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DISSERTATION

Complex L_p-intersection bodies

ausgeführt zum Zwecke der Erlangung des akademischen Grades
eines Doktors der technischen Wissenschaften unter der Leitung von

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von

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Kurzfassung

Das Hauptaugenmerk der Arbeit liegt auf der Erweiterung der dualen Brunn–Minkowski Theorie in komplexen Vektorräumen auf die L_p-Theorie.

Eine neue Familie von geometrischen Operatoren, die *komplexen L_p-Schnittkörper*, werden eingeführt. Sie sind inspiriert durch die Konstruktion der komplexen Projektionskörper. Wir zeigen, dass sie, ähnlich zu ihrem reellen Gegenstück, eine zwischen den komplexen Schwerpunktskörpern und komplexen Schnittkörpern interpolierende Familie darstellen. Außerdem untersuchen wir geometrische Eigenschaften, wie (Pseudo-)Konvexität und isoperimetrische Ungleichungen.

Wir widmen uns auch Fragen im Stile des Busemann–Petty Problems für den komplexen Projektionen- und L_p-Schnittkörper. Nachdem wir den Zusammenhang zu der sphärischen Fourier-Transformation hergestellt haben, können wir die Fragestellung einheitlich behandeln.

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Abstract

The main focus of this thesis lies on the extension of the dual Brunn–Minkowski theory on complex vector spaces to the L_p-setting.

A new family of geometric operators, the *complex* L_p*-intersection body* operators, is introduced which is motivated by the construction of complex projection bodies. It is shown that, similar to their real counterparts, they interpolate between the recently introduced intersection and centroid body operators on complex vector spaces. Moreover, geometric properties like (pseudo-)convexity as well as isoperimetric inequalities for them are examined.

Secondly, Busemann–Petty type problems for complex projection and L_p-intersection body operators are considered. After linking the spherical Fourier transform to the considered operators, one obtains a unified procedure to tackle these problems.

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1 Introduction

During the second half of the last century, the *Busemann–Petty problem* [9] has attracted a lot of attention (see, e.g., [15, 21, 33–37, 68, 70] for an overview). It may be posed as follows: Let K and L be origin-symmetric compact and convex subsets of $\mathbb{R}^n$ with the property that the volume of each central hyperplane section of K is less or equal than the corresponding one for L, does this imply that the volume of K is always less or equal than the volume of L? The tools that were developed on the way to the complete solution led to the development of the dual Brunn–Minkowski theory. The main object of interest for this problem is Lutwak's *intersection body* operator [47]. A big step forward to the solution has been taken by the observation that the answer to the problem is affirmative whenever K is an intersection body of a compact and star shaped set. Extending this principle and describing the image in terms of the Fourier transform culminated in the unified solution by Gardner, Koldobsky and Schlumprecht in [18]. It turns out that the answer is affirmative for $n \leq 4$ and negative for any higher dimension.

Already before the introduction of the intersection body, geometric properties for this object were established. Foremost, one should highlight Busemann's convexity theorem [8] and the famous intersection inequality [9, 52].

Starting from the examination of intersection bodies as geometric operators, several fruitful extensions were considered throughout the last years. Of special importance for this thesis is the family of L_p-intersection bodies which extended the notion of the L_p-centroid bodies (first defined in [49]) to the dual L_p-Brunn–Minkowski theory (see [4, 6, 17, 26, 28, 67]).

Recently, the investigation of geometric operators on complex vector spaces in the (dual) Brunn–Minkowski theory has arised. Starting with the construction of the family of *complex projection body* operators by Abardia and Bernig [2], several complex analogues of geometric operators on $\mathbb{R}^n$ were established in complex vector spaces. Just to name a few of these: the complex difference bodies [1], the complex centroid bodies [27] as well as the *complex intersection bodies* [40].

The main aim of this thesis lies in the further development of the dual Brunn–Minkowski theory in complex vector spaces. We introduce a new family of geometric operators, generalizing the concept of L_p-intersection bodies. The construction of these operators is inspired by the approach in the papers of Abardia and Bernig [2], Abardia [1] and Haberl [27].

The first part, which is joint work with Georg C. Hofstätter, deals with the definition of the *complex L_p-intersection body* operators. Besides proving basic properties like continuity, we also examine the operator in the same directions as it was done for its real counterpart. At first, we establish that a suitable normalization of the complex L_p-intersection bodies converge to the complex intersection body, which also justifies the name. We describe the operators via spherical harmonics to obtain statements about injectivity and in order to show (pseudo-)convexity of the image of sets with certain symmetry assumptions, we adapt

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techniques of Berck [6]. We close this section by linking real and complex L_p-intersection bodies by an operator which acts on each complex line. Together with the recent remarkable extension of Busemann's intersection inequality for real L_p-intersection bodies (see [4]), this allows us to lift the results to the complex case and obtain similar isoperimetric inequalities for a certain range of p.

In view of the Busemann–Petty problem, there have been several generalizations for geometric operators in real vector spaces (see [42, 60, 63, 64, 67]). In complex vector spaces, Koldobsky, König and Zymonopoulou [39] considered the corresponding problem by replacing real by complex hyperplane sections and origin-symmetry by $\mathbb{S}^1$-invariance.

We want to tackle similar questions for complex projection and L_p-intersection bodies in the second part of this thesis which is joint work with Georg C. Hofstätter. In this case we also make use of the fact that the answer to the problem is affirmative whenever one of the bodies is in the image of the considered operators. Similar to the approach used in [18, 60, 67], we set the stage towards a solution by proving a representation linking the complex projection and L_p-intersection body operators to the (spherical) Fourier transform. This also involves the notion of embedding into L_p (see Chapter 2). We provide a negative answer in the case $n \geq 2$, when $p > 0$ for the complex L_p-intersection and complex projection bodies, as well as a negative answer when $n \geq 3$ and $p < 0$. Finally, we can give an affirmative answer for the complex L_p-intersection bodies on $\mathbb{S}^1$-invariant convex bodies in dimension $n = 2$. The case $p = 1$ also contains the solution for the corresponding problem for complex centroid bodies.

The results of the first part can be found in [11] and the results concerning Busemann–Petty problems are to appear in [12].

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2 Background

In this chapter, we will fix notation and recall basic facts from convex geometry. Moreover, we will introduce two important concepts, that will occur at several points throughout this thesis: On the one hand, we will see that the complex structure on $\mathbb{C}^n$ also yields a finer decomposition of spherical harmonics into $\mathrm{U}(n)$-irreducible spaces and on the other hand, we will give a quick overview on the notion of embedding into L_p and its crucial connection to the Fourier transform. We refer to [16] and [62] for a general reference on convex geometry, to [3, 25, 54, 59] for more background on spherical harmonics and to [38, Ch. 6] and [57] for a detailed treatment of the notion of embedding into L_p.

2.1 Convex and Star Bodies

Let $\mathcal{K}(\mathbb{R}^n)$ denote the set of convex bodies in $\mathbb{R}^n$, that is all convex and compact subsets of $\mathbb{R}^n$. Every element of this space is uniquely determined by its support function

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}, \quad u \in \mathbb{R}^n \setminus \{0\},$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product. By one-homogeneity, this function is already determined by its values on the unit sphere $\mathbb{S}^{n-1}$ and the topology on the space $\mathcal{K}(\mathbb{R}^n)$ is uniform convergence of support functions.

Another description (unique up to translations) of a convex body K is given by its surface area measure S_K. For a Borel set $\omega \subseteq \mathbb{S}^{n-1}$ it is defined as the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K in ω. Strongly related to these measures is the mixed volume of $K, L \in \mathcal{K}(\mathbb{R}^n)$. It is defined by

$$\mathrm{V}(K, L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)dS_K(u),$$

see, e.g., [62, Sec. 5]. Clearly, $\mathrm{V}(K,\cdot)$ is monotone and $\mathrm{V}(K, K) = \mathrm{V}_n(K)$, where V_i denotes the i-dimensional volume. Minkowski's first inequality (see [62, Thm. 7.2.1]) states that

$$\mathrm{V}(K, L) \geq \mathrm{V}_n(K)^{\frac{n-1}{n}}\mathrm{V}_n(L)^{\frac{1}{n}} \tag{2.1}$$

for all $K, L \in \mathcal{K}(\mathbb{R}^n)$ with non-empty interior. Equality holds if and only if K and L are homothetic, that is, $K = \lambda L + x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^n$.

One major topic of convex geometry is the investigation of geometric operators on convex bodies. One example of these is given by the projection body ΠK of a convex body $K \in \mathcal{K}(\mathbb{R}^n)$ which is defined by (see, e.g., [16])

$$h_{\Pi K}(u) = \mathrm{V}_{n-1}(K|u^\perp), \quad u \in \mathbb{S}^{n-1},$$

where $K|u^\perp$ denotes the orthogonal projection of K onto $u^\perp$, the hyperplane perpendicular to u. This object sparked a lot of interest (see e.g. [23, 47, 53, 61, 63] and [16, Note 4.10] for a list of references) and we are going to have a look at the related *Shephard problem* (see Chapter 4).

Star bodies are considered to be the dual notion of convex bodies. These are compact subsets of $\mathbb{R}^n$, which are star shaped with respect to the origin that have a continuous radial function

$$\rho_K(u) = \sup\{\lambda \in \mathbb{R} : \lambda u \in K\}, \quad u \in \mathbb{R}^n \setminus \{0\}.$$

Let us denote the set of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_0(\mathbb{R}^n)$. Since this function is -1-homogeneous, it is already determined by its values on the unit sphere $\mathbb{S}^{n-1}$ and the topology on this space is uniform convergence of radial functions.

We will need the notion of dual mixed volume of $K, L \in \mathcal{S}_0(\mathbb{R}^n)$ which was introduced for non-zero $p \in \mathbb{R}$ in [46] as

$$\tilde{\mathrm{V}}_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^{n-p}(u) \rho_L^p(u) du.$$

Note that we always have $\tilde{\mathrm{V}}_p(K, K) = \mathrm{V}_n(K)$, by the polar formula for volume. If $L \subset M$, then $p\tilde{\mathrm{V}}_p(K, L) \leq p\tilde{\mathrm{V}}_p(K, M)$. Moreover, the dual L_p-Minkowski inequality states that (see, e.g., [62, (9.40)]) for $0 < p < n$ and $K, L \in \mathcal{S}_0(\mathbb{R}^n)$

$$\tilde{\mathrm{V}}_p(K, L) \leq \mathrm{V}_n(K)^{\frac{n-p}{n}} \mathrm{V}_n(L)^{\frac{p}{n}},$$

and for $p < 0$

$$\tilde{\mathrm{V}}_p(K, L) \geq \mathrm{V}_n(K)^{\frac{n-p}{n}} \mathrm{V}_n(L)^{\frac{p}{n}}.$$

The inequalities can be written in the following unified way for $p < 0$ or $0 < p < n$

$$p\tilde{\mathrm{V}}_p(K, L) \leq p\mathrm{V}_n(K)^{\frac{n-p}{n}} \mathrm{V}_n(L)^{\frac{p}{n}}. \tag{2.2}$$

Equality holds if and only if K and L are dilates of each other.

There are also several interesting operators in the dual setting, which were studied throughout the last decades. One important example is given by the intersection body $\mathrm{I}K \in \mathcal{S}_0(\mathbb{R}^n)$ of $K \in \mathcal{S}_0(\mathbb{R}^n)$, which was defined by Lutwak [47] as the unique, origin-symmetric star body satisfying

$$V_1(\mathrm{I}K \cap \mathrm{span}^{\mathbb{R}}\{u\}) = 2V_{n-1}(K \cap u^\perp), \qquad u \in \mathbb{S}^{n-1}, \tag{2.3}$$

where $\mathrm{span}^{\mathbb{R}}\{u\}$ is the linear span of u or equivalently by using its radial function by

$$\rho_{\mathrm{I}K}(u) = \mathrm{V}_{n-1}(K \cap u^\perp), \quad u \in \mathbb{S}^{n-1}.$$

This operator is related to the *Busemann–Petty problem* (see Chapter 4).

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A special connection between the notions of support and radial functions is given via polarity, i.e. given the polar body of $K \in \mathcal{K}_{(0)}(\mathbb{R}^n)$ (the set of all convex bodies that contain the origin in their interiors) which is defined by

$$K^\circ = \{z \in \mathbb{R}^n : \langle z, y\rangle \leq 1, \forall y \in K\}.$$

the radial function of K° is related to the support function by

$$\rho_{K^\circ}(x)^{-1} = h_K(x).$$

Note that we also have

$$\rho_{K^\circ}(x)^{-1} = \|x\|_{K^\circ},$$

where $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$ is the gauge function of K.

In the following, most of the time, we will work in the space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ endowed with a complex inner product $\cdot$, which is related to the Euclidean inner product on $\mathbb{R}^{2n}$ by

$$x \cdot u = \langle x, u\rangle + i\langle -ix, u\rangle, \quad x, u \in \mathbb{C}^n.$$

With this definition, we have $x \cdot (\lambda u) = \overline{\lambda}(x \cdot u)$, $x, u \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and consequently, by identifying $\mathbb{C} \cong \mathbb{R}^2$,

$$\langle c, x \cdot u\rangle = \langle cu, x\rangle, \quad x, u \in \mathbb{C}^n, c \in \mathbb{C}. \tag{2.4}$$

The unit disk in $\mathbb{C}$, i.e. the unit ball in the complex plane, is denoted by $\mathbb{D}$. We will identify the set of convex and star bodies respectively on $\mathbb{C}^n$ with the space of convex and star bodies on $\mathbb{R}^{2n}$ and in this sense, we can apply the results from above in dimension $2n$, when working in the space $\mathbb{C}^n$.

Also in complex vector spaces geometric operators have been considered. As an analogue of the projection body in real vector spaces, Abardia and Bernig [2] introduced the family of *complex projection bodies* which are defined by

$$h_{\Pi_C K}(u) = \frac{1}{2}\int_{\mathbb{S}^{2n-1}} h_{Cu}(v)dS_K(v), \quad K \in \mathcal{K}(\mathbb{C}^n), u \in \mathbb{S}^{2n-1}, \tag{2.5}$$

where $C \in \mathcal{K}(\mathbb{C})$ and $Cu = \{cu : c \in C\}$.

In the dual setting, the complex intersection body was recently introduced by Koldobsky, Paouris and Zymonopoulou [40]. For an $\mathbb{S}^1$-invariant star body $K \in \mathcal{S}_0(\mathbb{C}^n)$, that is, satisfying $cK = K$ for all $c \in \mathbb{S}^1 \subseteq \mathbb{C}$, the complex intersection body $\mathrm{I}_c K$ is defined as the unique $\mathbb{S}^1$-invariant star body satisfying

$$V_2(\mathrm{I}_c K \cap \mathrm{span}^{\mathbb{C}}\{u\}) = V_{2n-2}(K \cap u^{\perp,\mathbb{C}}), \quad u \in \mathbb{S}^{2n-1}, \tag{2.6}$$

where $\mathrm{span}^{\mathbb{C}}\{u\}$ denotes the complex line defined by u and $u^{\perp,\mathbb{C}}$ denotes the complex hyperplane perpendicular to $u \in \mathbb{S}^{2n-1}$ with respect to the complex inner product on $\mathbb{C}^n$. Note that, by $\mathbb{S}^1$-invariance, $\mathrm{I}_c K \cap \mathrm{span}^{\mathbb{C}}\{u\}$ is always a disk and (2.6) determines its radius.

We will introduce a new family of geometric operators in the dual setting in Chapter 3.

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2.2 Spherical Harmonics and Embedding into L_p

To exploit the complex structure in the space of spherical harmonics we will orient ourselves at the presentation in [3]. Let us first recall that the space $\mathcal{H}^{2n}$ consists of harmonic polynomials on $\mathbb{C}^n = \mathbb{R}^{2n}$ restricted to $\mathbb{S}^{2n-1}$ and naturally decomposes into $\mathrm{O}(2n)$-irreducible subspaces,

$$\mathcal{H}^{2n} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k^{2n},$$

where the subscript k indicates the degree of homogeneity of the polynomial in $\mathbb{C}^n$. We can decompose $\mathcal{H}_k^{2n}$ further into its $\mathrm{U}(n)$-irreducible subspaces $\mathcal{H}_{k,l}^{2n}$ of spherical harmonics of bi-degree (k,l). Here, a spherical harmonic $Y \in \mathcal{H}_{k,l}^{2n}$ has *bi-degree* $(k,l) \in \mathbb{N} \times \mathbb{N}$, if $Y(cu) = c^k \bar{c}^l Y(u)$ for all $u \in \mathbb{S}^{2n-1}$ and $c \in \mathbb{S}^1$.

Denoting by $\pi_{k,l}$ the orthogonal projection from $L_2(\mathbb{S}^{2n-1})$ (endowed with the standard L_2-inner product) onto $\mathcal{H}_{k,l}^{2n}$, every $f \in C(\mathbb{S}^{2n-1})$ is uniquely determined by its harmonic components $\pi_{k,l} f \in \mathcal{H}_{k,l}^{2n}$, $k,l \in \mathbb{N}$. Next, fixing a point $\bar{e} \in \mathbb{S}^{2n-1}$, there exists a unique spherical harmonic $\widetilde{P}_{k,l} \in \mathcal{H}_{k,l}^{2n}$, such that $\widetilde{P}_{k,l}(\bar{e}) = 1$ and $\widetilde{P}_{k,l}$ is invariant under the stabilizer $\mathrm{U}(n-1) \subseteq \mathrm{U}(n)$ of $\bar{e}$. The existence of $\widetilde{P}_{k,l}$ and some properties of it, that we will need later on, are the content of the following proposition from [31, Thm. 3.1(3)], see also [54, Prop. 4.2] for the formulation given here.

Proposition 2.2.1 ([31, Thm. 3.1(3)],[54, Prop. 4.2])**.** *Let $k,l \in \mathbb{N}$. Then $\mathcal{H}_{k,l}^{2n}$ contains a unique* $\mathrm{U}(n-1)$*-invariant spherical harmonic $\widetilde{P}_{k,l}$ with $\widetilde{P}_{k,l}(\bar{e}) = 1$, given by $\widetilde{P}_{k,l}(u) = P_{k,l}(\bar{e} \cdot u)$ for* Jacobi polynomial $P_{k,l} : \mathbb{D} \to \mathbb{C}$ *of order (k,l) , and satisfying*

1. *$P_{k,l}(\bar{z}) = \overline{P_{k,l}(z)}$, and*
2. *$P_{k,l}(z) = z^{|k-l|} Q_{\min\{k,l\}}(|k-l|, n-2, |z|^2)$, for all $z \in \mathbb{D}$,*

where $\{Q_l(a,b,\cdot) : l \in \mathbb{N}\}$ is the complete set of polynomials orthogonal on $[0,1]$ with respect to the L_2-inner product with weight $t^a(1-t)^b$ and $Q_l(a,b,1) = 1$, $a,b > -1$.

In analogy to their real counterparts (Legendre polynomials), Jacobi polynomials are very helpful in relation with transforms on $C(\mathbb{S}^{2n-1})$ given by a kernel ϕ, as the following *complex Funk–Hecke theorem* shows.

Theorem 2.2.2 ([54, Thm. 4.4])**.** *Suppose that $\phi \in L_2(\mathbb{D}, (1-|z|^2)^{n-2} dz)$ and let $Y_{k,l} \in \mathcal{H}_{k,l}^{2n}$. Then*

$$\int_{\mathbb{S}^{2n-1}} \phi(v \cdot u) Y_{k,l}(v) dv = \lambda_{k,l}[\phi] Y_{k,l}(u), \quad u \in \mathbb{S}^{2n-1},$$

with

$$\lambda_{k,l}[\phi] = (2n-2)\kappa_{2n-2} \int_{\mathbb{D}} \phi(z) \overline{P_{k,l}}(z) (1-|z|^2)^{n-2} dz. \tag{2.7}$$

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In general, a transform $T : C(\mathbb{S}^{2n-1}) \to C(\mathbb{S}^{2n-1})$ that satisfies

$$\pi_{k,l}(Tf) = \lambda_{k,l}[T]\pi_{k,l}f, \quad f \in C(\mathbb{S}^{2n-1}),$$

is called a *multiplier transform* with multipliers $\lambda_{k,l}[T] \in \mathbb{C}$. Note that since every $f \in C(\mathbb{S}^{2n-1})$ is completely determined by its projections $\pi_{k,l}f$, $k, l \in \mathbb{N}$, a multiplier transform is injective if and only if all of its multipliers are non-zero. We will give two examples of multiplier transforms below, which we will need in the following.

Example 2.2.3.

i) The *non-symmetric L_p-cosine transform* C_p^+, where $p > -1$ is non-zero is a well-known example of a multiplier transform. It is given by

$$(C_p^+ f)(u) = \int_{\mathbb{S}^{2n-1}\cap u^+} |\langle v, u\rangle|^p f(v)dv, \quad u \in \mathbb{S}^{2n-1}, \tag{2.8}$$

for every $f \in C(\mathbb{S}^{2n-1})$, writing $u^+ = \{v \in \mathbb{S}^{2n-1} : \langle v, u\rangle \geq 0\}$.

The multipliers of C_p^+ as a transform on a real vector space were calculated by different means by Rubin [56] (for dimension 3 and higher) and Haberl [26, Lem. 5] (also in dimension 2). Since $\mathcal{H}_{k,l}^{2n} \subseteq \mathcal{H}_{k+l}^{2n}$, the multipliers of C_p^+ when viewed as a transform on a complex vector space are equal to the corresponding real multipliers, that is,

$$\lambda_{k,l}[C_p^+] = \frac{\pi^n}{2^p} \frac{\Gamma(p+1)}{\Gamma\left(n + \frac{p+k+l}{2}\right)\Gamma\left(\frac{p-k-l}{2}+1\right)} \tag{2.9}$$

for non-zero $p > -1$ which is not an integer. In particular, $\lambda_{k,l}[C_p^+] \neq 0$ for all $k, l \in \mathbb{N}$, that is, C_p^+ is injective for $p \in (-1, \infty) \setminus \mathbb{N}$.

ii) The spherical Fourier transform $\mathbf{F}_p$ of degree p on $\mathbb{S}^{2n-1}$, $-2n < p < 0$, is defined by

$$(\mathbf{F}_q\varphi)(u) = (\widehat{\varphi_q})(u), \quad u \in \mathbb{S}^{n-1}, \tag{2.10}$$

where φ_p is the p-homogeneous extension of an even function $\varphi \in C^\infty(\mathbb{S}^{2n-1})$ and $\widehat{\cdot}$ denotes the usual Fourier transform, i.e.

$$\widehat{\phi}(x) = \int_{\mathbb{C}^n} \phi(y)e^{-i\langle x,y\rangle}dy, \quad x \in \mathbb{C}^n,$$

for every Schwartz function ϕ on $\mathbb{C}^n$. Note that, for this definition, it is important that $\widehat{\varphi_p}$ is again a smooth function (see [22], it is homogeneous of degree $-2n-p$) so that $\mathbf{F}_p$ defines a linear operator on even smooth functions. It can be extended to even distributions using that $\mathbf{F}_p$ is self-adjoint.

As $\mathbf{F}_p$ intertwines the $\mathrm{O}(2n)$-action on smooth functions, it acts as a multiplier transform on the spaces $\mathcal{H}_{k,l}^{2n}$, where $k+l$ is even. The multipliers were computed in [22, Lem. 3.4] (however, using a different parametrization),

$$\lambda_{k,l}[\mathbf{F}_p] = (-1)^{\frac{k+l}{2}} 2^{2n+p}\pi^n \frac{\Gamma\left(\frac{k+l+p}{2}+n\right)}{\Gamma\left(\frac{k+l-p}{2}\right)}. \tag{2.11}$$

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This formula extends analytically to all $p \in \mathbb{C}$ which are not even integers. Moreover, we can deduce from (2.11) that the inverse of $\mathbf{F}_p$ is given by

$$\mathbf{F}_{-2n-p}\mathbf{F}_p\varphi = (2\pi)^n\varphi. \tag{2.12}$$

Strongly connected to the spherical Fourier transform is the notion of embedding into L_p. Let K be a star body in $\mathbb{C}^n$. We say that the space $(\mathbb{C}^n, \|\cdot\|_K)$ *embeds (isometrically) in* L_p, $p > 0$, if and only if there exists a finite Borel measure μ on $\mathbb{S}^{2n-1}$, such that

$$\|x\|_K^p = \int_{\mathbb{S}^{2n-1}} |\langle x, u\rangle|^p d\mu(u), \quad x \in \mathbb{C}^n, \tag{2.13}$$

see, e.g., [38, Ch. 6] for details.

One can express (2.13) in different terms using the Fourier transform (see [38, Lem. 6.9]). With this at hand, the definition can be formally extended to negative values of p as follows.

Definition 2.2.4. [38, Def. 6.14] *Suppose that $K \in \mathcal{S}_0(\mathbb{C}^n)$ is origin-symmetric. Then the space $(\mathbb{C}^n, \|\cdot\|_K)$ is said to* embed *in L_p*

i) for $-n < p < 0$, if there exists a finite Borel measure μ on $\mathbb{S}^{2n-1}$ such that

$$\int_{\mathbb{C}^n} \|x\|_K^p \phi(x)dx = \int_{\mathbb{S}^{2n-1}} \left(\int_0^\infty r^{-p-1}\widehat{\phi}(ru)dr\right) d\mu(u), \tag{2.14}$$

for every even Schwartz function ϕ on $\mathbb{C}^n$ and

ii) for $p > 0$ that is not an even integer, if there exists a finite Borel measure μ on $\mathbb{S}^{2n-1}$ such that

$$\int_{\mathbb{C}^n} \|x\|_K^p \phi(x)dx = \frac{1}{\Gamma\left(-\frac{p}{2}\right)} \int_{\mathbb{S}^{2n-1}} \left(\int_0^\infty r^{-p-1}\widehat{\phi}(ru)dr\right) d\mu(u), \tag{2.15}$$

for every even Schwartz function ϕ on $\mathbb{C}^n$, whose Fourier transform is supported outside of the origin.

We denote the set of all star bodies that embed into L_p by $\{\hookrightarrow L_p\}$.

With the spherical Fourier transform of degree p, one can give an alternative characterization of when a body embeds into L_p. It was given in [32], see also [38, Thm. 6.10, Thm. 6.15] and [57, Thm. 5.2, Prop. 5.4] (formulated in terms of the spherical Fourier transform used here).

