# Analyticity with extra-logical information

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## Abstract

In this paper, a new approach to the issue of extra-logical information within analytic (i.e. obeying the sub-formula property) sequent systems is introduced. We prove that incorporating extra-logical axioms into a purely logical system can preserve analyticity, provided these axioms belong to a suitable class of formulas that can be decomposed into a set of equivalent initial sequents and are permutable over the cut rule. Our approach is applicable not only to first-order classical and intuitionistic logics, but also to substructural logics. Furthermore, we establish a limit for the augmented systems under analysis: exceeding the boundaries of their respective classes of extra-logical axioms leads to either a loss of analyticity or a loss of structural properties.

*Keywords*: Cut-elimination and analyticity, extra-logical axioms, classical and intuitionistic logics, substructural logics

## 1 Introduction

In the broadest possible sense, an extra-logical axiom is any formula that encodes mathematical, empirical or topic-specific information, and is not valid by purely logical reasoning. A century ago, Post proved that adding any extra-logical axiom to propositional classical logic, closed under arbitrary substitution, trivializes the resulting system [22]. This fundamental property was later dubbed 'Post completeness' (or 'completeness in the sense of Post') by Church[1] [4]. First-order classical logic with identity can be consistently extended with extra-logical axioms, as is the case with mathematical theories. Closure under substitution has two facets in first-order logic: axiom schemes are closed under substitution, such as the induction scheme in Peano arithmetic, and first-order axioms involving quantifiers are closed under substitution of terms for variables.

However, the addition of extra-logical axioms to any logical system (particularly to any sequent-based inference system) often results in a loss of analyticity. In some cases, the extended sequent calculus may not allow for a cut-elimination proof, due to the logical complexity of the added axioms. In other cases, it is possible to regain a cut-elimination theorem, but full analyticity is either lost and recovered in a weaker form, or the calculus loses some structural properties of the base. Therefore, full analyticity, which is essentially identified with the subformula property, is generally considered incompatible with extra-logicality.

Proof theory has proposed different solutions to the issue of extra-logical information (mathematical in particular) in a deductive framework. Gentzen himself contemplated two ways of extending sequent calculus by mathematical axioms. The first way involved adding the axioms as a context $\Gamma$, relativizing each theorem to $\Gamma$ and thus proving sequents of the form $\Gamma \vdash A$ [9]. The second way incorporated 'mathematical sequents' directly into the logical system, which are substitution instances of $P_1 \dots P_m \vdash Q_1 \dots Q_n$, with $P_i$ and $Q_i$ atomic [10]. However, both solutions

[1] As pointed out by a reviewer, the term 'extra-logical axiom' should refer only to axioms not closed under substitution. Intuitively, extra-logical axioms could be conceived as being intrinsically topic-dependent. In contrast, taking the closure under substitution of an axiom would amount to adding a logical axiom.

Vol. 00, No. 0, 
https://doi.org/10.1093/logcom/exae013

are accompanied by difficulties: the first allows for a proof of cut-elimination, but precludes the structural analysis of the proof and every theorem—including the logical ones—depends on the extra-logical axioms, whereas the second enables only a *partial* cut-elimination, wherein every derivation is transformed into another with cuts only on axioms.

In the 1980s, Girard brought attention back to the issue by claiming that the introduction of extra-logical axioms is incompatible with a full cut-elimination theorem [11]. He provided a frequently cited example where the addition of axioms $\vdash A$ and $\vdash A \to B$ leads, through cut, to a proof of $\vdash B$ as follows:

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}\,\text{Cut}}{A \vdash B}\,\text{Cut}}{\vdash B}\,\text{Cut}$$

Girard noticed that cut-elimination can be maintained using a system in which axioms—expressed as sequents—are closed under cut[2] . However, he argued that such a system would not be as interesting from a proof-theoretical perspective because the initial sequents would have a somewhat 'ad hoc' form. It is not entirely clear what Girard meant by 'ad hoc', but it is possible that he was referring to the fact that the closure under cut of universally quantified axioms necessitates an infinite number of cuts, leaving the form of an initial sequent effectively indeterminate.

Negri and von Plato proposed an approach in the late 1990s for treating proper extensions of purely logical calculi [19]. They introduced a method for converting mathematical axioms into rules of inference while preserving the admissibility of the structural rules and cut. However, using the conversion method, full analyticity is lost and only a restricted version of it can be recovered. In particular, every formula in a derivation of a system augmented by such rules is either a subformula of the end-sequent or an atomic formula. Additionally, the terms found in the derivation must be either eigenvariables or subterms of a term that appear in the conclusion.

Negri and von Plato's account has been generalized by Beckmann and Buss [3] in the light of *free-cut elimination*, originally adumbrated by Takeuti to transform derivations in Peano arithmetic into derivations involving only cuts in which at least one of the cut formulas is principal in an induction axiom [24]. Beckmann and Buss studied extensions of Gentzen's **LK** with non-logical axioms and used free-cut elimination to obtain computational complexity results in proof theory.

Another recent approach, mainly applied to propositional non-classical logics, is pursued by Ciabattoni, Galatos and Terui, who introduced a method for converting axioms into structural rules [5]. Their proposal is based on a hierarchy over formulas, similar to the arithmetic hierarchy $\Sigma_n$ and $\Pi_n$, and is extended to hypersequent calculi. The resulting calculi are fully analytic, but they fail to satisfy good structural properties. In particular, certain structural rules such as external contraction cannot be easily dispensed with.

In this paper, we introduce a novel approach to address the issue of extra-logicality within deductive systems. Specifically, we achieve the following results:

1. For a large class of sequent systems—including presentations of classical, intuitionistic and substructural logics—we characterize a set of axioms which can be added to the base system preserving the structural properties and ensuring the maintenance of full analyticity in the form of the subformula property.

[2] A set of axioms **X** is closed under cut if whenever $\Gamma \Rightarrow \Delta, A$ and $A, \Pi \Rightarrow \Sigma$ are in **X**, so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$.

2. If the starting set of axioms is finite and complies with certain syntactic constraints (which vary depending on the logic under consideration), the resultant system will contain a finite number of initial sequents.
3. We show that, in certain instances, such as the theory of equality and of partial orders, even when the set is infinite, it is feasible to provide an explicit definition of its constituent elements.

Therefore, given a logic with a sequent calculus **G**, we identify abstract criteria for the addition of extra-logical information. Another way to conceive the result is to say that we isolate three classes of formulas: one which is amenable to a fully analytic and finitary treatment, one which allows an analytic treatment but the shape of the resulting sequents is not explicitly definable and one which cannot be converted in initial sequents to be addeed while preserving analyticity.

Instead of a rule-based approach, we adopt and generalize what one may call the *decomposition-based* approach. In a nutshell, each logical system discussed is associated with a set of extra-logical axioms from a class of formulas which admit a complete decomposition into an equivalent set of initial sequents containing only atomic formulas. We then take the closure of the resulting set under cut to obtain a fully analytic calculus. Special care is required to deal with the interaction between the rules of cut and contraction (whenever present) due to the use of contexts as multisets instead of sets as in [20]. An analogous strategy was defined by Avron [2] and elaborated in [20] in relation to Makinson's logic of pivotal assumptions which extends classical propositional logic [16]. On the more conceptual side, the transformation of an extra-logical axiom into an initial sequent instead of an inference rule has the advantage that an initial sequent bears a closer similarity to the notion of axiom broadly construed. The underlying suggestion, indeed, is that rules and axioms are two distinct entities and they should be treated as such: rules involve a more complex structure and have an inferential and thus an intrinsically logical nature, whereas axioms are constructs outside logic delivering a new kind of information. The classes of decomposable formulas are individuated according to a hierarchy of formulas analogous to the one described by Ciabattoni, Galatos and Terui and based on the notion of polarity which has been mainly developed in the context of linear logic [1, 12, 15].

Unsurprisingly, our crucial technical device is the use of invertibility of logical rules, which is a very powerful tool in proof theory [14]. This property ensures that decomposing a formula does not result in a loss of information, so that invertibility naturally identifies a class of formulas which can be turned into initial extra-logical sequents. Also Negri and von Plato appealed to the decomposition of formulas via the invertibility of certain classes of rules. In particular, they hinted at the conversion of axioms of mathematical theories into rules for classical and intuitionistic base systems and they discussed the relations between different approaches to extra-logical axioms[3] [18, Chapter 6, §6.3]. The account we are going to present, however, needs to be distinguished from theirs in two main respects, apart from treating extra-logical axioms as initial sequents rather than rules. First, this method encompasses not only classical and intuitionistic logics, but it can be extended to the case of substructural logics. Secondly, this method allows one to regain the full analyticity of the logical systems considered in the form of a full subformula property and to preserve all the structural properties of their base calculi. Concerning the procedure of conversion of axioms into inference rules, the relation between Negri and von Plato's method and the proof-theoretically relevant notion of polarity has been recently explored in the paper [17].

[3] Classical and intuitionistic extra-logical axioms are called *proper axioms* in their works.

In order to obtain a sufficient criterion, the key point is combining invertibility with a condition representing the permutability of initial sequents over cuts. These considerations form the base for a general and abstract cut elimination theorem for axiomatic extensions of a cut-free sequent calculus. The result is particularly interesting because it does not refer to any specific language, but appeals exclusively to the structural properties of a given calculus.

This method is not circular since closure under cut is a preliminary step towards analyticity, and the cut-elimination theorem is established and proven for the new system. The axiomatic extension of a given system can easily be shown to be unique up to logical equivalence of the decomposed formula. Furthermore, the uniform extension of our methodology to a variety of logical systems is achieved in a modular fashion in the case of classical and intuitionistic (and, possibly, modal) logics by calibrating the set of formulas that can be decomposed in relation to the systems. The case of substructural logics is more delicate since the closure under cut of finite sets of sequents requires the extra-condition of acyclicity (see [6]). Lastly, our method arguably introduces a limit for full analyticity in the context of sequent systems. In fact, the decomposition of extra-logical axioms into initial sequents turns out to be the maximal extension of a calculus that can be achieved while preserving strong structural properties, especially cut-elimination, and at the same time maintaining the full-fledged analyticity represented by the subformula property[4] .

Finally, we consider some simple examples of first-order theories with axioms explicitly containing quantifiers as a case study. Specifically, we focus on theories of partial orders and equality. We demonstrate that in the cases considered here, it is possible to provide an explicit description of the set of extra-logical axioms that are closed under cut. The possibility of extending this approach and finding a general framework for theories is left as a theme for future research.

The paper is structured as follows. Section 3 provides an abstract sufficient criterion for cut-elimination in the presence of extra-logical axioms. Section 4 is devoted to the specific cases of classical and intuitionistic logic which explain the delicate interaction between the notions of invertibility and permutability. In Section 5, we extend the results to the first-order setting discussing as specific examples the ones of classical and intuitionistic logic. Section 6 deals with substructural logics and the connection between extra-logical axioms and the rules of contraction and weakening. Section 7 explores the possibility of giving an explicit presentation of this approach in the case of axioms involving quantifiers, and provide examples. Finally, we conclude with some remarks about future work in Section 8.

## 2 Preliminaries

In this section, we establish the terminology that will be used throughout the paper. In particular, we start by defining some notions which are commonly discussed in the literature on structural proof theory in an abstract way. The only assumption we make at this point is the difference between *atomic* and *compound* formulas. The final segment of the section will introduce the calculi that we will be working with along with their languages. As usual, a multiset of formulas is a collection of formulas in which repetitions count. We use upper case Greek letters to denote multisets $\Gamma, \Delta, \Pi, \dots$

[4] As pointed out by a referee, there are calculi enjoying cut-elimination but not satisfying the subformula property.

