\theta \mathrm{d}x$$
$$\geq -\frac{c_\kappa}{2} \int_\Omega \theta^2 |\nabla\theta|^2 \mathrm{d}x - C \int_\Omega \sum_{i=1}^{n} |\nabla\sqrt{\rho_i}|^2 \mathrm{d}x.$$

The integrals J_5 are J_6 are bounded from below since

$$J_5 \geq -\frac{\lambda}{4} \int_{\partial\Omega} \theta_0^2 \mathrm{d}s \geq -C(\lambda, \partial\Omega, \theta_0),$$

and the dominant term in J_6 is $\theta^2 \log\theta$, which is bounded from below by a negative constant. Finally, J_7 is nonnegative:

$$J_7 = \frac{\epsilon}{2} \int_\Omega \left(|\mathrm{D}^2\theta|^2 + \frac{|\nabla\theta|^4}{\theta^2} + \left| \mathrm{D}^2\theta - \frac{1}{\theta} \nabla\theta \otimes \nabla\theta \right|^2 \right) \mathrm{d}x \geq 0.$$

Collecting these estimates finishes the proof. □

6.2.4 Uniform estimates

Let $(w_1^k, \ldots, w_{n-1}^k, w^k)$ be a solution to the approximate scheme (6.37)–(6.38) with $(w_1^{k-1}, \ldots, w_{n-1}^{k-1}, w^{k-1}) = (\bar{w}_1, \ldots, \bar{w}_{n-1}, \bar{w})$. We set $\theta^k = \exp(w^k)$ and $\rho_i^k = \rho_i(w^k)$ determined from Lemma 27. Furthermore, we set $E^k = c_w \rho \theta^k$, recalling that $\rho = \sum_{i=1}^n \rho_i^0$. We introduce the piecewise constant in time functions

$$\rho_i^{(\tau)}(x,t) = \rho_i^k(x), \quad q_i^{(\tau)} = \frac{1}{m_i} \log\frac{\rho_i^k}{m_i} - c_w(\log\theta^k - 1) \quad \text{for } i = 1, \ldots, n,$$
$$\theta^{(\tau)}(x,t) = \theta^k(x), \quad E^{(\tau)}(x,t) = E^k(x), \quad w_i^{(\tau)}(x,t) = w_i^k(x) \quad \text{for } i = 1, \ldots, n-1,$$

where $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$, and $k = 1, \ldots, N$. At time $t = 0$, we set $\rho_i^{(\tau)}(0) = \rho_i^0$ and $\theta^{(\tau)}(0) = \theta^0$. Furthermore, we introduce the shift operator

$(\sigma_\tau\rho_i^{(\tau)})(x,t) = \rho_i^{k-1}(x)$ if $t \in ((k-1)\tau, k\tau]$. Then $(\boldsymbol{\rho}^{(\tau)}, \theta^{(\tau)})$ solves

$$
\begin{aligned}
0 &= \frac{1}{\tau}\int_0^T\int_\Omega (\rho_i^{(\tau)} - \sigma_\tau\rho_i^{(\tau)})\phi_i \mathrm{d}x\mathrm{d}t \\
&\quad + \epsilon\int_0^T\int_\Omega \big(\mathrm{D}^2 w_i^{(\tau)} : \mathrm{D}^2\phi_i + w_i^{(\tau)}\phi_i\big)\mathrm{d}x\mathrm{d}t \qquad (6.46)\\
&\quad + \int_0^T\int_\Omega\bigg(\sum_{j=1}^{n-1} A_{ij}(\boldsymbol{w}^{(\tau)})\nabla w_j^{(\tau)} + e^{-w^{(\tau)}}B_i(\boldsymbol{w}^{(\tau)})\nabla w^{(\tau)}\bigg)\cdot\nabla\phi_i \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{1}{\tau}\int_0^T\int_\Omega (E^{(\tau)} - \sigma_\tau E^{(\tau)})\phi_0 \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \kappa(\theta^{(\tau)})\nabla\theta^{(\tau)}\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega\sum_{i=1}^{n-1} B_j(\boldsymbol{w}^{(\tau)})\nabla w_i^{(\tau)}\cdot\nabla\phi_0\mathrm{d}x - \lambda\int_0^T\int_{\partial\Omega}(\theta_0 - \theta^{(\tau)})\phi_0\mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega\sum_{i,j=1}^{n}\frac{A_{ij}(\boldsymbol{w}^{(\tau)})}{m_i m_j}\nabla\theta^{(\tau)}\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t \qquad (6.47)\\
&\quad + \epsilon\int_0^T\int_\Omega(\theta_0 + \theta^{(\tau)})(\log\theta^{(\tau)} - \log\theta_0)\phi_0\mathrm{d}x\mathrm{d}t \\
&\quad + \epsilon\int_0^T\int_\Omega \theta^{(\tau)}\big(\mathrm{D}^2\log\theta^{(\tau)} : \mathrm{D}^2\phi_0 + |\nabla\log\theta^{(\tau)}|^2\nabla\log\theta^{(\tau)}\cdot\nabla\phi_0\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

The discrete entropy inequality in Lemma 33 and the temperature estimates in Lemma 34 yield, after summation over $k = 1, \ldots, N$,

$$
\begin{aligned}
&\sup_{0<t<T}\int_\Omega\Big(h(\rho_1^{(\tau)}(t), \ldots, \rho_{n-1}^{(\tau)}(t), \theta^{(\tau)}(t)) + \frac{c_w}{\theta_0}\rho\theta^{(\tau)}(t)\Big)\mathrm{d}x \qquad (6.48)\\
&\quad + \int_0^T\int_\Omega\big(|\nabla\log\theta^{(\tau)}|^2 + |\nabla\theta^{(\tau)}|^2\big)\mathrm{d}x\mathrm{d}t \\
&\quad + \epsilon C\int_0^T\big(\|\boldsymbol{w}^{(\tau)}\|_{H^2(\Omega)}^2 + \|\nabla w^{(\tau)}\|_{L^4(\Omega)}^4\big)\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega\sum_{i=1}^n\frac{\mu}{m_i^2}\big|2\nabla(\rho_i^{(\tau)})^{1/2} + (\rho_i^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)}\big|^2\mathrm{d}x\mathrm{d}t \\
&\le \int_\Omega\big(h(\rho_1^0, \ldots, \rho_{n-1}^0, \theta^0) + c_w\rho\theta^0\big)\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
&c_w\sup_{0<t<T}\int_\Omega\rho(\theta^{(\tau)})^2\mathrm{d}x + c_\kappa\int_0^T\int_\Omega(1 + (\theta^{(\tau)})^2)|\nabla\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}t \qquad (6.49)\\
&\quad \le C(T) + C\int_0^T\int_\Omega\sum_{i=1}^n|\nabla(\rho_i^{(\tau)})^{1/2}|^2\mathrm{d}x\mathrm{d}t + \frac{c_w}{2}\int_\Omega\rho(\theta^0)^2\mathrm{d}x.
\end{aligned}
$$