Proposition 2.2.5. *Let $K \in \mathcal{S}_0(\mathbb{C}^n)$ be origin-symmetric and let $-2n < p$ be non-zero. Then K embeds into L_p if and only if*

$$\frac{1}{\Gamma\left(-\frac{p}{2}\right)}\mathbf{F}_p\rho_K^{-p} \geq 0,$$

in the sense of distributions. In particular, $\frac{1}{\Gamma(-\frac{p}{2})}\mathbf{F}_p\rho_K^{-p}$ can be represented by a positive measure.

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In two dimensions, every origin-symmetric convex body satisfies the condition above. This is the content of the following lemma, which is a direct consequence of [38, Cor. 6.7 & 6.8], for $0 < p \leq 1$, and of [38, Thm. 6.17], for $-2 < p < 0$.

Lemma 2.2.6. *For every origin-symmetric $K \in \mathcal{K}_{(0)}(\mathbb{C})$, the space $(\mathbb{C}, \|\cdot\|_K)$ embeds in L_p for every non-zero $-2 < p \leq 1$.*

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3 Complex L_p-intersection bodies

For a star body K in $\mathbb{R}^n$, the intersection body $\mathrm{I}K$ (see (2.3)) was defined by Lutwak [47]. While intersection bodies played a key role in the solution of the famous Busemann–Petty problem (see e.g. [18] for an elegant unified solution and a comprehensive list of references), the origin of intersection bodies dates back to the pioneering works of Busemann on volume and area defined in Finsler spaces. Formulated in different terms, Busemann established his important convexity theorem [8], stating that the intersection body of an origin-symmetric convex body is convex, as well as his famous intersection inequality for convex bodies [9], which was extended to star bodies by Petty [52] in the following way: If K is a star body in $\mathbb{R}^n$, then

$$V_n(\mathrm{I}K)/V_n(K)^{n-1} \leq \kappa_{n-1}^n/\kappa_n^{n-2}, \tag{3.1}$$

where $\kappa_i = V_i(B^i)$ is the volume of the i-dimensional Euclidean unit ball B^i, and equality holds exactly for origin-symmetric ellipsoids.

Throughout the years, intersection bodies have sparked a lot of interest in a wide range of fields (see, e.g., [15, 21, 33–37, 45, 50, 58, 68, 70] for an overview). In particular, they played a central role in the development of the dual Brunn–Minkowski theory due to Lutwak [47], where their special role was revealed by characterizations of intersection bodies from a valuation-theoretic point of view by Ludwig [45]. In the emerging L_p-Brunn–Minkowski theory and its dual, the concept was extended to the L_p-intersection body I_pK, defined for $K \in \mathcal{S}_0(\mathbb{R}^n)$ and non-zero $p > -1$ by

$$\rho_{\mathrm{I}_pK}(u)^{-p} = \int_K |\langle x, u\rangle|^p dx, \quad u \in \mathbb{R}^n \setminus \{0\}, \tag{3.2}$$

Note that, for $p \geq 1$, this definition coincides (up to normalization) with the polar of the L_p-centroid body (first defined in [49], see also [17]). For $-1 < p < 1$, L_p-intersection bodies were studied in [6, 26, 28, 67]. Using well-known properties of the p-cosine transform, the L_p-intersection bodies relate to the classical intersection body by

$$\lim_{p \to -1^+} \left(\frac{1}{\Gamma(1+p)}\right)^{-1/p} \mathrm{I}_pK = 2 \cdot \mathrm{I}K, \quad K \in \mathcal{S}_0(\mathbb{R}^n), \tag{3.3}$$

see, e.g., [17, 24, 26], where convergence is with respect to the radial metric on $\mathcal{S}_0(\mathbb{R}^n)$. As a natural consequence, L_p-analogues of classical problems for intersection bodies were considered, leading to fruitful interactions and many new results, including a convexity theorem by Berck [6] and characterizations by Haberl and Ludwig [28]. The intersection inequality (3.1) was generalized in [49] for $p \geq 1$, leading to the discovery of an interpolating family of inequalities between the polar Busemann–Petty centroid inequality ($p = 1$) and

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the famous Blaschke–Santaló inequality ($p = \infty$). Very recently, this family of inequalities was extended in [4] to $0 < p < 1$ and to $-1 < p < 0$ with $n/|p| \in \mathbb{N}$, preceded by local inequalities including equality cases around the unit ball proved in [66].

A different generalization of intersection bodies was recently introduced by Koldobsky, Paouris and Zymonopoulou [40], based on a Busemann–Petty-type problem in complex vector spaces, first considered in [39]. Here, intersections by real hyperplanes are replaced by intersections by complex hyperplanes $u^{\perp,\mathbb{C}}$ perpendicular to $u \in \mathbb{S}^{2n-1}$ with respect to the complex inner product on $\mathbb{C}^n$, leading in a natural way to the definition of a complex intersection body (see (2.6)). Moreover, it was shown in [40] that $\mathrm{I}_c K$ is convex whenever K is the unit ball of a complex norm on $\mathbb{C}^n$, that is, when $K \in \mathcal{S}_0(\mathbb{C}^n)$ is convex and $\mathbb{S}^1$-invariant.

In this chapter, we combine the complex structure with the L_p-approach to define complex L_p-intersection bodies. We adapt a strategy used by Abardia and Bernig [2], Abardia [1] and Haberl [27] who introduced complex projection, difference and centroid bodies, respectively.

Replacing the support function $|\langle \cdot, u\rangle|$ of the interval $[-1,1]u$ by the support function of a convex body Cu, $C \in \mathcal{K}(\mathbb{C})$, then leads to the following definition.

Definition 3.1.1. *Suppose that $C \in \mathcal{K}(\mathbb{C})$ contains the origin in its relative interior, $\dim C > 0$, and $0 \neq p \in (-\dim C, 1)$. For $K \in \mathcal{S}_0(\mathbb{C}^n)$, the* complex L_p-intersection body *$\mathrm{I}_{C,p}K$ is the star body with radial function*

$$\rho_{\mathrm{I}_{C,p}K}(u)^{-p} = \int_K h_{Cu}(x)^p dx, \quad u \in \mathbb{S}^{2n-1}, \tag{3.4}$$

where $Cu = \{cu : c \in C\} \subseteq \mathbb{C}^n$.

Note that for $C = [-1,1]$ we recover the (real) L_p-intersection bodies defined in (3.2) and for $p = 1$, this equals the polar complex centroid body introduced by Haberl in [27]. For $\dim C = 2$ the range of admissible values for p extends to $(-2,1]$, see Section 3.2 for details and some basic properties of complex L_p-intersection bodies. Let us also point out that in [65] complex L_p-centroid (moment) bodies were defined in a similar way for $p \geq 1$.

As our first main result, we show that complex L_p-intersection bodies interpolate between the polar complex centroid body ($p = 1$) and the complex intersection body ($p = -2$), that is, we prove a complex analogue of (3.3), and thereby justify the name. To state the theorem, let us recall that $\mathcal{K}_{(0)}(\mathbb{C})$ denotes the set of convex bodies $K \in \mathcal{K}(\mathbb{C})$ that contain the origin in their interiors and therefore that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ implies $\dim(C) = 2$.

Theorem A. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$. Then there exists $k_C > 0$, such that*

$$\lim_{p \to -2^+} \left(\frac{1}{\Gamma(p+2)}\right)^{-1/p} \mathrm{I}_{C,p}K = k_C \cdot \mathrm{I}_c\left(K^{\mathbb{S}^1}\right),$$

for every $K \in \mathcal{S}_0(\mathbb{C}^n)$, where $K^{\mathbb{S}^1} \in \mathcal{S}_0(\mathbb{C}^n)$ is the star body with radial function

$$\rho_{K^{\mathbb{S}^1}}^{2n-2}(u) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \rho_K^{2n-2}(cu)dc, \quad u \in \mathbb{S}^{2n-1}.$$

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As before, convergence is with respect to the radial metric. Similar to the real setting, Theorem A is proved by showing by analytic continuation that a certain integral transform, used to define $\mathrm{I}_{C,p}$, converges in the strong operator topology to a multiple of the complex spherical Radon transform (see Section 3.3.1 for the definition), which defines I_c, as $p \to -2^+$. As a direct consequence, we obtain a simple formula for the multipliers of the complex spherical Radon transform, seen as a $\mathrm{U}(n)$-equivariant map on $C(\mathbb{S}^{2n-1})$, thereby partially recovering results (of higher generality) from Rubin [55] and showing that the complex intersection body map I_c, as well as the maps $\mathrm{I}_{C,p}$, are injective on $\mathbb{S}^1$-invariant star bodies. See Section 3.3.2 for the details of these calculations.

Next, we consider an analogue of the well-known convexity theorem by Busemann [8], as well as the following extension by Berck [6] to (real) L_p-intersection bodies:

Theorem 3.1.2 ([6])**.** *Let $p > -1$ be non-zero. If $K \in \mathcal{S}_0(\mathbb{R}^n)$ is convex and origin-symmetric, then $\mathrm{I}_p K$ is convex.*

Here, the condition that the body K is origin-symmetric, that is, K is the unit ball of a (real) norm cannot be omitted. Transferring the symmetry condition to the complex setting, real norms are naturally substituted by complex norms, that is, origin-symmetry is replaced by $\mathbb{S}^1$-invariance, leading to the complex convexity theorem in [40] for I_c.

Theorem 3.1.3 ([40])**.** *If $K \in \mathcal{S}_0(\mathbb{C}^n)$ is convex and $\mathbb{S}^1$-invariant, then $\mathrm{I}_c K$ is convex.*

It is a natural question to ask whether complex L_p-intersection bodies are convex. As our second main result, we extend Theorems 3.1.2 and 3.1.3 to complex L_p-intersection bodies of $\mathbb{S}^1$-invariant convex bodies in $\mathbb{C}^n$, weakening for $-2 < p < -1$ convexity to pseudo-convexity (see Section 3.4 for the definition). It is an interesting (open) question whether pseudo-convexity can be strengthened to convexity.

Theorem B. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$. If $K \in \mathcal{S}_0(\mathbb{C}^n)$ is convex and $\mathbb{S}^1$-invariant, then $\operatorname{int} \mathrm{I}_{C,p} K$ is pseudo-convex, if $-2 < p < -1$, and $\mathrm{I}_{C,p} K$ is convex, if $-1 \leq p \neq 0$.*

The proof of Theorem B is very much inspired by the techniques from [6] and relies for $p > -1$ on Theorem 3.1.2. Indeed, for $p > -1$, we actually show, using techniques from isometric embeddings into L_p-spaces, the following close relation.

Theorem C. *Suppose that $p > -1$ is non-zero and $C \in \mathcal{K}_{(0)}(\mathbb{C})$. Then there exists $d_{C,p} > 0$, such that $\mathrm{I}_{C,p} K = d_{C,p} \mathrm{I}_p K$ for every $\mathbb{S}^1$-invariant $K \in \mathcal{S}_0(\mathbb{C}^n)$.*

Let us also note that, in general, Theorem B without the assumption of $\mathbb{S}^1$-invariance is false, if $p \geq -1$, as we show in Section 3.4.4.

Turning now to inequalities for intersection bodies, our next main result relates the volume of complex L_p-intersection bodies with the volume of their real counterparts.

Theorem D. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ is origin-symmetric and $-1 \leq p < 1$ is non-zero. If $K \in \mathcal{S}_0(\mathbb{C}^n)$, then*

$$\mathrm{V}_{2n}(\mathrm{I}_{C,p} K)/\mathrm{V}_{2n}(\mathrm{I}_{C,p} B^{2n}) \leq \mathrm{V}_{2n}(\mathrm{I}_p K)/\mathrm{V}_{2n}(\mathrm{I}_p B^{2n}). \tag{3.5}$$

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Let us note that the equality cases of (3.5) can be completely described by a technical statement in terms of the convex body C and will be stated later in Section 3.5. By Theorem C, clearly $\mathbb{S}^1$-invariant bodies satisfy equality. From Theorem D, we deduce the following generalization of Busemann's intersection inequality (3.1) for $\mathrm{I}_{C,p}$ leading to affine isoperimetric inequalities in the following sense. Here, we call an ellipsoid E Hermitian, if $E = \varphi(B^{2n}) + t$ for $\varphi \in \mathrm{GL}(n, \mathbb{C}), t \in \mathbb{C}^n$.

Corollary E. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ is origin symmetric and $0 < p < 1$ or $-1 \leq p < 0$ and $n/|p| \in \mathbb{N}$. Among $K \in \mathcal{S}_0(\mathbb{C}^n)$, the ratio*

$$V_{2n}\left(\mathrm{I}_{C,p}K\right) / V_{2n}\left(K\right)^{2n+p}$$

is maximized by origin-symmetric Hermitian ellipsoids. If $p = -1$, these are the only maximizers.

Indeed, Theorem D shows that affine isoperimetric inequalities for real L_p intersection bodies are stronger than their complex counterparts. Corollary E therefore follows directly from the very recent breakthrough in [4], where the following inequality for L_p-intersection bodies was proved using methods from stochastic geometry.

Theorem 3.1.4 ([4])**.** *Suppose that $0 < p < 1$ or $-1 < p < 0$ and $n/|p| \in \mathbb{N}$. Among $K \in \mathcal{S}_0(\mathbb{R}^n)$, the ratio*

$$V_n(\mathrm{I}_p K)/V_n(K)^{n+p}$$

is maximized by origin-symmetric ellipsoids.

3.2 Definition and Basic Properties of Complex L_p-Intersection Bodies

In this section, we prove that by Definition 3.1.1 the complex L_p-intersection body map is well defined and show basic properties. We will deduce this from properties of a more general operator $\mathrm{J}_{C,p}$ on $C(\mathbb{S}^{2n-1})$.

Recalling the definition of support functions, $h_K(u) = \sup\{\langle x, u\rangle : x \in K\}$ of $K \in \mathcal{K}(\mathbb{C}^n)$, (2.4) directly implies for every $C \in \mathcal{K}(\mathbb{C})$,

$$h_{Cu}(x) = h_C(x \cdot u), \quad x, u \in \mathbb{C}^n. \tag{3.6}$$

For $K \in \mathcal{S}_0(\mathbb{C}^n)$, the complex parallel section function $\mathrm{A}^{\mathbb{C}}_{K,u}$ is defined by

$$\mathrm{A}^{\mathbb{C}}_{K,u}(z) = V_{2n-2}(K \cap \{x \in \mathbb{C}^n : x \cdot u = z\}), \quad u \in \mathbb{C}^n \setminus \{0\}, z \in \mathbb{C}. \tag{3.7}$$

Similarly, the real parallel section function $\mathrm{A}^{\mathbb{R}}_{K,u}$ is defined using intersections by real (affine) hyperplanes. $\mathrm{A}^{\mathbb{C}}_{K,u}$ can be written as complex Radon transform $\mathcal{R}^{\mathbb{C}}_u[\mathbb{1}_K]$ of the indicator function $\mathbb{1}_K$ of K, where for $\psi \in C(\mathbb{C}^n)$ with compact support,

$$\mathcal{R}^{\mathbb{C}}_u[\psi](z) = \int_{x \cdot u = z} \psi(x)dx, \quad u \in \mathbb{C}^n \setminus \{0\}, z \in \mathbb{C}.$$

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Moreover, by Fubini's theorem, $\mathrm{A}^{\mathbb{C}}_{K,u}$ can be used to express certain integrals over parallel complex hyperplanes, that is,

$$\int_K \varphi(x\cdot u)dx = \int_{\mathbb{C}} \varphi(z)\mathrm{A}^{\mathbb{C}}_{K,u}(z)dz \tag{3.8}$$

for every $\varphi \in C(\mathbb{C})$ and $u \in \mathbb{C}^n \setminus \{0\}$.

Suppose that $C \in \mathcal{K}(\mathbb{C})$ contains the origin in its relative interior, $\dim C > 0$ and let p be non-zero with $p > -\dim C$. For every $f \in C(\mathbb{S}^{2n-1})$, we define $\mathrm{J}_{C,p} f$ by

$$(\mathrm{J}_{C,p} f)(u) = \int_{\mathbb{S}^{2n-1}} h_C(v\cdot u)^p f(v)dv, \quad u \in \mathbb{S}^{2n-1}. \tag{3.9}$$

Rewriting Definition 3.1.1 in polar coordinates, shows that

$$\rho_{\mathrm{I}_{C,p}K}^{-p} = \frac{1}{2n+p}\mathrm{J}_{C,p}(\rho_K^{2n+p}), \tag{3.10}$$

for every $K \in \mathcal{S}_0(\mathbb{C}^n)$.

Lemma 3.2.1. *$\mathrm{J}_{C,p}$ is a well-defined operator on $C(\mathbb{S}^{2n-1})$, which is Lipschitz continuous with Lipschitz-constant $\|\mathrm{J}_{C,p}1\|_\infty$. Moreover, if $f \in C(\mathbb{S}^{2n-1})$ is strictly positive, so is $\mathrm{J}_{C,p} f$.*

Proof. First note that since $0 \in \operatorname{relint} C$, we have $h_C \geq 0$ and that $h_C(z) = 0$ if and only if z is orthogonal to $\operatorname{span}^{\mathbb{R}} C$. Hence, $h_C^p(v\cdot u)$ is well-defined and positive for all $v \in \mathbb{S}^{2n-1}$ that are not contained in the (proper) subspace defined by $v\cdot u \in (\operatorname{span}^{\mathbb{R}} C)^{\perp_{\mathbb{R}}}$, that is, for almost all $v \in \mathbb{S}^{2n-1}$, and we will interpret the integral in (3.9) accordingly. This readily implies that (assuming it is well-defined) $\mathrm{J}_{C,p} f$ is positive whenever f is positive.

Next, we distinguish the cases $\dim C = 2$ and $\dim C = 1$. In the first case, $\dim C = 2$, since $0 \in \operatorname{int} C$, there exist constants $d, D > 0$ such that $d\mathbb{D} \subseteq C \subseteq D\mathbb{D}$. A direct estimate then shows that

$$|h_C(v\cdot u)^p f(v)| \leq \|f\|_\infty \max\{d^p, D^p\} h_{\mathbb{D}}(v\cdot u)^p,$$

that is, by dominated convergence, $\mathrm{J}_{C,p} f$ is well-defined and continuous whenever $v \mapsto h_{\mathbb{D}}(v\cdot u)^p = |v\cdot u|^p$ is integrable with respect to the spherical Lebesgue measure on $\mathbb{S}^{2n-1}$ for some (and by invariance then every) $u \in \mathbb{S}^{2n-1}$. For this reason, let $u \in \mathbb{S}^{2n-1}$ be arbitrary and compute using polar coordinates (in $\mathbb{C}^n$) and (3.8),

$$\int_{\mathbb{S}^{2n-1}} |v\cdot u|^p dv = (2n+p)\int_{B^{2n}} |x\cdot u|^p dx = (2n+p)\int_{\mathbb{C}} |z|^p \mathrm{A}^{\mathbb{C}}_{B^{2n},u}(z)dz.$$

As $\mathrm{A}^{\mathbb{C}}_{B^{2n},u}$ is bounded by some $M > 0$ and has compact support contained in some ball $R\mathbb{D}$, $R > 0$, both uniformly in u, the latter integral can be estimated, using polar coordinates (in $\mathbb{C}$), by

$$(2n+p)\int_{\mathbb{C}} |z|^p \mathrm{A}^{\mathbb{C}}_{B^{2n},u}(z)dz \leq (2n+p)M\int_{R\mathbb{D}} |z|^p dz = (2n+p)M2\pi\int_0^R r^{p+1}dr,$$

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which is finite since $p+1 > -\dim C + 1 = -1$. We conclude that $\mathrm{J}_{C,p}f \in C(\mathbb{S}^{2n-1})$, whenever $\dim C = 2$.

In the second case, $\dim C = 1$, there exists an origin-symmetric interval $I \subseteq \mathbb{C}$ and constants $d, D > 0$ such that $dI \subseteq C \subseteq DI$, which reduces the claim to a similar calculation as in the previous case.

Finally, Lipschitz-continuity of $\mathrm{J}_{C,p}$ follows by a direct estimate, the Lipschitz constant is given by $\|\mathrm{J}_{C,p}1\|_\infty$. □

Indeed, the operators $\mathrm{J}_{C,p}f$ are jointly continuous in C and f. Before stating and proving this explicitly, we give two technical lemmas required in the proof.

Lemma 3.2.2. *Suppose that $C \in \mathcal{K}(\mathbb{C})$ with $C \neq \{0\}$ and $0 \in \operatorname{relint} C$, and let $p > -\dim C$. Then there exists $c(n,p) > 0$ such that*

$$(\mathrm{J}_{C,p}1)(u) = c(n,p) \int_{\mathbb{S}^1} h_C(v)^p dv, \quad u \in \mathbb{S}^{2n-1}.$$

Proof. A direct calculation using polar coordinates and (3.8) for the complex parallel section function $\mathrm{A}^{\mathbb{C}}_{B^{2n},u}$ yields for $u \in \mathbb{S}^{2n-1}$,

$$\begin{aligned}(\mathrm{J}_{C,p}1)(u) &= (2n+p)\int_{B^{2n}} h_C(x\cdot u)^p dx = (2n+p)\int_{\mathbb{C}} h_C(z)^p \mathrm{A}^{\mathbb{C}}_{B^{2n},u}(z)dz \\ &= (2n+p)\int_{\mathbb{S}^1} h_C(v)^p \int_0^\infty r^{p+1}\mathrm{A}^{\mathbb{C}}_{B^{2n},u}(rv)drdv \\ &= (2n+p)\int_0^\infty r^{p+1}\mathrm{A}^{\mathbb{C}}_{B^{2n},u}(r)dr \int_{\mathbb{S}^1} h_C(v)^p dv,\end{aligned}$$

where we used that $\mathrm{A}^{\mathbb{C}}_{B^{2n},u}(rv) = \mathrm{A}^{\mathbb{C}}_{B^{2n},u}(r)$ by the $\mathbb{S}^1$-invariance of B^{2n}. □

In the following lemma, convergence of convex bodies is, as always, in the Hausdorff-topology, that is, uniform convergence on $\mathbb{S}^{2n-1}$ of support functions.

Lemma 3.2.3. *Suppose that $(C_j)_{j\in\mathbb{N}} \subseteq \mathcal{K}_{(0)}(\mathbb{C})$ converges to $C_0 \in \mathcal{K}(\mathbb{C})$, with $C_0 \neq \{0\}$ and $0 \in \operatorname{relint} C_0$, and let $p > -\min\{\dim C_j : j = 0,1,\dots\}$ be non-zero. Then there exists $M > 0$ such that*

$$\int_{\mathbb{S}^1} h_{C_j}(u)^p du < M, \quad j \in \mathbb{N}. \tag{3.11}$$

Proof. First note that the integral in (3.11) is always finite as the case $n = 1$ of the previous Lemma 3.2.1 shows, that is, it remains to show that the integral can be uniformly bounded when j is large enough.

If $\dim C_0 = 2$, there exist $a, b > 0$ such that $a\mathbb{D} \subseteq C_j, C_0 \subseteq b\mathbb{D}$, and a direct estimate shows the claim. We are therefore left to prove the claim for $\dim C_0 = 1$. To this end, observe that the convergence $C_j \to C_0$ implies that $C_j \cap (-C_j) \to C_0 \cap (-C_0)$, as $j \to \infty$, and let $2d_0$ be the length of the maximal, origin-symmetric interval that is contained in $C_0 \cap (-C_0)$. As $0 \in \operatorname{relint} C_0$, $d_0 > 0$. Since

$$d_0 = \max_{u\in\mathbb{S}^1} h_{C_0\cap(-C_0)}(u),$$

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the convergence of $C_j \cap (-C_j)$ implies that for j sufficiently large, every $C_j \cap (-C_j)$ (and thus every C_j) contains an origin symmetric interval of length greater or equal $2d_0 - d_0 = d_0$.

Moreover, since C_j is a convergent sequence, there exists $D > 0$ such that $C_j \subseteq D\mathbb{D}$ for all $j \in \mathbb{N}$. Consequently, we have shown that, for every j large enough there exists $\xi_j \in \mathbb{S}^1$, such that $[-\frac{d_0}{2}\xi_j, \frac{d_0}{2}\xi_j] \subseteq C_j \subseteq D\mathbb{D}$, which implies that,

$$h_{C_j}(u)^p \leq \max\left\{ \left(\frac{d_0}{2}\right)^p |\langle \xi_j, u\rangle|^p, D^p|u|^p \right\}, \quad u \in \mathbb{S}^1. \tag{3.12}$$

Therefore the claim follows from the integrability of $|\cdot|^p$ and the fact that the (finite) integral of $|\langle w, \cdot\rangle|^p$ does not depend on the choice of $w \in \mathbb{S}^{2n-1}$. □

We are now in a position to prove the aforementioned joint continuity of $\mathrm{J}_{C,p}$.

Proposition 3.2.4. *Suppose that $p > -2$. Then the map*

$$\mathrm{J} : \{C \in \mathcal{K}(\mathbb{C}) : C \neq \{0\}, 0 \in \operatorname{relint} C, \dim C > -p\} \times C(\mathbb{S}^{2n-1}) \to C(\mathbb{S}^{2n-1}),$$

defined by $(C, f) \mapsto \mathrm{J}_{C,p} f$, is jointly continuous.

Proof. Suppose that $C_j \to C$, for $\{0\} \neq C_j, C \in \mathcal{K}(\mathbb{C})$ with $0 \in \operatorname{relint} C_j, C$ and $\dim C_j, C > -p$, and that $f_j \to f$ uniformly, $f_j, f \in C(\mathbb{S}^{2n-1})$, as $j \to \infty$. We need to show that $\mathrm{J}_{C_j,p} f_j \to \mathrm{J}_{C,p} f$ uniformly on $\mathbb{S}^{2n-1}$ as $j \to \infty$. To this end, we will first show pointwise convergence of $\mathrm{J}_{C_j,p} f_j$ and then use the Arzelà-Ascoli theorem to deduce uniform convergence.

