DISSERTATION

Large Deviations and Stochastic Volterra Equations

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Kurzfassung

Die vorliegende Arbeit besteht aus drei Teilen, die Resultate der Large Deviations Theory benutzen. In den letzten beiden Teilen spielen auch stochastische Volterra Integralgleichungen eine wichtige Rolle. Diese sind essenziell, wenn man in klassischen stochastischen Volatilitätsmodellen einen fraktionellen Kern in die Gleichung für die Instantaneous Variance einführen möchte, um eine „rough“ Version dieses Modells zu erzeugen. Large deviations werden oft benutzt, um in einer Vielzahl von Situation asymptotische Resultate herzuleiten. Diese Resultate sind besonders in der Finanzmathematik gefragt, sei es um finanzmathematische Größen, wie etwa Optionspreise, Implied Volatility, etc., qualitativ zu beurteilen oder auch z.B. für Fast Calibration Schemes.

Kapitel 1, *Preliminaries and some well-known results*, präsentiert einige aus der Literatur bekannte Resultate, die wir im weiteren auch verwenden werden. Auch die für diese Arbeit relevanten Definitionen und Resultate aus der Large Deviations Theory werden hier überblicksmäßig gezeigt.

Kapitel 2, *Large deviations related to the law of the iterated logarithm for Itô diffusions*, basiert auf einem Paper [GG20] zusammen mit Stefan Gerhold, das im Journal „Electronic Communications in Probability“ publiziert wurde. In diesem Teil der Arbeit zeigen wir ein Large Deviations Principle für das Supremum einer skalierten Itô Diffusion. Skaliert man eine Brownsche Bewegung wie beim Gesetz des iterierten Logarithmus, dann konvergiert das Supremum davon gegen Eins wenn die Zeit gegen Null geht. Upper Large Deviations dieses Prozesses bekommt man, indem man das Problem via Trefferzeiten einer stetigen Kurve formuliert und dann Resultate für Trefferzeiten von Strassen [Str67] anwendet. Wir erweitern das ganze zu einem small-time Large Deviations Principle für das Supremum einer skalierten Itô Diffusion, wobei die Hauptreferenz ein Resultat von Lerche [Ler86] ist, das auf Strassens Resultat aufbaut.

Kapitel 3, *Large deviations for fractional volatility models with non-Gaussian volatility driver*, basiert auf einem Paper [GGG21] gemeinsam mit Stefan Gerhold und Archil Gulisashvili, das im Journal „Stochastic Processes and their Applications“ publiziert wurde. In diesem Kapitel betrachten wir *non-Gaussian fractional stochastic volatility models.* Die Volatilität in solch einem Modell besteht aus einer positiven Funktion mit einem stochastischen Prozess als Argument. Dieser wird erzeugt durch eine fraktionelle Transformation der Lösung einer stochastischen Differentialgleichung, wobei die Yamada-Watanabe-Bedingung erfüllt sein muss. Derartige Modelle sind Verallgemeinerungen der fractional Version des Heston Modells aus Bäuerle und Desmettre [BD20]. Wir zeigen pfadweise Large Deviations und small-noise Large Deviations für den Log-Preis in einem solchen non-Gaussian Modell. Weiters zeigen wir in einem sehr vereinfachten Beispiel, wie man die Taylor Entwicklung zweiter Ordnung für die Rate Function dieser Large Deviations Principles berechnen könnte.

Kapitel 4, *Rough 3/2 – A truncated ansatz*, ist eine gemeinsame Arbeit mit Stefan Gerhold, die noch nicht publiziert wurde. In diesem Teil der Arbeit wird eine „rough“ Variante des bekannten *nicht*-affinen 3/2-Modells vorgeschlagen. Die Koeffizienten(funktionen) der zugrunde-

liegenden stochastischen Volterra Gleichung (SVE) werden in geeigneter Weise abgeschnitten, um die Existenz einer Lösung zu sichern. Es wird gezeigt, dass unser Modell die Voraussetzungen von Zhang [Zha10] erfüllt. Dennoch werden die notwendigen Abschätzungen für den Existenzbeweis ausführlich gezeigt, um sie einfach nachvollziehbar zu machen. Mithilfe von Jacquier und Pannier [JP20] bekommen wir, nachdem alle Voraussetzungen gezeigt wurden, pfadweise small-noise und small-time Large Deviations für die Instantaneous Variance und den Log-Preis Prozess. Für einen Teil davon bekommen wir dann auch Moderate Deviations Principles. Am Ende dieses Teils der Arbeit werden noch ein paar Anwendungsbeispiele gebracht, wie etwa Asymptotiken für Implied Volatility und Optionspreise auf Realized Variance. Weiters wird ein Implementierungsvorschlag mittels eines naiven Euler Schemes zur Simulation der Pfade präsentiert.

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Abstract

This thesis consists of three parts, all of which are related to large deviations theory. In the second and third part, stochastic Volterra integral equations play an important role. Stochastic Volterra equations are essential when it comes to rough volatility models, because a natural way of introducing roughness in a (classic) stochastic volatility model is to add a fractional (weakly singular) kernel to the integrand coefficient functions of an ordinary Itô diffusion. Large deviations results help, in a variety of situations, to establish asymptotic results. These can be used to develop asymptotic results for all kind of objects in the area of Financial Mathematics. These results can e.g. be used for investigating the qualitative behavior of these objects in a special situation or to develop fast calibration schemes.

Chapter 1, *Preliminaries and some well-known results*, presents some results from the literature and introduces notions and results from large deviations theory.

Chapter 2, *Large deviations related to the law of the iterated logarithm for Itô diffusions*, is based on the paper [GG20], a joint work with Stefan Gerhold, published in the journal "Electronic Communications in Probability". This work is a pure mathematics topic and does not provide applications in Finance. In this chapter, we establish a large deviations principle for the supremum of a scaled Itô diffusion. When a Brownian motion is scaled according to the law of the iterated logarithm, its supremum converges to one as time tends to zero. Upper large deviations of the supremum process can be quantified by writing the problem in terms of hitting times and applying a result of Strassen [Str67] on hitting time densities. We extend this to a small-time large deviations principle for the supremum of scaled Itô diffusions, using as our main tool a refinement of Strassen's result due to Lerche [Ler86].

Chapter 3, *Large deviations for fractional volatility models with non-Gaussian volatility driver*, is based on the paper [GGG21], a joint work with Stefan Gerhold and Archil Gulisashvili, published in the journal "Stochastic Processes and their Applications". In this chapter, we study non-Gaussian fractional stochastic volatility models. The volatility in such a model is described by a positive function of a stochastic process that is a fractional transform of the solution to an SDE satisfying the Yamada-Watanabe condition. Such models are generalizations of a fractional version of the Heston model considered in Bäuerle and Desmettre [BD20]. We establish sample path small-noise large deviation principles for the log-price process in a non-Gaussian model. We also illustrate how to compute the second order Taylor expansion of the rate function, in a simplified example.

Chapter 4, *Rough 3/2 – A truncated ansatz*, is based on a joint working paper with Stefan Gerhold. In this chapter, we come up with a rough variant of the well-known 3/2 model. The coefficient functions of the underlying stochastic Volterra integral equation (SVE) are truncated in an appropriate manner to ensure existence of the solution. The computations for showing existence according to Zhang [Zha10] are shown in detail such that they can easily be followed. With the help of Jacquier and Pannier [JP20] we establish sample path small-noise and small-time large deviations for the instantaneous variance and the log-price process. For some of

these we also get moderate deviation principles. Then, we present some applications of these large deviations results. Also a simple implementation using a naive Euler approach for the simulation of sample paths is given at the end of this chapter.

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I want to thank all of my friends and family. I was really happy when I realized a part of my study group stays at university as I did after my Master's. But I am also happy about all the new people I got to know during my journey. Doing research in mathematics as job is nice, but I think what I liked about it most was the people I was dealing with. I am very grateful for that. And, I am also very grateful for Lisa who I want to thank at this point. I really appreciate your unconditional support, especially in the time of writing this thesis, but also beyond. Thank you for being you and being here for me :)

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Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 1. April 2022

Christoph Gerstenecker

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1. Preliminaries and some well-known results

In this chapter we will cite some results that will be used throughout the thesis. An exhaustive introduction is omitted at this point, because separate introductory sections can be found in each of the subsequent chapters.

1.1. Preliminaries

Theorem 1.1.1 (Khinchin's law of the iterated logarithm). *Let B be a real-valued Brownian motion with continuous sample paths. Then*

$$P\left(\limsup_{t\nearrow\infty}\frac{B_t}{\sqrt{2t\log\log t}}=1\right)=1, \tag{1.1.1}$$

$$P\left(\limsup_{t\searrow 0}\frac{B_t}{\sqrt{2t\log\log\left(\frac{1}{t}\right)}}=1\right)=1. \tag{1.1.2}$$

Proof. See Theorem 9.23 in Chapter 2 of [KS91]. □

Theorem 1.1.2 (Reflection principle for Brownian motion). *Let B be a standard Brownian motion and $a>0$. Then*

$$\mathbb{P}\left(\sup_{0\le s\le t} B_s\ge a\right)=2\mathbb{P}(B_t\ge a). \tag{1.1.3}$$

Proof. See pp. 79–80 in [KS91]. □

Theorem 1.1.3 (Borell-TIS inequality). *Let $X=(X_t)_{t\in[0,1]}$ be a centered and continuous Gaussian process. Set $\sigma^2=\sup_{t\in[0,1]}\operatorname{Var}(X_t)$. Then $m:=E[\sup_{u\in[0,1]}X_u]$ is finite and we have, for all $x>m$,*

$$P\left(\sup_{u\in[0,1]} X_u\ge x\right)\le e^{-\frac{(x-m)^2}{2\sigma^2}}. \tag{1.1.4}$$

.

Proof. See Theorem 4.2 in [Nou12]. The proof there uses the continuity of X. In fact, we also get this inequality without continuity, but the proof gets harder. □

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Theorem 1.1.4 (Theorem 3.4.6 in [KS91]). *Let* $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\} \in \mathcal{M}^{c,\text{loc}}$ *satisfy* $\lim_{t\to\infty} \langle M\rangle_t = \infty$ *a.s.* P. *Define, for each* $0 \le s < \infty$, *the stopping time*

$$T(s) = \inf\{t \ge 0; \langle M\rangle_t > s\}. \tag{1.1.5}$$

Then, the time-changed process

$$B_s := M_{T(s)}, \quad \mathcal{G}_s := \mathcal{F}_{T(s)}; \quad 0 \le s < \infty \tag{1.1.6}$$

is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}_s\}$ *satisfies the usual conditions, and we have* P*-a.s.*

$$M_t = B_{\langle M\rangle_t}; \quad 0 \le t < \infty. \tag{1.1.7}$$

Corollary 1.1.5 (Problem 3.4.7 in [DZ98]). Show that if $P(S := \langle M\rangle_\infty < \infty) > 0$, it is still possible to define a Brownian motion B wor which (1.1.7) holds.

Theorem 1.1.6 (Kolmogorov continuity criterion). *Let* $\{X(t), t \ge 0\}$ *be an* $\mathbb{R}^d$*-valued stochastic process, and* τ *a bounded random time. Suppose that for some* $C_0, p > 0$ *and* $\delta > 1$,

$$E\left[|(X(t) - X(s)) \cdot \mathbb{1}_{\{s,t\in[0,\tau]\}}|^p\right] \le C_0|t - s|^\delta. \tag{1.1.8}$$

Then, there exist constants $C_1 > 0$ *and* $a \in (0, \frac{\delta-1}{p})$ *independent of* C_0 *and a continuous version* $\tilde{X}$ *of* X *such that*

$$E\left[\sup_{s\neq t\in[0,\tau]} \frac{|\tilde{X}(t) - \tilde{X}(s)|^p}{|t - s|^{ap}}\right] \le C_1 \cdot C_0. \tag{1.1.9}$$

Proof. See Theorem 2.10 in [Zha10]. Note that the formulation of the Kolmogorov continuity criterion in [Zha10] is for their generalized setting. Clearly, in our setting one can take $\mathbb{R}$ resp. $\mathbb{R}^2$ instead of their Banach space $\mathbb{X}$. □

Theorem 1.1.7 (Burkholder-Davis-Gundy inequality). *Let* M, $M_0 = 0$, *be a continuous local martingale with a.s. finite quadratic variation* $\langle M\rangle$ *and set* $M_t^* := \sup_{s\le t} |M_s|$. *Then, for every* $p > 0$ *there exist constants* c_p *and* C_p *such that for all stopping times* τ,

$$c_p E\left[\langle M\rangle_T^p\right] \le E\left[(M_T^*)^{2p}\right] \le C_P E\left[\langle M\rangle_T^p\right]. \tag{1.1.10}$$

Proof. See Theorem 3.28 in Chapter 3 of [KS91]. □

As a consequence, we can use the BDG inequality to estimate expectations of stochastic integrals without a stochastic driver.

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Corollary 1.1.8. Let X be a progressively measurable process with $E\left[\int_0^T X_t^2\,dt\right] < \infty$. Then, the stochastic integral with respect to Brownian motion is well-defined and its moments can be estimated by

$$E\left[\left|\int_0^t X_s\,dW_s\right|^p\right] \leq C \cdot E\left[\left(\int_0^t |X_s|^2\,ds\right)^{\frac{p}{2}}\right], \tag{1.1.11}$$

for any $p > 0$ and some $C > 0$.

Theorem 1.1.9 (Gronwall's inequality)**.** *Let $I := [a,b]$ and continuous functions $u, \alpha : I \to \mathbb{R}$ and $\beta : I \to [0,\infty)$. If the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\,ds \tag{1.1.12}$$

holds for all $t \in I$ then we have

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(\sigma)\,d\sigma}\,ds \tag{1.1.13}$$

for all $t \in I$.

Proof. See Theorem 1.3.2 in [Pac98]. $\square$

Theorem 1.1.10 (Minkovski inequality)**.** *Suppose that (S_1,μ_1) and (S_2,μ_2) are two σ-finite measure spaces and $F : S_1 \times S_2 \to \mathbb{R}$ is measurable. The Minkovski's integral inequality is*

$$\left[\int_{S_2}\left|\int_{S_1} F(x,y)\mu_1(dx)\right|^p \mu_2(dy)\right]^{\frac{1}{p}} \leq \int_{S_1}\left(\int_{S_2} |F(x,y)|^p\mu_2(dy)\right)^{\frac{1}{p}}\mu_1(dx). \tag{1.1.14}$$

with obvious modifications in the case $p = \infty$. If $p > 1$, and both sides are finite, then equality holds only if $|F(x,y)| = \varphi(x)\psi(y)$ a.e. for some non-negative measurable functions φ and ψ.

Proof. See Theorem 202 in [HLP88]. $\square$

1.2. Large and moderate deviations

In the following we cite some definitions and results for large deviations from [DZ98]. The *large deviation principle* (LDP) characterizes the limiting behavior, as $\epsilon \to 0$, of a family of probability measures $\{\mu_\epsilon\}$ on $(\mathcal{X},\mathcal{B})$ in terms of a *rate function*. This characterization is via asymptotic upper and lower exponential bounds on the values that $\{\mu_\epsilon\}$ assigns to measurable subsets of $\mathcal{X}$. Throughout, $\mathcal{X}$ is a topological space so that open and closed subsets of $\mathcal{X}$ are well-defined, and the simplest situation is when elements of $\mathcal{B}_{\mathcal{X}}$, the Borel σ-field on $\mathcal{X}$, are of interest. All probability spaces are assumed to have been completed, and, with some abuse of notations, $\mathcal{B}_{\mathcal{X}}$ always denotes the thus completed Borel σ-field.