Lemma 35. *There exists $C > 0$ not depending on (ϵ, τ) such that*

$$\|\boldsymbol{\rho}^{(\tau)}\|_{L^\infty(\Omega_T)} + \|\theta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{6.50}$$

$$\|\log \theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \|\theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{6.51}$$

$$\epsilon^{1/2}\|\boldsymbol{w}^{(\tau)}\|_{L^2(0,T;H^2(\Omega))} + \epsilon^{1/4}\|\nabla w^{(\tau)}\|_{L^4(\Omega_T)} \le C, \tag{6.52}$$

Proof. Estimates (6.50) and (6.52) are an immediate consequence of (6.48) and $\rho \ge \rho_* > 0$. Bound (6.48) also shows that $\sup_{(0,T)} \int_\Omega (-\log\theta^{(\tau)} + \theta^{(\tau)})\mathrm{d}x$ is uniformly bounded from above. Thus, $\log\theta^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Then the uniform bounds for $\nabla \log\theta^{(\tau)}$ and $\nabla\theta^{(\tau)}$ as well as the Poincaré–Wirtinger inequality yield bounds for $\log\theta^{(\tau)}$ and $\theta^{(\tau)}$ in $L^2(\Omega_T)$, proving (6.51). □

Lemma 36. *There exists $C > 0$ not depending on (ϵ, τ) such that for $i = 1, \dots, n$,*

$$\|(\rho_i^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\Omega))} + \|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{6.53}$$

$$\|\theta^{(\tau)}\|_{L^\infty(0,T;L^2(\Omega))} + \|(\theta^{(\tau)})^2\|_{L^2(0,T;H^1(\Omega))} + \|\theta^{(\tau)}\|_{L^{16/3}(\Omega_T)} \le C. \tag{6.54}$$

Proof. We infer from (6.48) that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla(\rho_i^{(\tau)})^{1/2}|^2\mathrm{d}x\mathrm{d}t &\le C\int_0^T\int_\Omega \left|2\nabla(\rho_i^{(\tau)})^{1/2}|^2 + (\rho_i^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)}\right|^2\mathrm{d}x\mathrm{d}t \\ &\quad + C\int_0^T\int_\Omega |\nabla\log\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}t \le C,\end{aligned}$$

and the $L^\infty(\Omega_T)$ bound (6.50) gives for $i = 1, \dots, n$,

$$\|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le 2\|\rho_i^{(\tau)}\|^{1/2}_{L^\infty(\Omega_T)}\|\nabla(\rho_i^{(\tau)})^{1/2}\|_{L^2(\Omega_T)} + \|\rho_i^{(\tau)}\|_{L^2(\Omega_T)} \le C.$$

Therefore, the right–hand side of (6.49) is uniformly bounded, which proves the first two estimates in (6.54). The remaining one is a consequence of the Gagliardo–

Nirenberg inequality with $\eta = 3/4$:

$$\|(\theta^{(\tau)})^2\|_{L^{8/3}(\Omega_T)}^{8/3} \le C \int_0^T \|(\theta^{(\tau)})^2\|_{H^1(\Omega)}^{8\eta/3} \|(\theta^{(\tau)})^2\|_{L^1(\Omega)}^{8(1-\eta)/3} \mathrm{d}t$$
$$\le \|\theta^{(\tau)}\|_{L^\infty(0,T;L^2(\Omega))}^{4/3} \int_0^T \|(\theta^{(\tau)})^2\|_{H^1(\Omega)}^2 \mathrm{d}t \le C.$$

This finishes the proof. □

The following lemma can be proved as in [9, Lemma 9].

Lemma 37. *There exists $C > 0$ not depending on (ϵ, τ) such that*

$$\|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} + \|\theta^{(\tau)} - \sigma_\tau \theta^{(\tau)}\|_{L^{16/15}(0,T;W^{2,16}(\Omega)^*)} \le C\tau. \qquad (6.55)$$

6.2.5 The limit $(\epsilon, \tau) \to 0$

The bounds (6.51), (6.53), and (6.55) allow us to apply the Aubin–Lions lemma in the version of [59]. There exist subsequences, which are not relabeled, such that as $(\epsilon, \tau) \to 0$,

$$\rho_i^{(\tau)} \to \rho_i, \quad \theta^{(\tau)} \to \theta \quad \text{strongly in } L^2(\Omega_T), \ i = 1, \dots, n-1.$$

The convergence also holds for $i = n$ since $\rho_n^{(\tau)} = 1 - \sum_{i=1}^{n-1} \rho_i^{(\tau)}$. Thanks to the $L^\infty(\Omega_T)$ bound for $\rho_i^{(\tau)}$ and the $L^{16/3}(\Omega_T)$ bound for $\theta^{(\tau)}$, we have

$$\rho_i^{(\tau)} \to \rho_i \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r < \infty,$$
$$\theta^{(\tau)} \to \theta \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r < 16/3.$$

We claim that $\rho_i > 0$ and $\theta > 0$ a.e. in Ω_T. The positivity of ρ_i is proved as in [9, p. 16]. The strong convergence of $(\theta^{(\tau)})$ implies a.e. convergence and in particular $\log \theta^{(\tau)} \to Z$ a.e. Thus, $\theta^{(\tau)} \to \exp(Z)$ a.e. We conclude that $\theta = \exp(Z) > 0$ a.e. in Ω_T.