Therefore, letting $u \in \mathbb{S}^{2n-1}$, a direct estimate yields

$$\begin{aligned} |(\mathrm{J}_{C_j,p} f_j - \mathrm{J}_{C,p} f)(u)| &\leq |(\mathrm{J}_{C_j,p} f_j - \mathrm{J}_{C_j,p} f)(u)| + |(\mathrm{J}_{C_j,p} f - \mathrm{J}_{C,p} f)(u)| \\ &\leq \|f_j - f\|_\infty |(\mathrm{J}_{C_j,p} 1)(u)| + \|f\|_\infty \int_{\mathbb{S}^{2n-1}} |(h_{C_j}^p - h_C^p)(v \cdot u)| dv. \end{aligned}$$

By Lemmas 3.2.2 and 3.2.3 the first term on the right-hand side is bounded by $M'\|f_j - f\|_\infty$, where $M' > 0$ is some constant independent of j. Moreover, arguing as in the proof of Lemma 3.2.3, see (3.12), the integrand in the second term has an integrable majorant. The uniform convergence of f_j and dominated convergence therefore imply that $\mathrm{J}_{C_j,p} f_j(u) \to \mathrm{J}_{C,p} f(u)$.

Next, since $(\mathrm{J}_{C_j,p} f_j)(u)$ is convergent for every $u \in \mathbb{S}^{2n-1}$, the sequence is uniformly bounded, that is, the family $(\mathrm{J}_{C_j,p} f_j)_{j \in \mathbb{N}}$ is pointwise bounded. In order to show equicontinuity, fix some arbitrary $u \in \mathbb{S}^{2n-1}$ and let $\eta \in \mathrm{U}(n)$ be a unitary linear map. The invariance of the Lebesgue measure on $\mathbb{S}^{2n-1}$ then yields,

$$(\mathrm{J}_{C_j,p} f_j)(\eta u) = \int_{\mathbb{S}^{2n-1}} h_{C_j}((\eta^{-1} v) \cdot u)^p f_j(v) dv = \int_{\mathbb{S}^{2n-1}} h_{C_j}(v \cdot u)^p f_j(\eta v) dv.$$

Letting $\varepsilon > 0$ arbitrary, by the equicontinuity of the f_j on the compact set $\mathbb{S}^{2n-1}$, there exists an open neighborhood U of the identity in $\mathrm{U}(n)$ such that $|f_j(v) - f_j(\eta v)| < \varepsilon$ for all $v \in \mathbb{S}^{2n-1}$, $\eta \in U$, $j \in \mathbb{N}$. Consequently, for all $\eta \in U$,

$$|(\mathrm{J}_{C_j,p} f_j)(u) - (\mathrm{J}_{C_j,p} f_j)(\eta u)| \leq \int_{\mathbb{S}^{2n-1}} h_{C_j}(v \cdot u)^p |f_j(v) - f_j(\eta v)| dv \leq \varepsilon |(\mathrm{J}_{C_j,p} 1)(u)|,$$

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which, by the previous estimate $|(\mathrm{J}_{C_j,p}1)(u)| < M'$ (independently of j) and since $\{\eta u : \eta \in U\}$ is an open neighborhood of $u \in \mathbb{S}^{2n-1}$ shows the equicontinuity of the family $(\mathrm{J}_{C_j,p} f_j)_{j\in\mathbb{N}}$.

The Arzelà-Ascoli theorem thus implies the existence of a uniformly convergent subsequence $(\mathrm{J}_{C_{j_k},p} f_{j_k})_{k\in\mathbb{N}}$. As the original sequence converges pointwise to $\mathrm{J}_{C,p} f$, we obtain $\mathrm{J}_{C_{j_k},p} f_{j_k} \to \mathrm{J}_{C,p} f$, and a standard argument (that is, starting with an arbitrary subsequence) implies that $\mathrm{J}_{C_j,p} f_j \to \mathrm{J}_{C,p} f$, which completes the proof. □

Note that, for $C \in \mathcal{K}_{(0)}(\mathbb{C})$, Proposition 3.2.4 can be proved directly by showing local Lipschitz-continuity of $\mathrm{J}_{C,p} f$ as a function in C.

It follows now directly that the complex L_p-intersection body body is well defined and continuous.

Corollary 3.2.5. *Suppose that $p > -2$. Then the map*

$$\mathrm{I} : \{C \in \mathcal{K}(\mathbb{C}) : C \neq \{0\}, 0 \in \operatorname{relint} C, \dim C > -p\} \times \mathcal{S}_0(\mathbb{C}^n) \to \mathcal{S}_0(\mathbb{C}^n),$$

defined by $(C, K) \mapsto \mathrm{I}_{C,p} K$, is well-defined and jointly continuous.

Proof. This follows directly from (3.10), Lemma 3.2.1, Proposition 3.2.4 and the fact that the maps $t \mapsto t^{2n+p}$ and $t \mapsto t^{-1/p}$ are locally Lipschitz-continuous for $t > 0$. Note that Lemma 3.2.1 asserts that $(\mathrm{J}_{C,p} \rho_K^{2n+p})^{-1/p}$ is positive and continuous and therefore a radial function of a star body in $\mathcal{S}_0(\mathbb{C}^n)$. □

Note that the proofs of Lemma 3.2.1 and Corollary 3.2.5 imply that for fixed $C \in \mathcal{K}(\mathbb{C})$ and non-zero $p > -\dim C$, the operator $\mathrm{I}_{C,p} : \mathcal{S}_0(\mathbb{C}^n) \to \mathcal{S}_0(\mathbb{C}^n)$ is locally Lipschitz-continuous.

In view of its importance for the real L_p-intersection body (see [28, 45]), we close the section with the following corresponding property for complex L_p-intersection bodies. The proof is a direct computation and will be omitted.

Lemma 3.2.6. *Suppose that $C \in \mathcal{K}(\mathbb{C})$ contains the origin in its relative interior and let $p > -\dim C$ be non-zero. Then $\mathrm{I}_{C,p} : \mathcal{S}_0(\mathbb{C}^n) \to \mathcal{S}_0(\mathbb{C}^n)$ is a $\mathrm{GL}(n, \mathbb{C})$-contravariant valuation with respect to L_{-p}-radial addition, that is,*

$$\rho^{-p}_{\mathrm{I}_{C,p}(K\cup L)} + \rho^{-p}_{\mathrm{I}_{C,p}(K\cap L)} = \rho^{-p}_{\mathrm{I}_{C,p}(K)} + \rho^{-p}_{\mathrm{I}_{C,p}(L)}, \quad K, L \in \mathcal{S}_0(\mathbb{C}^n),$$

and

$$\mathrm{I}_{C,p}(\Theta K) = |\det \Theta|^{-2/p} \Theta^{-*} \mathrm{I}_{C,p}(K), \quad K \in \mathcal{S}_0(\mathbb{C}^n), \Theta \in \mathrm{GL}(n, \mathbb{C}),$$

where $\Theta^{-*} = (\Theta^*)^{-1}$ denotes the inverse of the Hermitian adjoint $\Theta^* = \overline{\Theta}^T$.

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3.3 Proof of Theorem A and Injectivity

3.3.1 Proof of Theorem A

In this section, we will use several results from the previous section to give a proof of Theorem A, that is, to compute the limit of (a normalization of) $\mathrm{I}_{C,p}K$ for $p \to -2^+$, where $C \in \mathcal{K}_{(0)}(\mathbb{C})$ and $K \in \mathcal{S}_0(\mathbb{C}^n)$. To this end, we will first show a similar result for the operator $\mathrm{J}_{C,p}$ and then deduce from it Theorem A.

A key ingredient of the proof of the statements in this section will be the well-known fact that the familiy of distributions r_+^q,

$$\phi \mapsto \langle r_+^q, \phi\rangle = \int_0^\infty r^q \phi(r) dr \tag{3.13}$$

is analytic for every $q \in \mathbb{C}$ with $\Re q > -1$, and admits a meromorphic extension, with poles at $-\mathbb{N}$ (see, e.g., [19, Sec. 3.2]). Consequently,

$$\lim_{q\to 0} \int_0^\infty r^q \phi(r) dr = \int_0^\infty \phi(r) dr$$

and, as can be directly checked,

$$\lim_{q\to -1^+} \frac{1}{\Gamma(q+1)} \int_0^\infty r^q \phi(r) dr = \phi(0), \tag{3.14}$$

for every Schwartz function ϕ on $\mathbb{R}$. Moreover, since all distributions r_+^q, $\Re q > -1$, and their limit distribution can be applied to continuous functions with compact support, (3.14) holds for all $\phi \in C(\mathbb{R})$ with compact support (see, e.g., [30, Thm. 2.1.8]).

As the following proposition shows, the normalized operators $\mathrm{J}_{C,p}$ converge to a multiple of the *complex spherical Radon transform* $\mathcal{R}_c$,

$$(\mathcal{R}_c f)(u) = \int_{\mathbb{S}^{2n-1}\cap\{v\cdot u=0\}} f(v) dv, \quad u \in \mathbb{S}^{2n-1},$$

where $f \in C(\mathbb{S}^{2n-1})$.

Proposition 3.3.1. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$. Then there exists $k'_C > 0$ such that $\frac{1}{\Gamma(p+2)}\mathrm{J}_{C,p}$ converges to $k'_C\mathcal{R}_c$ in the strong operator topology, as $p \to -2^+$, that is,*

$$\frac{1}{\Gamma(p+2)} \mathrm{J}_{C,p} f \to k'_C \mathcal{R}_c f, \quad p \to -2^+, \tag{3.15}$$

uniformly on $\mathbb{S}^{2n-1}$ for every $f \in C(\mathbb{S}^{2n-1})$.

Proof. Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ and $f \in C(\mathbb{S}^{2n-1})$. In order to prove (3.15), we will first show that $\frac{1}{\Gamma(p+2)}\mathrm{J}_{C,p} f$ converges pointwise on $\mathbb{S}^{2n-1}$ and then use the Arzelà-Ascoli theorem to deduce uniform convergence.

To this end, we use polar coordinates (in $\mathbb{C}^n$), Fubini's theorem and again polar coordinates (in $\mathbb{C}$) to rewrite $\mathrm{J}_{C,p}f(u)$ for $u \in \mathbb{S}^{2n-1}$,

$$\begin{aligned}(\mathrm{J}_{C,p}f)(u) &= (2n+p)\int_{B^{2n}\setminus\{0\}} h_C(x\cdot u)^p f\left(\frac{x}{\|x\|}\right)dx\\ &= (2n+p)\int_{\mathbb{C}} h_C(z)^p \int_{x\cdot u=z} f\left(\frac{x}{\|x\|}\right)\mathbb{1}_{B^{2n}\setminus\{0\}}(x)dxdz\\ &= (2n+p)\int_0^\infty r^{p+1}\int_{\mathbb{S}^1} h_C(v)^p\int_{x\cdot u=rv} f\left(\frac{x}{\|x\|}\right)\mathbb{1}_{B^{2n}\setminus\{0\}}(x)dxdvdr.\end{aligned}$$

Letting $g_{u,v}(r) = \int_{x\cdot u=rv} f\left(\frac{x}{\|x\|}\right)\mathbb{1}_{B^{2n}\setminus\{0\}}(x)dx$, and using again Fubini's theorem, we arrive at

$$(\mathrm{J}_{C,p}f)(u) = (2n+p)\int_{\mathbb{S}^1} h_C(v)^p\int_0^\infty r^{p+1}g_{u,v}(r)drdv. \tag{3.16}$$

Next, noting that $g_{u,v}$ is continuous (by dominated convergence) and has compact support, we deduce by (3.14),

$$\lim_{p\to-2^+}\frac{1}{\Gamma(p+2)}\int_0^\infty r^{p+1}g_{u,v}(r)dr = g_{u,v}(0)$$

for every $v\in\mathbb{S}^1$ and $u\in\mathbb{S}^{2n-1}$. Consequently, the integrand in (3.16), normalized by $\Gamma(p+2)$, converges pointwise to $h_C(v)^{-2}g_{u,v}(0)$. As there exists $d\in(0,1)$ such that $d\mathbb{D}\subseteq C$, that is, $h_C(v)^p \le d^p \le d^{-2}$ for every $v\in\mathbb{S}^1$ and $-2<p<0$, and

$$\begin{aligned}\frac{1}{\Gamma(p+2)}\int_0^\infty r^{p+1}|g_{u,v}(r)|dr &\le \frac{\|f\|_\infty}{\Gamma(p+2)}\int_0^1 r^{p+1}\int_{x\cdot u=rv}\mathbb{1}_{B^{2n}\setminus\{0\}}(x)dxdr\\ &\le \frac{\|f\|_\infty}{\Gamma(p+2)(p+2)}\kappa_{2n-2} = \frac{\|f\|_\infty}{\Gamma(p+3)}\kappa_{2n-2},\end{aligned}$$

where $\Gamma(p+3)$ is continuous for $p\ge-2$, the integrand in (3.16) is bounded uniformly in p. Dominated convergence thus implies that

$$\begin{aligned}\lim_{p\to-2^+}\frac{1}{\Gamma(p+2)}(\mathrm{J}_{C,p}f)(u) &= (2n-2)\int_{\mathbb{S}^1} h_C(v)^{-2}g_{u,v}(0)dv\\ &= (2n-2)\int_{\mathbb{S}^1} h_C(v)^{-2}dv\int_{x\cdot u=0} f\left(\frac{x}{\|x\|}\right)\mathbb{1}_{B^{2n}\setminus\{0\}}(x)dx.\end{aligned}$$

Letting $k'_C = \int_{\mathbb{S}^1} h_C(v)^{-2}dv = 2V_2(C^\circ)$ and using polar coordinates in $x\cdot u = 0$, the latter expression is equal to

$$k'_C(2n-2)\int_{\mathbb{S}^{2n-1}\cap\{v\cdot u=0\}} f(v)dv\int_0^1 r^{2n-3}dr = k'_C(\mathcal{R}_cf)(u),$$

that is, $\frac{1}{\Gamma(p+2)}(\mathrm{J}_{C,p}f)(u) \to k'_C(\mathcal{R}_cf)(u)$, $u\in\mathbb{S}^{2n-1}$, $p\to-2^+$, as claimed.

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Next, since $\frac{1}{\Gamma(p+2)}|(\mathrm{J}_{C,p}f)(u)|$ is convergent for every $u \in \mathbb{S}^{2n-1}$, the sequence is bounded, that is, the family $(\frac{1}{\Gamma(p+2)}\mathrm{J}_{C,p}f)_{p>-2}$ is pointwise bounded. In order to show equicontinuity, we proceed as in the proof of Proposition 3.2.4 to conclude that for every $\varepsilon > 0$ and $u \in \mathbb{S}^{2n-1}$ there exists an open neighborhood U of u such that

$$|(\mathrm{J}_{C,p}f)(u) - (\mathrm{J}_{C,p}f)(w)| \le \varepsilon|(\mathrm{J}_{C,p}1)(u)|, \quad w \in U.$$

Hence, since $\frac{1}{\Gamma(p+2)}|(\mathrm{J}_{C,p}1)(u)|$ is convergent (for $p \to -2^+$) and thus bounded, the family $(\frac{1}{\Gamma(p+2)}\mathrm{J}_{C,p}f)_{p>-2}$ is equicontinuous.

The Arzelà-Ascoli theorem therefore implies the existence of a uniformly convergent subsequence, which, by pointwise convergence, must converge to $k_C'\mathcal{R}_cf$. A standard argument, finally, shows the uniform convergence of the whole sequence, which completes the proof. □

Theorem A is now a consequence of Proposition 3.3.1, since, by polar coordinates and $\mathbb{S}^1$-invariance, the radial function of the complex intersection body I_cK satisfies

$$\rho_{\mathrm{I}_cK}(u) = \left(\frac{1}{(2n-2)\pi}\mathcal{R}_c\rho_K^{2n-2}(u)\right)^{1/2}, \quad u \in \mathbb{S}^{2n-1}. \tag{3.17}$$

Proof of Theorem A. First observe that Proposition 3.3.1 readily implies that whenever $f_p \to f$ uniformly as $p \to -2^+$, $f_p, f \in C(\mathbb{S}^{2n-1})$, then $\frac{1}{\Gamma(p+2)}\mathrm{J}_{C,p}f_p$ converges uniformly to $k_C'\mathcal{R}_cf$. Indeed,

$$\begin{aligned}\left\|\frac{\mathrm{J}_{C,p}f_p}{\Gamma(p+2)} - k_C'\mathcal{R}_cf\right\|_\infty &\le \frac{\|\mathrm{J}_{C,p}(f_p-f)\|_\infty}{\Gamma(p+2)} + \left\|\frac{\mathrm{J}_{C,p}f}{\Gamma(p+2)} - k_C'\mathcal{R}_cf\right\|_\infty \\ &\le \|f_p - f\|_\infty\frac{\|\mathrm{J}_{C,p}1\|_\infty}{\Gamma(p+2)} + \left\|\frac{\mathrm{J}_{C,p}f}{\Gamma(p+2)} - k_C'\mathcal{R}_cf\right\|_\infty,\end{aligned}$$

where the right-hand side converges to zero by the uniform convergence of f_p to f and since $\frac{\|\mathrm{J}_{C,p}1\|_\infty}{\Gamma(p+2)}$ is bounded by Proposition 3.3.1 (for the first summand), and by Proposition 3.3.1 (for the second summand).

Next, note that for $K \in \mathcal{S}_0(\mathbb{C}^n)$ there exist $d > 0$ and $D > 1$ such that $d < \rho_K(u) < D$ for all $u \in \mathbb{S}^{2n-1}$. Since the map $p \mapsto t^{2n+p}$, $t > 0$, is differentiable with derivative $t^{2n+p}\ln(t)$, the mean value theorem of calculus implies for $-2 < p < 0$ and $u \in \mathbb{S}^{2n-1}$ that

$$\begin{aligned}|\rho_K(u)^{2n+p} - \rho_K(u)^{2n-2}| &\le \max_{q\in[-2,p]}\rho_K(u)^{2n+q}|\ln(\rho_K(u))||p+2| \\ &\le D^{2n}\max\{|\ln(d)|, |\ln(D)|\}|p+2|,\end{aligned}$$

that is, $\rho_K^{2n+p} \to \rho_K^{2n-2}$ uniformly as $p \to -2^+$. Hence, by (3.10) and the first part of the proof,

$$\lim_{p\to-2^+}\frac{1}{\Gamma(p+2)}\rho_{\mathrm{I}_{C,p}K}^{-p} = \lim_{p\to-2^+}\frac{\mathrm{J}_{C,p}\rho_K^{2n+p}}{(2n+p)\Gamma(p+2)} = \frac{k_C'}{2n-2}\mathcal{R}_c\rho_K^{2n-2}$$

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uniformly on $\mathbb{S}^{2n-1}$. Moreover, a direct estimate using $\rho_K(u) \in [d, D]$ shows that $\mathcal{R}_c\rho_K^{2n-2}(u) \in (2n-2)\kappa_{2n-2}[d^{2n-2}, D^{2n-2}]$. Consequently, by uniform convergence, there exist constants $d', D' > 0$ such that

$$d' < \frac{1}{\Gamma(p+2)}\rho_{\mathrm{I}_{C,p}K}(u)^{-p} < D', \quad u \in \mathbb{S}^{2n-1},$$

for all $p < 0$ sufficiently close to -2. Repeating the above argument for the differentiable function $p \mapsto t^{-1/p}$, $t > 0$, and using that the functions $t \mapsto t^{-1/p}$, $t \in [d', D']$ and $-2 < p < -1$, are Lipschitz-continuous with Lipschitz constants uniformly bounded by $d'^{-1/2}$, then yields

$$\lim_{p\to -2^+} \frac{\rho_{\mathrm{I}_{C,p}K}}{\Gamma(p+2)^{-1/p}} = \left(\frac{k'_C}{2n-2}\mathcal{R}_c\rho_K^{2n-2}\right)^{1/2} \tag{3.18}$$

uniformly on $\mathbb{S}^{2n-1}$.

Finally, as it is a direct computation that $\mathcal{R}_c f = \mathcal{R}_c f^{\mathbb{S}^1}$, where for $f \in C(\mathbb{S}^{2n-1})$,

$$f^{\mathbb{S}^1}(u) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(cu)dc, \quad u \in \mathbb{S}^{2n-1},$$

and by (3.17), the right-hand side of (3.18) is equal to $(\pi k'_C)^{1/2}\rho_{\mathrm{I}_c K^{\mathbb{S}^1}}$, which completes the proof by setting $k_C = (\pi k'_C)^{1/2}$. □

3.3.2 Spherical Harmonics and Injectivity

In this section, we will use spherical harmonics to show a criterion for the operators $\mathrm{J}_{C,p}$ to be injective and deduce that every $\mathrm{J}_{C,p}$ is injective on $\mathbb{S}^1$-invariant continuous functions. As a by-product, we will calculate the multipliers of $\mathrm{J}_{C,p}$ in terms of the Fourier coefficients of h_C^p, which leads (by taking limits) to a closed formula for the multipliers of the complex spherical Radon transform $\mathcal{R}_c$. All results for $\mathrm{J}_{C,p}$ directly translate to $\mathrm{I}_{C,p}$.

We are now ready to state the main proposition to prove injectivity of $\mathrm{J}_{C,p}$, calculating the multipliers of the transforms $\mathrm{J}_{C,p}$. In the statement of the proposition, we use the notation of the k^{th} *Fourier coefficient* $c_k(f)$ of $f \in C(\mathbb{S}^1)$,

$$c_0(f) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(c)dc \quad \text{and} \quad c_k(f) = \frac{1}{\pi}\int_{\mathbb{S}^1} f(c)c^k dc, \quad k \in \mathbb{Z}\setminus\{0\}. \tag{3.19}$$

Proposition 3.3.2. *Suppose that $C \in \mathcal{K}(\mathbb{C})$, with $C \neq \{0\}$ and $0 \in \operatorname{relint} C$, and let $p > -\dim C$ be non-zero. Then the multipliers of the transform $\mathrm{J}_{C,p}$ are given for $k, l \in \mathbb{N}$ by*

$$\lambda_{k,l}[\mathrm{J}_{C,p}] = \begin{cases} c_0(h_C^p) 2\alpha_{k,l}^{(n,p)}, & k = l, \\ c_{l-k}(h_C^p)\alpha_{k,l}^{(n,p)}, & k \neq l, \end{cases} \tag{3.20}$$

where

$$\alpha_{k,l}^{(n,p)} = \pi^n \frac{\Gamma\left(\frac{p+k-l}{2}+1\right)\Gamma\left(\frac{p-k+l}{2}+1\right)}{\Gamma\left(\frac{p+k+l}{2}+n\right)\Gamma\left(\frac{p-k-l}{2}+1\right)}.$$

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Proof. By (2.7), we calculate using polar coordinates and the properties of Jacobi polynomials from Proposition 2.2.1,

$$\frac{1}{(2n-2)\kappa_{2n-2}}\lambda_{k,l}[\mathrm{J}_{C,p}] = \int_{\mathbb{S}^1}\int_0^1 h_C(c)^p\overline{P_{k,l}}(rc)(1-r^2)^{n-2}r^{p+1}drdc$$
$$= \int_{\mathbb{S}^1} h_C(c)^p c^{l-k}dc \int_0^1 Q_{\min\{k,l\}}(|k-l|, n-2, r^2)(1-r^2)^{n-2}r^{p+1+|k-l|}dr,$$

where the second integral does not depend on C anymore. In particular, when $p > -1$, we can repeat the argument for $h_C^p(z)$ replaced by the kernel $h_{[-1,1]}^p(z)\mathbb{1}_{\Re z\geq 0}$ of the non-symmetric L_p-cosine transform C_p^+ to obtain

$$\lambda_{k,l}[\mathrm{J}_{C,p}] = \frac{\int_{\mathbb{S}^1} h_C(c)^p c^{l-k}dc}{\int_{\mathbb{S}^1}(\Re c)^p \mathbb{1}_{\Re c\geq 0}c^{l-k}dc}\lambda_{k,l}[C_p^+] = \frac{c_{l-k}(h_C^p)}{c_{l-k}((\Re c)^p\mathbb{1}_{\Re c\geq 0})}\lambda_{k,l}[C_p^+].$$

Next, (2.9) and direct computations using identities for the reciprocal beta function yield for $k \neq l$,

$$c_{l-k}((\Re c)^p\mathbb{1}_{\Re c\geq 0}) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(t)^p e^{i(l-k)t}dt = \frac{\Gamma(p+1)}{2^p\Gamma\left(\frac{p+k-l}{2}+1\right)\Gamma\left(\frac{p-k+l}{2}+1\right)}$$

and for $k = l$,

$$c_0((\Re c)^p\mathbb{1}_{\Re c\geq 0}) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\cos(t)^p dt = \frac{\Gamma(p+1)}{2^{p+1}\Gamma\left(\frac{p}{2}+1\right)^2}.$$

This proves the claim when $p > -1$. Noting, finally, that both sides of (3.20) are analytic functions in p (for $\Re(p) > -\dim C$) that coincide on the set $(-1, 0)$ and therefore on their domains, completes the proof. □

Note that $\alpha_{k,l}^{(n,p)} \neq 0$ for all $k, l \in \mathbb{N}$ and non-zero $p > -2$, $p \notin \mathbb{Z}$, as the gamma function has no zeros and its poles are exactly the non-positive integers. In particular, we have shown the following.

Corollary 3.3.3. *Suppose that $C \in \mathcal{K}(\mathbb{C})$, with $C \neq \{0\}$ and $0 \in \operatorname{relint} C$, and let $p > -\dim C$, $p \notin \mathbb{Z}$, be non-zero. Then $\mathrm{J}_{C,p}$ is injective if and only if $c_k(h_C^p) \neq 0$ for all $k \in \mathbb{Z}$.*

We turn now to $\mathbb{S}^1$-invariant functions on $\mathbb{S}^{2n-1}$. Here, the computation simplifies to the case $C = \mathbb{D}$, since

$$2\pi(\mathrm{J}_{C,p}f^{\mathbb{S}^1})(u) = \int_{\mathbb{S}^{2n-1}} h_C(v\cdot u)^p \int_{\mathbb{S}^1} f(cv)dcdv \tag{3.21}$$
$$= \int_{\mathbb{S}^{2n-1}}\int_{\mathbb{S}^1} h_C(\overline{c}(w\cdot u))^p dc f(w)dw = 2\pi(\mathrm{J}_{d\mathbb{D},p}f)(u) \tag{3.22}$$

for every $f \in C(\mathbb{S}^{2n-1})$, as the inner integral on the right-hand side can be written as $h_{d\mathbb{D}}(w\cdot u)^p$ for some $d > 0$ not depending on f.

