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A CONVERGENT FINITE-VOLUME SCHEME FOR NONLOCAL CROSS-DIFFUSION SYSTEMS FOR MULTI-SPECIES POPULATIONS

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Abstract. An implicit Euler finite-volume scheme for a nonlocal cross-diffusion system on the one-dimensional torus, arising in population dynamics, is proposed and analyzed. The kernels are assumed to be in detailed balance and satisfy a weak cross-diffusion condition. The latter condition allows for negative off-diagonal coefficients and for kernels defined by an indicator function. The scheme preserves the nonnegativity of the densities, conservation of mass, and production of the Boltzmann and Rao entropies. The key idea is to "translate" the entropy calculations for the continuous equations to the finite-volume scheme, in particular to design discretizations of the mobilities, which guarantee a discrete chain rule even in the presence of nonlocal terms. Based on this idea, the existence of finite-volume solutions and the convergence of the scheme are proven. As a by-product, we deduce the existence of weak solutions to the continuous cross-diffusion system. Finally, we present some numerical experiments illustrating the behavior of the solutions to the nonlocal and associated local models.

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1. Introduction

This paper is devoted to the design and analysis of structure-preserving finite-volume discretization of the following one-dimensional nonlocal cross-diffusion initial-value problem:

$$\partial_t u_i = \partial_x(\sigma\partial_x u_i + u_i\partial_x p_i(u)) \quad \text{in } \mathbb{T},\ t>0, \tag{1}$$

$$u_i(\cdot,0) = u_i^0 \quad \text{in } \mathbb{T},\ i=1,\ldots,n, \tag{2}$$

where $\sigma \geq 0$ is the diffusion coefficient, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus of unit measure, and p_i is the nonlocal operator

$$p_i(u)(x) := a_{ii}u_i(x) + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(B^{ij} * u_j)(x) = a_{ii}u_i(x) + \sum_{\substack{j=1\\ j\neq i}}^n \int_{\mathbb{T}} a_{ij}B^{ij}(x-y)u_j(y)\mathrm{d}y, \tag{3}$$

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where a_{ij} are some constants. The kernel functions $B^{ij} : \mathbb{T} \to \mathbb{R}$ are periodically extended to $\mathbb{R}$, and $u = (u_1, \ldots, u_n)$ is the solution vector. If we define $B^{ii} = \delta_0$, where $i \in \{1, \ldots, n\}$ and δ_0 is the Dirac measure, we can rewrite p_i as

$$p_i(u) = \sum_{j=1}^n a_{ij}(B^{ij} * u_j)(x). \tag{4}$$

Equation (1) with definition (4) and general kernels B^{ij} for $i, j = 1, \ldots, n$ can be derived from stochastic interacting particle systems in the many-particle limit [10].

We proved in [19] that the "full" nonlocal system, *i.e.* system (1) and (4), where $B^{ii} \neq \delta_0$ are general kernels, admits global weak solutions. Our analysis was based on the fact that this system possesses two Lyapunov functionals. More precisely, assume that there exist numbers $\pi_1, \ldots, \pi_n > 0$ such that the kernels B^{ij} satisfy the so-called detailed-balance condition

$$\pi_i a_{ij} B^{ij}(x-y) = \pi_j a_{ji} B^{ji}(y-x) \quad \text{for } i,j = 1,\ldots,n \text{ and a.e. } x, y \in \mathbb{T},$$

and the positive semi-definiteness condition

$$\sum_{i,j=1}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) v_j(y) v_i(x) \mathrm{d}y \mathrm{d}x \geq 0 \quad \text{for all } v_i, v_j \in L^2(\mathbb{T}). \tag{5}$$

Then we proved that the Boltzmann (type) and Rao (type) entropies, respectively,

$$H_B(u) = \sum_{i=1}^n \int_{\mathbb{T}} \pi_i u_i (\log u_i - 1) \mathrm{d}x,$$
$$H_R(u) = \frac{1}{2} \sum_{i,j=1}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) u_j(y) u_i(x) \, \mathrm{d}y \mathrm{d}x,$$

fulfill the following entropy dissipation inequalities:

$$\frac{dH_B}{\mathrm{d}t} + 4\sigma \sum_{i=1}^n \int_{\mathbb{T}} \pi_i |\partial_x \sqrt{u_i}|^2 \mathrm{d}x \leq - \sum_{i,j=1}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x) \mathrm{d}y \mathrm{d}x, \tag{6}$$

$$\frac{dH_R}{\mathrm{d}t} + \sum_{i=1}^n \int_{\mathbb{T}} \pi_i u_i |\partial_x p_i(u)|^2 \mathrm{d}x \leq -\sigma \sum_{i,j=1}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x) \mathrm{d}y \mathrm{d}x, \tag{7}$$

and the right-hand sides are nonpositive due to (5). The Boltzmann entropy is related to the thermodynamic entropy of the system, and the Rao entropy is a measure of the functional diversity of the species [23].

While this theoretical framework was suitable to prove the existence of weak solutions, condition (5) is cumbersome to check in practice. In Remark 1 from [19], we proved that (5) is satisfied for smooth kernels like the Gaussian one, *i.e.* $B^{ij}(x-y) = \exp(-(x-y)^2/2)$ for $i, j = 1, \ldots, n$. We also claimed that kernels B^{ij} of the type $B^{ij} = 1_K$ for some interval K around the origin satisfy (5). This claim is in fact not true, see the counterexample in Appendix B.

System (1) and (4), with local or nonlocal self-diffusion terms, describes the dynamics of a population with n species, where the evolution of each species is driven by nonlocal sensing [22]. In other words, each species has the capability to detect other species over a spatial neighborhood, specified by the kernel B^{ij}, and weighted by the strength of attraction ($a_{ij} < 0$) or repulsion ($a_{ij} > 0$). Thus, from a modeling point of view, the case $B^{ij} = 1_K$ is biologically meaningful. To include this case in our analysis (at the continuous or discrete level), we propose to slightly modify the model studied in [19] by considering (3) instead of (4).

For model (1)–(3), we impose the following assumptions. We assume that there exist numbers $\pi_1, \dots, \pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ for $i, j \in \{1, \dots, n\}$, that $B^{ji}(-x) = B^{ij}(x) \geq 0$ for a.e. $x \in \mathbb{T}$ and $i, j \in \{1, \dots, n\}$ (with $i \neq j$), and that for all $i, j \in \{1, \dots, n\}$ with $i < j$, the matrices

$$M^{ij}(x) := \begin{pmatrix} \pi_i a_{ii} & (n-1)\pi_i a_{ij} B^{ij}(x) \\ (n-1)\pi_j a_{ji} B^{ij}(x) & \pi_j a_{jj} \end{pmatrix} \tag{8}$$

are uniformly positive definite for a.e. $x \in \mathbb{T}$. In particular, we could choose some nonpositive off-diagonal coefficients. The possibility to analyze system (1)–(3) with nonpositive off-diagonal coefficients is a new and meaningful result. However, we notice that with these assumptions, the system is only "weakly" nonlocal, in the sense that the self-diffusion coefficients have to dominate the cross-diffusion terms.

We claim that the functionals H_B and H_R are still entropies for system (1)–(3), where of course now

$$H_R(u) = \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{T}} \pi_i a_{ii} |u_i(x)|^2 \mathrm{d}x + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) u_j(y) u_i(x) \mathrm{d}y \mathrm{d}x.$$

Both functionals satisfy some entropy dissipation inequalities similar to (6)–(7), where, if $i = j$, the terms on the right-hand side are simply given by the square of the $L^2(\mathbb{T})$ norm of $\partial_x u_i$. Under the above-mentioned assumptions, the entropy production term

$$Q := \sum_{i=1}^n \int_{\mathbb{T}} \pi_i a_{ii} |\partial_x u_i(x)|^2 \mathrm{d}x + \sum_{\substack{i,j=1 \\ i \neq j}}^n \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x) \mathrm{d}y \mathrm{d}x \tag{9}$$

is nonnegative; see Lemma A.1 in Appendix A. Therefore, at least formally, the functionals H_B and H_R are entropies for system (1)–(3). In this work, we will translate this property to the discrete level by analyzing a two-point flux approximation finite-volume scheme for (1)–(3).

In the literature, there are some works dealing with the design and analysis of numerical schemes for nonlocal cross-diffusion systems. The work [7] studies a positivity-preserving one-dimensional finite-volume scheme for (1) with $n = 2$ and additional local cross-diffusion terms, with a focus on segregated steady states, but without any numerical analysis. The convergence of this finite-volume scheme was proved in [8], still focusing on the two-species model. A converging finite-volume scheme for a nonlocal cross-diffusion system modeling either a food chain of three species or, when dropping the cross-diffusion, being an SIR model, was analyzed in [1,3]. In both models, the nonlocality comes from the dependence of the self-diffusion coefficients on the total mass of the corresponding species. A structure-preserving finite-volume scheme for the nonlocal Shigesada–Kawasaki–Teramoto system was suggested and analyzed in [17]. We also mention the paper [6] on a second-order finite-volume scheme for a nonlocal diffusion equation, which preserves the nonnegativity and fulfills a spatially discrete entropy inequality. Related works include a Galerkin scheme for a nonlocal diffusion equation with additive noise [21], a finite-volume discretization of a nonlocal Lévy–Fokker–Planck equation [2], and numerical schemes for nonlocal diffusion equations arising in image processing [14]. Up to our knowledge, there does not exist any numerical analysis of system (1)–(3).

In this paper, we propose a finite-volume scheme which preserves the structure of equations (1)–(3). Compared to [8], we allow for an arbitrary number of species, include linear diffusion $\sigma \geq 0$, and prove the preservation of the discrete Boltzmann and Rao entropies. Since we need the positive definiteness of the matrix $M^{ij}(x)$, self-diffusion is needed in our situation. Moreover, in contrast to [8], we impose periodic boundary conditions (instead of no-flux conditions). Our proofs rely on the discrete analog of the rule $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$, see (17), which allows for kernels B^{ij} that are not differentiable, while in [8] the kernels are required to be in $C_b^2(\mathbb{R})$. Compared to [17], our equations do not have a Laplacian structure, which was used in [17] to define the numerical scheme, and we allow for nonpositive off-diagonal coefficients.

Our main results can be sketched as follows (see Sect. 2.3 for details):

– We prove the existence of solutions to the finite-volume scheme, which are nonnegative componentwise, conserve the discrete mass, and satisfy discrete versions of the entropy inequalities (6) and (7).
– We show that the discrete solutions converge to a weak solution to (1)–(3) when the mesh size tends to zero. As a by-product, this proves the existence of a weak solution to (1)–(2).
– We illustrate numerically the rate of convergence (in space) in the L^p-norm as well as the rate of convergence in different metrics of the solution to the nonlocal system towards the solution of the local one (localization limit). Moreover, we illustrate the segregation phenomenon exhibited by the solutions to (1)–(3); see [4].

The paper is organized as follows. The numerical scheme and our main results are introduced in Section 2. We prove the existence of discrete solutions in Section 3, while the proof of the convergence of the scheme is presented in Section 4. In Section 5, numerical experiments are given, Appendix A contains some auxiliary results, and we show in Appendix B that indicator kernels generally do not fulfill inequality (5).

2. Notation and numerical scheme

2.1. Notation

A uniform mesh $\mathcal{T}$ of the torus $\mathbb{T}$ consists of N intervals (or cells) K_ℓ of length $\Delta x = 1/N$, given by $K_\ell = (x_{\ell-1/2}, x_{\ell+1/2})$ with end points $x_{\ell\pm1/2} = (\ell \pm 1/2)\Delta x$ and centers $x_\ell = \ell\Delta x$ for $\ell \in G = \mathbb{Z} \setminus N\mathbb{Z}$. For given end time $T > 0$, let $N_T \in \mathbb{N}$ and define the time step size $\Delta t = T/N_T$ and the time steps $t_k = k\Delta t$. A space-time discretization of $Q_T := \mathbb{T} \times (0,T)$ is denoted by $\mathcal{D}$; it consists of the space discretization $\mathcal{T}$ of $\mathbb{T}$ and the time discretization $(N_T, \Delta t)$ of $(0,T)$.

We introduce some function spaces. The space of piecewise constant (in space) functions is given by

$$\mathcal{V}_\mathcal{T} = \left\{ v : \mathbb{T} \to \mathbb{R} : \exists (v_\ell)_{\ell\in G} \subset \mathbb{R},\ v(x) = \sum_{\ell\in G} v_\ell 1_{K_\ell}(x) \right\},$$

where 1_{K_ℓ} is the indicator function of K_ℓ. We identify the function $v \in \mathcal{V}_\mathcal{T}$ and the numbers $(v_\ell)_{\ell\in G}$ by writing $v = (v_\ell)_{\ell\in G}$. For $q \in [1,\infty)$ and $v \in \mathcal{V}_\mathcal{T}$, we introduce the $L^q(\mathbb{T})$ norm, the discrete $W^{1,q}(\mathbb{T})$ seminorm, and the discrete $W^{1,q}(\mathbb{T})$ norm by, respectively,

$$\|v\|_{0,q,\mathcal{T}}^q = \sum_{\ell\in G} \Delta x |v_\ell|^q, \quad |v|_{1,q,\mathcal{T}}^q = \sum_{\ell\in G} \Delta x \left| \frac{v_{\ell+1} - v_\ell}{\Delta x} \right|^q,$$
$$\|v\|_{1,q,\mathcal{T}}^q = |v|_{1,q,\mathcal{T}}^q + \|v\|_{0,q,\mathcal{T}}^q.$$

We also define the discrete $L^\infty(\mathbb{T})$ norm by $\|v\|_{0,\infty,\mathcal{T}} = \max_{\ell\in G} |v_\ell|$. Note that $\|v\|_{0,q,\mathcal{T}} = \|v\|_{L^q(\mathbb{T})}$ for functions $v \in \mathcal{V}_\mathcal{T}$. We set

$$\mathrm{D}_\ell v := \frac{v_{\ell+1} - v_\ell}{\Delta x} \quad \text{and} \quad \mathrm{D}v := (\mathrm{D}_\ell v)_{\ell\in G}.$$

We recall the definition of the space $\mathrm{BV}(\mathbb{T})$ of functions of bounded variation. A function $v \in L^1(\mathbb{T})$ belongs to $\mathrm{BV}(\mathbb{T})$ if its total variation $\mathrm{TV}(v)$, given by

$$\mathrm{TV}(v) = \sup\left\{ \int_\mathbb{T} v(x)\partial_x\phi(x)\mathrm{d}x : \ \phi \in C_0^1(\mathbb{T}),\ |\phi(x)| \le 1 \quad \text{for all } x \in \mathbb{T} \right\},$$

is finite. We endow the space $\mathrm{BV}(\mathbb{T})$ with the norm

$$\|v\|_{\mathrm{BV}(\mathbb{T})} = \|v\|_{L^1(\mathbb{T})} + \mathrm{TV}(v) \quad \text{for all } v \in \mathrm{BV}(\mathbb{T}).$$

In particular, it holds $\|v\|_{\mathrm{BV}(\mathbb{T})} = \|v\|_{1,1,\mathcal{T}}$ for any $v \in \mathcal{V}_\mathcal{T} \cap \mathrm{BV}(\mathbb{T})$.