It follows that $\log\theta \in L^2(\Omega_T)$ and estimate (6.51) yields

$$\nabla \log \theta^{(\tau)} \rightharpoonup \nabla \log \theta \quad \text{weakly in } L^2(\Omega_T). \tag{6.56}$$

Furthermore, in view of (6.51), (6.53), and (6.55), up to subsequences,

$$\begin{aligned} \rho_i^{(\tau)} \rightharpoonup \rho_i, \quad \theta^{(\tau)} \rightharpoonup \theta \quad & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ \tau^{-1}(\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}) \rightharpoonup \partial_t \rho_i \quad & \text{weakly in } L^2(0,T;H^2(\Omega)^*), \\ \tau^{-1}(\theta^{(\tau)} - \sigma_\tau \theta^{(\tau)}) \rightharpoonup \partial_t \rho_i \quad & \text{weakly in } L^{16/15}(0,T;W^{2,16}(\Omega)^*), \end{aligned}$$

and the bounds (6.52) show that

$$\epsilon \log \theta^{(\tau)} \to 0, \quad \epsilon w_i^{(\tau)} \to 0 \quad \text{strongly in } L^2(0,T;H^2(\Omega)).$$

The embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact, giving $\theta^{(\tau)} \to \theta$ strongly in $L^2(0,T;L^2(\partial\Omega))$.

These convergences are sufficient to pass to the limit $(\epsilon,\tau) \to 0$ in (6.46)–(6.47), showing that $(\boldsymbol{\rho},\theta)$ solves the weak formulation (6.35)–(6.36). We only detail the limits in the terms $A_{ij}^{(\tau)} = A_{ij}(\boldsymbol{w}^{(\tau)})$ and $B_i^{(\tau)} = B_i(\boldsymbol{w}^{(\tau)})$. We know that $\nabla(\rho_i^{(\tau)})^{1/2} \rightharpoonup \nabla\rho_i^{1/2}$ weakly in $L^2(\Omega_T)$ and

$$\frac{A_{ij}^{(\tau)}}{m_j(\rho_j^{(\tau)})^{1/2}} = M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{(\rho_i^{(\tau)})^{1/2}}{m_j} \to M_{ij}^{BD}(\boldsymbol{\rho})\frac{\rho_i^{1/2}}{m_j} = \frac{A_{ij}}{m_j\rho_j^{1/2}}$$

strongly in $L^\gamma(\Omega_T)$ for all $\gamma < \infty$. Using (6.32) and (6.25), this implies that

$$\begin{aligned} \sum_{j=1}^{n-1} A_{ij}^{(\tau)} \nabla w_j^{(\tau)} &= \sum_{j=1}^{n} \frac{A_{ij}^{(\tau)}}{m_j} \nabla \log \frac{\rho_i^{(\tau)}}{m_j} = 2\sum_{j=1}^{n} M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{(\rho_i^{(\tau)})^{1/2}}{m_j}\nabla(\rho_j^{(\tau)})^{1/2} \\ &\rightharpoonup 2\sum_{j=1}^{n} M_{ij}^{BD}\frac{\rho_i^{1/2}}{m_j}\nabla\rho_j^{1/2} \quad \text{weakly in } L^s(\Omega_T),\ s<2. \end{aligned}$$

Since the sequence is bounded in $L^2(\Omega_T)$, this convergence also holds in this space. Similarly,

$$B_i^{(\tau)} e^{-w^{(\tau)}} \nabla w^{(\tau)} = \sum_{j=1}^n \frac{A_{ij}^{(\tau)}}{m_j} \nabla \log \theta^{(\tau)} \rightharpoonup \sum_{j=1}^n \frac{A_{ij}}{m_j} \nabla \log \theta \quad \text{weakly in } L^2(\Omega_T),$$

$$A_{ij}^{(\tau)} \nabla \theta^{(\tau)} = M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})(\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2} \nabla \theta^{(\tau)} \rightharpoonup A_{ij} \nabla \theta \quad \text{weakly in } L^2(\Omega_T),$$

and using $\theta^{(\tau)} \to \theta$ strongly in $L^r(\Omega_T)$ for $r < 16/3$,

$$\sum_{i=1}^{n-1} B_i^{(\tau)} \nabla w_i^{(\tau)} = 2 \sum_{i,j=1}^n \frac{M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})}{m_i m_j} \theta^{(\tau)} (\rho_j^{(\tau)})^{1/2} \nabla (\rho_i^{(\tau)})^{1/2} \rightharpoonup 2 \sum_{i=1}^n \frac{B_i}{m_i \rho_i^{1/2}} \nabla \rho_i^{1/2}$$

weakly in $L^s(\Omega_T)$ for $s < 16/11$, and since the right–hand side lies in $L^{16/11}(\Omega_T)$, this convergence also holds in $L^{16/11}(\Omega_T)$.

Next, we claim that $\rho_i(0)$ and $\theta(0)$ satisfy the initial data. The time derivative of the linear interpolant

$$\widetilde{\rho}_i^{(\tau)}(t) = \rho_i^k - \frac{k\tau - t}{\tau}(\rho_i^k - \rho_i^{k-1}) \quad \text{for } (k-1)\tau < t < k\tau$$

is bounded since, because of (6.55),

$$\|\partial_t \widetilde{\rho}_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le \tau^{-1} \|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le C.$$

Thus, $\widetilde{\rho}_i^{(\tau)}$ is uniformly bounded in $H^1(0,T;H^2(\Omega)^*) \hookrightarrow C^0([0,T];H^2(\Omega)^*)$ and we conclude for a subsequence that $\rho_i^0 = \widetilde{\rho}_i^{(\tau)}(0) \rightharpoonup r_i$ weakly in $H^2(\Omega)^*$ for some $r_i \in H^2(\Omega)^*$. It follows that $r_i = \rho_i^0$. As $\widetilde{\rho}_i^{(\tau)}$ and $\rho_i^{(\tau)}$ converge to the same limit,

$$\|\widetilde{\rho}_i^{(\tau)} - \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le \|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le C\tau \to 0,$$

this shows that $\rho_i^0 = r_i = \rho_i(0)$ in $H^2(\Omega)^*$. In an analogous way, we verify that $\theta(0) = \theta^0$ in $W^{2,16}(\Omega)^*$.