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This can also be seen in terms of spherical harmonics, since restricting to $\mathbb{S}^1$-invariant functions corresponds exactly to restricting to the spaces $\mathcal{H}^{2n}_{k,k}$, $k \in \mathbb{N}$. Indeed, the definition of bi-degree directly implies that a function $f \in C(\mathbb{S}^{2n-1})$ is $\mathbb{S}^1$-invariant if and only if $\pi_{k,l} f = 0$, $k \neq l \in \mathbb{N}$, see, e.g., [3, Lem. 4.8]. Consequently, $\mathrm{J}_{C,p}$ is completely determined on $\mathbb{S}^1$-invariant functions by $\lambda_{k,k}[\mathrm{J}_{C,p}]$, $k \in \mathbb{N}$, that is, using (3.20), by $c_0(h^p_C) \neq 0$. Thus, we conclude the following.

Corollary 3.3.4. *Suppose that $C \in \mathcal{K}(\mathbb{C})$, with $C \neq \{0\}$ and $0 \in \operatorname{relint} C$, and let $p > -\dim C$ be non-zero. Then $\mathrm{J}_{C,p}$ is injective on $\mathbb{S}^1$-invariant functions in $C(\mathbb{S}^{2n-1})$.*

Finally, as we have seen in Section 3.3.1, letting $p \to -2^+$, the operators $\mathrm{J}_{C,p}$ converge appropriately normalized (in the strong operator topology) to the complex spherical Radon transform $\mathcal{R}_c$ for which $\mathcal{R}_c f = \mathcal{R}_c f^{\mathbb{S}^1}$ holds. Consequently, the multipliers of $\mathcal{R}_c$ can be directly calculated from (3.20) by taking the limit.

Proposition 3.3.5. *The multipliers of the complex spherical Radon transform $\mathcal{R}_c$ are given by $\lambda_{k,l}[\mathcal{R}_c] = 0$ for $k \neq l$ and*

$$\lambda_{k,k}[\mathcal{R}_c] = (-1)^k 2\pi^{n-1} \frac{k!}{(n+k-2)!}, \quad k \in \mathbb{N}.$$

In particular, the complex intersection body map I_c is injective.

3.4 (Pseudo-)Convexity

In this section we first collect the definition and basic properties of pseudo-convex sets that are used to prove Theorem B. As a general reference for pseudo-convex sets and plurisubharmonic functions, we refer to [29, 43].

Next, we prove Theorem B following the ideas of Berck [6] for his convexity theorem for L_p-intersection bodies. More precisely, we first establish concavity properties for complex p-moments of convex bodies using inequalities of Brunn–Minkowski type, which are then used to show that the reciprocal radial functions of $\mathrm{I}_{C,p}K$ satisfy the sufficient conditions for pseudo-convexity in Theorem 3.4.2, where $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ is $\mathbb{S}^1$-invariant and has a smooth boundary. The general case then follows by approximation.

In the final part of this section we give examples in the range $-1 < p < 1$ of convex bodies K that are not $\mathbb{S}^1$-invariant, such that $\mathrm{I}_{C,p}K$ is not convex for some $C \in \mathcal{K}_{(0)}(\mathbb{C})$, showing that $\mathbb{S}^1$-invariance is a necessary condition.

3.4.1 Basic notions

First, recall that a function $\varphi : \Omega \to [-\infty, \infty)$, defined on an open subset $\Omega \subseteq \mathbb{C}^n$, is called *plurisubharmonic*, if

- φ is upper semi-continuous;
- for all $u, v \in \mathbb{C}^n$, the map $z \mapsto \varphi(u + zv)$ is subharmonic where it is defined,

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see, e.g., [29, Def. 1.6.1 and 2.6.1]. Examples are given by all subharmonic and, hence, by all convex functions on $\mathbb{C}^n$. Using this notion, pseudo-convex sets are defined as follows.

Definition 3.4.1 ([29, Def. 2.6.8])**.** *An open, connected set $K \subseteq \mathbb{C}^n$ is* pseudo-convex*, if there exists a continuous, plurisubharmonic function φ in K such that the sets*

$$\{z \in K : \varphi(z) < c\}, \quad c \in \mathbb{R},$$

are all relatively compact in K.

Note that this is also called *Hartogs pseudo-convex* and equivalent to K being a domain of holomorphy or holomorphically convex. For sets with more regular boundary, the *Levi condition* yields an equivalent statement, which is more accessible:

Theorem 3.4.2 ([29, Thm. 2.6.12])**.** *Suppose that $K \subseteq \mathbb{C}^n$ is an open set with C^2-boundary, given by*

$$K = \{u \in \mathbb{C}^n : \rho(u) < 0\},$$

where $\rho : \mathbb{C}^n \to \mathbb{R}$ is C^2 in a neighborhood of $\operatorname{cl} K$ and $\nabla\rho \neq 0$ on $\operatorname{bd} K$. Then K is pseudo-convex, if and only if

$$\Delta_z \rho(u + zv)|_{z=0} \geq 0,$$

for all $u \in \operatorname{bd} K$ and $v \in \mathbb{C}^n$ with $\nabla\rho(u) \cdot v = 0$.

The next theorem shows how to use approximation by sets with smooth boundaries to extend our results to sets with arbitrary boundaries.

Theorem 3.4.3 ([29, Thm. 2.6.9])**.** *Suppose that $K_i \subseteq \mathbb{C}^n$, $i \in I$, are pseudo-convex sets for an index set I. Then the interior of $\bigcap_{i\in I} K_i$ is pseudo-convex.*

3.4.2 Brunn–Minkowski inequalities for complex moments

For $K \in \mathcal{K}(\mathbb{C}^n)$, $v \in \mathbb{C}^n \setminus \{0\}$ and $p \geq 0$, the p-th asymmetric complex moment of K is defined by

$$\mathcal{M}^{\Re,+}_{p,v}(K) = \int_{K\cap v^+} \Re(x \cdot v)^p dx,$$

where $v^+ = \{x \in \mathbb{C}^n : \Re(x\cdot v) \geq 0\}$. Note that clearly $\mathcal{M}^{\Re,+}_{p,v}(K)$ is $(2n+p)$-homogeneous. A direct application of the Prékopa–Leindler inequality yields the following Brunn–Minkowski-type inequality for $\mathcal{M}^{\Re,+}_{p,v}$ by Berck [6].

Proposition 3.4.4. *Suppose that $p \geq 0$ and $v \in \mathbb{C}^n \setminus \{0\}$. Then*

$$\mathcal{M}^{\Re,+}_{p,v}(K_0 + K_1)^{\frac{1}{2n+p}} \geq \mathcal{M}^{\Re,+}_{p,v}(K_0)^{\frac{1}{2n+p}} + \mathcal{M}^{\Re,+}_{p,v}(K_1)^{\frac{1}{2n+p}},$$

for every $K_0, K_1 \in \mathcal{K}(\mathbb{C}^n)$, such that $K_0 \cap v^+, K_1 \cap v^+ \neq \emptyset$.

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Like the classical Brunn–Minkowski-inequality, Proposition 3.4.4 directly implies an analogue of Brunn's concavity theorem for the moments of parallel sections by complex hyperplanes $H_{u,z} = \{x \in \mathbb{C}^n : x \cdot u = z\}$, that is, for

$$\mathcal{M}_{p,v}^{\Re,+,u}(K,z) = \int_{v^+ \cap (K \cap H_{u,z})} \Re(x \cdot v)^p dx,$$

where $u, v \in \mathbb{C}^n \setminus \{0\}$ are not contained in the same complex line.

Corollary 3.4.5. *Suppose that $K \in \mathcal{K}(\mathbb{C}^n)$, $p > 0$ and let $u, v \in \mathbb{C}^n \setminus \{0\}$ with $v \notin \operatorname{span}^{\mathbb{C}}\{u\}$. Then the function*

$$z \mapsto \mathcal{M}_{p,v}^{\Re,+,u}(K,z)^{\frac{1}{2n-2+p}}$$

is concave on the set $\{z \in \mathbb{C} : K \cap H_{u,z} \cap v^+ \neq \emptyset\}$.

Proof. This is a direct consequence of Proposition 3.4.4 and the fact that $(1-\lambda)(K \cap H_{u,z_0}) + \lambda(K \cap H_{u,z_1}) \subseteq K \cap H_{u,(1-\lambda)z_0+\lambda z_1}$ by convexity. □

As a concave function on its compact support, $\mathcal{M}_{p,v}^{\Re,+,u}(K,\cdot)^{\frac{1}{2n-2+p}}$ thus attains a maximum, which is, by the monotonicity of $t \mapsto t^{2n-2+p}$, also true for $\mathcal{M}_{p,v}^{\Re,+,u}(K,\cdot)$. The following lemma shows that by replacing v by $v + \lambda u$, with $\lambda \in \mathbb{C}$ suitable, we can ensure that, for smooth K, $z = 0$ is a critical point of $\mathcal{M}_{p,v}^{\Re,+,u}(K,\cdot)$ and thus its maximum. Note that for smooth K, $\mathcal{M}_{p,v+\lambda u}^{\Re,+,u}(K,\cdot)$ is differentiable at $z = 0$, since the complex parallel section function of K is smooth at $z = 0$ (see, e.g., [38, Lem. 2.4]).

In the proof, we denote by $H_{u,t}^{\mathbb{R}}$ the (real) hyperplane $\{x \in \mathbb{C}^n : \langle x, u\rangle = t\}$, $u \in \mathbb{C}^n \setminus \{0\}, t \in \mathbb{R}$.

Lemma 3.4.6. *Suppose that $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ is $\mathbb{S}^1$-invariant and has smooth boundary, $p > 0$ and let $u, v \in \mathbb{C}^n \setminus \{0\}$ with $v \notin \operatorname{span}^{\mathbb{C}}\{u\}$. Then there exists $\lambda \in \mathbb{C}$ such that*

$$\nabla_z \mathcal{M}_{p,v+\lambda u}^{\Re,+,u}(K,z)\Big|_{z=0} = 0.$$

Proof. Without loss of generality, we may assume that $v \cdot u = 0$. Letting $K_0 = K \cap H_{iu,0}^{\mathbb{R}}$, by [6, Lem. 3.6] applied in $H_{iu,0}^{\mathbb{R}}$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$t \mapsto \int_{K_0 \cap H_{u,t}^{\mathbb{R}} \cap (v+\lambda_1 u)^+} \langle x, v + \lambda_1 u\rangle^p dx$$

and

$$t \mapsto \int_{K_0 \cap H_{u,t}^{\mathbb{R}} \cap (-iv+\lambda_2 u)^+} \langle x, -iv + \lambda_2 u\rangle^p dx$$

have critical points at $t = 0$. Set $\lambda = \lambda_1 + i\lambda_2$. Next, since for $x \cdot u = t \in \mathbb{R}$, we have $\Re(x \cdot (v + \lambda u)) = \Re(x \cdot (v + \lambda_1 u)) = \langle x, v + \lambda_1 u\rangle$ and $K \cap H_{u,t} = K_0 \cap H_{u,t}^{\mathbb{R}}$,

$$\mathcal{M}_{p,v+\lambda u}^{\Re,+,u}(K,t) = \int_{K \cap H_{u,t} \cap (v+\lambda u)^+} \Re(x \cdot (v + \lambda u))^p dx = \int_{K_0 \cap H_{u,t}^{\mathbb{R}} \cap (v+\lambda_1 u)^+} \langle x, v + \lambda_1 u\rangle^p dx,$$

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and, hence, $t \mapsto \mathcal{M}^{\Re,+,u}_{p,v+\lambda u}(K,t)$ has a critical point at zero. If $x \cdot u = it$, $t \in \mathbb{R}$, then $\Re(x \cdot (v + \lambda u)) = \langle x, v + \lambda_2 iu\rangle$ and, by the $\mathbb{S}^1$-invariance of K, $K \cap H_{u,it} = i(K_0 \cap H^{\mathbb{R}}_{u,t})$. Consequently, by letting $x = iy$,

$$\mathcal{M}^{\Re,+,u}_{p,v+\lambda u}(K,t) = \int_{K\cap H_{u,it}\cap(v+\lambda u)^+} \Re(x \cdot (v + \lambda u))^p dx = \int_{i(K_0\cap H^{\mathbb{R}}_{u,t}\cap(-iv+\lambda_2 u)^+)} \langle x, v + \lambda_2 iu\rangle^p dx$$
$$= \int_{K_0\cap H^{\mathbb{R}}_{u,t}\cap(-iv+\lambda_2 u)^+} \langle y, -iv + \lambda_2 u\rangle^p dy,$$

we conclude that also $t \mapsto \mathcal{M}^{\Re,+,u}_{p,v+\lambda u}(K,it)$ has a critical point at zero, which yields the claim. □

Using symmetries, Lemma 3.4.6 now directly translates to symmetric moments,

$$\mathcal{M}^{|\cdot|,u}_{2,v}(K,z) = \int_{K\cap H_{u,z}} |x \cdot v|^2 dx,$$

whenever K is $\mathbb{S}^1$-invariant.

Proposition 3.4.7. *Suppose that $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ is $\mathbb{S}^1$-invariant and let $u, v \in \mathbb{C}^n \setminus \{0\}$ with $v \notin \operatorname{span}^{\mathbb{C}}\{u\}$. Then there exists $\lambda \in \mathbb{C}$ such that*

$$z \mapsto \mathcal{M}^{|\cdot|,u}_{2,v+\lambda u}(K,z)$$

is maximal at $z = 0$.

Proof. First note that since K is $\mathbb{S}^1$-invariant,

$$\mathcal{M}^{|\cdot|,u}_{2,w}(K,z) = \mathcal{M}^{\Re,+,u}_{2,w}(K,z) + \mathcal{M}^{\Re,+,u}_{2,w}(K,-z) + \mathcal{M}^{\Re,+,u}_{2,w}(K,iz) + \mathcal{M}^{\Re,+,u}_{2,w}(K,-iz),$$

for every $w \in \mathbb{C}^n \setminus \{0\}$, and we need to choose $\lambda \in \mathbb{C}$ such that $\mathcal{M}^{\Re,+,u}_{2,v+\lambda u}(K,\cdot)$ attains its maximum at $z = 0$. However, by Corollary 3.4.5, $\mathcal{M}^{\Re,+,u}_{2,v+\lambda u}(K,\cdot)^{\frac{1}{2n}}$ is concave on $\{z \in \mathbb{C} : K \cap H_{u,z} \cap (v + \lambda u)^+ \neq \emptyset\}$, and by Lemma 3.4.6 (together with the chain rule), there exists $\lambda \in \mathbb{C}$ such that $\mathcal{M}^{\Re,+,u}_{2,v+\lambda u}(K,\cdot)^{\frac{1}{2n}}$ is maximal at $z = 0$ for smooth $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$. Hence, the claim follows from the monotonicity of $t \mapsto t^{2n}$ and by approximating a general $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ by smooth bodies. □

3.4.3 Proof of Theorem B

In this section, we compute the necessary derivatives required in order to apply Theorem 3.4.2 in the proof of Theorem B.

Recalling that the analytic family of distributions r^q_+, $\Re q > -1$, can be extended analytically to $-\Re q \notin \mathbb{N}_+$, and that, for $-2 < \Re q < -1$, this extension is given by

$$\langle r^q_+, \phi\rangle = \int_0^\infty r^q \left(\phi(r) - \phi(0) - r\phi'(0)\right) dr, \quad \phi \in C^\infty_c(\mathbb{C}), \tag{3.23}$$

which clearly can be extended to all $\phi \in C(\mathbb{C})$ with compact support, which are smooth in a neighborhood of zero. The main auxiliary result can then be stated as follows.

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Proposition 3.4.8. *Suppose that $p > -2$, $p \neq 0, -1$, and let $u, w \in \mathbb{C}^n \setminus \{0\}$ with $w \notin \operatorname{span}^{\mathbb{C}}\{u\}$. Then*

$$\Delta_z(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p})|_{z=0} = 2\pi p^2 \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|,u}(K, \cdot)\rangle \tag{3.24}$$

for every $\mathbb{S}^1$-invariant $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ with smooth boundary.

Proof. Assume first that $p > 0$. Since $z \mapsto \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}$ is a tempered distribution on $\mathbb{C}$, we can consider the Fourier transform (denoted by $\widehat{\cdot}$) of its Laplacian, applied to a Schwartz function φ on $\mathbb{C}$, that is,

$$\langle (\Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p})\widehat{}, \varphi\rangle = \langle \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}, \Delta_z \widehat{\varphi}\rangle = \langle \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}, -(\widehat{|\cdot|^2\varphi})\rangle.$$

By inserting the definition of $\rho_{\mathrm{I}_{\mathbb{D},p}K}$, exchanging the order of integration, and letting $c = z - \overline{(x \cdot u)/(x \cdot w)}$

$$\begin{aligned}
\langle \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}, \widehat{|\cdot|^2\varphi}\rangle &= \int_K \int_{\mathbb{C}} |x \cdot (u + zw)|^p (\widehat{|\cdot|^2\varphi})(z) dz dx \\
&= \int_K \int_{\mathbb{C}} |x \cdot w|^p |z + \overline{(x \cdot u)/(x \cdot w)}|^p (\widehat{|\cdot|^2\varphi})(z) dz dx \\
&= \int_K |x \cdot w|^p \int_{\mathbb{C}} |c|^p (\widehat{|\cdot|^2\varphi})(c - \overline{(x \cdot u)/(x \cdot w)}) dc dx.
\end{aligned}$$

Next, it is a direct computation that for $p \neq -2, -4, \dots$ (see, e.g., [19, Sec. II.3.3]),

$$|\cdot|^2 \widehat{(|\cdot|^p)} = -p^2 (\widehat{|\cdot|^{p-2}}),$$

and, consequently, the previous integral simplifies to

$$\begin{aligned}
-p^2 \int_K |x \cdot w|^p \int_{\mathbb{C}} |c|^{p-2} \widehat{\varphi}(c - \overline{(x \cdot u)/(x \cdot w)}) dc dx \\
= -p^2 \int_{\mathbb{C}} \int_K |x \cdot w|^2 |x \cdot (u + zw)|^{p-2} dx\, \widehat{\varphi}(z) dz.
\end{aligned}$$

By taking the inverse Fourier transform, we conclude that

$$\Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p} = p^2 \int_K |x \cdot w|^2 |x \cdot (u + zw)|^{p-2} dx \tag{3.25}$$

as tempered distributions. By

$$\langle \Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}, \varphi\rangle = \langle \rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}, \Delta_z \varphi\rangle,$$

$\varphi \in C_c^\infty(\mathbb{C})$, and since $\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}$ is analytic in p, the left-hand side of (3.25) is an analytic family of distributions (in $z \in \mathbb{C}$). Rewriting the right-hand side of (3.25) by

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Fubini's theorem, and using polar coordinates on $\mathbb{C}$ and the $\mathbb{S}^1$-invariance of K,

$$\begin{aligned} p^2 \int_K |x \cdot w|^2 |x \cdot (u+zw)|^{p-2} dx &= p^2 \int_{\mathbb{C}} |\zeta|^{p-2} \int_{K \cap H_{u+zw,\zeta}} |x \cdot w|^2 dx d\zeta \\ &= p^2 \int_0^\infty r^{p-1} \int_{\mathbb{S}^1} \int_{K \cap H_{u+zw,rc}} |x \cdot w|^2 dx dc dr \\ &= 2\pi p^2 \int_0^\infty r^{p-1} \int_{K \cap H_{u+zw,r}} |x \cdot w|^2 dx dr \\ &= 2\pi p^2 \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r, \end{aligned}$$

we conclude that also the right-hand side of (3.25) is an analytic family of distributions. The uniqueness of analytic continuation therefore implies that

$$\langle \Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u+zw)^{-p}, \varphi \rangle = 2\pi p^2 \langle \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r, \varphi \rangle_z,$$

for all $\varphi \in C_c^\infty(\mathbb{C})$ and $p > -2$, $p \neq 0, -1$. Note that since (for $p > 0$)

$$\begin{aligned} \langle \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r, \varphi \rangle_z &= \int_{\mathbb{C}} \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r \varphi(z) dz \\ &= \langle r_+^{p-1}, \langle \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r), \varphi \rangle_z \rangle_r \end{aligned}$$

the analytic continuation of $\langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r$ is given by

$$\begin{aligned} \langle \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r, \varphi \rangle_z &= \int_0^\infty r_+^{p-1} \Big(\langle \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r), \varphi \rangle_z \\ &\quad - \langle \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,0), \varphi \rangle_z - \langle r \left. \frac{\partial}{\partial r} \right|_{r=0} \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r), \varphi \rangle_z \Big) dr \\ &= \langle \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r, \varphi \rangle_z. \end{aligned}$$

Since $K = -K$, $\mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r)$ is even (in r), the derivative at $r = 0$ vanishes. Consequently,

$$\begin{aligned} \Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u+zw)^{-p} &= 2\pi p^2 \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) \rangle_r \\ &= 2\pi p^2 \int_0^\infty r^{p-1} \left(\mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,r) - \mathcal{M}_{2,w}^{|\cdot|, u+zw}(K,0) \right) dr, \end{aligned} \tag{3.26}$$

as distributions. Next, observe that since the operator $\mathrm{J}_{\mathbb{D},p}$ commutes with the action of $\mathrm{U}(n)$ on $\mathbb{S}^{2n-1}$, $\mathrm{J}_{\mathbb{D},p}$ maps $C^\infty(\mathbb{S}^{2n-1})$ to itself. Consequently, by (3.10) and as $\rho_{\mathrm{I}_{\mathbb{D},p}K}(x)$ is strictly positive for $x \neq 0$, $\rho_{\mathrm{I}_{\mathbb{D},p}K}$ is smooth in $\mathbb{C}^n \setminus \{0\}$, whenever $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ has a smooth boundary.

As the right-hand side of (3.26) is also continuous in z, both sides of (3.26) coincide as functions (as $u + zw \neq 0$ for all $z \in \mathbb{C}$, by assumption). Evaluating at $z = 0$, we obtain

$$\Delta_z \rho_{\mathrm{I}_{\mathbb{D},p}K}(u+zw)^{-p}|_{z=0} = 2\pi p^2 \langle r_+^{p-1}, \mathcal{M}_{2,w}^{|\cdot|, u}(K, \cdot) \rangle,$$

which yields the claim. □

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The last ingredient for the proof of Theorem B is the following result from elementary calculus, included for the reader's convenience.

Lemma 3.4.9. *Suppose that $F \in C^\infty(\mathbb{C}^n \setminus \{0\})$ is non-negative and $F(u) > 0$ for $u \neq 0$, one-homogeneous and $\mathbb{S}^1$-invariant, that is, $F(zw) = |z|F(w)$, $z \in \mathbb{C}, w \in \mathbb{C}^n$, and let $u, v \in \mathbb{C}^n \setminus \{0\}$ with $\nabla F(u) \cdot v = 0$. Then*

$$\Delta|_{z=0} F(u+zw)^p = p^2|\lambda|^2 F(u)^p + pF(u)^{p-1}\, \Delta|_{z=0} F(u+zv),$$

where $w = v + \lambda u$, $\lambda \in \mathbb{C}$, and the derivatives are with respect to $z \in \mathbb{C}$.

Proof. First note that by one-homogeneity and $\mathbb{S}^1$-invariance,

$$\langle \nabla F(u), u\rangle = F(u) \quad \text{and} \quad \langle \nabla F(u), iu\rangle = 0, \tag{3.27}$$

and, by differentiating the equalities in (3.27),

$$d^2F(u)u = 0 \quad \text{and} \quad d^2F(u)iu = i\nabla F(u). \tag{3.28}$$

Next, computing by the chain rule, for $x \in \mathbb{C}^n \setminus \{0\}$ arbitrary, yields

$$\frac{d^2}{dt^2}\bigg|_{t=0} F(u+tx)^p = p(p-1)F(u)^{p-2}\langle \nabla F(u), x\rangle^2 + pF(u)^{p-1}\langle x, d^2F(u)x\rangle.$$

Letting $x = v + \lambda u$ and applying (3.27), (3.28) and the assumptions on v,

$$\frac{d^2}{dt^2}\bigg|_{t=0} F(u+tw)^p = p(p-1)F(u)^p(\Re\lambda)^2 + pF(u)^{p-1}\langle v, d^2F(u)v\rangle + pF(u)^p(\Im\lambda)^2,$$

and for $x = i(v+\lambda u)$,

$$\frac{d^2}{dt^2}\bigg|_{t=0} F(u+tiw)^p = p(p-1)F(u)^p(\Im\lambda)^2 + pF(u)^{p-1}\langle iv, d^2F(u)iv\rangle + pF(u)^p(\Re\lambda)^2,$$

which yields the claim, when summed up. □

We are now ready to prove Theorem B.

Proof of Theorem B. By (3.21) and (3.22), we can assume without loss of generality that $C = \mathbb{D}$. Moreover, by Theorem C (which is proved independently in Section 3.5) and Theorem 3.1.2, we only need to consider $-2 < p < -1$.

Let now $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$ be $\mathbb{S}^1$-invariant and assume first that its radial function ρ_K is smooth in $\mathbb{C}^n \setminus \{0\}$. Noting, as before, that $\rho_{\mathrm{I}_{\mathbb{D},p}K}$ is smooth in $\mathbb{C}^n \setminus \{0\}$, and

$$\operatorname{int} \mathrm{I}_{\mathbb{D},p}K = \{u \in \mathbb{C}^n : \rho_{\mathrm{I}_{\mathbb{D},p}K}(u)^{-1} - 1 < 0\},$$

by Theorem 3.4.2, we need to show that

$$\Delta_z \left(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u+zv)^{-1}\right)|_{z=0} \geq 0$$

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for all $u \in \operatorname{bd} \mathrm{I}_{\mathbb{D},p}K$ and $\nabla(\rho^{-1}_{\mathrm{I}_{\mathbb{D},p}K})(u) \cdot v = 0$.