DEFINITION 1
A *sequent* is a pair of multisets, separated by a turnstile $\vdash$,

$$\Gamma \vdash \Delta$$

where $\Gamma$ is the antecedent and $\Delta$ is the succedent. A sequent where $\Gamma$ ($\Delta$) is allowed to contain at most one formula is called single-antecedent (succedent), otherwise we refer to it as multi-antecedent (succedent).

DEFINITION 2
A rule is an ordered pair $(X, \Gamma \vdash \Delta)$, where $X$ is a list of premises $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_m \vdash \Delta_m$ and $\Gamma \vdash \Delta$ is the conclusion of the rule.

Graphically, we represent rules as follows

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \ldots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta} \text{R}$$

If the set $X$ is empty, we say that the rule is a zeroary rule (or a rule with no premises).

DEFINITION 3
In a rule instance, a formula is said to be *principal* if it is the formula which gets decomposed in passing from the conclusion to the premise(s). For zeroary rules, which are essentially initial sequents, we refer to the atomic formulas displayed in the rule as *active*.

According to our earlier definition, principal formulas can only be compound formulas. In particular, the calculi we will examine exclusively feature logical $n$-ary rules with $n \geq 1$. Consequently, an atomic formula never serves as the principal element in a rule application.

DEFINITION 4
A *context* in a rule is a multiset of formulas that remains unaltered during the rule's application. A rule is categorized as *multiplicative* or *additive* depending on whether its contexts undergo modification or not when passing from the conclusions to the premises.

DEFINITION 5
A set $X$ of sequents, finite or infinite, is said to be *closed under cut* if, for every pair of sequents $\Gamma \vdash \Delta, p$ and $p, \Gamma' \vdash \Delta'$ the sequent $\Gamma, \Gamma' \vdash \Delta, \Delta'$ resulting from the application of the cut rule is contained in $X$.

A calculus **G** is a set of rules. We assume that **G** contains only rules whose principal formulas are compound. There are two types of initial sequents:

- *logical* initial sequents, which are the zeroary rules of **G**;
- *extra-logical* initial sequents in which the multisets contain only atomic formulas and they are not derivable in **G**.

REMARK 1

In the case of extra-logical sequents we limit ourselves to considering sequents in which the active formulas are multisets of atomic formulas. This is crucial in order to achieve cut-elimination. In case of logics which enjoy weakening admissibility the extra-logical sequents are allowed to contain arbitrary contexts, but active formulas are only allowed to be atomic.

We assume that a sequent calculus **G** contains exclusively initial sequents and logical rules, i.e. rules which govern connectives. Furthermore, we assume that in each rule only a single formula is principal.

DEFINITION 6

A rule is said to be *admissible* in **G** if, whenever the premises are derivable, so is the conclusion.

We recall the structural rules of weakening, contraction and the cut rule.

$$\frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{ Weak} \qquad \frac{\Gamma, \Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta}\text{ Ctr} \qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta, \Delta'}\text{ Cut}$$

We assume to be working with **G3**-style variants ([25]) in which active formulas in initial sequents are only allowed to be atomic and the structural rules of weakening and contraction are admissible and not explicit rules of the calculus.[5] To simplify the reading of this work, we shall introduce here all the propositional fragments of calculi which will be discussed in the rest of the paper (later we will detail their first-order extensions).

### *2.1* G3cp

The language contains $n$-ary predicate symbols ($n > 0$), a denumerable set, denoted by $At$ of propositional variables $P_0, P_1, P_2, \ldots$ and connectives $\bot, \wedge, \vee, \rightarrow$. Metavariables for formulas are $A, B, C, \ldots$

**Initial Sequents**

$$\frac{}{P, \Gamma \vdash \Delta, P}\,Ax \qquad \frac{}{\bot, \Gamma \vdash \Delta}\,L\bot$$

**Logical Rules**

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\,L\wedge \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R\wedge$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,L\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta}\,L\rightarrow \qquad \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash \Delta, A \rightarrow B}\,R\rightarrow$$

[5]This can be obtained adopting some strategies. In particular, weakening is absorbed in the initial sequents and in the design of the rules with context-restrictions. Contraction is hidden in the formulation of the logical rules: unary rules are multiplicative, binary rules are additive and, when needed, principal formulas are repeated in the premise(s) of the rules.

### *2.2* G3ip

Intuitionistic logic is presented via a Maehara's style multi-succedent sequent calculus. The language coincides with the one of classical logic and the calculus is obtained from **G3cp** by replacing rules $L\rightarrow$ and $R\rightarrow$ with the following. Notice that R→ has a restriction on the context in the succedent in the premise and the rule L→ has an explicit repetition of the principal formula in the left premise. In Section 5 we will add further details on the calculus **G3ip** and on our choice to work with a multi-succedent calculus for intuitionistic logic.

$$\frac{A\rightarrow B,\Gamma\vdash\Delta,A \qquad B,\Gamma\vdash\Delta}{A\rightarrow B,\Gamma\vdash\Delta}\,L\rightarrow \qquad \frac{A,\Gamma\vdash B}{\Gamma\vdash\Delta,A\rightarrow B}\,R\rightarrow$$

**MALL**

**MALL** is the propositional fragment of linear logic without exponentials. We consider a propositional language, containing a denumerable set $At$ of propositional variables $P,Q,R,\ldots$, additive ($\oplus,\&$), multiplicative ($⅋,\otimes$) connectives and the involutive negation ($^{\perp}$). In order to maintain the presentation uniform, let us consider **MALL** in its two-sided sequent calculus [13].

**Initial Sequents**

$$\frac{}{P\vdash P}\,ax.$$

**Logical Rules**

$$\frac{\Gamma\vdash\Delta,A}{\Gamma,A^{\perp}\vdash\Delta}\,L^{\perp} \qquad \frac{\Gamma,A\vdash\Delta}{\Gamma\vdash\Delta,A^{\perp}}\,R^{\perp}$$

$$\frac{\Gamma,A,B\vdash\Delta}{\Gamma,A\otimes B\vdash\Delta}\,L\otimes \qquad \frac{\Gamma\vdash\Delta,A \qquad \Gamma'\vdash\Delta',B}{\Gamma,\Gamma'\vdash\Delta,\Delta',A\otimes B}\,R\otimes$$

$$\frac{A,\Gamma\vdash\Delta \qquad B,\Gamma'\vdash\Delta'}{A⅋B,\Gamma,\Gamma'\vdash\Delta,\Delta'}\,L⅋ \qquad \frac{\Gamma\vdash\Delta,A,B}{\Gamma\vdash\Delta,A⅋B}\,R⅋$$

$$\frac{A,\Gamma\vdash\Delta \qquad B,\Gamma\vdash\Delta}{A\oplus B,\Gamma\vdash\Delta}\,\mathrm{L}\oplus \qquad \frac{\Gamma\vdash\Delta,A_i}{\Gamma\vdash\Delta,A_1\oplus A_2}\,\mathrm{R}\oplus_i,\ i\in\{1,2\}$$

$$\frac{A_i,\Gamma\vdash\Delta}{A_1\,\&\,A_2,\Gamma\vdash\Delta}\,\mathrm{L}\&_i,\ i\in\{1,2\} \qquad \frac{\Gamma\vdash\Delta,A \qquad \Gamma\vdash\Delta,B}{\Gamma\vdash\Delta,A\,\&\,B}\,R\&$$

## 3 Abstract cut-elimination

We characterize the conditions according to which a set of propositional axioms can be converted into a set of initial sequents and yield a system which is closed under cut. In particular, we introduce sufficient conditions in order for a class of formulas to be converted into initial sequent and closure under cut within a sequent calculus **G** with certain properties.

We assume that the sequent calculus **G** enjoys cut-elimination and that the strategy runs by induction on two parameters: the degree of the cut formula and the sum of the height of the derivations of the premises of cut. The general strategy we refer to is the one detailed in [25]. Essentially, one distinguishes two cases: the cut formula is principal in both premises of the cut or not: in the first case, one replaces the cut with cuts on formulas of lesser complexity, otherwise the cut is permuted upwards.

DEFINITION 7
A rule is *invertible* if for every application of the rule the derivability of the conclusion entails the derivability of each of the premises of the rule. An application of a rule is an *invertible step* if the derivability of the conclusion entails the derivability of each of the premises.

REMARK 2
According to the definition, we immediately get that whenever a rule is invertible, then all its applications are invertible steps. The converse does not hold, but there can be applications of non-invertible rules which are invertible steps. For example, in a multiple-succedent setting—such as Maehara's calculus (see 2.2)—the right rule for implication is invertible in **G3ip** whenever the context in the succedent is empty, i.e. $\Gamma, A \vdash B$ is derivable if and only if $\Gamma \vdash A \to B$ is derivable. So the rule R$\to$ is not invertible in **G3ip**, but if the multiset $\Delta$ in the conclusion is empty, an application of R$\to$ is an invertible step.

DEFINITION 8
Given a sequent calculus **G**, a closed formula $A$ is *decomposable* in **G** if the sequent $\vdash A$ can be reduced to a multiset of initial sequents containing atomic formulas using only invertible steps. We define the set $\mathcal{S}_A$ to be the set of the initial sequents thus obtained.

EXAMPLE 1
Consider the formula $P \to (Q \to R)$, where $P$, $Q$ and $R$ are atomic formulas, and the classical sequent calculus **G3c**, see Section 2.1. This formula is decomposed into the initial sequent $P, Q \vdash R$.

Given a set of decomposable propositional formulas $X$, we define $\mathcal{S}_X = \bigcup_{A \in X} \mathcal{S}_A$. It is immediate to observe that $\mathbf{G} + \{\vdash A\}$ is equivalent to $\mathbf{G} + \mathcal{S}_A$, where the system $\mathbf{G} + \{\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n\}$ is the system obtained by adding as initial sequents $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$.

REMARK 3
The notion of decomposability is strongly tied with invertibility, but does not coincide with it. Observe that if a formula is decomposable, then its reduction into extra-logical initial sequents is obtained only through invertible steps. However, the converse does not hold in general. Indeed, there are rules which are invertible, but do not allow for a decomposition of formulas such as the rule L$\to$ in **G3i**. Due to the repetition of the principal formula in the left premise, the rule is trivially invertible, but it does not allow for a decomposition of $A \to B$ in the antecedent.

REMARK 4
Since **G** does not have any rule whose principal formulas are atomic, the derivability of extra-logical sequents is decidable.

Note that in the context of classical and intuitionistic logic, extra-logical sequents coincide with the *complementary* ones, i.e. sequents made up only of atomic formulas and in which the intersection between antecedent and succedent is empty.

DEFINITION 9
Given an extralogical sequent $\Gamma \vdash \Delta$, its closure under contraction is the sequent $\Gamma' \vdash \Delta'$ where $P$ is in $\Gamma'$ ($\Delta'$) if and only if $P$ is in $\Gamma$ ($\Delta$) and $P$ occurs only once in $\Gamma'$ ($\Delta'$).

DEFINITION 10
The closure under cut of a set of extra-logical sequents $\mathcal{S}_X$ in a system **G** is obtained as follows:

- If contraction is admissible in **G**, take the closure under contraction of $\mathcal{S}_X$.
- If $\Gamma \vdash \Delta, P$ and $P, \Gamma' \vdash \Delta'$ are in $\mathcal{S}_X$, then if $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is not an initial sequent, add it.
- Repeat the two steps.