For any given $q \in [1,\infty)$, we associate to these norms a dual norm with respect to the $L^2(\mathbb{T})$ inner product by

$$\|v\|_{-1,q',\mathcal{T}} = \sup\left\{\left|\int_{\mathbb{T}} vw\mathrm{d}x\right| : w \in \mathcal{V}_{\mathcal{T}},\ \|w\|_{1,q,\mathcal{T}} = 1\right\},$$

where $1/q + 1/q' = 1$. Then the following estimate holds for all v, $w \in \mathcal{V}_{\mathcal{T}}$,

$$\left|\int_{\mathbb{T}} vwdx\right| \le \|v\|_{-1,q',\mathcal{T}}\|w\|_{1,q,\mathcal{T}}.$$

We also need the space of piecewise constant (in time) functions taking values in $\mathcal{V}_{\mathcal{T}}$:

$$\mathcal{V}_{\mathcal{D}} = \left\{v : \mathbb{T}\times(0,T]\to\mathbb{R} : \exists(v^k)_{k=1,\ldots,N_T},\ v(x,t) = \sum_{k=1}^{N_T} 1_{(t_{k-1},t_k]}(t)v^k(x)\right\},$$

and the discrete $L^p(0,T;W^{1,q}(\mathbb{T}))$ norm

$$\left(\sum_{k=1}^{N_T}\Delta t\|v^k\|^p_{1,q,\mathcal{T}}\right)^{1/p}, \quad \text{where } 1 \le p,q < \infty,\ v \in \mathcal{V}_{\mathcal{D}}.$$

2.2. Numerical scheme

The initial datum (2) is approximated by

$$u^0_{i,\ell} = \frac{1}{\Delta x}\int_{K_\ell} u^0_i(x)\mathrm{d}x \quad \text{for } \ell \in G,\ i = 1,\ldots,n. \tag{10}$$

For given $k \in \{1,\ldots,N_T\}$ and $u^{k-1} \in \mathcal{V}^n_{\mathcal{T}}$, the values $u^k = (u^k_{i,\ell})_{i=1,\ldots,n,\,\ell\in G}$ are determined by the implicit Euler finite-volume scheme

$$\frac{\Delta x}{\Delta t}(u^k_{i,\ell} - u^{k-1}_{i,\ell}) + \mathcal{F}^k_{i,\ell+1/2} - \mathcal{F}^k_{i,\ell-1/2} = 0, \quad i = 1,\ldots,n,\ \ell \in G, \tag{11}$$

with the numerical fluxes

$$\mathcal{F}^k_{i,\ell+1/2} = -\frac{\sigma}{\Delta x}(u^k_{i,\ell+1} - u^k_{i,\ell}) - \frac{u^k_{i,\ell+1/2}}{\Delta x}(p^k_{i,\ell+1} - p^k_{i,\ell}), \tag{12}$$

where the discrete nonlocal operators are given by

$$p^k_{i,\ell} = a_{ii}u^k_{i,\ell} + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G}\Delta x a_{ij}B^{ij}_{\ell-\ell'}u^k_{j,\ell'}, \quad B^{ij}_{\ell-\ell'} = \frac{1}{\Delta x}\int_{K_{\ell-\ell'}} B^{ij}(y)\mathrm{d}y, \tag{13}$$

for all $i,j = 1,\ldots,n$ and ℓ, $\ell' \in G$. We show in the proof of Lemma 4.4 that $p^k_{i,\ell} = a_{ii}u^k_i(x_\ell) + \sum_{j\neq i} a_{ij}(B^{ij} * u^k_j)(x_\ell)$ for $\ell \in G$, verifying the consistency of the discretization of $p^k_{i,\ell}$. The mobility $u^k_{i,\ell+1/2} = \widehat{F}(u^k_{i,\ell}, u^k_{i,\ell+1})$ is assumed to satisfy the following properties for all $u_{i,\ell}$, $u_{i,\ell+1}$:

- The function $\widehat{F} : [0,\infty)^2 \to [0,\infty)$ is continuous and satisfies $\widehat{F}(u_{i,\ell}, u_{i,\ell}) = u_{i,\ell}$ as well as $\min\{u_{i,\ell}, u_{i,\ell+1}\} \le \widehat{F}(u_{i,\ell}, u_{i,\ell+1}) \le \max\{u_{i,\ell}, u_{i,\ell+1}\}$.
- There exists $c_0 > 0$ such that the following discrete chain rule holds:

$$u_{i,\ell+1/2}(p_{i,\ell+1} - p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \ge c_0(p_{i,\ell+1} - p_{i,\ell})(u_{i,\ell+1} - u_{i,\ell}). \tag{14}$$

Remark 2.1 (Examples for mobilities)**.** Property (14) is satisfied if $u_{i,\ell}$ (we omit the superindex k) is defined by the upwind approximation

$$u_{i,\ell+1/2} = \begin{cases} u_{i,\ell+1} & \text{if } p_{i,\ell+1} - p_{i,\ell} \geq 0, \\ u_{i,\ell} & \text{if } p_{i,\ell+1} - p_{i,\ell} < 0, \end{cases} \tag{15}$$

or by the logarithmic mean

$$u_{i,\ell+1/2} = \begin{cases} \dfrac{u_{i,\ell+1} - u_{i,\ell}}{\log u_{i,\ell+1} - \log u_{i,\ell}} & \text{if } u_{i,\ell+1} > 0,\ u_{i,\ell} > 0, \text{ and } u_{i,\ell+1} \neq u_{i,\ell}, \\ u_{i,\ell} & \text{if } u_{i,\ell+1} = u_{i,\ell} > 0, \\ 0 & \text{else.} \end{cases} \tag{16}$$

We refer to Lemma A.2 in Appendix A for a proof. □

Remark 2.2 (Symmetry of discrete kernels)**.** Definition (13) of $B^{ij}_{\ell-\ell'}$ is consistent with the discrete analog of $B^{ji}(-x) = B^{ij}(x)$. Indeed, with the change of variables $y \mapsto -y$,

$$B^{ji}_{-\ell'} = \frac{1}{\Delta x}\int_{K_{-\ell'}} B^{ji}(y)\mathrm{d}y = \frac{1}{\Delta x}\int_{K_{\ell'}} B^{ji}(-y)\mathrm{d}y = \frac{1}{\Delta x}\int_{K_{\ell'}} B^{ij}(y)\mathrm{d}y = B^{ij}_{\ell'}.$$

Remark 2.3 (Discrete derivative of the convolution)**.** A shift of Δx in definition (13) of $B^{ij}_{\ell-\ell'}$ shows that $B^{ij}_{\ell-\ell'} = B^{ij}_{(\ell+1)-(\ell'+1)}$, which leads to

$$\begin{aligned} \sum_{\ell'\in G}\big(B^{ij}_{(\ell+1)-\ell'} - B^{ij}_{\ell-\ell'}\big)\, u_{j,\ell'} &= \sum_{\ell'\in G}\big(B^{ij}_{(\ell+1)-(\ell'+1)} u_{j,\ell'+1} - B^{ij}_{\ell-\ell'} u_{j,\ell'}\big) \\ &= \sum_{\ell'\in G} B^{ij}_{\ell-\ell'}(u_{j,\ell'+1} - u_{j,\ell'}) \end{aligned} \tag{17}$$

for all $\ell \in G$, $i,j = 1,\dots,n$. This is the discrete analog of the rule $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$. □

Remark 2.4 (Asymptotic-preserving scheme)**.** For $j \neq i$, let $B^{ij} = B^{ij}_\varepsilon$ for some parameter $\varepsilon > 0$ and $B^{ij}_\varepsilon \to \delta_0$ in the sense of distributions as $\varepsilon \to 0$. Let $p^{k,\varepsilon}_{i,\ell}$ be defined as in (13) with $B^{ij}(y)$ replaced by $B^{ij}_\varepsilon(y)$. Then, as $\varepsilon \to 0$,

$$p^{k,\varepsilon}_{i,\ell} \to \sum_{j=1}^n a_{ij}\,(\delta_0 * u_j)\,(x_\ell) = \sum_{j=1}^n a_{ij} u_{j,\ell}.$$

Thus, our numerical scheme is asymptotic-preserving in the sense that the method converges to a finite-volume scheme for the local system, which also preserves the nonnegativity, conserves the mass, and dissipates the Boltzmann and Rao entropies. □

2.3. Main results

We impose the following hypotheses:

(H1) Domain and parameters: $\mathbb{T}$ is a one-dimensional torus, $T > 0$, $\sigma \geq 0$, and $Q_T := \mathbb{T} \times (0,T)$.

(H2) Initial datum: $u^0 = (u^0_1,\dots,u^0_n) \in L^2(\mathbb{T};\mathbb{R}^n)$ satisfies $u^0_i \geq 0$ in $\mathbb{T}$.

(H3) Kernels: Let $B^{ij} \in L^\infty(\mathbb{T})$ for $j \neq i$ be a nonnegative function satisfying $B^{ji}(x) = B^{ij}(-x)$ for a.e. $x \in \mathbb{T}$. There exist numbers $\pi_1,\dots,\pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ (detailed-balance condition), and the matrices M^{ij}, defined in (8), are uniformly positive definite for a.e. $x \in \mathbb{T}$.

We consider the one-dimensional equations mainly for notational simplicity. In several space dimensions $d > 1$, we infer uniform estimates in spaces with weaker integrability than in one space dimension, because of Sobolev embeddings. Thanks to the positive definiteness condition on $M^{ij}_{\ell-\ell'}$, we obtain a bound for u_i in the discrete $L^2(0,T;H^1(\mathbb{T}))$ norm, which allows us to conclude, together with the Rao entropy estimate, by the discrete Gagliardo–Nirenberg inequality, a bound for u_i in $L^2(Q_T)$, which is sufficient to estimate the product $u_i\partial_x p_i(u)$. In the one-dimensional situation, this procedure simplifies; see Lemma 4.5. We discuss the multidimensional case in Remark 4.7.

Our results also hold if $\sigma = 0$, since the condition $\sigma > 0$ provides an estimate for u_i in the discrete norm of $L^2(0,T;W^{1,1}(\mathbb{T}))$, while the positive definiteness condition on $M^{ij}_{\ell-\ell'}$ allows us to conclude a stronger bound in the discrete norm of $L^2(0,T;H^1(\mathbb{T}))$. Notice that kernels of the type $B^{ij} = 1_K$ satisfy Hypothesis (H3) (for suitable π_i and a_{ij}).

Condition $u^0 \in L^2(\mathbb{T};\mathbb{R}^n)$ in Hypothesis (H2) is needed to obtain a finite initial Rao entropy $H_R(u^0)$. For the existence result, the assumption on the kernels can be weakened to $B^{ij} \in L^1(\mathbb{T})$. The boundedness condition on B^{ij} in Hypothesis (H3) is needed in the proof of the convergence of the scheme.

We introduce for a given nonnegative function $u \in \mathcal{V}^n_{\mathcal{T}}$ the discrete entropies

$$\mathcal{H}_B(u) = \sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i h(u_{i,\ell}), \quad h(s) = s(\log s - 1), \tag{18}$$
$$\mathcal{H}_R(u) = \frac{1}{2}\sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i a_{ii}|u_{i,\ell}|^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \pi_i a_{ij} B^{ij}_{\ell-\ell'} u_{j,\ell'} u_{i,\ell},$$

and the matrices

$$M^{ij}_{\ell-\ell'}(x) := \begin{pmatrix} \pi_i a_{ii} & (n-1)\pi_i a_{ij} B^{ij}_{\ell-\ell'}(x) \\ (n-1)\pi_j a_{ji} B^{ij}_{\ell-\ell'}(x) & \pi_j a_{jj} \end{pmatrix} \text{ for } i<j, \ \ell,\ell' \in G. \tag{19}$$

In view of Hypothesis (H3), they are symmetric and positive definite uniformly in $\ell,\ell' \in G$, $x \in \mathbb{T}$, *i.e.* $z^\top M^{ij}_{\ell-\ell'}(x) z \geq c_M|z|^2$ for $z \in \mathbb{R}^2$, $x \in \mathbb{T}$ and some $c_M > 0$.

Our first main result is the existence of discrete solutions.

Theorem 2.5 (Existence of discrete solutions)**.** *Let Hypotheses (H1)–(H3) hold. Then there exists a solution $u^k \in \mathcal{V}^n_{\mathcal{T}}$ to (10)–(13) for all $k = 1,\dots,N_T$, satisfying $u^k_{i,\ell} \geq 0$ for all $i = 1,\dots,n$, $\ell \in G$ and the discrete entropy inequalities*

$$\mathcal{H}_B(u^k) + \frac{c_0\Delta t}{n-1}\sum_{\substack{i,j=1\\ i<j}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u^k_i \\ \mathrm{D}_{\ell'} u^k_j \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u^k_i \\ \mathrm{D}_{\ell'} u^k_j \end{pmatrix} \tag{20}$$
$$+ 4\sigma\Delta t \sum_{i=1}^n \pi_i |(u^k_i)^{1/2}|^2_{1,2,\mathcal{T}} \leq \mathcal{H}_B(u^{k-1}),$$
$$\mathcal{H}_R(u^k) + \Delta t \sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i u^k_{i,\ell+1/2}\left(\frac{p^k_{i,\ell+1} - p^k_{i,\ell}}{\Delta x}\right)^2 \tag{21}$$
$$+ \frac{\sigma\Delta t}{(n-1)}\sum_{\substack{i,j=1\\ i<j}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u^k_i \\ \mathrm{D}_{\ell'} u^k_j \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u^k_i \\ \mathrm{D}_{\ell'} u^k_j \end{pmatrix} \leq \mathcal{H}_R(u^{k-1}).$$

Furthermore, the solution conserves the mass, $\sum_{\ell\in G} \Delta x u^k_{i,\ell} = \int_{\mathbb{T}} u^0_i(x)\mathrm{d}x$ *for all* $i = 1,\dots,n$, $k = 1,\dots,N_T$.

This theorem is proved by solving a fixed-point problem based on a topological degree argument, similar as in [18]. For this, we formulate (11) in terms of the entropy variable $w_i = \pi_i \log u_i$ and regularize the equations by adding the discrete analog of $-\varepsilon\Delta w_i + \varepsilon w_i$. The regularization ensures the coercivity in the variable w_i. After transforming back to the original variable $u_i = \exp(w_i/\pi_i)$, we obtain automatically the positivity of u_i (and nonnegativity after passing to the limit $\varepsilon \to 0$). Like on the continuous level, the derivation of the discrete entropy inequalities (20) and (21) relies on the detailed-balance condition $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j = 1, \dots, n$.

For our second main result, we need to introduce some notation. We define the "diamond" cell of the dual mesh $T_{\ell+1/2} = (x_\ell, x_{\ell+1})$ with center $x_{\ell+1/2}$. These cells define another partition of $\mathbb{T}$. The gradient of $v \in \mathcal{V}_{\mathcal{D}}$ is then defined by

$$\partial_x^{\mathcal{D}} v(x,t) = \mathrm{D}_\ell v^k = \frac{v_{\ell+1}^k - v_\ell^k}{\Delta x} \quad \text{for } x \in T_{\ell+1/2},\ t \in (t_{k-1}, t_k].$$

We also introduce a sequence of space-time discretizations $(\mathcal{D}_m)_{m\in\mathbb{N}}$ indexed by the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$ satisfying $\eta_m \to 0$ as $m \to \infty$. The corresponding spatial mesh is denoted by $\mathcal{T}_m$ with $G_m = \mathbb{Z} \setminus N_m\mathbb{Z}$ and the number of time steps by N_T^m. Finally, to simplify the notation, we set $\partial_x^m := \partial_x^{\mathcal{D}_m}$.

Theorem 2.6 (Convergence of the scheme)**.** *Let Hypotheses (H1)–(H3) hold and let $(\mathcal{D}_m)$ be a sequence of uniform space-time discretizations satisfying $\eta_m \to 0$ as $m \to \infty$. Let (u_m) be the solutions to (10)–(13) constructed in Theorem 2.5. Then there exists $u = (u_1, \dots, u_n)$ satisfying $u_i \geq 0$ in Q_T and, up to a subsequence, as $m \to \infty$,*

$$\begin{aligned} u_{i,m} &\to u_i \quad \textit{strongly in } L^2(Q_T), \\ \partial_x^m u_{i,m} &\rightharpoonup \partial_x u_i \quad \textit{weakly in } L^2(Q_T), \end{aligned}$$

and u is a weak solution to (1)–(2), i.e., it holds for all $\psi_i \in C_0^\infty(\mathbb{T} \times [0,T))$ and $i = 1, \dots, n$ that

$$\int_0^T \int_{\mathbb{T}} u_i \partial_t \psi_i \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}} u_i^0 \psi_i(\cdot, 0)\mathrm{d}x = \int_0^T \int_{\mathbb{T}} (\sigma\partial_x u_i + u_i \partial_x p_i(u))\partial_x \psi_i \mathrm{d}x\mathrm{d}t.$$

The proof of Theorem 2.6 is based on suitable estimates uniform with respect to Δx_m and Δt_m, derived from the discrete entropy inequalites. A discrete version of the Aubin–Lions lemma from [15] yields the strong convergence of a subsequence of solutions (u_m) to (11)–(13). The most technical part is the identification of the limit function as a weak solution to (1)–(2).

3. Proof of Theorem 2.5

Theorem 2.5 is proved by induction over $k = 1, \dots, N_T$. We first regularize the problem and prove the existence of an approximate solution by using a topological degree argument for the fixed-point problem. The discrete entropy inequalities yield *a priori* estimates independent of the approximation parameter. The de-regularization limit is performed thanks to the Bolzano–Weierstraß theorem.

Let $k \in \{1, \dots, N_T\}$ and $u^{k-1} \in \mathcal{V}_{\mathcal{T}}^n$ satisfying $u_{i,\ell}^{k-1} \geq 0$ for $i = 1, \dots, n$, $\ell \in G$ be given.

3.1. Solution to a linearized regularized scheme

We prove the existence of a unique solution to a linearized regularized problem, which allows us to define the fixed-point operator. Let $R > 0$, $\varepsilon > 0$ and define

$$Z_R = \big\{ w = (w_1, \dots, w_n) \in \mathcal{V}_{\mathcal{T}}^n : \|w_i\|_{1,2,\mathcal{T}} < R \text{ for } i = 1, \dots, n \big\}.$$

We introduce the mapping $F : Z_R \to \mathbb{R}^{nN}$, $w \mapsto w^\varepsilon$, where w^ε is the solution to the linear regularized problem

$$-\varepsilon \frac{w_{i,\ell+1}^\varepsilon - 2w_{i,\ell}^\varepsilon + w_{i,\ell-1}^\varepsilon}{\Delta x} + \varepsilon \Delta x w_{i,\ell}^\varepsilon = -\Delta x \frac{u_{i,\ell} - u_{i,\ell}^{k-1}}{\Delta t} - (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}), \tag{22}$$

where $i = 1, \ldots, n$, $\ell \in G$, $u_{i,\ell}$ is defined by $u_{i,\ell} = \exp(w_{i,\ell}/\pi_i)$, $\mathcal{F}_{i,\ell\pm1/2}$ is defined as in (12) with u_i^k replaced by u_i and $p_{i,\ell}^k$ replaced by

$$p_{i,\ell} = a_{ii}u_{i,\ell} + \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\ell'\in G} \Delta x a_{ij} B_{\ell-\ell'}^{ij} u_{j,\ell'}.$$

We claim that F is well defined. For this, we write (22) in the form

$$Mw^\varepsilon = v, \quad \text{where } v_{i,\ell} = -\Delta x \frac{u_{i,\ell} - u_{i,\ell}^{k-1}}{\Delta t} - (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}).$$

The matrix $M \in \mathbb{R}^{nN\times nN}$ is a block diagonal matrix with entries $M' \in \mathbb{R}^{N\times N}$, which are tridiagonal matrices such that $M'_{\ell,\ell} = \varepsilon\Delta x + 2\varepsilon/\Delta x$, $M'_{\ell+1,\ell} = M'_{\ell,\ell+1} = -\varepsilon/\Delta x$. We can decompose the full system $Mw^\varepsilon = v$ into the subsystems $M'w_i^\varepsilon = v_i$ for $i = 1, \ldots, n$. Since M' is strictly diagonally dominant, there exists a unique solution to $M'w_i^\varepsilon = v_i$ and consequently for $Mw^\varepsilon = v$ by setting $w^\varepsilon = (w_1^\varepsilon, \ldots, w_n^\varepsilon)$. We infer that the mapping F is well defined.

