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Global existence of weak solutions and weak–strong uniqueness for nonisothermal Maxwell–Stefan systems[*]

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Global existence of weak solutions and weak–strong uniqueness for nonisothermal Maxwell–Stefan systems[*]

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Abstract

The dynamics of multicomponent gas mixtures with vanishing barycentric velocity is described by Maxwell–Stefan equations with mass diffusion and heat conduction. The equations consist of the mass and energy balances, coupled to an algebraic system that relates the partial velocities and driving forces. The global existence of weak solutions to this system in a bounded domain with no-flux boundary conditions is proved by using the boundedness-by-entropy method. *A priori* estimates are obtained from the entropy inequality which originates from the consistent thermodynamic modelling. Furthermore, a conditional weak–strong uniqueness property is shown by using the relative entropy method.

1. Introduction

The dynamics of multicomponent gaseous mixtures with vanishing barycentric velocity and constant temperature can be described by the Maxwell–Stefan equations [24, 27]. The existence of local-in-time smooth and global-in-time weak solutions to these systems has been proved in [2, 13, 16, 21]. The analysis of *nonisothermal* gas mixtures is, however, incomplete. The existence of local-in-time solutions was shown in [19], while [15] investigated a special nonisothermal case. In this paper, we prove the existence of global-in-time weak solutions and a conditional weak–strong uniqueness property for a rather general nonisothermal Maxwell–Stefan system. The novelty of our approach is the consistent thermodynamic modelling.

1.1. Model equations

The evolution of the mass densities $\rho_i(x,t)$ of the ith gas component and the temperature $\theta(x,t)$ of the mixture is described by the mass and energy balances

$$\partial_t \rho_i + \operatorname{div} J_i = 0, \quad \partial_t (\rho e) + \operatorname{div} J_e = 0, \quad i = 1, \ldots, n, \tag{1}$$

$$J_i = \rho_i u_i, \quad J_e = -\kappa(\theta) \nabla \theta + \sum_{j=1}^{n} (\rho_j e_j + p_j) u_j \quad \text{in } \Omega, \, t > 0, \tag{2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, J_i and J_e are the diffusion and energy fluxes, respectively, u_i are the diffusional velocities, $\rho = \sum_{i=1}^{n} \rho_i$ is the total mass density, p_i the partial pressure with the total pressure $p = \sum_{i=1}^{n} p_i$, $\rho_i e_i$ the partial internal energy $\rho_i e_i$ with the total energy $\rho e = \sum_{i=1}^{n} \rho_i e_i$, and $\kappa(\theta)$ is the heat conductivity. Equations (1) and (2) are supplemented with the boundary and initial conditions

$$J_i \cdot \nu = 0, \quad J_e \cdot \nu = \lambda(\theta - \theta_0) \quad \text{on } \partial\Omega, \, t > 0, \tag{3}$$

$$\rho_i(0) = \rho_i^0, \quad \theta(0) = \theta^0 \quad \text{in } \Omega, \, i = 1, \ldots, n, \tag{4}$$

where ν is the exterior unit normal vector to $\partial\Omega$, $\theta_0 > 0$ is the given background temperature, and $\lambda > 0$ is a relaxation constant. The boundary conditions mean that the gas components cannot leave the domain, while heat exchange through the boundary is possible and proportional to the difference between the gas and background temperatures. To close the model, we need to determine u_i, $\rho_i e_i$, and p_i.

The velocities u_i are computed from the constrained algebraic Maxwell–Stefan system

$$-\theta \sum_{j=1}^{n} b_{ij} \rho_i \rho_j (u_i - u_j) = d_i \quad \text{for } i = 1, \ldots, n, \quad \sum_{i=1}^{n} \rho_i u_i = 0, \tag{5}$$

where the constant coefficients $b_{ij} = b_{ji} > 0$ model the interaction between the ith and jth components. The driving force d_i is given by

$$d_i = \rho_i \theta \nabla \frac{\mu_i}{\theta} - \theta (\rho_i e_i + p_i) \nabla \frac{1}{\theta}, \quad i = 1, \ldots, n, \tag{6}$$

where μ_i is the chemical potential. The constraint

$$\nabla p = 0 \quad \text{in } \Omega, \, t > 0, \tag{7}$$

is needed in order for our system to be thermodynamically consistent. We refer to section 2 for details.

The internal energies $\rho_i e_i$ and chemical potentials μ_i are determined from the Helmholtz free energy (see (17)), and the pressure is computed from the Gibbs–Duhem relation. As shown in section 2, these quantities are explicitly given by

$$\mu_i = \frac{\theta}{m_i} \log \frac{\rho_i}{m_i} - c_v \theta \left(\log \theta - 1 \right), \qquad \rho_i e_i = c_v \rho_i \theta,$$

$$\rho_i \eta_i = -\frac{\rho_i}{m_i} \left(\log \frac{\rho_i}{m_i} - 1 \right) + c_v \rho_i \log \theta, \qquad p_i = \frac{\rho_i \theta}{m_i}, \quad i = 1, \dots, n, \tag{8}$$

where $\rho_i \eta_i$ is the entropy density of the ith component and $c_v > 0$ is the heat capacity. Then the driving force d_i and energy flux J_e simplify to

$$d_i = \frac{\nabla(\rho_i \theta)}{m_i}, \quad J_e = -\kappa \nabla \theta + \theta \sum_{i=1}^{n} \frac{\rho_i u_i}{m_i}. \tag{9}$$

When dealing with weak solutions, the system presents three main mathematical challenges:

(a) To analyse (1) and (2), we need to invert the linear system in (5) for the diffusional velocities u_i. However, the linear system is singular, yielding infinitely many solutions. For this reason, the constraint $\sum_i \rho_i u_i = 0$, which ensures conservation of the total mass, rules out all non-physical solutions. This suggests that the inversion of the linear system needs to be performed in a particular subspace; see below for more details.

(b) The inversion of the constrained linear system generates velocities u_i that in general depend on all mass densities ρ_i and all density gradients $\nabla \rho_i$, meaning that each equation of the system will be coupled with all others in a nonlinear fashion. As a result, maximum principles cannot be applied and thus, bounds on the mass densities cannot be deduced.

(c) It is not clear *a priori* whether the inversion of the linear system generates the structure $\boldsymbol{u} = -A(\boldsymbol{\rho}) \nabla \boldsymbol{\rho}$, where $A(\boldsymbol{\rho})$ is positive (semi-)definite. Therefore, it is non-trivial to derive *a priori* estimates that will allow us to prove the global-in-time existence of solutions.

The matrix M associated to the algebraic system (5) is singular (since $\sum_{i=1}^{n} d_i = 0$) and thus not positive definite. However, we recall in section 3.1 that it is positive definite on the subspace $L = \{ \boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n : \sqrt{\boldsymbol{\rho}} \cdot \boldsymbol{y} = 0 \}$ (here, $\sqrt{\boldsymbol{\rho}}$ is the vector with components $\sqrt{\rho_i}$). Therefore, the Bott–Duffin inverse of M, denoted by $M^{BD} = M^{BD}(\boldsymbol{\rho})$, exists and is symmetric and positive definite on L. Moreover, we show in section 3.3 below that the fluxes can be expressed as a linear combination of the entropy variables (or thermo-chemical potentials) $\boldsymbol{\mu}/\theta = (\mu_1/\theta, \dots, \mu_n/\theta)$ and $-1/\theta$,

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -Q(\boldsymbol{\rho}, \theta) \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}, \quad \text{where } Q(\boldsymbol{\rho}, \theta) = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix}, \tag{10}$$

and $A = (A_{ij}) \in \mathbb{R}^{n \times n}, \boldsymbol{B} = (B_i) \in \mathbb{R}^n, a > 0$ are given by

$$A_{ij}(\boldsymbol{\rho}) = M_{ij}^{BD} \sqrt{\rho_i \rho_j}, \quad B_i(\boldsymbol{\rho}, \theta) = \theta \sum_{j=1}^{n} \frac{A_{ij}}{m_j}, \quad a(\boldsymbol{\rho}, \theta) = \theta^2 \left(\kappa + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \right). \tag{11}$$

Here, variables in bold font are n-dimensional vectors. The Onsager matrix Q turns out to be positive semidefinite (see (34)), which reveals the parabolic structure of equations (1) and (2).

1.2. State of the art

The isothermal Maxwell–Stefan equations can be derived from the multispecies Boltzmann equations in the diffusive approximation [6]. The high-friction limit in Euler (–Korteweg) equations reveals a formal gradient-flow form of the Maxwell–Stefan equations [17], leading to Fick–Onsager diffusion fluxes instead of (5). In fact, it is shown in [5] that the Fick–Onsager and generalised Maxwell–Stefan approaches are equivalent. A formal Chapman–Enskog expansion of the stationary nonisothermal model was given in [28]. Another nonisothermal Maxwell–Stefan system was derived in [1], but with a different energy flux than ours.

Maxwell–Stefan systems with nonvanishing barycentric velocities can be formulated in the framework of hyperbolic–parabolic systems, which allows one to perform a local-in-time existence analysis [13]. Global-in-time regular solutions around the constant equilibrium state were found to exist in [14]. An existence analysis for Maxwell–Stefan systems coupled to the incompressible Navier–Stokes equations for the barycentric velocity can be found in [8] for the isothermal and in [23] for the nonisothermal case. For the compressible case, we refer to [4]. For steady-state problems, see, e.g. [7, 25].

When the barycentric velocity vanishes, the (isothermal) Maxwell–Stefan equations can be solved by generalised parabolic theory. The existence of local-in-time classical solutions was proved in [2], while the existence of global-in-time weak solutions with general initial data was shown in [21]. Concerning weak solutions to the nonisothermal equations with zero mean flow, the only known result to the best of our knowledge is [15], where an existence analysis for global-in-time weak solutions was presented. However, this model has some modelling deficiencies explained below. Therefore, our first aim is to prove the global existence of weak solutions for a thermodynamically consistent nonisothermal model, like the one presented for example in [13].

The uniqueness of strong solutions to the isothermal Maxwell–Stefan equations was shown in [2, 16, 19], but the uniqueness of weak solutions for general coefficients b_{ij} is still unsolved. A very special case (the coefficients b_{ij} have two degrees of freedom only) was investigated in [9]. It was shown in [18] that strong solutions are unique in the class of weak solutions, which is known as the weak–strong uniqueness property. Our second aim is to prove this property for the nonisothermal case.

Let us detail the main differences of our work compared to [15]:

(i) The most important difference is the lack of validity of the Onsager reciprocity relations in the model of [15]. The relations imply the symmetry of the coefficients of the Onsager matrix; see (10). The choice in [15] leads to a cancelation in the entropy inequality, thus simplifying the estimation. Our results do not rely on this simplification; see remark 6 for further details.

(ii) The constraint (7) on the pressure is not taken into account in [15]. This condition is not necessary mathematically, but its lack creates an inconsistency with the assumption of vanishing barycentric velocity. Indeed, a difference in pressure induces a force difference, which can result in an acceleration according to Newton's second law, if there is no additional force to balance it.

(iii) According to Onsager's reciprocity relations, the Onsager matrix Q in (10) has to be positive semidefinite. We show that Q is in fact positive definite on the subspace $L = \{\boldsymbol{y} \in$

$\mathbb{R}^n : \boldsymbol{y} \cdot \sqrt{\boldsymbol{\rho}} = 0\}$. In [15], is *assumed* that this subspace equals $\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} \cdot \mathbf{1} = 0\}$. This is not consistent with the thermodynamic modelling.

(iv) We consider different molar masses m_i, while they are assumed to be the same in [15]. When we assume equal molar masses, the cross-terms cancel, and we end up with the simple heat flux $J_e = -\kappa \nabla \theta$ (see (9) and the constraint in (5)), thus decoupling the equations.

1.3. Main results

We impose the following assumptions:

(A1) Domain: $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary, and $T > 0$. We set $\Omega_T = \Omega \times (0, T)$ and $\mathbb{R}_+ = [0, \infty)$.

(A2) Data: $\rho_i^0 \in L^\infty(\Omega)$ satisfies $\rho_i^0 \geqslant 0$ in Ω and $0 < \rho_* \leqslant \sum_{i=1}^n \rho_i^0 \leqslant \rho^*$ in Ω for some $\rho_*, \rho^* > 0$ and for all $i = 1, \ldots, n$; $\theta^0 \in L^\infty(\Omega)$ satisfies $\inf_\Omega \theta^0 > 0$.

(A3) Coefficients: $b_{ij} = b_{ji} > 0$ for all $i, j = 1, \ldots, n$.

(A4) Heat conductivity: $\kappa \in C^0(\mathbb{R}_+^n \times \mathbb{R}_+)$ satisfies $c_\kappa(1 + \theta^2) \leqslant \kappa(\theta) \leqslant C_\kappa(1 + \theta^2)$ for some $c_\kappa, C_\kappa > 0$ and all $(\boldsymbol{\rho}, \theta) \in \mathbb{R}_+^n \times \mathbb{R}_+$.

The lower bound for the total mass density ρ is needed to derive uniform estimates for the temperature. The proof of lemma 10 in [18] shows that $M_{ij}^{BD}(\boldsymbol{\rho})$ is bounded for all $\boldsymbol{\rho} \in \mathbb{R}_+^n$. The growth condition for the heat conductivity is used to derive higher integrability bounds for the temperature, which are needed to derive a uniform estimate for the discrete time derivative of the temperature. We may also assume reaction terms R_i in (1) with the properties that the total reaction rate $\sum_{i=1}^n R_i$ vanishes and the vector of reaction rates R_i is derived from a convex, nonnegative potential [11, section 2.2].

The first main result is the existence of solutions.

Theorem 1 (Existence of weak solutions). *Let assumptions (A1)–(A4) hold. Then there exists a weak solution to (1)–(8) satisfying $\rho_i > 0$, $\theta > 0$ a.e. in $\Omega_T = \Omega \times (0, T)$ and*

$$\sqrt{\rho_i} \in L^\infty(\Omega_T) \cap C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \partial_t \rho_i \in L^2(0, T; H^1(\Omega)^*),$$

$$\theta \in C_w^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \partial_t(\rho \theta) \in L^{16/11}(0, T; W^{1, 16/11}(\Omega)^*),$$

$$\theta^2, \log \theta \in L^2(0, T; H^1(\Omega)), \quad i = 1, \ldots, n,$$

the weak formulation

$$\int_0^T \langle \partial_t \rho_i, \phi_i \rangle_{H^1(\Omega)^*} \, \mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=}^n M_{ij}^{BD}(2\nabla\sqrt{\rho_j} + \rho_j \nabla \log \theta) \, \mathrm{d}x\mathrm{d}t = 0,$$

$$\int_0^T \int_\Omega \langle \partial_t(\rho\theta), \phi_0 \rangle_{W^{1,16/5}(\Omega)^*} \, \mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^n \frac{\theta M_{ij}^{BD}}{m_i m_j} \sqrt{\rho_i}(2\nabla\sqrt{\rho_j} + \sqrt{\rho_j}\nabla \log \theta) \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \kappa \nabla \theta \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t = \lambda \int_0^T \int_{\partial\Omega} (\theta_0 - \theta)\phi_0 \mathrm{d}s\mathrm{d}t$$

holds for all $\phi_1, \ldots, \phi_n \in L^2(0, T; H^1(\Omega))$ and $\phi_0 \in L^{16/5}(0, T; W^{1, 16/5}(\Omega)^*)$, and the initial conditions (4) are satisfied in the sense $\rho_i(0) = \rho_i^0$ in $L^2(\Omega)$ and $\theta(0) = \theta^0$ weakly in $L^2(\Omega)$.

