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Scaling Limits and Critical Phenomena in Interacting Particle Systems

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Abstract

This thesis studies three different large-scale phenomena in statistical mechanics. The first phenomenon studied is superdiffusivity in two different models, a diffusion in a random environment, and a critical stochastic partial differential equation in Chapters 2 and 3 respectively. Both models are diffusive systems, which are perturbed by some external forcing. The effect of this forcing can be measured using the so-called *Diffusion coefficient* $D(t)$. In this part of the Thesis it is proven that $D(t)$ diverges like $(\log t)^{\frac{1}{2}}$ and $(\log t)^{\frac{2}{3}}$, respectively, up to Tauberian inversions. These results prove conjectures made for the corresponding models. The proofs use tools from Gaussian Analysis and an iterative estimation scheme to study the resolvent of the generator of the process.

The second phenomenon is a near-critical limit of a conformally invariant model, namely the dimer model in Chapter 4. The fluctuations of the planar dimer model in two dimensions are one of the few models from statistical mechanics where conformal invariance has been rigorously proven. This conformal invariance holds for certain *critical* weights and certain boundary conditions. In this chapter, we study the dimer model near criticality. We execute part of the program initiated in [N. Makarov and S. Smirnov, Off-critical lattice models and massive SLEs, 2009, Proceedings of ICMP 2009], by finding a scaling limit for the corresponding height functions, and connecting this scaling limit to massive SLE_2. As is typical for near-critical models, this limit is no longer conformally invariant but conformally covariant. The proof uses a connection to loop-erased random walks via Temperley's bijection and Wilson's algorithm. We also prove an exact discrete Girsanov identity for the triangular lattice, which might be of independent interest.

The third phenomenon is the almost sure convergence of the asymptotic speed of a second-class particle in an interacting particle system started from specific non-stationary initial conditions. In particular, we study the stochastic six-vertex model on the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with step initial conditions, i.e. every incoming edge from the left is occupied by a particle and every incoming edge from the bottom is unoccupied. We then add a single second-class particle coming in from below. The main theorem of Chapter 5 states that the speed $\frac{\boldsymbol{X}_T}{T}$ of this second-class particle converges almost surely to a random limit. This allows one to define the stochastic six-vertex speed process. We use tools from integrable probability to obtain precise bounds on the fluctuations of the height functions around its limit shape together with a novel result that allows us to control the behavior of an individual second-class particle by controlling the behavior of a larger number of third-class particles.

Zusammenfassung

In dieser Doktorarbeit werden drei verschiedene Phänomene aus der statistischen Mechanik. Das erste dieser Phänomene ist das Phänomen der 'superdiffusivity', das in zwei verschiedenen Modellen gezeigt wird. Die Modelle sind eine Diffusion in einer zufälligen Umgebung in Kapitel 2 und eine kritische stochastische partielle Differentialgleichung in Kapitel 3. Beide Modelle können als diffusive Systeme beschrieben werden, die durch eine externe Krafteinwirkung gestört werden. Der Effekt dieser Krafteinwirkung kann durch den sogenannten Diffusionskoeffizient $D(t)$ gemessen werden. In diesem Teil der Doktorarbeit wird bewiesen, dass $D(t)$ jeweils wie $(\log t)^{\frac{1}{2}}$ und $(\log t)^{\frac{2}{3}}$ divergieren (im Sinne einer tauberianischen Umkehr). Diese Ergebnisse beweisen Vermutungen für die jeweiligen Modelle. Die Beweise verwenden Werkzeuge der Gaußschen Analysis und ein iteratives Abschätzungsschema um die Resolventen der Generatoren von assoziierten Markovprozessen zu untersuchen.

Das zweite Phänomen ist das Phänomen fast-kritischer Skalierungsgrenzwerte. Insbesondere wird in Kapitel 4 ein fast-krtischer Grenzwert eines Dimer-Modells untersucht. Die Schwankungen des planaren Dimer-Modells in zwei Dimensionen sind eine der wenigen Modelle der statistischen Mechanik, für die konforme Invarianz rigoros gezeigt werden konnte. Diese konforme Invarianz gilt für gewisse *kritische* Gewichte und gewisse Randbedingungen. In Kapitel 4 wird das Dimer-Modell mit fast-kritischen Gewichten untersucht. Wir führen Teil des in [N. Makarov and S. Smirnov, Off-critical lattice models and massive SLEs, 2009, Proceedings of ICMP 2009] initiierten Programms aus, in dem wir einen Grenzwert der entsprechenden Höhenfunktion finden, und diesen Grenzwert mit massiver SLE_2 verbinden. Wie es für fast-kritische Modelle typisch ist, ist dieser Grenzwert nicht mehr konform-invariant, sondern konform-kovariant. Der Beweis verwendet Verknüpfungen mit dem 'loop-erased random walk' via Temperleys Bijektion und Wilsons Algorithmus. Wir beweisen dabei auch eine exakte Girsanov Identität für den Simple Random Walk auf dem Dreiecksgitter, die unabhängig vom Hauptresultat von Interesse sein könnte.

Das dritte Phänomen ist die fast-sichere Konvergenz der asymptotischen Geschwindigkeit eines Teilchens zweiter Klasse in einem Vielteilchensystem ausgehend von spezifischen nicht-stationären Startbedingungen. Spezifischer untersuchen wir das 'stochastic six-vertex model' auf dem Quadranten $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ mit 'step initial conditions', das heißt jede Kante am linken Rand ist besetzt, während jede Kante am unteren Rand leer ist. Wir fügen dann ein Teilchen zweiter Klasse am Ursprung, von unten kommend, hinzu. Das Haupttheorem von Kapitel 5 besagt, dass die asymptotische Geschwindigkeit dises Teilchens fast sicher gegen einen zufälligen Grenzwert konvergiert. Dies erlaubt es den 'stochastic six-vertex speed process' zu definieren. Um diesen Satz zu beweisen, verwenden wir Methoden aus der integrierbare Wahrscheinlichkeitstheorie, um präzise Abschätzungen der Verteilungsränder der assoziierten Höhenfunktion zu finden. Zusätzlich zeigen wir ein neues Resultat, das uns ermöglicht, das Verhalten eines einzelnen Teilchens zweiter Klasse mithilfe einer größeren Menge von Teilchen dritter Klasse zu kontrollieren.

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Contents

Chapter 1

Introduction

At the heart of statistical mechanics is the concept of **scales** and many phenomena are tied to them. One starts with a model on the **microscopic** scale and then tries to understand the large scale or **macroscopic** behavior. If this is successful it involves a certain vanishing of microscopic properties of the model. This gives rise to the phenomenon of **universality**. Many different microscopic models exhibit the same large-scale behavior and can be put into different **universality classes** according to their large-scale behavior. Another common phenomenon is the vanishing of randomness altogether, with the large-scale behavior of a system being deterministic. This is often called a **law of large numbers**, but in specific cases might also be called a **limit shape** or a **hydrodynamic limit**, in the cases of random curves or interacting particle systems respectively. In such cases, one can consider **mesoscopic** scales, which are between the micro- and macroscopic scales, and capture the fluctuations around the deterministic limit. All of these phenomena can occur in models **in** and **out-of-equlibrium**.

These phenomena will appear in various places in this thesis, which is structured into the following chapters.

- In Chapter 2 we study a two-dimensional Brownian particle in a divergence-free drift field, that is the solution of the SDE

  $$\mathrm{d}X_t = \omega(X_t)\mathrm{d}t + \mathrm{d}B_t\,,$$

  where ω is the curl of a mollified Gaussian free field. We prove that the (annealed) mean-square displacement of this particle at time t grows like $t\sqrt{\log t}$, proving a conjecture from [TV12]. This model is one of several where this type of $\sqrt{\log t}$ super-diffusive behavior is expected. To the best of the authors' knowledge, this is the first result that establishes such precise asymptotics. The proof uses an associated environment process from the theory of diffusions in random environment, as well as Gaussian analysis together with an estimation scheme based on the seminal [LQSY04] to understand the generator of this process.

  Chapter 2 is based on the article [CHST22], which is joint work with Fabio Toninelli and Giuseppe Cannizzaro.

- In Chapter 3 we study the stochastic Burgers equation, which was introduced in [vBKS85] as a continuous approximation of the fluctuations of the asymmetric simple exclusion process. Formally it is given by

  $$\partial_t\eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot\nabla(\eta^2) + \nabla\cdot\xi\,,$$

where ξ is d-dimensional space time white noise and $\mathfrak{w}$ is a fixed non-zero vector. We consider this equation in dimension $d = 2$ and at stationarity. We prove that its bulk diffusion coefficient behaves like $(\log t)^{\frac{2}{3}}$, which confirms a prediction made in [vBKS85]. This complements the recent results in [CGT24] where this equation was studied in dimensions 3 and higher and in dimension 2 in the weak-coupling regime. It can be seen as a continuous analog to [Yau04], which proved $(\log t)^{\frac{2}{3}}$ super-diffusivity for the two-dimensional asymmetric simple exclusion process. The overall structure of the proof is similar to Chapter 2, but the different types of models and the different universality classes change many details.

Chapter 3 is based on the preprint [DGHS24], which is joint work with Damiano De Gaspari.

- In Chapter 4 we study the dimer model on the square and hexagonal lattice with certain weights. For uniform weights, the dimer model has been shown to be conformally invariant, see [Ken00]. This can be seen as a critical point for the dimer model. The weights we consider make the model **near-critical**. The result is a non-trivial interpolation between the critical and non-critical models. The limit is no longer conformally invariant, but conformally covariant: it depends on a drift field α, which transforms under conformal maps in a specific way. More specifically we

  – connect this near-critical dimer model to massive SLE_2 as constructed by [MS10].

  – show convergence of the associated height function on arbitrary Temperleyan domains of the square and hexagonal lattice.

  – prove the conformal covariance of this limit.

  A novel tool in the proof is an exact discrete Girsanov identity for the triangular lattice which might be of independent interest.

  Chapter 4 is based on the preprint [BHS22], which is joint work with Nathanaël Berestycki.

  This article grew out of the author's master thesis, which causes there to be some overlap with said thesis in Section 4.3. All other parts of Chapter 4 have been significantly extended or are entirely new when compared to the Author's master thesis.

- In Chapter 5 we study the stochastic six-vertex model with step initial conditions. In particular, we study the model on the quadrant $\mathbb{Z}_{\geq}0 \times \mathbb{Z}_{\geq}0$ with all incoming positions from the left occupied and all incoming positions from the bottom empty. Adding a single second-class particle at the origin, entering from the bottom, we prove that the speed of this second-class particle converges almost surely to a random limiting speed. This allows us to construct the stochastic six-vertex speed process.

  To obtain this result we prove an effective hydrodynamic limit estimate using tools from integrable probability as well as a novel lemma, which allows us to control the position of an individual second-class particle by controlling a larger number of third-class particles. The "effective hydrodynamic limit estimate" is one that gives quantitative bounds on the fluctuations around the hydrodynamic limit at finite time S, on intervals of size sublinear in S.

  Chapter 5 is based on ongoing work with Hindy Drillick.

1.1 Superdiffusivity

In Chapters 2 and 3 we study two different models exhibiting a phenomenon known as superdiffusivity. In simple words, the phenomenon of superdiffusivity is the following. One starts with a system (usually described mathematically as a Markov process) that is in some sense diffusive. In our case, it will be a tracer particle driven by Brownian motion or a scalar fluctuation field driven by the heat equation with additive space-time white noise. Then one adds an external (possibly random) forcing, that makes the evolution irreversible. In the case of the tracer particle, the forcing typically consists of a random divergence-free vector field (a drift term in the equation). For the fluctuation field, the forcing usually takes the form of a non-linear term in the stochastic PDE. After this perturbation, it may happen (according to the spatial dimension, the nature of the nonlinearity, etc.) that correlations spread faster in space (as a function of time), or that the typical displacement of the tracer particle grows faster with time, with respect to the diffusive case. In this case, we say that the driven system behaves superdiffusively.

Concretely, the (super)diffusive behavior of our system is usually encoded in the so-called *Diffusion coefficient*, defined as

$$D(t) = \frac{1}{t}\int_{\mathbb{R}} |x|^2 S(x,t)\mathrm{d}x$$

where $S(x,t)$ is a suitable auto-correlation function. For instance, in the case of the tracer particle $S(x,t)$ is the probability density of finding the particle at position x at time t, given that the initial position is at $x = 0$ so that $D(t)$ is just the mean square displacement of the particle, divided by t. For driven lattice gases (interacting particle systems on $\mathbb{Z}^d$, for instance, the well-known Asymmetric Simple Exclusion Process), S usually has the form $S(x,t) = Cov[\eta(x,t), \eta(0,0)]$, with $\eta(x,t)$ the particle occupation variable at the lattice site x and Cov denoting the covariance with respect to the law of the stationary process.

In the case of interacting particle systems, $S(x,t)$ can be interpreted as the probability that a second-class particle that starts at position $x = 0$ at time $t = 0$, is at x at time t and the diffusion coefficient is $\frac{1}{t}$ times the mean square displacement of the particle.[1] If the system "behaves diffusively", the second-class particle or the tracer particle in the divergence-free drift field should behave like a Brownian motion at large scales. Since the typical displacement of a Brownian motion at time t grows linear in t, this means that diffusive behavior corresponds to a $D(t)$ that is uniformly of order 1 as $t \to \infty$, while superdiffusive behavior corresponds to $D(t)$ growing to infinity.

As already mentioned, whether superdiffusivity occurs or not depends crucially on the space dimension, as well as on the nature of the forcing. In many cases, for instance, for interacting particle systems of exclusion type, self-interacting polymers, and tracer particles in random divergence-free vector fields, the behavior of $D(t)$ depending on the dimension d is as follows (see [TV12, Yau04]):

$$D(t) \overset{t\to\infty}{\approx} \begin{cases} t^{\frac{1}{3}}, \text{ for } d = 1 \\ (\log t)^{\frac{1}{2}} \text{ or } (\log t)^{\frac{2}{3}}, \text{ for } d = 2 \\ C, \text{ for } d \geq 3 \end{cases}$$

These different cases can be seen as universality classes of superdiffusive systems. For $d \geq 3$, while the system still behaves diffusively, there is still some increase in the diffusivity,

[1] second-class particles also appear in Chapter 5, see Section 1.3.1 for a definition of the multi-class stochastic six-vertex model.

indicated here by the constant C, which is strictly larger than for the unperturbed diffusive system. In this case, often one can prove a Gaussian scaling limit, for instance, an invariance principle for the tracer particle, or convergence of the fluctuation field to a linear stochastic heat equation. This limit does depend on the forcing introduced, in particular when the forcing has an associated direction and strength, the limit will depend on these quantities, see [CGT24, LOV04, HTV12].

As is apparent from the formula above, space dimension $d = 2$ is in a sense marginal, in that it separates superdiffusive ($d < 2$) from diffusive ($d > 2$) behavior. For $d = 2$, superdiffusive effects are only logarithmic, and the exponent γ of the logarithm, $\gamma = 1/2$ or $\gamma = 2/3$, depends on the symmetries of the model. The two cases in dimension $d = 2$ correspond to **isotropic** or **anisotropic** superdiffusivity. In anisotropic superdiffusivity ($\gamma = 2/3$) there is a direction (e.g. the first coordinate axis) in which the forcing acts and the system behaves diffusively in the other direction. The setting of [Yau04] gives an easy example of this: It studies an exclusion process, in which particles move on $\mathbb{Z}^2$ and perform jumps at the rate given by

$$p(x) = \begin{cases} 1, \text{ if } x = e_1 \\ \frac{1}{2}, \text{ if } x = \pm e_2 \\ 0, \text{else.} \end{cases}$$

One can see that in the vertical direction, the particles perform a simple symmetric exclusion process, which is known to be diffusive. In the horizontal direction, they perform a totally asymmetric exclusion process, known to be superdiffusive. The behavior in the two directions is not independent, since it is one set of particles performing both horizontal and vertical jumps. Another example of this class is the Stochastic PDE treated in Chapter 3, i.e., the two-dimensional Stochastic Burgers equation. The nonlinearity is of transport type and only transports in the direction $\mathfrak{w}$, see (1.6).

In the isotropic cases, corresponding to the exponent $\gamma = 1/2$, the forcing acts in every direction, often isotropically. The stochastic differential equation (1.1) in Chapter 2, describing a tracer particle in two dimensions, subject to Brownian noise and to a quenched divergence-free driving field, is easily seen to be of this type since its (annealed) law is rotationally invariant. Another example of a system belonging to the isotropic ($\gamma = 1/2$) universality class is the stochastic PDE given formally by

$$\partial_t H = \Delta H + (\partial_{x_1} H)^2 - (\partial_{x_2} H)^2 + \xi$$

with ξ a space-time white noise, $H = H(x,t)$ a scalar field and $x = (x_1, x_2) \in \mathbb{R}^2$. Somewhat confusingly, this equation is known as the "Anisotropic KPZ equation" (AKPZ), in contrast to the usual two-dimensional KPZ equation whose non-linearity is $|\nabla H|^2$. While the AKPZ equation does indeed have two distinguished directions (distinguished by the minus sign in the non-linearity), the non-linearity (that is responsible for the super-diffusive behavior) acts in a sense symmetrically in the two directions, as can be seen by remarking the following symmetry: if

$$\tilde{H}(x_1, x_2) = -H(x_2, x_1)$$

then the process $\tilde{H}$ has the same law as H.

Here is a quick overview of results establishing $(\log t)^{\frac{2}{3}}$ or $(\log t)^{\frac{1}{2}}$ superdiffusivity in two-dimensional models:

- The first proof of $(\log t)^{\frac{2}{3}}$ superdiffusivity was given in [Yau04] for the 2-dimensional asymmetric exclusion process mentioned above.

- In [CHST22], which corresponds to Chapter 2 of this thesis, $(\log t)^{\frac{1}{2}}$ superdiffusivity is proved for the first time for a model in the $\gamma = 1/2$ universality class, confirming a conjecture made in [TV12]. The studied model is a Brownian tracer particle in a turbulent fluid and will be introduced in section 1.1.1 below. It is worth mentioning that the isotropic case requires, in some sense, finer estimates than the anisotropic case, see the discussion in Section 1.1.6 and in Chapter 2.

- In [dLFW24] an extended version of the above result was proved, where the drift field evolves in time as the solution of an independent (fractional) stochastic heat equation. If the environment evolves too quickly, corresponding to a fractional heat equation with Laplacian term $(-\Delta)^s$ with $0 \leq s < 1$, the system becomes diffusive, while for $s \geq 1$ the behavior is essentially unchanged. The authors also manage to interpolate between the two regimes with a somewhat artificial "logarithmically fractional" heat equation resulting in $(\log t)^\theta$ superdiffusivity for $\theta \in (0, \frac{1}{2})$, which falls outside of the two universality classes mentioned above.

- In [CMOW22] a different proof of the result from [CHST22] using methods from the homogenization theory of PDEs was given. A particular upside of these methods is that they were able to prove superdiffusivity "in real time", rather than just in a Tauberian (Laplace transform) sense, see also the discussion in Section 1.1.1.

- In the recent preprint [ABRK24] homogenization results were used to obtain a quenched central limit theorem for the model in [CHST22]. To do this the diffusion is rescaled not diffusively but like
$$(|\log \varepsilon|)^{-\frac{1}{4}} \varepsilon X_{t/\varepsilon^2} \,,$$
to remove the exploding variance as $\varepsilon \to 0$. This result achieves two previously unaccomplished goals. It proves superdiffusivity in a quenched sense, which the results in [CHST22, dLFW24, CMOW22] were not able to do, and it deals with the "strong coupling limit", see the discussion in section 1.1.8.

- In [CET23a] $(\log t)^{\frac{1}{2}}$ superdiffusivity was shown for the 2-dimensional AKPZ equation. This result was written more or less in parallel to [CHST22].

- In [DGHS24], which corresponds to Chapter 3 of this thesis, $(\log t)^{\frac{2}{3}}$ superdiffusivity is proven for the 2d stochastic Burgers equation. Compared to the result [Yau04] by Yau, this uses more modern techniques, which leads to much better estimates on the sub-dominant corrections to the $(\log t)^{\frac{2}{3}}$, and a more approachable presentation. In particular, it avoids splitting estimates into good and bad regions, which has been a major obstacle in replicating the success of [Yau04]. It is also the first $(\log t)^{\frac{2}{3}}$ superdiffusivity result for a continuous model and in particular for a critical SPDE.

It should be mentioned that before these results established $D(t) \approx (\log t)^\gamma$ with the precise exponent γ, there were several works giving rougher lower and upper bounds for $D(t)$, typically of order $\log\log t$ and $\log t$ respectively for isotropically superdiffusive models and of order $(\log t)^{\frac{1}{2}}$ and $\log t$ for anisotropically superdiffusive models. These results use a variational approach which we sketch in Section 1.1.4. Let us mention some results of this type here:

- Before establishing $(\log t)^{\frac{2}{3}}$ in [Yau04], Yau together with Landim, Quastel, and Salmhofer established sub-optimal superdiffusive bounds for the asymmetric exclusion process in dimension 1 and 2 in [LQSY04].

- In [TV12] super diffusive bounds were established for the model treated in Chapter 2 and for a self-interacting Brownian polymer. This self-interacting Brownian polymer is also conjectured to be $(\log t)^{\frac{1}{2}}$ superdiffusive, but this is still open. However, in [CG24] an annealed central limit theorem was shown for this model in the weak coupling limit. See Section 1.1.8 for a definition of the weak coupling regime and a discussion of such results.

- In [LRY05] suboptimal superdiffusive bounds were shown for a lattice gas, which consists of multiple interacting copies of the 2d ASEP model studied in [Yau04]. The way these multiple 2d ASEP models interact makes the model belong to the isotropic superdiffusivity class, and therefore the conjectured superdiffusivity is $(\log t)^{\frac{1}{2}}$. However, this is still open.

As we see, there are several cases where these suboptimal bounds can be obtained, but the full program of establishing $D(t) \approx (\log t)^\gamma$ with the correct exponent γ has not been carried out. It would be very interesting to see if there is some more fundamental obstacle in these cases preventing the method first used in [Yau04] from being applied.

In the rest of this section, we will introduce the two models from Chapters 2 and 3 and the main results obtained in this thesis about them and then give a joint outline of the respective proof found in those chapters, remarking on differences as we go along. We will make brief detours to discuss the variational method used to obtain the first diffusive bounds mentioned above. We will end this section by comparing with other types of results in this field, namely strong and weak coupling limits.

1.1.1 Brownian particle in the curl of the Gaussian Free Field

In Chapter 2 we study the following SDE

$$\mathrm{d}X(t) = \omega(X(t))\mathrm{d}t + \mathrm{d}B_t\,, \quad X(0) = 0\,, \tag{1.1}$$

where $B(t)$ is a standard two-dimensional Brownian motion ω is defined as

$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x)) = (\partial_{x_2}\xi(x), -\partial_{x_1}\xi(x)), \tag{1.2}$$

with ξ being the $2d$ Gaussian Free Field (GFF) about which, at this stage, we recall just that it is a centered Gaussian distribution, whose covariance function is proportional to the logarithm of the distance. This can be seen as a Brownian tracer particle moving in a turbulent incompressible fluid: it moves on its own as a Brownian motion ($\mathrm{d}B_t$), but is also pushed by the fluid ($\omega(X(t))\mathrm{d}t$). The fact that ω is by definition divergence-free ($\nabla \cdot \omega = 0$) can be interpreted as the incompressibility of the fluid, but is also essential for the analysis, see Proposition 1.1.2. The turbulence is reflected in the randomness of the drift. One can interpret the drift not changing in time as the particle moving much faster than the surrounding fluid. However, letting the drift evolve in the most natural way, namely by a stochastic heat equation, does not change the behavior of the system, see [dLFW24].

Equation (1.1) is formal, since the regularity of ω is too low for the classical solution theory of SDEs, or even for more modern techniques, e.g. [CC18, DD16]. We will regularize it by convolving ω with a smooth, radially symmetric bump function U, which satisfies $\int_{\mathbb{R}^2} U(x)\mathrm{d}x = 1$. Call this regularized drift field ω^U. As opposed to what one might expect we will not let this regularization go to 0, but rather we will look at large times t. This is actually equivalent, since the diffusively rescaled diffusion

$$X^\varepsilon_t = \varepsilon X_{t/\varepsilon^2} \tag{1.3}$$

satisfies

$$\mathrm{d}X_t^\varepsilon = \mathrm{d}\tilde{B}_t + \omega^{U^\varepsilon}(X_t^\varepsilon)\,,$$

where the law of $\tilde{B}_t$ is still standard 2d Brownian motion, and U^ε is given by

$$U^\varepsilon(x) = \varepsilon^{-2}U(\varepsilon^{-1}x)\,.$$

Note that $\int_{\mathbb{R}^2} U^\varepsilon(x)\mathrm{d}x = 1$ under this rescaling and the bump function becomes more and more concentrated around the origin as $\epsilon \to 0$, thus converging to a Dirac δ. This is closely connected to the fact that this equation is formally invariant under diffusive scaling. Considering a fixed time horizon and taking the regularization to 0 is equivalent to considering large times and fixing the regularization parameter; thus, we will restrict ourselves to the latter point of view.

As mentioned in the beginning, to prove superdiffusivity of a system, one first defines a diffusion coefficient $D(t)$. In this case, it will be simply the mean-squared displacement, i.e., $tD(t) = \mathbf{E}(|X(t)|^2)$ Our main theorem concerns the Laplace transform of this quantity (the reason we work in Laplace transform is explained at the end of this section):

$$\mathcal{D}(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}\mathbf{E}(|X(t)|^2)\mathrm{d}t, \text{ for } \lambda > 0\,.$$

Here, $\mathbf{E}$ denotes the double average (annealed average) with respect to the Brownian noise and the realization of the drift field. The main result of Chapter 2 is

Theorem 1.1.1 (informal version of Theorem 2.2.2 in Chapter 2)**.** *Up to multiplicative* $(\log\log\lambda)^{1+}$ *errors*

$$\mathcal{D}(\lambda) \stackrel{\lambda\to 0^+}{\approx} \frac{\lambda^2}{\sqrt{\log\lambda}}. \tag{1.4}$$

This result is meaningful because, as usual, the small-λ behavior of the Laplace transform of a function is related to the large-time behavior of the function itself. While there exists a whole set of tools (Abelian/Tauberian theorems, see [BGT89]) that address this correspondence under suitable assumptions, there exists no general theorem turning the bound on $\mathcal{D}(\lambda)$ for λ small into pointwise upper/lower bounds on $D(t)$. In our concrete case, however, a simple adaptation to our setting of an argument in [QV08] allows us to deduce from (1.4) the pointwise upper bound $D(t) \leq (1 + \log(1 + t))^{\frac{1}{2}+o(1)}$. For the lower bound, the same argument does not work but the general theory of Tauberian inversion [BGT89, Theorem 1.7.1]) gives the following:

$$\limsup_{t\to\infty} \frac{D(t)}{\sqrt{\log t}\,(\log\log t)^{-1-}} > 0\,.$$

Also, if we had an asymptotic equivalence of the form $\mathcal{D}(\lambda) \stackrel{\lambda\to 0^+}{\sim} C\frac{\lambda^2}{\sqrt{\log\lambda}}$ would imply

$$\frac{1}{T}\int_0^T tD(t)\mathrm{d}t \sim \frac{CT^2}{2}\sqrt{\log t}\,,$$

by general Tauberian inversion theorems, see [Fel91, Chapter XIII.5].

To adapt the general strategy of [Yau04] we first must change perspective on our process to see it as a stationary Markov process. This is done by looking at the **Environment as seen from the particle** ω_t defined as

$$\omega_t = \omega_t(x) \stackrel{\text{def}}{=} \omega(X(t) + x), \quad x \in \mathbb{R}^2\,.$$

The following is a well-known fact from the theory of diffusions in random environments, see [KLO12, Chapter 11], in particular Proposition 11.10.

Proposition 1.1.2. *Consider a diffusion in a divergence-free drift field, whose law is translation invariant and ergodic. Then the environment process, as defined above, is a stationary ergodic Markov process.*

Let us give some intuition for this result: Given a general random drift field $F : \mathbb{R}^2 \to \mathbb{R}^2$ it can be decomposed into a part that is divergence-free and a part that is of gradient type:

$$F(x) = \tilde{\nabla}\psi + \nabla\phi$$

where $\tilde{\nabla}$ is the rotated gradient as in (1.2) and $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are potentials. Given a realization of F the potentials ϕ and ψ are defined uniquely up to constants. Considering the SDE

$$\mathrm{d}X(t) = F(X(t))\mathrm{d}t + \mathrm{d}B(t), \quad X(0) = 0\,,$$

for a standard Brownian motion the two potentials have the following effect: The term $\tilde{\nabla}\psi$ pushes $X(t)$ along the level lines of ψ, while the term $\nabla\phi$ pushes $X(t)$ in the direction of steepest ascent with respect to ϕ. Thus the expected value of $\phi(X(t)) - \phi(0)$ will generally be positive (at least in the case where only the ϕ term is present.) This is immediately an obstacle to the stationarity of the environment process $F(x,t) \stackrel{\text{def}}{=} F(X_t + x)$. However, when only the ψ term is present, no such obstacle exists. For example, if there is no diffusion term (i.e. one considers instead of the SDE the associated ODE) then the equation with no $\nabla\phi$ will simply trace the level lines of ψ. Then $\psi(X(t)) - \psi(0)$ will just be 0 for all t since the process does not leave the value it started at. In our case, the drift field is $\tilde{\nabla}\xi$ with ξ being a mollified Gaussian free field, so the assumptions are easily satisfied and ω_t is indeed a stationary Markov process.

Writing the first component of the solution $X(t)$ of (1.1) in integral form, we obtain

$$X_1(t) = B_1(t) + \int_0^t \omega_1(X_s)\mathrm{d}s = B_1(t) + \int_0^t \phi(\omega_s)\mathrm{d}s\,,$$

where ϕ is the functional defined by

$$\phi(\omega) = \omega_1(0)\,,$$

i.e., the evaluation of the first component of the field at 0. Since the process is isotropic it suffices to study $\mathbf{E}[|X_1(t)|^2]$ and since $\mathbf{E}[|B_t|^2] = t$, any superdiffusivity must come from the term involving the integral of ϕ. The rewriting above has the advantage that the second term is an additive functional of the stationary Markov process ω_t. This implies as in [CES21, Lemma 5.1] that

$$\tilde{D}(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}\mathbf{E}\left[|\int_0^t \phi(\omega_s)\mathrm{d}s|^2\right]\mathrm{d}t = \frac{2}{\lambda^2}\mathbb{E}[\phi(\omega)(\lambda - \mathcal{G})^{-1}\phi(\omega)], \tag{1.5}$$

where $\mathcal{G}$ is the generator of the Markov process ω and $\mathbb{E}$ (as opposed to the double average $\mathbf{E}$) denotes the expectation with respect to the stationary measure of the environment process, i.e., the law of the GFF. Note that this allows us to rewrite a quantity associated with an expectation with respect to the process ω_s as an expectation with respect to just the stationary law. This is the main reason for performing the Laplace transform and the starting point for the method. Before we see how to deal with the resolvent $(\lambda - \mathcal{G})^{-1}$ we will now see how to get to this point for the model from Chapter 3.

1.1.2 The stochastic Burgers equation

In Chapter 3 we study the stochastic Burgers

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot\nabla(\eta^2) + \nabla\cdot\xi\,, \tag{1.6}$$

where η is a scaler field depending on space and time, $\mathfrak{w} \in \mathbb{R}^2 \neq 0$ is a fixed vector controlling the strength and direction of the non-linearity, and ξ is d-dimensional space-time white noise. Van Beijeren, Kutner, and Spohn introduced this equation in [vBKS85] as a toy example of a driven diffusive system with one conserved quantity. They already conjectured superdiffusive behavior in dimensions 1 and 2 with Diffusion coefficients $t^{\frac{1}{3}}$ and $(\log t)^{\frac{2}{3}}$ respectively, based on a mode-coupling heuristic.

We will restrict to dimension $d = 2$, and just briefly mention that the equation $d = 1$ is (at least formally) the gradient of the KPZ equation, and the case $d \geq 3$ was recently treated in [CGT24] where large-scale Gaussian fluctuations were shown. Dimension $d = 2$ is again scaling critical, in the sense that the equation above is formally invariant under the diffusive rescaling

$$\eta^N(t,x) = N\eta(N^2 t, Nx)\,. \tag{1.7}$$

Since any potential solution of equation (1.6) is too rough for the square in the non-linearity to be defined, we again need to regularize. One could try to mollify the noise ξ, but we found it more convenient to regularize the non-linearity with a Fourier cutoff Π_a acting in Fourier as

$$\widehat{\Pi_a \eta}(k) \stackrel{\text{def}}{=} \widehat{\eta}(k)\mathbf{1}_{|k|\leq a}.$$

To avoid integrability issues we will also work on a large torus $\mathbb{T}^2_N$ of side length $2\pi N$, instead of the whole plane. This leads to the following equation

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot\Pi_1\nabla(\Pi_1\eta)^2 + \nabla\cdot\xi\,. \tag{1.8}$$

Related to the formal scale invariance of the model, there is again an equivalence between looking at large scales and removing the regularization. By defining the rescaled solution η^N as in (1.7) one obtains a solution to the following equation on the fixed size torus $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{T}^2_1$:

$$\partial_t \eta^N = \frac{1}{2}\Delta\eta^N + \mathfrak{w}\cdot\Pi_N\nabla\left(\Pi_N\eta^N\right)^2 + \nabla\cdot\xi\,. \tag{1.9}$$

The diffusion coefficient will now depend on N and we will take a limit $N \to \infty$ in the main statement. Compared to the previous section, the definition of the diffusion coefficient is less obvious. We adopt the following definition of **bulk diffusion coefficient**:

$$D^N(t) = 1 + N^2\frac{2|\mathfrak{w}|^2}{t}\int_0^{\frac{t}{N^2}}\int_0^s\int_{\mathbb{T}^2_1}\mathbf{E}\left(\Pi_N{:}(\Pi_N\eta^N)^2{:}(r,x)\Pi_N{:}(\Pi_N\eta^N)^2{:}(0,0)\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\,. \tag{1.10}$$

The reader should feel free to skip this exact expression for now. The formula has a similar structure to the bulk diffusion coefficients for particle systems, e.g. in [Yau04]. For a heuristic derivation of this formula, see Section 3.B.

Remark 1.1.3. It might seem that we are removing the regularization of the non-linearity twice, once by letting N tend to ∞ and once by considering large times t. However, the first limit is letting the volume go to infinity. We make no claims of convergence of $\eta^N(x,t)$ as $N \to \infty$, and such a convergence would fail to hold. Working on the rescaled torus of size 1 is for notational convenience, and not necessary mathematically.

The main theorem of Chapter 3 again regards the Laplace transform of $tD(t)$,

$$\mathcal{D}^N(\lambda) = \int_0^\infty e^{-\lambda t} t D^N(t) \mathrm{d}t.$$

Given this definition, we can now formulate that main Theorem

Theorem 1.1.4 (informal version of Theorem 3.1.1). *For $\mathfrak{w} \neq 0$ and $\lambda > 0$ small*

$$\mathcal{D}^N(\lambda) \approx \frac{1}{\lambda^2} |\log \lambda|^{\frac{2}{3}},$$

uniformly in N, up to multiplicative errors that are polynomial in $\log\log|\log\lambda|$.

Again the same comments about Tauberian inversions as made after 1.1.1 apply. It is also worth noting that the errors are smaller than the ones obtained in the corresponding result in [Yau04], which are of order $\exp((\log\log|\log\lambda|)^2)$.

The expectation $\mathbf{E}$ in (1.10) is with respect to the stationary process. It can easily be checked (at least formally) that spatial white noise is stationary for the stochastic Burgers equation, and regularization by a cut-off in Fourier does not change the invariant measure in this case. This process is however not ergodic, since the equation is conservative, and, thus, the 0 mode $\widehat{\eta_t^N}(0) \int_{\mathbb{T}} \eta_t^N(x) dx$ is constant in time. Because of this, we use stationary white noise conditioned to have average 0 instead, which is still invariant, since the Fourier modes of white noise are independent. See also the discussion at the beginning of Section 3.2.2.

The Markov process describing the evolution of the stochastic Burgers equation has a generator $\mathcal{L}^N$ and the Laplace transformed bulk diffusivity $\mathcal{D}^N(\lambda)$ can again be expressed as an inner product using the resolvent:

Proposition 1.1.5 (Informal version of Proposition 3.3.1). *For any N and $\lambda > 0$ the Laplace transform of the bulk diffusivity satisfies*

$$\mathcal{D}^N(\lambda) = \frac{1}{\lambda^2} + \frac{C}{\lambda^2} \mathbb{E}\Big[\mathcal{N}^N[\eta^N](\lambda N^2 - \mathcal{L}^N)^{-1} \mathcal{N}^N[\eta]\Big]$$

where C is an explicit constant depending on $\mathfrak{w}$, and $\mathcal{N}^N[\eta^N]$ is an explicit quadratic observable.

Here, $\mathbb{E}$ denotes expectation with respect to the law of the spatial white noise. Using this, as in (1.5) for the diffusion in the curl of the Gaussian free field above, the proof of the main theorems has been reduced to estimating a term of the shape $\langle \phi, (\lambda - \mathcal{L})^{-1} \phi \rangle$ and the next section will be dedicated to developing the tools to do so.

1.1.3 Generators and Gaussian Chaos Decomposition

The task at hand for both models is understanding a resolvent acting on observables of a Gaussian field. Denote by $L^2(\mathbb{P})$ the space of square-integrable random variables with respect to the stationary law $\mathbb{P}$ which is the law either of the environment ω in the case of Chapter 2 or of white noise in the case of Chapter 3. We want to understand better how the generators $\mathcal{G}$ and $\mathcal{L}$ act on this space. Since in both cases $\mathbb{P}$ is Gaussian there is a powerful tool for this task at hand: the Wiener chaos decomposition. By [Nua06, Theorem 1.1.1] $L^2(\mathbb{P})$ admits and orthogonal Hilbert space decomposition given by

$$L^2(\mathbb{P}) = \bigoplus_{n \geq 0}^{\infty} H_n,$$

where H_n is the n-th chaos, which is intuitively given by observables of degree n. It can be defined as the closure of observables of the type

$$\int_{(\mathbb{R}^2)^n} f(x_{1:n}) {:}\eta(x_1)\dots\eta(x_n){:} \mathrm{d}x_{1:n}\,, \tag{1.11}$$

where f is in some class of test functions and $:X:$ denotes the Wick product associated with $\mathbb{P}$, which is essential for making the different chaoses orthogonal. We also used the convenient shorthand notation $x_{1:n}$ for $x_1,\dots,x_n$, which we will use throughout this introduction. For the white noise, (1.11) is slightly formal since we cannot evaluate η at points. However, it can be made rigorous in various ways, since the above expression can be interpreted as testing η against test functions, for example by using the polarization identity and Hermite polynomials, see Section 3.2.2 or the first Chapter of [Nua06] for more detail. While ω has been mollified, it is vector-valued, so the above expression needs to be modified by adding indices to the test functions and the field, see (2.17).

In both cases, there is an isomorphism from the n-th chaos to Fock-space, which is the space ΓL^2_n of symmetric mean-zero kernels in n variables, with an inner product specified by the specific structure of $\mathbb{P}$. In the case of white noise on the torus, this is just $n!$ times the standard L^2 inner product on $(\mathbb{T}^2)^n$, while for the mollified Gaussian free field it takes the form (in Fourier)

$$\langle \psi_n, \phi_n\rangle \stackrel{\text{def}}{=} n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \frac{\widehat{V}(p_j)}{|p_j|^2}\, \overline{\hat{\psi}_n(p_{1:n})}\, \hat{\phi}_n(p_{1:n})\, \mathrm{d}p_{1:n}\,, \tag{1.12}$$

where $\widehat{V} = U * U$ is the effect of the mollification and decays for large arguments, and the denominator is from the Gaussian free field. This is the space where all of our calculations will take place, and we freely identify operators acting on $L^2(\mathbb{P})$, on the chaoses, on ΓL^2, and the Fourier transformed kernels.

The generators of the two Markov processes have a structure that is particularly compatible with the Wiener chaos decomposition.

Proposition 1.1.6 (part of Lemma 3.2.3)**.** *The generator $\mathcal{L}$ can be written as*

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_+ + \mathcal{A}_-$$

where $-\mathcal{L}_0$ is symmetric, positive definite and leaves the chaos invariant, i.e. $\mathcal{L}_0(H_n) = H_n$, while $\mathcal{A}_+$ and $\mathcal{A}_-$ are minus the adjoint of one another and increase and decrease the chaos respectively, i.e. $\mathcal{A}_+(H_n) \subset H_{n+1}$.

The same decompositions hold for $\mathcal{G}$, whose symmetric part is called $-\Delta$ in Chapter 2. The way these operators act on kernels is completely explicit, here is the example for the stochastic Burgers equation (see (2.22)):

$$\begin{aligned}
\mathcal{F}\left(\mathcal{L}_0\varphi\right)(k_{1:n}) &= -\tfrac{1}{2}|k_{1:n}|^2\hat{\varphi}(k_{1:n})\\
\mathcal{F}\left(\mathcal{A}^N_+\varphi\right)(k_{1:n+1}) &= -\frac{\iota}{\pi(n+1)}\sum_{1\le i<j\le n+1} \mathbb{J}^N_{k_i,k_j}\left[\mathfrak{w}\cdot(k_i+k_j)\right]\hat{\varphi}\left(k_i+k_j, k_{\{1:n+1\}\setminus\{i,j\}}\right)\\
\mathcal{F}\left(\mathcal{A}^N_-\varphi\right)(k_{1:n-1}) &= -\frac{\iota\, n}{\pi}\sum_{j=1}^{n-1}(\mathfrak{w}\cdot k_j)\sum_{\ell+m=k_j}\mathbb{J}^N_{\ell,m}\,\hat{\varphi}\left(\ell,m,k_{\{1:n-1\}\setminus\{j\}}\right)\,,
\end{aligned}$$

where the indicator function $\mathbb{J}$ is given by

$$\mathbb{J}_{\ell,m} \stackrel{\text{def}}{=} \mathbf{1}_{\{0<|\ell|\le N, 0<|m|\le N, 0<|\ell+m|\le N\}}\,.$$

The exact expressions are not important for the sake of this introduction. We point out some relevant features (that are also shared among the two models):

- The operator $\mathcal{L}_0$ is not only diagonal in chaos but also in Fourier: If the kernel φ is concentrated on a single Fourier mode $k_{1:n}$, then so is $\mathcal{L}_0\varphi$. We say such an operator is acting via a **Fourier multiplier**, in this case $\sigma(k_{1:n}) = -\frac{1}{2}|k_{1:n}|^2$. Such operators are much easier to deal with than other operators, and the aim of the estimates below is often to estimate other operators with operators that act via a Fourier multiplier. In particular, such operators are easy to invert, since their inverses simply act with the Fourier multiplier $1/\sigma$.

- The operator $\mathcal{A}_+$ can be seen as a combination of a "creation" and a gradient operator. The term $\mathfrak{w} \cdot (k_i + k_j)$ corresponds to a derivative in direction $\mathfrak{w}$. The increase in chaos can be seen as splitting apart a particle with momentum m into all possible ways to write $m = k_i + k_j$.

- In this case (i.e., the case of the stochastic Burgers equation), the operators $\mathcal{A}_+$ and $\mathcal{A}_-$ preserve total momentum, in the sense that a kernel concentrated on momenta satisfying $\sum k_i = k$ for some fixed k will again be mapped to such kernel. While this property will be used to simplify a step, see Lemma 3.3.5 and in particular (3.18) below, this property is not satisfied by the corresponding operators for the diffusion in Chapter 2 and does not seem to be essential for this analysis.

- Both the symmetric part of this generator and the one of Chapter 2 can be seen as a generalized Laplacian. Indeed, in the first chaos, they act on kernels like a Laplacian. However, on higher chaoses they act quite differently: one acts via Fourier multiplier given by $\sum_{i=1}^n |k_i|^2$, while the other acts by multiplication with $|\sum_{i=1}^n k_i|^2$. This can be traced back to the randomness being injected into the system coming from white noise in the case of the Burgers equation, while it comes from a single Brownian motion in the case of the diffusion in the curl of the Gaussian free field. In general, the Laplacian coming from the diffusion is more difficult to handle, since it can vanish even for large momenta.

We are now ready to use this structure to estimate $\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle$. Before we develop the full setup using the truncated resolvent equation in Section 1.1.5, we will see in the next section how this setup can already be used to find the first superdiffusive lower and upper bounds.

1.1.4 Variational approach

One possible approach to estimating the term $\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle$ is via a variational formula.

Lemma 1.1.7. *For a selfadjoint A it holds that*

$$\langle \phi, A\phi \rangle = \sup_{\|\rho\|<\infty} \{ 2\langle \phi, \rho \rangle - \langle \rho, A^{-1}\rho \rangle \} \tag{1.13}$$

To apply this to $\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle$ we need to find the inverse of the symmetric part of $(\lambda - \mathcal{L})^{-1}$, which (by general linear algebra) is given by $(\lambda - \mathcal{L})^*(\lambda - \mathcal{L}_0)^{-1}(\lambda - \mathcal{L})$. Plugging this into (1.13) yields

$$\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle = \sup_{\|\rho\|<\infty} \{ 2\langle \phi, \rho \rangle - \langle \rho, (\lambda - \mathcal{L})^*(\lambda - \mathcal{L}_0)^{-1}(\lambda - \mathcal{L})\rho \rangle \}$$

$$= \sup_{\|\rho\|<\infty} \{2\langle\phi,\rho\rangle - \langle\rho,(\lambda-\mathcal{L}_0)\rho\rangle - \langle(\mathcal{A}_+ + \mathcal{A}_-)\rho,(\lambda-\mathcal{L}_0)^{-1}(\mathcal{A}_+ + \mathcal{A}_-)\rho\rangle\}, \tag{1.14}$$

where the second equality is obtained by expanding the product on the right and canceling a $\langle\rho,(\mathcal{A}_+ + \mathcal{A}_-)\rho\rangle$ with $\langle\rho,(\mathcal{A}_+ + \mathcal{A}_-)^*\rho\rangle$. To obtain a first upper bound from this one can drop the second term to obtain

$$\langle\phi,(\lambda-\mathcal{L})^{-1}\phi\rangle \le \sup_{\|\rho\|<\infty} \{2\langle\phi,\rho\rangle - \langle\rho,(\lambda-\mathcal{L}_0)\rho\rangle\} = \langle\phi,(\lambda-\mathcal{L}_0)^{-1}\phi\rangle\,.$$

Recalling that ϕ is an explicit element of either H_1 in Chapter 2 or of H_2 in Chapter 3 and that $\lambda - \mathcal{L}_0$ is easy to invert since it acts via a Fourier multiplier, this quantity is easily calculated. In both cases, one ends up with a quantity that behaves like the integral

$$\int_{\mathbb{R}^2} \frac{\mathbf{1}_{|p|\le 1}}{\lambda+|p|^2} = |\log(\lambda)| + O(1)$$

for $\lambda \to 0$, which yields the first upper bound.

In principle, to obtain a lower bound from the variational formula, one needs to choose a function ρ, and any such choice will lead to a lower bound. A priori, it is unclear how to choose such a test function in dependence on λ to obtain a good lower bound. As a first attempt, at least for the diffusion (i.e. Chapter 2) one can look for a test function ρ in the first chaos H_1. Using the fact that $\mathcal{A}_-\rho = 0$ for any ρ in the first chaos, one ends up having to upper bound the quantity

$$\langle\mathcal{A}_+\rho,(\lambda-\mathcal{L}_0)^{-1}\mathcal{A}_+\rho\rangle\,. \tag{1.15}$$

This can be done (and indeed will be the first step in our iterative estimation scheme, see Section 1.1.6). The upper bound one obtains is of the type

$$\langle\rho,\mathcal{S}\rho\rangle \tag{1.16}$$

where $\mathcal{S}$ is an operator acting via Fourier multiplier $\sigma(k) = |k|^2\log(1+\frac{1}{\lambda+|k|^2})$ in the isotropic case (i.e., the diffusion in the curl of the GFF) and $\sigma(k) = |\mathfrak{w}\cdot k|^2\log(1+\frac{1}{\lambda+|k|^2})$ in the anisotropic case (i.e., the stochastic Burgers equation)[2]. Either way, thanks to this estimate one can solve the modified variational problem (that is, where (1.15) is replaced by (1.16) and ρ is restricted to the lowest chaos, as mentioned above) to obtain an estimate of the type

$$\langle\phi,(\lambda-\mathcal{L})^{-1}\phi\rangle \ge \int_{\mathbb{R}^2} \frac{\mathbf{1}_{|p|\le 1}}{\lambda+|p|^2+\sigma(p)}dp \gtrsim \begin{cases} \log|\log(\lambda)|, \text{ in the isotropic case and} \\ |\log\lambda|^{\frac{1}{2}}, \text{ in the anisotropic case.} \end{cases} \tag{1.17}$$

See [TV12] for these calculations carried out for the diffusion in the curl of the GFF and a related anisotropic model. One can already see the difference between the anisotropic and isotropic universality classes, which will become even more apparent in Section 1.1.6 below.

This is how the superdiffusive upper and lower bounds are obtained in [LQSY04, LRY05, TV12]. In principle, there is no obstacle to improving at least the lower bound by taking ρ with values in larger and larger chaoses. However, from the analysis below it becomes clear, that the expressions for ρ would become increasingly complex, to the point where such calculations would become very hard to carry out. We did not proceed by immediately taking a specific test function and evaluating it inside 1.13, but instead first found the bound (1.16). This already indicates the general strategy. If such a bound was not available, but instead one only had a specific ρ which obtains the first lower bound (1.17) it would not be possible to proceed like below. Indeed, this is the case for the self-repelling Brownian polymer model first introduced in [TV12] and recently revisited by [CG24].

[2] In the anisotropic case we are dropping (for the sake of this informal discussion) some terms which need to be dealt with separately, namely the "off-diagonal" terms, see Lemma 3.4.7 for their definition and the iterative estimates in Section 1.1.6 to see how they are estimated.

1.1.5 Truncated resolvent equation

As we have seen in the previous section, we need more sophisticated tools to estimate $\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle$. The difficulty lies in inverting the operator $(\lambda - \mathcal{L})$. While ϕ is purely either in the first or second chaos, the solution to the generator equation $(\lambda - \mathcal{L})\psi = \phi$ has components in every chaos and is notoriously hard to find explicitly. To avoid this problem, we will truncate this equation. Let $I_{\leq n}$ be the projection onto the first n chaoses, i.e. onto $H_{\leq} n = \bigoplus_{0 \leq k \leq n} H_k$. Let $\psi^{(n)}$ be the solution to the **truncated generator equation**, that is,

$$I_{\leq n}(\lambda - \mathcal{L}) I_{\leq n} \psi^{(n)} = \phi. \tag{1.18}$$

This is useful due to the following Lemma, which was first proved in [LQSY04, Lemma 2.1].

Lemma 1.1.8. *For every $n \geq 1$, one has*

$$\langle \phi, \psi^{(2n)} \rangle \leq \langle \phi, (\lambda - L)^{-1}\phi \rangle = \langle \phi, \psi \rangle \leq \langle \phi, \psi^{(2n+1)} \rangle.$$

Furthermore, the upper and lower bounds both converge to $\langle \phi, \psi \rangle$.

The proof of this is not specific to the model, but rather only needs the basic properties of the decomposition $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_+ + \mathcal{A}_-$ and its interaction with the chaos decomposition:

- The operator $\mathcal{L}_0$ is symmetric and leaves the chaos invariant.
- The operator $\mathcal{A}_+$ and $\mathcal{A}_-$ satisfy $(\mathcal{A}_+)^* = -\mathcal{A}_-$, $\mathcal{A}_+$ increases the chaos by 1, while $\mathcal{A}_-$ decreases it by one.
- The observable ϕ is in finitely many chaoses.

The reason this is useful is that the truncated equation (1.18) is a finite triangular system:

$$\begin{cases} (\lambda - \mathcal{L}_0)\psi_n^{(n)} - \mathcal{A}_+\psi_{n-1}^{(n)} = 0, \\ (\lambda - \mathcal{L}_0)\psi_{n-1}^{(n)} - \mathcal{A}_+\psi_{n-2}^{(n)} + \mathcal{A}_+^*\psi_n^{(n)} = 0, \\ \dots \\ (\lambda - \mathcal{L}_0)\psi_2^{(n)} - \mathcal{A}_+\psi_1^{(n)} + \mathcal{A}_+^*\psi_3^{(n)} = \phi_2, \\ (\lambda - \mathcal{L}_0)\psi_1^{(n)} + \mathcal{A}_+^*\psi_2^{(n)} = \phi_1\,, \end{cases}$$

where in Chapter 3 ϕ_1 is zero, while in Chapter 2 ϕ_2 is zero. This system can be solved recursively starting from the top by expressing $\psi_n^{(n)}$ in terms of $\psi_{n-1}^{(n)}$ and then $\psi_{n-1}^{(n)}$ in terms of $\psi_{n-2}^{(n)}$ and so on. Doing so leads to the recursively defined operators

$$\begin{aligned} &\mathcal{H}_1 \stackrel{\text{def}}{=} 0\,, \\ &\mathcal{H}_{j+1} = \mathcal{A}_+^*(\lambda - \mathcal{L}_0 + \mathcal{H}_j)^{-1}\mathcal{A}_+\,, \qquad \text{for } j \geq 1\,. \end{aligned} \tag{1.19}$$

Note that the definition of the operators $\mathcal{H}_k$ in Chapters 2 and 3 differs by a shift in the index k by 1. This is a matter of convention and stems from the observable ϕ being in the first or the second chaos. For this introduction, we will pretend the operators start from index $j = 1$ in both cases for simplicity. Since ϕ is only in the first or second chaos, we only need to find $\psi_1^{(n)}$ or $\psi_2^{(n)}$. Using the operators $\mathcal{H}_j$ we can express this solution by

$$\langle \phi, \psi^{(n)} \rangle = \langle \phi, \psi_1^{(n)} \rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_n)^{-1}\phi \rangle \tag{1.20}$$

in the case of $\phi = \phi_1$, i.e. the diffusion in Chapter 2. In the case $\phi = \phi_2$ there is an additional term:

$$\langle \phi, \psi^{(n)} \rangle = \langle \phi, \psi_2^{(n)} \rangle = \langle \phi, (((\lambda - \mathcal{L}_0) + \mathcal{H}_k^N - \mathcal{A}_+^N (\lambda - \mathcal{L}_0)^{-1} \mathcal{A}_-^N)^{-1})^{-1} \phi \rangle .$$

In this case the operator $-\mathcal{A}_+^N (\lambda - \mathcal{L}_0)^{-1} \mathcal{A}_-^N$ does not actually give a contribution, see Lemma 3.3.5. This is however due to a special property of the bulk-diffusivity. For a general observable in the second chaos, one would have to estimate this term separately. This is possible and not fundamentally different from the estimates carried out here, see [CET23a, Section 3.2].

Thus, we have further reduced the task to understanding the operators $\mathcal{H}_k$. By induction, one can see that the H_k are self-adjoint, positive operators that leave the chaos invariant. However, their structure is still quite complicated. Unpacking the definition one can see that to write down $\mathcal{H}_k \phi$ for some specific $\phi \in H_1$ one encounters expressions in all chaoses up to H_k and has to repeatedly invert operators on these spaces. Since the only operators that we can easily invert are those that act by a Fourier multiplier (see the discussion after Proposition 1.1.6), the aim will be to use the recursive structure of the $\mathcal{H}_k$ to find such estimates. These recursive estimates will be outlined in the next section.

1.1.6 Iterative estimates

The goal of the iterative estimates is to bound the operators $\mathcal{H}_k$ with operators $\mathcal{Z}_k$ that act via a Fourier multiplier. We use the following partial operators

Definition 1.1.9. *Given two selfadjoint operators $\mathcal{A}$ and $\mathcal{B}$ on ΓL^2,*

$$\mathcal{A} \leq \mathcal{B} \text{ if and only if } \quad \forall n \, \forall \varphi \in \Gamma L_n^2 \quad \langle \mathcal{A}\varphi, \varphi \rangle \leq \langle \mathcal{B}\varphi, \varphi \rangle \quad \Leftrightarrow \quad \mathcal{B} - \mathcal{A} \geq 0 ,$$

where the last statement is taken to mean that $\mathcal{B} - \mathcal{A}$ is a positive operator.

For this partial ordering, the following well-known lemma holds

Lemma 1.1.10. *For any two operators $\mathcal{A}$ and $\mathcal{B}$ on ΓL^2 it holds that*

$$0 < \mathcal{A} \leq \mathcal{B} \quad \Leftrightarrow \quad 0 < \mathcal{B}^{-1} \leq \mathcal{A}^{-1} .$$

This means that, given a bound of the form

$$\mathcal{H}_k \leq \mathcal{Z}_k ,$$

by using the definition (1.19) this immediately implies the bound

$$\mathcal{H}_{k+1} \geq (\mathcal{A}_+)^* (\lambda - \mathcal{L}_0 + \mathcal{Z}_k)^{-1} \mathcal{A}_+ . \tag{1.21}$$

Since the first upper bound in (1.16) is already of this type, we have a starting point. However, finding a good form of the $\mathcal{Z}_k$ is far from trivial. If one tries to make all estimates as sharp as possible, the complexity of the expressions grows with each iteration, and it is unclear how to write down or prove a general upper or lower bound. On the other hand, if one loses too much at each iteration, the bounds one obtains will fail to determine the behavior of $\mathcal{D}(\lambda)$.

Let us look at the upper and lower bounds in Section 2.4.1. Recall that for this model $\mathcal{L}_0$ is written as $(-\Delta)$.

Upper and lower bounds for Chapter 2

For $k \in \mathbb{N}$, $x > 0$ and $z \geq 0$ we define L, LB_k and UB_k as follows

$$\mathrm{L}(x,z) = z + \log(1 + x^{-1}),$$
$$\mathrm{LB}_k(x,z) = \sum_{0 \leq j \leq k} \frac{(\frac{1}{2}\log \mathrm{L}(x,z))^j}{j!} \quad \text{and} \quad \mathrm{UB}_k(x,z) = \frac{\mathrm{L}(x,z)}{\mathrm{LB}_k(x,z)}$$

and for $k \geq 1$, σ_k, as

$$\sigma_k(x,z) = \begin{cases} \mathrm{UB}_{\frac{k-2}{2}}(x,z), & \text{if } k \text{ is even}, \\ \mathrm{LB}_{\frac{k-1}{2}}(x,z), & \text{if } k \text{ is odd}. \end{cases}$$

Note that $\sigma_1 \equiv 1$. All the properties we need on the functions $\mathrm{UB}_k, \mathrm{LB}_k$ are summarized in Lemma 2.A.1. Further, let

$$z_k(n) = K_1(n+k)^{2+2\varepsilon} \quad \text{and} \quad f_k(n) = K_2\sqrt{z_k(n)},$$

where K_1, K_2 are absolute constants and ε is the small positive constant that appears in the statement of Theorem 2.2.2.

Finally, for $k \geq 1$ let $\mathcal{S}_k$ be the operator whose multiplier is σ_k, i.e.

$$\mathcal{S}_k = \begin{cases} f_k(\mathcal{N})\,\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) & \text{if } k \text{ is even}, \\ \frac{1}{f_k(\mathcal{N})}\big(\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) - f_k(\mathcal{N})\big) & \text{if } k \text{ is odd}, \end{cases}$$

where $\mathcal{N}$ is the number operator acting on the n-th chaos as multiplication by n, i.e., $(\mathcal{N}\phi_n) = n\phi_n$ for $\phi_n \in H_n$. We are now ready to state the following theorem.

Theorem 1.1.11 (Theorem 2.4.2 from Chapter 2)**.** *For any $\varepsilon > 0$, the constants K_1, K_2 can be chosen such that the following holds. For $0 < \lambda \leq 1$ and $k \geq 1$, one has the operator bounds*

$$\mathcal{H}_{2k-1} \geq c_{2k-1}\,(-\Delta)\mathcal{S}_{2k-1}$$

and

$$\mathcal{H}_{2k} \leq c_{2k}\,(-\Delta)\mathcal{S}_{2k}$$

where $c_1 = 1$ and

$$c_{2k} = \frac{\pi}{c_{2k-1}}\left(1 + \frac{1}{k^{1+\varepsilon}}\right), \qquad c_{2k+1} = \frac{\pi}{c_{2k}}\left(1 - \frac{1}{(k+1)^{1+\varepsilon}}\right). \tag{1.22}$$

There is quite a bit to take in here, so let us notice several things about these bounds.

- Forgetting about the constants c_k and setting $f_k = z_k = 0$, the upper and lower bounds both converge to $(-\Delta)\log(\lambda - \Delta)^{\frac{1}{2}}$, which is coherent with the expected behaviour. It is interesting whether or not the operators $\mathcal{H}_k$ converge to a fixed point of (1.19) as $k \to \infty$. In [CGT24] an approximate version of such a fixed point is found in the weak-coupling regime.

- The terms z_k and f_k give polynomial additive and multiplicative terms in the upper and lower bounds. These are not problematic since we will apply Lemma 1.1.8 for k being of order $\log|\log \lambda|$. Thus these polynomial terms are of lower order, and indeed give the subdominant terms in the main statement.

- The general strategy is to split (1.21) into a "main" and several "error" terms. The main term is then shown to be close to the next diagonal estimate, while the error terms are absorbed into this main term by paying a multiplicative cost given by (1.22). The terms z_k and f_k are needed to make this cost smaller and smaller as the iteration progresses, resulting in the convergence of c_{2k} and c_{2k+1}. Otherwise, we would have multiplicative error terms of the form C^k, which would be of the same order as the main term and thus problematic.

- The polynomial terms f_k and z_k are applied to the number operator $\mathcal{N}$, and so they grow with both the iteration and the chaos. This is necessary to absorb the error terms since as we will see, they grow in number as the chaos increases.

- Note that the lower bounds are not actually positive, since their Fourier multipliers become negative for very large momenta. This is however not an issue, since the prefactor $\frac{1}{f_k}$ ensures that the $\lambda - \mathcal{L}_0 + \mathcal{H}_k$ is positive. This is an instance of a suboptimal bound, which reduces the complexity of the expression since the fact that the lower bound becomes negative is simply an effect of the integral in (2.67) as being interpreted as negative when $\lambda + |p|^2$ is larger than 1, however, the statement would also be true if we took it to be 0 in that case. But in the proof, it would not only introduce a case distinction in this case, but also have iterate less well in the next steps.

Before we compare Theorem 1.1.11 fore the diffusion in the curl of the GFF to the bounds obtained in Chapter 3 for the stochastic Burgers equation, we need to introduce a specific type of "error" term that appears in (1.21). Namely, we split into **diagonal** and **off-diagonal** terms in the following way. For any diagonal operator $\mathcal{Z}$ acting on ΓL^2 via the Fourier multiplier $\zeta = (\zeta_n)_{n\in\mathbb{N}}$ and $\varphi \in \Gamma L^2_n$ we can write

$$\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}\mathcal{A}^N_+\varphi, \varphi \right\rangle = \frac{n!\,n}{2\pi^2} \sum_{k_{1:n+1}} \zeta(k_{1:n+1}) \left| \sum_{1\le i<j\le n+1} (\mathfrak{w}\cdot(k_i+k_j))\hat{\varphi}(k_i+k_j, k_{1:n+1\setminus i,j}) \right|^2 . \tag{1.23}$$

Expanding the square as the product of two sums (one of them conjugated), the diagonal terms will be the sum of squares, i.e., of terms of the form

$$(\mathfrak{w}\cdot(k_1+k_2))^2 |\hat{\varphi}(k_1+k_2, k_{3:n+1})|^2 ,$$

while the off-diagonal terms are all the others. These are the ones one would obtain as an upper bound if one would apply the Cauchy inequality to the inner sum in (1.23). While that is too rough, since one picks up a factor depending on n that grows too quickly, these diagonal terms will still be the dominant ones. While the precise terms in Chapter 2 are of course different, the same type of decomposition with the same properties holds, see Lemma 2.4.4.

Let us look at the form the bounds take in Chapter 3.

Upper and lower bounds for Chapter 3

First, we need a few definitions. The skew Laplacian $\mathcal{L}_0^{\mathfrak{w}}$ is the linear operator whose action on Fock space is given, for every $\varphi \in \Gamma L^2_n$, by

$$\mathcal{F}\left(\mathcal{L}_0^{\mathfrak{w}}\varphi\right)(k_{1:n}) \stackrel{\text{def}}{=} -\tfrac{1}{2}(\mathfrak{w}\cdot k)^2_{1:n}\hat{\varphi}(k_{1:n}), \qquad \text{where} \qquad (\mathfrak{w}\cdot k)^2_{1:n} \stackrel{\text{def}}{=} \sum_{i=1}^n (\mathfrak{w}\cdot k_i)^2 .$$

The sequence of the exponents of the logarithm in the upper and lower bounds is defined recursively by

$$\theta_2 \stackrel{\text{def}}{=} 0 \qquad \text{and} \qquad \theta_{k+1} = 1 - \frac{\theta_k}{2} \qquad \text{for every } k \geq 3\,,$$

which can be seen to converge to $\frac{2}{3}$ exponentially fast. Let $k, N, n \in \mathbb{N}$ and $\delta \in (0,1)$. For $k \geq 2$, $x \in (0,\infty)$ and $z \in (1,\infty)$, we set

$$\mathrm{L}(x,z) \stackrel{\text{def}}{=} \log(1+x^{-1}) + z\,, \quad \mathrm{L}_k(x,z) \stackrel{\text{def}}{=} (\mathrm{L}(x,z))^{\theta_k}\,, \quad \mathrm{L}_k^N(x,z) \stackrel{\text{def}}{=} \mathrm{L}_k\left(\frac{x}{N^2}, z\right)\,.$$

For $k \geq 1$, let

$$z_k(n) \stackrel{\text{def}}{=} K(n+k)^{\frac{9}{2}+\frac{3}{2}\delta} \qquad \text{and} \qquad f_k(n) \stackrel{\text{def}}{=} 3(z_k(n))^{\frac{2}{3}}\,,$$

where K is a sufficiently large positive constant depending on $|\mathfrak{w}|$.

Definition 1.1.12. *For $\lambda > 0$ and $k \geq 2$*

$$\mathcal{S}_k^N \stackrel{\text{def}}{=} \begin{cases} f_k(\mathcal{N})\mathrm{L}_k^N(\lambda - \mathcal{L}_0, z_k(\mathcal{N})) & \text{if } k \text{ is odd,} \\ \frac{1}{f_k(\mathcal{N})}\left[\mathrm{L}_k^N(\lambda - \mathcal{L}_0, z_k(\mathcal{N})) - f_k(\mathcal{N})\right] & \text{if } k \text{ is even,} \end{cases}$$

where $\mathcal{N}$ is the number operator, acting on $\varphi \in \Gamma L_n$ by $\mathcal{N}\varphi = n\varphi$ for each $n \in \mathbb{N}$ and λ is the Laplace variable.

We can now state the bounds on the operators $\mathcal{H}_k$.

Theorem 1.1.13 (Theorem 3.4.5)**.** *For every $\delta \in (0,1)$ and for every $k \in \mathbb{Z}$, $k \geq 0$ we have*

$$\mathcal{H}_{2k+3} \leq c_{2k+3}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+3} + f_{2k+3}(\mathcal{N})(-\mathcal{L}_0)\right)\,, \tag{1.24}$$

$$\mathcal{H}_{2k+2} \geq c_{2k+2}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+2} - \frac{1}{(\mathcal{N}+k)^{1+\delta}}(-\mathcal{L}_0)\right)\,, \tag{1.25}$$

where the constants c_{2k+1} and c_{2k+2} are defined recursively by setting, for $k \geq 1$,

$$c_2 \stackrel{\text{def}}{=} \frac{1}{\pi(|\mathfrak{w}|^2 \vee 1)}\,, \quad c_{2k+1} = \frac{3}{2\pi|\mathfrak{w}|}\frac{\left(1+\frac{1}{2k^{1+\delta}}\right)}{c_{2k}} > 1\,, \quad c_{2k+2} = \frac{3}{2\pi|\mathfrak{w}|}\frac{\left(1-\frac{1}{2k^{1+\delta}}\right)}{\left(1+\frac{1}{2k^{1+\delta}}\right)c_{2k+1}} < 1\,.$$

Comparing the lower and upper bounds for Chapter 2 one can notice several things

- Again the lower and upper bound, if one ignores all multiplicative and additive errors, converge to the expected $\log^{\frac{2}{3}}$ behavior. Compared to the ones for the isotropic case, they do so much faster (because the convergence $\theta_k \to 2/3$ is exponentially fast. This will in particular mean that in the proof of the main theorem, k will be of order $\log\log|\log\lambda|$, which is the reason for the smaller multiplicative errors in Theorem 3.1.1 as compared to Theorem 2.2.2.

- The main terms of the upper and lower bounds are not multiplied by a full generator $(-\mathcal{L}_0)$ but by $(-\mathcal{L}_0^{\mathfrak{w}})$, which only acts in direction $\mathfrak{w}$.

- There are however error terms that are multiplied by the full $(-\mathcal{L}_0)$ and thus cannot easily be absorbed into the main term. They are small enough not to hinder the iteration, since in the expression $(\lambda - \mathcal{L}_0 + \mathcal{H}_k)^{-1}$, they can, in some sense, be absorbed into the $\mathcal{L}_0$.

1.1.7 Replacement Lemmas

The proof of the iterative estimates laid out in the previous section is quite technical. Here is a short guide to the structure of these estimates, for simplicity only given for Chapter 2.

- As already mentioned, the terms are split into a diagonal and an off-diagonal part. By relabeling variables, both of these can be reduced to estimating a certain type of two-dimensional integral, see Lemma 2.4.5 and Lemma 2.4.6 respectively. Lemma 2.4.6 uses a reweighted Cauchy inequality (see (2.39)), which seems to be somewhat model-independent, since it has also been used in [CET23a] and Chapter 3.

- For the off-diagonal terms, this integral is treated in Lemma 2.A.3. Since the off-diagonal terms are considered error terms, they are only estimated from above (and this estimate is subtracted from the main term in the lower bound.) The proof proceeds by splitting the integral into various regions and is not particularly illuminating. An essential part is using the $\sin(\cdot)$ in the denominator to deal with regions where q_1 and q_3 are anti-parallel. The fact that there does not appear to be a general method to deal with the off-diagonal terms seems to be one of the major obstacles for the models in which only the first lower and upper bounds have been achieved (e.g. the models in [LRY05, CG24]).

- For the diagonal parts, instead of immediately doing upper and lower bounds, one first proves a "replacement lemma". The idea is to show that the difference between the diagonal term and a term already amenable to the iteration is small (compared to the main term), see Lemma 3.A.2. This ensures that one does not lose any multiplicative constants in the estimate, which would prevent the prefactors c_k from converging. Typically, this proceeds in several steps of successively replacing an expression with the next. These are subtly different for the various models, even though there are also similarities.

  These include:

  - Replacing terms of the form $|p + q|$ with terms of the form $|p| + |q|$.
  - Re-bracketing terms such that the Fourier multiplier of the previous iterations multiplies the whole denominator (only for isotropic models).
  - Approximating a sum by an integral in a Riemann-type approximation for models that have been regularized in the infra-red by moving to the torus. This still involves ad-hoc arguments, see for example Step 5 in the proof of Lemma 3.A.2, where parts of the sum are excluded and treated by hand.

  After such a replacement step has been proven, typically the final term will be such that it either has an explicit anti-derivative or can at least be easily estimated by something that does.

- So far, we have accrued several additive errors from the replacement steps and the off-diagonal terms. In the proof of the iterative estimates (see Section 2.4.3 and the end of Section 3.4) these errors are collected and absorbed into the main term by introducing a small multiplicative constant, which then defines the recursion for the prefactors c_k.

- To finally use these estimates to prove the main theorem one plugs in the bounds obtained into (1.20), and needs to estimate a sum of the type already studied one last time. One obtains a bound dependent on k, and, after optimizing for the value of k, one obtains the main theorem.

1.1.8 Weak and strong coupling results

If one is so inclined, one could call the results of Chapter 2 and 3 negative results, in the following sense. They show that even though both models are formally invariant under diffusive scaling, after regularizing them and applying this scaling, the resulting models do not converge, as one might have hoped, as is the case in higher dimensions, see, e.g., [CGT24, CLO01, HTV12].

However, there are two possible ways to overcome this problem and obtain a scaling limit. One is to consider the **weak coupling** limit, where the strength of the non-linearity (or the drift in the case of diffusions) is tuned down. Typically, there is a certain scaling window for such a coupling constant such that a scaling limit exists, but is not identical to the linear equation obtained by just setting the non-linearity to zero. This has been successfully implemented for the 2d Burgers equation in [CGT24], for the AKPZ equation in [CET23b] and for the self-repellent Brownian polymer in [CG24]. In all of these cases, the limits are Gaussian, but with a covariance structure that depends on the strength and the form of the linearity. If one considers the linear system as "critical" (it is after all the one with the most symmetries) one can see this as a near-critical scaling limit in the sense discussed in Section 2.1.

The other option is changing the diffusive space-time scaling to compensate for the divergence. For the diffusion in the curl of the Gaussian free field, this would be done by considering instead of the diffusive rescaled process (1.3) the following rescaling:

$$X_t^\varepsilon = |\log\varepsilon|^{-\frac{1}{4}}\varepsilon X_{t/\varepsilon^2}\,.$$

or, equivalently,

$$X_t^\varepsilon = \varepsilon X_{\frac{t}{|\log\varepsilon|^{\frac{1}{2}}\varepsilon^2}}\,.$$

For a logarithmically superdiffusive SPDE, the correct rescaling to get a scaling limit should be

$$\eta^\varepsilon(x,t) = \eta\Big(\frac{x}{\varepsilon}, \frac{t}{|\log\varepsilon|^\gamma\varepsilon^2}\Big)\,.$$

This type of rescaling is called **strong coupling** limit, in contrast with the weak coupling limit mentioned above. The expected strong coupling limits are still Gaussian and invariant under diffusive scaling. This means one expects convergence to a limit process, under a scaling that does not leave this limit process invariant, which is certainly unusual. It is however consistent with the formal scaling invariance that these models have. There are only very few strong coupling results that have been proven in this context, see [ABRK24] and [MT16].

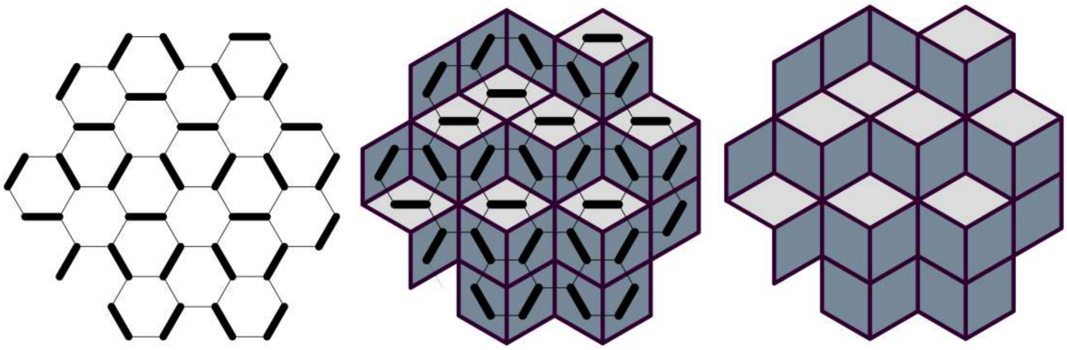


Figure 1.1: From left to right: a dimer configuration on a portion of the hexagonal lattice; the same dimer configuration overlayed with the corresponding lozenge tiling; the lozenge tiling, which can be viewed as a discrete surface.

1.2 The near-critical dimer model and massive SLE

The dimer model is a classical model from statistical mechanics. Given a finite graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}_+$, let a **dimer configuration** ω be a collection of edges such that each vertex in G is incident to exactly one edge in ω. Define the weight of such a configuration to be

$$w(\omega) = \prod_{e \in \omega} w(e)$$

and, if there is at least one dimer configuration on this graph, define the **dimer measure** to be the measure on dimer configurations proportional to their weights:

$$\mathbb{P}(\omega) = \frac{w(\omega)}{Z},$$

where the normalizing constant Z is known as the partition function. On embedded planar bipartite graphs this model can be interpreted as a discrete random surface model, see Figure 1.1

The planar dimer model was found to be "exactly solvable" by Kasteleyn and Fisher in the '60s, see [Kas61, Kas63, Fis61]. This exact solvability is based on a determinantal representation of the partition function of the model, via the Kasteleyn matrix. There has recently been renewed interest in the dimer model. On a bounded domain, the model can exhibit different phases (frozen, liquid, and gaseous) and limit shape phenomena separating the various phases, see [CEP00, KO06, BCJ18]. The model has an associated height function which converges to the minimizer of a certain variational problem, see [CKP00]. In the liquid (or massless) region the fluctuations of the height function around this limit shape converge to the Gaussian free field (after a certain change of coordinates), see [Ken01, Las21, CLR22]. This makes it one of the few statistical mechanics models for which conformal invariance has been shown at criticality, first by Kenyon in [Ken00].

We study a near-critical version of this model. Near-critical models have been considered for several models from statistical mechanics including spanning forests [BDW20], percolation [GPS18, NW09] and FK-Percolation[DCGP14]. We connect this near-critical dimer model to massive SLE, which was first constructed by Makarov and Smirnov in [MS10], where they initiate a programme to describe a wide variaty of near-critical scaling limits of planar statistical mechanics models in terms of massive SLE. A common feature of near-critical planar models is the loss of conformal invariance. From a theoretical physics perspective, this is because they are associated with massive Field theories and such theories can not be scale invariant (and therefore not conformally invariant). However, they are still conformally covariant, when the mass is seen as a variable covariant density. We prove such a conformal covariance for the scaling limit of the height function associated with this near-critical dimer model. We also conjecture a connection to the Sine-Gordon model, see Conjecture 4.1.9 below.

Instead of Kasteleyn theory, we use another approach to studying the dimer model via Temperley's bijection. For uniform (i.e. constant 1) weights Temperley's bijection maps the dimer model to the uniform spanning tree. The uniform spanning tree is further connected to the loop-erased random walk by Wilson's algorithm [Wil96]. For uniform weights, this loop-erased random walk has a conformally invariant scaling limit, given by SLE_2 [LSW01]. The height function can be connected to the winding of the loop-erased random walk, and this connection (with some additional work) carries through to the continuum, giving another proof of conformal invariance of the height fluctuations, and extending it to more general geometries, see [BLR20, BLR19, BLR22]. The starting point of Chapter 4 is that these connections do not only hold for uniform weights. They also hold for a set of weights (see Figure 1.2), where certain edges have weights different from one. They will depend on the mesh size δ, in a way that is analogous to the near-critical weights studied in [Chh12]. Our results can be summarized as follows

- Using Temperley's bijection and Wilson's algorithm we connect the near-critical dimer model on Temperleyan domains of the square and hexagonal lattice to the loop-erasure for a random walk with drift.
- Using a discrete Girsanov-Itô formula, we connect this loop-erased random walk to a loop-erased random walk with inhomogeneous mass.
- Using both the methods of Yadin and Yehudayoff from [YY11] and of [CW19] we show that this loop-erased random walk has a scaling limit, which is given by a massive SLE_2 as constructed in Makarov and Smirnov [MS10].
- Using this convergence together with results from [BLR20] gives the convergence of the uniform spanning tree as well as the convergence of the fluctuations of the height function.
- Finally we show a conformal covariance property of massive SLE_2 which implies a conformal covariance for the scaling limit of the height fluctuations.

Let us now define the key objects necessary to state our results. Some of our results we also prove for the square lattice, see Section 4.2.4. For the sake of brevity, this introduction focuses on the hexagonal lattice (and the corresponding directed triangular lattice), for which the full results are shown.

Near critical dimer model

The weights we consider on the hexagonal lattice are given in Figure 1.2. These weights can be described as follows. For a Temperleyan piece of the hexagonal lattice (see 4.1.2 for a full definition) there is a set of vertices V arranged in a periodic pattern, called the **primary** vertices. In Figure 1.2 these vertices are in bold. In Temperley's bijection, these vertices correspond to vertices of the directed triangular lattice $\mathbb{T}$ and we identify them with the vertices of $\mathbb{T}$. The three edges around a primary vertex v are assigned the weights $a_0(v), a_1(v), a_2(v)$. All edges not incident to a primary vertex are assigned weight 1. We

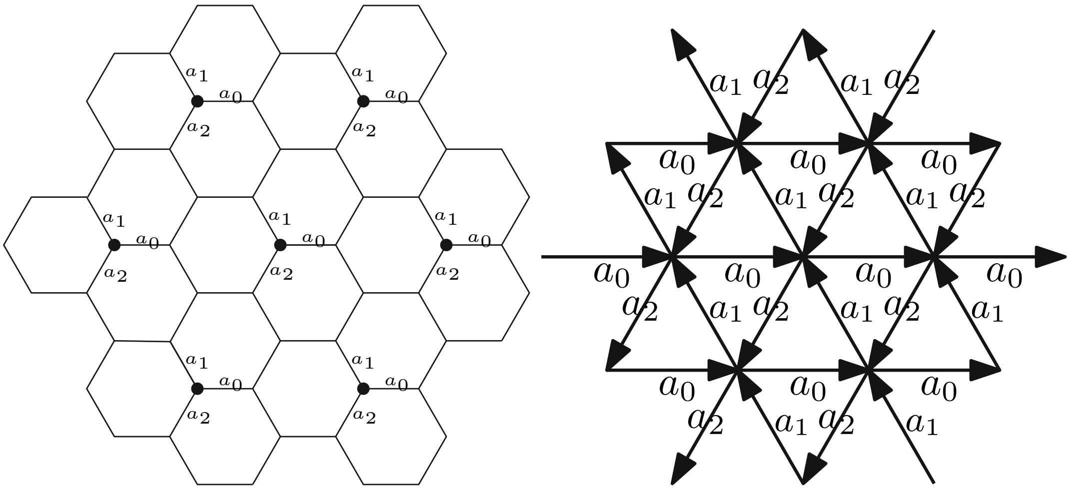


Figure 1.2: A piece of the hexagonal lattice and a corresponding piece of the directed triangular lattice $\mathbb{T}$. Unlabeled edges on the left have weight 1. The weights depend on the vertex.

parameterize these weights in the following way. Given weights $a_0(v), \dots, a_2(v)$ for each primal vertex v, let $\alpha_0(v), \dots, \alpha_2(v) \in \mathbb{R}$ and the **drift vector** $\alpha(v) \in \mathbb{R}^2$ be defined by

$$\exp(\alpha_i(v)) = a_i(v), \quad \text{and} \quad \alpha(v) = \frac{\sum_{s=0}^{2} \alpha_s(v)\tau^s}{3},$$

where $\tau = \exp(2\pi i/3)$ and we identify $\mathbb{C}$ with $\mathbb{R}^2$. Given a drift vector $\alpha(v)$, we can always find corresponding $a_1(v), \dots, a_2(v)$, but they are not defined uniquely, since $1 + \tau + \tau^2 = 0$. However, two sets of weights corresponding to the same drift vector will always be related by multiplication with a positive constant. Therefore, the corresponding dimer models are equivalent by a gauge transformation. We thus will take the model to be parameterized by $\alpha(v)$.

The drifted random walk

Via Temperley's bijection and Wilson's algorithm the above dimer model is connected to the random walk on the directed triangular lattice taking steps according to the following transition matrix

$$\mathbb{Q}^{(\alpha)}(v, v + \tau^k) = \frac{a_k(v)}{a(v)}, \quad k = 0, \dots, 2, \quad \text{with } a(v) = a_0(v) + \dots + a_2(v).$$

where the random walk takes values in $\mathbb{T}$, the **directed triangular lattice**, which can be seen on the right hand side of Figure 1.2. We will often consider this random walk until it leaves a domain $\Omega^\delta \subset \delta\mathbb{T}$, started from a vertex o, conditioned to leave Ω^δ through a boundary edge a and denote this law with $\mathbb{Q}^{(\alpha)}_{o \to a, \Omega^\delta}$.

The (inhomogeneous) massive random walk

Given a mass profile $m^2 : \Omega^\delta \to [0, 1]$, the massive random walk on $\mathbb{T}$ is the random walk with transition probabilities

$$\mathbb{Q}^{(m)}(v, v + \tau^k) = \frac{1 - m^2(v)}{3}, \quad k = 0, \dots, 2,$$

i.e. with probability $1-m^2(v)$ the random walk take a simple random walk step (respecting the orientation of the edges) and with probability $m^2(v)$ it "dies". This can be interpreted as going to an absorbing "cemetery" vertex, or as $\mathbb{Q}^{(m)}$ being a measure with total mass less than 1. Either way, if we again consider the random walk in a discrete domain Ω^δ, started from o, conditioned to leave through an edge a, we obtain a probability measure on such paths and denote it with $\mathbb{Q}^{(m)}_{o\to a,\Omega^\delta}$

The loop-erased random walk

For the drifted random walk above (or indeed any path $(x_0,\dots,x_n)$) one defines the loop-erasure by chronologically deleting loops as they appear, see Section 1.2.1 below. Wilson's algorithm states that the branches of the random spanning tree on a weighted graph are given by the results of such loop-erasures.

The following is a consequence of the discrete Girsanov-Itô formula in Corollary 4.1.4, which is discussed in Section 1.2.2.

Proposition 1.2.1. *If α is given by a discrete gradient, as in* (1.36) *and the $m^2(v)$ satisfies* (1.38)*, then the law of the loop erasure of the walk with law $\mathbb{Q}^{(\alpha)}_{o\to a,\Omega^\delta}$ coincides with the law of the loop-erasure of the walk with law $\mathbb{Q}^{(m)}_{o\to a,\Omega^\delta}$.*

Using this result, finding the scaling limit of the loop-erasure of the drifted random walk $Q^{(\alpha)}$ has been reduced to finding the scaling limit of the loop-erasure of the massive random walk $\mathbb{Q}^{(m)}$. Together with results from [BLR20] this then implies convergence of the height function of the associated dimer model.

Massive Brownian motion

The massive random walk without the loop-erasure has the following scaling limit, if the mass $m^2(v):\delta\mathbb{T}\to[0,1]$ is scaled like $m^2(v)=\frac{\delta^2\rho(v)}{2}+o(\delta^2)$ for some $\rho:\mathbb{R}^2\to[0,\infty)$.

Definition 1.2.2. *Brownian motion with mass profile ρ is a Brownian motion, which dies at rate $\rho(X_s)$ when at position X_s, i.e. it is a process which is absolutely continuous with respect to Brownian motion and has Radon Nikodym-derivative*

$$\left.\frac{d\mathbb{P}^{(\rho)}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t}=\exp\Big(-\int_0^t\rho(X_s)ds\Big). \tag{1.26}$$

Note that the total mass of $\mathbb{P}^{(\rho)}_x$ is less than one, so it is not a probability measure but a (finite) measure on paths.

Massive radial SLE

The following is the definition of massive SLE_2 as first given by Makarov and Smirnov in [MS10] as part of a general programme of using massive SLE_κ to describe near-critical scaling limits.

Definition 1.2.3 (inhomogeneous massive radial SLE)**.** *For a given domain $\Omega\subset\mathbb{C}$ and a bounded continuous mass profile $\rho:\Omega\to[0,\infty]$, inhomogeneous radial massive SLE_2 with mass profile ρ is defined to be the Loewner evolution associated to the driving function satsifying the SDE*

$$\mathrm{d}\xi_t=\sqrt{2}\mathrm{d}B_t+2\lambda_t\mathrm{d}t,\quad \lambda_t=\frac{\partial}{\partial g_t(a_t)}\log\frac{P^{(\rho)}_{\Omega_t}(o,a_t)}{P_{\Omega_t}(o,a_t)} \tag{1.27}$$

where $P^{(\rho)}_{\Omega_t}$ *is the massive Poisson kernel in* Ω_t *with mass profile* ρ*, which is defined in Section 4.4.1.*

The definition of a Loewner evolution is given by (1.30) below.

Main results

The following is the main theorem about convergence of the massive random walk to massive SLE_2.

Theorem 1.2.4 (Informal version of Theorem 4.4.1)**.** *Given a random walk on a planar embedded graph, dying at each step with a specified vertex- (and scaling-) dependent probability, such that this random walk converges to massive Brownian motion with mass profile* ρ *up to a time change, then the loop-erasure of this random walk converges to massive* SLE_2 *with mass profile* ρ *if additionally it satisfies the following assumptions:*

- *There exists a constant* C *such that for each discrete domain* Ω^δ *with diameter at most* R*, each interior point* $o \in \Omega^\delta$ *and each boundary edge* a *it holds that*

$$\mathbb{E}_{o\to a;\Omega^\delta}(\sigma_\delta) \leq C\delta^{-2}|\rho|_\infty\,,$$

  where σ_δ *is the number of steps until leaving the domain and the expectation is with respect to the law of the random walk started at* o *and conditioned to leave* Ω^δ *through* a*.*

- *The random walk satisfies some uniform crossing assumption.*

See Theorem 4.4.1 for the precise assumptions. This result generalizes the main theorem of Chelkak and Wan in [CW19] both to non-constant mass and more general graphs. Defining and showing the existence of the limiting objects is non-trivial and part of the result.

Using the discrete Girsanov-Itô formula discussed below in Section 1.2.2 this gives the following theorem for the drifted random walk on the triangular lattice.

Theorem 1.2.5 (Informal version of Theorem 4.1.6)**.** *Given a simply connected domain* Ω *and a drift field* $\alpha : \Omega \to \mathbb{R}^2$*, let* $\Omega^\delta \subset \mathbb{T}$ *approximate* Ω *and* $o^\delta \in \Omega^\delta \to o \in \Omega$ *and* $a^\delta \in \partial\Omega^\delta \to a \in \partial\Omega$*. Suppose that there is a smooth potential* φ *such that* $\nabla\varphi = \alpha$ *and further that*

$$\rho(x) \overset{def}{=} \frac{\Delta\varphi + \|\nabla\varphi\|^2}{2} \geq 0, \text{ for } x \in \Omega. \tag{1.28}$$

Then the loop erasure of the random walk $P^{(\varphi)}_{o^\delta\to a^\delta}$ *with transition rates given by* (1.32) *and* α^δ *given by the discrete gradient* $\nabla^\delta\varphi$ *converges to massive* SLE_2 *with mass profile* ρ*.*

The authors believe that the drift field α being of gradient-type is necessary for the limit to be described by massive SLE_2. However, the restriction that ρ is positive could be removed, if one is able to define massive SLE_2 for (some) negative masses.

Finally, for the dimer model, we prove the following theorem:

Theorem 1.2.6 (Informal version of Theorem 4.1.8)**.** *Consider the dimer model on domains* Ω^δ *of the hexagonal lattice with weights* $(a_0(v), \ldots, a_2(v))$ *deriving from a drift field* α *as above, which in turn is given by* $\nabla\varphi$*, satisfying* (1.28)*. Then the fluctuations of the height function on this domain converge to a scaling limit denoted by* $h^{(\alpha);\Omega}$*, satisfying the following identity in law*

$$h^{(\alpha);\Omega} \circ T^{-1} = h^{(\tilde\alpha);\tilde\Omega},$$

if T *is a conformal map from* Ω *to* $\tilde\Omega$ *with* $|{T^{-1}}'|$ *bounded and* $\tilde\alpha = \nabla\varphi \circ T^{-1}$*.*

Requiring that $|(T^{-1})'|$ is bounded should not be necessary, but simplifies the proofs, since it guarantees that the mass associated to the right hand side is bounded.

In summary we achieve the following goals

- We build on the programme initiated by [MS10] by building a further connection between massive SLE and another near-critical model and conjecturing a connection to a massive field theory.
- We answer a question asked in [Chh12] by giving a rigorous connection between the near-critical dimer model and an object associated with massive field theory.
- We find a discrete exact version of Girsanov's formula for the random walk on the directed triangular lattice, see Theorem 4.1.4, which might be of independent interest. The fact that this identity is exact is a convenient property specific to the triangular lattice. We find another such discrete Girsanov formula for the square lattice which only approximately connects the random walk with drift and the one with mass, see Lemma 4.2.9.
- We extend the result from [CW19] to a much more general setting, see Theorem 4.4.1, showing that the convergence of the massive loop-erased random walk to massive SLE_2 is universal and further implementing the programme set out by [MS10].
- We find a conformally invariant limiting object for the fluctuations of the height field, reminiscent of other near-critical scaling limits like [DCGP14, GPS18]. We conjecture it to be given by a variant of the Sine-Gordon model at the free fermionic point:

  Conjecture 1.2.7. *Let $\mathbb{P}^{(\varphi);\Omega}$ denote the law of the field $h^{(\varphi);\Omega}$ in Theorem 4.1.8. Then*

  $$\mathbb{P}^{(\varphi);\Omega}(dh) \propto \exp\left(z \int_\Omega \left\langle e^{\mathbf{i} h(x)/\chi}, \alpha(x) \right\rangle dx \right) \mathbb{P}^{\mathrm{GFF}}(dh)\,,$$

  where the inner product is taken by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

  Here $\alpha = \nabla\varphi$, and $\chi = 1/\sqrt{2}$ is the imaginary geometry constant associated to $\kappa = 2$. This expression is formal, since the Gaussian free field h cannot be point-evaluated, for more details on this see the discussion after in Section 4.1.9. In particular it is consistent with the recent results by [Mas22], in which the two-point correlation function of the near-critical dimer model on the square lattice in the full plane is shown to coincide with the two-point function for the Sine-Gordon model.

The remainder of this introduction is structured as follows. In Section 1.2.1 the connection between the critical dimer model and SLE_2 is recalled. In the following sections some ingredients to the main theorem are introduced, i.e. the discrete Girsanov-Itô in Section 1.2.2, the resolvent identities in Section 1.2.3 and finally the identification of the limit in Section 1.2.4.

1.2.1 The critical dimer model and SLE_2

In this section we will establish the connection represented in the diagram in Figure 1.3, in particular along the top and right edge, i.e. going from the dimer model to SLE_2.

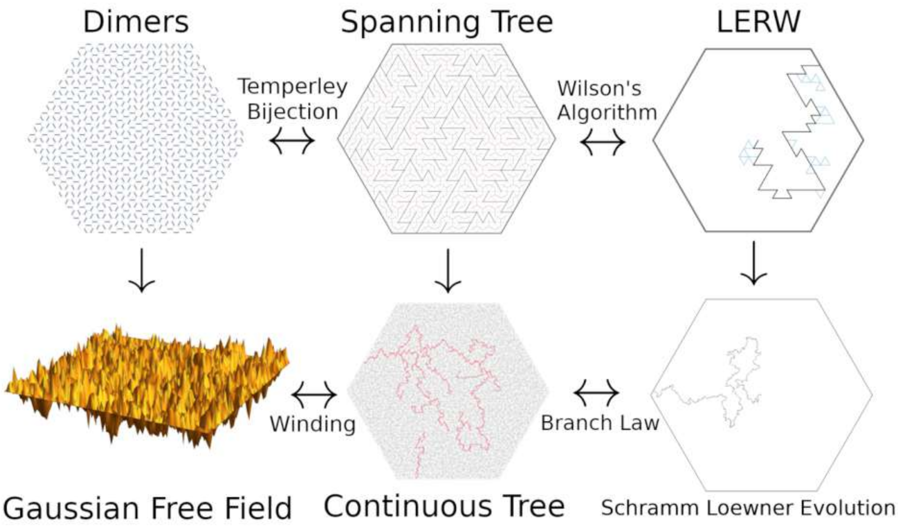


Figure 1.3: The connections discussed in Section 1.2.1. Vertical arrows are scaling limits, while horizontal arrows show connections, which are (at least in some sense) bidirectional.

Temperleys bijection

Temperley's bijection was found by Temperley in 1972. Reference surprisingly difficult to find It is a bijection between the dimer model on a rectangular region of the square lattice and a uniform spanning tree on an associated graph (in this case, a larger region of the square lattice). This bijection was then generalized by Kenyon, Propp and Wilson in [KPW00] to more general weighted graphs. Considering a finite, connected and directed graph $G = (V, E)$ together with edge weights $w : E \to \mathbb{R}_+$, define a directed spanning tree rooted at $v \in V$ to be a connected sub-graph T, such that each vertex except v has exactly one outgoing edge in T. Let the weight of a tree T be

$$w(T) = \prod_{e \in T} w(e)\,, \tag{1.29}$$

where we identify the sub-graph T with its edges.

Theorem 1.2.8 (Theorem 1 in [KPW00]). *Given a weighted graph G as above, as well as vertex v, there is a weight preserving bijection between spanning trees on G, and dimer configurations on a related weighted graph H.*

The graph H is obtained from the graph G by a specific procedure, involving superimposing the graph G with its dual, and then removing two vertices.[3] We will sometimes refer to H as the **dimer graph** and to G as the **tree graph**. If one wants to study the dimer model using this bijection, one can only do so on graphs H that can be obtained this way. Two specific examples that are already mentioned are regions of the square and of the hexagonal lattice, which come from taking G to be either a region of the square lattice, or a region of the directed triangular lattice $\mathbb{T}$, which is obtained from the triangular lattice by orienting

[3]The choice of graph H in this theorem is not unique, since it depends on a choice of embedding, and a face f incident to v.

each of the three types of edges in the directions $1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively, see the right hand-side of Figure 1.2. In these cases, the graph for the dimer model is not quite general. It is necessary for these graphs to have so called Temperleyan boundary conditions, see Section 4.1.2 below for a definition. In particular, these force the height function on the boundary to be of order 1. When all weights on the graph G are one, then the same is true for H and Temperley's bijection connects the uniform spanning tree on G with a uniformly chosen dimer configuration.

Wilson's algorithm

Now that we have connected the dimer model to a random (uniform) spanning tree, we need tools to study the uniform spanning tree. The uniform spanning tree is an interesting object in its own right, see, e.g., [Ald90, BLPS01, vEH23].

A very powerful (and beautiful) tool to study uniform spanning trees is the algorithm found by Wilson in [Wil96]. Given a directed graph $G = (V, E)$ with edge weights w as above, as well as a distinguished vertex, r the algorithm generates a directed spanning tree (or *arborescence*) with probability proportional to its weight as defined in (1.29). It does so using loop-erased random walks, so let us first define a random walk with respect to the weights w. This is the random walk which takes steps proportional to the weights w, i.e., its transition probabilities are given by

$$p(x, y) = \frac{w(x, y)}{\sum_{z \sim x} w(x, z)},$$

where the sum is over all vertices such that $(x, z) \in E$. Note that the graph is directed and in particular w is not assumed to be symmetric.

Given the trajectory $\gamma = (x_0, \ldots, x_l)$ of a random walk the loop erasure $\mathrm{LE}(\gamma)$ is obtained by deleting cycles in γ chronologically, i.e. as they appear. More precisely, let $u_0 \overset{\text{def}}{=} x_0$. Then find the last time i that the trajectory visits u_0, and set $u_1 = X_{i+1}$. Continue in this way by setting u_{k+1} always to the first vertex visited after the last visit to u_k as long as this is possible. Necessarily, this procedure gives a simple path $\mathrm{LE}(\gamma) \overset{\text{def}}{=} (u_0, \ldots, u_{l'})$ from $u_0 = x_0$ to $u_{l'} = x_l$.

Wilson's algorithm proceeds as follows: Let G be as above, i.e. a directed graph with positive weights w. Let r be a distinguished vertex of the graph, which will be the root of the weighted spanning trees. We will generate a growing sequence of trees T_i. To do this we first fix an arbitrary ordering of the vertices of G with the root r as the first vertex. Let $T_1 := r$. Then repeat the following for $i \geq 1$:

- If T_i is a spanning tree, we are done.
- Otherwise, find the earliest vertex x in our ordering which is not contained in T_i.
- Run a random walk on G starting at x until it hits T_i, independent of the walks in previous steps.
- Create T_{i+1} from T_i by adding the vertices and the directed edges of the loop erasure of this walk.

The main result of [Wil96] states that this algorithm indeed generates trees with the desired probabilities

Theorem 1.2.9. *Given any directed graph G with positive weights ω, Wilson's algorithm produces a directed spanning tree T with probability proportional to $\prod_{e \in T} \omega(e)$.*

A remarkable feature of the algorithm is that the arbitrary ordering of the vertices does not affect the law of the tree. This gives us the following corollary by choosing $v_2 = x$:

Corollary 1.2.10. *Let G be a directed weighted graph, x a vertex of G and T be a spanning tree associated to w. The law of the unique path γ_x from x to the root in T is given by the law of the loop-erased random walk started at x and run until it hits r.*

Schramm Loewner Evolution

The graphs that we will apply Wilson's algorithm to are of a special type. To obtain a piece of the hexagonal or square lattice for the dimer graph in Temperley's bijection in Theorem 1.2.8, the tree graph G must be **wired**, i.e. it will consist of a piece of either $\mathbb{Z}^2$ or $\mathbb{T}$, with the complement of the respective lattice all identified into one **boundary vertex**, see e.g. [KPW00, Figure 1]. This vertex will always be chosen as the root vertex in Wilson's algorithm. Doing so, the algorithm takes the form of repeatedly running a random walk until it leaves a (discrete) domain, and then removing the loop-erasure of this walk from the domain, after which one starts again, with a new starting point. Performing the loop-erasure never changes the endpoint of the trajectory. Therefore, one can condition the walk on leaving the domain through a. Since the exit distribution is given by the random walk, it is not difficult to remove this conditioning. This leads us to study the following object:

Definition 1.2.11. *Let Ω be a discrete simply connected domain, either in $\mathbb{Z}^2$ or $\mathbb{T}$ and o a vertex in Ω and a a boundary edge of Ω, i.e., an oriented edge (x, y) such that $x \in \Omega$ and $y \notin \Omega$. The loop-erased random walk from o to a in Ω is the loop-erasure of the random walk started at o, conditioned to leave Ω through a.*

In the seminal paper [LSW01] Lawler, Schramm and Werner proved the following

Theorem 1.2.12 (Informal version of Theorem 1 in [LSW01])**.** *For the simple random walk on $\delta\mathbb{Z}^2$, loop-erased random walk from $o^\delta \to o$ to $a^\delta \to a$ in a sequence of domains $\Omega^\delta \to \Omega \subset \mathbb{C}$ converges to radial SLE$_2$ from a to o as $\delta \to 0$.*

This result was later strengthened in [YY11] to require only minimal assumptions on the random walk and the graph G. Essentially it suffices that the random walk (after reparametrization) converges to Brownian motion and that the graph is planar.

To understand this result, we will now give a definition of radial SLE$_\kappa$. There are two versions of SLE: radial and chordal. One describes a random curve from a point on the boundary of a simply connected domain to a point inside the domain, and the other describes a random curve between two boundary points. We will focus on the radial case since it is more natural for the problem at hand. Note however that chordal SLE is also a very common scaling limit and in many cases easier to work with.

Let $D \subset \mathbb{C}$ with $0 \in D$ be a simply connected domain. By the Riemann mapping theorem, there exists a unique conformal homeomorphism ψ_D from D onto the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that $\psi_D(0) = 0$ and $\psi'_D(0) \in \mathbb{R}^+$. Consider a continuous simple curve $\eta\colon [0, \infty] \to \overline{\mathbb{D}}$ such that $\eta(0) \in \partial\mathbb{D}$, $\eta(t) \in \mathbb{D}$ for $t > 0$ and $\eta(\infty) = 0$. For each t let $K_t := \eta[0, t], D_t := \mathbb{D} \setminus K_t$ and $g_t := \psi_{D_t}$. By the Schwarz lemma (see e.g. [BN23, Lemma 2.8]) and because $D_t \subset \mathbb{D}$ the derivative $g'_t(0) \geq 1$. Then $\log({g_t}'(0))$ is called the capacity of K_t from 0. Furthermore, again by the Schwarz lemma $t \mapsto g'_t(0)$ is increasing and thus η can be reparametrized such that the capacity of K_t is t. Then, we say that η_t is parametrized by capacity. By [Pom92, Proposition 2.5] the limit

$$W(t) := \lim_{z \to \eta(t)} g_t(z)$$

exists for each $t \in [0,\infty]$, when z approaches $\eta(t)$ from inside D_t. It is true under these assumptions that $W\colon [0,\infty) \to \partial\mathbb{D}$ is continuous. Now, we can state Loewner's theorem:

Theorem 1.2.13. *Let $\eta\colon [0,\infty] \to \overline{\mathbb{D}}$ be as above and parametrized by capacity. Then the maps $g_t\colon D_t \to \mathbb{D}$ satisfy Loewner's differential equation:*

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z)+W(t)}{g_t(z)-W(t)}; \quad \text{for} \quad 0 \le t < \tau_z \tag{1.30}$$

with the initial value $g_0(z) = z$, where $\tau_z \stackrel{def}{=} \sup\{t \ge 0 : z \in D_t \in\}$.

We call $(W(t))_{t\ge 0}$ the **driving function** or Loewner transform of the curve η. Conversely the driving function $W(t)$ fully determines η. For every $z \in \mathbb{D}$ the Loewner equation has a unique solution up to the maximal time $\tau_z \in (0,\infty]$. If $\tau_z = \infty$ the ODE has a solution at z for all times and therefore $z \in D_t$ for all $t \in [0,\infty)$. If however $\tau_z < \infty$ then $\lim_{t\uparrow\tau_z} W(t) - g_t(z) = 0$ as this is the only singularity in (1.30). If W is obtained from a simple path as above then obtain η from W by setting $\eta(t) = g_t^{-1}(W(t))$. For an arbitrary continuous function W this might not work. One can still define $K_t = \{z \in \overline{\mathbb{D}} : \tau_z \le t\}$, but this might not be a simple path.

Radial Schramm Loewner evolution with parameter κ is the process obtained from setting the driving function to $W(t) := \exp(iB_{\kappa t})$, where $B\colon [0,\infty) \to \mathbb{R}$ is standard Brownian motion started most commonly at 0 or at a uniform point in $[0,2\pi]$. In [RS05] it was shown that for $\kappa \le 4$ almost surely K_t is a simple path, while for $\kappa > 4$ there is always a path $\eta(t)$ such that D_t is the connected component of $\mathbb{D} \setminus \eta[0,t]$ containing 0. To obtain radial SLE$_\kappa$ in a domain D from $o \in D$ to $a \in \partial D$, let ψ_D be the unique map from D to $\mathbb{D}$ with $\psi_D(o) = 0$ and apply ψ_D^{-1} to radial SLE$_\kappa$ on $\mathbb{D}$.

They are a large number of scaling limit results proving convergence to SLE$_\kappa$:

- the loop-erased random walk converges to SLE$_2$,[Sch99, LSW01],
- the Peano curve associated to a uniform spanning tree converges to SLE$_8$, [Sch99, LSW01],
- boundaries of critical percolation clusters converge to SLE$_6$,[Smi01],
- Critical Ising interfaces converge to SLE$_3$, [CDCH+14],
- contour lines of the 2d discrete Gaussian free field converge to SLE$_4$[SS06].

For lecture notes and a monograph on SLE see e.g. [Wer04, BN23, Law05].

Returning to Theorem 1.2.12, an essential part of the proof is played by the **martingale observables**. A key observation from [LSW01] is the following

Proposition 1.2.14. *Let Ω be a discrete domain and γ be the reversal of a loop-erased random walk from $o \in \Omega$ to $a \in \partial\Omega$ (and therefore a simple path from a to o). Let $\Omega_n \stackrel{def}{=} \Omega \setminus \gamma[0,n]$, and v a vertex in Ω and further $Z_{\Omega_n}(v,\gamma[n])$ be the probability that a random walk started from v exits Ω_n via $\gamma[n]$. Then*

$$M_n(v) = \frac{Z_{\Omega_n}(v,\gamma[n])}{Z_{\Omega_n}(o,\gamma[n])}$$

is a martingale (as long as $v \in \Omega_n$) with respect to the filtration generated by $\gamma[0,n]$.

To prove this proposition, one first notices that the reverse chronological loop-erasure (i.e., the reversal of the loop-erasure of the reversed path) of a random walk has the same law as the chronological loop-erasure defined above. Then the statement follows by simply summing over the possible values of $\gamma[n+1]$, see [LSW01, Remark 3.6]. These martingale observables converge to the Poisson kernel ratio, which is conformally invariant. This suffices to show that, at least if the limit is given by some SLE_κ, then κ must be 2, see e.g. [BN23, Proposition 7.7]

1.2.2 Discrete Girsanov-Itô

Let us recall the random walk we consider has transition probabilities given by

$$\mathbb{Q}^{(\alpha)}(v, v+\tau^k) = \frac{a_k(v)}{a(v)}, \quad k=0,\dots,2, \quad \text{with } a(v) = a_0(v)+\dots+a_2(v)\,. \tag{1.31}$$

Since we want to find a scaling limit we need to introduce a scaling parameter δ. We will do so by setting

$$a_k(v) = 1 + c_k(v)\delta \quad (k=0,\dots,2)$$

and now the random walk takes values on $\delta\mathbb{T}$, i.e.

$$\mathbb{Q}^{(\alpha)}(v, v+\delta\tau^k) = \frac{a_k(v)}{a(v)}, \quad k=0,\dots,2, \quad \text{with } a(v) = a_0(v)\,. \tag{1.32}$$

A first sign that this type of scaling is reasonable is the fact that with this scaling, if c_0, c_1 and c_2 are continuous functions evaluated at the vertices of δT, this random walk converges to the solution of the SDE

$$dX_t = \alpha(X_t)\mathrm{d}t + \mathrm{d}B_t\,, \tag{1.33}$$

where

$$\alpha(x) = \frac{2}{3}(c_0(x) + c_1(x)\tau + c_2(x)\tau^2)$$

and B_t is standard 2d Brownian motion.