3.2. Continuity of F

We fix $i \in \{1, \ldots, n\}$, multiply (22) by $w_{i,\ell}^\varepsilon$, and sum over $\ell \in G$:

$$\varepsilon \sum_{\ell\in G}\bigg(-\frac{w_{i,\ell+1}^\varepsilon - 2w_{i,\ell}^\varepsilon + w_{i,\ell-1}^\varepsilon}{\Delta x} + \Delta x w_{i,\ell}^\varepsilon\bigg) w_{i,\ell}^\varepsilon \\ = -\sum_{\ell\in G} \Delta x \frac{u_{i,\ell} - u_{i,\ell}^{k-1}}{\Delta t} w_{i,\ell}^\varepsilon - \sum_{\ell\in G} (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}) w_{i,\ell}^\varepsilon. \tag{23}$$

The left-hand side can be rewritten by using discrete integration by parts (or summation by parts):

$$\varepsilon \sum_{\ell\in G}\bigg(-\frac{(w_{i,\ell+1}^\varepsilon - w_{i,\ell}^\varepsilon) - (w_{i,\ell}^\varepsilon - w_{i,\ell-1}^\varepsilon)}{\Delta x} w_{i,\ell}^\varepsilon + \Delta x (w_{i,\ell}^\varepsilon)^2\bigg) \\ = \varepsilon \sum_{\ell\in G} \frac{(w_{i,\ell+1}^\varepsilon - w_{i,\ell}^\varepsilon)^2}{\Delta x} + \varepsilon \sum_{\ell\in G} \Delta x (w_{i,\ell}^\varepsilon)^2 = \varepsilon \|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2. \tag{24}$$

The first term on the right-hand side of (23) is estimated by the Cauchy–Schwarz inequality, taking into account that $w \in Z_R$, which implies a finite discrete $L^2(\mathbb{T})$ norm for $u_{i,\ell} = \exp(w_{i,\ell}/\pi_i)$:

$$\bigg| -\sum_{\ell\in G} \Delta x \frac{u_{i,\ell} - u_{i,\ell}^{k-1}}{\Delta t} w_{i,\ell}^\varepsilon \bigg| \le C(\Delta t) \|u_i - u_i^{k-1}\|_{0,2,\mathcal{T}} \|w_i^\varepsilon\|_{0,2,\mathcal{T}} \le C(\Delta t, R) \|w_i^\varepsilon\|_{1,2,\mathcal{T}},$$

where here and in the following $C > 0$, $C(\Delta t, R) > 0$, etc. are generic constants with values changing from line to line. We split the second term on the right-hand side of (23) into two parts:

$$-\sum_{\ell\in G} (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}) w_{i,\ell}^\varepsilon = I_1 + I_2, \quad \text{where}$$

$$I_1 = \sigma \sum_{\ell\in G} \bigg(\frac{u_{i,\ell+1} - u_{i,\ell}}{\Delta x} - \frac{u_{i,\ell} - u_{i,\ell-1}}{\Delta x} \bigg) w_{i,\ell}^\varepsilon,$$

$$I_2 = \sum_{\ell\in G} \bigg(u_{i,\ell+1/2} \frac{p_{i,\ell+1} - p_{i,\ell}}{\Delta x} - u_{i,\ell-1/2} \frac{p_{i,\ell} - p_{i,\ell-1}}{\Delta x} \bigg) w_{i,\ell}^\varepsilon.$$

For I_1, we use discrete integration by parts, the Cauchy–Schwarz inequality, and the fact that $w \in Z_R$:

$$\begin{aligned}|I_1| &= \bigg| -\sigma \sum_{\ell\in G} \Delta x \frac{u_{i,\ell+1}-u_{i,\ell}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \\ &\le \sigma\bigg(\sum_{\ell\in G}\Delta x\bigg|\frac{u_{i,\ell+1}-u_{i,\ell}}{\Delta x}\bigg|^2\bigg)^{1/2}\bigg(\sum_{\ell\in G}\Delta x\bigg|\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg|^2\bigg)^{1/2} \\ &= \sigma|u_i|_{1,2,\mathcal{T}}|w_i^\varepsilon|_{1,2,\mathcal{T}} \le C(R)\|w_i^\varepsilon\|_{1,2,\mathcal{T}}.\end{aligned}$$

Using discrete integration by parts, and definition (13) of $p_{i,\ell}$, we obtain

$$\begin{aligned}|I_2| &= \bigg| -\sum_{\ell\in G}\Delta x u_{i,\ell+1/2}\frac{p_{i,\ell+1}-p_{i,\ell}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \le I_{21}+I_{22}, \quad \text{where} \\ I_{21} &= \bigg|\sum_{\ell\in G}\Delta x u_{i,\ell+1/2}a_{ii}\frac{(u_{i,\ell+1}-u_{i,\ell})}{\Delta x}\frac{(w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell})}{\Delta x}\bigg|, \\ I_{22} &= \bigg|\sum_{\substack{j=1\\j\ne i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2 u_{i,\ell+1/2}a_{ij}\frac{B^{ij}_{\ell+1-\ell'}-B^{ij}_{\ell-\ell'}}{\Delta x}u_{j,\ell'}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg|.\end{aligned}$$

For I_{21}, because of the bound in Z_R, we can estimate $u_{i,\ell+1/2} \le \max\{u_{i,\ell+1},u_{i,\ell}\} \le C(R)$. Then, thanks to the Cauchy–Schwarz inequality, we obtain

$$I_{21} \le C(R)a_{ii}\,|u_i|_{1,2,\mathcal{T}}\,|w_i^\varepsilon|_{1,2,\mathcal{T}} \le C(R)\,\|w_i^\varepsilon\|_{1,2,\mathcal{T}}.$$

For I_{22}, applying the discrete analog (17) of the rule $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$,

$$\begin{aligned}I_{22} &= \bigg|\sum_{\substack{j=1\\j\ne i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2 u_{i,\ell+1/2}a_{ij}B^{ij}_{\ell-\ell'}\frac{u_{j,\ell'+1}-u_{j,\ell'}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \\ &= \bigg|\sum_{\substack{j=1\\j\ne i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2 u_{i,\ell+1/2}a_{ij}B^{ij}_{\ell-\ell'}(\mathrm{D}_{\ell'}u_j)(\mathrm{D}_\ell w_i)\bigg|,\end{aligned}$$

where we used the notation of Section 2.1. Similarly to I_{21}, we infer that

$$I_{22} \le C(R)\sum_{\substack{j=1\\j\ne i}}^n a_{ij}\sum_{\ell\in G}\Delta x\bigg(\sum_{\ell'\in G}\Delta x B^{ij}_{\ell-\ell'}\mathrm{D}_{\ell'}u_j\bigg)\mathrm{D}_\ell w_i.$$

Then, by the Cauchy–Schwarz inequality and the discrete convolution inequality from Lemma A.3 in Appendix A,

$$\begin{aligned}I_{22} &\le C(R)\sum_{\substack{j=1\\j\ne i}}^n\bigg\{\sum_{\ell\in G}\Delta x\bigg(\sum_{\ell'\in G}\Delta x B^{ij}_{\ell-\ell'}\mathrm{D}_{\ell'}u_j\bigg)^2\bigg\}^{1/2}|w_i|_{1,2,\mathcal{T}} \\ &\le C(R)\sum_{\substack{j=1\\j\ne i}}^n\|B^{ij}\|_{L^1(\mathbb{T})}|u_j|_{1,2,\mathcal{T}}|w_i|_{1,2,\mathcal{T}} \le C(R)\|w_i\|_{1,2,\mathcal{T}}.\end{aligned}$$

Combining these estimates, we deduce from (23) that $\varepsilon\|w_i^\varepsilon\|_{1,2,\mathcal{T}} \le C(\Delta t, R)$.

We can proceed to show the continuity of F. Let $(w^k)_{k\in\mathbb{N}}$ be such that $w^k \to w \in Z_R$ as $k \to \infty$ and set $w^{\varepsilon,k} := F(w^k)$. We have just proved that $(w^{\varepsilon,k})_{k\in\mathbb{N}}$ is bounded with respect to the $\|\cdot\|_{1,2,\mathcal{T}}$ norm. By the Bolzano–Weierstraß theorem, there exists a subsequence (not relabeled) such that $w^{\varepsilon,k} \to w^\varepsilon$ in Z_R as $k \to \infty$. Performing the limit $k \to \infty$ in (23), satisfied for $w^{\varepsilon,k}$, shows that w^ε solves scheme (23) with $u_{i,\ell} = \exp(w_i^\varepsilon/\pi_i)$. This means that $w^\varepsilon = F(w)$, which proves the continuity of F.

3.3. Existence of a fixed point

We show that $F : Z_R \to \mathbb{R}^{nN}$ admits a fixed point by using a topological degree argument. We recall that the Brouwer topological degree is a mapping $\deg : M \to \mathbb{Z}$, where

$$M = \big\{(f, Z, y) : f \in C^0(\mathbb{T}),\ Z \text{ is open, bounded, } y \notin f(\partial Z)\big\};$$

see Chapter 1, Theorem 3.1 from [11] for details and properties. If we show that any solution $(w^\varepsilon, \rho) \in \overline{Z}_R \times [0,1]$ to the fixed-point equation $w^\varepsilon = \rho F(w^\varepsilon)$ satisfies $(w^\varepsilon, \rho) \notin \partial Z_R \times [0,1]$ for sufficiently large values of $R > 0$, then we deduce from the invariance by homotopy that $\deg(I - \rho F, Z_R, 0)$ is invariant in ρ. Then, choosing $\rho = 0$, $\deg(I, Z_R, 0) = 1$ and, if $\rho = 1$, $\deg(I - F, Z_R, 0) = \deg(I, Z_R, 0) = 1$. This implies that there exists $w^\varepsilon \in Z_R$ such that $(I - F)(w^\varepsilon) = 0$, which is the desired fixed point.

Let (w^ε, ρ) be a fixed point of $w^\varepsilon = \rho F(w^\varepsilon)$. If $\rho = 0$, there is nothing to show. Therefore, let $\rho > 0$. Then w_i^ε solves

$$-\varepsilon\frac{w^\varepsilon_{i,\ell+1} - 2w^\varepsilon_{i,\ell} + w^\varepsilon_{i,\ell-1}}{\Delta x} + \varepsilon\Delta x w^\varepsilon_{i,\ell} = -\rho\bigg(\Delta x\frac{u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} + \mathcal{F}^\varepsilon_{i,\ell+1/2} - \mathcal{F}^\varepsilon_{i,\ell-1/2}\bigg) \tag{25}$$

for all $\ell \in G$ and $i = 1, \dots, n$, where $u^\varepsilon_{i,\ell} = \exp(w^\varepsilon_{i,\ell}/\pi_i)$, and the fluxes $\mathcal{F}^\varepsilon_{i,\ell\pm1/2}$ are defined as in (12) with $u^k_{i,\ell}$ replaced by $u^\varepsilon_{i,\ell}$. We multiply the previous equation by $\Delta t w^\varepsilon_{i,\ell}$, sum over $\ell \in G$, $i = 1, \dots, n$, and use discrete integration by parts as in (24):

$$\varepsilon\Delta t\sum_{i=1}^n \|w_i^\varepsilon\|^2_{1,2,\mathcal{T}} = -\rho\sum_{i=1}^n\sum_{\ell\in G}\big(\Delta x(u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell})w^\varepsilon_{i,\ell} + \Delta t(\mathcal{F}^\varepsilon_{i,\ell+1/2} - \mathcal{F}^\varepsilon_{i,\ell-1/2})w^\varepsilon_{i,\ell}\big). \tag{26}$$

For the first term on the right-hand side, we use $w^\varepsilon_{i,\ell} = \pi_i \log u^\varepsilon_{i,\ell}$ and the convexity of $h(s) = s(\log s - 1)$:

$$(u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell})\pi_i \log u^\varepsilon_{i,\ell} \ge \pi_i\big(h(u^\varepsilon_{i,\ell}) - h(u^{k-1}_{i,\ell})\big).$$

Recalling definition (18) of $\mathcal{H}_B$, this shows that

$$-\rho\sum_{i=1}^n\sum_{\ell\in G}\Delta x(u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell})w^\varepsilon_{i,\ell} \le -\rho\big(\mathcal{H}_B(u^\varepsilon) - \mathcal{H}_B(u^{k-1})\big).$$

Like in Section 3.2, we split the second term in (26) into two parts:

$$\begin{aligned} &-\rho\Delta t\sum_{i=1}^n\sum_{\ell\in G}(\mathcal{F}^\varepsilon_{i,\ell+1/2} - \mathcal{F}^\varepsilon_{i,\ell-1/2})w^\varepsilon_{i,\ell} = I_3 + I_4, \quad \text{where} \\ &I_3 = \rho\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\bigg(\frac{u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell}}{\Delta x} - \frac{u^\varepsilon_{i,\ell} - u^\varepsilon_{i,\ell-1}}{\Delta x}\bigg)w^\varepsilon_{i,\ell}, \\ &I_4 = \rho\Delta t\sum_{i=1}^n\sum_{\ell\in G}\bigg(u^\varepsilon_{i,\ell+1/2}\frac{p^\varepsilon_{i,\ell+1} - p^\varepsilon_{i,\ell}}{\Delta x} - u^\varepsilon_{i,\ell-1/2}\frac{p^\varepsilon_{i,\ell} - p^\varepsilon_{i,\ell-1}}{\Delta x}\bigg)w^\varepsilon_{i,\ell}. \end{aligned} \tag{27}$$

We use discrete integration by parts, the definition $w^\varepsilon_{i,\ell} = \pi_i \log u^\varepsilon_{i,\ell}$, and the elementary inequality $(a-b)(\log a - \log b) \ge 4(\sqrt{a}-\sqrt{b})^2$ for a, $b>0$ to estimate the first term:

$$\begin{aligned} I_3 &= -\rho\sigma\Delta t \sum_{i=1}^n \sum_{\ell\in G} \frac{u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell}}{\Delta x}(w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}) \\ &\le -4\rho\sigma\Delta t \sum_{i=1}^n \sum_{\ell\in G} \frac{\pi_i}{\Delta x}\big((u^\varepsilon_{i,\ell+1})^{1/2}-(u^\varepsilon_{i,\ell})^{1/2}\big)^2 = -4\rho\sigma\Delta t \sum_{i=1}^n \pi_i |(u^\varepsilon_i)^{1/2}|^2_{1,2,\mathcal{T}}. \end{aligned}$$

For the second term I_4, we use discrete integration by parts and $w^\varepsilon_{i,\ell} = \pi_i \log u^\varepsilon_{i,\ell}$ again as well as property (14) (discrete chain rule):

$$\begin{aligned} I_4 &= -\rho \frac{\Delta t}{\Delta x} \sum_{i=1}^n \sum_{\ell\in G} \pi_i u^\varepsilon_{i,\ell+1/2}(p^\varepsilon_{i,\ell+1}-p^\varepsilon_{i,\ell})(\log u^\varepsilon_{i,\ell+1}-\log u^\varepsilon_{i,\ell}) \\ &\le -\rho c_0 \frac{\Delta t}{\Delta x} \sum_{i=1}^n \sum_{\ell\in G} \pi_i (p^\varepsilon_{i,\ell+1}-p^\varepsilon_{i,\ell})(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell}). \end{aligned}$$

Then, inserting definition (3) of $p^\varepsilon_{i,\ell}$ and using the discrete analog (17) of $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$,

$$\begin{aligned} I_4 &\le -\rho c_0 \frac{\Delta t}{\Delta x}(I_{41}+I_{42}), \quad \text{where} \\ I_{41} &= \sum_{i=1}^n \sum_{\ell\in G} \pi_i a_{ii}(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell})^2, \\ I_{42} &= \sum_{\substack{i,j=1 \\ j\ne i}}^n \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ij} B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1}-u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell}). \end{aligned}$$

We insert $(n-1)^{-1}\sum_{j\ne i} 1 = 1$ and $\sum_{\ell'\in G}\Delta x = 1$ (note that $\mathrm{m}(\mathbb{T})=1$) in I_{41} and split the resulting sum into two parts:

$$I_{41} = \frac{1}{n-1}\sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ii}(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell})^2 + \frac{1}{n-1}\sum_{\substack{i,j=1 \\ i>j}}^n \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ii}(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell})^2.$$

We exchange i and j as well as ℓ and ℓ' in the second term, which leads to

$$I_{41} = \frac{1}{n-1}\sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell'\in G} \Delta x \big[\pi_i a_{ii}(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell})^2 + \pi_j a_{jj}(u^\varepsilon_{j,\ell'+1}-u^\varepsilon_{j,\ell'})^2\big].$$