Therefore, let $u \in \operatorname{bd} \mathrm{I}_{\mathbb{D},p}K$ be fixed and take $v \in \mathbb{C}^n \setminus \{0\}$ arbitrary such that $\nabla(\rho^{-1}_{\mathrm{I}_{\mathbb{D},p}K}) \cdot v = 0$. If $v = \zeta u$ for some $\zeta \in \mathbb{C}$, then, since $\rho_{\mathrm{I}_{\mathbb{D},p}K}(u) = 1$, the $\mathbb{S}^1$-invariance and homogeneity of the radial function imply that

$$\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zv)^{-1} = |1 + z\zeta|,$$

and one directly sees that $\Delta_z|1 + z\zeta| \geq 0$ at $z = 0$. If $v \notin \operatorname{span}^{\mathbb{C}}\{u\}$, by Lemma 3.4.9,

$$\Delta_z \left(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zv)^{-1}\right)|_{z=0} + p|\lambda|^2 = \frac{1}{p}\Delta_z \left(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zw)^{-p}\right)|_{z=0},$$

with $w = v + \lambda u$, for some $\lambda \in \mathbb{C}$ to be chosen later, which, by Proposition 3.4.8 is equal to

$$2\pi p \langle r_+^{p-1}, \mathcal{M}^{|\cdot|,u}_{2,w}(K, \cdot)\rangle.$$

Denoting $\Phi(r) = \mathcal{M}^{|\cdot|,u}_{2,w}(K, r)$, we conclude from (3.23) that

$$\Delta_z \left(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zv)^{-1}\right)|_{z=0} + p|\lambda|^2 = 2\pi p \int_0^\infty r^{p-1}(\phi(r) - \phi(0) - r\phi'(0))dr.$$

Since K is origin-symmetric, ϕ is even, and, hence, $\phi'(0) = 0$. Moreover, by Proposition 3.4.7, we can choose $\lambda \in \mathbb{C}$ such that $\phi(r) \leq \phi(0)$ for all $r > 0$. As $p < 0$, we conclude that

$$\Delta_z \left(\rho_{\mathrm{I}_{\mathbb{D},p}K}(u + zv)^{-1}\right)|_{z=0} \geq 0,$$

that is, $\operatorname{int} \mathrm{I}_{\mathbb{D},p}K$ is pseudo-convex.

For general $\mathbb{S}^1$-invariant $K \in \mathcal{K}_{(0)}(\mathbb{C}^n)$, we approximate K by smooth $\mathbb{S}^1$-invariant convex bodies K_j, $j \in \mathbb{N}$ such that $K \subseteq K_j$ for all $j \in \mathbb{N}$. By the first part of the proof and the monotonicity of $\mathrm{I}_{\mathbb{D},p}$,

$$\mathrm{I}_{\mathbb{D},p}K = \bigcap_{j \in \mathbb{N}} \mathrm{I}_{\mathbb{D},p}K_j,$$

where all $\operatorname{int} \mathrm{I}_{\mathbb{D},p}K_j$ are pseudo-convex, and the claim follows by Theorem 3.4.3. □

3.4.4 Counterexamples to convexity

In the proof of Theorem B, $\mathbb{S}^1$-invariance of the convex bodies played a critical role. It is therefore a natural question to ask whether this is a particular aspect of the proof or reflects an underlying principle. In this section, we give a (partial) answer to this by providing examples in the range $-1 \leq p < 1$ of convex bodies which are not $\mathbb{S}^1$-invariant and have non-convex, complex L_p-intersection bodies.

These examples are obtained by considering sequences of ellipsoids, whose complex L_p-intersection bodies converge to a non-convex star body. The key ingredient of this argument is the following generalization of (parts of) [24, Lem. 6.3], proved using similar arguments.

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Lemma 3.4.10. *Suppose that $p \geq -1$ and let $\bar{e} \in \mathbb{S}^{2n-1}$. Then there exists a sequence of origin-symmetric ellipsoids $E_j \subseteq \mathbb{C}^n, j \in \mathbb{N}$, such that*

$$\lim_{j\to\infty} \int_{\mathbb{S}^{2n-1}} f(u)\rho_{E_j}(u)^{2n+p}du = \frac{1}{2}(f(\bar{e}) + f(-\bar{e})), \tag{3.29}$$

for all $f \in C(\mathbb{S}^{2n-1})$.

Proof. First, without loss of generality, we may assume that $\bar{e}$ is the first standard unit vector in $\mathbb{C}^n$. Using generalized spherical coordinates $u = (u_1 \sin(t), u_2 \cos(t))$ for $u \in \mathbb{S}^{2n-1}$, with $u_1 \in \mathbb{S}^0 = \{\pm\bar{e}\}, u_2 \in \mathbb{S}^{2n-2}$ and $t \in [0, \pi/2]$, the radial function of the ellipsoid

$$E_{a,b} = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n : \frac{(\Re z_1)^2}{a^2} + \frac{(\Im z_1)^2 + |z_2|^2 + \cdots + |z_n|^2}{b^2} \leq 1\right\},$$

for $a, b > 0$ is given by

$$\rho_{E_{a,b}}(u_1 \sin(t), u_2 \cos(t)) = \left(\frac{\sin(t)^2}{a^2} + \frac{\cos(t)^2}{b^2}\right)^{-1/2}, \quad t \in [0, \pi/2].$$

Next, choose $b_j > 0$ by the intermediate value theorem, such that

$$\int_{\mathbb{S}^{2n-1}} \rho_{E_{j,b_j}}(u)^{2n+p}du = 1, \tag{3.30}$$

whenever $j \in \mathbb{N}$ is large enough. Note that $b_j \to 0$ as $j \to \infty$. Indeed, assume that $b_j \geq M$ for some constant $M > 0$. Writing (3.30) in generalized spherical coordinates (where $du = \cos(t)^{2n-2}du_1du_2dt$), denoting $C_n = 2(2n-2)\kappa_{2n-2}$,

$$\begin{aligned} 1 = \int_{\mathbb{S}^{2n-1}} \rho_{E_{j,b_j}}(u)^{2n+p}du &= C_n \int_0^{\pi/2} \cos(t)^{2n-2} \left(\frac{\sin(t)^2}{j^2} + \frac{\cos(t)^2}{b_j^2}\right)^{-(2n+p)/2} dt \\ &\geq C_n \int_0^{\pi/2} \cos(t)^{2n-2} \left(\frac{\sin(t)^2}{j^2} + \frac{\cos(t)^2}{M^2}\right)^{-(2n+p)/2} dt, \end{aligned}$$

and letting $j \to \infty$ yields (by monotone convergence)

$$1 \geq C_n M^{2n+p} \int_0^{\pi/2} \cos(t)^{-2-p}dt,$$

which contradicts the fact that $\cos(t)^{-2-p}$ is integrable only if $-2-p > -1$, that is $p < -1$. Since b_j is clearly monotonously decreasing, $b_j \to 0$.

Setting $E_j = E_{j,b_j}$, we claim that a subsequence of $(E_j)_{j\in\mathbb{N}}$ already satisfies (3.29). Indeed, observe that by (3.30) and since they are positive, the functions $\rho_{E_j}^{2n+p}$ all have norm 1, when seen as elements of the dual space of $C(\mathbb{S}^{2n-1})$. Consequently, by the Banach–Alaoglu theorem, there exists a subsequence (again denoted by $(\rho_{E_j}^{2n+p})_j$) converging in the weak-* topology to a Borel measure μ on $\mathbb{S}^{2n-1}$, that is,

$$\int_{\mathbb{S}^{2n-1}} f(u)\rho_{E_j}(u)^{2n+p}du \to \int_{\mathbb{S}^{2n-1}} f(u)d\mu(u), \quad j \to \infty,$$

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for every $f \in C(\mathbb{S}^{2n-1})$. Showing that $\mu = \frac{1}{2}(\delta_{\bar{e}} + \delta_{-\bar{e}})$, thus directly implies the claim.

To this end, suppose that $u \in \mathbb{S}^{2n-1} \setminus \{\pm\bar{e}\}$ and let $U \subseteq \mathbb{S}^{2n-1}$ be an open neighborhood of u not containing $\pm\bar{e}$ in its closure. Then there exists $\varepsilon > 0$ such that for all $u = (u_1 \sin(t), u_2 \cos(t)) \in U$, we have $\cos(t) > \varepsilon$, and, therefore

$$\left(\frac{\sin(t)^2}{j^2} + \frac{\cos(t)^2}{b_j^2}\right)^{-(2n+p)/2} \leq \left(\frac{\sin(t)^2}{j^2} + \frac{\varepsilon^2}{b_j^2}\right)^{-(2n+p)/2} \leq \frac{b_j^{2n+p}}{\varepsilon^{2n+p}}. \tag{3.31}$$

As $b_j \to 0$ for $j \to \infty$, a direct estimate for $f \in C(\mathbb{S}^{2n-1})$ concentrated on U, shows

$$\int_{\mathbb{S}^{2n-1}} f(c) d\mu(c) = \lim_{j\to\infty} \int_U f(c) \rho_{E_j}(c)^{2n+p} dc = 0,$$

that is, $\operatorname{supp}\mu \subseteq \mathbb{S}^{2n-1} \setminus U$ and, hence, $\operatorname{supp}\mu \subseteq \{\pm\bar{e}\}$, as u was arbitrary. Since $\mu(\mathbb{S}^{2n-1}) = 1$, by (3.30), and μ must be even (as weak-* limit of even measures), we conclude that $\mu = \frac{1}{2}(\delta_{\bar{e}} + \delta_{-\bar{e}})$, which completes the proof. □

The previous lemma for continuous functions on $\mathbb{S}^{2n-1}$ can be directly extended to functions with a specific type of pole.

Lemma 3.4.11. *Suppose that $p \geq -1$ and let $\bar{e} \in \mathbb{S}^{2n-1}$. Then there exists a sequence of origin-symmetric ellipsoids $E_j \subseteq \mathbb{C}^n, j \in \mathbb{N}$, such that*

$$\lim_{j\to\infty} \int_{\mathbb{S}^{2n-1}} h_C(v \cdot (c\bar{e}))^p \rho_{E_j}(u)^{2n+p} dv = \frac{1}{2}\left(h_C(\bar{c})^p + h_C(-\bar{c})^p\right), \quad c \in \mathbb{S}^1, \tag{3.32}$$

for all $C \in \mathcal{K}_{(0)}(\mathbb{C})$.

Proof. If $p > 0$, the function $g(v) = h_C(v \cdot (c\bar{e}))^p$ is continuous on $\mathbb{S}^{2n-1}$ and, the claim follows from Lemma 3.4.10. For $p < 0$, set $M = 2\max\{h_C(\bar{c})^p, h_C(-\bar{c})^p\}$, and consider the decomposition

$$g(v) = \min\{g(v), M\} + (\max\{g(v), M\} - M),$$

where the first function clearly is continuous and coincides with g on a neighborhood V of $\pm\bar{e}$, whereas the second function vanishes on the same neighborhood. Taking the ellipsoids E_j as in the previous lemma, the same estimate as in (3.31) implies that

$$\int_{\mathbb{S}^{2n-1}} (\max\{g(v), M\} - M) \rho_{E_j}(u)^{2n+p} dv \leq \frac{b_j^{2n+p}}{\varepsilon^{2n+p}} \int_{\mathbb{S}^{2n-1}\setminus V} (\max\{g(v), M\} - M) dv,$$

where the integral on the right-hand side is finite, since its absolute value is bounded by $\|\mathrm{J}_{C,p}1\|_\infty + M(2n-1)\kappa_{2n-1}$. Consequently, as $b_j \to 0$, the left-hand side converges to zero as $j \to \infty$. Hence, together with Lemma 3.4.10 for the first term $\min\{g(v), M\}$ and since $\min\{g(v), M\} = g(v)$ for $v = \pm\bar{e}$, the claim follows. □

We are now ready to state the aforementioned counterexample.

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Proposition 3.4.12. *Let $-1 \le p < 1$ be non-zero. Then there exists $C \in \mathcal{K}_{(0)}(\mathbb{C})$ and an origin-symmetric ellipsoid $K \subseteq \mathbb{C}^n$ such that $\mathrm{I}_{C,p}K$ is not convex.*

Proof. By (3.10) and Lemma 3.4.11, there exists $\bar{e} \in \mathbb{S}^{2n-1}$ and a sequence $(E_j)_{j\in\mathbb{N}}$ of origin-symmetric ellipsoids such that

$$\rho_{\mathrm{I}_{C,p}E_j}(c\bar{e})^{-p} \to \frac{1}{2(2n+p)}(h_C(\bar{c})^p + h_C(-\bar{c})^p), \quad j \to \infty,$$

for every $c \in \mathbb{S}^1$, $C \in \mathcal{K}_{(0)}(\mathbb{C})$ and $p \ge -1$. Note that, when choosing C to be, e.g., a suitable triangle, the function $c \mapsto (h_C(c)^p + h_{-C}(c)^p)^{-1/p}$ is not the radial function of a convex body (see [16, Sec. 6.1] for details) when $p < 1$. Consequently, the radial function of $\mathrm{I}_{C,p}E_j$ converges pointwise to the radial function of a non-convex star body as $j \to \infty$ and, hence, $\mathrm{I}_{C,p}E_j$ cannot be convex when j is sufficiently large. □

3.5 Proof of Theorems C and D

In this section we establish a representation of the radial function of $\mathrm{I}_{C,p}K$ for $K \in \mathcal{S}_0(\mathbb{C}^n)$ and origin-symmetric $C \in \mathcal{K}_{(0)}(\mathbb{C})$, and use it to prove Theorem D.

To this end, we require a lemma relating the Fourier transforms of the complex and the real Radon transform.

Lemma 3.5.1. *Suppose that $f \in C(\mathbb{C}^n)$ has compact support and, for $u \in \mathbb{C}^n \setminus \{0\}$, recall that for $z \in \mathbb{C}$ and $t \in \mathbb{R}$,*

$$(\mathcal{R}_u^{\mathbb{C}} f)(z) = \int_{x\cdot u = z} f(x)dx \quad \text{and} \quad (\mathcal{R}_u^{\mathbb{R}} f)(t) = \int_{\langle x,u\rangle = t} f(x)dx,$$

denote the complex and real Radon transforms *of f. Then*

$$\widehat{\mathcal{R}_u^{\mathbb{C}}} f(rc) = \widehat{\mathcal{R}_{cu}^{\mathbb{R}}} f(r), \quad r \in \mathbb{R}, c \in \mathbb{C},$$

for $u \in \mathbb{C}^n \setminus \{0\}$, where the left Fourier transform is on $\mathbb{C}$ and the right one on $\mathbb{R}$.

Proof. The claim follows by Fubini's theorem applied twice and (2.4),

$$\begin{aligned}\widehat{\mathcal{R}_u^{\mathbb{C}}} f(rc) &= \int_{\mathbb{C}} \int_{x\cdot u = z} f(x) e^{-i\langle rc, z\rangle} dx dz = \int_{\mathbb{C}^n} f(x) e^{-i\langle rc, x\cdot u\rangle} dx \\ &= \int_{\mathbb{C}^n} f(x) e^{-ir\langle cu, x\rangle} dx = \int_{\mathbb{R}} \int_{\langle x, cu\rangle = t} f(x) e^{-irt} dx dt = \widehat{\mathcal{R}_{cu}^{\mathbb{R}}} f(r).\end{aligned}$$

□

We are now in a position to prove the main proposition required in the proof of Theorem C.

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Proposition 3.5.2. *Let $C \in \mathcal{K}_{(0)}(\mathbb{C})$ be origin-symmetric and $-1 \le p < 1$ be non-zero. Then there exists a finite Borel measure $\mu_{C,p}$ on $\mathbb{S}^1$, such that*

$$\rho_{\mathrm{I}_{C,p}K}(u)^{-p} = \int_{\mathbb{S}^1} \rho_{\mathrm{I}_p K}(cu)^{-p} d\mu_{C,p}(c), \quad u \in \mathbb{S}^{2n-1}, \tag{3.33}$$

for every $K \in \mathcal{S}_0(\mathbb{C}^n)$, where, for $p=-1$, $d\mu_{C,-1} = \frac{1}{2}\rho_{iC^\circ}(c)dc$. In particular, if $K \in \mathcal{S}_0(\mathbb{C}^n)$ is $\mathbb{S}^1$-invariant, then $\mathrm{I}_{C,p}K = \mu_{C,p}(\mathbb{S}^1)\mathrm{I}_p K$.

Proof. We distinguish the cases $0 < p < 1$, $-1 < p < 0$ and $p=-1$. If $p > 0$, by Lemma 2.2.6 applied to $C^\circ \in \mathcal{K}_{(0)}(\mathbb{C})$, there exists a finite Borel measure $\mu_{C,p}$ on $\mathbb{S}^1$ such that

$$h_C(z)^p = \rho_{C^\circ}(z)^{-p} = \|z\|_{C^\circ}^p = \int_{\mathbb{S}^1} |\langle z, c\rangle|^p d\mu_{C,p}(c), \quad z \in \mathbb{C}. \tag{3.34}$$

Note that we identify $\mathbb{C} \cong \mathbb{R}^2$ here. Combining, for $u \in \mathbb{S}^{2n-1}$, (3.34) with the definition (3.4) of $\mathrm{I}_{C,p}K$, $K \in \mathcal{S}_0(\mathbb{C}^n)$, by (2.4), and interchanging the order of integration,

$$\begin{aligned}\rho_{\mathrm{I}_{C,p}K}(u)^{-p} &= \int_K h_C(x\cdot u)^p dx = \int_K \int_{\mathbb{S}^1} |\langle x\cdot u, c\rangle|^p d\mu_{C,p}(c) dx \\ &= \int_{\mathbb{S}^1}\int_K |\langle x, cu\rangle|^p dx d\mu_{C,p}(c) = \int_{\mathbb{S}^1} \rho_{\mathrm{I}_p K}(cu)^{-p} d\mu_{C,p}(c)\end{aligned}$$

we arrive at the claim.

In the second case, $-1 < p < 0$, Lemma 2.2.6, applied again to $C^\circ \in \mathcal{K}_{(0)}(\mathbb{C})$, implies the existence of a measure $\nu_{C,p}$ on $\mathbb{S}^1$ such that

$$\int_{\mathbb{C}} \rho_{C^\circ}(z)^{-p}\phi(z)dz = \int_{\mathbb{S}^1}\left(\int_0^\infty t^{-p-1}\hat{\phi}(tc)dt\right) d\nu_{C,p}(c), \tag{3.35}$$

for every even Schwartz function ϕ on $\mathbb{C}$. Note that $\nu_{C,p}$ can be chosen to be even. Since ϕ is even and (see, e.g., [38, Lem. 2.23])

$$\widehat{|t|^{-p-1}}(r) = 2\Gamma(-p)\sin\left(\frac{\pi(p+1)}{2}\right)|r|^p, \quad r \in \mathbb{R}, \tag{3.36}$$

we can rewrite the inner integral on the right-hand side to obtain

$$\begin{aligned}\int_0^\infty t^{-p-1}\hat{\phi}(tc)dt &= \frac{1}{2}\int_{\mathbb{R}} |t|^{-p-1}\hat{\phi}(tc)dt \\ &= \Gamma(-p)\sin\left(\frac{\pi(p+1)}{2}\right)\int_{\mathbb{R}} |r|^p\tilde{\phi}_c(r)dr = c_p\int_0^\infty r^p\tilde{\phi}_c(r)dr,\end{aligned}$$

where we denote by $\tilde{\phi}_c$ the Fourier transform in $\mathbb{R}$ of $t \mapsto \hat{\phi}(tc)$, and collect the constants into $c_p \in \mathbb{R}$.

If $f \in C^\infty(\mathbb{C}^n)$ is even and has compact support, then the complex Radon transform $\mathcal{R}_u^{\mathbb{C}} f$ is again even and smooth with compact support, and thus a Schwartz function. Taking now $\phi = \mathcal{R}_u^{\mathbb{C}} f$, then $\tilde{\phi}_c = \mathcal{R}_{cu}^{\mathbb{R}} f$, by Lemma 3.5.1, and, hence,

$$\int_0^\infty t^{-p-1}\widehat{\mathcal{R}_u^{\mathbb{C}} f}(tc)dt = c_p\int_0^\infty r^p\left(\mathcal{R}_{cu}^{\mathbb{R}} f\right)(r)dr.$$

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Equation (3.35) therefore implies

$$\int_{\mathbb{C}} \rho_{C^\circ}(z)^{-p} \left(\mathcal{R}_u^{\mathbb{C}} f\right)(z)dz = c_p \int_{\mathbb{S}^1} \int_0^\infty r^p \left(\mathcal{R}_{cu}^{\mathbb{R}} f\right)(r)dr\, d\nu_{C,p}(c),$$

for every even $f \in C^\infty(\mathbb{C}^n)$ with compact support. As ρ_{C° and $\nu_{C,p}$ are even, this equation clearly also holds for functions f that are not necessarily even. Moreover, by approximation, it holds for $f = \mathbb{1}_K$, $K \in \mathcal{S}_0(\mathbb{C}^n)$, where $\mathcal{R}_u^{\mathbb{C}} f = \mathrm{A}_{K,u}^{\mathbb{C}}$ and $\mathcal{R}_{cu}^{\mathbb{R}} f = \mathrm{A}_{K,cu}^{\mathbb{R}}$ are the complex and real parallel section functions (see (3.7) and the comment below it). Consequently, for $u \in \mathbb{S}^{2n-1}$,

$$\rho_{\mathrm{I}_{C,p}K}(u)^{-p} = \int_{\mathbb{C}} \rho_{C^\circ}(z)^{-p} \mathrm{A}_{K,u}^{\mathbb{C}}(z)dz = c_p \int_{\mathbb{S}^1} \int_0^\infty r^p \mathrm{A}_{K,cu}^{\mathbb{R}}(r)dr\, d\nu_{C,p}(c),$$

and, since $\nu_{C,p}$ is even, the right-hand side is equal to

$$\frac{c_p}{2} \int_{\mathbb{S}^1} \int_{\mathbb{R}} |r|^p \mathrm{A}_{K,cu}^{\mathbb{R}}(r)dr\, d\nu_{C,p}(c) = \frac{c_p}{2} \int_{\mathbb{S}^1} \rho_{\mathrm{I}_p K}(cu)^{-p}\, d\nu_{C,p}(c),$$

which yields the claim with $\mu_{C,p} = \frac{c_p}{2}\nu_{C,p}$.

For $p = -1$, finally, we first show that (3.35) holds for $d\nu_{C,-1} = \frac{1}{2\pi}\rho_{iC^\circ}(c)dc$. Indeed, let ϕ be an even Schwartz function on $\mathbb{C}$. Using polar coordinates, the homogeneity of radial functions and the parity of ϕ,

$$\int_{\mathbb{C}} \rho_{C^\circ}(x)\phi(x)dx = \int_{\mathbb{S}^1} \frac{1}{2}\rho_{C^\circ}(c) \int_{\mathbb{R}} \phi(rc)drdc,$$

where the inner integral equals $\widehat{\phi(\cdot c)}(0)$. By [38, Lem. 2.11], $\widehat{\mathcal{R}_c^{\mathbb{R}}\hat{\phi}}(s) = (2\pi)^2\phi(sc)$ for $s \in \mathbb{R}$, that is,

$$\int_{\mathbb{R}} \phi(rc)dr = \widehat{\phi(\cdot c)}(0) = \frac{1}{2\pi}\mathcal{R}_c^{\mathbb{R}}\hat{\phi}(0) = \frac{1}{2\pi}\int_{\langle x,c\rangle=0} \hat{\phi}(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(itc)dt.$$

Changing the outer integration and by $\rho_{C^\circ}(-ic) = \rho_{iC^\circ}(c)$ and the parity of $\hat{\phi}$, we arrive at (3.35) for $p = -1$ and $d\nu_{C,-1} = \frac{1}{2\pi}\rho_{iC^\circ}(c)dc$.

Next, we repeat the steps from the previous part ($-1 < p < 0$) to obtain

$$\int_{\mathbb{S}^1} \left(\int_0^\infty t^{-p-1}\widehat{\mathcal{R}_u^{\mathbb{C}} f}(tc)dt\right) d\nu_{C,-1}(c) = c_p \int_{\mathbb{S}^1} \int_0^\infty r^p \left(\mathcal{R}_{cu}^{\mathbb{R}} f\right)(r)dr\, d\nu_{C,-1}(c)$$

for every even $f \in C^\infty(\mathbb{C}^n)$ with compact support. Recalling the convergence (3.14) of the family of distributions r_+^p from (3.13) as $p \to -1^+$, we deduce by dominated convergence (as $\widehat{\mathcal{R}_u^{\mathbb{C}} f}$ is a Schwartz function and $\mathcal{R}_{cu}^{\mathbb{R}} f$ has support uniformly bounded in c) that

$$\int_{\mathbb{S}^1} \left(\int_0^\infty \widehat{\mathcal{R}_u^{\mathbb{C}} f}(tc)dt\right) d\nu_{C,-1}(c) = \pi \int_{\mathbb{S}^1} \left(\mathcal{R}_{cu}^{\mathbb{R}} f\right)(0)\, d\nu_{C,-1}(c), \tag{3.37}$$

where we also used that $c_p\Gamma(p+1) \to \pi$. Combining (3.37) with (3.35),

$$\int_{\mathbb{C}} \rho_{C^\circ}(x) \left(\mathcal{R}_u^{\mathbb{C}} f\right)(x)dx = \pi \int_{\mathbb{S}^1} \left(\mathcal{R}_{cu}^{\mathbb{R}} f\right)(0)\, d\nu_{C,-1}(c),$$

which, since $\nu_{C,-1}$ is even and by approximating $\mathbb{1}_K$ by smooth functions f, implies

$$\rho_{\mathrm{I}_{C,-1}K}(u) = \int_{\mathbb{C}} \rho_{C^\circ}(z)\mathrm{A}^{\mathbb{C}}_{K,u}(z)dz = \pi \int_{\mathbb{S}^1} \mathrm{A}^{\mathbb{R}}_{K,cu}(0)\, d\nu_{C,-1}(c) = \frac{1}{2}\int_{\mathbb{S}^1} \rho_{\mathrm{I}K}(cu)\rho_{iC^\circ}(c)dc$$

for every $K \in \mathcal{S}_0(\mathbb{C}^n)$ and $u \in \mathbb{S}^{2n-1}$, yielding the claim. □

Note that we had to consider the case $p = -1$ separately in the proof as we can not apply (3.36) for $p = -1$.

The proof Theorem C is now a direct consequence of Proposition 3.5.2.

Proof of Theorem C. Let $p > -1$ and $C \in \mathcal{K}_{(0)}(\mathbb{C})$. If $K \in \mathcal{S}_0(\mathbb{C}^n)$ is $\mathbb{S}^1$-invariant, then I_pK is $\mathbb{S}^1$-invariant as well, by $\mathrm{SL}(2n,\mathbb{R})$-contravariance. Consequently, $\rho_{\mathrm{I}_pK}(cu) = \rho_{\mathrm{I}_pK}(u)$ for all $u \in \mathbb{S}^{2n-1}$ and $c \in \mathbb{S}^1$, and by Proposition 3.5.2,

$$\rho_{\mathrm{I}_{C,p}K}(u)^{-p} = \int_{\mathbb{S}^1} \rho_{\mathrm{I}_pK}(cu)^{-p}d\mu_{C,p}(c) = \mu_{C,p}(\mathbb{S}^1)\rho_{\mathrm{I}_pK}(u)^{-p}, \quad u \in \mathbb{S}^{2n-1},$$

which yields the claim for $d_{C,p} = \mu_{C,p}(\mathbb{S}^1)^{-1/p}$ (noting that $\mu_{C,p}(\mathbb{S}^1) > 0$ as it is equal to the radius of $\mathrm{I}_{C,p}B$ for some suitably chosen ball $B \subseteq \mathbb{C}^n$). □

We continue by proving Theorem D, which we can now state with the (technical) equality conditions, depending on the measure $\mu_{C,p}$ from Proposition 3.5.2.