The idea of the proof is to apply the boundedness-by-entropy method, which automatically yields $L^\infty(\Omega_T)$ bounds [20]. More precisely, we formulate system (1) and (2) in terms of the relative entropy variables $(\mu_i - \mu_n)/\theta$ for $i = 1,\ldots,n-1$ and $\log\theta$. We show in lemma 3 that this defines the mass densities and temperature uniquely as a function of $(w_1,\ldots,w_{n-1},w)$. We introduce the mathematical entropy density

$$h(\boldsymbol{\rho}',\theta) = \sum_{i=1}^{n} \frac{\rho_i}{m_i} \left(\log \frac{\rho_i}{m_i} - 1 \right) - c_v \rho \log\theta,$$

where the nth partial mass density is computed from $\rho_n = \rho - \sum_{i=1}^{n-1} \rho_i$, i.e. h depends on $\boldsymbol{\rho}' = (\rho_1,\ldots,\rho_{n-1})$ and θ. Gradient estimates for $(\boldsymbol{\rho},\theta)$ are first derived from the entropy equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h(\boldsymbol{\rho}',\theta)\,\mathrm{d}x + \int_\Omega \frac{\kappa}{\theta^2} |\nabla\theta|^2 \mathrm{d}x + \sum_{i,j=1}^{n} \int_\Omega M_{ij}^{BD} \frac{d_i}{\theta\sqrt{\rho_i}} \frac{d_j}{\theta\sqrt{\rho_j}} \mathrm{d}x = 0,$$

which becomes an inequality for weak solutions. Second, as in [15], the energy balance equation (2) yields a bound for θ^2 in $L^2(0,T;H^1(\Omega))$. As mentioned before, the derivation of the entropy inequality differs from that one in [15], because the cross-term

$$I_5 = 2 \int_\Omega \sum_{i=1}^{n-1} \frac{B_i}{\theta} \nabla \frac{\mu_i - \mu_n}{\theta} \cdot \nabla \log\theta\,\mathrm{d}x,$$

which cancels out in [15], needs to be controlled. (We recall definition (11) of B_i.) This is done by observing that the sum $I_4 + I_5 + I_8$ (see (41)) is nonnegative,

$$I_4 + I_5 + I_8 = \int_\Omega \sum_{i,j=1}^{n} A_{ij} \nabla \left(\frac{\mu_i}{\theta} + \frac{1}{m_i} \log\theta \right) \cdot \nabla \left(\frac{\mu_j}{\theta} + \frac{1}{m_j} \log\theta \right) \mathrm{d}x \geqslant 0,$$

as (A_{ij}) is positive semidefinite due to (34).

From a technical viewpoint, we approximate equations (1) and (2) by replacing the time derivative by the implicit Euler discretisation to avoid issues with the time regularity and by adding a higher-order regularisation to achieve $H^2(\Omega)$ and hence $L^\infty(\Omega)$ regularity for the entropy variables. The approximation is chosen in such a way that a discrete entropy inequality can be derived, yielding uniform estimates for both the compactness of the fixed-point operator (to obtain a solution to the approximate problem) and the de-regularisation limit (to obtain a solution to the original problem).

Our second main result concerns a conditional weak–strong uniqueness property.

Theorem 2 (Weak–strong uniqueness). *Let the assumptions of theorem 1 hold, let $\lambda = 0$ in (3), let $(\boldsymbol{\rho},\theta)$ be a weak solution and $(\bar{\boldsymbol{\rho}},\bar{\theta})$ be a strong solution to (1)–(8). We assume that there exist $m,M > 0$ such that*

$$0 < \rho_i \leqslant \rho^*, \quad 0 < \theta \leqslant M, \quad 0 < \bar{\rho}_i \leqslant \rho^*, \quad 0 < m \leqslant \bar{\theta} \leqslant M \quad \text{in } \Omega_T.$$

Furthermore, we suppose that $\bar{u}_i$, $|\nabla \log\bar{\theta}| \in L^\infty(\Omega_T)$ for $i = 1,\ldots,n$ and that the thermal conductivity κ is Lipschitz continuous. If the initial data of $(\boldsymbol{\rho},\theta)$ and $(\bar{\boldsymbol{\rho}},\bar{\theta})$ coincide then $\boldsymbol{\rho}(x,t) = \bar{\boldsymbol{\rho}}(x,t)$ and $\theta(x,t) = \bar{\theta}(x,t)$ for a.e. $x \in \Omega$ and all $t > 0$.

By a strong solution, we understand a solution that has sufficient regularity to satisfy the entropy equality stated in lemma 14; see section 5. We require more regularity than provided by the existence theory, for instance the boundedness of the temperature θ, which is not proved in theorem 1, explaining the notion of 'conditional weak-strong uniqueness'. For more details on the regularity of strong solutions, we refer to [2] for the isothermal case and to [13, 19] for the nonisothermal one. The proof of theorem 2 is based on the relative entropy, defined by

$$
\begin{aligned}
H\left(\boldsymbol{\rho}, \theta \mid \bar{\boldsymbol{\rho}}, \bar{\theta}\right) &= \int_{\Omega}\left(h\left(\boldsymbol{\rho}, \theta\right) - h\left(\bar{\boldsymbol{\rho}}, \bar{\theta}\right) - \sum_{i=1}^{n} \frac{\partial h}{\partial \rho_i}\left(\bar{\boldsymbol{\rho}}, \bar{\theta}\right)\left(\rho_i - \bar{\rho}_i\right) - \frac{\partial h}{\partial E}\left(\bar{\boldsymbol{\rho}}, \bar{\theta}\right)\left(E - \bar{E}\right)\right) \mathrm{d}x \\
&= \int_{\Omega}\left\{\sum_{i=1}^{n} \frac{1}{m_i}\left(\rho_i \log \frac{\rho_i}{\bar{\rho}_i} - \left(\rho_i - \bar{\rho}_i\right)\right) - c_v \rho\left(\log \frac{\theta}{\bar{\theta}} - \left(\theta - \bar{\theta}\right)\right)\right\} \mathrm{d}x,
\end{aligned} \tag{12}
$$

where $E = c_v \rho \theta$ and $\bar{E} = c_v \rho \bar{\theta}$ are the internal energy densities. The idea is to compute the time derivative:

$$
\begin{aligned}
&\frac{\mathrm{d}H}{\mathrm{d}t}\left(\boldsymbol{\rho}, \theta \mid \bar{\boldsymbol{\rho}}, \bar{\theta}\right) + c \int_{\Omega} \sum_{i=1}^{n} \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x + c \int_{\Omega} |\nabla\left(\log \theta - \log \bar{\theta}\right)|^2 \mathrm{d}x \\
&\qquad \leqslant C \int_{\Omega}\left(\sum_{i=1}^{n}\left(\rho_i - \bar{\rho}_i\right)^2 + \left(\theta - \bar{\theta}\right)^2\right) \mathrm{d}x,
\end{aligned}
$$

where $c > 0$ is some constant and $C > 0$ depends on the $L^\infty(\Omega_T)$ norms of θ, $\bar{u}_i$, and $\nabla \log \bar{\theta}$, $i = 1, \ldots, n$. The difficulty is to estimate the expressions arising from the time derivative of the relative entropy in such a way that only $\bar{u}_i$ and $\bar{\theta}$ need to be bounded. Thanks to the positive lower bound for $\bar{\theta}$, we can bound the right-hand side in terms of the relative entropy,

$$
\int_{\Omega}\left(\sum_{i=1}^{n}\left(\rho_i - \bar{\rho}_i\right)^2 + \left(\theta - \bar{\theta}\right)^2\right) \mathrm{d}x \leqslant \int_{\Omega} H\left(\boldsymbol{\rho}, \theta \mid \bar{\boldsymbol{\rho}}, \bar{\theta}\right) \mathrm{d}x.
$$

Then Gronwall's lemma shows that $H((\boldsymbol{\rho}, \theta)(t) \mid (\bar{\boldsymbol{\rho}}, \bar{\theta})(t)) = 0$ for $t > 0$ and hence $(\boldsymbol{\rho}, \theta)(t) = (\bar{\boldsymbol{\rho}}, \bar{\theta})(t)$. Compared to [18], we include the temperature terms and combine them with the entropy variables w_i in such a way that the positive semidefiniteness of M^{BD} can be exploited.

The paper is organised as follows. We detail the thermodynamic modelling of equations (1)–(8) in section 2. The inversion of the Maxwell–Stefan system (5), the definition of the (relative) entropy variables, and the formulations of the fluxes in terms of the relative entropy variables, as well as the corresponding weak formulation is presented in section 3. Section 4 is concerned with the proof of theorems 1 and 2 is proved in section 5.

2. Modelling

We consider the following system of equations modelling the dynamics of a nonisothermal gas mixture of n components with mass diffusion and heat conduction:

$$
\partial_t \rho_i + \operatorname{div}\left(\rho_i\left(v + u_i\right)\right) = 0, \quad i = 1, \ldots, n, \tag{13}
$$

$$
\partial_t\left(\rho v\right) + \operatorname{div}\left(\rho v \otimes v\right) = \rho b - \nabla p, \tag{14}
$$

$$\partial_t \left(\rho e + \frac{1}{2} \rho |v|^2 \right) + \mathrm{div} \left(\left(\rho e + \frac{1}{2} \rho |v|^2 \right) v \right)$$

$$= \mathrm{div} \left(\kappa \nabla \theta \right) - \mathrm{div} \sum_{j=1}^{n} \left(\rho_i e_i + p_i \right) u_i - \mathrm{div} \left(pv \right) + \rho r + \rho b \cdot v + \sum_{i=1}^{n} \rho_i b_i \cdot u_i. \tag{15}$$

Besides of the variables introduced in the introduction, v denotes the barycentric velocity of the mixture. The quantities $\rho_i b_i$ are the body forces, where $\rho b = \sum_{i=1}^{n} \rho_i b_i$ is the total force exerted on the mixture, and ρr is the total heat supply due to radiation. The diffusional velocities u_i are determined by solving the linear system (5) with driving forces given by (16), while the partial internal energy densities $\rho_i e_i$, and the partial pressures p_i are determined from the free energy; see below.

Equations (13)–(15) correspond to a so-called class-I model. They can be derived either via an entropy invariant model reduction [3] or in the high-friction limit [12] from a class-II model, in which each component has its own velocity v_i. Equation (13) are the partial mass balances, (14) is the momentum balance, and (15) the energy balance. As proved in [12], system (13)–(15) and (5) fits into the general theory of hyperbolic–parabolic composite-type systems introduced in [22] and further explored in [26].

As mentioned in the introduction, system (1), (2) and (7) is supplemented by the constrained Maxwell–Stefan system (5) for the velocities u_i. These equations can be derived from a class-II model in the diffusion approximation [3, section 14, (210)] or in the high-friction limit [12, section 2, (2.50)] with the driving forces

$$d_i = -\frac{\rho_i}{\rho} \nabla p + \rho_i \theta \nabla \frac{\mu_i}{\theta} - \theta \left(\rho_i e_i + p_i \right) \nabla \frac{1}{\theta} + \rho_i \left(b - b_i \right), \tag{16}$$

where μ_i is the chemical potential of the ith component. Since the pressure is uniform in space, $\nabla p = 0$, and we have neglected external forces, the driving force becomes (6). Then equations (1), (2) and (7) are obtained by setting $v = 0$ and $r = b_i = 0$.

The internal energy densities $\rho_i e_i$, partial pressures p_i, and the chemical potential μ_i are determined from the Helmholtz free energy. We assume that the gas is a simple mixture, which implies that these quantities can be calculated from the partial free energy densities $\psi_i(\rho_i, \theta)$, $i = 1, \ldots, n$. We have

$$\mu_i = \frac{\partial \psi_i}{\partial \rho_i}, \quad \rho_i \eta_i = -\frac{\partial \psi_i}{\partial \theta}, \quad \rho_i e_i = \psi_i + \theta \rho_i \eta_i, \quad p_i = \rho_i \mu_i - \psi_i,$$

where $\rho_i \eta_i$ is the entropy density of the ith component and the equation for p_i is called the Gibbs–Duhem relation. Defining the partial Helmholtz free energy as

$$\psi_i = \theta \frac{\rho_i}{m_i} \left(\log \frac{\rho_i}{m_i} - 1 \right) - c_v \rho \theta \left(\log \theta - 1 \right), \quad i = 1, \ldots, n, \tag{17}$$

the thermodynamic quantities are given by (8). Moreover, the driving force d_i and enthalpy $h_i := \rho_i e_i + p_i$ read as

$$d_i = \frac{\nabla (\rho_i \theta)}{m_i}, \quad h_i = \left(c_v + \frac{1}{m_i} \right) \rho_i \theta, \quad i = 1, \ldots, n. \tag{18}$$

This corresponds to equation (9).

3. Preparations

3.1. Inversion of the Maxwell–Stefan system

We discuss the inversion of the Maxwell–Stefan system (5) following [12] and [18, section 2]. We write (5) equivalently as

$$-\theta\sqrt{\rho_i}\sum_{j=1}^{n}M_{ij}\sqrt{\rho_j}u_j = d_i, \quad i = 1,\dots,n, \tag{19}$$

where the matrix $M(\boldsymbol{\rho}) = (M_{ij}) \in \mathbb{R}^{n\times n}$ is given by

$$M_{ij} = \begin{cases} \sum_{k=1,\,k\neq i}^{n} b_{ik}\rho_k & \text{if } i = j, \\ -b_{ij}\sqrt{\rho_i\rho_j} & \text{if } i \neq j. \end{cases} \tag{20}$$

We wish to invert $M\boldsymbol{v} = \boldsymbol{w}$, where $v_i = \sqrt{\rho_i}u_i$ and $w_i = -d_i/(\theta\sqrt{\rho_i})$. Since (b_{ij}) is symmetric, $0 = (M\boldsymbol{v})_i = \sum_{i\neq j}b_{ij}\sqrt{\rho_j}(\sqrt{\rho_j}v_i - \sqrt{\rho_i}v_j)$ shows that the kernel of M consists of span$\{\sqrt{\boldsymbol{\rho}}\}$. Thus, we can invert M only on the subspace $L = \{\boldsymbol{y} \in \mathbb{R}^n : \sqrt{\boldsymbol{\rho}}\cdot\boldsymbol{y} = 0\}$. We define the projections P_L on L and $P_{L^\perp}$ on $L^\perp$ by

$$(P_L)_{ij} = \delta_{ij} - \rho^{-1}\sqrt{\rho_i\rho_j}, \quad (P_{L^\perp})_{ij} = \rho^{-1}\sqrt{\rho_i\rho_j} \quad \text{for } i,j = 1,\dots,n,$$

where δ_{ij} is the Kronecker symbol. The matrix $M = (M_{ij})$ is positive definite on L [18, lemma 4]:

$$\boldsymbol{z}^T M \boldsymbol{z} \geqslant \mu_M |P_L \boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{21}$$

where $\mu_M = \min_{i\neq j} b_{ij} > 0$. Since the matrix $MP_L + P_{L^\perp}$ is invertible [18, lemma 4], we can define the Bott–Duffin inverse of M with respect to L as $M^{BD} = P_L(MP_L + P_{L^\perp})^{-1}$. Hence, we can invert (19) by

$$\sqrt{\rho_i}u_i = -\sum_{j=1}^{n} M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}, \quad i = 1,\dots,n. \tag{22}$$

The matrix $M^{BD} = M^{BD}(\boldsymbol{\rho})$ is symmetric and positive definite on L [18, lemma 4],

$$\boldsymbol{z}^T M^{BD} \boldsymbol{z} \geqslant \mu |P_L \boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{23}$$

where $\mu = (2\sum_{i\neq j}(b_{ij} + 1))^{-1}$.

3.2. Entropy variables

The mathematical analysis becomes easier when formulating the system in terms of the so-called entropy variables. To this end, we introduce the mathematical entropy density

$$h = -\sum_{i=1}^{n}\rho_i\eta_i = \sum_{i=1}^{n}\frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_v\rho\log\theta, \tag{24}$$

which is the negative of the physical (total) entropy density (8). Summing the mass balances (1) over $i = 1,\dots,n$ and using the constraint $\sum_{i=1}^{n}\rho_i u_i = 0$ from (5), we obtain $\partial_t\rho = 0$. Thus, the total density is determined by the initial total density, $\rho(x,t) = \sum_{i=1}^{n}\rho_i^0(x)$ for $x \in \Omega$, and is independent of time. This suggests to compute only the first $n-1$ mass densities, since the

last one can be determined by $\rho_n = \rho - \sum_{i=1}^{n-1} \rho_i$. Then we interpret the entropy density h as a function of $(\boldsymbol{\rho}', \theta) := (\rho_1, \ldots, \rho_{n-1}, \theta)$:

$$h(\boldsymbol{\rho}', \theta) = \sum_{i=1}^{n-1} \frac{\rho_i}{m_i} \left(\log \frac{\rho_i}{m_i} - 1 \right) + \frac{\rho_n}{m_n} \left(\log \frac{\rho_n}{m_n} - 1 \right) - c_v \rho \log \theta$$

with the partial derivatives

$$\frac{\partial h}{\partial \rho_i} = \frac{1}{m_i} \log \frac{\rho_i}{m_i} - \frac{1}{m_n} \log \frac{\rho_n}{m_n}, \quad i = 1, \ldots, n-1, \quad \frac{\partial h}{\partial \theta} = -c_v \frac{\rho}{\theta}.$$

The Hessian matrix

$$\mathrm{D}^2 h = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0}^T & c_v \rho/\theta^2 \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad \text{where } R_{ij} = \frac{\delta_{ij}}{m_i \rho_i} + \frac{1}{m_n \rho_n},$$

is positive definite, showing that the entropy is convex.