The initial data are satisfied in better spaces. Indeed, going back to (6.35)–(6.36), the regularity of ρ_i implies that $\partial_t \rho_i \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*) \hookrightarrow C^0([0,T];L^2(\Omega))$ and thus $\rho_i(0) = \rho_i^0$ in the sense of $L^2(\Omega)$. The temperature satisfies $\theta \in L^\infty(0,T;L^2(\Omega))\cap C^0([0,T];W^{2,16}(\Omega)^*)$, which gives $\theta \in C^0_w([0,T];L^2(\Omega))$. Consequently, $\theta(0)=\theta^0$ weakly in $L^2(\Omega)$. Moreover, we deduce from $|\kappa\nabla\theta| \le C_\kappa(|\nabla\theta| + \theta|\nabla\theta^2|) \in L^{16/11}(\Omega_T)$ that $\partial_t\theta \in L^{16/11}(0,T;W^{1,16/11}(\Omega)^*)$. This completes the proof.

6.3 Weak–strong uniqueness

In this section we prove Theorem 26. Let $(\boldsymbol{\rho},\theta)$ be a weak solution and $(\bar{\boldsymbol{\rho}},\bar\theta)$ be a strong solution to (6.1)–(6.8). We introduce the entropy

$$H(\boldsymbol{\rho}(t),\theta(t)) = \int_\Omega \bigg(\sum_{i=1}^n \frac{\rho_i}{m_i}\Big(\log\frac{\rho_i}{m_i}-1\Big) - c_w\rho\log\theta\bigg)\mathrm{d}x.$$

Lemma 38 (Entropy equality for strong solutions)**.** *Let $(\bar{\boldsymbol{\rho}},\bar\theta)$ be a strong solution to* (6.1)–(6.8) *(in the sense mentioned after Theorem 26) with $\lambda=0$. Then*

$$\begin{aligned} H(\bar{\boldsymbol{\rho}}(t),\bar\theta(t)) &+ \int_0^t\int_\Omega \frac{\kappa(\bar\theta)}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x\mathrm{d}s \\ &+ \frac12\int_0^t\int_\Omega \sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j|\bar u_i-\bar u_j|^2\mathrm{d}x\mathrm{d}s = H(\bar{\boldsymbol{\rho}}(0),\bar\theta(0)). \end{aligned}$$

Proof. We use (6.1) and (6.2) and integrate by parts to obtain

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \int_\Omega\bigg(\sum_{i=1}^n \frac{\partial_t\bar\rho_i}{m_i}\log\frac{\bar\rho_i}{m_i} - \frac{c_w}{\rho}\partial_t(\rho\bar\theta)\bigg)\mathrm{d}x \\ &= \int_\Omega\bigg\{\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\nabla\log\frac{\bar\rho_i}{m_i} + \frac{\nabla\bar\theta}{\bar\theta^2}\bigg(-\bar\kappa\nabla\bar\theta + \bar\theta\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\bigg)\bigg\}dx \\ &= -\int_\Omega\frac{\bar\kappa}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x + \int_\Omega\sum_{i=1}^n\frac{\bar u_i}{m_i}\cdot(\nabla\bar\rho_i+\bar\rho_i\nabla\log\bar\theta)\mathrm{d}x \\ &= -\int_\Omega\frac{\bar\kappa}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x + \int_\Omega\sum_{i=1}^n\frac{1}{\bar\theta}\bar u_i\cdot\bar d_i\mathrm{d}x, \end{aligned}$$

where $\bar\kappa=\kappa(\bar\theta)$ and we used (6.18) in the last step. By the algebraic system (6.5) and the symmetry of (b_{ij}),

$$\sum_{i=1}^n \frac{1}{\bar\theta}\bar u_i \cdot \bar d_i = -\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j(\bar u_i - \bar u_j)\cdot \bar u_i = -\frac{1}{2}\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j|\bar u_i - \bar u_j|^2. \tag{6.57}$$

This shows the claim. □

Lemma 39 (Entropy inequality for weak solutions)**.** *Let $(\bar{\boldsymbol\rho},\bar\theta)$ be a weak solution to* (6.1)–(6.8) *with $\lambda = 0$. Then*

$$H(\boldsymbol\rho(t),\theta(t)) + \int_0^t\int_\Omega \frac{\kappa}{\theta^2}|\nabla\theta|^2 \mathrm{d}x\mathrm{d}s \\ + \frac{1}{2}\int_0^t\int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i-u_j|^2\mathrm{d}x\mathrm{d}s \le H(\boldsymbol\rho^0,\theta^0).$$

Proof. Let $(\boldsymbol\rho^k,\theta^k)$ for $k=1,\dots,N$ be a solution to the approximate problem (6.37)–(6.38), constructed in Section 6.2.2. According to (6.43), this solution satisfies

$$H(\boldsymbol\rho^k,\theta^k) + \tau\int_\Omega \kappa(\theta^k)|\nabla\log\theta^k|^2\mathrm{d}x \\ + \tau\int_\Omega\sum_{i,j=1}^n A_{ij}^k\nabla\left(q_i^k + \frac{w^k}{m_i}\right)\cdot\nabla\left(q_j^k+\frac{w^k}{m_j}\right)\mathrm{d}x \le H(\boldsymbol\rho^{k-1},\theta^{k-1}),$$

where the superindex k denotes the kth time step. By Corollary 29 as well as relations (6.22) and (6.57),

$$\sum_{i,j=1}^n A_{ij}^k\nabla\left(q_i^k + \frac{w^k}{m_i}\right)\cdot\nabla\left(q_j^k+\frac{w^k}{m_j}\right) = \sum_{i,j=1}^n (M_{ij}^{BD})^k\frac{d_i^k}{\theta^k(\rho_i^k)^{1/2}}\cdot\frac{d_j^k}{\theta^k(\rho_j^k)^{1/2}} \\ = -\sum_{i=1}^n\frac{1}{\theta^k}d_i^k\cdot u_i^k = \frac{1}{2}\sum_{i,j=1}^n b_{ij}\rho_i^k\rho_j^k|u_i^k-u_j^k|^2.$$

Therefore,

$$H(\boldsymbol\rho^k,\theta^k)+\tau\int_\Omega\kappa(\theta^k)|\nabla\log\theta^k|^2\mathrm{d}x+\frac{\tau}{2}\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^k\rho_j^k|u_i^k-u_j^k|^2\mathrm{d}x \le H(\boldsymbol\rho^{k-1},\theta^{k-1}).$$

We sum over $k=1,\dots,j$ with $t\in((j-1)\tau,j\tau]$ and use the notation of Section

6.2.4:

$$H(\boldsymbol{\rho}^{(\tau)}(t),\theta^{(\tau)}(t)) + \int_0^t\int_\Omega \kappa(\theta^{(\tau)})|\nabla\log\theta^{(\tau)}|^2 \mathrm{d}x\mathrm{d}s \\ + \frac{1}{2}\int_0^t\int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)} - u_j^{(\tau)}|^2 \mathrm{d}x\mathrm{d}s \le H(\boldsymbol{\rho}^0,\theta^0) \tag{6.58}$$

for a.e. $t\in(0,T)$.