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Definition 1.2.1 (Rate function, p. 4 [DZ98]). A *rate function* I is a mapping $I : \mathcal{X} \to [0, \infty]$ that is finite at least at one point and lower semicontinuous, i.e. for all $\alpha \in [0, \infty)$, the level set $\psi_I(\alpha) := \{x : I(x) \le \alpha\}$ is a closed subset of $\mathcal{X}$. A good rate function is a rate function for which all the level sets $\psi_I(\alpha)$ are compact subsets of $\mathcal{X}$. The effective domain of I, dented by $\mathcal{D}_I$, is the set of points in $\mathcal{X}$ of finite rate, namely, $\mathcal{D}_I := \{x : I(x) < \infty\}$. When no confusion occurs, we refer to $\mathcal{D}_I$ as the domain of I.

Definition 1.2.2 (Large deviation principle, p. 5 [DZ98]). We say that μ_ϵ satisfies the large deviation principle with a rate function I if, for all $\Gamma \in \mathcal{B}$,

$$- \inf_{x \in \Gamma^\circ} I(x) \le \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \le \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \le - \inf_{x \in \overline{\Gamma}} I(x). \tag{1.2.1}$$

Definition 1.2.3 (Cameron-Martin space, p. 261 in [GW16]). The Cameron-Martin space on $[0, T]$ is given by

$$\mathcal{H} := \left\{ h \in C([0,T], \mathbb{R}^n);\ h(t) = \int_0^t \dot{h}(s)\, ds,\ t \in [0,T];\ \|h\|_H^2 := \int_0^T |\dot{h}(t)|^2\, dt < \infty \right\}. \tag{1.2.2}$$

Definition 1.2.4 (Hölder Norm, p. 260 in [GW16]). For $0 \le \alpha < 1$ given, for each function ϕ, the Hölder norm $\|.\|_\alpha$ is defined by

$$\|\phi\|_\alpha = \sup_{s,t \in [0,T],\, s \neq t} \frac{|\phi(t) - \phi(s)|}{|t-s|^\alpha}. \tag{1.2.3}$$

We denote $C_0([0,T], \mathbb{R}^d)$ the space of continuous functions on $[0, T]$ with initial value 0, equipped the supremum-norm and set

$$C_0^\alpha([0,T], \mathbb{R}^d) = \left\{ \phi \in C_0([0,T], \mathbb{R}^d);\ \lim_{\delta \to 0} \sup_{|s-t| < \delta,\, s \neq t} \frac{|\phi(t) - \phi(s)|}{|t-s|^\alpha} = 0,\ \|\phi\|_\alpha < \infty \right\}. \tag{1.2.4}$$

An LDP is preserved under continuous mappings. We present the theorems of [DZ98] handling that.

Theorem 1.2.5 (Contraction principle, Theorem 4.2.1 in [DZ98]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous function. Consider a good rate function $I : \mathcal{X} \to [0, \infty]$.*

(a) *For each $y \in \mathcal{Y}$, define*

$$I'(y) := \inf\{I(x) : x \in \mathcal{X},\ y = f(x)\}. \tag{1.2.5}$$

Then I' is a good rate function on $\mathcal{Y}$, where, as usual, the infimum over the empty set is taken as ∞.

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(b) *If I controls the LDP associated with a family of probability measures μ_ϵ on $\mathcal{X}$, then I' controls the LDP associated with the family of probability measures $\{\mu_\epsilon \circ f^{-1}\}$ on $\mathcal{Y}$.*

In Section 4.2.2 of [DZ98], the authors present an extension of the contraction principle for *non*-continuous functions, if there is a "suitable" approximation of the process available.

Definition 1.2.6 (Definition 4.2.10 in [DZ98])**.** Let $(\mathcal{Y}, d)$ be a metric space. The probability measures $\{\mu_\epsilon\}$ and $\{\tilde{\mu}_\epsilon\}$ on $\mathcal{Y}$ are called *exponentially equivalent* if there exist probability spaces $\{(\Omega, \mathcal{B}_\epsilon, P_\epsilon)\}$ and two families of $\mathcal{Y}$-valued random variables $\{Z_\epsilon\}$ and $\{\tilde{Z}_\epsilon\}$ with joint laws $\{P_\epsilon\}$ and marginals $\{\mu_\epsilon\}$ and $\{\tilde{\mu}_\epsilon\}$, respectively, such that the following condition is satisfied: For each $\delta > 0$, the set $\{\omega : (\tilde{Z}_\epsilon, Z_\epsilon) \in \Gamma_\delta\}$ is $\mathcal{B}_\epsilon$ measurable, and

$$\limsup_{\epsilon \to 0} \epsilon \log P_\epsilon(\Gamma_{v_0}\delta) = -\infty, \tag{1.2.6}$$

where

$$\Gamma_\delta := \{(\tilde{y}, y) : d(\tilde{y}, y) > \delta\} \subset \mathcal{Y} \times \mathcal{Y}. \tag{1.2.7}$$

Theorem 1.2.7 (Theorem 4.2.13 in [DZ98])**.** *If an LDP with a good rate function $I(\cdot)$ holds for the probability measures $\{\mu_\epsilon\}$, which are exponentially equivalent to $\{\tilde{\mu}_\epsilon\}$, then the same LDP holds for $\{\tilde{\mu}_\epsilon\}$.*

Definition 1.2.8 (Definition 4.2.14 in [DZ98])**.** Let $\mathcal{Y}$ and Γ_δ be as in Definition 1.2.6. For each $\epsilon > 0$ and all $m \in \mathbb{Z}_+$, let $(\Omega, \mathcal{B}_\epsilon, P_{\epsilon,m})$ be a probability space, and let the $\mathcal{Y}$-valued random variables $\tilde{Z}_\epsilon$ and $Z_{\epsilon,m}$ be distributed according to the joint law $P_{\epsilon,m}$, with marginals $\tilde{\mu}_\epsilon$ and $\mu_{\epsilon,m}$, respectively. $\{Z_{\epsilon,m}\}$ are called *exponentially good approximations* of $\{\tilde{Z}_\epsilon\}$ if, for every $\delta > 0$, the set $\{\omega : (\tilde{Z}_\epsilon, Z_{\epsilon,m}) \in \Gamma_\delta\}$ is $\mathcal{B}_\epsilon$ measurable and

$$\lim_{m \to \infty} \limsup_{\epsilon \to 0} \epsilon \log P_{\epsilon,m}(\Gamma_\delta) = -\infty. \tag{1.2.8}$$

Similarly, the measures $\{\mu_{\epsilon,m}\}$ are exponentially good approximations of $\{\tilde{\mu}_\epsilon\}$ if one can construct probability spaces $\{(\Omega, \mathcal{B}_\epsilon, P_{\epsilon,m})\}$ as above.

Theorem 1.2.9 (Theorem 4.2.16 in [DZ98])**.** *Suppose that for every m, the family of measures $\{\mu_{\epsilon,m}\}$ satisfies the LDP with rate function $I_m(\cdot)$ and that $\{\mu_{\epsilon,m}\}$ are exponentially good approximations of $\{\tilde{\mu}_\epsilon\}$. Then*

(a) *$\{\tilde{\mu}_\epsilon\}$ satisfies a weak LDP with the rate function*

$$I(y) := \sup_{\delta > 0} \liminf_{m \to \infty} \inf_{z \in B_{y,\delta}} I_m(z), \tag{1.2.9}$$

where $B_{y,\delta}$ denotes the ball $\{z : d(y, z) < \delta\}$.

(b) *If $I(\cdot)$ is a good rate function and for every closed set F,*

$$\inf_{y \in F} I(y) \leq \limsup_{m \to \infty} \inf_{y \in F} I_m(y), \tag{1.2.10}$$

then the full LDP holds for $\{\tilde{\mu}_\epsilon\}$ with rate function I.

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Corollary 1.2.10 (Corollary 4.2.21 in [DZ98]). Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a continuous map from a Hausdorff topological space $\mathcal{X}$ to the metric space $(\mathcal{Y}, d)$ and that $\{\mu_\epsilon\}$ satisfy the LDP with the good rate function $I : \mathcal{X} \to [0, \infty]$. Suppose further that for all $\epsilon > 0$, $f_\epsilon : \mathcal{X} \to \mathcal{Y}$ are measurable maps such that for all $\delta > 0$, the set $\Gamma_{\epsilon,\delta} := \{x \in \mathcal{X} : d(f(x), f_\epsilon(x)) > \delta\}$ is measurable, and

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma_{\epsilon,\delta}) = -\infty. \tag{1.2.11}$$

Then the LDP with the good rate function $I'(\cdot)$ of (1.2.5) holds for the measures $\mu_\epsilon \circ f^{-1}$ on Y.

Theorem 1.2.11 (Theorem 4.2.23 in [DZ98]). *Let μ_ϵ be a family of probability measures that satisfies the LDP with a good rate function I on a Hausdorff topological space $\mathcal{X}$, and for $m = 1, 2, \ldots$, let $f_m : \mathcal{X} \to \mathcal{Y}$ be continuous functions, with $(\mathcal{Y}, d)$ a metric space. Assume there exists a measurable map $f : \mathcal{X} \to \mathcal{Y}$ such that for every $\alpha < \infty$,*

$$\limsup_{m \to \infty} \sup_{\{x : I(x) \le \alpha\}} d(f_m(x), f(x)) = 0. \tag{1.2.12}$$

Then any family of probability measures $\{\tilde{\mu}_\epsilon\}$ for which $\{\mu_\epsilon \circ f_m^{-1}\}$ are exponentially good approximations satisfies the LDP in $\mathcal{Y}$ with the good rate function $I'(y) = \inf\{I(x) : y = f(x)\}$.

Definition 1.2.12 (Definition 3.4 in [Zha08], Definition 1.2.2 in [DE97]). Let I be a good rate function on E. We say that $\{Z^\varepsilon, \varepsilon > 0\}$ satisfies the *Laplace principle* on E with rate function I if for all real bounded continuous functions g on E

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left(\exp \left[-\frac{g(Z^\varepsilon)}{\varepsilon} \right] \right) = -\inf_{f \in E} \{g(f) + I(f)\} \tag{1.2.13}$$

Theorem 1.2.13 (Theorem 1.2.3 in [DE97]). *The Laplace principle implies the large deviation principle with the same rate function. More precisely, if I is a rate function on $\mathcal{H}$ and the limit*

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left[\frac{h(X^\varepsilon)}{\varepsilon} \right] = -\inf_{x \in \mathcal{H}} \{h(x) + I(x)\} \tag{1.2.14}$$

is valid for all bounded continuous functions h, then $\{X^\varepsilon\}$ satisfies the large deviation principle on $\mathcal{H}$ with rate function I.

Theorem 1.2.14 (Schilder's theorem, Theorem 5.2.3 in [DZ98]). *Let W_t, $t \in [0, 1]$ denote a standard Brownian motion in $\mathbb{R}^d$. Consider the process*

$$W_\epsilon(t) = \sqrt{\epsilon} W_t,$$

and let ν_ϵ be the probability measure induced by $W_\epsilon(\cdot)$ on $C_0([0, 1])$, the space of all continuous functions $\phi : [0, 1] \to \mathbb{R}^d$ such that $\phi(0) = 0$, equipped with the supremum norm topology. Then, $\{\nu_\epsilon\}$ satisfies, in $C_0([0, 1])$, an LDP with good rate function

$$I_W(\phi) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{\phi}(t)|^2 \, dt, & \phi \in H_1, \\ \infty, & \textit{otherwise}, \end{cases}$$

where H_1 is the Cameron-Martin space.

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2. Large deviations related to the law of the iterated logarithm for Itô diffusions

2.1. Introduction and main results

This part of the thesis presents a work published together with Stefan Gerhold in 2020 [GG20]. Here, we are looking for small-time large deviations of Itô diffusions that have been scaled according to Khinchin's law of the iterated logarithm, see Theorem 1.1.1. Its original formulation is using large time. For our small-time considerations we need Brownian inversion and get for a standard Brownian motion B that

$$\limsup_{t\nearrow\infty} \frac{B_t}{\sqrt{2t\log\log t}} \stackrel{d}{=} \limsup_{t\nearrow\infty} \frac{t\cdot W_{1/t}}{\sqrt{2t\log\log t}} = \limsup_{t\nearrow\infty} \frac{W_{1/t}}{\sqrt{2\frac{1}{t}\log\log t}} = 1, \quad \text{a.s.}$$

Substituting $s = 1/t$ we get

$$\limsup_{s\searrow 0} \frac{W_s}{\sqrt{2s\log\log\frac{1}{s}}} = 1, \quad \text{a.s.}$$

Defining the logarithmic function $h : [0, e^{-1}] \to \mathbb{R}_+$

$$h(u) := \sqrt{2u\log\log\frac{1}{u}},$$

we have

$$\limsup_{t\searrow 0} \frac{W_t}{h(t)} = \lim_{t\searrow 0} \sup_{0<u<t} \frac{W_u}{h(u)} = 1 \quad \text{a.s.},$$

which can be extended to the diffusion case including a drift; see the proof of Proposition 2.1.2 below for some references. Note, that for small arguments of h the boundary function is increasing. The bounded domain is no problem here, because we consider small-time asymptotics. Here, we are interested in small-time large deviations of the supremum process

$$\sup_{0<u<t} \frac{X_u}{h(u)}$$

for an Itô diffusion X, i.e. the solution X of a stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,ds, \quad t \geq 0, \tag{2.1.1}$$

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with coefficient functions $b, \sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$.

For Brownian motion, a large deviations estimate follows from a result of Strassen [Str67], which gives precise tail asymptotics for the last (or, by time inversion, first) time at which a Brownian motion hits a smooth curve. Using this result we get for fixed $\varepsilon > 0$ that

$$P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon}\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+o(1))}, \quad t \searrow 0. \tag{2.1.2}$$

See Section 2.2 for details. For the readers not familiar with asymptotic notations, we refer to Appendix A. In Theorem 2.2.3 below, we cite an extension of Strassen's result due to Lerche [Ler86], which we will use when extending the estimate (2.1.2) to Itô diffusions. We make the following assumptions on our diffusion process. Simple sufficient conditions, just concerning smoothness and growth of b and σ, are given in Proposition 2.1.2.