Similarly, we distinguish between $i<j$ and $i>j$ in I_{42} and exchange i and j as well as ℓ and ℓ' in the sum over $i>j$, leading to

$$\begin{aligned} I_{42} &= \sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ij} B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1}-u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell}) \\ &\quad + \sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell'\in G} \Delta x \pi_j a_{ji} B^{ji}_{\ell'-\ell}(u^\varepsilon_{i,\ell+1}-u^\varepsilon_{i,\ell})(u^\varepsilon_{j,\ell'+1}-u^\varepsilon_{j,\ell'}). \end{aligned}$$

By Remark 2.2, we have $B^{ji}_{\ell'-\ell} = B^{ij}_{\ell-\ell'}$. Therefore,

$$I_{42} = \sum_{\substack{i,j=1\\i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x(\pi_i a_{ij} + \pi_j a_{ji}) B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell}).$$

The sum of I_{41} and I_{42} can be written as a quadratic form in $\mathrm{D}_\ell u^\varepsilon_i$ and $\mathrm{D}_{\ell'} u^\varepsilon_j$ with the matrix $M^{ij}_{\ell-\ell'}$, defined in (19). This shows that

$$I_4 \le -\frac{\rho c_0 \Delta t}{(n-1)} \sum_{\substack{i,j=1\\i<j}}^{n} \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix} \le 0.$$

Collecting the estimates for I_3 and I_4 in (27), we deduce from (26) the following regularized discrete entropy inequality:

$$\begin{aligned} &\rho\mathcal{H}_B(u^\varepsilon) + \varepsilon\Delta t \sum_{i=1}^{n} \|w^\varepsilon_i\|^2_{1,2,\mathcal{T}} + 4\rho\sigma\Delta t \sum_{i=1}^{n} \pi_i |(u^\varepsilon_i)^{1/2}|^2_{1,2,\mathcal{T}} \\ &\quad + \frac{\rho c_0 \Delta t}{(n-1)} \sum_{\substack{i,j=1\\i<j}}^{n} \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix} \le \rho\mathcal{H}_B(u^{k-1}). \end{aligned} \tag{28}$$

We proceed with the topological degree argument. We set $R = 1 + (\mathcal{H}_B(u^{k-1})/(\varepsilon\Delta t))^{1/2}$. Then (28) implies that

$$\varepsilon\Delta t \sum_{i=1}^{n} \|w^\varepsilon_i\|^2_{1,2,\mathcal{T}} \le \rho\mathcal{H}_B(u^{k-1}) \le \mathcal{H}_B(u^{k-1}) = \varepsilon\Delta t(R-1)^2 < \varepsilon\Delta t R^2,$$

and hence $w^\varepsilon \notin \partial Z_R$. We infer that $\deg(I-F, Z_R, 0) = 1$ and consequently, F admits a fixed point. Note that we did not use the estimate for u^ε_i in the seminorm $|\cdot|_{1,2,\mathcal{T}}$ at this point, such that $\sigma = 0$ is admissible here (and also in the following two subsections).

3.4. Limit $\varepsilon \to 0$

There exists a constant $C > 0$ such that $C(s-1) \le h(s)$ for all $s \ge 0$. Hence,

$$C\pi_i \Delta x(u^\varepsilon_{i,\ell} - 1) \le \pi_i \Delta x h(u^\varepsilon_{i,\ell}) \le \mathcal{H}_B(u^\varepsilon) \le \mathcal{H}_B(u^{k-1}),$$

for all $\ell \in G$, $i = 1,\ldots,n$. Thus, $(u^\varepsilon_{i,\ell})$ is bounded in ε and the Bolzano–Weierstraß theorem implies the existence of a subsequence (not relabeled) such that $u^\varepsilon_{i,\ell} \to u^k_{i,\ell} \ge 0$ as $\varepsilon \to 0$. It follows from (28) that $\varepsilon w^\varepsilon_{i,\ell} \to 0$. Thus, the limit $\varepsilon \to 0$ in (25) shows that u^k is a solution to the numerical scheme (11)–(13). Moreover, the limit $\varepsilon \to 0$ in (28) leads to the discrete entropy inequality (20).

3.5. Discrete Rao entropy inequality

We prove inequality (21). To this end, we multiply (11) by $\Delta t \pi_i p^k_{i,\ell}$ and sum over $\ell \in G$, $i = 1,\ldots,n$:

$$\sum_{i=1}^{n} \sum_{\ell\in G} \Delta x \pi_i (u^k_{i,\ell} - u^{k-1}_{i,\ell}) p^k_{i,\ell} + \sum_{i=1}^{n} \sum_{\ell\in G} \Delta t \pi_i (\mathcal{F}^k_{i,\ell+1/2} - \mathcal{F}^k_{i,\ell-1/2}) p^k_{i,\ell} = 0. \tag{29}$$

For the first term in (29), we use the definition of $p^k_{i,\ell}$:

$$\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i(u^k_{i,\ell}-u^{k-1}_{i,\ell})p^k_{i,\ell}=I_5+I_6,\quad\text{where}$$
$$I_5=\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i a_{ii}(u^k_{i,\ell}-u^{k-1}_{i,\ell})u^k_{i,\ell},$$
$$I_6=\sum_{\substack{i,j=1\\ j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2\pi_i a_{ij}B^{ij}_{\ell-\ell'}(u^k_{i,\ell}-u^{k-1}_{i,\ell})u^k_{j,\ell'}.$$

We rewrite I_5 and I_6 according to

$$I_5=\frac12\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i a_{ii}\big((u^k_{i,\ell})^2-(u^{k-1}_{i,\ell})^2\big)+\frac12\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i a_{ii}\big(u^k_{i,\ell}-u^{k-1}_{i,\ell}\big)^2,$$
$$I_6=\frac12\sum_{\substack{i,j=1\\ j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2\pi_i a_{ij}B^{ij}_{\ell-\ell'}(u^k_{i,\ell}u^k_{j,\ell'}-u^{k-1}_{i,\ell}u^{k-1}_{j,\ell'})$$
$$+\frac12\sum_{\substack{i,j=1\\ j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2\pi_i a_{ij}B^{ij}_{\ell-\ell'}(u^k_{i,\ell}-u^{k-1}_{i,\ell})(u^k_{j,\ell'}-u^{k-1}_{j,\ell'}).$$

Combining the second terms in I_5 and I_6, using similar computations as for I_4 in Section 3.3, and applying Hypothesis (H3) shows that the second term of I_5+I_6 is nonnegative leading to

$$I_5+I_6\geq\frac12\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i a_{ii}\big((u^k_{i,\ell})^2-(u^{k-1}_{i,\ell})^2\big)$$
$$+\frac12\sum_{\substack{i,j=1\\ j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2\pi_i a_{ij}B^{ij}_{\ell-\ell'}(u^k_{i,\ell}u^k_{j,\ell'}-u^{k-1}_{i,\ell}u^{k-1}_{j,\ell'}).$$

Then it holds that

$$\sum_{i=1}^n\sum_{\ell\in G}\Delta x\pi_i(u^k_{i,\ell}-u^{k-1}_{i,\ell})p^k_{i,\ell}\geq\mathcal{H}_R(u^k)-\mathcal{H}_R(u^{k-1}).$$

Now, we split the second term in (29) again into two parts:

$$\sum_{i=1}^n\sum_{\ell\in G}\Delta t\pi_i(\mathcal{F}^k_{i,\ell+1/2}-\mathcal{F}^k_{i,\ell-1/2})p^k_{i,\ell}=I_7+I_8,\quad\text{where}$$
$$I_7=-\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\pi_i\bigg(\frac{u^k_{i,\ell+1}-u^k_{i,\ell}}{\Delta x}-\frac{u^k_{i,\ell}-u^k_{i,\ell-1}}{\Delta x}\bigg)p^k_{i,\ell},$$
$$I_8=-\Delta t\sum_{i=1}^n\sum_{\ell\in G}\pi_i\bigg(u^k_{i,\ell+1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x}-u^k_{i,\ell-1/2}\frac{p^k_{i,\ell}-p^k_{i,\ell-1}}{\Delta x}\bigg)p^k_{i,\ell}.$$

We reformulate I_7 by using discrete integration by parts:

$$I_7=\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\pi_i\frac{u^k_{i,\ell+1}-u^k_{i,\ell}}{\Delta x}(p^k_{i,\ell+1}-p^k_{i,\ell}).$$

Then, with similar computations as for I_4 in Section 3.3, we obtain

$$I_7 = \frac{\sigma \Delta t}{(n-1)} \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell' \in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \geq 0.$$

Finally, the term I_8 can be rewritten as

$$I_8 = \Delta t \sum_{i=1}^{n} \sum_{\ell \in G} \pi_i u_{i,\ell+1/2}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x} (p_{i,\ell+1}^k - p_{i,\ell}^k) = \Delta t \sum_{i=1}^{n} \sum_{\ell \in G} \pi_i \Delta x \big| (u_{i,\ell+1/2}^k)^{1/2} \mathrm{D}_\ell p_i^k \big|^2.$$

Hence, we infer from (29) that

$$\begin{aligned} &\mathcal{H}_R(u^k) + \Delta t \sum_{i=1}^{n} \sum_{\ell \in G} \pi_i \Delta x \big| (u_{i,\ell+1/2}^k)^{1/2} \mathrm{D}_\ell p_i^k \big|^2 \\ &\quad + \frac{\sigma \Delta t}{(n-1)} \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell' \in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \leq \mathcal{H}_R(u^{k-1}), \end{aligned}$$

which proves (21).

Finally, conservation of mass follows from summing (11) over $\ell \in G$ and observing that the sum over the numerical fluxes vanishes. This ends the proof of Theorem 2.5.

4. Proof of Theorem 2.6

To prove the convergence of the scheme, we derive first some uniform estimates and then apply a discrete Aubin–Lions compactness lemma.

4.1. Uniform estimates

Let $(u_m)_{m \in \mathbb{N}}$ be a sequence of finite-volume solutions to (11)–(13) associated to the mesh $\mathcal{D}_m$ and constructed in Theorem 2.5. The conservation of mass and the discrete entropy inequalities (20) and (21) show that, after summing over $k = 1, \ldots, N_T^m$,

$$\max_{k=1,\ldots,N_T^m} \|u_i^k\|_{0,2,\mathcal{T}_m}^2 + \sum_{k=1}^{N_T^m} \Delta t_m \|(u_i^k)^{1/2}\|_{1,2,\mathcal{T}_m}^2 \leq C, \quad i = 1, \ldots, n, \tag{30}$$

where $C > 0$ denotes here and in the following a constant independent of the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$, but possibly depending on u^0 and T. Because of the positive definiteness of $M_{\ell-\ell'}^{ij}$, we conclude a bound for u_i^k in the norm $\|\cdot\|_{1,2,\mathcal{T}_m}$.

Lemma 4.1. *Let the assumptions of Theorem 2.6 hold. Then there exists $C > 0$ independent of η_m (but depending on the positive definiteness constant c_M) such that for all $m \in \mathbb{N}$, $i = 1, \ldots, n$,*

$$\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|_{1,2,\mathcal{T}_m}^2 \leq C. \tag{31}$$

Proof. We infer from (20) that

$$\frac{c_0}{n-1} \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell' \in G_m} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \leq \mathcal{H}_B(u^0),$$

Since $M^{ij}_{\ell-\ell'}$ is uniformly positive definite with constant $c_M > 0$,

$$\begin{aligned}
\frac{c_0}{n-1} &\sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G_m} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \\
&\geq \frac{c_M c_0}{n-1} \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G_m} (\Delta x)^2 \big(|\mathrm{D}_\ell u_i^k|^2 + |\mathrm{D}_{\ell'} u_j^k|^2\big) \\
&= c_M c_0 \sum_{i=1}^{n} \sum_{\ell\in G_m} \Delta x |\mathrm{D}_\ell u_i^k|^2 + c_M c_0 \sum_{j=1}^{n} \sum_{\ell'\in G_m} \Delta x |\mathrm{D}_{\ell'} u_j^k|^2 \\
&= 2 c_M c_0 \sum_{i=1}^{n} \sum_{\ell\in G_m} \Delta x |\mathrm{D}_\ell u_i^k|^2.
\end{aligned}$$

Together with the first bound in (30), this finishes the proof. ☐

Lemma 4.2. *Let the assumptions of Theorem 2.6 hold. Then there exists a constant $C > 0$ independent of η_m (but depending on σ) such that for all $m \in \mathbb{N}$, $i = 1, \ldots, n$,*

$$\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{1,1,\mathcal{T}_m} + \sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{0,\infty,\mathcal{T}_m} \leq C.$$

Moreover, there exists another constant, still denoted by $C > 0$ and independent of η_m, such that

$$\sum_{k=1}^{N_T^m} \Delta t_m |p_i^k|^2_{1,2,\mathcal{T}_m} \leq C. \tag{32}$$

Proof. As $\mathrm{m}(\mathbb{T}) = 1$, thanks to the Cauchy–Schwarz inequality,

$$|u_i^k|_{1,1,\mathcal{T}_m} = \sum_{\ell\in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k| \leq |u_i^k|_{1,2,\mathcal{T}_m}.$$

Using (31), this shows that

$$\begin{aligned}
\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{1,1,\mathcal{T}_m} &\leq 2 \sum_{k=1}^{N_T^m} \Delta t_m \big(\|u_i^k\|^2_{0,1,\mathcal{T}_m} + |u_i^k|^2_{1,1,\mathcal{T}_m}\big) \\
&\leq 2T \max_{k=1,\ldots,N_T^m} \|u_i^k\|^2_{0,1,\mathcal{T}_m} + 2 \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|^2_{1,2,\mathcal{T}_m} \leq C(u^0, T).
\end{aligned}$$

To show the discrete $L^\infty(\mathbb{T})$ bound, we apply the continuity of the embedding $\mathrm{BV}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ (in one space dimension). We conclude that, for $i = 1, \ldots, n$,

$$\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{0,\infty,\mathcal{T}_m} \leq C \sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{\mathrm{BV}(\mathbb{T})} = C \sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|^2_{1,1,\mathcal{T}_m} \leq C(u^0, T).$$

For the last part, we estimate as follows:

$$\begin{aligned}
|p_i^k|_{1,2,\mathcal{T}_m}^2 &= \sum_{\ell\in G_m} \Delta x_m \left| \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} \right|^2 \\
&\le C a_{ii}^2 |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \left| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} \frac{B_{\ell+1-\ell'}^{ij} - B_{\ell-\ell'}^{ij}}{\Delta x_m} u_{j,\ell'}^k \right|^2 \\
&\le C |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \left| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \frac{u_{j,\ell'+1}^k - u_{j,\ell'}^k}{\Delta x_m} \right|^2 \\
&\le C |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \left| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \right|^2 .
\end{aligned}$$

Then we deduce from the elementary inequality $(\sum_{j=1,\, j\neq i}^n a_j)^2 \le (n-1)\sum_{j=1,\, j\neq i}^n a_j^2$ for $a_j \in \mathbb{R}$ and the discrete Young convolution inequality in Lemma A.3 that

$$\begin{aligned}
&\sum_{\ell\in G_m} \Delta x_m \left| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \right|^2 \\
&\le (n-1) \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell\in G_m} \Delta x_m \left(\sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \right)^2 \le C \sum_{\substack{j=1\\ j\neq i}}^n \|B^{ij}\|_{L^2(\mathbb{T})}^2 |u_j^k|_{1,1,\mathcal{T}_m}^2 .
\end{aligned}$$

Summing over k, we infer that

$$\sum_{k=1}^{N_T^m} \Delta t_m |p_i^k|_{1,2,\mathcal{T}_m}^2 \le C \left\{ \sum_{i=1}^n \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 + \sum_{\substack{j=1\\ j\neq i}}^n \left(\|B^{ij}\|_{L^2(\mathbb{T})}^2 \sum_{k=1}^{N_T^m} \Delta t_m |u_j^k|_{1,1,\mathcal{T}_m}^2 \right) \right\} \le C,$$

where we used Lemma 4.2 for the last inequality. At this point, we need the discrete $L^2(0,T;H^1(\mathbb{T}))$ bound of $(u_{m,i})$. This ends the proof. ☐

Next, we show a uniform bound for the discrete time derivative.