To understand the random walk defined at the end of the previous section via (1.32), we will consider its Radon-Nikodym derivative with respect to the simple random walk. Since this will be an exact statement for the simple random walk, we will write it for the unscaled lattice $\mathbb{T}$, i.e., with transition probabilities (1.31). Let α_i be such that $\exp(\alpha_i) = a_i$, define $\alpha(v)$ by

$$\alpha(v) = \frac{2}{3}(\alpha_0(v) + \alpha_1(v)\tau + \alpha_2(v)\tau^2)$$

and define $\beta(v)$ by

$$\exp(-\beta(v)^2) = (a(v)/3)^{-3}\prod_{k=0}^{2} e^{\alpha_k(v)}, \tag{1.34}$$

Note that while α does not determine $\alpha_0,\dots,\alpha_2$, it does determine the law $\mathbb{Q}^{(\alpha)}$, since adding a constant to $\alpha_0,\dots\alpha_2$ is equivalent to multiplying a and $a_0,\dots,a_2$ with a constant.

Using these, we can conveniently state our discrete Girsanov theorem.

Theorem 1.2.15. *Let $\mathbb{P}_o$ be the law of the simple random walk and $\mathbb{Q}_o^{(\alpha)}$ be the law of the random walk with transition probabilities given by (1.31) both started at o and taking n steps. Then for any path $\gamma = (x_0,\dots,x_n)$ (with $x_0 = o$) the following identity holds:*

$$\frac{\mathbb{Q}_o^{(\alpha)}(\gamma)}{\mathbb{P}_o(\gamma)} = \exp(M_n - \tfrac{1}{2}V_n),$$

where

$$M_n = \sum_{s=0}^{n-1} \langle \alpha(x_s), \mathrm{d}x_s \rangle \quad \text{and} \quad V_n = \frac{2}{3} \sum_{s=0}^{n-1} \beta^2(x_s)\,,$$

where $\mathrm{d}x_s = x_{s+1} - x_s$.

This can be seen as a discrete analogue of the fact that a solution to the SDE (1.33) satisfies, by Girsanov,

$$\left.\frac{d\mathbb{Q}^{(\alpha)}}{d\mathbb{P}}\right|_t = \exp\left(\int_0^t \alpha(X_s) \cdot dX_s - \frac{1}{2} \int_0^t |\alpha(X_s)|^2 ds \right). \tag{1.35}$$

We further assume that α is of gradient type, i.e., given by $\nabla\varphi$ for some bounded C^2 potential φ. Under this assumption, in the continuum analogue (1.35),we can use Itô's formula to rewrite the stochastic integral above as:

$$\int_0^t \alpha(X_s) \cdot dX_s = \varphi(X_t) - \varphi(X_0) - \frac{1}{2} \int_0^t \Delta\varphi(X_s)\mathrm{d}s,$$

giving the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}_o}{d\mathbb{P}_o}\right|_t = \exp\left(\varphi(X_t) - \varphi(X_0) - \frac{1}{2} \int_0^t \Delta\varphi(X_s) + |\nabla\varphi|^2 \mathrm{d}s \right).$$

This fact has a discrete analogue. To formulate this, we need some notation: we say that the drift vector $\alpha = \alpha(v), v \in \Omega^\delta$ derives from a **potential function** $\Phi : \mathbb{T} \to \mathbb{R}$, when

$$\alpha(v) = \nabla^{\mathbb{T}}\Phi(v) := \frac{2}{3} \sum_{i=0}^{2} (\Phi(v + \tau^i) - \Phi(v))\tau^i; \tag{1.36}$$

in other words, $\alpha_{i+1}(v) = \Phi(v + \tau^i) - \Phi(v)$ for $0 \leq i \leq 2$. If α is of this form, the Radon-Nikodym derivative in Theorem 1.2.15 takes a particularly nice form:

Corollary 1.2.16. *Suppose α derives from a potential function Φ as above. Then*

$$\frac{\mathbb{Q}_o^{(\alpha)}(\gamma)}{\mathbb{P}_o(\gamma)} = \exp\left(\Phi(x_n) - \Phi(x_0) - A_n\right), \tag{1.37}$$

where

$$A_n = \sum_{s=0}^{n-1} \Delta^{\mathbb{T}}\Phi(x_s) + \frac{1}{3}\beta^2(x_s).$$

Here $\Delta^{\mathbb{T}}\Phi(x) = \frac{1}{3}\sum_{i=0}^{2} \Phi(x+\tau^i) - \Phi(x)$ is the usual graph Laplacian on the directed triangular lattice $\mathbb{T}$.

One can check that, introducing δ in all the right places makes under appropriate scaling (1.37) converges to (1.35). The important feature of (1.37) is that the exponent splits into two parts: one that depends only on the endpoint, and one that is simply a sum along the path γ. This second part can be interpreted as a mass term, in the following sense. Consider a simple random walk on $\mathbb{T}$, which at each step dies with probability $m^2(v) \in [0, 1)$. This gives a measure $\mathbb{Q}^{(m)}$ on paths with Radon-Nikodym derivative

$$\frac{\mathbb{Q}_o^{(\rho)}(\gamma)}{\mathbb{P}_o(\gamma)} = \prod_{s=0}^{n} (1 - m^2(v)).$$

Note that this is not a probability measure since its total mass is less than one. Choosing $m^2(v)$ such that

$$1 - m^2(v) = \exp(-(\Delta^{\mathbb{T}}\Phi(v) + \frac{1}{3}\beta^2(v)) \tag{1.38}$$

one sees this term agrees with the second part of (1.37).

Denote by $\mathbb{P}_{o\to a;\Omega^\delta}$, $\mathbb{Q}^{(\alpha)}_{o\to a;\Omega\Omega^\delta}$ and $\mathbb{Q}^{(m)}_{o\to a;\Omega\Omega^\delta}$ the random walks with laws $\mathbb{P}$, $\mathbb{Q}^{(\alpha)}$, and $\mathbb{Q}^{(m)}$ started from o and conditioned to leave Ω^δ via a. These are the laws of the simple, drifted or massive random walk in Ω started from o, conditioned to exit at a. A direct consequence of the formulas above is the following

Corollary 1.2.17. *If α derives from a potential Φ as in* (1.36) *and* (1.38) *holds for every v in Ω, then $\mathbb{Q}^{(\alpha)}_{o\to a;\Omega}$ and $\mathbb{Q}^{(m)}_{o\to a;\Omega}$ are identical.*

It is only possible to apply this corollary when there exists an m^2 such that (1.38) holds. This is the case iff $\Delta^{\mathbb{T}}\Phi(v) + \frac{1}{3}\beta^2(v) > 0$, which leads to condition (1.28) in Theorem 1.2.5. As opposed to the assumption that α is a gradient, we believe that this condition is not necessary for the existence of a Poisson kernel P^ρ, an associated massive SLE_2 with profile ρ and the statement of Theorem 1.2.5. While there is no random walk that produces $\mathbb{Q}^{(m)}_{o\to a;\Omega}$ for negative m, for certain m it might be seen as a measure connected to a **branching random walk**, and many of the steps below might still be possible in this setting.

In view of Corollary 1.2.17, the problem is reduced to finding a scaling limit for $\mathbb{Q}^{(m)}$. This proceeds in roughly three steps.

- Absolute continuity: The discrete Radon-Nikodym derivative of the loop-erasure is bounded, which follows from the assumption in Theorem 1.2.4 that the random walk under consideration leaves the domain after a number of steps of order δ^{-2}. This implies that the laws of the loop-erased random walks are tight, and any limit point is absolutely continuous with respect to SLE_2, see Section 4.4.3. It further implies that any limit point is given by a Loewner evolution, with driving function ξ_t satisfying the SDE
$$\mathrm{d}\xi_t = \sqrt{2}\mathrm{d}B_t + 2\lambda_t \mathrm{d}t\,.$$
The task at hand is thus to show that λ_t is given by (1.27) (and hence the same for any limit point).

- Convergence of discrete Poisson kernels: As in the critical case, there are martingale observables given by discrete massive Poisson kernel ratios. To use this to identify the laws of limit points, one needs to show that they converge to continuous Poisson kernel ratios. This is done by adapting arguments for the critical case in [YY11], see Section 4.4.2.

- Identification of the limit. Given the convergence of the massive martingale observables, one needs to show that these indeed identify the limit. To do so one needs to show that the involved massive kernels $P_t^{(\rho)}$ and $Q_t^{(\rho)}$ are semimartingales, and one needs to find their decomposition in martingale and finite variation part. In particular, this is done by relating the massive quantities to their non-massive counterparts via **resolvent identites**, see Proposition 4.4.7 and (4.4.10), and by proving a massive version of Hadamard's formula using these identities.

The next two sections expand on the third point.

1.2.3 Resolvent identities

As in the case with uniform weights (recalled in Proposition 1.2.14), the ratio

$$M_n^{(\rho)}(v) = \frac{Z_{\Omega_n}^{(\rho)}(v,\gamma[n])}{Z_{\Omega_n}^{(\rho)}(o,\gamma[n])}$$

is a martingale, where $Z_{\Omega_n}^{(\rho)}(v,\gamma[n])$ is the probability that the massive random walk started from v exits the domain $\Omega_n = \Omega^\delta \setminus \gamma[0,n]$ through $\gamma[n]$. By the arguments in Section 4.4.2 this converges, after using a Skorohod embedding to assume γ converges almost surely, and using $n = n(t)$ such that the limiting curve is parameterized by capacity, to the ratio

$$\frac{P_{\Omega_t}^{(\rho)}(x,a_t)}{P_{\Omega_t}^{(\rho)}(o,a_t)}$$

where $P_\Omega^{(\rho)}$ is the **massive Poisson kernel**, which we define in Section 4.4.1 to be given by

$$P_\Omega^{(\rho)}(x,a) = P_\Omega^{(0)}(x,a)\mathbb{E}_{x\to a}\left[\exp\left(-\int_0^\sigma \rho(X_s)ds\right)\right], \tag{1.39}$$

where $P_\Omega^{(0)}(x,a)$ is the non-massive Poisson kernel normalized to be 1 at o. In this section the domain Ω and the target point a will be fixed, so we suppress them from the notation, i.e., $P^{(\rho)}(x) = P_\Omega^{(\rho)}(x,a)$ and likewise for $P^{(0)}$.

In Section 4.4.4 we prove that P^ρ satisfies a **resolvent identity**. As was noted in [MS10], massive analogues of critical quantities solve certain boundary value problems of the type

$$\begin{aligned}(\frac{1}{2}\Delta - \rho)h &= 0, \text{ in } \Omega \\ h &= h_0 \text{ on } \partial\Omega\end{aligned}$$

with the critical counterparts solving the same boundary value problem with the operator $(\frac{1}{2}\Delta - \rho)$ being replaced with $\frac{1}{2}\Delta$, but with the same boundary data. Here ρ is identified with the operator acting by pointwise multiplication with ρ, i.e

$$(\rho h)(x) = \rho(x)h(x)$$

Considering $h = P^{(\rho)}$ we can write

$$(\frac{1}{2}\Delta - \rho)(P_\Omega^{(\rho)} - P_\Omega^{(0)}) = \rho P_\Omega^{(0)}$$

and using the fact that as an operator on Ω with Dirichlet boundary conditions $(\frac{1}{2}\Delta - \rho)$ is invertible, with the inverse given by convolution with (the negative of) the massive Green's function we obtain:

$$P_\Omega^{(\rho)}(x,a) = P_\Omega^{(0)}(x,a) - \int_\Omega G_\Omega^{(\rho)}(x,y)\rho(y)P_\Omega^{(0)}(y,a)\mathrm{d}y\,, \tag{1.40}$$

which is the content of Proposition 4.4.7 and a similar derivation can be given by for Proposition 4.4.10.

1.2.4 Identification of the limit

The strategy to identify the limit is as follows. To make use of the fact that $\frac{P_t^{(\rho)}(z)}{P_t^{(\rho)}(o)}$ is a martingale for each z, we need to use Itô's formula, for which we have to find $\mathrm{d}P_t^{(\rho)}$ using the resolvent identity. We will need the following additional ingredients

- From the critical case we know

$$\mathrm{d}P_t^{(0)}(x) = Q_t^{(0)}(x)\mathrm{d}\xi_t$$

  and by absolute continuity this is also true under the massive law.

- A massive Hadamard's formula (Lemma 4.4.12): $G_t^{(\rho)}$ is differentiable in t and satisfies

$$\frac{\partial}{\partial t} G_t^{(\rho)}(x,y) = -2\pi P_t^{(\rho)}(x) P_t^{(\rho)}(y) \,.$$

- A stochastic Fubini theorem (Lemma 4.4.15) based on estimates in Lemma 4.4.14 and Proposition 4.4.13.

Using this one can determine λ_t, see Section 4.4.6, where we show that

$$\lambda_t = \frac{Q_t^{(\rho)}(o, a_t)}{P_t^{(\rho)}(o, a_t)} \,.$$

This expression is used as the definition for the formal expression (1.27), see the discussion in Section 4.1.4. This means that the law of every limit point is given by a Loewner evolution with driving function satisfying the SDE

$$\mathrm{d}\xi_t = \sqrt{2}\mathrm{d}B_t + 2\lambda_t, \text{ where } \lambda_t = \frac{Q_t^{(\rho)}(o, a_t)}{P_t^{(\rho)}(o, a_t)} \,.$$

By checking Novikov's condition, one can see that this SDE has a unique strong solution, see Lemma 4.4.16. In particular, this means all limit points agree.

1.3 The stochastic six-vertex speed process

In Chapter 5 we study the stochastic six-vertex model, which is a specialization of the six-vertex model to specific weights This specialization allows one to view the model as a one-dimensional interacting particle system instead of a two-dimensional equilibrium lattice model.

The main result of Theorem 1.3.2 is that the speed of a single second-class particle placed at the origin, with all positions filled to the left of it and all positions to the right of it empty, will almost surely converge. The limiting speed is random and can be read from the hydrodynamic limit of this system. This convergence then allows one to define the **stochastic six-vertex speed process**, which is a translation invariant, ergodic probability measure assigning real-valued labels, (which we call classes) to each particle, which is stationary under the dynamics of the **multi-class** stochastic six-vertex model.

The proof of the main theorem uses a variety of tools. From the theory of particle systems it uses various couplings, in particular, the attractive coupling which defines the multi-class model and the approximate monotonicity recently shown in [ACH24]. To control the position of the second-class particle by the behavior of a larger number of third-class particles, we prove Proposition 5.1.7. Finally, to prove that the system is already close to its hydrodynamic limit at a large but finite time we use methods from integrable probability to obtain effective hydrodynamics in Theorem 5.4.1 via precise tail bounds on an associated point process. Using these methods we obtain a bound on the fluctuations around the eventual limiting speed, see Theorem 1.3.3.

Speed processes have been introduced and studied for some other interacting particle models, starting with TASEP in [AAV08] and since then for the ASEP and TAZRP (Totally Asymmetric Zero Range Process) in [ABGM21, ACG23]. The stochastic six-vertex models posed us with some particular issues. Its basic coupling has less convenient properties compared to other models (in particular it is not monotone). Additionally, there is no straightforward analogue for the stochastic six-vertex model of Rezakhanlou's coupling from [Rez95], which was used to control the position of an individual second-class particle by the behavior of a larger number of third-class particle for ASEP. Our result in Proposition 5.1.7 is not a coupling. However, it is significantly stronger than the bound obtained by using the coupling from [Rez95] and could be used to simplify the arguments of [ACG23], since it can be extended to ASEP.

In the rest of this introduction, the model in its simple and multi-class form will be defined and heuristics for the hydrodynamic limit and the main theorem based on an assumption of local equilibrium will be given. However, we do not use such an assumption, nor does the proof of Theorem 1.3.2 proceed via proving local equilibrium. After that, a sketch of the proof of our main Theorem 1.3.2 will be followed by a brief discussion of the different tools we use. Finally, the speed process and some of its basic properties will be introduced.

1.3.1 The model

The study of the six-vertex model goes back all the way to [Pau35], where it was proposed as a model for the residual entropy in water ice.

The six-vertex model is defined on a region D of $\mathbb{Z}^2$. A configuration consists of orienting each edge in one of the two possible directions, with the requirement that the number of edges oriented towards each vertex equals the number of edges oriented away from that vertex. This is sometimes called the **ice rule**. There are six different possible configurations around each vertex (hence the name of the model); we label the six possible arrow configurations at a vertex with a letter from a to f, see Figure 1.4. Given a region D of $\mathbb{Z}^2$ we call the edges with

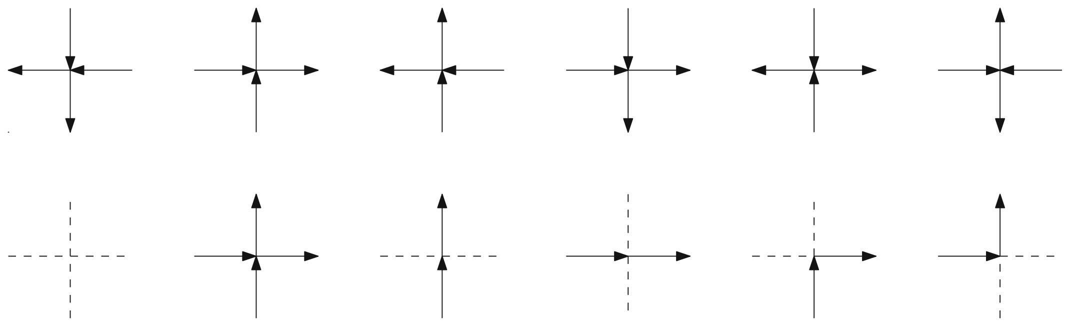


Figure 1.4: The six different configurations of the six-vertex, before and after moving to the up-right path version, and their weights.

one end point in D and one point outside D the **boundary** of D. A **boundary condition** is a given orientation on some or all edges of the boundary. If the domain D is finite, then there are finitely many configurations satisfying a given boundary condition. We call $a, ..., f$ the weight corresponding to the six possible vertex configurations $a, ..., f)$ and defining the weight of a configuration ω to be

$$w(\omega) = \prod_{v \in D} w(v) = a^{\#a\text{-vertices}} b^{\#b\text{-vertices}} c^{\#c\text{-vertices}} d^{\#d\text{-vertices}} e^{\#e\text{-vertices}} f^{\#f\text{-vertices}},$$

i.e., the product over all the vertex weights of the vertices in D, we can define a probability measure on configurations by setting

$$\mathbb{P}(\omega) = \frac{w(\omega)}{Z},$$

where Z is the partition function of the model, i.e. the normalizing constant that makes $\mathbb{P}(\omega)$ into a probability measure.

This is a classical model from equilibrium statistical mechanics. It is exactly solvable using a transfer matrix approach and the Bethe ansatz, see e.g. [Bax89, DCGH$^+$16a]. It exhibits many interesting phenomena including multiple phase transitions, limit shapes, a connection to the dimer model, FK percolation and alternating sign matrices and much more, see [CP10, DCKK$^+$22, DCKMO24, DCGH$^+$16b, Bre99] and the references therein.

We will consider the six-vertex model with specific weights, which were first introduced in [GS92]. Before introducing these weights we will make a combinatorial reformulation of the space of configurations. Since every edge has to be oriented in one of two ways it suffices to keep track of which edges are oriented up and right, see Figure 1.4. By doing so, one can see a configuration as a collection of up and right paths. The ice rule becomes a rule of local conservation: the number of incoming arrows must equal the number of outgoing arrows. This change allows us to think of the arrows as **particles**. If one thinks of two arrows meeting as bouncing off one another, each particle moves on its unique path, and these paths are non-crossing. The stochastic weights are now obtained by thinking of the two edges on the bottom and left of a specific vertex as the **input** and the two edges on the top and right as the **output** of this vertex. Given a specific input, we want to choose one of the possible outputs a random. Hence, we must choose weights such that the possible outputs sum up to 1. If the input consists of either two incoming or no incoming arrows, there is only one possible choice and the weight of these vertices must be one. If however there is only one incoming vertical arrow, there are two possible outputs, and therefore we assign one of them

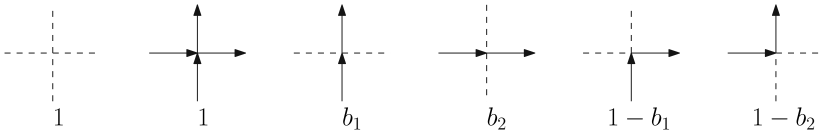


Figure 1.5: The stochastic weights for the six-vertex model

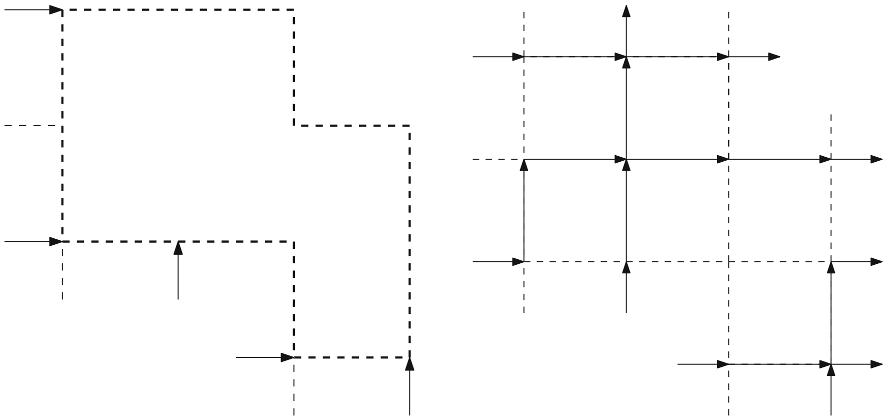


Figure 1.6: A finite domain with given boundary condition and a possible configuration on this domain.

weight b_1 and one of them weight $1-b_1$, and the same for an incoming horizontal arrow. See Figure 1.5 for the weights of the corresponding configurations. Let us define

$$q \stackrel{\text{def}}{=} \frac{b_1}{b_2} \qquad \kappa \stackrel{\text{def}}{=} \frac{1-b_1}{1-b_2}\,, \tag{1.41}$$

since these quantities appear quite often. For $q \in (0,1)$ and $\kappa \in (1,\infty)$ these two offer an alternative parameterization of the (b_1, b_2) satisfying $0 < b_1 < b_2 < 1$, which is the regime we are interested in.

Since we now think of arrows coming from the bottom and the left as "incoming" and arrows going to the right and the top as outgoing, it is natural to consider boundary conditions where only the incoming arrows along the boundary are specified, and the outgoing arrows are left free, see Figure 1.6. With this kind of boundary condition, one can see inductively that the partition function Z always equals 1 and that the configuration can be sampled **vertex by vertex**. This is done by choosing a vertex all of whose incoming vertices have been determined, and, if necessary, generating a random Bernoulli with parameter b_1 or b_2 to decide which type of vertex it is supposed to be. This stochastic sampling process can be used to define the stochastic six-vertex model on some infinite domains. Of particular interest to us will be the corner $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with so-called **step initial conditions**, that is, every incoming edge from the left has an arrow, while no incoming edge from the bottom has an arrow. The model one obtains is equivalent to taking the limit of Gibbs measures on rectangular domains, see [BCG16].

1.3.2 Shocks, rarefaction fans and 2nd class particles

We now want to study the behaviour of the model on the step initial conditions defined in the previous section. For reasons that will become clear later, we will associate the first axis with space and the variable x and the second axis with time and the variable t. The behaviour

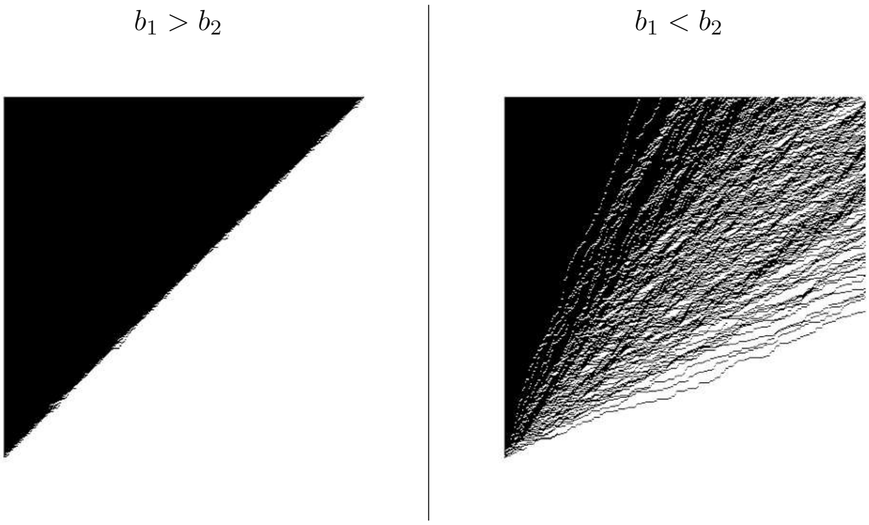


Figure 1.7: Two simulations of the stochastic six-vertex model on the corner with step initial conditions, one for $b_1 > b_2$ exhibiting a shock, the other for $b_1 < b_2$ exhibiting a rarefaction fan

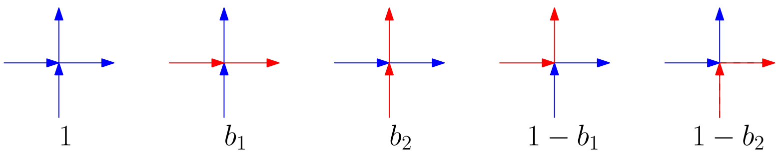


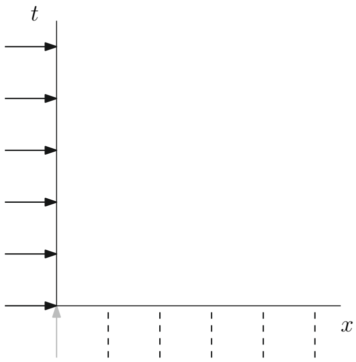
Figure 1.8: The vertex weights of the multi-class stochastic six-vertex model, where the colors blue and red correspond to classes i and j respectively with $i < j$.

of the model on step initial conditions depends strongly on which between b_1 or b_2 is bigger. See Figure 1.7 for two simulations of either type. If b_1 is bigger, particles on their own will want to move up, and therefore any particle that finds itself to the right of the line $x = t$ will quickly move back to it, and similarly any hole above the line will quickly be filled. This phenomenon is called a shock. On the other hand, for $b_1 < b_2$ the particles will quickly spread out and form a continuous decrease from density 1 to density 0. This is called a rarefaction fan. Both of these phenomena will become clearer once we look at the hydrodynamic limit of this model.

Chapter 5 studies the rarefaction fan phenomenon. To do so, we use the multi-class version of the stochastic six-vertex model. In this model, every edge (which we still call particle/arrow) has a label in $\mathbb{Z} \cup \{-\infty, \infty\}$, which we call its **class**. The classes are preserved at each vertex, so there are two possible outcomes if the classes of the incoming particles are distinct, and only one if the classes of the incoming particles are equal. In the case where there is a choice to be made, the probability of the two possible outcomes are determined by the classes: If the incoming classes are i and j and $i < j$, then the weight of the vertex is chosen as if j was a hole and i was a particle in the standard stochastic six-vertex model, see Figure 1.8.

The standard six-vertex model can be obtained as a special case of the multi-class stochastic six-vertex model where only two classes are present. In particular we will often identify the two models by assigning particles class 1 and holes class ∞.

What makes the multi-class model a useful tool to studying the single-class model is the



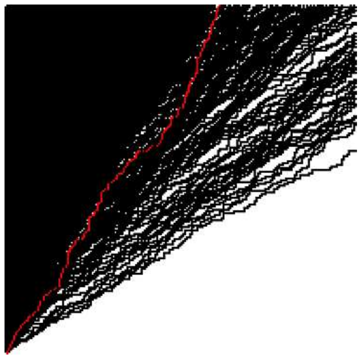


Figure 1.9: On the left: the initial condition with a single second-class particle. Black arrows denote first class particles, while the gray arrow denotes the second-class particles. Dashed lines denote holes. On the right: a simulation of this process on a 200 by 200 square with $b_1 = 0.3$ and $b_2 = 0.6$, with the second-class particle in red.

merging property.

Lemma 1.3.1. *If ω is a configuration of the multi-class stochastic six-vertex model, then for any monotone map $\phi : \mathbb{Z} \cup \{-\infty, \infty\} \to \mathbb{Z} \cup \{-\infty, \infty\}$, the configuration $\phi \circ \omega$ is a realization of the multi-class model, with boundary conditions changed by ϕ.*

A particular case is the following: Considering a multi-class stochastic six-vertex model with particles only of class $1, 2$ and ∞, i.e. first-class particles, second-class particles and holes. This can be mapped to the single-class stochastic six-vertex model in two (non-trivial) ways:

$$\phi_1(x) = \begin{cases} 1, \text{ if } x = 1 \text{ or } x = 2 \text{ and} \\ \infty, \text{ if } x = \infty \end{cases} \quad \text{and} \quad \phi_2(x) = \begin{cases} 1, \text{ if } x = 1 \text{ and} \\ \infty, \text{ if } x = 2 \text{ or } x = \infty. \end{cases}$$

I.e., one can treat the second-class particles either as particles or holes, and in either way obtains a single-class stochastic six-vertex model. By doing so one obtains a coupling of two stochastic six-vertex models, such that the occupied edges of one of the models is a superset of the occupied edges of the other model. This coupling has been used to study the model, see e.g. [BB19, Agg20a, ACH24].

We are now ready to state our main results, which we do in the next section.

1.3.3 Main results

Consider the following boundary conditions on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

- On all incoming edges from the left there is a particle of class 1.
- On the incoming edge from the bottom at $(0, 0)$ there is a particle of class 2.
- On all other incoming edges, there is no incoming particle (i.e., class ∞).

We call these boundary conditions **step initial conditions with a vertical second-class particle at the origin**, see Figure 1.3.3 for a diagram showing these initial conditions and a simulation. The local conservation of particles guarantees that for each horizontal row of edges, almost surely exactly one of them is occupied by a second-class particle. Denote by $\boldsymbol{X}_t$ the position of that edge for the row of edges $\{(x, t) - (x, t + 1) : x \in \mathbb{Z}_{\geq 0}\}$ and call it the position of the second-class particle at time t.

Theorem 1.3.2. *Let $b_1 < b_2$. Under step initial conditions with a vertical second-class particle at the origin, the speed of the second-class particle almost surely converges:*

$$\frac{\boldsymbol{X}_t}{t} \to \boldsymbol{U},$$

where U is an explicit continuous random variable on $[\frac{1}{\kappa}, \kappa]$, where κ is as defined in (1.41).

The law of $\boldsymbol{U}$ can be derived from the hydrodynamic limit, see Section 1.3.6 below. We also derive a bound on the fluctuations around the limit:

Theorem 1.3.3. *Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t as above and $\boldsymbol{U}$ its almost sure limit speed. Then for any $\delta > 0$, almost surely we have that*

$$\lim_{t\to\infty} |\boldsymbol{X}_t - t\boldsymbol{U}| t^{-(\frac{7}{9}+\delta)} = 0\,.$$

As mentioned in Remark 5.1.3, the exponent $\frac{7}{9}$ is not optimal, and rather an exponent $\frac{2}{3}$ is expected.

As a direct consequence of Theorem 1.3.2 we obtain the stochastic six-vertex speed process. We state this result for a version of the stochastic six-vertex model on $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$, with so-called **packed initial conditions**, meaning that at each vertex $(x, 0)$ there is an incoming particle from below with class x, see Section 1.3.4 for how this is defined.

Theorem 1.3.4. *Under packed initial conditions, denote by $\boldsymbol{X}_t(x)$ the position of the particle with class x at time t. Then*

$$\lim_{t\to\infty} \left(\frac{U_t(x)}{t}\right)_{x\in\mathbb{Z}} = U(x)$$

exists. Its law is translation invariant and ergodic, and the law of $-U(-x)$ is stationary under the multi-class stochastic six-vertex dynamics.

The only part of the above theorem that is not a direct consequence of Theorem 1.3.2 is the stationarity of this measure. This is proven using the recently developed color-position symmetry from [BB19].

To prove Theorems 1.3.2 and 1.3.3 we prove an the effective hydrodynamic limit estimate for the (single-class) stochastic six-vertex model with step initial conditions, which might be of independent interest. To state it, we need to first define the height function for step initial conditions. For a given configuration ω on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ define the height function $H(x, t; \omega)$ for $x, t \in \mathbb{R}_{\geq 0}$ by setting $H(x, 0; \omega) = 0$ for all x and increasing H whenever you cross a path in the vertical direction. There is a law of large numbers of this height function H. With probability one it holds that

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g(x, y), \qquad \forall x, y \in \mathbb{R}_{\geq 0}.$$

where for $b_1 \leq b_2$, we have

$$g(x,y) = \begin{cases} y - x & \frac{x}{y} \leq \frac{1}{\kappa} \\ \frac{\left(\sqrt{x} - \sqrt{\kappa y}\right)^2}{\kappa - 1} & \frac{1}{\kappa} \leq \frac{x}{y} \leq \kappa, \\ 0 & \frac{x}{y} \geq \kappa\,. \end{cases}$$

This was proven at the level of weak convergence in [BCG16] and [Agg20b], and we strengthen to almost sure convergence in [DL23].

Using tools from integrable probability we prove bounds on how much the height function H fluctuates around its limit shape g. Note that the exponent $T^{\frac{1}{3}}$ in this theorem is optimal, since at this scale Tracy-Widom fluctuations have been shown in [BCG16, Theorem 1.2].

Theorem 1.3.5. *Let $H(x,t)$ be the height function associated with step initial conditions. For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that the following holds. For any x, y such that $\kappa^{-1} + \varepsilon \le x, y \le \kappa - \varepsilon$, and for any $T > 1$, $s \in [0,T]$,*

$$\mathbb{P}\left[|H(Tx,T) - H(Ty,T) - (g(x) - g(y))\,T| \ge sT^{1/3}\right] \le c^{-1}e^{-cs}, \tag{1.42}$$

and the constant c can be chosen to decrease weakly in ε.

Now that we have stated the main results, in the remainder of this introduction we will describe some basic properties of the process in Sections 1.3.4 and 1.3.5, provide some heuristics in Section 1.3.6, give the ideas of the proof in Section 1.3.7 and finally mention some of the tools we use in Section 1.3.8. We then conclude with a definition of the speed process and mention some of its basic properties in Section 1.3.9.

1.3.4 The particle process

Until this point, we have treated the stochastic six-vertex model as a measure on configurations consisting of oriented edges. However, it is also natural to consider it as a particle system, as has already been quite noticeable in the language we have been using, and was already observed in [GS92]. Let us now introduce a notation that emphasises this connection. For a given configuration ω of the 6-vertex model, define $\eta_t(x)$ for $x \in \mathbb{Z}_{\ge 0}$ by

$$\eta_t(x) = \begin{cases} 1, \text{ if the incoming vertex at } (x,t) \text{ from below in } \omega \text{ is occupied and} \\ 0, \text{ else.} \end{cases}$$

Defined like this, $(\eta_t)_{t\in\mathbb{Z}_{\ge 0}}$ is a Markov process with values in $\{0,1\}^{\mathbb{Z}_{\ge 0}}$. The boundary conditions on the bottom give the initial condition η_0 and the boundary conditions on the left inject particles at specific times. The transition probabilities of this process can be described by particles staying in place with probability b_1 and performing jumps given by geometric random variables with parameter b_2 if they start moving. If they would jump across another particle, instead they stop and the other particle starts moving. See [BCG16, Section 2.2.] for these transition weights written out. To reflect this change of perspective denote the height function $h_t(x;\eta)$ satisfying the property

$$\begin{aligned} &h_t(x;\eta) - h_t(x+1;\eta) = \eta_t(x) \text{ and} \\ &h_{t+1}(0;\eta) - h_t(0;\eta) = \begin{cases} 1, \text{ if there is an incoming arrow from the left at } (0,t) \\ 0, \text{ else.} \end{cases} \end{aligned} \tag{1.43}$$

Given a configuration $(\eta_t)_{t\ge 0}$ these equations determine $(h_t)_{t\ge 0}$ up to a global shift. Unless otherwise specified, the height function is made unique by setting $h_0(0) = 0$, but in some places it might be convenient to choose some other initial condition h_0. One quickly recovers the definition of $H(x,t)$ above, since by (1.43) for step initial conditions $h_0(x) = 0$ for all x. By taking the discrete gradient of h_t one recovers the occupation variable η_t. In particular, h_t is a Markov process. The same change of perspective can also be applied to the multi-class model. Then $\eta_t : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, \infty\}$ maps the position x to the class of the particle at position x at time t, i.e. of the incoming arrow from below.

So far, all we did in this subsection is changing notation. However, for the definition of the speed process it is convenient to go one step further and to consider the particle system on the **line**, i.e., started from initial conditions $\eta_0 : \mathbb{Z} \to [0,1]$ and evolve it in a way that agrees with the description of the stochastic six-vertex model above in terms of Bernoulli random variables and geometric jumps. This is done for general initial conditions in [Agg20a]. For step initial conditions the two models are easily identified. Indeed for $(\eta_t(x))_{t\in\mathbb{Z}_{\geq 0},x\in\mathbb{Z}}$ a stochastic six-vertex process on the line started from the initial condition $\eta_0(x) = \mathbf{1}_{x<0}$, the law of the restriction $(\eta_t(x))_{t\in\mathbb{Z}_{\geq 0},x\in\mathbb{Z}_{\geq 0}}$ is exactly the stochastic six-vertex process on the quadrant $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ started from step inital data as described above.

One benefit of changing domain to $\mathbb{Z}$ is that the dynamics are now translation invariant. If one initial condition is obtained from translating another, then the corresponding processes can be coupled to be related by the same translation. Another advantage, that we however do not use, is that the dynamics are monotone, see [Agg20a, Proposition 2.6], which means that given two initial conditions with corresponding height functions, such that one is above the other, the two processes can be coupled such that this property is maintained. However, as opposed to the corresponding property of ASEP, one cannot (to the best of the author's knowledge) couple two height functions h^1 and h^2 such that the $\max_{x\in\mathbb{Z}}|h^1_t(x)-h^2_t(x)|$ is non-increasing. Because of this, we have to use an approximate version of this property proven in [ACH24, Lemma D.3], which states that given two initial conditions with at most N particles, then under the "basic coupling", if at time 0 the height functions satisfy $\max_{x\in\mathbb{Z}}|h^1_0(x) - h^2_0(x)| = K$, then at any later time $\max_{x\in\mathbb{Z}}|h^1_t(x) - h^2_t(x)|$ is at most $K + C(\log N)^2$, with high probability, see Proposition 5.2.6 for a precise statement.

1.3.5 Stationary measures

A first step in understanding the model is taking a look at the translation invariant measures. For any $b_1, b_2 \in (0,1)$ and $\rho_1, \rho_2 \in [0,1]$ denote by $P(b_1, b_2, \rho_1, \rho_2)$ the measure on configurations obtained by choosing the following random boundary conditions on the quadrant: Independently every edge coming in from the left is occupied with probability ρ_1 and every edge coming in from the bottom is occupied with probability ρ_2. Call these the **Bernoulli**$-(\rho_1, \rho_2)$ initial conditions.

As was conjectured in [GS92] and proven in [Agg16] under the condition

$$\frac{\rho_1}{1-\rho_1} = \kappa\frac{\rho_2}{1-\rho_2} \tag{1.44}$$

if $(\eta_t(x))_{t,x\in\mathbb{Z}_{\geq 0}}$ is the process sampled according to $P(b_1, b_2, \rho_1, \rho_2)$ then for any $s, y \in \mathbb{Z}_{\geq 0}$ $(\eta_{t+s}(x+y))_{t,x\in\mathbb{Z}_{\geq 0}}$ and $(\eta_t(x))_{t,x\in\mathbb{Z}_{\geq 0}}$ have the same law. Using this translation invariance one can obtain a translation invariant Gibbs measure in the full plane with densities ρ_1, ρ_2. Moving to the line as discussed in the previous section, the analogous statement is that the law of iid Bernoulli(ρ) random variables is stationary and translation invariant. These are the only extremal translation invariant stationary measures on the line, as was proven in [Agg20a, Theorem 3.6].

Recently, stationary measures for the multi-class stochastic six-vertex model were constructed in [ANP23] using the Yang-Baxter relation for higher spin vertex models.

1.3.6 Local equilibrium and the hydrodynamic limit: a heuristic

In this section, we will give a formal argument to show that the hydrodynamic limit of the stochastic six-vertex model is given by a Burgers equation, by assuming that the system satisfies a local equilibrium assumption.

Given a sequence of initial conditions $\eta^\delta_0 : \mathbb{Z} \to \{0,1\}$ such that

$$\delta\sum_{x=\delta^{-1}a}^{\delta^{-1}b}\eta^\delta_0(x) \approx \int_a^b \rho_0(x)\mathrm{d}x$$

for some continuous initial density profile $\rho_0 : \mathbb{R} \to [0,1]$, what do we expect when we let this process run for some time of the order $\delta^{-1}t$? A reasonable assumption is that the system is in **local equilibrium**, which would mean that

$$\tilde{\eta}_s^{\delta,x,t}(y) \overset{\text{def}}{=} \eta^{\delta}_{\delta^{-1}t+s}(x\delta^{-1}+y) \tag{1.45}$$

is close to a stationary stochastic six-vertex process with some density $\rho_t(x)$. This $\rho_t(x)$ describes the density of vertical arrows. By solving (1.44), one can see that the density of vertical and the density of horizontal arrows are connected via the function

$$\phi(\rho) \overset{\text{def}}{=} \frac{\kappa\rho}{(\kappa-1)\rho+1}\,.$$

As a quick sanity check one can see that this function is monotone and maps $[0,1]$ to itself. The local equilibrium assumption can now be stated as

$$\mathrm{Law}([\tilde{\eta}_s^{\delta,x,t}(y)]_{y\in\mathbb{Z},s\in\mathbb{Z}_{\geq 0}}) \approx P(b_1,b_2,\rho_t(x),\phi(\rho_t(x)))\,, \tag{1.46}$$

for $\delta \to 0$ and we recall that the object on the right-hand side is the process at stationarity from the previous section. Consider now a mesoscopic box around $\delta^{-1}x, \delta^{-1}t$ of side-length $2\varepsilon\delta^{-1}$. By the local conservation property of the stochastic six-vertex process, the number of arrows entering the box from the bottom and the left must equal the number of arrows leaving through the top and the right. By the assumption, the density of particles entering from the bottom is approximately $\rho_{t-\varepsilon}(x)$, while the density of particles entering from the left is approximately $\phi\circ\rho_t(x-\varepsilon)$ and similarly for the other two edges. The local conservation property then takes the form

$$\rho_{t-\varepsilon}(x) + \phi\circ\rho_t(x-\varepsilon) = \rho_{t+\varepsilon}(x) + \phi\circ\rho_t(x+\varepsilon) \iff$$
$$0 = \frac{\rho_{t+\varepsilon}(x)-\rho_{t-\varepsilon}(x)}{2\varepsilon} + \frac{\phi\circ\rho_t(x+\varepsilon)-\phi\circ\rho_t(x+\varepsilon)}{2\varepsilon}\,.$$

Taking the limit $\varepsilon \to 0$ we obtain that ρ satisfies the PDE

$$\frac{\partial}{\partial x}\rho + \frac{\partial}{\partial x}\phi\circ\rho = 0\,. \tag{1.47}$$

Weak convergence of η^δ to a solution of this equation is rigorously proven in [Agg20a, Theorem 1.1]. This PDE can develop singularities at finite times, which causes there to be multiple weak solutions. Among these there is a specific one called “entropy solution”, satisfying a certain inequality, which corresponds to the thermodynamic fact that the entropy increases in time. It is this entropy solution that η^δ converges to (in a weak sense).

For step initial conditions $\rho_0(x) = \mathbf{1}_{x<0}$, the solution of this equation is given by

$$\rho_t(x) = \begin{cases} 1 & \text{if } \frac{x}{t} \leq \kappa^{-1} \\ (\phi')^{-1}(\frac{x}{t}) = \frac{\sqrt{\kappa t/x}-1}{\kappa-1} & \text{if } \kappa^{-1} \leq \frac{x}{t} \leq \kappa \\ 0 & \text{if } \frac{x}{t} \geq \kappa\,. \end{cases} \tag{1.48}$$

From this, one can already guess the limiting law of the asymptotic speed $\boldsymbol{U}$ of the second-class particle $\frac{\boldsymbol{X}_t}{t}$. By the merging property discussed at the end of Section 1.3.1, the second-class particle tracks the difference between two stochastic six-vertex processes started from the initial conditions $\eta_0(x) = \mathbf{1}_{x<0}$ and $\xi_0(x) = \mathbf{1}_{x\leq 0}$. This implies that

$$\mathbb{P}(\boldsymbol{X}_t = x) = \mathbb{E}[\xi_t(x)] - \mathbb{E}[\eta_t(x)]$$

Since the initial condition of ξ is the initial condition of η shifted by 1, for large t the right-hand side is close to

$$\rho_1(\frac{x-1}{t}) - \rho_t(\frac{x}{t}) \approx -\frac{1}{t}\rho_1'(\frac{x}{t}).$$

Therefore, the distribution of $\boldsymbol{U}$ has the density

$$\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}} \mathbf{1}_{\kappa^{-1}\le x\le\kappa} \,. \tag{1.49}$$

This argument can be made rigorous to show that $\frac{\boldsymbol{X}_t}{t}$ converges to $\boldsymbol{U}$ in distribution, see Section 5.A.

The local equilibrium assumption also supplies a heuristic argument for Theorem 1.3.2. Consider the process run at equilibrium with density ρ, i.e., with initial conditions given by a sequence of i.i.d. Bernoulli(ρ) random variables. Adding a second-class particle, this will move with speed $x/t = \phi'(\rho)$, with fluctuations around this trajectory being of order $t^{\frac{2}{3}}$ (see [Agg16]).

Considering now again the process started from step initial conditions with a single second-class particle at the origin. After running this process for some long initial time S, the second-class particle is at some position $\boldsymbol{X}_S$. By the hydrodynamic limit, the density of particles around $\boldsymbol{X}_S$ is approximately $\rho_S(\boldsymbol{X}_S) = (\phi')^{-1}(\frac{\boldsymbol{X}_S}{S})$ Then by the local equilibrium assumption, the law of the first class particles around position $\boldsymbol{X}_S$ is close to the law of iid Bernoulli variables with this parameter. Therefore

$$\boldsymbol{X}_{S+t} \approx \boldsymbol{X}_S + t\phi'(\rho_S(\boldsymbol{X}_S)) + O(t^{\frac{2}{3}}) = \boldsymbol{X}_S + t\frac{\boldsymbol{X}_S}{S} + O(t^{\frac{2}{3}}) = (S+t)\frac{\boldsymbol{X}_S}{S} + O(t^{\frac{2}{3}})\,.$$

However, one cannot expect this to hold arbitrarily large t. After some time the fluctuations of the second-class particle are big enough, to cause the density observed around the particle to be noticeably different from the density observed at time S. How long such an approximation is reasonable depends on how close the environment around $\boldsymbol{X}_{\boldsymbol{S}}$ at time S is to equilibrium and therefore it depends on the initial time S. However, if one can show that such an approximation holds for a long enough time, dependent on S, one can use this iteratively to show almost sure convergence. Proposition 5.5.2 shows that this holds up to $t = S^\beta$ for $\beta < 1$.

Local equilibrium in the sense of convergence of the left hand side of (1.46) to the right handside on finite boxes was shown in [Agg20a, Theorem 1.3]. However to imply our result one would need not just convergence. Rather one would need a result showing e.g. that $[\tilde{\eta}_s^{\delta,x,t}(y)]_{y\in[-N,N],s\in[0,N]}$ can be coupled to the stationary process with high probability. We emphasize that the actual proof does not proceed via showing such a result. It is possible that such a proof would improve the exponent in Theorem 1.3.3.

1.3.7 Proof sketch for Theorem 1.3.2

As discussed in the previous section, the idea is to show that after some initial time has passed, the particle will stay close to the slope it has at the end of that initial time, with high probability. This is made precise by the following proposition

Proposition 1.3.6 (Proposition 5.5.2 below)**.** *For any integer $S > 2$ and $\beta \in (2/3, 1)$ let $T = S^\beta$. Also, define the following $\mathcal{F}_S$-measurable event, which depends on ε:*

$$P_S = \{\frac{\boldsymbol{X}_S}{S} \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]\} \tag{1.50}$$

and the $\mathcal{F}_{S+T}$-measurable events

$$E_S^{\geq} = \{\boldsymbol{X}_{S+T} - \boldsymbol{X}_S \geq \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma}\}$$
$$E_S^{\leq} = \{\boldsymbol{X}_{S+T} - \boldsymbol{X}_S \leq \frac{\boldsymbol{X}_S}{S} T + S^{1-\gamma}\}$$

and let $E_S = E_S^{\geq} \cap E_S^{\leq}$. For any positive $\alpha < \beta/2 - 1/3$ and for any $\varepsilon \in (0, \frac{1}{4})$, there is a $c = c(\varepsilon, \alpha) > 0$ and a $\mathcal{F}_S$-measurable event H_S such that for all $S > 2$ and for $\gamma = 5/6 - \beta/2 - \alpha$ we have

$$\mathbb{P}[H_S] \geq 1 - c^{-1} e^{-cS^{\alpha}} \quad \mathbb{P}[E_S | \mathcal{F}_S] \geq (1 - c^{-1} e^{-cT^{\alpha}}) \mathbf{1}_{P_s \cap H_s}. \tag{1.51}$$

Let us unpack the different parts of this proposition.

- The event P_S states that the particle is not too close to the boundary of the rarefaction fan. From a local equilibrium perspective it makes sense that this would be problematic, since the law of Bernoulli(ρ) random variables becomes degenerate in for ρ close to 0 or 1. One can also see that the effective hydrodynamics in Theorem 1.3.5 are only stated inside the rarefaction fan, bounded away from the boundary.

- The event E_S is exactly the event that X_{S+T} is not too far from the trajectory given by the slope at time S. One can easily see that it implies that

$$\left| \frac{X_S}{S} - \frac{X_{S+T}}{S+T} \right| \leq S^{-\gamma}. \tag{1.52}$$

- The event H_S, which we call the **hydrodynamic event**, is that the particle process at time S has not strayed to far from the hydrodynamic limit. This event allows us to prove that E_S has high probability, conditioned on the configuration at time S as long as the events P_S and H_S take place.

Once we have shown this proposition, we prove Theorem 1.3.2 by considering the sequence of times S_n defined by $S_{n+1} = S_n + S_n^{\beta}$ and show that with probability converging to 1 as $S_0 \to \infty$ all events E_{S_n} occur. This implies by (1.52) that the sequence $(\frac{X_{S_n}}{S_n})_n$ is a Cauchy sequence. For times t between the S_n, one can use the monotonicity of $\boldsymbol{X}_t$ to show convergence. This also gives Theorem 1.3.3 by bounding how fast this Cauchy sequence converges as well as bounding the deviations between times S_n. A tradeoff between those two effects gives the value $\beta = \frac{7}{9}$.

There are two key ingredients to obtaining Proposition 1.3.6.

- Theorem 1.3.5 gives very precise bounds on the fluctuations of the height function started from step initial conditions.

- Proposition 5.1.7 concerns the multi-class stochastic six-vertex model started from initial conditions with

  - Some first-class particles (potentially infinitely many),
  - a single second-class particle and
  - finitely many third-class particles, which are all initially to the left of the second-class particle.

  Given such an initial condition, Proposition 5.1.7 states that at any future time t, the amount of third-class particles that have passed the second-class particle is dominated by a geometric random variable with parameter $q \stackrel{\text{def}}{=} \frac{b_1}{b_2}$.

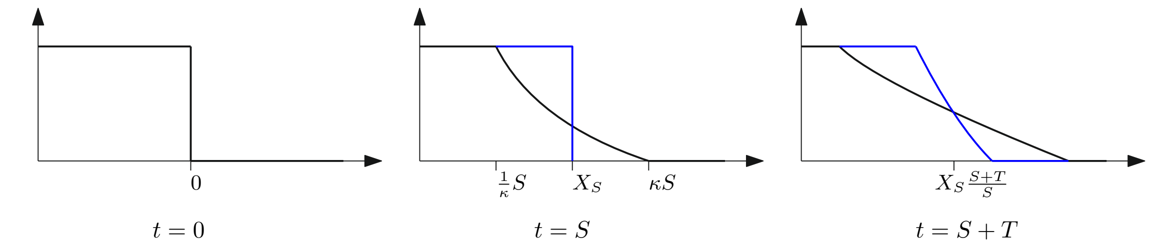


Figure 1.10: A sketch of the densities of the processes η^1 in black at times $0, S$ and $S+T$ and η^{step} at times S and $S+T$. At time S the process $\eta^{1,2}$ is given exactly by the maximum of the two processes η^1 and η^{step}, while at time $S+T$ it is at least the maximum of η^1 and η^{step}.

Let us explain how these two results can be used to show Proposition 1.3.6. Let us focus on showing the event $E_S^{\geq}$. We want to control the behaviour of the second-class particle after the initial time S. However, Theorem 1.3.5 only allows us to control the behaviour of a large number of particles, not of an individual one. To use this, we fill up all positions to the left of $\boldsymbol{X}_S$ with third-class particles. Then, Proposition 5.1.7 will guarantee that only a very small number of these third-class particles will be to the right of $\boldsymbol{X}_{S+T}$ at time $S+T$. It therefore suffices to prove that a large number of these particles are to the right of $\boldsymbol{X}_S + \frac{\boldsymbol{X}_S}{S}T - S^{1-\gamma}$ at time $S+T$. To do so denote by $\eta^{1,2}$ the (single-class) stochastic six-vertex model containing all first-, second- and third-class particles and with η^1 the one with only the first-class particles. Additionally, we introduce an auxiliary third process η^{step} which is started at time S from the initial condition $\eta_S^{\text{step}}(x) = \mathbf{1}_{x \leq \boldsymbol{X}_S}$. At time S these three processes satisfy

$$\eta_S^{1,2}(x) = \max(\eta_S^1(x), \eta_S^{\text{step}}(x))\,.$$

The multi-class stochastic six-vertex process allows us to couple $\eta^{1,2}$ and η^{step} such that at any later time $S+T$ it holds that $\eta_{S+t}^{1,2}(x) \geq \eta_{S+t}^{\text{step}}(x)$. Since $\eta^{1,2}$ and η^1 are already coupled in such a way, this implies for any $t \geq 0$

$$\eta_{S+t}^{1,2}(x) \geq \max(\eta_{S+t}^1(x), \eta_{S+t}^{\text{step}}(x))\,.$$

Note that this also couples η^1 and η^{step} in some non-trivial way. See Figure 1.10 for a sketch of the particle densities. By using the hydrodynamic estimate together with a recent approximate monotonicity result from [ACH24], we show that with high probability η^1 is still close to the hydrodynamic limit at time $S+T$. Since the process η^{step} is started from step initial conditions, it is also close to a hydrodynamic limit at time $S+T$, which is obtained by translating the hydrodynamic limit from standard step initial conditions in time and space. By the coupling above

$$\eta_{S+T}^{1,2}(x) - \eta_{S+T}^1(x) \geq \eta_{S+T}^{\text{step}}(x) - \eta_{S+T}^1(x).$$

By using Theorem 1.3.5 twice for the two processes on the right hand side, this gives a lower bound for the number of third-class particles to the right of $\frac{X_S}{S}(S+T) - S^{1-\gamma}$, as desired.

1.3.8 Tail bounds

To show Theorem 1.3.5 we prove the following two tail bounds for the height function.

Proposition 1.3.7. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa^{-1} - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs},$$

and c can be chosen to weakly decrease in ε.

Proposition 1.3.8. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa^{-1} - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \leq g(\mu)T - sT^{1/3}\right] \leq c^{-1}(e^{-cs} + e^{-cT}),$$

and c can be chosen to weakly decrease in ε.

Both of these are proved using methods from integral probability. For Proposition 1.3.8 we use the following result from [AB19], which gives a Fredholm-determinant formula for the q-Laplace transform of H, that is the quantity on the left-hand side of the formula in the next Proposition:

Proposition 1.3.9 (Part of Theorem 4.9. in [AB19])**.** *Let $H(X,T)$ be the height function associated to a stochastic six-vertex model, with parameters $0 \leq b_1 \leq b_2$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, then*

$$\mathbb{E}\left[\prod_{k\geq 0} \frac{1}{1 - \zeta q^{H(X,T)+k}}\right] = \det(1 + K_\zeta)\,,$$

where K_ζ is an explicit kernel given in (2.18) *and q is defined in* (1.41)*.*

This kernel can then be analysed by standard tools, extending what was already done in [AB19]. This is carried out in Sections 5.4.2 and 5.4.3.

For Proposition 1.3.7 we instead use a remarkable connection to Schur measures and the Meixner ensemble from [BO17]. The Schur measure is a measure on integer partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, with finitely many non-zero λ_i, depending on two finite sequences $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ of non-negative parameters. The Meixner ensemble is a determinantal point process with three parameters (N, β, ξ). It is a measure on N-point configurations in $\mathbb{Z}_{\geq 0}$, which is associated to the classical orthogonal Meixner polynomials. See Section 5.4.1 for definitions of these objects.

Proposition 1.3.10 (Proposition 8.4 in [BO17])**.** *Take any $0 < q < 1$ and $\kappa > 0$ and consider the stochastic six-vertex model on the quadrant parameterized by q and κ as defined in* (1.41)*. Consider any integers $M, N \geq 1$. Then for any $\xi \notin -q^{\mathbb{Z}_{\leq 0}}$ we have*

$$\mathbb{E}_{\mathbf{6v}} \prod_{i\geq 0} \frac{1}{1 + \xi q^{H(M,N)+i}} = \mathbb{E}_{\mathbf{SM}} \prod_{j\geq 0} \frac{1 + \xi q^{\lambda_{N-j}+j}}{1 + \xi q^j} \tag{1.53}$$

where in the right-hand side we assume that $q^{\lambda_{-m}} = 0$ for $m \geq 0$, and the right-hand expectation is with respect to the Schur measure $\mathbf{SM}(\underbrace{(\kappa q^{-1/2})^{-1}, \dots, (\kappa q^{-1/2})^{-1}}_{N}; \underbrace{q^{-1/2}, \dots, q^{-1/2}}_{M-1})$.

By [BO17, Proposition 8.2], the push-forward of the Schur measure

$$\mathbf{SM}(\underbrace{\kappa^{-1}q^{1/2}, \dots, \kappa^{-1}q^{1/2}}_{N}; \underbrace{q^{-1/2}, \dots, q^{-1/2}}_{M-1})$$

under the map

$$\lambda \mapsto \{\min(N, M-1) + \lambda_i - i\}_{i=1}^{\min(N,M-1)}$$

coincides with the point process Meixner$(\min(N, M-1), |N-M+1|+1, \kappa^{-1})$. Distinguishing the cases $M < N$ and $M \geq N$ this leads to the following proposition.

Proposition 1.3.11 (Corollary 8.5 of [BO17])**.** *For any $M, N \in \mathbb{Z}_{\geq 1}, 0 < q < 1, \kappa > 1$ and $\xi \notin -q^{\mathbb{Z}_{\leq 0}}$ We have*

$$\mathbb{E}_{\mathbf{SM}} \prod_{j \geq 0} \frac{1 + \xi q^{\lambda_{N-j}+j}}{1 + \xi q^j} = \mathbb{E}_X \prod_{x \in X} \frac{1}{1 + \xi q^x}. \tag{1.54}$$

where the right-hand expectation is with respect to the point process

$$X \sim \begin{cases} \text{Meixner}^\circ(N, M-N, \kappa^{-1}) & \text{if } M > N \\ N - (M-1) + \text{Meixner}^\circ(M-1, N-M+2, \kappa^{-1}) & \text{if } M \leq N \end{cases},$$

where, for a point process X, $n + X$ denotes the point process obtained by deterministically shifting over each particle in X by n and X° denotes the complement of X.

One can use this identity together with standard tools for dealing with q-Laplace transforms (see e.g. [ACG23, Lemma B.7]), to transfer estimates on the smallest hole of the Meixner ensemble to estimates on the stochastic six-vertex model. What we need is an upper bound on the probability that the **smallest** hole x_1 is atypically **large**. We use a Fredholm determinant formula from [Bor18, in the proof of Theorem 6.1.]

$$P(x_1 - N > h) = \det(1 - \widetilde{K})_{\ell^2(h,h-1,\dots)} = \det(1 - \Pi_h \widetilde{K} \Pi_h), \tag{1.55}$$

where Π_h denotes the projection onto the functions supported on $(h, h-1, \dots)$.

The smallest hole x_1 of the Meixner ensemble being unusually large is a **deviation into the bulk**. The holes of the Meixner ensemble have a certain limit shape, and the left edge of this limit shape corresponds to the typical value for the height function $H(M, N)$. The height function being unusually large corresponds to x_1 being inside the bulk. In both cases "corresponds" refers the connection via (1.53) and (1.54), i.e. it does not mean there is a bijection between the configurations, but rather that estimates on certain events can be transfered from one model to the other. Since in the bulk, the kernel K will be generally large, one cannot proceed as in the proof of Proposition 5.1.9. Instead we use the following "trick", first used by Widom in [Wid02, Lemma 1]. For a kernel K with eigenvalues in $[0, 1]$

$$\det(1 - K) \leq \exp(-\text{Tr}(K)). \tag{1.56}$$

To show that the kernel in (1.55) indeed only has eigenvalues in $[0, 1]$ we use the connection with the Meixner ensemble.

The upper bound on the probability $P(x_1 - N > h)$ is therefore reduced to a lower bound on the $\text{Tr}(\Pi_h \widetilde{K} \Pi_h)$, which can be done using standard complex analysis techniques. This is carried out in Section 5.4.1.

Combining Propositions 1.3.8 and 1.3.7 straightforwardly yields Theorem 1.3.5.

1.3.9 The speed process

As mentioned in Section 1.3.4 one can also consider the stochastic six-vertex model on the line, i.e. as a process $(\eta_t)_{t \in \mathbb{Z}_{\geq 0}}$ with $\eta_t : \mathbb{Z} \to \{0, 1\}$. The same extension also applies to the multi-class stochastic six-vertex model.

For the multi-class stochastic six-vertex model, consider the initial condition $\eta_0(x) = x$, i.e. every position on the boundary has an incoming arrow and every incoming arrow has a distinct class in $\mathbb{Z}$, which are in increasing order. These are known as **packed initial conditions**[4]. With these initial conditions, we can now define the stochastic six-vertex speed process.

Corollary 1.3.12 (Existence of the speed process)**.** *Consider packed initial conditions, i.e. $\eta_0(x) = x$, for $x \in \mathbb{Z}$. Denote by the $\boldsymbol{X}_t(x)$ the position of the unique particle of class x at time t. Then almost surely $(\frac{\boldsymbol{X}_t(x)}{t})_{x\in\mathbb{Z}}$ converges as $t \to \infty$. Denote the limit of this process as $(U(x)_{x\in/Z}$, the* ***stochastic six-vertex speed process****.*

Proof. By the merging property, see Lemma 1.3.1, the law of $\boldsymbol{X}_t(x)$ is identical to the law of the position of the second-class particle in the process started from initial condition

$$\eta_0(y) = \begin{cases} 1, \text{ if } y < x, \\ 2, \text{ if } y = x, \\ \infty, \text{ if } y > x\,, \end{cases}$$

i.e., there is a single second-class particle at x. All positions to the left of this particle are occupied with first-class particles, and all to the right of it are empty (recall that for the multi-class stochastic six-vertex process, holes are represented by class ∞). These are a translation of the "step initial conditions with a single second-class particle" in the statement of Theorem 1.3.2, and therefore $\frac{\boldsymbol{X}_t(x)}{t}$ converges almost surely. Since there are countably many particles, almost surely all slopes converge and form the speed process. □

This speed process has a number of properties that are immediate from its definition. It is translation invariant, ergodic, and the one-dimensional marginals are given by (1.49). Less immediate is the following property:

Proposition 1.3.13 (Proposition 5.8.4 below)**.** *Let U be a stochastic six-vertex speed process. Then the stochstic six-vertex process started from initial conditions $\eta_0(x) = -U(-x)$ is stationary.*

This follows from a certain symmetry of the stochastic six-vertex process recently proven in [BB19].

The speed process can also be defined on other domains, including the quadrant. To do so one assigns each incoming arrow along the boundary a distinct class, increasing along the boundary. See Figure 1.11 for a simulation of the stochastic six-vertex model with these initial conditions. One also sees the formation of "convoys", which has been shown for other speed processes.

[4]at least in [ACH24]

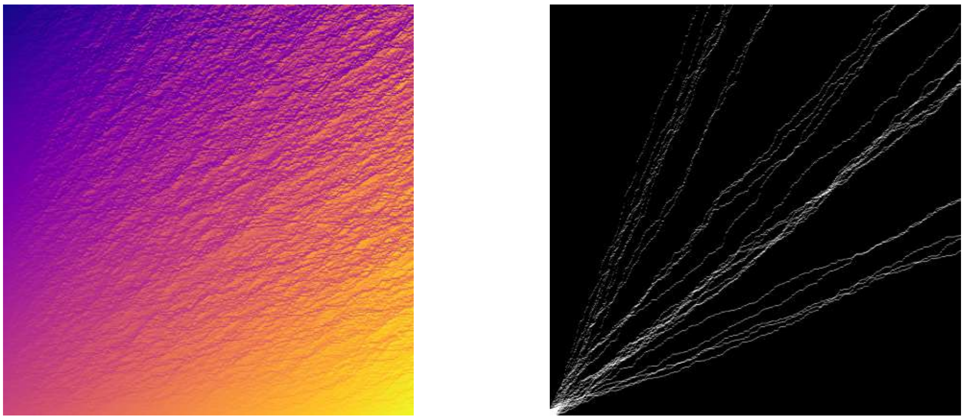


Figure 1.11: A simulation of the stochastic six-vertex model on the quadrant with packed initial conditions, and the traces of the 25 particles starting closest to the origin, both on a square of size 800×800. One can observe the phenomenon of “convoys”, i.e. several particles traveling at the same asymptotic speed.

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Chapter 2

Diffusion in the curl of the Gaussian free field

Abstract

The present work is devoted to the study of the large time behaviour of a critical Brownian diffusion in two dimensions, whose drift is divergence-free, ergodic and given by the curl of the 2-dimensional Gaussian Free Field. We prove the conjecture, made in [B. Tóth, B. Valkó, J. Stat. Phys., 2012], according to which the diffusion coefficient $D(t)$ diverges as $\sqrt{\log t}$ for $t \to \infty$. Starting from the fundamental work by Alder and Wainwright [B. Alder, T. Wainwright, Phys. Rev. Lett. 1967], logarithmically superdiffusive behaviour has been predicted to occur for a wide variety of out-of-equilibrium systems in the critical spatial dimension $d = 2$. Examples include the diffusion of a tracer particle in a fluid, self-repelling polymers and random walks, Brownian particles in divergence-free random environments, and, more recently, the 2-dimensional critical Anisotropic KPZ equation. Even if in all of these cases it is expected that $D(t) \sim \sqrt{\log t}$, to the best of the authors' knowledge, this is the first instance in which such precise asymptotics is rigorously established.

2.1 Introduction

In the present work, we study the motion of a Brownian particle in $\mathbb{R}^2$, subject to a random, time-independent drift ω given by the curl of the two-dimensional Gaussian Free Field (2d GFF). Namely, we look at the SDE which is (formally) given by

$$\mathrm{d}X(t) = \omega(X(t))\mathrm{d}t + \mathrm{d}B(t), \qquad X(0) = 0 \tag{2.1}$$

where $B(t)$ is a standard two-dimensional Brownian motion and

$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x))$$

is defined as

$$x = (x_1, x_2) \mapsto \omega(x) = (\partial_{x_2}\xi(x), -\partial_{x_1}\xi(x)), \tag{2.2}$$

with ξ the 2d GFF. As written, (2.1) is ill-posed due to the singularity of the drift ω. In fact, not only classical stochastic analytical tools would fail in characterising (even) its law but it would also be *critical* for the recent techniques established in [CC18, DD16] as its spatial regularity is way below the threshold identified therein[1]. Nevertheless, we are interested in

[1] Formally, the 2d GFF is in $\mathcal{C}^\alpha$, $\alpha < 0$, the latter being the space of Hölder distributions with regularity α (see [CC18] for the definition), so that $\omega \in \mathcal{C}^{\alpha-1}$. In the aforementioned works, the threshold regularity is $-2/3$ so that (2.1) falls indeed out of their scope.

its large time behaviour and hence we regularise ξ by convolving it with a C^∞ bump function (see Section 2.2 for details), so that ω is well-defined pointwise and smooth. Note that the vector field ω is everywhere orthogonal to the gradient of the field ξ, and therefore parallel to its level lines. As a consequence, the particle is subject to two very different mechanisms: the drift tends to push the motion *along the level lines of the GFF*, while the Brownian noise tends to make it diffuse *isotropically*. Our main theorem is a *sharp* superdiffusivity result: the mean square displacement $\mathbf{E}[|X(t)|^2]$ (under the joint law of the Brownian noise and of the random drift) is of order $t\sqrt{\log t}$ for $t \to \infty$, up to multiplicative loglog corrections. This proves a conjecture of B. Tóth and B. Valkó [TV12] and, in a broader perspective, it is the first proof of the $\sqrt{\log t}$- *superdiffusivity phenomenon* conjectured to occur in a large class of (self-)interacting diffusive systems in dimension $d = 2$ (see the discussion below).

To put the model and the result into context, let us observe first that the vector field ω is divergence-free and that its law is translation-invariant and ergodic. Brownian diffusions in ergodic, divergence-free vector fields have been introduced in the physics and mathematics literature as a (toy) model for a tracer particle evolving in an incompressible turbulent flow. If the energy spectrum of the vector field (i.e. the Fourier transform $e(p)$ of the trace of the covariance matrix $R(x-y) = \{\mathbb{E}(\omega_a(x)\omega_b(y))\}_{a,b\leq d}$, with d the space dimension) satisfies the integrability condition[2]

$$\int_{\mathbb{R}^d} \frac{e(p)}{|p|^2} \mathrm{d}p < \infty, \tag{2.3}$$

the behaviour of the particle is known to be diffusive on large scales [KO01, KLO12] (see also [KT17, Tó18] for analogous results obtained, via different methods, in the discrete setting of random walks in divergence-free random environments). In the robustly superdiffusive case, where the integral in (2.3) has a power-law divergence for small p, it turns out that $\mathbf{E}[|X(t)|^2]$ grows like t^ν for some $\nu > 1$ [KO02]. The case under consideration in this work instead, where $d = 2$ and ω is the curl of the GFF, is precisely at the boundary between the diffusive and the super-diffusive case: $e(p)$ is essentially constant for p small, the integral (2.3) diverges logarithmically at small momenta and logarithmic corrections to diffusivity are expected.

Logarithmic corrections to diffusivity in two-dimensional out-of-equilibrium systems have a long history. The seminal works [AW67, WAG71] of Alder and Wainwright lead the way, in that they predicted that the velocity auto-correlation of a tracer particle diffusing in a fluid behaves like $t^{-d/2}$ in dimension $d \geq 3$ and like $1/(t\sqrt{\log t})$ in the critical dimension $d = 2$. This translates into the fact that, in two dimensions, the mean square displacement $\mathbf{E}[|X(t)|^2]$ of the particle should grow like $tD(t)$ with

$$D(t) \approx \sqrt{\log t} \qquad \text{as} \quad t \to \infty. \tag{2.4}$$

The quantity $D(t)$ takes the name of (bulk) diffusion coefficient. The same prediction was obtained by Forster, Nelson and Stephen [FNS77] via Renormalization Group methods. Subsequently, anomalous logarithmic corrections as in (2.4) were conjectured to occur for several other two-dimensional (self-)interacting diffusions, including self-repelling random walks and Brownian polymers[3] [APP83, OP83, PP87, TV12], lattice gas models [LRY05], the diffusion (2.1) in the curl of the $2d$ GFF [TV12] and, more recently, the two-dimensional Anisotropic KPZ equation ($2d$ AKPZ) [CET20]. We emphasize that in all of these cases, it is known or conjectured that the analogous models behave diffusively ($D(t) \sim 1$) in dimension $d \geq 3$ (see for instance [HTV12] for the self-interacting random walks and Brownian polymers).

[2]The integral in the l.h.s. is known as “Péclet number” [KLO12].

[3]There has been some controversy in the physics literature as to the value of the exponent ζ of the logarithm in (2.4) for self-repelling random walks. The values $\zeta = 0.4$ and $\zeta = 1$ have been proposed [APP83, OP83, PP87], in addition to the $\zeta = 1/2$ prediction [TV12] based on the Alder-Wainwright argument.

From a rigorous viewpoint, results available so far fall short of the conjecture (2.4). Until recently, the best estimates obtained can be summarised into bounds of the form

$$\log\log t \lesssim D(t) \lesssim \log t \tag{2.5}$$

(see [TV12] for $2d$ self-repelling Brownian polymers and for the SDE (2.1), and [LRY05] for two-dimensional lattice fluids). More recently, two of the authors together with D. Erhard proved in [CET20] that, for the $2d$ AKPZ equation, one has

$$(\log t)^a \lesssim D(t) \lesssim (\log t)^{1-a} \tag{2.6}$$

for some sufficiently small $a > 0$; after the present work was completed, in a second version of [CET20] the result for the $2d$ AKPZ equation has been also improved to $a = 1/2$. (All the above cited results have been shown in the sense of Laplace transform.)

For the SDE (2.1) under consideration in the present work, we establish for the first time the conjectured behaviour (2.4), up to corrections that are polynomial in $\log\log t$ (see Theorem 2.2.2 below). The result holds again in the sense of Laplace transform - see, however, Remark 2.2.3 for its implications in real time.

Our argument is based on an iterative analysis of the resolvent of the generator of the Markov process given by the environment seen from the particle (see (2.9) below). This is inspired by the method employed by H.-T. Yau [Yau04] to prove $(\log t)^{2/3}$ corrections to the diffusivity of the two-dimensional Asymmetric Simple Exclusion Process ($2d$ ASEP) and, more closely, by the techniques developed in [CET20] to determine (2.6) for the $2d$ AKPZ equation. Note that the exponent $2/3$ of the logarithmic corrections of $2d$ ASEP is different from the exponent $1/2$ in (2.4), reflecting the fact that the two models belong to two different universality classes, as emphasized already in [LRY05, TV12]. From a technical point of view, a crucial difference between the two models is that for $2d$ ASEP the iterative method in [Yau04] provides, at each step k of the recursion, upper/lower bounds for $D(t)$ of the form $(\log t)^{\nu_k}$, with ν_k converging exponentially fast to $2/3$ as $k \to \infty$. In our case, on the other hand, at step k the method naturally provides lower (resp. upper) bounds of order $(\log\log t)^k/k!$ (resp. $k!\log t/(\log\log t)^k$) and we have to run the iteration for a number of steps of order $k = k(t) \approx \log\log t$ (instead of $k(t) \approx \log\log\log t$ as in [Yau04]) to reach the final result. As a consequence, in contrast with [Yau04], we cannot afford to lose a multiplicative constant at each step of the iteration (such multiplicative constants are responsible for the sub-optimal result (2.6) in the first version of [CET20]), and a much finer analysis of the resolvent is needed. Further, we get a significantly sharper control of sub-leading corrections to $D(t)$ with respect to $2d$ ASEP, namely, a multiplicative correction that is polynomial in $\log\log t$ (see Remark 2.2.3), to be compared with the corrections of order $\exp((\log\log\log t)^2)$ for $2d$ ASEP [Yau04].

Organization of the article

The rest of this work is organized as follows. In Section 2.2, we rigorously define the model (2.1) and state the main result. In Section 2.3, we introduce the main tools: we recall the generator of the environment seen from the particle process and we describe the space on which it acts. Section 2.4 is devoted to the analysis of the generator and the derivation of the crucial recursive bounds, while in Section 2.5, the proof of the main result is given. At last, in Appendix 2.A, we collect some technical estimates needed in Sections 2.4 and 2.5.

2.2 The model and main result

The Brownian diffusion in the curl of the 2-dimensional Gaussian Free Field is the stochastic process $t \mapsto X(t) \in \mathbb{R}^2$ given by the solution of the SDE (2.1) where $B(t)$ is a standard two-dimensional Brownian motion and

$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x))$$

is a smooth, divergence-free, random vector field on $\mathbb{R}^2$, given by the curl of (a smoothened version of) the two-dimensional Gaussian free field. To be more precise, let us introduce the following assumption which will be in place throughout the paper.

Assumption 2.2.1. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a radially symmetric bump function, i.e. a function such that there exists $U : \mathbb{R}^2 \to \mathbb{R}$ which is a smooth function in $C^\infty(\mathbb{R}^2)$, radially symmetric, decaying sufficiently (say, exponentially) fast at infinity and such that $\int_{\mathbb{R}^2} U(x)\mathrm{d}x = 1$, for which*

$$V = U * U\,. \tag{2.7}$$

Let V satisfy Assumption 2.2.1 and U be such that (2.7) holds. Then, ω is a centred Gaussian field whose law $\mathbb{P}$ (and corresponding expectation $\mathbb{E}$), is defined as follows. Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be the two-dimensional Gaussian Free Field convoluted with U, i.e. the centred Gaussian field with covariance

$$\mathbb{E}(\xi(x)\xi(y)) = V * g(x-y)\,, \qquad \text{for all } x,\, y \in \mathbb{R}^2$$

where $g(x) = -\log|x|$. Then, we define ω as the curl of the scalar field ξ, i.e. as in (2.2), which clearly satisfies for all $x,\, y \in \mathbb{R}^2$

$$\mathbb{E}(\omega_\ell(x)) = 0, \quad \mathbb{E}(\omega_k(x)\omega_\ell(y)) = -\tilde{\partial}_k\tilde{\partial}_\ell V * g(x-y), \quad k, \ell = 1, 2, \tag{2.8}$$

for $\tilde{\partial}_1 := \partial_{x_2}$ and $\tilde{\partial}_2 := -\partial_{x_1}$. Note that, while convolving the full-plane Gaussian free field with U is a somewhat formal operation (because the field is only defined up to a constant), the derivatives of the convolved field and therefore ω are (pointwise) defined without any ambiguity and are smooth with respect to x.

It is well known [KLO12, Chapter 11] that, since ω sampled from $\mathbb{P}$ is divergence-free, translation invariant and ergodic, the law $\mathbb{P}$ is stationary for the Markov process of the environment seen from the particle, i.e. the time-evolving field $t \mapsto \omega(t, \cdot)$ given by

$$\omega(t, x) \stackrel{\text{def}}{=} \omega(X(t) + x)\,, \qquad x \in \mathbb{R}^2\,. \tag{2.9}$$

2.2.1 Main result

Our main result is a sharp estimate on the super-diffusivity of the process X. For $t > 0$, let $\mathbf{E}(|X(t)|^2)$ denote the mean square displacement of X at time t - the expectation being taken with respect to the joint randomness of the vector field ω and of the Brownian noise B in (2.1). Let us remark that $\mathbf{E}(X(t)) = 0$ because the law of the environment is symmetric and $X(0) = 0$.

Throughout the present article we will be working with its Laplace transform, given by

$$D(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}\mathbf{E}(|X(t)|^2)\mathrm{d}t, \quad \lambda > 0. \tag{2.10}$$

Note that if in (2.1) there were no drift, one would trivially have $X(t) = B(t)$ so that $\mathbf{E}(|X(t)|^2) = 2t$ and $D(\lambda) = 2\lambda^{-2}$. In [TV12], it was conjectured that

$$D(\lambda) \overset{\lambda\to 0}{\approx} \lambda^{-2}\sqrt{|\log\lambda|}, \tag{2.11}$$

corresponding in real time to

$$\mathbf{E}(|X(t)|^2) \overset{t\to\infty}{\approx} t\sqrt{\log t}$$

(the diffusion coefficient mentioned in the abstract is $D(t) = t^{-1}\mathbf{E}(|X(t)|^2)$. From a rigorous point of view, in the aforementioned work it was proved that

$$C_1\lambda^{-2}\log|\log\lambda| \le D(\lambda) \le C_2\lambda^{-2}|\log\lambda|$$

for some positive constants C_1, C_2, for sufficiently small λ. In this work, we establish the conjecture (2.11) in full.

Theorem 2.2.2. *For every $\varepsilon > 0$ there exists constants $C_\pm(\varepsilon)$ such that, for every $0 < \lambda < 1$,*

$$C_-(\varepsilon)\big(\log|\log\lambda|\big)^{-1-\varepsilon} \le \lambda^2\frac{D(\lambda)}{\sqrt{|\log\lambda|}} \le C_+(\varepsilon)\big(\log|\log\lambda|\big)^{1+\varepsilon}. \tag{2.12}$$

where D is defined according to (2.10).

The exponent $1+\varepsilon$ in the sub-dominant corrections can presumably be improved by some additional technical work, but we do not pursue this here. The constants $C_\pm(\varepsilon)$ implicitly depend also on the choice of bump function V.

Remark 2.2.3. By a well-established argument (see [QV08]) the upper bound in (2.12) implies an upper bound for the diffusivity in real time of the form

$$\mathbf{E}(|X(t)|^2) \le O\left(t\sqrt{\log t}\,(\log\log t)^{1+\varepsilon}\right).$$

Deducing a pointwise (in time) lower bound on $\mathbf{E}(|X(t)|^2)$ from the behaviour for $\lambda \to 0$ of the Laplace transform is much more delicate. That said, one can easily get (applying for instance [BGT89, Theorem 1.7.1]) the following

$$\limsup_{t\to\infty}\frac{\mathbf{E}(|X(t)|^2)}{t\sqrt{\log t}\,(\log\log t)^{-1-\varepsilon}} > 0\,.$$

2.3 Preliminaries

By rotation invariance, one has $\mathbf{E}(|X(t)|^2) = \mathbf{E}(X_1(t)^2 + X_2(t)^2) = 2\mathbf{E}(X_1(t)^2)$, and we write

$$X_1(t) = B_1(t) + \int_0^t \phi(\omega_s)\mathrm{d}s =: B_1(t) + F_1(t) \tag{2.13}$$

where $t \mapsto \omega_t$ is the environment seen from the particle (recall (2.9)), and

$$\phi(\omega) \overset{\text{def}}{=} \omega_1(0)\,. \tag{2.14}$$

Recall that $\mathbf{E}X_1(t) = 0$. The first term on the r.h.s. of (2.13) has variance t, so to prove (2.12), it is sufficient to show

$$\frac{C_-(\varepsilon)}{\big(\log|\log\lambda|\big)^{1+\varepsilon}} \le \frac{\lambda^2}{\sqrt{|\log\lambda|}}\int_0^\infty e^{-\lambda t}\mathbf{E}[F_1(t)^2]\mathrm{d}t \le C_+(\varepsilon)\big(\log|\log\lambda|\big)^{1+\varepsilon}. \tag{2.15}$$

The starting point in the study of $\mathbf{E}[F_1(t)^2]$ is the understanding of the environment process $t \mapsto \omega_t$. As argued in [TV12], this is a Markov process, whose generator will be denoted by $\mathcal{G}$, on the Fréchet space of C^∞, divergence-free two-dimensional vector fields with derivatives growing slower than any power at infinity. As the field is stationary, ergodic and divergence-free, the probability measure $\mathbb{P}$ is stationary for the environment process [KLO12, Chapter 11]. This ensures that, as in [CES21, Lemma 5.1], we have

$$\tilde{D}(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}\mathbf{E}[F_1(t)^2]\mathrm{d}t = \frac{2}{\lambda^2}\mathbb{E}[\phi(\omega)(\lambda-\mathcal{G})^{-1}\phi(\omega)], \tag{2.16}$$

with ϕ defined in (2.14). Hence, our analysis will focus on the resolvent $(\lambda-\mathcal{G})^{-1}$. Recall that $\mathbb{E}$ is the expectation with respect to the stationary law of the environment.

A first necessary step is to describe how $\mathcal{G}$ acts on elements in $L^2(\mathbb{P})$, for which we need a more accurate description of the latter space. Since $\mathbb{P}$ is Gaussian (and given by the law of the curl of the smoothed Gaussian free field ω), $L^2(\mathbb{P})$ admits a Wiener chaos decomposition which we now briefly describe.

Let H_0 be the set containing constant random variables and H_n be the closure of the span of

$$\left\{\psi = \sum_{j_1,\dots,j_n=1}^{2}\int_{\mathbb{R}^{2n}} f_{\mathbf{j}}(x_{1:n}) : \prod_{l=1}^{n}\omega_{j_l}(x_l) : \mathrm{d}x_{1:n}\right\}, \tag{2.17}$$

where $x_{1:n}$ is a short-hand notation for $(x_1,\dots,x_n)$, $\mathbf{j} \stackrel{\text{def}}{=} (j_1,\dots,j_n)$, $:\cdots:$ denotes the Wick product associated to the measure $\mathbb{P}$ and the symmetric functions $f_{\mathbf{j}}$'s are such that

$$\hat{\psi}(p_{1:n}) \stackrel{\text{def}}{=} \sum_{j_1,\dots,j_n=1}^{2}\prod_{k=1}^{n}\tilde{p}_{k,j_k}\hat{f}_{\mathbf{j}}(p_{1:n}) \tag{2.18}$$

satisfies

$$\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}\frac{\widehat{V}(p_i)}{|p_i|^2}|\hat{\psi}(p_{1:n})|^2\mathrm{d}p_{1:n} < \infty\,. \tag{2.19}$$

Above, $\hat{f}_{\mathbf{j}}$ is the Fourier transform of $f_{\mathbf{j}}$, $\tilde{p}_{k,1} = (p_k)_2$ and $\tilde{p}_{k,2} = -(p_k)_1$, with $(p_k)_\ell$ the ℓ-th component, $\ell = 1,2$, of p_k. Also, $\widehat{V}$ is the Fourier transform of the bump function V.

Remark 2.3.1. The intuition behind (2.19) becomes clear upon noting that the components of ω are nothing but the derivatives of the smoothed Gaussian field ξ (see (2.2)). Indeed, by performing an n-fold integration by parts in (2.17), using the fact that, in Fourier space, $\partial_{(x_j)_\ell}$ corresponds to multiplication by $\iota(p_j)_\ell$ (with $\iota = \sqrt{-1}$) and writing the covariance (2.8) of the field ξ in Fourier variables, one sees that (2.19) is just the $L^2(\mathbb{P})$ norm squared of ψ in (2.17).

Remark 2.3.2. It is easy to see that the random variable $\phi(\omega) = \omega_1(0)$ belongs to H_1 and has kernel $\hat{\phi}(p) = p_2$.

Then, by [Nua06, Theorem 1.1.1], $L^2(\mathbb{P})$ can be orthogonally decomposed as

$$L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} H_n \tag{2.20}$$

and the expectation of the scalar product of F, $G \in L^2(\mathbb{P})$ satisfies

$$\mathbb{E}[FG] = \sum_{n=1}^{\infty}\langle\psi_n,\phi_n\rangle\,.$$

Above, ψ_n and ϕ_n are the kernels of the projections of F and G onto H_n and the scalar product appearing at the right hand side is given by

$$\langle \psi_n, \phi_n \rangle \stackrel{\text{def}}{=} n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \frac{\widehat{V}(p_j)}{|p_j|^2} \overline{\hat{\psi}_n(p_{1:n})}\, \hat{\phi}_n(p_{1:n})\, \mathrm{d}p_{1:n}\,. \tag{2.21}$$

Remark 2.3.3. In what follows, we will implicitly identify a random variable F in H_n of the form (2.17) with its kernel $\hat{\psi}_n$ in Fourier space, since this mapping is an isometry from H_n to the set $L^2_{sym}(\mathbb{R}^{2n})$ of symmetric functions on $\mathbb{R}^{2n}$ endowed with the scalar product $\langle \cdot, \cdot \rangle$ in (2.21). In the same spirit, we will identify linear operators acting on $L^2(\mathbb{P})$ with the corresponding linear operators acting on $\bigoplus_n L^2_{sym}(\mathbb{R}^{2n})$, and, with a slight abuse of notation, we will denote them using the same symbol.

Remark 2.3.4. With respect to [TV12, HTV12] we are using different normalization conventions in (2.17) and in the scalar product in (2.21). More specifically, in the conventions of [TV12, HTV12] there would be a factor $1/\sqrt{n!}$ in front of the integral in (2.17) and no factor $n!$ in (2.21). In other words, our kernels ψ_n equal those of [TV12, HTV12] times $1/\sqrt{n!}$. Our conventions are consistent with those of [CET20] and of [Jan97, Nua06]; we refer to these latter references for more details on Wiener chaos analysis.

We are now ready to move back to the analysis of the generator $\mathcal{G}$ of the environment process. As noted in [TV12], $\mathcal{G}$ can be written as

$$\mathcal{G} = -\Delta + \mathcal{A}_+ - \mathcal{A}_+^*,$$

where $-\Delta$ and $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{A}_+ - \mathcal{A}_+^*$ respectively denote the symmetric and anti-symmetric part of $\mathcal{G}$ with respect to $\mathbb{P}$, and $\mathcal{A}_+^*$ is the adjoint of $\mathcal{A}_+$ in $L^2(\mathbb{P})$. The action of $-\Delta$ and $\mathcal{A}$ in Fock space is explicit. First of all, Δ maps the n-th chaos H_n into itself while $\mathcal{A}_+$ (resp. $\mathcal{A}_+^*$) maps H_n into H_{n+1} (resp. H_{n-1}) and can therefore be interpreted as a "creation" (resp. annihilation) operator. Moreover, Δ is diagonal in Fourier space as it acts as a Fourier multiplier on the kernels, while $\mathcal{A}_+$ is not. Adopting the convention in Remark 2.3.3, one has (see also [TV12, Section 2.1])[4]

$$\begin{aligned} \widehat{(-\Delta)\psi}_n(p_{1:n}) &= |\sum_{i=1}^{n} p_i|^2 \widehat{\psi}_n(p_{1:n}) \\ \widehat{\mathcal{A}_+\psi_n}(p_{1:n+1}) &= \iota \frac{1}{n+1} \sum_{i=1}^{n+1} \Big(p_i \times \sum_{j=1}^{n+1} p_j \Big) \widehat{\psi_n}(p_{1:n+1\setminus i})\,, \end{aligned} \tag{2.22}$$

$\iota = \sqrt{-1}$, for $\psi_n \in H_n$. Above and throughout, we denote by $p_{1:n+1\setminus i}$ the collection $p_{1:n} = (p_1, \ldots, p_{n+1})$ with p_i removed. Also, for a, b two vectors in $\mathbb{R}^2$, by $a \times b$ we mean the scalar given by the vertical component of the usual cross product $a \times b$, with a, b viewed as vectors in $\mathbb{R}^3$. Explicitly, $a \times b = |a||b| \sin\theta$, where θ is the angle between a and b.

Remark 2.3.5. Observe that the "Laplacian" $-\Delta$ acting on H_n with $n \geq 2$ is different from the one appearing in [CET20], which acts instead as multiplication by $\sum_{i=1}^{n} |p_i|^2$. This represents a major technical difference which forces us to significantly modify the arguments therein.

[4] If we had adopted the normalization conventions analogous to those of [HTV12], the factor $1/(n+1)$ in (2.22) would be replaced by $1/\sqrt{n+1}$. This is due to the different definition of the kernels, see Remark 2.3.4 above.

At last, in light of the notations and conventions introduced above (see in particular Remark 2.3.3) we rewrite (2.16) in Fock space as

$$\int_0^\infty e^{-\lambda t}\mathbf{E}[F_1(t)^2]\mathrm{d}t = \frac{2}{\lambda^2}\langle \phi, (\lambda - \mathcal{G})^{-1}\phi\rangle, \tag{2.23}$$

where ϕ is the random variable $\phi(\omega)$ in (2.14) which lives in H_1 (see Remark 2.3.2).

2.4 The generator equation and the diffusivity

In order to obtain suitable bounds on the right hand side of (2.23) one should in principle solve the generator equation $(\lambda - \mathcal{G})\psi = \phi$ and then try to evaluate $\langle \phi, \psi\rangle$. While ϕ belongs to the first chaos, the operator $\mathcal{G}$ is not diagonal in the chaos decomposition and finding ψ explicitly is a rather challenging task. A way out was first devised in [LQSY04]. The idea is to truncate the generator $\mathcal{G}$ by defining $\mathcal{G}_n \stackrel{\text{def}}{=} I_{\le n}\mathcal{G}I_{\le n}$, with $I_{\le n}$ the orthogonal projection onto $H_{\le n} \stackrel{\text{def}}{=} \oplus_{k\le n} H_k$ (the chaoses up to order n), and then consider the solution $\psi^{(n)} \in H_{\le n}$ of the truncated generator equation

$$(\lambda - \mathcal{G}_n)\,\psi^{(n)} = \phi. \tag{2.24}$$

The advantage of this procedure is that it provides upper and lower bounds (depending on the parity of n) on (2.23). Indeed, the following lemma, which was first proven in [LQSY04, Lemma 2.1] (and whose proof straightforwardly carries out in the present case) holds.

Lemma 2.4.1. *For every $n \ge 1$, one has*

$$\langle \phi, \psi^{(2n)}\rangle \le \langle \phi, (\lambda - \mathcal{G})^{-1}\phi\rangle = \langle \phi, \psi\rangle \le \langle \phi, \psi^{(2n+1)}\rangle. \tag{2.25}$$

The equation (2.24) coincides with the following hierarchical system of n equations, one for each component $\psi_k^{(n)}$ of $\psi^{(n)}$,

$$\begin{cases} (\lambda - \Delta)\psi_n^{(n)} - \mathcal{A}_+\psi_{n-1}^{(n)} = 0, \\ (\lambda - \Delta)\psi_{n-1}^{(n)} - \mathcal{A}_+\psi_{n-2}^{(n)} + \mathcal{A}_+^*\psi_n^{(n)} = 0, \\ \dots \\ (\lambda - \Delta)\psi_2^{(n)} - \mathcal{A}_+\psi_1^{(n)} + \mathcal{A}_+^*\psi_3^{(n)} = 0, \\ (\lambda - \Delta)\psi_1^{(n)} + \mathcal{A}_+^*\psi_2^{(n)} = \phi\,. \end{cases} \tag{2.26}$$

Since ϕ belongs to the first chaos and different chaoses are orthogonal, in order to estimate the terms at the left and right hand side of (2.25) we only need to know $\psi_1^{(n)}$. The latter in turn can be obtained by solving the system (2.26) iteratively starting from $k = n$ so that we get

$$\langle \phi, \psi^{(n)}\rangle = \langle \phi, \psi_1^{(n)}\rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_n)^{-1}\phi\rangle \tag{2.27}$$

where the self-adjoint operators $\mathcal{H}_j$ are recursively defined as

$$\begin{aligned} &\mathcal{H}_1 \stackrel{\text{def}}{=} 0\,, \\ &\mathcal{H}_{j+1} = \mathcal{A}_+^*(\lambda - \Delta + \mathcal{H}_j)^{-1}\mathcal{A}_+\,, \qquad \text{for } j \ge 1\,. \end{aligned} \tag{2.28}$$

We remark that these operators are positive and leave each chaos invariant - that is $\mathcal{H}_j H_n \subset H_n$, for all j, $n \in \mathbb{N}$.

2.4.1 Operator recursive estimates

In view of (2.25) and (2.27), the proof of Theorem 2.2.2 must entail a good understanding of the operators $\mathcal{H}_j$'s in (2.28). In particular, we need to derive suitable (upper and lower) bounds on them and this is the content of the main result of this section, Theorem 2.4.2. To state it, we need a few preliminary definitions.

For $k \in \mathbb{N}$, $x > 0$ and $z \geq 0$ we define L, LB_k and UB_k as follows

$$\mathrm{L}(x,z) = z + \log(1 + x^{-1}), \tag{2.29}$$

$$\mathrm{LB}_k(x,z) = \sum_{0\leq j\leq k} \frac{(\frac{1}{2}\log \mathrm{L}(x,z))^j}{j!} \quad \text{and} \quad \mathrm{UB}_k(x,z) = \frac{\mathrm{L}(x,z)}{\mathrm{LB}_k(x,z)} \tag{2.30}$$

and for $k \geq 1$, σ_k, as

$$\sigma_k(x,z) = \begin{cases} \mathrm{UB}_{\frac{k-2}{2}}(x,z), & \text{if } k \text{ is even,} \\ \mathrm{LB}_{\frac{k-1}{2}}(x,z), & \text{if } k \text{ is odd.} \end{cases}$$

Note that $\sigma_1 \equiv 1$. All the properties we need on the functions $\mathrm{UB}_k, \mathrm{LB}_k$ are summarized in Lemma 2.A.1. Further let

$$z_k(n) = K_1(n+k)^{2+2\varepsilon} \quad \text{and} \quad f_k(n) = K_2\sqrt{z_k(n)}, \tag{2.31}$$

where K_1, K_2 are absolute constants (chosen sufficiently large, so that (2.43), (2.47) and (2.48) below, hold) and ε is the small positive constant that appears in the statement of Theorem 2.2.2.

Finally, for $k \geq 1$ let $\mathcal{S}_k$ be the operator whose multiplier is σ_k, i.e.

$$\mathcal{S}_k = \begin{cases} f_k(\mathcal{N})\,\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) & \text{if } k \text{ is even,} \\ \frac{1}{f_k(\mathcal{N})}\big(\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) - f_k(\mathcal{N})\big) & \text{if } k \text{ is odd,} \end{cases}$$

where $\mathcal{N}$ is the number operator acting on the n-th chaos as multiplication by n, i.e. $(\mathcal{N}\phi_n) = n\phi_n$ for $\phi_n \in H_n$. We are now ready to state the following theorem.

Theorem 2.4.2. *For any $\varepsilon > 0$, the constants K_1, K_2 in* (2.31) *can be chosen in such a way that the following holds. For $0 < \lambda \leq 1$ and $k \geq 1$, one has the operator bounds*

$$\mathcal{H}_{2k-1} \geq c_{2k-1}\,(-\Delta)\mathcal{S}_{2k-1} \tag{2.32}$$

and

$$\mathcal{H}_{2k} \leq c_{2k}\,(-\Delta)\mathcal{S}_{2k} \tag{2.33}$$

where $c_1 = 1$ and

$$c_{2k} = \frac{\pi}{c_{2k-1}}\left(1 + \frac{1}{k^{1+\varepsilon}}\right), \qquad c_{2k+1} = \frac{\pi}{c_{2k}}\left(1 - \frac{1}{(k+1)^{1+\varepsilon}}\right). \tag{2.34}$$

Remark 2.4.3. A crucial aspect we need to stress is that, as $j \to \infty$, c_{2j} tends to a *finite* constants larger than 1, while c_{2j+1} tends to a *strictly positive* constant smaller than 1.

2.4.2 Generalities about the operators

In this section we collect some preliminary facts and bounds concerning operators in Fock space. In all the statements herein, S will be a *diagonal operator*, meaning that S commutes with $\mathcal{N}$ (that is, it maps the n-th chaos H_n into itself) and is diagonal in the Fourier basis, i.e. it acts in Fourier space as multiplication by a function of the momenta. The Fourier multiplier of S will be denoted $\mathfrak{s}$, and actually $\mathfrak{s}$ is the collection $(\mathfrak{s}_n)_{n\geq 1}$, with $\mathfrak{s}_n$ the Fourier multiplier on H_n. It is understood that $\mathfrak{s}_n$ is a symmetric function of its n arguments.

Lemma 2.4.4. *Let S be a positive diagonal operator and let $\mathfrak{s}$ be its Fourier multiplier. For any element ψ of H_n we can write*

$$\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle = \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Diag}} + \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Off}}$$

where the "diagonal part" is defined as

$$\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Diag}} \stackrel{def}{=} n! \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 \mathfrak{s}_{n+1}(p_{1:n+1}) \left(p_{n+1} \times \sum_{j=1}^{n} p_j \right)^2 \mathrm{d}p_{1:n+1} \tag{2.35}$$

while the "off-diagonal part" is

$$\begin{aligned} \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Off}} \stackrel{def}{=} n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \overline{\hat{\psi}(p_{1:n})} \hat{\psi}(p_{1:n+1\setminus n}) \mathfrak{s}_{n+1}(p_{1:n+1}) \times \\ \times \left(p_{n+1} \times \sum_{i=1}^{n+1} p_i \right) \left(p_n \times \sum_{i=1}^{n+1} p_i \right) \mathrm{d}p_{1:n+1}. \end{aligned} \tag{2.36}$$

Proof. Expanding the inner product using (2.22) we obtain:

$$\langle \mathcal{A}_+ \psi_n, S \mathcal{A}_+ \psi_n \rangle = \frac{(n+1)!}{(n+1)^2} \int_{\mathbb{R}^{2(n+1)}} \prod_{i=1}^{n+1} \frac{\widehat{V}(p_i)}{|p_i|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \left| \sum_{i=1}^{n+1} \hat{\psi}(p_{1:n+1\setminus i}) \Big(p_i \times \sum_{j=1}^{n+1} p_j \Big) \right|^2 \mathrm{d}p_{1:n+1}.$$

The "diagonal" and "off-diagonal" refer to the squared sum. The former is the contribution of the squared summands while the latter comes from all the cross terms. Hence, the diagonal part is

$$\begin{aligned} &\frac{n!}{n+1} \int_{\mathbb{R}^{2(n+1)}} \prod_{i=1}^{n+1} \frac{\widehat{V}(p_i)}{|p_i|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \sum_{i=1}^{n+1} |\hat{\psi}(p_{1:n+1\setminus i})|^2 \Big(p_i \times \sum_{j=1}^{n+1} p_j \Big)^2 \mathrm{d}p_{1:n+1} \\ &= n! \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 \mathfrak{s}_{n+1}(p_{1:n+1}) \Big(p_{n+1} \times \sum_{j=1}^{n} p_j \Big)^2 \mathrm{d}p_{1:n+1}, \end{aligned}$$

where we pulled out the sum and used that $\hat{\psi}$ is symmetric in its arguments. For the off-diagonal part, one follows the same procedure. Since there are in total $n(n+1)$ summands, a factor n is left in front of the integral. □

The next two results will be used in the bounds on the diagonal and off-diagonal parts respectively. In order to appreciate them, note that at the right hand side of both (2.35) and (2.36), there appears the vector product.

Lemma 2.4.5. *Let S be a positive diagonal operator, and let $\mathfrak{s}$ be the associated Fourier multiplier. If for every integer n and for every $p_{1:n} \in \mathbb{R}^{2n}$ with $\sum_{k=1}^{n} p_k \neq 0$*

$$\int_{\mathbb{R}^2} \widehat{V}(q)(\sin\theta)^2 \mathfrak{s}_{n+1}(p_{1:n}, q)\mathrm{d}q \leq \tilde{\mathfrak{s}}_n(p_{1:n}) \tag{2.37}$$

with θ the angle between q and $\sum_{k=1}^{n} p_k$, then for every ψ

$$\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\mathrm{Diag}} \leq \langle \psi, (-\Delta)\tilde{S}\psi \rangle \tag{2.38}$$

where $\tilde{S}$ is the diagonal operator whose Fourier multiplier is $\tilde{\mathfrak{s}}$.

If the inequality in (2.37) *is reversed, then* (2.38) *holds with the reversed inequality.*

Proof. Starting from (2.35) and denoting $q = p_{n+1}$ we get that the left-hand side equals:

$$\begin{aligned} n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 \Big|\sum_{k=1}^{n} p_k\Big|^2 \int_{\mathbb{R}^2} \mathfrak{s}_{n+1}(p_{1:n}, q)\widehat{V}(q)(\sin\theta)^2 \mathrm{d}q\mathrm{d}p_{1:n} \\ \leq n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 \Big|\sum_{k=1}^{n} p_k\Big|^2 \tilde{\mathfrak{s}}_n(p_{1:n}) = \langle \psi, (-\Delta)\tilde{S}\psi \rangle, \end{aligned}$$

where we used that $a \times b = |a||b| \sin\theta$, θ being the angle between a and b, and (2.37). Since every step except the assumption is an equality, the other direction also holds. □

Lemma 2.4.6. *Let S be a diagonal, positive operator with Fourier multiplier $\mathfrak{s}$. If for every integer n and every $p_{1:n} \in \mathbb{R}^{2n}$ one has*

$$\Big|\sum_{i=1}^{n} p_i\Big| \int_{\mathbb{R}^2} \widehat{V}(q) \frac{(\sin\theta)^2 \mathfrak{s}_{n+1}(p_{1:n}, q)}{|q + \sum_{i=1}^{n-1} p_i|} \mathrm{d}q \leq \tilde{\mathfrak{s}}_n(p_{1:n})$$

with θ the angle between q and $\sum_{i=1}^{n} p_i$, then for every $n \in \mathbb{N}, \psi \in H_n$ one has

$$|\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\mathrm{Off}}| \leq n \langle \psi, (-\Delta)\tilde{S}\psi \rangle,$$

with $\tilde{S}$ the diagonal operator of Fourier multiplier $\tilde{\mathfrak{s}}$.

Proof. We start by bounding the left-hand side of (2.36) as

$$\begin{aligned} n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})||\hat{\psi}(p_{1:n+1\setminus n})| \mathfrak{s}_{n+1}(p_{1:n+1}) \\ \times \Big|p_{n+1} \times \sum_{k=1}^{n} p_k\Big|\Big|p_n \times \Big(\sum_{k=1}^{n-1} p_k + p_{n+1}\Big)\Big| \mathrm{d}p_{1:n+1} \\ = n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \Phi(p_{1:n+1}) \Phi(p_{1:n-1}, p_{n+1}, p_n) \\ \times \Big|\sum_{k=1}^{n-1} p_k + p_{n+1}\Big|\Big|\sum_{k=1}^{n} p_k\Big| \mathrm{d}p_{1:n+1} \end{aligned} \tag{2.39}$$

where

$$\Phi(p_{1:n+1}) = \frac{|\hat{\psi}(p_{1:n})||p_{n+1} \times \sum_{k=1}^{n} p_k|}{|\sum_{k=1}^{n-1} p_k + p_{n+1}|}.$$

We apply Cauchy-Schwartz and exploit symmetry of ψ to bound (2.39) from above by

$$n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \Phi(p_{1:n+1})^2 |\sum_{k=1}^{n-1} p_k + p_{n+1}|| \sum_{k=1}^{n} p_k | \mathrm{d}p_{1:n+1}.$$

Now set $s_1 = \sum_{k=1}^{n} p_k$, $s_2 = \sum_{k=1}^{n-1} p_k$ and $q = p_{n+1}$, which gives

$$\begin{aligned} & n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \frac{|\hat{\psi}(p_{1:n})|^2 |q \times s_1|^2}{|q + s_2|} |s_1| \mathrm{d}p_{1:n+1} \\ &= n!n \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \left(\frac{\widehat{V}(p_j)}{|p_j|^2} \mathrm{d}p_j \right) |\hat{\psi}(p_{1:n})|^2 |s_1|^3 \int_{\mathbb{R}^2} \mathfrak{s}_{n+1}(p_{1:n}, q) \frac{\widehat{V}(q)(\sin\theta)^2}{|q + s_2|} \mathrm{d}q \\ &\leq n!n \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 |s_1|^2 \tilde{\mathfrak{s}}_n(p_{1:n}) \mathrm{d}p_{1:n} = n\langle \psi, (-\Delta)\tilde{S}\psi \rangle, \end{aligned}$$

which concludes the proof. □

2.4.3 Proof of Theorem 2.4.2

This section is devoted to Theorem 2.4.2. We will first show the lower bound and then the upper bound, both by induction on k. The induction switches from lower to upper bounds and viceversa, as follows: For $k = 1$ the bound (2.32) will be trivial; given (2.32) for $k = 1$ we will deduce (2.33) with $k = 1$, then (2.32) with $k = 2$ and so on.

Proof of the lower bound (2.32). For $k = 1$ (2.32) trivially holds as $\mathcal{H}_1$ is by definition zero while $\mathcal{S}_1$ is non-positive if the constant K_2 in the definition (2.31) is large enough.

We need then to prove (2.32) with $k \geq 1$ and $2k - 1$ replaced by $2k + 1$. Assume by induction that (2.33) holds. Then, we have

$$\mathcal{H}_{2k+1} = \mathcal{A}_+^*(\lambda - \Delta + \mathcal{H}_{2k})^{-1}\mathcal{A}_+ \geq \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}\mathcal{A}_+ . \tag{2.40}$$

For $\psi \in H_n$, we apply Lemma 2.4.4 with $S = (\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}$ and we split

$$\langle \psi, \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}\mathcal{A}_+\psi \rangle \tag{2.41}$$

into diagonal and off-diagonal part. In order to control the former from below, we exploit Lemma 2.4.5 according to which it suffices to consider

$$\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2}{\lambda + |p + q|^2(1 + c_{2k}f_{2k}(n + 1)\mathrm{UB}_{k-1}(\lambda + |p + q|^2, z_{2k}(n + 1)))} \mathrm{d}q, \tag{2.42}$$

where $p = \sum_{i=1}^{n} p_i \neq 0$ and θ is the angle between p and q. Note that the functions f_{2k}, z_{2k} have argument $n+1$ because $\mathcal{A}_+\psi \in H_{n+1}$ but, by (2.31), $f_{2k}(n+1) = f_{2k+1}(n), z_{2k}(n+1) = z_{2k+1}(n)$. To lighten the notation, throughout the proof we will omit the argument n and write z_{2k+1}, f_{2k+1} instead of $z_{2k+1}(n), f_{2k+1}(n)$.