It remains to pass to the limit $(\epsilon,\tau)\to 0$ in (6.58). We deduce from the strong convergence of $(\boldsymbol{\rho}^{(\tau)})$ and $(\theta^{(\tau)})$ that

$$H(\boldsymbol{\rho}(t),\theta(t)) \le \liminf_{(\epsilon,\tau)\to 0} H(\boldsymbol{\rho}^{(\tau)}(t),\theta^{(\tau)}(t)).$$

We deduce from the strong convergence $\rho_i^{(\tau)}\to\rho_i$ in $L^q(\Omega_T)$ for any $q<\infty$ and the boundedness of M_{ij}^{BD} that $M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\to M_{ij}^{BD}(\boldsymbol{\rho})$ strongly in any $L^q(\Omega_T)$. In view of the weak convergences $\nabla\log\theta^{(\tau)}\rightharpoonup\nabla\log\theta$ from (6.56) and $\nabla(\rho_i^{(\tau)})^{1/2}\rightharpoonup\nabla\rho_i^{1/2}$ from (6.53) weakly in $L^2(\Omega_T)$, we have

$$2\nabla(\rho_i^{(\tau)})^{1/2} + \rho_i^{(\tau)}\nabla\log\theta^{(\tau)} \rightharpoonup 2\nabla\rho_i^{1/2} + \rho_i\nabla\log\theta \quad \text{weakly in } L^2(\Omega_T).$$

Hence, using (6.22),

$$(\rho_i^{(\tau)})^{1/2}u_i^{(\tau)} = \sum_{j=1}^n M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{1}{m_j}\big(2\nabla(\rho_i^{(\tau)})^{1/2} + \rho_i^{(\tau)}\nabla\log\theta^{(\tau)}\big) \\ \rightharpoonup \sum_{j=1}^n M_{ij}^{BD}(\boldsymbol{\rho})\frac{1}{m_j}\big(2\nabla\rho_i^{1/2} + \rho_i\nabla\log\theta\big) = \rho_i^{1/2}u_i.$$

weakly in $L^2(\Omega_T)$, where the last identity is the definition of u_i. Then, taking into account the boundedness of $\rho_i^{(\tau)}$ in $L^\infty(\Omega_T)$, for any $i,j=1,\ldots,n$,

$$(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2}u_i^{(\tau)} \rightharpoonup (b_{ij}\rho_i\rho_j)^{1/2}u_i \quad \text{weakly in } L^2(\Omega_T).$$

As the $L^2(\Omega_T)$ norm is weakly lower semicontinuous,

$$\begin{aligned}
&\int_0^T \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i - u_j|^2 \mathrm{d}x\mathrm{d}s \\
&\qquad\qquad \le \liminf_{(\epsilon,\tau)\to 0} \int_0^T \int_\Omega \sum_{i,j=1}^n \left|(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2}(u_i^{(\tau)} - u_j^{(\tau)})\right|^2 \mathrm{d}x\mathrm{d}s \\
&\qquad\qquad = \liminf_{(\epsilon,\tau)\to 0} \int_0^T \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)} - u_j^{(\tau)}|^2 \mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Finally, $\kappa(\theta^{(\tau)})^{1/2}\nabla \log\theta^{(\tau)} \to \kappa(\theta)^{1/2}\nabla\log\theta$ weakly in $L^1(\Omega_T)$ and, because of the uniform bounds, also in $L^2(\Omega_T)$. Hence,

$$\int_0^t \int_\Omega \frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2 \mathrm{d}x\mathrm{d}s \le \liminf_{(\epsilon,\tau)\to 0} \int_0^t \int_\Omega \frac{\kappa(\theta^{(\tau)})}{(\theta^{(\tau)})^2}|\nabla\theta^{(\tau)}|^2 \mathrm{d}x\mathrm{d}s.$$

Thus, applying the limit inferior $(\epsilon,\tau) \to 0$ to both sides of (6.58) yields the result. □

Lemma 40 (Relative entropy inequality)**.** *Let the assumptions of Theorem 26 hold and let $\rho_i(0) = \bar\rho_i(0)$ for $i = 1,\ldots,n$ and $\theta(0) = \bar\theta(0)$. Then*

$$\begin{aligned}
&H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t)) + \frac{\mu_M}{2}\int_0^t \int_\Omega \sum_{i=1}^n \rho_i|u_i - \bar u_i|^2 \mathrm{d}x\mathrm{d}s \\
&\quad + \frac{c_\kappa}{2}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar\theta)|^2 \mathrm{d}x\mathrm{d}s \qquad (6.59)\\
&\quad \le C\int_0^t \int_\Omega \left(\sum_{j=1}^n (\rho_j - \bar\rho_j)^2 + (\theta - \bar\theta)^2\right)\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

where the relative entropy $H(\boldsymbol{\rho},\theta|\bar{\boldsymbol{\rho}},\bar\theta)$ is defined in (6.12).

Proof. We use the test functions $\phi_i = m_i^{-1}\log(\bar\rho_i/m_i) - c_w\log\bar\theta$ and $\phi_0 = -1/\bar\theta$ in the weak formulations satisfied by $\rho_i - \bar\rho_i$ and $\rho(\theta - \bar\theta)$, respectively,

$$\int_\Omega (\rho_i - \bar\rho_i)(t)\phi_i(t)\mathrm{d}x = \int_0^t \int_\Omega (\rho_i - \bar\rho_i)\partial_t\phi_i \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega (\rho_i u_i - \bar\rho_i\bar u_i)\cdot\nabla\phi_i \mathrm{d}x\mathrm{d}s,$$

$$\int_\Omega c_w\rho(\theta-\bar\theta)(t)\phi_0(t)\mathrm{d}x = \int_0^t\int_\Omega c_w\rho(\theta-\bar\theta)\partial_t\phi_0\mathrm{d}x\mathrm{d}s$$
$$-\int_0^t\int_\Omega(\kappa\nabla\theta-\bar\kappa\nabla\bar\theta)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega\sum_{j=1}^n(h_ju_j-\bar h_j\bar u_j)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s,$$