Lemma 4.3. *Let the assumptions of Theorem 2.6 hold. Then there exists $C > 0$ independent of η_m such that for all $m \in \mathbb{N}$, $i = 1, \ldots, n$,*

$$\sum_{k=1}^{N_T^m} \Delta t_m \left\| \frac{u_i^k - u_i^{k-1}}{\Delta t_m} \right\|_{-1,2,\mathcal{T}_m}^{4/3} \le C.$$

Proof. Let $\phi = (\phi_\ell)_{\ell\in G_m} \in \mathcal{V}_{\mathcal{T}_m}$ be such that $\|\phi\|_{1,2,\mathcal{T}_m} = 1$. We multiply (11) by ϕ_ℓ, sum over $\ell \in G_m$, and use discrete integration by parts:

$$
\begin{aligned}
\sum_{\ell\in G_m}\Delta x_m\frac{u_{i,\ell}^k-u_{i,\ell}^{k-1}}{\Delta t_m}\phi_\ell &= \sigma\sum_{\ell\in G_m}\Big(\frac{u_{i,\ell+1}^k-u_{i,\ell}^k}{\Delta x_m}-\frac{u_{i,\ell}^k-u_{i,\ell-1}^k}{\Delta x_m}\Big)\phi_\ell \\
&\quad+\sum_{\ell\in G_m}\Big(u_{i,\ell+1/2}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}-u_{i,\ell-1/2}^k\frac{p_{i,\ell}^k-p_{i,\ell-1}^k}{\Delta x_m}\Big)\phi_\ell \\
&= -\sigma\sum_{\ell\in G_m}\Delta x_m\frac{u_{i,\ell+1}^k-u_{i,\ell}^k}{\Delta x_m}\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}-\sum_{\ell\in G_m}\Delta x_m u_{i,\ell+1/2}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m} \\
&=: I_9+I_{10}.
\end{aligned} \tag{33}
$$

By the Cauchy–Schwarz inequality,

$$
\begin{aligned}
|I_9| &\le \sigma\sum_{\ell\in G_m}\Delta x_m\big((u_{i,\ell+1}^k)^{1/2}+(u_{i,\ell}^k)^{1/2}\big)\left|\frac{(u_{i,\ell+1}^k)^{1/2}-(u_{i,\ell}^k)^{1/2}}{\Delta x_m}\right|\left|\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}\right| \\
&\le 2\sigma\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}|(u_i^k)^{1/2}|_{1,2,\mathcal{T}_m}|\phi|_{1,2,\mathcal{T}_m}.
\end{aligned}
$$

Furthermore, using $(u_{i,\ell+1/2}^k)^{1/2}\le\max\{(u_{i,\ell}^k)^{1/2},(u_{i,\ell+1}^k)^{1/2}\}\le\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}$,

$$
\begin{aligned}
|I_{10}| &\le \sum_{\ell\in G_m}\Delta x_m\big|(u_{i,\ell+1/2}^k)^{1/2}\big|\left|(u_{i,\ell+1/2}^k)^{1/2}\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\right|\left|\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}\right| \\
&\le \|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}\bigg(\sum_{\ell\in G_m}\Delta x_m\left|(u_{i,\ell+1/2}^k)^{1/2}\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\right|^2\bigg)^{1/2}|\phi|_{1,2,\mathcal{T}_m}.
\end{aligned}
$$

Applying the elementary inequality $(a+b)^r\le C(a^r+b^r)$ for all $a,b\ge 0$ and $r>1$, inserting the previous estimates into (33), and using Hölder's inequality, we find that

$$
\begin{aligned}
&\sum_{k=1}^{N_T^m}\Delta t_m\left\|\frac{u_i^k-u_i^{k-1}}{\Delta t_m}\right\|_{-1,2,\mathcal{T}_m}^{4/3} = \sum_{k=1}^{N_T^m}\Delta t_m\sup_{\|\phi\|_{1,2,\mathcal{T}_m}=1}\left|\sum_{\ell\in G_m}\Delta x_m\frac{u_{i,\ell}^k-u_{i,\ell}^{k-1}}{\Delta t_m}\phi_\ell\right|^{4/3} \\
&\le C\sum_{k=1}^{N_T^m}\Delta t_m\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}^{4/3}|(u_i^k)^{1/2}|_{1,2,\mathcal{T}_m}^{4/3} \\
&\quad+C\sum_{k=1}^{N_T^m}\Delta t_m\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}^{4/3}\bigg(\sum_{\ell\in G_m}\Delta x_m\left|(u_{i,\ell+1/2}^k)^{1/2}\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\right|^2\bigg)^{2/3} \\
&\le C\bigg(\sum_{k=1}^{N_T^m}\Delta t_m\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}^4\bigg)^{1/3}\bigg(\sum_{k=1}^{N_T^m}\Delta t_m|(u_i^k)^{1/2}|_{1,2,\mathcal{T}_m}^2\bigg)^{2/3} \\
&\quad+C\bigg(\sum_{k=1}^{N_T^m}\Delta t_m\|(u_i^k)^{1/2}\|_{0,\infty,\mathcal{T}_m}^4\bigg)^{1/3}\bigg(\sum_{k=1}^{N_T^m}\Delta t_m\sum_{\ell\in G_m}\Delta x_m\left|(u_{i,\ell+1/2}^k)^{1/2}\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\right|^2\bigg)^{2/3} \\
&\le C(u^0,T),
\end{aligned}
$$

and the last bound follows from Lemma 4.2 and the discrete Rao entropy inequality (21). □

4.2. Compactness

We claim that the estimates from Lemmas 4.2 and 4.3 are sufficient to conclude the relative compactness of $(u_m)_{m\in\mathbb{N}}$. In fact, the result follows from the discrete Aubin–Lions lemma Theorem 3.4 from [15] if the following two properties are satisfied:

– For any $(v_m)_{m\in\mathbb{N}} \subset \mathcal{V}_{\mathcal{T}_m}$ such that $\sup_{m\in\mathbb{N}} \|v_m\|_{1,2,\mathcal{T}_m} \le C$ for some $C > 0$, there exists $v \in L^2(\mathbb{T})$ satisfing, up to a subsequence, $v_m \to v$ strongly in $L^2(\mathbb{T})$. This property follows from Theorem 14.1 from [13].
– If $v_m \to v$ strongly in $L^2(\mathbb{T})$ and $\|v_m\|_{-1,2,\mathcal{T}_m} \to 0$ as $m \to \infty$, then $v = 0$. This property can be replaced by the condition that $\|\cdot\|_{1,2,\mathcal{T}_m}$ and $\|\cdot\|_{-1,2,\mathcal{T}_m}$ are dual norms with respect to the $L^2(\mathbb{T})$ norm, which is the case Remark 6 from [15]. A more detailed proof can be found in Proposition 10 from [18].

Hence, it follows Theorem 3.4 from [15] that there exists a subsequence, which is not relabeled, such that

$$u_{m,i} \to u_i \quad \text{strongly in } L^1(0,T;L^2(\mathbb{T})) \text{ as } m \to \infty.$$

Let us now adapt the Gagliardo–Nirenberg inequality to our situation. Let $k = 1, \dots, N_T^m$ be fixed. We first apply Lemma A.4 with $s = p = 2$:

$$\|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m} \le C\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^{1/2}\|u_{m,i}^k\|_{0,2,\mathcal{T}}^{1/2}.$$

Then it follows from the Hölder inequality

$$\|u_{m,i}^k\|_{0,6,\mathcal{T}_m} \le \|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m}^{2/3}\|(u_{m,i}^k)^{1/3}\|_{0,6,\mathcal{T}_m} = \|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m}^{2/3}\|u_{m,i}^k\|_{0,2,\mathcal{T}_m}^{1/3}$$

that

$$\|u_{m,i}^k\|_{0,6,\mathcal{T}_m} \le C\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^{1/3}\,\|u_{m,i}^k\|_{0,2,\mathcal{T}}^{2/3}.$$

Therefore,

$$\sum_{k=1}^{N_T}\Delta t_m\|u_{m,i}^k\|_{0,6,\mathcal{T}_m}^6 \le C \max_{k=1,\dots,N_T}\|u_{m,i}\|_{0,2,\mathcal{T}_m}^4 \sum_{k=1}^{N_T}\Delta t_m\,\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^2.$$

Recalling estimates (30) and (31), we conclude that $(u_{m,i})_{m\in\mathbb{N}}$ is uniformly bounded in $L^6(\mathbb{T})$. The convergence dominated theorem implies that, up to a subsequence, for every $p < 6$,

$$u_{m,i} \to u_i \quad \text{strongly in } L^p(Q_T) \text{ as } m \to \infty.$$

Lemma 4.2 implies that the sequence of discrete derivatives $(\partial_x^m u_{m,i})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Thus, there exists a subsequence (not relabeled) such that $\partial_x^m u_{m,i} \rightharpoonup v_i$ weakly in $L^2(Q_T)$, and the proof of Lemma 4.4 from [9] allows us to identify $v_i = \partial_x u_i$.

Lemma 4.4. *The following convergences hold, up to subsequences, as $m \to \infty$,*

$$\begin{aligned} p_{m,i} &\to p_i(u) \quad \textit{strongly in } L^2(Q_T),\\ \partial_x p_{m,i} &\rightharpoonup \partial_x p_i(u) \quad \textit{weakly in } L^2(Q_T), \quad i = 1,\dots,n. \end{aligned}$$

Proof. We follow the strategy of Corollary 14 from [17]. First, we rewrite $p_{i,\ell}^k$ defined in (13). By a change of variables, we have

$$\begin{aligned} p_{i,\ell}^k &= a_{ii}u_{m,i,\ell}^k + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} a_{ij}\left(\int_{K_{\ell-\ell'}} B^{ij}(y)\mathrm{d}y\right)u_{m,j,\ell'}^k\\ &= a_{ii}u_{m,i,\ell}^k + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} a_{ij}\int_{K_{\ell'}} B^{ij}(x_\ell - z)u_{m,j}^k(z)dz\\ &= a_{ii}u_{m,i}^k(x_\ell) + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(B^{ij} * u_{m,j}^k)(x_\ell). \end{aligned}$$

We introduce the piecewise constant function Q_m^{ij} by setting $Q_m^{ij} := (B^{ij} * u_{m,j})(x_\ell)$ in K_ℓ for $\ell \in G_m$. Then

$$p_i(u) - p_{m,i} = a_{ii}(u_i - u_{m,i}) + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(B^{ij} * u_j - Q_m^{ij}).$$

Since we know that $u_i - u_{m,i} \to 0$ strongly in $L^2(Q_T)$, it is sufficient to prove that $B^{ij} * u_j - Q_m^{ij} \to 0$ strongly in $L^2(Q_T)$. For this, we write

$$(B^{ij} * u_j - Q_m^{ij})(x,t) = B^{ij} * (u_j - u_{m,j})(x,t) + \int_{\mathbb{T}} (B^{ij}(x-y) - B^{ij}(x_\ell - y))u_{m,j}(y,t)\mathrm{d}y.$$

By Young's convolution inequality, we have

$$\|B^{ij} * (u_j - u_{m,j})\|_{L^2(Q_T)} \le \|B^{ij}\|_{L^1(\mathbb{T})}\|u_j - u_{m,j}\|_{L^2(Q_T)} \to 0.$$

Setting $\xi(x,y) = B^{ij}(x-y) - B^{ij}(x_\ell - y)$ for $x \in K_\ell$ and $y \in \mathbb{T}$, we estimate

$$\begin{aligned}\left\| \int_{\mathbb{T}} \xi(\cdot,y)u_{m,j}(y,t)\mathrm{d}y \right\|^2_{L^2(Q_T)} &\le \int_{\mathbb{T}} \|\xi(x,\cdot)\|^2_{L^2(\mathbb{T})}\mathrm{d}x \|u_{m,j}\|^2_{L^2(Q_T)} \\ &\le \sup_{|z|\le \Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}\|^2_{L^2(\mathbb{T})}\|u_{m,j}\|^2_{L^2(Q_T)}.\end{aligned}$$

Since $(u_{m,j})$ is bounded in $L^2(Q_T)$, it remains to verify that the first factor converges to zero as $\Delta x_m \to 0$. This follows from the density of continuous functions in $L^2(\mathbb{T})$. Indeed, let $\varepsilon > 0$ and B_ε^{ij} be continuous such that $\|B_\varepsilon^{ij} - B^{ij}\|_{L^2(\mathbb{T})} \le \varepsilon$. Then

$$\begin{aligned}\sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}\|_{L^2(\mathbb{T})} &\le \sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B_\varepsilon^{ij}(z+\cdot)\|_{L^2(\mathbb{T})} \\ &\quad + \sup_{|z|\le\Delta x_m} \|B_\varepsilon^{ij}(z+\cdot) - B_\varepsilon^{ij}\|_{L^2(\mathbb{T})} + \|B_\varepsilon^{ij} - B^{ij}\|_{L^2(\mathbb{T})} \\ &\le 2\varepsilon + \sup_{|z|\le\Delta x_m} \|B_\varepsilon^{ij}(z+\cdot) - B_\varepsilon^{ij}\|_{L^2(\mathbb{T})}.\end{aligned}$$

The last term is smaller than ε if we choose Δx_m sufficiently small. We have shown that $\sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}\|^2_{L^2(\mathbb{T})} \to 0$ as $m \to \infty$ and $B^{ij} * u_j - Q_m^{ij} \to 0$ strongly in $L^2(Q_T)$. This proves the first part of the lemma.

Thanks to (32), we have shown that $(\partial_x^m p_{m,i})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Hence, up to a subsequence, $\partial_x^m p_{m,i} \rightharpoonup z$ weakly in $L^2(Q_T)$. The first part of the proof shows that $z = \partial_x p_i(u)$, finishing the proof. □

4.3. Convergence of the scheme

We show that the limit $u = (u_1, \ldots, u_n)$ of the finite-volume solutions is a weak solution to (1)–(2). Let $i \in \{1,\ldots,n\}$ be fixed, let $\psi_i \in C_0^\infty(\mathbb{T}\times[0,T))$ be given, and let $\eta_m = \max\{\Delta x_m, \Delta t_m\}$. We set $\psi_{i,\ell}^k := \psi_i(x_\ell, t_k)$, multiply (11) by $\Delta t_m \psi_{i,\ell}^{k-1}$ and sum over $\ell \in G_m$, $k = 1,\ldots,N_T^m$. This yields $F_1^m + F_2^m + F_3^m = 0$, where

$$\begin{aligned}F_1^m &= \sum_{k=1}^{N_T^m}\sum_{\ell\in G_m} \Delta x_m (u_{i,\ell}^k - u_{i,\ell}^{k-1})\psi_{i,\ell}^{k-1}, \\ F_2^m &= -\sigma \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell\in G_m} \left(\frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} - \frac{u_{i,\ell}^k - u_{i,\ell-1}^k}{\Delta x_m} \right)\psi_{i,\ell}^{k-1}, \\ F_3^m &= -\sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell\in G_m} \left(u_{i,\ell+1/2}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} - u_{i,\ell-1/2}^k \frac{p_{i,\ell}^k - p_{i,\ell-1}^k}{\Delta x_m} \right)\psi_{i,\ell}^{k-1}.\end{aligned}$$

Furthermore, we introduce the terms

$$\begin{aligned} F_{10}^m &= -\int_0^T \int_{\mathbb{T}} u_{m,i} \partial_t \psi_i \mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}} u_{m,i}(x,0)\psi_i(x,0)\mathrm{d}x, \\ F_{20}^m &= \sigma \int_0^T \int_{\mathbb{T}} \partial_x^m u_{m,i} \partial_x \psi_i \mathrm{d}x\mathrm{d}t, \\ F_{30}^m &= \int_0^T \int_{\mathbb{T}} u_{m,i} \partial_x^m p_{m,i} \partial_x \psi_i \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Lemma 4.5. *Let the assumptions of Theorem 2.6 hold. Then it holds that, as $m \to \infty$,*

$$F_{10}^m \to -\int_0^T \int_{\mathbb{T}} u_i \partial_t \psi_i \mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}} u_i^0(x)\psi_i(x,0)\mathrm{d}x, \tag{34}$$

$$F_{20}^m \to \sigma \int_0^T \int_{\mathbb{T}} \partial_x u_i \partial_x \psi_i \mathrm{d}x\mathrm{d}t, \tag{35}$$

$$F_{30}^m \to \int_0^T \int_{\mathbb{T}} u_i \partial_x p_i(u) \partial_x \psi_i \mathrm{d}x\mathrm{d}t. \tag{36}$$

Proof. The strong convergence of $(u_{m,i})_{m\in\mathbb{N}}$ and the weak convergence of $(\partial_x^m u_{m,i})_{m\in\mathbb{N}}$ in $L^2(Q_T)$ as well as the fact that $u_{m,i}(x,0) = (\Delta x_m)^{-1}\int_{K_\ell} u_i^0(z)dz$ for $x \in K_\ell$ and $\ell \in G$ immediately show convergences (34) and (35). It remains to verify (36). We know from Lemma 4.4 that $\partial_x^m p_{m,i} \rightharpoonup \partial_x p_i(u)$ weakly in $L^2(Q_T)$. Since $u_{m,i} \to u_i$ strongly in $L^2(Q_T)$, this implies that

$$u_{m,i}\partial_x^m p_{m,i} \rightharpoonup u_i \partial_x p_i(u) \quad \text{weakly in } L^1(Q_T).$$

In fact, since $u_{m,i}^{1/2}\partial_x^m p_{m,i}$ is uniformly bounded in $L^2(Q_T)$ and $u_{m,i}^{1/2}$ is uniformly bounded in $L^\infty(0,T;L^4(\mathbb{T}))$, this weak convergence even holds in $L^2(0,T;L^{4/3}(\mathbb{T}))$. This proves (36) and ends the proof. ☐

Lemma 4.6. *Let the assumptions of Theorem 2.6 hold. Then it holds that, as $m \to \infty$,*

$$F_{10}^m - F_1^m \to 0, \quad F_{20}^m - F_2^m \to 0, \quad F_{30}^m - F_3^m \to 0.$$

The lemma implies that

$$\begin{aligned} F_{10}^m + F_{20}^m + F_{30}^m &= (F_{10}^m - F_1^m) + (F_{20}^m - F_2^m) + (F_{30}^m - F_3^m) + (F_1^m + F_2^m + f_3^m) \\ &= (F_{10}^m - F_1^m) + (F_{20}^m - F_2^m) + (F_{30}^m - F_3^m) \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Therefore, thanks to Lemma 4.5, we conclude that $u = (u_1, \ldots, u_n)$ is a weak solution to (1)–(2). This finishes the proof of Theorem 2.6, once Lemma 4.6 is proved.