The denominator in (2.42) is upper bounded by

$$c_{2k}f_{2k+1}\left(1 + \frac{1}{f_{2k+1}}\right)(\lambda + |p + q|^2\mathrm{UB}_{k-1}(\lambda + |p + q|^2, z_{2k+1})),$$

as c_{2k}, f_{2k+1} and UB_{k-1} are all larger than one. Thus we can concentrate on

$$\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2}{\lambda+|p+q|^2\mathrm{UB}_{k-1}(\lambda+|p+q|^2, z_{2k+1})}\mathrm{d}q.$$

For this we first apply Lemmas 2.A.2 and 2.A.4 to obtain the lower bound

$$\begin{aligned}&\frac{\pi}{2}\int_{\lambda+|p|^2}^{1}\frac{d\rho}{\rho\,\mathrm{UB}_{k-1}(\rho, z_{2k+1})} - C_{\mathrm{Diag}}\frac{\mathrm{LB}_{k-1}(\lambda+|p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}\\ &\geq \frac{\pi}{2}\int_{\lambda+|p|^2}^{1}\frac{d\rho}{(\rho+\rho^2)\mathrm{UB}_{k-1}(\rho, z_{2k+1})} - C_{\mathrm{Diag}}\frac{\mathrm{LB}_{k}(\lambda+|p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}\,.\end{aligned}$$

From Lemma 2.A.1 we have that the primitive of the integrand is $-2\mathrm{LB}_k(\rho, z_{2k+1})$, so that the last expression equals

$$\begin{aligned}&\pi\mathrm{LB}_k(\lambda+|p|^2, z_{2k+1}) - \pi\mathrm{LB}_k(1, z_{2k+1}) - C_{\mathrm{Diag}}\frac{\mathrm{LB}_{k}(\lambda+|p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}\\ &\geq \pi\mathrm{LB}_k(\lambda+|p|^2, z_{2k+1}) - \pi\frac{f_{2k+1}}{2} - C_{\mathrm{Diag}}\frac{\mathrm{LB}_{k}(\lambda+|p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}\,,\end{aligned}$$

where in the first inequality we need to choose K_2 large enough in (2.31) so that for all k and n,

$$\mathrm{LB}_k(1, z_{2k+1}) \leq \sqrt{\mathrm{L}(1, z_{2k+1})} = \sqrt{\log(2)+z_{2k+1}} \leq \frac{1}{2}f_{2k+1} \tag{2.43}$$

(see also (2.56)). Altogether, the diagonal part of (2.41) is lower bounded as $\langle\psi, (-\Delta)\tilde{S}\psi\rangle$, with

$$\tilde{S} = \left(1+\frac{1}{f_{2k+1}(1)}\right)^{-1}\frac{\pi}{c_{2k}}\left[\frac{\mathrm{LB}_k(\lambda-\Delta, z_{2k+1}(\mathcal{N}))}{f_{2k+1}(\mathcal{N})}\left(1-\frac{C_{\mathrm{Diag}}}{\pi\sqrt{z_{2k+1}(1)}}\right)-\frac{1}{2}\right] \tag{2.44}$$

(in two instances, we have lower bounded $z_{2k+1} = z_{2k+1}(n)$ and $f_{2k+1} = f_{2k+1}(n)$ with the same quantities for $n = 1$).

For the off-diagonal terms in (2.41) we use Lemma 2.4.6 so that, calling $p := \sum_{i=1}^n p_i$ and $p' := \sum_{i=1}^{n-1} p_i$, we have to upper bound

$$n|p|\int_{\mathbb{R}^2}\frac{\widehat{V}(q)(\sin\theta)^2}{(\lambda+|p+q|^2(1+c_{2k}f_{2k+1}\mathrm{UB}_{k-1}(\lambda+|p+q|^2, z_{2k+1})))|p'+q|}\mathrm{d}q\,. \tag{2.45}$$

Thanks to Lemmas 2.A.3 and 2.A.4, applied with $f(x,z) = c_{2k}f_{2k+1}\mathrm{UB}_{k-1}(x,z)$ and $g(x,z) = \frac{1}{c_{2k}f_{2k+1}}\mathrm{LB}_{k-1}(x,z)$, this expression is upper bounded by

$$\begin{aligned}&\frac{n\,C_{\mathrm{off}}}{c_{2k}f_{2k+1}z_{2k+1}}\mathrm{LB}_{k-1}(\lambda+|p|^2, z_{2k+1})\\ &\leq \frac{n\,C_{\mathrm{off}}}{c_{2k}f_{2k+1}z_{2k+1}}\mathrm{LB}_{k}(\lambda+|p|^2, z_{2k+1})\\ &\leq \frac{C_{\mathrm{off}}}{c_{2k}f_{2k+1}}\frac{1}{K_1(2k+1)^{1+\varepsilon}}\mathrm{LB}_{k}(\lambda+|p|^2, z_{2k+1}),\end{aligned} \tag{2.46}$$

where we used monotonicity properties of LB_k, the definition of $z_{2k+1} = z_{2k+1}(n)$ in (2.31) and in particular the fact that

$$\frac{n}{z_{2k+1}(n)} = \frac{n}{K_1(2k+1+n)^{2+2\varepsilon}} \leq \frac{1}{K_1(2k+1+n)^{1+\varepsilon}}.$$

Combining (2.44) and (2.46), together with Lemmas 2.4.5 and 2.4.6, we conclude that $\mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}\mathcal{A}_+$ is lower bounded by

$$(-\Delta)\frac{\pi}{c_{2k}}\left[\frac{\mathrm{LB}_k(\lambda-\Delta, z_{2k+1}(\mathcal{N}))}{f_{2k+1}(\mathcal{N})}A - B\right]$$

where A and B are given by

$$A = \left(1 - \frac{C_{\mathrm{Diag}}}{\pi\sqrt{z_{2k+1}(1)}}\right)\left(1 + \frac{1}{f_{2k+1}(1)}\right)^{-1} - \frac{C_{\mathrm{off}}}{\pi K_1(2k+1)^{1+\varepsilon}}$$
$$B = \frac{1}{2}\left(1 + \frac{1}{f_{2k+1}(1)}\right)^{-1}$$

and thanks to (2.40) the same lower bound holds for $\mathcal{H}_{2k+1}$. Note that, provided the constants K_1, K_2 in (2.31) are large, one has

$$A \geq 1 - \frac{1}{(k+1)^{1+\varepsilon}}, \qquad B \leq 1 - \frac{1}{(k+1)^{1+\varepsilon}}\,. \tag{2.47}$$

Therefore, we have proven (2.32) (with $2k+1$ instead of $2k-1$) with c_{2k+1} given by (2.34). □

Proof of the upper bound (2.33). For $k \geq 1$, again by the induction hypothesis we have

$$\mathcal{H}_{2k} = \mathcal{A}_+^*(\lambda - \Delta + \mathcal{H}_{2k-1})^{-1}\mathcal{A}_+ \leq \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k-1}\mathcal{S}_{2k-1}))^{-1}\mathcal{A}_+\,.$$

We split $\langle\psi, \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k-1}\mathcal{S}_{2k-1}))^{-1}\mathcal{A}_+\psi\rangle$ into diagonal and off-diagonal parts as in Lemma 2.4.4. By Lemma 2.4.5 for the diagonal part we need to upper-bound the integral

$$\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2}{\lambda + |p+q|^2(1 + \frac{c_{2k-1}}{f_{2k}}(\mathrm{LB}_{k-1}(\lambda + |p+q|^2, z_{2k}) - f_{2k}))}\mathrm{d}q$$
$$\leq \frac{f_{2k}}{c_{2k-1}}\int_{\mathbb{R}^2}\frac{\widehat{V}(q)(\sin\theta)^2}{\lambda + |p+q|^2\mathrm{LB}_{k-1}(\lambda + |p+q|^2, z_{2k})}\mathrm{d}q$$

where we used $f_{2k-1}(n+1) = f_{2k}(n)$, the same for z (and we suppressed the argument of both) and, in the second step, exploited the fact that $c_{2k-1} < 1$ and $f_{2k} > 1$. By Lemmas 2.A.2 and 2.A.4, the latter is bounded above by

$$\frac{f_{2k}\pi}{2c_{2k-1}}\left(\int_{\lambda+|p|^2}^1 \frac{\mathrm{d}\rho}{\rho\mathrm{LB}_{k-1}(\rho, z_{2k})} + \frac{C_{\mathrm{diag}}\mathrm{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{\sqrt{z_{2k}}}\right).$$

The integral can be controlled via Lemma 2.A.5, so that

$$\int_{\lambda+|p|^2}^1 \frac{\mathrm{d}\rho}{\rho\mathrm{LB}_{k-1}(\rho, z_{2k})} \leq \int_{\lambda+|p|^2}^1 \frac{\mathrm{d}\rho}{(\rho+\rho^2)\mathrm{LB}_{k-1}(\rho, z_{2k})} + C\frac{\mathrm{UB}_{k-1}(\lambda+|p|^2, z_{2k})}{z_{2k}}$$
$$\leq 2\mathrm{UB}_{k-1}(\lambda+|p|^2, z_{2k}) + C\frac{\mathrm{UB}_{k-1}(\lambda+|p|^2, z_{2k})}{z_{2k}}\,,$$

the last passage being a consequence of Lemma 2.A.1.

For the off-diagonal terms, we argue as in the analysis of (2.45), so that we need to control

$$n|p|\int_{\mathbb{R}^2}\frac{\widehat{V}(q)(\sin\theta)^2}{(\lambda + |p+q|^2(1 + \frac{c_{2k-1}}{f_{2k}}(\mathrm{LB}_{k-1}(\lambda+|p+q|^2, z_{2k}) - f_{2k})))|p'+q|}\mathrm{d}q\,.$$

Once again, we can pull out the factor $\frac{f_{2k}}{c_{2k-1}}$ and apply once more Lemmas 2.A.3 and 2.A.4, this time with $f(x,z) = \mathrm{LB}_{k-1}(x,z)$ and $g(x,z) = \mathrm{UB}_{k-1}(x,z)$. Hence we obtain

$$\frac{f_{2k}}{c_{2k-1}} \frac{n C_{\mathrm{off}} \mathrm{UB}_{k-1}(\lambda+|p|^2), z_{2k})}{z_{2k}} \le \frac{f_{2k}}{c_{2k-1}} \frac{C_{\mathrm{off}} \mathrm{UB}_{k-1}(\lambda+|p|^2, z_{2k})}{K_1(n+2k)^{1+2\varepsilon}}.$$

Collecting these upper bounds and using the fact that $z_{2k}(n) > z_{2k}(1)$, we conclude that $\mathcal{A}_+^*(\lambda - \Delta(1+c_{2k-1}\mathcal{S}_{2k-1}))^{-1}\mathcal{A}_+$ is upper bounded by

$$\frac{\pi}{c_{2k-1}} A'(-\Delta)\mathcal{S}_{2k}$$

where this time, upon choosing K_1 big enough, we have

$$A' = 1 + \frac{C_{\mathrm{Diag}}}{\pi\sqrt{K_1}(2k)^{1+\varepsilon}} + \frac{C}{\pi K_1 (2k)^{2+2\varepsilon}} + \frac{C_{\mathrm{off}}}{\pi K_1(2k)^{1+2\varepsilon}} \le 1 + \frac{1}{k^{1+\varepsilon}}. \tag{2.48}$$

It follows that (2.33) holds with c_{2k} satisfying (2.34).

Let us remark that constants K_1 and K_2 such that (2.43), (2.47) and (2.48) hold for all k, $n \in \mathbb{N}$ clearly exist, so that the proof of Theorem 2.4.2 is concluded. □

2.5 Proof of Theorem 2.2.2

This section is devoted to the proof of Theorem 2.2.2 and shows how to exploit the iterative bounds derived in the previous section. Recall from Section 2.3 that it suffices to prove (2.12) with $D(\lambda)$ replaced by $\tilde{D}(\lambda)$ defined in (2.16).

Proof of Theorem 2.2.2. Let us begin with the upper bound. By Lemma 2.4.1 and (2.23), we have

$$\frac{\lambda^2}{2}\tilde{D}(\lambda) \le \langle \phi, \psi^{(2k+1)}\rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_{2k+1})^{-1}\phi\rangle,$$

for ϕ such that $\hat{\phi}(q) = q_2$ (see Remark 2.3.2), which in turn, by Theorem 2.4.2, is bounded above by

$$\begin{aligned} \langle \phi, &(\lambda - \Delta(1 + c_{2k+1}\mathcal{S}_{2k+1}))^{-1}\phi\rangle \\ &= \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 \mathrm{d}q}{\lambda + |q|^2(1 + \frac{c_{2k+1}}{f_{2k+1}}(\mathrm{LB}_k(\lambda+|q|^2, z_{2k+1}) - f_{2k+1}))} \\ &\le \frac{f_{2k+1}}{c_{2k+1}} \int_{\mathbb{R}^2} \widehat{V}(q) \frac{\mathrm{d}q}{\lambda + |q|^2 \mathrm{LB}_k(\lambda+|q|^2, z_{2k+1})}. \end{aligned} \tag{2.49}$$

Note that, as $\phi \in H_1$, f_{2k+1} and z_{2k+1} are $f_{2k+1}(1)$ and $z_{2k+1}(1)$, that is, are constants depending only on k.

In view of (2.34), we can replace c_{2k+1} with its $k \to \infty$ limit. By Eq. (2.64) in Lemma 2.A.2, (2.49) is controlled, up to a multiplicative absolute constant, by

$$\begin{aligned} f_{2k+1}&\left[\int_\lambda^1 \frac{\mathrm{d}\rho}{\rho \mathrm{LB}_k(\rho, z_{2k+1})} + \frac{\mathrm{UB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}}\right] \\ &\lesssim f_{2k+1}\left[\int_\lambda^1 \frac{\mathrm{d}\rho}{(\rho+\rho^2)\mathrm{LB}_k(\rho, z_{2k+1})} + \frac{\mathrm{UB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}}\right] \\ &\lesssim f_{2k+1}\mathrm{UB}_k(\lambda, z_{2k+1}) \lesssim f_{2k+1}\frac{\mathrm{L}(\lambda,0) + z_{2k+1}}{\mathrm{LB}_k(\lambda, 0)}, \end{aligned} \tag{2.50}$$

where in the first inequality we applied Lemma 2.A.5, in the second Lemma 2.A.1 and in the last the monotonicity of $\mathrm{LB}_k(\cdot, z)$ with respect to z. We now recall that the central limit theorem, applied to Poisson random variables of rate one, gives that

$$\lim_{k\to\infty} \sum_{j=0}^{k} \frac{k^j}{j!} e^{-k} = \frac{1}{2}\,. \tag{2.51}$$

Hence, by choosing

$$k = k(\lambda) = \Big\lfloor \frac{\log \mathrm{L}(\lambda, 0)}{2} \Big\rfloor. \tag{2.52}$$

in (2.51) and recalling the definition of LB_k in (2.30), we have that for λ sufficiently small

$$\frac{e^{-k}}{\mathrm{LB}_k(\lambda, 0)e^{-k}} \lesssim \frac{1}{\sqrt{\mathrm{L}(\lambda, 0)}}\,. \tag{2.53}$$

Plugging this into (2.50) and using the definition of $z_{2k+1} = z_{2k+1}(1)$ and f_{2k+1} in (2.31), we ultimately get the upper bound

$$\lambda^2 \tilde{D}(\lambda) \lesssim (\log \mathrm{L}(\lambda, 0))^{1+\varepsilon} \sqrt{\mathrm{L}(\lambda, 0)}$$

which is the desired one, since

$$\mathrm{L}(\lambda, 0) = \log\left(1 + \frac{1}{\lambda}\right) \overset{\lambda \to 0}{\sim} |\log \lambda|.$$

For the lower bound, we argue similarly. Again by Lemma 2.4.1, we have

$$\frac{\lambda^2}{2} \tilde{D}(\lambda) \geq \langle \phi, \psi^{(2k)} \rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_{2k})^{-1} \phi \rangle,$$

for ϕ such that $\hat{\phi}(q) = q_2$, which in turn, by Theorem 2.4.2, is bounded below by

$$\begin{aligned} \langle \phi, (\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}\phi \rangle &\geq \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 \mathrm{d}q}{\lambda + |q|^2(1 + c_{2k} f_{2k} \mathrm{UB}_{k-1}(\lambda + |q|^2, z_{2k}))} \\ &\gtrsim \frac{1}{f_{2k}} \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 \mathrm{d}q}{\lambda + |q|^2 \mathrm{UB}_{k-1}(\lambda + |q|^2, z_{2k})}. \end{aligned} \tag{2.54}$$

We restrict the integral to the cone where $|q_2|^2 \geq (1/2)|q|^2$ and we get that (2.54) is lower bounded by

$$\frac{C}{f_{2k}} \int_{\mathbb{R}^2} \widehat{V}(q) \frac{\mathrm{d}q}{\lambda + |q|^2 \mathrm{UB}_{k-1}(\lambda + |q|^2, z_{2k})}$$

where now the integral is unrestricted because the integrand depends only on $|q|$. We can now apply again Eq. (2.64) in Lemma 2.A.2, so that overall (2.54) is lower bounded, up to a multiplicative absolute constant, by

$$\begin{aligned} &\frac{1}{f_{2k+1}} \left[\int_\lambda^1 \frac{\mathrm{d}\rho}{\rho \mathrm{UB}_{k-1}(\rho, z_{2k+1})} - \frac{\mathrm{LB}_{k-1}(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\quad \geq \frac{1}{f_{2k+1}} \left[\int_\lambda^1 \frac{\mathrm{d}\rho}{(\rho + \rho^2)\mathrm{UB}_{k-1}(\rho, z_{2k+1})} - \frac{\mathrm{LB}_{k-1}(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\quad \gtrsim \frac{1}{f_{2k+1}} \left[\mathrm{LB}_k(\lambda, z_{2k+1}) - \mathrm{LB}_k(1, z_{2k+1}) - \frac{\mathrm{LB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\quad \gtrsim \frac{1}{f_{2k+1}} \left[\mathrm{LB}_k(\lambda, z_{2k+1}) - f_{2k+1} \right] \end{aligned} \tag{2.55}$$

where in the second inequality we used Lemma 2.A.1, and $\mathrm{LB}_{k-1} \le \mathrm{LB}_k$, while in the last (2.43) and that, for k large enough, $1 - 1/\sqrt{z_{2k+1}}$ is bounded below by a strictly positive constant. Now, the $-f_{2k+1}$ just gives a constant contribution, which can be absorbed by decreasing the value of C if λ is small enough. Using the inequality in (2.53) for k as in (2.52), we see that

$$\mathrm{LB}_k(\lambda, 0) \gtrsim \sqrt{\mathrm{L}(\lambda, 0)}\,,$$

which, together with the definition of f_{2k+1} in (2.31), gives

$$\lambda^2 \tilde{D}(\lambda) \gtrsim (\log \mathrm{L}(\lambda, 0))^{-1-\varepsilon} \sqrt{\mathrm{L}(\lambda, 0)}\,.$$

Hence, (2.15) follows at once and, by (2.13) and the discussion thereafter, so does Theorem 2.2.2. □

2.A Technical estimates

Here we collect some the technical estimates about the integrals involved in the proofs. We also include some of the properties of the functions LB_k and UB_k from [CET20, Lemma C.3].

Lemma 2.A.1. *For $k \in \mathbb{N}$ let $\mathrm{L}, \mathrm{LB}_k$ and UB_k be the functions defined in (2.29) and (2.30). Then, L, LB_k and UB_k are decreasing in the first variable and increasing in the second. For any $x > 0$ and $z \ge 1$, the following inequalities hold*

$$\begin{aligned} &1 \le \mathrm{LB}_k(x,z) \le \sqrt{\mathrm{L}(x,z)}, \\ &1 \le \sqrt{z} \le \sqrt{\mathrm{L}(x,z)} \le \mathrm{UB}_k(x,z) \le \mathrm{L}(x,z). \end{aligned} \tag{2.56}$$

Moreover for any $0 < a < b$, we have

$$\int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{UB}_k(x,z)} = 2(\mathrm{LB}_{k+1}(a,z) - \mathrm{LB}_{k+1}(b,z)), \tag{2.57}$$

$$\int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{LB}_k(x,z)} \le 2(\mathrm{UB}_k(a,z) - \mathrm{UB}_k(b,z)). \tag{2.58}$$

At last, we also have

$$\begin{aligned} &\partial_x \mathrm{L}(x,z) = -\frac{1}{x^2+x}\,, \quad \partial_x \mathrm{LB}_k(x,z) = -\frac{1}{2(x^2+x)\mathrm{UB}_{k-1}(x,z)}\,, \\ &\partial_x \mathrm{UB}_k(x,z) = -\frac{1}{2(x^2+x)\mathrm{LB}_k(x,z)}\left(1 + \frac{(\frac{1}{2}\log \mathrm{L}(x,z))^k}{k!\mathrm{LB}_k(x,z)}\right). \end{aligned} \tag{2.59}$$

Proof. All of these properties were shown in [CET20, Lemma C.3]. For completeness, we add here the proof.

The two chains of inequalities in (2.56) are a direct consequence of the respective definitions. A computation of the partial derivative with respect to the second variable yields the desired monotonicity. Furthermore we have that

$$\partial_x \mathrm{L}(x,z) = -\frac{1}{x^2+x}, \qquad \partial_x \mathrm{LB}_k(x,z) = -\frac{1}{2}\frac{\mathrm{LB}_{k-1}(x,z)}{(x^2+x)\mathrm{L}(x,z)} \tag{2.60}$$

and

$$\begin{aligned}\partial_x \mathrm{UB}_k(x,z) &= -\frac{\mathrm{LB}_k(x,z) - \frac{1}{2}\mathrm{LB}_{k-1}(x,z)}{(x^2+x)(\mathrm{LB}_k(x,z))^2} \\ &= -\frac{1}{2(x^2+x)\mathrm{LB}_k(x,z)}\left[1 + \frac{\frac{(\frac{1}{2}\log \mathrm{L}(x,z))^k}{k!}}{\mathrm{LB}_k(x,z)}\right],\end{aligned} \tag{2.61}$$

which are all strictly negative for any $x > 0$ and $z \geq 1$. The above computation of the partial derivatives moreover reveals that

$$\int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{UB}_k(x,z)} = 2\int_b^a \partial_x \mathrm{LB}_{k+1}(x,z)\mathrm{d}x = 2\left[\mathrm{LB}_{k+1}(a,z) - \mathrm{LB}_{k+1}(b,z)\right],$$

which is (2.57). For (2.58), notice that

$$\begin{aligned}\int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{LB}_k(x,z)} &= \int_b^a \partial_x \mathrm{UB}_k(x,z)\mathrm{d}x + \frac{1}{2}\int_a^b \frac{\mathrm{LB}_{k-1}(x,z)}{(x^2+x)\mathrm{LB}_k(x,z)^2}\mathrm{d}x \\ &\leq \int_b^a \partial_x \mathrm{UB}_k(x,z)\mathrm{d}x + \frac{1}{2}\int_a^b \frac{1}{(x^2+x)\mathrm{LB}_k(x,z)}\mathrm{d}x,\end{aligned}$$

where the last inequality follows from the fact that all the terms are positive and for all x we have $\mathrm{LB}_{k-1}(x,z) \leq \mathrm{LB}_k(x,z)$. Bringing the last term to the left hand side gives the required estimate.

□

Lemma 2.A.2. *Let V be a bump function satisfying Assumption 2.2.1. Let $z > 1$, $f(\cdot,z) : [0,\infty) \mapsto [1,\infty)$ be a strictly decreasing, differentiable function such that*

$$-\frac{f(x)}{x} \leq f'(x) < 0 \text{ for all } x \in \mathbb{R} \tag{2.62}$$

and $g(\cdot,z) : [0,\infty) \mapsto [1,\infty)$ a strictly decreasing function such that $g(x,z)f(x,z) \geq z$. Then, there exists a constant $C_{\mathrm{Diag}} > 0$ such that for all $z > 1$, the following bound holds

$$\left|\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{\lambda + |p+q|^2 f(\lambda+|p+q|^2,z)} - \frac{\pi}{2}\int_{\lambda+|p|^2}^1 \frac{\mathrm{d}\rho}{\rho f(\rho,z)}\right| \leq C_{\mathrm{Diag}}\frac{g(\lambda+|p|^2,z)}{\sqrt{z}} \tag{2.63}$$

where $0 \neq p \in \mathbb{R}^2$, θ is the angle between p and q and it is understood that the second integral is zero if $\lambda + |p|^2 \geq 1$.

Moreover, for $\lambda \leq 1$,

$$\left|\frac{1}{2}\int_{\mathbb{R}^2} \frac{\widehat{V}(q)\mathrm{d}q}{\lambda + |q|^2 f(\lambda+|q|^2,z)} - \frac{\pi}{2}\int_\lambda^1 \frac{\mathrm{d}\rho}{\rho f(\rho,z)}\right| \leq C_{\mathrm{Diag}}\frac{g(\lambda,z)}{\sqrt{z}}. \tag{2.64}$$

Proof. As z is fixed throughout, we suppress the dependence of f and g on it. At first, we use the triangle inequality to split the left hand side of (2.63) into

$$\left|\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{\lambda + |p+q|^2 f(\lambda+|p+q|^2)} - \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{(\lambda + |p+q|^2)f(\lambda+|p+q|^2)}\right| \tag{2.65}$$

$$+\left|\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{(\lambda + |p+q|^2)f(\lambda+|p+q|^2)} - \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{(\lambda + |p|^2+|q|^2)f(\lambda+|p|^2+|q|^2)}\right| \tag{2.66}$$

$$+\left|\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin\theta)^2 \mathrm{d}q}{(\lambda+|p|^2+|q|^2)f(\lambda+|p|^2+|q|^2)} - \frac{\pi}{2}\int_{\lambda+|p|^2}^{1} \frac{\mathrm{d}\rho}{\rho f(\rho)}\right|. \tag{2.67}$$

We will bound these three terms separately. For the first, we re-write it as

$$\begin{aligned}\Big|\lambda\int_{\mathbb{R}^2} &\frac{\widehat{V}(q)(\sin\theta)^2(f(\lambda+|p+q|^2)-1)\mathrm{d}q}{(\lambda+|p+q|^2 f(\lambda+|p+q|^2))(\lambda+|p+q|^2)f(\lambda+|p+q|^2)}\Big| \\ &\le \lambda\int_{\mathbb{R}^2} \frac{\mathrm{d}q}{(\lambda+|p+q|^2 f(\lambda+|p+q|^2))(\lambda+|p+q|^2)}.\end{aligned}$$

The latter can be further split into two parts, corresponding to $|p+q|\le|p|$ and $|p+q|>|p|$. In the first case $f(\lambda+|p+q|^2)\ge f(\lambda+|p|^2)$ and thus we obtain the upper bound

$$\begin{aligned}&\frac{\lambda}{f(\lambda+|p|^2)}\int_{\mathbb{R}^2}\frac{\mathrm{d}q}{(\frac{\lambda}{f(\lambda+|p|^2)}+|p+q|^2)(\lambda+|p+q|^2)} \\ &\le \frac{\lambda}{f(\lambda+|p|^2)}\left(\int_{\mathbb{R}^2}\frac{\mathrm{d}q}{(\frac{\lambda}{f(\lambda+|p|^2)}+|p+q|^2)^2}\right)^{\frac12}\left(\int_{\mathbb{R}^2}\frac{\mathrm{d}q}{(\lambda+|p+q|^2)^2}\right)^{\frac12} \\ &= \frac{C\lambda}{f(\lambda+|p|^2)}\frac{f(\lambda+|p|^2)^{\frac12}}{\lambda} = \frac{C}{\sqrt{f(\lambda+|p|^2)}} \le C\frac{g(\lambda+|p|^2)}{\sqrt{z}},\end{aligned}$$

for some positive constant C. For the other case we use that $f(\lambda+|p+q|^2)\ge\frac{z}{g(\lambda+|p|^2)}$. Applying the same steps as above we get an upper bound of the form

$$\frac{C\sqrt{g(\lambda+|p|^2)}}{\sqrt{z}} \le \frac{Cg(\lambda+|p|^2)}{\sqrt{z}},$$

which holds as $g\ge1$.

Now we look at (2.66). First note that the restriction of each integral to the region $|q+p|<|p|$ has an upper bound of the desired form. Indeed for the first integral we can use $(\sin\theta)^2\le\frac{|p+q|^2}{|p|^2}$ (which holds for any q_1 and q_3 by elementary Euclidean geometry) to obtain

$$\begin{aligned}&\int_{|p+q|<|p|}\frac{\widehat{V}(q)(\sin\theta)^2\mathrm{d}q}{(\lambda+|p+q|^2)f(\lambda+|p+q|^2)} \\ &\qquad\le |p|^{-2}\int_{|p+q|<|p|}\frac{\mathrm{d}q}{f(\lambda+|p+q|^2)} \le \frac{C}{f(\lambda+|p|^2)} \le \frac{Cg(\lambda+|p|^2)}{\sqrt{z}}.\end{aligned}$$

For the second integral in (2.66), we can bound from above $|\sin\theta|\le1$, the denominator from below by $|p|^2 f(\lambda+5|p|^2)$ and notice that the area of integration is of order $|p|^2$.

As for the region $|q+p|\ge|p|$, define $h(x)=xf(x)$. By (2.62), $|h'(x)|\le2|f(x)|$, therefore

$$\begin{aligned}|h(\lambda+|p+q|^2)-&h(\lambda+|p|^2+|q|^2)| \\ &\le 2||p+q|^2-|p|^2-|q|^2|f(\min(\lambda+|p+q|^2,\lambda+|p|^2+|q|^2)),\end{aligned}$$

since f is positive and decreasing. Therefore, we get

$$\begin{aligned}&\int_{|p+q|\geq|p|}\frac{\widehat{V}(q)(\sin\theta)^2|h(\lambda+|p+q|^2)-h(\lambda+|p|^2+|q|^2)|\mathrm{d}q}{(\lambda+|p+q|^2)(\lambda+|p|^2+|q|^2)f(\lambda+|p+q|^2)f(\lambda+|p|^2+|q|^2)}\\&\lesssim\int_{|p+q|\geq|p|}\frac{\widehat{V}(q)(\sin\theta)^2|p||q||\cos\theta|\mathrm{d}q}{(\lambda+|p+q|^2)(\lambda+|p|^2+|q|^2)f(\max(\lambda+|p+q|^2,\lambda+|p|^2+|q|^2))}\\&\lesssim|p|\int_{|p+q|\geq|p|}\frac{|q|\mathrm{d}q}{(\lambda+|p+q|^2)(\lambda+|p|^2+|q|^2)f(\lambda+2|p|^2+2|q|^2)}\\&\lesssim\frac{g(\lambda+|p|^2)}{z}|p|\int_{|p+q|\geq|p|}\frac{|q|\mathrm{d}q}{(\lambda+|p|^2+|q|^2)(\lambda+|p+q|^2)}\leq C\frac{g(\lambda+|p|^2)}{\sqrt{z}}\end{aligned}$$

as can be seen by further splitting the last integral into the region where $|q|\geq 2|p|$ and the complementary one, and using $z>1$. This concludes the estimate of the second term.

For (2.67), we split the first integral into two regions, one such that $|q|^2\geq 1-(\lambda+|p|^2)$ and the other given by its (possibly empty) complement. Note that on the first $\lambda+|p|^2+|q|^2\geq 1$. Therefore, the integral can be bounded above by

$$\frac{1}{z}\int_{|q|^2\geq1-(\lambda+|p|^2)}\widehat{V}(q)g(\lambda+|p|^2+|q|^2)\mathrm{d}q\leq\frac{g(\lambda+|p|^2)}{z}\int_{\mathbb{R}^2}\widehat{V}(q)\mathrm{d}q=C\frac{g(\lambda+|p|^2)}{z}\,.$$

To treat the second, since $\widehat{V}(\cdot)$ is smooth and rotationally invariant, there is a constant C such that $|\widehat{V}(q)-\widehat{V}(0)|<C|q|^2$ for $|q|\leq 1$. We can now write the remaining integral as

$$\begin{aligned}&\int_{|q|^2<1-(\lambda+|p|^2)}\frac{(\sin\theta)^2\mathrm{d}q}{(\lambda+|p|^2+|q|^2)f(\lambda+|p|^2+|q|^2)}\\&\qquad+\int_{|q|^2<1-(\lambda+|p|^2)}\frac{(\widehat{V}(0)-\widehat{V}(q))(\sin\theta)^2\mathrm{d}q}{(\lambda+|p|^2+|q|^2)f(\lambda+|p|^2+|q|^2)}\,.\end{aligned}$$

By passing to polar coordinates and setting $\rho=\lambda+|p|^2+|q|^2$, the first summand can be immediately seen to equal the second integral in (2.67). The second summand instead can be controlled via

$$\begin{aligned}&\int_{|q|^2<1-(\lambda+|p|^2)}\frac{|\widehat{V}(0)-\widehat{V}(q)|(\sin\theta)^2\mathrm{d}q}{(\lambda+|p|^2+|q|^2)f(\lambda+|p|^2+|q|^2)}\\&\qquad\leq\frac{Cg(\lambda+|p|^2)}{z}\int_{|q|^2<1-(\lambda+|p|^2)}\frac{|q|^2\mathrm{d}q}{\lambda+|p|^2+|q|^2}\leq C\frac{g(\lambda+|p|^2)}{z}.\end{aligned}$$

Thus, collecting all the estimates obtained so far, (2.63) follows at once.

Finally, to see (2.64), we recall that (2.63) holds uniformly for $p\neq 0$. Letting $p\to 0$, the second integral and the r.h.s. of (2.63) tend to the analogous quantities in (2.64). As for the first integral in (2.63), for $p\to 0$ the integral over $|q|$ and θ factorizes, and we get the first integral in (2.64) times $1/2$ (coming from the average of $(\sin\theta)^2$). □

Lemma 2.A.3. *Let the assumptions of Lemma 2.A.2 be in place. Then, there exists a constant $C_{\mathrm{off}}>0$ such that, for every q_1,q_2,*

$$|q_1|\int_{\mathbb{R}^2}\frac{\widehat{V}(q_3)(\sin\theta)^2\mathrm{d}q_3}{(\lambda+|q_1+q_3|^2f(\lambda+|q_1+q_3|^2))|q_2+q_3|}\leq C_{\mathrm{off}}\frac{g(\lambda+|q_1|^2)}{z}$$

with θ the angle between q_1 and q_3.

Proof. Throughout the proof the constant C appearing in the bounds is independent of q_1, q_2 and q_3 and might change from line to line.

We split $\mathbb{R}^2$ into three regions, $\Omega_1 = \{q_3 \colon |q_1+q_3| < \frac{|q_1|}{2}\}$, $\Omega_2 = \{q_3 \colon |q_2+q_3| < \frac{|q_1|}{2}\}$ and $\Omega_3 = \mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Note that Ω_1 and Ω_2 might not be disjoint, but this is no issue as we are proving an upper bound.

In Ω_1, we exploit the monotonicity of f to bound $f(\lambda+|q_1+q_3|^2) \geq f(\lambda+\frac{1}{4}|q_1|^2)$. Moreover, we estimate $(\sin\theta)^2 \leq \frac{|q_3+q_1|^2}{|q_1|^2}$ and $\widehat{V}$ by a constant to get

$$
\begin{aligned}
|q_1| \int_{\Omega_1} & \frac{\widehat{V}(q_3)(\sin\theta)^2 \mathrm{d}q_3}{(\lambda+|q_1+q_3|^2 f(\lambda+|q_1+q_3|^2))|q_2+q_3|} \\
&\leq C|q_1|^{-1} \int_{\Omega_1} \frac{|q_1+q_3|^2 \mathrm{d}q_3}{(\lambda+|q_1+q_3|^2 f(\lambda+\frac{1}{4}|q_1|^2))|q_2+q_3|} \\
&\leq C\frac{|q_1|^{-1}}{f(\lambda+\frac{1}{4}|q_1|^2)} \int_{\Omega_1} \frac{\mathrm{d}q_3}{|q_2+q_3|} \\
&\leq \frac{C}{f(\lambda+\frac{1}{4}|q_1|^2)} \leq \frac{C}{f(\lambda+|q_1|^2)} \leq C\frac{g(\lambda+|q_1|^2)}{z} \,,
\end{aligned}
$$

the last step from the third to the fourth line being a consequence of the fact that, on Ω_1, $\frac{|q_1|}{2} < |q_3| < \frac{3}{2}|q_1|$.

For Ω_2 we estimate the sine differently, i.e.

$$
(\sin\theta)^2 \leq \frac{4|q_3+q_1|^2}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)}, \tag{2.68}
$$

which holds as, for $|q_2| \leq 2|q_1|$ this is just a weaker estimate than the previous one, while for $|q_2| \geq 2|q_1|$ we claim that, in the region Ω_2 the right hand side is always greater or equal to 1 (and thus the inequality holds as well). Indeed, notice that since $|q_2| \geq 2|q_1|$, we have

$$
\frac{4|q_3+q_1|^2}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)} = 16\frac{|q_3+q_1|^2}{|q_2|^2} \,.
$$

Assume by contradiction that $|q_3+q_1| < \frac{1}{4}|q_2|$. Then

$$
|q_3+q_2| \geq |q_2-q_1| - |q_1+q_3| > |q_2| - |q_1| - \frac{1}{4}|q_2| \geq \frac{1}{4}|q_2|
$$

where in the last step we used once again that $|q_2| \geq 2|q_1|$. Now, on Ω_2, $|q_2+q_3| < \frac{1}{2}|q_1|$, so that, in conclusion

$$
\frac{1}{4}|q_2| < |q_3+q_2| < \frac{1}{2}|q_1| \leq \frac{1}{4}|q_2|
$$

which is a contradiction. Hence, $|q_3+q_1| \geq \frac{1}{4}|q_2|$, from which (2.68) follows.

Plugging (2.68) into the integral we get

$$
\begin{aligned}
|q_1| \int_{\Omega_2} & \frac{\widehat{V}(q_3)(\sin\theta)^2 \mathrm{d}q_3}{(\lambda+|q_1+q_3|^2 f(\lambda+|q_1+q_3|^2))|q_2+q_3|} \\
&\leq C\frac{|q_1|}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)} \int_{\Omega_2} \frac{\mathrm{d}q_3}{|q_2+q_3| f(\lambda+|q_1+q_3|^2)}.
\end{aligned}
$$

Now we use the monotonicity of f to bound the previous integral from above by

$$\begin{aligned} &C\frac{|q_1|}{\Big(|q_1|^2 \vee (\frac{1}{4}|q_2|^2)\Big) f(\lambda + (\frac{3}{2}|q_1| + |q_2|)^2)} \int_{\Omega_2} \frac{dq_3}{|q_2+q_3|} \\ &\qquad = C\frac{|q_1|^2}{\Big(|q_1|^2 \vee (\frac{1}{4}|q_2|^2)\Big) f(\lambda + (\frac{3}{2}|q_1| + |q_2|)^2)}. \end{aligned} \tag{2.69}$$

We now bound this term by maximizing over $|q_2|$. It is easy to see that it is monotonically increasing for $|q_2| < 2|q_1|$. For $|q_2| \geq 2|q_1|$ we will prove that it is monotonically decreasing. Since f satisfies assumption (2.62), for any a, $b \geq 0$ we have

$$\begin{aligned} &\frac{d}{dr}\left(\frac{1}{r^2 f(a+(b+r)^2)}\right) = -\frac{2rf + 2r^2(b+r)f'}{r^4 f^2} \\ &= -\frac{2}{r^3 f^2}(f + r(b+r)f') \leq -\frac{2}{r^3 f}\left(1 - \frac{r(b+r)}{a+(b+r)^2}\right) < 0, \end{aligned}$$

where we suppressed the argument of f and f' because it does not change. Thus, the maximum over q_2 of the right hand side of (2.69) is achieved at $|q_2| = 2|q_1|$ and reads

$$\frac{C}{f(\lambda + (\frac{7}{2}|q_1|)^2)} \leq \frac{Cg(\lambda + (\frac{7}{2}|q_1|)^2)}{z} \leq \frac{Cg(\lambda + |q_1|^2)}{z}\,.$$

We are left to consider the integral over Ω_3. In this case, we first bound the $(\sin\theta)^2 \leq 1$ and then apply the Hölder inequality with exponents $\frac{3}{2}$ and 3, to the two functions $((\lambda + |q_1+q_3|^2) f(\lambda + |q_1+q_3|^2))^{-1}$ and $|q_2+q_3|^{-1}$ with respect to the measure $\widehat{V}(q_3)\mathrm{d}q_3$, so that we obtain

$$\begin{aligned} &|q_1|\int_{\Omega_3} \frac{\widehat{V}(q_3)(\sin\theta)^2 \mathrm{d}q_3}{(\lambda + |q_1+q_3|^2 f(\lambda + |q_1+q_3|^2))|q_2+q_3|} \\ &\leq |q_1|\left(\int_{\Omega_3} \frac{\widehat{V}(q_3)\mathrm{d}q_3}{(\lambda + |q_1+q_3|^2 f(\lambda+|q_1+q_3|^2))^{\frac{3}{2}}}\right)^{\frac{2}{3}} \left(\int_{\Omega_3}\frac{\widehat{V}(q_3)\mathrm{d}q_3}{|q_2+q_3|^3}\right)^{\frac{1}{3}}. \end{aligned} \tag{2.70}$$

The second integral is upper bounded by a constant factor times $|q_1|^{-1}$.

In the first integral of (2.70), we make the change of variables $q = q_1 + q_3$, bound the bump function $\widehat{V}$ by a constant and then pass to polar coordinates, hence we get

$$C\int_{\frac{|q_1|}{2}}^{\infty} \frac{\varrho \mathrm{d}\varrho}{(\lambda + \varrho^2 f(\lambda+\varrho^2))^{\frac{3}{2}}}\,. \tag{2.71}$$

We split the domain of integration into two parts, $\rho^2 > \lambda$ and $\rho^2 \leq \lambda$ (the second one might be empty). In the first, we note that

$$\lambda + \varrho^2 f(\lambda + \varrho^2) \geq \frac{1}{2}(\lambda + \varrho^2) f(\lambda + \varrho^2).$$

Using $f(x) \geq \frac{z}{g(x)}$ we obtain that this part of (2.71) is upper bounded by

$$C\int_{\frac{\sqrt{\lambda}}{2}}^{\infty} \frac{\varrho \mathrm{d}\varrho}{((\lambda+\varrho^2)f(\lambda+\varrho^2))^{\frac{3}{2}}} \leq C\int_{\frac{|q_1|}{2}}^{\infty} \frac{\varrho \mathrm{d}\varrho}{((\lambda+\varrho^2)f(\lambda+4\varrho^2))^{\frac{3}{2}}}$$

$$\leq C\left(\frac{g(\lambda+|q_1|^2)}{z}\right)^{\frac{3}{2}}\int_{\frac{|q_1|}{2}}^{\infty}\frac{\varrho \mathrm{d}\varrho}{(\lambda+\varrho^2)^{\frac{3}{2}}}$$
$$\leq C|q_1|^{-1}\left(\frac{g(\lambda+|q_1|^2)}{z}\right)^{\frac{3}{2}}, \tag{2.72}$$

where in the last step we estimated the integral by dropping λ from the denominator.

We now turn to the second part of the integral, where $\rho^2 \leq \lambda$. We use the following

$$\int_{\frac{|q_1|}{2}}^{\sqrt{\lambda}}\frac{\varrho \mathrm{d}\varrho}{(\lambda+\varrho^2 f(\lambda+\varrho^2))^{\frac{3}{2}}} \leq \frac{1}{f(2(\lambda+|q_1|^2))^{\frac{3}{2}}}\int_{\frac{|q_1|}{2}}^{2\sqrt{\lambda}}\frac{\mathrm{d}\varrho}{\varrho^2} \leq C|q_1|^{-1}\frac{g(\lambda+|q_1|^2)^{\frac{3}{2}}}{z^{\frac{3}{2}}}.$$

In conclusion, plugging these estimates into (2.70), we get that the integral over Ω_3 is upper bounded by

$$C\frac{g(\lambda+|q_1|^2)}{z},$$

and, collecting all the bounds derived so far, the statement follows at once. □

Lemma 2.A.4. *The functions* $\mathrm{UB}_k(\cdot,z)$ *and* $\mathrm{LB}_k(\cdot,z)$ *satisfy the conditions of the previous lemmas.*

Proof. By definition $\mathrm{UB}_k(\cdot,z)\mathrm{LB}_k(\cdot,z) = \mathrm{L}(\cdot,z) \geq z$. From Lemma 2.A.1 we get that $\mathrm{LB}_k(x,z) > 1$ and $\mathrm{UB}_k(x,z) > 1$ for all x and that their derivatives are both negative. Equation (2.59) implies that

$$\partial_x \mathrm{LB}_k(x,z) = -\frac{\mathrm{LB}_{k-1}(x,z)}{2(x^2+x)\mathrm{L}(x,z)} \geq -\frac{\mathrm{LB}_k(x,z)}{x}$$

and

$$\partial_x \mathrm{UB}_k(x,z) \geq -\frac{\mathrm{UB}_k(x,z)}{(x^2+x)\mathrm{L}(x,z)} \geq -\frac{\mathrm{UB}_k(x,z)}{x}$$

which gives (2.62). □

Lemma 2.A.5. *For any* $z \geq 1$, $\lambda \in \mathbb{R}_+$ *and* $p \in \mathbb{R}^2$ *such that* $\lambda + |p|^2 \leq 1$, *we have*

$$\left|\int_{\lambda+|p|^2}^{1}\frac{d\rho}{\rho \mathrm{LB}_k(\rho,z)} - \int_{\lambda+|p|^2}^{1}\frac{d\rho}{(\rho+\rho^2)\mathrm{LB}_k(\rho,z)}\right| \leq \frac{\mathrm{UB}_k(\lambda+|p|^2,z)}{z}.$$

Proof. Note that the difference of integrals equals

$$0 \leq \int_{\lambda+|p|^2}^{1}\frac{d\rho}{(1+\rho)\mathrm{LB}_k(\rho,z)} = \int_{\lambda+|p|^2}^{1}\frac{\mathrm{UB}_k(\rho,z)}{(1+\rho)\mathrm{L}(\rho,z)}d\rho \leq \frac{\mathrm{UB}_k(\lambda+|p|^2,z)}{z}$$

because $\mathrm{UB}_k(\cdot,z)$ is decreasing and $\mathrm{L}(\cdot,z) \geq z$. □

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Chapter 3

Stochastic Burgers equation

Abstract

The Stochastic Burgers equation was introduced in [H. van Beijeren, R. Kutner and H. Spohn, Excess noise for driven diffusive systems, PRL, 1985] as a continuous approximation of the fluctuations of the asymmetric simple exclusion process. It is formally given by

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w} \cdot \nabla(\eta^2) + \nabla \cdot \xi\,,$$

where ξ is d-dimensional space time white noise and $\mathfrak{w}$ is a fixed non-zero vector. In the critical dimension $d = 2$ at stationarity, we show that this system exhibits superdiffusive behaviour: more specifically, its bulk diffusion coefficient behaves like $(\log t)^{\frac{2}{3}}$, in a Tauberian sense, up to $\log\log\log t$ corrections. This confirms a prediction made in the physics literature and complements [G. Cannizzaro, M. Gubinelli, F. Toninelli, Gaussian Fluctuations for the stochastic Burgers equation in dimension $d \geq 2$, CMP, 2024], where the same equation was studied in the weak-coupling regime. Furthermore this model can be seen as a continuous analogue to [H.T. Yau, $(\log t)^{\frac{2}{3}}$ law of the two dimensional asymmetric simple exclusion process, Annals of Mathematics, 2004].

3.1 Introduction

We study the stochastic Burgers equation formally given by

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w} \cdot \nabla(\eta^2) + \nabla \cdot \xi\,, \tag{3.1}$$

where $\eta = \eta(x, t)$ is a scalar field depending on time t and space $x \in \mathbb{R}^d$, with $d \geq 1$, $\mathfrak{w} \in \mathbb{R}^d$ is a fixed vector controlling the strength and direction of the nonlinearity and $\xi = (\xi_1, \ldots, \xi_d)$ is d-dimensional space-time white noise. This equation was introduced in [vBKS85] as a proposed continuum analogue of the fluctuations of driven diffusive systems with one conserved quantity, like the Asymmetric Simple Exclusion Process. In dimension $d = 1$ this equation is equivalent to the space derivative of the Kardar-Parisi-Zhang (KPZ) equation, for which there is a local solution theory, the large scale statistics have been determined and the connection with the discrete models (in particular the Weakly Asymmetric Exclusion Process on $\mathbb{Z}$) is well established. See [BG97, HQ18, GPS20] for works connecting particle and growth models to one-dimensional KPZ. For a study of the bulk diffusivity and similar quantities for $d = 1$ see [BQS11]. In dimension $d \geq 3$ the recent work [CGT24] establishes Gaussian fluctuations at large scales. The analogous result for asymetric simple exclusion processes was proven before in [EMY94, LY97, CLO01]. For dimension $d = 2$ [CGT24] studies weak coupling regime, i.e. the size of the nonlinearity is scaled down while looking at larger and larger scales. In this regime they also find non-trivial Gaussian fluctuations, in the sense that the limiting equation is a stochastic heat equation with modified Laplacian, that depends on the nonlinearity. This result suggests superdiffusivity for the strong coupling case, i.e. when the nonlinearity is not scaled down, but it does not imply it.

Dimension $d = 2$ is of particular interest for several reasons. It is the *critical* dimension in the sense of scaling (and in the sense of regularity structures), as we will further discuss below. It is also the model which should describe the fluctuations of 2d ASEP, for which $(\log t)^{\frac{2}{3}}$ superdiffusivity was shown in [Yau04]. In general, bulk diffusion coefficients have been conjectured to diverge either like $(\log t)^{\frac{1}{2}}$ or like $(\log t)^{\frac{2}{3}}$ for a wide variety of models in the critical dimension, see e.g. [TV12, LRY05, WAG71]. Recent successes in proving $(\log t)^{\frac{1}{2}}$ superdiffusivity are [CET23, CHST22, dLFW24]. The distinction between these two classes is characterized by their symmetries. The models in the $(\log t)^{\frac{2}{3}}$ universality class have one direction in which the system behaves superdiffusively, while in the orthogonal direction it behaves diffusively. In our case this direction is given by the vector $\mathfrak{w}$. The models in the $(\log t)^{\frac{1}{2}}$ class, on the other hand, often have some kind of rotational symmetry and behave superdiffusively in every direction.

Let us now situate the present paper with respect to some other works on the topic.

- In [CGT24] the same equation is studied in the weak-coupling limit and the form of the renormalized Laplacian in the limiting linear equation (see Theorem 1.3 in [CGT24]) suggests the $\frac{2}{3}$ exponent.

- The iterative estimation scheme we use is inspired by the methods of [LQSY04, Yau04, CET23, CHST22]. However the expressions of our upper and lower bounds are different, see Theorem 3.4.5 and the definitions in that section. In particular, compared to [CET23, CHST22] we do not absorb the off-diagonal terms into the main term, but instead estimate them separately.

- Our result can be seen as an analog of [Yau04] in the continuum and is also the first critical SPDE for which $(\log t)^{\frac{2}{3}}$ superdiffusivity has been proven, to the best of the authors' knowledge. While our estimates remain technical, we manage to avoid the splitting of sums into various good and bad regions, which have been a main obstacle to replicating the success of [Yau04] to other models. Also, compared to [Yau04], the sub-leading corrections to the $(\log t)^{\frac{2}{3}}$ behavior are of lower order.

- In dimension 2 and greater, this equation falls outside the domain of applicability of both the method of regularity structures developed in [Hai14] and the paracontrolled distribution method of [GIP15, GP18].

- Gaussian Analysis has been succesfully used to understand a variety of critical and super-critical SPDEs and related models via their generator, see [JP24, CG24, GP20].

Finally let us summarize the structure of this paper. In the following Subsection 3.1.1 we rigorously define the equation and the bulk diffusion coefficient and state the main theorem. Then, in Section 3.2, we set up notation and recall elements of Gaussian analysis and the form of the generator. In Section 3.3 we reduce the problem to estimating certain operators on Fock space. Then, in Section 3.4, we prove iterative estimates of these operators, and finally use them in Section 3.5 to prove the main theorem.

3.1.1 Scaling, Regularization and Green-Kubo formula

As it is written, equation (3.1) is ill-posed, since any solution would be too irregular for the nonlinearity to be well-defined. Since we are interested in the large scale behaviour, we regularize the nonlinearity at small scales and then consider larger and larger scales. We do so by introducing Fourier cutoffs inside and outside the nonlinearity:

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot\Pi_1\nabla(\Pi_1\eta)^2 + \nabla\cdot\xi\,, \tag{3.2}$$

where for $a>0$, the Fourier cut-off Π_a acts on η in Fourier by cutting modes larger than a, i.e.

$$\widehat{\Pi_a\eta}(k) \stackrel{\text{def}}{=} \hat{\eta}(k)\mathbf{1}_{|k|\le a}\,.$$

Additionally, in order to avoid integrability issues arising in infinite volume, we study the equation on a large torus $\mathbb{T}^2_N$ of side-length $2\pi N$. We will later let N go to infinity, see Theorem 3.1.1. For equation (3.2) we define the bulk diffusivity using a Green-Kubo formula justified in Appendix 3.B:

$$D^N(t) \stackrel{\text{def}}{=} 1 + \frac{2|\mathfrak{w}|^2}{t}\int_0^t\int_0^s\int_{\mathbb{T}^2_N}\mathbf{E}\left(\Pi_1{:}(\Pi_1\eta)^2{:}(r,x)\Pi_1{:}(\Pi_N\eta)^2{:}(0,0)\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\,, \tag{3.3}$$

where $\mathbf{E}$ denotes the expectation with respect to the stationary solution started from mean-zero white noise, and $:X^2:$ denotes the Wick product, which in this case just subtracts the expectation, i.e. $:X^2: = X^2 - \mathbf{E}(X^2)$. Heuristically, the bulk diffusivity coefficient measures how correlations spread out in space as a function in time.

For convenience, we work on the torus with side-length 2π. To do so, define the rescaled solution $\eta^N : \mathbb{R}_+\times\mathbb{T}^2_1 \to \mathbb{R}$ by

$$\eta^N(t,x) = N\eta\left(N^2t, Nx\right)\,, \tag{3.4}$$

which solves the equation

$$\partial_t\eta^N = \frac{1}{2}\Delta\eta^N + \mathfrak{w}\cdot\Pi_N\nabla\left(\Pi_N\eta^N\right)^2 + \nabla\cdot\xi\,. \tag{3.5}$$

Expressing the bulk diffusivity from (3.3) in terms of η^N leads, after a suitable change of variables, to the expression

$$D^N(t) = 1 + N^2\frac{2|\mathfrak{w}|^2}{t}\int_0^{\frac{t}{N^2}}\int_0^s\int_{\mathbb{T}^2_1}\mathbf{E}\left(\Pi_N{:}(\Pi_N\eta^N)^2{:}(r,x)\Pi_N{:}(\Pi_N\eta^N)^2{:}(0,0)\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\,. \tag{3.6}$$

Our main theorem concerns the Laplace transform of D^N, defined by

$$\mathcal{D}^N(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}tD^N(t)\mathrm{d}t\,. \tag{3.7}$$

Note that this is the standard Laplace transform instead of the one used in [CET23], but the two definitions only differ by a factor of λ.

Theorem 3.1.1. *Let $\mathfrak{w}\neq 0$, and define the Laplace transform of the bulk diffusivity as in (3.7). Then, for every $\delta\in(0,1)$, there is a constant $C = C(|\mathfrak{w}|)$ such that, for all λ small enough,*

$$\limsup_{N\to\infty}\mathcal{D}^N(\lambda) \le \frac{C}{\lambda^2}\left(\log\log|\log\lambda|\right)^{3+\delta}|\log\lambda|^{\frac{2}{3}}$$

and

$$\liminf_{N\to\infty}\mathcal{D}^N(\lambda) \ge \frac{1}{C\lambda^2}\left(\log\log|\log\lambda|\right)^{-3-\delta}|\log\lambda|^{\frac{2}{3}}\,.$$

Note that by translating [QV08, Lemma 1] into our setting, the upper bound gives $D^N(t) \lesssim (1+\log(1+t))^{\frac{2}{3}+o(1)}$ as $t \uparrow \infty$. For the lower bound such a statement is not true in general. Note however that $\mathcal{D}^N(\lambda) \sim \frac{C}{\lambda^2}|\log \lambda|^{\frac{2}{3}}$ as $\lambda \downarrow 0$ would imply $\frac{2C}{T}\int_0^T tD^N(t)\mathrm{d}t \sim T(\log T)^{\frac{2}{3}}$ as $T \uparrow \infty$ by general Tauberian inversion theorems, see [Fel91, Chapter XIII.5]. Thus, the theorem says that $D(t)$ grows like $(\log t)^{\frac{2}{3}}$, at least in a weak Tauberian sense.

Note also that the correction terms $(\log\log|\log\lambda|)^{\pm 3\pm\delta}$ of Theorem 3.1.1 are of lower order with respect to the ones of the corresponding result in [Yau04], which are $e^{\pm\gamma(\log\log|\log\lambda|)^2}$ for some constant $\gamma > 0$.

3.2 Preliminaries

3.2.1 Notation

Recall that for $N > 0$ we denote by $\mathbb{T}^2_N$ the torus of side-length $2\pi N$. If $N = 1$ we write $\mathbb{T}^2$ instead of $\mathbb{T}^2_1$. Let $(e_k)_{k\in\mathbb{Z}^2}$ be the standard Fourier basis on $\mathbb{T}^2$, i.e. $e_k(x) = \frac{1}{2\pi}\exp(ik\cdot x)$, which constitute an orthonormal basis of $L^2(\mathbb{T}^2)$. The Fourier transform of a function $\varphi \in L^2(\mathbb{T}^2)$, denoted by $\mathcal{F}(\varphi)$ or $\hat{\varphi}$ interchangeably, is given by

$$\mathcal{F}(\varphi)(k) \stackrel{\text{def}}{=} \hat{\varphi}(k) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} \varphi(x)e_{-k}(x)\mathrm{d}x \qquad \text{for } k \in \mathbb{Z}^2\,.$$

Moreover, we denote by $k_{1:n}$ the sequence $(k_1,\ldots,k_n)$, where $k_i \in \mathbb{Z}^2$. For example, for an L^2 function f on $(\mathbb{T}^2)^n$, we write its Fourier transform as $\mathcal{F}(f)(k_{1:n}) = \mathcal{F}(f)(k_1,\ldots,k_n)$. Furthermore we define $|k_{1:n}|^2 \stackrel{\text{def}}{=} \sum_{i=1}^n |k_i|^2$.

We denote by $\mathbb{P}$ and $\mathbb{E}$ the law and the corresponding expectation of the stationary measure given by mean-zero spatial white noise, as will be defined in subsection 3.2.2. With $\mathbf{P}$ and $\mathbf{E}$ we denote instead the law and the corresponding expectation of the process given by the solution of (3.5) started from the aforementioned stationary measure.

Finally, given $A, B \in \mathbb{R}$, we write $A \lesssim B$ if there exists an absolute constant $c > 0$, independent of all variables on which A and B may depend, such that $A \le cB$. In particular, we will only use this notation if c is independent of $\mathfrak{w}$.

3.2.2 Chaos Decomposition

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and η be real-valued mean-zero spatial white noise on $\mathbb{T}^2$, i.e. η is the Gaussian field with covariance

$$\mathbb{E}\left(\eta(\varphi)\eta(\psi)\right) = \langle \varphi, \psi\rangle_{L^2(\mathbb{T}^2)}\,, \tag{3.8}$$

where φ and ψ belong to $L^2_0(\mathbb{T}^2)$, the space of square-integrable real-valued functions that integrate to 0. Since we work in Fourier, we also want to test η against complex valued functions, by setting, for $\varphi \in L^2(\mathbb{T}^2;\mathbb{C})$, $\eta(\varphi) = \eta(\mathrm{Re}(\varphi)) + \iota\eta(\mathrm{Im}(\varphi))$, which leads to considering the covariance function (which extends (3.8))

$$\mathbb{E}\left(\eta(\varphi)\overline{\eta(\psi)}\right) = \langle \varphi, \psi\rangle_{L^2(\mathbb{T}^2;\mathbb{C})}\,, \tag{3.9}$$

where the inner product is the standard sesquilinear inner product of square-integrable complex valued functions (and φ and ψ still integrate to 0). Note that η is still real-valued in the sense that $\overline{\eta(\varphi)} = \eta(\overline{\varphi})$, which would not be the case for complex-valued white noise, see e.g. [Jan97, Section 1.4]. Using this extension we define $\hat{\eta}(k) = \eta(e_{-k})$. These are complex valued Gaussian variables satisfying $\overline{\hat{\eta}(k)} = \hat{\eta}(-k)$ and $\mathbb{E}\left(\hat{\eta}(j)\overline{\hat{\eta}(k)}\right) = \delta_{j,k}$. Since we only test against mean-zero functions $\hat{\eta}(0)$ is not defined and we set it to 0.

Let $L^2(\eta)$ be the space of L^2 random variables on Ω measurable with respect to the σ-algebra generated by η. For $n \in \mathbb{N}$, let $\mathscr{H}_n$ be the *n-th homogeneous Wiener chaos*, i.e. the closed linear subspace of $L^2(\eta)$ generated by the random variables $H_n(\eta(h))$, where H_n is the n-th Hermite polynomial and h is a mean-zero test function of norm 1. By [Nua06, Theorem 1.1.1], $L^2(\eta) = \bigoplus_{n\geq 0} \mathscr{H}_n$ is an orthogonal Hilbert space decomposition of $L^2(\eta)$. Define also $\Gamma L^2 = \bigoplus_{n\geq 0} \Gamma L^2_n$, where ΓL^2_n is the n-fold symmetric tensor product of $L^2_0(\mathbb{T}^2)$, i.e. the space of symmetric L^2 functions f on $(\mathbb{T}^2)^n$ which are mean-zero in every variable, i.e. such that $\int_{\mathbb{T}^2} f(x, y_{1:n-1})\mathrm{d}x = 0$ for every $y_{1:n-1} \in (\mathbb{T}^2)^{n-1}$. By [Nua06, Proposition 1.1.1], there is a canonical isometry I between ΓL^2 and $L^2(\eta)$, whose restrictions I_n to ΓL^2_n are isometries between ΓL^2_n and $\mathscr{H}_n$. This gives the following correspondence. For every $F \in L^2(\eta)$ there is a family of kernels $(f_n)_{n\geq 0} \in \Gamma L^2$ such that $F = \sum_{n\geq 0} I_n(f_n)$ and

$$\mathbb{E}\left(F^2\right) = \|(f_n)_{n\geq 0}\|_{\Gamma L^2} \stackrel{\text{def}}{=} \sum_{n\geq 0} n! \|f_n\|^2_{L^2((\mathbb{T}^2)^n)} \, .$$

Here the right hand side also defines the ΓL^2 inner product.

Remark 3.2.1. By this isometry between $L^2(\eta)$ and the Fock space ΓL^2, we will identify throughout the paper operators acting on either space by composing them with I or I^{-1} as appropriate (and without mentioning that we are doing so).

Remark 3.2.2. It is not strictly necessary to take the white noise to be mean-zero, but it is natural since the dynamics of the system are conservative. If we start equation (3.5) from a standard white noise η_0 (i.e. $\hat{\eta}(0)$ is a standard Gaussian), then, for any future time t, we have $\hat{\eta}_t(0) = \hat{\eta}_0(0)$. Moreover, $\hat{\eta}_t(0)$ is independent of all other $\hat{\eta}_t(k)$. Therefore we can just set it to 0. In terms of Fourier kernels it means that for any $\varphi \in \mathscr{H}_n$ it holds that $\hat{\varphi}(k_{1:n})$ is 0 if any of the $k_1, \dots, k_n$ are 0.

3.2.3 The Generator

The following is (part of) Lemma 2.1 and Lemma 2.2. from [CGT24].

Lemma 3.2.3. *For any deterministic initial condition η_0 the solution of (3.5) exists globally in time and is a strong Markov process, as a $\alpha-$Hölder continuous process with values in tempered distributions on the torus. The generator of η^N_t can be written as $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_+ + \mathcal{A}_-$, where $\mathcal{L}_0$ is symmetric with respect to $\mathbb{P}$, $\mathcal{A}^*_+ = -\mathcal{A}_-$, again with respect to $\mathbb{P}$, and the operators $\mathcal{L}_0$, $\mathcal{A}_+$ and $\mathcal{A}_-$ act on $\varphi \in \mathscr{H}_n$ as:*

$$\begin{aligned}
\mathcal{F}\left(\mathcal{L}_0\varphi\right)(k_{1:n}) &= -\tfrac{1}{2}|k_{1:n}|^2 \hat{\varphi}(k_{1:n}) \qquad (3.10)\\
\mathcal{F}\left(\mathcal{A}^N_+\varphi\right)(k_{1:n+1}) &= -\frac{\iota}{\pi(n+1)} \sum_{1\leq i<j\leq n+1} \mathbb{J}^N_{k_i,k_j} \left[\mathfrak{w}\cdot(k_i+k_j)\right] \hat{\varphi}\left(k_i+k_j, k_{\{1:n+1\}\setminus\{i,j\}}\right)\\
\mathcal{F}\left(\mathcal{A}^N_-\varphi\right)(k_{1:n-1}) &= -\frac{\iota\, n}{\pi} \sum_{j=1}^{n-1} (\mathfrak{w}\cdot k_j) \sum_{\ell+m=k_j} \mathbb{J}^N_{\ell,m}\, \hat{\varphi}\left(\ell, m, k_{\{1:n-1\}\setminus\{j\}}\right),
\end{aligned}$$

where the indicator function $\mathbb{J}$ is given by

$$\mathbb{J}_{\ell,m} \stackrel{def}{=} \mathbf{1}_{\{0<|\ell|\leq N, 0<|m|\leq N, 0<|\ell+m|\leq N\}} \, . \qquad (3.11)$$

Additionally the law of mean-zero white noise as defined by (3.8) is stationary for this equation.

Note the change in sign in the Fourier multiplier of $\mathcal{A}_-$, this is a typo in [CGT24].

Remark 3.2.4. The spatial regularity in Lemma 3.2.3 is not optimal, it can be improved to some Besov space of negative regularity, see [CES21, Theorem 4.5] where it is done for the AKPZ equation.

3.3 Truncated Resolvent Equation

The following proposition allows to express $\mathcal{D}^N$ just in terms of the stationary measure and the generator. Recall η^N is the stationary solution to (3.5) started from mean-zero white noise.

Proposition 3.3.1. *The Laplace transform of the bulk diffusivity is given by*

$$\mathcal{D}^N(\lambda) = \frac{1}{\lambda^2} + \frac{8\pi^2}{\lambda^2}|\mathfrak{w}|^2\mathbb{E}\left(\tilde{\mathcal{N}}^N[\eta]\left(\lambda N^2 - \mathcal{L}^N\right)^{-1}\tilde{\mathcal{N}}^N[\eta]\right),$$

where $\tilde{\mathcal{N}}^N[\eta] \in \mathscr{H}_2$ is purely in the second chaos and given by

$$\tilde{\mathcal{N}}^N[\eta] \stackrel{def}{=} \frac{1}{2\pi}\int_{\mathbb{T}^2}\Pi_N{:}(\Pi_N\eta)^2{:}(x)\mathrm{d}x = \sum_{\substack{\ell+m=0\\ 0<|\ell|\leq N}} {:}\hat{\eta}(\ell)\hat{\eta}(m){:} \tag{3.12}$$

and its kernel $\mathfrak{n}^N = I^{-1}\left(\tilde{\mathcal{N}}^N[\eta]\right) \in \Gamma L^2_2$ is given in Fourier by

$$\mathcal{F}\left(\mathfrak{n}^N\right)(j_1,j_2) = \mathbf{1}_{\{0<|j_1|\leq N,\, j_1+j_2=0\}}\,. \tag{3.13}$$

Proof. Multiplying (3.6) by t yields

$$tD^N(t) = t + 2N^2|\mathfrak{w}|^2\int_0^{\frac{t}{N^2}}\int_0^s\int_{\mathbb{T}^2}\mathbf{E}\left(\Pi_N{:}(\Pi_N\eta^N)^2{:}(r,x)\Pi_N{:}(\Pi_N\eta^N)^2{:}(0,0)\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\,. \tag{3.14}$$

Since the process η^N is translation invariant in space, we can write the spatial integral in the expression above as

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\mathbf{E}\left(\Pi_N{:}(\Pi_N\eta^N)^2{:}(r,x+y)\Pi_N{:}(\Pi_N\eta^N)^2{:}(0,y)\right)\\
&\qquad = 4\pi^2\mathbf{E}\left(\left(\frac{1}{2\pi}\int_{\mathbb{T}^2}\Pi_N{:}(\Pi_N\eta^N)^2{:}(r,x)\mathrm{d}x\right)\left(\frac{1}{2\pi}\int_{\mathbb{T}^2}\Pi_N{:}(\Pi_N\eta^N)^2{:}(0,x)\mathrm{d}x\right)\right)\\
&\qquad = 4\pi^2\mathbf{E}\left(\tilde{\mathcal{N}}^N[\eta^N(r)]\tilde{\mathcal{N}}^N[\eta^N(0)]\right).
\end{aligned}$$

Using the stationarity of η^N we note

$$\int_0^t\int_0^s\mathbf{E}\left(\tilde{\mathcal{N}}^N[\eta^N(r)]\tilde{\mathcal{N}}^N[\eta^N(0)]\right)\mathrm{d}r\mathrm{d}s = \frac{1}{2}\mathbf{E}\left(\left(\int_0^t\tilde{\mathcal{N}}^N[\eta^N(s)]\mathrm{d}s\right)^2\right).$$

Using this to rewrite (3.14) and applying the Laplace transform gives:

$$\begin{aligned}
\mathcal{D}^N(\lambda) &= \int_0^\infty e^{-\lambda t}D^N(t)\mathrm{d}t\\
&= \int_0^\infty e^{-\lambda t}\left(t + 4\pi^2N^2|\mathfrak{w}|^2\mathbf{E}\left(\int_0^t\tilde{\mathcal{N}}^N[\eta^N(s)]\mathrm{d}s\right)^2\right)\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{\lambda^2} + |\mathfrak{w}|^2 \frac{8\pi^2}{\lambda^2} \mathbb{E}\left(\tilde{\mathcal{N}}^N (\lambda N^2 - \mathcal{L}^N)^{-1} \tilde{\mathcal{N}}^N \right),$$

where in the final step we used [CES21, Lemma 5.1], which allows us to go from an expectation with respect to the process to one just with respect to the stationary measure. □

By the isometry between $L^2(\eta)$ and ΓL^2, this expectation is equal to $\langle \mathfrak{n}^N, (\lambda - \mathcal{L}^N)^{-1} \mathfrak{n}^N \rangle$. To determine this we would need to invert the operator $\lambda - \mathcal{L}^N$, which is difficult because the presence operator $\mathcal{A}$ means such an inversion involves all chaoses, even though $\mathfrak{n}^N$ is purely in the second chaos. To overcome this we will apply a technique first used in [LQSY04]. This technique is based on truncating the resolvent equation: Let $P_{\leq k}$ be the projection onto $\Gamma L^2_{\leq k} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{k} \mathscr{H}_n$, i.e. onto the first k chaoses and $\mathcal{L}^N_k = P_{\leq k} \mathcal{L}^N P_{\leq k}$. Let $\mathfrak{h}^{N,k}$ be the solution to the truncated generator equation

$$\left(\lambda - \mathcal{L}^N_k\right) \mathfrak{h}^{N,k} = \mathfrak{n}^N. \tag{3.15}$$

The following lemma was first proved in [LQSY04, Lemma 2.1].

Lemma 3.3.2. *Let $\lambda > 0$. Then for every $k, N \in \mathbb{N}$ we have that*

$$\left\langle \mathfrak{n}^N, \mathfrak{h}^{N,2k+1} \right\rangle \leq \left\langle \mathfrak{n}^N, \left(\lambda - \mathcal{L}^N\right)^{-1} \mathfrak{n}^N \right\rangle \leq \left\langle \mathfrak{n}^N, \mathfrak{h}^{N,2k} \right\rangle,$$

furthermore both bounds monotonically converge to $\left\langle \mathfrak{n}^N, \left(\lambda - \mathcal{L}^N\right)^{-1} \mathfrak{n}^N \right\rangle$ as $k \to \infty$.

Equation (3.15) written chaos-by-chaos takes the form

$$\begin{aligned}
(\lambda - \mathcal{L}_0)\mathfrak{h}^{N,k}_k - \mathcal{A}_+ \mathfrak{h}^{N,k}_{k-1} &= 0, \\
(\lambda - \mathcal{L}_0)\mathfrak{h}^{N,k}_{k-1} - \mathcal{A}_+ \mathfrak{h}^{N,k}_{k-2} - \mathcal{A}_- \mathfrak{h}^{N,k}_k &= 0, \\
&\cdots \\
(\lambda - \mathcal{L}_0)\mathfrak{h}^{N,k}_2 - \mathcal{A}_+ \mathfrak{h}^{N,k}_1 - \mathcal{A}_- \mathfrak{h}^{N,k}_3 &= \mathfrak{n}^N, \\
(\lambda - \mathcal{L}_0)\mathfrak{h}^{N,k}_1 - \mathcal{A}_- \mathfrak{h}^{N,k}_2 &= 0.
\end{aligned} \tag{3.16}$$

This system of equations can be solved iteratively starting from the top, which leads to the following definition.

Definition 3.3.3. *For $k \geq 3$ we define the operators*

$$\mathcal{H}^N_2 \stackrel{\text{def}}{=} 0 \qquad \textit{and} \qquad \mathcal{H}^N_k = \left(\mathcal{A}^N_+\right)^* \left(\lambda - \mathcal{L}_0 + \mathcal{H}^N_{k-1}\right)^{-1} \mathcal{A}^N_+.$$

These operators are defined in an analogous way to the operators of the same name in [CET23] and thus share some basic properties.

Lemma 3.3.4 (Lemma 3.2 from [CET23])**.** *For $k \geq 3$, the operators $\mathcal{H}_k$ are positive definite and such that for all $n \in \mathbb{N}$ the operator $\mathcal{H}_k$ maps the n-th chaos into the n-th chaos.*

Solving the system of equations (3.16) we obtain

$$\mathfrak{h}^{N,k}_2 = \left((\lambda - \mathcal{L}_0) + \mathcal{H}^N_k - \mathcal{A}^N_+ (\lambda - \mathcal{L}_0)^{-1} \mathcal{A}^N_-\right)^{-1} \mathfrak{n}^N. \tag{3.17}$$

For the operator $-\mathcal{A}^N_+(\lambda - \mathcal{L}_0)^{-1}\mathcal{A}^N_-$ notice the following: consider the subspace $\mathscr{V}$ of ΓL^2 generated by ψ supported only on $k_{1:n}$ which satisfy $\sum_{i=1}^n k_i = 0$ (for arbitrary n). The operators $\mathcal{A}^N_+, \mathcal{A}^N_-$ and $\mathcal{L}^N_0$ all map $\mathscr{V}$ into $\mathscr{V}$ and the orthogonal complement of $\mathscr{V}$ into the orthogonal complement of $\mathscr{V}$. Additionally $\mathcal{A}_-^N$ vanishes on the intersection of $\mathscr{V}$ and $\mathscr{H}_2$. Since $\mathfrak{n}^N$ is in $\mathscr{V} \cap \mathscr{H}_2$ this implies

$$\big((\lambda - \mathcal{L}_0) + \mathcal{H}_k^N - \mathcal{A}_+^N(\lambda - \mathcal{L}_0)^{-1}\mathcal{A}_-^N\big)^{-1}\mathfrak{n}^N = \big((\lambda - \mathcal{L}_0) + \mathcal{H}_k^N\big)^{-1}\mathfrak{n}^N . \tag{3.18}$$

The following lemma summarises the result of this subsection.

Lemma 3.3.5. *For all $\lambda > 0$, $N \in \mathbb{N}$ and $k \geq 2$ it holds that*

$$\left\langle \mathfrak{n}^N, \mathfrak{h}^{N,k} \right\rangle = \left\langle \mathfrak{n}^N, \big((\lambda - \mathcal{L}_0) + \mathcal{H}_k^N\big)^{-1}\mathfrak{n}^N \right\rangle .$$

Proof. This follows from (3.15), (3.17) and (3.18). □

So all that remains is to estimate the operators $\mathcal{H}_k$'s.

3.4 Iterative Estimates

In this section we set up the iterative estimation scheme for the operators $\mathcal{H}_k$. In order to do so, we first need to give some definitions.

The skew Laplacian is the linear operator whose action on Fock space is given, for every $\varphi \in \Gamma L_n^2$, by

$$\mathcal{F}\left(\mathcal{L}_0^{\mathfrak{w}}\varphi\right)(k_{1:n}) \stackrel{\text{def}}{=} -\tfrac{1}{2}(\mathfrak{w}\cdot k)_{1:n}^2\hat{\varphi}(k_{1:n}) , \qquad \text{where} \qquad (\mathfrak{w}\cdot k)_{1:n}^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (\mathfrak{w}\cdot k_i)^2 . \tag{3.19}$$

The sequence of the exponents of the logarithm in the upper and lower bounds is defined recursively by

$$\theta_2 \stackrel{\text{def}}{=} 0 \qquad \text{and} \qquad \theta_{k+1} = 1 - \frac{\theta_k}{2} \qquad \text{for every } k \geq 3 ,$$

and admits the close formula

$$\theta_k = \frac{2}{3}\left(1 - \left(\frac{-1}{2}\right)^{k-2}\right) \qquad \text{for every } k \geq 2 . \tag{3.20}$$

We now introduce some elementary functions, that morally approximate a logarithm to the power two third. Let $k, N, n \in \mathbb{N}$ and $\delta \in (0,1)$. For $k \geq 2$, $x \in (0,\infty)$ and $z \in (1,\infty)$, we set

$$\mathrm{L}(x,z) \stackrel{\text{def}}{=} \log(1 + x^{-1}) + z , \quad \mathrm{L}_k(x,z) \stackrel{\text{def}}{=} (\mathrm{L}(x,z))^{\theta_k} , \quad \mathrm{L}_k^N(x,z) \stackrel{\text{def}}{=} \mathrm{L}_k\left(\frac{x}{N^2}, z\right) . \tag{3.21}$$

The functions above are accompanied by polynomial coefficients, that morally correspond to errors made in the estimates. For $k \geq 1$, those are given by

$$z_k(n) \stackrel{\text{def}}{=} K(n+k)^{\frac{9}{2}+\frac{3}{2}\delta} \qquad \text{and} \qquad f_k(n) \stackrel{\text{def}}{=} 3(z_k(n))^{\frac{2}{3}} , \tag{3.22}$$

where K is a sufficiently large positive constant depending on $|\mathfrak{w}|$.

Remark 3.4.1. The exact dependence of K on $|\mathfrak{w}|$ is not important for us. However, following the proof, it is not difficult to check that the lower bound that K must satisfy is of the kind $a\big(|\mathfrak{w}| \vee \frac{1}{|\mathfrak{w}|}\big)^b$ for some $a, b > 0$.

We also note the trivial identities

$$z_k(n+1) = z_{k+1}(n) \qquad \text{and} \qquad f_k(n+1) = f_{k+1}(n)\,. \tag{3.23}$$

Moreover, in the proofs we use the additional notation

$$\begin{aligned}\tilde\Gamma &\stackrel{\text{def}}{=} \tilde\Gamma(\ell,m,k_{2:n}) \stackrel{\text{def}}{=} \tfrac12\left(|\ell|^2+|m|^2+|k_{2:n}|^2\right)\,,\\ \tilde\Gamma^{\mathfrak{w}} &\stackrel{\text{def}}{=} \tilde\Gamma^{\mathfrak{w}}(\ell,m,k_{2:n}) \stackrel{\text{def}}{=} \tfrac12\left((\mathfrak{w}\cdot\ell)^2+(\mathfrak{w}\cdot m)^2+(\mathfrak{w}\cdot k)^2_{2:n}\right)\,,\end{aligned} \tag{3.24}$$

where $k_{2:n}$ means $k_1,\dots,k_n$ and ℓ, m, $k_2,\dots,k_n \in \mathbb{Z}^2$ are Fourier modes. This is coherent with the notation used in [CET23, Section 3]. By the Cauchy-Schwarz inequality, the symbols above can be compared as follows:

$$0 \le \tilde\Gamma^{\mathfrak{w}} \le |\mathfrak{w}|^2\tilde\Gamma\,. \tag{3.25}$$

We are finally ready to give the definitions of the operators used for the iterative bounds.

Definition 3.4.2. *For $\lambda > 0$ and $k \ge 2$*

$$\mathcal{S}_k^N \stackrel{def}{=} \begin{cases} f_k(\mathcal{N})\mathrm{L}_k^N\left(\lambda-\mathcal{L}_0, z_k(\mathcal{N})\right) & \text{if } k \text{ is odd,}\\ \frac{1}{f_k(\mathcal{N})}\left[\mathrm{L}_k^N(\lambda-\mathcal{L}_0, z_k(\mathcal{N})) - f_k(\mathcal{N})\right] & \text{if } k \text{ is even,}\end{cases}$$

where $\mathcal{N}$ is the number operator, acting on $\varphi \in \Gamma L_n$ by $\mathcal{N}\varphi = n\varphi$ for each $n \in \mathbb{N}$ and λ is the Laplace variable.

We will use the following (quite standard) partial ordering of operators:

Definition 3.4.3. *Given two self-adjoint operators $\mathcal{A}$ and $\mathcal{B}$ on ΓL^2,*

$$\mathcal{A} \le \mathcal{B} \quad \Leftrightarrow \quad \forall n\,\forall\varphi\in\Gamma L^2_n \quad \langle\mathcal{A}\varphi,\varphi\rangle \le \langle\mathcal{B}\varphi,\varphi\rangle \quad \Leftrightarrow \quad \mathcal{B}-\mathcal{A}\ge 0\,,$$

where the last statement is taken to mean that $\mathcal{B}-\mathcal{A}$ is a positive operator.

For this partial ordering the following well-known lemma holds

Lemma 3.4.4. *For any two operators $\mathcal{A}$ and $\mathcal{B}$ on ΓL^2 it holds that*

$$0 < \mathcal{A} \le \mathcal{B} \quad \Leftrightarrow \quad 0 < \mathcal{B}^{-1} \le \mathcal{A}^{-1}\,.$$

We can now state the bounds on the operators $\mathcal{H}_k$.

Theorem 3.4.5 (iterative bounds). *For every $\delta \in (0,1)$ and for every $k \in \mathbb{Z}$, $k \ge 0$ we have*

$$\mathcal{H}_{2k+3} \le c_{2k+3}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+3} + f_{2k+3}(\mathcal{N})(-\mathcal{L}_0)\right)\,, \tag{3.26}$$

$$\mathcal{H}_{2k+2} \ge c_{2k+2}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+2} - \frac{1}{(\mathcal{N}+k)^{1+\delta}}(-\mathcal{L}_0)\right)\,, \tag{3.27}$$

where the constants c_{2k+1} and c_{2k+2} are defined recursively by setting, for $k \ge 1$,

$$c_2 \stackrel{def}{=} \frac{1}{\pi(|\mathfrak{w}|^2\vee 1)}\,, \quad c_{2k+1} = \frac{3}{2\pi|\mathfrak{w}|}\frac{\left(1+\frac{1}{2k^{1+\delta}}\right)}{c_{2k}} > 1\,, \quad c_{2k+2} = \frac{3}{2\pi|\mathfrak{w}|}\frac{\left(1-\frac{1}{2k^{1+\delta}}\right)}{\left(1+\frac{1}{2k^{1+\delta}}\right)c_{2k+1}} < 1$$

and the δ explicitly appearing in (3.27) is the same as the one used for defining the $\mathcal{S}_k$'s and the c_k's.

Note that compared to the structure in previous works ([CHST22], [CET23]), the upper and lower bounds are now split into a part multiplying the skew Laplacian $\mathcal{L}_0^{\mathfrak{w}}$ and a part multiplying the full Laplacian $\mathcal{L}_0$. The second part is used to estimate the off-diagonals, i.e. we do not absorb them into the diagonal terms, but keep them separate.

Remark 3.4.6. The two inequalities on odd and even terms of the sequence $(c_k)_{k\geq 2}$ can be checked by induction, separately on odd and even terms, after distinguishing the two cases $|\mathfrak{w}| > 1$ and $|\mathfrak{w}| < 1$. Also, note that $\sum_k \frac{1}{2k^{1+\delta}}$ is summable. This implies that, as $k \to \infty$, c_{2k+1} and c_{2k+2} converge to two positive and finite limits, respectively larger and smaller than 1.

As explained in the above discussion, the main body of the present work consists in estimating certain scalar products on Fock space. We thus begin with some preliminary lemmas, which help in pinpointing and breaking down the exact expressions that one needs to estimate.

Lemma 3.4.7 (Decomposition in diagonal and off-diagonal terms)**.** *Let $\mathcal{Z}$ be a diagonal operator on ΓL^2 with Fourier multiplier $\zeta = (\zeta_n)_{n\in\mathbb{N}}$. Then, for every $\varphi \in \Gamma L^2_n$, the following decomposition holds:*

$$\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle = \left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{Diag}} + \sum_{i=1}^{2} \left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{off}_i},$$

where the diagonal terms, given by the first summand, are defined as

$$\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{Diag}} \stackrel{def}{=} \frac{n!\,n}{2\pi^2}\sum_{k_{1:n}}(\mathfrak{w}\cdot k_1)^2\overline{\hat\varphi(k_{1:n})}\hat\varphi(k_{1:n})\sum_{\ell+m=k_1}\mathbb{J}^N_{\ell,m}\zeta_{n+1}(\ell,m,k_{2:n}),$$

while the off-diagonal terms of type 1 *and* 2 *are respectively given by*

$$\begin{aligned}\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{off}_1} \stackrel{def}{=} \frac{n!\,n(n-1)}{\pi^2}&\sum_{k_{1:n+1}}(\mathfrak{w}\cdot(k_1+k_2))(\mathfrak{w}\cdot(k_1+k_3))\times \\ &\times\overline{\hat\varphi(k_1+k_2,k_3,k_{4:n+1})}\hat\varphi(k_1+k_3,k_2,k_{4:n+1})\mathbb{J}^N_{k_1,k_2}\mathbb{J}^N_{k_1,k_3}\zeta_{n+1}(k_{1:n+1})\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{off}_2} \stackrel{def}{=} \frac{n!\,n(n-1)(n-2)}{4\pi^2}&\sum_{k_{1:n+1}}(\mathfrak{w}\cdot(k_1+k_2))(\mathfrak{w}\cdot(k_3+k_4))\times \\ &\times\overline{\hat\varphi(k_1+k_2,k_{3:4},k_{5:n+1})}\hat\varphi(k_3+k_4,k_{1:2},k_{5:n+1})\mathbb{J}^N_{k_1,k_2}\mathbb{J}^N_{k_3,k_4}\zeta_{n+1}(k_{1:n+1})\,.\end{aligned} \tag{3.29}$$

The above decomposition is the same as the one used in the proof of [CGT24, Lemma 2.5] and we refer to [CET23, Lemma 3.6] for the combinatorics needed for the exact expressions of the multiplicative factors in front of the sums.

Lemma 3.4.8 (How to bound diagonal terms)**.** *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two positive operators on ΓL^2, diagonal both in chaos and in Fourier, with Fourier multipliers $\zeta^i = (\zeta^i_n)_{n\in\mathbb{N}}$, for $i = 1,2$. If for every $n \in \mathbb{N}$ and for every $k_{1:n} \in \mathbb{Z}^{2n}$*

$$\sum_{\ell+m=k_1}\mathbb{J}^N_{\ell,m}\zeta^1_{n+1}(\ell,m,k_{2:n}) \leq \zeta^2_n(k_{1:n}), \tag{3.30}$$

then for every $\varphi \in \Gamma L^2_n$

$$\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_1\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\mathrm{Diag}} \leq \frac{1}{\pi^2}\left\langle(-\mathcal{L}_0^{\mathfrak{w}})\mathcal{Z}_2\varphi,\varphi\right\rangle.$$

Moreover, a reverse inequality in the assumption implies a reverse inequality in the result.

Proof. Recalling the expressions of $\mathcal{A}_+^N$, $\left(\mathcal{A}_+^N\right)^*$ and $\mathcal{L}_0^{\mathfrak{w}}$, given in Lemma 3.2.3 and in (3.19) respectively, and using hyphothesis (3.30), we obtain:

$$\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_1 \mathcal{A}_+^N \varphi, \varphi \right\rangle_{\text{Diag}} = \frac{n!\,n}{2\pi^2} \sum_{k_{1:n}} (\mathfrak{w}\cdot k_1)^2 |\hat\varphi(k_{1:n})|^2 \sum_{\ell+m=k_1} \mathbb{J}_{l,m}^N \zeta_{n+1}^1(\ell,m,k_{2:n}) \leq$$

$$\leq \frac{n!\,n}{2\pi^2} \sum_{k_{1:n}} (\mathfrak{w}\cdot k_1)^2 |\hat\varphi(k_{1:n})|^2 \zeta_n^2(k_{1:n}) = \frac{n!}{\pi^2} \sum_{k_{1:n}} (-\mathcal{L}_0^{\mathfrak{w}}) |\hat\varphi(k_{1:n})|^2 \zeta_n^2(k_{1:n}) =$$

$$= \frac{1}{\pi^2} \langle (-\mathcal{L}_0^{\mathfrak{w}}) \mathcal{Z}_2 \varphi, \varphi \rangle\ .$$

The reverse inequality follows by repeating the exact same steps above with the inequality in the other direction. □

Lemma 3.4.9 (How to upper bound off-diagonal terms)**.** *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two positive operators on ΓL^2, diagonal both in chaos and in Fourier, with Fourier multipliers $\zeta^i = (\zeta_n^i)_{n\in\mathbb{N}}$, for $i = 1, 2$.*

If for every $n \in \mathbb{N}$ and for every $k_{1:n} \in \mathbb{Z}^{2n}$

$$|k_2| \sum_{\ell+m=k_1} \frac{1}{|m|} \mathbb{J}_{l,m}^N \zeta_{n+1}^1(\ell,m,k_{2:n}) \leq \zeta_n^2(k_{1:n})\,,$$

then for every $\varphi \in \Gamma L_n^2$

$$\left| \left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_1 \mathcal{A}_+^N \varphi, \varphi \right\rangle_{\text{off}_1} \right| \leq \frac{2|\mathfrak{w}|^2(n-1)}{\pi^2} \langle (-\mathcal{L}_0) \mathcal{Z}_2 \varphi, \varphi \rangle\ . \tag{3.31}$$

If for every $n \in \mathbb{N}$ and for every $k_{1:n} \in \mathbb{Z}^{2n}$

$$\sum_{\ell+m=k_1} \mathbb{J}_{\ell,m}^N \frac{\zeta_{n+1}^1(\ell,m,k_{2:n})}{|\ell||m|} \leq \frac{1}{|k_1|\sqrt{|k_{1:n}|^2}} \zeta_n^2(k_{1:n})\,,$$

then for every $\varphi \in \Gamma L_n^2$

$$\left| \left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_1 \mathcal{A}_+^N \varphi, \varphi \right\rangle_{\text{off}_2} \right| \leq \frac{|\mathfrak{w}|^2(n-1)(n-2)}{\pi^2} \langle (-\mathcal{L}_0) \mathcal{Z}_2 \varphi, \varphi \rangle\ . \tag{3.32}$$

Proof. Following ideas of [CET23], we define

$$\forall n \in \mathbb{N} \qquad \Phi(k_{1:n}) \stackrel{\text{def}}{=} \prod_{i=1}^n |k_i| |\hat\varphi(k_{1:n})|\,. \tag{3.33}$$

We start with the estimate for the off-diagonal terms of type 1. In order to have more easily readable expressions, we give a name to the coefficient (its exact expression is only used at the end of the proof):

$$c_{\text{off}_1}(n) \stackrel{\text{def}}{=} \frac{|\mathfrak{w}|^2 n!\, n(n-1)}{\pi^2}\,.$$

By expanding the left hand side of (3.31), applying the Cauchy-Schwarz inequality and using definition (3.33), we get

$$\left| \left\langle \mathcal{Z} \mathcal{A}_+^N \varphi, \mathcal{A}_+^N \varphi \right\rangle_{\text{off}_1} \right| \stackrel{\text{C-S}}{\leq} c_{\text{off}_1}(n) \sum_{k_{1:n+1}} |k_1+k_2||k_1+k_3| \times$$

$$\times |\hat\varphi(k_1+k_2,k_3,k_{4:n+1})||\hat\varphi(k_1+k_3,k_2,k_{4:n+1})|\mathbb{J}^N_{k_1,k_2}\mathbb{J}^N_{k_1,k_3}\zeta^1_{n+1}(k_{1:n+1})$$
$$= c_{\mathrm{off}_1}(n)\sum_{k_{1:n+1}}\frac{\Phi(k_1+k_2,k_3,k_{4:n+1})\Phi(k_1+k_3,k_2,k_{4:n+1})}{|k_2||k_3|\prod_{i=4}^{n+1}|k_i|^2}\mathbb{J}^N_{k_1,k_2}\mathbb{J}^N_{k_1,k_3}\zeta^1_{n+1}(k_{1:n+1})\,.$$

We now recall the elementary inequality $|ab|\leq a^2/2+b^2/2$, true for any $a,b\in\mathbb{R}$, and apply it with the choice $a=\Phi(k_1+k_2,k_3,k_{4:n+1})\mathbb{J}^N_{k_1,k_2}$ and b analogous. By symmetry, the second addend that we obtain by this procedure is actually equal to the first one, and so we obtain the upper bound

$$c_{\mathrm{off}_1}(n)\sum_{k_{1:n+1}}\frac{(\Phi(k_1+k_2,k_3,k_{4:n+1}))^2}{|k_2||k_3|\prod_{i=4}^{n+1}|k_i|^2}\mathbb{J}^N_{k_1,k_2}\zeta^1_{n+1}(k_{1:n+1})\,.$$

Expanding the definition of Φ, applying the change of variables $k_{1:n+1}\mapsto(\ell,m,k_{2:n})$ and finally using the hypothesis gives the desired upper bound:

$$\begin{aligned}
&c_{\mathrm{off}_1}(n)\sum_{k_{1:n+1}}\frac{|k_1+k_2|^2|k_3||\hat\varphi(k_1+k_2,k_3,k_{4:n+1})|^2}{|k_2|}\mathbb{J}^N_{k_1,k_2}\zeta^1_{n+1}(k_{1:n+1})\\
&= c_{\mathrm{off}_1}(n)\sum_{\ell,m,k_{2:n}}\frac{|\ell+m|^2|k_2||\hat\varphi(\ell+m,k_{2:n})|^2}{|m|}\mathbb{J}^N_{l,m}\zeta^1_{n+1}(\ell,m,k_{2:n})\\
&= c_{\mathrm{off}_1}(n)\sum_{k_{1:n}}|\hat\varphi(k_1,k_{2:n})|^2|k_1|^2|k_2|\sum_{\ell+m=k_1}\mathbb{J}^N_{l,m}\frac{\zeta^1_{n+1}(\ell,m,k_{2:n})}{|m|}\\
&\leq \frac{|\mathfrak{w}|^2 n!\,n(n-1)}{\pi^2}\sum_{k_{1:n}}|\hat\varphi(k_1,k_{2:n})|^2|k_1|^2\zeta^2_n(k_{1:n}) = \frac{2|\mathfrak{w}|^2(n-1)}{\pi^2}\langle(-\mathcal{L}_0)\mathcal{Z}_2\varphi,\varphi\rangle\,,
\end{aligned}$$

where the factor n was absorbed in the definition of $(-\mathcal{L}_0)$ (recall (3.10)).

We now prove the statement about the off-diagonal terms of type 2. We set

$$c_{\mathrm{off}_2}(n)\stackrel{\mathrm{def}}{=}\frac{|\mathfrak{w}|^2 n!\,n(n-1)(n-2)}{4\pi^2}$$

and follow the same steps already used for the off-diagonal terms of type 1, even though the expressions to which we apply them now are slightly different. More precisely, we consider the left hand side of (3.32), use the Cauchy-Schwarz inequality, $ab\leq a^2/2+b^2/2$ with $a=\Phi(k_1+k_2,k_{3:4},k_{5:n+1})\mathbb{J}^N_{k_1,k_2}$ and b analogous and the change of variables $k_{1:n+1}\mapsto(\ell,m,k_{2:n})$. Overall, this gives the upper bound

$$\left|\langle\mathcal{Z}\mathcal{A}^N_+\varphi,\mathcal{A}^N_+\varphi\rangle_{\mathrm{off}_2}\right|\leq c_{\mathrm{off}_2}(n)\sum_{k_{1:n}}|\hat\varphi(k_1,k_{2:n})|^2|k_1|^2|k_2||k_3|\sum_{\ell+m=k_1}\mathbb{J}^N_{\ell,m}\frac{\zeta^1_{n+1}(\ell,m,k_{2:n})}{|\ell||m|}\,.$$

Finally, by applying the hypothesis of the Lemma to innermost sum of the expression above, we obtain the upper bound

$$\frac{|\mathfrak{w}|^2 n!\,n(n-1)(n-2)}{4\pi^2}\sum_{k_{1:n}}|\hat\varphi(k_1,k_{2:n})|^2\frac{|k_1||k_2||k_3|}{\sqrt{|k_{1:n}|^2}}\zeta^2_n(k_{1:n})\,. \tag{3.34}$$

Using $|k_3|\leq\sqrt{|k_{1:n}|}$ the sum above is upper bounded by

$$\sum_{k_{1:n}}|k_1||k_2||\hat\varphi(k_{1:n})|^2\zeta^2_n(k_{1:n})\,.$$

Since both $\hat{\varphi}$ and ζ_n^2 are symmetric, we can replace $|k_1||k_2|$ in this sum by $\frac{1}{\binom{n}{2}}\sum_{i\neq j}|k_i||k_j|$, which we further estimate by

$$\frac{1}{\binom{n}{2}}\sum_{i\neq j}|k_i||k_j| \leq \frac{1}{2\binom{n}{2}}\sum_{i\neq j}|k_i|^2+|k_j|^2 = \frac{2}{n}\sum_{i=1}^{n}|k_i|^2\,.$$

Doing so we obtain that (3.34) is bounded by the right hand side of (3.32). This concludes the proof. □

We are now ready to prove Theorem 3.4.5. The proof is written separately for the upper and lower bounds.

Proof of Theorem 3.4.5, inequality (3.27). We proceed by induction. In this first part of the proof we show that the $(2k+2)$-th lower bound holds assuming that the $(2k+1)$-th upper bound does. In the next part, instead, we will assume the $(2k+2)$-th lower bound and prove that the $(2k+3)$-th upper bound holds.

As for all proofs by induction, we need an initial step. We take this to be the lower bound for $k=0$. More precisely, this consists in showing

$$0=\mathcal{H}_2 \geq c_2\left((-\mathcal{L}_0^{\mathfrak{w}})\frac{1}{f_2(\mathcal{N})}\left(1-f_2(\mathcal{N})\right)-\frac{1}{(\mathcal{N}+2)^{1+\delta}}(-\mathcal{L}_0)\right)\,.$$

Since $f_2(n)>1$ for every $n\in\mathbb{N}$, the right hand side above is negative and thus the inequality holds for any arbitrary choice of $c_2>0$ (uniformly in $n,k\geq 1$), so we may as well choose the one given in the statement of the theorem:

$$c_2 \stackrel{\text{def}}{=} \frac{1}{\pi(|\mathfrak{w}|^2\vee 1)}\,.$$

We now proceed to the inductive argument. Let $k\geq 1$. Assume by induction that (3.26) holds for $k-1$, i.e. assume the upper bound stated for $\mathcal{H}_{2k+1}$. We want to prove (3.27) for k, i.e. we want to prove the lower bound stated for $\mathcal{H}_{2k+2}$. We have:

$$\begin{aligned}\mathcal{H}_{2k+2} &= \left(\mathcal{A}_+^N\right)^*(\lambda-\mathcal{L}_0+\mathcal{H}_{2k+1})^{-1}\mathcal{A}_+^N\\ &\geq \left(\mathcal{A}_+^N\right)^*\left(\lambda-\mathcal{L}_0+c_{2k+1}\left[(-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+1}+f_{2k+1}(\mathcal{N})(-\mathcal{L}_0)\right]\right)^{-1}\mathcal{A}_+^N\\ &\stackrel{\text{def}}{=} \left(\mathcal{A}_+^N\right)^*\mathcal{Z}_{2k+1}\mathcal{A}_+^N\,,\end{aligned}$$

where we used the last equation to define the operator $\mathcal{Z}_{2k+1}$. We observe that $\mathcal{Z}_{2k+1}$ is diagonal in Fourier and thus, consistently with the notation already used in the other lemmas of this section, we denote by ζ^{2k+1} its Fourier multiplier.

By Remark 3.4.3, we set out to bound $\langle\left(\mathcal{A}_+^N\right)^*\mathcal{Z}_{2k+1}\mathcal{A}_+^N\varphi,\varphi\rangle$ from below, for $\varphi\in\Gamma L_n^2$. We recognize an expression of the type considered in Lemma 3.4.7, 3.4.8, and 3.4.9. First of all, we use Lemma 3.4.7 to split the scalar product above into diagonal and off-diagonal terms. Then we proceed to study them separately, starting with the diagonal ones.

Recall the definitions of $\tilde{\Gamma}$ and $\tilde{\Gamma}^{\mathfrak{w}}$ given in (3.24). In order to apply Lemma 3.4.8, we need a bound on the sum

$$\sum_{\ell+m=k_1}\frac{\mathbb{J}_{\ell,m}^N}{\lambda+\tilde{\Gamma}+c_{2k+1}f_{2k+2}\left[\tilde{\Gamma}^{\mathfrak{w}}\mathrm{L}_{2k+1}^N(\lambda+\tilde{\Gamma},z_{2k+2})+\tilde{\Gamma}\right]}\,, \tag{3.35}$$

where we first used property (3.23) to replace $z_{2k+1}(n+1)$ and $f_{2k+1}(n+1)$ with $z_{2k+2}(n)$ and $f_{2k+2}(n)$ respectively and then suppressed the argument of z_{2k+1} and f_{2k+1}, as it is constant throughout. Inside the sum above we recognize the Fourier multiplier ζ^{2k+1} multiplied by the product of indicator functions $\mathbb{J}^N_{\ell,m}$, as by hypothesis of Lemma 3.4.8.

The estimate of those kind of sums is carried out in Appendix 3.A. However, before invoking it, we do one additional step and lower bound it by:

$$\frac{1}{c_{2k+1}f_{2k+2}\left(1+\frac{1}{f_{2k+2}}\right)}\sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\tilde{\Gamma}+\tilde{\Gamma}^{\mathfrak{w}}\mathrm{L}^N_{2k+1}(\lambda+\tilde{\Gamma},z_{2k+2})}\,, \tag{3.36}$$

where we multiplied λ by $1+c_{2k+1}f_{2k+2}$ (a number larger than 1), replaced the $c_{2k+1}f_{2k+2}$ factor of $\tilde{\Gamma}\mathrm{L}^N_{2k+1}$ by $1+c_{2k+1}f_{2k+2}$ and finally factored $1+c_{2k+1}f_{2k+2}$ out and used

$$1+c_{2k+1}f_{2k+2}=c_{2k+1}f_{2k+2}\left(1+\frac{1}{c_{2k+1}f_{2k+2}}\right)\leq c_{2k+1}f_{2k+2}\left(1+\frac{1}{f_{2k+2}}\right).$$

As announced, we now apply Lemma 3.A.4 to the sum in (3.36) to lower bound the whole expression (3.36) by

$$\frac{B_{2k+2}}{f_{2k+2}}\frac{\pi}{|\mathfrak{w}|}\left[A_{2k+2}\mathrm{L}^N_{2k+2}\left(\lambda+|k_{1:n}|^2,z_{2k+2}\right)-\frac{4}{3}(z_{2k+2})^{\theta_{2k+2}}\right], \tag{3.37}$$

where, in order to increase readability, we set

$$\begin{aligned}A_{2k+2}(n)&\stackrel{\text{def}}{=}1-\left(|\mathfrak{w}|C_{\mathrm{Diag}}+2+\frac{3}{|\mathfrak{w}|}\right)\frac{1}{z^{\frac{\theta_{2k+1}}{2}}}\,,\\ B_{2k+2}(n)&\stackrel{\text{def}}{=}\frac{3}{2c_{2k+1}\left(1+\frac{1}{f_{2k+2}(n)}\right)}\,.\end{aligned} \tag{3.38}$$

We now proceed with two additional steps. In the first one, observing that $\theta_{2k+2}\leq\frac{2}{3}$ for every $k\geq0$ and recalling the definition of f_{2k+2} given in (3.22), we estimate the additive error in the square brackets of (3.37) by

$$\frac{4}{3}(z_{2k+2})^{\frac{2}{3}}\leq\frac{1}{2}f_{2k+2}\leq\left(1-\frac{1}{2k^{1+\delta}}\right)f_{2k+2}\,. \tag{3.39}$$

In the second step, instead, we observe that $\frac{\theta_{2k+1}}{2}\geq\frac{1}{3}$ for every $k\geq1$, so that

$$A_{2k+2}\geq1-\left(|\mathfrak{w}|C_{\mathrm{Diag}}+2+\frac{3}{|\mathfrak{w}|}\right)\frac{1}{(z_{2k+2})^{\frac{1}{3}}}\geq1-\frac{1}{2k^{1+\delta}}\,,$$

where the second inequality is true because of the lower bound $(z_{2k+2})^{\frac{1}{3}}\geq2K^{\frac{1}{3}}k^{1+\delta}$, assuming that K in definition (3.22) is large enough.

Summing up, (3.37) is lower bounded by

$$\frac{B_{2k+2}}{f_{2k+2}}\frac{\pi}{|\mathfrak{w}|}\left(1-\frac{1}{2k^{1+\delta}}\right)\left(\mathrm{L}^N_{2k+2}\left(\lambda+|k_{1:n}|^2,z_{2k+2}\right)-f_{2k+2}\right) \tag{3.40}$$

and by using the above as hypothesis in Lemma 3.4.8, we obtain the following bound on the diagonal term of the scalar product:

$$\left\langle\left(\mathcal{A}^N_+\right)^*\mathcal{Z}_{2k+1}\mathcal{A}^N_+\varphi,\varphi\right\rangle_{\mathrm{Diag}}\geq\left\langle\frac{B_{2k+2}(\mathcal{N})}{\pi|\mathfrak{w}|}\left(1-\frac{1}{2k^{1+\delta}}\right)(-\mathcal{L}^{\mathfrak{w}}_0)\mathcal{S}^N_{2k+2}\varphi,\varphi\right\rangle\quad\forall\,\varphi\in\Gamma L^2_n\,.$$

We now proceed to estimate the off-diagonal terms. We want to apply Lemma 3.4.9, whose hypotheses require estimates on two sums involving the Fourier multiplier ζ^{2k+1}, one for the off-diagonal terms of type 1 and one for the off-diagonal terms of type 2.

The sum that needs to be estimated for the off-diagonal terms of type 1 is

$$|k_2| \sum_{\ell+m=k_1} \frac{1}{|m|} \frac{\mathbb{J}^N_{\ell,m}}{\lambda + \tilde{\Gamma} + c_{2k+1} f_{2k+2} \left[\tilde{\Gamma}^{\mathfrak{w}} \mathrm{L}^N_{2k+1}(\lambda + \tilde{\Gamma}, z_{2k+2}) + \tilde{\Gamma}\right]}\,, \tag{3.41}$$

where, again, we first used property (3.23) and then suppressed the variable n. By dropping from the denominator of (3.41) $\lambda+\tilde{\Gamma}$ and all other terms involving the Fourier modes $k_{\{1:n\}\setminus\{2\}}$, and by lower bounding L^N_{2k+1} by 0, we obtain the upper bound

$$\frac{|k_2|}{c_{2k+1} f_{2k+2}} \sum_{\ell+m=k_1} \frac{1}{|m|} \frac{\mathbb{J}^N_{\ell,m}}{(|m|^2 + |k_2|^2)} \lesssim \frac{1}{c_{2k+1} f_{2k+2}}\,,$$

where we estimated the sum by the corresponding integral and applied Lemma 3.A.1 with $\beta = |k_2|^2$ and $\gamma = 1$.

Regarding the off-diagonal terms of type 2, instead, we need to estimate the sum

$$\sum_{\ell+m=k_1} \frac{1}{|\ell||m|} \frac{\mathbb{J}^N_{\ell,m}}{\lambda + \tilde{\Gamma} + c_{2k+1} f_{2k+2} \left[\tilde{\Gamma}^{\mathfrak{w}} \mathrm{L}^N_{2k+1}(\lambda + \tilde{\Gamma}, z_{2k+2}) + \tilde{\Gamma}\right]}\,. \tag{3.42}$$

We now observe that the condition $\ell + m = k_1$ implies that at least one between ℓ and m has norm larger than the one of $\frac{1}{2}k_1$. Using this to replace one $|\ell|^2$ by $|\frac{1}{2}k_1|^2$ and by applying arguments analogous to the ones just used in the estimate of (3.41), we obtain that (3.42) is upper bounded by

$$\frac{16}{c_{2k+1} f_{2k+2}} \sum_{\ell+m=k_1} \frac{1}{|k_1||m|} \frac{\mathbb{J}^N_{\ell,m}}{(|m|^2 + |k_{1:n}|^2)} \lesssim \frac{1}{c_{2k+1} f_{2k+2}} \frac{1}{|k_1|\sqrt{|k_{1:n}|^2}}\,,$$

where we estimated the sum by the corresponding integral and applied Lemma 3.A.1 with $\beta = |k_{1:n}|^2$ and $\gamma = 1$.