Theorem 3.5.3. *Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ is origin-symmetric and $-1 \le p < 1$ is non-zero. If $K \in \mathcal{S}_0(\mathbb{C}^n)$, then*

$$\frac{\mathrm{V}_{2n}(\mathrm{I}_{C,p}K)}{\mathrm{V}_{2n}(\mathrm{I}_{C,p}B^{2n})} \le \frac{\mathrm{V}_{2n}(\mathrm{I}_pK)}{\mathrm{V}_{2n}(\mathrm{I}_pB^{2n})}. \tag{3.38}$$

If $\mu_{C,p}$ has infinite support equality holds if and only if I_pK is $\mathbb{S}^1$-invariant, and for all other $\mu_{C,p}$ equality holds if and only if $\mathrm{I}_pK = c_1\overline{c_2}\mathrm{I}_pK$ whenever $c_1, c_2 \in \operatorname{supp}\mu_{C,p}$.

Proof. By Proposition 3.5.2,

$$\mathrm{V}_{2n}(\mathrm{I}_{C,p}K) = \frac{1}{2n}\int_{\mathbb{S}^{2n-1}} \left(\int_{\mathbb{S}^1} \rho_{\mathrm{I}_pK}(cu)^{-p}d\mu_{C,p}(c)\right)^{-2n/p} du.$$

We apply Jensen's inequality to the inner integral to obtain

$$\mathrm{V}_{2n}(\mathrm{I}_{C,p}K) \le \frac{\mu_{C,p}(\mathbb{S}^1)^{-1-2n/p}}{2n}\int_{\mathbb{S}^{2n-1}}\int_{\mathbb{S}^1} \rho_{\mathrm{I}_pK}(cu)^{2n}d\mu_{C,p}(c)du. \tag{3.39}$$

Changing the order of integration using Fubini's theorem,

$$\mathrm{V}_{2n}(\mathrm{I}_{C,p}K) \le \mu_{C,p}(\mathbb{S}^1)^{-1-2n/p}\int_{\mathbb{S}^1} \mathrm{V}_{2n}(\bar{c}\mathrm{I}_pK)d\mu_{C,p}(c) = \mu_{C,p}(\mathbb{S}^1)^{-2n/p}\mathrm{V}_{2n}(\mathrm{I}_pK),$$

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we arrive at the desired inequality, since $\rho_{\mathrm{I}_{C,p}B^{2n}} = \mu_{C,p}(\mathbb{S}^1)^{-1/p}\rho_{\mathrm{I}_pB^{2n}}$ by (3.33).

Equality holds in (3.38) if and only if it holds in (3.39), that is, by the equality conditions of Jensen's inequality, exactly if for almost every $u \in \mathbb{S}^{2n-1}$ there exists $d_u \in \mathbb{R}$ such that

$$\rho_{\mathrm{I}_pK}(cu)^{-p} = d_u, \qquad \text{for } \mu_{C,p}\text{-a.e. } c \in \mathbb{S}^1.$$

Note that, by the continuity of ρ_{I_pK}, this holds indeed for *all* $u \in \mathbb{S}^{2n-1}$. Consequently, $\overline{c_1}\mathrm{I}_pK = \overline{c_2}\mathrm{I}_pK$ for $\mu_{C,p}$-almost all $c_1, c_2 \in \mathbb{S}^1$, that is, again by continuity,

$$\mathrm{I}_pK = c_1\overline{c_2}\mathrm{I}_pK, \quad c_1, c_2 \in \operatorname{supp}\mu_{C,p}, \tag{3.40}$$

which yields the equality condition for $\mu_{C,p}$ with finite support. If $\operatorname{supp}\mu_{C,p}$ is infinite, then, by compactness of $\mathbb{S}^1$, for every $\varepsilon > 0$, there exist $c_1, c_2 \in \operatorname{supp}\mu_{C,p}$ such that $|c_1\overline{c_2} - 1| < \varepsilon$. Iterating (3.40), we obtain that the map $c \mapsto \rho_{\mathrm{I}_pK}(cu)$ is constant on the set $\{(\overline{c_1}c_2)^k : k \in \mathbb{N}\} \subseteq \mathbb{S}^1$, which is ε-close to every $c \in \mathbb{S}^1$. Since $\varepsilon > 0$ was arbitrary and ρ_{I_pK} is continuous, we conclude that I_pK must be $\mathbb{S}^1$-invariant. The converse follows easily from (3.33). □

Proof of Corollary E. Suppose that $C \in \mathcal{K}_{(0)}(\mathbb{C})$ is origin-symmetric and let $K \in \mathcal{S}_0(\mathbb{C}^n)$. Then, by inequality (3.38) and Theorem 3.1.4 (respectively Busemann's intersection inequality (3.1) for $p = -1$) it follows that

$$\frac{\mathrm{V}_{2n}(\mathrm{I}_{C,p}K)}{\mathrm{V}_{2n}(\mathrm{I}_{C,p}B^{2n})} \le \frac{\mathrm{V}_{2n}(\mathrm{I}_pK)}{\mathrm{V}_{2n}(\mathrm{I}_pB^{2n})} \le \frac{\mathrm{V}_{2n}(K)^{2n+p}}{\mathrm{V}_{2n}(B^{2n})^{2n+p}}. \tag{3.41}$$

Equality holds for $p = -1$ and $K \in \mathcal{S}_0(\mathbb{C}^n)$ if and only if there is equality in (3.38) and (3.1). Since $d\mu_{C,-1} = \frac{1}{2}\rho_{iC^\circ}(c)dc$ has infinite support (equal to $\mathbb{S}^1$), the equality cases of Theorem 3.5.3 imply that $\mathrm{I}K$ must be $\mathbb{S}^1$-invariant, whereas the equality cases of (3.1) imply that K must be an origin-symmetric ellipsoid. Consequently, as the intersection body map is injective on origin-symmetric star bodies (see, e.g., [16, Thm. 8.1.3]), we conclude that K is an $\mathbb{S}^1$-invariant ellipsoid, which is equivalent to K being an origin-symmetric Hermitian ellipsoid.

If, on the other hand K is an origin-symmetric Hermitian ellipsoid, then there clearly is equality in (3.41). □

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4 Busemann–Petty type problems on complex vector spaces

This chapter is devoted to questions in the spirit of the Busemann–Petty and Shephard's problem. To this end, let us formulate both problems properly, starting with the famous *Busemann–Petty problem* [10]: Suppose that $K, L \in \mathcal{K}(\mathbb{R}^n)$ are origin-symmetric with non-empty interior, and assume that

$$\mathrm{V}_{n-1}(K \cap u^\perp) \le \mathrm{V}_{n-1}(L \cap u^\perp), \quad \forall u \in \mathbb{S}^{n-1}. \tag{4.1}$$

Does this imply that $\mathrm{V}_n(K) \le \mathrm{V}_n(L)$? The answer is affirmative, if $n \le 4$, and negative, if $n \ge 5$. Many authors contributed to this solution (see [5, 7, 13, 14, 18, 20, 33, 44, 47, 51, 68, 69, 71]). A unified proof was given finally by Gardner, Koldobsky and Schlumprecht in [18].

The dual question to the Busemann–Petty problem is known as *Shephard's problem* [64]: Suppose that $K, L \in \mathcal{K}(\mathbb{R}^n)$ are origin-symmetric with non-empty interior, and assume that

$$\mathrm{V}_{n-1}(K|u^\perp) \le \mathrm{V}_{n-1}(L|u^\perp), \quad \forall u \in \mathbb{S}^{n-1}. \tag{4.2}$$

Does this imply that $\mathrm{V}_n(K) \le \mathrm{V}_n(L)$? Here, the full answer was given by Petty [53] and Schneider [61]. It is affirmative, if $n \le 2$, and negative, if $n \ge 3$.

The key step in the proof of both problems was to link them to the *intersection body* for the Busemann–Petty problem (as was noticed by Lutwak [47]), and the *projection body* for Shephard's problem. Indeed, the inequality for the volume holds, that is, the answer is affirmative, if, in (4.1), K is an intersection body, or if, in (4.2), L is a projection body. Moreover, if $K \in \mathcal{K}(\mathbb{R}^n)$ is origin-symmetric, sufficiently regular and *not* a projection body, then there exists an origin-symmetric $L \in \mathcal{K}(\mathbb{R}^n)$ such that (4.2) holds, but the inequality for the volume is reversed (see [61]). A similar statement holds for intersection bodies and (4.1) (see [47]). As a consequence, both problems were solved by an analysis of the classes of intersection respectively projection bodies.

Interestingly, both problems can also be stated in terms of intersection respectively projection bodies, translating the inequality condition into a set inclusion condition. Therefore, one can summarize these as instances of the following more general class of questions which we vaguely formulate for geometric operators on (subsets of) convex bodies with non-empty interiors and make more precise later.

Problem 1. *Let Φ be a geometric operator on (subsets of) convex bodies taking values in the set of star bodies and let $K, L \in \mathcal{K}(\Phi)$.*

$$\textit{Does} \qquad \Phi K \subseteq \Phi L \quad \Longrightarrow \quad \mathrm{V}_n(K) \le \mathrm{V}_n(L) \qquad \textit{hold?}$$

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Here, we denote by $\mathcal{K}(\Phi) \subset \mathcal{K}(\mathbb{R}^n)$ a subset of convex bodies with non-empty interior, for which we will consider Problem 1. Usually, $\mathcal{K}(\Phi)$ will be the injectivity set of the operator Φ, which turns out to be the largest subset for which Problem 1 is non-trivial. Problem 1 has been considered widely, e.g., for centroid bodies [48], for L_p-intersection bodies [67], for Minkowski valuations compatible with rotations [63], for complex intersection bodies [39], replacing real by complex hyperplanes in (4.1) and origin-symmetry by $\mathbb{S}^1$-invariance, as well as in a functional setting [41].

In the following, we will consider Problem 1 for the operators Π_C (see (2.5)) and $\mathrm{I}_{C,p}$ only for origin-symmetric $C \in \mathcal{K}(\mathbb{C})$ containing the origin in its interior. This will be a general condition in all statements below, together with the assumption that $-2 < p \leq 1$ is non-zero.

As in the real setting, Π_C and $\mathrm{I}_{C,p}$ behave in a dual way. In order to state the results in a unified way, we will therefore "dualize" the problem for Π_C by considering its polar operator Π_C°, $\Pi_C^\circ K = (\Pi_C K)^\circ$. As polarity is order-reversing, this reverses the inequality in Problem 1, but has no further effects.

Moreover, note that for $p > 0$, the volume inequality that we expect in Problem 1 is reversed. We will take care of this fact by multiplying both sides with $-p < 0$ and set $p = 1$ when $\Phi = \Pi_C^\circ$. The main result of this chapter reads

Theorem F. *Let $K, L \in \mathcal{K}(\Phi)$.*

- *If $p < 0$, then for $n = 2$ and $\Phi = \mathrm{I}_{\mathbb{D},p}$, the implication*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad -p\mathrm{V}_{2n}(K) \leq -p\mathrm{V}_{2n}(L)$$

  holds. For $n \geq 3$ and $\Phi = \mathrm{I}_{C,p}$, this implication does not hold in general.

- *If $p > 0$, then for $n \geq 2$ and $\Phi \in \{\Pi_C^\circ, \mathrm{I}_{C,p}\}$, the implication*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad -p\mathrm{V}_{2n}(K) \leq -p\mathrm{V}_{2n}(L)$$

  does not hold in general.

Let us point out that Theorem F implies a similar statement for the complex centroid body defined in [27] (see Remark 4.3.6 below).

Our proof of Theorem F follows a similar strategy as [42, 60, 67] for real Busemann–Petty type problems. In particular, it relies heavily on a connection to the spherical Fourier transform. It is our second main result of this chapter that the map $\mathrm{J}_{C,p}$ on even functions can be decomposed into the (distributional) spherical Fourier transform $\mathbf{F}_q$ and an integral operator on the body C.

Theorem G. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric containing the origin in its interior and let $-2 < p \leq 1$ be non-zero. Then there exists a finite even Borel measure $\nu_{C,p}$ on $\mathbb{S}^1$, such that $-p\,\nu_{C,p} \geq 0$, and*

$$\mathrm{J}_{C,p}\varphi(u) = \frac{1}{(2\pi)^2} \int_{\mathbb{S}^1} (\mathbf{F}_{-2n-p}\varphi)(cu)\, d\nu_{C,p}(c), \quad u \in \mathbb{S}^{2n-1}, \tag{4.3}$$

for all even $\varphi \in C^\infty(\mathbb{S}^{2n-1})$.

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For a more explicit description of the measure $\nu_{C,p}$, we refer to Section 4.2. Let us note here that Theorem G extends the well-known connection between the L_p-cosine transform and the spherical Fourier transform (see, e.g., [22]), which corresponds to the limit case $C = [-1, 1]$ with $\nu_{C,p} = c_p(\delta_1 + \delta_{-1})$ for some $c_p \in \mathbb{R}$.

Using Theorem G, the proof of Theorem F follows the steps that are usually taken in solutions of Problem 1. For the reader's convenience, we will explain them here in detail, including the exact statements proven in each step. We assume $n \geq 2$.

Determination of $\mathcal{K}(\Phi)$. The first step is to determine the maximal set $\mathcal{K}(\Phi)$ for which Problem 1 is non-trivial. As mentioned above, it is natural to define $\mathcal{K}(\Phi)$ as the injectivity set of Φ. More precisely, let $\mathcal{M}(\mathbb{S}^{2n-1})$ denote the set of all (signed) finite Borel measures on $\mathbb{S}^{2n-1}$ and denote by

$$\mathrm{Inj}(\mathrm{J}_{C,p}) = \{\mu \in \mathcal{M}(\mathbb{S}^{2n-1}) : \forall k, l \in \mathbb{N} : \pi_{k,l}(\mathrm{J}_{C,p}\mu) = 0 \implies \pi_{k,l}\mu = 0\}$$

the injectivity space of $\mathrm{J}_{C,p}$. The injectivity sets of Π_C and $\mathrm{I}_{C,p}$ are defined by

$$\mathcal{K}(\Pi_C) = \{K \in \mathcal{K}(\mathbb{C}^n) : \mathrm{int}\, K \neq \emptyset, S_K \in \mathrm{Inj}(\mathrm{J}_{C,1})\}, \tag{4.4}$$

$$\mathcal{K}(\mathrm{I}_{C,p}) = \{K \in \mathcal{K}(\mathbb{C}^n) : \mathrm{int}\, K \neq \emptyset, \rho_K^{2n+p} \in \mathrm{Inj}(\mathrm{J}_{C,p})\}. \tag{4.5}$$

In particular, if C is origin-symmetric, then every $K \in \mathcal{K}(\Phi)$ is origin-symmetric, and all $\mathbb{S}^1$-invariant convex bodies with non-empty interior are contained both in $\mathcal{K}(\Pi_C)$ and $\mathcal{K}(\mathrm{I}_{C,p})$.

Outside the injectivity sets, a perturbation argument using a specific spherical harmonic shows that the answer to Problem 1 is negative in general. It is therefore necessary and reasonable to restrict the study of Problem 1 to bodies in $\mathcal{K}(\Phi)$.

Proposition 4.1.1. *Let $\Phi \in \{\Pi_C^\circ, \mathrm{I}_{C,p}\}$ and suppose that $L \in \mathcal{K}(\mathbb{C}^n)$ is smooth with positive curvature. If $\mathcal{K}(\Phi)$ is a strict subset of the set of convex bodies with non-empty interior, then there is $K \notin \mathcal{K}(\Phi)$ with non-empty interior, such that*

$$\Phi K = \Phi L \qquad \textit{but} \qquad -p\mathrm{V}_{2n}(K) > -p\mathrm{V}_{2n}(L).$$

Affirmative Cases. The next step is to confirm the statement of Problem 1 in special cases, corresponding to intersection respectively projection bodies in the Busemann–Petty respectively Shephard problem. In contrast to the real problems, here, the answer is affirmative for bodies in the image of the *conjugated* operator $\Phi^\#$, where $\mathrm{I}_{C,p}^\# = \mathrm{I}_{\overline{C},p}$ and $\Pi_C^\# = \Pi_{\overline{C}}$. If $C = \overline{C}$, in particular, in the limit $C \to [-1, 1]$, this is consistent with the real case. We write $\mathrm{im}\, \Phi^\#$ for $\Pi_{\overline{C}}(\mathcal{K})$ respectively $\mathrm{I}_{\overline{C},p}(\mathcal{S}_0)$, where

$$\Pi_C(\mathcal{K}) = \{K \in \mathcal{K}(\Phi) : \exists L \in \mathcal{K}(\mathbb{C}^n) : \Pi_C L = K\},$$
$$\mathrm{I}_{C,p}(\mathcal{S}_0) = \{K \in \mathcal{K}(\Phi) : \exists L \in \mathcal{S}_0(\mathbb{C}^n) : \mathrm{I}_{C,p} L = K\}.$$

Let us point out here that we consider the image of all convex respectively star bodies, that is, we do not consider Φ as operator on $\mathcal{K}(\Phi)$ only. The statement then reads

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Proposition 4.1.2. *Let $\Phi \in \{\Pi_C^\circ, \mathrm{I}_{C,p}\}$. If $K \in \operatorname{im}\Phi^\#$ and $L \in \mathcal{K}(\Phi)$, then*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad -pV_{2n}(K) \leq -pV_{2n}(L).$$

Moreover, in this case, if $\Phi K \subseteq \Phi L$, then $\mathrm{V}_{2n}(K) = \mathrm{V}_{2n}(L)$ holds if and only if K is a translate of L, when $\Phi = \Pi_C^\circ$, and only if $K = L$, when $\Phi = \mathrm{I}_{C,p}$.

Proposition 4.1.2 is proved using an adjointness property of Π_C respectively $\mathrm{I}_{C,p}$ with respect to mixed volume respectively dual mixed volume, similar to the real case. In this sense, $\Phi^\#$ can be interpreted as the adjoint of Φ (while $\mathrm{J}_{\overline{C},p}$ is indeed the adjoint of $\mathrm{J}_{C,p}$).

Outside the image. To understand the image of Π_C° respectively $\mathrm{I}_{C,p}$, we will use the notion of a body embedding into L_p (see Definition 2.2.4 below), which already proved very useful for L_p-intersection bodies. Indeed, we will show that all bodies in $\Pi_C^\circ(\mathcal{K})$ respectively $\mathrm{I}_{C,p}(\mathcal{S}_0)$ embed into L_p.

Applying a known characterization of embedability by the spherical Fourier transform, a perturbation argument then shows that Problem 1, restricted to $\mathbb{S}^1$-invariant bodies, has a negative answer outside the images. Let us point out here that, in contrast to the usual treatment of real Busemann–Petty problems, we do not extend the maps $\mathrm{I}_{C,p}$ to measures. Therefore, technically, we have to consider the (weak) closure of the images (see Proposition 4.3.4 and Proposition 4.4.3 below).

Proposition 4.1.3. *Let $\Phi \in \{\Pi_C^\circ, \mathrm{I}_{C,p}\}$. If there is an $\mathbb{S}^1$-invariant body in $\mathcal{K}(\Phi)$, which is not in the (weak) closure of $\Pi_C^\circ(\mathcal{K}) \cap \mathcal{K}(\Pi_C)$ respectively $\mathrm{I}_{C,p}(\mathcal{S}_0) \cap \mathcal{K}(\mathrm{I}_{C,p})$, then there exist $\mathbb{S}^1$-invariant bodies $K, L \in \mathcal{K}(\Phi)$ such that*

$$\Phi K \subseteq \Phi L \qquad \textit{but} \qquad -p\mathrm{V}_{2n}(K) > -p\mathrm{V}_{2n}(L).$$

Counter examples in higher dimensions. Proposition 4.1.3 allows to give $\mathbb{S}^1$-invariant counter examples to Problem 1 in higher dimensions. These are known examples of bodies which do not embed into L_p (see [39]).

Proposition 4.1.4. *Let $\Phi \in \{\Pi_C^\circ, \mathrm{I}_{C,p}\}$. If $n \geq 3$, then there exists an $\mathbb{S}^1$-invariant body $K \in \mathcal{K}(\Phi)$, which is not in the (weak) closure of $\operatorname{im}(\Phi) \cap \mathcal{K}(\Phi)$. If $p > 0$, then the same is true also for $n = 2$.*

Affirmative answer in low dimension. The final step is to give an affirmative answer to Problem 1 in the remaining (complex) dimension $n = 2$. As we describe the image of $\mathrm{I}_{C,p}$ only in complete detail for $C = \mathbb{D}$, we prove

Proposition 4.1.5. *If $\Phi = \mathrm{I}_{\mathbb{D},p}$ and $-2 < p < 0$, then the answer to Problem 1 is affirmative in $\mathbb{C}^2$.*

4.2 Analysis of the operator $\mathrm{J}_{C,p}$

In this section, we study the operator $\mathrm{J}_{C,p}$ using the theory of spherical harmonics in complex vector spaces, and prove Theorem G.

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Combining (3.20) and (2.11), we deduce that (4.3) holds, if the integral transform defined by $\nu_{C,p}$ has multipliers $\lambda_{k,l}[\mathrm{J}_{C,p}]/(4\pi^2\lambda_{k,l}[\mathbf{F}_p])$. It remains to find a measure $\nu_{C,p}$ with this property and an appropriate sign. For this reason, we first describe the multipliers of operators T_μ of the form

$$\mathrm{T}_\mu f(u) = \int_{\mathbb{S}^1} f(cu)d\mu(c), \quad u \in \mathbb{S}^{2n-1}, f \in C(\mathbb{S}^{2n-1}), \tag{4.6}$$

where μ is a finite Borel measure on $\mathbb{S}^1$.

Lemma 4.2.1. *Let μ be a finite real Borel measure on $\mathbb{S}^1$ and let $f, g \in C(\mathbb{S}^{2n-1})$.*

i) Then $\langle \mathrm{T}_\mu f, g\rangle = \langle f, \mathrm{T}_{\mu^\#} g\rangle$, where $\mu^\#$ is the push-forward measure of μ by complex conjugation.

ii) The multipliers $\lambda_{k,l}[T_\mu]$, $k, l \in \mathbb{N}$, of T_μ are given by

$$\lambda_{k,l}[\mathrm{T}_\mu] = \begin{cases} 2\pi c_0[\mu] & k = l, \\ \pi c_{l-k}[\mu] & k \neq l. \end{cases}$$

In particular, $\lambda_{k,l}[\mathrm{T}_{\mu^\#}] = \pi c_{k-l}[\mu]$ for $k \neq l \in \mathbb{N}$.

Let us note that Lemma 4.2.1 generalizes a result from [3] for $\mu = S_K$, $K \in \mathcal{K}(\mathbb{C})$. By i), T_μ extends to measures.

Proof. Point i) follows by a direct computation

$$\begin{aligned}\langle \mathrm{T}_\mu f, g\rangle &= \int_{\mathbb{S}^{2n-1}} \int_{\mathbb{S}^1} f(cu)d\mu(c)\overline{g(u)}du = \int_{\mathbb{S}^{2n-1}} f(u) \int_{\mathbb{S}^1} \overline{g(\bar{c}u)}d\mu(c)du \\ &= \int_{\mathbb{S}^{2n-1}} f(u)\overline{\int_{\mathbb{S}^1} g(\bar{c}u)d\mu(c)}du = \langle f, \mathrm{T}_{\mu^\#} g\rangle .\end{aligned}$$

For ii), we use the $\mathrm{U}(n)$-invariance of the spherical Lebesgue measure and Fubini's theorem, to obtain for $Y_{k,l} \in \mathcal{H}^{2n}_{k,l}$

$$\begin{aligned}\langle Y_{k,l}, \mathrm{T}_\mu f\rangle = \langle \mathrm{T}_{\mu^\#} Y_{k,l}, f\rangle &= \int_{\mathbb{S}^{2n-1}} \int_{\mathbb{S}^1} Y_{k,l}(\bar{c}u)d\mu(c)\overline{f(u)}du \\ &= \int_{\mathbb{S}^{2n-1}} \int_{\mathbb{S}^1} \bar{c}^{k-l} Y_{k,l}(u)d\mu(c)\overline{f(u)}du = \pi c_{l-k}[\mu] \int_{\mathbb{S}^{2n-1}} Y_{k,l}(u)\overline{f(u)}du \\ &= \pi c_{l-k}[\mu] \langle Y_{k,l}, f\rangle ,\end{aligned}$$

for $l \neq k$, which yields the claim. The case $k = l$ is similar. □

We can now use that every suitable origin-symmetric body $C \subset \mathbb{C}$ embeds into L_p and, hence, by Proposition 2.2.5, we obtain a measure with the correct sign.

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Proposition 4.2.2. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric containing the origin in its interior, and $-2 < p \leq 1$. Then the distribution $\nu_{C,p} = \mathbf{F}_p h_C^p$ on $\mathbb{S}^1$ satisfies*

$$\frac{1}{\Gamma\left(-\frac{p}{2}\right)} \nu_{C,p} \geq 0,$$

and therefore is a measure. The multipliers of $\mathrm{T}_{\nu_{C,p}}$ are given by

$$\lambda_{k,l}[\mathrm{T}_{\nu_{C,p}}] = \begin{cases} 2\pi k_p \Gamma\left(\frac{p}{2}+1\right)^2 c_0[h_C^p], & k = l, \\ \pi k_p \Gamma\left(\frac{p+l-k}{2}+1\right)\Gamma\left(\frac{p-l+k}{2}+1\right) c_{l-k}[h_C^p], & k \neq l, \end{cases} \tag{4.7}$$

where $k_p = -2^{2+p}\sin\left(\frac{p\pi}{2}\right)$.

For $p = 1$, we have $\nu_{C,1} = -2\pi\mathcal{S}(iC,\cdot)$ and for $p = -1$, we have $d\nu_{C,-1} = 2\pi\rho_{iC^\circ}dc$.