Assumption 2.1.1. We assume that the following properties hold:

(i) The continuous one-dimensional stochastic process $X = (X_t)_{t\geq 0}$ satisfies the SDE

$$\begin{aligned} X_t &= \int_0^t b(X_u, u)\, du + \int_0^t \sigma(X_u, u)\, dW_u, \quad t > 0, \\ X_0 &= 0. \end{aligned} \tag{2.1.3}$$

(ii) The coefficients b and σ are continuous functions from $\mathbb{R} \times [0,\infty)$ to $\mathbb{R}$ with

$$\sigma_0 := \sigma(X_0, 0) = \sigma(0,0) > 0.$$

(iii) The process X satisfies a small-time sample path moderate deviations principle in Hölder space. More explicitly, for $1 \leq \lambda(\varepsilon) = o(\varepsilon^{-1/2})$ and $\alpha \in [0, \frac{1}{2})$, the family of processes $(\sqrt{\varepsilon}\lambda(\varepsilon))^{-1}(X_{\varepsilon t})_{t\in[0,1]}$ satisfies the LDP (large deviations principle) in $C_0^\alpha([0,1],\mathbb{R})$ as $\varepsilon \searrow 0$ with speed $\lambda^2(\varepsilon)$ and good rate function (see Definition 1.2.1)

$$\psi \mapsto \begin{cases} \|\psi\|_{\mathcal{H}}^2/(2\sigma_0^2) & \psi \in \mathcal{H}, \\ \infty & \psi \notin \mathcal{H}, \end{cases}$$

where $\mathcal{H}$ is the one-dimensional Cameron-Martin space (see Definition 1.2.3 and 1.2.4).

(iv) The process X satisfies the small-time law of the iterated logarithm, i.e.,

$$\limsup_{t\searrow 0} \frac{X_t}{h(t)} = \lim_{t\searrow 0} \sup_{0<u<t} \frac{X_u}{h(u)} = \sigma_0, \quad \text{a.s.}$$

By inspecting our proofs (see Lemma 2.3.2 and (2.3.9)), it is not hard to see that the continuity assumption (ii) can be slightly weakened. We do not make this explicit, since the available sufficient conditions implying the moderate deviations principle (iii) require much smoother coefficients. In part (iv), the second equality could be replaced by $\geq$. The following proposition gives sufficient conditions for Assumption 2.1.1.

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Proposition 2.1.2. *Suppose that the coefficients of the SDE (2.1.3) satisfy*

(i) $b : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ *is continuous, continuously differentiable on the interior of its domain, and has at most linear growth, i.e. there is some* $M > 0$ *such that*

$$b^2(x,t) \le M(1+x^2+t^2), \quad \textit{for all } (x,t) \in \mathbb{R}\times[0,\infty),$$

(ii) $\sigma : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ *is locally Lipschitz continuous and of at most linear growth. Furthermore,* $\sigma_0 := \sigma(0,0) > 0$.

Then, the diffusion equation (2.1.3) admits a unique strong solution, and all parts of Assumption 2.1.1 are satisfied.

Proof. It is well-known that Lipschitz and linear growth conditions (w.r.t. the space variable) imply strong existence and uniqueness, see e.g. Section 5.2 in [KS91]. The coefficients b and σ satisfy (A.1)–(A.3) from [GW16], and so (iii) follows from Corollary 4.1 in [GW16]. Part (iv) is a special case of the functional law of the iterated logarithm in Theorem 4.3 in [GW16]. See also p. 57 in [McK69] and p. 11 in [Car98]. □

Theorem 2.1.3. *Under Assumption 2.1.1, the process* $\sup_{0<u<t} X_u/h(u)$ *satisfies a small-time large deviations principle with speed* $\log\log(1/t)$ *and rate function*

$$J(x) := \begin{cases} (x/\sigma_0)^2 - 1 & x \ge \sigma_0, \\ \infty & x < \sigma_0. \end{cases}$$

This means that

$$\liminf_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) \ge -J(O) \tag{2.1.4}$$

for any open set O *and*

$$\limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in C\right) \le -J(C) \tag{2.1.5}$$

for any closed set C*, where* $J(M) := \inf_{x\in M} J(x)$.

Obviously, J is a *good rate function* in the sense of Definition 1.2.1, i.e. the level sets $\{J \le c\}$, $c \in \mathbb{R}$, are compact. The main estimate needed to prove Theorem 2.1.3 is contained in the following result, which is the generalization of (2.1.2) for Itô processes.

Theorem 2.1.4. *Under parts (i)–(iii) of Assumption 2.1.1, for* $\varepsilon > 0$ *we have*

$$\begin{aligned} P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \ge \sigma_0\sqrt{1+\varepsilon}\right) &= e^{-\varepsilon(\log\log\frac{1}{t})(1+o(1))} \\ &= \left(\log\frac{1}{t}\right)^{-\varepsilon+o(1)}, \quad t\searrow 0. \end{aligned}$$

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After some preparations, the proofs of Theorems 2.1.3 and 2.1.4 are given at the end of Section 2.3. We note that part (iv) of Assumption 2.1.1 is not needed to prove the lower bound (2.1.4). Moreover, note that our approach does not easily extend to the case of a *multi-dimensional* diffusion, and so we left this for future research. Even the case of two correlated Brownian motions is not trivial. Let B, W be independent standard Brownian motions and $\rho \in (0,1)$. While a joint LDP for the independent processes $\sup B/h$ and $\sup W/h$ clearly holds, it is not obvious how to treat the joint process

$$\Big(\sup_{u\le t} \frac{B_u}{h(u)}, \sup_{u\le t} \frac{\rho B_u + \sqrt{1-\rho^2}W_u}{h(u)}\Big).$$

2.2. Brownian motion

For the sake of simplicity we start with the Brownian motion case to show the fundamental ideas. Then, this can be extended to the general diffusion case including a drift. We want to show that the asymptotics given in (2.1.2) hold.

We can quickly see that there are positive constants γ_1, γ_2 (depending on ε) such that

$$e^{-\gamma_1(\log\log\frac{1}{t})(1+o(1))} \le P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \ge \sqrt{1+\varepsilon}\right) \le e^{-\gamma_2(\log\log\frac{1}{t})(1+o(1))}, \quad t \searrow 0. \tag{2.2.1}$$

As for the lower estimate, note that $h(u)$ increases for small $u>0$, and thus

$$P\left(\sup_{0<u<t} \frac{|W_u|}{h(u)} \ge \sqrt{1+\varepsilon}\right) = P\left(\sup_{0<u<t} |W_u| \ge \sqrt{1+\varepsilon}\, h(t)\right), \quad t \text{ small.}$$

From this and the reflection principle (see Theorem 1.1.2), it is very easy to see that we can take $\gamma_1 = \varepsilon + 1$ in (2.2.1). The upper estimate in (2.2.1) follows from applying the Borell inequality (Theorem 1.1.3) to the centered Gaussian process $(W_u/h(u))_{0<u<t}$, but neither of these estimates is sharp. To get the optimal constants $\gamma_1 = \gamma_2 = \varepsilon$, we use a result of Strassen [Str67] on boundary crossings (which is *not* directly related to Strassen's well-known functional law of the iterated logarithm).

Definition 2.2.1. Let B be a Brownian motion and φ a positive function on $\mathbb{R}_+$ such that $t^{-1/2}\varphi(t)$ increases with t. Then, we call

$$T_\varphi := \sup\{t : B_t \ge \varphi(t)\}$$

the *last passage time of the smooth curve* φ. According to p. 316 in [Str67] a well-known zero-one law assures that T_φ is either a random variable or $T_\varphi = \infty$ a.s.

Theorem 2.2.2 (Theorem 1.2 [Str67])**.** *Let φ be a positive function on $\mathbb{R}_+$ with a continuous derivative and such that $t^{-\delta}\varphi(t)$ increases in t for some $\delta > 0$. Assume that*

$$\frac{\varphi'(s)}{\varphi'(t)} \to 1, \quad \textit{as } t \nearrow \infty \textit{ and } \frac{s}{t} \to 1$$

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and that $T_\varphi < \infty$ a.s. Then, the stopping time T_φ has a continuous density D_φ (except possibly for some mass at 0) and

$$D_\varphi(t) \sim \varphi'(t)(2\pi t)^{-1/2} e^{-\varphi(t)^2/2t}, \quad \text{as } t \nearrow \infty.$$

By time inversion, we have

$$\begin{aligned} P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon}\right) &= P\big(\inf\{u : W_u \geq \sqrt{1+\varepsilon}\, h(u)\} \leq t\big) \\ &= P\left(\sup\{v : W_v \geq \sqrt{1+\varepsilon}\, v h(1/v)\} \geq \frac{1}{t}\right). \end{aligned}$$

Define $\varphi(v) = \sqrt{1+\varepsilon}\, v h(1/v)$. Now, by Theorem 2.2.2, the random variable $\sup\{v : W_v \geq \varphi(v)\}$ has a density $D_\varphi(s)$ (except possibly for some mass at zero, which is irrelevant for our asymptotic estimates), which satisfies

$$D_\varphi(s) \sim \varphi'(s)(2\pi s)^{-1/2} \exp\left(-\varphi(s)^2/2s\right), \quad s \nearrow \infty.$$

From this, the estimate (2.1.2) easily follows, very similarly as in the proof of Theorem 2.2.5 below. That theorem strengthens (2.1.2), replacing ε by some quantity that converges to ε. To prove it, we apply the following theorem due to Lerche.

Theorem 2.2.3 (Theorem 4.1 in [Ler86], p. 60)**.** *Let $T_a := \inf\{u > 0 : W_u \geq \psi_a(u)\}$ for some positive, increasing, continuously differentiable function $u \mapsto \psi_a(u)$, which depends on a positive parameter a. Assume that there are $0 < t_1 \leq \infty$ and $0 < \alpha < 1$ such that*

(i) *$P(T_a < t_1) \to 0$ as $a \nearrow \infty$,*

(ii) *$\psi_a(u)/u^\alpha$ is monotone decreasing in u for each a,*

(iii) *for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all a*

$$\left|\frac{\psi_a'(s)}{\psi_a'(u)} - 1\right| < \varepsilon \quad \text{if} \quad \left|\frac{s}{u} - 1\right| < \delta,$$

for $s, u \in (0, t_1)$.

Then the density of T_a satisfies

$$p_a(u) = \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big)\,(1 + \mathrm{o}(1)) \tag{2.2.2}$$

uniformly on $(0, t_1)$ as $a \nearrow \infty$. Here, n is the Gaussian density

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and Λ_a is defined by

$$\Lambda_a(u) := \psi_a(u) - u\psi_a'(u).$$

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Remark 2.2.4. Note that in [Ler86] we have a different situation than in Definition 2.2.1, because here T_a is the *first* passage time of smooth curve and *not* the last. This needs to be taken into account.

We can now prove the following variant of Theorem 2.1.4, where X is specialized to Brownian motion, but ε is generalized to $\varepsilon + \mathrm{o}(1)$.

Theorem 2.2.5. *Let $d(t)$ be a deterministic function with $d(t) = \mathrm{o}(1)$ as $t \searrow 0$. Then, for $\varepsilon > 0$,*

$$P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq \sqrt{1+\varepsilon+d(t)}\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0. \tag{2.2.3}$$

Proof. We put

$$q(t) := \sqrt{1+\varepsilon+d(t)} \tag{2.2.4}$$

and $a = 1/t$, to make the notation similar to [Ler86]. We can write the probability in (2.2.3) as a boundary crossing probability,

$$\begin{aligned} P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq q(t)\right) &= P\left(\inf\{u>0 : W_u \geq q(1/a)h(u)\} < \frac{1}{a}\right) \\ &= P\left(\inf\{au>0 : W_u \geq q(1/a)h(u)\} < 1\right) \\ &= P\left(\inf\left\{s>0 : W_{s/a} \geq q(1/a)h(s/a)\right\} < 1\right) \\ &= P\left(\inf\left\{s>0 : \sqrt{a}W_{s/a} \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right) \\ &= P\left(\inf\left\{s>0 : W'_s \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right), \end{aligned} \tag{2.2.5}$$

where W' is again a Brownian motion, using the scaling property. We will verify in Lemma 2.2.6 below that the function

$$\psi_a(u) := q(1/a)\sqrt{a}h(u/a) \tag{2.2.6}$$

satisfies the assumptions of Theorem 2.2.3. By (2.2.5) and the uniform estimate (2.2.2), we thus obtain

$$P\left(\sup_{0<u<t} \frac{W_u}{h(u)} \geq q(t)\right) \sim \int_0^1 \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big) du, \quad a = \frac{1}{t} \nearrow \infty.$$

Note, that the asymptotics is given uniformly on $(0, t_1)$ which is essential to get the asymptotics for the integral. An easy calculation shows that

$$\Lambda_a(u) \sim const \cdot \sqrt{u \log\log \frac{a}{u}}, \quad a \nearrow \infty$$

uniformly in $u \in (0,1)$, and so

$$\begin{aligned} \int_0^1 \frac{\Lambda_a(u)}{u^{3/2}} n\Big(\frac{\psi_a(u)}{\sqrt{u}}\Big) du &\sim const \cdot \int_0^1 \frac{1}{u}\sqrt{\log\log\frac{a}{u}}\Big(\log\frac{a}{u}\Big)^{-(1+\varepsilon+d(t))} du \\ &= const \cdot \int_a^\infty \frac{1}{x}\sqrt{\log\log x}\,(\log x)^{-(1+\varepsilon+d(t))} dx \\ &= const \cdot \int_a^\infty \frac{1}{x}(\log x)^{-(1+\varepsilon+\mathrm{o}(1))} dx \\ &= const \cdot (\log a)^{-\varepsilon+\mathrm{o}(1)} = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}. \end{aligned}$$

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As for the third line, note that

$$\log\log x = (\log x)^{\frac{\log\log\log x}{\log\log x}},$$

and that the exponent is $o(1)$ for $x \geq a$ and $a \nearrow \infty$. □

Lemma 2.2.6. *The function ψ_a defined in (2.2.6), with q defined in (2.2.4), satisfies the assumptions of Theorem 2.2.3.*

Proof. To verify condition (ii) of Theorem 2.2.3, it suffices to note that $h(u)/u^\alpha$ decreases for small u and $\alpha \in (\frac{1}{2}, 1)$. The continuity condition (iii) easily follows from

$$\log(t) \sim \log(T), \quad t/T \nearrow 1, \quad t, T \nearrow \infty.$$

It remains to show condition (i), i.e., that

$$\begin{aligned} P(T_a < 1) &= P\left(\inf\left\{s > 0 : W'_s \geq q(1/a)\sqrt{a}h(s/a)\right\} < 1\right) \\ &= P\left(\sup_{0<s\leq 1} \frac{W'_s}{\sqrt{2s\log\log\frac{a}{s}}} \geq q(1/a)\right) \end{aligned} \tag{2.2.7}$$

converges to zero as $a \nearrow \infty$. Choose $a_0 > 0$ such that

$$q(1/a) \geq \sqrt{1 + \frac{2}{3}\varepsilon}, \quad a \geq a_0. \tag{2.2.8}$$

By the law of the iterated logarithm for Brownian motion, we have

$$\lim_{s_0 \searrow 0} \sup_{0<s\leq s_0} \frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} = 1 \quad \text{a.s.}$$

From this we get that there exists an $s_0 > 0$ such that

$$\sup_{0<s\leq s_0} \frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} \leq \sqrt{1 + \frac{1}{2}\varepsilon} \quad \text{a.s.}$$

By monotonicity w.r.t. a, we obtain

$$\frac{|W'_s|}{\sqrt{2s\log\log\frac{a}{s}}} \leq \frac{|W'_s|}{\sqrt{2s\log\log\frac{a_0}{s}}} \leq \sqrt{1 + \frac{1}{2}\varepsilon}, \quad a \geq a_0,\ s \in (0, s_0] \quad \text{a.s.} \tag{2.2.9}$$

For $s \in [s_0, 1]$, note that the first factor of

$$\frac{W'_s}{\sqrt{2s}} \cdot \frac{1}{\sqrt{\log\log\frac{a}{s}}}$$

is bounded pathwise, and that the second factor satisfies

$$\frac{1}{\sqrt{\log\log\frac{a}{s}}} = \frac{1}{\sqrt{\log\log a + o(1)}} \to 0, \quad a \nearrow \infty,$$

uniformly on $[s_0, 1]$. From this and (2.2.9), we get

$$\limsup_{a\nearrow\infty} \sup_{0<s\leq 1} \frac{W'_s}{\sqrt{2s\log\log\frac{a}{s}}} \leq \sqrt{1 + \frac{1}{2}\varepsilon},$$

and together with (2.2.8) this implies that (2.2.7) converges to zero. □

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2.3. Itô diffusions

We now show that our results about Itô diffusions can be reduced to the case of Brownian motion, which was handled in the preceding section. The following easy consequence of the sample path moderate deviations principle will be used repeatedly.