Using Lemma 3.4.9 with the estimates above as hypotheses, we conclude that

$$\begin{aligned}
&\left|\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle_{\mathrm{Off}_1}\right| + \left|\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle_{\mathrm{Off}_2}\right| \\
&\le C_{\mathrm{off}} \left\langle \left(\frac{2|\mathfrak{w}|^2(\mathcal{N}-1)}{c_{2k+1} f_{2k+2}(\mathcal{N})} + \frac{|\mathfrak{w}|^2(\mathcal{N}-1)(\mathcal{N}-2)}{c_{2k+1} f_{2k+2}(\mathcal{N})} \right) (-\mathcal{L}_0)\varphi, \varphi \right\rangle \\
&\le C_{\mathrm{off}} \left\langle \frac{|\mathfrak{w}|^2 \mathcal{N}^2}{c_{2k+1} f_{2k+2}(\mathcal{N})} (-\mathcal{L}_0)\varphi, \varphi \right\rangle\,,
\end{aligned}$$

where C_{off} is an absolute constant independent of all variables at play (see (3.54) for more details). With this we have come to the last step of the proof of the iterative lower bounds. Recall that we were trying to estimate $\langle (\mathcal{A}^N_+)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \rangle$ from below, for $\varphi \in \Gamma L^2_n$. In order to do so, we split it into diagonal and off-diagonal terms and we bounded each of them separately. We now put those estimates together:

$$\begin{aligned}
&\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle \\
&\ge \left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle_{\mathrm{Diag}} - \left|\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle_{\mathrm{off}_1}\right| - \left|\left\langle \left(\mathcal{A}^N_+\right)^* \mathcal{Z}_{2k+1} \mathcal{A}^N_+ \varphi, \varphi \right\rangle_{\mathrm{off}_2}\right|
\end{aligned}$$

$$\geq \left\langle \left[\frac{B_{2k+2}(\mathcal{N})}{\pi|\mathfrak{w}|} \left(1 - \frac{1}{2k^{1+\delta}}\right) (-\mathcal{L}_0^{\mathfrak{w}}) \mathcal{S}_{2k+2}^N - \frac{C_{\text{off}}|\mathfrak{w}|^2\mathcal{N}^2}{c_{2k+1}f_{2k+2}(\mathcal{N})}(-\mathcal{L}_0) \right] \varphi, \varphi \right\rangle$$
$$\geq \left\langle \frac{B_{2k+2}(\mathcal{N})\left(1 - \frac{1}{2k^{1+\delta}}\right)}{\pi|\mathfrak{w}|} \left((-\mathcal{L}_0^{\mathfrak{w}}) \mathcal{S}_{2k+2}^N - \frac{8\pi C_{\text{off}}|\mathfrak{w}|^3\mathcal{N}^2}{3f_{2k+2}(\mathcal{N})}(-\mathcal{L}_0) \right) \varphi, \varphi \right\rangle,$$

where we first used the triangular inequality and then the bounds established in proof. Finally, the last inequality is obtained by factoring out the coefficient of $(-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+2}^N$ and upper bounding both $\left(1 + \frac{1}{f_{2k+2}}\right)$ and $\left(1 - \frac{1}{2k^{1+\delta}}\right)^{-1}$ by 2.

At this point, the proof is almost complete. We just need a few more estimates on the coefficients of the operators appearing in the scalar product above, so that it becomes exactly the one lower-bounding $\mathcal{H}_{2k+2}$ in the inequality (3.27) of Theorem 3.4.5. First of all, by recalling the definition of f_{2k+2} given in (3.22) and taking K large enough, we estimate the coefficient of $(-\mathcal{L}_0)$ by above by

$$\frac{8\pi C_{\text{off}}|\mathfrak{w}|^3 n^2}{3f_{2k+2}} = \frac{8\pi C_{\text{off}}|\mathfrak{w}|^3 n^2}{9K^{2/3}(n+2k+2)^{3+\delta}} \leq \frac{1}{(n+k)^{1+\delta}}.$$

Finally, we consider the coefficient in front of the square bracket and observe that

$$\frac{B_{2k+2}\left(1 - \frac{1}{2k^{1+\delta}}\right)}{\pi|\mathfrak{w}|} = \frac{3\left(1 - \frac{1}{2k^{1+\delta}}\right)}{2\pi|\mathfrak{w}|c_{2k+1}\left(1 + \frac{1}{f_{2k+2}}\right)} \geq c_{2k+2},$$

simply by expanding the definition of B_{2k+2} given at (3.38), lower bounding f_{2k+2} by $2k^{1+\delta}$ and recalling the definition of c_{2k+2} given in the statement of Theorem 3.4.5. This concludes the proof. □

Proof of Theorem 3.4.5, inequality (3.26). In this second part of the proof we show that the $(2k+3)$-th upper bound holds assuming that the $(2k+2)$-th lower bound does. In the hope of making the reading easier, we note that the general structure of the two parts is similar.

Let $k \geq 0$. Assume by induction that (3.27) holds for k, i.e. assume the lower bound stated for $\mathcal{H}_{2k+2}$. We want to prove that also (3.26) holds for k, i.e. we want to prove the upper bound stated for $\mathcal{H}_{2k+3}$. Then:

$$\begin{aligned}
\mathcal{H}_{2k+3} &= \left(\mathcal{A}_+^N\right)^* (\lambda - \mathcal{L}_0 + \mathcal{H}_{2k+2})^{-1} \mathcal{A}_+^N \\
&\leq \left(\mathcal{A}_+^N\right)^* \left(\lambda - \mathcal{L}_0 + c_{2k+2}\left[(-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+2} - \frac{1}{(\mathcal{N}+k)^{1+\delta}}(-\mathcal{L}_0)\right]\right)^{-1} \mathcal{A}_+^N \\
&\stackrel{\text{def}}{=} \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N,
\end{aligned}$$

where we the last equation defines $\mathcal{Z}_{2k+2}$. We denote by ζ^{2k+2} its Fourier multiplier.

By Remark 3.4.3, our aim is to bound $\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\rangle$ from above, for $\varphi \in \Gamma L_n^2$. We use 3.4.7 to split the scalar product into diagonal and off-diagonal terms. We start by studying the diagonal ones. Recall the definitions of $\tilde{\Gamma}$ and $\tilde{\Gamma}^{\mathfrak{w}}$ given in (3.24). In order to apply Lemma 3.4.8 we need a bound on

$$\sum_{\ell+m=k_1} \frac{\mathbb{J}_{\ell,m}^N}{\lambda + \tilde{\Gamma} + c_{2k+2}\left[\frac{\tilde{\Gamma}^{\mathfrak{w}}}{f_{2k+3}}\left(\mathrm{L}_{2k+2}^N(\lambda+\tilde{\Gamma}, z_{2k+3}) - f_{2k+3}\right) - \frac{\tilde{\Gamma}}{(n+1+k)^{1+\delta}}\right]}, \tag{3.43}$$

where we first used property (3.23) to replace $z_{2k+1}(n+1)$ and $f_{2k+1}(n+1)$ with $z_{2k+2}(n)$ and $f_{2k+2}(n)$ respectively and then suppressed the argument of z_{2k+1} and f_{2k+1}, as it is constant throughout. The plan is to estimate this sum by using Lemma 3.A.4, but before being able to do so we need to manipulate it a bit. While this was also the case for the proof of the iterative lower bounds, this time the process is a bit more involved, because not all addends in the denominator are positive. We start by expanding the denominator and applying inequality (3.25):

$$\begin{aligned}&\sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\tilde\Gamma+\frac{c_{2k+2}}{f_{2k+3}}\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}(\lambda+\tilde\Gamma,z_{2k+3})-c_{2k+2}\tilde\Gamma^{\mathfrak{w}}-\frac{c_{2k+2}}{(n+1+k)^{1+\delta}}\tilde\Gamma}\\&\leq\sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^{1+\delta}}\right)\tilde\Gamma+\frac{c_{2k+2}}{f_{2k+3}}\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}(\lambda+\tilde\Gamma,z_{2k+3})}\,.\end{aligned}\tag{3.44}$$

Now the goal is to factor out the coefficients of $\tilde\Gamma$ and of $\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}$, in the same fashion in which expression (3.36) was obtained. In order to be able to do this, we need some control on those coefficients. We start by estimating c_{2k+2} as follows:

$$|\mathfrak{w}|^2c_{2k+2}=|\mathfrak{w}|^2\frac{\prod_{j=1}^k\left(1-\frac{1}{2j^{1+\delta}}\right)}{\prod_{j=1}^k\left(1+\frac{1}{2j^{1+\delta}}\right)^2}c_2\leq|\mathfrak{w}|^2\frac{1}{\pi(|\mathfrak{w}|^2\vee 1)}\leq\frac{1}{\pi}\,.\tag{3.45}$$

Thus expression (3.44) is upper bounded by the following:

$$\sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\left(1-\frac{1}{\pi}-\frac{1}{\pi(n+1+k)^{1+\delta}}\right)\tilde\Gamma+\frac{c_{2k+2}}{f_{2k+3}}\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}(\lambda+\tilde\Gamma,z_{2k+3})}\,.\tag{3.46}$$

In particular, we observe that the coefficient of $\tilde\Gamma$ is positive. Moreover, by (3.45) above and for a large enough K, we have that $\frac{c_{2k+2}}{f_{2k+3}}\leq 1-\frac{2}{\pi}$. This means that replacing the coefficient of $\tilde\Gamma$ by the one of $\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}$ gives an upper bound. Further multiplying λ by c_{2k+2}/f_{2k+3} and factoring out finally provides us with an upper bound of the kind we were looking for:

$$\frac{f_{2k+3}}{c_{2k+2}}\sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\tilde\Gamma+\tilde\Gamma^{\mathfrak{w}}\,\mathrm{L}^N_{2k+2}(\lambda+\tilde\Gamma,z_{2k+3})}\,.$$

We are finally ready to apply Lemma 3.A.4 to the sum in the expression above. This gives us the upper bound

$$\frac{3\pi}{2|\mathfrak{w}|}\left(1+\frac{|\mathfrak{w}|C_{\mathrm{Diag}}}{(z_{2k+3})^{\theta_{2k+3}}}\right)\mathrm{L}^N_{2k+3}\left(\lambda+|k_{1:n}|^2,z_{2k+3}\right)\,,$$

which, by choosing K large enough, can be further upper bounded by

$$\frac{3\pi}{2|\mathfrak{w}|}\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)\mathrm{L}^N_{2k+3}\left(\lambda+|k_{1:n}|^2,z_{2k+3}\right)\,.\tag{3.47}$$

Using the bound provided by expression (3.47) in the hypothesis of Lemma 3.4.8 (and recalling the coefficient that was in front of the sum before invoking Appendix 3.A), we obtain the following upper bound on the diagonal part:

$$\begin{aligned}&\left\langle\left(\mathcal{A}^N_+\right)^*\mathcal{Z}_{2k+2}\mathcal{A}^N_+\varphi,\varphi\right\rangle_{\mathrm{Diag}}\\&\leq\left\langle\frac{f_{2k+3}(\mathcal{N})}{c_{2k+2}}\frac{3}{2\pi|\mathfrak{w}|}\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)(-\mathcal{L}^{\mathfrak{w}}_0)\mathrm{L}^N_{2k+3}\left(\lambda+(-\mathcal{L}_0),z_{2k+3}(\mathcal{N})\right)\varphi,\varphi\right\rangle\end{aligned}$$

$$= \left\langle \frac{3\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)}{2\pi|\mathfrak{w}|c_{2k+2}}(-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+3}^N\varphi,\varphi \right\rangle .$$

Let us now estimate the off-diagonal terms. We want to apply Lemma 3.4.9, whose hypotheses require estimates on two sums involving the Fourier multiplier ζ^{2k+2}, one for the off-diagonal terms of type 1 and one for the off-diagonal terms of type 2.

The sum that need to be estimated for the off-diagonal terms of type 1 is

$$|k_2| \sum_{\ell+m=k_1} \frac{1}{|m|} \frac{\mathbb{J}_{\ell,m}^N}{\lambda+\tilde{\Gamma}+c_{2k+2}\left[\frac{\tilde{\Gamma}^{\mathfrak{w}}}{f_{2k+3}}\left(\mathrm{L}_{2k+2}^N(\lambda+\tilde{\Gamma},z_{2k+3})-f_{2k+3}\right)-\frac{\tilde{\Gamma}}{(n+1+k)^{1+\delta}}\right]}, \qquad (3.48)$$

where, again, we first used property (3.23) and then suppressed the variable n. As was the case for the diagonal terms, we will first manipulate this expression a bit and then apply to it a lemma proved in the appendix. By dropping from the denominator of (3.48) the Laplace variable λ and all terms involving the Fourier modes $k_{\{1:n\}\setminus\{2\}}$, by lower bounding L_{2k+2}^N by 0 and by applying Cauchy-Schwarz (3.25), we obtain the upper bound

$$\frac{4|k_2|}{\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^\delta}\right)} \sum_{\ell+m=k_1} \frac{1}{|m|}\frac{\mathbb{J}_{\ell,m}^N}{(|m|^2+|k_2|^2)} \lesssim \frac{1}{\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^\delta}\right)},$$

where we estimated the sum by the corresponding integral and applied Lemma 3.A.1 with $\beta=|k_2|^2$ and $\gamma=1$.

Regarding the off-diagonal terms of type 2, instead, we need to estimate the sum

$$\sum_{\ell+m=k_1} \frac{1}{|\ell||m|} \frac{\mathbb{J}_{\ell,m}^N}{\lambda+\tilde{\Gamma}+c_{2k+2}\left[\frac{\tilde{\Gamma}^{\mathfrak{w}}}{f_{2k+3}}\left(\mathrm{L}_{2k+2}^N(\lambda+\tilde{\Gamma},z_{2k+3})-f_{2k+3}\right)-\frac{\tilde{\Gamma}}{(n+1+k)^{1+\delta}}\right]}. \qquad (3.49)$$

We now observe that the condition $\ell+m=k_1$ implies that at least one between ℓ and m has norm larger than the one of $\frac{1}{2}k_1$. Using this to replace $|\ell|^2$ by $|\frac{1}{2}k_1|^2$ and by applying arguments analogous to the ones just used in the estimate of (3.48), we obtain that (3.49) is upper bounded by

$$\begin{aligned}&\frac{16}{\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^\delta}\right)} \sum_{\ell+m=k_1} \frac{1}{|k_1||m|}\frac{\mathbb{J}_{\ell,m}^N}{(|m|^2+|k_{1:n}|^2)} \\ &\lesssim \frac{1}{|k_1|\sqrt{|k_{1:n}|^2}} \frac{\pi^2}{\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^\delta}\right)},\end{aligned}$$

where in the last inequality we estimated the sum by the corresponding integral and applied Lemma 3.A.1 with $\beta=|k_{1:n}|^2$ and $\gamma=1$.

Using Lemma 3.4.9 with the estimates above as hypotheses, we conclude that:

$$\begin{aligned}&\left|\left\langle \left(\mathcal{A}_+^N\right)^*\mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi,\varphi\right\rangle_{\mathrm{Off}_1}\right| + \left|\left\langle \left(\mathcal{A}_+^N\right)^*\mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi,\varphi\right\rangle_{\mathrm{Off}_2}\right| \\ &\le C_{\mathrm{off}}\left\langle \left(\frac{2|\mathfrak{w}|^2(\mathcal{N}-1)}{1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(\mathcal{N}+1+k)^\delta}} + \frac{|\mathfrak{w}|^2(\mathcal{N}-1)(\mathcal{N}-2)}{1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(\mathcal{N}+1+k)^\delta}}\right)(-\mathcal{L}_0)\varphi,\varphi\right\rangle \\ &\le C_{\mathrm{off}}\left\langle \frac{|\mathfrak{w}|^2\mathcal{N}^2}{1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(\mathcal{N}+1+k)^\delta}}(-\mathcal{L}_0)\varphi,\varphi\right\rangle,\end{aligned}$$

where C_{off} is an absolute constant independent of all variables at play (see (3.54) for more details).

With this we have come to the last part of the proof of the iterative upper bounds. Recall that we were trying to estimate $\langle (\mathcal{A}_+^N)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\rangle$ from above, for $\varphi \in \Gamma L^2_n$. In order to do so, we split it into diagonal and off-diagonal terms and we bounded each of them separately. We now put those estimates together:

$$
\begin{aligned}
&\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\right\rangle \\
&\le \left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\text{Diag}} + \left|\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\text{off}_1}\right| + \left|\left\langle \left(\mathcal{A}_+^N\right)^* \mathcal{Z}_{2k+2}\mathcal{A}_+^N\varphi, \varphi\right\rangle_{\text{off}_2}\right| \\
&\le \left\langle \left(\frac{3\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)}{2\pi|\mathfrak{w}|c_{2k+2}} \mathcal{S}^N_{2k+3} + \frac{C_{\text{off}}|\mathfrak{w}|^2\mathcal{N}^2}{1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(\mathcal{N}+1+k)^\delta}}(-\mathcal{L}_0)\right)\varphi, \varphi\right\rangle \\
&\le \left\langle \frac{3\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)}{2\pi|\mathfrak{w}|c_{2k+2}} \left(\mathcal{S}^N_{2k+3} + \frac{\pi C_{\text{off}}|\mathfrak{w}|^3 c_{2k+2}\mathcal{N}^2}{1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(\mathcal{N}+1+k)^\delta}}(-\mathcal{L}_0)\right)\varphi, \varphi\right\rangle, \qquad (3.50)
\end{aligned}
$$

where we first used the triangular inequality, then the bounds established in the proof and finally factored out the coefficient of $\mathcal{S}^N_{2k+3}$, together with the estimate $\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)^{-1} < 1$.

At this point, the proof is almost complete. We just need a few more estimates on the coefficient of $(-\mathcal{L}_0)$. First we multiply and divide it by f_{2k+3} and then we use the estimate

$$
\frac{\pi C_{\text{off}}|\mathfrak{w}|^3 c_{2k+2}\, n^2}{\left(1-|\mathfrak{w}|^2c_{2k+2}-\frac{c_{2k+2}}{(n+1+k)^\delta}\right) f_{2k+3}} \le \frac{C_{\text{off}}|\mathfrak{w}|^3}{3K^{2/3}} \frac{n^2}{(n+k)^{3+\delta}} \le 1,
$$

which holds for K large enough. This tells us that expression (3.50) is upper bounded by

$$
\left\langle \frac{3\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)}{2\pi|\mathfrak{w}|c_{2k+2}} \left(\mathcal{S}^N_{2k+3} + f_{2k+3}(\mathcal{N})(-\mathcal{L}_0)\right)\varphi, \varphi\right\rangle.
$$

Since the fraction in the above expression is exactly the definition of c_{2k+3} given in the statement of Theorem 3.4.5, the proof is complete. □

3.5 Proof of the main theorem

Proof of Theorem 3.1.1. The strategy of the proof is the following. First of all we apply Proposition 3.3.1, so to reduce our problem to the one of finding estimates from above and from below on the quantity $\langle \mathfrak{n}^N, (\lambda N^2 - \mathcal{L}^N)^{-1})\,\mathfrak{n}^N\rangle$. This is done by using the upper and lower bounds provided by Lemma 3.3.2, which we first simplify thanks to Lemma 3.3.5 and then further estimate with Theorem 3.4.5.

We start with the upper bound. We have

$$
\begin{aligned}
&\left\langle \mathfrak{n}^N, \left(\lambda N^2 - \mathcal{L}^N\right)^{-1}\mathfrak{n}^N\right\rangle \\
&\le \left\langle \mathfrak{n}^N, \left((\lambda N^2 - \mathcal{L}_0) + \mathcal{H}^N_{2k+2}\right)^{-1}\mathfrak{n}^N\right\rangle \\
&\le \left\langle \mathfrak{n}^N, \left((\lambda N^2 - \mathcal{L}_0) + c_{2k+2}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+2} - \frac{1}{(\mathcal{N}+k)^{1+\delta}}(-\mathcal{L}_0)\right)\right)^{-1}\mathfrak{n}^N\right\rangle,
\end{aligned}
$$

where the first inequality follows from Lemma 3.3.2 and Lemma 3.3.5 and the second one from estimate (3.27) on $\mathcal{H}_{2k+2}$ given by Theorem 3.4.5. Recalling the Fourier expression of $\mathfrak{n}^N$ given in (3.13), the above scalar product is exactly twice the sum (3.43), written for $n=2$, $k_{2:n}=0$, $k_1=0$ and Laplace variable λN^2. Following exactly the same steps performed there (compare with (3.47) and include the factor that multiplies the sum to which Lemma 3.A.4 is applied), we obtain the upper bound

$$\frac{f_{2k+3}(2)}{c_{2k+2}}\frac{3\pi}{|\mathfrak{w}|}\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)\mathrm{L}^N_{2k+3}\left(\lambda N^2, z_{2k+3}(2)\right) .$$

By recalling the definition of L^N_k, f_{2k+3} and z_{2k+3} given by (3.21) and (3.22), we can further estimate it by

$$\begin{aligned}&\frac{9K^{\frac{2}{3}}(2k+5)^{3+\delta}}{c_{2k+2}}\frac{\pi}{|\mathfrak{w}|}\left(1+\frac{1}{2(k+1)^{1+\delta}}\right)\mathrm{L}_{2k+3}\left(\lambda, K(2k+5)^{\frac{9}{2}+\frac{3}{2}\delta}\right)\\&\lesssim C(|\mathfrak{w}|)k^{3+\delta}\left(\left(\log\left(1+\lambda^{-1}\right)\right)^{\theta_{2k+3}}+k^{\frac{9}{2}+\frac{3}{2}\delta}\right)\\&=C(|\mathfrak{w}|)\left[k^{3+\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{\theta_{2k+3}-\frac{2}{3}}+k^{\frac{15}{2}+\frac{5}{2}\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{-\frac{2}{3}}\right]\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}}\end{aligned} \tag{3.51}$$

where in the inequality we used both that c_{2k+2} is bounded away from 0 and infinity uniformly in k. Expression (3.51) provides us with a valid upper bound for each value of k, with the best one being the one that minimizes the factor in front of $\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}}$. We choose

$$k=k(\lambda)=\left\lfloor(\log 4)^{-1}\log\log\log\left(1+\lambda^{-1}\right)\right\rfloor , \tag{3.52}$$

which is greater than or equal to 0 if λ is such that $1+\lambda^{-1}\geq e^e$. Recalling the close formula for θ_k given in (3.20), this gives us the estimates

$$\begin{aligned}\theta_{2k(\lambda)+3}&=\frac{2}{3}+\frac{1}{3}\left(\frac{1}{4}\right)^{\left\lfloor(\log 4)^{-1}\log\log\log\left(1+\lambda^{-1}\right)\right\rfloor}\leq\frac{2}{3}+\frac{1}{3\log\log\left(1+\lambda^{-1}\right)} ,\\ k^{3+\delta}&\leq\left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{3+\delta} ,\\ k^{\frac{15}{2}+\frac{5}{2}\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{-\frac{2}{3}}&\leq\left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{\frac{15}{2}+\frac{5}{2}\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{-\frac{2}{3}}\lesssim 1 ,\\ \left(\log\left(1+\lambda^{-1}\right)\right)^{\theta_{2k+3}-\frac{2}{3}}&\leq\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{1}{3\log\log\left(1+\lambda^{-1}\right)}}=\sqrt[3]{e} ,\end{aligned}$$

where the second inequality in the second to last line is justified by the fact that the left hand side goes to 0 as $\lambda\to 0$.

Summing up, upper bounding expression (3.51) by the estimates above and recalling the expression of $\mathcal{D}^N(\lambda)$ derived in Proposition 3.3.1, we obtain

$$\begin{aligned}\mathcal{D}^N(\lambda)&=\frac{1}{\lambda^2}+\frac{8\pi^2}{\lambda^2}|\mathfrak{w}|^2\left\langle\mathfrak{n}^N,\left(\lambda N^2-\mathcal{L}^N\right)^{-1}\mathfrak{n}^N\right\rangle\\&\lesssim\frac{C(|\mathfrak{w}|)}{\lambda^2}\left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{3+\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}} .\end{aligned}$$

Since the above inequality holds for every $N\in\mathbb{N}$, taking $\limsup_{N\to\infty}$ on both sides and observing that $\log\left(1+\lambda^{-1}\right)\sim_{\lambda\to 0}|\log(\lambda)|$ proves the upper bound of Theorem 3.1.1.

We now proceed to the lower bound. We have

$$\left\langle\mathfrak{n}^N,\left(\lambda N^2-\mathcal{L}^N\right)^{-1}\mathfrak{n}^N\right\rangle$$

$$\geq \left\langle \mathfrak{n}^N, \left((\lambda N^2 - \mathcal{L}_0) + \mathcal{H}^N_{2k+1}\right)^{-1} \mathfrak{n}^N \right\rangle$$
$$\geq \left\langle \mathfrak{n}^N, \left((\lambda N^2 - \mathcal{L}_0) + c_{2k+1}\left((-\mathcal{L}_0^{\mathfrak{w}})\mathcal{S}_{2k+1} + f_{2k+1}(\mathcal{N})(-\mathcal{L}_0)\right)\right)^{-1} \mathfrak{n}^N \right\rangle ,$$

where the first inequality follows again from Lemma 3.3.2 and Lemma 3.3.5 and the second one from estimate (3.26) on $\mathcal{H}_{2k+1}$ given by Theorem 3.4.5. Again, recalling the Fourier expression of $\mathfrak{n}^N$ given in (3.13), the above scalar product is exactly twice the sum (3.35), written for $n = 2$, $k_{2:n} = 0$, $k_1 = 0$ and Laplace variable λN^2. Following exactly the same steps performed there (compare with (3.40)), we obtain the lower bound

$$\frac{3\pi}{|\mathfrak{w}|}\frac{1}{c_{2k+1}}\left(1 - \frac{1}{2k^{1+\delta}}\right)\frac{1}{f_{2k+2}(2)}\left[\mathrm{L}^N_{2k+2}\left(\lambda N^2, z_{2k+2}(2)\right) - f_{2k+2}(2)\right] .$$

Recalling again the definitions of L^N_k, f_{2k+3} and z_{2k+3} given by (3.21) and (3.22), and using the fact that c_{2k+1} is bounded away from 0 and infinity uniformly in k, the above is further lower bounded by

$$\frac{\pi}{|\mathfrak{w}| c_{2k+1} K^{\frac{2}{3}}(2k+4)^{3+\delta}}\left[\mathrm{L}_{2k+2}\left(\lambda, K(2k+4)^{\frac{9}{2}+\frac{3}{2}\delta}\right) - 3K^{\frac{2}{3}}(2k+4)^{3+\delta}\right]$$
$$\gtrsim \frac{C(|\mathfrak{w}|)}{k^{3+\delta}}\left(\log\left(1+\lambda^{-1}\right) + k^{\left(\frac{9}{2}+\frac{3}{2}\delta\right)}\right)^{\theta_{2k+2}}$$
$$\geq \frac{C(|\mathfrak{w}|)}{k^{3+\delta}} \frac{1}{\left(\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}-\theta_{2k+2}} + k^{\left(\frac{9}{2}+\frac{3}{2}\delta\right)\left(\frac{2}{3}-\theta_{2k+2}\right)}\right)}\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}} , \qquad (3.53)$$

where we have used that fact that L_{2k+2} goes to infinity as $\lambda \to 0$ to absorb the -1 in the multiplicative constant. As before, this gives a valid lower bound for each choice of k, this time with the best one being the one that maximizes the factor in front of $\log^{\frac{2}{3}}\left(1+\lambda^{-1}\right)$. We use the same choice made for the upper bound, namely (3.52). Recalling the close formula for θ_k given in (3.20), this gives us the estimates

$$\theta_{2k(\lambda)+2} = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{4}\right)^{\left\lfloor (\log 4)^{-1}\log\log\log\left(1+\lambda^{-1}\right)\right\rfloor} \geq \frac{2}{3} - \frac{1}{3\log\log\left(1+\lambda^{-1}\right)} ,$$
$$\left(\log\left(1+\lambda^{-1}\right)\right)^{\left(\frac{2}{3}-\theta_{2k+2}\right)} \leq \left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{1}{3\log\log\left(1+\lambda^{-1}\right)}} = \sqrt[3]{e} ,$$
$$k^{\left(\frac{9}{2}+\frac{3}{2}\delta\right)\left(\frac{2}{3}-\theta_{2k+2}\right)} \leq \left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{\frac{9+3\delta}{6\log\log\left(1+\lambda^{-1}\right)}} \leq \sqrt[6]{e} ,$$
$$k^{3+\delta} \leq \left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{3+\delta} ,$$

with which we can upper bound all terms that appear in the denominator of (3.53).

Summing up, lower bounding expression (3.53) by the estimates above and recalling the expression of $\mathcal{D}^N(\lambda)$ derived in Proposition 3.3.1, we obtain

$$\mathcal{D}^N(\lambda) = \frac{1}{\lambda^2} + \frac{8\pi^2}{\lambda^2}|\mathfrak{w}|^2\left\langle \mathfrak{n}^N, \left(\lambda N^2 - \mathcal{L}^N\right)^{-1}\mathfrak{n}^N\right\rangle$$
$$\gtrsim \frac{C(|\mathfrak{w}|)}{\lambda^2}\left(\log\log\log\left(1+\lambda^{-1}\right)\right)^{-3-\delta}\left(\log\left(1+\lambda^{-1}\right)\right)^{\frac{2}{3}} .$$

Since the above inequality holds for every $N \in \mathbb{N}$, taking $\liminf_{N\to\infty}$ on both sides and observing that $\log\left(1+\lambda^{-1}\right) \sim_{\lambda\to 0} |\log(\lambda)|$ proves the lower bound of Theorem 3.1.1. □

3.A Replacement Lemmas

The present Appendix is devoted to estimating sums corresponding to the hypothesis of Lemma 3.4.8 in the context of the proof of Theorem 3.4.5.

We start by stating some useful identities and setting up some notation.

Lemma 3.A.1. *For every $\beta > 0$ and $\gamma > 0$*

$$\int_0^{+\infty} \frac{1}{\beta+\gamma r^2}\mathrm{d}r = \frac{\pi}{2\sqrt{\beta\gamma}}\,, \qquad \int_0^{\pi} \frac{1}{\beta+\gamma(\cos\theta)^2}\mathrm{d}\theta = \frac{\pi}{\sqrt{\beta(\beta+\gamma)}}\,.$$

Since this Appendix concerns expressions involving a large number of variables, and it is of technical nature anyway, let us list them all here once and for all, together with their range:

$$\lambda\in(0,+\infty)\,,\quad z\in(1,+\infty)\,,\quad \ell,m,k_1,\dots,k_n\in\mathbb{Z}_0^2\,,\quad N\in\mathbb{N}\,,\quad k\in\mathbb{N},\ k\geq 2\,,\quad \mathfrak{w}\in\mathbb{R}^2\,. \tag{3.54}$$

In particular, the constant that we omit when using the notation $\lesssim$, which was introduced in subsection 3.2.1, is independent of all variables listed above.

Let us start by introducing some additional notation:

$$\alpha \stackrel{\text{def}}{=} \alpha(\lambda,k_{1:n}) \stackrel{\text{def}}{=} \lambda+|k_{1:n}|^2\,, \qquad \alpha_N \stackrel{\text{def}}{=} \frac{\alpha}{N^2}\,,$$

$$\Gamma \stackrel{\text{def}}{=} \Gamma(\ell,m,k_{2:n}) \stackrel{\text{def}}{=} |\ell|^2+\tfrac{1}{2}\left(|k_{1:n}|^2\right)\,, \qquad \Gamma^{\mathfrak{w}} \stackrel{\text{def}}{=} \Gamma^{\mathfrak{w}}(\ell,m,k_{2:n}) \stackrel{\text{def}}{=} (\mathfrak{w}\cdot\ell)^2+\tfrac{1}{2}\left((\mathfrak{w}\cdot k)^2_{1:n}\right)\,.$$

Recall also the definitions of $\tilde\Gamma$ and $\tilde\Gamma^{\mathfrak{w}}$ given in (3.24). It is useful to observe that

$$\tilde\Gamma=\Gamma-\ell\cdot k_1\,,\qquad \tilde\Gamma^{\mathfrak{w}}=\Gamma^{\mathfrak{w}}-(\mathfrak{w}\cdot k_1)(\mathfrak{w}\cdot\ell)\,,\qquad \frac{1}{2}\leq\Gamma\lesssim\tilde\Gamma\lesssim\Gamma\,. \tag{3.55}$$

Finally, we take note of the following derivatives, that will be needed later on:

$$\partial_x\mathrm{L}(x,z)=-\frac{1}{x(x+1)}\,,\qquad \partial_x\mathrm{L}^N_k(x,z)=-\theta_k\left(\mathrm{L}^N_k(x,z)\right)^{\theta_k-1}\frac{N^2}{x(x+N^2)}\,.$$

Our goal is to study sum (3.56) below, which is the one that appears in the estimates of the diagonal terms in Theorem 3.4.5. First, in Lemma 3.A.2, we replace the sum with an integral. Then, in Lemma 3.A.4, we replace this integral with another one, which admits an explicit primitive. The first replacement comes at the price of an additive constant, the second one at the price of a lower-order term.

We set

$$\tilde{\mathrm{S}} \stackrel{\text{def}}{=} \tilde{\mathrm{S}}(\lambda,N,k,k_{1:n},\mathfrak{w}) \stackrel{\text{def}}{=} \sum_{\ell+m=k_1}\frac{\mathbb{J}^N_{\ell,m}}{\lambda+\tilde\Gamma+\tilde\Gamma^{\mathfrak{w}}\mathrm{L}^N_k\left(\lambda+\tilde\Gamma,z\right)}\,, \tag{3.56}$$

$$\begin{aligned}\mathrm{I} &\stackrel{\text{def}}{=} \mathrm{I}(\lambda,N,k,k_{1:n},\mathfrak{w})\\ &\stackrel{\text{def}}{=} \int_0^{\pi}\int_0^1\frac{1}{(r+\alpha_N)(r+\alpha_N+1)\left(1+|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}^N_k\left(N^2(r+\alpha_N),z\right)\right)}\mathrm{d}r\mathrm{d}\theta\,.\end{aligned} \tag{3.57}$$

The definition of I is motivated by the change of variables (3.72) below.

Lemma 3.A.2 (From sum to integral)**.** *There exists a constant $C_{\mathrm{Diag}}>0$ such that*

$$|\tilde{\mathrm{S}}-\mathrm{I}|\leq C_{\mathrm{Diag}}$$

for all in $\lambda\in(0,+\infty)$, $z\in(1,+\infty)$, $k_{1:n}\in(\mathbb{Z}_0^2)^n$, $N\in\mathbb{N}$, $k\in\mathbb{N}$ such that $k\geq 2$ and $\mathfrak{w}\in\mathbb{R}^2$.

Remark 3.A.3. The proof of Theorem 3.4.5 would work even if C_{Diag} depended on $|\mathfrak{w}|$ (up to changing the constant K). In the following proof, however, showing that C_{Diag} does not depend on $\mathfrak{w}$ does not come with any significant additional difficulties.

Proof. The proof proceeds through a number of steps, each of them consisting in slightly modifying the expression of $\tilde{\mathrm{S}}$, at the price of an additive constant, so that it becomes closer to the one of $\tilde{\mathrm{I}}$. More precisely, set $\mathrm{S}_0 \stackrel{\mathrm{def}}{=} \tilde{\mathrm{S}}$ and $\mathrm{S}_6 \stackrel{\mathrm{def}}{=} \mathrm{I}$. Then for all $i \in \{1, \ldots, 6\}$, step i consists in showing $|\mathrm{S}_{i-1} - \mathrm{S}_i| \leq C_i$, where the S_i's for $i \in \{1,2,3,4,5\}$ will be defined in the proof below and C_i's are some absolute constants independent of all variables at play. Compared to [CGT24, Appendix A], we are faced with some additional technical difficulties, coming from the fact that our equation is in the strong coupling regime.

Step 1 We define S_1 by replacing the condition $|k_1| \leq N$, contained in $\mathbb{J}^N_{\ell,m}$, by $|k_1| \leq N/2$. In doing so, we lose all summands of $\tilde{\mathrm{S}}$ corresponding to $\ell + m = k_1 \in [N/2, N]$. Without loss of generality, suppose $|\ell| \geq N/4$. The computation

$$|\tilde{\mathrm{S}} - \mathrm{S}_1| \leq 2 \sum_{\substack{\ell+m=k_1 \\ \frac{N}{2} \leq |k_1| \leq N}} \frac{\mathbf{1}_{\{N/4 \leq |\ell| \leq N\}} \mathbf{1}_{\{|k_1-\ell| \leq N\}}}{\lambda + \tilde{\Gamma} + \tilde{\Gamma}^{\mathfrak{w}} \mathrm{L}^N_k \left(\lambda + \tilde{\Gamma}, z\right)} \lesssim \sum_{\substack{\ell \in \mathbb{Z}^2_0 \\ \frac{N}{4} \leq |\ell| \leq N}} \frac{1}{|\ell|^2} \leq C_1$$

completes step 1.

Step 2 We define S_2 by

$$\mathrm{S}_2 \stackrel{\mathrm{def}}{=} \sum_{\ell+m=k_1} \frac{\mathbb{J}^N_{\ell,m} \mathbf{1}_{\{1 \leq |k_1| \leq N/2\}}}{\lambda + \Gamma + \Gamma^{\mathfrak{w}} \mathrm{L}^N_k (\lambda + \Gamma, z)}.$$

With respect to S_1, we replaced $\tilde{\Gamma}$ and $\tilde{\Gamma}^{\mathfrak{w}}$ by Γ and $\Gamma^{\mathfrak{w}}$ respectively. Using relationship (3.55) on $\tilde{\Gamma}^{\mathfrak{w}}$ and the triangular inequality, we get that

$$|\mathrm{S}_1 - \mathrm{S}_2| \leq \sum_{\substack{\ell+m=k_1 \\ 1 \leq |\ell|, |m| \leq N,\, |k_1| \leq \frac{N}{2}}} \frac{A + B + D}{\left[\lambda + \tilde{\Gamma} + \tilde{\Gamma}^{\mathfrak{w}} \mathrm{L}^N_k \left(\lambda + \tilde{\Gamma}, z\right)\right] \left[\lambda + \Gamma + \Gamma^{\mathfrak{w}} \mathrm{L}^N_k (\lambda + \Gamma, z)\right]}, \tag{3.58}$$

where

$$A \stackrel{\mathrm{def}}{=} \left|\Gamma - \tilde{\Gamma}\right|, \quad B \stackrel{\mathrm{def}}{=} \left|\Gamma^{\mathfrak{w}} \left(\mathrm{L}^N_k (\lambda + \Gamma, z) - \mathrm{L}^N_k \left(\lambda + \tilde{\Gamma}, z\right)\right)\right|,$$
$$D \stackrel{\mathrm{def}}{=} \left|(\mathfrak{w} \cdot k_1)(\mathfrak{w} \cdot \ell) \mathrm{L}^N_k \left(\lambda + \tilde{\Gamma}, z\right)\right|.$$

We estimate (3.58) by considering the terms with A, B and D separately.

First, by dropping some terms from the denominator of (3.58) (they are all positive) and using (3.55) for $\tilde{\Gamma}$, we obtain the following upper bound for the A term:

$$\sum_{\substack{\ell+m=k_1 \\ 1 \leq |\ell|, |m| \leq N,\, |k_1| \leq \frac{N}{2}}} \frac{|\ell \cdot k_1|}{\tilde{\Gamma}\, \Gamma} \lesssim \sum_{|\ell| \leq N} \frac{|\ell \cdot k_1|}{(|\ell|^2 + |k_1|^2)^2} \lesssim 1 \tag{3.59}$$

where one can check that the constant on the right hand side of (3.59) above is independent of k_1 by splitting the sum into the two regions $|\ell| \geq |k_1|$ and $|\ell| < |k_1|$.

Then, by the mean value theorem applied to the function L_k^N and the interval $[a,b]$, where $a \stackrel{\text{def}}{=} (\lambda+\tilde\Gamma)\wedge(\lambda+\Gamma)$ and $b \stackrel{\text{def}}{=} (\lambda+\tilde\Gamma)\vee(\lambda+\Gamma)$, we obtain

$$B \lesssim \Gamma^{\mathfrak{w}} \sup_{y\in[a,b]} \left|\frac{N^2}{y(y+N^2)}\right| \left|\tilde\Gamma - \Gamma\right| \lesssim \Gamma^{\mathfrak{w}} \frac{1}{\Gamma} |\ell\cdot k_1|\,,$$

where we estimated the derivative of L_k^N by first using $\theta_k - 1 < 0$ and $\mathrm{L}_k^N \geq 1$ and then lower bounding $y + N^2 \geq N^2$, so that N^2 cancels. Finally, we used (3.55) again. Thus, by dropping λ and Γ from the second factor in the denominator of (3.58) and lower bounding L_k^N by 1, the term of sum (3.58) corresponding to B is upper bounded by

$$\sum_{|\ell|\leq N} \frac{\Gamma^{\mathfrak{w}}|\ell\cdot k_1|}{\Gamma\,\tilde\Gamma\,\Gamma^{\mathfrak{w}}} = \sum_{|\ell|\leq N} \frac{|\ell\cdot k_1|}{\Gamma\,\tilde\Gamma} \lesssim 1\,,$$

where the last inequalities follows from the same argument used in estimate (3.59).

Finally, we need to bound the sum corresponding to the D term. First of all, observe that if $\mathfrak{w}\cdot k_1 = 0$, then D is identically 0 and so in the following we can assume $\mathfrak{w}\cdot k_1 \neq 0$. In particular, this guarantees that for every ℓ (and for every θ when we will write the integral) the denominator in the following expressions does not vanish. We then drop some terms from the denominator, so to obtain an upper bound in which the function L_k^N has simplified:

$$\sum_{\substack{\ell+m=k_1\\ 1\leq|\ell|,|m|\leq N,\,|k_1|\leq\frac{N}{2}}} \frac{\left|(\mathfrak{w}\cdot k_1)(\mathfrak{w}\cdot\ell)\mathrm{L}_k^N\left(\lambda+\tilde\Gamma,z\right)\right|}{\left[\tilde\Gamma^{\mathfrak{w}}\mathrm{L}_k^N\left(\lambda+\tilde\Gamma,z\right)\right]\Gamma} \lesssim \sum_{1\leq|\ell|\leq N} \frac{|(\mathfrak{w}\cdot k_1)(\mathfrak{w}\cdot\ell)|}{[(\mathfrak{w}\cdot\ell)^2+(\mathfrak{w}\cdot k_1)^2]\,|\ell|^2}\,.$$

Finally, we check that the right hand side of the above can be upper bounded by a convergent series whose sum, as usual, does not depend on any of the variables at play. We do this by passing to an integral. This is justified after excluding ℓ such that $|\mathfrak{w}\cdot\ell| \leq |\mathfrak{w}|$, which can be treated separately. For more details see Step 5, where this is done carefully for the main term. We write this integral using polar coordinates:

$$\begin{aligned}\int_0^{2\pi}\int_0^N \frac{|(\mathfrak{w}\cdot k_1)||\mathfrak{w}||\cos(\theta)|r^2}{[|\mathfrak{w}|^2(\cos\theta)^2r^2+(\mathfrak{w}\cdot k_1)^2]\,r^2}\mathrm{d}r\mathrm{d}\theta &= \int_0^{2\pi}\int_0^N \frac{|(\mathfrak{w}\cdot k_1)||\mathfrak{w}||\cos(\theta)|}{|\mathfrak{w}|^2(\cos\theta)^2r^2+(\mathfrak{w}\cdot k_1)^2}\mathrm{d}r\mathrm{d}\theta\\ &\leq \frac{\pi}{2}\int_0^{2\pi} \frac{|(\mathfrak{w}\cdot k_1)||\mathfrak{w}||\cos(\theta)|}{\sqrt{|\mathfrak{w}|^2(\cos\theta)^2(\mathfrak{w}\cdot k_1)^2}}\mathrm{d}r\mathrm{d}\theta = \pi^2\,,\end{aligned}$$

where we used Lemma 3.A.1 to compute the integral in r.

Thus step 2 is completed with C_2 equal to the sum of the three constants with which we have estimated the sum corresponding to the A, B and D terms.

Step 3 We define S_3 by

$$\mathrm{S}_3 \stackrel{\text{def}}{=} \sum_{\ell+m=k_1} \frac{\mathbb{J}_{\ell,m}^N \mathbf{1}_{\{1\leq|k_1|\leq N/2\}}}{\lambda+\Gamma+(\mathfrak{w}\cdot\ell)^2\mathrm{L}_k^N(\lambda+\Gamma,z)}\,.$$

With respect to S_2, we have replaced $\Gamma^{\mathfrak{w}}$ by $(\mathfrak{w}\cdot\ell)^2$. If $(\mathfrak{w}\cdot k)_{1:n}^2 = 0$, we do not have anything to prove. Otherwise, we estimate

$$|\mathrm{S}_2 - \mathrm{S}_3| \lesssim \sum_{\substack{\ell+m=k_1\\ 1\leq|\ell|,|m|\leq N,\,|k_1|\leq\frac{N}{2}}} \frac{(\mathfrak{w}\cdot k)_{1:n}^2\,\mathrm{L}_k^N(\lambda+\Gamma,z)}{\left[\lambda+\Gamma+\Gamma^{\mathfrak{w}}\mathrm{L}_k^N(\lambda+\Gamma,z)\right]\left[\lambda+\Gamma+(\mathfrak{w}\cdot\ell)^2\mathrm{L}_k^N(\lambda+\Gamma,z)\right]}$$

$$\lesssim \sum_{1\le|\ell|\le N} \frac{(\mathfrak{w}\cdot k)_{1:n}^2}{\left[(\mathfrak{w}\cdot\ell)^2+(\mathfrak{w}\cdot k)_{1:n}^2\right]\left[|\ell|^2+|k_{1:n}|^2\right]}$$
$$\lesssim \int_0^N\int_0^{2\pi} \frac{(\mathfrak{w}\cdot k)_{1:n}^2 r}{(|\mathfrak{w}|^2r^2(\cos\theta)^2+(\mathfrak{w}\cdot k)_{1:n}^2)(r^2+|k_{1:n}|^2)}\mathrm{d}\theta\mathrm{d}r$$
$$\lesssim \int_0^N \frac{(\mathfrak{w}\cdot k)_{1:n}^2}{r^2+|k_{1:n}|^2}\frac{r}{\sqrt{(\mathfrak{w}\cdot k)_{1:n}^2((\mathfrak{w}\cdot k)_{1:n}^2+|\mathfrak{w}|^2r^2)}}\mathrm{d}r\,,$$

where we first dropped $\lambda+\Gamma$ and $(\mathfrak{w}\cdot\ell)^2\mathrm{L}_k^N$ from the first and second factor in the denominator respectively, then simplified L_k^N and finally used Lemma 3.A.1 to estimate the integral in θ. Before moving from the sum to the integral, one once again needs to exclude ℓ such that $|\mathfrak{w}\cdot\ell|\le|\mathfrak{w}|$. These can again easily be treated separately. We now simplify the multiplicative factor $(\mathfrak{w}\cdot k)_{k_{1:n}}^2$ in the denominator and drop the additive one, so that we can simplify r and obtain the upper bound

$$\int_0^N \frac{|\mathfrak{w}\cdot k|_{1:n}}{r^2+|k_{1:n}|^2}\frac{1}{|\mathfrak{w}|}\mathrm{d}r \lesssim \frac{|\mathfrak{w}||k_{1:n}|}{|\mathfrak{w}|}\frac{1}{\sqrt{|k_{1:n}|^2}}=1\,,$$

where we applied the Cauchy-Schwarz inequality to the numerator and estimated the integral by using Lemma 3.A.1. This concludes step 3.

Step 4 We define S_4 by

$$\mathrm{S}_4 \stackrel{\mathrm{def}}{=} \sum_{\ell+m=k_1} \frac{\mathbf{1}_{\{1\le|\ell|\le N\}}\mathbf{1}_{\{1\le|k_1|\le N/2\}}}{\lambda+\Gamma+(\mathfrak{w}\cdot\ell)^2\mathrm{L}_k^N(\lambda+\Gamma,z)}\,.$$

With respect to S_3, we have removed the constraint $\mathbf{1}_{\{|m|\le N\}}$. More precisely, we are adding to S_3 the terms indexed by the set

$$\left\{\ell,m\in\mathbb{Z}_0^2 : \ell+m=k_1,\ |k_1|\le \tfrac{N}{2},\ |\ell|\le N,\ |m|>N\right\}\,,$$

which is contained (thanks to the extra condition on $|k_1|$ imposed in step 1) in

$$\left\{\ell,m\in\mathbb{Z}_0^2 : \ell+m=k_1,\ |k_1|\le \tfrac{N}{2},\ |\ell|\le N,\ |\ell|\ge \tfrac{N}{2}\right\}\,.$$

The sum over this last index set can be bounded as done in step 1.

Step 5 We define S_5 by

$$\mathrm{S}_5 \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2} \frac{\mathbf{1}_{\{|x|\le 1\}}}{\alpha_N+|x|^2+(\mathfrak{w}\cdot x)^2\mathrm{L}_k^N(N^2(\alpha_N+|x|^2),z)}\mathrm{d}x\,. \tag{3.60}$$

We set $Q_\ell^N \stackrel{\mathrm{def}}{=} \frac{1}{N}[\ell-\frac{1}{2},\ell-\frac{1}{2}]^2\subset\mathbb{R}^2$ and by multiplying and dividing S_4 by $\frac{1}{N^2}$ we obtain

$$\mathrm{S}_4 = \sum_{\ell+m=k_1}\mathbf{1}_{\{1\le|\ell|\le N\}}\mathbf{1}_{\{1\le|k_1|\le N/2\}}\int_{Q_\ell^N} I_N\left(\tfrac{\ell}{N}\right)\mathrm{d}x\,, \tag{3.61}$$

where we denoted by I_N the integrand of S_5 (without the indicator function $\mathbf{1}_{\{|x|\le1\}}$). To show $|\mathrm{S}_4-\mathrm{S}_5|\le C_5$, we write S_5 as the sum in ℓ of the integrals over Q_ℓ^N. Since for large N the summand $I_N(x)$ changes very rapidly when x and $\mathfrak{w}$ are almost orthogonal, we will treat this case separately. Note first that

$$\sum_{\ell+m=k_1}\frac{\mathbf{1}_{\{1\le|\ell|\le N\}}\mathbf{1}_{\{1\le|k_1|\le N/2\}}\mathbf{1}_{\{|\ell\cdot\mathfrak{w}|\le|\mathfrak{w}|\}}}{\lambda+\Gamma+(\mathfrak{w}\cdot\ell)^2\mathrm{L}_k^N(\lambda+\Gamma,z)} \lesssim \sum_{\ell\neq0}\frac{\mathbf{1}_{\{|\ell\cdot\mathfrak{w}|<|\mathfrak{w}|\}}}{|\ell|^2}\lesssim 1\,.$$

For the integral note that $\bigcup_{|\ell\cdot\mathfrak{w}|\le\mathfrak{w}} Q_\ell^N$ is contained in $\{x\in\mathbb{R}^2 : |x\cdot\mathfrak{w}|\le \frac{2}{N}|\mathfrak{w}|\}$. Using this we see

$$\sum_{\substack{1\le|\ell|\le N\\ |\mathfrak{w}\cdot\ell|\le|\mathfrak{w}|}} \int_{Q_\ell^N} I_N(x)\mathrm{d}x \le \int_{|\mathfrak{w}\cdot x|\le\frac{2}{N}|\mathfrak{w}|} I_N(x)\mathrm{d}x \le \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{1}{\alpha+|x|^2}\mathrm{d}x_1\mathrm{d}x_2 \lesssim 1\,,$$

where we used a change of variables in x (i.e. scaling by N) and the fact that $\alpha\ge 1$ as well as Lemma 3.A.1. Also note that the x appearing in the rewriting of S_4 (3.61) but not in S_5, are contained in $\{x : 1\le|x|\le 1+\frac{\sqrt{2}}{2N}\}$ and

$$\int_{1\le|x|\le1+\frac{1}{2N}} I_N(x)\mathrm{d}x \lesssim \sup_{1\le|x|\le 1+\frac{1}{2N}} |I_N(x)| \lesssim 1\,.$$

It thus remains to show that

$$\sum_{\substack{1\le|\ell|\le N\\ |\ell\cdot\mathfrak{w}|\ge|\mathfrak{w}|}} \int_{Q_\ell^N} \left|I_N\left(\tfrac{\ell}{N}\right) - I_N(x)\right| \mathrm{d}x \lesssim 1\,. \tag{3.62}$$

In order to prove (3.62), we estimate, by the mean value theorem applied to the function I_N and the line segment $[\frac{\ell}{N}, x]$,

$$\left|I_N\left(\tfrac{\ell}{N}\right) - I_N(x)\right| \le \sup_{y\in Q_\ell^N} |\nabla I_N(y)|\left|\tfrac{\ell}{N}-x\right| \lesssim \frac{1}{N}(E+F+G)\,, \tag{3.63}$$

where E, F and G are the suprema over Q_ℓ^N of the the norms of the three terms in the expression of the gradient below:

$$-\frac{1}{2}\nabla I_N(x) = \frac{x + (\mathfrak{w}\cdot x)\mathrm{L}_k^N(N^2(|x|^2+\alpha_N),z)\mathfrak{w} + \frac{(-\theta_k)(\mathfrak{w}\cdot x)^2(\mathrm{L}^N(N^2(|x|^2+\alpha_N),z))^{1-\theta_k}x}{(|x|^2+\alpha_N)(|x|^2+\alpha_N+1)}}{\left(\alpha_N+|x|^2+(\mathfrak{w}\cdot x)^2\mathrm{L}_k^N(N^2(|x|^2+\alpha_N),z)\right)^2}\,. \tag{3.64}$$

Since each Q_ℓ^N has an area of $\frac{1}{N^2}$, we need to show that

$$\sum_{\substack{1\le|\ell|\le N\\ |\ell\cdot\mathfrak{w}|\ge|\mathfrak{w}|}} E+F+G \lesssim N^3\,. \tag{3.65}$$

Note first that for all ℓ such that $|\ell\cdot\mathfrak{w}|\ge|\mathfrak{w}|$ and $|\ell|\ge1$ and for all $x\in Q_\ell^N$ it holds that

$$\begin{gathered}\left(1-\frac{\sqrt{2}}{2}\right)\left|\frac{\ell}{N}\right| \le |x| \le \left(1+\frac{\sqrt{2}}{2}\right)\left|\frac{\ell}{N}\right|\,,\\ \left(1-\frac{\sqrt{2}}{2}\right)\left|\mathfrak{w}\cdot\frac{\ell}{N}\right| \le |\mathfrak{w}\cdot x| \le \left(1+\frac{\sqrt{2}}{2}\right)\left|\mathfrak{w}\cdot\frac{\ell}{N}\right|\,.\end{gathered} \tag{3.66}$$

For E note that

$$\sup_{x\in Q_\ell^N} \frac{|x|}{\left(\alpha_N+|x|^2+(\mathfrak{w}\cdot x)^2\mathrm{L}_k^N(N^2(|x|^2+\alpha_N),z)\right)^2} \lesssim \frac{N^3}{|\ell|^3}\,.$$

For F note that

$$\sup_{x\in Q_\ell^N} \frac{|\mathfrak{w}\cdot x|\mathrm{L}_k^N(N^2(|x|^2+\alpha_N),z)|\mathfrak{w}|}{\big(\alpha_N+|x|^2+(\mathfrak{w}\cdot x)^2\mathrm{L}_k^N(N^2(|x|^2+\alpha_N),z)\big)^2} \lesssim \frac{N^3|\mathfrak{w}|}{|\ell|^2|\mathfrak{w}\cdot\ell|}\,,$$

where we used one of the factors of the denominator to cancel the L_k^N in the numerator. Now note again by (3.66) that

$$\sum_{\substack{1\le|\ell|\le N\\ |\ell\cdot\mathfrak{w}|\ge|\mathfrak{w}|}} \frac{|\mathfrak{w}|}{|\ell|^2|\mathfrak{w}\cdot\ell|} \lesssim \int_{|x\cdot\mathfrak{w}|\ge(1-\frac{\sqrt{2}}{2})|\mathfrak{w}|} \frac{|\mathfrak{w}|}{|x|^2|\mathfrak{w}\cdot x|}\mathrm{d}x \lesssim 1\,.$$

Finally for G by similar arguments $G \lesssim \frac{N^3}{\ell^3}$, (note that $0\le\theta_k\le 1$).

This completes step 5, with C_5 equal to the sum of the three constants with which we have estimated the sum corresponding to the E, F and G terms.

Step 6 Let $\theta_\mathfrak{w}$ be the angle from the first coordinate axis to $\mathfrak{w}$. By successively performing the change of variables $x\mapsto r(\cos\theta,\sin\theta)$ and $r^2\mapsto r$, we first rewrite S_5 as follows.

$$\begin{aligned}
\mathrm{S}_5 &= \int_0^{2\pi}\int_0^1 \frac{r}{r^2+\alpha_N+r^2|\mathfrak{w}|^2\cos^2(\theta-\theta_\mathfrak{w})\mathrm{L}_k^N\left(N^2(r^2+\alpha_N),z\right)}\mathrm{d}r\mathrm{d}\theta\\
&= \int_0^{2\pi}\frac{1}{2}\int_0^1 \frac{1}{r+\alpha_N+r|\mathfrak{w}|^2\cos^2(\theta-\theta_\mathfrak{w})\mathrm{L}_k^N\left(N^2(r+\alpha_N),z\right)}\mathrm{d}r\mathrm{d}\theta\\
&= \int_0^{\pi}\int_0^1 \frac{1}{r+\alpha_N+r|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N\left(N^2(r+\alpha_N),z\right)}\mathrm{d}r\mathrm{d}\theta\,,
\end{aligned}\tag{3.67}$$

where in the last equality we used the π-periodicity of the integrand as a function of θ. We then recall the definition of $\mathrm{S}_6 \overset{\text{def}}{=} \mathrm{I}$ given in (3.57). We observe that the absolute value of the difference between (3.67) and (3.57), after the simplifications that occur, is upper bounded by

$$\int_0^{\pi}\int_0^1 \frac{(r+\alpha_N)^2(1+|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N)+\alpha_N|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N}{\left[r+\alpha_N+r|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N\right]\left[(r+\alpha_N)(r+\alpha_N+1)(1+|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N)\right]}\mathrm{d}r\mathrm{d}\theta\,,\tag{3.68}$$

where we omitted the argument of L_k^N for ease of reading. We thus study the two terms corresponding to the two summand of the numerator separately. The first one can be estimated by

$$\int_0^{\pi}\int_0^1 \frac{(r+\alpha_N)}{[r+\alpha_N]\left[(r+\alpha_N+1)\right]}\mathrm{d}r\mathrm{d}\theta \le \int_0^{\pi}\int_0^1 \frac{1}{r+\alpha_N+1}\mathrm{d}r\mathrm{d}\theta \lesssim 1$$

and the second one by

$$\begin{aligned}
&\int_0^{\pi}\int_0^1 \frac{\alpha_N|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N}{[r+\alpha_N]\left[(r+\alpha_N)(r+\alpha_N+1)|\mathfrak{w}|^2(\cos\theta)^2\mathrm{L}_k^N\right]}\mathrm{d}r\mathrm{d}\theta\\
&= \int_0^{\pi}\int_0^1 \frac{\alpha_N}{(r+\alpha_N)^2(r+\alpha_N+1)}\mathrm{d}r\mathrm{d}\theta\\
&\lesssim \alpha_N\int_0^1 \frac{1}{(r+\alpha_N)^2}\mathrm{d}r = \alpha_N\left(\frac{1}{\alpha_N}-\frac{1}{1+\alpha_N}\right)\le 1\,.
\end{aligned}$$

This concludes step 6 and with it also the proof of Lemma 3.A.2. □

Lemma 3.A.4 (From integral to estimate)**.** *Recall the definition of* $\tilde{\mathrm{S}}$ *given in (3.56) (in particular that it depends on* k*). There exists a constant* C_{Diag} *such that, for any even* $k \geq 2$*,*

$$\tilde{\mathrm{S}} \leq \frac{3\pi}{2|\mathfrak{w}|}\left(1+\frac{|\mathfrak{w}|C_{\mathrm{Diag}}}{z^{\theta_{k+1}}}\right)\mathrm{L}^N_{k+1}\left(\lambda+|k_{1:n}|^2, z\right), \tag{3.69}$$

whereas, for any odd $k \geq 3$*,*

$$\tilde{\mathrm{S}} \geq \frac{3\pi}{2|\mathfrak{w}|}\left[\left(1-\left(|\mathfrak{w}|C_{\mathrm{Diag}}+2+\frac{3}{|\mathfrak{w}|}\right)\frac{1}{z^{\frac{\theta_k}{2}}}\right)\mathrm{L}^N_{k+1}(\alpha, z)-\frac{4}{3}z^{\theta_{k+1}}\right], \tag{3.70}$$

uniformly in $\lambda \in (0, +\infty)$*,* $z \in (1, +\infty)$*,* $k_{1:n} \in (\mathbb{Z}^2_0)^n$*,* $N \in \mathbb{N}$ *and* $\mathfrak{w} \in \mathbb{R}^2$*.*

Proof. By applying Lemma 3.A.2, we immediately get

$$|\tilde{\mathrm{S}} - \mathrm{I}| \leq C_{\mathrm{Diag}}\,. \tag{3.71}$$

The task now is to obtain bounds on I, whose definition was given in (3.57).

We start by proving the statement for $k \geq 4$, which corresponds to $\theta_k \in \left[\frac{1}{2}, \frac{3}{4}\right]$. We first transform integral I with the change of variables

$$u = \mathrm{L}^N\left(N^2(r+\alpha_N), z\right) = \log\left(1+\frac{1}{r+\alpha_N}\right)+z\,, \quad \mathrm{d}u = \frac{-1}{(r+\alpha_N)(r+\alpha_N+1)}\mathrm{d}r\,, \tag{3.72}$$

which gives

$$\mathrm{I} = \int_0^\pi \int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \frac{1}{1+|\mathfrak{w}|^2(\cos\theta)^2 u^{\theta_k}}\mathrm{d}\theta\mathrm{d}u\,. \tag{3.73}$$

By then integrating in θ, using Lemma 3.A.1 with $\gamma = |\mathfrak{w}|^2 u^{\theta_k}$ and $\beta = 1$, we get

$$\int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \frac{\pi}{\sqrt{1+|\mathfrak{w}|^2 u^{\theta_k}}}\mathrm{d}u = \frac{\pi}{|\mathfrak{w}|}(\mathrm{I}_1 - \mathrm{I}_2)\,,$$

where $\mathrm{I}_1, \mathrm{I}_2 > 0$ are defined by

$$\mathrm{I}_1 \stackrel{\text{def}}{=} \int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \frac{1}{\sqrt{u^{\theta_k}}}\mathrm{d}u\,, \qquad \mathrm{I}_2 \stackrel{\text{def}}{=} \int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \left(-\frac{1}{\sqrt{|\mathfrak{w}|^{-2}+u^{\theta_k}}}+\frac{1}{\sqrt{u^{\theta_k}}}\right)\mathrm{d}u\,.$$

The integral I_1 is the one announced in the general strategy explained before Lemma 3.A.2. Indeed, it can be computed explicitly:

$$\mathrm{I}_1 = \frac{\left(\mathrm{L}^N(\alpha, z)\right)^{\theta_{k+1}}}{\theta_{k+1}} - \frac{\left(\mathrm{L}^N(N^2(1+\alpha_N), z)\right)^{\theta_{k+1}}}{\theta_{k+1}} = \frac{\mathrm{L}^N_{k+1}(\alpha, z)}{\theta_{k+1}} - \frac{\mathrm{L}^N_{k+1}(N^2(1+\alpha_N), z)}{\theta_{k+1}}\,. \tag{3.74}$$

It is precisely this computation that gives upper and lower bounds of the form logarithm to the power θ_k, with the sequence of powers $(\theta_k)_{k\geq 2}$ converging to $2/3$. The integral I_2, instead, is regarded as an error term and can be estimated by

$$\begin{aligned}\mathrm{I}_2 &= \int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \frac{\sqrt{|\mathfrak{w}|^{-2}+u^{\theta_k}}-\sqrt{u^{\theta_k}}}{\sqrt{u^{\theta_k}}\sqrt{|\mathfrak{w}|^{-2}+u^{\theta_k}}}\mathrm{d}u \\ &\leq \int_{\mathrm{L}^N(N^2(1+\alpha_N), z)}^{\mathrm{L}^N(\alpha, z)} \frac{\sqrt{u^{\theta_k}}+\sqrt{|\mathfrak{w}|^{-2}}-\sqrt{u^{\theta_k}}}{\sqrt{u^{\theta_k}}\sqrt{u^{\theta_k}}}\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&= |\mathfrak{w}|^{-1} \int_{\mathrm{L}^N(N^2(1+\alpha_N),z)}^{\mathrm{L}^N(\alpha,z)} \frac{1}{u^{\theta_k}} \mathrm{d}u \\
&= \frac{1}{|\mathfrak{w}|(1-\theta_k)} \left[\left(\mathrm{L}^N(\alpha,z)\right)^{1-\theta_k} - \left(\mathrm{L}^N(N^2(1+\alpha_N),z)\right)^{1-\theta_k} \right] .
\end{aligned} \tag{3.75}
$$

The inequality

$$
\theta_{k+1} - (1-\theta_k) = \frac{\theta_k}{2} > 0
$$

shows that $\mathrm{I}_2(N) = o_{N\to\infty}(\mathrm{I}_1(N))$, so that I_2 is indeed of lower-order in N with respect to I_1.

We now have everything we need to conclude the proof in the case $k \geq 4$. We first show the upper bound (3.69). Inequality (3.71) and the above steps give

$$
\tilde{\mathrm{S}} \leq \frac{\pi}{|\mathfrak{w}|}(\mathrm{I}_1 - \mathrm{I}_2) + C_{\mathrm{Diag}} . \tag{3.76}
$$

Then, by plugging (3.74) and (3.75) into (3.76) above and dropping the negative terms, we obtain

$$
\begin{aligned}
\tilde{\mathrm{S}} &\leq \frac{\pi}{|\mathfrak{w}|\theta_{k+1}} \left(\mathrm{L}^N(\alpha,z)\right)^{\theta_{k+1}} + C_{\mathrm{Diag}} \\
&\leq \frac{\pi}{|\mathfrak{w}|\theta_{k+1}} \left(\mathrm{L}^N(\alpha,z)\right)^{\theta_{k+1}} + C_{\mathrm{Diag}} \frac{\left(\mathrm{L}^N(\alpha,z)\right)^{\theta_{k+1}}}{z^{\theta_{k+1}}} \\
&\leq \frac{3\pi}{2|\mathfrak{w}|} \left(1 + \frac{|\mathfrak{w}|C_{\mathrm{Diag}}}{z^{\theta_{k+1}}}\right) \mathrm{L}^N_{k+1}(\alpha,z) ,
\end{aligned} \tag{3.77}
$$

where we observed that, for k even, $\theta_{k+1} \in \left[\frac{2}{3}, 1\right]$. What we obtained is exactly the claimed (3.69).

Finally, we prove the lower bound (3.70). Inequality (3.71) and the above steps give

$$
\tilde{\mathrm{S}} \geq \frac{\pi}{|\mathfrak{w}|}(\mathrm{I}_1 - \mathrm{I}_2) - C_{\mathrm{Diag}} .
$$

By dropping the second term in the square brackets in (3.75) and using $1+\alpha_N \leq 2$ to estimate the negative term in (3.74) by $\mathrm{L}_{k+1}(1+\alpha_N, z) \leq (\log 2 + z)^{\theta_{k+1}} \leq (\log 2)^{\theta_{k+1}} + z^{\theta_{k+1}}$, we obtain

$$
\begin{aligned}
\tilde{\mathrm{S}} &\geq \frac{\pi}{|\mathfrak{w}|} \left[\frac{\left(\mathrm{L}^N(\alpha,z)\right)^{\theta_{k+1}}}{\theta_{k+1}} - \frac{(\log 2)^{\theta_{k+1}}}{\theta_{k+1}} - \frac{z^{\theta_{k+1}}}{\theta_{k+1}} - \frac{1}{|\mathfrak{w}|(1-\theta_k)} \left(\mathrm{L}^N(\alpha,z)\right)^{1-\theta_k} \right] - C_{\mathrm{Diag}} \\
&\geq \frac{3\pi}{2|\mathfrak{w}|} \left[\left(1 - \frac{|\mathfrak{w}|C_{\mathrm{Diag}} + 2}{z^{\theta_{k+1}}} - \frac{3}{|\mathfrak{w}| z^{\frac{\theta_k}{2}}}\right) \mathrm{L}^N_{k+1}(\alpha,z) - \frac{4}{3} z^{\theta_{k+1}} \right] \\
&\geq \frac{3\pi}{2|\mathfrak{w}|} \left[\left(1 - \left(|\mathfrak{w}|C_{\mathrm{Diag}} + 2 + \frac{3}{|\mathfrak{w}|}\right) \frac{1}{z^{\frac{\theta_k}{2}}}\right) \mathrm{L}^N_{k+1}(\alpha,z) - \frac{4}{3} z^{\theta_{k+1}} \right] ,
\end{aligned} \tag{3.78}
$$

where for the first inequality we observed that, for k odd, $\theta_{k+1} \in \left[\frac{1}{2}, \frac{2}{3}\right]$ and then applied similar steps as the ones used to obtain (3.77), whereas in the second one we used that $\theta_{k+1} \geq \frac{\theta_k}{2}$. What we obtained is exactly the claimed (3.70).

If $k = 3$, we proceed as above until we reach we reach (3.73). At this point, we do not need to split I into I_1 and I_2, since in this case I already admits an explicit primitive.

Indeed, by first using Lemma 3.A.1 to integrate in θ and then applying the change of variables $v = 1 + |\mathfrak{w}|^2 u$, we obtain

$$\mathrm{I} = \frac{2\pi}{|\mathfrak{w}|^2}\sqrt{1+|\mathfrak{w}|^2 u}\Big|_{\mathrm{L}^N(N^2(1+\alpha_N),z)}^{\mathrm{L}^N(\alpha,z)},$$

so that

$$\begin{aligned}\tilde{\mathrm{S}} &\geq \mathrm{I} - C_{\mathrm{Diag}}\\ &\geq \frac{3\pi}{2|\mathfrak{w}|}\left[\left(1-\frac{|\mathfrak{w}|C_{\mathrm{Diag}}}{z^{\theta_4}}\right)\mathrm{L}_4^N(\alpha,z) - \frac{4}{3}\left(\frac{1}{|\mathfrak{w}|^2}+\log(2)+z\right)^{\theta_4}\right]\\ &\geq \frac{3\pi}{2|\mathfrak{w}|}\left[\left(1-\left(|\mathfrak{w}|C_{\mathrm{Diag}}+2+\frac{3}{|\mathfrak{w}|}\right)\frac{1}{z^{\theta_4}}\right)\mathrm{L}_4^N(\alpha,z) - \frac{4}{3}z^{\theta_4}\right],\end{aligned}$$

where we recalled that $\theta_4 = \frac{1}{2}$ and followed steps similar to the ones with which we obtained (3.78). The claimed 3.70 immediately follows by the trivial $\theta_4 \geq \frac{\theta_4}{2}$.

If $k = 2$, $\mathrm{L}_k^N \equiv 1$, because $\theta_2 = 0$ by definition. In this somewhat degenerate case, we estimate from above I, defined in 3.57, using the following steps. First apply Lemma 3.A.1 to integrate it in θ. Then lower bound the resulting factor $(r + \alpha_N + 1)$ in the denominator by 1 and integrate in r. The upper bound obtained in this way is

$$\frac{\pi}{|\mathfrak{w}|}\log\left(1+\frac{1}{\alpha_N}\right) \leq \frac{\pi}{|\mathfrak{w}|}\mathrm{L}_3^N\left(\lambda+|k_{1:n}|^2, z\right). \tag{3.79}$$

Finally, the stated (3.69) follows from (3.71) and steps analogous to the ones that concluded the case $k \geq 4$. □

Remark 3.A.5. In inequalities (3.77) and (3.78), we first go from an additive error to a multiplicative one, which is easier to iterate, but increases complexity, and then decrease complexity by estimating $\mathrm{L}^N(\alpha, z)$ by z. This is quite rough, but otherwise the iteration would give more and more complicated bounds at each step.

3.B Heuristic Derivation of the Green Kubo Formula

In this section we give a heuristic derivation of the bulk diffusivity formula (3.3).

Consider the equation on the full space regularized with Fourier cutoff 1 with regularized white noise:

$$\partial_t\eta = \frac{1}{2}\Delta\eta + \mathfrak{w}\cdot\Pi_1\nabla(\Pi_1\eta)^2 + \nabla\cdot\Pi_a\xi\,, \tag{3.80}$$

where $\Pi_a\xi$ is a space time white noise, regularized in space by a cut-off in Fourier at level $a \in (1, \infty)$. It can be seen using techniques adopted in [CES21] that this equation still has a unique solution, existing for all time, and this solution is a strong Markov process invariant under translations in space and time. The invariant measure is given by regularized spacial white noise η^a, regularized by the same cutoff as $\Pi_a\xi$. We consider the equation run at stationarity, i.e. started from η^a. Since the noise is regularized, η will be a continuous function and therefore we can evaluate it at space-time points. This allows us to define the correlation function

$$S(x,t) = \mathbf{E}(\eta(x,t)\eta(0,0))\,,$$

for $t \geq 0$ and $x \in \mathbb{R}^2$.

The bulk diffusivity is commonly defined (see e.g. [Yau04, Spo12] for examples from discrete systems and [BQS11] for a continuous example in $d = 1$) as the matrix $(D_{ij}(t))_{1\leq i,j\leq 2}$ with entries given by

$$D_{ij}(t) = \frac{1}{2t}\int_{\mathbb{R}^2} x_i x_j S(x,t)\mathrm{d}x\,.$$

Without loss of generality assume that $\mathfrak{w}_2 = 0$, then the reflection symmetry of the system in the second component gives $D_{12}(t) = D_{21}(t) = 0$. We will work with the bulk diffusivity as defined in [CET23]

$$D(t) = \frac{1}{2t}\int_{\mathbb{R}^2} |x|^2 S(x,t)\mathrm{d}x\,,$$

which can be interpreted as ($1/t$ times) the variance of $S(\cdot,t)$ seen as a density. In a particle system this would be the density of a second class particle started at the origin. This definition of the bulk diffusivity can be connected to the bulk diffusivity matrix above by taking the trace, see also the remark at the end of the section.

We now want to show that this definition of the bulk diffusivity is heuristically consistent with (3.3). To do this assume that $S(x,t)$ decays fast in $|x|$, noting that for the linear case ($w = 0$) it is Gaussian. Also assume that $S(\cdot,t)$ integrates to 1 for every t. At time $t = 0$ this is true by the law of the stationary measure, since cutting Fourier-modes larger than 1 is equivalent to convolving with a mass 1 bump function. For later time it formally follows from the conservative nature of (3.80):

$$\begin{aligned}\int_{\mathbb{R}^2} S(x,t)\mathrm{d}x &= \int_{\mathbb{R}^2}\mathbf{E}(\eta(x,t)\eta(0,0))\mathrm{d}x = \mathbf{E}\left(\int_{\mathbb{R}^2}\eta(x,t)\mathrm{d}x\,\eta(0,0)\right)\\ &= \mathbf{E}\left(\int_{\mathbb{R}^2}\eta(x,0)\mathrm{d}x\,\eta(0,0)\right) = \int_{\mathbb{R}^2} S(x,0)\mathrm{d}x = 1\,,\end{aligned}$$

where the second to last equality follows from an integration by parts, because the entire right hand side of (3.80) can be put in divergence form. Integrating (3.80) in time, multiplying by $\eta(0,0)$ and taking expectations we obtain:

$$S(x,t) = S(x,0) + \frac{1}{2}\int_0^t \Delta S(x,s)\mathrm{d}s + \int_0^t \mathcal{N}(\eta)(x,s)\eta(0,0)\mathrm{d}s\,, \tag{3.81}$$

where the noise term disappears because it has zero space-average and $\mathcal{N}$ is

$$\mathcal{N}(\eta) = \mathfrak{w}\cdot\nabla\left(\Pi_1\eta\right)^2\,.$$

We will integrate the terms on the left hand side against $|x|^2$ and divide them by $2t$ one by one. The first one doesn't depend on time before dividing by $2t$ and so will vanish for large t. The second one is

$$\frac{1}{4t}\int_0^t\int_{\mathbb{R}^2}|x|^2\Delta S(x,t)\mathrm{d}x = \frac{1}{t}\int_0^t\int_{\mathbb{R}^2} S(x,t)\mathrm{d}x = 1\,.$$

Finally let's consider the third one. Using that $\mathcal{N}$ is quadratic in η and η is Gaussian we see that

$$\mathbf{E}\left(\mathcal{N}(\eta)(s,x)\eta(s,0)\right) = 0\,. \tag{3.82}$$

and we rewrite

$$\int_{\mathbb{R}^2}|x|^2\mathbf{E}\left(\mathcal{N}(\eta)(s,x)\eta(0,0)\right)\mathrm{d}x = \int_{\mathbb{R}^2}|x|^2\mathbf{E}\left(\mathcal{N}(\eta)(s,x)(\eta(0,0)-\eta(s,0)\right)\mathrm{d}x$$

$$= \int_{\mathbb{R}^2} |x|^2 \mathbb{E}\left(\mathcal{N}(\xi^a)(x) \tilde{\mathbf{E}}_{\xi^a}(\tilde{\eta}(s,0) - \tilde{\eta}(0,0))\right) \mathrm{d}x\,,$$

where the $\tilde{\mathbf{E}}_{\xi^a}$ is the law of $\tilde{\eta}(r,x) \stackrel{\text{def}}{=} \eta(s-r,x)$ and we used translation invariance of ξ^a. The time reversed process $\tilde{\eta}$ satisfies the equation (3.80) with a changed sign in front of the Laplacian and the nonlinearity and a different noise with the same law, that is also independent of ξ^a. Using this we get

$$\mathbb{E}\left(\mathcal{N}(\xi^a)(x) \tilde{\mathbf{E}}_{\xi^a}(\tilde{\eta}(s,0) - \tilde{\eta}(0,0))\right) = \int_0^s \mathbb{E}\left(\mathcal{N}(\xi^a)(x) \tilde{\mathbf{E}}_{\xi^a}(-\Delta\tilde{\eta}(r,0) - \mathcal{N}(\tilde{\eta})(r,0)\right) \mathrm{d}r\,, \tag{3.83}$$

where we used that the noise term vanishes under the expectation. The term with the Laplacian, after being integrated against $|x|^2$, using the translation invariance to move the x to the $\tilde{\eta}$ and an integration by parts becomes

$$\begin{aligned} -\int_0^s \int_{\mathbb{R}^2} \mathbb{E}\left(\mathcal{N}(\xi^a)(0) \tilde{\mathbf{E}}_{\xi^a}(\tilde{\eta}(r,x)\right) \mathrm{d}x\mathrm{d}r &= -\int_0^s \int_{\mathbb{R}^2} \mathbb{E}\left(\mathcal{N}(\xi^a)(0)\eta(s-r,-x)\right) \mathrm{d}x\mathrm{d}r \\ &= -\int_0^s \int_{\mathbb{R}^2} \mathbb{E}\left(\mathcal{N}(\xi^a)(0)\eta(0,-x)\right) \mathrm{d}x\mathrm{d}r = 0\,, \end{aligned}$$

where we used that the dynamics are conservative and then again (3.82).

The second term in (3.83) also integrated against $|x|^2$ becomes

$$\begin{aligned} &-\int_{\mathbb{R}^2} \int_0^s |x|^2 \mathbb{E}\left(\mathcal{N}(\xi^a)(x) \tilde{\mathbf{E}}_{\xi^a}(\mathcal{N}(\tilde{\eta})(r,0))\right) \mathrm{d}r\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^2} \int_0^s |x|^2 \mathbf{E}\left(\mathcal{N}(\eta)(s,x)\mathcal{N}(\eta)(s-r,0)\right) \mathrm{d}r\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^2} \int_0^s |x|^2 \mathbf{E}\left(\mathcal{N}(\eta)(r,x)\mathcal{N}(\eta)(0,0)\right) \mathrm{d}r\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^2} \int_0^s |x|^2 \mathbf{E}\left(((\mathfrak{w}\cdot\nabla)\Pi^1{:}(\Pi^1\eta)^2{:})(r,x)((\mathfrak{w}\cdot\nabla)\Pi^1{:}(\Pi^1\eta)^2{:})(0,0)\right) \mathrm{d}r\mathrm{d}x \\ &\quad = 2|\mathfrak{w}|^2 \int_{\mathbb{R}^2} \int_0^s \mathbf{E}\left((\Pi^1{:}(\Pi^1\eta)^2{:})(r,x)(\Pi^1{:}(\Pi^1\eta)^2{:})(0,0)\right) \mathrm{d}r\mathrm{d}x\,, \end{aligned}$$

where in the last step we first performed integration by parts on the gradient from the first factor, and then used translation invariance to move the x to the second factor, after which we perform another integration by parts. Each integration by parts gives a factor -1, as well as an additional -1, since the x becomes a $-x$ when moved to the second factor. These integration by parts are not rigorous, since we cannot exchange the integral and the expectation. However the terms in each line are well-defined assuming the decay in S mentioned above. Here the Wick squares ${:}X^2{:}$ simply subtract the expectations, i.e. ${:}X^2{:} = X^2 - \mathbf{E}(X^2)$. They are necessary since otherwise the integrand would not decay in space. Collecting all the terms in equation 3.81 we obtain:

$$D(t) = 1 + \frac{2|\mathfrak{w}|^2}{t} \int_0^t \int_0^s \int_{\mathbb{R}^2} \mathbf{E}\left((\Pi^1{:}(\Pi^1\eta)^2{:})(0,0)(\Pi^1{:}(\Pi^1\eta)^2{:})(r,x)\right) \mathrm{d}x\mathrm{d}r\mathrm{d}s + o(1)\,.$$

Dropping the $o(1)$ term and replacing $\mathbb{R}^2$ with a large torus $\mathbb{T}^2_N$ we obtain exactly formula (3.3).

If we had chosen instead to analyse $D_{ij}(t)$ the same steps would have given (for $\mathfrak{w}$ parallel to the first coordinate axis)

$$D_{11}(t) = D(t) - 1/2, \quad D_{12}(t) = D_{21}(t) = 0, \quad D_{22}(t) = 1/2\,.$$

As we see the asymptotic behaviours of D_{11} and D are equivalent.

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Chapter 4

Near-critical dimers and massive SLE_2

abstract

We consider the dimer model on the square and hexagonal lattices with doubly periodic weights. The purpose of this paper is threefold: (a) we establish a rigourous connection with the massive SLE_2 constructed by Makarov and Smirnov [MS10] (and recently revisited by Chelkak and Wan [CW19]); (b) we show that the convergence takes place in *arbitrary* bounded domains subject to Temperleyan boundary conditions, and that the scaling limit is universal; and (c) we prove conformal covariance of the scaling limit. For this we introduce an inhomogeneous near-critical dimer model, corresponding to a drift for the underlying random walk which is a smoothly varying vector field or alternatively to an inhomogeneous mass profile. When the vector field derives from a potential satisfying a certain nonnegativity assumption we prove that the corresponding loop-erased random walk has a universal scaling limit. Our techniques rely on an exact discrete Girsanov identity on the triangular lattice which may be of independent interest. We complement our results by stating precise conjectures making connections to a generalised Sine-Gordon model at the free fermion point.

4.1 Introduction

Makarov and Smirnov initiated in [MS10] a programme to describe near-critical scaling limits of planar statistical mechanics models in terms of massive SLE and/or Gaussian free field. To quote from their paper:

The key property of SLE is its conformal invariance, which is expected in 2D lattice models only at criticality, and the question naturally arises: Can SLE success be replicated for off-critical models? In most off-critical cases to obtain a non-trivial scaling limit one has to adjust some parameter [...], sending it at an appropriate speed to the critical value. Such limits lead to massive field theories, so the question can be reformulated as whether one can use SLEs to describe those. Massive CFTs are no longer conformally invariant, but are still covariant when mass is considered as a variable covariant density [...].

As part of this programme, Makarov and Smirnov introduced a massive version of SLE_2, which will be defined more precisely in Section 4.1.4. As established rigourously recently by Chelkak and Wan [CW19], this can be seen as the scaling limit of the loop-erasure of a massive random walk, i.e., a random walk which has a fixed probability of being killed at every step, and which is conditioned to leave the domain before being killed. Makarov and Smirnov also listed a number of fascinating questions, many of which remain open today.

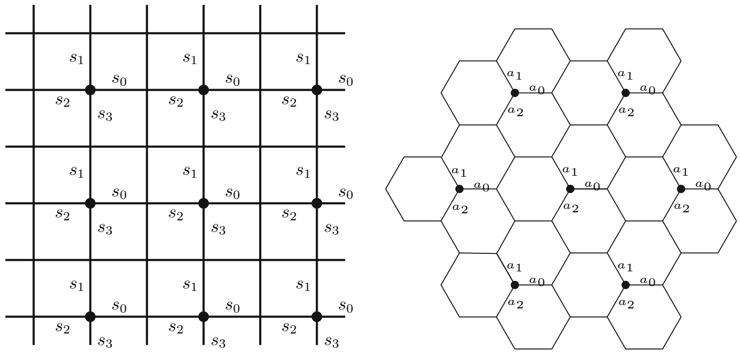

Figure 4.1: Doubly periodic weights on the square and hexagonal lattices. Black vertices of type 1 are marked with disks. The weights $s_0, \ldots, s_3$ or $a_0, \ldots, a_2$ are periodically repeated around every black vertex of type 1. Every other edge weight is equal to 1.

In this paper we carry out part of this programme for the near-critical dimer model. The dimer model is one of the most classical models of statistical mechanics, and is equivalent to random matchings on a planar bipartite graph. That is, given such a (finite) graph G, we associate to every dimer covering (or perfect matching) $\mathbf{m}$ (a subset of the edges such that every vertex is covered exactly once) the Gibbs weight

$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{e \in \mathbf{m}} w_e,$$

where $w_e > 0$ are given edge weights and Z is a normalisation constant (partition function). The model is also equivalent to tilings (in particular to lozenge tilings if the underlying graph is the hexagonal lattice; see [Gor21] for a recent superb introduction). The study of the dimer model goes back to the pioneering work of Temperley and Fisher [TF61] and Kasteleyn [Kas61], who computed its partition function, and noted that it is equal (up to a sign or more generally a complex number of modulus one) to the determinant of a matrix now called the Kasteleyn matrix, which is a suitably weighted adjacency matrix. This identity is the starting point of a far-reaching theory which eventually led Kenyon to prove convergence (subject to so-called Temperleyan boundary conditions, described below) of the associated height function to a Gaussian free field in a sequence of two landmark papers [Ken00], [Ken01] when all edge weights are equal. This was the first proof of conformal invariance for a planar model of statistical mechanics.

4.1.1 Off-critical dimer model.

In this paper we are concerned with an off-critical model, which can be defined either on the square lattice or on the hexagonal lattice when the edge weights are assumed to be doubly periodic, in the following sense. We start with the square lattice. Let $s_0, \ldots, s_3 > 0$. We divide the square lattice into the usual black and white vertices in checkboard fashion, and the black vertices are themselves divided into two alternating classes B_1 and B_2 (as in [Ken00]). We declare that around every B_1 vertex, the edge weights are respectively $s_0, \ldots, s_3$ as we move in the clockwise direction starting from the east (thus s_k corresponds to the direction $\mathbf{i}^k = e^{\mathbf{i}k\pi/2}, k = 0, \ldots, 3$; here $\mathbf{i} = \sqrt{-1}$). All other edge weights are set to 1. See Figure 4.1 for an illustration. We will further specify the weights s_k so as to be in the near-critical regime in (4.3).

A similar construction can be applied to the hexagonal lattice. Consider the usual black and white colouring of the vertices of the hexagonal lattice $\mathbb{H}$. Black vertices at distance two apart in $\mathbb{H}$ form a triangular lattice, which is a tripartite graph. So all black vertices in $\mathbb{H}$ belong one of three possible classes, B_1, B_2, B_3, say. We declare that the edge weights around a B_1 vertex are respectively a_0, a_1 and a_2 going counter-clockwise starting from the east direction See Figure 4.1.

This model was first considered in the work of Chhita [Chh12] in the case of the square lattice, who called it the "drifted" dimer model, for reasons that will become clear later. Suppose $s_k = 1 + c_k\delta$, where δ tends to zero (we will later identify δ with the mesh size). This scaling will be enforced throughout the paper. As already noted in [Chh12], this choice of scaling essentially corresponds to studying the **liquid-gas boundary** of the dimer phases. When applying the treatment of Kenyon [Ken00] to this model, if K denote the associated Kasteleyn matrix then one can easily check that $L = K^*K$, viewed as an operator on the black vertices, is approximately the negative of a massive Laplacian: indeed, on the B_1 vertices, the diagonal entry is of the form $s_0^2 + \ldots + s_3^2$, while the sum of the off-diagonal entries is $-2s_1s_3 - 2s_0s_2$. (The reason why this is only an approximation is because terms of the form $L(b_1, b_2)$ are not all exactly zero when $b_1 \in B_1, b_2 \in B_2$; they are simply lower order than $L(b_1, b_1')$ for $b_1, b_1' \in B_1$). (In fact, after a suitable transformation, the inverse Kasteleyn matrix can be related to a modified Kasteleyn matrix which corresponds exactly to the Green function of a massive random walk, see Section 3 of [Chh12]).

From this it is perhaps natural to conjecture that the height function, suitably rescaled, converges to the **massive Gaussian free field**, which is (informally) the Gaussian field whose covariance matrix is the massive Green function. Surprisingly, however, [Chh12] showed that while there is a scaling limit for the height function as $\delta \to 0$ in the full plane, the limit cannot be the massive Gaussian free field since its moments do not even satisfy the Wick relation, hence it is not even Gaussian.

The purpose of this paper is threefold:

- First, we extend the results of [Chh12] in several different ways: we consider not only the square lattice but also the hexagonal lattice; furthermore our results are not only valid in the whole plane but in arbitrary simply connected domains subject to Temperleyan boundary conditions (these are perhaps the nicest boundary conditions from the combinatorial point of view and are defined immediately below in Section 4.1.2).

- Second, we show for the first time a connection to massive models and more specifically to the massive SLE_2, constructed by Makarov and Smirnov [MS10] and revisited recently by Chelkak and Wan [CW19].

- Finally, we show that the scaling limit of the height function obeys a certain conformal covariance rule. This is reminiscent of other near-critical scaling limits previously obtained e.g. for percolation [GPS18]. Interestingly however, the covariance rule involves not only the modulus of the derivative of the conformal map but also its argument.

Last but not least, this will be complemented by some novel conjectures attempting to make a connection with a generalised Sine-Gordon model (which will be introduced below) at its free fermion point. Along the way we identify a larger and more interesting family of near-critical dimer models which give an intuitively transparent explanation for why and how the Sine-Gordon model is connected to near-critical dimers; these models are characterised by the fact that the mass (or equivalently the drift) is inhomogeneous.

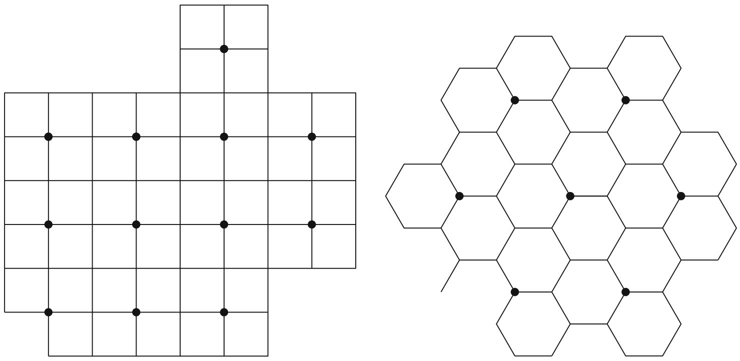

Figure 4.2: A Temperleyan domain on the square lattice and a Temperleyan domain on the hexagonal lattice. In both the black vertices of type B_1 have been highlighted and a (non-B_1) black vertex on the lower left boundary has been removed.

At the technical level a key contribution of this paper will be an exact discrete Girsanov identity on the triangular lattice as well as a proof that the loop-erasure of a random walk with drift which may vary with the position has a scaling limit.

4.1.2 Temperleyan boundary conditions.

To make the connection to massive SLE and state our results, we will now define precisely the type of boundary conditions we impose on the model, which in the case of the square grid are known as Temperleyan. We recall the definition in this case first. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain of the complex plane. Let $\Gamma_\delta = (v(\Gamma_\delta), E(\Gamma_\delta))$ be a sequence of graphs in $\delta\mathbb{Z}^2$ approximating Ω: that is, Γ_δ is a planar graph with vertex set $v(\Gamma_\delta) \subset \Omega \cap (\delta\mathbb{Z}^2)$ and edge set $E(\Gamma_\delta)$ such that if $x, y \in v(\Gamma_\delta)$ and $x \sim y$ in $\delta\mathbb{Z}^2$, then $(x, y) \in E(\Gamma_\delta)$ if and only if $[x, y] \subset \Omega$. We assume that the *vertex boundary* of Γ_δ, i.e. the vertices $v \in v(\Gamma_\delta)$ which have at least one neighbour w of the full plane square lattice not in $v(\Gamma_\delta)$, is within $O(\delta)$ of $\partial\Omega$. We also assume that Γ_δ is **Temperleyan**: namely, all corners (be them convex or concave) are of type B_2, and one further such corner has been removed. See e.g. Figure 4.2. Equivalently, along the vertex boundary, all black vertices are of type B_2, i.e, the boundary alternates between B_2 and white vertices (except at the removed corner).

We make a similar definition in the hexagonal case. We say that the domain Γ_δ whose vertices are in $\delta\mathbb{H}$ is Temperleyan if the boundary does not contain any B_1 vertices (i.e., consists only of B_2 and B_3 and white vertices), and a vertex of type B_2 or B_3 has been removed. Figure 4.2 shows examples of a Temperleyan domain on both the square and hexagonal lattices.

4.1.3 Temperley's bijection.

Temperley's bijection is a powerful tool which relates the dimer model on the Temperleyan graph Γ_δ to a pair of spanning trees on a different graph. As it turns out, the Temperleyan boundary conditions described above are such that both dimer models (i.e., on the square and hexagonal lattices respectively) are equivalent to a certain spanning tree on a (possibly directed) graph Ω^δ whose vertices are the B_1 vertices of Γ_δ (or, equivalently, to a pair of dual spanning trees on Ω^δ and its planar dual). In the square lattice (and for rectangles) this goes back to the original paper of Temperley and Fisher [TF61]. This was considerably generalised and strengthened in many subsequent works, in particular, [KPW00]. That paper included the perhaps lesser well known case of the hexagonal lattice, which we will use in this paper and will be recalled in more detail in Section 4.2.1; in that case, the corresponding graph Ω^δ of the spanning tree is the directed triangular lattice with mesh size δ.

As developed in the sequence of papers [BLR20, BLR19, BLR22], Temperley's bijection can be used to describe the scaling limit of the height function fluctuations via a random geometric approach. Essentially these papers reduce the problem of finding the scaling limit of the dimer height function to the (easier) problem of finding a scaling limit for the associated Tempereleyan tree in the Schramm topology: in other words, to the question of the scaling limit of a single branch of that tree. In turn, by Wilson's algorithm, this boils down to the scaling limit of the loop-erasure of the random walk on the (possibly directed) graph Ω^δ.

4.1.4 Massive SLE$_2$.

As already mentioned, the construction of massive SLE$_2$ was sketched by Makarov and Smirnov in [MS10] and recently revisited by Chelkak and Wan [CW19] (see also [BBK08] for a mathematical physics perspective). We will describe it in the radial case for ease of comparison with the situation which is of interest to us, though one should note that Chelkak and Wan's paper actually deals with the chordal case.

A massive random walk (on the square lattice, say) is a walk which has a chance of order δ^2 to be killed at every time step (the constant of proportionality is by definition $m^2/2$, where $m \geq 0$ is the mass), and otherwise moves like ordinary walk. Massive radial SLE$_2$ describes the scaling limit of the loop-erasure of a massive random walk from o to a (where a is on the boundary of a simply connected domain Ω, and o is in the interior of Ω), conditioned on not getting killed before reaching a. In fact, it is more convenient to define massive SLE$_2$ by its associated Loewner flow, which in the radial case is defined by Loewner's equation (parametrised by capacity)

$$\frac{dg_t(z)}{dt} = -\varphi'(z)g_t(z)\frac{g_t(z)+\zeta_t}{g_t(z)-\zeta_t}; z \in \Omega_t$$

where φ is a fixed conformal map sending Ω to $\mathbb{D}$ and o to 0, Ω_t denotes the slit domain $\Omega \setminus \gamma([0,t])$ (since $\kappa = 2$ we do not need to remove more than that), g_t is the Loewner map from Ω_t to $\mathbb{D}$, and if we write the driving function in the form $\zeta_t = e^{i\xi_t}$, then ξ solves the Stochastic Differential Equation:

$$d\xi_t = \sqrt{2}dB_t + 2\lambda_t dt; \lambda_t = \frac{\partial}{\partial g_t(a_t)} \log \frac{P^{(m)}_{\Omega_t}(o, a_t)}{P_{\Omega_t}(o, a_t)}. \tag{4.1}$$

Here $a_t = \gamma(t)$, and $P^{(m)}_{\Omega_t}$ and P_{Ω_t} are the Poisson kernels for the Brownian motion with mass m, and regular Brownian motion respectively, in Ω_t. By Brownian motion with mass m we mean the law of a standard Brownian motion, killed at rate m^2 (which defines a subprobability measure)[1].

The above expression for λ_t is that in [CW19] which is somewhat similar to the one appearing in Makarov and Smirnov [MS10, (9)]. However we feel it deserves a few explanations, notably considering the meaning of the differentiation $\frac{\partial}{\partial g_t(a_t)}$. This should be understood as a spatial differentiation with respect to $x \in \mathbb{R}$ after mapping Ω_t to $\mathbb{D}$, setting $a_t = e^{ix}$, and

[1] Note that the probability with which the above discrete random walk is killed at each step (namely, $(1/2)m^2\delta^2$) is chosen so that in the scaling limit, we obtain a Brownian motion killed at rate m^2. Indeed, if X^δ is an ordinary random walk on $\delta\mathbb{Z}^2$, then $(X_{2\delta^{-2}t})_{t\geq 0}$ converges weakly uniformly on compacts to a standard planar Brownian motion. In [CW19] the killing probability is chosen to be $\delta^2 m^2$ instead of $\delta^2 m^2/2$ but this appears to be a typo.

evaluating the result at $x = \xi_t$. That is, let $\rho = m^2$, and let ρ_t denote the (squared) mass profile in $\mathbb{D}$ that corresponds to the constant mass m in Ω_t; i.e., after mapping to $\mathbb{D}$ (and applying the relevant time change) we obtain a Brownian motion killed at rate $\rho_t(x)$). Then

$$\lambda_t = \frac{\partial}{\partial x} \log \frac{P_{\mathbb{D}}^{(\rho_t)}(0, e^{ix})}{P_{\mathbb{D}}^{(0)}(0, e^{ix})}\bigg|_{x=\xi_t}.$$

By choice of normalisation, the denominator in the fraction is simply equal to 1, so that this logarithmic derivative can also be written in the form:

$$\lambda_t = \frac{\frac{\partial}{\partial x} P_{\mathbb{D}}^{(\rho_t)}(0, e^{ix})}{P_{\mathbb{D}}^{(\rho_t)}(0, e^{ix})}\bigg|_{x=\xi_t}.$$

The spatial derivative of the Poisson kernel, i.e., the numerator of this fraction, is a quantity which can be shown to correspond to what Chelkak and Wan [CW19] denote by $Q_{\mathbb{D}}^{(\rho_t)}(x, a_t)$. Their result in fact establishes convergence with

$$\lambda_t = \frac{Q_{\mathbb{D}}^{(\rho_t)}(0, e^{i\xi_t})}{P_{\mathbb{D}}^{(\rho_t)}(0, e^{i\xi_t})} = \frac{Q_t^{(\rho)}(o, a_t)}{P_t^{(\rho)}(o, a_t)}. \tag{4.2}$$

Proving this chain of identities would require some arguments. This is circumvented by defining λ_t (i.e., the right hand side of (4.1)) as in the right hand side of (4.2).

The description above is then a theorem proved in the chordal case and on the square lattice by [CW19] (the radial case is briefly discussed as being analogue to, and in fact a little simpler than, the chordal case). See Theorem 1.1 in [CW19] for a precise statement, and see [Law05] as well as [BN23] for general references on SLE.

4.1.5 Main results

Our first result below concerns the branches of the Temperleyan tree for an off-critical dimer model on a graph (defined more precisely below, which may be a piece either of the square lattice or of the hexagonal lattice, scaled by δ) with Temperleyan boundary conditions, as explained above. The result shows that the scaling limit exists, and furthermore gives a connection to massive models. On the square lattice, suppose that the weights $s_0, \dots, s_3$ satisfy

$$s_k = 1 + c_k\delta \quad (k = 0, \dots, 3) \tag{4.3}$$

counterclockwise from the east direction, while on the hexagonal lattice we assume that the weights $a_0, \dots, a_2$ satisfy

$$a_k = 1 + c_k\delta \quad (k = 0, \dots, 2) \tag{4.4}$$

also counterclockwise from the east direction. We consider the associated rescaled drift vector α defined respectively by

$$\alpha = \frac{1}{2}\sum_{k=0}^{3} c_k \mathbf{i}^k; \alpha = \frac{2}{3}\sum_{k=0}^{2} c_k \tau^k, \tag{4.5}$$

where $\mathbf{i} = \sqrt{-1} = e^{\mathbf{i}\pi/2}$ and $\tau = e^{2\mathbf{i}\pi/3}$ are the fourth and third roots of unity, respectively. The scaling factors in front of these expressions are chosen to guarantee that in the scaling limit, a random walk with the above weights converges to Brownian motion with drift α: that is, if X^δ denotes this random walk on either $\delta\mathbb{Z}^2$ or $\delta\mathbb{T}$, then $(X^\delta_{2\delta^{-2}t})_{t\geq 0}$ converges weakly, uniformly on compacts, to $(B_t + \alpha t)_{t\geq 0}$, where B is a standard planar Brownian motion.

We also assume

$$c_0 + c_2 = c_1 + c_3 \tag{4.6}$$

in the square lattice case. See Remark 4.2.14 for a discussion of this condition.

We suppose we are given a Temperleyan lattice domain Γ_δ as in Section 4.1.2 and a dimer model on Γ_δ. Applying the Temperleyan bijection leads to a pair of dual trees respectively on Ω^δ and its dual, where Ω^δ is a subgraph of either $\delta\mathbb{T}$ or $\delta\mathbb{Z}^2$. Note that there is natural edge boundary $\partial\Omega^\delta$ on Ω^δ, corresponding to pair of vertices (y_1, y_2) of the lattice such that y_1 and y_2 are neighbours in the lattice, at least one of y_1 or y_2 is a vertex of Ω^δ but not both. With a slight abuse of notation we will still refer to y_δ as a point on the boundary, identify it in calculations with y_2 and say that $y_\delta \to y$ if y_2 converges to y as $\delta \to 0$ (or equivalently y_1). Likewise, we will often consider the random walk $(X_n, \geq 0)$ on Ω^δ. With an abuse of notation we will refer to the first time τ that the walk leaves Ω^δ as the smallest $n \geq 1$ such that (X_{n-1}, X_n) is a boundary edge. We will also identify, with an abuse of notation, the position X_τ with the boundary edge $(X_{\tau-1}, X_\tau)$, and denote it by Y_δ in the following.

Theorem 4.1.1. *Consider a near-critical dimer model as above in a domain Γ_δ with Temperleyan boundary conditions. Let o^δ be a vertex on the primal lattice and let γ^δ be the path starting from o^δ in the associated Temperleyan tree on Ω^δ. Let $Y_\delta \in \partial\Omega^\delta$ denote the endpoint of this path. Then conditional on $Y_\delta = a^\delta$, if $o^\delta \to o \in \Omega$ and $a^\delta \to a \in \partial\Omega$, then also the path γ^δ converges to radial massive SLE$_2$ (see* (4.1)*) from a to o with mass $m = \|\alpha\|/\sqrt{2}$ (associated with a standard planar Brownian motion killed at rate $\|\alpha\|^2/2$), where $\|\alpha\|$ denotes the Euclidean norm of the drift vector α defined in* (4.5).

In fact, the distribution of Y_δ converges weakly to a distribution $\mu_z^{(\alpha)}$ on $\partial\Omega$, which is the exit law from Ω of Brownian motion with unit covariance matrix and drift vector α. We therefore obtain the following result.

Theorem 4.1.2. *Let $\mathcal{T}_\delta$ denote the Temperleyan tree associated with the dimer configuration in Γ_δ (either in the hexagonal or square lattice case). Then as $\delta \to 0$, the tree $\mathcal{T}_\delta$ converges in the Schramm sense to a continuum limit tree $\mathcal{T}$. Each branch of this tree from a point $z \in \Omega$ has the law described in Theorem 4.1.1: that is, sample a according to $\mu_z^{(\alpha)}$; given a, the branch of $\mathcal{T}$ from z to a has the law of massive radial SLE$_2$ with mass $m = \|\alpha\|/\sqrt{2}$ (associated with a Brownian motion killed at rate $\|\alpha\|^2/2$).*

A key result from [BLR20] (see also [BLR19]) is that the convergence of the Temperleyan tree implies the convergence of the dimer height function. This requires only a uniform crossing estimate and some basic estimates such as polynomial decay on the probability for the loop-erasure to visit a small ball, and control on the moments of winding close to a point (these estimates are a fairly simple consequence of our work, and are written explicitly at the end of Section 4.5 in a more general context). We obtain the following corollary:

Corollary 4.1.3. *In the setup of Theorem 4.1.1 or 4.1.2, the centered height function $h^\delta - \mathbb{E}(h^\delta)$ converges to a limit as $\delta \to 0$ whose law depends only on the vector α defined in* (4.5).

4.1.6 Exact Girsanov identity

To establish these results, we observe that the law of a branch in the Temperleyan tree may be described via Wilson's algorithm as the loop-erasure of a random walk on Ω^δ with near-critical weights defined by (4.3) on the square lattice and (4.4) on the directed triangular lattice respectively. The random walk corresponding to these weights is one which has a drift: as the mesh size $\delta \to 0$, the random walk converges to a Brownian motion with drift vector

α defined in (4.5). Furthermore, using a discrete Girsanov transform (which will be detailed below), we relate the corresponding random walks to massive ones on the same lattices; the above result then intuitively follows by the known convergence of the massive LERW to the massive SLE$_2$ of Makarov and Smirnov (proved rigourously by Chelkak and Wan recently in [CW19]).

We now describe our Girsanov identity. As this holds independent of any scaling limit consideration we formulate it in the unscaled triangular lattice $\mathbb{T}$ and square lattice $\mathbb{Z}^2$. The Girsanov identity takes a slightly different form in each case. Although both are exact formulas, the connection between massive and drifted walk is only exact on the triangular case (Corollary 4.2.5) whereas it is approximate in the case of the square lattice (Corollary 4.2.12). On the other hand, the application of the results of Chelkak and Wan [CW19] in the directed triangular case needs additional arguments because of the lack of reversibility. As we believe this result is of independent interest, we state it below on the triangular lattice where the statement is the simplest. We consider a Markov chain on the (directed) triangular lattice $\mathbb{T}$ where the jump probabilities are allowed to depend on the position of the vertex v of the triangular lattice $\mathbb{T}$. That is, suppose given for any $v \in \mathbb{T}$, a collection of parameters $(\alpha_0(v), \alpha_1(v), \alpha_2(v)) \in \mathbb{R}^3$, and let $\mathbb{Q}$ denote the law of a Markov chain such that if the walk is at the vertex v, then the jump probabilities are given by

$$\mathbb{Q}(v, v+\tau^k) = \frac{e^{\alpha_k(v)}}{a(v)}, \quad k = 0, \dots, 2, \quad \text{with } a(v) = e^{\alpha_0(v)} + \dots + e^{\alpha_2(v)}. \tag{4.7}$$

Let also Y denote the position of the random walk when it hits $\partial\Omega$ and let $\mathbb{Q}(\cdot|Y = y)$ denote the conditional law given the exit point is y. We also let $\mathbb{P} = \mathbb{P}^{(0)}$ denote the law of the usual simple random on the directed triangular lattice $\mathbb{T}$.

Fix $\gamma = (x_0, \dots, x_n)$ a given path on the triangular lattice, starting from some point $x_0 = z \in \Omega$ of some length $n = N(\gamma)$. Let $\mathrm{d}x_s = x_{s+1} - x_s \in \{1, \tau, \tau^2\} \subset \mathbb{R}^2$, for $s = 0, \dots, n-1$ denote the discrete derivative of γ_s at time s. Define $\beta(v) > 0$ by

$$\exp(-\beta(v)^2) = (\tfrac{a(v)}{3})^{-3} \prod_{k=0}^{2} e^{\alpha_k(v)}, \tag{4.8}$$

which is well-defined by the arithmetic-geometric mean inequality. Let $\alpha = \alpha(v) = \frac{2}{3}(\alpha_0 + \alpha_1\tau + \alpha_2\tau^2)$, which is a complex number (identified with a vector in $\mathbb{R}^2$) associated to every vertex v of the triangular lattice $\mathbb{T}$.