According to thermodynamics [3], the entropy variables equal $(\mu_1/\theta, \ldots, \mu_n/\theta, -1/\theta)$. We set

$$q_i = \frac{\mu_i}{\theta} = \frac{1}{m_i} \log \frac{\rho_i}{m_i} - c_v (\log \theta - 1) \quad \text{for } i = 1, \ldots, n. \tag{25}$$

Since the nth partial density is determined by the densities $\rho_1, \ldots, \rho_{n-1}$, we prefer to work with the relative entropy variables

$$w_i = q_i - q_n = \frac{\mu_i - \mu_n}{\theta} = \frac{\partial h}{\partial \rho_i}, \quad i = 1, \ldots, n-1. \tag{26}$$

Setting additionally $w = \log \theta$, our new set of variables is $(w_1, \ldots, w_{n-1}, w)$. The following lemma states that the mapping $(\rho_1, \ldots, \rho_n, \theta) \mapsto (w_1, \ldots, w_{n-1}, w)$ is invertible.

Lemma 3. *Let* $(w_1, \ldots, w_{n-1}, w) \in \mathbb{R}^n$ *and* $\rho > 0$ *be given. Then there exists a unique* $(\rho_1, \ldots, \rho_n, \theta) \in \mathbb{R}_+^{n+1}$ *with* $\rho_i > 0$ *for* $i = 1, \ldots, n$ *satisfying* $\sum_{i=1}^{n} \rho_i = \rho$, $w_i = \partial h/\partial \rho_i$ *for* $i = 1, \ldots, n-1$, *and* $w = \log \theta$.

Proof. The proof is similar to [8, lemma 6] with some small changes. Given $w \in \mathbb{R}$, the temperature equals $\theta = \exp(w) > 0$. The function

$$f(s) = \sum_{i=1}^{n-1} m_i e^{m_i w_i} \left(\frac{\rho - s}{m_n} \right)^{m_i/m_n} \quad \text{for } s \in [0, \rho],$$

is strictly decreasing and $0 = f(\rho) < f(s) < f(0)$ for $s \in (0, \rho)$. By continuity, there exists a unique fixed point $s_0 \in (0, \rho)$. Then $\rho_i := m_i \exp(m_i w_i)((\rho - s_0)/m_n)^{m_i/m_n}$ for $i = 1, \ldots, n$ satisfies $\rho_i > 0$ and $\sum_{i=1}^{n-1} \rho_i = f(s_0) = s_0 < \rho$. Consequently, $\rho_n := \rho - \sum_{i=1}^{n-1} \rho_i = \rho - s_0 > 0$ and $\rho_i/m_i = \exp(m_i w_i)(\rho_n/m_n)^{m_i/m_n}$ is equivalent to

$$w_i = \frac{1}{m_i} \log \frac{\rho_i}{m_i} - \frac{1}{m_n} \log \frac{\rho_n}{m_n} = \frac{\partial h}{\partial \rho_i}$$

for $i = 1, \ldots, n-1$, which finishes the proof. $\qquad \square$

3.3. Formulation of the fluxes and parabolicity

We can compute the fluxes as a linear combination of $\nabla(w_1, \ldots, w_{n-1}, w)$ or $\nabla(q_1, \ldots, q_n, -1/\theta)$.

Lemma 4. *It holds for $i = 1, \ldots, n$ that*

$$J_i = -\sum_{j=1}^{n-1} A_{ij} \nabla w_j - \frac{B_i}{\theta} \nabla w = -\sum_{j=1}^{n} A_{ij} \nabla q_j - B_i \nabla\left(-\frac{1}{\theta}\right), \tag{27}$$

$$J_e = -\kappa\theta\nabla w - \sum_{j=1}^{n-1} B_j \nabla w_j - \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \nabla w \tag{26}$$

$$= -\sum_{j=1}^{n} B_j \nabla q_j - \theta^2 \left(\kappa + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j}\right) \nabla\left(-\frac{1}{\theta}\right), \tag{28}$$

where the coefficients

$$A_{ij} = M_{ij}^{BD} \sqrt{\rho_i \rho_j}, \quad B_i = \theta \sum_{j=1}^{n} A_{ij}\left(c_v + \frac{1}{m_j}\right) = \theta \sum_{j=1}^{n} \frac{A_{ij}}{m_j} \tag{29}$$

for $i, j = 1, \ldots, n$ depend on $(\boldsymbol{\rho}, \theta)$ and satisfy the relations

$$\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = \sum_{i=1}^{n} B_i = 0. \tag{30}$$

Proof. We wish to express the driving force $d_j = \nabla(\rho_j \theta)/m_j$ from (9) in terms of $\nabla q_j = \nabla \log \rho_j / m_j - c_v \nabla \log \theta$. A computation, using $w = \log \theta$, yields

$$d_j = \rho_j \theta \nabla q_j + \rho_j \theta \left(c_v + \frac{1}{m_j}\right) \nabla w. \tag{31}$$

Therefore, by (22), for $i = 1, \ldots, n$,

$$J_i = \rho_i u_i = -\sqrt{\rho_i} \sum_{j=}^{n} M_{ij}^{BD} \frac{d_j}{\theta \sqrt{\rho_j}} = -\sum_{j=1}^{n} M_{ij}^{BD} \sqrt{\rho_i \rho_j} \left\{\nabla q_j + \left(c_v + \frac{1}{m_j}\right) \nabla w\right\}$$

$$= -\sum_{j=1}^{n} A_{ij} \nabla q_j - \sum_{j=1}^{n} A_{ij}\left(c_v + \frac{1}{m_j}\right) \nabla \log \theta = -\sum_{j=1}^{n} A_{ij} \nabla q_j - \frac{B_i}{\theta} \nabla \log \theta.$$

This shows the second relation in (27). The first relation then follows from (30) (which is proved below), since, using $q_j = w_j + q_n$ for $j = 1, \ldots, n-1$ (see (26)),

$$\sum_{j=1}^{n} A_{ij} \nabla q_j = \sum_{j=1}^{n-1} A_{ij}(\nabla w_j + \nabla q_n) + A_{in} \nabla q_n = \sum_{j=1}^{n-1} A_{ij} \nabla w_j. \tag{32}$$

Next, we compute the energy flux defined in (2). We use (18), (22) and (31):

$$
\begin{aligned}
J_e &= -\kappa\theta\nabla w + \sum_{i=1}^{n}\sqrt{\rho_i}\,\theta\left(c_v+\frac{1}{m_i}\right)\sqrt{\rho_i}u_i\\
&= -\kappa\theta\nabla w - \theta\sum_{i,j=1}^{n}\sqrt{\rho_i}\left(c_v+\frac{1}{m_i}\right)M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}\\
&= -\kappa\theta\nabla w - \theta\sum_{i,j=1}^{n}\left(c_v+\frac{1}{m_i}\right)M_{ij}^{BD}\sqrt{\rho_i\rho_j}\left\{\nabla q_j+\left(c_v+\frac{1}{m_j}\right)\nabla w\right\}\\
&= -\kappa\theta\nabla w - \sum_{j=1}^{n}B_j\nabla q_j - \theta\sum_{i,j=1}^{n}A_{ij}\left(c_v+\frac{1}{m_i}\right)\left(c_v+\frac{1}{m_j}\right)\nabla w\\
&= -\kappa\theta\nabla w - \sum_{j=1}^{n}B_j\nabla q_j - \theta\sum_{i,j=1}^{n}\frac{A_{ij}}{m_im_j}\nabla w,
\end{aligned}
$$

where the last equation follows from (30). Moreover, because of

$$
\sum_{j=1}^{n}B_j\nabla q_j = \sum_{j=1}^{n-1}B_j\nabla\left(w_j+q_n\right)+B_n\nabla q_n = \sum_{j=1}^{n-1}B_j\nabla w_j, \tag{33}
$$

we have proved (28).

It remains to verify (30). We recall the property $P_L(MP_L+P_{L^\perp})^{-1}P_{L^\perp}=0$ from [29, lemma 2], which implies that $M^{BD}P_{L^\perp}=0$. Hence, $L^\perp \subset \ker M^{BD}$ and since $L^\perp = \mathrm{span}\{\sqrt{\rho}\}$, we conclude that $\sum_{j=1}^{n}M_{ij}^{BD}\sqrt{\rho_j}=0$. This shows that, by the definition of A_{ij},

$$
\sum_{j=1}^{n}A_{ij} = \sqrt{\rho_i}\sum_{j=1}^{n}M_{ij}^{BD}\sqrt{\rho_j}=0.
$$

The symmetry of (A_{ij}) immediately gives $\sum_{i=1}^{n}A_{ij}=0$. Finally, by the definition of B_i,

$$
\sum_{i=1}^{n}B_i = \theta\sum_{i,j=1}^{n}M_{ij}^{BD}\sqrt{\rho_i\rho_j}\left(c_v+\frac{1}{m_j}\right)=\theta\sum_{j=1}^{n}\left(c_v+\frac{1}{m_j}\right)\sum_{i=1}^{n}A_{ij}=0.
$$

This finishes the proof. $\square$

The previous proof shows that we can formulate the diffusion fluxes in different ways.

Corollary 5. *It holds for $i=1,\ldots,n$ that*

$$
J_i = \rho_i u_i = -\sum_{j=1}^{n}A_{ij}\nabla\left(q_j+\frac{w}{m_j}\right) = -\sqrt{\rho_i}\sum_{j=1}^{n}M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}.
$$

We claim that the Onsager matrix $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ in (10) is positive semidefinite. Let $a = \theta(\kappa + \sum_{i,j=1}^{n} A_{ij}/(m_i m_j))$. We compute for $\xi \in \mathbb{R}^{n+1}$:

$$
\begin{aligned}
\xi^T Q \xi &= \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j + 2\sum_{i=1}^{n} B_i \xi_i \xi_{n+1} + a\xi_{n+1}^n \\
&= \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j + 2\theta\sum_{i,j=1}^{n} \frac{A_{ij}}{m_j}\xi_i\xi_{n+1} + \theta^2\left(\kappa + \sum_{i,j=1}^{n}\frac{A_{ij}}{m_i m_j}\right)\xi_{n+1}^2 \\
&= \sum_{i,j=1}^{n} A_{ij}\left(\xi_i + \frac{\theta\xi_{n+1}}{m_i}\right)\left(\xi_j + \frac{\theta\xi_{n+1}}{m_j}\right) + \kappa\theta^2\xi_{n+1}^2 \geqslant 0, \quad (34)
\end{aligned}
$$

where the nonnegativity follows from the positive semidefiniteness (23) of M^{BD}. This reveals the parabolicity of our system in terms of the entropy variables.

3.4. Weak formulation

The previous subsection shows that we can write our system as the mass and energy balances (1), (2) with the fluxes (27) and (28). The weak formulation in the relative entropy variables (26) reads as

$$
\int_0^T \langle \partial_t \rho_i, \phi_i \rangle \mathrm{d}t + \int_0^T \int_\Omega \left(\sum_{j=1}^{n-1} A_{ij}\nabla w_j + e^{-w}B_i\nabla w\right)\cdot\nabla\phi_i\,\mathrm{d}x\mathrm{d}t = 0, \quad (35)
$$

$$
\int_0^T \langle \partial_t E, \phi_0 \rangle \mathrm{d}t + \int_0^T \int_\Omega e^w\left(\kappa + \sum_{i,j=1}^{n}\frac{A_{ij}}{m_i m_j}\right)\nabla w \cdot \nabla\phi_0\,\mathrm{d}x\mathrm{d}t
$$

$$
+ \int_0^T \int_\Omega \sum_{j=1}^{n-1} B_j \nabla w_j \cdot \nabla\phi_0\,\mathrm{d}x\mathrm{d}t = \lambda \int_0^T \int_{\partial\Omega} (\theta_0 - \theta)\,\phi_0\,\mathrm{d}s\mathrm{d}t \quad (36)
$$

for test functions $\phi_1, \ldots, \phi_n \in L^2(0,T;H^1(\Omega))$ and $\phi_0 \in L^\infty(0,T;W^{1,\infty}(\Omega))$. According to (8), the energy is given by $E = c_v \rho\theta$. Moreover, ρ_i, A_{ij}, B_i, and E are interpreted as functions of $(w_1, \ldots, w_{n-1}, w)$.

4. Proof of theorem 1

The proof follows the lines of [15, section 3], which is based on the boundedness-by-entropy method [20], but some details are different. We approximate equations (35) and (36) by replacing the time derivative by the implicit Euler scheme and adding a higher-order regularisation in w_i. The existence of solutions to the approximate system is shown by means of the Leray–Schauder fixed-point theorem, where the compactness of the fixed-point operator is obtained by the approximate entropy inequality. This inequality yields estimates uniform in the regularisation parameters, allowing for the de-regularisation limit via the Aubin–Lions compactness lemma.

Let $\varepsilon \in (0,1)$, $N \in \mathbb{N}$, and $\tau = T/N$. We set $w_0 = \log \theta_0$ and $\boldsymbol{w} = (w_1, \dots, w_{n-1}, w)$. Let $\bar{\boldsymbol{w}} = (\bar{w}_1, \dots, \bar{w}_{n-1}, \bar{w}) \in L^\infty(\Omega; \mathbb{R}^n)$ be given. We define for test functions $\phi_i \in H^2(\Omega)$, $i = 0, \dots, n-1$, the approximate scheme

$$
\begin{aligned}
0 = & \frac{1}{\tau} \int_\Omega \left(\rho_i(\boldsymbol{w}) - \rho_i(\bar{\boldsymbol{w}}) \right) \phi_i \mathrm{d}x + \int_\Omega \left(\sum_{j=1}^{n-1} A_{ij} \nabla w_j + e^{-w} B_i \nabla w \right) \cdot \nabla \phi_i \mathrm{d}x \\
& + \varepsilon \int_\Omega \left(\mathrm{D}^2 w_i : \mathrm{D}^2 \phi_i + w_i \phi_i \right) \mathrm{d}x,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
0 = & \frac{1}{\tau} \int_\Omega \left(E(\boldsymbol{w}) - E(\bar{\boldsymbol{w}}) \right) \phi_0 \mathrm{d}x + \int_\Omega e^w \left(\kappa(e^w) + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \right) \nabla w \cdot \nabla \phi_0 \mathrm{d}x \\
& + \int_\Omega \sum_{i=1}^{n-1} B_i \nabla w_i \cdot \nabla \phi_0 \mathrm{d}x - \lambda \int_{\partial\Omega} (e^{w_0} - e^w) \phi_0 \mathrm{d}s \\
& + \varepsilon \int_\Omega (e^{w_0} + e^w)(w - w_0) \phi_0 \mathrm{d}x + \varepsilon \int_\Omega e^w \left(\mathrm{D}^2 w : \mathrm{D}^2 \phi_0 + |\nabla w|^2 \nabla w \cdot \nabla \phi_0 \right) \mathrm{d}x, \quad (38)
\end{aligned}
$$

where $\mathrm{D}^2 w_i$ is the Hesse matrix of w_i, the double point ':' denotes the Frobenius matrix product, we recall that $E(\boldsymbol{w}) = c_v \rho \theta$, and A_{ij} and B_i are interpreted as functions of $\boldsymbol{w}$. The higher-order regularisation yields solutions $w_i, w \in H^2(\Omega)$, and the $W^{1,4}(\Omega)$ regularisation allows us to estimate the higher-order terms when using the test function $e^{-w_0} - e^{-w}$ (see the estimate of I_{11} below). The lower-order regularisation $(e^{w_0} - e^w)(w - w_0)$ provides an ε-dependent $L^2(\Omega)$ bound for w.