Formally put, the above definition can be conceived as the definition of an operator $\mathbb{C}$ on the powerset of the set extra-logical sequents. Since $\mathbb{C}$ can be easily seen to be monotone and the powerset of the set extra-logical sequents is a complete lattice, we get (by Knaster–Tarski's theorem) the existence of the fixed point of the operator $\mathbb{C}$ which is the closure under cut of $\mathcal{S}_{\mathbf{X}}$. Let us consider a calculus **G** of a set **X** of decomposable formulas; we define the closure under cut of $\mathcal{S}$ as $\mathcal{S}_X^*$.

REMARK 5
The set $\mathcal{S}_X^*$ is, in general, infinite. In some cases, when $X$ is finite, $\mathcal{S}_X^*$ will be finite, but this is not always the case, as detailed in Section 6, devoted to substructural logics.

If weakening is admissible in **G**, we require that, whenever $\Gamma \vdash \Delta \in \mathcal{S}_X^*$, then $\Gamma, \Gamma' \vdash \Delta, \Delta' \in \mathcal{S}_X^*$ for any $\Gamma'$ and $\Delta'$.

For every set of decomposable formulas $X$, we now prove some general results concerning the addition of extra-logical axioms $\mathcal{S}_X^*$—we denote them as $\overline{ax}$—to a base sequent calculus **G**. Let us call the resulting system $\mathbf{GE}_{\mathcal{S}_X^*}$.

LEMMA 1
The following statements hold:

1. If weakening is admissible in **G**, then it is admissible in $\mathbf{GE}_{\mathcal{S}_X^*}$.
2. If a rule is invertible in **G**, then so is in $\mathbf{GE}_{\mathcal{S}_{\mathbf{X}}^*}$.
3. If contraction is admissible in **G**, then it is admissible $\mathbf{GE}_{\mathcal{S}_X^*}$.

PROOF. The properties need to be checked only with respect to the initial extra-logical sequents.

The proof of item 1. is immediate by the definition of $\mathcal{S}_X^*$. With respect to item 3., if we have a derivation of the initial sequent $P, P, \Gamma \vdash \Delta$ which is an extra-logical axiom, then, by definition of the operator $\mathcal{S}_X^*$, $P, \Gamma \vdash \Delta$ is an initial sequent too.

The proof of item 2. is immediate by noticing that compound formulas are never principal in extra-logical initial sequents. □

In the next definition we exploit the fact that atomic formulas are never principal in the rules of the calculi under consideration.

DEFINITION 11
An extra-logical sequent can be left permuted over a rule $\rho$ of **G** whenever the cut between it and every premise of the rule in which the cut formula occurs commute, in the sense that the derivation:

$$\dfrac{\dfrac{\ldots \quad \Gamma' \vdash \Delta', P \quad \ldots}{\Gamma \vdash \Delta, P} \qquad \dfrac{}{P, \Pi \vdash \Sigma}\,\overline{ax}}{\Gamma, \Pi \vdash \Delta, \Sigma}\,Cut$$

where $P$ is atomic (and active in $\overline{ax}$), can be transformed into

$$\dfrac{\dots \qquad \dfrac{\Gamma' \vdash \Delta', P \qquad \dfrac{}{P, \Pi' \vdash \Sigma'}\,\overline{ax}}{\Gamma', \Pi' \vdash \Delta', \Sigma'}\,Cut \qquad \dots}{\Gamma, \Pi' \vdash \Delta, \Sigma'}$$

where $P, \Pi' \vdash \Sigma'$ is in $\mathcal{S}^*_X$ and $\Pi'(\Sigma') \subseteq \Pi(\Sigma)$ (this is the case if **G** is closed under weakening). The notion of left permutation is defined symmetrically.

REMARK 6

Notice that if the cut formula $P$ is not present in a premise of the rule $\rho$, then the extra-logical sequent is trivially permutable.

DEFINITION 12

An extra-logical sequent $\Gamma \vdash \Delta$ is *permutable* whenever it is left and right permutable over the rules of the calculus.

In our approach the term 'permutable' applies to sequents and should not be confused or associated with permutation rules, as found e.g. in the linear logic literature.

THEOREM 1

Let **X** be a set of decomposable formulas. If $\mathcal{S}^*_X$ is a set of permutable sequents, the calculus $\mathbf{GE}_{\mathcal{S}^*_X}$ admits cut-elimination.

PROOF. The proof runs by double induction as in the case of the base system **G**. The only cases to check are the ones in which one of the two premises of the cut is an extra-logical initial sequents. Suppose, without loss of generality, that the left premise of the cut is an extra-logical sequent (the other case is symmetric), then we have

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, P}\,\overline{ax} \qquad P, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

We consider the right premise of the cut and we distinguish cases:

1. If $P, \Gamma' \vdash \Delta'$ is an initial sequent, then we distinguish two subcases. Either it is an initial tautological sequent or it is an extra-logical sequent.

    - If it is a tautological initial sequent, then $P, \Gamma' \vdash \Delta'$ is of the shape $P, \Gamma' \vdash \Delta', P$. If the logic contains weakening, then the conclusion is obtained as follows by admissibility of weakening:

    $$\dfrac{\Gamma \vdash \Delta, P}{\Gamma, \Gamma' \vdash \Delta, \Delta', P}\,\text{Weak}$$

    - Otherwise, $P, \Gamma' \vdash \Delta'$ is $P \vdash P$ and thus $\Gamma \vdash \Delta, P$ is the desired conclusion.
    - If it is an extra-logical initial sequent, then we distinguish two subcases, either $P$ is active in $P, \Gamma' \vdash \Delta'$ or not. In the first case, then it part of a weakened context and the conclusion is also an extra-logical sequent. If $P$ is active, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is an extra-logical sequent too due to the procedure of closure under cut.

2. If the height of $P, \Gamma' \vdash \Delta'$ is $> 0$, then the cut is permuted upwards and removed invoking the induction on the sum of the height of the derivations of the premises of the cut by the property of permutability, because atomic formulas are never principal in any rule of the system **G**. Since by assumption the cut elimination proof in **G** runs by a double induction on the degree and the height of the sum of the derivation of the premises of the cut, we observe that the addition of extra-logical axioms does not require new inductive measures. □

As observed by a referee, in the cut-elimination theorem it is crucial that cuts between extra-logical axioms are always redundant, because we have taken the closure under cut of the set of extra-logical sequents. As an immediate corollary of the cut-elimination theorem, we get the equivalence of the systems $\mathbf{G} + A$ with the cut rule and $\mathbf{GE}_{\mathcal{S}^*_{\mathbf{A}}}$.

COROLLARY 1
$\mathbf{G} + A + Cut$ proves $\Gamma \vdash \Delta$ if and only if $\mathbf{GE}_{\mathcal{S}^*_{\mathbf{A}}}$ proves $\Gamma \vdash \Delta$.

PROOF. From left to right we use Theorem 1 and the derivability of the formula $A$ in $\mathbf{GE}_{\mathcal{S}^*_{\mathbf{A}}}$ (immediate by a root-first application of the rules). In the other direction, we observe that—by definition—the decomposability of the formula $A$ is obtained using only invertible steps which preserve the derivability of sequents. To generate all the sequents in $\mathcal{S}^*_A$ we use the cut rule (and, possibly, the contraction rule) and this concludes the proof. □

Furthermore, we get the uniqueness of the axiomatic extensions. Indeed, we can establish the following general result.

PROPOSITION 1
Let $X, Y$ be sets of decomposable and permutable formulas. If $\mathbf{GE}_{\mathcal{S}^*_X} = \mathbf{GE}_{\mathcal{S}^*_Y}$, then $\mathcal{S}^*_X = \mathcal{S}^*_Y$.

PROOF. Let us assume towards a contradiction that $\mathcal{S}^*_X \neq \mathcal{S}^*_Y$, hence there is, without loss of generality, a sequent $\Gamma \vdash \Delta \in \mathcal{S}^*_X$ which is not in $\mathcal{S}^*_Y$. Formulas occurring in $\Gamma$ and $\Delta$ are only atomic formula and clearly $\mathbf{GE}_{\mathcal{S}^*_X}$ derives it. By assumption, $\mathbf{GE}_{\mathcal{S}^*_X} = \mathbf{GE}_{\mathcal{S}^*_Y}$, so $\mathbf{GE}_{\mathcal{S}^*_Y}$ derives it, which yields a contradiction. □

REMARK 7
As pointed out by a reviewer, it is possible for extra-logical initial sequents to contain multiple principal formulas. Nevertheless, there is no necessity to explicitly account for this scenario in the cut-elimination proof. This is because each application of the cut rule eliminates only one formula, thereby ensuring the comprehensiveness of our argument.

REMARK 8
The requirement of permutability explains the reason why invertibility is a necessary criterion to achieve cut-elimination, but not a sufficient one. Indeed, let us consider a concrete case of this phenomenon. The calculus **G3ip** (see 2.2) has an invertible rule for the disjunction. Hence, if we consider the set $\{\vdash Q_1 \vee Q_2\}$, where $Q_1$ and $Q_2$ are atomic formulas, the sequent contains only decomposable formulas. However, $\vdash Q_1, Q_2$ is not permutable. Consider e.g.

$$\dfrac{\dfrac{}{\vdash Q_1, Q_2}\,\overline{ax} \qquad \dfrac{\dfrac{}{Q_1, R \vdash Q_1}\,\overline{ax}}{Q_1 \vdash R \to Q_1}\,R\to}{\vdash R \to Q_1, Q_2}\,Cut$$

The end-sequent cannot be derived without cut. This shows why the failure of permutability impairs cut-elimination.

In what follows we will analyse some concrete cases to which we apply the procedure introduced above.

# 4 Classical and intuitionistic logic

In this section we deal with the propositional fragments of classical and intuitionistic logic. The treatment of first-order extensions is postponed to the next section. We recall that we denote with **G3cp** and **G3ip** the propositional fragments of the sequent calculi for classical and intuitionistic logic, respectively.

## *4.1 Classical logic*

Let us consider the calculus **G3cp** absorbing weakening and contraction into axiom and rules.

THEOREM 2
In **G3cp** the rules of weakening and contraction are height-preserving admissible, every rule is height-preserving invertible and cut is admissible.

PROOF. The reader is referred to [25]. □

The cut-elimination theorem runs by double induction with main induction on the degree of the cut formula and secondary induction hypothesis on the sum of the height of the derivations of the premises of the cut [25]. In the case of classical propositional logic any formula can be converted in a set of initial sequents. Indeed, applying backwards the rules of the calculus **G3cp**, every formula is decomposable into sequents of the shape: $\Gamma \vdash \Delta$ where $\Gamma, \Delta$ are multisets of atomic formulas. Furthermore, every such sequent is permutable (this will be shown below in the cut elimination theorem).

Indeed, such formulas are built up from invertible connectives, so they satisfy the decomposability requirement. Furthermore, it can be easily checked that permutability of cuts for extra-logical sequents holds with respect to the rules of the calculus **G3cp**. Given a set $X$ of extra-logical axioms, we denote by $\mathbf{G3cp}_{\mathcal{S}^*_X}$ the calculus obtained by adding as extra-logical initial sequents the sequents in the closure under cut of $X$, i.e. $\mathcal{S}^*_X$.

LEMMA 2
The rule of weakening and contraction are height-preserving admissible in $\mathbf{G3cp}_{\mathcal{S}^*_X}$.

PROOF. Immediate. □

THEOREM 3
For every finite subset $X$ of formulas, the calculus $\mathbf{G3cp}_{\mathcal{S}^*_X}$ admits cut-elimination.