Note that while α does not uniquely determine the α_i, it does determine $\mathbb{Q}$, since the transition probabilities in (4.7) do not change under a shift $(\alpha_1, \alpha_2, \alpha_3) \to (\alpha_1 + x, \alpha_2 + x, \alpha_3 + x)$ and all $(\alpha_1, \alpha_2, \alpha_3)$ corresponding to a specific α are related to one another by such a shift. Thus given a vector $\alpha(v) \in \mathbb{R}^2$, there is a unique choice of $\alpha_0(v), \alpha_1(v), \alpha_2(v) \in \mathbb{R}$ summing to zero such that $\alpha(v) = \frac{2}{3}(\alpha_0(v) + \alpha_1(v)\tau + \alpha_2(v)\tau^2)$. The following gives us an exact value for the global Radon–Nikodym derivative of the law $\mathbb{Q}$ compared to $\mathbb{P}^{(0)}$.

Theorem 4.1.4.

$$\frac{\mathbb{Q}_x(\gamma)}{\mathbb{P}_x(\gamma)} = \exp(M_n - \tfrac{1}{2}V_n), \tag{4.9}$$

where

$$M_n = \sum_{s=0}^{n-1} \langle \alpha(x_s), \mathrm{d}x_s \rangle \quad \textit{and} \quad V_n = \frac{2}{3}\sum_{s=0}^{n-1} \beta^2(x_s). \tag{4.10}$$

Of particular relevance in this article will be the case where the drift vector $\alpha = \alpha(v), v \in \Omega^\delta$ derives from a **potential function** $\Phi : \mathbb{T} \to \mathbb{R}$, i.e., when

$$\alpha(v) = \nabla^{\mathbb{T}}\Phi(v) := \frac{2}{3}\sum_{i=0}^{2}(\Phi(v+\tau^i) - \Phi(v))\tau^i; \tag{4.11}$$

in other words, $\alpha_{i+1}(v) = \Phi(v+\tau^i) - \Phi(v)$ for $0 \le i \le 2$. If α is of this form, the Radon–Nikodym derivative in Theorem 4.1.4 takes a particularly nice form:

Corollary 4.1.5. *Suppose α derives from a potential function Φ as above. Then*

$$\frac{\mathbb{Q}_x(\gamma)}{\mathbb{P}_x(\gamma)} = \exp\left(\Phi(x_n) - \Phi(x_0) - A_n\right);$$

where

$$A_n = \sum_{s=0}^{n-1} \Delta^{\mathbb{T}}\Phi(x_s) + \frac{1}{3}\beta^2(x_s).$$

Here $\Delta^{\mathbb{T}}\Phi(x) = \frac{1}{3}\sum_{i=0}^{2}\Phi(x+\tau^i) - \Phi(x)$ is the usual graph Laplacian on the directed triangular lattice $\mathbb{T}$.

To understand the formulas in Theorem 4.1.4 and Corollary 4.1.5, we now explain how both should be viewed as the discrete analogues of Girsanov's theorem followed by an application of Itô's formula. Indeed, in the continuum, if $\mathbb{Q}$ is the law of the solution of the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mathrm{d}B_t + \alpha(X_t)\mathrm{d}t\,; \quad \text{where} \quad \alpha(x) = \nabla\varphi(x) \tag{4.12}$$

and where φ is a smooth Lipschitz function on $\mathbb{R}^2$, then

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \exp\left(\int_0^t \alpha(X_s)\cdot dX_s - \frac{1}{2}\int_0^t \|\alpha(X_s)\|^2 ds\right) \tag{4.13}$$

$$= \exp\left(\varphi(X_t) - \varphi(X_0) - \frac{1}{2}\int_0^t \Delta\varphi(X_s) + \|\nabla\varphi(X_s)\|^2 ds\right). \tag{4.14}$$

Thus the two terms M_n and V_n in (4.9) are the discrete analogues of the two terms on the right hand side of (4.13). The term A_n in Corollary 4.1.5 is the direct discrete analogue of the integral in (4.14).

4.1.7 Conformal covariance; loop-erased random walk with drift

A fundamental feature of critical models in two-dimensional models of statistical mechanics is that they display conformal invariance. In the near-critical regimes that are under consideration in this paper, we cannot of course expect conformal invariance but rather a change of conformal coordinates rule known as **conformal covariance** which, roughly speaking, says that the transformation needs to be corrected by suitable powers of the derivative of the conformal map. This has been established in particular in the case of near-critical percolation in the paper [GPS18] (where this follows from analogous covariance rules for the limit of the uniform measure on pivotal points proved earlier in the remarkable work [GPS13]). To state such a result we need to extend the setup slightly, by allowing the drift vector α to depend continuously on the point $z \in \Omega$.

Thus, let us fix $\alpha : \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$ a locally Lipschitz, bounded **vector field** (identified with a complex-valued function) on Ω; for $z \in \Omega$, $\alpha(z) \in \mathbb{R}^2$ will represent the drift at position z.

Our results pertain only to the case where α **derives from a potential**, i.e. there exists a $\mathcal{C}^1$ function $\varphi : \bar{\Omega} \to \mathbb{R}$ such that $\alpha = \nabla\varphi$. (We will also make additional assumptions on φ in the theorem.)

Given such a bounded, continuous vector field, we associate weights on the (scaled) directed triangular lattice Ω^δ as follows:

$$\alpha_i^\delta(v) = \varphi(z + \delta\tau^i) - \varphi(z); \quad i = 0, \dots, 2 \tag{4.15}$$

and, as before, these parameters define a Markov chain on Ω^δ (which we will refer to later as random walk on Ω^δ with drift α) given by:

$$\mathbb{P}^{(\varphi)}(v, v + \delta\tau^i) = \frac{e^{\alpha_i^\delta(v)}}{a(v)}, \quad i = 0, \dots, 2, \qquad \text{with } a(v) = e^{\alpha_0^\delta(v)} + \dots + e^{\alpha_2^\delta(v)}. \tag{4.16}$$

Thus the weights are defined by the gradient of the potential φ, computed locally at each point $z \in \Omega^\delta$. An easy application of the Stroock–Varadhan theorem shows that as $\delta \to 0$, the position of a random walk starting from $o^\delta \to o \in \Omega$, after scaling time by $2\delta^{-2}$, converges to the solution of the Stochastic Differential Equation $\mathrm{d}X_t = \mathrm{d}B_t + \alpha(X_t)\mathrm{d}t$. Since α derives from a potential, the previous SDE takes the form

$$\mathrm{d}X_t = \mathrm{d}B_t + \nabla\varphi(X_t)\mathrm{d}t, \tag{4.17}$$

known as a **Langevin** SDE or diffusion.

We will show that Theorems 4.1.1 and 4.1.2 can be generalised to this more general setup both for the case of a general drift vector field. The first step is the construction of a scaling limit for the loop-erased random walk with drift (i.e., with weights as above) when the drift vector field derives from a potential satisfying a certain condition.

Theorem 4.1.6. *Let Ω be a simply connected domain and $\alpha : \Omega \to \mathbb{R}^2$ be given. Fix $o \in \Omega$ and let $o^\delta \in \Omega^\delta$ such that $o^\delta \to o$ as $\delta \to 0$. Let $a \in \partial\Omega$ and let a^δ be a sequence of vertices on the boundary of Ω^δ such that $a^\delta \to a$.*

Suppose the vector field α derives from a smooth potential $\varphi : \bar{\Omega} \to \mathbb{R}$ and suppose also that

$$\rho(x) = \frac{1}{2}\Delta\varphi(x) + \frac{1}{2}\|\nabla\varphi(x)\|^2 \geq 0; x \in \Omega. \tag{4.18}$$

Let $(X_t^\delta, t = 0, 1, \dots)$ be a random walk on Ω^δ with drift $\alpha = \nabla\varphi$, i.e., a sample from $\mathbb{P}^{(\varphi)}_{o^\delta}$ defined in (4.16). Let σ^δ denote the first time at which X^δ leaves Ω and consider the the loop erasure $\mathsf{LE}(X^\delta)$ of the walk up until this time. Then conditionally on $X^\delta_{\sigma^\delta} = a^\delta$, $\mathsf{LE}(X^\delta)$ converges weakly to a radial Loewner evolution γ starting from $\gamma_0 = a$, whose driving function $\zeta_t = e^{i\xi_t}$ (when parametrised by capacity) satisfies the stochastic differential equation

$$d\xi_t = \sqrt{2}dB_t + \lambda_t dt, \quad \lambda_t = \frac{\partial}{\partial g_t(a_t)} \log\left(\frac{P^{(\rho)}_{\Omega_t}(o, a_t)}{P_{\Omega_t}(o, a_t)}\right), \tag{4.19}$$

where $a_t = \gamma(t)$, $\Omega_t = \Omega \setminus \gamma([0, t])$ is the slitted domain at time t, g_t is the Loewner map from Ω_t to $\mathbb{D}$ and $P^{(\rho)}_{\Omega_t}$ and P_{Ω_t} are the Poisson kernels for massive Brownian motion with (squared) mass profile $\rho = \frac{1}{2}\Delta\varphi + \frac{1}{2}\|\nabla\varphi\|^2$, and regular Brownian motion respectively, in Ω_t: that is, the Brownian motion is killed at instantaneous rate $\rho(x)$ when in $x \in \Omega$.

As above, the drift term in (4.19) has to be understood appropriately, and will really be defined as $Q_t^{(\rho)}(o)/P_t^{(\rho)}(o)$ where these terms will be defined carefully in Section 4.4. In particular, the construction of the Poisson kernel $P^{(\rho)}_{\Omega_t}$ appearing in the result above is not trivial and will also be described in Section 4.4. That there is also a unique strong solution to the SDE (4.19) is also not obvious; this will follow from the estimate in Lemma 4.4.16.

Discussion of the assumption (4.18)**.** The theorem above relies on the condition (4.18) which plays a technical but important role. We do not believe this assumption is necessary, but it greatly simplifies the analysis leading to the result. Essentially, our discrete Girsanov theorem allows us to relate random walk with drift to **random walk with variable mass**. The corresponding limiting (squared) mass function is then given by the formula

$$\rho(x) := \frac{1}{2}\Delta\varphi(x) + \frac{1}{2}\|\nabla\varphi(x)\|^2. \tag{4.20}$$

Thus our assumption (4.18) amounts to requiring the killing rate to be nonnegative. One can already intuit the emergence of this function from (4.14).

Although this condition could appear somewhat artificial, we note that this condition is actually **invariant under conformal transformations**. More precisely, fix $T : \Omega \to \tilde{\Omega}$ a conformal isomorphism of simply connected domains, and let X be a solution of the Langevin SDE (4.17), where we assume $\frac{1}{2}\Delta\varphi + \frac{1}{2}\|\nabla\varphi\|^2 \geq 0$. Then $T(X_t)$ is, up to a time-change, a solution of the SDE:

$$dY_t = d\tilde{B}_t + \nabla\tilde{\varphi}(Y_t)dt. \tag{4.21}$$

This is also a Langevin SDE (4.17), where the new potential $\tilde{\varphi}$ is simply given by

$$\tilde{\varphi}(y) = \varphi(T^{-1}(y)).$$

From there it is not hard to see that

$$\Delta\tilde{\varphi}(T(x)) = \Delta\varphi(x) \cdot |T'(x)|^2;$$

(this is best seen by computing the Laplacian via Wirtinger derivatives). Since

$$\|\nabla\tilde{\varphi}(T(x))\|^2 = |T'(x)|^2\|\nabla\varphi(x)\|^2,$$

we deduce that the associated mass function $\tilde{\rho}$ satisfies

$$\tilde{\rho}(y) = |T'(x)|^2\rho(x); \quad y = T(x). \tag{4.22}$$

Thus $\rho \geq 0$ if and only if $\tilde{\rho} \geq 0$.

Physically, the assumption (4.18) corresponds to a potential that tends to push the diffusion towards the boundary. In particular, if φ is convex then this condition is satisfied. (Note that our φ follows an unusual sign convention: the Langevin diffusion is pushed towards higher potential instead of the more commonly adopted convention of lower potentials).

Remark 4.1.7. The relation (4.22) is a **conformal covariance relation for the mass functions**.

We now address the consequence of Theorem 4.1.6 for the dimer model. Let $\alpha = \nabla\varphi$ be a vector field deriving from a smooth potential $\varphi : \bar{\Omega} \to \mathbb{R}$ satisfying (4.18). To the weights $e^{\alpha_k(v)}$ in (4.16) we can associate edge weights on Γ_δ in a bipartite fashion similar to (4.1). The only difference with what was discussed in Section 4.1.1 is that now the weights a_0, a_1, a_2 depend on the point v. We call this the **inhomogeneous massive dimer model**.

Nevertheless, Temperley's bijection still applies: thus dimer configurations on Γ_δ are in (measure-preserving) bijection with wired spanning trees on Ω^δ. Using results from [BLR20], we deduce from Theorem 4.1.6 that the height function $h^{(\alpha)}_\delta$ of the corresponding dimer model converges to a scaling limit (this generalises Corollary 4.1.3 to the variable drift setting). Furthermore, under the additional technical restriction that the conformal map $T : \Omega \to \tilde{\Omega}$ extends analytically to a neighbourood of Ω, the limit is conformally covariant.

Theorem 4.1.8. *Fix Ω and $\alpha = \nabla\varphi$ satisfying (4.18) as above. The height function $h_\delta^{(\alpha)}$, of the corresponding biperiodic dimer model just described, has a scaling limit which we denote by $h^{(\alpha);\Omega}$. Furthermore, let $T : \Omega \to \tilde{\Omega}$ denote a conformal isomorphism of bounded simply connected domains, and suppose that T extends analytically to a neighbourhood of Ω. Then we have the identity in law,*

$$h^{(\alpha);\Omega} \circ T^{-1} = h^{(\tilde{\alpha});\tilde{\Omega}}$$

where at a point $w \in \tilde{\Omega}$,

$$\tilde{\alpha}(w) = \overline{(T^{-1})'(w)} \cdot \alpha(T^{-1}(w)). \tag{4.23}$$

The product above refers to the multiplication of complex numbers; and this drift vector field $\tilde{\alpha}$ derives from the potential $\varphi \circ T^{-1}$.

To explain the theorem, we point out that the new drift vector field $\tilde{\alpha}$ in $\tilde{\Omega}$ has an amplitude which, compared to that of α in Ω, has been scaled by 1 over the modulus of the derivative of the conformal map going from Ω to Ω', and the vector has been rotated (in the positive direction) by the argument of its derivative. This is the desired conformal covariance rule. Once again, we point out that this formula may be simply understood in terms of conformal covariance of Langevin diffusions. Simply put, the above rule describes (by Itô's formula and the Cauchy–Riemann equations) the change of coordinates for a Brownian motion with drift $\alpha = \nabla\phi$.

4.1.8 Comments and open problems

1. The limiting height function $h^{(\alpha);\Omega}$ is determined implicitly from the scaling limit of the associated Temperleyan tree. A natural question would be to identify its law explicitly. For this the **Coleman correspondence** (see [BW20] which establishes a rigorous version) is a natural starting point. Briefly speaking, the Coleman correspondence can be viewed as a massive extension of the boson-fermion correspondence, embodied (in the critical case) by the convergence of the dimer (= fermionic) height function to the Gaussian (= bosonic) free field. This suggests that the height function $h^{(\alpha)}$ should be related in the scaling limit to the so-called **Sine-Gordon model** at the free fermion point, from quantum field theory. The latter is one of the most canonical quantum (yet not conformal) field theories. Despite its non-conformal nature, it enjoys a great deal of integrability. Informally, the sine-Gordon field is defined (in the whole plane) by the law

$$\mathbb{P}^{\mathrm{SG}}(\mathrm{d}h) \propto \exp\left(z \int_{\mathbb{C}} \cos(\sqrt{\beta} h(x)) dx\right) \mathbb{P}^{\mathrm{GFF}\#}(\mathrm{d}h), \tag{4.24}$$

where $\mathbb{P}^{\mathrm{GFF}\#}(\mathrm{d}h)$ corresponds to the law of a Gaussian free field in Ω (with Dirichlet boundary conditions) but normalised so that the whole plane Green function satisfies $G^{\#}_{\mathbb{C}}(x, y) = -(2\pi)^{-1} \log |x - y|$.

The above expression is however purely formal, as the cosine of (multiples) of the GFF is ill-defined. While this can be made sense of using the theory of imaginary chaos ([JSW18]) for all $\beta < 4\pi$, the free fermion point (corresponding to $\beta = 4\pi$) falls just outside the regime where this theory yields a nontrivial object.

We conjecture however that the Sine-Gordon field above describes the limit of the dimer height function only in the case of constant mass/drift (and assuming also that the drift vector field points to the right, or that Ω is the full plane). More generally, we conjecture the following description for the limiting height function for the inhomogeneous massive dimer model (with weights (4.16)), given any vector field deriving from a smooth potential $\varphi : \bar{\Omega} \to \mathbb{R}$.

Conjecture 4.1.9. *Let $\mathbb{P}^{(\alpha);\Omega}$ denote the law of the field $h^{(\alpha);\Omega}$ in Theorem 4.1.8. Then*

$$\mathbb{P}^{(\alpha);\Omega}(\mathrm{d}h) \propto \exp\left(z_0 \int_\Omega \left\langle e^{\mathbf{i}h(x)/\chi}, \alpha(x) \right\rangle dx \right) \mathbb{P}^{\mathrm{GFF}}(\mathrm{d}h). \tag{4.25}$$

Again this expression is informal and assigning it a meaning is itself nontrivial. The factor z_0 in front of the integral comes from conventions such as the normalisation of the Laplacian and that of the limiting drift α. But note how the expression (4.25) reduces to that in (4.24) at the free fermion point when Ω is replaced by the whole plane. Indeed, first of all the normalisation of the GFF in $\mathbb{P}^{\mathrm{GFF}}$ and $\mathbb{P}^{\mathrm{GFF}\#}$ differ by a factor of $\sqrt{2\pi}$. Thus $h^{\#} = (2\pi)^{-1/2}h$, so that $\sqrt{\beta}h^{\#} = (1/\chi)h$ when $\beta = 4\pi$ and $\chi = 1/\sqrt{2}$ is the imaginary geometry constant associated to $\kappa = 2$.

Furthermore, when $\Omega = \mathbb{C}$ then by rotational invariance, then $\langle e^{\mathbf{i}h(x)/\chi}, \alpha(x)\rangle$ has the same law as $\|\alpha(x)\| \cos(\sqrt{2}h(x))$. Thus taking $\alpha(x) = \alpha$ to be constant the expression (4.25) indeed boils down to (4.24) with $z = z_0\|\alpha\|$. Since z is the mass parameter of the Sine-Gordon model, this is entirely consistent with our Theorem 4.1.1 (and with $z_0 = 1/\sqrt{2}$ in our choice of conventions for the normalisation of the Laplacian).

The above conjecture is informally supported by the imaginary geometry approach to the dimer model ([BLR20]). Informally, this conjecture says that massive SLE_2 is (in some sense) a **flow line** of the Sine-Gordon field at the free fermion point. We do not know whether this should hold away from the free fermion point, but it is tempting to conjecture so. (Recall that for $\beta < 4\pi$ the Sine-Gordon field is absolutely continuous with respect to a GFF so that the notion of flow line is at least well defined).

2. A possible approach to the above (which is also of independent interest in its own right) is the following: can an axiomatic characterisation of this field be given in the manner of [BPR20, BPR21, AP21]? (This last question is due to Christophe Garban who asked it in a slightly different form.)

3. A separate line of enquiry concerns the possible implications of our results to the study of the Ising model. By bosonization, it is known that the critical Ising model is related to the critical dimer model ([Dub11]). This correspondence remains at least partly valid in the near-critical regime studied here, but we do not know whether the corresponding Ising model is near-critical in the sense of commonly studied perturbations of the critical Ising model (see in particular, [DCGP14], [CIM21], [Par18], [CJN20] and references therein).

4. Finally we have developed a near-critical dimer theory on the square and hexagonal lattices using the symmetries of these lattices, but it would be of considerable interest to have a theory in some more general setting, e.g., for double isoradial graphs (i.e., superposition of an isoradial graph and its dual) since we know for instance that the Temperleyan bijection extends to this setting ([KPW00]).

Updates. Since the paper was first put on arXiv we can report on a few developments in the direction of the above conjectures.

1. On the one hand, Mason [Mas22] showed that in the full plane and in the case of constant drift, the two-point correlation of the limiting massive dimer height function coincides with that of the free fermion Sine-Gordon field.

2. Separately, Papon considered the case $\kappa = 4$ of Makarov and Smirnov's programme ([Pap23a, Pap23b]). Roughly speaking, she shows in these articles the following results: convergence of the massive harmonic explorer to massive SLE_4, and a conformal invariance property analogous to Theorem 4.1.8. Furthermore, level lines of the massive GFF are given by massive variant of CLE_4, and the occupation field of a massive Brownian loop soup coincides with the square of the massive GFF. The pairwise relations between these three objects hold simultaneously, as in the work of Qian and Werner for the non-massive case ([QW19]).

3. Finally, Rey [Rey24] has developed a Girsanov identity for isoradial graphs and applied it to massive dimer models, thereby generalising the results of this paper.

While it is not the purpose of this paper to give an extensive overview of recent works on near-critical models, we feel it is appropriate to conclude this introduction by mentioning some which are at least in spirit motivated by similar questions albeit for different models. These include, beyond the already mentioned works on near-critical percolation [GPS18] and the near-critical Ising model [CIM21], [Par18], [CJN20] and [DCGP14], the work of Duminil-Copin and Manolescu on scaling relations in the random cluster model [DCM20]), the work of Benoist, Dumaz and Werner [BDW20] on near-critical spanning forests, and Camia's work on off-critical Brownian loop soup [Cam13].

4.1.9 Notation and Scaling

The triangular lattice $\mathbb{T}$ always refers to the *directed* triangular lattice, in which each edge has been directed in the respective direction $1, \tau$ or τ^2, i.e. when we speak of a random walk on this lattice only steps in those three directions are allowed.

We recall that Ω^δ is the graph on which all random walk paths will leave (a scaled copy of the triangular lattice or the square lattice, approximating Ω), which is often identified with its vertex set. In this paper several measures on such lattice paths appear. For the convenience of the reader we collect the most important ones here.

- The simple random walk measure $\mathbb{P}^{(0)}$, which takes all possible steps with equal probability. Note that if X^δ has law $\mathbb{P}^{(0)}$ then $(X_{2\delta^{-2}t})_{t\geq 0}$ converges to a standard planar Brownian motion (this holds both on $\delta\mathbb{T}$ and on $\delta\mathbb{Z}^2$).

- Given a function $\rho^\delta : \Omega^\delta \to [0,1]$, the massive random walk measure $\mathbb{P}^{(\rho^\delta)}$ is the massive random walk, dying at each step with probability $\rho^\delta(v)$ if it is in position $v \in \Omega^\delta$ (and otherwise jumping to one of its neighbours with equal probability).

- Consider the triangular lattice case. Given $\alpha^\delta : \Omega^\delta \to \mathbb{R}^2$ a discrete vector field, the random walk with variable drift $\mathbb{P}^{(\alpha^\delta)}$ takes steps according to (4.7), i.e., the walk jumps from v to $v + \delta\tau^k$ with probability proportional to $e^{\alpha_k(v)}$ $(k = 0, 1, 2)$, where $\alpha_0(v), \alpha_1(v), \alpha_2(v) \in \mathbb{R}$ are uniquely defined by the requirements $\sum_{k=0}^2 \alpha_k(v) = 0$ and $\alpha(v) = \frac{2}{3}\sum_{k=0}^2 \alpha_k(v)\tau^k$.

Here we have defined the law of random walk with drift α^δ, $\mathbb{P}^{(\alpha^\delta)}$, only in the case of the triangular lattice. Obviously, an analogous definition can be given in the case of the square lattice too; this will be made explicit in Section 4.2.4 when it is needed.

The weights $\alpha^\delta(v) = \alpha^\delta(v)$ typically depend on both δ and v. When there is no risk of ambiguity we will sometimes write $\mathbb{P}^{(\alpha)}$ for $\mathbb{P}^{(\alpha^\delta)}$, and likewise we will write $\mathbb{P}^{(\rho)}$ for $\mathbb{P}^{(\rho^\delta)}$ if there is no risk of confusion.

Usually we view them as measures on the canonical path space, whose corresponding random variable is denoted by $X^\delta_t, t = 0, 1, \ldots$. Sometimes however, given a discrete path $\gamma^\delta = (x_0, \ldots, x_n)$ we write $\mathbb{P}(\gamma^\delta)$ for $\mathbb{P}((X^\delta_s)_{s=0,\ldots,n} = (\gamma^\delta_s)_{s=0,\ldots,n})$, where $\mathbb{P}$ is any of the laws above.

Note that $\mathbb{P}^{(0)}$ is a special case of all of these measures, setting the respective parameters to 0.

For the appropriate weights these random walks have scaling limits. For instance, if $\alpha^\delta(v) = \delta F(v) + o(\delta)$ for some bounded Lipschitz-continuous F, then this random walk converges to the solution of the SDE

$$\mathrm{d}X_t = F(X_t)\mathrm{d}t + \mathrm{d}B_t\,.$$

let $\varphi : \Omega \to \mathbb{R}$ be a smooth function. Noting the fact that $\nabla^{\delta\mathbb{T}}\varphi(v) = \delta\nabla\varphi(v) + o(\delta)$ (recall our conventions for the discrete gradient in (4.11)), this implies that if $\alpha^\delta(v) = \nabla^{\delta\mathbb{T}}(\varphi)$, then the random walk corresponding to $\mathbb{P}^{\alpha^\delta}$ converges in the scaling to the Langevin diffusion

$$\mathrm{d}X_t = \nabla\varphi(X_t)\mathrm{d}t + \mathrm{d}B_t\,.$$

That is, $(X^\delta_{2\delta^{-2}t})_{t\geq 0}$ converges weakly under $\mathbb{P}^{(\alpha^\delta)}$ to the above Langevin diffusion.

Likewise, in the massive case, suppose that $\rho^\delta(v) = \delta^2\rho(v)/2 + o(\delta^2)$. Then the random walk corresponding to $\mathbb{P}^{(\rho^\delta)}$, converges (under the same scaling) to massive Brownian motion with profile ρ, i.e. its law converges to the measure $\mathbb{P}^{(\rho)}_x$ whose Radon–Nikodym derivative with respect to Brownian motion is given by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\rho)}_x}{\mathrm{d}\mathbb{P}_x}\right|_t = \exp\left(-\int_0^t \rho(X_s)\mathrm{d}s\right)\,.$$

Organisation of the paper. In Section 4.2 we state and prove the discrete Girsanov identities and explain the implication for the connection between drifted and massive walks which lies at the heart of this paper. In Section 4.3 we extend Chelkak and Wan's result about the convergence of the massive LERW to massive SLE_2 to the directed triangular case; the additional difficulty compared to their setup is the lack of reversibility. At this stage Theorems 4.1.1 and 4.1.2 are proved.

In Sections 4.4 and 4.5 we show how to get the existence of a scaling limit for loop-erased random walk on graphs where the drift is a variable function of the vertices given by the gradient of a potential (in particular, the scaling limit of the random walk is given by a Langevin diffusion). Finally in Section 4.5 we transfer results about convergence of trees to convergence of height function (which implies in particular the conformal covariance of Theorem 4.1.8).

4.2 Girsanov identity; proof of Theorems 4.1.1 and 4.1.2

In this section we start with a proof of Theorem 4.1.1, which we prove separately in the case of the square and hexagonal lattices. As mentioned the result will follow from applying a form of Temperley's bijection and studying the scaling limit of the corresponding loop-erased random walk (which describe branches in the spanning tree by Wilson's algorithm). Since Temperley's bijection is not so well known in the case where Γ_δ is a subgraph of the hexagonal lattice, we start by explaining the bijection in this case, which can also be found (albeit somewhat informally) in Section 2 of [KPW00]; see in particular their Figure 2.

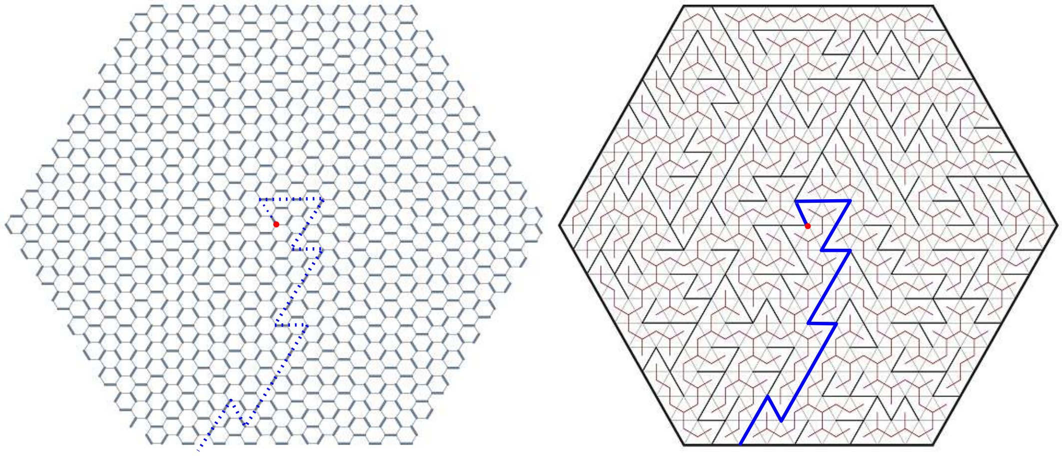

Figure 4.3: A dimer configuration on the hexagonal lattice, and its associated pair of dual spanning trees from Tempereley's bijection. The outer (i.e., boundary) vertex is represented as a black hexagon for convenience. The unique path connecting a vertex v (at the centre of the hexagon) to the boundary has been highlighted on the tree; the corresponding path on the dimer graph appears as a dotted line. Each dimer on this path can be viewed as the first half of the corresponding tree edge. Conversely, we obtain the tree by multiplying by two each dimer emanating from a B_1 vertex, in the direction from black to white.

4.2.1 Temperley's bijection on the hexagonal lattice

As mentioned in Section 4.1.3, Temperley's bijection relates rooted spanning trees on a graph Ω^δ (already discussed in Section 4.1.2) to dimers on the graph Γ_δ, a Temperleyan subgraph of the hexagonal lattice. We start by describing how Ω^δ and Γ_δ are related to one another. The bijection itself will be stated in Theorem 4.2.1 and is illustrated in Figure 4.3.

Consider the triangular lattice, that is the graph whose vertices are given by $a+b\tau$, where $a, b \in \delta\mathbb{Z}$ are integers (times δ) and $\tau = e^{2\mathbf{i}\pi/3}$ is the third root of unity, and where each pair of vertices at distance δ is connected by an edge. We will give each edge an orientation, such that it is oriented in direction 1, τ or τ^2 and a weight, which is a_1, a_2 or a_3 accordingly. This gives a directed graph in which each vertex has three outgoing and three incoming edges. We will call this graph the directed triangular lattice and denote it by $\mathbb{T}$ throughout this article.

Let us now choose a simply connected set of vertices of $\mathbb{T}$ and identify all other vertices as a single **outer vertex**. We call the resulting graph Ω^δ. A **spanning tree of Ω^δ rooted at the outer vertex** is a spanning set of edges containing no cycle. By orienting the edges to wards the root of the tree (the outer vertex), any such tree is equivalent to a spanning *arborescence*, i.e., a collection of directed edges such that there is exactly one outgoing edge from each non-root vertex (and none at the root), and having no cycle (irrespective of the orientation). This point of view is useful in Temperley's bijection since edges come with a natural orientation.

(Sometimes such a tree is called an arborescence). By definition we assign a weight to a rooted spanning tree given by the product of the weights of the edges in the tree.

Now consider the superposition graph H^* obtained in the following way. The vertices of H^* are the vertices, edges and faces of Ω^δ. To avoid terminological confusion, call the vertices of H^* nodes and call them vertex-nodes, edge-nodes and face-nodes depending on their counterpart in Ω^δ. The edges of H^* are called links and are defined as follows: connect a vertex-node v and an edge-node e if e is an outgoing edge of v in Ω^δ and give this link the same weight as e in Ω^δ. Also connect an edge-node e and a face-node f if e is adjacent to f in Ω^δ, and assign weight 1 to such links. Finally obtain Γ_δ from H^* by deleting the vertex-node corresponding to the outer vertex and one face-node for a face adjacent to the the outer vertex. Note that the vertex nodes of H^* are the B_1 vertices of the hexagonal lattice, while edge-nodes are white. (The face nodes of H^* are either of type B_2 or B_3.) For an illustration of this procedure see Figure 4.3.

The graph Γ_δ obtained this way is exactly a Temperleyan domain of the hexagonal lattice as defined in Section 4.1.2, and by choosing Ω^δ as the directed triangular lattice formed by the B_1 vertices in such a domain, it is also clear that each Temperleyan subgraph of the hexagonal lattice can be obtained in this way. The weights on this graph are as in Figure 4.1. The relevant version of Temperley's bijection is then the following:

Theorem 4.2.1 ([KPW00])**.** *There is a weight preserving bijection between spanning trees of Ω^δ rooted at the outer vertex (i.e., spanning arborescences rooted at the outer vertex) and dimer configuration on Γ_δ.*

The bijection is easier to describe in the direction "dimers" to "trees": given a dimer configuration $\mathbf{m}$ on Γ_δ, define a collection $\mathcal{T}$ of oriented edges in Ω^δ as follows: for every dimer occupying a link between a vertex-node $v \in v(\Omega^\delta)$ and an edge node $e \in E(\Omega^\delta)$, include the outgoing edge e from v to $\mathcal{T}$. One can check that the resulting collection of edges $\mathcal{T}$ is a spanning tree in the sense above. (Essentially, to every vertex $v \in v(\Omega^\delta)$ there is a unique outgoing edge containing v in $\mathcal{T}$ by definition of the dimer model and of $\mathcal{T}$; following the outgoing edges from a given vertex $v \in v(\Omega^\delta)$ may not result in a cycle by duality considerations, and thus necessarily ends at the outer vertex – this is the unique path to the outer vertex in the definition). Once again, we refer to Figure 4.3 for illustration.

4.2.2 Proof of Theorem 4.1.4 and relation to massive walk

We consider first the case of the triangular lattice and give the proof of Theorem 4.1.4, and recall that here we work on the unscaled lattice $\mathbb{T}$ rather than the scaled lattice $\delta\mathbb{T}$.

Proof of Theorem 4.1.4. Let $n_0 = n_0(v), n_1 = n_1(v)$ and $n_2 = n_2(v)$ be the number of steps taken by γ from v in the directions $1, \tau$ and τ^2 respectively. Then

$$\begin{aligned}
&\mathbb{P}_x^{(\alpha)}(\gamma)\\
&= \prod_{v\in v(\Omega)} \prod_{k=0}^{2} (\frac{e^{\alpha_k}}{a})^{n_k} = 3^{-n} \prod_{v\in v(\Omega)} \left[((a/3)^{-(n_0+n_1+n_2)} \prod_{k=0}^{2} (e^{\alpha_k})^{\frac{n_0+n_1+n_2}{3}} \prod_{k=0}^{2} (e^{\alpha_k})^{n_k - \frac{n_0+n_1+n_2}{3}} \right]\\
&= 3^{-n} \prod_{v\in v(\Omega)} e^{-\beta(v)^2 \frac{n_0+n_1+n_2}{3}} \exp\left(\sum_{k=0}^{2} \alpha_k (n_k - \tfrac{n_0+n_1+n_2}{3}) \right)\\
&= 3^{-n} e^{-\frac{1}{2}V_n} \exp\left(\sum_v \alpha_0 (\tfrac{2n_0-n_1-n_2}{3} + \alpha_1 (\tfrac{2n_1-n_0-n_2}{3}) + \alpha_2 (\tfrac{2n_2-n_0-n_1}{3}) \right)\\
&= 3^{-n} e^{-\frac{1}{2}V_n} \exp\left(\tfrac{2}{3} \sum_v \langle \alpha_0 + \alpha_1\tau + \alpha_2\tau^2, n_0 + n_1\tau + n_2\tau^2 \rangle \right)
\end{aligned}$$

where we have used in the last line that $\langle 1, \tau \rangle = \langle 1, \tau^2 \rangle = \langle \tau, \tau^2 \rangle = -1/2$. To conclude, simply observe that each $\mathrm{d}x_s$ contributes exactly 1, τ or τ^2 exactly n_0, n_1 or n_2 times respectively. Therefore,

$$\frac{2}{3} \sum_v \langle \alpha_0 + \alpha_1\tau + \alpha_2\tau^2, n_0 + n_1\tau + n_2\tau^2 \rangle = \sum_{s=0}^{n-1} \langle \alpha(x_s), \mathrm{d}x_s \rangle = M_n, \tag{4.26}$$

so that

$$\frac{\mathbb{P}_x^{(\alpha)}(\gamma)}{\mathbb{P}_x^{(0)}(\gamma)} = \exp(M_n - \tfrac{1}{2}V_n),$$

as desired. □

Before we proceed with the case of constant drift let us first consider the case of drift of gradient type.

Proof of Corollary 4.1.5. Consider a single summand of M_n. Let j be such that $x_{s+1}-x_s = \tau^j$

$$\langle \nabla^{\mathbb{T}}\Phi(x_s), x_{s+1} - x_s\rangle = \frac{2}{3}\sum_{i=0}^{2}(\Phi(x_s + \tau^i) - \Phi(x_s))\langle \tau^i, \tau^j\rangle = \tag{4.27}$$

$$\frac{2}{3}\left(\Phi(x_s+\tau^j) - \Phi(x_s) - \frac{1}{2}(\Phi(x_s+\tau^{j+1}) - \Phi(x_s)) - \frac{1}{2}(\Phi(x_s+\tau^{j+2}) - \Phi(x_s))\right) = \tag{4.28}$$

$$\Phi(x_{s+1}) - \Phi(x_s) - \Delta^{\mathbb{T}}\Phi(x_s). \tag{4.29}$$

Telescoping the first term gives the desired result. □

Remark 4.2.2. Since Theorem 4.1.4 and Corollary 4.1.5 are just statements about the random walk on the triangular lattice, i.e. independent of the embedding of this graph, we chose to state it for the unscaled lattice $\mathbb{T}$. However, for convenience let us describe what these results become when we scale the triangular lattice, as this will be the situation of interest in the rest of the article. Thus, let us assume that we are given $\alpha^\delta : \delta\mathbb{T} \to \mathbb{R}$, such that $\alpha^\delta(v) = \delta\alpha(v) + o(\delta)^2$. Let $\alpha_0^\delta, \alpha_1^\delta, \alpha_2^\delta$ be associated weights such that $\alpha^\delta = \frac{2}{3}\sum_{k=0}^{2}\alpha_k^\delta\tau^k$ (these are defined only up to a common additive constant, as $1+\tau+\tau^2 = 0$).

Then, as will be checked in Lemma 4.2.4, the corresponding factor $\beta(v)$ (which does not depend on the choice of the above constant) will be of order δ^2. The statement of Theorem 4.1.4 remains unchanged except that one has

$$M_n = \sum_{s=0}^{n-1}\langle \delta^{-1}\alpha^\delta(x_s), \mathrm{d}x_s\rangle\,. \tag{4.30}$$

The additional factor δ^{-1} compared to (4.26), comes from the scaling of the triangular lattice: in (4.26) we had used that $\mathrm{d}x_s \in \{1, \tau, \tau^2\}$, but on the scaled triangular lattice one has instead $\frac{\mathrm{d}x_s}{\delta} \in \{1,\tau,\tau^2\}$, so we need to add a factor of δ^{-1} to compensate. The fact that the terms in the sum defining M_n in (4.30) are each of order δ, while the summands in the sum defining V_n are of order δ^2 is consistent with the fact that M_n converges to a stochastic integral and V_n converges to a finite variation integral (with n of order δ^{-2} in both cases).

A similar remark applies to Corollary 4.1.5 when we scale the triangular lattice. The assumption $\alpha(v) = \nabla^{\mathbb{T}}\Phi(v)$ becomes

$$\alpha^\delta(v) = \nabla^{\delta\mathbb{T}}\varphi(v) := \frac{2}{3}\sum_{s=0}^{2}(\varphi(v+\tau^s) - \varphi(v))\tau^s = \delta\nabla\varphi(v) + o(\delta^2)\,,$$

for $v \in \delta\mathbb{T}$, and smooth $\varphi : \Omega \to \mathbb{R}$. Using

$$\Delta^{\delta\mathbb{T}}\varphi(v) := \frac{1}{3}\sum_{s=0}^{2}\varphi(v+\tau^s) - \varphi(v) = \frac{\delta^2}{4}\Delta\varphi(v) + o(\delta^2)$$

in place of $\Delta^{\mathbb{T}}$ the statement remains unchanged.

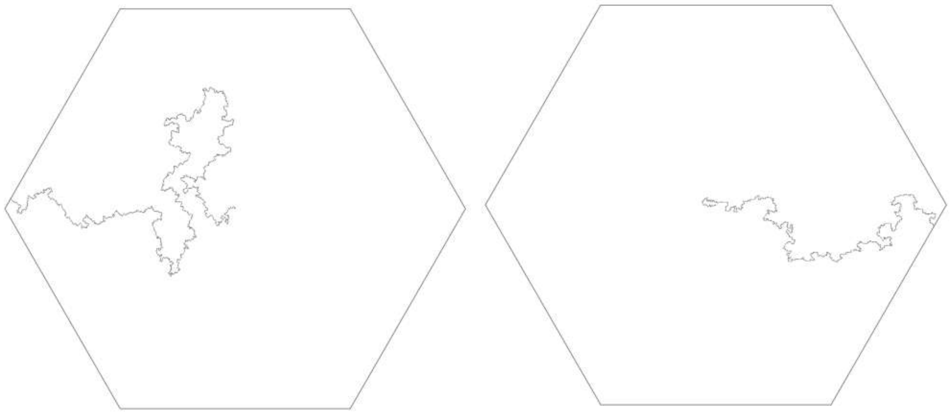

Figure 4.4: Two samples of loop-erased random walks on the triangular lattice in a hexagon of side-length 500. Left: no drift. Right: small drift to the right.

4.2.3 Statement of the theorem about LERW

We may now state the theorem needed for the proof of Theorem 4.1.1. Let Γ_δ be as in Theorem 4.1.1 and let Ω^δ denote the embedded graph on which the tree obtained from the Temperleyan bijection lives; thus Ω^δ is either a portion of the scaled square lattice or of the (directed) triangular lattice, and is embedded within the domain Ω. With an abuse of notation, we often identify the vertex set $v(\Omega^\delta)$ of Ω^δ with Ω^δ itself. Consider the random walk on Ω^δ arising from the weights (4.4) (resp. (4.3)). Observe that in either case, the corresponding law is of the form $\mathbb{P}^{(\alpha^\delta)}$ with $\alpha^\delta = \delta\alpha + o(\delta)$ does not depend on v, where α is as in (4.5). For instance, in the case of the triangular lattice, we define α_k^δ $(0 \le k \le 2)$ by

$$\exp(\alpha_k^\delta) = a_k = 1 + c_k\delta \tag{4.31}$$

so $\alpha^\delta = \frac{2}{3}\sum_{k=0}^2 \alpha_k^\delta \tau^k = \frac{2}{3}\delta \sum_{k=0}^2 c_k \tau^k + o(\delta) = \delta\alpha + o(\delta)$.

Theorem 4.2.3. *Suppose Ω is bounded and suppose $\alpha^\delta : \Omega \to \mathbb{R}^2$ is independent of v and satisfies $\alpha^\delta = \delta\alpha + o(\delta)$ for some fixed $\alpha \in \mathbb{R}^2$. Let $o \in \Omega$ and let o^δ denote a lattice point on Ω^δ which converges to o as $\delta \to 0$. Let $(\gamma_0^\delta, \dots, \gamma_T^\delta)$ denote the loop-erasure of a random walk sampled from $\mathbb{P}_{o^\delta}^{(\alpha^\delta)}$, starting from o^δ killed when leaving Ω^δ, and identify γ^δ with its linear interpolation to get a continuous path on $[0, T]$. Then as $\delta \to 0$,*

$$\gamma^\delta \to \gamma^0,$$

where γ^0 has the following law: first, its endpoint a has the law $\mu_o^{(\alpha)}$ which is the hitting distribution of $\partial\Omega$ by a Brownian motion with drift α starting from o; furthermore, conditionally given a, γ^0 is a massive radial SLE$_2$ from a to o in Ω with mass $\|\alpha\|/\sqrt{2}$. Here the convergence is in the sense of uniform convergence up to reparametrisation.

Note that Theorems 4.1.1 and 4.1.2 follow directly from Theorem 4.2.3 and Temperley's bijection (Theorem 4.2.1). The rest of Section 4.2 will be devoted to a proof of Theorem 4.2.3. We will separate the case of the square and triangular lattices as the proofs are a little different in each case. We first outline the main ideas. Essentially, we are able to relate at the discrete level the loop-erasure of random walk on Ω^δ with that of a massive random walk. The relation is exact in the case of the triangular lattice and approximate in the case of the square lattice. On the square lattice, we know by the results of Makarov and Smirnov [MS10] (as clarified by the more recent work of Chelkak and Wan [CW19]) that the massive LERW converges to massive SLE$_2$. Combined with the above-mentioned approximate relation on the square lattice, this gives a proof of Theorem 4.2.3 in this case. The theorem of Chelkak and Wan is however only stated for the square lattice and we will verify that their approach can be extended to cover the directed triangular lattice as well. The lack of reversibility is a difficulty in that case.

Let us now begin the proof of Theorem 4.2.3 for the triangular lattice, with a proof of the fact that the loop-erased random walk has the same law as the loop-erasure of a massive walk, once we condition on the endpoint.

Fix α^δ as in the theorem, and write $a_k = a_k^\delta = e^{\alpha_k^\delta}$ $(k = 0, \ldots, 2)$, and $a = a_0 + a_1 + a_2$. Let $\beta(v) = \beta^\delta(v)$ be implicitly defined by (4.8), which as we will soon see is of order δ^2, and clearly does not depend on v. We will want to compare our walk $\mathbb{P}^{(\alpha)}$ with an appropriate massive walk. Let $m = m^\delta > 0$ be defined by

$$\frac{1}{3}(1 - \frac{m^2\delta^2}{2}) = \frac{\sqrt[3]{a_0a_1a_2}}{a}. \tag{4.32}$$

(Note that m is well defined by the arithmetic-geometric mean inequality.) The mass m can also be related to the factor β^2 previously introduced in (4.8): that is,

$$1 - \frac{m^2\delta^2}{2} = \exp(-\beta^2/3).$$

We now show that the mass $m = m^\delta$ is non degenerate in the limit, and in fact simply equals the norm of the drift vector α (up to a factor 1/2).

Lemma 4.2.4. *Let α^δ be as in Theorem 4.2.3 and β be as above. Then*

$$\beta^2(v) = \frac{3}{4}\delta^2\|\alpha\|^2 + o(\delta^2), \tag{4.33}$$

Equivalently, if $m = m^\delta$ be as in (4.32) then m^δ converges as $\delta \to 0$ to $\frac{\|\alpha\|}{2}$.

Proof. In fact we will directly prove the result on m. This will come from a careful second order expansion (note however that our assumption about α^δ implies only $\alpha^\delta = \delta\alpha + o(\delta)$). For $k = 0, 1, 2$, let us write $a_k = e^{\alpha_k^\delta} = 1 + c_k\delta$, and let $s = c_0 + c_1 + c_2$, so that with these notations $a = a_0 + a_1 + a_2 = 3 + s\delta$. Then starting from the identity

$$\frac{\sqrt[3]{(1 + c_0\delta)(1 + c_1\delta)(1 + c_2\delta)}}{3 + s\delta} = \frac{1}{3}(1 - \frac{m^2\delta^2}{2}),$$

and expanding the product before doing a Taylor expansion of the left hand side as $\delta \to 0$, we find

$$\frac{1 + \frac{s\delta}{3} + (\frac{c_0c_1 + c_1c_2 + c_2c_0}{3} - \frac{1}{9}s^2)\delta^2 + o(\delta^2)}{3 + s\delta} = \frac{1}{3}(1 - \frac{m^2\delta^2}{2})$$

in other words, writing $\kappa = \frac{c_0c_1 + c_1c_2 + c_2c_0}{3} - \frac{1}{9}s^2$,

$$\frac{1}{3} + \frac{\kappa}{3}\delta^2 + o(\delta^2) = \frac{1}{3}(1 - \frac{m^2\delta^2}{2})$$

from which it follows that

$$m^2 = -2\kappa + o(1).$$

Let us call $p = c_0c_1 + c_1c_2 + c_2c_0$, so that

$$\kappa = \frac{p}{3} - \frac{s^2}{9} = \frac{p}{3} - \frac{1}{9}(\sum_{k=0}^{2} c_k^2 + 2p) = \frac{p}{9} - \frac{1}{9}\sum_{k=0}^{2} c_k^2$$

Now observe that since $\langle 1,\tau\rangle = \langle \tau,\tau^2\rangle = \langle 1,\tau^2\rangle = -1/2$,

$$\begin{aligned}\|\alpha\|^2 &= \frac{4}{9}\langle c_0 + c_1\tau + c_2\tau^2, c_0 + c_1\tau + c_2\tau^2\rangle \\ &= -\frac{4}{9}p + \frac{4}{9}\sum_{k=0}^{2} c_k^2 = -\kappa.\end{aligned}$$

Therefore,

$$m^2 = \frac{1}{2}\|\alpha\|^2 + o(1),$$

as desired. □

Let $\rho = \rho^\delta = m^2\delta^2/2$, and let $\mathbb{P}^{(\rho)} = \mathbb{P}^{(\rho^\delta)}$ denote the law of massive random walk, which has jump probabilities

$$\mathbb{P}^{(\rho)}(v, v+\delta\tau^k) = \frac{1}{3}(1 - \frac{m^2\delta^2}{2}) = \frac{\sqrt[3]{a_0a_1a_2}}{a}; \quad k = 0,\dots,2$$

and which jumps to an additional ghost or cemetery vertex with probability $\rho^\delta = m^2\delta^2/2$ (in which case say that the path has died). Let $\mathbb{P}^{(\rho)}(\cdot|Y^\delta = a^\delta)$ denote the conditional law of massive random walk, given that the walk does not die before leaving Ω^δ and that the exit point is a^δ.

From Theorem 4.1.4 we get the following corollary:

Corollary 4.2.5. *For each $\delta > 0$, for each $o^\delta \in \Omega^\delta$ and $a^\delta \in \partial\Omega^\delta$, we have*

$$\mathbb{P}_{o^\delta}^{(\alpha^\delta)}(\cdot|Y_\delta = a_\delta) = \mathbb{P}_{o^\delta}^{(\rho^\delta)}(\cdot|Y_\delta = a_\delta).$$

Proof. Since α^δ does not depend on v, the discrete stochastic integral can be written as

$$M_n = \sum_{s=0}^{n-1}\langle \delta^{-1}\alpha^\delta(x_s), \mathrm{d}x_s\rangle = \langle \delta^{-1}\alpha^\delta, a_\delta - o_\delta\rangle \tag{4.34}$$

and so does not depend on the path γ^δ subject to the condition $Y^\delta = a^\delta$. Furthermore the mass m has been chosen so that the quadratic variation part cancels the mass term exactly: that is,

$$\exp(-\tfrac{1}{2}V_n) = (1 - \frac{m^2\delta^2}{2})^n.$$

Hence the ratio of the left hand side to the right hand side is a constant, independent of the path γ^δ, therefore this constant is one since both probability measures sum up to one when we sum over all paths. □

Corollary 4.2.6. *Let $\alpha = \alpha^\delta$ be as in Theorem 4.2.3 and consider the mass m as in* (4.32) *and $\rho = \rho^\delta = m^2\delta^2/2$. Suppose $o^\delta \to o \in \Omega, a^\delta \to a \in \partial\Omega$,*

$$\frac{\mathbb{P}_{o^\delta}^{(\alpha)}}{\mathbb{P}_{o^\delta}^{(\rho)}}(\gamma^\delta) \to \exp(\langle \alpha, a - o\rangle).$$

Proof. This follows from our exact expression for $(\mathbb{P}_{o^\delta}^{(\alpha)}/\mathbb{P}_{o^\delta}^{(0)})(\gamma^\delta)$, (4.34), the already observed fact that the quadratic variation part cancels exactly with the mass, and the fact that $\delta^{-1}\alpha^\delta$ converges to α. □

Note that this is stated without conditioning the massive walk to hit the boundary before dying. (This conditioning would simply add a term to the Radon-Nikodym derivative of the previous lemma, corresponding to the probability to hit the boundary before dying.)

From Corollary 4.2.6, in particular we see that when γ^δ is a loop then $\mathbb{P}^{(\alpha)}(\gamma^\delta) = \mathbb{P}^{(\rho)}(\gamma^\delta)$. Although we do not need this here, this implies that the loop measures associated with the drifted walk $\mathbb{P}^{(\alpha)}$ and $\mathbb{P}^{(\rho)}$ are identical. Since these loop measures can be used to identify the law of loop-erased random walk (see, e.g., [LL10, Chapter 9.5]), we can use this observation to deduce that the expression obtained in Lemma 4.2.6 can be transferred at the level of the loop-erasure. In fact this can be proved directly as follows. If X is a lattice path, let $\mathsf{LE}(X)$ denote the chronological loop-erasure of X considered up until its hitting time of $\partial\Omega^\delta$ (if the path never reaches $\partial\Omega^\delta$ – for instance if it dies before reaching the boundary – then $\mathsf{LE}(X)$ is by convention the empty path).

Lemma 4.2.7. *Let γ^δ denote a fixed (sequence of)* ***simple*** *lattice paths from $o^\delta \in \Omega^\delta$ to $a^\delta \in \partial\Omega^\delta$, with $o^\delta \to o \in \Omega$, $a^\delta \to a \in \partial\Omega$. Then*

$$\frac{\mathbb{P}^{(\alpha)}(\mathsf{LE}(X) = \gamma^\delta)}{\mathbb{P}^{(\rho)}(\mathsf{LE}(X) = \gamma^\delta)} \to \exp(\langle \alpha, a - o \rangle).$$

as $\delta \to 0$.

Proof. This follows directly from Corollary 4.2.6 by summing over all ways to obtain γ^δ as a loop-erasure, and noting that the expression for the Radon-Nikodym derivative in Corollary 4.2.6 depends only on the endpoints of the path, and not the rest of the path itself. □

As was mentioned in the introduction, the scaling limit of massive LERW is rather well understood, at least on the square lattice. Although the existing proofs of convergence to massive SLE$_2$ do not cover the case of the triangular lattice, it is possible with a bit of effort to extend these methods to cover this case (the main issue is the lack of reversibility which is needed to establish the crucial "resolvent identity" at the discrete level). We state the result here, but defer its proof until later, and see how this can be used to deduce Theorem 4.2.3.

Theorem 4.2.8. *Let $\Omega^\delta \subset \delta\mathbb{T}$ approximate Ω with $o^\delta \in \Omega^\delta \to o \in \Omega$, and let a^δ be a boundary point of Ω^δ such that $a^\delta \to a \in \partial\Omega$. Consider the loop-erasure of a random walk sampled from $\mathbb{P}^{(\rho^\delta)}_{o^\delta}$, started at o^δ and conditioned to hit the boundary at a^δ before dying, with mass $\rho^\delta = m^2\delta^2/2$, where $m = m^\delta \to m$, converges in law to radial massive SLE$_2$ from a to o with mass m.*

Since the exit distribution of massive LERW from Ω^δ, conditional on exiting this domain before dying, has a limit as $\delta \to 0$ (the "massive harmonic measure" on $\partial\Omega$), and since the law of radial massive SLE$_2$ from $a \in \partial\Omega$ to $o \in \Omega$ is continuous with respect to a, we deduce from this theorem that the scaling limit holds even if we do not condition on the exit point a^δ of the random walk, and simply condition on not dying before reaching the boundary.

The proof of Theorem 4.2.8 is deferred to Section 4.3. For now, we see how this immediately implies Theorem 4.2.3 for the triangular lattice.

Proof of Theorem 4.2.3. This will follow rather simply from Lemma 4.2.7, the fact that the expression for the Radon-Nikodym derivative is well-behaved, and the fact that the extinction probability for the massive walk converges to some nontrivial probability bounded away from zero and one as $\delta \to 0$. Indeed, since Ω is bounded, the function $\exp(\langle \alpha, a - o \rangle)$, viewed as a function of the endpoint $a \in \partial\Omega$, is a bounded continuous functional on path space.

Therefore, if F is another arbitrary such functional, then letting σ^δ be the hitting time of $\partial\Omega^\delta$ by X^δ,

$$\mathbb{E}_{o^\delta}^{(\alpha^\delta)}[F(\mathsf{LE}(X^\delta))] = \mathbb{E}_{o^\delta}^{(\rho^\delta)}\left[F(\mathsf{LE}(X^\delta))\exp(\langle\alpha+o(1), Y_\delta - o^\delta\rangle)\middle|\,\sigma^\delta<\infty\right]\mathbb{P}_{o^\delta}^{(\rho^\delta)}(\sigma^\delta<\infty)$$
$$\to \mathbb{E}_o^{\mathrm{mSLE}_2}\left[F(\gamma)\exp(\langle\alpha, Y-o\rangle)\right]p(o),$$

where $\mathbb{E}_o^{\mathrm{mSLE}_2}$ denote the law of a massive radial SLE$_2$ started from massive harmonic measure on $\partial\Omega$ (a point which we denote by Y), towards o, and $p(o)$ is the survival probability for massive Brownian motion in Ω starting from o, killed at rate $\rho = m^2 = \|\alpha\|^2/2$, i.e., $p(o) = \mathbb{P}_o^{(\rho)}(\sigma<\infty) = \mathbb{E}_o^{\mathrm{BM}}(\exp(-\|\alpha\|^2\sigma/2))$ with σ the exit time from Ω. The rest of the result follows immediately by specifying F to be a continuous function of the endpoint Y_δ. □

4.2.4 Discrete Girsanov on the square lattice

Now let us consider the case of the square lattice, so Ω^δ is a portion of $\delta\mathbb{Z}^2$ which approximates Ω in the sense discussed above. Our first task is to define precisely what we mean by $\mathbb{P}^{(\alpha^\delta)}$. Let $c_k\colon \Omega^\delta \to \mathbb{R}$ be bounded functions for $k = 0,\dots,3$, satisfying $c_0 + c_2 = c_1 + c_3$ at each vertex $v \in \Omega^\delta$. Then $\mathbb{P}^{(\alpha^\delta)}$ is the law of the Markov chain on Ω^δ whose jump probabilities from the vertex $v \in \Omega^\delta$ are given by

$$\mathbb{P}^{(\alpha^\delta)}(v, v+\delta\mathbf{i}^k) = \frac{a_k(v)}{a(v)} \quad k = 0,\dots,3\,, \tag{4.35}$$

where

$$a(v) = \sum_{k=0}^{3} a_k(v) \text{ and } a_k(v) = 1 + c_k(v)\delta, \text{ for } k = 0,\dots,3, v\in\Omega^\delta.$$

And define $\alpha = \alpha^\delta$ via

$$\alpha^\delta(v) = \frac{1}{2}\sum_{k=0}^{3} a_k(v)\mathbf{i}^k = \frac{\delta}{2}\sum_{k=0}^{3} c_k(v)\mathbf{i}^k \in \mathbb{C}\simeq\mathbb{R}^2.$$

Again α^δ does not determine the c_k uniquely, but only up to global shift, which does not influence the limit of the law, so that our notation $\mathbb{P}^{(\alpha^\delta)}$ is justified.

Together these assumptions guarantee that, if $c_k(v)$ is given by some fixed Lipschitz function $c_k : \Omega \to \mathbb{R}$ evaluated at $v \in \Omega^\delta \subset \Omega$, then $\alpha^\delta = \delta\alpha + o(\delta)$, where $\alpha = (1/2)\sum_{k=0}^{3} c_k\mathbf{i}^k$. (This is in particular the situation of interest for Theorem 4.1.1, where c_k are in fact constant). Thus α is itself a Lipschitz vector field defined on all of Ω; this random walk converges to a Brownian motion with drift α under the same scaling as discussed in Section 4.1.9.

Again fix $\gamma^\delta = (x_0,\dots,x_n)$ a given path, this time on the square lattice, starting from some point $x_0 = o^\delta \in \Omega^\delta$ of some length $n = N(\gamma^\delta)$. Define $\alpha_k^\delta(v)\in\mathbb{R}, k = 0,\dots,3$ by

$$\exp(\alpha_k) = a_k = 1 + c_k\delta. \tag{4.36}$$

Define also (for $i = 1,2$), $\beta_i = \beta_i^\delta(v) \geq 0$ by

$$\exp(-\beta_i^2) = \frac{a_{i-1}a_{i+1}}{(a/4)^2}.$$

Note that β_1 is well defined by the arithmetic-geometric mean inequality since for $i = 1,2$,

$$\frac{a}{4} = 1 + \frac{1}{4}\sum_{k=0}^{3} c_k\delta = 1 + \tfrac{1}{2}(c_{i-1}\delta + c_{i+1}\delta) = \frac{a_{i-1}+a_{i+1}}{2},$$

where we used the assumption that $c_0 + c_2 = c_1 + c_3$. We will then denote by $\beta \in \mathbb{R}^2$ the vector $\beta = (\beta_1, \beta_2)$.

The next lemma gives the Girsanov identity in the case of the square lattice, which for conciseness we only give on the scaled lattice.

Lemma 4.2.9. *On the square lattice, we have*

$$\frac{\mathbb{P}_z^{(\alpha^\delta)}(\gamma^\delta)}{\mathbb{P}_z^{(0)}(\gamma^\delta)} = \exp(M_n - \tfrac{1}{2}V_n), \tag{4.37}$$

where M_n and V_n can be written as

$$M_n = \sum_{s=0}^{n-1} \delta^{-1}\langle \alpha(x_s); \mathrm{d}x_s \rangle \text{ and } V_n = \sum_{s=0}^{n-1} \delta^{-2}\|\beta(x_s) \odot \mathrm{d}x_s\|^2$$

where $a \odot b$ is the Hadamard product of the vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$, whose coordinate are $a_i b_i$ $(i = 1, 2)$. Explicitly, $V_n = \delta^{-2}\sum_{s=0}^{n-1} \beta_1(x_s)^2|\mathrm{d}x_s^1|^2 + \beta_2(x_s)^2|\mathrm{d}x_s^2|^2$.

Proof. Denote for a given path γ^δ of length n whose starting point is z, by $n_0(v), \ldots, n_3(v)$ the number of steps of the walk from v and going in the direct $1, \mathbf{i}, -1, -\mathbf{i}$ respectively.

$$\begin{aligned}
&\mathbb{P}_z^{(\alpha)}(\gamma^\delta) \qquad (4.38)\\
&= \prod_{v \in v(\Omega^\delta)} a_0^{n_0(v)} a_1^{n_1(v)} a_2^{n_2(v)} a_3^{n_3(v)} a^{-n}\\
&= 4^{-n} \prod_{v \in v(\Omega^\delta)} \left(\frac{a_0}{a_2}\right)^{\frac{n_0-n_2}{2}} \left(\frac{a_1}{a_3}\right)^{\frac{n_1-n_3}{2}} \left(\frac{a_0 a_2}{(a/4)^2}\right)^{\frac{n_0+n_2}{2}} \left(\frac{a_1 a_3}{(a/4)^2}\right)^{\frac{n_1+n_3}{2}}\\
&= 4^{-n} \exp\left(\sum_{v \in v(\Omega^\delta)} (\alpha_0 - \alpha_2)\frac{n_0 - n_2}{2} + (\alpha_1 - \alpha_3)\frac{n_1 - n_3}{2} - \beta_1^2\frac{n_0 + n_2}{2} - \beta_2^2\frac{n_1 + n_3}{2}\right)\\
&= 4^{-n} \exp(M_n - \tfrac{1}{2}V_n), \qquad (4.39)
\end{aligned}$$

where in the last step we used that for each step of the walk in direction $1, i, -1$ or $-i$ the left two summands contribute $1/2$ times $\alpha_0 - \alpha_2, \alpha_1 - \alpha_3, \alpha_2 - \alpha_0$ or $\alpha_3 - \alpha_1$ respectively, whereas the right two summands contribute $1/2$ times β_1^2 or β_2^2 depending on whether the displacement is horizontal or vertical. This leads to the expressions for M_n and V_n respectively. The negative powers of δ in the expressions of M_n and V_n compensate the length of $\mathrm{d}x_s$. □

We can again compare $\mathbb{P}^{(\alpha)}$ with an appropriate massive random walk. Let $m = m^\delta = m^\delta(v) > 0$ be defined by:

$$\tfrac{1}{4}(1 - \frac{m^2\delta^2}{2}) = \frac{\sqrt[4]{a_0 a_1 a_2 a_3}}{a}$$

Note that this choice of m also satisfies:

$$(1 - \frac{m^2\delta^2}{2}) = \exp(-\frac{\beta_1^2 + \beta_2^2}{4}).$$

We will see below that in the situation of interest for Theorem 4.2.3 (and so in particular for Theorem 4.1.1) β_i^2 will indeed be of order δ^2.

Let $\rho = \rho^\delta(v) = m^2\delta^2/2$ and let $\mathbb{P}^{(\rho)}$ be the law of the massive random walk, which has jump probabilities

$$\mathbb{P}^{(\rho)}(v, v + \delta\mathbf{i}^k) = \tfrac{1}{4}(1 - \frac{m^2\delta^2}{2}), \text{ for } k = 0\ldots, 3$$

and which jumps to an additional ghost or cemetery vertex with probability $\frac{m^2\delta^2}{2}$ (in which case we say that the path has died). Let $\mathbb{P}^{(\rho)}(\cdot|Y_\delta = a^\delta))$ denote the conditional law of the massive random walk, given that the walk does not die before leaving Ω^δ and that the exit point is a^δ.

Lemma 4.2.10. *Suppose α^δ does not depend on v and $\alpha^\delta = \delta\alpha + o(\delta)$, or equivalently c_k converges as $\delta \to 0$ for $k = 0, \dots, 3$. Then for $i = 1, 2$*

$$\beta_i^2 = \tfrac{1}{4}(c_{i-1} - c_{i+1})^2\delta^2 + o(\delta^2).$$

In particular $\delta^{-2}\|\beta\|^2$ converges as $\delta \to 0$. Furthermore,

$$m^2 \to \tfrac{\|\alpha\|^2}{2}.$$

Proof. Note that by definition, for $i = 1, 2$,

$$e^{-\beta_i^2} = \frac{a_{i-1}a_{i+1}}{(a/4)^2} = \frac{(1 + c_{i-1}\delta)(1 + c_{i+1}\delta)}{(1 + \frac{c_{i-1}+c_{i+1}}{2}\delta)^2} = 1 - \tfrac{1}{4}(c_{i-1} - c_{i+1})^2\delta^2 + o(\delta^2),$$

as desired. Now,

$$1 - \frac{m^2\delta^2}{2} = \exp(-\tfrac{\beta_1^2+\beta_2^2}{4}) = \exp\left(-\tfrac{\delta^2}{16}\left((c_2 - c_0)^2 + (c_3 - c_1)^2\right)\right)$$

so that

$$m^2 = \tfrac{1}{8}\left((c_2 - c_0)^2 + (c_3 - c_1)^2\right) + o(\delta^2).$$

On the other hand, $\alpha = \frac{1}{2}\sum_{k=0}^{3} c_k \mathbf{i}^k = ((c_2 - c_0)/2, (c_3 - c_1)/2)$ so that

$$\|\alpha\|^2 = \tfrac{1}{4}((c_2 - c_0)^2 + (c_3 - c_1)^2),$$

which concludes the proof. □

While an exact connection between massive random walk and the random walk with drift conditioned on the exit point holds only for the triangular lattice, a similar statement holds asymptotically also for the square lattice. To establish the connection between the two random walks we first prove the following lemma:

Lemma 4.2.11. *Fix $\alpha^\delta = \delta\alpha + o(\delta)$, where $\alpha \in \mathbb{R}^2$ is fixed. Let $\sigma = \sigma^\delta$ be the first time the random walk leaves the domain Ω^δ and $\theta < 1$. Then uniformly over $z_\delta \in \Omega^\delta$:*

$$\mathbb{P}_{z_\delta}^{(\alpha^\delta)}(|V_\sigma - \sigma\frac{\|\beta\|^2}{2}| > \delta^\theta) \to 0,$$

as $\delta \to 0$. In particular this holds also under $\mathbb{P}_{z_\delta}^{(0)}$.

Proof. Since $(X^\delta_{2\delta^{-2}t})_{t\geq 0}$ converges to Brownian motion with drift $\alpha \in \mathbb{R}^2$, $\sigma = \sigma^\delta$ is of order δ^{-2} and fluctuates on that scale, i.e. the distribution of $\delta^2\sigma^\delta$ has a nontrivial weak limit, which simply is the law of the exit time σ of Ω by a Brownian motion with drift α (let $\mathbb{P}^{(\alpha)}$ denote its law).

Let $\epsilon > 0$. Choose K large enough that $\mathbb{P}_z^{(\alpha)}(\sigma > K) < \epsilon$ uniformly over $z \in \Omega$. For δ small enough it follows by compactness that

$$\mathbb{P}_{z_\delta}^{(\alpha^\delta)}(\sigma > K\delta^{-2}) < 2\epsilon.$$

At each step, the walk takes a horizontal or a vertical step, each with probability $\frac{1}{2}$ (since $c_0 + c_2 = c_1 + c_3$). Therefore $Q_n := V_n - n\frac{\beta_1^2+\beta_2^2}{2} = V_n - n\|\beta\|^2/2$ is a martingale with increment jumps uniformly bounded by $O(\delta^2)$ by Lemma 4.2.10.

Hence $q_n = \delta^{-2}Q_n$ is a martingale with bounded increments, and we are interested in the terminal value of Q_n at the stopping time $\sigma = \sigma^\delta$. Using Freedman's martingale inequality (Proposition (2.1) in [Fre75]), we conclude

$$\begin{aligned}\mathbb{P}_{z_\delta}^{(\alpha^\delta)}(|Q_{\sigma^\delta}| > \delta^\theta) &\le \mathbb{P}_{z_\delta}^{(\alpha^\delta)}(\sigma > K\delta^{-2}) + \mathbb{P}_{z_\delta}^{(\alpha^\delta)}(|q_\sigma| > \delta^{\theta-2}; \sigma \le K\delta^{-2}) \\ &\le 2\epsilon + \exp\left(-\frac{\delta^{2\theta-4}}{2(C\delta^{\theta-2} + K\delta^{-2})}\right) \\ &\le 2\epsilon + \exp(-c\delta^{2\theta-2}),\end{aligned}$$

where c depends only on K and θ (and hence only on ϵ and θ) but not on δ. The lemma follows since $\theta < 1$. □

This allows us to prove the analogue of Corollary 4.2.6:

Corollary 4.2.12. *Let γ^δ be a path in Ω^δ from $o^\delta \in \Omega^\delta$. Then if $o^\delta \to o \in \Omega$ and $a^\delta \to a \in \partial\Omega$*

$$\frac{\mathbb{P}_{o^\delta}^{(\alpha^\delta)}(\gamma^\delta)}{\mathbb{P}_{o^\delta}^{(\rho^\delta)}(\gamma^\delta)} \to \exp(\langle (a-o), \alpha\rangle),$$

in probability as δ goes to 0, under either the law $\mathbb{P}^{(\rho^\delta)}(\cdot|Y_\delta = a^\delta)$ or $\mathbb{P}^{(\alpha^\delta)}(\cdot|Y_\delta = a^\delta)$.

Proof. By 4.2.9 we have that the ratio satisfies:

$$\frac{\mathbb{P}_{o^\delta}^{(\alpha^\delta)}(\gamma^\delta)}{\mathbb{P}_{o^\delta}^{(\rho^\delta)}(\gamma^\delta)} = \exp(M_n - \tfrac{1}{2}V_n + n\frac{\|\beta\|^2}{4}) = \exp(M_n)\exp\left(\tfrac{1}{2}(n\frac{\|\beta\|^2}{2} - V_n)\right).$$

Since we are only considering paths that do not die before reaching their endpoint, Lemma 4.2.11 applies and the second term converges in probability to 1 with respect to $\mathbb{P}^{(\alpha^\delta)}$ since the term in the exponential converges to 0 in probability.

On the other hand, as in the triangular case, since α^δ does not depend on v, $M_n = \langle \delta^{-1}\alpha^\delta, a^\delta - o^\delta\rangle$, which converges to $\langle \alpha, a-o\rangle$ under our assumptions. □

Remark 4.2.13. Note in particular that if G_δ is the good event

$$G_\delta := \{|V_\sigma - \sigma\frac{\|\beta\|^2}{2}| \le \delta^\theta\},$$

then we have learnt that on G_δ we may write

$$\frac{\mathbb{P}_{o^\delta}^{(\alpha^\delta)}(\gamma^\delta)}{\mathbb{P}_{o^\delta}^{(\rho^\delta)}(\gamma^\delta)} = (1 + o(1))\exp(\langle (a-o), \alpha\rangle),$$

where the $o(1)$ term is nonrandom. Note that since $\mathbb{P}^{(\rho^\delta)}(G_\delta|Y_\delta = a^\delta) \to 1$ this implies that Lemma 4.2.7 also holds on the square lattice.

With this proposition we can now conclude to the proof of Theorem 4.2.3 in the case of the square lattice.

Proof of Theorem 4.2.3, square lattice case. Let F be a bounded continuous functional on curves in Ω (for the topology of uniform convergence of paths up to reparametrisation). Let o, a and o^δ, a^δ be as in Corollary 4.2.12. Let γ^δ denote the random walk with jump probabilities given by (4.35) and let σ^δ denote the first time γ^δ leaves Ω^δ. Let $\mathsf{LE}(\gamma^\delta)$ denote the chronological loop-erasure of γ^δ. We want to show that

$$\mathbb{E}_{o^\delta}^{(\alpha^\delta)}[F(\mathsf{LE}(\gamma^\delta))] \to \int_{a\in\partial\Omega} \mathbb{E}_{o;a}^{\mathrm{mSLE}_2}[F(\gamma)]\mu_o^{(\alpha)}(\mathrm{d}a) \tag{4.40}$$

where $\mathbb{E}_{o;a}^{\mathrm{mSLE}_2}$ is the law of massive radial SLE_2 between a and o in Ω, with mass $m = \|\alpha\|/\sqrt{2}$, and $\mu_o^{(\alpha)}(dy)$ denote the hitting distribution of Brownian motion with drift α of $\partial\Omega$ from o. Then

$$\begin{aligned}\mathbb{E}_{o^\delta}^{(\alpha^\delta)}[F(\mathsf{LE}(\gamma^\delta))] &= \mathbb{E}_{o^\delta}^{(\alpha^\delta)}[F(\mathsf{LE}(\gamma^\delta)1_{G_\delta}] + o(1)\\ &= \mathbb{E}_{o^\delta}^{(\rho^\delta)}[F(\mathsf{LE}(\gamma^\delta))1_{G_\delta\cap\{\sigma^\delta<\infty\}}(1+o(1))\exp(\langle\alpha, \gamma_{\sigma^\delta}^\delta - o^\delta\rangle)] + o(1)\\ &= (1+o(1))\mathbb{E}_{o^\delta}^{(\rho^\delta)}[F(\mathsf{LE}(\gamma^\delta)1_{\{\sigma^\delta<\infty\}}\exp(\langle\alpha, \gamma_{\sigma^\delta}^\delta - o^\delta\rangle)] + o(1)\end{aligned}$$

by Remark 4.2.13. Now, using Lemma 4.2.10, by [CW19, Theorem 1.1], and since $\gamma_{\sigma^\delta}^\delta$ is a bounded, a.s. continuous functional of γ^δ (when Ω is bounded), we find

$$\mathbb{E}_{o^\delta}^{(\alpha^\delta)}[F(\mathsf{LE}(\gamma^\delta))] \to p^{(\rho)}(o)\int_{a\in\partial\Omega} \mathbb{E}_{o;a}^{\mathrm{mSLE}_2}[F(\gamma)\exp(\langle\alpha, a-o\rangle)]\mu_o^{(\rho)}(\mathrm{d}a),$$

where $\mu_o^{(\rho)}$ is the law of X_σ under $\mathbb{P}_o^{(\rho)}$, conditioned on $\sigma < \infty$ and $p^{(\rho)}(o)$ is the probability of this event.

Taking F to be a function of $\gamma_{\sigma^\delta}^\delta$ only, we see that

$$\int_{a\in\partial\Omega} F(a)\mu_o^{(\alpha)}(\mathrm{d}a) = p^{(\rho)}(o)\int_{a\in\partial\Omega} F(a)\exp(\langle\alpha, a-o\rangle)\mu_o^{(\rho)}(\mathrm{d}a),$$

so that

$$\mu_o^{(\alpha)}(\mathrm{d}a) = p^{(\rho)}(o)\exp(\langle\alpha, a-o\rangle)\mu_o^{(\rho)}(\mathrm{d}a)$$

almost everywhere with respect to $\mu_o^{(\rho)}$. This proves (4.40) and hence Theorem 4.2.3 in the case of the square lattice. □

Remark 4.2.14. If we had not assumed $c_0 + c_2 = c_1 + c_3$ we could not write the "quadratic variation term" V_n in the form of a sum along the path of positive terms of type β_k^2, $k = 1, 2$. Even if we don't insist on the positivity of these terms and try to analyse the limiting behaviour, we find that V_n is the sum of terms of order δ rather than δ^2. The first order contribution however cancels out on the large scale and we do get a term of order 1 when n is of order δ^{-2}, but it does not seem that this term can easily be interpreted as a massive term; in particular it seems it might not be concentrated at a fixed time $n \approx t\delta^{-2}$. In other words, the Radon-Nikodym dervative of the random walk with drift with respect to the massive random walk picks up a non-trivial contribution due to the walk taking more horizontal or vertical steps, even though the proportion of those steps behaves like $\frac{1}{2} + c\delta$.

4.3 Convergence of massive LERW on the triangular lattice

In [LSW01] Lawler, Schramm and Werner proved that the scaling limit of the loop-erased random walk in a simply connected domain on the square lattice converges to radial SLE_2.

While the proof is written for the LERW on the square grid, in the last chapter it is mentioned that the proof can be adapted to more general setups; the random walk on the directed triangular lattice is explicitly mentioned as an example of an *irreversible* random walk to which the proof applies. In [MS10] Makarov and Smirnov proposed a strategy for proving convergence of the massive LERW to massive SLE$_2$ building in part on ideas coming from Conformal Field Theory (see [BBC09, BBK08]). This strategy was then successfully followed by Chelkak and Wan in [CW19], using a framework for convergence to SLE developed by Kemppainen and Smirnov in [KS17] and a recent addition [Kar18] by Karilla. We show in this section that the arguments of Chelkak and Wan in [CW19] can be adapted to the directed triangular lattice which will imply a proof of Theorem 4.2.8. The additional difficulty here is the lack of reversibility, which is crucially used to derive a discrete "resolvent identity" and is the heart of the proof in [CW19]; see in particular Proposition 4.3.12 below. We note that a more general proof (but requiring quite a bit more work) will be given in Section 4.4, so that this section could be skipped by the reader.

In order to stay close to the notations of [CW19] we will use, in this section only, the notation $\mathbb{P}^{(m)}$ (instead of $\mathbb{P}^{(\rho)}$) for the massive random walk which dies with probability $m^2\delta^2/2$ at each step; likewise partition functions will be denoted e.g. by $Z^{(m)}$, as we will see below.

4.3.1 Convergence of domains and curves

For each discrete domain $\Omega^\delta \subset \delta\mathbb{T}$ we associate a polygonal domain $\hat{\Omega^\delta} \subset \mathbb{C}$ which is the union of open hexagons with side length δ centered at vertices of Ω^δ. Notice that vertices of $\delta\mathbb{T}$ on the boundary of $\hat{\Omega^\delta}$ are exactly vertices on the outer vertex boundary of Ω^δ.

We will assume that $\hat{\Omega^\delta}$ converges to Ω in the *Carathéodory* topology and if this is the case write, that Ω^δ approximates Ω. This means that each inner point of Ω belongs to $\hat{\Omega^\delta}$ for small enough δ and each boundary point of Ω can be approximated by boundery points of Ω^δ, see, e.g., [Pom92]. Further, we assume that $0 \in \Omega^\delta$ for each δ and we have a point $a^\delta \in \partial\Omega^\delta$ which converges to $a \in \partial\Omega$. Let $\psi_{\hat{\Omega^\delta}} \colon \hat{\Omega^\delta} \to \mathbb{D}$ be the unique conformal map such that $\psi_{\hat{\Omega^\delta}}(o) = 0$ and $\psi_{\hat{\Omega^\delta}}(a^\delta) = 1$. Then it can be seen (see, e.g., [Pom92]) that Carathéodory convergence is equivalent to the uniform convergence on compacts of $\psi_{\hat{\Omega^\delta}}$ and $\psi^{-1}_{\hat{\Omega^\delta}}$ to ψ_Ω and ψ_Ω^{-1} respectively.

The main theorem of [KS17] states that if a family Σ of measures of random curves satisfies a certain annulus crossing condition, then the family is tight and furthermore, if $\mathbb{P}_n \in \Sigma$ is a weakly converging subsequence then its limit is a random Loewner chain. Moreover if $(W^{(n)})_{n\in\mathbb{N}}$ are the driving processes of the random curves $(\gamma^{(n)})_{n\in\mathbb{N}}$ that satisfy the annulus crossing condition which are parametrized by capacity then:

- $(W^{(n)})_{n\in\mathbb{N}}$ is tight in the space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets.

- $(\gamma^{(n)})_{n\in\mathbb{N}}$ is tight in the space of curves up to reparametrization with the supremum norm.

If the sequence converges in either of the topologies it also converges in the other one and the limit of the driving processes is the driving process of the limiting random curve.

That the annulus crossing condition is satisfied is checked for a chordal loop-erased random walk in [KS17, Section 4.5] with a remark that the radial case is equivalent to calculations in [LSW01].

4.3.2 Absolute continuity with respect to classical SLE$_2$

Let $0 < \delta < m^{-1} \leq \infty$. Here, m is the mass, which we allow to be zero and δ is the scale. We consider subgraphs Ω^δ of the scaled triangular lattice $\delta\mathbb{T}$, which approximate some domain $\Omega \in \mathbb{C}$. Given such δ, m, Ω^δ as well as two vertices w^δ, z^δ we define the partition function of the massive random walk:

$$Z^{(m)}_{\Omega^\delta}(w^\delta, z^\delta) := \sum_{\pi^\delta \in S(w^\delta, z^\delta)} \left(\frac{1}{3}(1 - \frac{m^2\delta^2}{2}) \right)^{\#\pi^\delta}, \tag{4.41}$$

where the sum is over all possible paths π^δ from w^δ to z^δ remaining in Ω^δ. If $m = 0$ this corresponds to the classical random walk and we drop the superscript (m); thus

$$Z_{\Omega^\delta}(w^\delta, z^\delta) = Z^{(0)}_{\Omega^\delta}(w^\delta, z^\delta).$$

If w^δ is an interior vertex and z^δ is a vertex on the boundary, this is the probability that a random walk with killing rate $\frac{m^2\delta^2}{2}$ started at w^δ leaves the boundary at z^δ without any conditioning. More generally, $Z_{\Omega^\delta}(w^\delta, z^\delta)$ is the **discrete massive Green function**, i.e. the expected number of visits to z^δ starting from w^δ before hitting the boundary or being killed. Note that, because of the directed edges in general $Z^{(m)}_{\Omega^\delta}(w^\delta, z^\delta) \neq Z^{(m)}_{\Omega^\delta}(z^\delta, w^\delta)$. In the limit however, we will see (in section 4.3.3) that equality holds.

To apply the tightness results to the massive case we first need some estimates on this partition function, which are similar (but easier in some respects) as Lemma 2.4 and Proposition 2.5 in [CW19].

Proposition 4.3.1. *For each domain Ω^δ with $\Omega^\delta \subset B(0,1)$, for each $\varepsilon > 0$ there exists $c > 0$ (depending only on $\varepsilon > 0$) such that the following holds. For each interior point v^δ at distance at least $\varepsilon > 0$, and for each boundary point b^δ, $\delta \leq \frac{1}{2}m^{-1}$, one has*

$$\frac{Z^{(m)}_{\Omega^\delta}(v^\delta, b^\delta)}{Z_{\Omega^\delta}(v^\delta, b^\delta)} \geq \exp(-cm^2).$$

Proof. We proceed as in the proof of Proposition 2.5 in [CW19]. By Jensen's inequality (since $1 - \frac{m^2\delta^2}{2} \geq 0$):

$$\frac{Z^{(m)}_{\Omega^\delta}(v^\delta, b^\delta)}{Z_{\Omega^\delta}(v^\delta, b^\delta)} = \mathbb{E}\left((1 - \frac{m^2\delta^2}{2})^{\#\pi^\delta} \right) \geq (1 - \frac{m^2\delta^2}{2})^{\mathbb{E}(\#\pi^\delta)},$$

where the expectation is for a classical random walk π started at v^δ conditioned to leave at b^δ. Therefore it suffices to show

$$\mathbb{E}(\#\pi^\delta) \leq \text{const} \cdot \delta^{-2}. \tag{4.42}$$

In the chordal context of [CW19], where this needs to be proved for a random walk excursion from the boundary point a^δ to the boundary point b^δ, this is the content of Lemma 2.4 in [CW19] (in fact that Lemma is even more precise, since it bounds the expected time spent at any given point by a constant). This is done by referencing [Che16], which also works in the directed triangular lattice, as this random walk also satisfies conditions (S) and (T) in [Che16]. Rather than adapting the arguments of [Che16] to our radial context, let us give a brief argument which shows how the chordal estimate (Lemma 2.4 in [CW19]) implies the desired radial estimate (4.42). Let $\varepsilon > 0$ be such that v^δ is at distance at least $\varepsilon > 0$ from the boundary. Let u^δ be another interior point, also at distance at least $\varepsilon > 0$ from the boundary. Let $\mathbb{P}_{u^\delta \to b^\delta}$ denote the law of simple random random walk, conditioned to leave Ω^δ through b^δ. Our first observation is that

$$\frac{d\mathbb{P}_{u^\delta \to b^\delta}}{d\mathbb{P}_{v^\delta \to b^\delta}} \leq C \tag{4.43}$$

for a constant C depending only on $\varepsilon > 0$. Indeed, by first computing the Radon–Nikodym derivative with respect to simple random walk as a Doob h-transform, we see that the left hand side is simply $\mathbb{P}_{u^\delta}(X_{\sigma_{\Omega^\delta}} = b^\delta)/\mathbb{P}_{v^\delta}(X_{\sigma_{\Omega^\delta}} = b^\delta)$, where σ_{Ω^δ} is the hitting time of the boundary. This ratio is easily seen to be bounded by a constant depending only on ε (but otherwise independent of u^δ, v^δ). Indeed, it suffices to show that the walks beginning at v^δ and at u^δ can be coupled by requiring the walk from u^δ to make a loop around v^δ without leaving Ω^δ (an event of positive probability even when we condition on $X_{\sigma_{\Omega^\delta}} = b^\delta$, see e.g. Corollary 4.5 in [BLR20] – here we use the fact that Ω^δ is assumed to be contained in the unit ball). This immediately implies (4.43).

Let us now see how (4.43) and Lemma 2.4 in [CW19] imply (4.42). Fix a boundary point a^δ at distance at least $\varepsilon > 0$ from b^δ. Let σ_ε denote the first time that a given trajectory (we will use the random walk excursion Y from a^δ to b^δ) is at distance ε from the boundary, and note that $\mathbb{P}_{a^\delta \to b^\delta}(\sigma_\varepsilon < \infty) \geq c$ uniformly (which also follows from Lemma 2.4 in [CW19], where it is noted explicitly that the expected amount of time spent at a point is comparable to the probability to visit a macroscopic ball). Furthermore, given $\sigma_\varepsilon < \infty$, and given $Y_{\sigma_\varepsilon} = u^\delta$, the Markov property (for the excursion Y) implies that the remainder of the trajectory of Y is distributed according to $\mathbb{P}_{u^\delta \to b^\delta}$. We deduce (by neglecting the amount of time spent by Y until σ_ε):

$$\begin{aligned}
\mathbb{E}_{a^\delta \to b^\delta}(\sigma_\Omega^\delta) &\geq \mathbb{E}_{a^\delta \to b^\delta}(\sigma_\Omega^\delta; \sigma_\varepsilon < \infty) \\
&\geq \mathbb{E}_{a^\delta \to b^\delta}[1_{\sigma_\varepsilon < \infty} \mathbb{E}_{u^\delta \to b^\delta}(\sigma_\Omega^\delta)|_{u^\delta = Y_{\sigma_\varepsilon}}] \\
&\geq \mathbb{P}_{a^\delta \to b^\delta}(\sigma_\varepsilon < \infty) C^{-1} \mathbb{E}_{v^\delta \to b^\delta}(\sigma_\Omega^\delta) \\
&\geq c\mathbb{E}_{v^\delta \to b^\delta}(\sigma_\Omega^\delta)
\end{aligned}$$

where C is as in (4.43) and the value of c changes from line to line, but always depends only on ε. Since the left hand side is bounded above by $c^{-1}\delta^{-2}$, (4.42) follows. □

From this (just as in [CW19, Section 2.5]) it follows that the densities of massive LERW with respect to classical LERW are uniformly bounded from above by $\exp(cm^2R^2)$ and thus the tightness of the law of massive LERW follows. Also, (as in [CW19, Section 2.6]) it follows that each subsequential limit of $\mathbb{P}^{(m)}_{\Omega^\delta}$ is absolutely continuous with respect to the SLE$_2$ on Ω. Thus we can use Girsanov's theorem to find the driving term of ξ_t of the Loewner evolution under $\mathbb{P}^{(m)}_{\Omega^\delta}$.

4.3.3 Convergence of the Green function

In this section we prove the convergence of $Z^{(m)}_{\Omega^\delta}(u, v)$ to a multiple of the massive Green function $G^{(m)}(u, v)$. To do so we will show that $G^{(m)}(u, \cdot)$ is precompact in a suitable space of functions, and we will show that any subsequential limit must satisfy the following three properties:

$$G^{(m)}(u, \cdot) = 0 \text{ on the boundary of } \Omega, \tag{4.44}$$

$$(-\tfrac{1}{2}\Delta + m^2)G^{(m)}(u, \cdot) = 0 \text{ away from u, and} \tag{4.45}$$

$$G^{(m)}(u, v) = \frac{1}{\pi}\log(|u - v|^{-1}) + O(1) \text{ as } v \to u. \tag{4.46}$$

As we will see, these three properties uniquely characterise the the (continuous) **massive Green function**; from this the desired convergence will follow immediately. The second condition is that $G(u,\cdot)$ is a massive harmonic function. It will be useful to appeal to the discrete notion of massive harmonicity: given $m \geq 0$ we call a function H massive discrete harmonic at $v \in \delta\mathbb{T}$ if

$$H(v) = \frac{1}{3}(1 - \frac{m^2\delta^2}{2}) \sum_{w \in \delta\mathbb{T}: w \sim v} H(w). \tag{4.47}$$

Remark 4.3.2. Note that discrete massive harmonic functions with mass m correspond to massive harmonic functions in the sense of (4.45). Indeed, the graph Laplacian approximates $\frac{1}{4}\Delta$ as $\delta \to 0$ and thus a limit h of massive harmonic functions h^δ on $\delta\mathbb{T}$ satisfies

$$(\frac{1}{4}\Delta - \frac{m^2}{2})h = \frac{1}{2}(\frac{1}{2}\Delta - m^2)h = 0\,.$$

This is precisely the reason for the factor $\frac{1}{2}$ in the definitions of the massive random walk.

H being a discrete massive harmonic function is equivalent to being discrete harmonic on the augmented graph where every vertex is connected to an additional cemetery point, where the transition probability to the cemetery is $\frac{m^2\delta^2}{2}$ from every point; and the value of H at the cemetery point being 0. We immediately deduce:

Lemma 4.3.3. *Let Ω^δ be a bounded domain in $\delta\mathbb{T}$ and $(X_n)_{n\in\mathbb{N}}$ be a massive random walk with mass $\frac{m^2\delta^2}{2}$. Let H be a bounded real valued function defined on $\Omega^\delta \cup \partial\Omega^\delta$ and massive discrete harmonic at every point of Ω^δ. Denote by $\mathbb{P}_v^{(m)}$ the law of this walk started at v and by $\mathbb{E}_v^{(m)}$ the corresponding expectation. Let σ_{Ω^δ} be the hitting time of the boundary and let σ^* denote the killing time, or hitting time of the cemetery state. Then*

$$H(v) = \mathbb{E}_v^{(m)}\Big(H(X_{\sigma_{\partial\Omega^\delta}})1_{\{\sigma^* > \sigma_{\Omega^\delta}\}}\Big).$$

The above statement needs to be interpreted carefully as we defined the boundary $\partial\Omega^\delta$ to be the edge boundary, that is, pairs (y_1, y_2) of vertices such that exactly one of these vertices (say y_1) lies in Ω^δ. In the above statement, we abusively identify $\partial\Omega^\delta$ with the outer vertex boundary (i.e., the vertices of the form y_2 where (y_1, y_2) is a boundary edge such that $y_1 \in \Omega^\delta$ but $y_2 \notin \Omega^\delta$). Now we can prove the uniqueness of the Green function:

Lemma 4.3.4. *For each $u \in \Omega$ and $k \in \mathbb{R}^+$ there is exactly one function $G(u,\cdot)\colon \Omega \to \mathbb{R}$ that is massive harmonic away from u, 0 on the boundary, and satisfies*

$$G(u,\cdot) = k\log(|u-v|^{-1}) + o(\log|u-v||) \ \textit{as}\ v \to u.$$

Proof. Let h and g be two such functions. Then $f := h - g$ is a massive harmonic function that is massive harmonic away from u, 0 on the boundary, and

$$f(v) = o(\log(|u-v|))$$

as $v \to u$. Fix $x \neq u \in \Omega$ and let $\mathbb{P}_x^{(m)}$ be the law of massive Brownian motion with mass m started at x: thus if σ^* denote an exponential random variable with rate m^2 then by definition

$$\mathbb{E}_x^{(m)}(f(B_t)) = \mathbb{E}_x(f(B_t)1_{\{\sigma^* > t\}}).$$

Since f is massive harmonic, $M_t = f(B_t)1_{\{\sigma^* > t\}}$ is a $\mathbb{P}_x^{(m)}$-local martingale. Let $r > 0$, $B(u,r)$ be the disk of radius r, σ_r the hitting time of $B(u,r)$ and σ_Ω the hitting time of $\partial\Omega$. It is a well known fact about Brownian motion that the probability that

$$\mathbb{P}_x(\sigma_r < \sigma_\Omega) \lesssim 1/\log(1/r),$$

as $r \to 0$. (This can be seen by applying the optional stopping theorem to the $\mathbb{P}_x$-local martingale $\log|B_t - u|$, see for example [LG16]). By applying the optional stopping theorem to M under $\mathbb{P}_x^{(m)}$ (which is justified since f is smooth and hence bounded away from u, as Ω is bounded) we obtain:

$$f(x) = \mathbb{E}^{(m)}(M_{\sigma_r \wedge \sigma_\Omega}).$$

The only contribution comes from the event $\sigma_r < \min(\sigma^*, \sigma_\Omega)$ since if either of these two stopping times occur before σ_r then the martingale is equal to zero. Hence

$$f(x) = \mathbb{E}_x(f(B_{\sigma_r}) 1_{\sigma_r < \min(\sigma^*, \sigma_\Omega)})$$

But $f(B_{\sigma_r}) = o(\log(r))$ by assumption on f, and

$$\mathbb{P}_x(\sigma_r < \min(\sigma^*, \sigma_\Omega)) \leq \mathbb{P}_x(\sigma_r < \sigma_\Omega) \lesssim 1/\log(1/r).$$

Hence letting $r \to 0$ we see that $f(x) = 0$. Since x was arbitrary, we deduce $f = 0$ and hence $g = h$, as desired. □

(The existence of a function satisfying (4.44), (4.45) and (4.46) follows from the result in [CW19], or the convergence result below.) In order to prove convergence of the discrete Green function $Z_{\Omega^\delta}^{(m)}(u, v)$ to $G^{(m)}(u, v)$ we will show precompactness and identify the limit ultimately via Lemma 4.3.4. The following lemma will be useful for the existence of subsequential limits:

Lemma 4.3.5. *There are constants C and β depending on m such that for all positive massive harmonic functions H defined in $B(v_0, 2r) \cap \delta\mathbb{T}$ with $r \leq m^{-1}$ and for all $v_1, v_2 \in B(v_0, r) \cap \delta\mathbb{T}$ one has:*

$$|H(v_1) - H(v_2)| \leq C(|v_2 - v_1|/r)^\beta \max(H(v)).$$

Proof. Essentially, one can follow the argument of [CW19, Lemma 3.10]. Its proof relies on the following estimate: for any annulus $A = A(v_0, r, 2r)$, let $E(A)$ be the event that X_n makes a non-trivial loop around in the annulus before leaving it and before dying, i.e. there are $0 < s < t < \sigma_A$ such that $X[s, t]$ disconnects v_0 from ∞; and $\sigma^* > \sigma_A$. Then there exists a positive constant $c > 0$independent of δ, r, v_0, and v such that:

$$\mathbb{P}_v^{(m)}(E(A(v_0, r, 2r))) \geq c, \tag{4.48}$$

for all $8\delta < r \leq m^{-1}$ and all $v \in \delta\mathbb{T}$ such that $\frac{3}{2}r - \delta \leq |v_0 - v| \leq \frac{3}{2}r + \delta$. This needs to be established in our directed context, which is not covered explicitly by [CW19]. To see this, simply observe that we can in fact also require $\sigma_A \leq Mr^2\delta^{-2}$ for some large M. Then

$$\begin{aligned}\mathbb{P}_v^{(m)}(E(A)) &\geq \mathbb{P}_v^{(m)}(E(A); \sigma_A \leq M\delta^{-2}r^2) \\ &\geq \mathbb{P}_v^{(0)}(E(A); \sigma_A \leq M\delta^{-2}r^2)(1 - \frac{m^2\delta^2}{2})^{M\delta^{-2}r^2} \\ &\geq \exp(-(M/2)r^2)[\mathbb{P}^{(0)}(E(A)) - \mathbb{P}^{(0)}(\sigma_A > M\delta^{-2}r^2)]\end{aligned}$$

It is well known and easy to see that $\mathbb{P}^{(0)}(E(A))$ is bounded away from 0 (by convergence to Brownian motion) and the second term can be made arbitrarily small by choosing M sufficiently large. The result follows. □

Finally, for the estimate we also need the following lemma about convergence of the conditioned (non-massive) random walk to a Brownian bridge:

Lemma 4.3.6. *Let $t > 0$. Let X_n^δ be the simple random walk on $\delta\mathbb{T}$ started at x^δ converging to x. Let $y^\delta \in \delta\mathbb{T}$ approximate y in such a way that for any $\delta > 0$ it is always possible to go from x^δ to y^δ in $\lfloor \delta^{-2} t \rfloor$ steps with positive probability. Then the law of $(X^\delta_{\lfloor \delta^{-2} s \rfloor})_{s\in[0,t]}$ conditioned on $X^\delta_{\lfloor \delta^{-2} c\rfloor} = y^\delta$ converges to the law of the Brownian bridge $(b_s)_{s\in[0,t]}$ from x to y of duration $t > 0$.*

Proof. We interpolate linearly between vertices to consider $(X^\delta_{\lfloor \delta^{-2} s \rfloor})_{s\in[0,t]}$ as a continuous function on $[0,t]$. Let $(S_s^\delta)_{0\le s\le t}$ be this interpolation. Fix $u = 2t/3$, and let us first show that $(S^\delta)_{0\le s\le u}$ converges to $(b_s)_{0\le s\le u}$. Fix $F\colon C([0,u]) \to \mathbb{R}$ be a bounded continuous functional. Then the conditioning $S_t^\delta = y^\delta$ weights every path $(S_s^\delta)_{s\in[0,u]}$ by how likely it is to go to y^δ from S_u^δ. Thus the conditional expectation of the functional can be rewritten as:

$$\mathbb{E}_{x^\delta}(F((S_s^\delta)_{s\in[0,u]})|S_t^\delta = y^\delta) = \mathbb{E}_{x^\delta}\left(F((S_s^\delta)_{s\in[0,u]})\frac{\mathbb{P}_{x^\delta}(S_t^\delta = y^\delta|S_u^\delta)}{\mathbb{P}_{x^\delta}(S_t^\delta = y^\delta)}\right). \tag{4.49}$$

The probability in the numerator can be written as $\mathbb{P}_{z^\delta}(S^\delta_{t/3} = y^\delta)$, with $z^\delta = S_u^\delta$. The ratio of probabilities therefore converges and the limit is

$$\mathbb{E}(F((B_s)_{s\in[0,u]})\frac{\varphi(\frac{y-B_u}{t-u})}{\varphi(\frac{y}{t})}) = \mathbb{E}(F((b_s)_{0\le s\le u})),$$

where φ is the density of a two-dimensional standard normal random variable. Applying the same argument but in the other direction of time (from t to $t-u = t/3$), the time-reversed random walk $\hat{S}$ is distinct but the same argument applies to it. We deduce that

$$\begin{aligned}\mathbb{E}_{x^\delta}(F((S^\delta_{t-s})_{s\in[0,u]})|S_t^\delta = y^\delta) &= \mathbb{E}_{y^\delta}[F(\hat{S}_s^\delta)_{0\le s\le u}|\hat{S}_t^\delta = x^\delta) \\ &\to \mathbb{E}[F((\hat{b}_s)_{0\le s\le u})] = \mathbb{E}[F((b_{t-s})_{0\le s\le u})]\end{aligned}$$

where $\hat{b}$ is a Brownian bridge of duration t from y to x, and we used the reversibility of Brownian bridge. Altogether this proves the lemma. □

We will use this to approximate the probability that a random walk conditioned on the point at time n leaves a domain by the corresponding probability for the Brownian motion.

Corollary 4.3.7. *Let Ω^δ approximate a domain $\Omega \in \mathbb{C}$ and x^δ, y^δ approximate x, y in Ω. Let $\mathbb{P}_{x\to y;t}$ denote the law of a Brownian bridge of duration t from x to y. For any $t > 0$,*

$$\mathbb{P}^\delta_{x^\delta}(\sigma_{\Omega^\delta} > t\delta^{-2}|X^\delta_{\lfloor t\delta^{-2}\rfloor} = y^\delta) \to P_{x,y}(t) := \mathbb{P}_{x\to y;t}(\sigma_\Omega > t)$$

Suppose x, y are fixed. When t is small the Brownian bridge of duration t is close to a straight line segment $[x,y]$. If the latter is contained in Ω then it is very likely that the bridge did not leave Ω by time t. This can be made rigorous through the following lemma.

Lemma 4.3.8. *Let $P_{x,y}(t)$ be as above. Assume that the line between x and y is in Ω. Then:*

$$\lim_{t\to 0} P_{x,y}(t) = 1.$$

Furthermore, $P_{x,y}$ is a continuous function of t.

Proof. Let $(b_s)_{s\in[0,t]}$ be the Brownian bridge from x to y of duration t. A well known representation of the Brownian bridge is $b_s = x + (y-x)\frac{s}{t} + W_s - \frac{s}{t}W_t$, where $(W_s)_{s\in[0,t]}$ is a standard two dimensional Brownian motion started at 0. By rescaling the time to the interval $[0,1]$ we get $\hat{b}_t = b_{tc}$ for t in $[0,1]$, which satisfies:

$$\hat{b}_s = x + (y-x)s + W_{st} - sW_t.$$

As $t \to 0$ the second term $W_{st} - sW_t$ converges to 0 in probability as $t \to 0$, uniformly in s. Since Ω is an open set and hence also contains an open set around the line from x to y this implies that $P_{x,y}(t)$ converges to 1. □

It is also useful to recall the following elementary estimate which can be obtained e.g. by Stirling's approximation (or from computing the Fourier transform):

Lemma 4.3.9. *Let x^δ and $y^\delta \in \delta\mathbb{T}$ be sequences of lattice points. Then there exists a constant $C < \infty$ independent of $x^\delta, y^\delta, \delta$ and n such that*

$$\mathbb{P}_{x^\delta}(X_n = y^\delta) \leq \frac{C}{n} \tag{4.50}$$

for some universal constant $C > 0$.

Lemma 4.3.10. *Let $x, y \in \mathbb{C}$ and $x^\delta, y^\delta = x^\delta + a\delta + b\delta\tau \in \Omega^\delta$ such that $x^\delta \to x$ and $y^\delta \to y$ and that $n-a-b$ is divisible by 3. Then*

$$3^{-n}\binom{n}{n-a-b,\, n-a+2b,\, n+2a-b} = \frac{\sqrt{27}}{2\pi n}\exp\Big(-\frac{|x-y|^2}{\delta^2 n}\Big)(1+O(\delta)). \tag{4.51}$$

where the error is uniform in x^δ, y^δ and δ, n such that $|x-y|^2\delta^{-2} < n < M\delta^{-2}$ for some constant M.

Proof. Since $y^\delta - x^\delta \to y - x$ we have that a and b are of order δ^{-1}. Because the domains are bounded they are uniformly of this order. Therefore all entries in the multinomial coefficient are uniformly of order δ^{-2} and we can apply Stirling's approximation to all appearing factorials to obtain that the multinomial coefficient equals:

$$\begin{aligned}
&\frac{n^n\sqrt{2\pi n}}{(\frac{n-a-b}{3})^{\frac{n-a-b}{3}}(\frac{n+2a-b}{3})^{\frac{n+2a-b}{3}}(\frac{n-a+2b}{3})^{\frac{n-a+2b}{3}}(\sqrt{2\pi n/3})^3 3^n}(1+O(\delta^2))\\
=&\frac{\sqrt{27}}{2\pi n}\left((1+\frac{-a-b}{n})(1+\frac{2a-b}{n})(1+\frac{-a+2b}{n})\right)^{-\frac{n}{3}}\times\\
&(1+\frac{-a-b}{n})^{-\frac{-a-b}{3}}(1+\frac{2a-b}{n})^{-\frac{2a-b}{3}}(1+\frac{-a+2b}{n})^{-\frac{-a+2b}{3}}(1+O(\delta^2))\\
=&\frac{\sqrt{27}}{2\pi n}\left(1+\frac{-3(a^2-ab+b^2)}{n^2}+O(\delta^3)\right)^{-\frac{n}{3}}\times\\
&(1+\frac{-a-b}{n})^{-\frac{-a-b}{3}}(1+\frac{2a-b}{n})^{-\frac{2a-b}{3}}(1+\frac{-a+2b}{n})^{-\frac{-a+2b}{3}}(1+O(\delta^2))\\
=&\frac{\sqrt{27}}{2\pi n}\exp(\frac{a^2-ab+b^2}{n})\exp(-\frac{(-a-b)^2+(2a-b)^2+(2b-a)^2}{3n})(1+O(\delta))\\
=&\frac{\sqrt{27}}{2\pi n}\exp(-\frac{|x-y|^2}{\delta^2 n})(1+O(\delta)).
\end{aligned}$$

In the last step we used that $\delta^2(a^2-ab+b^2) = |a\delta + b\delta\tau|^2 = |x-y|^2 + o(\delta)$. □

Lemma 4.3.11. *Let $\Omega^\delta \subset \delta\mathbb{T}$ be a sequence of lattice domains satisfying $\Omega^\delta \subset B(0,R)$ for some $R > 0$ independent of δ. Let x^δ and $y^\delta \in \Omega^\delta T$ be a sequences of lattice points. Then there exists a constants $c > 0$ depending on R, but not on δ, n, x^δ or y^δ such that for all $n \geq 1$:*

$$\mathbb{P}^{(0)}_{x^\delta}(\tau_{\Omega^\delta} > n | X^\delta_n = y^\delta) < \exp(-cn\delta^2). \tag{4.52}$$

Proof. This can easily be deduced from the fact that the Radon–Nikodym derivative of the conditioned random walk compared to an unconditional random walk, restricted to $[0, n/2]$, is bounded (see, e.g., (4.49)), and the analogous (and straightforward) bound for unconditional random walk. Details are left to the reader. □

Now we state the main result of this section:

Proposition 4.3.12. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and $x, y \in \Omega$ be two distinct points of Ω. Assume that discrete domains $\Omega^\delta \subset \delta\mathbb{T}$ approximate Ω. Then*

$$Z^{(m)}_{\Omega^\delta}(x^\delta, y^\delta) \to \sqrt{3}G^{(m)}_\Omega(x, y).$$

Proof. Fix $r > 0$ and assume that $|x - y| \geq r$. We will need to obtain estimates that do not depend on $r > 0$. To begin we rewrite the Green function as

$$Z^{(m)}_{\Omega^\delta}(x^\delta, y^\delta) = \sum_{n=0}^{\infty} \mathbb{P}^{(0)}_{x_\delta}(X_n = y^\delta)(1 - \frac{m^2\delta^2}{2})^n \mathbb{P}^{(0)}_{x^\delta}(\tau_{\partial\Omega} > n | X_n = y^\delta). \tag{4.53}$$

We split this sum into three parts: First the sum from $n = 0$ to $\lfloor |x - y|^2\delta^{-2} \rfloor$, then from $n = \lfloor |x - y|^2\delta^{-2} \rfloor + 1$ to $\lfloor M\delta^{-2} \rfloor$ (where M is a large constant chosen suitably later), then larger values of n. We will call these sums I, II and III and estimate them separately.