4.1. Solution of the linearised approximate problem

Let $\boldsymbol{w}^* \in W^{1,4}(\Omega; \mathbb{R}^n)$ and $\sigma \in [0,1]$. We want to find a solution $\boldsymbol{w} \in H^2(\Omega; \mathbb{R}^n)$ to the linear problem

$$
a(\boldsymbol{w}, \boldsymbol{\phi}) = \sigma F(\boldsymbol{\phi}) \quad \text{for } \boldsymbol{\phi} = (\phi_1, \dots, \phi_{n-1}, \phi_0) \in H^2(\Omega; \mathbb{R}^n),
\tag{39}
$$

where

$$
\begin{aligned}
a(\boldsymbol{w}, \boldsymbol{\phi}) = & \int_\Omega \kappa\left(e^{w^*}\right) e^{w^*} \nabla w \cdot \nabla \phi_0 \mathrm{d}x + \varepsilon \int_\Omega \sum_{i=1}^{n-1} \left(\mathrm{D}^2 w_i : \mathrm{D}^2 \phi_i + w_i \phi_i \right) \mathrm{d}x \\
& + \varepsilon \int_\Omega \left(e^{w_0} + e^{w^*} \right) w \phi_0 \mathrm{d}x + \varepsilon \int_\Omega e^{w^*} \left(\mathrm{D}^2 w : \mathrm{D}^2 \phi_0 + |\nabla w^*|^2 \nabla w \cdot \nabla \phi_0 \right) \mathrm{d}x, \\
F(\boldsymbol{\phi}) = & -\int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w}^*) \nabla w_j^* \cdot \nabla \phi_i \mathrm{d}x - \int_\Omega e^{w^*} \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w}^*)}{m_i m_j} \nabla w^* \cdot \nabla \phi_0 \mathrm{d}x \\
& - \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*) e^{-w^*} \nabla w^* \cdot \nabla \phi_i \mathrm{d}x - \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*) \nabla w_i^* \cdot \nabla \phi_0 \mathrm{d}x \\
& - \frac{1}{\tau} \int_\Omega \sum_{i=1}^{n-1} (\rho_i^* - \bar{\rho}_i) \phi_i \mathrm{d}x - \frac{1}{\tau} \int_\Omega (E^* - \bar{E}) \phi_0 \mathrm{d}x + \lambda \int_{\partial\Omega} \left(e^{w_0} - e^{w^*} \right) \phi_0 \mathrm{d}s \\
& + \varepsilon \int_\Omega \left(e^{w_0} + e^{w^*} \right) w_0 \phi_0 \mathrm{d}x,
\end{aligned}
$$

where we abbreviated $\rho_i^* = \rho_i(\boldsymbol{w}^*)$, $\bar{\rho}_i = \rho_i(\bar{\boldsymbol{w}})$, $E^* = c_v \rho e^{w^*}$, and $\bar{E} = c_v \rho e^{\bar{w}}$. The bilinear form a is clearly coercive on $H^2(\Omega; \mathbb{R}^n)$, and both a and F are continuous on this space. By the Lax–Milgram lemma, there exists a unique solution $\boldsymbol{w} \in H^2(\Omega; \mathbb{R}^n)$ to (39).

4.2. Solution of the approximate problem

The solution $\boldsymbol{w} \in H^2(\Omega; \mathbb{R}^n)$ to (39) defines the fixed-point operator $S : W^{1,4}(\Omega; \mathbb{R}^n) \times [0, 1] \to W^{1,4}(\Omega; \mathbb{R}^n)$, $S(\boldsymbol{w}^*, \sigma) = \boldsymbol{w}$. The operator is continuous, compact (because of the compact embedding $H^2(\Omega; \mathbb{R}^n) \hookrightarrow W^{1,4}(\Omega; \mathbb{R}^n)$), and it satisfies $S(\boldsymbol{w}^*, 0) = 0$ for all $\boldsymbol{w}^* \in W^{1,4}(\Omega; \mathbb{R}^n)$. It remains to find a uniform bound for all fixed points of $S(\cdot, \sigma)$. Let $\boldsymbol{w} \in H^2(\Omega; \mathbb{R}^n)$ be such a fixed point. Then $\boldsymbol{w}$ solves (39) with $\boldsymbol{w}^* = \boldsymbol{w}$. We choose the test functions $\phi_i = w_i$ for $i = 1, \ldots, n-1$ and $\phi_0 = e^{-w_0} - e^{-w}$ in (39):

$$
\begin{aligned}
0 = &\frac{\sigma}{\tau} \int_\Omega \sum_{i=1}^{n-1} (\rho_i - \bar{\rho}_i) w_i \mathrm{d}x + \frac{\sigma}{\tau} \int_\Omega (E - \bar{E}) \left(-e^{-w} \right) \mathrm{d}x + \frac{\sigma}{\tau} \int_\Omega (E - \bar{E}) e^{-w_0} \mathrm{d}x \\
&+ \sigma \int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w}) \nabla w_i \cdot \nabla w_j \mathrm{d}x + 2\sigma \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}) e^{-w} \nabla w_i \cdot \nabla w \mathrm{d}x \\
&+ \int_\Omega \kappa(e^w) |\nabla w|^2 \mathrm{d}x + \varepsilon \int_\Omega \sum_{i=1}^{n-1} \left(|\mathrm{D}^2 w_i|^2 + w_i^2 \right) \mathrm{d}x + \sigma \int_\Omega \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w})}{m_i m_j} |\nabla w|^2 \mathrm{d}x \\
&- \sigma \lambda \int_{\partial\Omega} (e^{w_0} - e^w) (e^{-w_0} - e^{-w}) \mathrm{d}s + \varepsilon \int_\Omega (e^{w_0} + e^w) (e^{-w_0} - e^{-w}) (w - \sigma w_0) \mathrm{d}x \\
&+ \varepsilon \int_\Omega \left(|\mathrm{D}^2 w|^2 - \mathrm{D} w : (\nabla w \otimes \nabla w) + |\nabla w|^4 \right) \mathrm{d}x =: I_1 + \cdots + I_{11}. \quad (40)
\end{aligned}
$$

We estimate the terms $I_1, \ldots, I_{11}$ step by step. First, by the convexity of the entropy and arguing similarly as in [15, section 3, Step 2],

$$
\begin{aligned}
I_1 + I_2 &= \frac{\sigma}{\tau} \int_\Omega \sum_{i=1}^{n-1} \left((\rho_i - \bar{\rho}_i) \frac{\partial h}{\partial \rho_i} + (\theta - \bar{\theta}) \frac{\partial h}{\partial \theta} \right) \mathrm{d}x \\
&\geqslant \frac{\sigma}{\tau} \int_\Omega \left(h(\rho_1, \ldots, \rho_{n-1}, \theta) - h(\bar{\rho}_1, \ldots, \bar{\rho}_{n-1}, \bar{\theta}) \right) \mathrm{d}x,
\end{aligned}
$$

where we have set $\theta = e^w$ and $\bar{\theta} = e^{\bar{w}}$. Definition (26) of w_i, definition (29) of B_i, and the relations

$$
\sum_{j=1}^{n-1} A_{ij}(\boldsymbol{w}) \nabla w_j = \sum_{j=1}^{n} A_{ij}(\boldsymbol{w}) \nabla q_j, \quad \sum_{i=1}^{n-1} B_i(\boldsymbol{w}) \nabla w_i = \sum_{j=1}^{n} B_i(\boldsymbol{w}) \nabla q_i
$$

from (32) and (33) allow us to rewrite the sum $I_4 + I_5 + I_8$ as

$$
I_4 + I_5 + I_8 = \sigma \int_\Omega \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{w}) \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) \mathrm{d}x. \quad (41)
$$

This expression is nonnegative because of the positive semidefiniteness of $A_{ij} = M_{ij}^{BD}\sqrt{\rho_i\rho_j}$; see (23). Furthermore, since $\sinh(z)/z \geqslant 1$ for $z \in \mathbb{R}$, $z \neq 0$,

$$I_9 = \sigma\lambda \int_{\partial\Omega} e^{-w-w_0}\left(e^w - e^{w_0}\right)^2 \mathrm{d}x \geqslant 0,$$

$$I_{10} = 2\varepsilon \int_\Omega \sinh(w - w_0)(w - \sigma w_0)\,\mathrm{d}x = 2\varepsilon \int_\Omega (w - w_0)(w - \sigma w_0)\frac{\sinh(w - w_0)}{w - w_0}\mathrm{d}x$$

$$= \varepsilon \int_\Omega w^2 \frac{\sinh(w - w_0)}{w - w_0}\mathrm{d}x + \varepsilon \int_\Omega \left(w^2 - 2(1 + \sigma)ww_0 + 2\sigma w_0^2\right)\frac{\sinh(w - w_0)}{w - w_0}\mathrm{d}x$$

$$\geqslant \varepsilon \int_\Omega w^2\mathrm{d}x + \varepsilon \int_\Omega \left(w^2 - 2(1 + \sigma)ww_0 + 2\sigma w_0^2\right)\frac{\sinh(w - w_0)}{w - w_0}\mathrm{d}x.$$

We claim that there exists $m = m(w_0, \sigma) > 0$ such that for all $w \in \mathbb{R}$,

$$g(w) = \left(w^2 - 2(1 + \sigma)ww_0 + 2\sigma w_0^2\right)\frac{\sinh(w - w_0)}{w - w_0} \geqslant -m,$$

where $w_0 \in \mathbb{R}$ and $\sigma \in (0, 1]$ are given. Indeed, this follows from $g(w) \to \infty$ as $|w| \to \infty$ and $g((1 + \sigma)w_0) < 0$ (unless $w_0 = 0$). We conclude that

$$I_{10} \geqslant \varepsilon \int_\Omega w^2\mathrm{d}x - \varepsilon m.$$

Finally, we can estimate

$$I_{11} = \frac{\varepsilon}{2}\int_\Omega \left(|\mathrm{D}^2 w|^2 + |\mathrm{D}^2 w - \nabla w \otimes \nabla w|^2 + |\nabla w|^4\right)\mathrm{d}x \geqslant \frac{\varepsilon}{2}\int_\Omega \left(|\mathrm{D}^2 w|^2 + |\nabla w|^4\right)\mathrm{d}x.$$

Summarising these estimates, we find that

$$\frac{\sigma}{\tau}\int_\Omega \left(h(\rho_1, \ldots, \rho_{n-1}, \theta) + Ee^{-w_0}\right)\mathrm{d}x + \varepsilon C\left(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\right)$$

$$+ \int_\Omega \kappa(e^w)|\nabla w|^2\mathrm{d}x \leqslant \frac{\sigma}{\tau}\int_\Omega \left(h(\bar\rho_1, \ldots, \bar\rho_{n-1}, \bar\theta) + \bar E e^{-w_0}\right)\mathrm{d}x + \varepsilon m. \qquad (42)$$

The right-hand side is bounded since $\bar{\boldsymbol{w}} \in L^\infty(\Omega; \mathbb{R}^n)$ by assumption, implying that $(\bar\rho_1, \ldots, \bar\rho_{n-1}, \bar\theta) \in L^\infty(\Omega; \mathbb{R}^n)$. The first term on the left-hand side is bounded from below since, by definition (24) of h and $Ee^{-w_0} = c_v\rho\theta/\theta_0$,

$$h(\rho_1, \ldots, \rho_{n-1}, \theta) + Ee^{-w_0} = \sum_{i=1}^n \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_v\rho\left(\log\theta - \frac{\theta}{\theta_0}\right).$$

Thus, we obtain a uniform bound for $\boldsymbol{w}$ in $H^2(\Omega; \mathbb{R}^n)$ and consequently also in $W^{1,4}(\Omega; \mathbb{R}^n)$. We can apply the Leray–Schauder fixed-point theorem to conclude the existence of a fixed point of $S(\cdot, 1)$. This, in turn, shows that $\boldsymbol{w}$ is a weak solution to the approximate problem (37) and (38).

Remark 6 (Treatment of the cross-terms). In the paper [15], the fluxes are given by

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} \boldsymbol{M} & -\boldsymbol{G} \\ \boldsymbol{G}^T & \kappa\theta^2 \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},$$

where $M = M(\boldsymbol{\rho}, \theta) \in \mathbb{R}^{n \times n}$ and $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{\rho}, \theta) \in \mathbb{R}^n$. A multiplication of this equation by $\nabla(\boldsymbol{\mu}/\theta, -1/\theta)$ shows that the cross-terms cancel out,

$$
-\nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^n M_{ij} \nabla \frac{\mu_i}{\theta} \cdot \nabla \frac{\mu_j}{\theta} + \kappa |\nabla \log \theta|^2 \geqslant 0,
$$

since M is assumed to be positive semidefinite in [15]. In the present work, we have

$$
\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},
$$

and the cross-terms do not cancel. This is compensated by the sum $\sum_{i,j=1}^n A_{ij}/(m_i m_j)$. Indeed, a computation shows that (also see (41))

$$
-\nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^n A_{ij} \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) + \kappa |\nabla \log \theta|^2 \geqslant 0,
$$

since A is positive semidefinite because of (34). $\qquad\qquad\square$

4.3. Discrete entropy inequality

We derive some estimates from (40) with $\sigma = 1$, which are uniform in (ε, τ), by exploiting the sum $I_4 + I_5 + I_8$, which we have neglected in (42). Taking into account that the estimate of I_{10} becomes for $\sigma = 1$

$$
I_{10} = 2\varepsilon \int_\Omega \sinh(w - w_0)(w - w_0)\,\mathrm{d}x \geqslant 2\varepsilon \int_\Omega (w - w_0)^2 \,\mathrm{d}x \geqslant 0,
$$

we obtain the discrete entropy inequality

$$
\begin{aligned}
\frac{\sigma}{\tau} \int_\Omega &\big(h(\rho_1, \ldots, \rho_{n-1}, \theta) + E e^{-w_0} \big) \mathrm{d}x + \varepsilon C \left(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4 \right) \\
&+ \int_\Omega \kappa(e^w) |\nabla w|^2 \mathrm{d}x + \int_\Omega \sum_{i,j=1}^n A_{ij} \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) \mathrm{d}x \\
&\leqslant \frac{\sigma}{\tau} \int_\Omega \big(h(\bar{\rho}_1, \ldots, \bar{\rho}_{n-1}, \bar{\theta}) + \bar{E} e^{-w_0} \big) \mathrm{d}x.
\end{aligned}
\tag{43}
$$

Lemma 7. *It holds that*

$$
\int_\Omega \sum_{i,j=1}^n A_{ij} \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) \mathrm{d}x \geqslant \int_\Omega \sum_{i=1}^n \frac{\mu}{m_i^2} |2\nabla \sqrt{\rho_i} + \sqrt{\rho_i} \nabla w|^2 \mathrm{d}x,
\tag{44}
$$

where $\mu > 0$ is defined in (23).