PROOF. Cuts between extra-logical axioms are redundant, because the conclusion is already an extra-logical initial sequent. Hence, by the general cut-elimination theorem, we need to check the permutability of the rules. We consider the case of unary rules, the case of binary rules is analogous. Let $\rho$ be any unary rule of **G3cp**, we have

$$\dfrac{\dfrac{\Gamma'' \vdash \Delta'', P_1}{\Gamma \vdash \Delta, P_1} \qquad \dfrac{}{\Gamma', P_1, P_2, \ldots, P_m \vdash Q_1, \quad , Q_n, \Delta'}\,\overline{ax}}{,\Gamma', P_2, \ldots, P_m \vdash Q_1, \ldots, Q_n, \Delta, \Delta'}\,\text{Cut}$$

The cut is permuted as follows:

$$\dfrac{\dfrac{\Gamma'' \vdash \Delta'', P_1 \qquad \overline{\Gamma', P_1, P_2, \ldots, P_m \vdash Q_1, \ldots, Q_n, \Delta'}\,{\scriptstyle ax}}{\Gamma, \Gamma'', P_2, \ldots, P_m \vdash Q_1, \ldots, Q_n, \Delta, \Delta''}\,{\scriptstyle \mathrm{Cut}}}{\Gamma, \Gamma', P_2, \ldots, P_m \vdash Q_1, \ldots, Q_n, \Delta, \Delta'}$$

□

REMARK 9
We observe that we could add to the system a countable set of extra-logical axioms **X**. In that case, $\mathcal{S}_{\mathbf{X}}$ is countable. Indeed, the decomposition of a formula $A \in \mathbf{X}$ gives a finite set of initial sequents and so $\mathcal{S}_{\mathbf{X}}$, being a countable union of finite sets, is countable. Furthermore, every initial sequent in the closure under contraction and cut of $\mathcal{S}_{\mathbf{X}}$ corresponds to a formula of the language and is therefore enumerable.

### *4.2 Intuitionistic logic*

Only a subset of propositional formulas can be turned into a set of extra-logical sequents with preservation of cut admissibility as in the case of classical logic. This is due to the lack of full invertibility of the logical rules in intuitionistic logic. The following hierarchy of formulas was introduced in [5].

DEFINITION 13
The classes $\mathcal{N}_n$ and $\mathcal{P}_n$ of formulas are inductively defined as follows.

- $\mathcal{P}_0 = \mathcal{N}_0 = At$
- $\mathcal{P}_{n+1} = \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}$
- $\mathcal{N}_{n+1} = \bot \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1}$

The stratified definition of the classes $\mathcal{N}_n$ and $\mathcal{P}_n$ is defined so as to take into account the polarity of the connectives in a formula and the invertibility of their corresponding rule. The set of extra-logical axioms of intuitionistic logic is $\mathcal{N}_1$.

LEMMA 3
Every formula $A$ in $\mathcal{N}_1$ is equivalent to a (finite) conjunction of formulas of the shape $P_1 \wedge \cdots \wedge P_n \to Q$, where $P_1, \ldots, P_n, Q$ are atomic formulas or $\bot$.

PROOF. By induction on the construction of the class $\mathcal{N}_1$ exploiting the following intuitionistically provable equivalences:

1. $((A \vee B) \to C) \leftrightarrow (A \to C) \wedge (B \to C)$
2. $(A \to B \wedge C) \leftrightarrow (A \to B) \wedge (A \to C)$
3. $(A \to (B \to C)) \leftrightarrow (A \wedge B \to C)$

If $A$ is $\bot$ or an atomic formula, the proof is immediate. Otherwise, it is a conjunction of two formulas $A_1, A_2$ in $\mathcal{N}_1$. In this case, we exploit the equivalence 1. If $A$ is $A_1 \to A_2$, then $A_1 \in \mathcal{P}_1$ and $A_2 \in \mathcal{N}_1$. By induction hypothesis $A_2$ is equivalent to a formula of the shape $P_1 \wedge \cdots \wedge P_n \to Q$. So the desired conclusion follows using the equivalences 2. and 3. □

In short, each formula in $\mathcal{N}_1$ can be transformed into a provably equivalent conjunction of Horn clauses, i.e. implications whose antecedent is a finite conjunction of atomic formulas and the succedent is a single atomic formula. Thus, every formula in $\mathcal{N}$ can be transformed into a set of initial sequents of the shape $P_1, \dots, P_n \vdash Q$.

In particular, we decompose the formulas by exploiting the invertibility of logical rules $L\wedge, R\wedge, L\vee$ and $R \rightarrow$ in the case in which the context in the succedent is empty.

LEMMA 4
The following statements hold relative to $\mathbf{G3ip}_{\mathcal{S}^*_X}$:

1. The rule of weakening is height-preserving admissible.
2. Every rule except for $R \rightarrow$ is height-preserving invertible.
3. The rules of contraction are height-preserving admissible.

PROOF. The proofs follow the pattern of the Section 4.1. □

The main structural property that is preserved under the addition of extra-logical axioms is cut-elimination. By Theorem 1 it suffices to show permutability of the extra-logical sequents.

THEOREM 4
For every subset $X$ of $\mathcal{N}_1$, the cut rule is admissible in $\mathbf{G3ip}_{\mathcal{S}^*_X}$.

PROOF. The proof runs by double induction, with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the heights of the derivations of the premises of the cut. We only need to check the permutability with respect to the rules of the calculus. We do it by showing that the sequents are right and left permutable. Suppose the left premise of the cut is an extra-logical initial sequent.

$$\dfrac{\dfrac{}{\Gamma, P_1, \dots, P_n \vdash \Delta, Q}\,\overline{ax} \qquad Q, \Gamma' \vdash \Delta'}{\Gamma, \Gamma', P_1, \dots, P_n \vdash \Delta, \Delta'}\,\text{Cut}$$

We can always permute the cut upwards to the premises and then remove it invoking the secondary induction hypothesis. The critical case occurs when the last rule applied is $R \rightarrow$:

$$\dfrac{\dfrac{}{\Gamma, P_1, \dots, P_n \vdash \Delta, Q}\,\overline{ax} \qquad \dfrac{B, Q, \;' \vdash C}{Q, \Gamma' \vdash \Delta'', B \rightarrow C}\,R\rightarrow}{\Gamma, \Gamma', P_1, \dots, P_n \vdash Q, \Delta, \Delta'', B \rightarrow C}\,\text{Cut}$$

In this case we proceed as follows:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, P_1, \dots, P_n \vdash Q}\,\overline{ax} \qquad B, Q, \Gamma' \vdash C}{\Gamma, \Gamma', B, P_1, \dots, P_n \vdash C}\,\text{Cut}}{\Gamma, \Gamma', P_1, \dots, P_n \vdash Q, B \rightarrow C, \Delta, \Delta''}\,R\rightarrow$$

Notice that the restriction to extra-logical axioms with only a single active formula in the succedent is crucial as in the case of rule R$\rightarrow$, otherwise the permutation would not be feasible.

Let us now consider the case in which the right premise of the cut is an extra-logical initial sequent instead. We can assume that the cut formula is one of the active formulas in it (otherwise the reduction is trivial).

$$\dfrac{\Gamma \vdash \Delta, P_1 \qquad \overline{P_1,\ldots,P_n,\Gamma' \vdash \Delta', Q}\;ax}{\Gamma, P_2,\ldots,P_n,\Gamma' \vdash \Delta,\Delta', Q}\;\text{Cut}$$

The most interesting case is when the last rule applied is $R\rightarrow$. Suppose we have

$$\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma \vdash \Delta'', A \rightarrow B, P_1}\;R\forall \qquad \overline{P_1,\ldots,P_n,\Gamma' \vdash \Delta', Q}\;ax}{\Gamma, P_2,\ldots,P_n,\Gamma' \vdash \Delta,\Delta', Q}\;\text{Cut}$$

Here the requirement of permutability vacuously holds by definition, since the cut formula does not occur in the premise. Hence we simply apply again rule $R\rightarrow$ and then by weakening admissibility we obtain the desired conclusion. □

The cut-elimination result shows that, whenever we restrict ourselves to considering Horn clauses, i.e. implications in which the antecedent is a finite conjunction of atomic formulas and the conclusion is an atomic formula, we can avoid employing rules to obtain a system enjoying cut elimination. Furthermore, our method yields a strongly analytic system which obeys the subformula property.

COROLLARY 2
The calculus $\mathbf{G3ip}_{\mathcal{S}^*_X}$ satisfies the subformula property.

PROOF. Straightforward by induction on the height of derivations. □

The result is optimal, in the sense that every formula which is not in $\mathcal{N}_1$ cannot be transformed into an equivalent set of initial sequents without losing the admissibility of cut.

THEOREM 5 (Optimality).
The class $\mathcal{N}_1$ is the only class of formulas among the $\mathcal{P}_j$ and $\mathcal{N}_k$ that allows the conversion into extra-logical sequents with preservation of cut admissibility in **G3ip**.

PROOF. It suffices to prove that in every other class there are formulas whose addition to **G3ip** results in a failure of the cut-elimination theorem.

Since for every $n \in \mathbb{N}$, $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ and $\mathcal{P}_1 \subset \mathcal{N}_2$, this boils down to show that in $\mathcal{P}_1$ there are formulas which cannot be transformed into initial sequents.

To prove this, observe that despite the invertibility of rule $R\vee$, we cannot add an initial sequent with a disjunction in the consequent. In fact, suppose we wish to transform an axiom $Q_1 \vee Q_2$ into an equivalent initial sequent $\vdash Q_1, Q_2$. The entry of such axiom breaks the cut-elimination algorithm□

REMARK 10
The optimality result highlights the connection between the class $\mathcal{N}_1$ and the property of analyticity. In particular, on the rule-based approach described in [19] it is indeed possible to convert a formula of the form $P \rightarrow Q \vee R$ (which is not in $\mathcal{N}_1$) into a rule preserving cut-elimination. However, the rule

$$\dfrac{Q, P,\ \Gamma \vdash \Delta \qquad R, P,\ \Gamma \vdash \Delta}{P,\ \Gamma \vdash \Delta}$$

does not, in general, preserve the subformula property of the calculus. Thus, if compared with the method by Negri and von Plato, we acquire something in terms of expressivity, but we lose something in terms of analyticity.

We now show that the result of uniqueness of axiomatic extension also holds for intuitionistic logic provided that we restrict ourselves to formulas in the class $\mathcal{N}_1$. A few additional remarks are in order. As mentioned by one of the reviewers, there are many interesting first-order intuitionistic mathematical theories whose axioms lie beyond the class $\mathcal{N}_1$. However, the $\mathcal{N}_1$ isolates the class of formulas which can be converted in initial sequents and added to the base calculus fully preserving analyticity as subformula property. The individuation of $\mathcal{N}_1$ stems from the intersection of the requirements of decomposability and permutability and, in this sense, the choice is not arbitrary and we claim that $\mathcal{N}_1$ represents the maximal class of extra-logical axioms which admit a strongly analytic treatment for intuitionistic logic.

### *4.3 Metalogical properties and translations*

THEOREM 6 (Glivenko).
Let $X$ be a class of axioms in $\mathcal{N}_1$. If $\mathbf{G3cp}_{\mathcal{S}^*_X}$ proves $\vdash A$, then $\mathbf{G3ip}_{\mathcal{S}^*_X}$ proves $\vdash \neg\neg A$.

PROOF. The proof runs by induction on the height of the derivations, proving the stronger statement:

$$\text{If } \mathbf{G3cp}_{\mathcal{S}^*_X} \text{ proves } \Gamma \vdash \Delta, \text{then } \mathbf{G3ip}_{\mathcal{S}^*_X} \text{ proves } \Gamma^{\neg\neg}, \Delta^{\neg} \vdash$$

where $\Pi^{\neg} = \{\neg A \mid A \in \Pi\}$.