Bounding I. To estimate the first part of the sum we compare $\mathbb{P}^{(0)}_{x^\delta}(X_n = y^\delta)$ with the same probability for points that are closer to x, as follows. Depending on the residue of n modulo 3 a different set of vertices is reachable from x^δ. Assuming that a point is reachable and is at least twice as close to x than y^δ in the Euclidean sense, then it is easier to reach that point than y^δ:

Lemma 4.3.13. *Fix $n \geq 0$. For any vertex z such that $\mathbb{P}^{(0)}_{x^\delta}(X_n = z) > 0$ and satisfies $|z - x^\delta| < \frac{1}{2}|y^\delta - x^\delta|$, we claim that*

$$\mathbb{P}^{(0)}_{x^\delta}(X_n = z) \geq \mathbb{P}^{(0)}_{x^\delta}(X_n = y^\delta).$$

Proof. Since the number of steps n is fixed this is just about comparing multinomial coefficients. It is easy to check that for any n and any a_1, a_2, a_3 such that $n = a_1 + a_2 + a_3$ and $a_1 > a_2$ it holds that:

$$\binom{n}{a_1, a_2, a_3} \leq \binom{n}{a_1 - 1, a_2 + 1, a_3}.$$

Assume without loss of generality that $y^\delta - x^\delta = a_1 + a_2\tau + a_3\tau^2$, such that $a_1 + a_2 + a_3 = n$ and $a_1 \geq a_2 \geq a_3$. The above inequality implies that for any z reachable from y by repeatedly reducing one of the a_i and increasing another a_j subject to $a_i > a_j$ satisfies: $\mathbb{P}^{(0)}_{x^\delta}(X_n = z) \geq \mathbb{P}^{(0)}_{x^\delta}(X_n = y^\delta)$. It is clear that in this way only points z can be be obtained that are also reachable in n steps from x^δ.

Claim All z reachable from x^δ in n steps, which are in the quadrilateral descriped by the lines through x^δ in the directions 1 and τ and through y^δ in the directions orthogonal to 1 and τ are reachable through these operations. See figure 4.3.3.

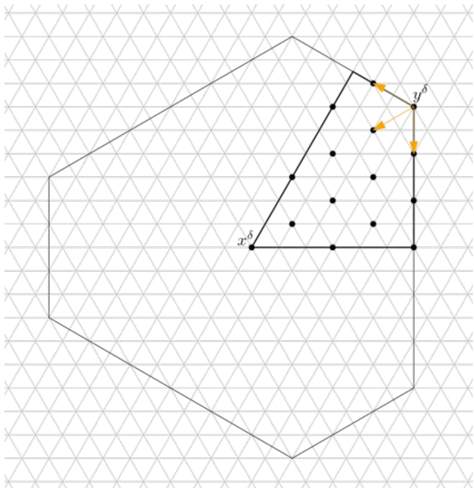

Figure 4.5: Situation of Lemma 4.3.13, the bold line marks the relevant quadrilateral, the marked points are the points reachable with n steps from x and the orange arrows are the three steps possible from y^δ.

Proof of the claim:

By applying the step of reducing a_1 and increasing a_2 we see that all such points on the line through y^δ orthogonal to 1 are reachable and the same by reducing a_2 and increasing a_3 for the line orthogonal to τ. By choosing the correct starting point on these lines any other point in the quadrilateral is reachable by applying the step of reducing a_1 and increasing a_3. This proves the claim.

By mirroring this quadrilateral on the lines in directions $1, \tau$ and τ^2 through x^δ we obtain that also all z in the resulting hexagon satisfy $\mathbb{P}^{(0)}_{x^\delta}(X_n = z) \geq \mathbb{P}^{(0)}_{x^\delta}(X_n = y^\delta)$. The vertices of this hexagon are the reflections of y^δ along those lines. The points on the boundary of this hexagon which are closest to x^δ are the projections of y^δ onto the lines through x in directions 1 and τ (and their respective reflections). Since the angle between those lines is $\frac{1}{3}\pi$ both of those points have distance from x^δ of at least $\frac{1}{2}|x^\delta - y^\delta|$, therefore the disk of radius $\frac{1}{2}|x^\delta - y^\delta|$ is contained in the hexagon. This proves the Lemma. Note that the extreme case of this being the largest disk that fits inside the hexagon is obtained exactly when $y^\delta - x^\delta$ is a multiple of $1, \tau$ or τ^2. $\square$

There are approximately $C|x-y|^2\delta^{-2}$ points verifying the conditions of Lemma 4.3.13, where $C = \frac{1}{6\sqrt{3}}\pi$. Consequently we have:

$$\begin{aligned}\lfloor |x-y|^2\delta^{-2} \rfloor &= \sum_z \sum_{n=0}^{\lfloor |x-y|^2\delta^{-2} \rfloor} \mathbb{P}^{(0)}_{x^\delta}(X_n = z) \\ &\geq (C + o(1))|x-y|^2\delta^{-2} I\end{aligned}$$

Which implies that $I < 1/C + o(1)$ and thus I is bounded independently of r.

Bounding III**.** From Lemma 4.3.11, we see that

$$\mathbb{P}^{(0)}_{x^\delta}(\tau_{\partial\Omega} > n | X_n = y^\delta) \leq \exp(-cn\delta^2).$$

By Lemma 4.3.13, $\mathbb{P}^{(0)}_{x_\delta}(X_n = y^\delta) \leq \mathbb{P}^{(0)}_{x^\delta}(X_n = x^\delta) \leq C/n$ by Lemma 4.3.9. Hence, crudely bounding $(1-\frac{m^2\delta^2}{2})^n$ by 1 in the sum III we get

$$\begin{aligned} III &\leq \sum_{n\geq M\delta^{-2}} \mathbb{P}^{(0)}_{x_\delta}(X_n = y^\delta)\mathbb{P}^{(0)}_{x^\delta}(\tau_{\partial\Omega} > n | X_n = y^\delta) \\ &\leq \sum_{n\geq M\delta^{-2}} \frac{C}{n}\exp(-cn\delta^2) = \sum_{k\geq 1}\sum_{n=2^k M\delta^{-2}}^{2^{k+1}M\delta^{-2}-1} \frac{C}{n}\exp(-cM2^k) \\ &= \sum_{k\geq 1} C\exp(-cM2^k), \end{aligned}$$

which is bounded independently of r, as desired.

Estimating II. For II we estimate the number of lattice paths using Stirling's formula. Assume without loss of generality that $y^\delta = x^\delta + a + be^{2\pi i/3} = a + b\tau$ with $a = a^\delta, b = b^\delta \in \{0, 1, \ldots\}$ (other cases are similar), then the number of paths from x^δ to y^δ is 0 if $n-a-b$ is not divisible by 3. If $n-a-b$ is divisible by 3, the number of paths is given by the multinomial coefficient:

$$\binom{n}{\frac{n-a-b}{3}, \frac{n+2a-b}{3}, \frac{n-a+2b}{3}}.$$

Now, in the regime II, Applying Lemma 4.3.10 we find

$$\mathbb{P}^{(0)}_{x^\delta}(X_n = y^\delta) = \frac{\sqrt{27}}{2\pi n}\exp(-\frac{|x-y|^2}{\delta^2 n})(1+O(\delta)) \tag{4.54}$$

Recall that $P_{x,y}(t) = \mathbb{P}_{x\to y;t}(\tau_{\partial\Omega} > t)$. By Corollary 4.3.7 we get:

$$\mathbb{P}_x(\tau_{\partial\Omega} > n | X^\delta_n = y) = P_{x,y}(n\delta^2)(1+o_\delta(1)),$$

where $o_\delta(1)\to 0$ when $\delta\to 0$, uniformly in n such that $|x-y|^2\delta^{-2} \leq n \leq M\delta^{-2}$. Using this we get:

$$II = \frac{\sqrt{27}}{6\pi}\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor M\delta^{-2}\rfloor} \frac{1}{n}\exp(-\frac{|x-y|^2}{\delta^2 n})P_{x,y}(n\delta^2)(1-\frac{m^2\delta^2}{2})^n(1+O(\delta))(1+o_\delta(1)) \tag{4.55}$$

$$= \frac{\sqrt{3}}{2\pi}(1+o_\delta(1))\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} \frac{1}{n}\exp(-\frac{|x-y|^2}{\delta^2 n})(1-\frac{m^2\delta^2}{2})^n P_{x,y}(n\delta^2), \tag{4.56}$$

where the fact 1/3 in the first line comes from the fact that only one in three terms contribute to the sum (owing to periodicity).

This can be transformed into a Riemann sum, from which we deduce:

$$II = \frac{\sqrt{3}}{2\pi}(1+o_\delta(1))\int_{|x-y|^2}^{M} \frac{P_{x,y}(s)\exp(-\frac{|x-y|^2}{s})\exp(-m^2 s)}{s}ds \tag{4.57}$$

The convergence of the Riemann sum is guaranteed by the fact that the continuity of the integrand over the relevant interval.

From (4.57) and our bounds on I and III note that $Z^{(m)}_{\Omega^\delta}(x^\delta,\cdot)$ is uniformly bounded in δ on compacts of $\Omega\setminus\{x\}$. Using Lemma 4.3.5 we deduce that $Z^{(m)}_{\Omega^\delta}(x^\delta,\cdot)$ has subsequential limits in every compact of $\Omega\setminus\{x\}$. By considering a countable number of such compacts (e.g. $\Omega_n = \{y\in\Omega : d(y,x)\wedge d(y,\partial\Omega)\geq 1/n\}$) and a standard diagonalisation argument we may assume that there are subsequential limits in all of these compact domains simultaneously, which are necessarily consistent with one another. Let $h(x,\cdot)$ denote any such limit. We aim to identify h uniquely.

As we are interested in the behaviour when y is close to x we can assume that the straight line from x to y is in Ω and therefore lemma 4.3.8 applies and $P_{x,y}(s)$ approaches 1 as s goes to 0. Elementary computations give the asymptotic behaviour of this integral as $-2\log(|x-y|)+O(1)$ when $|x-y|\to 0$.

It is elementary to check that $Z^{(m)}_{\Omega^\delta}(x^\delta,\cdot)$ is a discrete massive harmonic function in the sense of Lemma 4.3.3. Since the convergence to the limit in the chosen subsequence is uniform, it is not hard to see that we can pass to the limit in the solution of the massive Dirichlet problem of Lemma 4.3.3, and deduce that $h(x,\cdot)$ is massive harmonic away from x. Furthermore, from our estimates above it follows that

$$h(x,y) = -\frac{\sqrt{3}}{\pi}\log(|x-y|)+O(1). \tag{4.58}$$

Thus h is the unique function satisfying the desired properties. Therefore all subsequential limits are the same which proves the desired convergence of the discrete massive Green functions. □

Remark 4.3.14. The factor $\sqrt{3}$ can be explained as follows: Just as in the discrete case, the expected time spent by Brownian motion in a disk B is given by the integral of the Green's function. The random walk considered in this section converges to Brownian motion under the scaling $X_{\lfloor 2t\delta^{-2}\rfloor}$. Thus, the expected amount of time spent in B of the discrete walk on the scaled lattice should satisfy:

$$\tfrac{1}{2}\delta^2\mathbb{E}(|\{n : X_n\in B\}|) = \tfrac{1}{2}\delta^2\sum_{y^\delta\in B\cap\delta\mathbb{T}} Z^m_{\Omega^\delta}(x^\delta,y^\delta)\to\int_B G^m_\Omega(x,y)dy\,.$$

This is indeed the case, since the density of points in the square lattice is $\frac{2}{\sqrt{3}}$ and thus the sum converges to the integral after cancelling the $\frac{1}{2}$ from the time change with $\frac{2}{\sqrt{3}}$ from the lattice and the $\sqrt{3}$ from the statement of Proposition 4.3.12. Therefore the factor $\sqrt{3}$ in the right hand side of Proposition 4.3.12 is consistent with the above.

4.3.4 Convergence of discrete massive Poisson kernel

Given a domain Ω, an interior point $z\in\Omega$ and a boundary point $a\in\partial\Omega$ (thought of as a prime end of Ω), we define the **continuous massive Poisson kernel** as:

$$P^{(m)}_\Omega(z,a) := P_\Omega(z,a) - m^2\int_{\Omega_t} G^{(m)}_\Omega(z,w)P_\Omega(w,a)dA(w). \tag{4.59}$$

where $P_\Omega(w,a)$ is the (non-massive) continuous Poisson kernel.

This definition is motivated by the following crucial identity for the discrete massive Green function (this is the discrete counterpart of the *resolvent identities* to which we will return in Section 4.4.4, which related massive and non-massive harmonic functions, as already observed in the work of Makarov and Smirnov [MS10]):

Lemma 4.3.15.

$$(1-\frac{m^2\delta^2}{2})Z^{(m)}_{\Omega^\delta}(w^\delta,z^\delta) = Z_{\Omega^\delta}(w^\delta,z^\delta) - \frac{m^2\delta^2}{2}\sum_{v^\delta\in\operatorname{Int}\Omega^\delta} Z^{(m)}_{\Omega^\delta}(w^\delta,v^\delta)Z_{\Omega^\delta}(v^\delta,z^\delta), \tag{4.60}$$

Proof. We prove this by splitting each trajectory in the definition of Z_Ω into two parts, and summing over all possible ways to do so:

$$\begin{aligned}\sum_{v^\delta \in \mathrm{Int}\Omega^\delta} Z^{(m)}_{\Omega^\delta}(w^\delta, v^\delta) Z_{\Omega^\delta}(v^\delta, z^\delta) &= \sum_{v^\delta \in \mathrm{Int}\Omega^\delta} \sum_{k\geq 0} \sum_{\substack{\pi: w\to z,\\ \pi_k = v}} (\frac{1}{3}(1-\frac{m^2\delta^2}{2}))^k (\frac{1}{3})^{(\#\pi)-k} \\ &= \sum_{\pi: w\to z} (\frac{1}{3})^{\#\pi} \sum_{k=0}^{\#\pi} (1-\frac{m^2\delta^2}{2})^k \\ &= \sum_{\pi: w\to z} (\frac{1}{3})^{\#\pi} \frac{1-(1-\frac{m^2\delta^2}{2})^{(\#\pi)+1}}{\frac{m^2\delta^2}{2}} \\ &= \frac{Z_{\Omega^\delta}(w^\delta, z^\delta) - (1-\frac{m^2\delta^2}{2}) Z^{(m)}_{\Omega^\delta}(w^\delta, z^\delta)}{\frac{m^2\delta^2}{2}}.\end{aligned}$$

Rearranging the terms gives the desired result. □

The importance of the Poisson kernel stems from the well known martingale observable of Lawler, Schramm and Werner [LSW01]. Namely, let γ^δ be a massive LERW between b^δ in Ω^δ and $a^\delta \in \partial\Omega^\delta$. We parametrise γ^δ from b^δ to a^δ. For a vertex $v^\delta \in \Omega^\delta$, define the **massive martingale observable** as:

$$M_n^{(m)}(v^\delta) := \frac{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(v^\delta, \gamma^\delta(n))}{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(b^\delta, \gamma^\delta(n))}. \tag{4.61}$$

Since $\gamma^\delta(n)$ is on the boundary of $\Omega^\delta \setminus \gamma^\delta[0,n]$, this is also simply equal to the ratio of hitting probabilities of $\gamma^\delta(n)$ from v^δ vs. b^δ. Proceeding exactly as in [LSW01, Remark 3.6], one can check that for every $\delta > 0$ and every fixed vertex v^δ, the sequence $(M_n^{(m)}(v^\delta))_{0\leq n\leq T(b^\delta)}$ gives a martingale (see also [Law13, Lemma 7.2.1]).

The strategy of the proof of convergence of this martingale observable to its continuum limit in Chelkak and Wan [CW19] is to:

- first, prove the convergence of the non-massive martingale observable in the non-massive case (something which was in fact already proved in the radial case by Lawler, Schramm and Werner [LSW01] and generalised by Yadin and Yehudayoff [YY11], but in the chordal context of [CW19] requires some additional justifications); this was proved in Proposition 3.5 and Corollary 3.6 in [CW19] (and put in the correct chordal framework in Proposition 3.14)
- second, prove that the ratio of massive Green function to non-massive Green function converges to its continuum limit, which is Proposition 3.15 in [CW19].

The first step follows directly from the work of Yadin and Yehudayoff [YY11], which holds for arbitrary planar graphs subject to convergence of random walk to Brownian motion (which we know is true on the directed triangular lattice). Therefore only the second step needs to be justified, this is the content of the next lemma (which is the analogue of Proposition 3.15 in [CW19]).

Lemma 4.3.16. *In the setup above for any $z \in \Omega_t$ and $z^\delta \to z$ as $\delta \to 0$, one has:*

$$\frac{Z^{(m)}_{\Omega^\delta{}_t}(z^\delta, a_t^\delta)}{Z_{\Omega^\delta{}_t}(z^\delta, a_t^\delta)} \to \frac{P^{(m)}_{\Omega_t}(z, a_t)}{P_{\Omega_t}(z, a_t)} = 1 - m^2 \int_{\Omega_t} \frac{P_{\Omega_t}(w, a_t)}{P_{\Omega_t}(z, a_t)} G^{(m)}_{\Omega_t}(z, w) dA(w).$$

Proof. The proof in [CW19] works also for the directed triangular lattice, as besides the convergence results of the last section it only requires the identity above, and estimates on the massive Green function, which follow from convergence to Brownian motion. One might at first be worried as the right-hand side of (4.60) is not linear in Z, whereas the limit of the discrete Green function for the triangular lattice is $\sqrt{3}$ times massive Green function (see Proposition 4.3.12. However, the factor $\sqrt{3}$ combines with the $\frac{1}{2}$ in (4.60) to make the sum over the triangular lattice to converge to a Lebesgue integral, see the remark after Proposition 4.3.12. □

As a corollary we obtain the following convergence of martingale observables. Fix a subsequential limit $(\gamma_t)_{t\geq 0}$ of massive LERW on the directed triangular lattice, which a priori we know to be a simple curve (by absolute continuity with standard LERW), and parametrise it by capacity. Let $\Omega_t = \Omega \setminus \gamma([0,t])$. Let $a_t = \gamma(t)$ denote the tip of the curve at time t, which is on the boundary of Ω_t.

Corollary 4.3.17. *Fix $r > 0$. Suppose $v^\delta \in B(b^\delta, r/2)$. For $t \leq \log(1/r)$, let n_t denote the first n such that the capacity of $\gamma^\delta([0,n])$ viewed from b^δ exceeds t (equivalently, the conformal radius of b^δ in $\Omega \setminus \gamma([0,n])$ is less than e^{-t}).*

$$M^{(m)}_{n_t}(v^\delta) \to \frac{P^{(m)}_{\Omega_t}(v, a_t)}{P^{(m)}_{\Omega_t}(b, a_t)} =: M^{(m)}_{\Omega_t}(v),$$

almost surely along the underlying subsequential limit $\delta \to 0$.

4.3.5 Proof of the main statement

We are now ready to prove convergence to massive SLE$_2$, as stated in Theorem 4.2.8.

Proof of Theorem 4.2.8. As discussed in Section 4.3.2 the laws of the massive loop-erased random walks are tight and all subsequential limits are absolutely continuous with respect to classical SLE$_2$. This justifies the application of Girsanov's theorem which in particular implies that the driving function ξ_t is a semi-martingale under $\mathbb{P}^{(m)}$.

Moreover, the discrete martingales of (4.61) have continuous limits as shown in Proposition 4.3.17. Writing the martingale in the form

$$M^{(m)}_n(v^\delta) = \frac{Z^{(m)}_{\Omega^\delta_n}(v^\delta, a^\delta_n)}{Z_{\Omega^\delta_n}(v^\delta, a^\delta_n)} \left(\frac{Z^{(m)}_{\Omega^\delta_n}(b^\delta, a^\delta_n)}{Z_{\Omega^\delta_n}(b^\delta, a^\delta_n)} \right)^{-1} \frac{Z_{\Omega^\delta_n}(v^\delta, a^\delta_n)}{Z_{\Omega^\delta_n}(b^\delta, a^\delta_n)}$$

with $a^\delta_n = \gamma^\delta(n)$ and $\Omega_n = \Omega \setminus \gamma^\delta([0,n])$, we see that $M^{(m)}(v^\delta)$ is uniformly bounded: the first term is trivially bounded by 1, the second is bounded by Proposition 4.3.1 (and Koebe's one-quarter theorem), and the third one is bounded for $n \leq n_t$ by (uniform) convergence to the continuous Poisson kernel (here we use the strength of the result of Yadin and Yehudayoff [YY11]) and conformal invariance of the latter. Hence the limit in Corollary 4.3.17 must also be a martingale (see Remark 2.3 and (2.14) in [CW19] for the argument).

Standard Itô calculations together with Hadamard's formula (as outlined in [MS10] and written out in [CW19], see Section 4.3 and more specifically Lemma 4.9) for this family of martingales imply that the law of the driving function ξ_t under $\mathbb{P}^{(m)}$ is uniquely determined. Changing from the chordal to radial setting does not change this argument. In Sections 4.4.4 to 4.4.6 we will perform these calculations for a variable mass, one can also refer to them for the (very minor) changes to the radial setting. □

4.4 Convergence of massive LERW on general planar graphs

In the previous section it was proven that on a triangular and on a square lattice the loop erasure of a random walk with certain weights converges to SLE_2 with drift given by a re-weighting of massive SLE_2. This also extended a result from [CW19] which proved that the loop erasure of a massive random walk on the square lattice converges to massive SLE_2 as conjectured in [MS10].

We now want to extend this result to a general mass profile ρ and to a more general planar graph. To do this we will combine techniques from [YY11] and [CW19]. In particular we will first argue similarly to [CW19] that the laws of the LERW are tight and any limit point is absolutely continuous with respect to the law of SLE_2. Then we use the strategy employed by [YY11] to show that the discrete Poisson kernel ratios converge to the continuous ones and finally use this convergence to identify the limiting law again as in [CW19].

We will use the following convergence of paths. For two continuous curves $\alpha, \beta : [0,1] \to \mathbb{C}$, consider the norm $\inf_g \sup_{x\in[0,1]} |\alpha(x) - \beta \circ g(x)|$, where the infimum is over all continuous increasing bijections $g : [0,1] \to [0,1]$. This is a norm on equivalence classes of continuous curves under reparametrization. A law μ^δ on continuous curves is said to converge weakly to a law μ, if it converges weakly in the topology of this norm.

Recall that a Brownian motion with mass profile ρ is a Brownian motion, which dies at rate $\rho(X_s)$ when at position X_s, i.e. it is a process which is absolutely continuous with respect to Brownian motion and has Radon Nikodym-derivative

$$\left.\frac{d\mathbb{P}_x^{(\rho)}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \rho(X_s)ds\right). \tag{4.62}$$

Note that the total mass of $\mathbb{P}_x^{(\rho)}$ is less than one, so it is not a probability measure but a (finite) measure on paths.

Fix a domain Ω and two bounded, smooth functions $\rho, \tilde{\rho} : \Omega \to [0, \infty)$. Let Ω^δ be a sequence of planar graphs embedded in the complex plane, with weights w^δ and a discrete mass function ρ^δ from the vertices of Ω^δ to $\mathbb{R}_+$. Denote the partition function of the massive random walk with mass ρ^δ as $Z_{\Omega^\delta}^{(\rho)}(x^\delta, y^\delta)$. We need to assume the following properties of these objects:

1. Let $(X_t^\delta, t = 0, 1, \ldots)$ be the massive random walk on Ω^δ, started at a specified vertex $o^\delta \to o$ of Ω^δ, with transition probabilities proportional to the directed weights w^δ and probability to die at each step given by $\rho^\delta(v) = \frac{\delta^2}{2}\tilde{\rho}(v) + o(\delta^2)$, where the $o(\delta^2)$ term needs to be uniform in v. The law of $(X_{\delta^{-2}t}^\delta, t \geq 0)$ (interpolated continuously between time steps) converges weakly (in the above sense) as $\delta \to 0$, to the law (measure) $\mathbb{P}_o^{(\rho)}$ of a Brownian motion with mass profile ρ.

2. Consider the non-massive random walk on with transition probabilities proportional to the directed weights w^δ in a domain $\Omega^\delta \subset B(0, R)$ started at o^δ and conditioned to leave at a^δ. Denote with σ^δ the number of steps before hitting the boundary $\partial\Omega^\delta$. There is a constant c_0 uniform in R and Ω^δ, such that

$$\mathbb{E}^{(0)}_{\Omega^\delta, o^\delta \to a^\delta}(\sigma^\delta) \leq c_0 R^2 \delta^{-2} \tag{4.63}$$

3. The random walk satisfies a **uniform crossing** assumption: Let $\mathcal{R}$ be the horizontal rectangle $[0,3] \times [0,1]$ and $\mathcal{R}'$ be the vertical rectangle $[0,1] \times [0,3]$. Let $B_1 := B((1/2,1/2),1/4)$ be the *starting ball* and $B_2 := B((5/2,1/2),1/4)$ be the *target ball*.

Let $\mathcal{R}_r = r\mathcal{R} + z$ (resp. $\mathcal{R}'_r = r\mathcal{R}' + z$) for some $r > 0$ and $z \in \Omega$, and suppose that $\mathcal{R}_r$ (resp. $\mathcal{R}'_r$) $\subset \Omega \subset R\mathbb{D}$. Let B_1^r, B_2^r be the corresponding scaled starting and target balls. Let Cross_r denote the event that the walk hits B_2^r before leaving the rectangle $\mathcal{R}_r$ (respectively $\mathcal{R}'_r$) or getting killed. We will say that the graphs (Ω^δ) satisfying the uniform crossing estimate if there is a constant $c > 0$ such that, uniformly over $z \in B_1^r$, uniformly over $r \leq R$,

$$\mathbb{P}_z^{(0)}(\mathsf{Cross}_r) \geq c. \tag{4.64}$$

Assumption (i) is essentially an assumption about the fact that random walk converges to a Brownian motion, potentially up to a time-change (as in [YY11]). Our assumption basically requires that this time-change is not too rough, since before time change the rate of dying is $\tilde{\rho}(x)$ at a point x, whereas after this time-change the rate of dying is given by $\rho(x)$, and both are assumed smooth and bounded.

The main theorem of this section is:

Theorem 4.4.1. *Let Ω and $\rho, \tilde{\rho}$ be as above, and let Ω^δ, ρ^δ and w^δ be such that the assumptions above are satisfied. Let again $(X_t^\delta, t = 0, 1, \dots)$ be the random walk on Ω^δ, started at a specified vertex $o^\delta \to o$ of Ω^δ, with transition probabilities proportional to the directed weights w^δ and dying at each step with probability ρ^δ. Let σ_δ denote the first time at which X^δ leaves Ω (with $\sigma_\delta = \infty$ if X^δ dies before leaving the domain) and consider the loop erasure $\mathsf{LE}(X^\delta)$ of the walk up until this time. Then conditionally on $\sigma_\delta < \infty$ and $X_{\sigma_\delta}^\delta = a^\delta$, $\mathsf{LE}(X^\delta)$ converges weakly to a radial Loewner evolution γ, whose driving function $\zeta_t = e^{i\xi_t}$ (when parametrised by capacity) satisfies the stochastic differential equation*

$$\mathrm{d}\xi_t = \sqrt{2}\mathrm{d}B_t + \lambda_t \mathrm{d}t, \quad \lambda_t = \frac{\partial}{\partial g_t(a_t)} \log\left(\frac{P_{\Omega_t}^{(\rho)}(o, a_t)}{P_{\Omega_t}(o, a_t)}\right), \tag{4.65}$$

where $a_t = \gamma(t)$, $\Omega_t = \Omega \setminus \gamma([0,t])$ is the slitted domain at time t, g_t is the Loewner map from Ω_t to $\mathbb{D}$ and $P_{\Omega_t}^{(\rho)}$ and P_{Ω_t} are the Poisson kernels for the Brownian motion with mass profile ρ, and regular Brownian motion respectively, in Ω_t.

The definition of the Poisson kernel $P_{\Omega_t}^{(\rho)}(o, a_t)$ appearing in the theorem is not *a priori* obvious (its construction will be explained in Section 4.4.1). We will obtain more explicit expressions for the drift term λ_t in the course of the proof, which will show in particular that when the domains Ω_t are contiunous with respect to the Carathéodory topology (which must be the case a.s. here), $t \mapsto \lambda_t$ is itself continuous. Furthermore, we will see as a result of Lemma 4.4.16 that $\int_0^\infty \lambda_t^2 \mathrm{d}t \leq C$ for some constant $C > 0$, thereby showing that λ_t satisfies the Novikov condition and ensuring existence and pathwise uniqueness for solutions to the SDE (4.65).

4.4.1 Poisson kernel for Brownian motion with mass

To describe the scaling limit of loop-erased random walk when the walk itself does not converge to a Brownian motion, but rather to a Brownian motion with mass, it is necessary to first define the Poisson kernel ratio of the latter, and describe a few of its properties.

Recall that the Poisson kernel of Brownian motion in a domain Ω, when it exists, is the density $h_\Omega(x, a)$ of harmonic measure in Ω viewed from $x \in \Omega$, with respect to the natural length measure $|da|$ on $\partial\Omega$, evaluated at the point $a \in \partial\Omega$. For a fixed $o \in \Omega$, the Poisson kernel *ratio* is then the quantity

$$\lambda_\Omega(x, a) = \lambda_\Omega^{(0)}(x, a) = \frac{h_\Omega(x, a)}{h_\Omega(o, a)}.$$

As Brownian motion is conformally invariant, this makes sense even when the boundary of the domain is not smooth but Ω is simply connected; in that case we still have

$$\frac{\mathbb{P}_x(X_\sigma \in I)}{\mathbb{P}_o(X_\sigma \in I)} \to \lambda_\Omega(x,a) \tag{4.66}$$

as $I \subset \partial\Omega$ and $I \downarrow a$, where $\mathbb{P}_x$ is the law of Brownian motion X starting from x, and σ is the first time that X leaves Ω. To be more precise, I should be thought of as a decreasing sequence of compacts in the Martin boundary of Ω, whose intersection is $\{a\}$, where a is also viewed as an element of the Martin boundary (or, equivalently, a prime end). Equivalently, we may parameterize Ω by the unit disc $\mathbb{D}$ via its Riemann mapping $\psi : \mathbb{D} \to \Omega$; then $I = \psi(\tilde{I})$, where $\tilde{I}$ is an arc of $\partial\mathbb{D}$ shrinking to $\tilde{a} \in \partial\mathbb{D}$. When Ω has a locally connected boundary (which will a.s. hold in the cases where we apply the results below) then the Riemann map ψ may be extended from $\bar{\mathbb{D}}$ to $\bar{\Omega}$. For future reference we also denote by $\mathbb{P}_{x\to a} = \mathbb{P}^{(0)}_{x\to a;\Omega}$ the conditional law P_x, given $X_\sigma = a$. (This is obtained by mapping a Brownian motion $\tilde{B}$ in $\mathbb{D}$, conditioned so that $\tilde{B}_{\tilde{\sigma}} = \tilde{a}$, where $\psi(\tilde{a}) = a$, and $\tilde{\sigma}$ is the first exit time from $\mathbb{D}$ by $\tilde{B}$, and performing the appropriate time-change).

Now suppose $\rho : \bar{\Omega} \to \mathbb{R}$ is a given smooth real valued function on $\bar{\Omega}$, and consider the law $\mathbb{P}^{(\rho)}$ of the associated massive Brownian motion with profile ρ, i.e. a Brownian motion dying with probability $\rho(X_s)ds$ at each time step. Without conformal invariance, some arguments are required to construct this Poisson kernel ratio for arbitrary simply connected domains Ω. In fact, various constructions are possible, which we summarise:

- the approach of Yadin and Yehudayoff [YY11] can be used directly to show that the left hand side of (4.66) forms a Cauchy sequence, and it would be possible to obtain some mild regularity this way.
- we could use the so-called resolvent identity to *define* the Poisson kernel ratio (this will be discussed in much greater detail below, but going back to the work of Makarov and Smirnov [MS10] and also used extensively by Chelkak and Wan [CW19]).
- by multiplying the standard Poisson kernel ration by the appropriate change of measure ("Girsanov") terms.

In fact all these approaches will play a role in the arguments below and part of the work will be to show these various definitions coincide with one another. While in [CW19] the second option is chosen, we have found it simplest to start from the Girsanov approach which gives us a continuous object to work with and for which some minimal regularity can be easily shown. From this we can connect to the discrete picture and separately show that it obeys the appropriate resolvent identity, see Proposition 4.4.7.

Theorem 4.4.2. *Suppose Ω is bounded and simply connected. As $I \subset \partial\Omega$ shrinks to $a \in \partial\Omega$ (thought of as a prime end or a point on the Martin boundary)*

$$\frac{\mathbb{P}^{(\rho)}_x(\sigma < \infty, X_\sigma \in I)}{\mathbb{P}^{(\rho)}_o(\sigma < \infty, X_\sigma \in I)} \to \frac{P^{(\rho)}_\Omega(x,a)}{P^{(\rho)}_\Omega(o,a)} \tag{4.67}$$

converges to a limit, which we denote by $\lambda^{(\rho)}_\Omega(x,a)$, the Poisson kernel ratio of the corresponding massive Brownian motion. Furthermore,

$$P^{(\rho)}_\Omega(x,a) = \lambda^{(0)}_\Omega(x,a)\mathbb{E}_{x\to a}\left[\exp\left(-\int_0^\sigma \rho(X_s)ds\right)\right], \tag{4.68}$$

where $\lambda^{(0)}_\Omega(x,a)$ is the above (non-massive) Poisson kernel ratio.

Remark 4.4.3. $x \mapsto P_\Omega^{(\rho)}(x,a)$ thus coincides up to a constant with $x \mapsto \lambda_\Omega^{(\rho)}(x,a)$ but this choice of normalisation will turn out to be more adapted to the resolvent identity below.

Proof of Theorem 4.4.2. For $I \subset \partial\Omega$ (viewed as a subset of the Martin boundary),

$$\mathbb{P}_x^{(\rho)}(\sigma < \infty, X_\sigma \in I) = \mathbb{E}_x^{(0)}\left[1_{\{X_\sigma \in I\}} \exp\left(-\int_0^\sigma \rho(X_s)ds\right)\right]$$
$$= \mathbb{P}_x^{(0)}(X_\sigma \in I)\mathbb{E}_x\left[\exp\left(-\int_0^\sigma \rho(X_s)ds\right)\middle| X_\sigma \in I\right] \tag{4.69}$$
$$= \mathbb{P}_x^{(0)}(X_\sigma \in I)\int_I \mathbb{E}_{x\to y}\left[\exp\left(-\int_0^\sigma \rho(X_s)ds\right)\right] h_I^\#(x,dy) \tag{4.70}$$

where $h_I^\#(x,dy)$ is the harmonic measure of Brownian motion restricted to I, viewed as a measure on the Martin boundary, and normalised so that it gives I unit mass.

Suppose now that $I \downarrow a$ in the above sense of Martin boundaries. The integrand $y \in \partial\Omega \mapsto \mathbb{E}_{x\to y}[\exp(-\int_0^\sigma \rho(X_s)ds]$ is clearly continuous with respect to the natural topology on the Martin boundary of Ω because $\psi : \bar{\mathbb{D}} \to \bar{\Omega}$ is continuous. Using the dominated convergence theorem (as $\rho \geq 0$), the integral converges to $\mathbb{E}_{x\to a}[\exp(-\int_0^\sigma \rho(X_s)ds]$, as $I \downarrow a$. Taking the ratio of the right hand side of (4.70) with the same expression but for the starting point b, and using (4.66), we conclude that the limit (4.67) exists, and is equal to the right hand side of (4.68), as desired. □

Note that from the formula (4.68) a number of features of the Poisson kernel ratio are immediately obvious, such as its continuity with respect to x or a, or continuity with respect to the domain in the Carathéodory sense.

4.4.2 Convergence of discrete Poisson kernel

Recall that $Z_{\Omega^\delta}^{(\rho)}(x^\delta, a^\delta)$ denotes the total mass of random walk paths going from x^δ to a^δ without being killed: that is, $Z_{\Omega^\delta}^{(\rho)}(x^\delta, a^\delta) = \mathbb{P}_{x^\delta}^{(\rho)}(\sigma^\delta < \infty, X_{\sigma^\delta}^\delta = a^\delta)$.

Lemma 4.4.4. *For $r > 0$ and Ω^δ a sequence of subgraphs approximating Ω all containing a ball of radius r around $o^\delta \to o$ and marked boundary points a^δ, and $x^\delta \to x \in B(o, \frac{1}{2}r)$ it holds that*

$$\frac{Z_{\Omega^\delta}^{(\rho)}(x^\delta, a^\delta)}{Z_{\Omega^\delta}^{(\rho)}(o^\delta, a^\delta)} \to \frac{P_\Omega^{(\rho)}(x,a)}{P_\Omega^{(\rho)}(o,a)}$$

Essentially this is an adapation of the arguments in [YY11]. We will content ourselves with describing the instances where changes are needed. Because of this, we feel it is useful to first give a simplified overview of the arguments in [YY11], as it may otherwise prove difficult to see why the instances below are indeed the only arguments that need to be changed. In order to go through this we first map Ω to the unit disc $\mathbb{D}$ (this is both because the proof of convergence to SLE requires mapping everything to a reference domain, and in order to avoid issues related to the distinction between prime ends of a domain and the actual boundary). Thus let $\phi = \psi^{-1}$ be the map from Ω to $\mathbb{D}$ sending o to 0.

The first observation of Yadin and Yehudayoff is that "the exit probabilities are correct": given a small macroscopic arc $\tilde{I}$ on $\partial\mathbb{D}$ and $I = \psi(\tilde{I}) \subset \partial\Omega$, then the ratio of the probabilities $\mathbb{P}_x^\delta(X_{\sigma^\delta} \in I)/\mathbb{P}_o^\delta(X_{\sigma^\delta} \in I)$ converges to what one would expect, namely $\mathbb{P}_x(B_\sigma \in I)/\mathbb{P}_o(B_\sigma \in I)$. This is the content of their Lemma 4.8 and is a more or less obvious consequence of the assumption that random walk converges to Brownian motion, together with planarity. When the arc I (or rather $\tilde{I} \subset \partial\mathbb{D}$) is small, this ratio is itself close to the continuum Poisson kernel ratio $\lambda(x, o; \Omega)$ (essentially by definition of the latter).

Next for a boundary point $a \in \partial\Omega$ (understood as a prime end) and an interior point x, set $\tilde{a} = \phi(a)$, $\tilde{x} = \phi(x)$ and $\tilde{X} = \phi(X)$. They fix a small boundary arc $\tilde{I} \subset \partial\mathbb{D}$ centered around $\tilde{a}$ and write

$$H^\delta(x, a, \Omega) = \mathbb{P}^\delta_{\tilde{x}}(\tilde{X}_{\sigma^\delta} = \tilde{a} | \tilde{X}_{\sigma^\delta} \in \tilde{I}) \mathbb{P}^\delta_{\tilde{x}}(\tilde{X}_{\sigma^\delta} \in \tilde{I}),$$

so that it suffices to prove that the ratio

$$\frac{\mathbb{P}^\delta_{\tilde{x}}(\tilde{X}_{\sigma^\delta} = \tilde{a} | \tilde{X}_{\sigma^\delta} \in \tilde{I})}{\mathbb{P}^\delta_{o}(\tilde{X}_{\sigma^\delta} = \tilde{a} | \tilde{X}_{\sigma^\delta} \in \tilde{I})} \approx 1 \tag{4.71}$$

is close to 1, in the sense that

$$\limsup_{\tilde{I} \downarrow \tilde{a}} \limsup_{\delta \to 0} \left| \frac{\mathbb{P}^\delta_{\tilde{x}}(\tilde{X}_{\sigma^\delta} = \tilde{a} | \tilde{X}_{\sigma^\delta} \in \tilde{I})}{\mathbb{P}^\delta_{o}(\tilde{X}_{\sigma^\delta} = \tilde{a} | \tilde{X}_{\sigma^\delta} \in \tilde{I})} - 1 \right| = 0 \tag{4.72}$$

The key argument for this is a multiscale coupling, which is implicitly described in Propositions 5.4 – 5.6. The idea is to consider exponentially growing scales $R_j, j = 1, \ldots, N$ (from microscopic to macroscopic) and points ξ_j in the unit disc at distance of order R_j from both $\tilde{a}$ and the unit circle, with $R_j \approx e^j r$, and r being the width of the arc $\tilde{I} = \phi(I)$. At the smallest scale $j = 1$, ξ_j is thus at a distance of order r from $\tilde{a}$ itself, while at the largest scale $j = N$, ξ_j is at macroscopic distance from $\tilde{a}$. They condition both walks starting from x and o respectively to leave Ω through I. At each successive scale, there is a positive chance that when the walks get to that scale, they will go and visit the *same* predetermined small ball, chosen to be centered around ξ_j and to have a radius proportional to R_j times a very small constant. Once that is the case, the conditional chances of exiting through a specifically rather than anywhere else in I are necessarily essentially the same for both walks, which proves (4.71). Essentially, Proposition 5.4 shows that the coupling succeeds with positive probability at each scale independently of previous attempts. Proposition 5.5 shows that the ratio in (4.71) is bounded even in the unlikely event that the coupling never succeeded, and Proposition 5.6 quantifies how close to 1 the ratio in (4.71) once there is a success.

At the discrete level, the only properties of the walks that are needed are planarity (which of course always holds for the random walks considered in this paper) as well as crossing estimates (i.e., (4.64)) and simple consequences of it, such as Beurling estimates. These will be discussed briefly in Appendix 4.B. At the continuum level the required estimates are described (without proof) in Section 3 of [YY11], mostly Proposition 3.3 to Lemma 3.10. One can see that with very few exceptions, these estimates are properties of Brownian motion which are concerned with typical events of Brownian motion that can additionally be required to hold in a short time scale. In such cases the change of measure between massive (or drifted) and ordinary Brownian motion is harmless, hence these properties also obviously hold true in our situation. The lone exception is Proposition 3.3 (recalled below as Lemma 4.A.1, which concerns the probability to hit a very small ball); since this is not a typical event for Brownian motion, one needs to consider the effect of the change of measure and more specifically one needs to check that conditioning on the atypical event does not cause the change of measure to degenerate. This will be carried out in Appendix 4.A. This concludes our discussion of the proof of Lemma 4.4.4.

4.4.3 Density and absolute continuity with respect to classical SLE$_2$

In this section we will use assumption 2 to show that the massive SLE$_2$ with profile ρ is mutually absolutely continuous with respect to classical SLE$_2$ and the Radon Nikodym derivative is bounded.

Proposition 4.4.5. *There is a constant c_0 such that for every R and every domain $\Omega^\delta \subset B(0,R)$ and internal point x^δ and boundary point a^δ,*

$$1 \geq \frac{Z^{(\rho)}_{\Omega^\delta}(x^\delta, a^\delta)}{Z^{(0)}_{\Omega^\delta}(x^\delta, a^\delta)} \geq \exp(-c_0 R^2 \|\widetilde{\rho}\|_\infty)\,. \tag{4.73}$$

Proof. The inequality on the left hand-side is obvious since $\rho \geq 0$. For the equality on the right hand side note that the ratio can be written as

$$\begin{aligned} \mathbb{E}^{(0)}_{\Omega^\delta, v^\delta \to a^\delta}\left(\prod_{s=0}^{\sigma^\delta - 1}(1-\rho^\delta(X_s))\right) &\geq \mathbb{E}^{(0)}_{\Omega^\delta, v^\delta \to a^\delta}\left[\left(1-\delta^2\|\widetilde{\rho}\|_\infty + o(\delta^2)\right)^{\sigma^\delta}\right] \\ &\geq \left(1-\delta^2\|\widetilde{\rho}\|_\infty + o(\delta^2)\right)^{\mathbb{E}^{(0)}_{\Omega^\delta, v^\delta \to a^\delta}(\sigma^\delta)} \\ &\geq \exp(-c_0 R^2 \|\widetilde{\rho}\|_\infty)\,, \end{aligned}$$

where the second inequality follows from Jensen's inequality, and the last inequality is a direct consequence of assumption 2. □

Note that since

$$\frac{Z^{(\rho)}_{\Omega^\delta}(x^\delta, a^\delta)}{Z^{(\rho)}_{\Omega^\delta}(o^\delta, a^\delta)} = \frac{Z^{(\rho)}_{\Omega^\delta}(x^\delta, a^\delta)}{Z^{(0)}_{\Omega^\delta}(x^\delta, a^\delta)} \frac{Z^{(0)}_{\Omega^\delta}(o^\delta, a^\delta)}{Z^{(\rho)}_{\Omega^\delta}(o^\delta, a^\delta)} \frac{Z^{(0)}_{\Omega^\delta}(x^\delta, a^\delta)}{Z^{(0)}_{\Omega^\delta}(o^\delta, a^\delta)}$$

we deduce from this lemma that the left hand side (which is our discrete massive martingale observable) is also bounded provided that Ω contains a ball of radius r around o: indeed, the first two fractions are bounded by the previous Lemma, and the third one is the classical (non-massive) martingale observable, which is bounded as long as Ω contains a ball of radius r as in [CW19], say.

Let $\mathbb{P}^{(\rho)}_{o^\delta \to a^\delta}$ denote the random walk starting from o^δ with mass ρ, conditioned so that $\sigma^\delta < \infty$ and $X^\delta_{\sigma^\delta} = a^\delta$.

Proposition 4.4.6. *Let*

$$D^{(\rho)}_{\Omega^\delta}(\gamma^\delta) := \frac{\mathbb{P}^{(\rho)}_{o^\delta \to a^\delta}(\mathrm{LE}(X^\delta) = \gamma^\delta)}{\mathbb{P}^{(0)}_{o^\delta \to a^\delta}(\mathrm{LE}(X^\delta) = \gamma^\delta)}$$

be the Radon Nikodym derivative of the Loop erasure of the ρ-massive random walk with respect to the loop-erasure of the regular random walk, conditioned to leave Ω^δ at a^δ. Then

$$D^{(\rho)}_{\Omega^\delta}(\gamma^\delta) \leq \exp(c_0\, \mathrm{Diam}(\Omega)\|\rho\|_\infty)$$

for each γ^δ. Furthermore

$$\mathbb{E}^{(0)}_{o^\delta \to a^\delta}[\log(D^{(\rho)}_{\Omega^\delta}(\mathrm{LE}(X^\delta)))] \geq -c_0\, \mathrm{Diam}(\Omega)^2\|\rho\|_\infty. \tag{4.74}$$

Proof. Note that by definition

$$D^{(\rho)}_{\Omega^\delta}(\gamma^\delta) = \frac{\sum_{X^\delta:\mathrm{LE(X^\delta)}=\gamma^\delta} w^{(\rho)}(X^\delta)}{\sum_{X^\delta:\mathrm{LE(X^\delta)}=\gamma^\delta} w^{(0)}(X^\delta)} \cdot \frac{Z^{(0)}_{\Omega^\delta}(o^\delta, a^\delta)}{Z^{(\rho)}_{\Omega^\delta}(o^\delta, a^\delta)}. \tag{4.75}$$

The upper bound follows from the fact that the first fraction is less than 1 and the bound in Proposition 4.4.5. For the lower bound note that the second fraction is bigger than 1 and the first one is equal to $\mathbb{E}^{(0)}_{\Omega^\delta}[\prod_{s=0}^{\sigma^\delta-1}(1-\rho^\delta(X^\delta_s))|\mathrm{LE}(X^\delta) = \gamma^\delta]$. Taking the logarithm and applying Jensen gives the lower bound

$$\log(D^{(\rho)}_{\Omega^\delta}(\gamma^\delta) \geq \mathbb{E}^{(0)}_{o^\delta\to a^\delta}\left[\log(\prod_{s=0}^{\sigma^\delta-1}(1-\rho^\delta(X^\delta_s)))\middle|\mathrm{LE}(X^\delta)=\gamma^\delta\right].$$

Taking the expectation in γ^δ with respect to the measure of the loop-erasure removes the conditioning and gives

$$\mathbb{E}^{(0)}_{o^\delta\to a^\delta}[\log(D^{(\rho)}_{\Omega^\delta}(\mathrm{LE}(X^\delta)))] \geq \mathbb{E}^{(0)}_{o^\delta\to a^\delta}[\log(\prod_{s=0}^{\sigma^\delta-1}(1-\rho^\delta(X^\delta_s)))] \geq -c_0\,\mathrm{Diam}(\Omega)^2\|\widetilde{\rho}\|_\infty\,,$$

again by assumption 2. □

This last proposition implies that the laws of the Loop-erasure of the massive random walks are tight, and any limit point is mutually absolutely continuous with respect to SLE_2. Furthermore, by (4.74) and Girsanov's theorem, the Loewner transform of a limit point is driven by a process of the form $\xi_t = \sqrt{2}B_t + 2\lambda_t$ and therefore our goal is to identify λ_t. See Section 2.6 of [CW19] for more details.

4.4.4 Resolvent identity

We fix a boundary point $y = a$, and consider the Poisson kernel $P^{(\rho)}_D(x) = P^{(\rho)}(x, a)$ associated just with the mass profile ρ from the previous section. On a subdomain D (which will later be Ω_t) with a marked boundary point a' (which will later be a_t), we consider $P^{(\rho)}_D(x) = P^{(\rho)}_D(x, a')$.

We aim to establish the following **resolvent identity** for $P^{(\rho)}_D(x)$:

$$P^{(\rho)}_D(x) = P^{(0)}_D(x) - \int_D G^{(\rho)}_D(x,y)\rho(y)P^{(\rho)}_D(y)dy, \tag{4.76}$$

where $G^{(\rho)}_D(x, y) = G^{(\rho)}_D(x, y)$ is an appropriate Green function, more precisely the Green function of the Brownian motion with mass profile ρ with Dirichlet boundary conditions in D: that is,

$$G^{(\rho)}_D(x,y) = \int_0^\infty p^{(\rho)}_t(x,y)dt$$

where for $t \geq 0$,

$$p^{(\rho)}_t(x,y) := p_t(x,y)\mathbb{E}_{x\to y;t}[\exp(-\int_0^t \rho(X_s)ds)1_{\{X[0,t]\subset D\}}] \tag{4.77}$$

and $p_t(x, y)$ denotes the (full plane) transition probabilities for standard Brownian motion.

A trivial but essential property of the massive Green function is that it is bounded by the standard (non-massive) Green function, i.e.

$$G_D^{(\rho)}(x,y) \leq G_D^{(0)}(x,y)$$

for all $x, y \in D$. This allows us to estimate many integrals simply with their non-massive counterparts.

We are now in a position to prove the resolvent identity (4.76).

Proposition 4.4.7. *We have*

$$P_D^{(\rho)}(x) = P_D(x) - \int_D G_D^{(\rho)}(x,y)\rho(y)P_D(y)dy\,. \tag{4.78}$$

Proof. It is clear from the expression of the potential kernel $P^{(\rho)}(x)$ in Theorem 4.4.2 that this is C^2 in Ω; the definition as a limit of hitting probabilities shows that it is harmonic with respect to the generator of the massive Brownian motion, i.e. $\mathcal{L}_\rho := \frac{1}{2}\Delta - \rho$.

Let us consider the function

$$f(x) = P_D^{(\rho)}(x) - P_D(x).$$

Our goal is to show that $f(x) = \int_D G_D^{(\rho)}(x,y)\rho(y)P_D(y)dy$ for all $x \in D$.

Note that from (4.68) f is clearly continuous (in fact twice differentiable) in D. Furthermore if $x = x_n$ tends to a point $x' \in \partial D$ with $x' \neq a$, both terms in $f(x_n)$ tend to zero. We now claim that $f(x)$ is "negligible compared to the probability of leaving by a" as $x \to a$. Let us explain what we mean by this. Recall the map $\phi : D \to \mathbb{D}$ which is the conformal isomorphism sending o to 0 and a to 1. For small $r > 0$, let $A_r^{\mathbb{D}}$ be the set of points in $\mathbb{D}$ at distance r from 1 and let $A_r = \phi^{-1}(A_r^{\mathbb{D}})$. Let D_r be the connected complement of $D \setminus A_r$ not adjacent to a.

Lemma 4.4.8. *For $x \in A_r$ we have $f(x) = o(1/r)$ uniformly. On the other hand, for fixed $x \in D$, as $r \to 0$ (assume without loss of generality that r is small enough that $x \in \Omega_r$), if $\sigma_r = \inf\{t \geq 0 : X_t \notin D_r\}$ then there is a constant $C = C(x, \Omega, \varphi)$ such that*

$$\mathbb{P}_x^{(0)}(X_{\sigma_r} \in A_r) \leq Cr, \quad r > 0.$$

Proof. Recall that $P_D^{(\rho)}(x) = P_D(x)\mathbb{E}_{x\to a}[\exp(\int_0^\sigma \rho(X_s)ds)]$. Note that $P_D(x)$ is exactly conformally invariant (it is equal to the Poisson kernel ratio $\lambda_D^{(0)}(x,a)$ for Brownian motion) and so $P_D(x) \leq O(1/r)$ uniformly on A_r (using the exact value of the Poisson kernel in the unit disc and the fact that by definition A_r is mapped to $A_r^{\mathbb{D}}$ by ϕ which lies at distance r from 1). Furthermore, uniformly for $x \in A_r$, the expectation $\mathbb{E}_x(\exp(\int_0^\sigma \rho(X_s)ds)) \to 1$ as $x \to a$ (i.e., as $r \to 0$) by dominated convergence since $\sigma \to 0$ in probability (indeed, D is simply connected hence has a regular boundary) and $\rho(x) \geq 0$. Thus

$$f(x) = P_D(x)\left(\mathbb{E}_x[\exp\int_0^\sigma \rho(X_s)ds] - 1\right) = o(1/r)$$

as $r \to 0$, as desired.

For the probability to leave through A_r, we simply note that using conformal invariance of harmonic measure for Brownian motion, we have that

$$\mathbb{P}_x^{(0)}(X_{\sigma_r} \in A_r) = \mathbb{P}_{\phi(x)}^{(0)}(X_{\sigma_r^{\mathbb{D}}} \in A_r^{\mathbb{D}}) \leq C(x,\Omega)r$$

which concludes the proof. □

For the proof of Proposition 4.4.7 we will need to find a suitable martingale. Recall that $f(x) = P_D^{(\rho)}(x) - P_D(x)$. Note that since, $P_D^{(\rho)}$ is $\mathcal{L}_\rho$ harmonic,

$$\begin{aligned}\mathcal{L}_\rho f &= \mathcal{L}_\rho P_D^{(\rho)} - \mathcal{L}_\rho P_D \\ &= 0 - (\frac{1}{2}\Delta - \rho)P_D \\ &= \rho P_D.\end{aligned}$$

We will call $g(x) = -\rho(x)P_D(x)$, so that the above identity reads

$$\mathcal{L}_\rho f = -g. \tag{4.79}$$

Lemma 4.4.9. *For $t \le \sigma$, let $I_t = \int_0^t \rho(X_s)ds$, and let*

$$M_t = f(X_t)e^{-I_t} + \int_0^t g(X_s)e^{-I_s}ds.$$

Then for each $r > 0$, $(M_{t\wedge\sigma_r}, t \ge 0)$ is a continuous local martingale under $\mathbb{P}_x^{(0)}$.

Proof. We apply Itô's formula:

$$\begin{aligned}dM_t &= -dI_t e^{-I_t} f(X_t) + e^{-I_t} df(X_t) + g(X_t)e^{-I_t} \\ &= \text{mart.} + e^{-I_t}\left[-\rho(X_t)f(X_t)dt + \frac{1}{2}\Delta f(X_t)dt + g(X_t)dt\right] \\ &= \text{mart.} + e^{-I_t}[\mathcal{L}_\rho f(X_t) + g(X_t)]dt\end{aligned}$$

and so, since $\mathcal{L}_\rho f + g = 0$ by (4.79), M_t is indeed a continuous local martingale up to time σ_r. □

We are now ready to derive a proof of the resolvent identity. To do this we apply the optional stopping for M_t at time $t \wedge \sigma_r$ (until which M remains bounded, so the application is justified: indeed f is continuous on Ω_r and has zero boundary conditions on $\partial\Omega_r$ except on A_r where it is uniformly bounded by $o(1/r)$ by the Lemma 4.4.8, a similar justification applies to g also). We find, since $M_0 = f(x)$,

$$f(x) = \mathbb{E}_x\left[f(X_{t\wedge\sigma_r})e^{-I_{t\wedge\sigma_r}}\right] + \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} g(X_s)e^{-I_s}ds\right] \tag{4.80}$$

We will now let $t \to \infty$ and *then* $r \to 0$ in both terms separately to obtain the resolvent identity. We start with the first term, for which we claim the limit as $t \to \infty$ and then $r \to 0$ is simply zero. Indeed, since $r > 0$ is fixed and f is bounded $\bar{D}_r$, by the dominated convergence theorem we get

$$\lim_{t\to\infty} \mathbb{E}_x\left[f(X_{t\wedge\sigma_r})e^{-I_{t\wedge\sigma_r}}\right] \le \mathbb{E}_x^{(0)}[f(X_{\sigma_r})].$$

Furthermore recall that $f(x) = 0$ for $x \in \partial\Omega$ with $x \ne a$, so the only contribution comes from the event where $X_{\sigma_r} \in A_r$: thus

$$\begin{aligned}|\mathbb{E}_x^{(0)}[f(X_{\sigma_r})]| &\le \mathbb{P}_x(X_{\sigma_r} \in A_r) \sup_{x\in A_r} |f(x)| \\ &= O(r)o(1/r) \to 0\end{aligned}$$

by Lemma 4.4.8.

It remains to deal with the second term, which is of the form

$$\int_{D_r} g(y) G^{(\rho)}_{t,r}(x,y)dy,$$

where $G^{(\rho)}_{t,r}(x,y)$ is the Green function for Brownian motion up to time t, weighted by $\exp(-I_t)$, and stopped when leaving D_r, i.e.,

$$G^{(\rho)}_{t,r}(x,y) = \int_0^t p^{(\rho)}_{s,r}(x,y)ds; \quad p^{(\rho)}_{s,r}(x,y) = p_s(x,y)\mathbb{E}_{x\to y;s}[e^{I_s} 1_{\{X[0,s]\subset D_r\}}]$$

Letting $t \to \infty$, there is no problem (by monotone convergence) in showing that $G^{(\rho)}_{t,r}$ converges pointwise to $G^{(\rho)}_r(x,y)$, the massive Green function in the domain D_r. Subsequently letting $r \to 0$, there is for the same reason no problem in showing that this converges monotonically to $G^{(\rho)}_D(x,y)$. Thus, for all $t < \infty$ and $r > 0$,

$$G^{(\rho)}_{t,r}(x,y) \le G^{(\rho)}_D(x,y) \le G^{(0)}_D(x,y)\,.$$

Furthermore, observe that since $|g(y)| = \rho(y)P_D(y)$ and ρ is bounded,

$$\int_D G^{(0)}_D(x,y)|g(y)|dy \le C\int_D G^{(0)}_D(x,y)P_D(y)dy < \infty. \tag{4.81}$$

To see the finiteness of the right hand side, observe that both terms in the integral on the right hand side are conformally invariant, and that after mapping by the conformal isomorphism ϕ, $P_{\mathbb{D}}(\phi(y)) \sim 2/|\phi(y)-1|$ as $y \to a$, while $G^{(0)}_{\mathbb{D}}(\phi(x),\phi(y)) \sim |1-\phi(y)|$, so that the integrand is bounded in the neghbourhood of $y = a$. Elsewhere, $P_D(y)$ is bounded, and the Green function $G_D(x,y)$, while having a singularity at $y = x$, is clearly integrable over D.

Consequently the assumptions of the dominated convergence theorem are satisfied, and we deduce that

$$\lim_{r\to 0}\lim_{t\to\infty} \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} g(X_s)e^{-I_s}ds\right] = \int_D G^{(\rho)}_D(x,y)g(y)dy.$$

Plugging into (4.80), we therefore obtain:

$$f(x) = \int_D G^{(\rho)}_D(x,y)g(y)dy,$$

which is the desired identity. □

There is also a resolvent identity for the Green function $G^{(\rho)}$ itself.

Proposition 4.4.10. *We have*

$$G^{(\rho)}_D(x,z) = G_D(x,z) - \int_D G_D(x,y)\rho(y)G^{(\rho)}_D(y,z)dy. \tag{4.82}$$

Proof. Since G^ρ is also $\mathcal{L}_\rho$-harmonic in both variables, and G is harmonic, the proof proceeds essentially along the same lines. The difference is that we need to replace Lemma 4.4.8 by the following control over $f(z) := G^{(\rho)}_D(x,z) - G^{(0)}_D(x,z)$ near $z = x$:

Lemma 4.4.11. *Let f be as above. Then as $z \to x$,*

$$f(z) = o(\log|x-z|^{-1}).$$

Proof. Since $G_\Omega^{(\rho)}(x,z) \le G_\Omega^{(0)}(x,z)$ we only need a lower bound on the massive Green function. This is easily obtained: for any $\varepsilon > 0$,

$$\begin{aligned} G_\Omega^{(\rho)}(x,z) &= \int_0^\infty p_t^{(\rho)}(x,z)dt \\ &\ge \int_0^\varepsilon p_t^{(0)}(x,z)e^{-I_t}dt \\ &\ge e^{-\varepsilon\|\rho\|_\infty}\left(G^{(0)}(x,z) - \int_\varepsilon^\infty p_t^{(0)}(x,z)dt\right) \\ &\ge (1-\varepsilon\|\rho\|_\infty)\left(G^{(0)}(x,z) - O(1)\right) \end{aligned}$$

where we used the easy consequence of Beurling's estimate that for a simply connected domain Ω, $p_t(x,z) \le t^{-1-\eta}$ for some $\eta > 0$. The lemma follows. □

The rest of the argument proceeds exactly as in the proof of Proposition 4.4.7. □

From this we can deduce a massive version of Hadamard's formula (see [CW19, Lemma 4.7] for the case of constant mass). Let $(K_t)_{t\ge 0}$ be a growing family of compact Ω-hulls in Ω, growing from a to the inner point $o \in \Omega$ and having the locality property (see e.g. [BN23] for a definition of these terms), generated by a continuous curve γ_t growing in Ω from a to o. Let $\Omega_t = \Omega \setminus K_t$, which is a monotone decreasing family of subdomains of Ω_t, and let a_t be the point on the (Martin) boundary of Ω_t corresponding to γ_t. We will assume that $\gamma[0,\infty)$ has Lebesgue measure zero.

Let $G_t^{(\rho)} = G_{\Omega_t}^{(\rho)}$ be the massive Green function in Ω_t and let $P_t^{(\rho)} = P_{\Omega_t}^{(\rho)}$ be the Poisson kernel (with normalisation specified by Proposition 4.4.2 associated with the boundary point a_t). Since Ω_t is monotone decreasing, it is obvious that for each fixed $z \ne x$ in Ω, $t \mapsto G_t^{(\rho)}(x,z)$ is monotone decreasing until the first time t such that either one of x and z are in K_t. The massive Hadamard identity expresses the derivative of $G_t^{(\rho)}$ in terms of product of Poisson kernels. Intuitively, this is because the paths from x to z that are lost between times t and $t+dt$ can be decomposed into two portions, one from x and one from z, which go via the tip of the curve a_t. (In the case of constant mass, this is stated without proof in the proof of Proposition 3.1 by Makarov and Smirnov [MS10]; the argument below is close to the proof given by Chelkak and Wan [CW19, Lemma 4.7]).

Proposition 4.4.12. *For each fixed $x \ne z$, such that $x, z \in \Omega_t$, the massive Green function $s \mapsto G_s^{(\rho)}(x,z)$ is differentiable at $s = t$, and*

$$\partial_t G_t^{(\rho)}(x,z) = -\pi P_t^{(\rho)}(x) P_t^{(\rho)}(z), \tag{4.83}$$

where $P_t^{(\rho)}$ is defined above.

Proof. Since the mass ρ is nonnegative (or more precisely by (4.77)), for $s < t$,

$$0 \le G_s^{(\rho)}(x,z) - G_t^{(\rho)}(x,z) \le G_s^{(0)}(x,z) - G_t^{(0)}(x,z)$$

so the increments of $G^{(\rho)}$ are bounded by those of $G^{(0)}$. Since $s \mapsto G_s^{(0)}(x,z)$ is differentiable by the classical (non-massive) Hadarmard formula (see, e.g., [SS13]) we deduce by monotonicity that $s \mapsto G_s^{(\rho)}$ itself is differentiable.

Therefore $\partial_t G_t^{(\rho)}(x,z)$ exists for all $x, z \in \Omega_t$ and it also holds that

$$0 \le -\partial_t G_t^{(\rho)}(x,y) \le -\partial_t G_t^{(0)}(x,y) = \pi P_t(x) P_t(y) < \infty. \tag{4.84}$$

Recall from the Green function resolvent identity that

$$G_t^{(\rho)}(x,z) = G_t^{(0)}(x,z) - \int_{\Omega_t} G_t^{(0)}(x,y)\rho(y)G_t^{(\rho)}(y,z)dy$$

Since $\gamma[0,t]$ has Lebesgue measure equal to zero we can replace the domain of integration in the above integral by Ω.

Differentiating this identity (4.82), using the classical Hadamard formula and the resolvent identity for the massive Poisson kernel (Proposition 4.4.7), we obtain:

$$\begin{aligned}\partial_t G_t^{(\rho)}(x,z) &= -\pi P_t(x)P_t(z) + \pi \int_{\Omega} P_t(x)P_t(y)\rho(y)G_t^{(\rho)}(y,z)dy \\ &\quad - \int_{\Omega} G_D(x,y)\rho(y)\partial_t G_t^{(\rho)}(y,z)dy \\ &= -\pi P_t(x)P_t^{(\rho)}(y) - \int_{\Omega} G_D(x,y)\rho(y)\partial_t G_t^{(\rho)}(y,z)dy. \end{aligned} \tag{4.85}$$

Differentiation under the integral is justified because both G_t and $G_t^{(\rho)}$ are decreasing in t so their product is also monotone, and we can then use the positive case of the Fubini theorem (i.e., the Tonnelli theorem) as well as the fundamental theorem of calculus to conclude.

Consider now the integral operators $\mathfrak{G}_t$ and $\mathfrak{G}_t^{(\rho)}$ acting on an arbitrary function $h : \Omega_t \to \mathbb{R}$ by

$$(\mathfrak{G}_t h)(x) := \int_{\Omega} G_t(x,y)\rho(y)h(y)dy \tag{4.86}$$

$$(\mathfrak{G}_t^{(\rho)} h)(x) := \int_{\Omega} G_t^{(\rho)}(x,y)\rho(y)h(y)dy. \tag{4.87}$$

whenever the integrals above converge. Using these operators we can rewrite (4.85) as

$$(\mathrm{Id} + \mathfrak{G}_t)(\partial_t G_t^{(\rho)}(\cdot,z)) = -\pi P_t(\cdot)P_t^{(\rho)}(z). \tag{4.88}$$

Again by the resolvent identity (4.82) we will see that

$$(\mathrm{Id} - \mathfrak{G}_t^{(\rho)})(\mathrm{Id} + \mathfrak{G}_t)h = h, \tag{4.89}$$

whenever all involved integrals are absolutely convergent. Indeed the left hand side, evaluated at x, expands as

$$\begin{aligned} &h(x) - \int_{\Omega} G_t^{(\rho)}(x,y)\rho(y)h(y)\mathrm{d}y + \int_{\Omega} G_t(x,y)\rho(y)h(y)\mathrm{d}y \\ &\quad - \iint_{\Omega\times\Omega} G_t^{(\rho)}(x,z)\rho(z)G_t(z,y)\rho(y)h(y)\mathrm{d}y\mathrm{d}z \\ &= h(x) + \int_{\Omega} \rho(y)h(y)\left(G_t(x,y) - G_t^{(\rho)}(x,y) - \int_{\Omega} G_t^{(\rho)}(x,z)\rho(z)G_t(z,y)\mathrm{d}z\right)\mathrm{d}y \\ &= h(x), \end{aligned}$$

since the bracketed term in the integral vanishes due to the Green function resolvent identity (4.82) and reversibility of $G_t(z,y)$.

For $h(x) = \partial_t G_t^{(\rho)}(x,z)$ the absolute convergence of the integrals is justified by, respectively: (4.84) and the finiteness of the integral in (4.81), and an application of (4.6) in [CW19] (also recalled below explicitly in (4.92)) together with the fact that $\int_\Omega G_t(z,y)dy$ is bounded uniformly in $z \in \Omega$ by a constant depending only on the diameter of Ω.

Applying (4.88) and Proposition 4.4.7 yields

$$\begin{aligned}\partial_t G_t^{(\rho)}(\cdot, z) &= (\mathrm{Id} - \mathfrak{G}_t^{(\rho)})(\mathrm{Id} + \mathfrak{G}_t)\partial_t G_t^{(\rho)}(\cdot, z) \\ &= (\mathrm{Id} - \mathfrak{G}_t^{(\rho)})(-\pi P_t(\cdot) P_t^{(\rho)}(z)) \\ &= -\pi P_t^{(\rho)}(\cdot) P_t^{(\rho)}(z).\end{aligned}$$

This proves the statement. □

4.4.5 Derivative resolvent identity

We continue with the setup introduced above Proposition 4.4.12. Thus, K_t is a growing family of hulls generated by a continuous curve γ_t, and $\Omega_t = \Omega \setminus K_t$. This curve will later be either be non-massive or massive SLE_2.

We define the (radial) derivative kernel Q_t in Ω_t by setting for $y \in \Omega_t$,

$$Q_t(y) = Q_t(y, a_t) = \mathrm{Im}\left(\frac{2\phi_t(y)}{(1-\phi_t(y))^2}\right),$$

where that $\phi_t : \Omega_t \to \mathbb{D}$ is the unique conformal isomorphism mapping o to 0 and a_t to 1. (This is the radial analogue of (4.1) in [CW19]). The reason for introducing this radial derivative kernel Q_t is that, writing P_t for the potential kernel ratio P_{Ω_t},

$$dP_t(x) = Q_t(x) d\xi_t.$$

In the chordal case this is an easy application of Itô's formula which can be seen for instance from Proposition 7.7 in [BN23]. In the radial case this calculation is slightly more involved but essentially similar. To calculate $\mathrm{d}P_t^{(\rho)}(x)$ we need to define massive version of Q_t. For this it is simpler to *define* $Q_t^{(\rho)}$ via its associated resolvent identity: namely, we set

$$Q_t^{(\rho)}(x) = Q_t(x) - \int_\Omega G_t^{(\rho)}(x,y)\rho(y)Q_t(y)dy. \tag{4.90}$$

To make this definition we need to check that the integral appearing on the right hand side is finite; we will check this is a.s. the case for almost every time. Since $G_t^{(\rho)}(x,y) \le G_t(x,y)$ it suffices to prove the same with $G_t^{(\rho)}$ replaced with the ordinary Green function G_t in Ω_t. This is done in [CW19, Corollary 4.6] for the chordal case. What remains to be checked is that these arguments can be carried over to the radial case, but for completeness we will also repeat the rest of the argument.

Proposition 4.4.13. *For any fixed Loewner chain we have the following estimate. For all $x \in \Omega$, for almost every time $t \ge 0$, $\int_\Omega G_t(x,y)Q_t(y) < \infty$. In particular this holds almost surely for the Loewner chain driven by $(\xi_t)_{t\ge 0}$.*

Proof. Using [CW19, Lemma 4.1] and expressing our Q in the upper-half plane via a suitable Möbius map (and conformal invariance of P), it is easy to check that

$$\left|\frac{P_t(y)}{P_t(x)} - 1\right| G_t(x,y) \le C, \quad \left|\frac{Q_t(y)}{P_t(y)} - \frac{Q_t(x)}{P_t(x)}\right| \frac{G_t(x,y)}{P_t(x)} \le C \tag{4.91}$$

for some uniform constant $C > 0$ independent of anything. In particular,

$$P_t(y)G_t(y,x) \le P_t(x)G_t(y,x) + CP_t(x) \tag{4.92}$$

and

$$|Q_t(y)|G_t(x,y) \leq CP_t(x)P_t(y) + |Q_t(x)|G_t(x,y) + C|Q_t(x)|. \tag{4.93}$$

When we integrate over $y \in \Omega$, the third term does not depend on y so is integrable, the second term depends on y only through $G_t(x,y)$ but using the fact that $G_t(x,y) \leq G_\Omega(x,y)$ which has only a logarithmic singularity at x, it is easy to see that this term too is integrable for all $t \geq 0$. The problematic term is the first term. The proposition follows if we can prove

$$\int_\Omega P_t(y)dy < \infty \tag{4.94}$$

for almost every $t \geq 0$, almost surely. In fact, we will check

$$\int_0^\infty \left[\int_{\Omega_t} P_t(y)dy\right]^2 dt \leq C(\Omega) < \infty \tag{4.95}$$

where $C(\Omega)$ depends only on Ω (this is the analogue of Corollary 4.6 (i) in [CW19]). This will obviously imply (4.94) and thus Proposition 4.4.13. This however follows immediately from the classical Hadamard formula since

$$\left[\int_{\Omega_t} P_t(y)dy\right]^2 = \iint_\Omega P_t(x)P_t(y)dxdy = -\frac{1}{\pi}\iint_\Omega \partial_t G_t(x,y)dxdy.$$

Thus integrating over $t > 0$ we get

$$\int_{t>0} \left[\int_{\Omega_t} P_t(y)dy\right]^2 dt = \frac{1}{\pi}\iint_\Omega G_\Omega(x,y) - G_\infty(x,y)dxdy \leq \frac{1}{\pi}\iint G_\Omega(x,y)dxdy < \infty.$$

□

For our subsequent use of the radial derivative kernel in Itô's formula we need a strengthening of (4.95), which is the analogue of Corollary 4.6 (ii) in [CW19]. This is the key estimate, and requires us to make one additional assumption compared to the general setup introduced above Proposition 4.4.12, We will stop assuming that $(\gamma_t)_{t\geq 0}$ is deterministic and arbitrary, and instead assume it is random, absolutely continuous on compact intervals of time $[0,T]$ with respect to SLE$_\kappa$ for some $\kappa \leq 4$. (In our application in this article, γ will be either SLE$_2$ or the inhomogeneous massive SLE$_2$, so these assumptions will be satisfied).

Lemma 4.4.14. *Almost surely, for any fixed $T > 0$,*

$$\int_0^T \int_{\Omega_t} P_t(x)^2 \mathrm{d}x\mathrm{d}t < \infty$$

In view of the nature of the singularity of $P_t(x)$ near a_t, such a result might seem surprising initially.

Proof. Fix $x \in \Omega$ and suppose $t < \tau_x$. Let y be sufficiently close to x that $y \in \Omega_t$, but $y \neq x$. We know that we may write the Green function

$$G_t(x,y) = -\frac{1}{\pi}\log|x-y| + h_t(y),$$

where $h_t(y)$ is harmonic in $y \in \Omega_t$ (including at $y = x$) and $h_t(x) = 1/(\pi)\log \operatorname{crad}(x,\Omega_t)$ (see, e.g., Theorem 1.23 in [BP23]). The left hand side is (for instance by Hadamard's formula) differentiable in t; since in the right hand side only $h_t(y)$ depends on t we get that $t \mapsto h_t(y)$ is differentiable and

$$\partial_t h_t(y) = \partial_t G_t(x, y) = -\pi P_t(x) P_t(y).$$

We want to take $y \to x$ and this requires an exchange of derivation of limit. This can be done using the fact that $h_t(x)$ is harmonic and thus satisfies the mean value property:

$$\frac{h_{t+\delta}(x) - h_t(x)}{\delta} = \int_y \frac{h_{t+\delta}(y) - h_t(y)}{\delta} s(\mathrm{d}y)$$

where $s(\mathrm{d}y)$ is the uniform law on some given circle centred at x of arbitrary sufficiently small positive radius. Pointwise, as $\delta \to 0$, the terms in the integral converge to $P_t(x)P_t(y)$ by the above. The assumptions of the dominated convergence theorem are satisfied by the mean value theorem and the fact that $t \mapsto P_t(x)P_t(y)$ is continuous at a given time t so long as $y \in \Omega_t$. We deduce, using harmonicity of $P_t(y)$:

$$\frac{1}{\pi}\partial_t \log \mathrm{crad}(x, \Omega_t) = -\pi P_t(x)^2,$$

for any $x \in \Omega_t$. Therefore,

$$\int_0^T P_t(x)^2 \mathrm{d}t = \frac{1}{\pi^2} \log \frac{\mathrm{crad}(x, \Omega)}{\mathrm{crad}(x, \Omega_T)},$$

for any $x \in \Omega_T$ (note the unimportant difference of a factor of $\pi/2$ with respect to the proof of Corollary 4.6(ii) in [CW19], which comes from a different choice of normalisation of the Laplacian and what appears to be a typo).

This can be integrated over $x \in \Omega_T$ and even $x \in \Omega$ when we set the integrand to be infinity on K_T; since K_T has Lebesgue measure a.s. equal to zero this does not make a difference. We get

$$\int_{\Omega_T} \int_0^T P_t(x)^2 \mathrm{d}t\mathrm{d}x = \frac{1}{\pi^2} \int_{\Omega_T} \log \frac{\mathrm{crad}(x, \Omega)}{\mathrm{crad}(x, \Omega_T)} dx.$$

Using Fubini's theorem (since the integrand is positive) we can exchange the space and time integration on the left hand side. Taking expectations, we further obtain:

$$\mathbb{E}\left[\int_0^T \int_{\Omega_T} P_t(x)^2 \mathrm{d}x\right] = \frac{1}{\pi^2} \int_{\Omega} \mathbb{E}\left[1_{x \in \Omega_T} \log \frac{\mathrm{crad}(x, \Omega)}{\mathrm{crad}(x, \Omega_T)}\right] \mathrm{d}x$$

In the left hand side there is do difference if we replace Ω_T by Ω_t (since the difference has zero Lebesgue measure a.s.) and in the right hand side for the same reason we can ignore the indicator. The result follows since the expectation on the right hand side is finite. Indeed, much stronger bounds are known than (negative) logarithmic moments for $\mathrm{crad}(x, \Omega_T)$: it is known that $\mathbb{P}(\mathrm{dist}(w, K_T) \le \varepsilon) \le C(T)\varepsilon^{1-\kappa/8}$ with $\kappa \le 4$ (see, e.g., Proposition 4 in [Bef08] for the chordal case, but the argument easily generalises to the radial case). This gives polynomial hence the desired logarithmic moments using the Koebe one-quarter theorem. □

This result implies a lemma ("stochastic Fubini") corresponding to Lemma 4.8 in [CW19]:

Lemma 4.4.15. *The process* $t \mapsto \int_\Omega G_t^{(\rho)}(x, y) Q_t(y) \mathrm{d}y$ *is a local semi martingale in the filtration of the driving function* $(\xi_t)_{t \ge 0}$. *Moreover almost surely, for all* $T > 0$ *the following identity is satisfied:*

$$\int_\Omega \int_0^T G_t^{(\rho)}(x, y) Q_t(y) \mathrm{d}\xi_t \mathrm{d}y = \int_0^T \int_\Omega G_t^{(\rho)}(x, y) Q_t(y) \mathrm{d}y \mathrm{d}\xi_t \tag{4.96}$$

This follows from Proposition 4.4.13 and Lemma 4.4.14 in the same way as in [CW19].

4.4.6 Identification of LERW limit: proof of Theorem 4.4.1

In this section we complete the proof of Theorem 4.4.1 by showing that the limit of loop-erased random walk on the triangular lattice, in the case where the walk itself converges to inhomogeneous massive Brownian motion, exists and is given by a Lowener evolution whose driving function ξ satisfies (4.19).

Let us summarise the situation at this stage. As already mentioned at the end of Section 4.4.3, we know that subsequential limits of the loop-erasure exist (i.e., the laws of the loop-erasure are tight), and it suffices to identify any subsequential limit uniquely. We also know, again from the same discussion, that any subsequential limit is absolutely continuous with respect to radial SLE$_2$, and may be described by a radial Loewner evolution whose driving function ξ satisfies

$$\mathrm{d}\xi_t = \sqrt{2} dB_t + 2\lambda_t \mathrm{d}t. \tag{4.97}$$

Our goal is thus to identify λ_t (we will show that $\lambda_t = \frac{Q_t^{(\rho)}(o)}{P_t^{(\rho)}(o)}$) and show that the above SDE has a unique weak solution (we will in fact get strong pathwise uniqueness).

Proof of Theorem 4.4.1. We know by a classical argument, see [LSW01, Remark 3.6], that for every vertex x^δ, we get a **discrete martingale observable** $M_n^{(\rho)}(x^\delta)$ defined by

$$M_n^{(\rho)}(x^\delta) = \frac{Z^{(\rho)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(x^\delta, \gamma^\delta(n))}{Z^{(\rho)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(o^\delta, \gamma^\delta(n))}. \tag{4.98}$$

Applying Lemma 4.4.4 (which, as already explained, extends to our setup one of the main results of Yadin and Yehudayoff [YY11]), we see that if we take $x^\delta \to x \in \Omega$ and parameterize $\gamma^\delta[0,n]$ by capacity (which requires taking $n = n^\delta(t)$ for any given $t > 0$) then, assuming $z \in \Omega_t$, we have as $\delta \to 0$, for each $t > 0$,

$$\frac{Z^{(\rho)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(x^\delta, \gamma^\delta(n))}{Z^{(\rho)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(o^\delta, \gamma^\delta(n))} \to \frac{P^{(\rho)}_{\Omega_t}(x, a_t)}{P^{(\rho)}_{\Omega_t}(o, a_t)} \tag{4.99}$$

where a_t denotes the tip of γ_t, viewed as a prime end in Ω_t. This is the analogue of Proposition 3.16 in [CW19]. The right hand side is continuous in $t \geq 0$, as remarked at the end of Section 4.4.1. Furthermore, as argued in Section 2.4 of [CW19], the discrete martingales $M_n^{(\rho)}(x^\delta)$ yield continuous martingales in any subsequential limit (see in particular Remark 2.3 in [CW19]), hence we deduce that for every $x \in \Omega$, every $r > 0$,

$$M_t^\rho(x) := \frac{P^{(\rho)}_{\Omega_t}(x, a_t)}{P^{(\rho)}_{\Omega_t}(o, a_t)}; t \wedge \tau_r$$

is a martingale, where for every $r > 0$ the stopping time τ_r is defined as $\inf\{t > 0 : |\gamma_t - b| \wedge |\gamma_t - x| \leq r\}$.

Now we explain how these martingales can be used to identify the drift λ_t uniquely. To see this, first recall that $P_t^{(\rho)} := P^{(\rho)}_{\Omega_t}$ satisfies the resolvent identity (Proposition 4.4.7), namely,

$$P_t^{(\rho)}(x, a_t) = P_t(z, a_t) + \int_\Omega G_t^{(\rho)}(x, y)\rho(y) P_t(y, a_t) \mathrm{d}y.$$

Now we know that in the critical ($\rho = 0$) case, one has as a direct application of Loewner's equation

$$\mathrm{d}P_t(x, a_t) = Q_t(x) \mathrm{d}\xi_t.$$

Since this is an a.s. identity, this same identity remains true by absolute continuity for our subsequential limit. Together with the Hadamard formula (Lemma 4.4.12), the resolvent identity for $P_t^{(\rho)}$ (Proposition 4.4.7) implies

$$\begin{aligned} \mathrm{d}P_t^{(\rho)}(x) &= Q_t(x)\mathrm{d}\xi_t + \int_\Omega G_t^{(\rho)}(x,y)\rho(y)Q_t(y)\mathrm{d}\xi_t\mathrm{d}y - \pi P_t^{(\rho)}(x)\int_\Omega P_t^{(\rho)}(y)\rho(y)P_t(y)\mathrm{d}y\mathrm{d}t \\ &= Q_t^{(\rho)}(x)\mathrm{d}\xi_t - \pi P_t^{(\rho)}(x)\int_\Omega P_t^{(\rho)}(y)\rho(y)P_t(y)\mathrm{d}y\mathrm{d}t, \end{aligned} \tag{4.100}$$

where we used Lemma 4.4.15 to exchange $\mathrm{d}\xi_t$ and $\mathrm{d}y$.

Since we know that $\frac{P_t^{(\rho)}(x)}{P_t^{(\rho)}(o)}$ is a bounded martingale for any $x \in B(o, \frac{1}{2}r)$ until time τ_r, we consider:

$$\begin{aligned} \mathrm{d}\frac{P_t^{(\rho)}(x)}{P_t^{(\rho)}(o)} =& P_t^{(\rho)}(o)^{-1}\mathrm{d}P_t^{(\rho)}(x) + P_t^{(\rho)}(x)\mathrm{d}\left(P_t^{(\rho)}(o)^{-1}\right) + \mathrm{d}\left\langle P_t^{(\rho)}(o)^{-1}, P_t^{(\rho)}(x)\right\rangle_t \\ =& P_t^{(\rho)}(x)\left(\mathrm{d}\left(P_t^{(\rho)}(o)^{-1}\right) - \pi P_t^{(\rho)}(o)^{-1}\int_{\Omega_t}\rho(y)P_t(y)P_t^{(\rho)}(y)\mathrm{d}y\mathrm{d}t\right) \qquad (4.101) \\ &+ Q_t^{(\rho)}(x)\left(P_t^{(\rho)}(o)^{-1}\mathrm{d}\xi_t + \mathrm{d}\left\langle \xi_t, P_t^{(\rho)}(o)^{-1}\right\rangle\right). \qquad (4.102) \end{aligned}$$

Since this is a martingale for any x, and $P_t^{(\rho)}$ and $Q_t^{(\rho)}$ are clearly linearly independent functions of x, both of (4.101) and (4.102) (and thus each bracket in these two lines) must be local martingales and thus have vanishing finite variation parts. By standard stochastic calculus arguments (applying Itô's formula to describe $\mathrm{d}P_t^{(\rho)}(o)^{-1}$ from (4.100)), the finite variation part of the second bracket is

$$\frac{2}{(P_t^{(\rho)}(o))}\lambda_t - 2\frac{Q_t^{(\rho)}(o)}{(P_t^{(\rho)}(o))^2}, \tag{4.103}$$

this implies that $\lambda_t = \frac{Q_t^{(\rho)}(o)}{P_t^{(\rho)}(o)}$, as desired.

The following lemma together with continuity of λ_t implies the uniqueness of solutions to the SDE (4.97) by Novikov's condition.

Lemma 4.4.16. *There is a constant $C = C(\|\rho\|_\infty, \mathrm{Diam}(\Omega)) < \infty$ such that the drift λ_t almost surely satisfies*

$$\int_0^\infty |\lambda_t|^2 \leq C, .$$

Proof. We start by noting that

$$\begin{aligned} P_t^{(\rho)}(o) = \mathbb{E}_{o\to a_t;\Omega_t}[\exp(-\int_0^\sigma \rho(X_s)\mathrm{d}s] &\geq \exp(-c_0\|\rho\|_\infty \mathrm{Diam}(\Omega)^2) \\ &\geq \exp[-\|\rho\|_\infty \mathbb{E}_{o\to a_t;\Omega_t}(\sigma)]. \end{aligned}$$

We claim that $\mathbb{E}_{o\to a_t;\Omega_t}(\sigma) \leq c_0 \mathrm{Diam}(\Omega)^2$. To see this, note that (for instance using the Doob transform description of Brownian motion conditioned to leave Ω_t by a_t),

$$\mathbb{E}_{o\to a_t;\Omega_t}(\sigma) = \int_{\Omega_t} G_{\Omega_t}(o,y)\frac{P_t(y,a_t)}{P_t(o,a)}\mathrm{d}y.$$

Moreover, one can deduce from (4.91) and conformal invariance that

$$G_t(o,y)\frac{P_t(y,a_t)}{P_t(o,a_t)} \le c_0(1+G_t(o,y)) \le c_0(1+G_\Omega(o,y))$$

so that

$$\mathbb{E}_{o\to a_t;\Omega_t}(\sigma) \le c_o(\int_\Omega 1+G_\Omega(o,y)\mathrm{d}y) \le c_0' \operatorname{Diam}(\Omega)^2$$

as claimed.

Furthermore, by the resolvent equation for $Q_t^{(\rho)}$ (4.90) and $Q_t(o)=0$ by definition, we have

$$Q_t^{(\rho)}(o) = -\int_\Omega G_t^{(\rho)}(o,y)\rho(y)Q_t(y)\mathrm{d}y \tag{4.104}$$

Combining these two estimates together and using (4.93), we get

$$\int_0^\infty \left|\frac{Q_t^{(\rho)}(o)}{P^{(\rho_t)}(o)}\right|^2 \mathrm{d}t \lesssim \int_0^\infty \left(\int_\Omega P_t(z)\mathrm{d}x\right)^2 \mathrm{d}t \le \frac{1}{\pi}\int_\Omega\int_\Omega G_\Omega(z,w)\mathrm{d}z\mathrm{d}w\,,$$

as shown in the proof of (4.95), and where the hidden constant depends only on $\operatorname{Diam}(\Omega)$ and $\|\rho\|_\infty$ and the final integral is bounded by a constant only depending on $\operatorname{Diam}(\Omega)$. □

This concludes the proof of Theorem 4.4.1. □

4.5 Scaling limit of the LERW with drift and conformal covariance

In this section we explain how Theorem 4.4.1 implies Theorem 4.1.6. We also discuss why this implies convergence of the height function in the associated dimer models and why these satisfy conformal covariance (Theorem 4.1.8). To do this we will rely on our discrete Girsanov theorem (more precisely Corollary 4.1.5). We will in particular need to check that the assumptions of Theorem 4.4.1 are satisfied not only on the directed triangular lattice, but also on the **image** of this lattice under a conformal map.

Remark 4.5.1. While in principle possible, trying to work directly with quantities associated to the random walk with drift poses serious difficulty since the formal analogues of many statements (e.g. 4.90) have well-posedness issues stemming from worse regularity properties of the operator $\frac{1}{2}\Delta + \alpha\cdot\nabla$ compared to $\frac{1}{2}\Delta - \rho$.

Let us begin with the proof for the triangular lattice.

Proof of Theorem 4.1.6. We want to apply Theorem 4.4.1. Recall that $\mathbb{P}^{(\varphi)}$ denotes the law of a random walk on Ω^δ with drift $\alpha(v) = \alpha^\delta(v)$ where $\alpha_k^\delta(v) = \varphi(v+\delta\tau^k) - \varphi(v)$ (the transition probabilities of the walk are described in (4.7)). By Corollary 4.1.5 the law $\mathbb{P}^{(\varphi)}$, conditioned so that $X^\delta_{\sigma^\delta} = a^\delta$, has the same law as the random walk $\mathbb{P}^{(\rho^\delta)}$ with mass

$$\rho^\delta = \Delta^{\delta\mathbb{T}}\varphi + \frac{1}{3}\beta^2,$$

also conditioned so that $X^\delta_\sigma = a^\delta$ (and in particular to survive until doing so).

We need to check that this random walk $\mathbb{P}^{(\rho^\delta)}$ satisfies the conditions of the theorem 4.4.1. First note that $\rho^\delta(v) = \delta^2\rho(v)/2 + o(\delta^2)$ uniformly in v, where $\rho(z) = \frac{1}{2}(\Delta\varphi(z) + \|\nabla\varphi(z)\|^2)$ as in the statement of Theorem 4.1.6.

As already noted in Section 4.1.9, this implies that $(X^\delta_{2t\delta^{-2}}))_{t\geq 0}$ converges weakly, uniformly on compacts, to the law $\mathbb{P}^{(\rho)}$ of Brownian motion killed at the instantaneous rate $\rho(x)$ when in position $x \in \Omega$.

Secondly we need to check that there is a constant c_0 such that

$$\mathbb{E}^{(0)}_{\Omega^\delta, o^\delta \to a^\delta}(\sigma^\delta) \leq c_0 \delta^{-2} R^2 \,. \tag{4.105}$$

This is an estimate purely for the simple random walk on the triangular lattice and follows as in [CW19, Corrollary 2.8].

It remains to check the uniform crossing assumption (4.64). In order to prove that $\mathbb{P}^{(\rho^\delta)}_z(\mathsf{Cross}_r) \geq c$ for some uniform constant $c > 0$ we will in fact consider the restricted event $G = \mathsf{Cross}_r \cap \{\tau \leq \delta^{-2}\}$, where τ is the stopping time at which the walk first leaves the relevant rectangle.

$$\mathbb{P}^{(\rho^\delta)}_x(\mathsf{Cross}_r) \geq \mathbb{P}^{(\rho^\delta)}_x(G) = \mathbb{E}^{(0)}_z(1_G \prod_{s=0}^{\tau}(1-\rho^\delta(X_s)) \tag{4.106}$$

$$\geq \mathbb{P}^{(0)}_z(G)(1-\|\rho^\delta\|_\infty)^{\delta^{-2}} \tag{4.107}$$

$$= (1+o(1))\exp(-c\|\rho\|_\infty)\mathbb{P}^{(0)}_x(G). \tag{4.108}$$

The statement follows since $\mathbb{P}^{(0)}_z(G)$ is uniformly bounded below. Thus the assumptions of Theorem 4.4.1 are fulfilled and Theorem 4.1.6 follows. □

Now to prove 4.1.8 we will first show conformal covariance for massive SLE$_2$.

Theorem 4.5.2 (Conformal covariance for SLE$_2$ with mass profile)**.** *Let Ω be a simply connected domain, and $\rho : \Omega \to [0,\infty)$ be a bounded and continuous mass profile. Let $T : \Omega \to \tilde{\Omega}$ be a conformal map such that $|T'|$ is uniformly bounded away from 0 and ∞ on Ω. Then the image of radial massive SLE$_2$ from $a \in \partial\Omega$ to $o \in \Omega$, with mass profile ρ, under T is given by radial massive SLE$_2$ from $T(a)$ (seen as an element of the Martin boundary) to $T(o)$ with mass profile $|(T^{-1})'(\cdot)|^2\rho(T^{-1}(\cdot))$.*

Proof. The strategy of this proof is as follows: Let X^δ be the random walk on the directed triangular lattice with mass profile ρ^δ approximating ρ as in the previous theorem. Consider the random walk $T(X^\delta)$, which is a random walk on the **image** of the directed triangular lattice $T(\Omega^\delta)$ under T. Note that this is also a planar graph. We will aim to apply Theorem 4.4.1 to this walk and so need to check that the conditions of the theorems are also fulfilled.

Noting that condition 2 on the expected time to leave the domain, does not depend on the embedding of the graph, so it follows directly from what we proved above. It remains to check the other two assumptions. The fact that it converges to a time changed massive Brownian motion follows from the standard conformal invariance of Brownian motion. Indeed if B_t is standard Brownian motion, then so is $\tilde{B}_t := T(B_{\xi^{-1}(t)})$, where $\xi(t) = \int_0^t |T'(B_s)|^2 \mathrm{d}s$. Let σ_Ω be the time at which B_t leaves Ω and $\tilde{\sigma}_{\tilde{\Omega}}$ be the time at which $\tilde{B}_t$ leaves $\tilde{\Omega}$. By definition, $\xi(\tau_\Omega) = \tilde{\tau}_{\tilde{\Omega}}$. Now consider the Radon–Nikodym derivative of a massive Brownian motion in Ω with profile ρ with respect to standard Brownian motion and rewrite to in terms of $\tilde{B}$:

$$\exp\left(-\int_0^{\tau_\Omega} \rho(B_s)\mathrm{d}s\right) = \exp\left(-\int_0^{\tau_\Omega} \rho(T^{-1}(\tilde{B}_{\xi(s)})\mathrm{d}s\right) \tag{4.109}$$

$$= \exp\left(-\int_0^{\tau_{\tilde{\Omega}}} \rho(T^{-1}(\tilde{B}_s))|T'(T^{-1}(\tilde{B}_s)|^2\mathrm{d}s\right) \tag{4.110}$$

$$= \exp\left(-\int_0^{\tau_{D'}} \rho(T^{-1}(\tilde{B}_s))|((T^{-1})'(\tilde{B}_s)|^{-2}\mathrm{d}s\right). \tag{4.111}$$

Since the condition in Theorem 4.4.1 is convergence of paths up to time reparametrization, this shows that assumption 1 holds. The uniform crossing estimate (3) follows the fact that this assumption is invariant under conformal maps using the Koebe one-quarter theorem and the uniform control over $|T'|$ in our assumption. □

To prove Theorem 4.1.8 we first see how Theorem 4.1.6 implies convergence of the dimer height function when the weights are given by (4.4).

Proposition 4.5.3. *Consider the directed triangular lattices with weights* (4.4)*. Let $h^{(\alpha),\delta}$ denote the height function of the biperiodic dimer on the dimer graph G^δ (a piece of the hexagonal lattice). Then $h^{(\alpha),\delta}$ converges, in the sense that if f is a test function, then*

$$(h^{(\alpha),\delta}, f) \to (h^{(\alpha)}, f)$$

converges in law and in the sense of moments. Here $h^{(\alpha),\delta}$ is identified with a function defined on all Ω which is constant on each face of G^δ, and the inner product above is simply the L^2 inner product of square integrable functions.

Proof. The convergence of the loop-erased random walk in Theorem 4.1.6, applied iteratively using Wilson's algorithm, implies the convergence of the uniform spanning tree $\mathcal{T}$ with weights (4.4) in the Schramm topology ([Sch99]). Recall that this tree is identical to the tree one obtains from applying the Temperley bijection to the biperiodic dimer model with weights (4.4). We apply a general theorem (Theorem 8.1 in [BLR19]) in order to deduce convergence of the height function. The theorem, which follows the approach originating in [BLR20], is particularly simple to apply on simply connected domains, which is our situation. The assumptions of that theorem in this simplified situations are as follows:

- There exists $c > 0$ such that the following holds. For any vertex $v \in v(\Omega^\delta)$, for any interior point $z \in \Omega$, if $r = |v - z| \wedge \text{dist}(v, \partial\Omega) \wedge \text{dist}(z, \partial\Omega)$ and if γ is the loop-erasure of the random walk starting from v and killed when it leaves Ω, then for any $0 < \epsilon < 1$,

$$\mathbb{P}_v^{(\varphi)}(\gamma \cap B(z, r\epsilon) \neq \emptyset) \leq \epsilon^c, \tag{4.112}$$

  in other words γ is polynomially unlikely to enter a small ball near z.

- There exists $C, c > 0$ and for every $k \geq 1$ there is a constant M_k such that the following holds. For any $v \in v(\Omega^\delta)$, let γ denote the loop-erasure of the random walk starting from v and killed when it leaves Ω, parameterized from v to $\partial\Omega$. For all $r > 0$, let θ_r denote the first time it leaves $B(v, r)$ and σ_r the last time it is in $B(v, er)$. For $s < t$, let $W(\gamma[s, t])$ denote the intrinsic winding of the path $\gamma([s, t])$ (that is, on a graph where all edges are straight, the sum of the turning angles of γ during that interval of time). Then for every $k \geq 1$,

$$\mathbb{E}_v^{(\varphi)}[\sup_{\theta_r \leq s \leq t \leq \sigma_r} |W(\gamma[s, t])|^k] \leq M_k, \tag{4.113}$$

  in other words the winding of the path γ at any scale r is of order one.

The proofs in [BLR20] of both these facts for the random walk on Ω^δ relies on nothing but the uniform crossing estimate of (4.64); in fact Proposition 4.4 of [BLR20] and Proposition 4.12 of [BLR20] are stated for general random walks on embedded planar graphs subject to the uniform crossing estimate (convergence to Brownian motion is also assumed throughout that section, but plainly that assumption is only used to identify the law of the limit of loop-erased random walk). Hence Proposition 4.4 of [BLR20] applies and yields (4.112); and Proposition 4.12 of [BLR20] also applies and yields uniform stretched exponential tails hence (4.113). This completes the proof of Proposition 4.5.3.

It is also possible to deduce (4.112) and (4.113) from the Proposition 4.4 of [BLR20] and Proposition 4.12 of [BLR20] (applied to the usual driftless random walk on the square lattice) and the fact that the Radon-Nikodym derivative in Corollary 4.1.5 is uniformly bounded by $\exp(2\sup_{x\in\Omega}|\varphi(x)|)$.

Let us see how this may be used to finish the proof of (4.112) and (4.113). Consider for instance (4.112).

$$\begin{aligned}\mathbb{P}_v^{(\varphi)}(\gamma\cap B(z,r\epsilon)\neq\emptyset) &= \mathbb{E}_v^{(0)}[1_{\{\gamma\cap B(z,r\epsilon)\neq\emptyset\}}e^{\varphi(X_\sigma)-\varphi(X_0)-\frac{1}{2}A_\sigma}]\\ &\leq \mathbb{P}_v^{(0)}[\gamma\cap B(z,r\epsilon)\neq\emptyset]\exp(2\sup_{x\in\Omega}|\varphi(x)|)\end{aligned}$$

so using Proposition 4.4 of [BLR20] we obtain (4.112). The same argument also implies (4.113).

This concludes the proof of Proposition 4.5.3. □

Proof of Theorem 4.1.8. We are now ready to finish the proof of Theorem 4.1.8. All that remains to prove is the conformal covariance of the limiting height function $h^{(\alpha);\Omega}$ (here we write explicitly the dependence on the domain Ω in order to avoid confusions). Let $\tilde{\Omega}$ be another bounded simply connected domain and let $T:\Omega\to\tilde{\Omega}$ be a conformal map with bounded derivative. Recall that we wish to show

$$h^{(\alpha);\Omega}\circ T^{-1}=h^{(\tilde{\alpha});\tilde{\Omega}}$$

where at a point $w\in\tilde{\Omega}$,

$$\tilde{\alpha}(w)=\overline{(T^{-1})'(w)}\cdot\alpha(T^{-1}(w)). \tag{4.114}$$

The idea is to use the same approach as in Theorem 4.5.2, i.e. using both the convergence as in Proposition 4.5.3 and the same type of result on the lattice obtained by the image of G^δ under T. Indeed since the connection with the massive random walk (i.e. Theorem 4.1.5) does not depend on the embedding, the analogue of Theorem 4.1.6 for the random walk on the deformed triangular lattice is an immediate consequence of Theorem 4.5.2. The scaling limit of the corresponding random walk is necessarily the image by T of a Brownian motion with drift α in Ω. Applying Itô's formula and the Cauchy–Riemann equations, one checks that α and $\tilde{\alpha}$ are related via (4.114).

Likewise (4.112) and (4.113) are trivially verified in $\tilde{\Omega^\delta}$ because they are verified in Ω^δ and T has bounded derivative. The dimer model associated to $T(G^\delta)$ is the image by T of the dimer model on G^δ and has a height function which necessarily converges to $h^{(\alpha);\Omega}\circ T^{-1}$ in $\tilde{\Omega}$. On the other hand, the law of the limiting Temperleyan tree is uniquely determined by the law of its branches, which by Theorem 4.1.6 are off-critical radial SLE_2 with limiting drift vector field $\tilde{\alpha}$, as described in (4.19). We conclude that, in law,

$$h^{(\alpha);\Omega}\circ T^{-1}=h^{(\tilde{\alpha}),\tilde{\Omega}},$$

as desired. □

4.A Continuum hitting probabilities

The following well-known proposition is recalled as Proposition 3.3 of [YY11] and can be proved using the fact that for two dimensional Brownian motion $\log(|B_t|)$ is a local martingale and the inequality $\log(1-r)\leq -r$.

Lemma 4.A.1. *Let $\mathbb{D}$ be the unit disc and let $x \in \mathbb{D}$ be different from 0. Let $0 < \epsilon < |x|$. Let σ be the exit time of X_t from the unit disc $\mathbb{D}$. Then*

$$\mathbb{P}_x(\exists t \in [0, \sigma] : |X_t| < \epsilon) \geq \frac{1 - |x|}{\log(1/\epsilon)}, \tag{4.115}$$

We need to replace this with a suitable analogue for massive Brownian motion.

Lemma 4.A.2. *Suppose $\Omega = \mathbb{D}$ is the unit disc. There exists a constant $c > 0$ such that the following holds. Let $x \in \Omega$ be different from 0 and $0 < \epsilon < |x|$. Let σ be the exit time of X_t from the disc. Then*

$$\mathbb{P}_x^{(\rho)}(\exists t \in [0, \sigma \wedge \sigma_*] : |X_t| < \epsilon) \geq c \frac{1 - |x|}{\log(1/\epsilon)}. \tag{4.116}$$

Proof. Suppose without loss of generality that $\varepsilon = e^{-N}$ for some $N \geq 1$. Writing down the Radon–Nikodym derivative with respect to ordinary Brownian motion, and letting σ_ϵ being the first time the trajectory enters $B(0, \epsilon)$, we get

$$\begin{aligned} \mathbb{P}_x^{(m)}(\sigma_\epsilon < \sigma) &= \mathbb{E}_x^{(0)}\left(1_{\{\sigma_\epsilon < \sigma\}} \exp(-\int_0^{\sigma_\epsilon} \rho(X_s) ds)\right) \\ &\geq \mathbb{E}_x^{(0)}\left(\exp(-\sigma_\epsilon \|\rho\|_\infty) \Big| \sigma_\epsilon < \sigma\right) \mathbb{P}_x(\sigma_\epsilon < \sigma). \end{aligned}$$

Thus it remains to show

$$\mathbb{E}_x\left(\exp(-M^2 \sigma_\epsilon) \Big| \sigma_\epsilon < \sigma\right) \geq c, \tag{4.117}$$

for some constant c, where $M^2 = \|\rho\|_\infty$. A priori, the difficulty is that conditioning the Brownian motion to hit a very small ball might cause the process to waste a lot of time and thus make it highly likely to be killed (or equivalently make the exponential term very small). We will see this is not the case; essentially, when we condition planar Brownian motion to hit zero before leaving the unit disc, it does so in an a.s. finite time.

Let $\sigma_0 = \inf\{t > 0 : |B_t| = e^k \text{ for some } k \in \mathbb{Z}\}$, and define inductively a sequence of stopping times σ_n by setting

$$\sigma_{n+1} = \inf\{t > \sigma_n : |B_t| = e^k \text{ for some } k \in \mathbb{Z} \text{ with} |B_t| \neq |B_{\sigma_n}|\}.$$

In words, the sequence σ_n corresponds to the sequence of times at which $|B_t|$ is of the form e^k for some distinct k.

Let $M_n = \log_r(|B_{\sigma_n}|)$. Because $\log |x|$ is a harmonic function on $\mathbb{R}^2$ and rotational invariance of Brownian motion, it is easy to see that M_n is nothing but simple random walk on $\mathbb{Z}$ with a possibly random initial value M_0 which however differs from $\log |x|$ by at most 1. Let θ_ϵ denote the first n such that $M_n \leq -N$ (recall that we have assumed $\epsilon = e^{-N}$, so θ_ϵ corresponds to Brownian motion entering $B(0, \epsilon)$). Let θ be the smallest n such that $M_n \geq 0$ (which corresponds to Brownian motion leaving the unit disc).

Now let us describe the effect of conditioning on $\sigma_\epsilon < \sigma$ (or equivalently $\sigma_\epsilon < \sigma$). The conditional transition probabilities are well known and easy to compute (this can be viewed as an elementary version of Doob's h-transform). Writing $\tilde{\mathbb{P}}$ for the conditional probability measure given $\theta_\epsilon < \theta$, we obtain for $-N + 1 \leq k \leq -1$,

$$\tilde{\mathbb{P}}(M_{n+1} = k \pm 1 | M_n = k) = \frac{1}{2}(1 \mp \frac{1}{|k|}). \tag{4.118}$$

Note that this description is actually independent of N (or equivalently ϵ). The formalism of electrical networks is useful to describe the conditional walk defined by (4.118) (which, up to the sign, is essentially a discrete version of a three-dimensional Bessel process, and is in particular transient). To put it in this framework, note that (4.118) coincides with the walk on the network with conductances $c(k,k-1) = \binom{|k|+1}{2}$. Indeed in that case the corresponding stationary measure is then

$$\pi(k) = \binom{|k|+1}{2} + \binom{|k|}{2} = k^2$$

after simplification, so that $c(k,k-1)/\pi(k)$ coincides with (4.118) as desired. The corresponding unit current voltage $v(k) = \frac{2}{|k|}$ (if we set zero voltage at $-\infty$ and unit voltage at 1), which means that the expected number of visits to k is exactly $2|k|$ if we let the conditioned walk (4.118) live forever. We deduce that

$$\tilde{\mathbb{E}}(\#\{n \le \theta : M_n = k\}) \le 2|k|. \tag{4.119}$$

(This can also be computed directly using elementary computations based on the gambler's ruin probability, and considering the probability from k that the conditioned walk ever returns to k).

Now let us decompose

$$\sigma_\epsilon - \sigma_0 = \sum_{n=0}^{\theta_\epsilon - 1}(\sigma_{n+1} - \sigma_n) = \sum_{j=1}^{N-1}\sum_{m=1}^{\infty} 1_{\{N_j^m < \theta_\epsilon\}}(\sigma_{N_j^m+1} - \sigma_{N_j^m}) \tag{4.120}$$

where for $1 \le j \le N-1$ and $m \ge 1$, $n = N_j^m$ is the time of the mth visit to level $-j$ by the martingale M_n. We will check that the conditional expectation of the left hand side, given $\theta_\varepsilon < \theta$, remains finite as $\varepsilon \to 0$.