where $h_j=(c_w+1/m_j)\rho_j\theta$, $\bar h_j=(c_w+1/m_j)\bar\rho_j\bar\theta$, and $\kappa=\kappa(\theta)$, $\bar\kappa=\kappa(\bar\theta)$. Strictly speaking, we cannot use ϕ_i as a test function since $\log\bar\rho_i$ and $1/\bar\theta$ may be not integrable. However, we can use a density argument similarly as in the proof of [10, Lemma 8]. Then, summing over $i=1,\dots,n$,

$$\int_\Omega\bigg\{\sum_{i=1}^n(\rho_i-\bar\rho_i)(t)\bigg(\frac{1}{m_i}\log\frac{\bar\rho_i}{m_i}-c_w\log\bar\theta\bigg)(t)-c_w\rho\frac{\theta-\bar\theta}{\bar\theta}(t)\bigg\}\mathrm{d}x$$
$$=\int_0^t\int_\Omega\bigg\{\sum_{i=1}^n\bigg((\rho_i-\bar\rho_i)\frac{\partial_t\bar\rho_i}{m_i\bar\rho_i}+(\rho_iu_i-\bar\rho_i\bar u_i)\cdot\frac{\nabla\bar\rho_i}{m_i\bar\rho_i}\bigg)$$
$$+c_w\rho(\theta-\bar\theta)\partial_t\bigg(-\frac{1}{\bar\theta}\bigg)\bigg\}\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega(\kappa\nabla\theta-\bar\kappa\nabla\bar\theta)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_\Omega\sum_{j=1}^n(h_ju_j-\bar h_j\bar u_j)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s.$$

We subtract this identity and the entropy equality from Lemma 38 for $(\bar{\boldsymbol\rho},\bar\theta)$ from the entropy inequality for $(\boldsymbol\rho,\theta)$ obtained in Lemma 39 and insert equations (6.1)–(6.2) to replace the time derivatives $\partial_t\bar\rho_i$ and $\partial_t(-1/\bar\theta)$. A computation shows that

$$H((\boldsymbol\rho,\theta)(t)|(\bar{\boldsymbol\rho},\bar\theta)(t))\le K_1\cdots+K_5, \tag{6.60}$$

where

$$K_1=-\int_0^t\int_\Omega\big(\kappa|\nabla\log\theta|^2-\bar\kappa|\nabla\log\bar\theta|^2\big)\mathrm{d}x\mathrm{d}s+\int_0^t\int_\Omega\bar\kappa\nabla\bar\theta\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2}-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_\Omega(\kappa\nabla\theta-\bar\kappa\nabla\bar\theta)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s,$$

$$K_2=-\int_0^t\int_\Omega\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\cdot\nabla\bigg(\frac{\rho_i}{\bar\rho_i}\bigg)\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i=1}^n\frac{\nabla\bar\rho_i}{m_i\bar\rho_i}\cdot(\rho_iu_i-\bar\rho_i\bar u_i)\mathrm{d}x\mathrm{d}s,$$

$$K_3 = -\int_0^t \int_\Omega \sum_{i=1}^n \bar{h}_i \bar{u}_i \cdot \nabla\left(\frac{\theta}{\bar{\theta}^2} - \frac{1}{\bar{\theta}}\right) \mathrm{d}x\mathrm{d}s$$
$$- \int_0^t \int_\Omega \sum_{i=1}^n (h_i u_i - \bar{h}_i \bar{u}_i) \cdot \nabla\left(-\frac{1}{\bar{\theta}}\right) \mathrm{d}x\mathrm{d}s,$$
$$K_4 = -\frac{1}{2} \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \rho_i \rho_j |u_i - u_j|^2 \mathrm{d}x\mathrm{d}s$$

and

$$K_5 = \frac{1}{2} \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j \left|\bar{u}_i - \bar{u}_j\right|^2 \mathrm{d}x\mathrm{d}s.$$

The term K_1 can be rewritten as

$$K_1 = -\int_0^t \int_\Omega \frac{1}{\bar{\theta}} (\kappa\bar{\theta} - \bar{\kappa}\theta) \nabla(\log\theta - \log\bar{\theta}) \cdot \nabla \log\bar{\theta} \mathrm{d}x\mathrm{d}s$$
$$- \int_0^t \int_\Omega \kappa \left|\nabla(\log\theta - \log\bar{\theta})\right|^2 \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_\Omega \frac{\theta - \bar{\theta}}{\bar{\theta}} \nabla \log\bar{\theta} \cdot (\kappa \nabla \log\theta - \bar{\kappa} \nabla \log\bar{\theta}) \mathrm{d}x\mathrm{d}s =: K_{11} + K_{12} + K_{13}.$$

The algebraic system (6.5) with $d_i = \nabla(\rho_i\theta)/m_i$ can be formulated as

$$-m_i \sum_{j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j (\bar{u}_i - \bar{u}_j) - \bar{\rho}_i \nabla \log\bar{\theta} = \nabla\bar{\rho}_i.$$

This allows us to rewrite K_2:

$$K_2 = \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \rho_i \rho_j (u_i - u_j) \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \rho_i \bar{\rho}_j (\bar{u}_i - \bar{u}_j) \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \rho_i \bar{\rho}_j (\bar{u}_i - \bar{u}_j) \cdot u_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j (\bar{u}_i - \bar{u}_j) \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i} \rho_i \nabla \log\theta \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i} \rho_i \nabla \log\bar{\theta} \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i} \rho_i \nabla \log\bar{\theta} \cdot u_i \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i} \bar{\rho}_i \nabla \log\bar{\theta} \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s$$
$$=: K_{21} + \cdots + K_{28}.$$