The only interesting case occurs when $\Gamma \vdash \Delta$ is an extra-logical initial sequent. By the hypothesis it is of the form $\Gamma', P_1, \dots, P_n \vdash Q, \Delta'$. In that case $\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, \neg Q, \Delta'^{\neg} \vdash$ is derivable:

$$\cfrac{\cfrac{}{\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, P_1, \dots, P_n, \neg Q, \Delta'^{\neg} \vdash Q}\,\overline{ax}}{\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, P_1, \dots, P_n, \neg Q, \Delta'^{\neg} \vdash}\, L\to$$

$$\vdots$$

$$\cfrac{\cfrac{\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, P_1, \neg Q, \Delta'^{\neg} \vdash}{\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, \neg Q, \Delta'^{\neg} \vdash \neg P_1}\, R\to}{\Gamma'^{\neg\neg}, \neg\neg P_1, \dots, \neg\neg P_n, \neg Q, \Delta'^{\neg} \vdash}\, L\to$$

□

For a related and recent development, the reader can also consult the results contained in the note [23].

THEOREM 7
The following propositions hold:

1. $\mathbf{G3cp}_{\mathcal{S}^*_X}$ has terminating proof search.
2. Consistency of any multiset of extra-logical axioms $\Gamma$ is decidable both in classical and in intuitionistic propositional logic.

PROOF. Item 1: easily follows by induction on the height of derivations, by noting that each application of the rules in $\mathbf{G3cp}_{\mathcal{S}^*_X}$ strictly reduces the number of logical symbols in the sequent.

Item 2: follows by combining the Glivenko-style result with item 1. In fact, for every multiset of extra-logical axioms $\Gamma$ we have that $\Gamma$ is inconsistent in intuitionistic logic if and only if $\Gamma \vdash$ is derivable in **G3ip** if and only if $\bigwedge \Gamma \vdash$ is derivable in **G3ip** if and only if $\vdash \neg \bigwedge \Gamma$ is derivable in **G3ip**. By the extended Glivenko theorem we can deduce that this holds if and only if $\vdash \neg \bigwedge \Gamma$ is derivable in $\mathbf{G3cp}_{\mathcal{S}^*_\Gamma}$. But proof search terminates in the latter calculus and thus the problem is decidable. □

REMARK 11
Theorem 7 shows that by combining **G3**-style calculi and the method of decomposition of axioms into initial sequents it is possible to obtain simple (and purely syntactic) decision procedures for the consistency of classical and intuitionistic extra-logical clusters of propositional formulas. This could be also related to the treatment of the logic of default assumptions discussed in [16].

## 5 Going first-order

In this section we extend the results for intuitionistic and classical logic to the first-order setting. We first recall that the full sequent calculus for first-order classical logic, **G3c**, is obtained by adding to **G3cp** the following four rules:

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\, L\exists,\ y \text{ fresh} \qquad \frac{\Gamma \vdash \Delta, \exists x A, A[t/x]}{\Gamma \vdash \Delta, \exists x A}\, R\exists$$

$$\frac{\forall x A, A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\, L\forall \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \Delta, \forall x A}\, R\forall,\ y \text{ fresh}$$

where [t/x] is the substitution operation, defined as usual. The sequent calculus for first-order intuitionistic logic is obtained by adding to **G3ip** the following rules:

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\, L\exists,\ y \text{ fresh} \qquad \frac{\Gamma \vdash \Delta, \exists x A, A[t/x]}{\Gamma \vdash \Delta, \exists x A}\, R\exists$$

$$\frac{\forall x A, A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\, L\forall \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \Delta, \forall x A}\, R\forall,\ y \text{ fresh}$$

As the reader may observe the difference with respect to the quantifers' rule is the design of the right rule for the universal quantifier. The choice to consider a multisuccedent calculus instead of a single-succedent one for intuitionistic logic serves two purposes:

1. Bring closer classical and intuitionistic logic w.r.t. their sequent calculi presentation;
2. Highlight the difference between decomposability and permutability of formulas[6] .

The classes of formulas which can be converted into initial sequents to be added to the sequent calculi preserving their structural properties are the universal closure of their propositional counterpart. In other words, in classical logic any formula of the shape:

$$\forall \bar{x}(P_1 \wedge \ldots \wedge P_m \to Q_1 \vee \ldots \vee Q_n)$$

can be converted into a countable set of initial sequents

$$\{P_1(\bar{t}_1), \ldots, P_m(\bar{t}_m) \vdash Q_1(\bar{s}_1), \ldots, Q_n(\bar{s}_n) \mid \bar{t}_i, \bar{s}_j \in TER^k\}$$

[6]For instance the rule R$\vee$ is invertible and allows for the decomposability of the sequent $\vdash P \vee Q$ also in intuitionistic logic, but it is not permutable.

where *TER* is the set of terms of the language. Analogously, the decomposable and permutable formulas of intuitionistic logic are defined as

$$\forall \bar{x}(P_1 \wedge \ldots \wedge P_m \rightarrow Q)$$

Each of these formulas is transformed into an (countably) infinite set of initial sequents $\{P_1(\bar{t}_1), \ldots, P_m(\bar{t}_m) \vdash Q(\bar{s}) \mid \bar{t}_i, \bar{s} \in TER^k\}$, where $\bar{t}_i$ denotes $t_{i1}, \ldots, t_{in_i}$ for $n_i$ terms where $n_i$ is the arity of the predicate $P_i$. We now spell out the formal details of the procedure.

Definition 14
The language contains $n$-ary predicate symbols ($n > 0$), a denumerable set of variables $x, y, z, \ldots$, $m$-ary function symbols ($m \geq 0$), connectives $\perp, \wedge, \vee, \rightarrow$ and quantifiers $\forall$ and $\exists$.

As hinted above, a specific difference concerning the structural analysis of the calculi for first-order classical and intuitionistic logics is the presence of the rule of substitution which enables to replace a variable with a term of the language. The rule of substitution for every term and variable is defined as follows:

$$\frac{\Gamma \vdash \Delta}{\Gamma[t/x] \vdash \Delta[t/x]} \; Sub[t/x]$$

where $x$ is an individual variable and $t$ is an arbitrary term of the language.

### *5.1 Abstract cut elimination for first-order languages*

We recall that our working framework is a sequent calculus **G** which satisfies cut-elimination and in which the proof is obtained by a double induction with main induction on the degree of the cut formula and secondary induction on the sum of the height of the derivations of the premises of the cut.

We recall that a closed formula $A$ in a system **G** is decomposable if it can be reduced to a multiset of initial sequents containing only atomic formulas using only invertible steps.

Example 2
We now show some examples of this phenomenon.

- The formula $\forall x(P(x) \rightarrow Q(x))$ is decomposable in **G3c**. Indeed it can be reduced to the initial sequent $P(y) \vdash Q(y)$ for some free variable $y$.
- The formula $\exists x(P(x) \rightarrow Q(x))$ is not decomposable in **G3c**. Indeed, the rule R$\exists$ is invertible, but the formula $\exists x(P(x) \rightarrow Q(x))$ never gets decomposed and is copied in every premise when applying bottom up the rule[7] .

Definition 15
Given a formula containing a free variable $x$, we denote its closure under substitution with respect to $x$ as the set of formulas $\{A[t/x] \mid t \in TER\}$. The closure under substitution of a formula is the set $\{A[t_1/x_1; \ldots ; t_n/x_n] \mid t_1, \ldots, t_n \in TER\}$ where $x_1, \ldots, x_n$ is the set of free variables of $A$.

[7] A reviewer suggested that the rules R$\exists$ and L$\forall$ are not invertible. Actually, they are trivially invertible, because the premise is obtained from the conclusion via an application of weakening due to the repetition of the principal formula in the premise. However, for this reason they do not allow for a decomposition of the formulas.

The definition naturally extends to extra-logical initial sequents. Hence, in a first-order setting we define the set $\mathcal{S}_A$ to be the closure under substitution of the free variables in the initial sequents thus obtained. The closure under cut and contraction is defined in the same way of the propositional systems. We now need to check that given a sequent calculus **G** with certain structural properties, i.e. the admissibility of substitution, weakening, contraction and cut, the addition of an extra-logical axiom $A$ (transformed into the set $\mathcal{S}^*_A$) preserves these properties. Let us once again $\mathbf{GE}_{\mathcal{S}^*_X}$ stand for the system augmented with the permutable initial sequents in $\mathcal{S}^*_X$ for some set of decomposable formulas $X$.

LEMMA 5
The following statements hold:

1. If substitution is admissible in **G**, then it is admissible in $\mathbf{GE}_{\mathcal{S}^*_X}$.
2. If weakening is admissible in **G**, then it is admissible in $\mathbf{GE}_{\mathcal{S}^*_X}$.
3. If a rule is invertible in **G**, then so is in $\mathbf{GE}_{\mathcal{S}^*_\mathbf{X}}$.
4. If contraction is admissible in **G**, then it is admissible $\mathbf{GE}_{\mathcal{S}^*_X}$.

PROOF. The properties need to be checked only with respect to the initial extra-logical sequents. The proof of item 1 follows from the fact that the initial sequents in $\mathcal{S}^*_X$ are closed under substitution.

The proofs of items 2 and 4 are immediate by the definition of $\mathcal{S}^*_X$. For example, with respect to item 3, if we have a derivation of the initial sequent $P, P, \Gamma \vdash \Delta$ which is an extra-logical axiom, then, by definition of the operator $\mathcal{S}^*_X$, $P, \Gamma \vdash \Delta$ is an initial sequent too.

The proof of item 3 is immediate by noticing that compound formulas are never principal in extra-logical initial sequents. □

The proof of cut-elimination does not change when we move to a first-order setting. Indeed, as far as the formulas considered are decomposable and permutable, the resulting calculus preserves all its original structural properties. Summing up:

- Cuts between extra-logical axioms are redundant insofar as the conclusion is already an extra-logical axioms.
- In all the remaining cases, if one of the premises of the cut rule is an extra-logical axiom and the other one is the conclusion of a certain rule, we can permute the cut upwards due to the property of permutability.

Hence we get the following theorem.

THEOREM 8
Let **X** be a set of first-order decomposable formulas. If $\mathcal{S}^*_\mathbf{X}$ is a set of permutable sequents, the calculus $\mathbf{GE}_{\mathcal{S}^*_\mathbf{X}}$ admits cut-elimination.

In the next two subsections we will detail the specific cases of classical and intuitionistic logic, showing how the general cut-elimination theorem can be adapted to these specific cases.

### *5.2 First-order classical logic*

With respect to first-order classical logic, the class of formulas which are amenable to a fully analytic treatment is analogous to the class $\Pi^0_1$ in the arithmetical hierarchy, i.e. containing formulas which are the universal closure of quantifier-free formulas.

Given a set $X$ of extra-logical axioms in $\Pi^0_1$, we denote by $\mathbf{G3c}_{\mathcal{S}^*_X}$ the calculus obtained by adding as extra-logical initial sequents the sequents in the closure under cut of $X$, i.e. $\mathcal{S}^*_X$.

We would like to conclude the discussion on classical logic by stressing a relevant point concerning the results of analyticity for the class $\Pi^0_1$. Indeed, whenever the sequents closed under cut form an infinite set, the system is analytic, but the set of extra-logical sequents is not *a priori* explicitly defined. An explicit definition of the set of initial sequents to be added becomes attainable when the set is finite. We now show that the result is optimal in the sense that if a formula falls beyond the scope of $\Pi^0_1$ it cannot be added to **G3c** while preserving its structural properties.

THEOREM 9
For every formula $A$, if $\mathbf{G3c}_{\mathcal{S}^*_A}$ admits cut-elimination, then $A \in \Pi^0_1$.