Let $\mathcal{F}$ denote the σ-algebra generated by all the random variables of the form $X_{\sigma_n}, 0 \le n \ne N$. Note that the event $\theta_\varepsilon < \theta$ is measurable with respect to $\mathcal{F}$, and that given $\mathcal{F}$, the trajectory of $(X_t, 0 \le \sigma_\varepsilon)$ may be split in pieces of the form $X[\sigma_n, \sigma_{n+1}]$, which are *independent* of one another, and where each piece may be described as a Brownian motion starting from X_{σ_n} conditioned to exit a certain annulus $A_n = B(0, e^{M_n+1}) \setminus B(0, e^{M_n-1})$ through $X_{\sigma_{n+1}}$. Now, if A is any annulus of the form $B(0, e^{k+1}) \setminus B(0, e^{k-1})$ and $y \in A$ is any interior point, $z \in \partial A$ is any point on the boundary of the annulus A, then it is not hard to see for some constant $C > 0$, by Brownian scaling,

$$\mathbb{E}_y(\sigma_A | X_{\sigma_A} = z) \le Ce^{2k} \tag{4.121}$$

where σ_A is the time at which X leaves A, and this estimate is uniform in $y \in A, z \in \partial A$, and $k \in \mathbb{Z}$. Consequently,

$$\mathbb{E}(\sigma_{n+1} - \sigma_n | \mathcal{F}) \le Ce^{2M_n}. \tag{4.122}$$

This implies that $\tilde{\mathbb{E}}(\sigma_0) \leq C < \infty$. Furthermore, using (4.120)

$$\begin{aligned}\tilde{\mathbb{E}}(\sigma_\epsilon - \sigma_0) &= \sum_{j=1}^{N-1}\sum_{m=1}^{\infty} \tilde{\mathbb{E}}\left[1_{\{N_j^m < \theta_\epsilon\}}(\sigma_{N_j^m+1} - \sigma_{N_j^m})\right] \\ &= \sum_{j=1}^{N-1}\sum_{m=1}^{\infty} \tilde{\mathbb{E}}\left[1_{\{N_j^m < \theta_\epsilon\}}\tilde{\mathbb{E}}[(\sigma_{N_j^m+1} - \sigma_{N_j^m})|\mathcal{F}]\right] \\ &\leq \sum_{j=1}^{N-1}\sum_{m=1}^{\infty} \tilde{\mathbb{E}}\left[1_{\{N_j^m < \theta_\epsilon\}}Ce^{-2j}\right] \\ &\leq C\sum_{j=1}^{N-1} e^{-2j}\tilde{\mathbb{E}}(\#\{n \leq \theta : M_n = j\}) \\ &\leq C\sum_{j=1}^{N-1} j^2 e^{-2j}.\end{aligned}$$

Here we used (4.122) in the third line, and (4.119) in the last line. The right hand side is uniformly bounded in N (or equivalently ε). We deduce that $\mathbb{E}(\sigma_\epsilon|\sigma_\epsilon < \sigma) \leq C$ for some constant C independent of x. Therefore, using Jensen's inequality and convexity of $x \mapsto e^{-x}$, we get

$$\mathbb{E}_x\Big(\exp(-M^2\sigma_\epsilon)\Big|\sigma_\epsilon < \sigma\Big) \geq \exp(-M^2\mathbb{E}_x(\sigma_\epsilon|\sigma_\epsilon < \sigma)) \geq \exp(-M^2 C),$$

which proves (4.117). This concludes the proof of Lemma 4.A.2. □

4.B Discrete crossing, Beurling estimates

To end this section we conclude with the remaining missing discrete estimates required for the proof of Theorems 4.1.6 and 4.4.1. The first one concerns disconnection events: for $z \in \Omega$, and $r > 0$ such that $B(z, 10r) \subset \Omega$, let us write $x[0,t] \circlearrowleft^{(r)} z$ for the event that the path $x[0,t]$ disconnects $B(z,r)$ from $B(z,5r)^c$ (or, equivalently, makes a noncontractible loop in the corresponding annulus); this is the notation from [YY11]. The next lemma corresponds to Proposition 3.4 in [YY11] although there it is only stated for Brownian motion, although we will need its random walk version.

Lemma 4.B.1. *For every R there exists a z such that the following holds: Let $0 < r \leq R$ and let $z \in \mathbb{C}$. Let T be the exit time of $X(\cdot)$ from $B(z,r)$. Then for every $x^\delta \in B(z, r/2)$,*

$$\mathbb{P}^{(\rho)}_{x^\delta}(X(0,T) \circlearrowleft^{(r)} z) \geq c.$$

Proof. Encircling a point at scale r contains the intersection of ten box-crossing events (see Figure 4.B). We conclude using our crossing assumption (4.64). □

The last missing piece is a Beurling estimate (corresponding to Proposition 4.1. in [YY11]), which shows that a walk starting close to the boundary of a domain is very likely to leave this domain in a short time, without going far from its starting point. Actually what is needed is the version of this estimate in which we want to ensure the random walk will hit a given curve which is close to its starting point; of course, this makes no difference. Such an estimate is well known in the critical case where the walk converges to Brownian motion. This remains true in the off-critical regime thanks to the following observation: while of course the off-critical Brownian motions are not scale invariant, this effect disappears at small scales. In

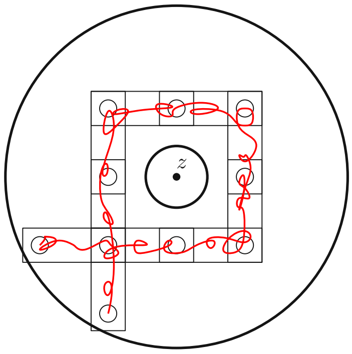

Figure 4.6: Making a loop by crossing rectangles.

fact, making loops at any scale above that separating the curve from the starting point guarantees an intersection, and so we can get a uniform bound using the previous observations. Also, since we assume that the original domain Ω is bounded, we do not need to consider arbitrarily large scales and can therefore obtain uniform bounds for all domains which have diameter less than some constant R.

The desired estimate is formulated in [YY11] after applying a conformal map to the unit disc (let ϕ denote the unit conformal map from Ω to $\mathbb{D}$ such that $\phi(o) = 0$ and $\phi'(o) > 0$). This is initially a little worrying, since we did not assume uniform crossing after applying the conformal map ϕ but instead only in Ω itself. (Note that this uniform crossing estimate could in fact fail to hold for $\phi(\Omega^\delta)$ if the domain Ω is not very nice). Thankfully, we will see that thanks to Koebe's one quarter theorem we can get the required estimate.

Lemma 4.B.2. *For all $\alpha, R > 0$, there exists an $\eta > 0$ such that for all $\tilde\epsilon > 0$, for all simply connected domains Ω such that $0 \in \Omega \subset B(0, R)$, and for all $\tilde a \in (1 - \tilde\epsilon)\not\leqslant$, there exists a δ_0 such that the following holds for all $\delta < \delta_0$:*

Let $y \in v(\Omega^\delta) \cap \phi^{-1}(\rho(\tilde a, \eta\tilde\epsilon)) \in \Omega$. Let X^δ denote random walk on Ω^δ starting from y. Then, for every continuous curve g starting in $B(\tilde a, \eta\tilde\epsilon)$ and ending outside of $B(\tilde a, \tilde\epsilon)$,

$$\mathbb{P}^\delta_y(\phi(X[0, T]) \cap [g] = \emptyset) \leq \alpha$$

where $[g]$ is the range of g and T is the time at which $\phi(X)$ leaves $B(\tilde a, \tilde\epsilon)$.

Proof. Let $\tilde\epsilon > 0$ and let $\tilde a \in (1 - \tilde\varepsilon)\mathbb{D}$. Let $a = \phi^{-1}(\tilde a) \in \Omega$, and let $\varepsilon = |(\phi^{-1})'(\tilde a)|\tilde\varepsilon$; note that we have no control over the actual size of ε since it depends on the conformal map near $\tilde a$. Nevertheless, applying the Koebe 1/4-theorem (twice), it is easy to see that the image of curve g under ϕ^{-1} starts from a ball of radius $4\eta\varepsilon$ around a, and ends outside of a ball of radius $\varepsilon/4$ around a. For $\phi(X^\delta[0, T])$ to avoid g, $X^\delta[0, T]$ must therefore avoid making loops at *all* scales between $4\eta\varepsilon$ and $\varepsilon/4$ (this corresponds to a *fixed* number of scales, even though ε itself is variable). Furthermore, using the strong Markov property, all the events $\circlearrowleft^{(r)}$ *a* occur with fixed positive probability (by Lemma 4.B.1) and independently of one another. By choosing η small enough, this probability can therefore be made smaller than α, uniformly over all the parameters. □

Together these results conclude the convergence of the discrete Poisson kernel and therefore the proof of Theorem 4.4.1.

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Chapter 5

The stochastic six-vertex speed process

abstract

For the stochastic six-vertex model on the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with step initial conditions and a single second-class particle at the origin, we show almost sure convergence of the speed of the second-class particle to a random limit. This allows us to define the stochastic six-vertex speed process, whose law we show to be ergodic and stationary for the dynamics of the multi-class stochastic six-vertex process. The proof follows the scheme developed in [ACG23] for ASEP and requires the development of precise bounds on the fluctuations of the height function of the stochastic six-vertex model around its limit shape using methods from integrable probability. We also obtain a novel result that allows us to control the behavior of an individual second-class particle by controlling the behavior of a geometric number of third-class particles.

5.1 Introduction

5.1.1 Preface

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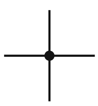
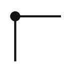
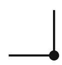
Configuration						
Weight	1	1	b_1	$1 - b_1$	b_2	$1 - b_2$

Figure 5.1: The six allowed configurations for the stochastic six-vertex model

Figure 5.2: A possible sampling of the stochastic six-vertex model on the quadrant with step initial data. The height function is denoted in blue.

The stochastic six-vertex model was first introduced by Gwa and Spohn in [GS92] as a specialization of the six-vertex model, which is a classical model from equilibrium statistical mechanics going back to [Pau35]. Recently there has been a lot of interest in this model. It is connected via a suitable limit degeneration to ASEP [Agg17], the Kardar-Parisi-Zhang equation [CT17, Lin20, CGST20], the stochastic telegraph equation [BG19] and lies in the (one-dimensional) KPZ universality class [GS92, BCG16, ACH24], in particular exhibiting $N^{\frac{1}{3}}$ fluctuations and $N^{\frac{2}{3}}$ correlations on a domain of size N, see [Agg16, AB19, Bor18, BCG16, CD18]. Furthermore, it can be put into the more general setting of higher-spin vertex models, see [CP16, Agg18, BP18].

To define the stochastic six-vertex model we need to specify two parameters $b_1, b_2 \in [0,1]$. Given a subset D of $\mathbb{Z}^2$, a configuration of the stochastic six-vertex model is given by a subset of the edges incident to the vertices in D such that at each vertex a local conservation law is satisfied, namely that the number of edges to the left and the bottom of that vertex equals the number of edges to the top and right. See Figure 5.1 for the six possible configurations at a given vertex. We call the bottom and left edges incident to a vertex its **incoming** edges and the top and right edges its **outgoing** edges. We say that edges are **occupied** if they are in the selected subset, and we sometimes refer to occupied edges as particles or arrows oriented from top to bottom and left to right. Each of the six possibilities is assigned a weight, see Figure 5.1, and the weight of a configuration is given by the product of its vertex weights. For finite D, a configuration is then sampled proportional to its weight, after perhaps specifying some edges as a boundary condition.

We will study this model on the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. On $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the model can be taken to be defined via the following stochastic sampling algorithm, which coincides with taking a limit of the model on finite boxes $[0,N] \times [0,M] \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, see [GS92, BCG16]. First one needs to specify a boundary configuration on the edges incoming from the left at the vertices $\{0\} \times \mathbb{Z}_{\geq 0}$ and from the bottom at the vertices $\mathbb{Z}_{\geq 0} \times \{0\}$. Choose any vertex where both the left and bottom edges have already been determined. In the beginning, the only such vertex is $(0,0)$, but later there will be potentially many such vertices. The law does not depend on this choice.

- If there are two incoming particles then there is only one possibility for the outgoing edges. Set the outgoing edges to be occupied as well, as in configuration **I**. Similarly, if there are zero incoming particles, then set the outgoing edges to be unoccupied as in configuration **II**. Continue by selecting the next vertex.

- If there is a single incoming vertical particle, there are two possible configurations: **III** and **IV**. Choose **III** with probability b_1 and **IV** with probability $1-b_1$.

- Similarly, if there is a single horizontal incoming particle choose configuration **V** with probability b_2 and **VI** with probability $1-b_2$.

If one chooses which vertices to update in an antidiagonal way (i.e. ordered by $x+y$) every vertex will eventually be updated and this defines a law on configurations of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

There is an alternative parameterization of the model by parameters $q, \kappa > 0$ defined as

$$q := \frac{b_1}{b_2} \qquad \kappa := \frac{1-b_1}{1-b_2}\,.$$

This parameterization will be quite useful to us, and these variables will appear in many formulas throughout the paper.

The most common boundary condition that we will work with is one where all incoming edges from the left boundary of the quadrant are occupied and all incoming edges from the bottom boundary are empty. We will refer to this boundary condition as **step initial conditions** in analogy with analogous initial conditions in interacting particle systems. For a given configuration ω of the stochastic six-vertex model with step initial conditions, we define the height function $H(x,t) = H(x,t;\omega)$ for $x,t \in \mathbb{R}_{\geq 0}$ by setting $H(x,0;\omega) = 0$ for all x and increasing H whenever one crosses a path in the vertical direction, see Figure 5.2.

The model exhibits two very different behaviors depending on whether b_1 or b_2 is larger. If $b_1 > b_2$, then particles prefer moving up to moving to the right. Since the upper part of the quadrant is already packed, this leads to a sharp transition between a region with density 1 and a region with density 0, whose boundary stays close to the line $x = t$. This behavior is known as a **shock**. On the other hand if $b_1 < b_2$, particles want to move right more than up, and thus they spread out. Three regions form: one above the line $x = \kappa^{-1}t$, where the density of particles is 1, one below the line $x = \kappa t$, where the density is 0, and one in between, where the density decreases continuously from 1 to 0 (See the right-hand side of Figure 5.4 for a simulation). The middle section is known as the **rarefaction fan**. Both the shock and rarefaction fan regimes are interesting in their own rights, but our results concern the latter: from now on we always assume $b_1 < b_2$.

We will now introduce the multi-class stochastic six-vertex model. Instead of every edge being either occupied or unoccupied it will now be assigned a **class** from $\mathbb{R} \cup \{-\infty, \infty\}$. The classes assigned to the two outgoing edges equal the classes of the incoming edges, and the weight of a vertex depends on the classes, see Figure 5.3. Intuitively if $i < j$ then a particle of class i treats particles of class j as **holes**. The single-class stochastic six-vertex model can be obtained from the multi-class one by setting the class of unoccupied edges to 1 and the class of occupied edges to ∞.

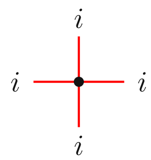
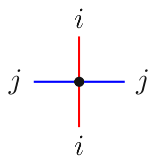
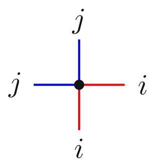
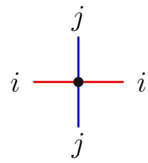
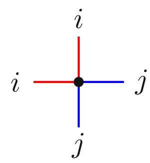
Configuration					
Weight	1	b_1	$1 - b_1$	b_2	$1 - b_2$

Figure 5.3: The allowed configurations for the multi-class stochastic six-vertex model, where red lines represent class i and blue lines represent class j for $i < j$.

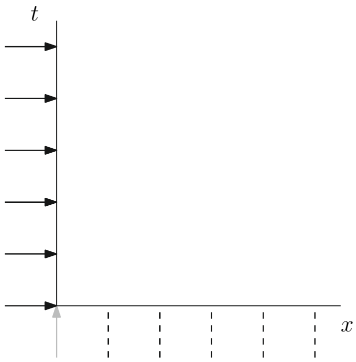
Our main theorem concerns the following variant of the step initial condition, which we will call **step initial conditions with a vertical second-class particle at the origin**. All particles coming in from the left have class 1, there is a single particle coming in from the bottom at $(0, 0)$, and all other incoming particles from the bottom have class ∞, i.e. are holes, see Figure 5.4. By the conservative property of the model, for every t there is exactly one x such that the vertical arrow leaving (x, t) has class 2. We call this x the **position of the second-class particle at time** t and denote it by $\boldsymbol{X}_t$. Our main result states that the speed $\frac{\boldsymbol{X}_t}{t}$ of the second-class particle converges a.s. to a random limit:

Theorem 5.1.1. *Let $0 < b_1 < b_2 < 1$ and consider the stochastic six-vertex model with step initial positions with a vertical second-class particle at the origin. Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t. Then almost surely*

$$\lim_{t \to \infty} \frac{\boldsymbol{X}_t}{t} = \boldsymbol{U} \tag{5.1}$$

where $\boldsymbol{U}$ is a continuous random variable taking values in $[\kappa^{-1}, \kappa]$ with density $\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}}$.

Even the weak convergence of the speed of the second-class particle has not been stated in the literature, to the best of the authors' knowledge. However, it follows readily from the hydrodynamic limit proved in [Agg20] in the same way as for ASEP using the arguments from [FK95]. For the convenience of the reader we adapt this argument to our setting in Appendix 5.A.



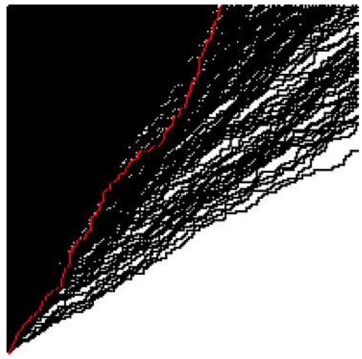


Figure 5.4: On the left: step initial conditions with a vertical second-class particle at the origin. Black arrows denote first-class particles, while the grey arrow denotes the second-class particles. Dashed lines denote holes. On the right: a simulation of this process on a 200 by 200 square with $b_1 = 0.3$ and $b_2 = 0.6$ and with the second-class particle in red.

For TASEP with step initial conditions, the weak convergence of the speed of a second-class particle at the origin was first proven in [FK95] and a.s. convergence was proven in [MG05] (see also [FP05] and [FMP09] for alternative proofs). For the Hammersley process, a.s. convergence of the speed of a second-class particle at the origin was proven in [CP07] under suitable initial conditions, and for the totally asymmetric zero range process (TAZRP), this was proven in [Gon14]. All of these proofs rely crucially on connections between the models under consideration and last passage percolation models (for example, TASEP can be coupled with exponential last passage percolation). Since this no longer holds for ASEP, new tools were required to prove the analogous result for ASEP under step initial conditions, and this was done in [ACG23] using inputs from integrable probability as well as a coupling due to Rezakhanlou [Rez95]. Since for the stochastic six-vertex model last passage methods also do not apply, our proof strategy for Theorem 5.1.1 is inspired by the ideas in [ACG23]. The speed of second-class particles for ASEP and the Hammersley process has also been studied for other classes of initial conditions in [CP13, GSZ19, FGN19].

We also derive a bound on the fluctuations around the limiting speed:

Theorem 5.1.2. *Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t as above and $\boldsymbol{U}$ its almost sure limit. Then for any $\delta > 0$, almost surely we have that*

$$\lim_{t\to\infty} |\boldsymbol{X}_t - t\boldsymbol{U}| t^{-(\frac{7}{9}+\delta)} = 0\,. \tag{5.2}$$

Remark 5.1.3. For the stochastic six-vertex model with stationary initial conditions, the fluctuations are of order $t^{\frac{2}{3}}$, see e.g. [Agg16, LS23], so the best exponent one could achieve in the above expression is $-(\frac{2}{3}+\delta)$, see also Remark 5.5.9.

For ASEP and TASEP the fluctuations at stationarity are also of order $t^{\frac{2}{3}}$ [QV07, BS10], but the fluctuations of the speed of a second-class particle around its eventual limit speed are also not known. Our techniques can also be used for ASEP, where they would give an analogous result to Theorem 5.1.2 for ASEP.

Going beyond adding a single second-class particle into our model, we can consider initial conditions where each incoming particle has a different class in $\mathbb{Z} \cup \{-\infty, \infty\}$. Individually, each particle will have an asymptotic speed given by Theorem 5.1.1. By considering the joint speeds of all the particles simultaneously, we can construct the **stochastic six-vertex model speed process**. Speed processes have previously been constructed and studied for TASEP [AAV08], TAZRP [ABGM21], and ASEP [ACG23]. To define the speed process, we first need to discuss how we can view the stochastic six-vertex model as a particle system, as was first done in [GS92], see also [BCG16, Section 2.2].

5.1.2 The stochastic six-vertex model as an interacting particle system

Until this point, we have treated the stochastic six-vertex model as a measure on configurations consisting of oriented edges. However, it is also natural to consider it as a particle system, as has already been quite noticeable in the language we have been using and was already observed in [GS92]. Let us now introduce notation that emphasizes this connection. For a given configuration ω on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ define $\eta_t(x)$ for $x \in \mathbb{Z}_{\geq 0}$ by

$$\eta_t(x) = \begin{cases} 1 & \text{if the incoming vertex at } (x,t) \text{ from below in } \omega \text{ is occupied} \\ 0 & \text{else.} \end{cases}$$

Defined like this $(\eta_t)_{t \in \mathbb{Z}_{\geq 0}}$ is a Markov process with values in $\{0,1\}^{\mathbb{Z}_{\geq 0}}$. We call this a **stochastic six-vertex process**. The boundary conditions on the bottom give the initial condition η_0 and the boundary conditions on the left inject particles at specific times. The transition probabilities of this process can be described as follows: Particles stay in place with probability b_1 and start moving to the right with probability $(1-b_1)$. If a particle starts moving, the amount it moves is the minimum of a $\mathrm{Geo}(b_2)$ distributed random variable and the distance to the nearest particle to its right. If it moves to the location of the neighboring particle to the right, that other particle then starts moving, following the above described rules. See [BCG16, Section 2.2] for these transition weights written out in more detail.

We now define the height function in this setting and show that it generalizes the definition of $H(x,t)$ above for the case of step initial conditions.

Definition 5.1.4 (Height Function)**.** *For a given stochastic six-vertex process $(\eta_t)_{t \geq 0}$, the height function $h_t(x) = h_t(x;\eta)$ is the unique function (up to a global shift) that satisfies*

$$h_t(x;\eta) - h_t(x+1;\eta) = \eta_t(x) \tag{5.3}$$

$$h_{t+1}(0;\eta) - h_t(0;\eta) = \begin{cases} 1 & \textit{if there is an incoming arrow from the left at } (0,t) \\ 0 & \textit{else.} \end{cases} \tag{5.4}$$

Since the height function is only unique up to a global shift, unless otherwise specified the choice of height function is made by setting $h_0(0) = 0$, but in some places it will be convenient to choose some other $h_0(0)$. For a configuration ω of the stochastic six-vertex model with step initial conditions, one recovers the definition of $H(x,t)$ above, since by (5.3), $h_0(x) = 0$ for all x.

Definition 5.1.5. *As shown in [Agg20], these dynamics can be extended to processes $\eta_t : \mathbb{Z} \to \{0,1\}$. We call this the* ***stochastic six-vertex process on the line****.*

Given an initial condition $\eta_0 : \mathbb{Z} \to \{0,1\}$ that satisfies $\eta_0(x) = \mathbf{1}_{x<0}$, the restriction $(\eta_t(x))_{x,t \in \mathbb{Z}_{\geq 0}}$ of the stochastic six-vertex process on the line to $x \geq 0$ agrees with the process on the quadrant with step initial conditions. It is this process that we will be considering in Sections 5.2 to 5.6. The height function is still defined by (5.3) and (5.4).

This extension is also compatible with the multi-class stochastic six-vertex process. While the single-class processes $\eta_t : \mathbb{Z} \to \{0,1\}$ have occupation variables in $\{0,1\}$ with 0 encoding holes and 1 encoding particles, we will let the multi-class processes have occupation variables in $\mathbb{Z} \cup \{\infty\}$, with ∞ encoding holes and all other values encoding particles of different classes. In other words, we define the **multi-class stochastic six-vertex process on the line** as

$\eta_t : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$, where $\eta_t(x) = i$ if at time t, there is a particle of class i at position x To avoid confusion, we will always specify in the text whether we are considering a single- or multi-class process.

We can now define the speed process whose existence will be obtained as a corollary of Theorem 5.1.1.

Corollary 5.1.6 (Existence of the speed process)**.** *Consider the multi-class stochastic six-vertex model on the line with initial conditions $\eta_0(x) = x$ for all $x \in \mathbb{Z}$, i.e. at position x there is a particle of class x. We call this* ***packed initial conditions****. Denote by $\boldsymbol{X}_t(x)$ the position of the unique particle of class x at time t. Then the process $\left(\frac{\boldsymbol{X}_t(x)}{t}\right)_{x\in\mathbb{Z}}$ converges a.s. as $t \to \infty$ to a process $U(x)$. We call $U(x)$ the* ***stochastic six-vertex model speed process****.*

Now that the stochastic six-vertex model speed process is defined, we can study some of its properties. In Section 5.8, we will prove that the speed process is ergodic and stationary with respect to the dynamics of the multi-class stochastic six-vertex model. Assuming uniqueness of multi-class stationary measures with a given marginal for the stochastic six vertex model, this, together with recent results from [ANP23] implies that the stochastic six vertex speed process is related to the ASEP speed process by a deterministic map.

There are also many avenues for further work on these processes. In particular the article [BSS22] shows that the suitably rescaled TASEP speed process converges weakly to a process known as the stationary horizon. The stationary horizon was first introduced in [Bus23] and is expected to be a universal scaling limit for multi-class invariant measures of models in the KPZ universality class. Then in [BSS24], they develop a more general framework to show convergence to the stationary horizon. In particular, they show that if a model converges to the directed landscape under suitable rescaling, then the stationary measures of the associated multi-class process converge to the stationary horizon at the level of finite-dimensional projections. In [ACH24] they prove the convergence of the stochastic six-vertex model and ASEP to the directed landscape, and hence using the results from [BSS24], they obtain as a corollary [ACH24, Corollary 2.14] that the stationary measures for the multi-class ASEP converge to marginals of the stationary horizon. By the above discussion, these stationary measures are the same as for the multi-class stochastic six-vertex model. It is still an open problem to prove convergence of the ASEP and stochastic six-vertex model speed processes to the stationary horizon in the space $D(\mathbb{R}, C(\mathbb{R}))$.

5.1.3 Proof Ideas

The proof of the main theorem uses a variety of tools. We follow the general strategy developed in [ACG23], which requires certain model-specific inputs that have not yet been developed for the stochastic six-vertex model. In particular, we need the following two ingredients, which are the key novelties of this paper:

- A statement that a second-class particle to the right of any number of third-class particles will at any fixed time be overtaken by at most a geometric number of third-class particles.

- Effective hydrodynamic estimates that quantify how close the height function of the stochastic six-vertex model started from step initial conditions will be to its limit shape.

These results will be used in the following way. We want to control the behavior of a single second-class particle. Hydrodynamic theory allows us to control the bulk behavior of many particles, so we augment our system by filling up all empty positions to the left of $\boldsymbol{X}_t$ with third-class particles. We then use our effective hydrodynamic estimates to control the union of the second- and third- class particles. Finally, we can revert this back to an estimate on the position of the second-class particle since we know that our second-class particle is to the left of at most a geometric number of the third-class particles. A similar argument can be made to bound the position of the second-class particle from the left.

We now state these two results in detail. The first will be the content of Proposition 5.1.7 and the second, the content of Propositions 5.1.8 and 5.1.9.

5.1.4 Controlling a Second-Class Particle by Third-Class Particles

The following proposition allows us to control the behavior of a single second-class particle by controlling the behavior of a large number of third-class particles inserted to the left of the second-class particle.

Recall that $q = \frac{b_1}{b_2}$. By $X \sim \mathrm{Geo}(q)$ we denote the law given by

$$\mathbb{P}[X = k] = (1 - q)q^k \text{ for } k \geq 0\,.$$

Proposition 5.1.7. *Let $(\eta_t)_{t\geq 0}$ be a multi-class stochastic six-vertex process on the line with parameters $0 < b_1 < b_2 < 1$ and with the following initial conditions:*

- *There are some first-class particles (finitely or infinitely many).*
- *There is a single second-class particle.*
- *There are M third-class particles, all to the left of the second-class particle.*

Let $\boldsymbol{Z}_t(0) > \boldsymbol{Z}_t(1) > \cdots > \boldsymbol{Z}_t(M)$ be the ordered positions of the second- and third-class particles at time t. Further, let L_t be number of third-class particles to the right of the second-class particle at time t. Then for any t the law of L_t, conditioned on both $\boldsymbol{Z}$ and the space-time history of the first-class particle is dominated by $\mathrm{Geo}(q)$.

Let us briefly compare this result with Rezakhanlou's coupling from [Rez95], which was used to control a second-class particle in ASEP in [ACG23]. In [Rez95] an auxiliary label process on the second and third-class particles is defined, which has the following properties.

- Every second- and third-class particle has a unique label from 0 to M, which can change over time.
- The law of this labeling process at any fixed time is that of a uniform permutation, and it is stationary.
- It is coupled to the dynamics of the multiclass ASEP, such that at any time, the particle with label 1 is to the left of the single second-class particle.

This allows us to control the second-class particle with a uniformly chosen third-class particle, see [AB19, (5.4)].

One can construct an analogous coupling for the stochastic six-vertex model,[1] but only for the case $b_1 < \frac{1}{2}$. Proposition 5.1.7 takes a different approach and works for all $b_1 < b_2$. There are two key differences between these approaches: Firstly, Proposition 5.1.7 does not

[1]Such a coupling was presented by Ivan Corwin at the 2022 PIMS-CRM Summer School in Probability.

proceed via a coupling. Secondly, the bound in Proposition 5.1.7 is significantly stronger for large M. Intuitively, the result from [Rez95] shows that the number of third-class particles that *do not* pass the second-class particle grows linearly in the number of third-class particles, while Proposition 5.1.7 shows that the number that *do* pass is of order 1.

Since the statement of Proposition 5.1.7 is entirely insensitive to scaling time or space, it can be carried over to ASEP, with $q = \frac{b_1}{b_2}$ fixed. For ASEP this result could also be obtained from the censoring inequality [PW13].

5.1.5 Tail Bounds for the Height Function

In this subsection, we state effective hydrodynamic estimates for the fluctuations of the height function $H(x,t)$ of the stochastic six-vertex model with step initial conditions. To do so we first state the law of large numbers for H.

With probability one it holds that

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g(x,y), \qquad \forall x,y \in \mathbb{R}_{\geq 0}. \tag{5.5}$$

where for $b_1 \leq b_2$, we have

$$g(x,y) = \begin{cases} y - x & \text{if } \frac{x}{y} \leq \kappa^{-1} \\ \frac{\left(\sqrt{x}-\sqrt{\kappa y}\right)^2}{\kappa - 1} & \text{if } \kappa^{-1} < \frac{x}{y} < \kappa \\ 0 & \text{if } \frac{x}{y} \geq \kappa \end{cases} \tag{5.6}$$

and for $b_1 \geq b_2$, we have

$$g(x,y) = \begin{cases} 0 & \text{if } x \geq y \\ y - x & \text{if } x \leq y. \end{cases}$$

This was proven at the level of weak convergence in [BCG16] and [Agg20] and was strengthened to almost sure convergence in [DL23].

Let $g(x) := g(x,1)$. We prove the following two tail bounds on the fluctuations of the height function H around its limit shape g.

Proposition 5.1.8. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa^{-1} - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1} e^{-cs^{\frac{3}{2}}}, \tag{5.7}$$

and c can be chosen to weakly decrease in ε.

Proposition 5.1.9. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa^{-1} - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \leq g(\mu)T - sT^{1/3}\right] \leq c^{-1}(e^{-cs} + e^{-cT}),$$

and c can be chosen to weakly decrease in ε.

Remark 5.1.10. The power $T^{\frac{1}{3}}$ on the left-hand side of Propositions 5.1.8 and 5.1.9 is optimal, since on this scale the fluctuations of the height function have been shown to converge to the Tracy-Widom GUE distribution, see [BCG16, Theorem 1.2]. The optimal exponents on the right-hand side however, are expected to be s^3 for Proposition 5.1.8 and $s^{3/2}$ for Proposition 5.1.9 as was obtained for the longest increasing subsequence of a permutation in [LM01, LMR02]. The parameters μ_1 and μ_2 need to be bounded away from the edge of the rarefaction fan in order to obtain a uniform constant $c(\varepsilon)$.

We call Proposition 5.1.8 a “lower tail” bound since it corresponds to the lower tail of the Tracy-Widom distribution. Similarly, we call Proposition 5.1.9 an “upper tail” bound. The reason that the upper tail decays more slowly than the lower tail is because in order for the height function to be smaller than expected, we just need the position of the right-most path/particle in the stochastic six-vertex model to be small. On the other hand for the height function to be larger than expected, we must have that the positions of many paths/particles are large. Since this requires more deviations to occur, it has a smaller probability.

We prove the lower tail bound by using an identity from [Bor18] that expresses the q-Laplace transform of the height function in terms of an expectation with respect to the law of the Meixner ensemble. This identity allows us to bound the upper tail for the height function by the lower tail of the position of the smallest hole in the Meixner ensemble. The Meixner ensemble is a determinantal point process, so this tail can be expressed as a Fredholm determinant, which we then bound using Widom's trick [Wid02]. The upper tail bound is more straightforward. We directly express the q-Laplace transform of the height function in terms of a Fredholm determinant and use Fredholm determinant estimates from [AB19].

Tail estimates for the height function of the stochastic six-vertex model have previously been obtained in [LS23] for stationary initial conditions. The recent work [DLM24] obtains a large deviation principle for the stochastic six-vertex model with step initial conditions, whereas our results are in the “moderate deviation” regime There is also an upcoming work [GS] that will prove tight tail bounds in the moderate deviations regime.

5.1.6 Proof Sketch

We now sketch the proof of Theorems 5.1.1 and 5.1.2 using the above two ingredients. To show that the speed $\frac{\boldsymbol{X}_t}{t}$ converges a.s., we will introduce a sequence of times S_n and prove that as long as we are not too close to the edge of the rarefaction fan, then with high probability,

$$\left|\frac{\boldsymbol{X}_{S_n}}{S_n} - \frac{\boldsymbol{X}_{S_{n+1}}}{S_{n+1}}\right| \leq S_n^{-\gamma} \tag{5.8}$$

for some positive γ.

For this to imply convergence of the sequence $\frac{\boldsymbol{X}_{S_n}}{S_n}$, we need the right-hand side to be summable. For general times $S_n \leq t \leq S_{n+1}$, one can then use the monotonicity of $\boldsymbol{X}_t$ to bound $\left|\frac{\boldsymbol{X}_t}{t} - \frac{\boldsymbol{X}_{S_n}}{S_n}\right|$ as long as the sequence S_n does not grow too quickly. We will take the sequence $S_{n+1} = S_n + T(S_n) := S_n + S_n^{\frac{7}{9}}$ and prove (5.8) for this sequence in Proposition 5.5.2.

To prove Proposition 5.5.2, we want to control the behavior of the second-class particle after some large initial time S_0. However, the effective hydrodynamic bounds in Propositions 5.1.8 and 5.1.9 only allow us to control the behavior of a large number of particles, not of an individual one since they are mesoscopic statements as opposed to microscopic ones. Therefore, we fill up all empty positions to the left of $\boldsymbol{X}_S$ with third-class particles and control the union of the second- and third-class particles by Propositions 5.1.8 and 5.1.9. Letting $T = T(S) = S^{\frac{7}{9}}$, Proposition 5.1.7 will guarantee that only a small number of these third-class particles will be to the right of $\boldsymbol{X}_{S+T}$ at time $S+T$, so that controlling the union of the second- and third-class particles gives us a bound on $\boldsymbol{X}_{S+T}$.

We split the proof of (5.8) into an upper and a lower bound, which are treated analogously. Proposition 5.1.7 reduces the lower bound to showing that a large number of these second- and third-class particles are to the right of $\boldsymbol{X}_S + \frac{\boldsymbol{X}_S}{S}T - S^{1-\gamma}$ at time $S+T$. To do so denote by $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ the augmented (single-class) stochastic six-vertex model containing the union of all first-, second- and third-class particles and by $\boldsymbol{\mathcal{B}}^{(1)}$ the process with only the first-class

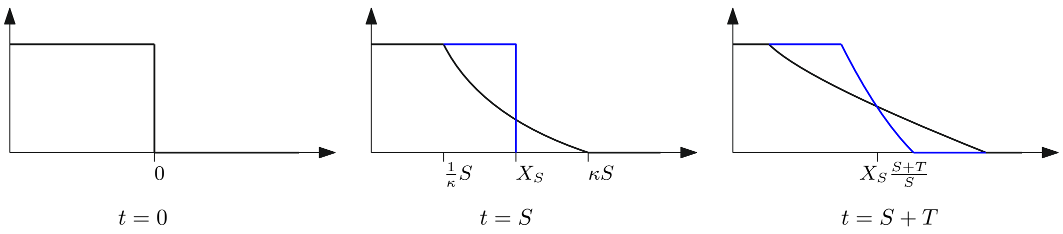


Figure 5.5: A sketch of the densities of the processes $\boldsymbol{\mathcal{B}}^{(1)}$ in black at times $0, S$ and $S+T$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ in blue at times S and $S+T$. At time S the process $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ is given exactly by the maximum of the two processes $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$, while at time $S+T$ it is at least the maximum of $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$.

particles. Additionally, we introduce an auxiliary third process $\boldsymbol{\mathcal{B}}^{\text{step}}$ which is started at time S from the initial condition $\boldsymbol{\mathcal{B}}^{\text{step}}_S(x) = \mathbf{1}_{x \le \mathbf{X}_S}$. At time S these three processes satisfy

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_S(x) = \max(\boldsymbol{\mathcal{B}}^{(1)}_S(x), \boldsymbol{\mathcal{B}}^{\text{step}}_S(x)) \,. \tag{5.9}$$

The multi-class stochastic six-vertex process allows us to couple $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ such that at any later time $S+T$ it holds that $\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+t}(x) \ge \boldsymbol{\mathcal{B}}^{\text{step}}_{S+t}(x)$. Since $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ and $\boldsymbol{\mathcal{B}}^{(1)}$ are already coupled in such a way, this implies that for any $t \ge 0$

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+t}(x) \ge \max(\boldsymbol{\mathcal{B}}^{(1)}_{S+t}(x), \boldsymbol{\mathcal{B}}^{\text{step}}_{S+t}(x)) \,. \tag{5.10}$$

Note that this also couples $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ in some non-trivial way. See Figure 5.5 for a sketch of the particle densities of the processes $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ at times $0, S$, and $S+T$.

By using the effective hydrodynamic estimates together with a recent approximate monotonicity result from [ACH24], we show that with high probability $\boldsymbol{\mathcal{B}}^{(1)}$ is still close to the hydrodynamic limit at time $S+T$, uniformly over all possible configurations of $\boldsymbol{\mathcal{B}}^1_S$ when on a certain event H_S, which also occurs with high probability. Since the process $\boldsymbol{\mathcal{B}}^{\text{step}}$ is started from step initial conditions, it is also close to a hydrodynamic limit at time $S+T$, which is obtained by translating the hydrodynamic limit for standard step initial conditions. By the coupling above

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+T}(x) - \boldsymbol{\mathcal{B}}^{(1)}_{S+T}(x) \ge \boldsymbol{\mathcal{B}}^{\text{step}}_{S+T}(x) - \boldsymbol{\mathcal{B}}^{(1)}_{S+T}(x). \tag{5.11}$$

Using the hydrodynamic estimates for the two processes on the right-hand side, this gives a lower bound for the number of third-class particles to the right of $\frac{X_S}{S}(S+T) - S^{1-\gamma}$, as desired.

Remark 5.1.11. While the general strategy outlined above is similar to the strategy employed in [ACG23], we would like to highlight the following differences:

- The choice of time steps S_n is different than the choice in [ACG23] and is optimized to allow us to also prove the more refined fluctuation result in Theorem 5.1.2. See Remark 5.5.9 for further discussion.

- The fact that Proposition 5.1.7 does not get worse with the number of particles (as compared to Rezakhanlou's coupling) allows us to fill in all empty positions to the left of the second-class particle with third-class particles. In [ACG23] only a small number of positions were filled, which made it necessary to deal with more complicated "φ-distributed" Bernoulli initial conditions and introduced a further approximation step.

- For ASEP, monotonicity is a straightforward consequence of the basic coupling. However, for the stochastic six-vertex model, the basic coupling is not monotone, and we instead need to use a recent result from [ACH24] which gives an approximate form of monotonicity for the basic coupling, see Proposition 5.2.6.

5.1.7 Structure

In section 5.2, we recall some couplings and properties of the stochastic six-vertex model, including the approximate monotonicity result from [ACH24] which is stated in Proposition 5.2.6. The two core ingredients are proved in Sections 5.3 and 5.4 respectively—in Section 5.3 we prove Proposition 5.1.7 and in Section 5.4 we prove Propositions 5.1.8 and 5.1.9.

These results are then used in Sections 5.5, 5.6 and 5.7 to prove the main theorem. In order these sections show that

- the main theorem follows if one can show that with high probability the second-class particle does not deviate too much from its current speed in a given time frame,
- which follows if one can show that the augmented progress with additional third-class particles does not deviate too much from its hydrodynamic limit with high probability,
- which follows from the effective hydrodynamics from Section 5.4 together with approximate monotonicity.

Finally in Section 5.8 the existence of the speed process is deduced from Theorem 5.1.1, and some of its properties are found using recent results from [BB19, ANP23].

5.1.8 Notation

Throughout the paper, many floor functions are dropped when we consider large integers. We use

$$[\![A, B]\!] = [A, B] \cap \mathbb{Z}$$

for intervals of integers.

Our convention for geometric random variables is that a random variable $X \sim \mathrm{Geo}(q)$ satisfies

$$\mathbb{P}[X = k] = (1 - q)q^k \,.$$

We consider both single-class and multi-class processes by considering their occupation variables. Single-class processes have occupation variables in $\{0, 1\}$ with 0 encoding holes and 1 encoding particles, while multi-class processes have occupation variables in $\mathbb{Z} \cup \{\infty\}$, with ∞ encoding holes and all other values encoding particles of different classes.

The parameters b_1 and b_2 are fixed throughout the paper and therefore all constants can depend on them freely even if this is not explicitly mentioned.

5.2 The basic coupling

We consider the following construction of the single-classt stochastic six-vertex model, which also allows us to couple multiple stochastic six-vertex models with varying boundary conditions. We will first state it on the quadrant.

Definition 5.2.1 (Basic Coupling). *We will construct a coupling using two independent families $(\chi^1(x,t))_{x,t\geq 0}$ and $(\chi^2(x,t))_{x,t\geq 0}$ of i.i.d. Bernoulli(b_1) and Bernoulli(b_2) random variables respectively. Given such random variables, we can sample the stochastic six-vertex* *model in the following way. If at a given vertex there are either two incoming arrows or no incoming arrows then there is only a single possible outcome. If there is a single incoming vertical arrow at (x,t) and $\chi^1(x,t)=1$, then the the outgoing arrow is vertical. If $\chi^1(x,t)=0$, then the outgoing arrow is horizontal. Similarly, if there is a single incoming vertical arrow and $\chi^2(x,t)=1$, then the outgoing arrow is vertical. If $\chi^2(x,t)=0$, then the outgoing arrow is horizontal.*

Given boundary conditions on the left and bottom edge of $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ the random variables $(\chi^1(x,t))_{x,t\geq 0}$ and $(\chi^2(x,t))_{x,t\geq 0}$ uniquely define a configuration, which can be obtained by updating the vertices along the anti-diagonal lines $\{(x,t): x+t=k\}$ with increasing k. Note also that the order of updates does not matter. Using the same $(\chi^1(x,t))_{x,t\geq 0}$ and $(\chi^2(x,t))_{x,t\geq 0}$ for different boundary conditions gives a coupling of stochastic six-vertex models, which we call the ***basic coupling****.*

This coupling was used in [ACH24]. Before we recall several properties of this coupling, let us show how it can be used to define the stochastic six-vertex-process on the line, in a way that is similar to both the construction in [Agg20, Section 2.1] using a different coupling of the stochastic six-vertex model and to the graphical construction of Harris for ASEP on $\mathbb{Z}$ in [Har78].

Proposition 5.2.2 (Extension to $\mathbb{Z}$)**.** *The construction in Definition 5.2.1 can be extended to the domain $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$. More specifically given two independent families $(\chi^1(x,t))_{x\in\mathbb{Z},t\geq 0}$ and $(\chi^2(x,t))_{x\in\mathbb{Z},t\geq 0}$ of i.i.d. Bernoulli(b_1) and Bernoulli(b_2) and any boundary conditions on the incoming edges of $\mathbb{Z}\times\{0\}$, there is almost surely a unique configuration on $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ that is coherent with the boundary conditions and that at each vertex satisfies the rules in Definition 5.2.1, i.e. if there is only one incoming arrow, the configuration at the vertex (x,t) is given by the values of $\chi^1(x,t)$ and $\chi^2(x,t)$. Furthermore, the law of this unique configuration is given by the stochastic six-vertex model.*

Proof. We will construct the configuration line by line. Consider first the random variables $\chi^1(x,0)$ and $\chi^2(x,0)$. We call a vertex $(x,0)$ such that $\chi^1(x,0)=\chi^2(x,0)=0$ a **cut-vertex**. Almost surely, there are infinitely many cut-vertices both to the left and to the right of the origin since each vertex $(x,0)$ has an independent positive probability of $(1-b_1)(1-b_2)$ to be a cut-vertex. Notice that at a cut-vertex, the outgoing horizontal edge is occupied if and only if the incoming vertical edge is occupied, and the outgoing vertical edge is occupied if and only if the incoming horizontal edge is occupied. Therefore, if $(x_0,0)$ and $(x_1,0)$ with $x_0<x_1$ are cut-vertices, the configuration of all vertices $(x,0)$ with $x_0<x\leq x_1$ is determined by the incoming arrows at these vertices and the Bernoulli variables $\chi^1(x,0)$ and $\chi^2(x,0)$ for $x_0\leq x\leq x_1$. Therefore on the probability 1 event that there are cut-vertices infinitely far to the left, the configuration is uniquely determined. □

Again, using the same Bernoulli random variables for different initial conditions gives a coupling of stochastic six-vertex processes. Let us now consider several properties of this coupling starting with attractivity.

As mentioned in the introduction, we will use the notation $(\eta_t(x))_{x\in\mathbb{Z},t\geq 0}$ for the occupation variables, i.e. $\eta_t(x)=1$ if the vertical incoming edge is occupied. The initial conditions are then given by a function $\eta_0(x):\mathbb{Z}\to\{0,1\}$.

Lemma 5.2.3 (Attractivity)**.** *Given a collection of initial conditions η_0^k for $k=1,\dots,n$, such that $\eta_0^i(x)\leq\eta_0^j(x)$ for $i\leq j$ and all $x\in\mathbb{Z}$, under the basic coupling it will hold that $\eta_t^i(x)\leq\eta_t^j(x)$ for all $t\in\mathbb{Z}_{\geq 0}$ and $x\in\mathbb{Z}$.*

Proof. Let us consider η^i and η^j. Assume that the desired property is true until updating a specific vertex. If at this vertex the incoming arrows are identical for η^i and η^j, by the coupling the outgoing arrows will also be identical. If they are not, since the property holds for all the previous steps, either there are two incoming arrows in η^i or no incoming arrows in η^j. In either case the the outgoing arrows will also still satisfy the desired condition. □

Remark 5.2.4. Note that the basic coupling with initial conditions η_0^k for $k = 1, \ldots, n$, such that $\eta_0^i(x) \le \eta_0^j(x)$ for $i \le j$ and all $x \in \mathbb{Z}$, exactly corresponds to the $n+1$-class stochastic six-vertex model with classes $\{1, \ldots, n, \infty\}$ in the following way. Define

$$\eta_t^{\text{mult}}(x) = \min\{i \in \{1, \ldots n\} : \eta_t^i(x) = 1\}\,,$$

where the convention is used that the minimum of the empty set is ∞. By considering the possible situations at a single vertex, one easily checks that η_t^{mult} is a multi-class stochastic six vertex process.

The attractivity property also has the following analogue for the multi-class process.

Lemma 5.2.5 (Merging)**.** *Let $(\eta_t)_{t\in\mathbb{Z}_{\ge 0}}$ be a multi-class stochastic six-vertex model with classes in $\mathbb{Z} \cup \{-\infty, \infty\}$, i.e. $\eta_t : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, \infty\}$. Then for any weakly increasing function $\phi : \mathbb{Z} \cup \{-\infty, \infty\} \to \mathbb{Z} \cup \{-\infty, \infty\}$, the process $(\phi \circ \eta_t)_{t\in\mathbb{Z}_{\ge 0}}$ is also a multi-class stochastic six-vertex model.*

Proof. This is an immediate consequence of the weights in Figure 5.3 only depending on the incoming classes i and j via their ordering. Consider a vertex for which an update is about to be performed. If the two incoming classes i and j are equal, they will also be equal after applying the map, and in either case, there is exactly one outcome which then of course has probability 1. If the two incoming classes i and j are different, i.e. $i < j$ (note that we do not assume whether i is the horizontal or vertical incoming arrow), then either $\phi(i) < \phi(j)$ or $\phi(i) = \phi(j)$. In the first case, there are two possible outcomes for both a vertex with incoming arrows i and j and a vertex with incoming arrows $\phi(i)$ and $\phi(j)$ and the probabilities match, since the relative order of the incoming arrows is the same. In the second case there are two possible outcomes before applying ϕ but only one outcome after applying ϕ. Since the two possibilities before applying ϕ are complementary, their probabilities sum up to 1, which is the probability of the one possible outcome after applying ϕ. □

Recall that given a stochastic six-vertex process $(\eta_t(x))_{x\in\mathbb{Z},t\ge 0}$, there is a height function $h_t(x;\eta)$ defined up to a global shift defined in Definition 5.1.4 The following proposition is Lemma D.3 of [ACH24].

Proposition 5.2.6 (Approximate Monotonicity)**.** *Consider two single-class initial conditions $\eta^1 : \mathbb{Z} \to \{0,1\}$ and $\eta^2 : \mathbb{Z} \to \{0,1\}$ both with at most N particles . Further consider height functions $h_t(x;\eta^1)$ and $h_t(x;\eta^2)$ satisfying $h_t(x;\eta^1) = h_t(x;\eta^2) = 0$ for x large enough. If $M \ge (logN)^2$ and $|h_0(x;\eta^1) - h_0(x;\eta^2)| < K$ for all $x \in \mathbb{Z}$, and $t \ge 0$, then with probability at least $1 - c^{-1}e^{-cM}$, and for all $x \in \mathbb{Z}$ it holds that*

$$|h_t(x;\eta^1) - h_t(x;\eta^2)| \le K + M\,.$$

Remark 5.2.7. In [ACH24] this is stated without the absolute value. However, the basic coupling has the following property: If (η^1, η^2) are two stochastic six-vertex processes coupled using the basic coupling so are (η^2, η^1). (This is a property that the monotone coupling in [Agg20, Proposition 2.6] does not have). Additionally the conditions on η^1 and η^2 are symmetric and therefore the statement with the absolute value follows from the statement without the absolute value by a simple union bound.

Another property that we will need is a special case of [ACH24, Lemma D.4], and the proof is quite similar to [Agg20, Proposition 2.17].

Proposition 5.2.8 (Finite Speed of Discrepancies)**.** *There exists a constant $c = c(b_2) > 0$ depending only on b_2 such that the following holds. Consider two particle configurations η_0 and ξ_0 with height function $h_0(x;\eta)$ and $h_0(x;\xi)$which are equal on some interval $[\![A, B]\!]$. Then, under the basic coupling, with probability at least $1 - c^{-1}e^{-cT}$ it holds that $h_t(x;\eta) = h_t(x;\xi)$ for all $t \le T$ and all $x \in \left[\!\!\left[A + \frac{2T}{1-b_2} + 1, B\right]\!\!\right]$.*

Using Propositions 5.2.8 and Propositions 5.2.6 together, we can show that given two initial conditions with height functions close on an interval, the height functions will stay close on a smaller interval for some time.

Lemma 5.2.9 (Approximate Monotonicity on Intervals)**.** *There exists a constant $c = c >$ 0, depending only on $b_1, b_2 \in (0, 1)$, such that the following holds. Consider two particle configurations η_0 and ξ_0 with height functions $h(x;\eta_0)$ and $h(x;\xi_0)$ such that for $x \in [\![A, B]\!]$ we have $|h_0(x;\xi) - h_0(x;\eta)| \le K$. Let $M \ge \log(B - A)^2$. Then we can couple them such that with probability at least $1 - c^{-1}(e^{-cT} + e^{-cM})$ it holds that $|h_T(x;\xi) - h_T(x;\eta)| \le 3K + M$ for all $x \in [\![A + \frac{2T}{1-b_2} + 1, B]\!]$.*

Proof. This will follow from Propositions 5.2.6 and 5.2.8. Let $\widetilde{\eta}_0$ be the particle configuration obtained from η_0 by setting

$$\widetilde{\eta}_0(x) = \begin{cases} 0 & \text{if } x < A \\ \eta(x) & \text{if } x \in [\![A, B]\!] \\ 0 & \text{if } x > B\,, \end{cases} \tag{5.12}$$

and define $\widetilde{\xi}_0$ in the same way. Couple $\eta, \xi, \widetilde{\eta}$ and $\widetilde{\xi}$ all with one basic coupling (i.e. all using the same iid Bernoulli random variables). Let the height functions $h_0(x;\widetilde{\eta})$ and $h_0(x;\widetilde{\xi})$ be chosen such that $h_0(B;\widetilde{\eta}) = h_0(B;\widetilde{\xi}) = 0$, i.e. $h_0(x;\widetilde{\eta}) = h_0(x;\eta) - h_0(B,\eta)$ for $x \in [\![A, B]\!]$ and the same for ξ. Note that $h_0(x;\eta) - h_0(B,\eta)$ is a height function for η_0, and therefore by applying Proposition 5.2.8 twice, once for η and once for ξ and a union bound, we obtain that

$$h_t(x,\widetilde{\eta}) = h_t(x,\eta) - h_0(B,\eta) \text{ and } h_t(x,\widetilde{\xi}) = h_t(x,\xi) - h_0(B,\xi), \text{ for} \tag{5.13}$$

holds for all $t \le T$ and all $x \in [\![A + \frac{2T}{1-b_2}, B]\!]$ with probability at least $1 - c^{-1}e^{-cT}$.

Further note that at time 0, for all x

$$|h_0(x;\widetilde{\eta}) - h_0(x;\widetilde{\xi})| = |(h_0(x;\eta) - h_0(B,\eta)) - (h_0(x;\xi) - h_0(B;\xi))| \le 2K\,. \tag{5.14}$$

Therefore we can apply Proposition 5.2.6 to $\widetilde{\xi}$ and $\widetilde{\eta}$ since they are coupled with the basic coupling. Indeed both $\widetilde{\xi}$ and $\widetilde{\eta}$ have at most $B - A$ particles each, so we will have with probability at least $1 - c^{-1}e^{-cM}$ that

$$|h_T(x;\widetilde{\eta}) - h_T(x;\widetilde{\xi})| \le 2K + M \text{ for all } x \in \mathbb{Z}. \tag{5.15}$$

By a union bound, with probability at least $1 - c^{-1}(e^{-cT} + e^{-cM})$ both events (5.13) and (5.15) take place. On this event it holds for all $x \in [\![A + \frac{2T}{1-b_2} + 1, B]\!]$ that

$$\begin{aligned} |h_T(x,\eta) - h_T(x,\xi)| &\le |h_T(x,\eta) - h_T(x,\widetilde{\eta}) + h_T(x,\widetilde{\xi}) - h_T(x,\xi) + h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\ &\le |h_T(x,\eta) - h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi}) + h_T(x,\xi)| + |h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\ &= |-h_0(B,\eta) + h_0(B,\xi)| + |h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\ &\le K + 2K + M = 3K + M\,, \end{aligned}$$

where we are used a triangular inequality, (5.13), (5.15) and that $|-h_0(B,\eta)+h_0(B,\xi)| < K$, by the assumption on the height functions at time 0. □

Remark 5.2.10. The factor 3 in the term $3K + M$ in the previous step is an artifact of Proposition 5.2.6 being only stated only for height functions which are 0 far enough to the right. This restriction could easily be removed, which would remove the factor 3. However, for our purposes the above is sufficient.

This property will be used in Proposition 5.7.2, to show that if a stochastic six-vertex process η is close to its hydrodynamic limit at time S, it will still be close to its hydrodynamic limit at time $S + T$ with high probability, even conditioned on its full configuration at time S.

Finally, the stochastic six-vertex model has the following two symmetries which are often used together.

Proposition 5.2.11 (Particle-Hole Inversion)**.** *If we interchange all particles and holes in a stochastic six-vertex process, we obtain another stochastic six-vertex process, but with b_1 and b_2 swapped.*

Proposition 5.2.12 (Space Inversion)**.** *If we exchange the two coordinate axes in a stochastic six-vertex process, we obtain another stochastic six-vertex process, but with b_1 and b_2 swapped.*

Proof. Both of these can be seen by looking at what happens to the six configurations in Figure 5.1 under this inversion. □

Using both of these symmetries on the quadrant, which is symmetric with respect to the line $x = t$, we obtain a symmetry of one stochastic six-vertex model with itself. In particular one can see that the law of the stochastic six-vertex model started from step initial conditions on the quadrant is invariant after applying both inversions. Furthermore, the step initial condition with a single particle coming in at the origin from the left is dual to step initial conditions with a single particle coming in at the origin from the bottom. Therefore it suffices to prove the main theorem for this kind of initial condition.

5.3 Number of overtaking third-class particles

The purpose of this subsection is to prove Proposition 5.1.7 which will allow us to control an individual second-class particle by controlling a large number of third-class particles.

Proof. As stated we will condition both on the paths of the first-class particles and on $\boldsymbol{Z}$ and prove the statement for any given realization of these. After conditioning on the paths of the first-class particles, one can run the stochastic six-vertex dynamics as follows: Assume all vertices (x,t) with $t \leq t_0$ and $x \in \mathbb{Z}$ have already been updated. Let $x_{\min} := Z_{t_0}(M)$ and $x_{\max} := Z_{t_0}(0)$ be the position of left-most and the right-most second- or third-class particles, respectively. Since we have conditioned on the paths of the first-class particles, the configurations of the vertices $(x, t_0 + 1)$ for $x < x_{\min}$ are already determined. Starting with $x = x_{\min}$ one can update each vertex $(x, t_0 + 1)$ sequentially. The only time the result of this update is random is when the incoming particles are the second-class particle and a third-class particle. All other updates are determined by either the paths of the first-class particles or $\boldsymbol{Z}$. Therefore after updating $(x_{\max}, t_0 + 1)$ all remaining vertices $(x, t_0 + 1)$ are determined. One can then continue with the next line $(x, t_0 + 2)$.

Using this system of updating we will redefine L_t to refer to the number of third-class particles to the right of the second-class particle after t updates have been performed. The sequence L_t where t now refers to an update is a refinement of the original sequence L_t indexed by times t since there are (potentially) multiple updates performed between times t and $t+1$. Therefore, proving that L_t is dominated by $\mathrm{Geo}(q)$ for the refined sequence of updates will give the desired result for times t.

How can L_t change when updating a vertex? It will only change when the incoming particles are exactly a second- and a third-class particle. In this case, the two incoming particles must be in positions $Z_t(k)$ and $Z_t(k+1)$ for some $0 \le k \le M-1$ and L_t is either k or $k+1$. Then L_t changes in the following way:

- If $L_t = k$, then $L_{t+1} = k+1$ with probability b_1 and $L_{t+1} = k$ with probability $1-b_1$.
- If $L_t = k+1$, then $L_{t+1} = k$ with probability b_2 and $L_{t+1} = k+1$ with probability $1-b_2$.

Now let us see how the law of L_t evolves. We will identify laws on $\{0, 1, \ldots, M\}$ with vectors in $\mathbb{R}^{M+1}$ and write $(e_i)_{0\le i\le M}$ for the standard coordinate basis of this space. The law of L_0 is given by e_0 since L_0 is deterministically 0. Let $S(t)$ denote the collection of updates at which the two incoming particles are both either second- or third-class particles, which is given by $\boldsymbol{Z}$. By the above observation, the law of L_t is given by

$$\left(\prod_{k\in S(t)} P_k\right) e_0 \,, \tag{5.16}$$

where the matrices P_k are given by the transition rates above, i.e.

$$P_k = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1-b_1 & b_2 & & \\ & & b_1 & 1-b_2 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} . \tag{5.17}$$

To understand this product we introduce a new basis $(\mathbf{v}_i)_{i=0}^M$. Let $\mathbf{v}_i$ be the vector corresponding to the law of the random variable $\min(i, G)$ where $G \sim \mathrm{Geo}(q)$, i.e.

$$(\mathbf{v}_i)_k := \mathbb{P}\left[\min(i, G) = k\right] = \begin{cases} (1-q)q^k & \text{if } k < i \\ q^i & \text{if } k = i \\ 0, & \text{if } k > i. \end{cases} \tag{5.18}$$

This basis satisfies the following relation with the matrices P_k for all $0 \le k \le M-1$ and $0 \le j \le M$:

$$P_k \mathbf{v}_j = \begin{cases} \mathbf{v}_j & \text{if } j \ne k, k+1 \\ (1-b_2)\mathbf{v}_k + b_2 \mathbf{v}_{k+1} & \text{if } j = k \\ b_1 \mathbf{v}_k + (1-b_1)\mathbf{v}_{k+1} & \text{if } j = k+1\,. \end{cases} \tag{5.19}$$

To see this, first recall that $q = \frac{b_1}{b_2}$ so that $qb_2 = b_1$. We now check each of the three cases in (5.19):

1. $j \ne k, k+1$: Since P_k is equal to the identity matrix in all rows except k and $k+1$ $(P_k\mathbf{v}_j)_i = (\mathbf{v}_j)_i$ for $i \ne k, k+1$. For $j < k$, we have $(\mathbf{v}_j)_k = (\mathbf{v}_j)_{k+1} = 0$ and therefore also $(P_k\mathbf{v}_j)_i = (\mathbf{v}_j)_i$ for $i = k, k+1$. For $j > k+1$, we have $(\mathbf{v}_j)_{k+1} = q(\mathbf{v}_j)_k$ and therefore:

$$(P_k v_j)_k = (1-b_1)(\mathbf{v}_j)_k + b_2(\mathbf{v}_j)_{k+1} = (1-b_1+qb_2)(\mathbf{v}_j)_k = (\mathbf{v}_j)_k \tag{5.20}$$

and

$$(P_k\mathbf{v}_j)_{k+1} = b_1(\mathbf{v}_j)_k + (1-b_2)(\mathbf{v}_j)_{k+1} = \left(\frac{b_1}{q} + 1 - b_2\right)(\mathbf{v}_j)_{k+1} = (\mathbf{v}_j)_{k+1}. \tag{5.21}$$

2. $j = k$: We have $(\mathbf{v}_j)_k = q^k$ and $(\mathbf{v}_j)_{k+1} = 0$. Therefore,

$$\begin{pmatrix} (P_k\mathbf{v}_j)_k \\ (P_k\mathbf{v}_j)_{k+1} \end{pmatrix} = \begin{pmatrix} 1-b_1 & b_2 \\ b_1 & 1-b_2 \end{pmatrix} \begin{pmatrix} q^k \\ 0 \end{pmatrix} \tag{5.22}$$

$$= \frac{b_1}{q} \begin{pmatrix} (1-q)q^k \\ q^{k+1} \end{pmatrix} + \left(1 - b_1 - \frac{b_1(1-q)}{q}\right) \begin{pmatrix} q^k \\ 0 \end{pmatrix} \tag{5.23}$$

$$= b_2 \begin{pmatrix} (1-q)q^k \\ q^{k+1} \end{pmatrix} + (1-b_2) \begin{pmatrix} q^k \\ 0 \end{pmatrix}. \tag{5.24}$$

This suffices since $(\mathbf{v}_k)_j = (\mathbf{v}_{k+1})_j$ for j different from $k, k+1$.

3. $j = k+1$: The calculation is similar to the one above and we omit the details.
Returning to the law of L_t, we can write it as

$$\sum_{i=0}^{M} \lambda_t(i)\mathbf{v}_i \tag{5.25}$$

for some random coefficients $\lambda_t(i)$. Using (5.16) together with (5.19), we see that for any time t, the vector λ_t is the law of a random variable on $\{0, 1, \ldots, N\}$. Letting X be a random variable with this law independent of $G \sim \mathrm{Geo}(q)$, we see that L_t is equal in distribution to $\min(X, G)$, and therefore is dominated stochastically by G. This proves the statement.

Note that M being finite was only used to define the vertex by vertex updates. This assumption can easily be removed. □

Remark 5.3.1. The proof shows that the law of L_t is equal in distribution to the law of the minimum between a geometric random variable and a process X_t, which behaves in the same way as L_t, except that b_1 and b_2 are reversed. This seems to be some kind of duality statement. It would be interesting to see if this is a specific case of some more general duality.

We can also obtain a dual statement to Proposition 5.1.7:

Corollary 5.3.2. *Let $(\eta_t)_t$ be a multi-class stochastic six-vertex process with the following initial conditions:*

- *There are some first-class particles (finitely or infinitely many).*
- *There are M second-class particles.*
- *There is a single third-class particle, to the left of all second-class particles.*

Let L_t be the number of second-class particles to the left of the third-class particle. Then conditioned on the paths of the first-class particles, and the joint paths of the second- and third-class particles, for any $t \geq 0$ the random variable L_t is stochastically dominated by Geo(q).

Proof. In the initial configuration, there are four classes of particles: $\{1, 2, 3, \infty\}$, (recall that holes are considered particles of class ∞). We invert the order of classes so that particles of class 1 become holes, holes become particles of class 1, and the second and third-class particles swap class. Doing this and swapping the x and t coordinates, we obtain a stochastic six-vertex model with the same parameters b_1 and b_2 by Propositions 5.2.11 and 5.2.12. This is now a stochastic six-vertex process on the domain $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$, i.e. the right half plane, which can be defined in the same way as the stochastic six-vertex model on the line. The boundary conditions obtained after these transformations satisfy the hypothesis of Proposition 5.1.7, with the third-class particles being above the second-class particle. The proof then goes through without any changes. □

Remark 5.3.3. While we stated Proposition 5.1.7 and Corollary 5.3.2 for the stochastic six-vertex process on the line, they can also be stated for the stochastic six-vertex model on domains whose boundary is a down-right path. Since the proof takes a vertex-by-vertex approach, it will carry through with minimal changes.

5.4 Effective hydrodynamic estimates

The purpose of this section is to prove Propositions 5.1.8 and 5.1.9. Before doing that, we combine them to prove the following theorem:

Recall that $H(X,T)$ refers to the height function of a stochastic six-vertex model on the quadrant with step initial conditions and that $g(x) = g(x,1)$ is the limit shape of the height function (see (5.5)).

Theorem 5.4.1. *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that the following holds. For any $\mu_1, \mu_2 \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$, and for any $T \geq 1$, $s \in [0,T]$,*

$$\mathbb{P}\left[|H(T\mu_1,T) - H(T\mu_2,T) - (g(\mu_1) - g(\mu_2))\,T| \geq sT^{1/3}\right] \leq c^{-1}e^{-cs}. \tag{5.26}$$

Furthermore, the constant c can be chosen to weakly decrease in ε.

Proof of Theorem 5.4.1. For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ we have the following two bounds from Propositions 5.1.8 and 5.1.9, respectively. There exists a c (that will change from line to line) such that

$$\mathbb{P}\left[H(T\mu,T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{2}{3}}}$$
$$\mathbb{P}\left[H(T\mu,T) \leq g(\mu)T - sT^{1/3}\right] \leq c^{-1}(e^{-cs} + e^{-cT}) \leq 2c^{-1}e^{-cs}.$$

Combining these two bounds, we obtain

$$\mathbb{P}\left[|H(T\mu,T) - g(\mu)T| \geq sT^{1/3}\right] \leq c^{-1}e^{-cs}.$$

It follows from a union bound that

$$\begin{aligned}
&\mathbb{P}\left[|H(T\mu_1,T) - H(T\mu_1,T) - (g(\mu_1) - g(\mu_2))\,T| \geq sT^{1/3}\right] \\
&\leq \mathbb{P}\left[|H(T\mu_1,T) - g(\mu_1)T| \geq \frac{s}{2}T^{1/3}\right] + \mathbb{P}\left[|H(T\mu_2,T) - g(\mu_2)T| \geq \frac{s}{2}T^{1/3}\right] \\
&\leq c^{-1}e^{-cs}.
\end{aligned}$$

This finishes the proof of Theorem 5.4.1. The constant c can be chosen to be weakly decreasing in ε, since this is the case for both Proposition 5.1.8 and Proposition 5.1.9. □

We immediately obtain the following corollary of Theorem 5.4.1:

Corollary 5.4.2. *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that the following holds. For any $T \geq 1$ and for any $s \in [0, T]$,*

$$\mathbb{P}\left[\max_{\mu_1,\mu_2\in[\kappa^{-1}+\varepsilon,\kappa-\varepsilon]} |H(T\mu_1, T) - H(T\mu_2, T) - (g(\mu_1) - g(\mu_2))\, T| \geq sT^{1/3}\right] \leq c^{-1}T^2e^{-cs}, \tag{5.27}$$

and c can be chosen to weakly decrease in ε.

Proof. Notice that there are only finitely many μ_i satisfying $\kappa^{-1} + \varepsilon \leq \mu_i \leq \kappa - \varepsilon$ and such that $T\mu_i$ is an integer. In fact, there are at most κT of them, giving at most $O(T^2)$ possible pairs (μ_1, μ_2). Taking a union bound of (5.26) over all such pairs yields the result. □

Finally, we can quickly extend Proposition 5.1.8 to the case of step Bernoulli boundary conditions, i.e. the incoming arrows from the left are given by i.i.d. Bernoulli(ρ) random variables, while the incoming positions from the bottom are all still empty. Denote these boundary conditions as $(\rho, 0)$-Bernoulli boundary conditions. Even though we don't need this result to prove our main theorem, we state it as a corollary for completeness.

Corollary 5.4.3. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: Let $\rho \in [\varepsilon, 1]$ and let $H^\rho(x, y)$ be the height function for the stochastic six-vertex model on the quadrant with $(\rho; 0)$-Bernoulli boundary conditions. For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H^\rho(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{3}{2}}}. \tag{5.28}$$

Proof. This is a straightforward consequence of the attractivity of the stochastic six-vertex model, by which we can couple the model with $(\rho; 0)$-Bernoulli initial data with the model with step initial data. In this coupling the height function of the model with $(\rho; 0)$-Bernoulli initial data is smaller at every point, and thus the statement follows from Proposition 5.1.8. □

Remark 5.4.4. Proposition 5.1.9 can also be extended to the case of $(\rho, 0)$-Bernoulli boundary conditions as follows: For $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$

$$\mathbb{P}\left[H^\rho(T\mu, T) \leq g(\mu)T - sT^{2/3}\right] \leq c^{-1}e^{-cs}$$

by following the same steps as in [ACG23, Appendix B].

Note that we only obtain $T^{\frac{2}{3}}$ fluctuations here as opposed to the $T^{\frac{1}{3}}$ fluctuations obtained in Proposition 5.1.9. The reason for this is that for $(\rho, 0)$-Bernoulli boundary conditions, the rarefaction fan is $\left\{\frac{x}{t} \in \left[\frac{\kappa}{(\kappa\rho-\rho+1)^2}, \kappa\right]\right\}$, which is smaller than the rarefaction fan for step initial conditions. Therefore, for small μ, the vertex $(T\mu, T)$ is outside the rarefaction fan. Instead, it is in a region where the process is very close to the stationary process given by i.i.d. Bernoulli(ρ) random variables. In such a region the fluctuations of the process are Gaussian, so it is expected that the $\frac{2}{3}$ exponent could be improved to a $\frac{1}{2}$. If one restricts μ to the smaller interval $\left[\frac{\kappa}{(\kappa\rho-\rho+1)^2} + \varepsilon, \kappa - \varepsilon\right]$, the same bound as in Proposition 5.1.9 can be obtained and the proof barely changes.

5.4.1 Proof of Proposition 5.1.8

To prove Proposition 5.1.8, we will make use of a remarkable exact identity that relates the height function of the stochastic six-vertex model to the holes of the Meixner ensemble. We can then reduce the question of studying the tail of the height function to studying the tail for the position of the smallest hole in this determinantal point process. We study this tail by taking asymptotics of the associated kernel.

In this subsection, we define the Meixner ensemble, which is a determinantal point process on $\mathbb{Z}$. We will then relate the q-Laplace transform of the stochastic six-vertex model height function to an expectation with respect to the Meixner ensemble.

We give a brief introduction to the theory of discrete determinantal point processes. Let $\mathfrak{X}$ denote the state space of a single particle, which we will take to be a countable set (for the Meixner ensemble, we will take $\mathfrak{X} = \mathbb{Z}_{\geq 0}$). A subset $X \subseteq \mathfrak{X}$ is called a *point configuration*, and we define $\mathrm{Conf}(\mathfrak{X}) = 2^{\mathfrak{X}}$ to be the set of all possible point configurations.

We define the following Borel sigma algebra for $\mathrm{Conf}(\mathfrak{X})$:

$$\mathcal{B} := \sigma\left(\{X \in \mathrm{Conf}(\mathfrak{X}) : |A \cap X| = n\} : n \in \mathbb{N}, A \subseteq \mathfrak{X} \text{ compact }\right).$$

A probability measure $\mathbb{P}$ on $(\mathrm{Conf}(\mathfrak{X}), \mathcal{B})$ is called a *random point process*. From now on, we will use X to denote this random point process by setting $X : \mathrm{Conf}(\mathfrak{X}) \to \mathrm{Conf}(\mathfrak{X})$, $X(\omega) = \omega$.

We define the *n-point correlation function* as follows: for $A = \{x_1, ..., x_n\} \subseteq \mathfrak{X}$, let

$$\rho_n(A) = \rho_n(x_1, ..., x_n) := \mathbb{P}\left[A \subseteq X\right].$$

Definition 5.4.5 (Determinantal Point Process)**.** *A random point process X is determinantal if there exists a kernel $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ such that for all $n \geq 1$ and for all $x_1, \ldots, x_n$ with $x_i \neq x_j$ for $i \neq j$, we have*

$$\rho_n(x_1, ..., x_n) = \det\left(K(x_i, x_j)_{i,j=1}^n\right) \tag{5.29}$$

Let $W(x) : \mathfrak{X} \to \mathbb{R}$ be a weight function, and let $P_0, P_1, \ldots$ be the family of orthonormal polynomials with respect to W, i.e.,

$$\int_{\mathfrak{X}} P_i(x) P_j(x) W(x) dx = \mathbb{1}_{i=j}.$$

The corresponding *N-point ensemble* (a random point process where $\mathbb{P}$ is supported on configurations with exactly N particles) is given by

$$\mathbb{P}(x_1, \ldots, x_N) \propto \det(V(x_1, \ldots, x_N))^2 \prod_{i=1}^{N} W(x_i),$$

where $V(x_1, \ldots, x_N) = (x_i^{j-1})_{i,j=1}^N$ is the Vandermonde matrix, and $\det(V(x_1, \ldots, x_N)) = \prod_{i<j}(x_j - x_i)$ is the Vandermonde determinant. An N-point ensemble generated in this way is determinantal with the *Christoffel-Darboux* kernel

$$K_N(x, y) = (W(x)W(y))^{\frac{1}{2}} \sum_{n=0}^{N-1} P_n(x) P_n(y). \tag{5.30}$$

The Meixner polynomials are a family of orthogonal polynomials on $\mathbb{Z}_{\geq 0}$. We fix two parameters: $\beta > 0$ and $\xi \in (0, 1)$, and then define the weight function $W : \mathbb{Z}_{\geq 0} \to \mathbb{R}$:

$$W(x) = \frac{\Gamma(\beta + x)}{\Gamma(\beta)x!}\xi^x. \tag{5.31}$$

We can then define the Meixner polynomials to be the family of orthogonal polynomials with respect to the weight function W. Using these orthogonal polynomials, we can define the Meixner ensemble Meixner(N, β, ξ) to be the corresponding N-point ensemble.

We use the term *particles* to refer to the elements of a point process X and use the term *holes* to refer to elements of $\mathfrak{X} \setminus X$. *Particle-hole involution* is an involution from $\mathrm{Conf}(\mathfrak{X}) \to \mathrm{Conf}(\mathfrak{X})$ that exchanges particles with holes. In other words, $X \mapsto X^\circ := \mathfrak{X} \setminus X$. If we start with an N-point ensemble, then particle-hole involution yields a point process with infinitely many particles. Suppose that X is a determinantal point process with kernel K. Then X° is a determinantal point process with kernel $1 - K$.

Next, we give a brief overview of Schur measures. An integer partition is denoted as $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $l(\lambda)$ denotes the number of nonzero λ_i in the partition λ. Let $\mathbb{Y}$ denote the set of all integer partitions. Let $\mathbf{x} = (x_1, x_2 \ldots), \mathbf{y} = (y_1, y_2, \ldots)$ be two sets of nonnegative variables. For fixed $\mathbf{x}$ and $\mathbf{y}$, we define the Schur measure $\mathbf{SM}(\mathbf{x}; \mathbf{y})(\lambda)$ as a measure on partitions λ as follows:

$$\mathbf{SM}(\mathbf{x}; \mathbf{y})(\lambda) := \frac{s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})}{\Pi(\mathbf{x}; \mathbf{y})} \tag{5.32}$$

where s_λ is the Schur symmetric function indexed by λ and $\Pi(\mathbf{x}; \mathbf{y}) = \sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})$ is the partition function. We need to assume that $\Pi(\mathbf{x}; \mathbf{y}) < \infty$ for our choice of $\mathbf{x}$ and $\mathbf{y}$ for this to define a valid probability measure.

The Meixner ensemble can be obtained as a pushforward of the Schur measure as follows: Consider the Schur measure of the form $\mathbf{SM}(x, \ldots, x; y \ldots, y)$ where we take n copies of x and m copies of y. Using standard properties of Schur functions, it follows that this measure is supported on $\mathbb{Y}^{(\min\{m,n\})}$, which is the set of partitions with $l(\lambda) \leq \min\{m, n\}$. Finally, consider the map from $\mathbb{Y}^{(\min\{m,n\})} \to \mathrm{Conf}(\mathbb{Z}_{\geq 0})$ such that $\lambda \mapsto \{\min\{m, n\} + \lambda_i - i\}_{i=1}^{\min\{m,n\}}$. Then the pushforward of $\mathbf{SM}(x, \ldots, x; y \ldots, y)$ to a measure on $\mathrm{Conf}(\mathbb{Z}_{\geq 0})$ gives us the Meixner ensemble Meixner$(\min\{m, n\}, |m - n| + 1, xy)$. This can be checked directly, see Proposition 8.2 in [BO17].

The following identity originates from [Bor18], although we state a version written in [BO17]: Let $\mathbb{E}_{\mathbf{6v}}$ refer to the expectation with respect to the stochastic six-vertex model and let $\mathbb{E}_{\mathbf{SM}}$ denote the expectation with respect to a specified Schur measure.

Proposition 5.4.6 (Proposition 8.4 in [BO17])**.** *Take any $0 < q < 1$ and $\kappa > 1$ and consider the stochastic six-vertex model on the quadrant parameterized by q and κ. Consider any integers $M, N \geq 1$. Then for any $\xi \notin -q^{\mathbb{Z}_{\leq 0}}$ we have*

$$\mathbb{E}_{\mathbf{6v}} \prod_{i \geq 0} \frac{1}{1 + \xi q^{H(M,N)+i}} = \mathbb{E}_{\mathbf{SM}} \prod_{j \geq 0} \frac{1 + \xi q^{\lambda_{N-j}+j}}{1 + \xi q^j} \tag{5.33}$$

where in the right-hand side we assume that $q^{\lambda_{-m}} = 0$ for $m \geq 0$, and the expectation is with respect to the Schur measure $\mathbf{SM}(\underbrace{\kappa^{-1}q^{1/2}, \ldots, \kappa^{-1}q^{1/2}}_{N}; \underbrace{q^{-1/2}, \ldots, q^{-1/2}}_{M-1})$.

If $M > N$, then the Schur measure in (5.33) is supported on $\mathbb{Y}^{(N)}$. We can obtain the N particles of the Meixner$(N, M - N, \kappa^{-1})$ ensemble by taking the above-mentioned pushforward of the Schur measure so that the particles in the Meixner ensemble are given by

$\{\lambda_i + N - i\}_{i=1}^{N} = \{\lambda_{N-j} + j\}_{j=0}^{N-1}$. On the other hand, if $M \le N$, then the Schur measure is supported on $\mathbb{Y}^{(M-1)}$. We now have that

$$\{\lambda_{N-j} + j\}_{j=0}^{N-1} = \{0, \cdots, N-M\} \sqcup \{\lambda_i + (N-M) + M - i\}_{i=1}^{M-1}.$$

This gives us the $M-1$ particles in Meixner$(M-1, N-M+2, \kappa^{-1})$ each shifted over deterministically by $N-(M-1)$ along with the addition of particles packed from 0 to $N-M$. In either case, we can obtain the following identity:

Proposition 5.4.7. *Take any $0 < q < 1$ and $\kappa > 1$ and consider the stochastic six-vertex model on the quadrant parameterized by q and κ. Consider any integers $M, N \ge 1$. Then for any $\xi \notin -q^{\mathbb{Z}_{\le 0}}$ we have*

$$\mathbb{E}_{\mathbf{SM}} \prod_{j \ge 0} \frac{1 + \xi q^{\lambda_{N-j}+j}}{1 + \xi q^j} = \mathbb{E}_X \prod_{x \in X} \frac{1}{1 + \xi q^x}. \tag{5.34}$$

where in the left-hand side we assume that $q^{\lambda_{-m}} = 0$ for $m \ge 0$ and the right-hand expectation is with respect to the point process

$$X \sim \begin{cases} \text{Meixner}^{\circ}(N, M-N, \kappa^{-1}) & \text{if } M > N \\ N-(M-1) + \text{Meixner}^{\circ}(M-1, N-M+2, \kappa^{-1}) & \text{if } M \le N\,, \end{cases} \tag{5.35}$$

where for a point process X, $n + X$ denotes the point process obtained by deterministically shifting over each particle in X by n.

Proof. The proof of this follows from crossing out each term in the denominator that equals one of the nontrivial terms in the numerator (i.e., a term corresponding to one of the particles in the Meixner ensemble). All terms that remain in the denominator will correspond to holes of the associated Meixner ensemble. □

We now explain how we go from Propositions 5.4.6 and 5.4.7 to proving Proposition 5.1.8. We will first need the following definition and lemma:

Definition 5.4.8 (q-Pochhammer symbol). *For any complex numbers q and a such that $|q| < 1$, we define $(a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$.*

The following Lemma is taken from [ACG23] and it allows us to connect the q-Laplace transform of H (the left-hand side of (5.33)) with the tail probability of H.

Lemma 5.4.9 (Lemma B.7 in [ACG23]). *Let $\mathbf{A}$ be a real-valued random variable, $q \in [0, 1)$ and $b \in \mathbb{R}$. Then,*

$$\mathbb{P}[\mathbf{A} \le 0] \le 2 \cdot \left(1 - \mathbb{E}\left[\left(-q^{\mathbf{A}}; q\right)_\infty^{-1}\right]\right), \tag{5.36}$$

$$\mathbb{E}\left[\left(-q^{\mathbf{A}}; q\right)_\infty^{-1}\right] \ge e^{q^b/(q-1)} \cdot \mathbb{P}[\mathbf{A} \ge b], \tag{5.37}$$

$$\mathbb{E}\left[\left(1 + q^{\mathbf{A}}\right)^{-1}\right] \le \mathbb{P}[\mathbf{A} > -b] + q^b \cdot \mathbb{P}[\mathbf{A} \le -b]. \tag{5.38}$$

Remark 5.4.10. While in the statement of Proposition 5.1.8 we consider a height function of the form $H(T\mu, T)$, for the remainder of this section we will work with the more general form $H(T\mu, T\nu)$ in order to highlight that many of the formulas that we will use in our analysis will have some symmetries in μ and ν. In the end, we will simply take $\nu = 1$. This does not actually reduce generality, since any appropriate $H(M,N)$ can be obtained by taking $T=N$ and $\mu = M/T$.

The constants in this section are allowed to depend on κ freely, but can be chosen to be uniform in μ and ν as long as $\kappa^{-1}+\varepsilon \le \mu \le \kappa-\varepsilon$ and $\nu = 1$ (Any other compact set bounded away from the two lines $\frac{\mu}{\nu} = \kappa^{-1}$ and $\frac{\mu}{\nu} = \kappa$ would also work). In particular this will also be true for all implicit constants hidden in big O notation terms.

Take $M = T\mu$, $N = T\nu$, and $\xi = q^{g(\mu,\nu)T - sT^{1/3}}$. Then using (5.37), with $\mathbf{A} = H(T\mu, T\nu) - g(\mu,\nu)T - sT^{1/3}$ and $b = 0$, we obtain

$$\mathbb{P}\left[H(T\mu,T\nu) \ge g(\mu,\nu)T + sT^{1/3}\right] \le e^{-1/(q-1)} \cdot \mathbb{E}_{\mathbf{6v}} \prod_{i\ge 0} \frac{1}{1+\xi q^{H(M,N)+i}} \tag{5.39}$$

$$= e^{-1/(q-1)} \cdot \mathbb{E}_X \prod_{x\in X} \frac{1}{1+\xi q^{x}} \tag{5.40}$$

where the point process X is defined as in Proposition 5.4.7. We can estimate the last product by dropping all terms in the product except for that corresponding to the smallest hole x_1. More precisely, since all the terms in the product are at most 1, we have

$$\mathbb{E}_X \prod_{x\in X} \frac{1}{1+\xi q^{x}} \le \frac{1}{1+\xi q^{x_1}}. \tag{5.41}$$

Using (5.38) with $\mathbf{A} = x_1 - g(\mu,\nu)T - sT^{1/3}$ and $b = \frac{sT^{1/3}}{2}$, we see that

$$\frac{1}{1+\xi q^{x_1}} \le \mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}} \mathbb{P}\left[x_1 \le g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] \tag{5.42}$$

$$\le \mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}}. \tag{5.43}$$

So in order to obtain an upper bound on $\mathbb{P}\left[H(T\mu,T\nu) \ge g(\mu,\nu)T + sT^{1/3}\right]$, it will suffice to obtain an upper bound on $\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right]$. Let us denote the holes of the Meixner ensemble by $x_1, x_2, \ldots$. We know that λ has at most N nonzero parts. If there are only k nonzero parts, then $\lambda_{k+1}, \ldots, \lambda_N = 0$, so there are $N-k$ Meixner particles at positions $0, \ldots, N-k-1$. Therefore, the smallest hole x_1 will occur at position $N-k$. It follows that $\ell(\lambda) = N - x_1$. Equivalently, we have

$$x_1 = N - \ell(\lambda). \tag{5.44}$$

It follows from (5.44) that

$$\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] = \mathbb{P}\left[-\ell(\lambda) > (g(\mu,\nu)-\nu)\,T + \frac{sT^{1/3}}{2}\right]. \tag{5.45}$$

According to [Bor18, In the proof of Theorem 6.1] we can represent the tail probability $\mathbb{P}\left[-\ell(\lambda) > h\right]$ as a Fredholm determinant. We first recall the definition of a Fredholm determinant, see e.g. [AB19, Definition A.1]

Definition 5.4.11 (Fredholm Determinant). *Fix a contour $\mathcal{C} \subset \mathbb{C}$ in the complex plane. Let $K : \mathcal{C} \times \mathcal{C} \to \mathbb{C}$ be a meromorphic function with no poles on $\mathcal{C} \times \mathcal{C}$. We define the Fredholm determinant*

$$\det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = 1 + \sum_{k=1}^{\infty} \frac{1}{(2\pi \mathrm{i})^k k!} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \det \left[K\left(x_i, x_j\right)\right]_{i,j=1}^{k} \prod_{j=1}^{k} dx_j .$$

We then have

$$\mathbb{P}[-\ell(\lambda) > h] = \det(1 - \widetilde{K})_{\ell^2(h,h-1,\dots)} = \det(1 - \Pi_h \widetilde{K} \Pi_h). \tag{5.46}$$

where $\widetilde{K}$ is a correlation kernel obtained as a dual of the kernel for the Meixner ensemble and Π_h is the projection from $\ell^2(\mathbb{Z})$ to $\ell^2(h, h-1, \dots)$.

We can write out $\widetilde{K}$ explicitly as (see [Bor18, Equation (6.1) and the subsequent paragraph in the reference])

$$\widetilde{K}(x, y) = \frac{1}{(2\pi i)^2} \oint \oint \frac{(\sqrt{\kappa} - z^{-1})^N}{(\sqrt{\kappa} - z)^{M-1}} \frac{(\sqrt{\kappa} - w)^{M-1}}{(\sqrt{\kappa} - w^{-1})^N} \frac{dz dw}{(w - z) z^{x+1} w^{-y}} \tag{5.47}$$

where $x, y \in \mathbb{Z}$ and the integrals are taken over positively oriented circular contours with $1/\sqrt{\kappa} < |z| < 1 < |w| < \sqrt{\kappa}$. Note that our integrand has poles at $0, 1/\sqrt{\kappa}$ and $\sqrt{\kappa}$, so the contours are chosen so that they do not pass through the poles.

To estimate the Fredholm determinant in (5.47) we will use a technique known as Widom's trick first used in [Wid02, Lemma 1]. It consists in the observation that for a kernel K with eigenvalues in $[0, 1]$ it holds that

$$\det(1 - K) \leq \exp(-\mathrm{Tr}(K)) . \tag{5.48}$$

The following lemma checks that the operator $\Pi_h \widetilde{K} \Pi_h$ satisfies this condition.

Lemma 5.4.12. *The operator $\Pi_h \widetilde{K} \Pi_h$ has real eigenvalues $(\mu_j)_{j \geq 0}$ all of which are in $[0, 1]$ and hence*

$$\det(1 - \Pi_h \widetilde{K} \Pi_h) \leq \exp(-Tr(\Pi_h \widetilde{K} \Pi_h)) . \tag{5.49}$$

Proof. Let $I(x, y) = \mathbf{1}_{x=y}$ be the identity operator and K as in [Bor18, Equation (6.1)]. As noted in [Bor18, Below Equation (6.1)] the operator $\widetilde{K}$ satisfies $\widetilde{K} = I - K$. The operator K is related to the Christoffel-Darboux kernel K_N (see (5.30)) associated to the Meixner ensemble via a gauge transformation, see Theorem 3.3 and Lemma 3.5 in [BO06]. Since I is invariant under gauge transformations, this also means that $\widetilde{K}$ is related to $I - K_N$ via a gauge transformation. The operator K_N is a projection operator since it is a Christoffel-Darboux kernel. Therefore, $I - K_N$ is also a projection operator and finally $\widetilde{K}$ as well, since a gauge transform of a projection operator is a projection operator. After the gauge transformation, $\Pi_h \widetilde{K} \Pi_h$ becomes self-adjoint, so the eigenvalues are real and non-negative. Since we have now also seen that this operator is a composition of projections, at most 1. Since $1 - x \leq \exp(-x)$ for $x \in [0, 1]$ this implies (5.49). □

Therefore, obtaining an upper bound on $\mathbb{P}[-\ell(\lambda) > h]$ reduces to obtaining a lower bound for $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h)$. Similar kinds of bounds were obtained for other kernels in e.g. [Wid02, BFP14]. We first compute this trace in the following lemma.

Lemma 5.4.13. *We have*

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \frac{1}{(2\pi i)^2} \oint \oint \exp(T(G_x(z) - G_x(w))) \frac{dz dw}{(w - z)^2} , \tag{5.50}$$

where

$$G_x(z) = \nu \ln\left(\sqrt{\kappa} - z^{-1}\right) - \mu \ln\left(\sqrt{\kappa} - z\right) - x\ln(z) \tag{5.51}$$

and we have reparameterized M, N and h as

$$\mu = \frac{M-1}{T}; \quad \nu = \frac{N}{T}; \quad x = \frac{h}{T}.$$

Proof. The trace is given by

$$\begin{aligned}\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) &= \sum_{j=-\infty}^{h} \widetilde{K}(j,j) \\ &= \sum_{j=-\infty}^{h} \frac{1}{(2\pi i)^2} \oint\oint \frac{(\sqrt{\kappa}-z^{-1})^N}{(\sqrt{\kappa}-z)^{M-1}} \frac{(\sqrt{\kappa}-w)^{M-1}}{(\sqrt{\kappa}-w^{-1})^N} \left(\frac{w}{z}\right)^j \frac{dzdw}{(w-z)z}.\end{aligned}$$

Since $|w/z| > 1$ by our choice of contours, we can sum w/z from $-\infty$ to h which yields

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \frac{1}{(2\pi i)^2} \oint\oint \frac{(\sqrt{\kappa}-z^{-1})^N}{(\sqrt{\kappa}-z)^{M-1}} \frac{(\sqrt{\kappa}-w)^{M-1}}{(\sqrt{\kappa}-w^{-1})^N} \left(\frac{w}{z}\right)^h \frac{dzdw}{(w-z)^2}.$$

Finally, we can rewrite the integrand in exponential form to obtain (5.50). □

The function G has the following two critical points:

$$z_c^{\pm} = \frac{\mu+\nu+(\kappa+1)x \pm \sqrt{-4\kappa(\mu+x)(\nu+x) + (\mu+\nu+(\kappa+1)x)^2}}{2\sqrt{\kappa}(\mu+x)} \tag{5.52}$$

We can see that these two critical points are equal if we choose $x = x_c^{\pm}$ where

$$x_c^{\pm} = \frac{(\sqrt{\mu} \pm \sqrt{\kappa\nu})^2}{\kappa - 1} - \nu. \tag{5.53}$$

Note that $x_c^- = g(\mu,\nu) - \nu$. For $x = x_c^-$, we have

$$z_c^+ = z_c^- = \frac{\sqrt{\mu} - \sqrt{\kappa\nu}}{\sqrt{\kappa\mu} - \sqrt{\nu}}.$$

Denote this value as z_c. For general x, we can rewrite the formula for $z_c^{\pm}$ as

$$z_c^{\pm} = \frac{\mu+\nu+(\kappa+1)x \pm \sqrt{(\kappa-1)^2(x-x_c^+)(x-x_c^-)}}{2\sqrt{\kappa}(\mu+x)}. \tag{5.54}$$

If $x_c^- < x < x_c^+$, then the two critical points $z_c^{\pm}$ are not real. Then it holds that

$$|z_c^{\pm}| = \sqrt{\frac{\nu+x}{\mu+x}}. \tag{5.55}$$

The following lemma describes how this function behaves around (x_c^-, z_c).

Lemma 5.4.14. *The function $G_x(z)$ satisfies:*

$$G'''_{x_c^-}(z_c) = 2\frac{\sqrt{\kappa\mu\nu}(\sqrt{\kappa} - \sqrt{\mu/\nu})^2(\sqrt{\kappa} - \sqrt{\nu/\mu})^2}{z_c^3(\kappa-1)^3}.$$

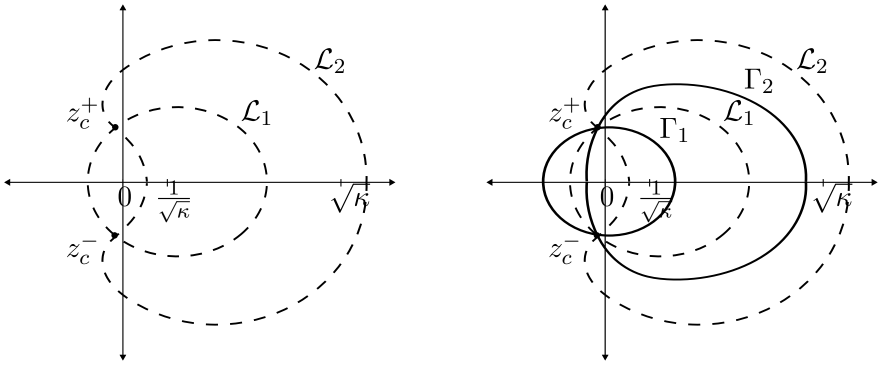


Figure 5.6: Left panel: the level lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Right panel: the contours Γ_1 and Γ_2.

As $x \to x_c^-$ from above we have the following:

$$z_c^+ = z_c + \frac{i\left(1-\kappa^{-1}\right)^{\frac{1}{2}}\left(\kappa^{-1}\mu\nu\right)^{\frac{1}{4}}\sqrt{x-x_c^-}}{\sqrt{\kappa}(\kappa\mu+\nu+2\sqrt{\kappa\mu\nu})} + O(x-x_c^-) \text{ and} \tag{5.56}$$

$$G_x''(z_c^+) = \frac{iG_{x_c^-}'''(z_c)(1-\kappa^{-1})^{\frac{1}{2}}(\kappa^{-1}\mu\nu)^{\frac{1}{4}}\sqrt{x-x_c^-}}{\sqrt{\kappa}(\kappa\mu+\nu+2\sqrt{\kappa\mu\nu})} + O(x-x_c^-)\,, \tag{5.57}$$

where the implicit constant in the big O term can be chosen independently of $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $x \in [x_c^-, x_c^+]$, i.e. it depends only on κ and ε.

Proof. The first two equalities are calculations, the third one is the Taylor expansion of G'' in x and z around (x_c^-, z_c). □

Now that we have established all the variables at play we can state an estimate on the trace.

Proposition 5.4.15. *Define $s = 2(x-x_c^-)T^{\frac{2}{3}}$. For any ε there exist s_0, T_0 and C such that for any $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $\nu = 1$, any $x \in \left[x_c^- + \frac{s_0}{2}T^{-2/3}, 0\right]$ and any $T > T_0$ it holds that:*

$$\operatorname{Tr}(\Pi_h \widetilde{K} \Pi_h) \geq C s^{3/2}.$$

Here s is seen as function of h via the two equations $x = \frac{h}{T}$ and $s = (x-x_c^-)T^{\frac{2}{3}}$.

To prove Proposition 5.4.15 we will deform the contours in (5.50). To do so we need to understand the level lines of $\operatorname{Re}(G_x(z))$ which pass through the critical points $z_c^\pm$. The following proposition describes the properties of these level lines, which are depicted in the left panel in Figure 5.6:

Proposition 5.4.16. *Let x be such that $x_c^- < x < x_c^+$. Then there are two smooth curves $\mathcal{L}_1$ and $\mathcal{L}_2$ such that:*

1. *The two curves only intersect at the critical points, i.e. $\mathcal{L}_1 \cap \mathcal{L}_2 = \{z_c^\pm\}$.*
2. *The two curves are the level lines through the critical points, i.e. $\operatorname{Re}(G_x(z)) = \operatorname{Re}(G_x(z_c^+))$ iff $z \in \mathcal{L}_1 \cup \mathcal{L}_2$.*
3. *Both curves are bounded simple loops.*
4. *$\mathcal{L}_1$ contains 0 and $1/\sqrt{\kappa}$ but not $\sqrt{\kappa}$, while $\mathcal{L}_2$ contains $\sqrt{\kappa}$ and $1/\sqrt{\kappa}$ but not 0.*

Proof. Note that while the logarithms in the definition of $G(z)$ require a choice of branch cut, $\mathrm{Re}(\log(z))$ is defined and smooth everywhere except 0 and does not depend on the choice of branch cut.

Let us consider the level lines through the critical points. Since the function is critical at this point and the second derivative does not vanish, there are exactly two level lines emerging, which intersect each other at those points. These cannot intersect at any other points, since the intersection points would again be critical points of G_x. For z with $|z|$ large it holds that

$$\mathrm{Re}(G_x(z)) = -(\mu + x)\ln(|z|) + \nu \ln\left(\sqrt{\kappa}\right) + o(1). \tag{5.58}$$

Since for $x \geq x_c^-$ we have $\mu + x > 0$, the level lines must be bounded. Considering $\mathrm{Re}(G_x(z))$ on the real line we see poles at 0, $1/\sqrt{\kappa}$ and $\sqrt{\kappa}$, where this function converges to $+$, $-$ and $+\infty$ respectively. Between two consecutive poles, the level lines can only cross once, since otherwise between two crossings there would be another critical point of G. This means there are exactly four points $d_1, \ldots, d_4$ along the real line such that $\mathrm{Re}(G_x(d_i)) = \mathrm{Re}(G(z_c^{\pm}))$ which satisfy $d_1 < 0 < d_2 < 1/\sqrt{\kappa} < d_3 < \sqrt{\kappa} < d_4$. Each of the four half-lines emanating from one of the critical points will intersect the real line at exactly one of those four points. Indeed the only other option would be for two of these lines to meet, but that would create a closed level-line loop containing no pole, which would force the function to be constant by harmonicity. A brief consideration shows that the only way to connect the half-lines gives the description in the fourth point.

Finally, there cannot be any other points z for which $\mathrm{Re}(G_x(z)) = \mathrm{Re}(G(z_c^{\pm}))$, since each of those would need to lie on a closed level-line, and such a level line would need to surround a pole and therefore also intersect the real line. But all points on the real line with value $\mathrm{Re}(G(z_c^{\pm}))$ already lie on the two level lines through the critical points.

□

Using these properties of the level lines we can choose contours Γ_1 and Γ_2, as depicted in the right panel in Figure 5.6.

Proposition 5.4.17. *Let x be such that $x_c^- < x < x_c^+$. Then there are two simple curves Γ_1 and Γ_2 such that:*

- *The two curves only intersect at the critical points, i.e. $\Gamma_1 \cap \Gamma_2 = \{z_c^+, z_c^-\}$.*
- *At the critical points the two curves intersect perpendicularly and in the direction of steepest ascent and descent respectively.*
- *The two curves only intersect the level lines $\mathcal{L}_1$ and $\mathcal{L}_2$ at the critical points.*
- *Both curves contain 0 and $1/\sqrt{\kappa}$ but not $\sqrt{\kappa}$.*
- *On Γ_1, the function G_x is always larger than $G_x(z_c^+(x))$, on Γ_2 it is always smaller.*
- *There exists an $r = r(\kappa)$ such that for $w \in \Gamma_1$ and $z \in \Gamma_2$, the inequality $|w - z| \leq r(x - x_c^-)^{\frac{1}{2}}$ implies that either*

$$|w - z_c^+| < 2r(x - x_c^-)^{\frac{1}{2}} \text{ and } |z - z_c^+| < 2r(x - x_c^-)^{\frac{1}{2}}$$

or

$$|w - z_c^-| < 2r(x - x_c^-)^{\frac{1}{2}} \text{ and } |z - z_c^-| < 2r(x - x_c^-)^{\frac{1}{2}}.$$

Furthermore r can be chosen such that $4r(x - x_c^-)^{\frac{1}{2}} < |z_c^+ - z_c^-|$ for all $x \in [x_c^-, 0]$.

Proof. Let us first consider the steepest descent/ascent curves through the critical points. These are given by the level lines of $\mathrm{Im}(G_x(z))$. They cannot cross $\mathcal{L}_1$ or $\mathcal{L}_2$ at points other than the critical points $z_c^{\pm}$. Since along these curves, the real part is strictly increasing/decreasing, these curves must end at the poles of $G_x(z)$ which are at $0, \kappa^{-\frac{1}{2}}$, and $\kappa^{\frac{1}{2}}$. By considering the signs of the poles one can see that the steepest descent curve (which is in the region where $\mathrm{Re}(G_x)$ is positive) connects the pole at 0 to the pole at $\kappa^{\frac{1}{2}}$. The steepest ascent curve connects $\kappa^{-\frac{1}{2}}$ to ∞.

By considering small circles $K_0, K_{\kappa^{-\frac{1}{2}}}, K_{\kappa^{\frac{1}{2}}}$ around each pole and a large circle K_∞ around the origin, we can construct the contours as follows: The curve Γ_1 is given by the steepest descent curves through the critical points until those hit the circles K_0 and $K_{\kappa^{\frac{1}{2}}}$. Then it follows those circles such that it contains 0 but not $\kappa^{\frac{1}{2}}$. The curve Γ_2 is given by the steepest ascent curve until it hits K_∞ and $K_{\kappa^{-\frac{1}{2}}}$, where it similarly follows the circles such that it includes 0 and $\kappa^{-\frac{1}{2}}$.

By considering (5.58) and (5.55), one can see that the choice of circle can be made independently of μ and x. Indeed one can see that $\mathrm{Re}(G(z_c))$ depends continuously on $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $x \in [x_c^-, x_c^+]$ and is therefore bounded uniformly in absolute value, with the bound depending only on κ. Around each of the poles, one can also find a uniform lower or upper bound depending on the sign of the pole. For example, around 0 one can bound:

$$\mathrm{Re}(G_x(z)) = \nu \ln(|\sqrt{\kappa} - z^{-1}|) - \mu \ln(|\sqrt{\kappa} - z|) - x \ln(|z|) \gtrsim (\nu + x)\ln(|z|^{-1}), \tag{5.59}$$

where for $|z|$ small enough the implicit constant depends only on κ. The prefactor $\nu + x = \frac{(\sqrt{\mu} \pm \sqrt{\kappa\nu})^2}{\kappa - 1}$ is bounded below by a constant which only depends on κ and ε. Therefore one can find a radius small enough, depending only on κ and ε such that for K_0 a circle of this radius and $z \in K_0$, $\mathrm{Re}(G_x(z)) > \mathrm{Re}(G(z_c)) + 1$ for all $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $x \in [x_c^-, x_c^+]$. In particular, this ensures that z_c^+ is outside this ball around 0. With very similar arguments one can determine the radii of $K_{\kappa^{-\frac{1}{2}}}, K_{\kappa^{\frac{1}{2}}}$ and K_∞, such that for all x and μ, the values of $\mathrm{Re}(G_x(z))$ on these circles is respectively larger, larger and smaller than the value of $\mathbb{R}(G_x(z_c^+))$.

For the last point, we will actually show the following stronger statement: There exists an $r_0 = r_0(\kappa)$ such that for all $r < r_0$, $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$, and $x \in [x_c^-, 0]$ it holds that for $w \in \Gamma_1$ and $z \in \Gamma_2$, the inequality $|w - z| \le r$ implies that either

$$|(w - z_c^+| < 2r \text{ and } |z - z_c^+| < 2r) \text{ or } (|w - z_c^-| < 2r \text{ and } |z - z_c^-| < 2r)\,.$$

This clearly implies the desired statement. For the last point, first consider fixed $x \in (x_c^-, 0]$ and $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$. Since the two curves only intersect at z_c^+ and z_c^- and intersect there perpendicularly, there exists an $\widetilde{r}_0 = \widetilde{r}_0(x, \mu, \kappa)$ such that for all $r < \widetilde{r}_0$ the statement holds. For $x = x_c^-$ the two critical points merge into a double critical point, and Γ_1 and Γ_2 deform in the following way. The part of Γ_1 that connects the critical points to $K_{\kappa^{\frac{1}{2}}}$ deforms into a piece-wise continuous curve, which has a $2\pi/3$ angle at the z_c and leaves this point in the directions $e^{\pi i/3}$ and $e^{-\pi i/3}$ The part of Γ_1 which connects the critical points to K_0 becomes straight lines connecting z_c to K_0, parallel to the horizontal axis. Similarly Γ_2 deforms into a straight line segment connecting z_c to $K_{\kappa^{-1/2}}$ and a piece-wise continuous curve which goes through z_c at an $2\pi/3$ angle, in the directions $e^{2\pi i/3}$ and $e^{4\pi i/3}$. See Figure 5.8 for the level lines of a double critical point, which appears in the other tail bound. Since these curves still only meet at z_c and there they do so at a $\pi/3$ angle, there also clearly is a $\widetilde{r}_0$ such that the statement holds for $x = x_c$. Since Γ_1 and Γ_2 deform continuously in $x \in [x_c^-, 0]$

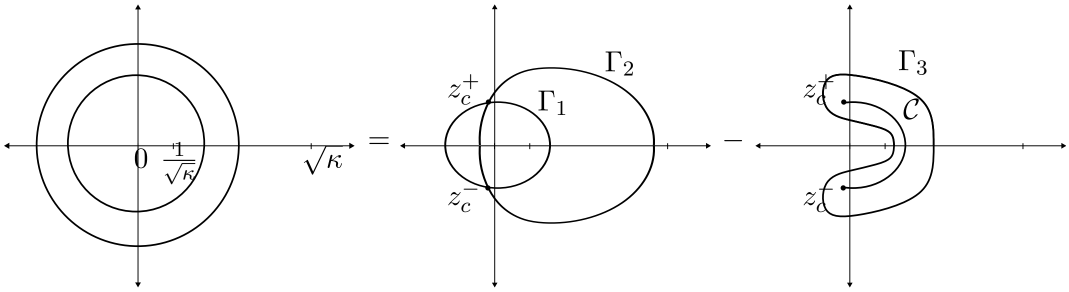


Figure 5.7: An illustration of how we pick up residues when deforming the original circular contours into Γ_1 and Γ_2.

and $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$, one can find r by taking a minimum over all r_0. Since the factor $(x-x_c^-)^{\frac{1}{2}}$ is

Decreasing r further one can obtain $4r(x-x_c^-)^{\frac{1}{2}} < |z_c^+ - z_c^-|$, which guarantees that w and z are indeed close to the same critical point when $|w-z| \leq r(x-x_c^-)^{\frac{1}{2}}$. □

Proof of Proposition 5.4.15. For clarity, we divide the proof into several steps:

Step 1. Decomposing the trace into two parts: Recall that in our original definition of $\widetilde{K}$ in (5.47), we started off with two positively oriented circular contours for w and z such that $1/\sqrt{\kappa} < |z| < 1 < |w| < \sqrt{\kappa}$. We will now deform these two circular contours into our new choice of contours Γ_1 and Γ_2, respectively.