Lemma 2.3.1. *Suppose that parts (i) and (iii) of Assumption 2.1.1 hold. Define*

$$\mathcal{A}_t := \{|X_u| \leq u^{1/4},\ u \leq t\}.$$

Then there is $c > 0$ such that for $t > 0$ sufficiently small

$$P\left(\mathcal{A}_t^{\mathrm{c}}\right) \leq \exp(-c/\sqrt{t}).$$

Proof. For $\frac{1}{4} < \alpha < \frac{1}{2}$, it is easy to see that the map $\Phi : C_0^\alpha \to [0, \infty)$ defined by

$$\Phi(f) := \sup_{0<u\leq 1} |f(u)| u^{-1/4}$$

is continuous. Using part (iii) of Assumption 2.1.1 with $\lambda(\varepsilon) = \varepsilon^{-1/4}$ and the contraction principle (see Theorem 1.2.5), we get that the family of random variables

$$\sup_{0<u\leq 1} \frac{|X_{\varepsilon u}|}{(\varepsilon u)^{1/4}}, \quad \varepsilon > 0,$$

satisfies an LDP with speed $\varepsilon^{-1/2}$. The assertion now follows from

$$P\left(\mathcal{A}_t^{\mathrm{c}}\right) \leq P\left(\sup_{u\leq t} \frac{|X_u|}{u^{1/4}} \geq 1\right) = P\left(\sup_{u\leq 1} \frac{|X_{\varepsilon u}|}{(\varepsilon u)^{1/4}} \geq 1\right)\bigg|_{\varepsilon=t} \leq e^{-\frac{I'(1,\infty)}{2\sqrt{t}}}, \quad t \text{ small},$$

where $I'(\cdot)$ is the rate function (1.2.5) coming from the contraction principle using continuous map Φ from above. □

The drift of X can be easily controlled by continuity and the preceding lemma. Define

$$D_t := \sup_{0<u<t} \frac{|\int_0^u b(X_v, v)\, dv|}{h(u)}. \tag{2.3.1}$$

Lemma 2.3.2. *Under parts (i)–(iii) of Assumption 2.1.1, there is $c > 0$ such that for $t > 0$ sufficiently small*

$$P\left(D_t > \sqrt{t}\right) \leq \exp(-c/\sqrt{t}). \tag{2.3.2}$$

Proof. By the continuity of b,

$$c_1 := \sup\left\{|b(x,v)| : |x| \leq 1,\ v \leq 1\right\} < \infty.$$

Therefore, for small t we have

$$\left|\int_0^u b(X_v, v)\, dv\right| \leq c_1 u, \quad u \leq t,\ \text{on } \mathcal{A}_t.$$

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Using that $u \mapsto \dfrac{u}{h(u)}$ is increasing for its domain $[0, e^{-1}]$ this implies

$$D_t \leq \sup_{0<u<t} \frac{c_1 u}{h(u)} = c_1 \sqrt{\frac{t}{2 \log\log \frac{1}{t}}} \quad \text{on } \mathcal{A}_t,$$

and thus

$$P\left(D_t > \sqrt{t},\ \mathcal{A}_t\right) = 0, \quad t \text{ small.}$$

Then Lemma 2.3.1 implies the result. □

Note that the decay rate in (2.3.2) is clearly negligible in comparison to (2.1.2). The next step in the proof of Theorem 2.1.4 is contained in Lemma 2.3.4, which allows us to deal with the local martingale part, after expressing it as a time-changed Brownian motion. We will require the following well-known result.

Theorem 2.3.3 (Lévy modulus of continuity, Theorem 2.9.25 in [KS91])**.** *Defining the function* $f(\delta) := \sqrt{2\delta \log(1/\delta)}$, *we have*

$$\limsup_{\delta \searrow 0} \frac{1}{f(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ |t-s| \leq \delta}} |W_t - W_s| = 1 \quad a.s.$$

Lemma 2.3.4. *Suppose that parts* (i)–(iii) *of Assumption 2.1.1 hold. Let* $\widehat{W}$ *be a standard Brownian motion, and* $d(t)$ *a deterministic function satisfying* $d(t) = \mathrm{o}(1)$ *as* $t \searrow 0$*. Then*

$$P\left(\sup_{0<u<t} \frac{\left|\widehat{W}_{\langle X \rangle_u}\right|}{h(u)} \geq \sigma_0 \sqrt{1+\varepsilon} + d(t)\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \tag{2.3.3}$$

$$P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X \rangle_u}}{h(u)} \geq \sigma_0 \sqrt{1+\varepsilon} + d(t)\right) = e^{-\varepsilon(\log\log\frac{1}{t})(1+\mathrm{o}(1))}, \quad t \searrow 0. \tag{2.3.4}$$

Proof. Define

$$g(u) := \sup_{\substack{|x| \leq u^{1/4} \\ s<u}} \left|\sigma^2(x,s) - \sigma_0^2\right| = \mathrm{o}(1), \quad u \searrow 0,$$

where convergence is attained using the local Lipschitz condition for σ. Since

$$\langle X \rangle_u = \int_0^u \sigma^2(X_v, v)\, dv,$$

we conclude from the mean value theorem that

$$|\langle X \rangle_u - \sigma_0^2 u| = u|\sigma^2(X_{\hat{u}}, \hat{u}) - \sigma_0^2| \leq u g(u), \quad u \leq t, \tag{2.3.5}$$

on the event $\mathcal{A}_t$ from Lemma 2.3.1. The mean value theorem also implies

$$\langle X \rangle_u = u\sigma^2(X_{\tilde{u}}, \tilde{u}) \leq 2u\sigma_0^2. \tag{2.3.6}$$

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Note at this point that we have a typo in (3.5) and (3.6) in our paper [GG20], where we have once written $\hat{u}$ instead of u when applying the mean value theorem. However, this does not change anything for the estimates that follow. This estimate and (2.3.7)–(2.3.9) below hold for t sufficiently small and $u \le t$ on the event $\mathcal{A}_t$. Putting $(s,t) = (x,y)/(2\sigma_0^2 u)$ in Theorem 2.3.3, and using Brownian scaling, we obtain

$$\max_{\substack{0\le x<y\le 2\sigma_0^2 u\\ |y-x|\le 2\sigma_0^2 u\delta}} |\widehat{W}_y - \widehat{W}_x| \le \sigma_0\sqrt{2u}\sqrt{3\delta\log\left(\frac{1}{\delta}\right)} \tag{2.3.7}$$

for $\delta > 0$ sufficiently small. In particular, with $\delta := \frac{g(u)}{2\sigma_0^2}$ we get

$$\max_{\substack{0\le x<y\le 2\sigma_0^2 u\\ |y-x|\le ug(u)}} |\widehat{W}_y - \widehat{W}_x| \le \sqrt{3ug(u)\left(\log\frac{1}{g(u)} + \log(2\sigma_0^2)\right)}. \tag{2.3.8}$$

Together with (2.3.5) and (2.3.6), this estimate implies

$$\begin{aligned}\sup_{0<u<t} &\frac{|\widehat{W}_{\langle X\rangle_u} - \widehat{W}_{\sigma_0^2 u}|}{h(u)} \\ &\le \sup_{0<u<t} \frac{\sqrt{3ug(u)\left(\log\left(\frac{1}{g(u)}\right) + \log(2\sigma_0^2)\right)}}{h(u)} =: r(t) = \mathrm{o}(1), \quad t \searrow 0, \end{aligned}\tag{2.3.9}$$

on the event $\mathcal{A}_t$. We conclude that, for small t,

$$\begin{aligned} P\left(\sup_{0<u<t}\frac{|\widehat{W}_{\langle X\rangle_u}|}{h(u)} \ge \sigma_0\sqrt{1+\varepsilon} + d(t),\ \mathcal{A}_t\right) \\ &\le P\left(\sup_{0<u<t}\frac{|\widehat{W}_{\sigma_0^2 u}|}{h(u)} \ge \sigma_0\sqrt{1+\varepsilon} + d(t) - r(t)\right) \\ &= P\left(\sup_{0<u<t}\frac{|\widetilde{W}_u|}{h(u)} \ge \sqrt{1+\varepsilon} + \frac{d(t)-r(t)}{\sigma_0}\right) \\ &\le 2P\left(\sup_{0<u<t}\frac{\widetilde{W}_u}{h(u)} \ge \sqrt{1+\varepsilon} + \frac{d(t)-r(t)}{\sigma_0}\right). \end{aligned}$$

where $\widetilde{W}$ is again a Brownian motion. Now the upper estimate in (2.3.3) follows from Theorem 2.2.5 and Lemma 2.3.1. To complete the proof of the lemma, a lower estimate for the left-hand side of (2.3.4) is needed. We have

$$\sup_{0<u<t}\frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \ge \sup_{0<u<t}\frac{\widehat{W}_{\sigma_0^2 u}}{h(u)} - \sup_{0<u<t}\frac{|\widehat{W}_{\langle X\rangle_u} - \widehat{W}_{\sigma_0^2 u}|}{h(u)},$$

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and thus, by (2.3.9),

$$
\begin{aligned}
P\Bigg(&\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + d(t),\ \mathcal{A}_t\Bigg) \\
&\geq P\left(\sup_{0<u<t} \frac{\widehat{W}_{\sigma_0^2 u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + d(t) + r(t),\ \mathcal{A}_t\right) \\
&\geq P\left(\sup_{0<u<t} \frac{\widetilde{W}_u}{h(u)} \geq \sqrt{1+\varepsilon} + \frac{d(t)+r(t)}{\sigma_0}\right) - P\left(\mathcal{A}_t^{\mathrm{c}}\right), \qquad (2.3.10)
\end{aligned}
$$

using $P(A\cap B) \geq P(A) - P(B^{\mathrm{c}})$. The first probability in (2.3.10) can be estimated by Theorem 2.2.5, and the second probability in (2.3.10) is asymptotically smaller by Lemma 2.3.1. □

We now conclude this chapter by proving our main results, Theorem 2.1.4 and its consequence, Theorem 2.1.3.

Proof of Theorem 2.1.4. Recalling the definition of D_t in (2.3.1), we have

$$
P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon}\right) \leq P\left(\sup_{0<u<t} \frac{\left|\int_0^u \sigma(X_v, v)\, dW_v\right|}{h(u)} + D_t \geq \sigma_0\sqrt{1+\varepsilon}\right). \qquad (2.3.11)
$$

By the Dambis-Dubins-Schwarz theorem (Theorem 1.1.4 and Corollary 1.1.5), the local martingale can be written as

$$
\int_0^u \sigma(X_v, v)\, dW_v = \widehat{W}_{\langle X\rangle_u} \qquad (2.3.12)
$$

with a Brownian motion $\widehat{W}$. The upper estimate thus follows from applying Lemma 2.3.2 and (2.3.3) to (2.3.11). We proceed with the lower estimate in Theorem 2.1.4. From

$$
\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sup_{0<u<t} \frac{\int_0^u \sigma(X_v, v)\, dW_v}{h(u)} - \sup_{0<u<t} \frac{\left|\int_0^u b(X_v, v)\, dv\right|}{h(u)}
$$

and (2.3.12), we get

$$
P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon}\right) \geq P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + D_t\right).
$$

Since we need a lower bound, we can intersect with the event $D_t \leq \sqrt{t}$. Using $P(A\cap B) \geq P(A) - P(B^{\mathrm{c}})$, we obtain

$$
\begin{aligned}
P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + D_t\right) &\geq P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + \sqrt{t},\ D_t \leq \sqrt{t}\right) \\
&\geq P\left(\sup_{0<u<t} \frac{\widehat{W}_{\langle X\rangle_u}}{h(u)} \geq \sigma_0\sqrt{1+\varepsilon} + \sqrt{t}\right) - P(D_t > \sqrt{t}).
\end{aligned}
$$

The lower estimate now follows from Lemma 2.3.2 and (2.3.4). □

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Proof of Theorem 2.1.3. First, let $C \subseteq \mathbb{R}$ be a closed set. Then, the increasing process $\sup_{0<u<t} X_u/h(u)$ converges to σ_0 as $t \searrow 0$ by part (iv) of Assumption 2.1.1, and hence its values are $\geq \sigma_0$ a.s.; note that this is the only place where part (iv) is used. Moreover, the rate function satisfies $J(C) = J(C \cap [\sigma_0, \infty))$. We may thus assume $C \subseteq [\sigma_0, \infty)$. If $\inf C = \sigma_0$, then $J(C) = 0$, and it suffices to estimate the probability in (2.1.5) by 1. Otherwise, let $\sigma_0\sqrt{1+\kappa} := \inf C$ with $\kappa > 0$. Then, by Theorem 2.1.4,

$$\begin{aligned}\limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in C\right) &\\ &\leq \limsup_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{1+\kappa}\right)\\ &= -\kappa = -J(C).\end{aligned}$$

Now, consider an open set $O \neq \emptyset$, and define $\tilde{O} := O \cap [\sigma_0, \infty)$. It is clear that $J(O) = J(\tilde{O})$. If $\tilde{O} = \emptyset$, then $J(O) = J(\tilde{O}) = \infty$, and so the lower bound is trivial. Hence we may suppose that $\tilde{O} \neq \emptyset$. For arbitrary $\lambda > 0$, we can pick $x > 1$ and $\delta > 0$ such that

$$\inf \tilde{O} < \sigma_0\sqrt{x-\delta} < \sigma_0\sqrt{x+\delta} < \inf \tilde{O} + \lambda$$

and

$$\left(\sigma_0\sqrt{x-\delta}, \sigma_0\sqrt{x+\delta}\right) \subseteq \tilde{O}.$$

Then,

$$\begin{aligned}P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) &\geq P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in \left(\sigma_0\sqrt{x-\delta}, \sigma_0\sqrt{x+\delta}\right)\right)\\ &= P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{x-\delta}\right) - P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \geq \sigma_0\sqrt{x+\delta}\right)\\ &= e^{-(x-\delta-1)(\log\log\frac{1}{t})(1+o(1))}, \quad t \searrow 0,\end{aligned}$$

by Theorem 2.1.4. Therefore,

$$\begin{aligned}\liminf_{t\searrow 0} \frac{1}{\log\log\frac{1}{t}} \log P\left(\sup_{0<u<t} \frac{X_u}{h(u)} \in O\right) &\geq -(x-\delta-1)\\ &\geq -\Big(\frac{\inf \tilde{O} + \lambda}{\sigma_0}\Big)^2 + 1 = -J(\tilde{O}) + O(\lambda),\ \lambda \searrow 0.\end{aligned}$$

As $J(O) = J(\tilde{O})$, this yields (2.1.4). $\square$

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3. Large deviations for fractional volatility models with non-Gaussian volatility driver

3.1. Introduction

This part of the thesis presents a work published together with Stefan Gerhold and Archil Gulisashvili in 2021 [GGG21]. Here, we introduce and study a general class of non-Gaussian stochastic volatility models. The main building block of the volatility in such a model is a Volterra type integral transform of the solution to a stochastic differential equation satisfying the Yamada-Watanabe condition, while the volatility is described by a positive function of such an integral transform. Interesting special cases of non-Gaussian models are the models in which the kernels appearing in the integral transforms possess certain fractional features. Examples of such kernels are the kernels of fractional Brownian motion, the Riemann-Liouville fractional Brownian motion, or the fractional Ornstein-Uhlenbeck process. We call the corresponding models non-Gaussian fractional stochastic volatility models. Our class of models is related to the fractional Heston model (see [BD20, GJRS18]), as explained in Section 3.4.