PROOF. We argue by contraposition assuming that $A \notin \Pi^0_1$. Due to the prenex normal form, any formula in classical logic can be transformed into a formula of the shape:

$$Qx_1 \dots Q_{xn}B,$$

where $Qx_1 \dots Q_{xn}$ is an alternation of universal and existential quantifiers and $B$ is a formula not containing quantifiers. If $A$ is not in the class $\Pi^0_1$, then its normal form contains an existential quantifier front of $B$. As a consequence, the formula $A$ is not decomposable, as it cannot be reduced to a multiset of extra-logical initial sequents due to the repetition of the principal formula in the rule R∃. □

Hence, we can categorize axioms into three distinct types:

- Axioms which are not amenable to a fully analytic treatment (those which do not respect the requirements of decomposability or permutability), i.e. not included in $\Pi^0_1$.
- Axioms included in $\Pi^0_1$, hence decomposable and permutable, but whose closure under cut is infinite (these are formulas of the shape $\forall xA$, where $A$ is quantifier-free).
- Axioms which are decomposable, permutable and the resulting set of sequents is finite (these are quantifier-free closed formulas).

*First-order intuitionistic logic*

To define the class of formulas which is amenable to an analytic treatment, we start by broadening our definition of the hierarchy introduced above which now includes quantifiers. With a slight abuse of notation we adopt the same notation $\mathcal{N}_n, \mathcal{P}_n$ to denote the classes of first-order formulas in intuitionistic logic.

DEFINITION 16
The classes $\mathcal{N}_n$ and $\mathcal{P}_n$ of formulas are inductively defined as follows.

- $\mathcal{P}_0 = \mathcal{N}_0 = At$
- $\mathcal{P}_{n+1} = \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \exists x \mathcal{P}_{n+1}$
- $\mathcal{N}_{n+1} = \bot \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \forall x \mathcal{N}_{n+1}$

Once again, the classes $\mathcal{N}_n, \mathcal{P}_n$ are individuated according to the properties of the rules of the calculus.

LEMMA 6
Every closed formula in $\mathcal{N}_1$ is equivalent to a (finite) conjunction of formulas of the shape $\forall \overline{x}(P_1 \wedge \dots \wedge P_n \to Q)$, where $P_1, \dots, P_n, Q$ are atomic formulas or $\bot$.

PROOF. The proof is analogous to the one for propositional intuitionistic logic and thus we omit the details. The only difference consists in exploiting the following intuitionistically provable equivalences:

- $\forall x(B \wedge C) \leftrightarrow \forall x B \wedge \forall x C$
- $\exists x(B \vee C) \leftrightarrow \exists x B \vee \exists x C$
- $\forall x(A \to B(x)) \leftrightarrow (A \to \forall x B(x))$, with $x$ not free in $A$
- $(\exists x A(x) \to B) \leftrightarrow \forall x(A(x) \to B)$, with $x$ not free in $B$

which involve quantifiers and we omit the details. □

We now give a concrete example of the transformation.

EXAMPLE 3
The formula $\exists x(P(x) \vee Q(x)) \to \forall y R(y)$ where $P, Q, R$ are unary predicates is in $\mathcal{N}_1$. Indeed, according to our definition, $P(x) \vee Q(x)$ is in $\mathcal{P}_1$ and so is $\exists x(P(x) \vee Q(x))$, whereas $\forall y R(y)$ is in $\mathcal{N}_1$ and thus $\exists x(P(x) \vee Q(x)) \to \forall y R(y)$ is in $\mathcal{N}_1$. Using the previous lemma we have the following chain of equivalences:

1. $\exists x(P(x) \vee Q(x)) \to \forall y R(y)$
2. $\exists x P(x) \vee \exists x Q(x) \to \forall y R(y)$
3. $(\exists x P(x) \to \forall y R(y)) \wedge (\exists x Q(x) \to \forall y R(y))$
4. $\forall x(P(x) \to \forall y R(y)) \wedge \forall x(Q(x) \to \forall y R(y))$
5. $\forall x \forall y(P(x) \to R(y)) \wedge \forall x \forall y(Q(x) \to R(y))$

The formula $\forall x \forall y(P(x) \to R(y)) \wedge \forall x \forall y(Q(x) \to R(y))$ can be decomposed in two different initial sequents $P(x) \vdash R(y)$ and $Q(x) \vdash R(y)$. The closure under substitution yields two infinite sets of initial sequents $\{P(t) \vdash R(s) \mid t, s \in TER\} \cup \{Q(t) \vdash R(s) \mid t, s \in TER\}$.

The structural properties are preserved under the addition of extra-logical axioms in the class $\mathcal{N}_1$. In particular, the closure under substitution of the set of initial extra-logical sequents ensures that the rule of substitution is admissible in the extended system.

LEMMA 7
The following statements hold relative to $\mathbf{G3i}_{\mathcal{S}^*_X}$:

1. The rule of substitution is height-preserving admissible.
2. The rule of weakening is height-preserving admissible.
3. Every rule except for $R\to$ and $R\forall$ is height-preserving invertible.
4. The rules of contraction are height-preserving admissible.

We finally prove the cut-elimination result.

THEOREM 10
The cut rule is admissible in $\mathbf{G3i}_{\mathcal{S}^*_{\mathbf{X}}}$ for $\mathbf{X} \subseteq \mathcal{N}_\mathbf{1}$.

PROOF. The proof runs by double induction, with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the heights of the derivations of the premises of the cut. We only need to check the permutability with respect to the new rules for the quantifiers.

Suppose the left premise of the cut is an extra-logical initial sequent.

$$\dfrac{\overline{\Gamma, P_1, \ldots, P_n \vdash \Delta, Q}^{\,ax} \qquad Q, \Gamma' \vdash \Delta'}{\Gamma, \Gamma', P_1, \ldots, P_n \vdash \Delta, \Delta'}\,\text{Cut}$$

With respect to the propositional case, the new case arises when the last rule applied is $R\forall$:

$$\dfrac{\dfrac{}{\Gamma, P_1, \dots, P_n \vdash \Delta, Q}\,\overline{ax} \qquad \dfrac{Q, \Gamma' \vdash B[y/x]}{Q, \Gamma' \vdash \Delta'', \forall x B}\,R\forall}{\Gamma, \Gamma', P_1, \dots, P_n \vdash Q, \Delta, \Delta'', \forall x B}\,\text{Cut}$$

We can assume that the variable $y$ does not occur in the left premise of the cut. Otherwise, we replace it by a new fresh variable, say $u$, via height-preserving admissibility of substitution. In this case we proceed as follows:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, P_1, \dots, P_n \vdash Q}\,\overline{ax} \qquad Q, \Gamma' \vdash B[y/x]}{\Gamma, \Gamma', P_1, \dots, P_n \vdash B[y/x]}\,\text{Cut}}{\Gamma, \Gamma', P_1, \dots, P_n \vdash Q, \forall x B, \Delta, \Delta''}\,R\forall$$

Notice that the restriction to extra-logical axioms with only a single active formula in the succedent is crucial, otherwise the permutation would not be feasible in the case of the rule R$\forall$.

Let us now consider the case in which the right premise of the cut is an extra-logical initial sequent instead. We can assume that the cut formula is one of the active formulas in it (otherwise the reduction is trivial).

$$\dfrac{\Gamma \vdash \Delta, P_1 \qquad \dfrac{}{P_1, \dots, P_n, \Gamma' \vdash \Delta', Q}\,\overline{ax}}{\Gamma, P_2, \dots, P_n, \Gamma' \vdash \Delta, \Delta', Q}\,\text{Cut}$$

The new case is when the last rule applied is $R\forall$. Suppose we have

$$\dfrac{\dfrac{\Gamma \vdash B[y/x]}{\Gamma \vdash \Delta'', \forall x B, P_1}\,R\forall \qquad \dfrac{}{P_1, \dots, P_n, \Gamma' \vdash \Delta', Q}\,\overline{ax}}{\Gamma, P_2, \dots, P_n, \Gamma' \vdash \Delta', \Delta'', \forall x B, Q}\,\text{Cut}$$

Here the requirement of permutability vacuously holds by definition, since the cut formula does not occur in the premise. Hence we simply apply again rule $R\forall$ and then by weakening admissibility we obtain the desired conclusion. □

The optimality result from the section on propositional logic obviously continues to hold with respect to first-order extensions.

REMARK 12

We would like to draw a comparison between our method and the one developed by Ciabattoni, Galatos and Terui [5]. Their method and ours differ for three reasons. First, their approach transforms axioms into rules of inference, whereas we remain closer to the original axiomatic structure considering initial sequents instead of rules. Secondly, they work with a concrete calculus and then consider its extensions, whereas we establish our result for an abstract sequent calculus **G** with certain general properties. Finally, they focus on propositional logic, whereas we study axiomatic extensions even in first-order languages.

## 6 Extra-logicality and linearity

The goal of this section is to study our methodology in relation to substructural ones. Confining ourselves to propositional languages is aligned with the trend in the literature to study substructural logics in their propositional format as a consequence of the fact that the structural rules, acting on formulas as 'blocks', essentially have a propositional nature. However, as it will be clear from the exposition, the results we will present can be also expanded to include the first-order extensions of these logics.

We shall discuss the cases of the multiplicative and additive fragment of linear logic **MALL**, see 2.2, a system wherein contraction and weakening rules are dropped. This system motivates the generality of the definition of extra-logical axiom which will no longer coincide with complementary sequents. In particular, we will show how certain structural properties can be recovered in the form of extra-logical information.

We use $At$ to denote the set of atomic formulas. Without loss of generality, we focus on atomic initial sequents, so as to maximize the invertibility of the logical rules. As usual, $A \multimap B$ abridges $A^{\perp} \parr B$.

DEFINITION 17
The classes $\mathcal{NL}_n$ and $\mathcal{PL}_n$ of formulas are inductively defined as follows.

- $\mathcal{PL}_0 = \mathcal{NL}_0 = At$
- $\mathcal{PL}_{n+1} = \mathcal{NL}_n \mid (\mathcal{NL}_{n+1})^{\perp} \mid \mathcal{PL}_{n+1} \otimes \mathcal{PL}_{n+1} \mid \mathcal{PL}_{n+1} \oplus \mathcal{PL}_{n+1}$
- $\mathcal{NL}_{n+1} = \mathcal{PL}_n \mid (\mathcal{PL}_{n+1})^{\perp} \mid \mathcal{NL}_{n+1} \,\&\, \mathcal{NL}_{n+1} \mid \mathcal{NL}_{n+1} \parr \mathcal{NL}_{n+1}$

This definition is connected to the works in the field of linear logic [1, 12, 15], which focus on positive and negative formulas (according to the invertibility of the rules of the calculus) with a stratification to measure their complexity, i.e. the nestings of invertible and non invertible connectives in a formula. In **MALL** the class of axioms $\mathbb{EA}_{MALL}$ which can be translated into extra-logical initial sequents is simply defined as $\mathcal{NL}_1$.

LEMMA 8
Rule $R\parr$, $R\&$, $L\otimes$ and $L\oplus$ are height-preserving invertible.

PROOF. Let us discuss, as an example, the case of $R\parr$. If $n = 0$, then $\Gamma, A \parr B$ is an initial sequent and so is $\Gamma, A, B$. If $n > 0$ and $A \parr B$ is principal we take the premise. If $n > 0$ and $A \parr B$ is not principal, we apply the induction hypothesis to the premise(s) and then the rule again. □

LEMMA 9
Every extra-logical axiom $A$ in $\mathbb{EA}_{MALL}$ can be transformed into an equivalent formula $B$, i.e. $A \multimapboth B$, of the form: $B_1 \,\&\cdots\&\, B_m$, where $B_i \equiv P_1 \otimes \cdots \otimes P_{n_i} \multimap Q_1 \parr \ldots \parr Q_{m_i}$, with $P's$ and $Q's$ being atomic formulas.