Originally, the z contour is nested inside of the w contour. When we deform the z contour into Γ_2, part of it will cross through the w contour, see Figure 5.7. Therefore we will pick up some residues since our integrand has a pole of order 2 at $z = w$ due to the term $\frac{1}{(w-z)^2}$. After doing this deformation, we can decompose $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h)$ into two parts as follows:

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \left[\frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \exp(T(G_x(z) - G_x(w))) \frac{dzdw}{(w-z)^2}\right] + \left[-\frac{1}{(2\pi i)^2} \oint_{\mathcal{C}} \oint_{\Gamma_3} \exp(T(G_x(z) - G_x(w))) \frac{dzdw}{(w-z)^2}\right], \quad (5.60)$$

where Γ_1 and Γ_2 are the contours given in Proposition 5.4.17, $\mathcal{C}$ is an arc connecting z_c^+ and z_c^- and intersecting the real line between $\kappa^{-\frac{1}{2}}$ and $\kappa^{\frac{1}{2}}$, and Γ_3 is a contour around this arc, intersecting the real line only between $\kappa^{-\frac{1}{2}}$ and $\kappa^{\frac{1}{2}}$ (i.e. not including any pole other than $z = w$). Denote the first expression in (5.60) as I_1 and the second one as I_2, such that $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = I_1 + I_2$. Here I_1 needs to be interpreted as a principal value integral due to the quadratic singularity at the intersection points. I_2 accounts for the residues picked up in the above-described deformation.

Step 2: Estimating I_2: In this step we show that there exists a constant $C = C(\varepsilon)$, such that $I_2 > Cs_0^{3/2}$. Let $f_w(z) := \frac{\exp(T(G_x(z)-G_x(w)))}{(w-z)^2}$. Using Cauchy's residue theorem for a pole of order 2, we can compute

$$I_2 = -\frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \oint_{\Gamma_3} f_w(z) dzdw = -\frac{1}{2\pi i} \int_{\mathcal{C}} \mathrm{Res}(f_w, w) dw \quad (5.61)$$

$$= -\frac{1}{2\pi i} \int_{\mathcal{C}} T G_x'(w) dw \quad (5.62)$$

$$= -\frac{T(G_x(z_c^+) - G_x(z_c^-))}{2\pi i}. \quad (5.63)$$

Since $z_c^\pm(y)$ are critical points, and since only one term in G_y depends explicitly on y, we can compute

$$\frac{d}{dy} G_y(z_c^\pm(y)) = \ln(z_c^\pm(y)).$$

At x_c^-, we have $z_c^+ = z_c^-$, so we can add and subtract $G_{x_c^-}(z_c^+) = G_{x_c^-}(z_c^-)$ to get

$$G_x(z_c^+) - G_x(z_c^-) = (G_x(z_c^+) - G_{x_c^-}(z_c^+)) - (G_x(z_c^-) - G_{x_c^-}(z_c^-)) \tag{5.64}$$

$$= \int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) dy. \tag{5.65}$$

This integrand is purely imaginary since we are taking the log of the ratio of complex conjugates (which has modulus 1). Therefore, when we divide by $2\pi i$ we will get something real. We now estimate the integrand. Since the modulus is 1, the integrand is just the argument of $\frac{z_c^+(y)}{z_c^-(y)}$, which varies along the unit circle clockwise starting at 0.

Let us define the new variable $v = \sqrt{(y - x_c^-)}$ and also define

$$Z(v) := \frac{z_c^+(y)}{z_c^-(y)} = \frac{\mu + \nu + (\kappa+1)y + \sqrt{(\kappa-1)^2(y - x_c^+)(y - x_c^-)}}{\mu + \nu + (\kappa+1)y - \sqrt{(\kappa-1)^2(y - x_c^+)(y - x_c^-)}}$$

$$= \frac{\mu + \nu + (\kappa+1)(v^2 + x_c^-) + v\sqrt{(\kappa-1)^2(v^2 + x_c^- - x_c^+)}}{\mu + \nu + (\kappa+1)(v^2 + x_c^-) - v\sqrt{(\kappa-1)^2(v^2 + x_c^- - x_c^+)}}.$$

Note that $Z(0) = 1$, so that

$$Z(v) = 1 + Z'(0)v + O(v^2)$$

where

$$Z'(0) = \frac{-i2(\kappa-1)^{3/2}\kappa^{1/4}(\frac{\mu}{\nu})^{1/4}}{\nu^{1/2}\left[\kappa^{1/2}(\frac{\mu}{\nu})^{1/2}(\kappa+1) - (1+\frac{\mu}{\nu})\kappa\right]}.$$

We can see that the numerator above is bounded for $\frac{\mu}{\nu} \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$. We can also see that the denominator is zero precisely when $\frac{\mu}{\nu} \to \kappa^{-1}$ or $\frac{\mu}{\nu} \to \kappa$ and is positive between those two values. Finally, note that $Z'(0)$ is purely imaginary and $iZ'(0) > 0$. It follows that there exists $C = C(\varepsilon)$ such that $iZ'(0) > C$ for all $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and $\nu = 1$.

It follows that

$$\ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) = Z'(0)v + O(v^2)$$

$$= Z'(0)\sqrt{(y - x_c^-)} + O(y - x_c^-)$$

and that

$$\int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) dy = Z'(0)(x - x_c^-)^{3/2} + O((x - x_c^-)^2),$$

where the $O(x - x_c^-)^2$ is uniform in μ. Plugging this back into (5.63) and (5.65), we conclude that

$$I_2 = -\frac{T}{2\pi i}\int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right)dy > CT(x-x_c^-)^{3/2} \\ = CT(sT^{-2/3})^{3/2} \\ = Cs^{3/2} \tag{5.66}$$

for some positive real constant C depending on ε, which changes from line to line.

Step 3: Estimating I_1. In this section, we show that the integral I_1 defined above is bounded uniformly in x, i.e., there exists constants C, T_0, s_0 depending on ε, such that $I_1 < C$ for all x in $[x_c^- + s_0 T^{-\frac{2}{3}}, 0]$ and $T > T_0$.

Consider first the part of the integral I_1 where $|z-w| \geq r(x-x_c)^{\frac{1}{2}}$, where r is from Proposition 5.4.17.

$$\frac{1}{(2\pi i)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\exp(T(G_x(z)-G_x(w)))\frac{\mathbf{1}_{|z-w|\geq r(x-x_c)^{\frac{1}{2}}}dzdw}{(w-z)^2}.$$

This integral we can bound by taking absolute values and the triangle inequality to obtain

$$\frac{1}{(x-x_c)r^2(2\pi)^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\exp(T(\mathrm{Re}(G_x(z)-G_x(w))))dzdw$$

This integral has no singularities and can be split into the product of two integrals, each of which can be treated using the method of steepest descent. Each of them gives a contribution $\frac{C}{\sqrt{TG_x''(z_c^+)}}$ for T large enough. Combined with the prefactor $\frac{1}{(x-x_c)r^2(2\pi)^2}$ we obtain the upper bound $T^{-1}|G_x''(z_c^+)|^{-1}r^{-2}(x-x_c)^{-1}$. By Lemma 5.4.14, this is of order $\frac{(x-x_c)^{-\frac{3}{2}}}{T} \lesssim s_0^{-3/2}$ and therefore $O(1)$.

As observed in Proposition 5.4.17, since the distance of the two critical points z_c^+ and z_c^- is of order $\sqrt{x-x_c^-}$ by Lemma 5.4.14 we have the following. For small but fixed r, the only $w \in \Gamma_1$ and $z \in \Gamma_2$ such that $|w-z| \leq r(x-x_c)^{\frac{1}{2}}$ satisfy either

$$|w-z_c^+| < 2r(x-x_c)^{\frac{1}{2}} \text{ and } |z-z_c^+| < 2r(x-x_c)^{\frac{1}{2}}$$

or

$$|w-z_c^-| < 2r(x-x_c)^{\frac{1}{2}} \text{ and } |z-z_c^-| < 2r(x-x_c)^{\frac{1}{2}}$$

i.e. they are both close to the same critical point.

By symmetry it suffices to consider both w and z in the ball around z_c^+ of radius $2r(x-x_c)^{\frac{1}{2}}$. Denote this ball by B. Let us Taylor expand around our integrand.

$$\exp(T(G_x(z))-G_x(w)) = \\ \exp\left(\frac{1}{2}TG_x''(z_c^+)((z-z_c^+)^2-(w-z_c^+)^2)\right)(1+O(T|z-z_c^+|^3+|w-z_c^+|^3)), \tag{5.67}$$

where the big O constant depends on ε, but not on x or T. Let us first consider the contribution of the big O term on the right. After taking absolute values we have to bound

$$\oint_{\Gamma_1\cap B}\oint_{\Gamma_2\cap B} T(|z-z_c^+|^3+|w-z_c^+|^3)\exp\left(\frac{1}{2}T\mathrm{Re}(G_x''(z_c^+)((z-z_c^+)^2-(w-z_c^+)^2))\right)\frac{dzdw}{|w-z|^2}.$$

Since our contours meet at a right angle at the critical point we have $\frac{1}{|z-w|^2} = O\left(\frac{1}{|z-z_c^+|^2+|w-z_c^+|^2}\right)$ and the integral is bounded by

$$T\int_{\mathbb{R}}\int_{\mathbb{R}}(|z|+|w|)\exp\left(-\frac{1}{2}T|G_x''(z_c^+)|(-z^2-w^2)\right)dzdw \lesssim \frac{T}{(TG_x''(z_c^+))^{\frac{3}{2}}} = O(s_0^{-\frac{3}{4}}).$$

For the other part of (5.67) we use a change of variables $z = z_c^+ + \xi_1/\sqrt{TG_x''(z_c^+)}$ and $w = z_c^+ + \xi_2/\sqrt{TG_x''(z_c^+)}$ which yields the principal value integral:

$$\int_{\widetilde{\Gamma}_1}\int_{\widetilde{\Gamma}_2}\frac{\exp(\xi_1^2-\xi_2^2)}{(\xi_1-\xi_2)^2}d\xi_1 d\xi_2\,,$$

where $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are contours crossing at the origin with $\widetilde{\Gamma}_1$ vertical and $\widetilde{\Gamma}_2$ horizontal there. This is clearly bounded away from the origin, and close to the origin the exponential can be estimated by 1 up to an $O(1)$ error, and the resulting principal value integral is also of order $O(1)$.

Combining the above estimates, one obtains that I_1 is $O(1)$. Combining this with (5.66) one obtains that

$$\mathrm{Tr}(\Pi_h\widetilde{K}\Pi_h) = I_1 + I_2 > Cs^{\frac{3}{2}}$$

for a different constant C, using that $s \geq s_0$ to absorb the $O(1)$ term. □

We now prove Proposition 5.1.8:

Proof of Proposition 5.1.8. Combining Equations (5.40)-(5.46) and (5.49) and setting $\nu = 1$ and h as

$$h = (g(\mu)-1)T + \frac{sT^{1/3}}{2} = x_c^- T + \frac{sT^{1/3}}{2}\,.$$

we have that

$$\mathbb{P}\left[H(T\mu,T) \geq g(\mu)T + sT^{1/3}\right] \leq e^{-1/(q-1)}\left(\mathbb{P}\left[-\ell(\lambda) > (g(\mu)-1)\,T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}}\right) \tag{5.68}$$

$$= e^{-1/(q-1)}\left(\det(1-\Pi_h\widetilde{K}\Pi_h) + q^{\frac{sT^{1/3}}{2}}\right) \tag{5.69}$$

$$\leq e^{-1/(q-1)}\left(\exp(-\mathrm{Tr}(\Pi_h\widetilde{K}\Pi_h)) + q^{\frac{sT^{1/3}}{2}}\right)\,, \tag{5.70}$$

By Proposition 5.4.15 there exist constants C, T_0, and s_0 depending on ε such that for $s > s_0, T > T_0$ and $x \in [x_c^- + \frac{s_0}{2}T^{-2/3}, 0]$:

$$\mathrm{Tr}(\Pi_h\widetilde{K}\Pi_h) \geq Cs^{3/2}\,.$$

Noting that x is given by

$$x = \frac{h}{T} = x_c^- + \frac{sT^{-2/3}}{2}\,,$$

the restriction $x \in [x_c^- + \frac{s_0}{2}T^{-2/3}, 0]$ is equivalent to $s \in [s_0, -2T^{\frac{2}{3}}x_c^-]$. For s in this range, applying Proposition 5.4.15 to (5.70) yields

$$\mathbb{P}\left[H(T\mu,T) \geq g(\mu)T + sT^{1/3}\right] \leq e^{-1/(q-1)}\left(\exp(-Cs^{\frac{3}{2}}) + q^{\frac{sT^{1/3}}{2}}\right) \leq c^{-1}\exp(-cs^{\frac{3}{2}})\,. \tag{5.71}$$

For $s > -2T^{\frac{2}{3}}x_c^-$ we have

$$g(\mu)T + sT^{\frac{1}{3}} > (x_c^- + 1)T - 2Tx_c^- = T(1 - x_c^-) > T\,.$$

Since $H(\mu T, T)$ can be at most T, for such s the left-hand side of (5.71) is 0 and therefore (5.7) is trivially satisfied.

In summary, we have proved Proposition 5.1.8 for all $s \geq s_0$ and all $T \geq T_0$. For fixed $T < T_0$, the statement is trivial for s large enough as the left-hand side is 0 if $g(\mu)T + sT^{\frac{1}{3}} > T$. Thus by increasing s_0 the statement holds for all $T \geq 1$ and $s > s_0$. By decreasing c such that $c^{-1}e^{-cs} \leq 1$ for $s < s_0$ the statement holds for all $T \geq 1$ and $s \geq 0$. That c can be chosen weakly decreasing in ε is easily checked by checking that all constants in the above estimates depend continuously on ε. □

5.4.2 Proof of Proposition 5.1.9

The goal of this section is to prove Proposition 5.1.9.

Remark 5.4.18. Again, it suffices to prove the statement for $T \geq T_0$ and $s \geq s_0$ for some T_0 and s_0 large enough. For fixed T, the left-hand side becomes 0 for s large enough since the height function is always non-negative. We can also alter the constant c to be small enough so that the right-hand side becomes greater than 1 for all $s < s_0$.

In this section, we will closely follow the results in [AB19]. We start with an identity that relates the q-Laplace transform of the stochastic six-vertex model under step Bernoulli initial data, to a Fredholm determinant of some kernel.

Recall that $(\rho, 0)$-Bernoulli boundary conditions denotes the boundary condition in which the incoming arrows from the left are given by i.i.d. Bernoulli(ρ) random variables, while the incoming positions from the bottom are all empty. The following proposition is stated for $\rho \in (0, 1)$ as Proposition 5.1 in [AB19]. We will ultimately need statements for $\rho = 1$. What we will do here is first prove the desired tail bound for $\rho < 1$ and extend to $\rho = 1$ by attractivity. An alternative proof is to notice that the following proposition itself can be extended to $\rho = 1$ either using the continuity of the involved formulas in the parameter ρ. This case is also a consequence of Lemma 4.18 in [BCG16].

Proposition 5.4.19 (Prop 5.1 in [AB19])**.** *Fix $b_1, b_2 \in (0, 1)$; $\rho \in (0, 1]$; $x \in \mathbb{Z}$; and $p \in \mathbb{R}$. Denote $\beta = \rho/(1 - \rho)$.*

Let $\Gamma \subset \mathbb{C}$ be a positively oriented, star-shaped contour in the complex plane containing 0, but leaving outside $-q\kappa$ and $q\beta$. Let $\mathcal{C} \subset \mathbb{C}$ be a positively oriented, star-shaped contour contained inside $q^{-1}\Gamma$; that contains $0, -q$, and Γ; but that leaves outside $q\beta$.

Let $\mathbb{E}_{\mathbf{6v}}$ denote the expectation with respect to the stochastic six-vertex model with left jump probability b_1, right jump probability b_2, and $(\rho, 0)$-Bernoulli initial data. Then, we have that

$$\mathbb{E}_{\mathbf{6v}}\left[\frac{1}{\left(-q^{H^\rho(X,T)+p}; q\right)_\infty}\right] = \det\left(\mathrm{Id} + K^{(p)}\right)_{L^2(\mathcal{C})} \tag{5.72}$$

where

$$K^{(p)}\left(w, w'\right) = \frac{1}{2\mathbf{i}\log q}\sum_{j=-\infty}^{\infty}\oint_\Gamma \frac{\left(\kappa^{-1}w + q\right)^{X-1}}{\left(\kappa^{-1}v + q\right)^{X-1}}\frac{(v+q)^T}{(w+q)^T}\frac{\left(q^{-1}\beta^{-1}v; q\right)_\infty}{\left(q^{-1}\beta^{-1}v; q\right)_\infty} \cdot \frac{v^{p-1}w^{-p}}{\sin\left(\frac{\pi}{\log q}(\log v - \log w + 2\pi\mathbf{i}j)\right)}\frac{dv}{w' - v}, \tag{5.73}$$

Once we choose p appropriately, (5.72) implies a bound on the probability $\mathbb{P}[H^\rho(X,T) \leq -p]$ by applying (5.36). Let $\mu = \frac{X-1}{T}$ and for $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ we define

$$f_\mu = \frac{(\sqrt{\kappa\mu}-1)^{2/3}(\kappa-\sqrt{\kappa\mu})^{2/3}}{(\kappa-1)\kappa^{1/6}\mu^{1/6}}.$$

The function f_μ appears as the scaling factor in the convergence to Tracy-Widom GUE fluctuations, see [BCG16, Theorem 1.2]. We then define for $s \geq 0$:

$$p_T = sf_\mu T^{1/3} - g(\mu)T.$$

We will now study asymptotics of the kernel K^{p_T}, closely following section 6 of [AB19], but adding in control of the decay in s as well, since in [AB19] they treat s as a constant.

5.4.3 Fredholm Determinant Estimates

Our first step is to rewrite the formula for the kernel $K^{(p_T)}$ in an exponential form that utilizes the explicit form we chose for p_T. Plugging $p = p_T$ into (5.73), we obtain

$$\begin{aligned} K^{(p_T)}(w,w') =& \frac{1}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \oint_\Gamma \frac{\exp\left(T\left(G(w)-G(v)\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j+\log v-\log w)\right)} \\ &\times \frac{\left(q^{-1}\beta^{-1}v;q\right)_\infty}{\left(q^{-1}\beta^{-1}w;q\right)_\infty} \\ &\times \left(\frac{v}{w}\right)^{sf_\mu T^{1/3}} \frac{dv}{v\left(w'-v\right)}, \end{aligned}$$

where we define

$$G(z) = \mu\log\left(\kappa^{-1}z+q\right) - \log(z+q) + g(\mu)\log z.$$

Next, we Taylor expand G around its critical point: We compute its derivative

$$G'(z) = \frac{(\sqrt{\kappa\mu}-1)^2}{\kappa-1}\frac{(z-\psi)^2}{z(z+q\kappa)(z+q)}$$

with

$$\psi = \frac{q(\kappa-\sqrt{\kappa\mu})}{\sqrt{\kappa\mu}-1}.$$

Therefore, ψ is a critical point of G, and we have $G''(\psi)=0$. We also have

$$G'''(\psi) = \frac{2(\sqrt{\kappa\mu}-1)^5}{q^3(\kappa-1)^3(\kappa-\sqrt{\kappa\mu})\sqrt{\kappa\mu}} = 2\left(\frac{f_\mu}{\psi}\right)^3.$$

Putting this all together, the Taylor expansion of G can be written as

$$G(z) - G(\psi) = \frac{1}{3}\left(\frac{f_\mu(z-\psi)}{\psi}\right)^3 + R\left(\frac{f_\mu(z-\psi)}{\psi}\right), \tag{5.74}$$

where R is the remainder. By Taylor's remainder theorem, we have

$$R\left(\frac{f_\mu(z-\psi)}{\psi}\right) = \mathcal{O}\left(|z-\psi|^4\right) \quad \text{as } |z-\psi| \to 0.$$

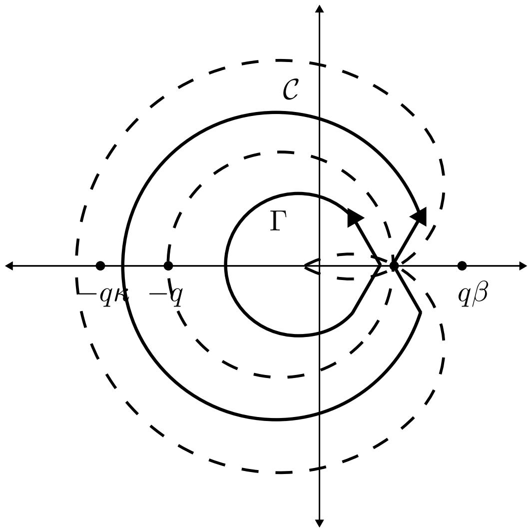


Figure 5.8: The solid lines represent the contours Γ and $\mathcal{C}$, and the dashed curves represent the level lines of $\mathrm{Re}(G(z))$.

This remainder can be bounded uniformly for $\kappa^{-1}+\varepsilon \leq \mu \leq \kappa-\varepsilon$. Note that $G'''(\psi)=0$ at $\mu=\kappa^{-1}$ and $G'''(\psi))=\infty$ at $\mu=\kappa^{-1}$, so μ being bounded away from κ^{-1} and κ is really necessary for uniformity.

We now need to choose contours Γ and $\mathcal{C}$. We use the contours defined in [AB19, Definitions 6.2-6.5], which will take the following shape. The contour Γ will consist of two parts: a piecewise linear part $\Gamma^{(1)}$ and a round part $\Gamma^{(2)}$ that connects the endpoints of $\Gamma^{(1)}$. Similarly $\mathcal{C}$ will consist of two parts $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, where $\mathcal{C}^{(1)}$ is piecewise linear and $\mathcal{C}^{(2)}$ is a round part connecting the end points of $\mathcal{C}^{(1)}$.

Definition 5.4.20. *For a real number $r\in\mathbb{R}$ and a positive real number $\varpi>0$ (possibly infinite), let $\mathfrak{W}_{r,\varpi}$ denote the piecewise linear curve in the complex plane that connects $r+\varpi e^{-\pi\mathbf{i}/3}$ to r to $r+\varpi e^{\pi\mathbf{i}/3}$. Similarly, let $\mathfrak{V}_{r,\varpi}$ denote the piecewise linear curve in the complex plane that connects $r+\varpi e^{-2\pi\mathbf{i}/3}$ to r to $r+\varpi^{2\pi\mathbf{i}/3}$.*

The contours $\mathcal{C}$ and Γ look as follows:

- $\mathcal{C}^{(1)}=\mathfrak{W}_{\psi,\varpi}$ and $\Gamma^{(1)}=\mathfrak{V}_{\psi-\psi f_\mu^{-1}T^{-1/3},\varpi}$, for some sufficiently small ϖ (independently of T).
- $\mathcal{C}^{(2)}$ is a positively oriented contour from the top endpoint $\psi+\varpi e^{\pi\mathbf{i}/3}$ of $\mathcal{C}^{(1)}$ to the bottom endpoint $\psi+\varpi e^{-\pi\mathbf{i}/3}$ of $\mathcal{C}^{(1)}$, and $\Gamma^{(2)}$ is a positively oriented contour from the top endpoint $\psi-\psi f_\mu^{-1}T^{-1/3}+\varpi e^{2\pi\mathbf{i}/3}$ of $\Gamma^{(1)}$ to the bottom endpoint $\psi+\psi f_\mu^{-1}T^{-1/3}+\varpi e^{-2\pi\mathbf{i}/3}$ of $\Gamma^{(1)}$.
- We take $\mathcal{C}=\mathcal{C}^{(1)}\cup\mathcal{C}^{(2)}$ and $\Gamma=\Gamma^{(1)}\cup\Gamma^{(2)}$.

See Figure 5.8 for a depiction of these contours.

Proposition 5.4.21. *The contours Γ and $\mathcal{C}$ satisfy the following properties: The contour Γ is positively oriented and star-shaped; it contains 0 , but leaves outside $-q\kappa$ and $q\beta$. Furthermore, $\mathcal{C}$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma$; that* *contains* $0, -q$ *and* Γ*; but that leaves outside* $q\beta$*. Furthermore, there exists some positive real number* $c_1 > 0$*, independent of* T*, such that*

$$\max\left\{\sup_{\substack{w\in\mathcal{C}\\ v\in\Gamma^{(2)}}} \mathrm{Re}(G(w)-G(v)),\ \sup_{\substack{w\in\mathcal{C}^{(2)}\\ v\in\Gamma}} \mathrm{Re}(G(w)-G(v))\right\} < -c_1\,,$$

where c_1 *depends on* ε*, but is uniform in* μ*.*

Proof. These properties are all stated in [AB19, Definition 6.3, Lemma 6.6, and Lemma 6.13] except for the uniformity of c_1. This uniformity follows from the uniformity of G and ψ in $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$. □

By Proposition 5.4.21, Γ and $\mathcal{C}$ satisfy the necessary conditions stated in Proposition 5.4.19. We can now analyze the kernel $K := K^{(p_T)}$ for the different cases where v, w belong to the different components of these contours. The first case is where $w \in \mathcal{C}^{(1)}$ and $v \in \Gamma^{(1)}$. In this case, both w and v are close to ψ. The second case is where either $w \in \mathcal{C}^{(2)}$ or $v \in \Gamma^{(2)}$. Let $\widetilde{K}(w,w')$ be the same kernel as $K(w,w')$, but where we replace the contour Γ with $\Gamma^{(1)}$.

We perform a change of variables to zoom in around ψ: Let $\sigma = \psi f_\mu^{-1} T^{-1/3}$ and set

$$w = \psi + \sigma\widehat{w}; \qquad w' = \sigma + \sigma\widehat{w}';$$
$$v = \psi + \sigma\widehat{v}; \qquad \widehat{K}(\widehat{w},\widehat{w}') = \sigma\widetilde{K}(w,w')$$
$$\overline{K}(\widehat{w},\widehat{w}') = \sigma K(w,w').$$

For any contour $\mathcal{D}$, set $\widehat{\mathcal{D}} = \sigma^{-1}(\mathcal{D}-\psi)$. In particular, we have

$$\widehat{\mathcal{C}^{(1)}} = \mathfrak{W}_{0,\varpi/\sigma}, \qquad \widehat{\Gamma^{(1)}} = \mathfrak{V}_{-1,\varpi/\sigma}.$$

Notice that as $T\to\infty$, we have

$$\widehat{\mathcal{C}^{(1)}} \to \mathfrak{W}_{0,\infty}, \qquad \widehat{\Gamma^{(1)}} \to \mathfrak{V}_{-1,\infty/\sigma},$$

each of which consists of a pair of rays emanating from 0 and -1 respectively.

The only difference between $\widehat{K}$ and $\overline{K}$ is that for $\widehat{K}$ we are integrating over the contour $\widehat{\Gamma}^{(1)}$ and for $\overline{K}$ we are integrating over the contour $\widehat{\Gamma}$. The following lemma will deal with estimating these two kernels.

Lemma 5.4.22. *There exist positive constants* c, C *and* T_0 *all depending on* ε *such that for* $T \geq T_0$ *we have*

1. $$\left|\overline{K}(\widehat{w},\widehat{w}') - \widehat{K}(\widehat{w},\widehat{w}')\right| < c^{-1}\exp\left(-c\left(T+|\widehat{w}|^3\right)\right)$$

   for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$

2. $$\left|\overline{K}(\widehat{w},\widehat{w}')\right| < c^{-1}\exp\left(-c\left(T+|\widehat{w}|^3\right)\right)$$

   for all $\widehat{w} \in \widehat{\mathcal{C}^{(2)}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$.

3.

$$|\widehat{K}(\widehat{w},\widehat{w}')| \le \frac{C}{1+|\widehat{w}'|}\exp(-c|\hat{w}|^3 - cs)$$

for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}}$.

Combining the above items, we can conclude that

$$\left|\overline{K}(\widehat{w},\widehat{w}')\right| \le c^{-1}\exp\left(-c|\widehat{w}|^3 - cs\right) + c^{-1}\exp\left(-c|\widehat{w}|^3 - cT\right) \tag{5.75}$$

for all $\widehat{w} \in \widehat{\mathcal{C}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}}$.

Proof. The proof of the first two items is the content of Corollary 6.14 in [AB19]. For the proof of Item 3 we write out the formula for kernel $\widehat{K}(\widehat{w},\widehat{w}')$ in terms of the variables $\widehat{w},\widehat{w}'$, and $\widehat{v}$, using the fact that $\sigma = \psi f_\mu^{-1}T^{-1/3}$ as well as the Taylor estimates in (5.74). We have

$$\begin{aligned}
\widehat{K}(\widehat{w},\widehat{w}') &= \sigma\widetilde{K}(w,w') \\
&= \frac{\sigma}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \oint_{\widehat{\Gamma^{(1)}}} \frac{\exp\left(T\left(G(\psi+\sigma\widehat{w}) - G(\psi+\sigma\widehat{v})\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log(\psi+\sigma\widehat{v}) - \log(\psi+\sigma\widehat{w})\right))} \\
&\times \frac{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q\right)_\infty}{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q\right)_\infty} \\
&\times \left(\frac{\psi+\sigma\widehat{v}}{\psi+\sigma\widehat{w}}\right)^{sf_\mu T^{1/3}} \frac{\sigma d\widehat{v}}{(\psi+\sigma\widehat{v})\,(\sigma\widehat{w}'-\sigma\widehat{v})} \\
&= \oint_{\widehat{\Gamma^{(1)}}} \frac{\sigma\psi^{-1}}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \frac{\exp\left(\frac{\widehat{w}^3-\widehat{v}^3}{3} + T\left(R\left(T^{-1/3}\widehat{w}\right) - R\left(T^{-1/3}\widehat{v}\right)\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log(1+\sigma\psi^{-1}\widehat{v}) - \log(1+\sigma\psi^{-1}\widehat{w})\right))} \\
&\times \frac{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q\right)_\infty}{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q\right)_\infty} \\
&\times \left(\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right)^{s\psi\sigma^{-1}} \frac{d\widehat{v}}{(1+\sigma\psi^{-1}\widehat{v})\,(\widehat{w}'-\widehat{v})}.
\end{aligned}$$

Next, we estimate each of the terms in the integrand, using Proposition 5.4.23. Multiplying the seven bounds in Proposition 5.4.23 together, we obtain that the integrand is bounded in absolute value by $\frac{c^{-1}}{1+|\widehat{w}'|}\exp\left(cs\mathrm{Re}\widehat{v} - c(|\widehat{w}|^3 + |\widehat{v}|^3)\right)$. Noting that $\mathrm{Re}\widehat{v} \le -1$ for $\widehat{v} \in \widehat{\Gamma^{(1)}}$, we obtain

$$\begin{aligned}
|\widehat{K}(\widehat{w},\widehat{w}')| &\le \oint_{\widehat{\Gamma^{(1)}}} \frac{c^{-1}}{1+|\widehat{w}'|}\exp\left(cs\mathrm{Re}\widehat{v} - c(|\widehat{w}|^3+|\widehat{v}|^3)\right)d\widehat{v} \\
&\le \oint_{\widehat{\Gamma^{(1)}}} \frac{c^{-1}}{1+|\widehat{w}'|}\exp\left(-cs - c(|\widehat{w}|^3+|\widehat{v}|^3)\right)d\widehat{v} \\
&\le \frac{C}{1+|\widehat{w}'|}\exp(-c|\hat{w}|^3 - cs).
\end{aligned}$$

□

Proposition 5.4.23. *In this proposition, we prove estimates for the terms in the integrand of the kernel* $\widehat{K}$*. There exists* $c = c(\varepsilon) > 0$ *such that for* T *large enough, the following seven bounds hold for all* $\widehat{w} \in \widehat{\mathcal{C}^{(1)}} = \mathfrak{W}_{0,\varpi/\sigma}$, $\widehat{w}' \in \widehat{\mathcal{C}}$, *and* $\widehat{v} \in \widehat{\Gamma^{(1)}} = \mathfrak{V}_{-1,\varpi/\sigma}$*:*

$$\left|\frac{1}{1+\sigma\psi^{-1}\widehat{v}}\right| \leq c^{-1}; \tag{5.76}$$

$$\left|\frac{1}{\widehat{w}'-\widehat{v}}\right| \leq \frac{c^{-1}}{1+|\widehat{w}'|}; \tag{5.77}$$

$$\left|\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right|^{s\psi\sigma^{-1}} \leq c^{-1}\exp\left(cs\mathrm{Re}(\widehat{v})\right); \tag{5.78}$$

$$\frac{\sigma\psi^{-1}}{|\log q|}\sum_{j\neq 0}\left|\frac{1}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j+\log(1+\sigma\psi^{-1}\widehat{v})-\log(1+\sigma\psi^{-1}\widehat{w}))\right)}\right| \leq c^{-1}T^{-1/3}; \tag{5.79}$$

$$\left|\frac{\sigma\psi^{-1}}{\log q\sin\left(\pi(\log q)^{-1}(\log(1+\sigma\psi^{-1}\widehat{v})-\log(1+\sigma\psi^{-1}\widehat{w}))\right)}\right| \leq c^{-1}; \tag{5.80}$$

$$\left|\frac{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q\right)_\infty}{\left(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q\right)_\infty}\right| \leq c^{-1}\exp\left(c^{-1}\left(|\widehat{w}|+|\widehat{v}|\right)\right) \tag{5.81}$$

$$\left|\exp\left(\frac{\widehat{w}^3-\widehat{v}^3}{3}+T\left(R\left(T^{-1/3}\widehat{w}\right)-R\left(T^{-1/3}\widehat{v}\right)\right)\right)\right| \leq c^{-1}\exp\left(-\frac{1}{5}(|\widehat{w}|^3+|\widehat{v}|^3)\right) \tag{5.82}$$

Proof. The proof of all of these inequalities except for (5.78) can be found in [AB19, Proof of Lemma 6.12] without uniformity in μ. Uniformity in μ is checked in [ACG23, Proof of Lemma C.9]. In [AB19], s is fixed, and therefore they do not need estimates that depend on s. The proof of (5.78) is as follows:

First note that $|1+\sigma\psi^{-1}\widehat{w}| \geq 1$ for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$, so that we have

$$\left|\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right|^{s\psi\sigma^{-1}} \leq \left|1+\sigma\psi^{-1}\widehat{v}\right|^{s\psi\sigma^{-1}}.$$

For fixed $\widehat{v}$, we have $\lim_{T\to\infty}\left|1+\sigma\psi^{-1}\widehat{v}\right|^{s\psi\sigma^{-1}} = \exp\left(s\mathrm{Re}\widehat{v}\right)$. Without loss of generality let us suppose that we choose $\widehat{v}$ to be on the upper half plane so that $\widehat{v} = -1+\frac{r\varpi}{\sigma}e^{2\pi i/3} = -1-\frac{r\varpi}{2\sigma}+\frac{r\sqrt{3}\varpi i}{2\sigma}$ for some $0 \leq r \leq 1$. Then we have

$$1+\sigma\psi^{-1}\widehat{v} = 1-\sigma\psi^{-1}-\frac{r\varpi\psi^{-1}}{2}+\frac{r\sqrt{3}\varpi\psi^{-1}i}{2}.$$

If we choose ϖ small enough then there exists a T_0 such that for $T \geq T_0$, $\left|1+\sigma\psi^{-1}\widehat{v}\right| < 1$. Therefore, $\left|1+\sigma\psi^{-1}\widehat{v}\right|^{s\psi\sigma^{-1}}$ is increasing monotonically in T for $T \geq T_0$ and is therefore bounded by $\exp\left(s\mathrm{Re}\widehat{v}\right)$.

□

We will need the following lemma about Fredholm determinants from [AB19]:

Lemma 5.4.24 (Lemma A.4 in [AB19])**.** *We have*

$$\left|\det(\mathrm{Id}+K)_{L^2(\mathcal{C})}-1\right| \leq \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\prod_{i=1}^{k}\left|\frac{1}{k}\sum_{j=1}^{k}\left|K\left(x_i,x_j\right)\right|^2\right|^{1/2}\prod_{i=1}^{k}dx_i. \tag{5.83}$$

Proof of Proposition 5.1.9. Combining Lemma 5.4.24 with Lemma 5.4.22, we obtain the following (allowing the constant c to change between lines):

$$\begin{aligned}
&\left|\det(\mathrm{Id} + K^{(p_T)})_{L^2(\mathcal{C})} - 1\right| \\
&= \left|\det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})} - 1\right| \\
&\leq \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{(k-1)!} \int_{\widehat{\mathcal{C}}} \cdots \int_{\widehat{\mathcal{C}}} \prod_{i=1}^{k} \left| \frac{1}{k} \sum_{j=1}^{k} |K(x_i, x_j)|^2 \right|^{1/2} \prod_{i=1}^{k} dx_i \\
&\leq \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{(k-1)!} \int_{\widehat{\mathcal{C}}} \cdots \int_{\widehat{\mathcal{C}}} \prod_{i=1}^{k} \left| \frac{1}{k} \sum_{j=1}^{k} \left(c^{-1} \exp\left(-c|x_i|^3 - cs\right) + c^{-1} \exp\left(-c|x_i|^3 - cT\right)\right)^2 \right|^{1/2} \prod_{i=1}^{k} dx_i \\
&= \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{(k-1)!} \left(\int_{\widehat{\mathcal{C}}} \left(c^{-1} \exp\left(-c|x|^3 - cs\right) + c^{-1} \exp\left(-c|x|^3 - cT\right)\right) dx \right)^k \\
&= \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{(k-1)!} \left(c^{-1} e^{-cs} + c^{-1} e^{-cT}\right)^k \left(\int_{\widehat{\mathcal{C}}} c^{-1} \exp\left(-c|x|^3\right) dx \right)^k
\end{aligned}$$

This last integral is bounded above by a constant, and we can also bound $(k-1)! \geq 8^{-k} k^k$. We then obtain

$$\left|\det(\mathrm{Id} + K^{(p_T)})_{L^2(\mathcal{C})} - 1\right| \leq \sum_{k=1}^{\infty} \frac{16^k}{k^{k/2}} \left(c^{-1} e^{-cs} + c^{-1} e^{-cT}\right)^k \leq c^{-1}(e^{-cs} + e^{-cT}). \tag{5.84}$$

Combining (5.36), (5.72), and (5.84) we have the following: There exists $c = c(\varepsilon)$ such that for all T large enough and for all $s \geq 0$ we have that for $\mu \in \left[\frac{\kappa}{(\kappa\rho - \rho + 1)^2} + \varepsilon, \kappa - \varepsilon\right]$:

$$\begin{aligned}
\mathbb{P}[H^\rho(T\mu, T) \leq g(\mu)T - s f_\mu T^{1/3}] &\leq 2\left(1 - \det(\mathrm{Id} + K^{(p_T)})_{L^2(\mathcal{C})}\right) \\
&\leq c^{-1}(e^{-cs} + e^{-cT}).
\end{aligned}$$

Due to the fact that f_μ is bounded uniformly for $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$, we can absorb the constant f_μ into s by simply substituting $s \to f_\mu^{-1} s$. We then get the desired bound on $\mathbb{P}[H^\rho(T\mu, T) \leq g(\mu)T - sT^{1/3}]$ for all $\rho \in (0, 1)$. Finally, we use attractivity to obtain the same bound for step initial data (i.e. $\rho = 1$). Chose ρ such that $[\kappa^{-1} + \varepsilon, \kappa - \varepsilon] \subset \left[\frac{\kappa}{(\kappa\rho - \rho + 1)^2} + \frac{\varepsilon}{2}, \kappa - \frac{\varepsilon}{2}\right]$. Then we have that for all $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$:

$$\mathbb{P}[H(T\mu, T) \leq g(\mu)T - sT^{1/3}] \leq \mathbb{P}[H^\rho(T\mu, T) \leq g(\mu)T - sT^{1/3}] \leq c^{-1}(e^{-cs} + e^{-cT})$$

as desired. □

5.5 From linear trajectories to the proof of the main theorem

We can now begin to prove the main theorem. Let us recall the setup of Theorem 5.1.1. We start a stochastic six-vertex process from step initial conditions with a vertical second-class particle at the origin. In this section, and in Sections 5.6 and 5.7, we always view the stochastic six-vertex model as a particle system on the line (see Definition 5.1.5). On the line, the above initial conditions are given by a first-class particle at every position $x < 0$, a second-class particle at position 0 and holes at positions $x > 0$. Denote by $(\boldsymbol{\mathcal{A}}_t)_{t\geq 0}$ the single-class stochastic six-vertex process given by the first-class particles in this process and by $\boldsymbol{X}_t$ the position of the second-class particle at time t. These processes are started from the initial conditions

$$\boldsymbol{\mathcal{A}}_0(x) = \mathbf{1}_{x<0} \quad \text{and} \quad X_0 = 0\,,$$

and $(\boldsymbol{\mathcal{A}}, \boldsymbol{X})$ contains all the information of the multi-class process. Let $\mathcal{F}_s$ denote the σ-algebra generated by $(\boldsymbol{\mathcal{A}}, \boldsymbol{X})$ until time s, for $s \in \mathbb{Z}_{\geq 0}$.

Let us now define some events, which will be vital to the proof of the main theorem.

Definition 5.5.1. *Fix positive integers S and T. We define the following $\mathcal{F}_S$-measurable event, which depends on some $\varepsilon > 0$:*

$$P_S = \left\{ \frac{\boldsymbol{X}_S}{S} \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon] \right\}. \tag{5.85}$$

We also define the following $\mathcal{F}_{S+T}$-measurable events, which depend on some $\gamma \in [0, 1]$:

$$E_S^{\geq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \geq \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma} \right\} \tag{5.86}$$

$$E_S^{\leq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \leq \frac{\boldsymbol{X}_S}{S} T + S^{1-\gamma} \right\}.$$

Finally, we let $E_S := E_S^{\geq} \cap E_S^{\leq}$.

On the event P_S, the speed $\frac{\boldsymbol{X}_S}{S}$ is bounded strictly away from the edge of the rarefaction fan so that the effective hydrodynamic bounds in Corollary 5.4.2 will apply. On event E_S, we control how much the speed of the second-class particle at time $S + T$ deviates from the speed at time S. In the following proposition, we choose appropriate time steps S and T and show the existence of a high-probability $\mathcal{F}_S$-measurable *hydrodynamic event* H_S upon which E_S will hold with high probability at time $S + T$. We call H_s the hydrodynamic event since it is the event upon which at time S the height function of $\boldsymbol{\mathcal{A}}$ has not deviated too much from the hydrodynamic limit. We will show that the same is true at time $S + T$ with high probability, for any possible configuration in H_S.

Proposition 5.5.2. *For any integer $S \geq 1$, let $T = S^\beta$ for some $\beta \in (\frac{2}{3}, 1)$. For any positive $\alpha < \frac{\beta}{2} - 1/3$ and for any $\varepsilon \in (0, \frac{1}{4})$, there is a $c = c(\varepsilon, \alpha) > 0$ and an $\mathcal{F}_S$-measurable event H_S such that for all $S \geq 1$ and for $\gamma = \frac{5}{6} - \frac{\beta}{2} - \alpha$ we have*

$$\mathbb{P}[H_S] \geq 1 - c^{-1} e^{-cS^\alpha} \quad \mathbb{P}[E_S | \mathcal{F}_S] \geq (1 - c^{-1} e^{-cT^\alpha}) \mathbf{1}_{P_s \cap H_s}. \tag{5.87}$$

Note that since $T < S$, the bound on $\mathbb{P}[H_S]$ remains true if we replace S with T.

Remark 5.5.3. The parameters α, β and γ have the following meaning. The parameter β controls the size of the time steps, with β closer to 1 giving larger time steps. The parameter γ determines our control on the trajectory of the particle, with larger γ giving tighter bounds. For each β we can prove the statement of the proposition for each $\gamma < \frac{5}{6} - \frac{\beta}{2}$ and the difference between the two is α. Therefore, small α are of interest. There is a trade-off for the value of β. Bigger β gives better control along the sequence of time steps. To see this note that $S^{1-\gamma} = T^{\frac{1}{6}\beta^{-1} + \frac{1}{2} + \alpha\beta^{-1}}$, and this exponent is minimized for $\beta = 1$. However, between the time steps, bigger β leads to less control, since we use a rough bound based on monotonicity between time steps. We will ultimately set $\beta = \frac{7}{9}$, where the value $\beta = \frac{7}{9}$ is obtained by an optimization balancing these two effects.

Before proving this proposition, let us see how it implies Theorems 5.1.1 and 5.1.2. This largely follows [ACG23, Section 4], especially Proposition 5.5.8, except that there a different choice of time sequence was made, see Remark 5.5.9.

Definition 5.5.4. *For $S_0 \geq 2$ define $(S_m)_{m\geq 1}$ and $(T_m)_{m\geq 0}$ as follows. Let $T(S) = S^\beta$. Let $T_m = T(S_m)$ and $S_{m+1} = S_m + T_m$. Note that $T(S)$ is strictly increasing for all $S \geq 1$.*

Lemma 5.5.5. *There exists constants $z_- = z_-(\beta) > 0$ and $z_+ = z_+(\beta) > 0$, such that $z_- m^{\frac{1}{1-\beta}} \leq S_m \leq z_+ m^{\frac{1}{1-\beta}}$ for all $m \geq 1$.*

Proof. This follows from a straightforward induction argument, using the fact that $z_- x^{\frac{1}{1-\beta}} + (z_- x^{\frac{1}{1-\beta}})^\beta - z_-(x+1)^{\frac{1}{1-\beta}}$ is both positive at $x = 2$ and strictly increasing on $[2, \infty]$ for z_- small enough (but still positive). The upper bound also follows by induction, for z^+ bigger than 1 and such that the inequality is true for S_1. □

Lemma 5.5.6. *For each ε there exists a $D = D(\varepsilon, \beta, \alpha)$ such that for $S_0 \geq D$ the following holds*

$$\sum_{m\geq 0} S_m^{-\gamma} \leq \varepsilon/9; \quad \sum_{m\geq 0} c^{-1} e^{-cT_m^\alpha} < \varepsilon/2; \quad \kappa \frac{T_m}{S_m} < \varepsilon/9, \text{ for all } m \geq 0\,. \tag{5.88}$$

Proof. By Lemma 5.5.5 and the monotonicity of S_m in S_0 mentioned in Definition 5.5.4, it holds that $S_m \geq \max(S_0, z_- m^{\frac{1}{1-\beta}})$. The term $(z_- m^{\frac{1}{1-\beta}})^{-\gamma} = z_-^{-\gamma} m^{\frac{-\frac{5}{6}+\frac{\beta}{2}+\alpha}{1-\beta}}$ is summable since the exponent is less than -1 by the assumption $\alpha < \frac{\beta}{2} - \frac{1}{3}$. Therefore by the dominated convergence theorem as $D \to \infty$ this sum goes to 0. The second and third sums are convergent since S_m and T_m grow faster than linear in m, and therefore the sums can be bounded by geometric series. By the same dominated convergence argument, the statement follows. Finally $\kappa \frac{T_m}{S_m} < \varepsilon/4$ simply follows from the fact that $\frac{T_m}{S_m}$ goes to 0 as S_0 goes to ∞. □

Definition 5.5.7. *Define ε_m by setting $\varepsilon_0 = \varepsilon \in (0, \frac{1}{4})$ and*

$$\varepsilon_{m+1} = \varepsilon_m - S_m^{-\gamma}$$

and note that for $S_0 > D(\varepsilon, \beta, \alpha)$ all ε_m are positive (in fact they are greater than $\frac{8\varepsilon}{9}$) by Lemma 5.5.6. Define further the event

$$L_{S_0}^\varepsilon(k) = \bigcap_{m=0}^{k-1} P_{S_m}^{\varepsilon_m} \cap H_{S_m}^{\varepsilon_m} \cap E_{S_m} \tag{5.89}$$

and let $L_{S_0}^\varepsilon = L_{S_0}^\varepsilon(\infty)$.

Since the constant c in Proposition 5.5.2 can be taken to be weakly decreasing in ε, we can assume that the statement of Proposition 5.5.2 holds with the same c for all ε_m in the definition above.

Proposition 5.5.8. *There exists a constant d such that for all $\varepsilon \in (0, 1/4)$ there is a constant D such that for all $S_0 \geq D$ the probability of $L_{S_0}^\varepsilon$ is at least $1 - (d+1)\varepsilon$.*

Proof. Note first that the event $P^{\varepsilon_m}_{S_m} \cap E_{S_m}$ is contained in the event $P^{\varepsilon_{m+1}}_{S_{m+1}}$, since on the event E_{S_m} it holds that $\left|\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right| \leq S_m^{-\gamma}$. Using this we obtain

$$\mathbb{P}[L^{\varepsilon}_{S_0}(k)] = \mathbb{P}[P^{\varepsilon_0}_{S_0}] - \sum_{m=0}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c] \tag{5.90}$$

$$- \sum_{m=0}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap (E^{\varepsilon_m}_{S_m})^c] - \sum_{m=1}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap (P^{\varepsilon_m}_{S_m})^c] \tag{5.91}$$

$$\geq \mathbb{P}[P^{\varepsilon_0}_{S_0}] - \sum_{m\geq 0} \mathbb{P}[P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c] - \mathbb{P}[P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap (E_{S_m})^c]\,, \tag{5.92}$$

where we used the fact noted above to observe that $\mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap (P^{\varepsilon_m}_{S_m})^c] = 0$. By Proposition 5.A.4, $\frac{\boldsymbol{X}_t}{t}$ converges in law to a continuous random variable on $[\kappa^{-1}, \kappa]$, with density $\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}}$. This density is bounded, and therefore there is a constant d such that for all S_0 large enough $\mathbb{P}[P^{\varepsilon_0}_{S_0}] \geq 1 - d\varepsilon$, for all ε_0. Note further that by Proposition 5.5.2, both $\mathbb{P}[P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c]$ and $\mathbb{P}[P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap (E_{S_m})^c])$ are less than $c^{-1}e^{-cT^{\alpha}_m}$, and thus by Lemma 5.5.6 the right-hand side of (5.90) is at least $1 - (d+1)\varepsilon$. □

Proof of Theorem 5.1.1. We already have weak convergence to the desired distribution from Proposition 5.A.4, so it remains to prove a.s. convergence. We will show that with probability at least $1 - (d+1)\varepsilon$ the difference between the limit superior and the limit inferior of $\frac{\boldsymbol{X}_t}{t}$ is less than ε. This immediately implies that this probability is indeed 1 since these events form a decreasing family as ε goes to 0. Since this holds for any ε, the conclusion follows as the limit superior and the limit inferior must then be equal with probability 1.

Claim: Fix $\varepsilon > 0$. There exists D such that for $s, s' > D$, with probability at least $1 - (d+1)\varepsilon$:

$$\left|\frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{s'}}{s'}\right| \leq \varepsilon\,. \tag{5.93}$$

Proof of Claim: Let D be large enough such that both Lemma 5.5.6 and Proposition 5.5.8 hold. Let $S_0 = D$. For $S_0 < s < s'$ let m be the largest integer such that $S_m < s$ and m' be the smallest integer such that $s' < S_{m'}$. By Proposition 5.5.8 the event $L^{\varepsilon}_{S_0}$ holds with probability at least $1-(d+1)\varepsilon$. Assume now that the event $L^{\varepsilon}_{S_0}$ takes place, so that in particular E_{S_m} takes place for all $m \geq 0$. Then

$$\left|\frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right| \leq \left|\frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{S_{m+1}}}{s}\right| + \left|\frac{\boldsymbol{X}_{S_{m+1}}}{s} - \frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}}\right| + \left|\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right| \tag{5.94}$$

$$\leq \frac{|\boldsymbol{X}_{S_m} - \boldsymbol{X}_{S_{m+1}}|}{S_m} + \frac{|\boldsymbol{X}_{S_{m+1}}|}{S_{m+1}} \frac{S_{m+1} - S_m}{S_m} + S_m^{-\gamma} \tag{5.95}$$

$$\leq 2\left(\kappa \frac{T_m}{S_m} + S_m^{-\gamma}\right) \leq \tfrac{4}{9}\varepsilon\,, \tag{5.96}$$

where in the second inequality we used monotonicity of $\boldsymbol{X}_s$ for the first two terms, and for the third term we used that on E_{S_m} it holds that $|\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}| \leq S_m^{-\gamma}$. For the third inequality we used the event E_{S_m} to control $|\boldsymbol{X}_{S_m} - \boldsymbol{X}_{S_{m+1}}|$ and that $\frac{\boldsymbol{X}_{S_m}}{S_m} < \kappa$, since $P^{\varepsilon_m}_{S_m}$ holds. The final inequality then follows from Lemma 5.5.6. By the same argument $\left|\frac{\boldsymbol{X}_{s'}}{s'} - \frac{\boldsymbol{X}_{S_{m'}}}{S_{m'}}\right| \leq \frac{4}{9}\varepsilon$. Finally, it holds that

$$\left|\frac{\boldsymbol{X}_{S_m}}{S_m} - \frac{\boldsymbol{X}_{S_{m'}}}{S_{m'}}\right| \leq \sum_{n=m}^{m'-1} \left|\frac{\boldsymbol{X}_{S_n}}{S_n} - \frac{\boldsymbol{X}_{S_{n+1}}}{S_{n+1}}\right| \leq \sum_{n=m}^{m'-1} S_n^{-\gamma} \leq \tfrac{1}{9}\varepsilon \tag{5.97}$$

since E_{S_n} holds for all $m \leq n \leq m'$ and Lemma 5.5.6. It follows that with probability at least $1-(d+1)\varepsilon$,

$$\left|\frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{s'}}{s'}\right| \leq \varepsilon\,, \tag{5.98}$$

for all $s, s' > S_0$ which was the claim.

This now implies that with probability at least $1-(d+1)\varepsilon$,

$$\left|\limsup_{t\to\infty}\frac{\boldsymbol{X}_t}{t} - \liminf_{t\to\infty}\frac{\boldsymbol{X}_t}{t}\right| < \varepsilon, \tag{5.99}$$

and the proof of Theorem 5.1.1 follows.

□

Note that the conditions on β and α together with the definition of γ imply

$$\beta + \gamma > 1 \tag{5.100}$$

Proof of Theorem 5.1.2. For time steps given by $T(S) = S^\beta$, again for each ε there exists a $D(\varepsilon, \beta, \alpha)$ such that the event $L^\varepsilon_{S_0}$ occurs with probability at least $1-(d+1)\varepsilon$ for all $S_0 > D$. Letting C be a constant that can depend on β and α and that can change from line to line, we note that on this event we have

$$\begin{aligned}
|\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| = & \left|\boldsymbol{X}_{S_n} - S_n\left(\frac{\boldsymbol{X}_{S_n}}{S_n} + \sum_{m\geq n}\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right)\right| && (5.101)\\
=& S_n\left|\sum_{m\geq n}\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right| && (5.102)\\
\leq & S_n\sum_{m\geq n} S_m^{-\gamma} && (5.103)\\
\leq & CS_n\sum_{m\geq n} m^{-\gamma\frac{1}{1-\beta}} && (5.104)\\
\leq & CS_n n^{-\gamma\frac{1}{1-\beta}+1} && (5.105)\\
\leq & CS_n^{2-\gamma-\beta}. && (5.106)
\end{aligned}$$

In (5.104) and (5.106), we used Lemma 5.5.5 to bound S_n from above and below by constants times $n^{\frac{1}{1-\beta}}$ and we used the fact that the $-\gamma\frac{1}{1-\beta} \leq -1$ by (5.100) to ensure that p-series in (5.104) converges and can be bounded. It remains to check the behavior times s that are not of the form S_n. For s between S_n and S_{n+1} we get

$$\begin{aligned}
|\boldsymbol{X}_s - s\boldsymbol{U}| &\leq |\boldsymbol{X}_s - \boldsymbol{X}_{S_n}| + |\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| + |S_n\boldsymbol{U} - s\boldsymbol{U}| && (5.107)\\
&\leq |\boldsymbol{X}_{S_{n+1}} - \boldsymbol{X}_{S_n}| + |\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| + \kappa|S_{n+1} - S_n| && (5.108)\\
&\leq C(S_n^{1-\gamma} + S_n^{2-\gamma-\beta} + S_n^{\beta}) \leq C(s^{2-\gamma-\beta} + s^\beta),\,, && (5.109)
\end{aligned}$$

where in the second inequality, we used monotonicity of $\boldsymbol{X}_s$ in the first term and that U is bounded by κ in the third term. For the third inequality, we used that the event E_n holds for the first term, (5.106) for the second term, and the definition of S_{n+1} for the third term. Setting $\beta = \frac{7}{9}$ gives $\gamma = \frac{4}{9} - \alpha$, which gives $2-\gamma-\beta = \frac{7}{9} + \alpha$, therefore showing that on an event of probability $1-(d+1)\varepsilon$ the limit of $|\boldsymbol{X}_S - s\boldsymbol{U}|s^{-\frac{7}{9}-2\alpha}$ is 0, which concludes the proof. □

Remark 5.5.9. While we used the sequence $S_{n+1} = S_n + S_n^\beta$, which grows polynomially, in [ACG23], they used the sequence $S_{n+1} = S_n + \frac{S_n}{\log S_n}$, which grows like $e^{\sqrt{n}}$. The latter sequence would have sufficed to prove Theorem 5.1.1. Furthermore, this time scale gives

$$|X_{S_n} - S_n U| \le S_n^{\frac{2}{3}+}$$

which is the expected order of fluctuations. However, it grows too quickly to prove the finer statement in Theorem 5.1.2 as in (5.109), we crucially used the fact that $|S_{n+1} - S_n| \le s^\beta$ to bound the fluctuations between the times S_n.

5.6 From hydrodynamic events to linear trajectories

The purpose of this section is to prove Proposition 5.5.2. We write the full proof only for $E_S^{\ge}$, since $E_S^{\le}$ can be treated similarly. We will outline the proof for $E^{\le}$ at the end of Section 5.7.

Let us now proceed with the proof for the process $E^{\ge}$. We couple the process $(\boldsymbol{\mathcal{A}}, \boldsymbol{X})$ to a new multi-class stochastic six-vertex process $\boldsymbol{\mathcal{B}}$ by filling in every position to the left of $\boldsymbol{X}_S$ with particles. Then Proposition 5.1.7 allows us to control the position of $\boldsymbol{X}_{S+T}$ by controlling a large number of these additional particles.

Definition 5.6.1. *Define a new multi-class process $(\boldsymbol{\mathcal{B}}_t)_{t\ge0}$, with first-, second- and third-class particles, depending on $\mathcal{A}_S$ in the following way:*

- *It has the same parameters b_1 and b_2.*
- *At time 0 each site $j \in \mathbb{Z}$ in $\boldsymbol{\mathcal{B}}_0$ is occupied by a first-class particle at time 0 if it is occupied by a particle in $\boldsymbol{\mathcal{A}}_S$,*
- *the site $\boldsymbol{X}_S$ is occupied by a second-class particle,*
- *and each site to the left of $\boldsymbol{X}_S$ not occupied by a particle in $\boldsymbol{\mathcal{A}}_S$ is also occupied by a third-class particle.*

Further let $\boldsymbol{M}$ be the number of third-class particles in $\boldsymbol{\mathcal{B}}$, which is finite, since it is at most $\boldsymbol{X}_S$. Further, let $\boldsymbol{Z}_t(0) > \boldsymbol{Z}_t(1) > \cdots > \boldsymbol{Z}_t(\boldsymbol{M})$ denote the ordered positions of the second-class particle and the third-class particles in $\boldsymbol{\mathcal{B}}_t$. At time 0 we have $\boldsymbol{Z}_0(0) = \boldsymbol{X}_S$, but at later times, $\boldsymbol{X}_{S+t}$ can be any of the positions $\boldsymbol{Z}_t$.

We will also consider the two single-class processes obtained by merging the second-class and third-class particles in $\boldsymbol{\mathcal{B}}$ with either the holes or the first-class particles. That is, let $(\boldsymbol{\mathcal{B}}_t^{(1)})_{t\ge0}$ be the single-class process given by just the first-class particles in $\boldsymbol{\mathcal{B}}$ and let $(\boldsymbol{\mathcal{B}}_t^{(1,2,3)})_{t\ge0}$ be the single-class process of the first-, second- and third-class particles in $\boldsymbol{\mathcal{B}}$ forgetting their classes. The triplet $(\boldsymbol{\mathcal{B}}_t^{(1)}, \boldsymbol{\mathcal{B}}_t^{(1,2,3)}, \boldsymbol{X}_{S+t})_{t\ge0}$ contains all the information of $\boldsymbol{\mathcal{B}}$.

The process $\boldsymbol{\mathcal{B}}$ depends both on $\boldsymbol{\mathcal{A}}_S$ and $\boldsymbol{X}_S$. We will often use a union bound over possible values of $\boldsymbol{X}_S$. To do so it will be convenient to introduce a version of $\boldsymbol{\mathcal{B}}$ in which the position of the second-class particle is replaced by a deterministic position X. For X in the interval $[S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \cap \mathbb{Z}$, define $\boldsymbol{\mathcal{B}}^X$ as the process obtained from $\boldsymbol{\mathcal{A}}_S$ by adding a second-class particle at X if that position is empty and filling all empty positions $k < X$ with third-class particles (note that $\boldsymbol{\mathcal{B}}^X$ may not have a second-class particle if there is a particle at position X in $\boldsymbol{\mathcal{A}}_S$). Let further $\boldsymbol{\mathcal{B}}^{(1,2,3),X}$ be the single-class process of the first- and second-class particles in $\boldsymbol{\mathcal{B}}^X$. Note that it possible to couple all the processes $\boldsymbol{\mathcal{B}}^X$, $\boldsymbol{\mathcal{B}}$,

$\boldsymbol{\mathcal{A}}$ and $\boldsymbol{X}$ such that the first-class particles in $\boldsymbol{\mathcal{B}}_t^X$ and $\boldsymbol{\mathcal{B}}_t$ are given by $\boldsymbol{\mathcal{A}}_{S+t}$, and such that substituting $\boldsymbol{X_S}$ for X it holds that

$$\boldsymbol{\mathcal{B}}^{\boldsymbol{X}_S} = \boldsymbol{\mathcal{B}}\,. \tag{5.110}$$

Note however that the law of $\boldsymbol{\mathcal{B}}$ conditioned on $\boldsymbol{X}_S = X$ is not given by $\boldsymbol{\mathcal{B}}^X$, since $\boldsymbol{X}_S$ and the first-class particles in $\boldsymbol{\mathcal{B}}$ are non-trivially correlated. However, by showing that certain events happen for all $\boldsymbol{\mathcal{B}}^X$ with exponentially small probability, one can use (5.110) to show that they also happen for $\boldsymbol{\mathcal{B}}$ with exponentially small probability.

Lemma 5.6.2. *Let $\boldsymbol{G} \sim \mathrm{Geo}(q)$ be independent of $\mathcal{F}_S$ and $\boldsymbol{Z}$. Then for any $y \in \mathbb{Z}$ and any $S, T \geq 1$ it holds that*

$$\mathbb{P}[\boldsymbol{X}_{S+T} \geq \boldsymbol{X}_S + y | \mathcal{F}_S] \geq \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{G} \wedge \boldsymbol{M}) \geq \boldsymbol{X}_S + y|\mathcal{F}_S]\,. \tag{5.111}$$

Proof. Denote the number of third-class particles that are to the right of the second-class particle in $\boldsymbol{\mathcal{B}}_T$ by $\boldsymbol{K}$, such that $\boldsymbol{X}_{S+T} = \boldsymbol{Z}_T(\boldsymbol{K})$. By Proposition 5.1.7, the law of $\boldsymbol{K}$ conditioned on $\mathcal{F}_S$ and $\boldsymbol{Z}_T$ is dominated by $\mathrm{Geo}(q)$. Therfore $\boldsymbol{K}$ can be coupled to a random variable $\boldsymbol{G}^* \sim \mathrm{Geo}(q)$ independent of $\mathcal{F}_S$ and $\boldsymbol{Z}_T$ such that $\boldsymbol{K} \leq \boldsymbol{G}^*$ almost surely. Thus we obtain

$$\mathbb{P}[\boldsymbol{X}_{S+T} \geq \boldsymbol{X}_S + y|\mathcal{F}_S] = \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{K}) \geq \boldsymbol{X}_S + y|\mathcal{F}_S] \tag{5.112}$$

$$\geq \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{G}^* \wedge \boldsymbol{M}) \geq \boldsymbol{X}_S + y|\mathcal{F}_S] \tag{5.113}$$

where in (5.113) we used $\boldsymbol{K} \leq \boldsymbol{G}^* \wedge \boldsymbol{M}$ and the ordering of $\boldsymbol{Z}_T$. Note that the right-hand side does not depend on the coupling between $\boldsymbol{K}$ and $\boldsymbol{G}^*$ since $\boldsymbol{G}^*$ is independent of $\mathcal{F}_S$ and $\boldsymbol{Z}$. Therefore, we can replace $\boldsymbol{G}^*$ by $\boldsymbol{G}$ in (5.113).

□

Let $\boldsymbol{L}$ be defined as

$$\boldsymbol{L} = \#\left\{\text{second- and third-class class particles in } \boldsymbol{\mathcal{B}}_T \text{ to the right of } \frac{\boldsymbol{X}_S}{S}(S+T) - S^{1-\gamma}\right\}. \tag{5.114}$$

Using Lemma 5.6.2, we can reduce the proof of Proposition 5.5.2 to the following lemma which states that $\boldsymbol{L}$ is of order at least $S^{\frac{1}{3}}$ with high probability.

Lemma 5.6.3. *For any positive $\varepsilon < \frac{1}{4}$, and for T, α and γ as in Proposition 5.5.2, there is a constant $c > 0$ and an $\mathcal{F}_S$-measurable event H_S such that for all $S \geq 1$*

$$\mathbb{P}(H_S) \geq 1 - c^{-1}e^{-cS^\alpha} \tag{5.115}$$

and

$$\mathbb{P}[\boldsymbol{L} \geq S^{\frac{1}{3}}|\mathcal{F}_S] \geq (1 - c^{-1}e^{-cT^\alpha})\mathbf{1}_{P_S \cap H_S}\,. \tag{5.116}$$

Before proving this proposition let us see how it implies Proposition 5.5.2.

Proof of Proposition 5.5.2. As said above, we only prove the statement for $E_S^{\gtrless}$. Condition on $\mathcal{F}_S$ and assume that $H_S \cap P_S$ holds. Let $\boldsymbol{G} \sim \mathrm{Geo}(q)$ be independent of $\boldsymbol{Z}$ and $\mathcal{F}_S$, as above. Define the events

$$F_S = \left\{\boldsymbol{Z}_T(\boldsymbol{G} \wedge \boldsymbol{M}) \geq \frac{\boldsymbol{X}_S}{S}(S+T) + S^{1-\gamma}\right\} \quad \text{and} \quad G_S = \left\{\boldsymbol{L} > S^{\frac{1}{3}}\right\} \tag{5.117}$$

and recall the definition of the event $E_S^{\geq}$ is

$$E_S^{\geq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \geq \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma} \right\} .$$

Setting $y = \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma}$, it follows from Lemma 5.6.2 that $\mathbb{P}[E_S^{\geq} | \mathcal{F}_S] \geq \mathbb{P}[F_S | \mathcal{F}_S]$. By the distribution of $\boldsymbol{G}$,

$$\mathbb{P}[F_S | G_S, \mathcal{F}_S] \geq \mathbb{P}[\boldsymbol{G} \leq S^{1/3}] \geq 1 - q^{S^{\frac{1}{3}}} .$$

Combining this with the statement of Lemma 5.6.3 gives

$$\begin{aligned} \mathbb{P}[E_S^{\geq} | \mathcal{F}_S] &\geq \mathbb{P}[F_S | \mathcal{F}_S] \\ &\geq \mathbb{P}[F_S | G_S, \mathcal{F}_S] \mathbb{P}[G_S | \mathcal{F}_S] \\ &\geq \left(1 - q^{S^{\frac{1}{3}}} \right) \left(1 - c^{-1} e^{-cT^{\alpha}} \right) \mathbf{1}_{H_S \cap P_S} \\ &\geq (1 - c^{-1} e^{-cT^{\alpha}}) \mathbf{1}_{H_S \cap P_S} . \end{aligned}$$

The final inequality is obtained by decreasing c so that we can absorb the term $q^{S^{\frac{1}{3}}}$ (since $\alpha \leq 1/3$ and $S \geq T$). □

5.7 From effective hydrodynamics to hydrodynamic events

The purpose of this section is to prove Lemma 5.6.3. To this end let $(\boldsymbol{\mathcal{B}}^{\text{step},X})_{t \geq 0}$ be a stochastic six-vertex process started from step initial conditions shifted by X, i.e. $\boldsymbol{\mathcal{B}}_0^{\text{step},X}(x) = \mathbf{1}_{x \leq X}$. Clearly, at time 0 we have that $\boldsymbol{\mathcal{B}}_0^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_0^{\text{step},X}(x)$ for all $x \in \mathbb{Z}$, so by attractivity (Proposition 5.2.3) we can couple them so that this holds for any later time as well. Note that $\boldsymbol{\mathcal{B}}^{(1,2,3),X}$ and $\boldsymbol{\mathcal{B}}^{(1)}$ are already coupled such that $\boldsymbol{\mathcal{B}}_t^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_t^{(1)}(x)$ for any time t by their relation to the multi-class process $\boldsymbol{\mathcal{B}}^X$. This gives a coupling of the three processes $\boldsymbol{\mathcal{B}}^{(1,2,3),X}, \boldsymbol{\mathcal{B}}^{(1)}, \boldsymbol{\mathcal{B}}^{\text{step},X}$ such that at all times t it holds that

$$\boldsymbol{\mathcal{B}}_t^{(1,2,3),X}(x) \geq \max(\boldsymbol{\mathcal{B}}_t^{(1)}(x), \boldsymbol{\mathcal{B}}_t^{\text{step},X}(x)) \text{ for all } x \in \mathbb{Z}.$$

The process $\boldsymbol{\mathcal{B}}^{\text{step},X}$ is a stochastic six-vertex process started from step initial conditions, thus we can use Corollary 5.4.2 to control its height function. Understanding $\boldsymbol{\mathcal{B}}^{(1)}$ is more intricate since its initial conditions are given by $\boldsymbol{\mathcal{A}}_S$. We will not be able to control $\boldsymbol{\mathcal{B}}^{(1)}$ for all values of $\boldsymbol{\mathcal{A}}_S$, and instead find an $\mathcal{F}_S$-measurable event H_S, which we call the *hydrodynamic event*, that holds with high probability. On this event, with high probability, $\boldsymbol{\mathcal{B}}_T^{(1)}$ is close to the hydrodynamic profile of a stochastic six-vertex process started from step initial conditions and evaluated at time $S+T$.

To simplify notation throughout this section, we will define $h_t([X,Y];\eta) := h_t(X;\eta) - h_t(Y;\eta)$. Let us now define the following shorthand notation for the event that a process is close to its hydrodynamic limit.

Definition 5.7.1. *For a single-class stochastic six-vertex process $(\boldsymbol{\mathcal{A}}_t)_{t \geq 0}$ started from step initial conditions, a time t and $\alpha, \varepsilon > 0$ define the event $C_t^{\alpha,\varepsilon}(\boldsymbol{\mathcal{A}})$ as*

$$\left\{ \left| h_t([X,Y];\boldsymbol{\mathcal{A}}) - t(g(\tfrac{X}{t}) - g(\tfrac{Y}{t})) \right| \leq t^{\frac{1}{3}+\alpha}, \text{ for all } X \text{ and } Y \text{ in } [t(\kappa^{-1}+\varepsilon), t(\kappa-\varepsilon)] \right\} . \tag{5.118}$$

Note that this event does not depend on the choice of height function $h(x,\boldsymbol{\mathcal{A}})$, since it only concerns height function differences.

Using this definition, Corollary 5.4.2 with $s = t^\alpha$ gives $\mathbb{P}[C_t^{\alpha,\varepsilon}(\boldsymbol{\mathcal{A}})] \geq 1 - c^{-1}t^2 e^{-ct^\alpha}$, with the constant $c > 0$ depending on ε.

Proposition 5.7.2. *For $T = T(S)$ as in Proposition 5.5.2, for $\alpha, \varepsilon > 0$, and for $S \geq 1$, there is a constant $c = c(\varepsilon)$ and an $\mathcal{F}_S$-measurable event H_S that holds with probability $1 - c^{-1}e^{-cS^\alpha}$ such that*

$$\mathbb{P}\left[C_{S+T}^{\alpha,\varepsilon}(\boldsymbol{\mathcal{A}})|\mathcal{F}_S\right] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha}) . \tag{5.119}$$

Proof. By making c smaller the right-hand side of (5.118) can be made non-positive for S small, so it suffices to consider S large enough. Let $(\widetilde{\boldsymbol{\mathcal{A}}}_t)_{0\leq t\leq S}$ be an independent copy of $(\boldsymbol{\mathcal{A}}_t)_{0\leq t\leq S}$, i.e. a stochastic six-vertex process started from step initial conditions and run until time S. After time S we will couple these two processes, such that they are no longer independent.

Define $H_S = C_S^{\frac{\alpha}{2},\frac{\varepsilon}{2}}(\boldsymbol{\mathcal{A}})$. This event has the desired probability by Corollary 5.4.2 and is $\mathcal{F}_S$-measurable. Further, let $\widetilde{H}_S = C_S^{\frac{\alpha}{2},\frac{\varepsilon}{2}}(\widetilde{\boldsymbol{\mathcal{A}}})$. On the intersection of H_S and $\widetilde{H}_S$ it holds that

$$\left|h_S([X,Y];\boldsymbol{\mathcal{A}}) - S\left(g\left(\tfrac{X}{S}\right) - g\left(\tfrac{Y}{S}\right)\right)\right| \leq S^{\frac{1}{3}+\frac{\alpha}{2}}, \text{ for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)] \tag{5.120}$$

and

$$\left|h_S([X,Y];\widetilde{\boldsymbol{\mathcal{A}}}) - S\left(g\left(\tfrac{X}{S}\right) - g\left(\tfrac{Y}{S}\right)\right)\right| \leq S^{\frac{1}{3}+\frac{\alpha}{2}}, \text{ for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)]. \tag{5.121}$$

These events do not depend on the choice of height function and therefore we can choose the height functions which satisfy $h_S(S(\kappa - \frac{\varepsilon}{2}), \boldsymbol{\mathcal{A}}) = h_S(S(\kappa - \frac{\varepsilon}{2}), \widetilde{\boldsymbol{\mathcal{A}}}) = 0$. With this choice of height function and setting $Y = S(\kappa - \frac{\varepsilon}{2})$ in (5.120) and (5.121), it follows that on the intersection $H_S \cap \widetilde{H}_S$ we have

$$|h_S(X;\boldsymbol{\mathcal{A}}) - h_S(X;\widetilde{\boldsymbol{\mathcal{A}}})| \leq 2S^{\frac{1}{3}+\frac{\alpha}{2}} \text{ for } X \in [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)] .$$

Thus we can couple $(\widetilde{\boldsymbol{\mathcal{A}}}_t)_{t\geq S}$ with $(\boldsymbol{\mathcal{A}}_t)_{t\geq S}$ via the coupling given in Lemma 5.2.9 with $M = S^{\frac{1}{3}}$ and $K = 2S^{\frac{1}{3}+\frac{\alpha}{2}}$. Denote the event that this coupling succeeds, i.e.

$$\left\{\left|h_{S+T}(X;\widetilde{\boldsymbol{\mathcal{A}}}) - h_{S+T}(X;\boldsymbol{\mathcal{A}})\right| < 6S^{\frac{1}{3}+\frac{\alpha}{2}} + S^{\frac{1}{3}} \text{ for } X \in [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa - \tfrac{\varepsilon}{2})]\right\} \tag{5.122}$$

as D. Then the statement of Lemma 5.2.9 gives

$$\mathbb{P}[D|H_S \cap \widetilde{H}_S, \mathcal{F}_S] \geq 1 - c^{-1}e^{-cT} - c^{-1}e^{-cS^{\frac{1}{3}}} ,$$

which implies that

$$\mathbb{P}[D|\mathcal{F}_S] \geq \mathbb{P}[D|H_S \cap \widetilde{H}_S, \mathcal{F}_S]\mathbb{P}[H_S \cap \widetilde{H}_S|\mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha}) , \tag{5.123}$$

where we used that $\widetilde{H}_S$ is independent of $\mathcal{F}_S$ and both $c^{-1}e^{-cT}$ and $c^{-1}e^{-cS^{\frac{1}{3}}}$ were absorbed into $c^{-1}e^{-cS^\alpha}$ by decreasing c.

Consider now the event

$$D \cap C_{S+T}^{\frac{\alpha}{2},\frac{\varepsilon}{2}}(\widetilde{\boldsymbol{\mathcal{A}}}) . \tag{5.124}$$

On this event, it holds that

$$|h_{S+T}([X,Y];\boldsymbol{\mathcal{A}}) - (S+T)(g(\tfrac{X}{S+T}) - g(\tfrac{Y}{S+T}))| \leq 13S^{\frac{1}{3}+\frac{\alpha}{2}} + 2S^{\frac{1}{3}}$$
$$\text{for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa-\tfrac{\varepsilon}{2})] , \tag{5.125}$$

by repeated use of the triangle inequality. For large enough S we have

$$[S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \subset [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa-\tfrac{\varepsilon}{2})]$$

since $T = S^\beta$ for some $\beta < 1$. Furthermore, for large enough S

$$13S^{\frac{1}{3}+\frac{\alpha}{2}} + 2S^{\frac{1}{3}} \le S^{\frac{1}{3}+\alpha}.$$

Therefore, for large enough S, the event $C^{\alpha,\varepsilon}_{S+T}(\boldsymbol{\mathcal{A}})$ contains $D \cap C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\boldsymbol{\mathcal{A}}})$. Using (5.123) and noting that $C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\boldsymbol{\mathcal{A}}})$ is independent of $\mathcal{F}_S$ we obtain that

$$\begin{aligned}
\mathbb{P}[C^{\alpha,\varepsilon}_{S+T}(\boldsymbol{\mathcal{A}})|\mathcal{F}_S] &\ge \mathbb{P}[D \cap C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\boldsymbol{\mathcal{A}}})|\mathcal{F}_S] && (5.126)\\
&\ge (\mathbb{P}[D|\mathcal{F}_S] - \mathbb{P}[C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\boldsymbol{\mathcal{A}}})^c|\mathcal{F}_S]) \vee 0 && (5.127)\\
&\ge (1 - c^{-1}e^{-cS^\alpha} - c^{-1}(S+T)^2 e^{-c(S+T)^\alpha})\mathbf{1}_{H_S} && (5.128)\\
&\ge (1 - c^{-1}e^{-cS^\alpha})\mathbf{1}_{H_S}, && (5.129)
\end{aligned}$$

where $(c^{-1}(S+T)^2 e^{-c(S+T)^\alpha})\mathbf{1}_{H_S}$ is absorbed by increasing c. □

We can now prove Lemma 5.6.3.

Proof of Lemma 5.6.3. We can decrease c such that (5.116) is trivial for S small and therefore it suffices to consider large enough S. First fix some $X \in [S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \cap \mathbb{Z}$. By Proposition 5.7.2 we have

$$\mathbb{P}[C^{\alpha,\varepsilon/2}_{S+T}(\boldsymbol{\mathcal{A}})|\mathcal{F}_S] \ge \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha}).$$

The event $C^{\alpha,\varepsilon/2}_{S+T}(\boldsymbol{\mathcal{A}})$ states a bound for all pairs of points in $[(S+T)(\kappa^{-1}+\varepsilon/2), (S+T)(\kappa-\varepsilon/2)]$. For S large enough both $\frac{X}{S}(S+T) - S^{1-\gamma}$ and $\frac{X}{S}(S+T)$ are in this interval since $\frac{X}{S} \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $\frac{S^{1-\gamma}}{S+T} \le \varepsilon/2$ for S large enough. Since the law of $\boldsymbol{\mathcal{B}}^{(1)}_T$ is equal to the law of $\boldsymbol{\mathcal{A}}_{S+T}$, we have

$$\mathbb{P}\Big[\big|h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{(1)}) - (S+T)(g(\tfrac{X}{S} - \tfrac{S^{1-\gamma}}{S+T}) - g(\tfrac{X}{S}))\big| \le (S+T)^{\frac{1}{3}+\alpha}\big|\mathcal{F}_S\Big] \ge (1 - c^{-1}e^{-cS^\alpha})\mathbf{1}_{H_S}. \quad (5.130)$$

The process $\boldsymbol{\mathcal{B}}^{\text{step},X}$ is also started from step initial data translated by X. For large enough S both $\frac{X}{S}(S+T) - S^{1-\gamma} - X$ and $\frac{X}{S}(S+T) - X$ are in $[T(\kappa^{-1}+\varepsilon/2), T(\kappa-\varepsilon/2)]$. Indeed $\frac{X}{S}(S+T) - X = \frac{X}{S}T$, and $\frac{S^{1-\gamma}}{T} < \varepsilon/2$ since $T = S^\beta$ and $\beta + \gamma > 1$ (see (5.100)). Thus we can apply Theorem 5.4.1 to obtain:

$$\mathbb{P}\Big[\big|h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{\text{step},X}) - T(g(\tfrac{X}{S} - \tfrac{S^{1-\gamma}}{T}) - g(\tfrac{X}{S}))\big| \le T^{\frac{1}{3}+\alpha}\Big] \ge 1 - c^{-1}e^{-cT^\alpha}. \quad (5.131)$$

Let us compare the limit shape terms in (5.130) and (5.131) by bounding their difference

$$\Delta = T\left(g\left(\frac{X}{S} - \frac{S^{1-\gamma}}{T}\right) - g\left(\frac{X}{S}\right)\right) - (S+T)\left(g\left(\frac{X}{S} - \frac{S^{1-\gamma}}{S+T}\right) - g\left(\frac{X}{S}\right)\right), \quad (5.132)$$

By the explicit form of $g(x) = \frac{(\sqrt{x}-\sqrt{\kappa})^2}{\kappa-1}$, one can easily see that on the interval $[\kappa^{-1},\kappa]$ both the first and second derivative of g are uniformly bounded and in particular $g''(x) \geq C$ for some C depending only on κ, for all $x \in [\kappa^{-1},\kappa]$. Considering the second-order Taylor expansion of g at $\frac{X}{S}$, we have

$$\begin{aligned} T(g(\tfrac{X}{S} - \tfrac{S^{1-\gamma}}{T}) - g(\tfrac{X}{S})) &= -T(g'(\tfrac{X}{S})\tfrac{S^{1-\gamma}}{T} + \tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{1-\gamma}}{T})^2 + O(\tfrac{S^{1-\gamma}}{T})) \\ &= -g'(\tfrac{X}{S})S^{1-\gamma} + \tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{2-2\gamma}}{T}) + O(\tfrac{S^{3-3\gamma}}{T^2}). \end{aligned}$$

Similarly for the other term in (5.132) we obtain

$$(S+T)(g(\tfrac{X}{S} - \tfrac{S^{1-\gamma}}{S+T}) - g(\tfrac{X}{S})) = -g'(\tfrac{X}{S})S^{1-\gamma} + \tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{2-2\gamma}}{S+T}) + O(\tfrac{S^{3-3\gamma}}{(S+T)^2})$$

For large enough S the error terms are smaller than the second order terms, since $\frac{S^{1-\gamma}}{T} \to 0$ and we obtain

$$\Delta \geq CS^{2-2\gamma}\left(\frac{1}{T} - \frac{1}{S+T}\right) \geq \frac{C}{2}S^{2-2\gamma}T^{-1} \geq \frac{C}{2}S^{1/3+2\alpha} \tag{5.133}$$

where the linear terms cancel each other out and we use that $T = S^\beta$ and $\gamma = \frac{5}{6} - \frac{\beta}{2} - \alpha$.

Let $\boldsymbol{L}^X$ be defined as

$$\boldsymbol{L}^X = \#\{\text{second- and third-class particles in } \boldsymbol{\mathcal{B}}_T^X \text{ to the right of } \frac{X}{S}(S+T) - S^{1-\gamma}\}$$

and note that

$$\boldsymbol{L}^X = \sum_{x \geq \frac{\boldsymbol{X}_S}{S}(S+T)-S^{1-\gamma}} \boldsymbol{\mathcal{B}}_T^{(1,2,3),X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x) \geq \sum_{x=\frac{\boldsymbol{X}_S}{S}(S+T)-S^{1-\gamma}}^{\frac{X}{S}(S+T)} \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x)\,,$$

since $\boldsymbol{\mathcal{B}}_T^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x)$ for every $x \in \mathbb{Z}$. If the events in (5.130) and (5.131) take place this sum can be bounded from below by

$$\begin{aligned} &\sum_{x=\frac{\boldsymbol{X}_S}{S}(S+T)-S^{1-\gamma}}^{\frac{X}{S}(S+T)} \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x) \\ &= h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{\text{step},X}) - h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{(1)})) \\ &\geq CS^{\frac{1}{3}+2\alpha} - (S+T)^{\frac{1}{3}+\alpha} - T^{\frac{1}{3}+\alpha} \geq S^{\frac{1}{3}}\,, \end{aligned} \tag{5.134}$$

for S large enough. Therefore

$$\mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} | \mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cT^\alpha})\,,$$

where we use that the event (5.131) is independent of $\mathcal{F}_S$, since it only depends on $\boldsymbol{\mathcal{B}}^{\text{step},X}$, which is only coupled to $\boldsymbol{\mathcal{B}}$ after time S. Using a union bound we obtain

$$\mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} \text{ for all } X \in [S(\kappa^{-1}+\varepsilon), S(\kappa+\varepsilon)] \cap \mathbb{Z} | \mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - Sc^{-1}e^{-cT^\alpha})$$

and we further can absorb S into $c^{-1}e^{-cT^\alpha}$ by decreasing c. By the definition of $\boldsymbol{L}$ in (5.114), the definition of P_S in (5.85), and the observation (5.110) we have that

$$\begin{aligned} \mathbb{P}[\boldsymbol{L} \geq S^{\frac{1}{3}} | \mathcal{F}_S] &\geq \mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} \text{ for all } X \in [S(\kappa^{-1}+\varepsilon), S(\kappa+\varepsilon)] \cap \mathbb{Z} | \mathcal{F}_S]\mathbf{1}_{P_S} \\ &\geq \mathbf{1}_{H_S \cap P_S}(1 - c^{-1}e^{-cT^\alpha}) \end{aligned}$$

as desired. □

Let us now sketch what needs to be changed for $E^{\leq}$. We need to show that on the hydrodynamic event (which is the same) the second-class particle does not deviate too much to the right. To do so we will delete all particles to the right of the second-class particle, which corresponds to a multi-class particle system $(\boldsymbol{\mathcal{B}}_t)_{t\geq 0}$ with the following initial conditions

- A first-class particle in every position that is occupied by a first-class particle in $\boldsymbol{\mathcal{A}}_S$ and is to the left of $\boldsymbol{X}_S$,
- A second-class particle in every position that is occupied by a first-class particle in $\boldsymbol{\mathcal{A}}_S$ that is to the right of $\boldsymbol{X}_S$ and
- A third-class particle in the position $\boldsymbol{X}_S$.

This process satisfies the conditions of Corollary 5.3.2 and by an argument analogous to Lemma 5.6.3 it suffices to show that there are a large number of second-class particles in $\boldsymbol{\mathcal{B}}_T$ to the left of $\frac{\boldsymbol{X}_S}{S}(S+T)+S^{1-\gamma}$. Denoting by $\boldsymbol{\mathcal{B}}^{(1)}$ the process of the first-class particles in $\boldsymbol{\mathcal{B}}$, by $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ the process of the first, second- and third-class particles, and by $\boldsymbol{\mathcal{B}}^{\text{step}}$ a stochastic six-vertex process with step initial conditions translated to position $\boldsymbol{X}_t$. At time 0 we have

$$\boldsymbol{\mathcal{B}}_0^{(1)}(x) \leq \min(\boldsymbol{\mathcal{B}}_0^{(1,2,3)}(x), \boldsymbol{\mathcal{B}}_0^{\text{step}}(x))\,,$$

so $\boldsymbol{\mathcal{B}}^{\text{step}}$ can be coupled to $\boldsymbol{\mathcal{B}}^{(1)}$ such that $\boldsymbol{\mathcal{B}}_t^{(1)}(x) \leq \boldsymbol{\mathcal{B}}_t^{\text{step}}(x)$ at all later times t as well. Note that $\boldsymbol{\mathcal{B}}_t^{(1)}(x) \leq \boldsymbol{\mathcal{B}}_t^{(1,2,3)}(x)$ already holds by definition. By Proposition 5.7.2, $\boldsymbol{\mathcal{B}}_T^{(1,2,3)} = \boldsymbol{\mathcal{A}}_{S+T}$ is close to the hydrodynamic limit at time $S+T$ with high probability. Since $\boldsymbol{\mathcal{B}}^{\text{step}}$ is also a stochastic six-vertex model started from step initial conditions we can use Theorem 5.4.1 to say that it is also close to the hydrodynamic limit at time T translated by $\boldsymbol{X}_T$ with high probability. These two results, together with a union bound over all possible values of $\boldsymbol{X}_t$ and a calculation similar to (5.133), yield the desired result.

Remark 5.7.3. For ASEP one could have simply used particle-hole duality to obtain the proof for $E^{\leq}$ as a corollary of the proof for $E^{\geq}$. After exchanging first-class particles with holes, and reversing space, one again obtains a multi-class ASEP, and the events $E^{\geq}$ and $E^{\leq}$ are exchanged. For the stochastic six-vertex model, this is not the case. Applying the particle-hole duality for the stochastic six-vertex model exchanges the two axes and therefore maps the event $E^{\geq}$ into an event that concerns the times at which the second-class particle hits positions S and $S+T$. This is clearly not the same as the event $E^{\leq}$. A different choice of $E^{\geq}$ and $E^{\leq}$ such that they are symmetric with respect to the particle-hole symmetry of the stochastic six-vertex model could be considered.

5.8 Symmetry and stationarity of the speed process

In this section, we will prove Corollary 5.1.6 and discuss various properties of the stochastic six-vertex model speed process.

Proof of Corollary 5.1.6. By the color merging property, the law of $(\boldsymbol{X}_t(x))_{t\geq 0}$ is equal to that of $(x+\boldsymbol{X}_t(0))_{t\geq 0}$. By Theorem 5.1.1 the speed $\frac{\boldsymbol{X}_t(0)}{t}$ converges almost surely, and therefore so does each speed $\frac{\boldsymbol{X}_t(x)}{t}$. Since there are countably many particles, this also implies that almost surely all of the speeds converge. □

An immediate consequence of the construction is the ergodicity of the speed process. In this section *ergodic* always refers to ergodicity with respect to translations of $\mathbb{Z}$, i.e. of space.

Proposition 5.8.1 (Ergodicity for the Speed Process)**.** *The stochastic six-vertex speed process is ergodic.*

Proof. This is immediately inherited from the fact that the process $(\boldsymbol{X}_t(x))_{x\in\mathbb{Z},t\geq 0}$ can be constructed by sampling i.i.d. pairs of Bernoulli(b_1) and Bernoulli(b_2) random variables at every vertex, which are clearly ergodic under the shift. □

To obtain stationarity of the speed process we need the following symmetry, which is a special case of [BB19, Corollary 7.1.]

Proposition 5.8.2 (Color Position Symmetry in Finite Domains)**.** *Consider the stochastic six-vertex model on an $M \times N$ box, with particles of class 1 to $M+N$ coming in on the left and lower boundaries such that from the top left to the bottom right the classes are in increasing order. Enumerate the outgoing positions along the top and right boundary with $\{1,\dots,M+N\}$ in descending order, first from left to right along the top and then from top to bottom along the right edge. Denote by π the (random) permutation of $\{1,\dots,M+N\}$ obtained by letting $\pi(x)$ is the class of the particle at position x. Then π and π^{-1} are equal in law.*

Proof. This follows from [BB19, Corollary 7.1] by specializing the Ferrer diagram S to a rectangle and using the fact that rectangles are invariant under point reflections. □

We now extend this to the stochastic six-vertex model on the line.

Proposition 5.8.3 (Color Position Symmetry on the Line)**.** *Consider the random bijection $\pi_N : \mathbb{Z} \to \mathbb{Z}$ obtained by running the stochastic six-vertex model from packed initial conditions until time N (i.e. on a box of infinite width and height $N+1$) and letting $\pi_N(x)$ be the class of the particle exiting at the vertex (x, N). Then $(\pi_N(x))_{x\in\mathbb{Z}}$ and $(-\pi_N^{-1}(-x))_{x\in\mathbb{Z}}$ are equal in law.*

Proof. Consider the box $[\![-M, M]\!] \times [\![0, N]\!]$. Consider the boundary conditions consisting of the incoming arrows from the left with $\{-M-N-1,\dots,-M-1\}$, in increasing order from top to bottom, and incoming arrows from the bottom with classes $\{-M, M\}$ from the bottom, again in increasing order. Enumerate the outgoing positions on the top and right boundary with $\{-M-N-1,\dots,M\}$, again in clockwise order, i.e. starting with $-M-N-1$ in the bottom right corner and ending with M in the top left corner. Again let $\pi_{M,N}(x)$ be the class of the outgoing particle at position x. Note that there is the following relation between $\pi_{M,N}$ and π_N for all $x, y \in [-M, M]$

$$\mathbb{P}(\pi_N(x) = y) = \mathbb{P}(\pi_{M,N}(x) = -y)\,. \tag{5.135}$$

Indeed, by using the merging property (Proposition 5.2.5), one can see that the trace of the particle of class y inside the box $[\![-M, M]\!] \times [\![0, N]\!]$ is the same for both models since all particles coming from the left have a smaller class than y. Thus the probability to exit the box through a specific vertex along the top edge is the same in both models. The negative sign on the right-hand side is due to the outgoing boundary positions being enumerated in descending order. Using this twice along with Proposition 5.8.2, we obtain

$$\mathbb{P}(\pi_N(x) = y) = \mathbb{P}(\pi_{M,N}(x) = -y) = \mathbb{P}(\pi_{M,N}(-y) = x) = \mathbb{P}(\pi_N(-y) = -x)\,, \tag{5.136}$$

which proves the statement. □

Now note that $\pi_N^{-1}(x) = \boldsymbol{X}_N(x)$, since it is the position of the particle of class x at time N. We can use this to prove that the speed process is stationary with respect to the dynamics of the multi-class stochastic six-vertex process.

Proposition 5.8.4 (Stationarity of the Speed Process). *Let U be sampled from the stochastic six-vertex speed process. Consider the multi-class stochastic six-vertex model with initial conditions given by $(-U(-x))_{x\in\mathbb{Z}}$. This process is stationary.*

Proof. Start with packed initial conditions and run the process until times N and $N+1$. Since $\pi_N(x)$ equals in distribution $-\pi_N^{-1}(-x)$, which equals $-\boldsymbol{X}_N(-x)$, we know that both $(\pi_N(x)/N)_{x\in\mathbb{Z}}$ and $(\pi_{N+1}(x)/N)_{x\in\mathbb{Z}}$ converge in law to $(-U(-x))_{x\in\mathbb{Z}}$ by Corollary 5.1.6. Let μ_N be the law of $(\pi_N(x)/N)_{x\in\mathbb{Z}}$ on the space $\mathbb{R}^{\mathbb{Z}}$ and ν_N the law of $(\pi_{N+1}(x)/N)_{x\in\mathbb{Z}}$. The laws μ_{N+1} and ν_N only differ by multiplying the corresponding random variables with a factor $\frac{N}{N+1}$. Since the dynamics do not change under monotone relabeling of classes, and division by N is such a monotone relabeling, we have for any bounded function f on $\mathbb{R}^{\mathbb{Z}}$

$$\int f(\eta)d\nu_N(\eta) = \int P_1 f(\eta)d\mu_N(\eta) \tag{5.137}$$

where P_1 is the one-step evolution operator for the process. Since both μ_N and ν_N converge to the law of $(-U(-x))_{x\in\mathbb{Z}}$, this proves the statement. □

The following conjecture is known for multi-class TASEP and ASEP respectively proven in [Lig76] and [FKS91] respectively. However, for the stochastic six-vertex model no proof of this seems to be in the literature, see also [ANP23, Remark 7.9].

Conjecture 5.8.5 (Uniqueness of Stationary Translation-invariant Measures). *For $\lambda_k \in (0,1)$ with $\sum_{k=0}^{n}\lambda_k = 1$, there is a unique ergodic stationary measure for the n-class stochastic six-vertex process on the line with $\mathbb{P}(\eta_0(x)=k)=\lambda_k$.*

The existence can be derived abstractly from a compactness argument. Recently in [ANP23] such measures have also been constructed in a way that is amenable to calculating marginals. It should be possible to prove uniqueness in a similar way to the analogous results for ASEP and TASEP, using the corresponding result for the single-class process, which is proven in [Agg20].

Given this conjecture, one can conclude the following

Proposition 5.8.6. *Given Conjecture 5.8.5 the ergodic stationary measures for the stochastic six vertex model (on the line) are the ergodic stationary measures for the multi-class ASEP speed process.*

Proof. This follows from Conjecture 5.8.5 together with the observation that the stationary measures constructed for both ASEP and the stochastic six vertex model in [ANP23] are actually identically. This can be seen by noticing that the stationary measures in [ANP23, Theorem 3.3] when specialized to ASEP, as done in [ANP23, Section 4.2] and considered on the line instead of the cylinder, are exactly the stationary measures in [ANP23, Section 7.3] for the stochastic six vertex model. □

This result implies the following connection between the stochastic six vertex speed process and the ASEP speed process.

Theorem 5.8.7. *Let f be the unique increasing map from $[-\kappa,-\kappa^{-1}]$ to $[-1,1]$, which maps a random variable with density $\frac{\sqrt{\kappa}}{2(\kappa-1)}|x|^{-\frac{3}{2}}\mathbf{1}_{x\in[-\kappa,-\kappa^{-1}]}$ to uniform random variable in $[-1,1]$. Given Conjecture 5.8.5, if U is sampled according to the stochastic six vertex speed process. Then*

$$(f(-U(-x)))_{x\in\mathbb{Z}}$$

has the same law as the ASEP speed process.

Proof. Note first that, since f is increasing, and the dynamics only considers the relative ordering of the labels, $(f(-U(-x)))_{x\in\mathbb{Z}}$ is still stationary for the stochastic six-vertex process. Given any $(\lambda_k)_{k=1,\dots,n}$ which satisfy $\sum_{k=1}^n \lambda_k = 1$, consider the increasing map ϕ from $[-1,1]$ to $\{1,\dots,n\}$ such that ϕ^k is an interval of length $2\lambda_k$. By the merging property $(\phi_\lambda \circ f(-U(-x)))_{x\in\mathbb{Z}}$ is a stationary measure for the n-class stochastic six-vertex process. By the definition of ϕ_λ and f it also satisfies

$$\mathbb{P}[\phi_\lambda \circ f(-U(0)) = k] = \lambda_k\,.$$

Ergodicity is also inherited from U. By Conjecture 5.8.5 there is only one such measure, and by Proposition 5.8.6 this is the same as the unique ASEP stationary measure with the same densities. If one denotes by $\widetilde{U}$ a sample of the ASEP speed process, this implies that

$$(\phi_\lambda \circ f(-U(-x)))_{x\in\mathbb{Z}} = (\phi_\lambda \circ \widetilde{U}(x))_{x\in\mathbb{Z}},\ \text{in law}.$$

This equality is satisfied for all ϕ_λ, and this set of functions is sufficiently large to determine all the marginals of the speed processes, as was outlined in [AAV08]. It follows that

$$(f(-U(-x)))_{x\in\mathbb{Z}} = (\widetilde{U}(x))_{x\in\mathbb{Z}},\ \text{in law},$$

as desired. □

5.A Hydrodynamic limit and weak convergence

In this section we will give a summary of the hydrodynamic limit and local statistics for the stochastic six-vertex model proved in [Agg20], specialized to the step initial conditions on the corner. We will then use these results to show the weak convergence of $\frac{\boldsymbol{X}_t}{t}$, which mirrors the arguments for ASEP from [FK95].

We will work with the single-class stochastic six-vertex model on the quadrant, i.e. as process $(\eta_t(x))_{x,t\in\mathbb{Z}_{\geq 0}}$ taking values in $\{0,1\}$.

To state the hydrodynamic limit, we define the function φ as

$$\varphi(\rho) := \frac{\kappa\rho}{(\kappa-1)x+1}\,,$$

where we recall that $\kappa = \frac{1-b_1}{1-b_2}$. This function encodes the “slope relation” of the stochastic six-vertex model (see [Agg22]). In particular, if the stochastic six-vertex model is run from i.i.d. Bernoulli(ρ) initial conditions on the bottom and i.i.d. Bernoulli($\phi(\rho)$) from the left, the process is stationary under space time shifts. Furthermore the asymptotic speed of a single second-class particle added to such initial conditions will be given by $\varphi'(\rho)$.

We can now state the general hydrodynamic limit in [Agg20, Theorem 1.1]. This is stated for the stochastic six-vertex model on the Torus. Let $\mathbb{T}^N$ be the discrete torus $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{T}$ the torus $\mathbb{R}/\mathbb{Z}$.

Theorem 5.A.1 (Theorem 1.1 from [Agg20])**.** *Consider initial conditions $\eta_0^N(x) : \mathbb{T}^N \to \{0,1\}$ which approximate a profile $\rho_0 : \mathbb{T} \to [0,1]$, in the sense that*

$$\lim_{N\to\infty} \sup_{x,y} \left| \frac{1}{N} \sum_{i=\lfloor Nx \rfloor}^{\lfloor Ny \rfloor} \eta_0^N(i) - \int_x^y \rho_0(x) \right| = 0\,.$$

Let $(\rho_t(x))_{t\geq 0, x\in\mathbb{R}}$ be the entropy solution of the partial differential equation

$$\frac{\partial}{\partial_t}\rho_t(x) + \frac{\partial}{\partial_x}\varphi(\rho_t(x)) = 0 \tag{5.138}$$

with initial condition given by ρ_0. Then uniformly on compact sets, we have the following convergence in probability:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{t=0}^{NT} \sum_{x=0}^{NX} \eta_t^N(x) = \int_0^T \int_0^X \rho_t(x) dx dt \,.$$

This theorem gives the expected density of particles at large times and scales. It is supplemented by the following local statistics result, given in [Agg20, Theorem 1.3], which states that whenever $\rho_t(x)$ is continuous at (x,t) the system will be approximately at equilibrium, that is the microscopic behavior around $\eta_{Nt}(Nx)$ will approach a stationary measure.

Theorem 5.A.2 (Theorem 1.3 from [Agg20])**.** *In the setting of Theorem 5.A.1, consider a point (t,x) such that $\rho_t(x)$ is continuous at (t,x), and fix an integer $k \geq 1$. Then the law of*

$$[\eta^N_{\lfloor Nt \rfloor + s}(\lfloor Nx \rfloor + y)]_{y\in[\![-k,k]\!], s\in[\![0,k]\!]}$$

converges in law to the stationary process started from i.i.d. Bernoulli$(\rho_t(x))$ random variables and restricted to the rectangle $[\![-k,k]\!] \times [\![0,k]\!]$.

To prove the weak convergence we will now need the following consequence of these theorems.

Corollary 5.A.3. *Let $\eta_t : \mathbb{Z} \to \{0,1\}$, be the stochastic six vertex process on the line started from $\eta_0(x) = \mathbf{1}_{x<0}$ and α be a positive real number. Then*

$$\lim_{t\to\infty} \mathbb{P}[\eta_t(\alpha t) = 1] = \rho_1(\alpha) \,,$$

where

$$\rho_t(x) = \begin{cases} 1 & \text{if } \frac{x}{t} < \kappa^{-1} \\ \frac{\sqrt{\kappa t/x}-1}{\kappa-1} & \text{if } \kappa^{-1} \leq \frac{x}{t} \leq \kappa \\ 0 & \text{if } \frac{x}{t} > \kappa \end{cases} \tag{5.139}$$

is the unique weak solution to (5.138) from $\rho_0(x) = \mathbf{1}_{x<0}$.

Proof. Since Theorems 5.A.1 and 5.A.2 are stated on the torus, we need to connect the process on the line with the process on a torus. Let B be some integer, which will later be chosen to be large. Let η_t be the stochastic six-vertex model on the line started from step initial conditions and $\eta_t^N : \mathbb{T}^{2B^3N+1} \to \{0,1\}$, be the stochastic six-vertex process on $\mathbb{T}^{2B^3N+1}$ started from the initial conditions

$$\eta_t^{N,\mathbb{T}}(x) = \begin{cases} 1, \text{ for } & -B^2N \leq x \leq 0 \\ 0 \text{ else,} \end{cases}$$

where we identify the torus $\mathbb{T}^{2B^3N+1}$ with the set $[\![-B^3N, B^3N]\!]$. Then by [Agg20, Proposition 5.7] the processes η and η^N can be coupled to agree on the interval $[\![-BN, BN]\!]$ until time BN with probability $1 - \exp -BN$, if B is large enough such that $B^2N - \frac{4BN}{1-b_2} > BN$. Assuming further that B is large than α, we obtain

$$|\mathbb{P}[\eta_N(\alpha N)] - \mathbb{P}[\eta_N^{N,\mathbb{T}}(\alpha N)]| \leq \exp -BN \,.$$

Therefore it remains to apply Theorem 5.A.2 to η^N. Note that the PDE (5.138) is invariant under scaling space and time by the same factor, and we can therefore also consider ρ_t defined on the torus of size $2B^3$, which simplifies the notation. The initial condition of η^N

approximate initial conditions $\rho_0^{\mathbb{T}}(x) = \mathbf{1}_{x \in [-B^2, 0]}$ on the torus of side-length $2B^3$ identified with $[-B^3, B^3]$. By Proposition 5.3 and Remark 5.4 of [Agg20], the solution $\rho_t^{\mathbb{T}}$ of (5.138) for these initial conditions and ρ_t (the solution for step initial conditions on $\mathbb{R}$) agree at time t on $[-B^2 + ct, B^2 - ct]$, where $c = \max_{x \in [\kappa^{-1}, \kappa]} |\phi'(x)|$. Given B large enough such that $B^2 - c > \alpha$, this implies that

$$\rho_1^{\mathbb{T}}(\alpha) = \rho_1(\alpha)\,.$$

Applying Theorem 5.A.2 with $k = 0, t = 1$ and $x = \alpha$ gives the desired result. □

Proposition 5.A.4 (Weak Convergence of the Speed of the Second-Class Particle)**.** *Let $\boldsymbol{X}_t$ be the position of the second-class particle under step initial conditions with a single second-class particle at the origin, as in Theorem 5.1.1 and let $\rho_t(x)$ be given by (5.139). Then the asymptotic speed of the second-class particle $\frac{\boldsymbol{X}_t}{t}$ converges weakly to a random variable with density*

$$\frac{\sqrt{\kappa}}{2(\kappa - 1)} x^{-\frac{3}{2}} \mathbf{1}_{\kappa^{-1} \le x \le \kappa}\,. \tag{5.140}$$

Proof. Consider step initial conditions (for the single-class model), i.e. $\eta_0(x) = \mathbf{1}_{x<0}$, and let $_0$ be the same initial conditions shifted by 1 to the right i.e. $\widetilde{\eta}_0(x) = \mathbf{1}_{x \le 0}$. Let $(\eta_t)_{t \ge 0}$ and $(\widetilde{\eta}_t)_{t \ge 0}$ be the two stochastic six-vertex processes started from these initial conditions, with the height functions made unique by the choice that $h_0(1; \eta) = h_0(1; \widetilde{\eta}) = 0$. In particular $h_0(0; \eta) = 0$, but $h_0(0; \widetilde{\eta}) = 1$. We will couple them in two different ways. The first coupling π_1 is given by the multi-class stochastic six-vertex model with step initial conditions and a single second-class particle at the origin, i.e. the setup of Theorem 5.1.1 and of this proposition. Denote the position of the second-class particle at time t with $\boldsymbol{X}_t$, as above. The second coupling π_2 is given by the deterministic shift i.e.

$$\mathbb{P}_{\pi_2}[\widetilde{\eta}_t(x) = \eta_t(x-1)] = 1.$$

We will now calculate $\mathbb{E}[h_t(x; \widetilde{\eta}) - h_t(x; \eta)]$ under both of these couplings. Under π_1 the configurations are identical except for the second-class particle and so the height functions agree for all (x,t) such that $x > \boldsymbol{X}_t$ and $h_t(x; \eta) = h_t(x; \widetilde{\eta}) - 1$ for (x,t) such that $x \le \boldsymbol{X}_t$. This gives

$$\mathbb{E}_{\pi_1}[h_t(x; \widetilde{\eta}) - h_t(x; \eta)] = \mathbb{P}[x \le \boldsymbol{X}_t]\,.$$

Under π_2 the height functions are related by deterministic shift $h_t(x; \widetilde{\eta}) = h_t(x-1; \eta)$ and therefore

$$\mathbb{E}_{\pi_2}[h_t(x; \widetilde{\eta}) - h_t(x; \eta)] = \mathbb{E}_{\pi_2}[h_t(x-1; \eta) - h_t(x; \eta)] = \mathbb{E}[\eta_t(x-1)]\,.$$

By the linearity of expectations, $\mathbb{E}[h_t(x; \widetilde{\eta}) - h_t(x; \eta)]$ does not depend on the coupling we take and therefore we obtain the following identity:

$$\mathbb{P}[x \le \boldsymbol{X}_t] = \mathbb{E}[\eta_t(x-1)]\,.$$

Choosing $x = \lfloor \alpha t \rfloor$ we obtain

$$\mathbb{P}\left[\frac{\boldsymbol{X}_t}{t} \ge \alpha\right] = \mathbb{E}[\eta_t(\lfloor \alpha t \rfloor - 1)].$$

By Corollary 5.A.3 the right hand side converges to $\rho_1(\alpha)$. Therefore $\frac{\boldsymbol{X}_t}{t}$ converges weakly to the random variable with density

$$-\rho_1'(x) = \frac{\sqrt{\kappa}}{2(\kappa - 1)} x^{-\frac{3}{2}} \mathbf{1}_{\kappa^{-1} \le x \le \kappa}\,,$$

which proves the proposition. □

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