In a Gaussian model, the stochastic volatility is described by a positive function of a Volterra Gaussian process. Such models have recently become popular objects of study. Numerous examples of Gaussian models are given in [GS17, Gul18, Gul20]. The non-Gaussian stochastic volatility models are less studied. To our knowledge, the general class of models introduced in this work has never been considered before the publication of [GGG21].

The main results obtained in in this part of the thesis are Theorems 3.1.6 and 3.1.7. In these theorems, small-noise and sample path large deviation principles are established for the log-price process in a non-Gaussian stochastic volatility model. In the proofs of Theorems 3.1.6 and 3.1.7, we use on the one hand known techniques form the general theory of large deviations, and on the other hand also employ new techniques. For example, a part of our proof of Theorem 3.1.7 is based on the results of Chiarini and Fischer (see [CF14]) concerning small-noise large deviations for Itô processes. Although we cannot use heavy machinery of the theory of Gaussian processes in the non-Gaussian case, we still borrow some techniques employed in [FZ17, Gul18, Gul20] in the proofs of large deviation theorems for Gaussian models. In Section 3.5, we show how to obtain a Taylor expansion of the rate function in a simplified example.

Recently, there has been a surge of interest in using stochastic Volterra equations for financial modeling. While small-noise large deviations for such equations are well studied in the case of Lipschitz coefficients (see [LWYZ17, NR00, Zha08, Zha10]), similar LDPs for equations in which non-Lipschitz functions are used in the description of the dynamics are scarce. In the papers [FGS21] and [GJRS18], concrete models with finite-dimensional parameter spaces are considered, whereas [CP21, FZ17, Gul18, Gul20, Gul21] deal with large deviation principles for

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Gaussian models. Here, we assume that the volatility process is a positive function σ of the following process:

$$\hat{V}_t = \int_0^t K(t,s)U(V_s)\,ds, \tag{3.1.1}$$

where U is a continuous non-negative function, assumptions on the kernel K will be specified below, and V solves a one-dimensional SDE, driven by a Brownian motion B and satisfying the Yamada-Watanabe condition. A (semi-)explicit generating function, as is available in the rough resp. fractional Heston models considered in [FGS21, GJRS18], is not required. Also, our process $\hat{V}$ is clearly non-Gaussian in general, which sets our results apart from the related papers with Gaussian drivers mentioned above. While our setup allows a lot of freedom in choosing the diffusion V and the other ingredients, we note that truly rough models are not covered, because (3.1.1) is a Lebesgue integral and not an integral w.r.t. Brownian motion. However, the models that we are considering may be rough at $t = 0$ (see Remark 4.2). The asset price is given by

$$\begin{aligned} dS_t &= S_t\sigma(\hat{V}_t)(\bar{\rho}\,dW_t + \rho\,dB_t), \quad 0 \le t \le T, \\ S_0 &= 1. \end{aligned} \tag{3.1.2}$$

Here, B, W are independent standard Brownian motions, $\rho \in (-1,1)$ and $\bar{\rho} = \sqrt{1-\rho^2}$. The extension to arbitrary $S_0 > 0$ is straightforward. We now specify the conditions under which our main results, Theorems 3.1.6 and 3.1.7 below, are valid. Assumptions 3.1.1, 3.1.3 and 3.1.4 formulated below are in force throughout this chapter.

Assumption 3.1.1. Throughout this chapter, K is a kernel on $[0,T]^2$ satisfying the following conditions:

(a)

$$\sup_{t\in[0,T]} \int_0^T K(t,s)^2\,ds < \infty. \tag{3.1.3}$$

(b) The modulus of continuity of the kernel K in the space $L^2[0,T]$ is defined as follows:

$$M(h) = \sup_{\{t_1,t_2\in[0,T]:|t_1-t_2|\le h\}} \int_0^T |K(t_1,s) - K(t_2,s)|^2\,ds, \quad 0 \le h \le T. \tag{3.1.4}$$

It is assumed that there exist constants $c > 0$ and $r > 0$ such that

$$M(h) \le ch^r \tag{3.1.5}$$

for all $h \in [0,T]$.

(c) $K(t,s) = 0$ for all $0 \le t < s \le T$.

The function K is a Volterra kernel in the sense of [Gul18] and [Gul20]. The conditions in Assumption 3.1.1 have been used earlier; e.g., (b) and (c) are parts of the definition of a

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Volterra type Gaussian process in [Hul03a, Hul03b]. It is a standard fact that the associated integral operator

$$\mathcal{K}(h)(t) = \int_0^T K(t,s)h(s)\,ds \tag{3.1.6}$$

is compact from $L^2[0,T]$ into $C[0,T]$ if the kernel's modulus of continuity tends to zero; see e.g. Lemma 2 of [Gul18] for a proof. A standard example of a kernel satisfying Assumption 3.1.1 is the fractional kernel $\Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-1/2}$, $0 \le s \le t$, with Hurst parameter $H \in (0,1)$. We note that Γ denotes the gamma function here, whereas later we will use the letter Γ for the solution map of the ODE (3.1.16) below.

Definition 3.1.2. Let ω be an increasing modulus of continuity on $[0,\infty)$, that is $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $\omega(0) = 0$ and $\lim_{s\to 0} \omega(s) = 0$. A function h defined on $\mathbb{R}$ is called locally ω-continuous, if for every $\delta > 0$ there exists a number $L(\delta) > 0$ such that for all $x, y \in [-\delta, \delta]$

$$|h(x) - h(y)| \le L(\delta)\omega(|x-y|). \tag{3.1.7}$$

Assumption 3.1.3. The function $U : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and σ is a positive function on $\mathbb{R}_+$ that is locally ω-continuous for some modulus of continuity ω as in Definition 3.1.2.

The process V in (3.1.1) is assumed to solve the SDE

$$\begin{aligned} dV_t &= \bar{b}(V_t)\,dt + \bar{\sigma}(V_t)\,dB_t, \quad 0 \le t \le T, \\ V_0 &= v_0 > 0, \end{aligned} \tag{3.1.8}$$

where $\bar{\sigma}$ and $\bar{b}$ satisfy the Yamada-Watanabe condition in Assumption 3.1.4 below. A well-known example is the CIR process, where $\bar{\sigma}$ is the square root function.

Assumption 3.1.4.

(R1) The dispersion coefficient $\bar{\sigma} : \mathbb{R} \to [0,\infty)$ is locally Lipschitz continuous on $\mathbb{R}\backslash\{0\}$, has sub-linear growth at ∞, and $\bar{\sigma}(0) = 0$, while $\bar{\sigma}(x) > 0$ for all $x \neq 0$. Moreover, there exists a continuous increasing function $\gamma : (0,\infty) \to (0,\infty)$ such that

$$\int_{0+}^{\infty} \frac{du}{\gamma(u)^2} = \infty \tag{3.1.9}$$

and

$$|\bar{\sigma}(x) - \bar{\sigma}(y)| \le \gamma(|x-y|) \quad \text{for all } x, y \in \mathbb{R},\ x \neq y.$$

Here, the sub-linear growth at ∞ is understood in the sense that for every x_0 there exists a μ such that for all $x > x_0$ we have

$$|\bar{\sigma}(x)|^2 \le \mu(1 + |x|^2).$$

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(R2) The drift coefficient $\bar{b} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, has sub-linear growth at ∞, and $\bar{b}(0) > 0$.

The process V is non-negative (see the remark after Theorem 3.2.2). Next, introducing a small-noise parameter $\varepsilon > 0$, we define the scaled version V^ε of the process V by

$$\begin{aligned} dV_t^\varepsilon &= \bar{b}(V_t^\varepsilon)\,dt + \sqrt{\varepsilon}\bar{\sigma}(V_t^\varepsilon)\,dB_t, \\ V_0^\varepsilon &= v_0 > 0, \end{aligned} \tag{3.1.10}$$

and the scaled asset price process by

$$dS_t^\varepsilon = \sqrt{\varepsilon} S_t^\varepsilon \sigma(\hat{V}_t^\varepsilon)(\bar{\rho}\,dW_t + \rho\,dB_t). \tag{3.1.11}$$

Here, we write $\hat{V}^\varepsilon$ for the process

$$\hat{V}_t^\varepsilon = \int_0^t K(t,s)U(V_s^\varepsilon)\,ds. \tag{3.1.12}$$

The scaled log-price process $X^\varepsilon = \log S^\varepsilon$, which is the process of interest for our large deviations analysis, is now given by

$$X_t^\varepsilon = -\frac{1}{2}\varepsilon \int_0^t \sigma(\hat{V}_s^\varepsilon)^2\,ds + \sqrt{\varepsilon}\int_0^t \sigma(\hat{V}_s^\varepsilon)\,d(\bar{\rho}W_s + \rho B_s), \quad 0 \le t \le T. \tag{3.1.13}$$

Definition 3.1.5. In addition to $\mathcal{K}$ from (3.1.6), we define the integral operators

$$\begin{aligned} \hat{\cdot} &: C[0,T] \to C[0,T], \\ \check{\cdot} &: H_0^1[0,T] \to C[0,T] \end{aligned}$$

by

$$\hat{f}(t) = \int_0^t K(t,s)U(f(s))\,ds, \quad t \in [0,T], \tag{3.1.14}$$

$$\check{g}(t) = \int_0^t K(t,s)U(v(s))\,ds, \quad t \in [0,T], \tag{3.1.15}$$

where v is the solution of the ODE

$$\dot{v} = \bar{b}(v) + \bar{\sigma}(v)\dot{g}, \quad v(0) = v_0, \tag{3.1.16}$$

and $H_0^1[0,T]$ is the Cameron-Martin space on $[0,T]$, see Definition 1.2.3.

Clearly, we have $\check{g} = \hat{v}$, where v solves the ODE (3.1.16). Moreover, $\hat{f} = \mathcal{K}(U \circ f)$ and $\check{g} = \mathcal{K}(U \circ \Gamma(g))$, where Γ maps g to the solution of (3.1.16). By Assumption 3.1.1 the integral operators of Definition 3.1.5 are well-defined. In fact, for our kernel K, we get that $\mathcal{K} : L^2[0,T] \to C[0,T]$. Note that for $h \in H_0^1[0,T]$, we have $h \in C[0,T]$. Further, for $f \in H_0^1[0,T]$ we have $U \circ f \in L^2[0,T]$ and for $g \in H_0^1[0,T]$ we have $U \circ v \in L^2[0,T]$. This can be easily seen using the fact that U is continuous and the input functions are continuous on a bounded interval and hence bounded themselves.

We can now state our main results.

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Theorem 3.1.6. *The family X_T^ε satisfies the small-noise large deviation principle (LDP) with speed ε^{-1} and good rate function I_T given by*

$$I_T(x) = \inf_{f \in H_0^1} \Big[\frac{T}{2} \frac{\big(x - \rho\langle \sigma(\mathcal{K}(U \circ \Gamma(f))), \dot{f}\rangle\big)^2}{\bar{\rho}^2 \langle \sigma(\mathcal{K}(U \circ \Gamma(f)))^2, 1\rangle} + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle \Big] \tag{3.1.17}$$

for all $x \in \mathbb{R}$, wherever this expression is finite. The validity of the LDP means that for every Borel subset $\mathcal{A}$ of $\mathbb{R}$, the following estimate holds, where $\mathcal{A}^\circ$ and $\bar{\mathcal{A}}$ denote the interior resp. the closure of $\mathcal{A}$:

$$-\inf_{x \in \mathcal{A}^\circ} I_T(x) \le \liminf_{\varepsilon \searrow 0} \varepsilon \log P(X_T^\varepsilon \in \mathcal{A}) \le \limsup_{\varepsilon \searrow 0} \varepsilon \log P(X_T^\varepsilon \in \mathcal{A}) \le -\inf_{x \in \bar{\mathcal{A}}} I_T(x). \tag{3.1.18}$$

Theorem 3.1.7. *The family of processes X^ε satisfies the sample path LDP with speed ε^{-1} and good rate function Q given by*

$$Q(g) = \inf_{f \in H_0^1} \Big[\frac{1}{2} \int_0^T \Big(\frac{\dot{g}(t) - \rho\sigma(\mathcal{K}(U \circ \Gamma(f))(t))\dot{f}(t)}{\bar{\rho}\sigma(\mathcal{K}(U \circ \Gamma(f))(t))} \Big)^2 dt + \frac{1}{2} \int_0^T |\dot{f}(t)|^2 \, dt \Big]$$

for all $g \in H_0^1[0,T]$, and by $Q(g) = \infty$, for all $g \in C[0,T] \backslash H_0^1[0,T]$. The validity of the LDP means that for every Borel subset $\mathcal{A}$ of $C[0,T]$, the following estimate holds:

$$-\inf_{g \in \mathcal{A}^\circ} Q(g) \le \liminf_{\varepsilon \searrow 0} \varepsilon \log P(X^\varepsilon \in \mathcal{A}) \le \limsup_{\varepsilon \searrow 0} \varepsilon \log P(X^\varepsilon \in \mathcal{A}) \le -\inf_{g \in \bar{\mathcal{A}}} Q(g). \tag{3.1.19}$$

Using the definition of $\mathcal{K}$, the rate functions in Theorems 3.1.6 and 3.1.7 can be equivalently written as

$$I_T(x) = \inf_{f \in H_0^1} \left[\frac{T}{2} \frac{\big(x - \rho \int_0^T \sigma(\int_0^t K(t,s) U(\Gamma(f)(s))\, ds) \dot{f}(t)\, dt\big)^2}{\bar{\rho}^2 \int_0^T \sigma(\int_0^t K(t,s) U(\Gamma(f)(s))\, ds)^2 \, dt} + \frac{1}{2} \int_0^T \dot{f}(t)^2 \, dt \right]$$

and

$$Q(g) = \inf_{f \in H_0^1} \left[\frac{1}{2} \int_0^T \left(\frac{\dot{g}(t) - \rho\sigma(\int_0^t K(t,s) U(\Gamma(f)(s))\, ds) \dot{f}(t)}{\bar{\rho}\sigma(\int_0^t K(t,s) U(\Gamma(f)(s))\, ds)} \right)^2 dt + \frac{1}{2} \int_0^T \dot{f}(t)^2 \, dt \right],$$

respectively.