We deduce from assumption (A4) that $\kappa(e^w)|\nabla w|^2 \geqslant c_\kappa |\nabla w|^2$, and in view of (43), this quantity is bounded in $L^2(\Omega)$. Therefore, lemma 7 yields a gradient bound for $\sqrt{\rho_i}$ in $L^2(\Omega)$, since

$$
4|\nabla \sqrt{\rho_i}|^2 \leqslant |2\nabla \sqrt{\rho_i} + \sqrt{\rho_i} \nabla w|^2 + \rho_i |\nabla w|^2.
$$

Proof of lemma 7. It follows from (25) and (30) that

$$\sum_{i,j=1}^{n} A_{ij} \nabla q_i = \sum_{i,j=1}^{n} A_{ij} \frac{\nabla \log \rho_i}{m_i} - c_v \sum_{i,j=1}^{n} A_{ij} \nabla w = \sum_{i,j=1}^{n} A_{ij} \frac{\nabla \rho_i}{m_i \rho_i}$$

and therefore, in view of the definition $A_{ij} = M_{ij}^{BD} \sqrt{\rho_i \rho_j}$ and the positive definiteness (23) on the subspace L,

$$\sum_{i,j=1}^{n} A_{ij} \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) = \sum_{i,j=1}^{n} A_{ij} \left(\frac{\nabla \rho_i}{m_i \rho_i} + \frac{\nabla w}{m_i} \right) \cdot \left(\frac{\nabla \rho_j}{m_j \rho_j} + \frac{\nabla w}{m_j} \right)$$

$$= \sum_{i,j=1}^{n} M_{ij}^{BD} \frac{1}{m_i} \left(\frac{\nabla \rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i} \nabla w \right) \cdot \frac{1}{m_j} \left(\frac{\nabla \rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j} \nabla w \right)$$

$$\geqslant \mu \left| P_L \left(\frac{1}{m_i} \left(\frac{\nabla \rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i} \nabla w \right) \right)_{i=1}^{n} \right|^2.$$

We insert the definition of the projection matrix P_L:

$$\left[P_L \left(\frac{1}{m_j} \left(\frac{\nabla \rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j} \nabla w \right) \right)_{j=1}^{n} \right]_i = \sum_{j=1}^{n} \left(\delta_{ij} - \frac{\sqrt{\rho_i \rho_j}}{\rho} \right) \frac{1}{m_j} \left(\frac{\nabla \rho_j}{\sqrt{\rho_j}} + \sqrt{\rho_j} \nabla w \right)$$

$$= \frac{1}{m_i} \left(\frac{\nabla \rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i} \nabla w \right) - \frac{\sqrt{\rho_i}}{\rho} \sum_{j=1}^{n} \frac{1}{m_j} \left(\nabla \rho_j + \rho_j \nabla w \right)$$

$$= \frac{1}{m_i} \left(\frac{\nabla \rho_i}{\sqrt{\rho_i}} + \sqrt{\rho_i} \nabla w \right).$$

The last step follows from the pressure constraint (7). Indeed, by (8),

$$\sum_{j=1}^{n} \frac{1}{m_j} \left(\nabla \rho_j + \rho_j \nabla w \right) = \frac{1}{\theta} \sum_{j=1}^{n} \frac{\nabla (\rho_j \theta)}{m_j} = \frac{1}{\theta} \nabla p = 0. \tag{45}$$

We have shown that

$$\sum_{i,j=1}^{n} A_{ij} \nabla \left(q_i + \frac{w}{m_i} \right) \cdot \nabla \left(q_j + \frac{w}{m_j} \right) \geqslant \sum_{i=1}^{n} \frac{\mu}{m_i^2} \left| 2 \nabla \sqrt{\rho_i} + \sqrt{\rho_i} \nabla w \right|^2,$$

which equals (44) after integration over Ω. $\square$

Remark 8. We observe that the sum (45) vanishes even without requiring the constraint (7). Indeed, by (18),

$$\sum_{j=1}^{n} \frac{1}{m_j} \left(\nabla \rho_j + \rho_j \nabla w \right) = \frac{1}{\theta} \sum_{j=1}^{n} \frac{1}{m_j} \nabla (\rho_j \theta) = \frac{1}{\theta} \sum_{j=1}^{n} d_j = 0.$$

The fact that $\sum_{j=1}^{n} d_j$ vanishes is a necessary condition for the invertibility of the linear system (19).

In view of lemma 7 and the lower bound $\kappa \geqslant c_\kappa (1 + \theta^2)$, we conclude from (43) the following discrete entropy inequality.

Lemma 9 (Discrete entropy inequality). *It holds that*

$$\frac{1}{\tau} \int_{\Omega} \left(h(\rho_1, \ldots, \rho_{n-1}, \theta) + E e^{-w_0} \right) dx + \varepsilon C \left(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4 \right)$$

$$+ \int_{\Omega} \left(|\nabla w|^2 + |\nabla \theta|^2 \right) dx + \int_{\Omega} \sum_{i=1}^{n} \frac{\mu}{m_i^2} \left| 2\nabla\sqrt{\rho_i} + \sqrt{\rho_i}\nabla w \right|^2 dx$$

$$\leqslant \frac{1}{\tau} \int_{\Omega} \left(h(\bar{\rho}_1, \ldots, \bar{\rho}_{n-1}, \bar{\theta}) + \bar{E} e^{-w_0} \right) dx.$$

Finally, we derive an estimate for the temperature.

Lemma 10. *There exists a constant $C > 0$, only depending on λ, Ω, $\partial\Omega$, and θ^0 such that*

$$\frac{c_v}{2\tau} \int_{\Omega} \rho\theta^2 dx + \frac{c_\kappa}{2} \int_{\Omega} \left(1 + \theta^2 \right) |\nabla\theta|^2 dx \leqslant C + C \int_{\Omega} \sum_{i=1}^{n} |\nabla\sqrt{\rho_i}|^2 dx + \frac{c_v}{2\tau} \int_{\Omega} \rho\bar{\theta}^2 dx.$$

Proof. We use θ as a test function in the approximate energy equation (38). Observing that $\nabla w_i = \nabla\rho_i/(m_i\rho_i) - \nabla\rho_n/(m_n\rho_n)$ by (26) and $\sum_{i=1}^{n} B_i \nabla w_i = \sum_{i=1}^{n} B_i (m_i\rho_i)^{-1} \nabla\rho_i$ by (30), we find that

$$0 = \frac{c_v}{\tau} \int_{\Omega} \rho\left(\theta - \bar{\theta}\right) dx + \int_{\Omega} \kappa(\theta) |\nabla\theta|^2 dx + \int_{\Omega} \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} |\nabla\theta|^2 dx \int_{\Omega} \sum_{i=1}^{n} \frac{B_i}{m_i\rho_i} \nabla\rho_i \cdot \nabla\theta dx$$

$$- \lambda \int_{\partial\Omega} (\theta_0 - \theta)\theta ds + \varepsilon \int_{\Omega} (\theta_0 + \theta)(\log\theta - \log\theta_0)\theta dx$$

$$+ \varepsilon \int_{\Omega} \left(|D^2\theta|^2 - \frac{1}{\theta} D^2\theta : (\nabla\theta \otimes \nabla\theta) + \frac{|\nabla\theta|^4}{\theta^2} \right) dx = J_1 + \cdots + J_7.$$

We deduce from Young's inequality and assumption (A4) on κ that

$$J_1 \geqslant \frac{c_v}{2\tau} \int_{\Omega} \rho\left(\theta^2 - \bar{\theta}^2\right) dx, \quad J_2 \geqslant c_\kappa \int_{\Omega} \left(1 + \theta^2 \right) |\nabla\theta|^2 dx.$$

Furthermore, $J_3 \geqslant 0$. Definition (29) of B_i and A_{ij} as well as the bound $\rho_j \leqslant \rho^*$ show that

$$J_4 = \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j \rho_i} \nabla\rho_i \cdot \nabla\theta dx = \theta \sum_{i,j=1}^{n} \frac{M_{ij}^{BD}}{m_i m_j} \frac{\sqrt{\rho_j}}{\sqrt{\rho_i}} \nabla\rho_i \cdot \nabla\theta dx$$

$$\geqslant -\frac{c_\kappa}{2} \int_{\Omega} \theta^2 |\nabla\theta|^2 dx - C \int_{\Omega} \sum_{i=1}^{n} |\nabla\sqrt{\rho_i}|^2 dx.$$

The integrals J_5 are J_6 are bounded from below since

$$J_5 \geqslant -\frac{\lambda}{4} \int_{\partial\Omega} \theta_0^2 ds \geqslant -C(\lambda, \partial\Omega, \theta_0),$$

and the dominant term in J_6 is $\theta^2 \log\theta$, which is bounded from below by a negative constant. Finally, J_7 is nonnegative:

$$J_7 = \frac{\varepsilon}{2} \int_{\Omega} \left(|D^2\theta|^2 + \frac{|\nabla\theta|^4}{\theta^2} + \left| D^2\theta - \frac{1}{\theta}\nabla\theta \otimes \nabla\theta \right|^2 \right) dx \geqslant 0.$$

Collecting these estimates finishes the proof. $\qquad\square$

4.4. Uniform estimates

Let $(w_1^k, \ldots, w_{n-1}^k, w^k)$ be a solution to the approximate scheme (37), (38) with $(w_1^{k-1}, \ldots, w_{n-1}^{k-1}, w^{k-1}) = (\bar{w}_1, \ldots, \bar{w}_{n-1}, \bar{w})$. We set $\theta^k = \exp(w^k)$ and $\rho_i^k = \rho_i(w^k)$ determined from lemma 3. Furthermore, we set $E^k = c_v \rho \theta^k$, recalling that $\rho = \sum_{i=1}^n \rho_i^0$. We introduce the piecewise constant in time functions

$$\rho_i^{(\tau)}(x,t) = \rho_i^k(x), \quad q_i^{(\tau)} = \frac{1}{m_i} \log \frac{\rho_i^k}{m_i} - c_v \left(\log \theta^k - 1\right) \quad \text{for } i = 1, \ldots, n,$$

$$\theta^{(\tau)}(x,t) = \theta^k(x), \quad E^{(\tau)}(x,t) = E^k(x), \quad w_i^{(\tau)}(x,t) = w_i^k(x) \quad \text{for } i = 1, \ldots, n-1,$$

where $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$, and $k = 1, \ldots, N$. At time $t = 0$, we set $\rho_i^{(\tau)}(0) = \rho_i^0$ and $\theta^{(\tau)}(0) = \theta^0$. Furthermore, we introduce the shift operator $(\sigma_\tau \rho_i^{(\tau)})(x,t) = \rho_i^{k-1}(x)$ if $t \in ((k-1)\tau, k\tau]$. Then $(\boldsymbol{\rho}^{(\tau)}, \theta^{(\tau)})$ solves

$$0 = \frac{1}{\tau} \int_0^T \int_\Omega \left(\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\right) \phi_i \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^T \int_\Omega \left(\mathrm{D}^2 w_i^{(\tau)} : \mathrm{D}^2 \phi_i + w_i^{(\tau)} \phi_i\right) \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left(\sum_{j=1}^{n-1} A_{ij}\left(\boldsymbol{w}^{(\tau)}\right) \nabla w_j^{(\tau)} + e^{-w^{(\tau)}} B_i\left(\boldsymbol{w}^{(\tau)}\right) \nabla w^{(\tau)}\right) \cdot \nabla \phi_i \mathrm{d}x\mathrm{d}t, \qquad (46)$$

$$0 = \frac{1}{\tau} \int_0^T \int_\Omega \left(E^{(\tau)} - \sigma_\tau E^{(\tau)}\right) \phi_0 \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \kappa\left(\theta^{(\tau)}\right) \nabla \theta^{(\tau)} \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \sum_{i=1}^{n-1} B_j\left(\boldsymbol{w}^{(\tau)}\right) \nabla w_i^{(\tau)} \cdot \nabla \phi_0 \mathrm{d}x - \lambda \int_0^T \int_{\partial\Omega} \left(\theta_0 - \theta^{(\tau)}\right) \phi_0 \mathrm{d}s\mathrm{d}t$$

$$+ \int_0^T \int_\Omega \sum_{i,j=1}^n \frac{A_{ij}\left(\boldsymbol{w}^{(\tau)}\right)}{m_i m_j} \nabla \theta^{(\tau)} \cdot \nabla \phi_0 \mathrm{d}x\mathrm{d}t$$

$$+ \varepsilon \int_0^T \int_\Omega \left(\theta_0 + \theta^{(\tau)}\right) (\log \theta^{(\tau)} - \log \theta_0) \phi_0 \mathrm{d}x\mathrm{d}t$$

$$+ \varepsilon \int_0^T \int_\Omega \theta^{(\tau)} \left(\mathrm{D}^2 \log \theta^{(\tau)} : \mathrm{D}^2 \phi_0 + |\nabla \log \theta^{(\tau)}|^2 \nabla \log \theta^{(\tau)} \cdot \nabla \phi_0\right) \mathrm{d}x\mathrm{d}t. \qquad (47)$$

The discrete entropy inequality in lemma 9 and the temperature estimates in lemma 10 yield, after summation over $k = 1, \ldots, N$,

$$\sup_{0 < t < T} \int_\Omega \left(h\left(\rho_1^{(\tau)}(t), \ldots, \rho_{n-1}^{(\tau)}(t), \theta^{(\tau)}(t)\right) + \frac{c_v}{\theta_0} \rho \theta^{(\tau)}(t)\right) \mathrm{d}x$$

$$+ \int_0^T \int_\Omega \left(|\nabla \log \theta^{(\tau)}|^2 + |\nabla \theta^{(\tau)}|^2\right) \mathrm{d}x\mathrm{d}t$$

$$+ \varepsilon C \int_0^T \left(\|\boldsymbol{w}^{(\tau)}\|_{H^2(\Omega)}^2 + \|\nabla w^{(\tau)}\|_{L^4(\Omega)}^4\right) \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \sum_{i=1}^n \frac{\mu}{m_i^2} \left|2\nabla \left(\rho_i^{(\tau)}\right)^{1/2} + \left(\rho_i^{(\tau)}\right)^{1/2} \nabla \log \theta^{(\tau)}\right|^2 \mathrm{d}x\mathrm{d}t$$

$$\leqslant \int_\Omega \left(h\left(\rho_1^0, \ldots, \rho_{n-1}^0, \theta^0\right) + c_v \rho \theta^0\right) \mathrm{d}x, \qquad (48)$$

$$c_v \sup_{0<t<T} \int_\Omega \rho \left(\theta^{(\tau)}\right)^2 \mathrm{d}x + c_\kappa \int_0^T \int_\Omega \left(1 + \left(\theta^{(\tau)}\right)^2\right) |\nabla \theta^{(\tau)}|^2 \mathrm{d}x\mathrm{d}t$$

$$\leqslant C(T) + C \int_0^T \int_\Omega \sum_{i=1}^n |\nabla \left(\rho_i^{(\tau)}\right)^{1/2}|^2 \mathrm{d}x\mathrm{d}t + \frac{c_v}{2} \int_\Omega \rho(\theta^0)^2 \mathrm{d}x. \tag{49}$$

Lemma 11. *There exists $C > 0$ not depending on (ε, τ) such that*

$$\|\boldsymbol{\rho}^{(\tau)}\|_{L^\infty(\Omega_T)} + \|\theta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C, \tag{50}$$

$$\|\log \theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \|\theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leqslant C, \tag{51}$$

$$\varepsilon^{1/2}\|\boldsymbol{w}^{(\tau)}\|_{L^2(0,T;H^2(\Omega))} + \varepsilon^{1/4}\|\nabla w^{(\tau)}\|_{L^4(\Omega_T)} \leqslant C, \tag{52}$$

Proof. Estimates (50) and (52) are an immediate consequence of (48) and $\rho \geqslant \rho_* > 0$. Bound (48) also shows that $\sup_{(0,T)} \int_\Omega (-\log \theta^{(\tau)} + \theta^{(\tau)})\mathrm{d}x$ is uniformly bounded from above. Thus, $\log \theta^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Then the uniform bounds for $\nabla \log \theta^{(\tau)}$ and $\nabla \theta^{(\tau)}$ as well as the Poincaré–Wirtinger inequality yield bounds for $\log \theta^{(\tau)}$ and $\theta^{(\tau)}$ in $L^2(\Omega_T)$, proving (51). $\qquad \square$

Lemma 12. *There exists $C > 0$ not depending on (ε, τ) such that for $i = 1, \ldots, n$,*

$$\| \left(\rho_i^{(\tau)}\right)^{1/2} \|_{L^2(0,T;H^1(\Omega))} + \|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leqslant C, \tag{53}$$

$$\|\theta^{(\tau)}\|_{L^\infty(0,T;L^2(\Omega))} + \| \left(\theta^{(\tau)}\right)^2 \|_{L^2(0,T;H^1(\Omega))} + \|\theta^{(\tau)}\|_{L^{16/3}(\Omega_T)} \leqslant C. \tag{54}$$

Proof. We infer from (48) that

$$\int_0^T \int_\Omega |\nabla \left(\rho_i^{(\tau)}\right)^{1/2}|^2 \mathrm{d}x\mathrm{d}t \leqslant C \int_0^T \int_\Omega |2\nabla \left(\rho_i^{(\tau)}\right)^{1/2}|^2 + \left(\rho_i^{(\tau)}\right)^{1/2} \nabla \log \theta^{(\tau)}|^2 \mathrm{d}x\mathrm{d}t$$

$$+ C \int_0^T \int_\Omega |\nabla \log \theta^{(\tau)}|^2 \mathrm{d}x\mathrm{d}t \leqslant C,$$

and the $L^\infty(\Omega_T)$ bound (50) gives for $i = 1, \ldots, n$,

$$\|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \leqslant 2\|\rho_i^{(\tau)}\|_{L^\infty(\Omega_T)}^{1/2}\|\nabla \left(\rho_i^{(\tau)}\right)^{1/2}\|_{L^2(\Omega_T)} + \|\rho_i^{(\tau)}\|_{L^2(\Omega_T)} \leqslant C.$$

Therefore, the right-hand side of (49) is uniformly bounded, which proves the first two estimates in (54). The remaining one is a consequence of the Gagliardo–Nirenberg inequality with $\eta = 3/4$:

$$\| \left(\theta^{(\tau)}\right)^2 \|_{L^{8/3}(\Omega_T)}^{8/3} \leqslant C \int_0^T \| \left(\theta^{(\tau)}\right)^2 \|_{H^1(\Omega)}^{8\eta/3} \| \left(\theta^{(\tau)}\right)^2 \|_{L^1(\Omega)}^{8(1-\eta)/3} \mathrm{d}t$$

$$\leqslant \|\theta^{(\tau)}\|_{L^\infty(0,T;L^2(\Omega))}^{4/3} \int_0^T \| \left(\theta^{(\tau)}\right)^2 \|_{H^1(\Omega)}^2 \mathrm{d}t \leqslant C.$$

This finishes the proof. $\qquad \square$

The following lemma can be proved as in [15, lemma 9].