Furthermore, it follows from $h_i = (c_w + 1/m_i)\rho_i\theta$ and $\sum_{i=1}^n \rho_i u_i = \sum_{i=1}^n \bar{\rho}_i \bar{u}_i = 0$

that

$$
\begin{aligned}
K_3 &= -\int_0^t \int_\Omega \sum_{i=1}^n \bar{h}_i \bar{u}_i \cdot \nabla\Big(\frac{\theta}{\bar{\theta}^2}\Big) \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n h_i u_i \cdot \nabla\Big(-\frac{1}{\bar{\theta}}\Big) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^t \int_\Omega \sum_{i=1}^n \Big(c_w + \frac{1}{m_i}\Big) \bar{\theta} \bar{\rho}_i \bar{u}_i \cdot \nabla\Big(\frac{\theta}{\bar{\theta}^2}\Big) \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \sum_{i=1}^n \Big(c_w + \frac{1}{m_i}\Big) \theta \rho_i u_i \cdot \nabla\Big(-\frac{1}{\bar{\theta}}\Big) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^t \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{\theta}}{m_i} \bar{u}_i \cdot \nabla\Big(\frac{\theta}{\bar{\theta}^2}\Big) \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{\rho_i \theta}{m_i} u_i \cdot \nabla\Big(-\frac{1}{\bar{\theta}}\Big) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i}{m_i \bar{\theta}} \cdot \nabla\theta \mathrm{d}x\mathrm{d}s + 2\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i \theta}{m_i \bar{\theta}^2} \cdot \nabla\bar{\theta} \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega \sum_{i=1}^n \frac{\rho_i u_i \theta}{m_i \bar{\theta}^2} \cdot \nabla\bar{\theta} \mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We reformulate K_4 as

$$
\begin{aligned}
K_4 &= -\frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j \big|(u_i - \bar{u}_i) - (u_j - \bar{u}_j)\big|^2 \mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j |\bar{u}_i - \bar{u}_j|^2 \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j (u_i - u_j)\cdot(\bar{u}_i - \bar{u}_j) \mathrm{d}x\mathrm{d}s =: K_{41} + K_{42} + K_{43}.
\end{aligned}
$$

A long but straightforward computation shows that

$$
\begin{aligned}
&K_{21} + K_{22} + K_{23} + K_{24} + K_{42} + K_{43} + K_5 \\
&\quad = -\int_0^T \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i(\rho_j - \bar{\rho}_j)(u_i - \bar{u}_i)\cdot(\bar{u}_i - \bar{u}_j) \mathrm{d}x\mathrm{d}s =: L_1
\end{aligned}
$$

and

$$K_{25}+K_{26}+K_{27}+K_{28}+K_3=\int_0^T\int_\Omega\sum_{i=1}^n\frac{1}{m_i}(\rho_i-\bar\rho_i)(\nabla\log\theta-\nabla\log\bar\theta)\cdot\bar u_i\mathrm{d}x\mathrm{d}s$$
$$+\int_0^T\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\bar\rho_i\bar u_i\cdot(\nabla\log\theta-\nabla\log\bar\theta)\Big(1-\frac{\theta}{\bar\theta}\Big)\mathrm{d}x\mathrm{d}s$$
$$+\int_0^T\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\rho_i(u_i-\bar u_i)\cdot\nabla\log\bar\theta\Big(1-\frac{\theta}{\bar\theta}\Big)\mathrm{d}x\mathrm{d}s$$
$$+\int_0^T\int_\Omega\sum_{i=1}^n\frac{1}{m_i}(\rho_i-\bar\rho_i)\bar u_i\cdot\nabla\log\bar\theta\Big(1-\frac{\theta}{\bar\theta}\Big)\mathrm{d}x\mathrm{d}s$$
$$=:L_2+L_3+L_4+L_5.$$

Inserting these expressions into (6.60), putting K_{12} on the left–hand side, and rearranging the terms, we find that

$$H((\boldsymbol\rho,\theta)(t)|(\bar{\boldsymbol\rho},\bar\theta)(t))+\frac12\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2\mathrm{d}x\mathrm{d}s \tag{6.61}$$
$$+\int_0^t\int_\Omega\kappa|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s\le K_{11}+K_{13}+L_1+\cdots+L_5.$$

The second term on the left–hand side can be bounded from below. Indeed, it follows from the symmetry of (b_{ij}), definition (6.20) of M_{ij}, and the positive definiteness (6.21) of M on L that

$$\frac12\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2$$
$$=\sum_{i=1}^n\bigg(\sum_{j=1,\,j\neq i}^n b_{ij}\rho_j\bigg)\rho_i|u_i-\bar u_i|^2-\sum_{i,j=1,\,i\neq j}^n b_{ij}\rho_i\rho_j(u_i-\bar u_i)\cdot(u_j-\bar u_j)$$
$$=\sum_{i,j=1}^n M_{ij}\sqrt{\rho_i}(u_i-\bar u_i)\cdot\sqrt{\rho_j}(u_j-\bar u_j)\ge\mu_M|P_L\boldsymbol Y|^2,$$

where $Y_j=\sqrt{\rho_j}(u_j-\bar u_j)$. The norm of the projection is computed according to

$$|P_L\boldsymbol Y|^2=|\boldsymbol Y|^2-|P_{L^\perp}\boldsymbol Y|^2=\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2-\sum_{i=1}^n\frac{\rho_i}{\rho^2}\bigg|\sum_{j=1}^n\rho_j(u_j-\bar u_j)\bigg|^2$$
$$=\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2-\frac{1}{\rho}\bigg|\sum_{j=1}^n(\rho_j-\bar\rho_j)\bar u_j\bigg|^2\ge\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2-C_1\sum_{j=1}^n(\rho_j-\bar\rho_j)^2,$$

where we used $\sum_{i=1}^n \rho_i u_i = 0$ in the third equality, and $C_1 > 0$ depends on ρ_* and the $L^\infty(\Omega_T)$ norms of $\bar{u}_j$, $j = 1, \ldots, n$. Consequently,

$$\begin{aligned} &\frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j \big|(u_i - \bar{u}_i) - (u_j - \bar{u}_j)\big|^2 \mathrm{d}x\mathrm{d}s \\ &\qquad \geq \mu_M \int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x\mathrm{d}s - C_2 \int_0^t \int_\Omega \sum_{j=1}^n (\rho_j - \bar{\rho}_j)^2 \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{6.62}$$