The structure of this chapter is as follows. In Section 3.2, we recall small-noise large deviations for SDEs satisfying the Yamada-Watanabe condition. In Section 3.3, we prove the main results, i.e. the small-noise LDP for the log-price. In Section 3.4 we clarify the relation of a special case of our setup to fractional Heston models considered in the literature. In Section 3.5 we compute the coefficients in the second-order Taylor expansion of the rate function in Theorem 3.1.6 for a special, simplified example. As was mentioned above, Assumptions 3.1.1, 3.1.3 and 3.1.4 are supposed to be satisfied throughout the rest of this chapter.

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3.2. LDPs for the driving processes

3.2.1. Sample path LDP for the diffusion

We apply a result of [CF14], which is based on a representation formula for functionals of Brownian motion obtained in [BD98], to obtain an LDP for $(\sqrt{\varepsilon}B, V^\varepsilon)$. While the Yamada-Watanabe condition from Assumption 3.1.4 covers virtually all one-dimensional diffusions that have been suggested in financial modelling, we note that Assumption 3.1.4 could still be weakened, if desired, e.g. by inspecting the proof of Theorem 4.3 in [BD98].

If assumptions (H1)–(H6) of [CF14] hold, then the family of processes $(\sqrt{\varepsilon}B, V^\varepsilon)$. which satisfy the two-dimensional SDE

$$\begin{pmatrix} \sqrt{\varepsilon} dB_t \\ dV_t^\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{b}(V_t^\varepsilon) \end{pmatrix} dt + \sqrt{\varepsilon} \begin{pmatrix} 1 \\ \bar{\sigma}(V_t^\varepsilon) \end{pmatrix} dB_t, \tag{3.2.1}$$

admits an LDP due to Theorem 1 in [CF14]. For the one-dimensional process V^ε, (H1)–(H6) have been checked in [CF14, pp. 1143–1144]. For $(\sqrt{\varepsilon}B, V^\varepsilon)$, the proofs are similar. The assumptions (H1)–(H3) are clearly satisfied. Let us check condition (H4), namely unique solvability of the control equation (7) in [CF14]. Here, it is

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ \bar{b}(\varphi_2(s)) \end{pmatrix} ds + \int_0^t \begin{pmatrix} 1 \\ \bar{\sigma}(\varphi_2(s)) \end{pmatrix} f(s)\, ds, \tag{3.2.2}$$

where $f \in L^2[0,T]$ is the control function. We also have $\varphi_1, \varphi_2 \in C[0,T]$. It follows that the unique solution of (3.2.2) is given by $\bar{\Gamma}_{v_0}(f) = \begin{pmatrix} \int_0^\cdot f(s)\, ds \\ \varphi_2 \end{pmatrix}$, where the function φ_2 is the unique solution of the equation

$$\varphi_2(t) = v_0 + \int_0^t \bar{b}(\varphi_2(s))\, ds + \int_0^t \bar{\sigma}(\varphi_2(s)) f(s)\, ds, \quad t \in [0,T], \tag{3.2.3}$$

which exists, and is positive, by [CF14, Proposition 1]. This establishes condition (H4) in our setting. Note at this point, that the ODE (3.2.3) above is formulated for $f \in L^2[0,T]$ to match the notation of [CF14]. Alternatively it can also be written, with a $g \in H_0^1$, and $\dot{g}$ instead of f, see (3.1.16). Condition (H5) for the second component of $\bar{\Gamma}_{v_0}$ was checked in [CF14, p. 1144]. For the first component, (H5) is true by the following simple fact.

Lemma 3.2.1. *The map $f \mapsto \int_0^\cdot f(s)\, ds$ is continuous from $\mathcal{B}_r$ into $C[0,T]$, where $\mathcal{B}_r$ is the closed ball of radius $r > 0$ in $L^2[0,T]$ endowed with the weak topology.*

Proof. If $f_n \in \mathcal{B}_r$ converges weakly to f, then the convergence is uniform on compact subsets of $L^2[0,T]$. Since $\{\mathbb{1}_{[0,t]} : 0 \le t \le T\}$ is compact, we have

$$\sup_{t \in [0,T]} \left| \int_0^t f(u)\, du - \int_0^t f_n(u)\, du \right| \to 0, \quad n \to \infty. \tag{3.2.4}$$

□

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The tightness assumption (H6) can be established as in [CF14]. The verification, which is based on the sub-linear growth of $\bar{b}$ and $\bar{\sigma}$ and the uniform moment estimate in Lemma A.2 of [CF14], is found on pp. 1137–1138 of [CF14]. See also Section 4.2 of [CF14]. Now, Theorem 1 of [CF14] implies the following assertion, in fact a Laplace principle (see Definition 1.2.12). But since the rate function is a good rate function in the sense of Definition 1.2.1 (which is shown in [CF14]), we also get an LDP with the same rate function. See Theorems 1.2.1 and 1.2.3 of [DE97] resp. Definition 1.2.12 and Theorem 1.2.13.

Theorem 3.2.2. *The family of processes* $(\sqrt{\varepsilon}B, V^{\varepsilon})$ *satisfies an LDP in the space* $C[0,T]^2$ *with speed* ε^{-1} *and good rate function* $I : C[0,T]^2 \to [0,\infty]$ *given by*

$$I(\varphi_1,\varphi_2) = \inf_{\left\{f\in L^2[0,T]:\,\bar{\Gamma}_{v_0}(f)=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\}} \frac{1}{2}\int_0^T f(t)^2\,dt, \tag{3.2.5}$$

whenever $\left\{f \in L^2[0,T] : \bar{\Gamma}_{v_0}(f) = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\} \neq \emptyset$, *and* $I(\varphi_1,\varphi_2) = \infty$ *otherwise. Here,* $\bar{\Gamma}_{v_0}(f)$ *maps* f *to the solution of* (3.2.2).

Note that the positivity of the solution of (3.2.3) shows that $I(\varphi_1,\varphi_2) = \infty$ whenever φ_2 is negative at some point. Thus, Theorem 3.2.2 implies that V is a non-negative process, as noted after Assumption 3.1.4.

The condition $\bar{\Gamma}_{v_0}(f) = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ implies that $\int_0^t f(s)\,ds = \varphi_1(t)$, or $f(t) = \dot{\varphi}_1(t)$. Therefore,

$$\dot{\varphi}_2(t) = \bar{b}(\varphi_2(t)) + \bar{\sigma}(\varphi_2(t))\dot{\varphi}_1(t),$$

and hence (recall that φ_2 is positive by [CF14, Proposition 1])

$$\dot{\varphi}_1(t) = \frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}. \tag{3.2.6}$$

Therefore, the following statement holds:

Corollary 3.2.3. For every φ_2 that is absolutely continuous on $[0,T]$ with $\varphi_2(0) = v_0$

$$I\Big(\int_0^{\cdot} \frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}\,dt, \varphi_2\Big) = \frac{1}{2}\int_0^T \Big(\frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}\Big)^2\,dt, \tag{3.2.7}$$

if the integral is finite, and $I(\varphi_1,\varphi_2) = \infty$ in all the remaining cases.

3.2.2. Sample path LDP for $(\sqrt{\varepsilon}B, \hat{V}^{\varepsilon})$

In this subsection we lift the sample path LDP in Theorem 3.2.2 to one for the family of processes we get when applying the “hat" operator defined in (3.1.12) to V^{ε}.

Lemma 3.2.4. *The mapping* $f \mapsto \hat{f}$ *is continuous from the space* $C[0,T]$ *into itself.*

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Proof. For $f \in C[0,T]$ and all $t_1, t_2 \in [0,T]$,

$$|\hat{f}(t_1) - \hat{f}(t_2)| \leq M(|t_1 - t_2|)^{\frac{1}{2}} \Big(\int_0^T U(f(s))^2 \, ds \Big)^{\frac{1}{2}} \leq C_f |t_1 - t_2|^{\frac{r}{2}}.$$

The number $r > 0$ in the exponent of the last term comes from an estimate for the modulus of continuity of the kernel given by (3.1.5). Here we used the local boundedness of the continuous function U, and also (3.1.4). Now, it is clear that the function $\hat{f}$ is continuous on $[0,T]$. It remains to prove the continuity of the mapping $f \mapsto \hat{f}$ on $C[0,T]$. Suppose $f_k \to f$ in $C[0,T]$. Then we have

$$\|\hat{f} - \hat{f}_k\|_{C[0,T]} \leq \Big(\int_0^T |U(f(s)) - U(f_k(s))|^2 \, ds \Big)^{\frac{1}{2}} \sup_{t \in [0,T]} \Big(\int_0^T K(t,s)^2 \, ds \Big)^{\frac{1}{2}}. \tag{3.2.8}$$

Moreover,

$$C_0 = \max \big\{ \|f\|_{C[0,T]}, \sup_k \|f_k\|_{C[0,T]} \big\} < \infty.$$

It follows from Assumption 3.1.1 and (3.2.8) that there exists a constant C_1 for which

$$\|\hat{f} - \hat{f}_k\|_{C[0,T]} \leq C_1 \sup_{s \in [0,T]} \big| U(f(s)) - U(f_k(s)) \big|, \tag{3.2.9}$$

and the previous expression converges to zero by the uniform continuity of U on $[-C_0, C_0]$. This completes the proof. □

The next assertion establishes the LDP for $(\sqrt{\varepsilon} B, \hat{V}^\varepsilon)$.

Theorem 3.2.5. *The family of processes $(\sqrt{\varepsilon} B, \hat{V}^\varepsilon)$ satisfies an LDP in the space $C[0,T]^2$ with speed ε^{-1} and good rate function given by*

$$\tilde{I}\big(\psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{1}{2} \int_0^T \dot{\psi}_1(t)^2 \, dt, \tag{3.2.10}$$

if the expression in (3.2.6) exists, and $\tilde{I}(\psi_1, \psi_2) = \infty$ otherwise. As above, Γ is the solution map of the one-dimensional ODE (3.1.16), which means that $\varphi = \Gamma(\psi_1)$ solves the ODE $\dot{\varphi} = \bar{b}(\varphi) + \bar{\sigma}(\varphi)\dot{\psi}_1$.

Proof. We know that $(\sqrt{\varepsilon} B, V^\varepsilon)$ satisfies the LDP in Theorem 3.2.2. The mapping $(\varphi_1, \varphi_2) \mapsto (\varphi_1, \hat{\varphi}_2)$ of $C[0,T]^2$ into itself is continuous due to Lemma 3.2.4. Hence, we can use the contraction principle, which gives

$$\tilde{I}(\psi_1, \psi_2) = \inf_{\{(\varphi_1, \varphi_2) \in C[0,T]^2 : \, (\psi_1, \psi_2) = (\varphi_1, \hat{\varphi}_2)\}} I(\varphi_1, \varphi_2) = \inf_{\hat{\varphi}_2 = \psi_2} I(\psi_1, \varphi_2).$$

The necessary condition under which we have $I(\psi_1, \varphi_2) < \infty$ is $\dot{\psi}_1 = \frac{\dot{\varphi}_2 - \bar{b}(\varphi_2)}{\bar{\sigma}(\varphi_2)}$ (see Corollary 3.2.3). □

Since B and W are independent, the following result is an immediate consequence of Theorem 3.2.5 and Schilder's theorem.

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Corollary 3.2.6. (i) The family $(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ satisfies an LDP with speed ε^{-1} and rate function

$$\hat{I}\big(y, \psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{T}{2}y^2 + \frac{1}{2}\int_0^T \dot{\psi}_1^2(t)dt, \tag{3.2.11}$$

for $y \in \mathbb{R}$ and $\psi_1 \in H_0^1[0,T]$, if all the expressions are finite, and $\hat{I}(y, \psi_1, \psi_2) = \infty$ otherwise.

(ii) The family of processes $(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ satisfies an LDP with speed ε^{-1} and rate function

$$\hat{I}\big(\psi_0, \psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{1}{2}\int_0^T \dot{\psi}_0(t)^2\,dt + \frac{1}{2}\int_0^T \dot{\psi}_1^2(t)dt, \tag{3.2.12}$$

for $\psi_0, \psi_1 \in H_0^1[0,T]$, if all the expressions are finite, and $\hat{I}(\psi_0, \psi_1, \psi_2) = \infty$ otherwise.

3.3. Proof of the LDP for the log-price

3.3.1. Proof of Theorem 3.1.6 (one-dimensional LDP)

It is clear that the one-dimensional LDP in Theorem 3.1.6 is a special case of the sample path LDP in Theorem 3.1.7. For the reader's convenience, though, it seemed better to us to first prove Theorem 3.1.6, and then refer to some parts of this proof in the proof of Theorem 3.1.7 below. We build on some ideas of [Gul18]. To match the notation there, we note that $\varepsilon^H\hat{B}$ from [Gul18] corresponds to our process $\hat{V}^\varepsilon$ as defined in (3.1.12). In the original proof of [Gul18] the author first supposes $T = 1$. Here, for convenience, we immediately allow a general $T > 0$. By the following lemma, it suffices to prove an LDP for the drift-less process

$$d\hat{X}_t^\varepsilon = \sqrt{\varepsilon}\sigma(\hat{V}_t^\varepsilon)(\bar{\rho}\,dW_t + \rho\,dB_t), \quad 0 \le t \le T. \tag{3.3.1}$$

Lemma 3.3.1. *The families $(X_T^\varepsilon)_{\varepsilon>0}$ and $(\hat{X}_T^\varepsilon)_{\varepsilon>0}$ are exponentially equivalent, i.e. for every $\delta > 0$, the following equality holds:*

$$\limsup_{\varepsilon\searrow 0} \varepsilon \log P(|X_T^\varepsilon - \hat{X}_T^\varepsilon| > \delta) = -\infty. \tag{3.3.2}$$

Proof. By the same reasoning as in Section 5 of [Gul18], there is a strictly increasing continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\lim_{u\nearrow\infty} \eta(u) = \infty$ and $\bar{\sigma}(u)^2 \le \eta(u)$ for all $u \in \mathbb{R}$. Let $\eta^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function. Replacing $\sqrt{\varepsilon}\hat{B}$ in [Gul18] by $\hat{V}^\varepsilon$, we get the estimate

$$\begin{aligned}
P(|X_T^\varepsilon - \hat{X}_T^\varepsilon| > \delta) &= P\Big(\frac{1}{2}\varepsilon\int_0^T \sigma(\hat{V}_s^\varepsilon)^2\,ds > \delta\Big) \le P\Big(\frac{1}{2}\varepsilon\int_0^T \eta(\hat{V}_s^\varepsilon)\,ds > \delta\Big)\\
&\le P\Big(\frac{1}{2}\varepsilon\int_0^T \eta(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon|)\,ds > \delta\Big) = P\Big(\frac{1}{2}\varepsilon T\eta(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon|) > \delta\Big)\\
&= P\Big(\eta(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon|) > \frac{2\delta}{\varepsilon T}\Big) = P\Big(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon| > \eta^{-1}\Big(\frac{2\delta}{\varepsilon T}\Big)\Big)\\
&\le \exp\Big(-\frac{\varepsilon^{-1}}{2}J(A)\Big),
\end{aligned} \tag{3.3.3}$$

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where J is the rate function of $\sup_{0\le t\le T}|\hat{V}_t^\varepsilon|$, and $A=(\eta^{-1}(\frac{2\delta}{\varepsilon T}),\infty)$. Since J is a good rate function, we know that $J(x,\infty)\nearrow\infty$ as $x\nearrow\infty$, so we get (3.3.2). $\square$

We will next reason as in [Gul18], p. 1121, using the LDP for $(\sqrt{\varepsilon}W_T,\sqrt{\varepsilon}B,\hat{V}^\varepsilon)$ in Corollary 3.2.6. Analogously to [Gul18], we define the functional Φ on the space $M=\mathbb{R}\times C[0,T]^2$ by

$$\Phi(y,f,g)=\bar{\rho}\Big(\int_0^T\sigma(g(s))^2\,ds\Big)^{1/2}y+\rho\int_0^T\sigma(g(s))\dot{f}(s)\,ds, \tag{3.3.4}$$

if $(f,g)=(f,\check{f})$ with $f\in H_0^1[0,T]$, and $\Phi(y,f,g)=0$ otherwise (recall the definition (3.1.15)). Further, for any integer $m\ge 1$, define a functional on M by

$$\Phi_m(y,h,l)=\bar{\rho}\Big(\int_0^T\sigma(l(s))^2\,ds\Big)^{1/2}y+\rho\sum_{k=0}^{m-1}\sigma(l(t_k))\big(h(t_{k+1})-h(t_k)\big), \tag{3.3.5}$$

where $t_k:=\frac{kT}{m}$ for $k\in\{0,\dots,m\}$. The following approximation property is the key to applying the extended contraction principle (see (1.2.12)).