Proof of Lemma 4.6. The limit $F_{10}^m - F_1^m \to 0$ is shown in Theorem 5.2 from [9]. For the convergence of $F_{20}^m - F_2^m$, we use discrete integration by parts:

$$\begin{aligned} F_2^m &= \sigma \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell\in G_m} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m}(\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}) \\ &= \sigma \sum_{k=1}^{N_T^m} \sum_{\ell\in G_m} \int_{x_\ell}^{x_{\ell+1}} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} \int_{t_{k-1}}^{t_k} \frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{\Delta x_m} \mathrm{d}x\mathrm{d}t, \\ F_{20}^m &= \sigma \sum_{k=1}^{N_T^m} \sum_{\ell\in G_m} \int_{t_{k-1}}^{t_k} \int_{x_\ell}^{x_{\ell+1}} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} \partial_x \psi_i \mathrm{d}x\mathrm{d}t. \end{aligned}$$

By the mean-value theorem,

$$\bigg|\int_{t_{k-1}}^{t_k}\frac{1}{\Delta x_m}\int_{x_\ell}^{x_{\ell+1}}\bigg(\frac{\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1}}{\Delta x_m}-\partial_x\psi_i\bigg)\mathrm{d}x\mathrm{d}t\bigg|\le C\Delta t_m\eta_m.$$

This shows that, as $m\to\infty$,

$$\begin{aligned}|F_2^m-F_{20}^m|&\le\sigma\sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}\bigg|\int_{t_{k-1}}^{t_k}\int_{x_\ell}^{x_{\ell+1}}\bigg(\frac{\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1}}{\Delta x_m}-\partial_x\psi_i\bigg)\frac{u_{i,\ell+1}^k-u_{i,\ell}^k}{\Delta x_m}\mathrm{d}x\mathrm{d}t\bigg|\\&\le C\eta_m\sum_{k=1}^{N_T^m}\Delta t_m\sum_{\ell\in G_m}|u_{i,\ell+1}^k-u_{i,\ell}^k|=C\eta_m\sum_{k=1}^{N_T^m}\Delta t_m|u_i^k|_{1,1,\mathcal{T}_m}\to 0,\end{aligned}$$

where we used the uniform discrete $L^2(0,T;W^{1,1}(\mathbb{T}))$ bound from Lemma 4.2.

It remains to prove that $|F_{30}^m-F_3^m|\to 0$. First, using discrete integration by parts, we rewrite F_3^m as well as F_{30}^m as

$$\begin{aligned}F_3^m&=\sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}\int_{t_{k-1}}^{t_k}u_{i,\ell+1/2}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}(\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1})\mathrm{d}t,\\F_{30}^m&=\sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}\int_{t_{k-1}}^{t_k}\bigg(\int_{x_\ell}^{x_{\ell+1/2}}u_{i,\ell}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\partial_x\psi_i\mathrm{d}x\\&\quad+\int_{x_{\ell+1/2}}^{x_{\ell+1}}u_{i,\ell+1}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\partial_x\psi_i\mathrm{d}x\bigg).\end{aligned}$$

Then we find that

$$\begin{aligned}|F_3^m-F_{30}^m|=\bigg|&\sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}(u_{i,\ell+1/2}^k-u_{i,\ell}^k)\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\\&\times\int_{t_{k-1}}^{t_k}\bigg(\frac{\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1}}{2}-\int_{x_\ell}^{x_{\ell+1/2}}\partial_x\psi_i(x)\mathrm{d}x\bigg)\mathrm{d}t\\&+\sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}(u_{i,\ell+1/2}^k-u_{i,\ell+1}^k)\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}\\&\times\int_{t_{k-1}}^{t_k}\bigg(\frac{\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1}}{2}-\int_{x_{\ell+1/2}}^{x_{\ell+1}}\partial_x\psi_i(x)\mathrm{d}x\bigg)\mathrm{d}t\bigg|.\end{aligned}$$

Thanks to the regularity of ψ_i, there exists a constant C independent of η_m such that

$$\bigg|\int_{t_{k-1}}^{t_k}\bigg(\frac{\psi_{i,\ell+1}^{k-1}-\psi_{i,\ell}^{k-1}}{2}-\int_{x_\ell}^{x_{\ell+1/2}}\partial_x\psi_i(x)\mathrm{d}x\bigg)\mathrm{d}t\bigg|\le C\eta_m\Delta t_m.$$

We obtain a similar expression if we integrate $\partial_x\psi_i$ over $(x_{\ell+1/2},x_{\ell+1})$. Thus, since

$$\begin{aligned}|u_{i,\ell+1/2}^k-u_{i,\ell}^k|&\le|u_{i,\ell+1}^k-u_{i,\ell}^k|\quad\text{and}\\|u_{i,\ell+1/2}^k-u_{i,\ell+1}^k|&\le|u_{i,\ell}^k-u_{i,\ell+1}^k|,\end{aligned}$$

we have

$$\begin{aligned}|F_3^m - F_{30}^m| &\le 2C\eta_m \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell\in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k||\mathrm{D}_\ell\, p_i^k| \\ &\le 2C\eta_m \bigg(\sum_{i=1}^n a_{ii} \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell,\ell'\in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k||a_{ij}(B_{\ell+1-\ell'}^{ij} - B_{\ell-\ell'}^{ij})u_{j,\ell'}^k| \bigg).\end{aligned}$$

It follows for $j\in\{1,\ldots,n\}$ with $j\neq i$, using the discrete analog (17) of $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$, that

$$\begin{aligned}\max_{\ell\in G_m} \bigg(\sum_{\ell'\in G_m} |a_{ij}(B_{\ell+1-\ell'}^{ij} - B_{\ell-\ell'}^{ij})u_{j,\ell'}^k| \bigg) &= \max_{\ell\in G_m} \bigg(\sum_{\ell'\in G_m} \Delta x_m |a_{ij}||B_{\ell-\ell'}^{ij}||\mathrm{D}_{\ell'} u_j^k| \bigg) \\ &\le |a_{ij}|\|B^{ij}\|_{L^\infty(\mathbb{T})} |u_j^k|_{1,1,\mathcal{T}_m}.\end{aligned}$$

At this point, we need the regularity condition $B^{ij}\in L^\infty(\mathbb{T})$ from Hypothesis (H3). Hence, it holds that

$$|F_3^m - F_{30}^m| \le 2C\eta_m \bigg(\sum_{i=1}^n \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 + \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,1,\mathcal{T}_m} \sum_{\substack{j=1\\ j\neq i}}^n |u_j^k|_{1,1,\mathcal{T}_m} \bigg).$$

It remains to apply the Cauchy–Schwarz inequality to conclude that

$$\begin{aligned}|F_3^m - F_{30}^m| \le 2C\eta_m \bigg\{ &\sum_{i=1}^n \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 \\ &+ \sum_{\substack{j=1\\ j\neq i}}^n \bigg(\sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,1,\mathcal{T}_m}^2 \bigg)^{1/2} \bigg(\sum_{k=1}^{N_T^m} \Delta t_m |u_j^k|_{1,1,\mathcal{T}_m}^2 \bigg)^{1/2} \bigg\}.\end{aligned}$$

Finally, we infer from Lemma 4.2 that $|F_3^m - F_{30}^m| \to 0$ as $m\to\infty$. Here, we need the discrete $L^2(0,T;H^1(\mathbb{T}))$ bound for u_i, which follows if $a_{ii}>0$. This concludes the proof of Lemma 4.6. □

Remark 4.7 (Multidimensional case)**.** Theorems 2.5 and 2.6 also hold in the multidimensional situation. The proof of Theorem 2.5 does not change, but the Sobolev embeddings in the proof of Theorem 2.6 change because of their dependence on the space dimension. We only sketch the changes. We consider a uniform mesh on $\mathbb{T}^d$ by taking the tensor product of the mesh $\mathcal{T}$ introduced in Section 2.1. The cells K_ℓ are then d-dimensional cubes with cell centers $\ell=(\ell_1,\ldots,\ell_d)$ and measure $\mathrm{m}(K_\ell)=(\Delta x)^d$. We write $\varsigma = K_\ell|K_{\ell'}$ for the edge (or hyper-face) ς between the neighboring cells K_ℓ and $K_{\ell'}$ and $\mathcal{E}_\ell$ for the set of edges of the cell K_ℓ. Finally, for every $\varsigma = K_\ell|K_{\ell'}$, we define the transmissibility coefficient $\tau_\varsigma := \mathrm{m}(\varsigma)/\mathrm{d}_\varsigma$ with $\mathrm{m}(\varsigma)=(\Delta x)^{d-1}$ and d_ς being the Euclidean distance between the cell centers. The numerical scheme (11)–(12) changes to

$$\mathrm{m}(K_\ell)\frac{u_{i,\ell}^k - u_{i,\ell}^{k-1}}{\Delta t} + \sum_{\varsigma\in\mathcal{E}_\ell} \mathcal{F}_{i,\ell,\varsigma}^k = 0, \quad i=1,\ldots,n,\ \ell\in G^d, \tag{37}$$

$$\mathcal{F}_{i,\ell,\varsigma}^k = -\sigma\,\tau_\varsigma\,\mathrm{D}_{\ell,\varsigma} u_i^k - \tau_\varsigma\, u_{i,\varsigma}^k\,\mathrm{D}_{\ell,\varsigma} p_i^k, \tag{38}$$

where we have set $\mathrm{D}_{\ell,\varsigma} v = v_{\ell'} - v_\ell$ for edges $\varsigma = K_\ell | K_{\ell'}$, the mobilities are defined by $u^k_{i,\varsigma} = \widehat{F}(u^k_{i,\ell}, u^k_{i,\ell'})$ with $\widehat{F}$ as in Section 2.2, and the discrete nonlocal operators are given by

$$p^k_{i,\ell} = a_{ii} u^k_{i,\ell} + \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\ell' \in G^d} \mathrm{m}(K_{\ell'}) a_{ij} B^{ij}_{\ell,\ell'} u^k_{j,\ell'}, \quad B^{ij}_{\ell,\ell'} = \frac{1}{\mathrm{m}(K_{\ell-\ell'})} \int_{K_{\ell-\ell'}} B^{ij}(y) \mathrm{d}y. \tag{39}$$

Let u_m be a solution to (37)–(39) associated to some space-time discretization indexed by the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$ satisfying $\eta_m \to 0$ as $m \to \infty$. The corresponding spatial mesh is denoted by $\mathcal{T}^d_m$ and the number of time steps by N^m_T. The uniform estimates (30) and (31) also hold for $d \geq 2$, but the regularity obtained in Lemma 4.2 is slightly weaker. Indeed, the embedding $\mathrm{BV}(\mathbb{T}^d) \hookrightarrow L^{d/(d-1)}(\mathbb{T}^d)$ (with $d/(d-1) = \infty$ if $d = 1$) yields

$$\sum_{k=1}^{N^m_T} \Delta t_m \|u^k_i\|^2_{1,1,\mathcal{T}^d_m} + \sum_{k=1}^{N^m_T} \Delta t_m \|u^k_i\|^2_{0,d/(d-1),\mathcal{T}^d_m} \leq C,$$

see for instance [5, 18] for the definitions of the discrete norms. Then Hölder's inequality $\|v\|_{0,2d/(2d-1),\mathcal{T}^d_m} \leq \|v^{1/2}\|_{0,2d/(d-1),\mathcal{T}^d_m} \|v^{1/2}\|_{0,2,\mathcal{T}^d_m}$ for $v \in \mathcal{V}_{\mathcal{T}^d_m}$ gives the following bound on the discrete time derivative (replacing the estimate in Lem. 4.3):

$$\sum_{k=1}^{N^T_M} \Delta t_m \left\| \frac{u^k_i - u^{k-1}_i}{\Delta t_m} \right\|^{4/3}_{-1,2d/(2d-1),\mathcal{T}^d_m} \leq C.$$

Similarly as in the one-dimensional case, we conclude from Theorem 3.4 from [15] the existence of a subsequence (which is not relabeled) such that $u_{m,i} \to u_i$ strongly in $L^1(0,T; L^{2d/(2d-1)}(\mathbb{T}^d))$ as $m \to \infty$. We deduce from the discrete Gagliardo–Nirenberg inequality Lemma 3.1 from [5]

$$\|u^k_i\|_{0,2d/(d-1),\mathcal{T}^d_m} \leq C \|u^k_i\|^{1/2}_{1,2,\mathcal{T}^d_m} \|u^k_i\|^{1/2}_{0,2,\mathcal{T}^d_m},$$

that the strong convergence $u_{m,i} \to u_i$ holds in $L^p(Q_T)$ for every $p < 2d/(d-1)$ (instead of $p < 6$ in the one-dimensional case) and in particular in $L^2(Q_T)$. Thus, the statement of Lemma 4.4 holds, and we have $\nabla^m p_{m,i} \rightharpoonup \nabla p_i(u)$ weakly in $L^2(Q_T)$, where ∇^m denotes the discrete gradient. In particular, $u_{m,i} \nabla^m p_{m,i} \rightharpoonup u_i \nabla p_i(u)$ weakly in $L^{4/3}(Q_T)$ as in the one-dimensional case. From this point on, the convergence of the scheme follows the lines of Section 4.3. □

5. Numerical experiments

In this section, we present several numerical experiments to illustrate the behavior of the scheme. The scheme was implemented in one space dimension using Matlab. In all the subsequent numerical tests, we choose the upwind mobility (15). The code is available at `https://gitlab.tuwien.ac.at/asc/nonlocal-crossdiff`. Our code is an adaptation of that one developed in [17] for the approximation of the nonlocal SKT system. We refer the reader to Section 6.1 of [17] for a complete presentation of the different methods used to implement the scheme.

5.1. Test case 1. Rate of convergence in space for various L^p-norms, convolution kernels, and initial data

We investigate the rate of convergence in space of the scheme at final time $T = 1$. In all test cases of this section, we consider $n = 2$ species, $\sigma = 10^{-4}$, the coefficient matrix $A = (a_{ij})_{1\leq i,j\leq 2}$ given by

$$A = \begin{pmatrix} 0.1251 & 0.25 \\ 1 & 2 \end{pmatrix},$$

TABLE 1. Orders of convergence in the L^1 and L^∞ norms in space at final time $T = 1$ for different kernels and initial data.

Kernel → / Initial Data ↓	Indicator (43)		Triangle (44)		Gaussian (45)	
(40)	**Testcase 13**		**Testcase 16**		**Testcase 19**	
	L^1-order:	1.1741	L^1-order:	1.1741	L^1-order:	1.0109
	L^1-error:	$9.76 \cdot 10^{-4}$	L^1-error:	$9.76 \cdot 10^{-4}$	L^1-error:	$3.20 \cdot 10^{-3}$
	L^∞-order:	1.14	L^∞-order:	1.1331	L^∞-order:	0.98437
	L^∞-error:	$1.49 \cdot 10^{-3}$	L^∞-error:	$1.68 \cdot 10^{-3}$	L^∞-error:	$2.45 \cdot 10^{-2}$
(41)	**Testcase 14**		**Testcase 17**		**Testcase 20**	
	L^1-order:	1.0948	L^1-order:	1.0336	L^1-order:	0.93381
	L^1-error:	$1.81 \cdot 10^{-5}$	L^1-error:	$2.78 \cdot 10^{-5}$	L^1-error:	$2.35 \cdot 10^{-3}$
	L^∞-order:	1.0486	L^∞-order:	1.0092	L^∞-order:	0.91831
	L^∞-error:	$4.73 \cdot 10^{-5}$	L^∞-error:	$8.57 \cdot 10^{-5}$	L^∞-error:	$8.87 \cdot 10^{-3}$
(42)	**Testcase 15**		**Testcase 18**		**Testcase 21**	
	L^1-order:	0.97752	L^1-order:	0.97495	L^1-order:	0.9611
	L^1-error:	$6.39 \cdot 10^{-5}$	L^1-error:	$5.35 \cdot 10^{-5}$	L^1-error:	$9.27 \cdot 10^{-4}$
	L^∞-order:	0.99787	L^∞-order:	0.99741	L^∞-order:	0.9761
	L^∞-error:	$1.74 \cdot 10^{-4}$	L^∞-error:	$11.48 \cdot 10^{-4}$	L^∞-error:	$3.69 \cdot 10^{-3}$

and $\pi_1 = 4$, $\pi_2 = 1$. We consider various initial data and kernels. More precisely, we choose

$$u_1^0(x) = \mathbf{1}_{[1/4,3/4]}(x), \quad u_2^0(x) = \mathbf{1}_{[0,1/4]}(x) + \mathbf{1}_{[3/4,1]}(x), \tag{40}$$

$$u_1^0(x) = \cos(2\pi x) + 1, \quad u_2^0(x) = \sin(2\pi x - \pi/2) + 1, \tag{41}$$

$$u_1^0(x) = \max(1 - |1 - 2x|, 0), \quad u_2^0(x) = \max(1 - 2|x|, 0) \tag{42}$$

and the kernels

$$B^{ij}(z) = \mathbf{1}_{[-0.3,0.3]}(z), \tag{43}$$

$$B^{ij}(z) = 2\max(1 - |z|/0.3, 0), \tag{44}$$

$$B^{ij}(z) = \exp\left(-|z|^2/2\varepsilon^2\right)/\sqrt{2\pi\varepsilon^2}, \ \varepsilon = 10^{-3}. \tag{45}$$

First, we consider a mesh of $N_{\text{init}} = 32$ cells and the time step size $\Delta t_{\text{init}} = 1/64$. Then, starting from this initial mesh, we refine the mesh in space by doubling the number of cells and halving the time step size, *i.e.* $N_{\text{new}} = 2N_{\text{old}}$ and $\Delta t_{\text{new}} = \Delta t_{\text{old}}/2$. This refinement of the meshes is in agreement with the first-order convergence rate of the Euler discretization in time and the expected first-order convergence rate in space of the scheme, due to the choice of the upwind mobility in the numerical fluxes. As exact solutions to system (1)–(3) are not explicitly known, we refine the mesh in space and time until $N_{\text{end}} = 2048$ and $\Delta t_{\text{end}} = 1/4096$, and we consider the solutions of the scheme obtained for N_{end} and Δt_{end} as reference solutions. The error is computed between the reference solutions and the solutions obtained for $N = 1024$ cells and $\Delta t = 1/2048$ at final time $T = 1$. Finally, using linear regression in logarithmic scale, we present in Table 1 the experimental order of convergence in the L^1 and L^∞-norms. As expected, we observe a rate of convergence around one. In Table 1, the numbers in bold letters denote the number of the test case available in our code (see the file loadTestcase.m).