Proof. By Lemma 2.2.6, every origin-symmetric convex body C with non-empty interior embeds into L_p for every non-zero $-2 < p \leq 1$. Now, Proposition 2.2.5 implies that $\mathbf{F}_p h_C^p = \mathbf{F}_p \rho_{C^\circ}^{-p}$ is up to the sign of $\Gamma\left(-\frac{p}{2}\right)$ a positive and even measure, yielding the first claim.

Next, noting that $\mathcal{H}_k^2$ is spanned by the maps $c \mapsto c^k, c^{-k}$, we have for even $k \neq 0$

$$\pi c_k[\mathbf{F}_p h_C^p] = \langle \pi_{k,0}\mathbf{F}_p h_C^p, c^k\rangle = \lambda_{k,0}[\mathbf{F}_p]\langle \pi_{k,0} h_C^p, c^k\rangle = \pi\lambda_{k,0}[\mathbf{F}_p] c_k[h_C^p],$$

and thus, by (2.11) and a similar computation for $k = 0$,

$$c_k[\mathbf{F}_p h_C^p] = \lambda_{k,0}[\mathbf{F}_p] c_k[h_C^p] = (-1)^{\frac{k}{2}} 2^{2+p}\pi \frac{\Gamma\left(\frac{k+p}{2}+1\right)}{\Gamma\left(\frac{k-p}{2}\right)} c_k[h_C^p]. \tag{4.8}$$

Applying Euler's reflection formula, we obtain for even k,

$$c_k[\mathbf{F}_p h_C^p] = -2^{2+p}\sin\left(\frac{p\pi}{2}\right)\Gamma\left(\frac{p+k}{2}+1\right)\Gamma\left(\frac{p-k}{2}+1\right) c_k[h_C^p],$$

which, by Lemma 4.2.1, yields the second claim as the multipliers for $k + l$ odd vanish.

For $p = 1$ and even k, (4.8) reduces to

$$c_k[\nu_{C,1}] = (-1)^{\frac{k}{2}} 8\pi \frac{\Gamma\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)} c_k[h_C] = 2\pi(-1)^{\frac{k}{2}}(k^2-1)c_k[h_C].$$

As, by [3, Lem. 4.6], the Fourier coefficients of $S_C(\cdot)$ are given by $(1-k^2)c_k[h_C]$, this yields the claim for $p = 1$. The statement for $p = -1$ follows directly. □

Corollary 4.2.3. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric, $0 \in \operatorname{int} C$, and $-2 < p \leq 1$ non-zero. Then $\nu_{\overline{C},p} = \nu_{C,p}^{\#}$ and, thus, $\mathrm{T}_{\nu_{\overline{C},p}}$ and $\mathrm{T}_{\nu_{C,p}}$ are adjoint.*

Proof. As $c_k[\mu^{\#}] = \overline{c_k[\mu]}$ for a real measure μ, and $c_k[h_{\overline{C}}^p] = \overline{c_k[h_C^p]}$, the first claim follows from (4.8). The second claim follows from Lemma 4.2.1i). □

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We are now ready to prove Theorem G.

Proof of Theorem G. By the origin-symmetry of C, $c_{l-k}[h_C^p] = 0$ when $k+l$ is odd, and thus $\lambda_{k,l}[\mathrm{J}_{C,p}] = 0$, that is, we only have to consider $k+l$ even. By (3.20) and (2.11), we obtain using $k_p = -2^{2+p}\sin(p\pi/2)$

$$\lambda_{k,l}[\mathrm{J}_{C,p}] = \begin{cases} \lambda_{k,l}[\mathbf{F}_{-2n-p}]\frac{k_p}{2\pi}\Gamma\left(\frac{p}{2}+1\right)^2 c_0[h_C^p], & k=l, \\ \lambda_{k,l}[\mathbf{F}_{-2n-p}]\frac{k_p}{4\pi}\Gamma\left(\frac{p+k-l}{2}+1\right)\Gamma\left(\frac{p-k+l}{2}+1\right)c_{l-k}[h_C^p], & k<l. \end{cases}$$

Consequently, by (4.7), $(2\pi)^2\lambda_{k,l}[\mathrm{J}_{C,p}] = \lambda_{k,l}[\mathbf{F}_{-2n-p}]\lambda_{k,l}[T_{\nu_{C,p}}]$, yielding the claim. □

Let us point out that the spherical Fourier transform maps smooth even functions to smooth even functions on $\mathbb{S}^{2n-1}$. However, $\mathbf{F}_p f$ for $f \in C(\mathbb{S}^{2n-1})$, defined in the distributional sense, in general only yields a distribution. As we want to apply Theorem G in the following for radial functions of general star respectively convex bodies, we need to argue that, in this case, $\mathrm{T}_{\nu_{C,p}} \circ \mathbf{F}_{-2n-p}$ maps continuous functions to continuous functions.

Proposition 4.2.4. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric containing the origin in its interior, and $-2 < p \leq 1$ non-zero. Then for $\nu_{C,p} = \mathbf{F}_p h_C^p$ we have*

$$\rho_{\mathrm{I}_{C,p}K}^{-p} = \frac{1}{(2\pi)^2(2n+p)}\mathrm{T}_{\nu_{C,p}} \circ \mathbf{F}_{-2n-p}\rho_K^{2n+p},$$

for every $K \in \mathcal{S}_0(\mathbb{C}^n)$ and

$$h_{\Pi_C K} = -\frac{1}{4\pi}\mathrm{T}_{S_{iC}} \circ \mathbf{F}_{-2n-1}S_K,$$

for every $K \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior.

Before proving Proposition 4.2.4, we note that

$$\rho_{\mathrm{I}_{C,p}K}^{-p} = \frac{1}{2n+p}\mathrm{J}_{C,p}(\rho_K^{2n+p}) \quad \text{and} \quad h_{\Pi_C K} = \frac{1}{2}\mathrm{J}_{C,1}(S_K) \tag{4.9}$$

for $K \in \mathcal{S}_0(\mathbb{C}^n)$ respectively $K \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior. Moreover, let us point out that the order of the transforms on the right-hand side of the equations in Proposition 4.2.4 is interchangeable as they are all multiplier transforms.

Proof of Proposition 4.2.4. First note that since C is origin-symmetric, and thus $\nu_{C,p}$ is even, both sides of (4.3) in Theorem G vanish for odd $\varphi \in C^\infty(\mathbb{S}^{2n-1})$. Hence, we conclude that the statement of Theorem G holds for all $\varphi \in C^\infty(\mathbb{S}^{2n-1})$.

Next, let $\varphi \in C^\infty(\mathbb{S}^{2n-1})$. By (4.9), the fact that $\mathrm{J}_{\overline{C},p}$ is the adjoint of $\mathrm{J}_{C,p}$ and Theorem G,

$$\begin{aligned}\left\langle \rho_{\mathrm{I}_{C,p}K}^{-p}, \varphi \right\rangle &= \frac{1}{2n+p}\left\langle \mathrm{J}_{C,p}\rho_K^{2n+p}, \varphi \right\rangle = \frac{1}{2n+p}\left\langle \rho_K^{2n+p}, \mathrm{J}_{\overline{C},p}\varphi \right\rangle \\ &= \frac{1}{(2\pi)^2(2n+p)}\left\langle \rho_K^{2n+p}, \mathrm{T}_{\nu_{\overline{C},p}} \circ \mathbf{F}_{-2n-p}\varphi \right\rangle.\end{aligned}$$

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Since the involved operators are multiplier transforms, they commute and therefore

$$\left\langle \rho_{\mathrm{I}_{C,p}K}^{-p}, \varphi \right\rangle = \frac{1}{(2\pi)^2(2n+p)} \left\langle \rho_K^{2n+p}, \mathbf{F}_{-2n-p} \circ \mathrm{T}_{\nu_{\overline{C},p}} \varphi \right\rangle.$$

Consequently, as φ was arbitrary, and $\mathrm{T}_{\nu_{C,p}} \circ \mathbf{F}_{-2n-p}$ is, by Corollary 4.2.3, the adjoint of $\mathbf{F}_{-2n-p} \circ \mathrm{T}_{\nu_{\overline{C},p}}$, we obtain $\rho_{\mathrm{I}_{C,p}K}^{-p} = \frac{1}{(2\pi)^2(2n+p)} \mathrm{T}_{\nu_{C,p}} \circ \mathbf{F}_{-2n-p} \rho_K^{2n+p}$ in the sense of distributions. However, since the left hand side is a continuous function on the sphere, the equality indeed holds as equality of continuous functions.

Repeating the argument above and noting that $-2\pi h_{\Pi K} = \mathbf{F}_{-2n-1} S_K$ (see [38, Thm. 8.1]) is a continuous map on $\mathbb{S}^{2n-1}$, the second equality follows since $\nu_{C,1} = -2\pi \mathcal{S}(iC, \cdot)$, by Proposition 4.2.2. □

4.3 Busemann–Petty problem for complex L_p-intersection bodies

In this section, we prove Theorem F for the complex L_p-intersection body maps $\mathrm{I}_{C,p}$ following the steps outlined at the beginning of this chapter.

4.3.1 Determination of $\mathcal{K}(\Phi)$

First, we prove that the injectivity set $\mathcal{K}(\mathrm{I}_{C,p})$ defined in (4.5) is maximal in the sense that the answer to Problem 1 is negative when the bodies are not in $\mathcal{K}(\mathrm{I}_{C,p})$.

Before doing so, let us point out that a body $K \in \mathcal{K}(\mathbb{C}^n)$ is in $\mathcal{K}(\mathrm{I}_{C,p})$, if $\pi_{k,l}\rho_K^{2n+p} = 0$ whenever $\lambda_{k,l}[\mathrm{J}_{C,p}] = 0$ or, equivalently by (3.20), $c_{l-k}[h_C^p] = 0$. In particular, for $K, L \in \mathcal{K}(\mathrm{I}_{C,p})$, $\mathrm{I}_{C,p}K = \mathrm{I}_{C,p}L$ implies $K = L$. Moreover, as $c_k[h_{\overline{C}}^p] = \overline{c_k[h_C^p]}$, we directly obtain that $\mathcal{K}(\mathrm{I}_{\overline{C},p}) = \mathcal{K}(\mathrm{I}_{C,p})$.

In the following, we will also work with the set of $\mathbb{S}^1$-invariant convex bodies with non-empty interior. As a function $f \in C(\mathbb{S}^{2n-1})$ is $\mathbb{S}^1$-invariant, if and only if $\pi_{k,l} f = 0$ for all $k \neq l$, and as $c_0[h_C^p] > 0$, all such bodies are contained in $\mathcal{K}(\mathrm{I}_{C,p})$.

For the proof of Proposition 4.1.1, we will use the following convexity statement for small perturbations.

Lemma 4.3.1 ([38, p. 96])**.** *Let L be a smooth convex body with strictly positive curvature and $p \in \mathbb{R} \setminus \{0\}$. Then for any $\varphi \in C^\infty(\mathbb{S}^{n-1})$, the star body K defined by*

$$\rho_K^{-p} = \rho_L^{-p} + \varepsilon\varphi$$

is convex for all $\varepsilon > 0$ sufficiently small.

Proof of Proposition 4.1.1 for $\mathrm{I}_{C,p}$. As the set $\mathcal{K}(\mathrm{I}_{C,p})$ is a strict subset, there exist indices $(k, l) \in \mathbb{N} \times \mathbb{N}$ such that $c_{l-k}[h_C^p] = 0$. Denoting by $\widetilde{P}_{k,l} \in \mathcal{H}_{k,l}^{2n}$ the Jacobi polynomial of bi-degree (k, l), we define

$$\rho_K^{2n+p} = \rho_L^{2n+p} + \varepsilon \widetilde{P}_{k,l}$$

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where $|\varepsilon|$ is sufficiently small such that $K \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior (using Lemma 4.3.1), and the sign of ε is chosen in a way such that

$$-p\varepsilon \left\langle \widetilde{P}_{k,l}, \rho_L^{-p} \right\rangle \geq 0. \tag{4.10}$$

By our choice of (k, l), we have $\mathrm{T}_{\nu_{C,p}} \widetilde{P}_{k,l} = 0$ and thus, by Proposition 4.2.4 and since multiplier transforms commute, $\mathrm{I}_{C,p}K = \mathrm{I}_{C,p}L$.

Next, note that by (4.10),

$$-2np\tilde{\mathrm{V}}_{-p}(K, L) = -p \left\langle \rho_K^{2n+p}, \rho_L^{-p} \right\rangle = -2np\mathrm{V}_{2n}(L) - p\varepsilon \left\langle \widetilde{P}_{k,l}, \rho_L^{-p} \right\rangle \geq -2np\mathrm{V}_{2n}(L),$$

and, consequently, by (2.2),

$$-p\mathrm{V}_{2n}(L) \leq -p\tilde{\mathrm{V}}_{-p}(K, L) \leq -p\mathrm{V}_{2n}(K)^{\frac{2n+p}{2n}} \mathrm{V}_{2n}(L)^{-\frac{p}{2n}},$$

that is, $-p\mathrm{V}_{2n}(L) \leq -p\mathrm{V}_{2n}(K)$. As equality holds in (2.2) only when K, L are dilates, and, by construction, this is not the case for K and L here, the inequality is strict, concluding the proof. □

4.3.2 Affirmative Cases

Next, we will show that Problem 1 has a positive answer, whenever $K \in \mathcal{K}(\mathrm{I}_{C,p})$ is in the image set $\mathrm{I}_{\overline{C},p}(\mathcal{S}_0)$.

Proof of Proposition 4.1.2 for $\mathrm{I}_{C,p}$. By Proposition 4.2.4 and Corollary 4.2.3, we obtain for every $K, L \in \mathcal{S}_0(\mathbb{C}^n)$

$$\begin{aligned} 2n\tilde{\mathrm{V}}_{-p}(K, \mathrm{I}_{\overline{C},p}L) &= \frac{1}{2n+p} \left\langle \rho_K^{2n+p}, \mathrm{J}_{\overline{C},p}\rho_L^{2n+p} \right\rangle \\ &= \frac{1}{(2\pi)^2(2n+p)} \left\langle \rho_K^{2n+p}, \mathrm{T}_{\nu_{\overline{C},p}} \mathbf{F}_{-2n-p}\rho_L^{2n+p} \right\rangle \\ &= \frac{1}{(2\pi)^2(2n+p)} \left\langle \mathrm{T}_{\nu_{C,p}} \mathbf{F}_{-2n-p}\rho_K^{2n+p}, \rho_L^{2n+p} \right\rangle = 2n\tilde{\mathrm{V}}_{-p}(L, \mathrm{I}_{C,p}K). \end{aligned} \tag{4.11}$$

Next, let $K \in \mathrm{I}_{\overline{C},p}(\mathcal{S}_0)$, that is, $K = \mathrm{I}_{\overline{C},p}K_0$ for some $K_0 \in \mathcal{S}_0(\mathbb{C}^n)$, and assume that $\mathrm{I}_{C,p}K \subseteq \mathrm{I}_{C,p}L$ for $L \in \mathcal{K}(\mathrm{I}_{C,p})$. Since $-p\tilde{\mathrm{V}}_{-p}(K_0, \cdot)$ is monotone, (4.11) together with the inclusion implies

$$\begin{aligned} -p\mathrm{V}_{2n}(K) = -p\tilde{\mathrm{V}}_{-p}(K, \mathrm{I}_{\overline{C},p}K_0) &= -p\tilde{\mathrm{V}}_{-p}(K_0, \mathrm{I}_{C,p}K) \\ &\leq -p\tilde{\mathrm{V}}_{-p}(K_0, \mathrm{I}_{C,p}L) = -p\tilde{\mathrm{V}}_{-p}(L, K) \end{aligned}$$

Applying the dual L_p-Minkowski inequality (2.2),

$$-p\mathrm{V}_{2n}(K) \leq -p\tilde{\mathrm{V}}_{-p}(L, K) \leq -p\mathrm{V}_{2n}(L)^{\frac{2n+p}{2n}} \mathrm{V}_{2n}(K)^{-\frac{p}{2n}},$$

that is, $-p\mathrm{V}_{2n}(K) \leq -p\mathrm{V}_{2n}(L)$, as claimed.

The statement when $\mathrm{V}_{2n}(K) = \mathrm{V}_{2n}(L)$ follows directly from the equality cases of the dual L_p-Minkowski inequality. □

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4.3.3 Outside the image

In the next step, we show that, if there are bodies that are not in the image of $\mathrm{I}_{C,p}$, then Problem 1 has a negative answer. To this end, we will give a characterization of $\mathrm{I}_{C,p}(\mathcal{S}_0)$ in terms of the spherical Fourier transform, similar to the real setting. Indeed, we first show that every body in the image embeds into L_p, that is, by [57, Prop. 5.4], is a real L_p-intersection body.

Proposition 4.3.2. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric with non-empty interior, and $-2 < p \le 1$ be non-zero. If $K \in \mathrm{I}_{C,p}(\mathcal{S}_0)$, then*

$$\frac{1}{\Gamma\left(-\frac{p}{2}\right)} \mathbf{F}_p \rho_K^{-p} \ge 0.$$

In particular, $\mathrm{I}_{C,p}(\mathcal{S}_0) \subseteq \{\hookrightarrow L_p\} \cap \{K \in \mathcal{S}_0(\mathbb{C}^n) : \rho_K^{-p} \in \operatorname{im} \mathrm{T}_{\nu_{C,p}}\}$.

Proof. Let $K_0 \in \mathcal{K}(\mathrm{I}_{C,p})$ such that $K = \mathrm{I}_{C,p} K_0$. By Proposition 4.2.4 and since multiplier transforms commute, we get

$$\mathbf{F}_p \rho_K^{-p} = \mathbf{F}_p \rho_{\mathrm{I}_{C,p}K_0}^{-p} = \frac{(2\pi)^{-2}}{(2n+p)} \mathbf{F}_p \circ \mathbf{F}_{-2n-p} \circ \mathrm{T}_{\nu_{C,p}} \rho_{K_0}^{2n+p} = \frac{(2\pi)^{2n-2}}{2n+p} \mathrm{T}_{\nu_{C,p}} \rho_{K_0}^{2n+p}.$$

By Proposition 4.2.2, $\frac{1}{\Gamma(-\frac{p}{2})} \nu_{C,p}$ is a positive measure and, hence,

$$\frac{1}{\Gamma\left(-\frac{p}{2}\right)} \mathbf{F}_p \rho_K^{-p} = \frac{(2\pi)^{2n-2}}{(2n+p)\Gamma\left(-\frac{p}{2}\right)} \mathrm{T}_{\nu_{C,p}} \rho_{K_0}^{2n+p} \ge 0,$$

which by Proposition 2.2.5 is equivalent to K embedding into L_p. □

Note that, in general, the image of $\mathrm{T}_{\nu_{C,p}}$ can be quite complicated and is only described by spherical harmonics. However, as it turns out, if the statement of Problem 1 is negative, it already fails when restricted to $\mathbb{S}^1$-invariant bodies. In the following, we will therefore concentrate on $\mathbb{S}^1$-invariant bodies, which we denote by a superscript $(\cdot)^{\mathbb{S}^1}$, that is, e.g., $\mathrm{I}_{C,p}(\mathcal{S}_0)^{\mathbb{S}^1}$.

Let us point out that, if $C = \mathbb{D}$, then $\operatorname{im} \mathrm{T}_{\nu_{\mathbb{D},p}} = C(\mathbb{S}^{2n-1})^{\mathbb{S}^1}$, as can be seen, e.g., by (4.7). Next, we show that on $\mathbb{S}^1$-invariant bodies, the image of $\mathrm{I}_{C,p}$ does not depend on C and coincides (up to closure) with $\mathbb{S}^1$-invariant bodies embedding into L_p.

Proposition 4.3.3. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric with non-empty interior, and $-2 < p \le 1$ be non-zero. Then*

$$\mathrm{I}_{C,p}(\mathcal{S}_0)^{\mathbb{S}^1} = \mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0) \quad \textit{and} \quad \mathrm{cl}_w\left(\mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0)\right) = \{\hookrightarrow L_p\}^{\mathbb{S}^1},$$

where we denote by cl_w the closure with respect to weak convergence of ρ_K^{-p}.

Proof. Let $K \in \mathrm{I}_{C,p}(\mathcal{S}_0)^{\mathbb{S}^1}$, that is, $\mathbb{S}^1$-invariant and $K = \mathrm{I}_{C,p} K_0$ for some $K_0 \in \mathcal{S}_0(\mathbb{C}^n)$. By the $\mathbb{S}^1$-invariance, $\mathrm{T}_{\nu_{C,p}} \rho_K^{-p} = \nu_{C,p}(\mathbb{S}^1) \rho_K^{-p}$ for all C, and, by Proposition 4.2.4,

$$\nu_{\mathbb{D},p}(\mathbb{S}^1) \rho_K^{-p} = \mathrm{T}_{\nu_{\mathbb{D},p}} \rho_K^{-p} = \frac{1}{(2\pi)^2(2n+p)} \mathrm{T}_{\nu_{\mathbb{D},p}} \circ \mathbf{F}_{-2n-p} \circ \mathrm{T}_{\nu_{C,p}} \rho_{K_0}^{2n+p}.$$

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We can therefore define $\widetilde{K}_0 \in \mathcal{S}_0(\mathbb{C}^n)$ by

$$\rho_{\widetilde{K}_0}^{2n+p} = \frac{1}{\nu_{\mathbb{D},p}(\mathbb{S}^1)} \mathrm{T}_{\nu_{C,p}} \rho_{K_0}^{2n+p}$$

to see that $K = \mathrm{I}_{\mathbb{D},p}\widetilde{K}_0$ and thus $K \in \mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0)$. Let us point out here that we used that $\nu_{\mathbb{D},p}(\mathbb{S}^1)^{-1}\nu_{C,p} \geq 0$, by Proposition 4.2.2, in order to get $\rho_{\widetilde{K}_0} > 0$. Repeating the argument with C and $\mathbb{D}$ replaced yields the other inclusion of $\mathrm{I}_{C,p}(\mathcal{S}_0)^{\mathbb{S}^1} = \mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0)$.

Next, by Proposition 4.3.2 and the comment below it, $\mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0) \subseteq \{\hookrightarrow L_p\}^{\mathbb{S}^1}$. As the condition in Proposition 2.2.5 is closed with respect to the weak topology, it therefore remains to see that $\{\hookrightarrow L_p\}^{\mathbb{S}^1} \subseteq \mathrm{cl}_w\left(\mathrm{I}_{\mathbb{D},p}(\mathcal{S}_0)\right)$.

To this end, let $K \in \{\hookrightarrow L_p\}^{\mathbb{S}^1}$. By Propositions 2.2.5 and 4.2.2, $\mu := \nu_{\mathbb{D},p}(\mathbb{S}^1)^{-1}\mathbf{F}_p\rho_K^{-p}$ is a positive measure on $\mathbb{S}^{2n-1}$. We can therefore find a sequence of positive, $\mathbb{S}^1$-invariant functions $f_k \in C^\infty(\mathbb{S}^{2n-1})$, $k \in \mathbb{N}$, which convergences to μ in the weak topology. For $k \in \mathbb{N}$, define $K_k \in \mathcal{S}_0(\mathbb{C}^n)$ by $\rho_{K_k}^{2n+p} = f_k$. Then, by $\mathbb{S}^1$-invariance,

$$\rho_{\mathrm{I}_{\mathbb{D},p}K_k}^{-p} = \frac{\nu_{\mathbb{D},p}(\mathbb{S}^1)}{(2\pi)^2}\mathbf{F}_{-2n-p}\rho_{K_k}^{2n+p} \quad \overset{w}{\longrightarrow} \quad \frac{\nu_{\mathbb{D},p}(\mathbb{S}^1)}{(2\pi)^2}\mathbf{F}_{-2n-p}\mu = (2\pi)^{2n-2}\rho_K^{-p},$$

concluding the proof. □

Next, we will use the characterization in Proposition 4.3.3 to prove Proposition 4.1.3. For the reader's convenience, we repeat the statement using the notation of Proposition 4.3.3.

Proposition 4.3.4. *Let $-2 < p \leq 1$ be non-zero. If $\mathrm{cl}_w(\mathrm{I}_{C,p}(\mathcal{S}_0)) \cap \mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1}$ is a strict subset of $\mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1}$, then there exist $\mathbb{S}^1$-invariant $K, L \in \mathcal{K}(\mathrm{I}_{C,p})$ such that*

$$\mathrm{I}_{C,p}K \subseteq \mathrm{I}_{C,p}L \qquad \text{but} \qquad -p\mathrm{V}_{2n}(K) > -p\mathrm{V}_{2n}(L).$$

Let us point out that $\mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1}$ consists precisely of all $\mathbb{S}^1$-invariant convex bodies with non-empty interior (in particular, does not depend on C).

For the proof of Proposition 4.3.4, we need the following perturbation statement, which is a slight generalization of [38, Lem. 4.10]. It follows from the fact that the condition in Proposition 2.2.5 is stable under suitable approximations.

Lemma 4.3.5. *Let $K \in \mathcal{K}(\mathbb{R}^n)$ be origin-symmetric with $0 \in \operatorname{int} K$, and suppose that K does not embed into L_p. Then there exists a sequence of smooth origin-symmetric convex bodies $(K_k)_{k\in\mathbb{N}}$ with strictly positive curvature such that $\rho_{K_k}^{-p} \to \rho_K^{-p}$ and each K_k does not embed into L_p.*

Proof of Proposition 4.3.4. Let $\widetilde{L} \in \mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1}$ not in $\mathrm{cl}_w\left(\mathrm{I}_{C,p}(\mathcal{S}_0)\right) = \{\hookrightarrow L_p\}^{\mathbb{S}^1}$. By Lemma 4.3.5, there exists a smooth $L \in \mathcal{K}(\mathbb{C}^n)$ with strictly positive curvature which does not embed into L_p. Moreover, we can assume that L is $\mathbb{S}^1$-invariant and, thus, $L \in \mathcal{K}(\mathrm{I}_{C,p})$.

As ρ_L is smooth on $\mathbb{S}^{2n-1}$, so is $\mathbf{F}_p\rho_L^{-p}$. By Proposition 2.2.5, $-p\mathbf{F}_p\rho_L^{-p}$ is negative on some $\mathbb{S}^1$-invariant open set $U \subseteq \mathbb{S}^{2n-1}$. Let $0 \neq \varphi \in C^\infty(\mathbb{S}^{2n-1})$ be non-negative and $\mathbb{S}^1$-invariant, such that φ is supported inside U, and set $\psi = \mathbf{F}_p\varphi$. Since $\mathbf{F}_p$ intertwines rotations, ψ is also $\mathbb{S}^1$-invariant.