Lemma 3.3.2. *For every $\alpha>0$,*

$$\limsup_{m\to\infty}\sup_{\{f\in H_0^1[0,T]:\frac{T}{2}y^2+\frac{1}{2}\int_0^T\dot{f}(s)^2\,ds\le\alpha\}}|\Phi(y,f,\check{f})-\Phi_m(y,f,\check{f})|=0. \tag{3.3.6}$$

Proof. The proof is similar to that of Lemma 21 in [Gul18]. We need to change the range of the integrals and suprema to $[0,T]$ instead of $[0,1]$. Hence, the grid points for h_m are $t_k:=\frac{Tk}{m}$ for $k\in\{0,\dots,m\}$, like in (3.3.5). We use a different integral operator than [Gul18], and so we have to show that the set $E_\beta=\{\check{f}:f\in D_\beta\}$ is precompact in $C[0,T]$ for $D_\beta=\{f\in H_0^1[0,T]:\int_0^T\dot{f}(s)^2\,ds<\beta\}$. For $f\in D_\beta$, we have $\dot{f}\in L^2[0,T]$ and therefore can use Eq. (16) of [CF14] to estimate the solution of the ODE

$$v=v_0+\int_0^{\cdot}\bar{b}(v(s))\,ds+\int_0^{\cdot}\bar{\sigma}(v(s))\dot{f}(s)\,ds$$

as follows:

$$\sup_{0\le s\le T}|v(s)|^2\le\big(3|v_0|^2+6\mu^2T^2+6\mu^2T\|\dot{f}\|_2^2\big)e^{6\mu^2T(T+\|\dot{f}\|_2^2)}=:C_\beta^2.$$

Here, μ comes from the sub-linear growth condition for the coefficient functions of the diffusion equation for V in Assumption 3.1.4. Since the continuous function U is bounded on the interval $[-C_\beta,C_\beta]$,

$$\{U\circ v: f\in D_\beta,\ \dot{v}=\bar{b}(v)+\bar{\sigma}(v)\dot{f}\} \tag{3.3.7}$$

is a bounded subset of $C[0,T]$. The compact operator $\mathcal{K}$, as defined in (3.1.6), maps the set in (3.3.7) to a precompact set in $C[0,T]$. So we can conclude that E_β is precompact. After that, the proof continues like in [Gul18]. $\square$

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Definition 3.3.3. Let $t \in [0,T]$ be fixed. Consider the grid $t_k := T\frac{k}{m}$ for $k \in \{0,\dots,m\}$. There is a k such that $t \in [t_k, t_{k+1})$. Denote by $\Xi(t)$ the left end t_k of the previous interval. Explicitly, we put

$$\Xi(t) := \frac{T}{m}\left[\frac{mt}{T}\right], \tag{3.3.8}$$

where $[a]$ stands for the integer part of the number $a \in \mathbb{R}$. For $T = 1$, this reduces to $\Xi(t) = \frac{[mt]}{m}$.

We will next prove that $\Phi_m(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ is an exponentially good approximation as $m \nearrow \infty$ to $(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$. We start with an auxiliary result.

Lemma 3.3.4. *For every* $y > 0$,

$$\limsup_{m\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon \log P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) = -\infty. \tag{3.3.9}$$

Proof. This corresponds to Lemma 23 in [Gul18], but we need to adjust some estimates in the proof, since we do *not* have Gaussianity in our setting. As in [Gul18] we use

$$P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) \le P\Big(\sup_{\substack{t_1,t_2\in[0,T]\\ |t_2-t_1|\le T/m}} |\hat{V}_{t_2}^\varepsilon - \hat{V}_{t_1}^\varepsilon| > y\Big). \tag{3.3.10}$$

Then, for $|s-t| \le T/m$, we have

$$\begin{aligned} |\hat{V}_t^\varepsilon - \hat{V}_s^\varepsilon| &= \Big|\int_0^T (K(t,v) - K(s,v))U(V_v^\varepsilon)\,dv\Big| \\ &\le \sqrt{M\Big(\frac{T}{m}\Big)} \sup_{v\in[0,T]} |U(V_v^\varepsilon)| \\ &\le \Big(\frac{cT}{m}\Big)^{r/2} \sup_{v\in[0,T]} |U(V_v^\varepsilon)|, \end{aligned}$$

where M is the modulus of continuity of the kernel function in Assumption 3.1.1. We know that V^ε satisfies an LDP, by Theorem 3.2.2. Using this, we can estimate

$$\begin{aligned} P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) &\le P\Big(\sup_{s\in[0,T]} |U(V_s^\varepsilon)| > yc^{-r/2}T^{-r/2}m^{r/2}\Big) \\ &\le \exp\Big(-\frac{\varepsilon^{-1}}{2} \cdot \tilde{J}\big((y\Big(\frac{m}{cT}\Big)^{\frac{r}{2}}, \infty)\big)\Big), \end{aligned}$$

for ε small enough. Here, $\tilde{J}$ is the good rate function corresponding to $\sup_{s\in[0,T]} |U(V_s^\varepsilon)|$, which satisfies an LDP, as seen from applying the contraction principle Theorem 1.2.5 to the continuous mapping $f \mapsto \sup_{s\in[0,T]} |U(f(s))|$. From this, we can write

$$\limsup_{\varepsilon\searrow 0} \varepsilon \log P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) \le -\frac{1}{2}\tilde{J}\Big(\big(y\Big(\frac{m}{cT}\Big)^{\frac{r}{2}}, \infty)\Big). \tag{3.3.11}$$

Since $\tilde{J}$ has compact level sets, the term on the right-hand side explodes for $m \nearrow \infty$. □

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Next, we show that the discretization functionals Φ_m yield an exponentially good approximation.

Lemma 3.3.5. *For every $\delta > 0$,*

$$\lim_{m\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon \log P\Big(\big|\Phi(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon) - \Phi_m(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)\big| > \delta \Big) = -\infty. \tag{3.3.12}$$

Proof. This lemma corresponds to Lemma 22 in [Gul18]. As in the proof of that lemma, it suffices to show

$$\lim_{m\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon \log P\left(\sqrt{\varepsilon}|\rho| \sup_{t\in[0,T]} \left| \int_0^t \sigma_s^{(m)}\, dB_s \right| > \delta \right) = -\infty, \tag{3.3.13}$$

where $\sigma_t^{(m)} = \sigma(\hat{V}_t^\varepsilon) - \sigma(\hat{V}^\varepsilon_{\Xi(t)})$. We have to redefine $\xi_\eta^{(m)}$ in order to take a general $T > 0$ into account:

$$\xi_\eta^{(m)} = \inf\left\{ t \in [0,T] : \frac{\eta}{q(\eta)}|\hat{V}^\varepsilon| + |\hat{V}_t^\varepsilon - \hat{V}^\varepsilon_{\Xi(t)}| > \eta \right\} \wedge T.$$

Note that we use the convention $\inf \emptyset = \infty$ here. The equations (55)–(65) in [Gul18] remain the same, except that we replace $\varepsilon^H \hat{B}$ by $\hat{V}^\varepsilon$ and use our redefined versions of $\sigma^{(m)}$ and $\xi_\eta^{(m)}$. Thus, formula (65) in [Gul18] can be applied. The estimates (66) and (67) have to be replaced by

$$P\left(\sqrt{\varepsilon}|\rho| \sup_{t\in[0,T]} \left| \int_0^t \sigma_s^{(m)}\, dB_s \right| > \delta \right) \le P(\xi_\eta^{(m)} < T) + P\left(\sqrt{\varepsilon}|\rho| \sup_{t\in[0,\xi_\eta^{(m)}]} \left| \int_0^t \sigma_s^{(m)}\, dB_s \right| > \delta \right)$$

and

$$\begin{aligned} P(\xi_\eta^{(m)} < T) &\le P\Big(\sup_{t\in[0,T]} \big(\frac{\eta}{q(\eta)}|\hat{V}_t^\varepsilon| + |\hat{V}_t^\varepsilon - \hat{V}^\varepsilon_{\Xi(t)}|\big) > \eta \Big) \\ &\le P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) + P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}^\varepsilon_{\Xi(t)}| > \frac{\eta}{2} \Big). \end{aligned} \tag{3.3.14}$$

Using Lemma 3.3.4, we can handle the second term, and so it remains to find an appropriate estimate for the first term. Here we need to adapt the reasoning in [Gul18] because of the lack of Gaussianity. By the LDP for $\hat{V}^\varepsilon$ and the contraction principle applied to the continuous mapping $f \mapsto \sup_{t\in[0,T]} |f(t)|$, we get

$$P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) \le \exp\Big(-\frac{\varepsilon^{-1}}{2} \cdot I_{\sup}\big((\tfrac{1}{2}q(\eta), \infty)\big) \Big), \tag{3.3.15}$$

for $\varepsilon > 0$ small enough, where $I_{\sup}$ is the rate function of $\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon|$. Note that $q(\eta) \nearrow \infty$ for $\eta \searrow 0$. So, we get

$$\limsup_{\eta\searrow 0} \limsup_{\varepsilon\searrow 0} \varepsilon \log P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) = -\infty. \tag{3.3.16}$$

Using (3.3.9) and (3.3.16), we get (73) and (74) of [Gul18]. Finally, we can complete the proof as in [Gul18]. $\square$

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Let us continue the proof of Theorem 3.1.6. Lemma 3.3.2 states that condition (1.2.12) is satisfied. Furthermore, due to Lemma 3.3.5, we know that $\Phi_m(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ is an exponentially good approximation of $\Phi(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ as $m \nearrow \infty$. Hence, we can use the extended contraction principle (Theorem 1.2.11), and get that $\hat{X}^\varepsilon_T$ satisfies an LDP with good rate function I and speed ε^{-1}. We know from Lemma 3.3.1 that $\hat{X}^\varepsilon_T$ and X^ε_T are exponentially equivalent, and so we finally arrive at Theorem 3.1.6.

According to the extended contraction principle, we have

$$I_T(x) = \inf\big\{\hat{I}(y,f,g) : x = \Phi(y,f,g)\big\}.$$

The rate function $\hat{I}$ is only finite for

$$\hat{I}\big(y,f,\mathcal{K}(U\circ\Gamma(f))\big) = \frac{T}{2}y^2 + \frac{1}{2}\langle \dot{f},\dot{f}\rangle.$$

Recall that Γ is the one-dimensional solution map that takes f to the solution of the ODE $\dot{v} = \bar{b}(v) + \bar{\sigma}(v)\dot{f}$, $v(0) = v_0$, and that the function Φ can be written as

$$\Phi(y,f,g) = \bar{\rho}\sqrt{\langle \sigma(g)^2, 1\rangle}\,y + \rho\langle \sigma(g), \dot{f}\rangle.$$

Hence, if $x = \Phi(y,f,g)$, then

$$y = \frac{x - \rho\langle \sigma(g), \dot{f}\rangle}{\bar{\rho}\sqrt{\langle \sigma(g)^2, 1\rangle}}.$$

Inserting this into the rate function obtained through the contraction principle, we get

$$\begin{aligned} I_T(x) &= \inf\big\{\hat{I}(y,f,g) : x = \Phi(y,f,g),\ f \in H^1_0,\ g = \mathcal{K}(U\circ\Gamma(f))\big\} \\ &= \inf\Big\{\frac{T}{2}y^2 + \frac{1}{2}\langle \dot{f},\dot{f}\rangle : y = \frac{x - \rho\langle \sigma(\mathcal{K}(U\circ\Gamma(f))), \dot{f}\rangle}{\bar{\rho}\sqrt{\langle \sigma(\mathcal{K}(U\circ\Gamma(f)))^2, 1\rangle}}, f \in H^1_0\Big\} \\ &= \inf_{f\in H^1_0}\Big\{\frac{T}{2}\Big(\frac{x - \rho\langle \sigma(\mathcal{K}(U\circ\Gamma(f))), \dot{f}\rangle}{\bar{\rho}\sqrt{\langle \sigma(\mathcal{K}(U\circ\Gamma(f)))^2, 1\rangle}}\Big)^2 + \frac{1}{2}\langle \dot{f},\dot{f}\rangle\Big\}. \end{aligned} \tag{3.3.17}$$

3.3.2. Proof of Theorem 3.1.7 (a sample path LDP)

We adapt the arguments on pp. 3655–3658 in [Gul20]. As in the preceding section, our starting point is that we already have an LDP for $(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$, see Corollary 3.2.6. We redefine the functions Φ and Φ_m so that they map $C[0,T]^3$ to $C[0,T]$. For $l \in H^1_0[0,T]$ and $(f,g) \in C[0,T]^2$ such that $f \in H^1_0[0,T]$ and $g = \check{f}$,

$$\Phi(l,f,g)(t) = \bar{\rho}\int_0^t \sigma(\check{f}(s))\dot{l}(s)\,ds + \rho\int_0^t \sigma(\check{f}(s))\dot{f}(s)\,ds, \quad 0 \le t \le T. \tag{3.3.18}$$

In addition, for all the remaining triples (l,f,g), we set $\Phi(l,f,g)(t) = 0$ for all $t \in [0,T]$. By the following lemma, we can remove the drift term.