Lemma 13. *There exists $C > 0$ not depending on (ε, τ) such that*

$$\|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} + \|\theta^{(\tau)} - \sigma_\tau \theta^{(\tau)}\|_{L^{16/15}(0,T;W^{2,16}(\Omega)^*)} \leqslant C\tau. \tag{55}$$

4.5. The limit (ε, τ)

The bounds (51), (53) and (55) allow us to apply the Aubin–Lions lemma in the version of [10]. There exist subsequences, which are not relabelled, such that as $(\varepsilon, \tau) \to 0$,

$$\rho_i^{(\tau)} \to \rho_i, \quad \theta^{(\tau)} \to \theta \quad \text{strongly in } L^2(\Omega_T), \, i = 1, \ldots, n-1.$$

The convergence also holds for $i = n$ since $\rho_n^{(\tau)} = 1 - \sum_{i=1}^{n-1} \rho_i^{(\tau)}$. Thanks to the $L^\infty(\Omega_T)$ bound for $\rho_i^{(\tau)}$ and the $L^{16/3}(\Omega_T)$ bound for $\theta^{(\tau)}$, we have

$$\rho_i^{(\tau)} \to \rho_i \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r < \infty,$$
$$\theta^{(\tau)} \to \theta \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r < 16/3.$$

We claim that $\rho_i > 0$ and $\theta > 0$ a.e. in Ω_T. The positivity of ρ_i is proved as in [15, p 16]. The strong convergence of $(\theta^{(\tau)})$ implies a.e. convergence and in particular $\log \theta^{(\tau)} \to Z$ a.e. Thus, $\theta^{(\tau)} \to \exp(Z)$ a.e. We conclude that $\theta = \exp(Z) > 0$ a.e. in Ω_T.

It follows that $\log \theta \in L^2(\Omega_T)$ and estimate (51) yields

$$\nabla \log \theta^{(\tau)} \rightharpoonup \nabla \log \theta \quad \text{weakly in } L^2(\Omega_T). \tag{56}$$

Furthermore, in view of (51), (53) and (55), up to subsequences,

$$\rho_i^{(\tau)} \rightharpoonup \rho_i, \quad \theta^{(\tau)} \rightharpoonup \theta \quad \text{weakly in } L^2\left(0, T; H^1(\Omega)\right),$$
$$\tau^{-1}\left(\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\right) \rightharpoonup \partial_t \rho_i \quad \text{weakly in } L^2\left(0, T; H^2(\Omega)^*\right),$$
$$\tau^{-1}\left(\theta^{(\tau)} - \sigma_\tau \theta^{(\tau)}\right) \rightharpoonup \partial_t \rho_i \quad \text{weakly in } L^{16/15}\left(0, T; W^{2,16}(\Omega)^*\right),$$

and the bounds (52) show that

$$\varepsilon \log \theta^{(\tau)} \to 0, \quad \varepsilon w_i^{(\tau)} \to 0 \quad \text{strongly in } L^2\left(0, T; H^2(\Omega)\right).$$

The embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact, giving $\theta^{(\tau)} \to \theta$ strongly in $L^2(0, T; L^2(\partial\Omega))$.

These convergences are sufficient to pass to the limit $(\varepsilon, \tau) \to 0$ in (46) and (47), showing that $(\boldsymbol{\rho}, \theta)$ solves the weak formulation (35) and (36). We only detail the limits in the terms $A_{ij}^{(\tau)} = A_{ij}(\boldsymbol{w}^{(\tau)})$ and $B_i^{(\tau)} = B_i(\boldsymbol{w}^{(\tau)})$. We know that $\nabla(\rho_i^{(\tau)})^{1/2} \rightharpoonup \nabla\rho_i^{1/2}$ weakly in $L^2(\Omega_T)$ and

$$\frac{A_{ij}^{(\tau)}}{m_j\left(\rho_j^{(\tau)}\right)^{1/2}} = M_{ij}^{BD}\left(\boldsymbol{\rho}^{(\tau)}\right) \frac{\left(\rho_i^{(\tau)}\right)^{1/2}}{m_j} \to M_{ij}^{BD}(\boldsymbol{\rho}) \frac{\rho_i^{1/2}}{m_j} = \frac{A_{ij}}{m_j\rho_j^{1/2}}$$

strongly in $L^\gamma(\Omega_T)$ for all $\gamma < \infty$. Using (32) and (25), this implies that

$$\sum_{j=1}^{n-1} A_{ij}^{(\tau)} \nabla w_j^{(\tau)} = \sum_{j=1}^{n} \frac{A_{ij}^{(\tau)}}{m_j} \nabla \log \frac{\rho_i^{(\tau)}}{m_j} = 2\sum_{j=1}^{n} M_{ij}^{BD}\left(\boldsymbol{\rho}^{(\tau)}\right) \frac{\left(\rho_i^{(\tau)}\right)^{1/2}}{m_j} \nabla\left(\rho_j^{(\tau)}\right)^{1/2}$$

$$\rightharpoonup 2\sum_{j=1}^{n} M_{ij}^{BD} \frac{\rho_i^{1/2}}{m_j} \nabla \rho_j^{1/2} \quad \text{weakly in } L^s(\Omega_T), \, s < 2.$$

Since the sequence is bounded in $L^2(\Omega_T)$, this convergence also holds in this space. Similarly,

$$B_i^{(\tau)} e^{-w^{(\tau)}} \nabla w^{(\tau)} = \sum_{j=1}^{n} \frac{A_{ij}^{(\tau)}}{m_j} \nabla \log \theta^{(\tau)} \rightharpoonup \sum_{j=1}^{n} \frac{A_{ij}}{m_j} \nabla \log \theta \quad \text{weakly in } L^2(\Omega_T),$$

$$A_{ij}^{(\tau)} \nabla \theta^{(\tau)} = M_{ij}^{BD}\left(\boldsymbol{\rho}^{(\tau)}\right) \left(\rho_i^{(\tau)} \rho_j^{(\tau)}\right)^{1/2} \nabla \theta^{(\tau)} \rightharpoonup A_{ij} \nabla \theta \quad \text{weakly in } L^2(\Omega_T),$$

and using $\theta^{(\tau)} \to \theta$ strongly in $L^r(\Omega_T)$ for $r < 16/3$,

$$\sum_{i=1}^{n-1} B_i^{(\tau)} \nabla w_i^{(\tau)} = 2 \sum_{i,j=1}^{n} \frac{M_{ij}^{BD}\left(\boldsymbol{\rho}^{(\tau)}\right)}{m_i m_j} \theta^{(\tau)} \left(\rho_j^{(\tau)}\right)^{1/2} \nabla \left(\rho_i^{(\tau)}\right)^{1/2} \rightharpoonup 2 \sum_{i=1}^{n} \frac{B_i}{m_i \rho_i^{1/2}} \nabla \rho_i^{1/2}$$

weakly in $L^s(\Omega_T)$ for $s < 16/11$, and since the right-hand side lies in $L^{16/11}(\Omega_T)$, this convergence also holds in $L^{16/11}(\Omega_T)$.

Next, we claim that $\rho_i(0)$ and $\theta(0)$ satisfy the initial data. The time derivative of the linear interpolant

$$\widetilde{\rho}_i^{(\tau)}(t) = \rho_i^k - \frac{k\tau - t}{\tau}\left(\rho_i^k - \rho_i^{k-1}\right) \quad \text{for } (k-1)\tau < t < k\tau$$

is bounded since, because of (55),

$$\|\partial_t \widetilde{\rho}_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \leqslant \tau^{-1}\|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \leqslant C.$$

Thus, $\widetilde{\rho}_i^{(\tau)}$ is uniformly bounded in $H^1(0,T;H^2(\Omega)^*) \hookrightarrow C^0([0,T];H^2(\Omega)^*)$ and we conclude for a subsequence that $\rho_i^0 = \widetilde{\rho}_i^{(\tau)}(0) \rightharpoonup r_i$ weakly in $H^2(\Omega)^*$ for some $r_i \in H^2(\Omega)^*$. It follows that $r_i = \rho_i^0$. As $\widetilde{\rho}_i^{(\tau)}$ and $\rho_i^{(\tau)}$ converge to the same limit,

$$\|\widetilde{\rho}_i^{(\tau)} - \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \leqslant \|\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \leqslant C\tau \to 0,$$

this shows that $\rho_i^0 = r_i = \rho_i(0)$ in $H^2(\Omega)^*$. In an analogous way, we verify that $\theta(0) = \theta^0$ in $W^{2,16}(\Omega)^*$.

The initial data are satisfied in better spaces. Indeed, going back to (35) and (36), the regularity of ρ_i implies that $\partial_t \rho_i \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*) \hookrightarrow C^0([0,T];L^2(\Omega))$ and thus $\rho_i(0) = \rho_i^0$ in the sense of $L^2(\Omega)$. The temperature satisfies $\theta \in L^\infty(0,T;L^2(\Omega)) \cap C^0([0,T];W^{2,16}(\Omega)^*)$, which gives $\theta \in C_w^0([0,T];L^2(\Omega))$. Consequently, $\theta(0) = \theta^0$ weakly in $L^2(\Omega)$. Moreover, we deduce from $|\kappa \nabla \theta| \leqslant C_\kappa(|\nabla \theta| + \theta|\nabla \theta^2|) \in L^{16/11}(\Omega_T)$ that $\partial_t \theta \in L^{16/11}(0,T;W^{1,16/11}(\Omega)^*)$. This completes the proof.

5. Proof of theorem 2

Let $(\boldsymbol{\rho},\theta)$ be a weak solution and $(\bar{\boldsymbol{\rho}},\bar{\theta})$ be a strong solution to (1)–(8). We introduce the entropy

$$H(\boldsymbol{\rho}(t),\theta(t)) = \int_\Omega \left(\sum_{i=1}^{n} \frac{\rho_i}{m_i}\left(\log \frac{\rho_i}{m_i} - 1\right) - \rho \log \theta\right) dx.$$

Lemma 14 (Entropy equality for strong solutions). *Let $(\bar{\boldsymbol{\rho}}, \bar{\theta})$ be a strong solution to (1)–(8) (in the sense mentioned after theorem 2) with $\lambda = 0$. Then*

$$H\left(\bar{\boldsymbol{\rho}}(t), \bar{\theta}(t)\right) + \int_0^t \int_\Omega \frac{\kappa(\bar{\theta})}{\bar{\theta}^2} |\nabla \bar{\theta}|^2 \mathrm{d}x \mathrm{d}s + \frac{1}{2} \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j |\bar{u}_i - \bar{u}_j|^2 \mathrm{d}x \mathrm{d}s = H\left(\bar{\boldsymbol{\rho}}(0), \bar{\theta}(0)\right).$$

Proof. We use (1) and (2) and integrate by parts to obtain

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_\Omega \left(\sum_{i=1}^n \frac{\partial_t \bar{\rho}_i}{m_i} \log \frac{\bar{\rho}_i}{m_i} - \frac{\ }{\rho} \partial_t \left(\rho\bar{\theta} \right) \right) \mathrm{d}x \\
&= \int_\Omega \left\{ \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i}{m_i} \nabla \log \frac{\bar{\rho}_i}{m_i} + \frac{\nabla\bar{\theta}}{\bar{\theta}^2} \left(-\bar{\kappa} \nabla\bar{\theta} + \bar{\theta} \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i}{m_i} \right) \right\} \mathrm{d}x \\
&= -\int_\Omega \frac{\bar{\kappa}}{\bar{\theta}^2} |\nabla\bar{\theta}|^2 \mathrm{d}x + \int_\Omega \sum_{i=1}^n \frac{\bar{u}_i}{m_i} \cdot \left(\nabla\bar{\rho}_i + \bar{\rho}_i \nabla \log \bar{\theta} \right) \mathrm{d}x \\
&= -\int_\Omega \frac{\bar{\kappa}}{\bar{\theta}^2} |\nabla\bar{\theta}|^2 \mathrm{d}x + \int_\Omega \sum_{i=1}^n \frac{1}{\bar{\theta}} \bar{u}_i \cdot \bar{d}_i \mathrm{d}x,
\end{aligned}$$

where $\bar{\kappa} = \kappa(\bar{\theta})$ and we used (18) in the last step. By the algebraic system (5) and the symmetry of (b_{ij}),

$$\sum_{i=1}^n \frac{1}{\bar{\theta}} \bar{u}_i \cdot \bar{d}_i = -\sum_{i,j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j \left(\bar{u}_i - \bar{u}_j \right) \cdot \bar{u}_i = -\frac{1}{2} \sum_{i,j=1}^n b_{ij} \bar{\rho}_i \bar{\rho}_j |\bar{u}_i - \bar{u}_j|^2. \qquad (57)$$

This shows the claim. $\qquad\qquad\square$

Lemma 15 (Entropy inequality for weak solutions). *Let $(\bar{\boldsymbol{\rho}}, \bar{\theta})$ be a weak solution to (1)–(8) with $\lambda = 0$. Then*

$$H(\boldsymbol{\rho}(t), \theta(t)) + \int_0^t \int_\Omega \frac{\kappa}{\theta^2} |\nabla\theta|^2 \mathrm{d}x \mathrm{d}s + \frac{1}{2} \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij} \rho_i \rho_j |u_i - u_j|^2 \mathrm{d}x \mathrm{d}s \leqslant H\left(\boldsymbol{\rho}^0, \theta^0\right).$$

Proof. Let $(\boldsymbol{\rho}^k, \theta^k)$ for $k = 1, \ldots, N$ be a solution to the approximate problem (37) and (38), constructed in section 4.2. According to (43), this solution satisfies

$$H\left(\boldsymbol{\rho}^k, \theta^k\right) + \tau \int_\Omega \kappa\left(\theta^k\right) |\nabla \log \theta^k|^2 \mathrm{d}x + \tau \int_\Omega \sum_{i,j=1}^n A_{ij}^k \nabla \left(q_i^k + \frac{w^k}{m_i} \right) \cdot \nabla \left(q_j^k + \frac{w^k}{m_j} \right) \mathrm{d}x \leqslant H\left(\boldsymbol{\rho}^{k-1}, \theta^{k-1}\right),$$

where the superindex k denotes the kth time step. By corollary 5 as well as relations (22) and (57),

$$\begin{aligned}
\sum_{i,j=1}^n A_{ij}^k \nabla \left(q_i^k + \frac{w^k}{m_i} \right) \cdot \nabla \left(q_j^k + \frac{w^k}{m_j} \right) &= \sum_{i,j=1}^n \left(M_{ij}^{BD} \right)^k \frac{d_i^k}{\theta^k \left(\rho_i^k \right)^{1/2}} \cdot \frac{d_j^k}{\theta^k \left(\rho_j^k \right)^{1/2}} \\
&= -\sum_{i=1}^n \frac{1}{\theta^k} d_i^k \cdot u_i^k = \frac{1}{2} \sum_{i,j=1}^n b_{ij} \rho_i^k \rho_j^k |u_i^k - u_j^k|^2.
\end{aligned}$$

Therefore,

$$H\left(\boldsymbol{\rho}^k,\theta^k\right) + \tau\int_\Omega \kappa\left(\theta^k\right)|\nabla\log\theta^k|^2\mathrm{d}x + \frac{\tau}{2}\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^k\rho_j^k|u_i^k - u_j^k|^2\mathrm{d}x \leqslant H\left(\boldsymbol{\rho}^{k-1},\theta^{k-1}\right).$$

We sum over $k = 1,\ldots,j$ with $t \in ((j-1)\tau,j\tau]$ and use the notation of section 4.4:

$$H\left(\boldsymbol{\rho}^{(\tau)}(t),\theta^{(\tau)}(t)\right) + \int_0^t\int_\Omega \kappa\left(\theta^{(\tau)}\right)|\nabla\log\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}s$$

$$+ \frac{1}{2}\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)} - u_j^{(\tau)}|^2\mathrm{d}x\mathrm{d}s \leqslant H\left(\boldsymbol{\rho}^0,\theta^0\right) \qquad (58)$$

for a.e. $t \in (0,T)$.