PROOF. One proceeds by applying the invertibility of the rules proved in Lemma 8 to decompose the sequent $\vdash A$. Once the process is over one obtains top-sequents $S_1, \ldots, S_m$ of the form $P_{1j}, \ldots, P_{nj} \vdash Q_{1j}, \ldots, Q_{kj}$, where $P_{ij}, Q_{lj}$ are an atomic propositional formula, with $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, $l \in \{1, \ldots, k\}$. Take the formula

$$B \equiv (\bigparr_{1 \leqslant i_1 \leqslant n_1} P^{\perp}_{i_1 1} \parr \bigparr_{1 \leqslant l_1 \leqslant k_1} Q_{l_1 1}) \,\&\cdots\&\, (\bigparr_{1 \leqslant i_m \leqslant n_m} P^{\perp}_{i_m m} \parr \bigparr_{1 \leqslant l_m \leqslant k_m} Q_{l_m m})$$

Clearly, $\vdash A, B^{\perp}$ and $\vdash A^{\perp}, B$ are both provable, so the claim of the theorem holds. □

For every set $X$ of formulas in $\mathbb{EA}_{MALL}$, let $\mathbf{MALL}_{\mathcal{S}^*_X}$ denote the calculus obtained by supplementing **MALL** with the initial extra-logical sequents in $\mathcal{S}^*_X$. Notice that with respect to **MALL** the sequent $P \vdash P, Q$ meets the requirements of extra-logicality (being atomic and underivable), yet it is not complementary.

Before we proceed with the closure under cut we need to stress a substantial point concerning logics without weakening rules. In the cases of classical and intuitionistic logic given a finite set of initial sequents, its closure under cut is finite as well. In the case of logics without weakening this is not always the case. Let us consider the following toy example.

EXAMPLE 4
Consider the sequents $Q \vdash P, Q$ and $\vdash Q, R$. If we perform a cut, we get

$$\dfrac{\vdash Q, R \qquad Q \vdash P, Q}{\vdash P, Q, R}\ \text{Cut}$$

The sequent $\vdash P, Q, R$ needs to be cut again with $Q \vdash P, Q$. This yields $\vdash P, P, Q, R$. It is not difficult to realize that the procedure does not terminate, leading to infinitely many sequents $\vdash P^n, Q, R$ for every natural number $n$, where $P^n$ denotes the repetition of the formula $P$ n-times.

The phenomenon stems from the fact that sequents of the form $P, \Gamma \vdash \Delta, P$, with either $\Gamma$ or $\Delta$ not being empty, represent tautological initial sequents in both intuitionistic and classical logic. However, in the absence of weakening, these sequents are underivable and so extra-logical. The term 'cyclic' is employed in this context to convey the idea that in the procedure of closure under cut they generate infinitely many sequents which differ only with respect to repetitions of the same formulas. The term should not be confused with other developments in the field of linear logic, such as cyclic linear logic [7]. A similar difficulty was pointed out in [6].

DEFINITION 18
An extra-logical sequent of the form $P, \Gamma \vdash \Delta, P$, where $\Gamma$ or $\Delta$ is not empty, is called a *cyclic* sequent. A sequent which is not cyclic is *acyclic*.

We present now a formal proof of the termination of the closure under cut for acyclic sequents in **MALL**. In particular, for any finite set of sequents $X$, we have that $\mathcal{S}^*_X$ is finite if and only if it is acyclic.

THEOREM 11
For any set of extra-logical sequents $X$, $\mathcal{S}^*_X$ is finite if and only if $\mathcal{S}^*_X$ does not contain cyclic sequents.

PROOF. The right to left direction is immediate. In fact, suppose an extra-logical sequent $\Gamma, P \vdash \Delta, P$ is in $\mathcal{S}^*_X$, then $(\Gamma)^n, P \vdash (\Delta)^n, P$ is in $\mathcal{S}^*_X$ for every $n$.

For the left to right direction we consider the set $X$ of extra-logical sequents. Let $P$ be a propositional variable occurring in $X$. We denote by $X_P$ and $X_{P^\perp}$ the subsets of $X$ in which $P$ occurs in the succedent (antecedent), respectively.

By acyclicity, we get $X_P \cap X_{P^\perp} = \emptyset$. Furthermore, since by assumption $\mathcal{S}^*$ does not contain cyclic sequents, if a sequent in $X_P$ ($X_{P^\perp}$) contains more than one occurrence of $P$, then sequents in $X_{P^\perp}$ ($X_P$) will contain exactly one occurrence of $P$. Let us assume without loss of generality that $X_P$ contains sequents with possibly more than one occurrence of $P$. So $X_P = \{\Pi \vdash \Delta, P^m \mid \text{ for some } m \geq 1\}$.

We operate all possible cuts on $P$ and we denote by $X_P^+$ the set obtained by adding the set

$$CUT_P = \{\Gamma^n, \Pi \vdash \Delta, \Sigma^n, P^{m-n} \mid \Pi \vdash \Delta, P^m \in X_P \text{ and} \Gamma, P \vdash \Sigma \in X_{P^\perp}\}$$

to $X$. Since by acyclicity $\Gamma^n \vdash \Sigma^n$ does not contain $P$, we observe that for every $m$ $\Pi, \Gamma^n \vdash \Delta, \Sigma^n, P^{m-n}$ contains a strictly lower number of occurrences of $P$ than $\Pi \vdash \Delta, P^m$. Therefore the set $CUT_P$ is finite. We pick a new variable $Q$ and we repeat the same operation.

We claim that the set $(X_P^+)_Q^+$ is closed under cuts on $P$, i.e. if $\Gamma \vdash \Delta, P$ and $P, \Gamma' \vdash \Delta'$ are in $(X_P^+)_Q^+$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is in $(X_P^+)_Q^+$. The desired conclusion easily follows by noticing that cuts on atomic formulas permute. In fact the two cuts:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, P \qquad P, \Pi \vdash \Sigma, R}{\Gamma, \Pi \vdash \Sigma, \Delta, R}\,\text{Cut} \qquad R, \Theta \vdash \Lambda}{\Gamma, \Pi, \Theta \vdash \Sigma, \Lambda, \Delta}\,\text{Cut}$$

can be rearranged as

$$\dfrac{\Gamma \vdash \Delta, P \quad \vdash \Sigma \qquad \dfrac{P, \Pi \vdash \Sigma, R \qquad R, \Theta \vdash \Lambda}{P, \Pi, \Theta \vdash \Sigma, \Lambda}\,\text{Cut}}{\Gamma, \Pi, \Theta \vdash \Sigma, \Lambda, \Delta}\,\text{Cut}$$

Therefore, we can in general push upwards all the cuts on the same atomic formula. As a consequence, we get $(X_P^+)_Q^+ = X_{P,Q}^+$.

The procedure is iterated for every atomic formula. The atomic formulas are finite, therefore the procedure terminates. Since every step consists in adding a finite number of sequents to a finite set, the output will be a finite set as well. As usual, cut-elimination is **MALL** is obtained via a double induction with main induction hypothesis on the degree of the cut formulas and secondary induction hypothesis on the sum of the height of the derivation of the premises of the cut. □

THEOREM 12
The rule:

$$\dfrac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Sigma}{\Gamma, \Pi \vdash \Delta, \Sigma}\,\text{Cut}$$

is admissible in $\mathbf{MALL}_{\mathcal{S}_X^*}$.

PROOF. Immediate by checking the permutability of the class of extra-logical axioms. For example, we have

$$\dfrac{\dfrac{}{P_1, \ldots, P_m \vdash Q_1, \ldots, Q_m}\,\overline{ax} \qquad \dfrac{Q_m, \Gamma \vdash \Delta, A \qquad \Pi \vdash \Sigma, B}{Q_m, \Pi, \Gamma \vdash \Delta, \Sigma, A \otimes B}\,\otimes}{P_1, \ldots, P_m, \Pi, \Gamma \vdash \Delta, \Sigma, Q_1, \ldots, Q_{m-1}, A \otimes B}\,\text{Cut}$$

which is replaced by the following cut of lesser height:

$$\dfrac{\dfrac{\dfrac{}{P_1, \ldots, P_m \vdash Q_1, \ldots, Q_m}\,\overline{ax} \qquad Q_m, \Gamma \vdash \Delta, A}{\Gamma, P_1, \ldots, P_m \vdash \Delta, Q_1, \ldots, Q_{m-1}, A}\,\text{Cut} \qquad \Pi \vdash \Sigma, B}{\Gamma, P_1, \ldots, P_m, \Pi \vdash \Delta, \Sigma, Q_1, \ldots, Q_{m-1}, A \otimes B}\,\otimes$$

The other cases are dealt with analogously. □

### *6.0.1 Regaining monotonicity*

Let us consider the formula $P \multimap (Q \multimap P)$, with $P, Q$ atomic. This is an instance of the classically (and intuitionistically) valid schema $A \to (B \to A)$, corresponding to the weakening rule in sequent calculus and forbidden in linear logic. On our methodology it is possible to convert such formula in the initial sequent $P, Q \vdash P$. The closure under cut yields the infinite set of sequents $\{P, Q^n \vdash P \mid n \leq 1\}$. By adding it to our base calculus **MALL** we are in the position to recover the admissibility of a restricted form of weakening for the formula $Q$. This can be understood as expressing the fact that $Q$ is compatible with the information conveyed by $P$; we prove this fact in a more general form.

PROPOSITION 2
Let $A$ be a formula in the class $\mathbb{EA}_{MALL}$ and $B$ a formula containing only $\otimes$ and $\oplus$. Then, the rule

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, B \vdash \Delta, A}\ \mathrm{W}$$

is admissible in $\mathbf{MALL}_{\mathcal{S}_{\{\mathcal{A} \multimap (\mathcal{B} \multimap \mathcal{A})\}}}$.

PROOF. The formula $A \multimap (B \multimap A)$ corresponds to $A^{\perp} \parr (B^{\perp} \parr A)$ which is in $\mathbb{EA}_{MALL}$ by definition of the class of formulas. Hence we can decompose it into a certain number of initial sequents. Consider now the following derivation:

$$\frac{\Gamma \vdash \Delta, A \qquad A, B \vdash A}{\Gamma, B \vdash \Delta}\ \mathrm{Cut}$$

The rightmost top-sequent is derivable in $\mathbf{MALL}_{\mathcal{S}_{\{\mathcal{A} \multimap (\mathcal{B} \multimap \mathcal{A})\}}}$ and cut can be eliminated by the cut-elimination algorithm. □

REMARK 13
In affine linear logic, i.e. linear logic extended with weakening, cyclic sequents can be discarded and so the closure under cut of any set of initial sequents is indeed finite.

## 7 Order theories and equality: a case study

We have considered the addition of axioms belonging to certain classes of formulas to base calculi. We have seen that considering a finite cluster of extra-logical axioms not containing quantifiers yields a finite set of initial sequents in the case of intuitionistic and classical logics. Therefore it is possible to give a complete description of the shape of the set of initial extra-logical sequents. We call such a set *explicit*. On the contrary, when we deal with axioms containing universal quantifiers, the closure under substitution imposes the need to consider infinitely many sequents, therefore we may not be able to give an explicit presentation of the set of initial sequents. A natural question thus arises:

> Is it possible to give an explicit description of the set of initial sequents closed under cut for axioms containing quantifiers?