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Next, we define $K \in \mathcal{K}(\mathbb{C}^n)$ by

$$\rho_K^{2n+p} = \rho_L^{2n+p} - \varepsilon\psi,$$

for $\varepsilon > 0$ sufficiently small, using Lemma 4.3.1. Note that, by construction, K is again $\mathbb{S}^1$-invariant and therefore in $\mathcal{K}(\mathrm{I}_{C,p})$. Since $(2\pi)^{2n}\varphi = \mathbf{F}_{-2n-p}\psi$, we get that $\mathbf{F}_{-2n-p}\psi$ is positive and, thus, by Theorem G and using that $-p\nu_{C,p} \geq 0$,

$$\begin{aligned}-p(2\pi)^2\mathrm{J}_{C,p}\rho_K^{2n+p} &= -p\mathrm{T}_{\nu_{C,p}}\mathbf{F}_{-2n-p}\left(\rho_L^{2n+p} - \varepsilon p\psi\right) \\ &= -p(2\pi)^2\mathrm{J}_{C,p}\rho_L^{2n+p} - \varepsilon(-p)\nu_{C,p}(\mathbb{S}^1)\mathbf{F}_{-2n-p}\psi \leq -p(2\pi)^2\mathrm{J}_{C,p}\rho_L^{2n+p}.\end{aligned}$$

Consequently, $\mathrm{I}_{C,p}K \subseteq \mathrm{I}_{C,p}L$. As φ is positive on an open subset of U and $-p\mathbf{F}_p\rho_L^{-p}$ is negative on U, we obtain by (2.12) and the self-adjointness of $\mathbf{F}_p$,

$$\begin{aligned}-2np\tilde{\mathrm{V}}_{-p}(K,L) = -p\left\langle \rho_K^{2n+p}, \rho_L^{-p}\right\rangle &= -2np\mathrm{V}_{2n}(L) + p\varepsilon\left\langle (2\pi)^{-2n}\mathbf{F}_{-2n-p}\psi, \mathbf{F}_p\rho_L^{-p}\right\rangle \\ &= -2np\mathrm{V}_{2n}(L) - \varepsilon\left\langle \varphi, (-p)\mathbf{F}_p\rho_L^{-p}\right\rangle > -2np\mathrm{V}_{2n}(L).\end{aligned}$$

By the dual L_p-Minkowski inequality (2.2),

$$-p\mathrm{V}_{2n}(L) < -p\tilde{\mathrm{V}}_{-p}(K,L) \leq -p\mathrm{V}_{2n}(K)^{\frac{2n+p}{2n}}\mathrm{V}_{2n}(L)^{-\frac{p}{2n}},$$

that is, $-p\mathrm{V}_{2n}(L) < -p\mathrm{V}_{2n}(K)$, concluding the proof. □

4.3.4 Counter examples in higher dimensions

By Proposition 4.1.3 respectively Proposition 4.3.4, it suffices to show that not every $\mathbb{S}^1$-invariant convex body is contained in the closure of $\mathrm{I}_{C,p}(\mathcal{S}_0)^{\mathbb{S}^1}$ in order to obtain a negative answer to Problem 1. Using examples from [39] of bodies that do not embed into some L_p, Proposition 4.3.2 then completes the proof of Proposition 4.1.4 for $\mathrm{I}_{C,p}$.

Proof of Proposition 4.1.4 for $\mathrm{I}_{C,p}$. By [39, Thm. 4], the unit ball of the complex space ℓ_n^q, i.e.

$$B_q = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^q \leq 1 \right\},$$

where $q > 2$ does not embed into L_p for $p \in (-2n+4, 0)$. In particular, B_q does not embed into L_p for $p \in (-2, 0)$ if $n \geq 3$. Moreover, by [38, Thm. 6.17], B^q cannot embed into L_p for $p \in (0, 2]$ if $n \geq 3$. By [39, Thm. 6], finally, there exist 2-dimensional, $\mathbb{S}^1$-invariant convex bodies that do not embed into L_p for any $p > 0$.

Hence, by Proposition 4.3.2, $\mathrm{cl}_w(\mathrm{I}_{C,p}(\mathcal{S}_0)) \cap \mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1} \subsetneq \mathcal{K}(\mathrm{I}_{C,p})^{\mathbb{S}^1}$. □

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4.3.5 Affirmative answer in low dimension

In the case $n = 2$, $C = \mathbb{D}$ and $p \in (-2, 0)$, embedding results from [39] yield a positive answer. Note that we need to take similar steps as in the proof of Proposition 4.1.2 here, as we did not extend the notion of complex L_p-intersection body (and thereby the statement of Proposition 4.1.2) to measures (that is, taking the closure) as was done in the real setting.

Proof of Proposition 4.1.5. As $\mathcal{K}(\mathrm{I}_{\mathbb{D},p})$ consists precisely of all $\mathbb{S}^1$-invariant convex bodies with non-empty interior, by [39, Thm. 3], every $K \in \mathcal{K}(\mathrm{I}_{\mathbb{D},p})$ in $\mathbb{C}^2$ embeds into L_p for $(-4, 0)$. In particular, by Proposition 2.2.5, $\mathbf{F}_p\rho_K^{-p} \geq 0$ for all $K \in \mathcal{K}(\mathrm{I}_{\mathbb{D},p})$, as $\Gamma\left(-\frac{p}{2}\right) > 0$ for $p < 0$.

Suppose now that $K, L \in \mathcal{K}(\mathrm{I}_{\mathbb{D},p})$ satisfy $\mathrm{I}_{\mathbb{D},p}K \subseteq \mathrm{I}_{\mathbb{D},p}L$. As K and L are $\mathbb{S}^1$-invariant, by Proposition 4.2.4,

$$\frac{\nu_{\mathbb{D},p}(\mathbb{S}^1)}{(2\pi)^2(4+p)}\mathbf{F}_{-4-p}\rho_K^{4+p} = \rho_{\mathrm{I}_{\mathbb{D},p}K}^{-p} \leq \rho_{\mathrm{I}_{\mathbb{D},p}L}^{-p} = \frac{\nu_{\mathbb{D},p}(\mathbb{S}^1)}{(2\pi)^2(4+p)}\mathbf{F}_{-4-p}\rho_L^{4+p},$$

that is, $\mathbf{F}_{-4-p}\rho_K^{4+p} \leq \mathbf{F}_{-4-p}\rho_L^{4+p}$, as $\nu_{\mathbb{D},p} \geq 0$ by Proposition 4.2.2. By (2.12) and the self-adjointness of the spherical Fourier transform, we thus obtain

$$\begin{aligned} 4\mathrm{V}_4(K) = 4\tilde{\mathrm{V}}_{-p}(K,K) &= \left\langle \rho_K^{4+p}, \rho_K^{-p} \right\rangle = (2\pi)^4 \left\langle \mathbf{F}_{-4-p}\rho_K^{4+p}, \mathbf{F}_p\rho_K^{-p} \right\rangle \\ &\leq (2\pi)^4 \left\langle \mathbf{F}_{-4-p}\rho_L^{4+p}, \mathbf{F}_p\rho_K^{-p} \right\rangle = \left\langle \rho_L^{4+p}, \rho_K^{-p} \right\rangle = 4\tilde{\mathrm{V}}_{-p}(L,K). \end{aligned}$$

The dual L_p-Minkowski inequality (2.2) therefore implies

$$\mathrm{V}_4(K) \leq \tilde{\mathrm{V}}_{-p}(L,K) \leq \mathrm{V}_4(L)^{\frac{4+p}{4}}\mathrm{V}_4(K)^{-\frac{p}{4}},$$

that is, $\mathrm{V}_4(K) \leq \mathrm{V}_4(L)$, concluding the proof. □

We conclude the treatment of complex L_p-intersection bodies by a remark on the complex centroid body.

Remark 4.3.6. The *complex centroid body* map Γ_C was defined in [27] for all $K \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior by

$$h_{\Gamma_C K}(u) = \frac{1}{\mathrm{V}_{2n}(K)}\int_K h_{Cu}(x)dx, \quad u \in \mathbb{S}^{2n-1}.$$

Clearly, by (3.4), $(\Gamma_C K)^\circ = \mathrm{V}_{2n}(K)\mathrm{I}_{C,1}K = \mathrm{I}_{C,1}(\mathrm{V}_{2n}(K)^{-1/(2n+1)}K)$, and, thus, $\mathcal{K}(\Gamma_C^\circ) = \mathcal{K}(\mathrm{I}_{C,1})$ and the images of Γ_C° and $\mathrm{I}_{C,1}$ coincide (as $\mathrm{I}_{C,1}$ is homogeneous).

Moreover, since

$$\Gamma_C K \subseteq \Gamma_C L \quad \Leftrightarrow \quad \Gamma_C^\circ L \subseteq \Gamma_C^\circ K,$$

the conditions in Problem 1 are equivalent, with K and L replaced. Proposition 4.1.2 therefore implies

$$\mathrm{V}_{2n}(\mathrm{V}_{2n}(K)^{-\frac{1}{2n+1}}K) \leq \mathrm{V}_{2n}(\mathrm{V}_{2n}(L)^{-\frac{1}{2n+1}}L),$$

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that is, $\mathrm{V}_{2n}(K) \leq \mathrm{V}_{2n}(L)$, whenever $L \in \mathrm{I}_{\overline{C},1}(\mathcal{S}_0) \cap \mathcal{K}(\mathrm{I}_{C,1})$.

In a similar way, by Proposition 4.1.3 for $\mathrm{I}_{C,1}$, we can find bodies K and L with $\Gamma_C K \subseteq \Gamma_C L$ but $\mathrm{V}_{2n}(L) < \mathrm{V}_{2n}(K)$, if there are bodies outside the image, as is the case for all $n \geq 2$ by the same statement for $\mathrm{I}_{C,1}$, solving Problem 1 for Γ_C as well.

4.4 Busemann–Petty problem for complex projection bodies

In this section, we prove Theorem F for the complex projection body map Π_C. As for $\mathrm{I}_{C,p}$, we will follow the steps outlined at the beginning of this chapter.

4.4.1 Determination of $\mathcal{K}(\Phi)$

Determined by the multipliers (3.20) of the operator $\mathrm{J}_{C,1}$, the injectivity set $\mathcal{K}(\Pi_C)$, defined in (4.4), is given by

$$\mathcal{K}(\Pi_C) = \left\{K \in \mathcal{K}(\mathbb{C}^n) : \operatorname{int} K \neq \emptyset \text{ and } c_{l-k}[h_C^p] = 0 \Rightarrow \pi_{k,l} S_K = 0\right\}.$$

In particular, we have $\Pi_C K = \Pi_C L$ for $K, L \in \mathcal{K}(\Pi_C)$ only if $S_K = S_L$, that is, $K = L$ up to translations. Moreover, $\mathcal{K}(\Pi_C)$ contains the set of $\mathbb{S}^1$-invariant convex bodies with non-empty interior.

We can now prove Proposition 4.1.1 using again a small perturbation.

Proof of Proposition 4.1.1 for Π_C°. Let $L \in \mathcal{K}(\mathbb{C}^n)$ be smooth with positive curvature, that is, S_L has a smooth, positive density $s_L \in C^\infty(\mathbb{S}^{2n-1})$. As $\mathcal{K}(\Pi_C)$ is a proper subset, there exists $(k,l) \in \mathbb{N}$ such that $c_{l-k}[h_C] = 0$. Using the Jacobi polynomial $\widetilde{P}_{k,l} \in \mathcal{H}^{2n}_{k,l}$, we define

$$\varphi = s_L + \varepsilon \widetilde{P}_{k,l},$$

where $|\varepsilon|$ is sufficiently small such that $\varphi > 0$ and the sign of ε is chosen such that

$$\varepsilon \left\langle \widetilde{P}_{k,l}, h_L \right\rangle \leq 0 \tag{4.12}$$

By Minkowski's existence theorem [62, Thm. 8.2.2], there exists a convex body $K \in \mathcal{K}(\mathbb{C}^n)$ such that $S_K = \varphi$.

By the choice of (k,l), $\mathrm{T}_{S_{iC}} \widetilde{P}_{k,l} = 0$ by (4.7), and we conclude by Proposition 4.2.4

$$-4\pi h_{\Pi_C K} = T_{S_{iC}} \mathbf{F}_{-2n-1} S_K = T_{S_{iC}} \mathbf{F}_{-2n-1} S_L = -4\pi h_{\Pi_C L},$$

that is, $\Pi_C K = \Pi_C L$. Consequently, by (4.12),

$$2n\mathrm{V}(K,L) = \langle S_K, h_L \rangle = 2n\mathrm{V}_{2n}(L) + \varepsilon \left\langle \widetilde{P}_{k,l}, h_L \right\rangle \leq 2n\mathrm{V}_{2n}(L).$$

Minkowski's first inequality (2.1) then implies

$$\mathrm{V}_{2n}(L) \geq \mathrm{V}(K,L) \geq \mathrm{V}_{2n}(K)^{\frac{2n-1}{2n}} \mathrm{V}_{2n}(L)^{\frac{1}{2n}},$$

that is, $\mathrm{V}_{2n}(L) \geq \mathrm{V}_{2n}(K)$. As K and L are not homothetic, the equality cases of (2.1) imply that the inequality is strict, concluding the proof. □

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4.4.2 Affirmative Cases

As in case of $\mathrm{I}_{C,p}$, Problem 1 for Π_C has an affirmative answer, when one of the bodies is in the image of the adjoint operator $\Pi_{\overline{C}}$. The proof idea is exactly the same as before, namely using an adjointness property for mixed volumes and applying Minkowski's first inequality.

Proof of Proposition 4.1.2 for Π_C. First, we apply Proposition 4.2.4 and use that $\mathbf{F}_{-2n-1}$ is self-adjoint and, by Lemma 4.2.1, $\mathrm{T}_{S_{iC}}$ is the adjoint operator of $\mathrm{T}_{S_{i\overline{C}}}$, to obtain

$$\begin{aligned} 2n\mathrm{V}(K,\Pi_{\overline{C}}L) &= \left\langle S_K, h_{\Pi_{\overline{C}}L}\right\rangle = -\frac{1}{4\pi}\left\langle S_K, \mathrm{T}_{S_{i\overline{C}}}\mathbf{F}_{-2n-1}S_L\right\rangle \\ &= -\frac{1}{4\pi}\left\langle \mathrm{T}_{S_{iC}}\mathbf{F}_{-2n-1}S_K, S_L\right\rangle = \left\langle h_{\Pi_C K}, S_L\right\rangle = 2n\mathrm{V}(L,\Pi_C K), \end{aligned} \tag{4.13}$$

for every $K, L \in \mathcal{K}(\Pi_C)$. Let now $K, L \in \mathcal{K}(\Pi_C)$ with $K = \Pi_{\overline{C}}K_0$ for some $K_0 \in \mathcal{K}(\Pi_C)$ and assume that $\Pi_C^\circ K \subseteq \Pi_C^\circ L$, equivalently, $\Pi_C L \subseteq \Pi_C K$. Then, by (4.13) and the monotonicity of mixed volume,

$$\mathrm{V}_{2n}(K) = \mathrm{V}(K,\Pi_{\overline{C}}K_0) = \mathrm{V}(K_0,\Pi_C K) \geq \mathrm{V}(K_0,\Pi_C L) = \mathrm{V}(L,\Pi_{\overline{C}}K_0) = \mathrm{V}(L,K).$$

Minkowski's first inequality (2.1) finally implies that

$$\mathrm{V}_{2n}(K) \geq \mathrm{V}(L,K) \geq \mathrm{V}_{2n}(L)^{\frac{2n-1}{2n}}\mathrm{V}_{2n}(K)^{\frac{1}{2n}},$$

that is, $\mathrm{V}_{2n}(K) \geq \mathrm{V}_{2n}(L)$ as claimed. Equality $\mathrm{V}_{2n}(K) = \mathrm{V}_{2n}(L)$ holds here only if there is equality in Minkowski's first inequality, that is, K and L are homothetic. As they have the same volume, they must be translates of each other. □

4.4.3 Outside the image

The next step is to show that the answer to Problem 1 is negative if there are bodies which are not contained in the image of Π_C. Again, we will start with a description of $\Pi_C(\mathcal{K})$ in terms of the spherical Fourier transform.

Proposition 4.4.1. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric with non-empty interior. If $K \in \Pi_C(\mathcal{K})$, then*

$$\frac{1}{\Gamma\left(-\frac{1}{2}\right)}\mathbf{F}_1 h_K \geq 0.$$

In particular, $K^\circ \in \{\hookrightarrow L_1\} \cap \{L \in \mathcal{K}(\mathbb{C}^n) : \rho_L^{-1} \in \operatorname{im}\mathrm{T}_{S_{iC}}\}$.

Proof. First note that $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$. Combining Proposition 4.2.4 and (2.12), using that multiplier transforms commute, we obtain

$$-\mathbf{F}_1 h_K = \frac{1}{4\pi}\mathbf{F}_1\mathrm{T}_{S_{iC}}\mathbf{F}_{-2n-1}S_K = \frac{(2\pi)^{2n}}{4\pi}\mathrm{T}_{S_{iC}}S_K \geq 0,$$

yielding the first claim. The second claim follows directly from Proposition 2.2.5. □

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As before, it is sufficient to consider only $\mathbb{S}^1$-invariant bodies. In the following, we will denote by $\Pi_C^\circ(\mathcal{K}) = \{K^\circ : K \in \Pi_C(\mathcal{K})\}$.

Proposition 4.4.2. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric with non-empty interior. Then*

$$\Pi_C(\mathcal{K})^{\mathbb{S}^1} = \Pi_{\mathbb{D}}(\mathcal{K}) \quad \textit{and} \quad \mathrm{cl}_w\left(\Pi_{\mathbb{D}}^\circ(\mathcal{K})\right) = \{\hookrightarrow L_1\}^{\mathbb{S}^1},$$

where we denote by cl_w the closure with respect to weak convergence of ρ_K^{-1}.

Proof. Let $K \in \Pi_C(\mathcal{K})^{\mathbb{S}^1}$, that is, $\mathbb{S}^1$-invariant and such that $K = \Pi_C K_0$ for some $K_0 \in \mathcal{K}(\Pi_C)$. By the $\mathbb{S}^1$-invariance, $\mathrm{T}_{S_{iC}} h_K = S_{iC}(\mathbb{S}^1) h_K$ for all C and, by Proposition 4.2.4,

$$S_{i\mathbb{D}}(\mathbb{S}^1) h_K = \mathrm{T}_{S_{i\mathbb{D}}} h_K = -\frac{1}{4\pi} \mathrm{T}_{S_{i\mathbb{D}}} \mathbf{F}_{-2n-1} \mathrm{T}_{S_{iC}} S_{K_0}.$$

Setting $\mu = S_{i\mathbb{D}}(\mathbb{S}^1)^{-1} \mathrm{T}_{S_{iC}} S_{K_0}$, we note that μ is a centered (even), non-negative measure, which is not concentrated in a subsphere. Hence, by Minkowski's existence theorem [62, Thm. 8.2.2], there exists an origin-symmetric body $L_0 \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior such that $S_{L_0} = \mu$.

Consequently, $K = \Pi_{\mathbb{D}} L_0$, that is, $K \in \Pi_{\mathbb{D}}(\mathcal{K})$. Repeating the argument with C and $\mathbb{D}$ replaced yields the other inclusion.

Next, by Proposition 4.4.1, all bodies in $\Pi_{\mathbb{D}}^\circ(\mathcal{K})$ embed into L_1 and are $\mathbb{S}^1$-invariant. As $\{\hookrightarrow L_1\}^{\mathbb{S}^1}$ is closed under weak convergence, it remains to show that $\{\hookrightarrow L_1\}^{\mathbb{S}^1} \subseteq \mathrm{cl}_w\left(\Pi_{\mathbb{D}}^\circ(\mathcal{K})\right)$.

To this end, let $K \in \{\hookrightarrow L_1\}^{\mathbb{S}^1}$. By Proposition 2.2.5, $\mu = -S_{i\mathbb{D}}(\mathbb{S}^1)^{-1} \mathbf{F}_1 \rho_K^{-1}$ is a non-negative measure. We can therefore find a sequence of positive, $\mathbb{S}^1$-invariant functions $f_k \in C^\infty(\mathbb{S}^{2n-1})$, $k \in \mathbb{N}$, which converges to μ in the weak topology. As every f_k is, in particular, even and positive, Minkowski's existence theorem implies that there are convex bodies $K_k \in \mathcal{K}(\mathbb{C}^n)$ with non-empty interior, such that $S_{K_k} = f_k$, $k \in \mathbb{N}$. Then, by $\mathbb{S}^1$-invariance,

$$\rho_{\Pi_{\mathbb{D}}^\circ K_k}^{-1} = -\frac{S_{i\mathbb{D}}(\mathbb{S}^1)}{4\pi} \mathbf{F}_{-2n-1} S_{K_k} \quad \xrightarrow{w} \quad -\frac{S_{i\mathbb{D}}(\mathbb{S}^1)}{4\pi} \mathbf{F}_{-2n-1} \mu = \frac{(2\pi)^{2n}}{4\pi} \rho_K^{-1},$$

concluding the proof. □

Next, we will use the characterization in Proposition 4.4.2 to prove Proposition 4.1.3. For the reader's convenience, we repeat the statement using the notation of Proposition 4.4.2.

Proposition 4.4.3. *If $\mathrm{cl}_w(\Pi_C^\circ(\mathcal{K})) \cap \mathcal{K}(\Pi_C)^{\mathbb{S}^1}$ is a strict subset of $\mathcal{K}(\Pi_C)^{\mathbb{S}^1}$, then there exist $\mathbb{S}^1$-invariant $K, L \in \mathcal{K}(\Pi_C)$ such that*

$$\Pi_C K \subseteq \Pi_C L \qquad \textit{but} \qquad \mathrm{V}_{2n}(K) > \mathrm{V}_{2n}(L).$$

Let us point out that $\mathcal{K}(\Pi_C)^{\mathbb{S}^1}$ consists precisely of all $\mathbb{S}^1$-invariant convex bodies with non-empty interior (in particular, does not depend on C and is invariant under taking polars).

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Proof. Let $\widetilde{L}^\circ \in \mathcal{K}(\Pi_C)^{\mathbb{S}^1}$ not in $\mathrm{cl}_w(\Pi_C^\circ(\mathcal{K})) = \{\hookrightarrow L_1\}^{\mathbb{S}^1}$. By Lemma 4.3.5, there exists a smooth $L^\circ \in \mathcal{K}(\mathbb{C}^n)$ with strictly positive curvature which does not embed into L_1. Moreover, we can assume that L° is $\mathbb{S}^1$-invariant and, thus, $L \in \mathcal{K}(\Pi_C)$.

As h_L is smooth on $\mathbb{S}^{2n-1}$, so is $\mathbf{F}_1 h_L$. By Proposition 2.2.5 for L°, $-\mathbf{F}_1 h_L$ is negative on some $\mathbb{S}^1$-invariant open set $U \subset \mathbb{S}^{2n-1}$. Let $0 \neq \varphi \in C^\infty(\mathbb{S}^{2n-1})$ be non-negative and $\mathbb{S}^1$-invariant, such that φ is supported inside U, and set $\psi = \mathbf{F}_1\varphi$. Since $\mathbf{F}_1$ intertwines rotations, ψ is $\mathbb{S}^1$-invariant.

Next, let s_L denote the smooth density of S_L and define

$$f = s_L - \varepsilon\psi$$

for $\varepsilon > 0$ sufficiently small such that f is still positive. Then f is, in particular, even and positive and by Minkowski's existence theorem there exists $K \in \mathcal{K}(\Pi_C)^{\mathbb{S}^1}$ such that $S_K = f du$. Since $\mathbf{F}_{-2n-1}\psi = (2\pi)^{2n}\varphi \geq 0$, and by Proposition 4.2.4,

$$\begin{aligned} -4\pi h_{\Pi_C K} = \mathrm{T}_{S_{iC}}\mathbf{F}_{-2n-1}S_K &= -4\pi h_{\Pi_C L} - \varepsilon \mathrm{T}_{S_{iC}}\mathbf{F}_{-2n-1}\psi \\ &= -4\pi h_{\Pi_C L} - \varepsilon S_{iC}(\mathbb{S}^1)(2\pi)^{2n}\varphi \leq -4\pi h_{\Pi_C L}, \end{aligned}$$

that is, $\Pi_C K \supseteq \Pi_C L$. As φ is positive on an open subset of U and $-\mathbf{F}_1 h_L$ is negative on U, we obtain by (2.12) and the self-adjointness of $\mathbf{F}_1$

$$\begin{aligned} 2n\mathrm{V}(K, L) = \langle S_K, h_L\rangle &= 2n\mathrm{V}_{2n}(L) - \varepsilon \left\langle (2\pi)^{-2n}\mathbf{F}_{-2n-1}\psi, \mathbf{F}_1 h_L\right\rangle \\ &= 2n\mathrm{V}_{2n}(L) + \varepsilon \langle \varphi, -\mathbf{F}_1 h_L\rangle < 2n\mathrm{V}_{2n}(L). \end{aligned}$$

By Minkowski's first inequality (2.1),

$$\mathrm{V}_{2n}(L) > \mathrm{V}(K, L) \geq \mathrm{V}_{2n}(K)^{\frac{2n-1}{2n}}\mathrm{V}_{2n}(L)^{\frac{1}{2n}},$$

that is, $\mathrm{V}_{2n}(L) > \mathrm{V}_{2n}(K)$, concluding the proof. □

4.4.4 Counter examples in higher dimensions

As in the case of $\mathrm{I}_{C,p}$, by Proposition 4.1.3 respectively Proposition 4.4.3, it suffices to show that not every $\mathbb{S}^1$-invariant convex body is contained in the closure of $\Pi_C^\circ(\mathcal{K})$ in order to solve Problem 1. Again, we will use examples from [38, 39] of bodies not embedding into L_1.

Proof of Proposition 4.1.4 for Π_C. In the proof of Proposition 4.1.4 for $\mathrm{I}_{C,p}$ it was deduced from [38, 39] that the unit ball of the complex space ℓ_n^q, $q > 2$, does not embed into L_1 for $n \geq 2$. Hence the claim follows by Proposition 4.4.3. □

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