It remains to pass to the limit $(\varepsilon,\tau) \to 0$ in (58). We deduce from the strong convergence of $(\boldsymbol{\rho}^{(\tau)})$ and $(\theta^{(\tau)})$ that

$$H(\boldsymbol{\rho}(t),\theta(t)) \leqslant \liminf_{(\varepsilon,\tau)\to0} H\left(\boldsymbol{\rho}^{(\tau)}(t),\theta^{(\tau)}(t)\right).$$

We deduce from the strong convergence $\rho_i^{(\tau)} \to \rho_i$ in $L^q(\Omega_T)$ for any $q < \infty$ and the boundedness of M_{ij}^{BD} that $M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)}) \to M_{ij}^{BD}(\boldsymbol{\rho})$ strongly in any $L^q(\Omega_T)$. In view of the weak convergences $\nabla\log\theta^{(\tau)} \rightharpoonup \nabla\log\theta$ from (56) and $\nabla(\rho_i^{(\tau)})^{1/2} \rightharpoonup \nabla\rho_i^{1/2}$ from (53) weakly in $L^2(\Omega_T)$, we have

$$2\nabla\left(\rho_i^{(\tau)}\right)^{1/2} + \rho_i^{(\tau)}\nabla\log\theta^{(\tau)} \rightharpoonup 2\nabla\rho_i^{1/2} + \rho_i\nabla\log\theta \quad\text{weakly in } L^2(\Omega_T).$$

Hence, using (22),

$$\left(\rho_i^{(\tau)}\right)^{1/2}u_i^{(\tau)} = \sum_{j=1}^n M_{ij}^{BD}\left(\boldsymbol{\rho}^{(\tau)}\right)\frac{1}{m_j}\left(2\nabla\left(\rho_i^{(\tau)}\right)^{1/2} + \rho_i^{(\tau)}\nabla\log\theta^{(\tau)}\right)$$

$$\rightharpoonup \sum_{j=1}^n M_{ij}^{BD}(\boldsymbol{\rho})\frac{1}{m_j}\left(2\nabla\rho_i^{1/2} + \rho_i\nabla\log\theta\right) = \rho_i^{1/2}u_i.$$

Weakly in $L^2(\Omega_T)$, where the last identity is the definition of u_i. Then, taking into account the boundedness of $\rho_i^{(\tau)}$ in $L^\infty(\Omega_T)$, for any $i,j = 1,\ldots,n$,

$$\left(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}\right)^{1/2}u_i^{(\tau)} \rightharpoonup (b_{ij}\rho_i\rho_j)^{1/2}u_i \quad\text{weakly in } L^2(\Omega_T).$$

As the $L^2(\Omega_T)$ norm is weakly lower semicontinuous,

$$\int_0^T\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i - u_j|^2\mathrm{d}x\mathrm{d}s \leqslant \liminf_{(\varepsilon,\tau)\to0}\int_0^T\int_\Omega\sum_{i,j=1}^n\left|\left(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}\right)^{1/2}\left(u_i^{(\tau)} - u_j^{(\tau)}\right)\right|^2\mathrm{d}x\mathrm{d}s$$

$$= \liminf_{(\varepsilon,\tau)\to0}\int_0^T\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)} - u_j^{(\tau)}|^2\mathrm{d}x\mathrm{d}s.$$

Finally, $\kappa(\theta^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)} \to \kappa(\theta)^{1/2}\nabla\log\theta$ weakly in $L^1(\Omega_T)$ and, because of the uniform bounds, also in $L^2(\Omega_T)$. Hence,

$$\int_0^t \int_\Omega \frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2\mathrm{d}x\mathrm{d}s \leqslant \liminf_{(\varepsilon,\tau)\to 0} \int_0^t \int_\Omega \frac{\kappa(\theta^{(\tau)})}{(\theta^{(\tau)})^2}|\nabla\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}s.$$

Thus, applying the limit inferior $(\varepsilon,\tau)\to 0$ to both sides of (58) yields the result. □

Lemma 16 (Relative entropy inequality). *Let the assumptions of theorem 2 hold and let $\rho_i(0) = \bar\rho_i(0)$ for $i = 1,\ldots,n$ and $\theta(0) = \bar\theta(0)$. Then*

$$H\big((\boldsymbol{\rho},\theta)(t)\,|\,(\bar{\boldsymbol{\rho}},\bar\theta)(t)\big) + \frac{\mu_M}{2}\int_0^t \int_\Omega \sum_{i=1}^n \rho_i|u_i - \bar u_i|^2\mathrm{d}x\mathrm{d}s$$

$$+ \frac{c_\kappa}{2}\int_0^t \int_\Omega |\nabla(\log\theta - \log\bar\theta)|^2\mathrm{d}x\mathrm{d}s \leqslant C\int_0^t \int_\Omega \left(\sum_{j=1}^n (\rho_j - \bar\rho_j)^2 + (\theta - \bar\theta)^2\right)\mathrm{d}x\mathrm{d}s, \quad (59)$$

where the relative entropy $H(\boldsymbol{\rho},\theta|\bar{\boldsymbol{\rho}},\bar\theta)$ is defined in (12).

Proof. We use the test functions $\phi_i = m_i^{-1}\log(\bar\rho_i/m_i) - c_v\log\bar\theta$ and $\phi_0 = -1/\bar\theta$ in the weak formulations satisfied by $\rho_i - \bar\rho_i$ and $\rho(\theta - \bar\theta)$, respectively,

$$\int_\Omega (\rho_i - \bar\rho_i)(t)\,\phi_i(t)\,\mathrm{d}x = \int_0^t \int_\Omega (\rho_i - \bar\rho_i)\,\partial_t\phi_i\mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega (\rho_i u_i - \bar\rho_i\bar u_i)\cdot\nabla\phi_i\,\mathrm{d}x\mathrm{d}s,$$

$$\int_\Omega c_v\rho(\theta - \bar\theta)(t)\,\phi_0(t)\,\mathrm{d}x = \int_0^t \int_\Omega c_v\rho(\theta - \bar\theta)\,\partial_t\phi_0\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega (\kappa\nabla\theta - \bar\kappa\nabla\bar\theta)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \sum_{j=1}^n (h_j u_j - \bar h_j\bar u_j)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s,$$

where $h_j = (c_v + 1/m_j)\rho_j\theta$, $\bar h_j = (c_v + 1/m_j)\bar\rho_j\bar\theta$, and $\kappa = \kappa(\theta)$, $\bar\kappa = \kappa(\bar\theta)$. Strictly speaking, we cannot use ϕ_i as a test function since $\log\bar\rho_i$ and $1/\bar\theta$ may be not integrable. However, we can use a density argument similarly as in the proof of [18, lemma 8]. Then, summing over $i = 1,\ldots,n$,

$$\int_\Omega \left\{\sum_{i=1}^n (\rho_i - \bar\rho_i)(t)\left(\frac{1}{m_i}\log\frac{\bar\rho_i}{m_i} - c_v\log\bar\theta\right)(t) - c_v\rho\frac{\theta - \bar\theta}{\bar\theta}(t)\right\}\mathrm{d}x$$

$$= \int_0^t \int_\Omega \left\{\sum_{i=1}^n \left((\rho_i - \bar\rho_i)\frac{\partial_t\bar\rho_i}{m_i\bar\rho_i} + (\rho_i u_i - \bar\rho_i\bar u_i)\cdot\frac{\nabla\bar\rho_i}{m_i\bar\rho_i}\right) + c_v\rho(\theta - \bar\theta)\,\partial_t\left(-\frac{1}{\bar\theta}\right)\right\}\mathrm{d}x\mathrm{d}s$$

$$- \int_0^t \int_\Omega (\kappa\nabla\theta - \bar\kappa\nabla\bar\theta)\cdot\nabla\left(-\frac{1}{\bar\theta}\right)\mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega \sum_{j=1}^n (h_j u_j - \bar h_j\bar u_j)\cdot\nabla\left(-\frac{1}{\bar\theta}\right)\mathrm{d}x\mathrm{d}s.$$

We subtract this identity and the entropy equality from lemma 14 for $(\bar{\boldsymbol{\rho}},\bar\theta)$ from the entropy inequality for $(\boldsymbol{\rho},\theta)$ obtained in lemma 15 and insert equations (1), (2) to replace the time derivatives $\partial_t\bar\rho_i$ and $\partial_t(-1/\bar\theta)$. A computation shows that

$$H\left((\boldsymbol{\rho},\theta)(t)\,|\,(\bar{\boldsymbol{\rho}},\bar{\theta})(t)\right) \leqslant K_1 \cdots + K_5, \quad \text{where}$$

$$K_1 = -\int_0^t \int_\Omega \left(\kappa|\nabla\log\theta|^2 - \bar{\kappa}|\nabla\log\bar{\theta}|^2\right)\mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega \bar{\kappa}\nabla\bar{\theta}\cdot\nabla\left(\frac{\theta}{\bar{\theta}^2} - \frac{1}{\bar{\theta}}\right)\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \left(\kappa\nabla\theta - \bar{\kappa}\nabla\bar{\theta}\right)\cdot\nabla\left(-\frac{1}{\bar{\theta}}\right)\mathrm{d}x\mathrm{d}s,$$

$$K_2 = -\int_0^t \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i\bar{u}_i}{m_i}\cdot\nabla\left(\frac{\rho_i}{\bar{\rho}_i}\right)\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{\nabla\bar{\rho}_i}{m_i\bar{\rho}_i}\cdot\left(\rho_i u_i - \bar{\rho}_i\bar{u}_i\right)\mathrm{d}x\mathrm{d}s,$$

$$K_3 = -\int_0^t \int_\Omega \sum_{i=1}^n \bar{h}_i\bar{u}_i\cdot\nabla\left(\frac{\theta}{\bar{\theta}^2} - \frac{1}{\bar{\theta}}\right)\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \left(h_i u_i - \bar{h}_i\bar{u}_i\right)\cdot\nabla\left(-\frac{1}{\bar{\theta}}\right)\mathrm{d}x\mathrm{d}s,$$

$$K_4 = -\frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i - u_j|^2\mathrm{d}x\mathrm{d}s,$$

$$K_5 = \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j\left|\bar{u}_i - \bar{u}_j\right|^2\mathrm{d}x\mathrm{d}s. \tag{60}$$

The term K_1 can be rewritten as

$$K_1 = -\int_0^t \int_\Omega \frac{1}{\bar{\theta}}\left(\kappa\bar{\theta} - \bar{\kappa}\theta\right)\nabla\left(\log\theta - \log\bar{\theta}\right)\cdot\nabla\log\bar{\theta}\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \kappa\left|\nabla\left(\log\theta - \log\bar{\theta}\right)\right|^2\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \frac{\theta - \bar{\theta}}{\bar{\theta}}\nabla\log\bar{\theta}\cdot\left(\kappa\nabla\log\theta - \bar{\kappa}\nabla\log\bar{\theta}\right)\mathrm{d}x\mathrm{d}s =: K_{11} + K_{12} + K_{13}.$$

The algebraic system (5) with $d_i = \nabla(\rho_i\theta)/m_i$ can be formulated as

$$-m_i\sum_{j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j\left(\bar{u}_i - \bar{u}_j\right) - \bar{\rho}_i\nabla\log\bar{\theta} = \nabla\bar{\rho}_i.$$

This allows us to rewrite K_2:

$$K_2 = \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j\left(u_i - u_j\right)\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\bar{\rho}_j\left(\bar{u}_i - \bar{u}_j\right)\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\bar{\rho}_j\left(\bar{u}_i - \bar{u}_j\right)\cdot u_i\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\bar{\rho}_i\bar{\rho}_j\left(\bar{u}_i - \bar{u}_j\right)\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\rho_i\nabla\log\theta\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\rho_i\nabla\log\bar{\theta}\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\rho_i\nabla\log\bar{\theta}\cdot u_i\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\bar{\rho}_i\nabla\log\bar{\theta}\cdot\bar{u}_i\mathrm{d}x\mathrm{d}s$$

$$=: K_{21} + \cdots + K_{28}.$$

Furthermore, it follows from $h_i = (c_v + 1/m_i)\rho_i\theta$ and $\sum_{i=1}^n \rho_i u_i = \sum_{i=1}^n \bar{\rho}_i \bar{u}_i = 0$ that

$$
\begin{aligned}
K_3 &= -\int_0^t \int_\Omega \sum_{i=1}^n \bar{h}_i \bar{u}_i \cdot \nabla\left(\frac{\theta}{\bar{\theta}^2}\right) \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n h_i u_i \cdot \nabla\left(-\frac{1}{\theta}\right) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^t \int_\Omega \sum_{i=1}^n \left(c_v + \frac{1}{m_i}\right) \bar{\theta}\bar{\rho}_i \bar{u}_i \cdot \nabla\left(\frac{\theta}{\bar{\theta}^2}\right) \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \sum_{i=1}^n \left(c_v + \frac{1}{m_i}\right) \theta\rho_i u_i \cdot \nabla\left(-\frac{1}{\theta}\right) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^t \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{\theta}}{m_i} \bar{u}_i \cdot \nabla\left(\frac{\theta}{\bar{\theta}^2}\right) \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{\rho_i \theta}{m_i} u_i \cdot \nabla\left(-\frac{1}{\theta}\right) \mathrm{d}x\mathrm{d}s \\
&= -\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i}{m_i \bar{\theta}} \cdot \nabla\theta \mathrm{d}x\mathrm{d}s + 2\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar{\rho}_i \bar{u}_i \theta}{m_i \bar{\theta}^2} \cdot \nabla\bar{\theta}\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega \sum_{i=1}^n \frac{\rho_i u_i \theta}{m_i \bar{\theta}^2} \cdot \nabla\bar{\theta}\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We reformulate K_4 as

$$
\begin{aligned}
K_4 &= -\frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j \left|(u_i - \bar{u}_i) - (u_j - \bar{u}_j)\right|^2 \mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j |\bar{u}_i - \bar{u}_j|^2 \mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j (u_i - u_j) \cdot (\bar{u}_i - \bar{u}_j) \mathrm{d}x\mathrm{d}s =: K_{41} + K_{42} + K_{43}.
\end{aligned}
$$

A long but straightforward computation shows that

$$
\begin{aligned}
&K_{21} + K_{22} + K_{23} + K_{24} + K_{42} + K_{43} + K_5 \\
&= -\int_0^T \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i (\rho_j - \bar{\rho}_j)(u_i - \bar{u}_i) \cdot (\bar{u}_i - \bar{u}_j) \mathrm{d}x\mathrm{d}s =: L_1
\end{aligned}
$$

and

$$
\begin{aligned}
K_{25} + K_{26} + K_{27} + K_{28} + K_3 &= \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}(\rho_i - \bar{\rho}_i)\left(\nabla\log\theta - \nabla\log\bar{\theta}\right) \cdot \bar{u}_i \mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\bar{\rho}_i \bar{u}_i \cdot \left(\nabla\log\theta - \nabla\log\bar{\theta}\right)\left(1 - \frac{\theta}{\bar{\theta}}\right) \mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\rho_i (u_i - \bar{u}_i) \cdot \nabla\log\bar{\theta}\left(1 - \frac{\theta}{\bar{\theta}}\right) \mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}(\rho_i - \bar{\rho}_i)\bar{u}_i \cdot \nabla\log\bar{\theta}\left(1 - \frac{\theta}{\bar{\theta}}\right) \mathrm{d}x\mathrm{d}s \\
&=: L_2 + L_3 + L_4 + L_5.
\end{aligned}
$$