In this section we deal with some simple examples of first-order theories whose axioms contain quantifiers in which the method of axioms as initial sequent can be applied to obtain an explicitly defined set of sequents closed under the cut rule. We limit ourselves to considering the case of theories over classical logic for the sake of brevity, but analogous results could be obtained for intuitionistic logic. We deal with the theory of equality and with the theory of partial orders. The first theory is based on a language with the language $=$ and it is axiomatized by the two axioms (the second is actually an axiom schema):

**EQ1** $\forall x(x = x)$ Reflexivity
**EQ2** $\forall xy(x = y \land A(x) \to A(y))$ Replacement for every formula $A$

The second one is defined by the following axioms with the language $<$:

**PO1** $\forall x \neg(x < x)$ Irreflexivity
**PO2** $\forall xyz(x < y \land y < z \to x < z)$ Transitivity

Let us now introduce the notion of chain which will be crucial to our development[8] .

DEFINITION 19
An ordering chain $\mathcal{C}$ from $a$ to $b$, in symbols $a\mathcal{C}^{<}b$, is a finite multiset of atomic formulas $a < c_1, \dots, c_n < b$ where $c_i < c_{i+1}$ for every $\{1, \dots, n-1\}$.

DEFINITION 20
An equality chain $\mathcal{C}$ from $a$ to $b$, in symbols $a\mathcal{C}^{=}b$, is a finite multiset of atomic formulas $c_0 = c_1, \dots, c_{n-1} = c_n$ where $c_i = c_{i+1}$ for every $\{0, \dots, n-1\}$ and $a$ is $c_k$ and $b$ is $c_j$ for some $k, j \in \{0, \dots, n-1\}$.

### *7.1 Partial orders*

The key point consists in finding a set of initial sequents which is closed under cut. This is not trivial, because *prima facie* there is not a way to identify such set and to give an explicit definition of it. A heuristics is to observe the shape of the cuts between instances of the transitivity axiom:

$$\dfrac{a < b, b < c \vdash a < c \qquad a < c, c < d \vdash a < d}{a < b, b < c, c < d \vdash a < d}\ \text{Cut}$$

In a way, it can be argued that the cut replaces the atom $a \leq c$ in the antecedent of the right premise of the cut with the chain from $a$ to $c$. Hence we consider the following class of initial sequents:

$$\dfrac{}{a\mathcal{C}^{<}a,\ \Gamma \vdash \Delta}\ \mathit{Irref} \qquad \dfrac{}{\Gamma, a\mathcal{C}^{<}b \vdash \Delta, a < b}\ \mathit{Trs}$$

where $a$ and $b$ are arbitrary terms of the language. Let us consider the sequent system obtained by adding the axioms **PO1**, **PO2** and the structural rules of weakening, contraction and cut. Let **G3PO** be the system consisting of the initial sequents displayed above and the rules of the calculus **G3c**. The admissibility of weakening and contraction is easily established.

PROPOSITION 3
The following statements hold for the calculus **G3PO**:

1. The rule of substitution is height-preserving admissible.
2. The rule of weakening is height-preserving admissible.

[8] Although the terminology may be reminiscent of the homonymous second-order notion, the definition is here introduced to refer to a multiset of order-relation atomic formulas enjoying certain properties.

3. Every rule is height-preserving invertible.
4. The rule of contraction is height-preserving admissible.

PROOF. The proofs are straightforward and thus we omit the details. □

In the next theorem we show that the initial sequents *Trs* and *Irref* coincide with the closure under cut of the axioms **PO1** and **PO2**, $\mathcal{S}^*_{\{\mathbf{PO1},\mathbf{PO2}\}}$. This and the requirement of permutability of the extra-logical initial sequents ensure cut-elimination.

THEOREM 13
The cut rule is admissible in **G3PO**.

PROOF. We essentially need to consider two cases: the cut between two instances of *Trs* and between an instance of *Trs* and an instance of *Irref*.

- In the first case we have

$$\dfrac{\dfrac{}{\Gamma, a\mathcal{C}^{<}b \vdash \Delta, a < b}\,\overline{ax} \qquad \dfrac{}{a < b, \Gamma', c\mathcal{C}^{<}d \vdash \Delta, c < d}\,\overline{ax}}{\Gamma, \Gamma', a\mathcal{C}^{<}b, c\mathcal{C}^{<}d \vdash \Delta, \Delta', c < d}\,\text{Cut}$$

- There are two subcases to consider: either $a < b$ is in the chain from $c$ to $d$ or not. If not, then clearly $\Gamma', c\mathcal{C}^{<}d \vdash \Delta, c < d$ is an initial sequent and so is the conclusion. If it is, then we have

$$c\mathcal{C}^{<}d \equiv c < c_1, \ldots, c_i < c_{i+1}, \ldots, c_n < d$$

- with $c_i \equiv a$ and $c_{i+1} \equiv b$. Hence we observe that $a\mathcal{C}^{<}b$ replaces it, i.e.

$$a\mathcal{C}^{<}b, c\mathcal{C}^{<}d \equiv c < c_1, \ldots, a\mathcal{C}^{<}b, \ldots, c_n < d$$

- is a chain from $c$ to $d$. Therefore $\Gamma, \Gamma', a\mathcal{C}^{<}b, c\mathcal{C}^{<}d \vdash \Delta, \Delta', c < d$ is an initial sequent.
- In the second case we have

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, a < b}\,\mathit{Trs} \qquad \dfrac{}{\Gamma', a < b \vdash \Delta'}\,\mathit{Irref}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

- We may assume that $a < b$ is principal in *Irref* and *Trs*, otherwise the conclusion easily follows. Hence $a < b$ is in the chain $a\mathcal{C}^{<}a$ and we have

$$\dfrac{\dfrac{}{\Gamma, a\mathcal{C}^{<}b \vdash \Delta, a < b}\,\mathit{Trs} \qquad \dfrac{}{\Gamma', a < b, .a\mathcal{C}^{<}a/' \vdash \Delta'}\,\mathit{Irref}}{\Gamma, \Gamma', a\mathcal{C}^{<}b\ (a\mathcal{C}^{<}a)' \vdash \Delta, \Delta'}\,\text{Cut}$$

- Repeating the argument from above we observe that $a\mathcal{C}^{<}b, (a\mathcal{C}^{<}a)'$ is a chain from $a$ to $a$ and so $\Gamma, \Gamma', a\mathcal{C}^{<}b, (a\mathcal{C}^{<}a)' \vdash \Delta, \Delta'$ is an instance of *Irref*.

Permutability with respect to the rules of the calculus is easily shown and we leave the details to the reader. □

### 7.2 Equality

To deal with equality we need to suitably modify the initial sequents. The key intuition comes again from the shape of cuts and from the use of equality chains. The initial sequents which encode the equality axioms are all the sequents of the shape:

$$\frac{}{\Gamma \vdash \Delta, t = t}\ \mathit{Ref} \qquad \frac{}{t\mathcal{C}^{=}s, P(t)\Gamma \vdash \Delta, P(s)}\ \mathit{Repl}$$

Notice that this includes also the instances of *Ref* and *Trs*. Let us denote by **G3EQ** the system obtained by adding to **G3c** the extra-logical sequents just defined. Once again the rules of weakening and contraction are immediately shown to be admissible.

PROPOSITION 4
The following statements hold w.r.t. the calculus **G3EQ**:

1. The rule of substitution is height-preserving admissible.
2. The rule of weakening is height-preserving admissible.
3. Every rule is height-preserving invertible.
4. The rule of contraction is height-preserving admissible.

PROOF. The proofs are straightforward and thus we omit the details. □

As regards the cut rule, we need to argue more carefully.

THEOREM 14
The cut rule is admissible in **G3EQ**

PROOF. Once again it suffices to show that this kind of cuts can be removed:

$$\frac{\dfrac{}{\Gamma \vdash \Delta, a = b}\ \overline{ax} \qquad \dfrac{}{a = b, \Gamma \vdash \Delta}\ \overline{ax}}{\Gamma \vdash \Delta}\ \text{Cut}$$

We may assume that the equality atom $a = b$ is principal in both premises, otherwise the reduction is trivial. Therefore we have two subcases to consider: $\Gamma \vdash \Delta, a = b$ is an instance of *Ref* or an instance of *Repl*.

- In the first subcase the atom $a = b$ would be of the shape $a = a$ and so it would not be principal in $a = b, \Gamma \vdash \Delta$ by the design of the *Repl* initial sequents.
- In the second subcase we apply the same argument used to establish cut elimination for the system **G3PO** in the case in which the two initial extra-logical sequents are instances of *Trs*.

Permutability is also easily seen to hold. □

It remains to show that the restriction to atomic initial sequents is enough to obtain the derivability of the axioms of equality. We limit ourselves to showing the derivability of the generalized replacement schema. We prove a strengthening of the replacement schema in order to use a stronger induction hypothesis.

PROPOSITION 5
The sequents $t = s, A(t) \vdash A(s)$ and $t = s, A(s) \vdash A(t)$ are derivable in **G3EQ** for every formula $A$.

PROOF. The proof is by induction on the degree of $A$. The atomic case is an instance of *Repl*. If $A$ is a compound formula, we need to distinguish cases according to its shape. We detail the case in which it is of the shape $B \to C$, the other ones are analogous.

$$\dfrac{\dfrac{t = s, B(s) \vdash C(s), B(t) \qquad t = s, C(t), B(s) \vdash C(s)}{t = s, (B \to C)(t), B(s) \vdash C(s)}\ \text{L}\!\to}{t = s, (B \to C)(t) \vdash (B \to C)(s)}\ \text{R}\!\to$$

The topmost sequents are derivable via admissibility of weakening and an application of the induction hypothesis. Notice that this case is the only one which requires the strengthening of the induction hypothesis (to derive the sequent $t = s, B(s) \vdash C(s), B(t)$) because the rules for implication shift the polarities of formulas. The derivability of $t = s, (B \rightarrow C)(s) \vdash (B \rightarrow C)(t)$ is shown symmetrically and thus we omit the details. □

We have shown that the axiomatic calculi can be embedded in the corresponding sequent calculus. The other direction is obtained by showing that the axiomatic calculus can derive the initial sequents, but this is immediate.

## 8 Conclusion

The general result presented in this paper concerns the introduction of a uniform method for injecting extra-logical information in pure deductive systems with preservation of the full subformula property. This method enables a modular treatment of extra-logicality, by identifying for each logical system discussed (ranging from classical and intuitionistic to substructural ones) a suitable class of well-behaved formulas.

Furthermore, we argue that this work casts light on the relation between extra-logicality and full analyticity, by yielding a limit relative to the latter in pure deductive systems. In fact, with respect to the systems here examined, it is clear that going beyond the limits of the corresponding classes leads either to a loss of analyticity or to a loss of structural properties.

Furthermore, we have spelled out a general criterion to ensure cut-elimination in a system extended with extra-logical initial sequents. The result is interesting as the requirements to fulfil it are double: the availability of a decomposition algorithm and the possibility to permute the cuts upwards.

We wish to indicate three issues which might be object of future research. First, we would like to give an explicit definition of the infinite set of sequents generated by the closure under cut of cyclic sequents. Secondly, it would be interesting to investigate the connection between our approach and the one detailed by Makinson in the context of supraclassical logics [16]. Finally, we are interested in studying the interaction between the addition of extra-logical axioms and control sets as presented in [8, 21]. In our opinion, the combination of negative information, i.e. prohibitions expressed by control sets, and positive information brought by extra-logical axioms can applied to the study of non-monotonicity in a first-order setting.

## Acknowledgements

We wish to thank two anonymous reviewers for their helpful comments which enhanced the quality of the paper. Mario Piazza acknowledges the Italian Ministry of University and Research for contribution to the development of the project ‘Understanding public data: experts, decisions, epistemic values’ as part of the PRO3 joint programme. Matteo Tesi received funding from the FWF project I 6372-N Logical methods for Deontic Explanations (LoDEx).

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Received 19 April 2023