5.2. Test case 2. Rate of convergence of the localization limit in various metrics

In the second test case, following [17], we evaluate numerically the rate of convergence of the localization limit. More precisely, for some sequences of kernels converging towards the Dirac measure δ_0, we compute the rate of convergence in different metrics of the solutions to scheme (10)–(13) towards its local version, *i.e.* $B^{ij} = \delta_0$ for all $i, j = 1, \dots, n$. At the continuous level, one can show, by adapting the approach of [19], that the localization limit holds thanks to a compactness method; see also [12] for the SKT system. However, so far no explicit rate of convergence is available. The goal of this numerical test is to obtain a better insight into this rate of convergence. Besides, it also illustrates Remark 2.4.

We consider the following parameters (for all 6 test cases of this section): $n = 3$ species, diffusion parameter $\sigma = 10^{-4}$, coefficient matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.125 \\ 0.4 & 1 & 0.2 \\ 0.25 & 0.2 & 1 \end{pmatrix},$$

and $\pi_1 = 4$, $\pi_2 = 2$, $\pi_3 = 2$. We choose the final time $T = 1$, a mesh of $N = 512$ cells, and the time step size $\Delta t = 10^{-3}$. Furthermore, we take the nonsmooth initial data

$$u_1^0(x) = \mathbf{1}_{[3/6,5/6]}(x), \quad u_2^0(x) = \mathbf{1}_{[0,1/6]}(x) + \mathbf{1}_{[5/6,1]}(x), \quad u_3^0(x) = \mathbf{1}_{[1/6,3/6]}(x), \tag{46}$$

and the smooth initial data

$$\begin{gathered} u_1^0(x) = \cos(2\pi x) + 1, \quad u_2^0(x) = \sin(2\pi x) + 1, \\ u_3^0(x) = (\cos(2\pi x) + \sin(2\pi x) + 2)/2. \end{gathered} \tag{47}$$

The kernels are chosen according to

$$B_\alpha^{ij}(z) = \mathbf{1}_{[-\alpha,\alpha]}(z)/2\alpha, \tag{48}$$

$$B_\alpha^{ij}(z) = \max(1 - |z|/\alpha, 0)/\alpha, \tag{49}$$

$$B_\alpha^{ij}(z) = \exp\left(-|z|^2/2\alpha^2\right)/\sqrt{2\pi\alpha^2}. \tag{50}$$

In our experiments, starting from $\alpha_{\text{init}} = 2^7 \Delta x$, we successively halve α until we reach $\alpha = \Delta x$. For each value of α, we compute the solutions to the nonlocal scheme (10)–(13) at final time. We evaluate the L^1, L^∞, and Wasserstein distance W_1 between the solution to the nonlocal scheme and the solution to the local one (for this, it is enough to set $\alpha = 0$ in our code). Since we work in one space dimension, we can explicitly compute the Wasserstein distance W_1; see Chapter 2 from [24]. The rates of convergence are estimated by linear regression (in log scale) and the results are presented in Table 2. Surprisingly, we observe a slightly better rate of convergence in the case of nonsmooth initial data. As before, the names in bold letters in Table 2 denote the name of the test case available in our code (see the file loadTestcase.m).

5.3. Test case 3. Segregation phenomenon

In this numerical experiment, we set $\sigma = 0$. Under the assumptions $n = 2$ species, $a_{ij} = 1$, and $B^{ij} = \delta_0$ for $i, j = 1, 2$, it has been shown in [4] that if the initial data are segregated (initial data with disjoint supports) then the solutions remain segregated (*i.e.*, they have disjoint supports) for all time. The main goal of this subsection is to illustrate the segregation pattern due to the nonlocal terms, *i.e.* $B^{ij} \neq \delta_0$. We expect that the solutions to the nonlocal model, given segregated initial data, are completely segregated, and that there exists a small region, *i.e.* a "gap" between the supports of the species, with a size that is related to the radius of the interaction kernels. Let us notice that in the subsequent test cases, Hypothesis (H3) is never satisfied. However, we did not encounter any numerical issues with our code.

We launched the code for a mesh of 512 cells and the time step size $\Delta t = 10^{-4}$. In the case of $n = 2$ species, we considered the initial data

$$u_1^0(x) = \mathbf{1}_{[0.1,0.4]}(x), \quad u_2^0(x) = \mathbf{1}_{[0.6,0.8]}(x),$$

TABLE 2. Rates of convergence of the localization limit in the L^1, L^∞ and W_1 metric for different initial data and kernels.

Kernel → / Initial Data ↓	(48)		(49)		(50)	
nonsmooth (46)	**Testcase NLTL2**		**Testcase NLTL4**		**Testcase NLTL6**	
	L^1-order:	1.8280	L^1-order:	1.8709	L^1-order:	1.7386
	L^∞-order:	1.8271	L^∞-order:	1.8698	L^∞-order:	1.7379
	W_1-order:	1.8306	W_1-order:	1.8724	W_1-order:	1.7426
smooth (47)	**Testcase NLTL3**		**Testcase NLTL5**		**Testcase NLTL7**	
	L^1-order:	1.7430	L^1-order:	1.8240	L^1-order:	1.5991
	L^∞-order:	1.7462	L^∞-order:	1.8261	L^∞-order:	1.6038
	W_1-order:	1.7451	W_1-order:	1.8252	W_1-order:	1.6023

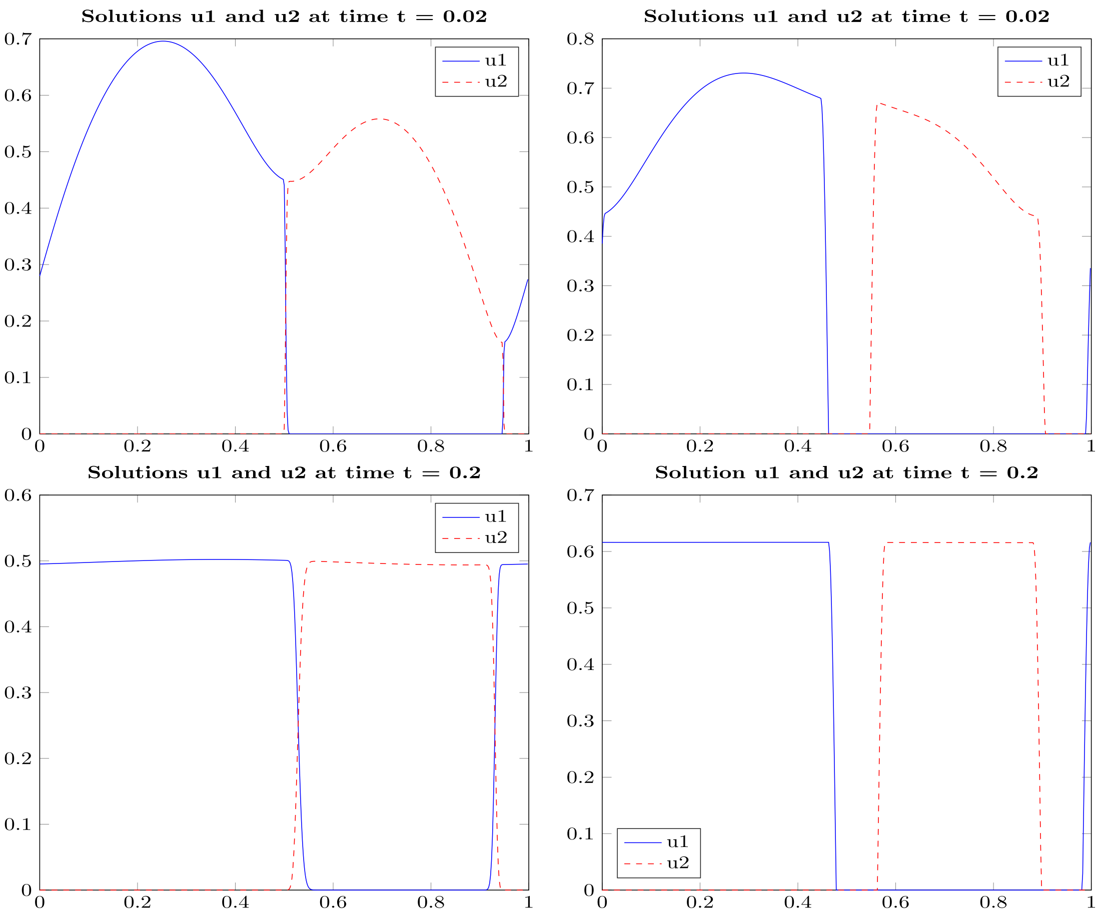

FIGURE 1. Comparison of the segregation pattern for *two* species at times $t = 0.02$ (top) and $t = 0.2$ (bottom) obtained from the local model (left) and nonlocal model (right). The solutions are almost in the steady state at $t = 0.2$.

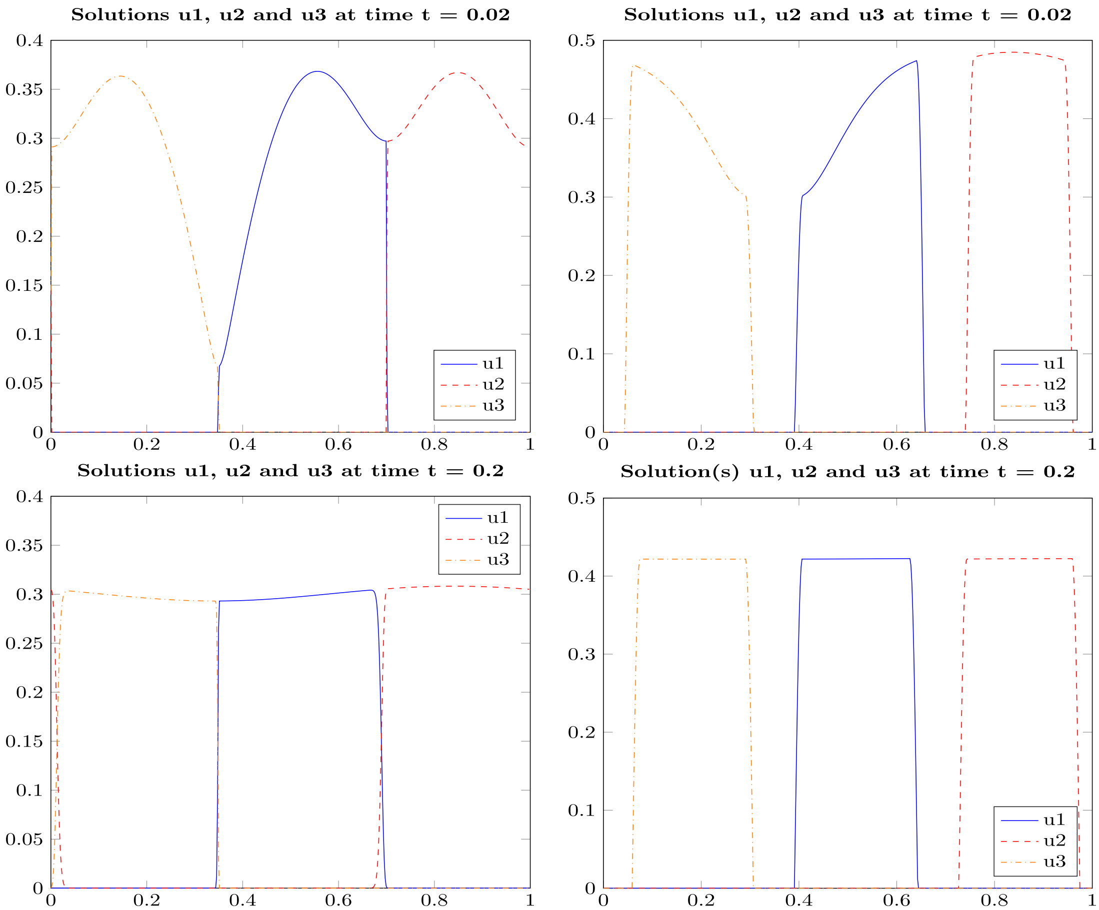

FIGURE 2. Comparison of the segregation patterns for *three* species at times $t = 0.02$ (top) and $t = 0.2$ (bottom) obtained from the local model (left) and nonlocal model (right). The solutions are almost in the steady state at $t = 0.2$.

while for $n = 3$ species, we have taken

$$u_1^0(x) = \mathbf{1}_{[0.5,0.6]}(x), \quad u_2^0(x) = \mathbf{1}_{[0.8,0.9]}(x), \quad u_3^0(x) = \mathbf{1}_{[0.1,0.2]}(x).$$

In both cases, we set $a_{ij} = 1$ for all $i, j = 1, \ldots, n$.

In Figures 1 and 2, we present the segregation pattern at time $t = 0.02$ and $t = 0.2$ obtained for the local model, $B^{ij} = \delta_0$, and the nonlocal model with

$$B^{ij}(z) = 100 \cdot \mathbf{1}_{[-0.1,0.1]}(z).$$

For small times, the support of the species extends until reaching the support of another species. In the local model, the species slightly mix (due to numerical diffusion), while we observe a "gap" between the supports of the solutions in the nonlocal model. This "gap" is of order 0.1 which is the size of the radius of the kernels B^{ij}. Similar numerical results have been observed in Section 6 of [8] but using different kernel functions and two species only.

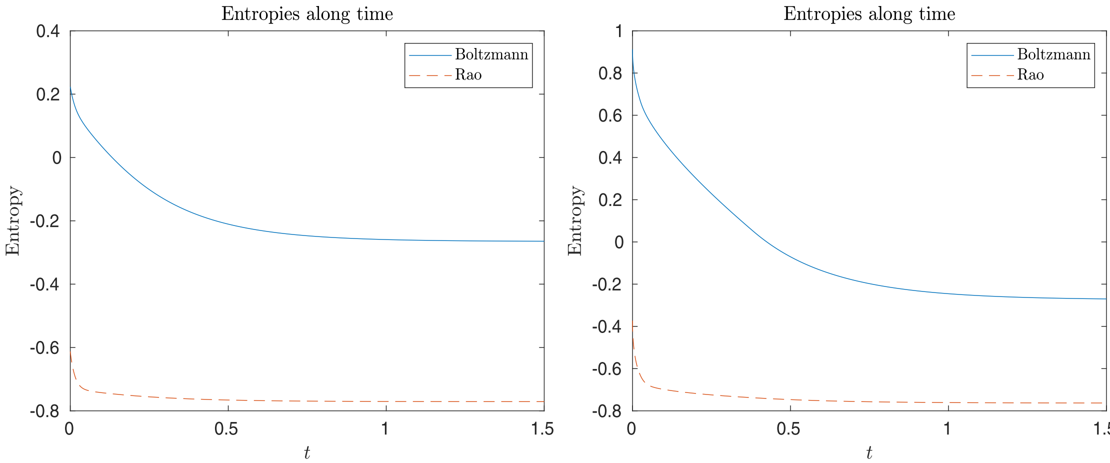

FIGURE 3. Temporal decay of the Boltzmann and Rao entropies for test cases 15 (left) and 16 (right) in semi-logarithmic scale.

5.4. Test case 4. Dissipation of entropy

In the last numerical experiment, we plot the two entropies $\mathcal{H}_B(u(t))$ and $\mathcal{H}_R(u(t))$ over time in semi-logarithmic scale to illustrate the entropy production as proved in Theorem 2.5. We set $T = 1.5$, $\Delta t = 10^{-4}$, use a mesh of $N = 512$ cells, and choose $n = 2$ species. The remaining parameters are taken as in Section 5.1; see Table 1 and the test cases therein. As expected, the entropies are decreasing functions of time. The Rao entropy decays first quickly but then stabilizes slowly, while the Boltzmann entropy takes more time to stabilize.