We turn to the estimation of the terms on the right–hand side of (6.61). By the Lipschitz continuity of κ and Young's inequality, K_{11} is estimated as

$$\begin{aligned} K_{11} &= -\int_0^t \int_\Omega \frac{1}{\bar{\theta}}\Big(\kappa(\bar{\theta} - \theta) + (\kappa - \bar{\kappa})\theta\Big)\nabla\bar{\theta} \cdot \nabla(\log\theta - \log\bar{\theta})\mathrm{d}x\mathrm{d}s \\ &\leq \frac{c_\kappa}{8}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar{\theta})|^2 \mathrm{d}x\mathrm{d}s + C_3 \int_0^t \int_\Omega (\theta - \bar{\theta})^2 \mathrm{d}x\mathrm{d}s, \end{aligned}$$

and $C_3 > 0$ depends on c_κ (see Assumption (A4)), and the $L^\infty(\Omega_T)$ norms of θ and $\nabla\log\bar{\theta}$. A similar estimate shows that

$$\begin{aligned} K_{13} &= -\int_0^t \int_\Omega \frac{\theta - \bar{\theta}}{\bar{\theta}}\big(\kappa\nabla(\log\theta - \log\bar{\theta}) + (\kappa - \bar{\kappa})\nabla\log\bar{\theta}\big) \cdot \nabla\log\bar{\theta}\mathrm{d}x\mathrm{d}s \\ &\leq \frac{c_\kappa}{8}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar{\theta})|^2 \mathrm{d}x\mathrm{d}s + C_4 \int_0^t \int_\Omega (\theta - \bar{\theta})^2 \mathrm{d}x\mathrm{d}s, \\ L_2 &\leq \frac{c_\kappa}{8}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar{\theta})|^2 \mathrm{d}x\mathrm{d}s + C_5 \int_0^t \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x\mathrm{d}s, \\ L_3 &\leq \frac{c_\kappa}{8}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar{\theta})|^2 \mathrm{d}x\mathrm{d}s + C_6 \int_0^t \int_\Omega (\theta - \bar{\theta})^2 \mathrm{d}x\mathrm{d}s, \end{aligned}$$

observing that C_4 depends on c_κ, δ and the $L^\infty(\Omega_T)$ norms of θ, $\nabla\log\bar{\theta}$, and $\bar{u}_i$, C_5 depends on the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$, and C_6 depends on c_κ, ρ^*, δ, and the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$ $(i = 1, \ldots, n)$. Moreover, by Young's inequality again,

$$\begin{aligned} L_1 &\leq \frac{\mu_M}{4}\int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x\mathrm{d}s + C_7 \int_0^t \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x\mathrm{d}s, \\ L_4 &\leq \frac{\mu_M}{4}\int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x\mathrm{d}s + C_8 \int_0^t \int_\Omega (\theta - \bar{\theta})^2 \mathrm{d}x\mathrm{d}s, \end{aligned}$$

where C_7 depends on ρ^*, μ_M, and the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$ $(i = 1, \ldots, n)$, while

C_8 depends on δ, ρ^*, and the $L^\infty(\Omega_T)$ norm of $\nabla \log \bar\theta$. Finally,

$$L_5 \le C_9 \int_0^T \int_\Omega \sum_{i=1}^n (\rho_i - \bar\rho_i)^2 \mathrm{d}x\mathrm{d}s + C_{10} \int_0^T \int_\Omega (\theta - \bar\theta)^2 \mathrm{d}x\mathrm{d}s,$$

where $C_9 > 0$ depends on the $L^\infty(\Omega_T)$ norms of $\bar u_i$ $(i = 1, \dots, n)$, and C_{10} depends on δ and the $L^\infty(\Omega_T)$ norm of $\nabla \log \bar\theta$.

Summarizing the previous estimations, we infer from (6.61), (6.62), and the lower bound for κ (see Assumption (A4)) the conclusion. □

It remains to estimate the right–hand side of (6.59) in terms of the relative entropy. For this, we observe that, by [10, Lemma 16],

$$\int_\Omega \sum_{i=1}^n \frac{1}{m_i}\Big(\rho_i \log\frac{\rho_i}{\bar\rho_i} - (\rho_i - \bar\rho_i)\Big)\mathrm{d}x \ge C \int_\Omega \sum_{i=1}^n (\rho_i - \bar\rho_i)^2 \mathrm{d}x.$$

Furthermore, for all functions $f \in C^1(\mathbb{R})$ with $f'(1) = 0$,

$$\begin{aligned} f(s) - f(1) &= (s-1)\int_0^1 f'(\sigma(s-1)+1)\mathrm{d}\sigma = (s-1)\int_0^1 f'(\tau(s-1)+1)\big|_{\tau=0}^{\sigma}\mathrm{d}\sigma \\ &= (s-1)^2 \int_0^1 \int_0^\sigma f''(\tau(s-1)+1)\mathrm{d}\tau\mathrm{d}\sigma. \end{aligned}$$

This yields, choosing $f(s) = -\log s + s - 1$ and $s = \theta/\bar\theta$,

$$\int_\Omega c_w\rho\Big(-\log\frac{\theta}{\bar\theta} + \frac{1}{\bar\theta}(\theta - \bar\theta)\Big)\mathrm{d}x \ge \int_\Omega c_w\rho\frac{(\theta-\bar\theta)^2}{\max\{\theta,\bar\theta\}^2}\mathrm{d}x \ge C\int_\Omega (\theta-\bar\theta)^2\mathrm{d}x,$$

where $C > 0$ depends on the lower bound for $\bar\theta$ in Ω_T. By definition of the relative entropy, we conclude from Lemma 40 that

$$\begin{aligned} H((\boldsymbol\rho,\theta)(t)|(\bar{\boldsymbol\rho},\bar\theta)(t)) &+ \frac{\mu_M}{2}\int_0^t\int_\Omega\sum_{i=1}^n \rho_i|u_i - \bar u_i|^2\mathrm{d}x\mathrm{d}s \\ &+ \frac{c_\kappa}{2}\int_0^t\int_\Omega |\nabla(\log\theta - \log\bar\theta)|^2\mathrm{d}x\mathrm{d}s \le C\int_0^t H(\boldsymbol\rho,\theta|\bar{\boldsymbol\rho},\bar\theta)\mathrm{d}s. \end{aligned}$$

Gronwall's lemma shows that $H((\boldsymbol\rho,\theta)(t)|(\bar{\boldsymbol\rho},\bar\theta)(t)) = 0$ and hence $\boldsymbol\rho(t) = \bar{\boldsymbol\rho}(t)$ and $\theta(t) = \bar\theta(t) = 0$ in Ω for $t > 0$. This finishes the proof.

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