Inserting these expressions into (60), putting K_{12} on the left-hand side, and rearranging the terms, we find that

$$H\left((\boldsymbol{\rho},\theta)\left(t\right)|\left(\bar{\boldsymbol{\rho}},\bar{\theta}\right)\left(t\right)\right) + \frac{1}{2}\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\left|\left(u_i-\bar{u}_i\right)-\left(u_j-\bar{u}_j\right)\right|^2\mathrm{d}x\mathrm{d}s$$

$$+\int_0^t\int_\Omega \kappa|\nabla\left(\log\theta-\log\bar{\theta}\right)|^2\mathrm{d}x\mathrm{d}s \leqslant K_{11}+K_{13}+L_1+\cdots+L_5. \qquad (61)$$

The second term on the left-hand side can be bounded from below. Indeed, it follows from the symmetry of (b_{ij}), definition (20) of M_{ij}, and the positive definiteness (21) of M on L that

$$\frac{1}{2}\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\left|\left(u_i-\bar{u}_i\right)-\left(u_j-\bar{u}_j\right)\right|^2$$

$$=\sum_{i=1}^n\left(\sum_{j=1,j\neq i}^n b_{ij}\rho_j\right)\rho_i|u_i-\bar{u}_i|^2 - \sum_{i,j=1,i\neq j}^n b_{ij}\rho_i\rho_j\left(u_i-\bar{u}_i\right)\cdot\left(u_j-\bar{u}_j\right)$$

$$=\sum_{i,j=1}^n M_{ij}\sqrt{\rho_i}\left(u_i-\bar{u}_i\right)\cdot\sqrt{\rho_j}\left(u_j-\bar{u}_j\right)\geqslant\mu_M|P_L\boldsymbol{Y}|^2,$$

where $Y_j=\sqrt{\rho_j}(u_j-\bar{u}_j)$. The norm of the projection is computed according to

$$|P_L\boldsymbol{Y}|^2 = |\boldsymbol{Y}|^2 - |P_{L^\perp}\boldsymbol{Y}|^2 = \sum_{i=1}^n\rho_i|u_i-\bar{u}_i|^2 - \sum_{i=1}^n\frac{\rho_i}{\rho^2}\left|\sum_{j=1}^n\rho_j\left(u_j-\bar{u}_j\right)\right|^2$$

$$=\sum_{i=1}^n\rho_i|u_i-\bar{u}_i|^2 - \frac{1}{\rho}\left|\sum_{j=1}^n\left(\rho_j-\bar{\rho}_j\right)\bar{u}_j\right|^2 \geqslant \sum_{i=1}^n\rho_i|u_i-\bar{u}_i|^2 - C_1\sum_{j=1}^n\left(\rho_j-\bar{\rho}_j\right)^2,$$

where we used $\sum_{i=1}^n\rho_i u_i = 0$ in the third equality, and $C_1 > 0$ depends on ρ_* and the $L^\infty(\Omega_T)$ norms of $\bar{u}_j$, $j=1,\dots,n$. Consequently,

$$\frac{1}{2}\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j\left|\left(u_i-\bar{u}_i\right)-\left(u_j-\bar{u}_j\right)\right|^2\mathrm{d}x\mathrm{d}s$$

$$\geqslant \mu_M\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar{u}_i|^2\mathrm{d}x\mathrm{d}s - C_2\int_0^t\int_\Omega\sum_{j=1}^n\left(\rho_j-\bar{\rho}_j\right)^2\mathrm{d}x\mathrm{d}s. \qquad (62)$$

We turn to the estimation of the terms on the right-hand side of (61). By the Lipschitz continuity of κ and Young's inequality, K_{11} is estimated as

$$K_{11} = -\int_0^t\int_\Omega\frac{1}{\bar{\theta}}\left(\kappa\left(\bar{\theta}-\theta\right)+\left(\kappa-\bar{\kappa}\right)\theta\right)\nabla\bar{\theta}\cdot\nabla\left(\log\theta-\log\bar{\theta}\right)\mathrm{d}x\mathrm{d}s$$

$$\leqslant \frac{c_\kappa}{8}\int_0^t\int_\Omega|\nabla\left(\log\theta-\log\bar{\theta}\right)|^2\mathrm{d}x\mathrm{d}s + C_3\int_0^t\int_\Omega\left(\theta-\bar{\theta}\right)^2\mathrm{d}x\mathrm{d}s,$$

and $C_3 > 0$ depends on c_κ (see assumption (A4)), and the $L^\infty(\Omega_T)$ norms of θ and $\nabla \log \bar{\theta}$. A similar estimate shows that

$$
\begin{aligned}
K_{13} &= -\int_0^t \int_\Omega \frac{\theta - \bar{\theta}}{\bar{\theta}} \left(\kappa \nabla \left(\log \theta - \log \bar{\theta} \right) + (\kappa - \bar{\kappa}) \nabla \log \bar{\theta} \right) \cdot \nabla \log \bar{\theta} \mathrm{d}x \mathrm{d}s \\
&\leqslant \frac{c_\kappa}{8} \int_0^t \int_\Omega |\nabla \left(\log \theta - \log \bar{\theta} \right)|^2 \mathrm{d}x \mathrm{d}s + C_4 \int_0^t \int_\Omega \left(\theta - \bar{\theta} \right)^2 \mathrm{d}x \mathrm{d}s, \\
L_2 &\leqslant \frac{c_\kappa}{8} \int_0^t \int_\Omega |\nabla \left(\log \theta - \log \bar{\theta} \right)|^2 \mathrm{d}x \mathrm{d}s + C_5 \int_0^t \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x \mathrm{d}s, \\
L_3 &\leqslant \frac{c_\kappa}{8} \int_0^t \int_\Omega |\nabla \left(\log \theta - \log \bar{\theta} \right)|^2 \mathrm{d}x \mathrm{d}s + C_6 \int_0^t \int_\Omega \left(\theta - \bar{\theta} \right)^2 \mathrm{d}x \mathrm{d}s,
\end{aligned}
$$

observing that C_4 depends on c_κ, δ and the $L^\infty(\Omega_T)$ norms of θ, $\nabla \log \bar{\theta}$, and $\bar{u}_i$, C_5 depends on the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$, and C_6 depends on c_κ, ρ^*, δ, and the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$ ($i = 1, \ldots, n$). Moreover, by Young's inequality again,

$$
\begin{aligned}
L_1 &\leqslant \frac{\mu_M}{4} \int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x \mathrm{d}s + C_7 \int_0^t \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x \mathrm{d}s, \\
L_4 &\leqslant \frac{\mu_M}{4} \int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 \mathrm{d}x \mathrm{d}s + C_8 \int_0^t \int_\Omega \left(\theta - \bar{\theta} \right)^2 \mathrm{d}x \mathrm{d}s,
\end{aligned}
$$

where C_7 depends on ρ^*, μ_M, and the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$ ($i = 1, \ldots, n$), while C_8 depends on δ, ρ^*, and the $L^\infty(\Omega_T)$ norm of $\nabla \log \bar{\theta}$. Finally,

$$
L_5 \leqslant C_9 \int_0^T \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x \mathrm{d}s + C_{10} \int_0^T \int_\Omega \left(\theta - \bar{\theta} \right)^2 \mathrm{d}x \mathrm{d}s,
$$

where $C_9 > 0$ depends on the $L^\infty(\Omega_T)$ norms of $\bar{u}_i$ ($i = 1, \ldots, n$), and C_{10} depends on δ and the $L^\infty(\Omega_T)$ norm of $\nabla \log \bar{\theta}$.

Summarising the previous estimations, we infer from (61), (62) and the lower bound for κ (see assumption (A4)) the conclusion. □

It remains to estimate the right-hand side of (59) in terms of the relative entropy. For this, we observe that, by [18, lemma 16],

$$
\int_\Omega \sum_{i=1}^n \frac{1}{m_i} \left(\rho_i \log \frac{\rho_i}{\bar{\rho}_i} - (\rho_i - \bar{\rho}_i) \right) \mathrm{d}x \geqslant C \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i)^2 \mathrm{d}x.
$$

Furthermore, for all functions $f \in C^1(\mathbb{R})$ with $f'(1) = 0$,

$$
\begin{aligned}
f(s) - f(1) &= (s-1) \int_0^1 f' \left(\sigma(s-1) + 1 \right) \mathrm{d}\sigma = (s-1) \int_0^1 f' \left(\tau(s-1) + 1 \right) \big|_{\tau=0}^\sigma \mathrm{d}\sigma \\
&= (s-1)^2 \int_0^1 \int_0^\sigma f'' \left(\tau(s-1) + 1 \right) \mathrm{d}\tau \mathrm{d}\sigma.
\end{aligned}
$$

This yields, choosing $f(s) = -\log s + s - 1$ and $s = \theta/\bar{\theta}$,

$$\int_\Omega c_v \rho \left(-\log \frac{\theta}{\bar{\theta}} + \frac{1}{\bar{\theta}} \left(\theta - \bar{\theta} \right) \right) dx \geqslant \int_\Omega c_v \rho \frac{\left(\theta - \bar{\theta} \right)^2}{\max\left\{ \theta, \bar{\theta} \right\}^2} dx \geqslant C \int_\Omega \left(\theta - \bar{\theta} \right)^2 dx,$$

where $C > 0$ depends on the lower bound for $\bar{\theta}$ in Ω_T. By definition of the relative entropy, we conclude from lemma 16 that

$$H\left((\boldsymbol{\rho}, \theta)(t) \,|\, (\bar{\boldsymbol{\rho}}, \bar{\theta})(t) \right) + \frac{\mu_M}{2} \int_0^t \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar{u}_i|^2 dx ds$$

$$+ \frac{c_\kappa}{2} \int_0^t \int_\Omega |\nabla \left(\log \theta - \log \bar{\theta} \right)|^2 dx ds \leqslant C \int_0^t H\left(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar{\theta} \right) ds.$$

Gronwall's lemma shows that $H((\boldsymbol{\rho}, \theta)(t)|(\bar{\boldsymbol{\rho}}, \bar{\theta})(t)) = 0$ and hence $\boldsymbol{\rho}(t) = \bar{\boldsymbol{\rho}}(t)$ and $\theta(t) = \bar{\theta}(t) = 0$ in Ω for $t > 0$. This finishes the proof.

Data availability statement

No new data were created or analysed in this study.

References

[1] Anwasia B, Bisi M, Salvarani F and Soares A J 2020 On the Maxwell–Stefan diffusion limit for a reactive mixture of polyatomic gases in non-isothermal setting *Kinet. Relat. Models* **13** 63–95
[2] Bothe D *et al* 2011 On the Maxwell–Stefan equations to multicomponent diffusion *Parabolic Problems. Progress in Nonlinear Differential Equations and Their Applications* ed J Escher (Springer) pp 81–93
[3] Bothe D and Dreyer W 2015 Continuum thermodynamics of chemically reacting fluid mixtures *Acta Mech.* **226** 1757–805
[4] Bothe D and Druet P-E 2021 Mass transport in multicomponent compressible fluids: local and global well-posedness in classes of strong solutions for general class-one models *Nonlinear Anal.* **210** 112389
[5] Bothe D and Druet P-E 2023 On the structure of continuum thermodynamical diffusion fluxes – a novel closure scheme and its relation to the Maxwell–Stefan and the Fick–Onsager approach *Int. J. Eng. Sci.* **184** 103818
[6] Boudin L, Grec B, Pavić M and Salvarani F 2013 Diffusion asymptotics of a kinetic model for gaseous mixtures *Kinet. Relat. Models* **6** 137–57
[7] Buliček M, Jüngel A, Pokorný M and Zamponi N 2022 Existence analysis of a stationary compressible fluid model for heat-conducting and chemically reacting mixtures *J. Math. Phys.* **63** 051501
[8] Chen X and Jüngel A 2015 Analysis of an incompressible Navier–Stokes–Maxwell–Stefan system *Commun. Math. Phys.* **340** 471–97
[9] Chen X and Jüngel A 2018 A note on the uniqueness of weak solutions to a class of cross-diffusion systems *J. Evol. Equ.* **18** 805–20
[10] Dreher M and Jüngel A 2012 Compact families of piecewise constant functions in $L^p(0, T; B)$ *Nonlinear Anal.* **75** 3072–7
[11] Dreyer W, Druet P-E, Gajewski P and Guhlke C 2020 Analysis of improved Nernst–Planck–Poisson models of compressible isothermal electrolytes *Z. Angew. Math. Phys.* **71** 119
[12] Georgiadis S and Tzavaras A E 2023 Asymptotic derivation of multicomponent compressible flows with heat conduction and mass diffusion *ESAIM: Math. Model. Numer. Anal.* **57** 69–106
[13] Giovangigli V and Massot M 1998 The local Cauchy problem for multicomponent reactive flows in full vibrational non-equilibrium *Math. Methods Appl. Sci.* **21** 1415–39

[14] Giovangigli V and Massot M 1998 Asymptotic stability of equilibrium states for multicomponent reactive flows *Math. Models Methods Appl. Sci.* **8** 251–97

[15] Helmer C and Jüngel A 2021 Analysis of Maxwell–Stefan systems for heat conducting fluid mixtures *Nonlinear Anal. Real World Appl.* **59** 103263

[16] Herberg M, Meyries M, Prüss J and Wilke M 2017 Reaction-diffusion systems of Maxwell–Stefan type with reversible mass-action kinetics *Nonlinear Anal.* **159** 264–84

[17] Huo X, Jüngel A and Tzavaras A 2019 High-friction limits of Euler flows for multicomponent systems *Nonlinearity* **32** 2875–913

[18] Huo X, Jüngel A and Tzavaras A 2022 Weak–strong uniqueness for Maxwell-Stefan systems *SIAM J. Math. Anal.* **54** 3215–52

[19] Hutridurga H and Salvarani F 2018 Existence and uniqueness analysis of a non-isothermal cross-diffusion system of Maxwell–Stefan type *Appl. Math. Lett.* **75** 108–13

[20] Jüngel A 2016 *Entropy Methods for Diffusive Partial Differential Equations* (SpringerBriefs in Mathematics)

[21] Jüngel A and Stelzer I V 2013 Existence analysis of Maxwell–Stefan systems for multicomponent mixtures *SIAM J. Math. Anal.* **45** 2421–40

[22] Kawashima S and Shizuta Y 1988 On the normal form of the symmetric hyperbolic–parabolic systems associated with the conservation laws *Tohoku Math. J.* **40** 449–64

[23] Marion M and Temam R 2015 Global existence for fully nonlinear reaction–diffusion systems describing multicomponent reactive flows *J. Math. Pures Appl.* **104** 102–38

[24] Maxwell C 1866 On the dynamical theory of gases *Phil. Trans. R. Soc.* **157** 49–88

[25] Mucha P, Pokorný M and Zatorska E 2015 Heat-conducting, compressible mixtures with multicomponent diffusion: construction of a weak solution *SIAM J. Math. Anal.* **47** 3747–97

[26] Serre D 2010 The structure of dissipative viscous system of conservation laws *Physica D* **239** 1381–6

[27] Stefan J 1871 Über das gleichgewicht und bewegung, insbesondere die diffusion von gasgemengen *Sitzungsberichte Kaiserl. Akad. Wiss. Wien* **63** 63–124

[28] Takata S and Aoki K 1999 Two-surface problems of a multicomponent mixture of vapors and noncondensable gases in the continuum limit in the light of kinetic theory *Phys. Fluids* **11** 2743–56

[29] Yonglin C 1990 The generalized Bott–Duffin inverse and its applications *Linear Algebr. Appl.* **134** 71–91