APPENDIX A. SOME AUXILIARY RESULTS

Lemma A.1. *Under Hypothesis (H3), the entropy dissipation Q, defined in (9), is nonnegative.*

Proof. We follow the approach of [12] and write $Q = Q_1 + \cdots + Q_3$, where

$$Q_1 = \frac{1}{n-1}\sum_{i,j=1,\,i<j}^{n}\int_{\mathbb{T}}\pi_i a_{ii}|\partial_x u_i(x)|^2 \mathrm{d}x + \frac{1}{n-1}\sum_{i,j=1,\,i>j}^{n}\int_{\mathbb{T}}\pi_i a_{ii}|\partial_x u_i(y)|^2 \mathrm{d}y,$$

$$Q_2 = \sum_{i,j=1,\,i<j}^{n}\int_{\mathbb{T}}\int_{\mathbb{T}}\pi_i a_{ij}B^{ij}(x-y)\partial_x u_j(y)\partial_x u_i(x)\mathrm{d}y\mathrm{d}x,$$

$$Q_3 = \sum_{i,j=1,\,i>j}^{n}\int_{\mathbb{T}}\int_{\mathbb{T}}\pi_i a_{ij}B^{ij}(x-y)\partial_x u_j(y)\partial_x u_i(x)\mathrm{d}y\mathrm{d}x.$$

Exchanging i and j in the second integral of Q_1 and using $\mathrm{m}(\mathbb{T}) = 1$, we have

$$Q_1 = \frac{1}{n-1}\sum_{i,j=1,\,i<j}^{n}\int_{\mathbb{T}}\int_{\mathbb{T}}\big(\pi_i a_{ii}|\partial_x u_i(x)|^2 \mathrm{d}x + \pi_j a_{jj}|\partial_x u_j(y)|^2\big)\mathrm{d}y\mathrm{d}x.$$

Exchanging i and j as well as x and y in Q_3 gives

$$\begin{aligned} Q_3 &= \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_j a_{ji} B^{ji}(y-x)\partial_x u_j(y)\partial_x u_i(x)\mathrm{d}y\mathrm{d}x \\ &= \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_j a_{ji} B^{ij}(x-y)\partial_x u_j(y)\partial_x u_i(x)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

We collect these expressions to obtain

$$Q = \frac{1}{(n-1)} \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}}\int_{\mathbb{T}} \begin{pmatrix} \partial_x u_i(x) \\ \partial_x u_j(y) \end{pmatrix}^{\top} M^{ij}(x-y) \begin{pmatrix} \partial_x u_i(x) \\ \partial_x u_j(y) \end{pmatrix} \mathrm{d}y\mathrm{d}x \geq 0,$$

where M^{ij} is defined in (8), and the last inequality follows from Hypothesis (H3). □

Lemma A.2. *The upwind approximation* (15) *and the logarithmic mean* (16) *satisfy property* (14) *of the mobilities* $u_{i,\sigma}$.

Proof. The proof is based on the following inequalities for the logarithmic mean:

$$\min\{a,b\} \leq \frac{a-b}{\log a - \log b} \leq \max\{a,b\} \quad \text{for all } a,b>0. \tag{A.1}$$

They imply the linear growth $u_{i,\ell+1/2} \leq \max\{u_{i,\ell}, u_{i,\ell+1}\}$ for the logarithmic mean, which also holds, by definition, for the upwind approximation. We show that property (14) is satisfied for the upwind approximation (15). Let $p_{i,\ell+1} - p_{i,\ell} \geq 0$. Then, by (A.1),

$$\begin{aligned} u_{i,\ell+1/2}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) &= u_{i,\ell+1}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \\ &\geq (p_{i,\ell+1}-p_{i,\ell})(u_{i,\ell+1}-u_{i,\ell}). \end{aligned}$$

On the other hand, if $p_{i,\ell+1} - p_{i,\ell} < 0$, again by (A.1),

$$\begin{aligned} u_{i,\ell+1/2}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) &= u_{i,\ell}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \\ &\geq (p_{i,\ell+1}-p_{i,\ell})(u_{i,\ell+1}-u_{i,\ell}). \end{aligned}$$

Property (14) follows immediately after inserting definition (16) of the logarithmic mean. This ends the proof. □

Lemma A.3 (Discrete Young convolution inequality)**.** *Let* $1 \leq p,q \leq \infty$ *and* $1 \leq r \leq \infty$ *be such that* $1+1/r = 1/p+1/q$ *and let* $B \in L^p(\mathbb{T})$ *and* $v = (v_\ell)_{\ell\in G} \in \mathcal{V}_{\mathcal{T}}$. *Furthermore, let* $B_{\ell-\ell'} = (\Delta x)^{-1}\int_{K_{\ell-\ell'}} B(y)\mathrm{d}y$ *for every* ℓ *and* $\ell' \in G$. *Then*

$$\left(\sum_{\ell\in G}\Delta x\left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'}\, v_{\ell'}\right|^r\right)^{1/r} \leq \|B\|_{L^p(\mathbb{T})}\|v\|_{0,q,\mathcal{T}}.$$

Proof. First, let $\ell \in G$ be fixed. Then

$$\left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'} v_{\ell'}\right| \leq \sum_{\ell'\in G}\Delta x\big(|B_{\ell-\ell'}|^p|v_{\ell'}|^q\big)^{1/r}|B_{\ell-\ell'}|^{(r-p)/r}|v_{\ell'}|^{(r-q)/r}.$$

Thanks to the assumption $1 = 1/p + 1/q - 1/r$, we can apply Hölder's inequality with exponents r, $pr/(r-p)$, and $qr/(r-q)$ to obtain

$$\begin{aligned}\bigg|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'}v_{\ell'}\bigg| &\le \bigg(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\bigg)^{1/r}\bigg(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p\bigg)^{(r-p)/pr} \\ &\quad\times\bigg(\sum_{\ell'\in G}\Delta x|v_{\ell'}|^q\bigg)^{(r-q)/qr} \\ &= \bigg(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\bigg)^{1/r}\|B\|_{0,p,\mathcal{T}}^{(r-p)/r}\|v\|_{0,q,\mathcal{T}}^{(r-q)/r}.\end{aligned}$$

Then, taking the exponent r and summing over $\ell \in G$,

$$\begin{aligned}\sum_{\ell\in G}\Delta x\bigg|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'}v_{\ell'}\bigg|^r &\le \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\bigg(\sum_{\ell\in G}\Delta x\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\bigg) \\ &\le \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\bigg(\sum_{\ell'\in G}\Delta x|v_{\ell'}|^q\sum_{\ell\in G}\Delta x|B_{\ell-\ell'}|^p\bigg) \\ &\le \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\|v\|_{0,q,\mathcal{T}}^{q}\|B\|_{0,p,\mathcal{T}}^{p} = \|B\|_{0,p,\mathcal{T}}^{r}\|v\|_{0,q,\mathcal{T}}^{r}.\end{aligned}$$

Finally, it holds that

$$\begin{aligned}\|B\|_{0,p,\mathcal{T}}^p &\le \sum_{\ell\in G}\Delta x\bigg|\frac{1}{\Delta x}\int_{K_\ell}B(y)\mathrm{d}y\bigg|^p \le \sum_{\ell\in G}\bigg(\int_{K_\ell}|B(y)|^p\mathrm{d}y\bigg)\bigg(\int_{K_\ell}\frac{\mathrm{d}x}{\Delta x}\bigg)^{p-1} \\ &\le \sum_{\ell\in G}\int_{K_\ell}|B(y)|^p\mathrm{d}y = \|B\|_{L^p(\mathbb{T})}^p,\end{aligned}$$

which concludes the proof. ∎

Lemma A.4. *Let $s > 1$ and $p > 1$. Then for any sequence $u = (u_\ell)_{\ell\in G}$, there exists a constant $C > 0$ only depending on s such that*

$$\|u\|_{0,\infty,\mathcal{T}} \le C\|u\|_{1,p,\mathcal{T}}^{1/s}\|u\|_{0,(s-1)p/(p-1),\mathcal{T}}^{1-1/s}.$$

Proof. We adapt the proof of Lemma 4.1 from [5] to the one-dimensional case. By the embedding $\mathrm{BV}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ applied to the sequence $(|u_\ell|^s)_{\ell\in G}$,

$$\|u\|_{0,\infty,\mathcal{T}}^s \le C\bigg(\|u\|_{0,s,\mathcal{T}}^s + \sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big|\bigg). \tag{A.2}$$

Since $s > 1$, we have

$$\sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big| \le s\sum_{\ell\in G}\big(|u_\ell|^{s-1} + |u_{\ell+1}|^{s-1}\big)|u_\ell - u_{\ell+1}|.$$

We apply Hölder's inequality with exponents p and $p/(p-1)$:

$$\sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big| \le 2s\bigg(\sum_{\ell\in G}\frac{|u_\ell - u_{\ell+1}|^p}{\Delta x^{p-1}}\bigg)^{1/p}\bigg(\sum_{\ell\in G}\Delta x|u_\ell|^{\frac{(s-1)p}{p-1}}\bigg)^{(p-1)/p}.$$

Besides, using again Hölder's inequality (with the same exponents), we find that

$$\|u\|_{0,s,\mathcal{T}} = \left(\sum_{\ell\in G}\Delta x|u_\ell||u_\ell|^{s-1}\right)^{1/s} \leq \|u\|_{0,p,\mathcal{T}_m}^{1/s}\|u\|_{0,(s-1)p/(p-1),\mathcal{T}}^{(s-1)/s}.$$

Then, inserting the last two inequalities into (A.2) yields the desired result. This concludes the proof of Lemma A.4. ☐

Appendix B. Counter-example

We claim that there exist kernels B^{ij}, being indicator functions, and piecewise constant functions $u_1,\ldots,u_n$ such that the positive semi-definiteness condition

$$J := \sum_{i,j=1}^n\int_{\mathbb{T}}\int_{\mathbb{T}}\pi_i a_{ij}B^{ij}(x-y)u_j(y)u_i(x)\mathrm{d}y\mathrm{d}x \geq 0,$$

is *not* satisfied. For this statement, we assume that the matrix $(\pi_i a_{ij}) \in \mathbb{R}^{n\times n}$ is (symmetric and) positive definite. With the notation of Section 2.1, we set $\Delta x = 1/N$ for some even number $N > 5$ and choose $r = 3\Delta x/2$ as well as the kernels

$$B^{ij}(x) = 1_{(-r,r)}(x) \quad \text{for } x\in\mathbb{T}.$$

Let $u_i = (u_{i,\ell})_{\ell\in G} \in \mathcal{V}_{\mathcal{T}}$ for $i=1,\ldots,n$. Then we can write J as

$$J = \sum_{i,j=1}^n\sum_{\ell,\ell'\in G}\pi_i a_{ij}\widehat{M}^{ij}_{\ell,\ell'}u_{j,\ell'}u_{i,\ell}, \quad \text{where } \widehat{M}^{ij}_{\ell,\ell'} = \int_{K_\ell}\int_{K_{\ell'}}B^{ij}(x-y)\mathrm{d}y\mathrm{d}x. \tag{B.3}$$

A straightforward, but rather tedious computation shows that the matrix $\widehat{M}^{ij} = (\widehat{M}^{ij}_{\ell,\ell'})_{\ell,\ell'\in G} \in \mathbb{R}^{N\times N}$ is pentadiagonal with entries

$$M^{ij}_{\ell,\ell'} = (\Delta x)^2, \quad M^{ij}_{\ell,\ell\pm1} = \frac{7}{8}(\Delta x)^2, \quad M^{ij}_{\ell,\ell\pm2} = \frac{1}{8}(\Delta x)^2.$$

This matrix possesses the eigenvector $w\in\mathbb{R}^N$, defined by $w_\ell = 1$ for ℓ odd and $w_\ell = -1$ for ℓ even, associated with the negative eigenvalue $\lambda = -4(\Delta x)^2$.

Let $v_1,\ldots,v_n\in\mathbb{R}^n$ be the eigenvectors of the symmetric matrix $(\pi_i a_{ij})_{i,j=1,\ldots,n}$ associated with the eigenvalues $0<\nu_1\leq\ldots\leq\nu_n$, respectively. We define the $nN\times nN$ matrix $\widehat{M} = (\pi_i a_{ij}\widehat{M}^{ij})$ consisting of the $N\times N$ blocks $\pi_i a_{ij}\widehat{M}^{ij}$. It can be verified that the matrix $\widehat{M}$ possesses the eigenvector $z=(z_1,\ldots,z_n)\in\mathbb{R}^{nN}$ with $z_i = v_{n,i}w\in\mathbb{R}^N$ for $i=1,\ldots,n$ associated with the eigenvalue $\lambda\nu_n = -4(\Delta x)^2\nu_n$. Then, choosing $u_i = z_i$ in (B.3), we find that

$$J = \sum_{i,j=1}^n\pi_i a_{ij}z_i^\top\widehat{M}^{ij}z_j = -4(\Delta x)^2\nu_n\sum_{i=1}^n|z_i|^2 < 0.$$

This provides the desired counter-example.

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Data availability statement

The code used in this paper is available online on the Github repository https://gitlab.tuwien.ac.at/asc/nonlocal-crossdiff; see [20]. It is based on the code developed in [17], which is available online on the Github repository https://gitlab.inria.fr/herda/nonlocal-skt; see [16].

References

[1] V. Anaya, M. Bendahmane and M. Sepúlveda, Numerical analysis for a three interacting species model with nonlocal and cross diffusion. *ESAIM:M2AN* **49** (2015) 171–192.

[2] N. Ayi, M. Herda, H. Hivert and I. Tristani, On a structure-preserving numerical method for fractional Fokker–Planck equations. *Math. Comput.* **92** (2023) 635–693.

[3] M. Bendahmane and M. Sepúlveda, Convergence of a finite volume scheme for nonlocal reaction-diffusion systems modelling an epidemic disease. *Discrete Contin. Dyn. Syst. B* **11** (2009) 823–853.

[4] M. Bertsch, M.E. Gurtin, D. Hilhorst and L.A. Peletier, On interacting populations that disperse to avoid crowding: preservation of segregation. *J. Math. Biol.* **23** (1985) 1–13.

[5] M. Bessemoulin-Chatard, C. Chainais-Hillairet and F. Filbet, On discrete functional inequalities for some finite volume schemes. *IMA J. Numer. Anal.* **35** (2015) 1125–1149.

[6] J.A. Carrillo, A. Chertock and Y. Huang, A finite-volume method for nonlinear nonlocal equations with a gradient flow structure. *Commun. Comput. Phys.* **17** (2015) 233–258.

[7] J.A. Carrillo, Y. Huang and M. Schmidtchen, Zoology of a nonlocal cross-diffusion model for two species. *SIAM J. Appl. Math.* **78** (2018) 1078–1104.

[8] J.A. Carrillo, F. Filbet and M. Schmidtchen, Convergence of a finite volume scheme for a system of interacting species with cross-diffusion. *Numer. Math.* **145** (2020) 473–511.

[9] C. Chainais-Hillairet, J.-G. Liu and Y.-J. Peng, Finite volume scheme for multi-dimensional drift-diffusion equations and convergence analysis. *ESAIM:M2AN* **37** (2003) 319–338.

[10] L. Chen, E. Daus and A. Jüngel, Rigorous mean-field limit and cross diffusion. *Z. Angew. Math. Phys.* **70** (2019) 21.

[11] K. Deimling, Nonlinear Functional Analysis. Springer, Berlin (1985).

[12] H. Dietert and A. Moussa, Persisting entropy structure for nonlocal cross-diffusion systems. To appear in *Ann. Fac. Sci. Toulouse* (2024) Preprint: arXiv:2101.02893.

[13] R. Eymard, T. Gallouët and R. Herbin, Finite Volume Methods, edited by P.G. Ciarlet and J.-L. Lions. In: *Handbook of Numerical Analysis* **7** (2000) 713–1018.

[14] G. Galiano, Error analysis of some nonlocal diffusion discretization schemes. *Comput. Math. Appl.* **103** (2021) 40–52.

[15] T. Gallouët and J.-C. Latché, Compactness of discrete approximate solutions to parabolic PDEs–Application to a turbulence model. *Commun. Pure Appl. Anal.* **11** (2012) 2371–2391.

[16] M. Herda and A. Zurek, Matlab code for "Study of a structure preserving finite volume scheme for a nonlocal cross-diffusion system" (2022). https://gitlab.inria.fr/herda/nonlocal-skt

[17] M. Herda and A. Zurek, Study of a structure preserving finite volume scheme for a nonlocal cross-diffusion system. *ESAIM:M2AN* **57** (2023) 1589–1617.

[18] A. Jüngel and A. Zurek, A convergent structure-preserving finite-volume scheme for the Shigesada–Kawasaki–Teramoto population system. *SIAM J. Numer. Anal.* **59** (2021) 2286–2309.

[19] A. Jüngel, S. Portisch and A. Zurek. Nonlocal cross-diffusion systems for multi-species populations and networks. *Nonlinear Anal.* **219** (2022) 26.

[20] A. Jüngel, S. Portisch and A. Zurek, Matlab code for "A convergent finite-volume scheme for nonlocal cross-diffusion systems for multi-species populations" (2023). https://gitlab.tuwien.ac.at/asc/nonlocal-crossdiff

[21] G. Medvedev and G. Simpson, A numerical method for a nonlocal diffusion equation with additive noise. *Stoch. PDE: Anal. Comput.* **11** (2023) 1433–1469.

[22] J.R. Potts and M.A. Lewis, Spatial memory and taxis-driven pattern formation in model ecosystems. *Bull. Math. Biol.* **81** (2019) 2725–2747.

[23] C. Rao, Diversity and dissimilarity coefficients: a unified approach. *Theor. Popul. Biol.* **21** (1982) 24–43.

[24] F. Santambrogio, Optimal transport for applied mathematicians. Calculus of variations, PDEs, and modeling. *Prog. Nonlinear Differ. Equ. Appl.*. Cham, Birkhäuser, Springer (2015).