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Lemma 3.3.6. *The families of processes X^ε and $\hat{X}^\varepsilon$ are exponentially equivalent, i.e. for every $\delta > 0$, the following equality holds:*

$$\limsup_{\varepsilon \searrow 0} \varepsilon \log P(\|X^\varepsilon - \hat{X}^\varepsilon\|_{C[0,T]} > \delta) = -\infty. \tag{3.3.19}$$

Here, $\hat{X}^\varepsilon$ is defined in (3.3.1).

Proof. By taking into account the proof of Lemma 3.3.1, we see that just one additional estimate is needed, namely

$$\|X^\varepsilon - \hat{X}^\varepsilon\|_{C[0,T]} = \sup_{0\le t\le T} |X_t^\varepsilon - \hat{X}_t^\varepsilon| \le \frac{1}{2}\varepsilon T \eta\big(\sup_{0\le t\le T} |\hat{V}_t^\varepsilon|\big).$$

Then we directly get

$$P(\|X^\varepsilon - \hat{X}^\varepsilon\| > \delta) \le P\Big(\frac{1}{2}\varepsilon T\eta\big(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon|\big) > \delta\Big) = P\Big(\sup_{0\le t\le T}|\hat{V}_t^\varepsilon| > \eta^{-1}\Big(\frac{2\delta}{\varepsilon T}\Big)\Big),$$

which is exactly the same expression as in the proof of (3.3.2). □

The sequence of functionals $(\Phi_m)_{m\ge 1}$ from $C[0,T]^3$ to $C[0,T]$ is given for $(r,h,l) \in C[0,T]^3$ and $t \in [0,T]$ by

$$\begin{aligned}\Phi_m(r,h,l)(t) = \bar{\rho}\Bigg(&\sum_{k=0}^{[\frac{mt}{T}-1]} \sigma(l(t_k))\Big(r(t_{k+1}) - r(t_k)\Big) + \sigma\big(l(\Xi(t))\big)\Big(r(t) - r(\Xi(t))\Big)\Bigg)\\ +\rho\Bigg(&\sum_{k=0}^{[\frac{mt}{T}-1]} \sigma(l(t_k))\Big(h(t_{k+1}) - h(t_k)\Big) + \sigma\big(l(\Xi(t))\big)\Big(h(t) - h(\Xi(t))\Big)\Bigg).\end{aligned} \tag{3.3.20}$$

It is not hard to see that for every $m \ge 1$, the mapping Φ_m is continuous.

Lemma 3.3.7. *For every $\zeta > 0$ and $y > 0$,*

$$\limsup_{m\nearrow\infty} \sup_{\{(r,f)\in H_0^1[0,T]^2:\, \frac{1}{2}\int_0^T \dot{r}(s)\,ds + \frac{1}{2}\int_0^T \dot{f}(s)\,ds \le \zeta\}} \|\Phi(r,f,\check{f}) - \Phi_m(r,f,\check{f})\|_{C[0,T]^2} = 0. \tag{3.3.21}$$

Proof. Lemma 3.3.7 can be obtained from the proofs of Lemma 3.3.2, Lemma 21 in [Gul18] and Lemma 2.13 in [Gul20]. The only difference here is that the supremum is taken over two functions from $D_\eta = \{w \in H_0^1[0,T] : \int_0^T \dot{w}^2\,ds \le \eta\}$. By the uniform bound in the proof of Lemma 21 of [Gul18], this is actually irrelevant. □

Next, we will show that the family $\Phi_m(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ is an exponentially good approximation for $\Phi(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$, as $m \nearrow \infty$.

Lemma 3.3.8. *For every $\delta > 0$*

$$\lim_{m\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon \log P(\|\Phi(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon) - \Phi_m(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)\|_{C[0,T]} > \delta) = -\infty. \tag{3.3.22}$$

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Proof. In the proof of Lemma 3.3.5, the estimate (3.3.13) was formulated stronger than needed. We can directly use this to show (2.13) of [Gul20]. We can also get (2.14) of [Gul20] this way. The ingredients of (55)–(65) in [Gul18] do in fact depend on the Brownian motion B via the process $\hat{V}^\varepsilon$. However, the reasoning for the estimate

$$P\Big(\sup_{t\in[0,\xi_\eta^{(m)}]} \varepsilon^H \Big| \int_0^t \sigma_s^{(m)}\, dB_s \Big| > \delta\Big) \leq \exp\Big(-\frac{\delta^2}{2\varepsilon^{2H} L(q(\eta))^2 \omega(\eta)^2}\Big) \tag{3.3.23}$$

in [Gul18] stays the same if we replace the driving Brownian motion B by W. The rest of the proof from here on is essentially the same as in the proof of Theorem 2.9 in [Gul20]. □

Just as in the preceding section, we combine Lemmas 3.3.6–3.3.8 to see that Theorem 3.1.7 follows from the extended contraction principle (Theorem 1.2.11). We have

$$Q(g) = \inf\{\hat{I}(\psi_0, \psi_1, \psi_1) : g = \Phi(\psi_0, \psi_1, \psi_2)\}.$$

The rate function $\hat{I}$ is only finite for

$$\hat{I}(\psi_0, \psi_1, \psi_2) = \frac{1}{2}\langle \dot{\psi}_0, \dot{\psi}_0\rangle + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle,$$

where $\psi_1 = f$ and $\psi_2 = \mathcal{K}(U \circ \Gamma(f))$ for some $f \in H_0^1[0,T]$. Recall that the function Φ is given by

$$\Phi(l, f, g)(t) = \bar{\rho} \int_0^t \sigma(g(s))\dot{l}(s)\, ds + \rho \int_0^t \sigma(g(s))\dot{f}(s)\, ds,$$

hence we can write

$$\dot{l} = \frac{\partial_t(\Phi(l, f, g)) - \rho\sigma(g)\dot{f}}{\bar{\rho}\sigma(g)}.$$

Finally, we get the rate function as follows:

$$\begin{aligned}
Q(g) &= \inf\{\hat{I}(\psi_0, \psi_1, \psi_2) : g = \Phi(\psi_0, \psi_1, \psi_2)\} \\
&= \inf\Big\{\frac{1}{2}\langle \dot{\psi}_0, \dot{\psi}_0\rangle + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle : f \in H_0^1,\ \psi_1 = f,\ \psi_2 = \mathcal{K}(U \circ \Gamma(f)), \\
&\qquad\qquad \dot{\psi}_0 = \frac{\partial_t(\Phi(\psi_0, \psi_1, \psi_2)) - \rho\sigma(\psi_2)\dot{\psi}_1}{\bar{\rho}\sigma(\psi_2)},\ g = \Phi(\psi_0, \psi_1, \psi_2)\Big\} \\
&= \inf\Big\{\frac{1}{2}\langle \dot{\psi}_0, \dot{\psi}_0\rangle + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle : f \in H_0^1,\ \dot{\psi}_0 = \frac{\dot{g} - \rho\sigma(\mathcal{K}(U \circ \Gamma(f)))\dot{f}}{\bar{\rho}\sigma(\mathcal{K}(U \circ \Gamma(f)))}\Big\} \\
&= \inf_{f \in H_0^1}\Big\{\frac{1}{2}\int_0^T \Big(\frac{\dot{g}(t) - \rho\sigma(\mathcal{K}(U \circ \Gamma(f))(t))\dot{f}(t)}{\bar{\rho}\sigma(\mathcal{K}(U \circ \Gamma(f))(t))}\Big)^2 dt + \frac{1}{2}\int_0^T |\dot{f}(t)|^2\, dt\Big\}.
\end{aligned} \tag{3.3.24}$$

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3.4. An example – Fractional CIR stochastic volatility

We describe an example of a model that fits our assumptions, and has already been studied in the literature on fractional volatility modelling [BD20]. Let V be a CIR process with positive parameters κ, θ and σ_{CIR}, satisfying $2\kappa\theta > \sigma_{\text{CIR}}^2$. In this case,

$$\bar{b}(x) = \kappa(\theta - x) \quad \text{and} \quad \bar{\sigma}(x) = \sigma_{\text{CIR}}\sqrt{x},$$

and the dynamics of V are

$$dV_t = \kappa(\theta - V_t)dt + \sigma_{\text{CIR}}\sqrt{V_t}dB_t.$$

We choose the fractional kernel $K(t,s) = \Gamma(\alpha)^{-1}(t-s)^{\alpha-1}$, $0 \le s \le t$, and $U = \text{id}$, so that the process $\hat{V}$ defined in (3.1.1) is the Riemann-Liouville integral of order α of the process V. We assume $\alpha \in (\frac{1}{2}, \frac{3}{2})$, which overlaps with the parameter range $\alpha \in (0,1)$ considered in Section 2 of [BD20], and implies our assumption (3.1.3). The definition of the model is completed by putting

$$\sigma(x) = \sqrt{\sigma_0^2 + x}, \quad x \ge 0,$$

where $\sigma_0 > 0$ is the initial value of the stochastic volatility process $\sigma(\hat{V}_t)$. Note a small difference in notation compared to [BD20]: We write $v_0 = V_0$ for the initial value of V, and not for the initial value of the variance process $\sigma(\hat{V}_t)^2$ of the stock, which we denote by σ_0^2. Unlike [BD20], which is a paper on portfolio optimization, we set the drift of the stock to zero, because the application we have in mind is approximate option pricing in the small-noise regime.

The advantages of using a fractional CIR process instead of the classical CIR process are described in [BD20], Section 2, and the references given there. The model captures volatility persistence, in particular, steep implied volatility smiles for long maturity options and the comovement between implied and realized volatility. The paper [BD20] also gives a formula that makes the long-range dependence of the variance process explicit.

The model we just described is also closely related to the fractional Heston model used in [GJRS18]. The main difference, besides the zero correlation assumption imposed in [GJRS18], is the range of α. They assume $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, whereas we have $\alpha \in (\frac{1}{2}, \frac{3}{2})$. Thus, the models we consider in this chapter could be seen as a complement to the fractional Heston model of [GJRS18], with positive correlation and rather general functions $\bar{b}(\cdot)$, $\bar{\sigma}(\cdot)$ and $\sigma(\cdot)$, but at the price of losing roughness of the volatility paths.

Remark 3.4.1. The paths of the CIR process V are $(\frac{1}{2}-\delta)$-Hölder continuous for any $\delta \in (0, \frac{1}{2})$ (see Lemma 7.1 in [BD20]). If we choose the fractional kernel $K(t,s) = \Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-1/2}$, $H \in (0,1)$, in the model considered in the present section, then the paths of $\hat{V}$ are in the Hölder space $\mathcal{H}^{H+1-\delta}$. See Definition 1.1.6 (p. 6) and Corollary 1.3.1 (p. 56) in [SKM93]. In particular, since $H+1-\delta > 1$ for small δ, the paths of $\hat{V}$ are C^1 on $(0,T)$. By modifying the model, using $U(x) = |x - V_0|^\kappa$ with $\kappa \in (0,1]$ instead of $U = \text{id}$, the paths of $\hat{V}$ become less smooth, namely $(\frac{1}{2}\kappa + H + \frac{1}{2} - \delta)$-Hölder continuous. In addition, if $\sigma(x) = \sigma_0(1 + x^\beta)$, $\beta \in (0,1)$, then the volatility paths $t \mapsto \sigma_0(1 + (\hat{V}_t)^\beta)$ are $(\frac{1}{2}\kappa\beta + (H+\frac{1}{2})\beta - \delta)$-Hölder continuous on $[0,T]$, for any small enough $\delta > 0$. While this Hölder exponent can be smaller than $\frac{1}{2}$, the volatility process is *not* rough, because $\sigma(\cdot)$ is smooth away from zero, and so “roughness” occurs only at time zero. Note that in truly rough models, the volatility process is constructed using stochastic integrals $\int_0^t K(t,s)dW_s$ or related processes, which is not the case in our setup.

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3.5. Second order Taylor expansion of the rate function

In order to compute the rate function, a certain variational problem needs to be solved numerically. It might be preferable to use the Taylor expansion of the rate function instead, if it can be computed in closed form. In principle, this can be done using the approach used in [BFG+19], but would involve rather cumbersome calculations. We therefore illustrate the method by the example $V = B$ (a Brownian motion; thus $\bar{b} \equiv 0$ and $\bar{\sigma} \equiv 1$), $U(x) = x^2$, $v_0 = 0$. It is very easy to see that our main results hold for this example. Indeed, the required results from [CF14], for which we made our assumptions on the SDE for V, trivially hold here. The control ODE is degenerate, and its solution mapping Γ is just the identity map. The statement of Theorem 3.2.5 follows from Schilder's theorem and the contraction principle, and the transfer to the log-price is a simplified version of the arguments in Section 3.3.

Proposition 3.5.1. *Let $U(x) = x^2$ and $V = B$. Furthermore, assume that σ is smooth (at least locally around 0). Suppose that the rate function I is also smooth locally around 0. Then, with $\sigma_0 = \sigma(0)$, its Taylor expansion is*

$$\begin{aligned} I(x) &= I(0) + I'(0)x + I''(0)x^2 + O(x^3) \\ &= I''(0)x^2 + O(x^3) \\ &= \frac{1}{2\sigma_0^2}x^2 + O(x^3). \end{aligned} \tag{3.5.1}$$

Remark 3.5.2. Formula (3.5.1) gives the second order Taylor expansion. However, the ideas in the proof of Proposition 3.5.1 can be used for higher orders. Clearly, the computations for the expansions get even more cumbersome in the latter case.

3.5.1. Proof of Proposition 3.5.1

The proof is very similar to the one of Theorem 3.1 in [BFG+19]. In the following, we will outline at which points adjustments are needed. Note that for the special we are treating we have $U(x) = x^2$ and $\Gamma \equiv \mathrm{id}$. To simplify computations in the proof, we put $T = 1$ and write $I = I_1$ for the rate function. In Proposition 5.1 of [BFG+19], there is a representation of the rate function that coincides with ours, except that different integral transforms are used. For our special case, we have

$$I(x) = \inf_{f \in H_0^1} \Big[\frac{(x - \rho \tilde{G}(f))^2}{2\bar{\rho}^2 \tilde{F}(f)} + \frac{1}{2}\tilde{E}(f) \Big] = \inf_{f \in H_0^1} \mathcal{I}_x(f), \tag{3.5.2}$$

where

$$\tilde{G}(f) := \int_0^1 \sigma((\mathcal{K}(f^2))(s)) \dot{f}(s)\, ds = \langle \sigma(\mathcal{K}(f^2)), \dot{f} \rangle, \tag{3.5.3}$$

$$\tilde{F}(f) := \int_0^1 \sigma((\mathcal{K}(f^2))(s))^2\, ds = \langle \sigma^2(\mathcal{K}(f^2)), 1 \rangle, \tag{3.5.4}$$

$$\tilde{E}(f) := \int_0^1 |\dot{f}(s)|^2\, ds = \langle \dot{f}, \dot{f} \rangle. \tag{3.5.5}$$

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Recall that $\mathcal{K}f = \int_0^\cdot K(\cdot,s)f(s)\,ds$. In [BFG+19] the authors use the same integral transform as used in [Gul18, Gul20], i.e. $\mathcal{K}\dot{f}$. We have to adjust this to our case of $\mathcal{K}(f^2)$. Here, $\mathcal{I}_x$ denotes the functional that needs to be minimized to get the value of the rate function at x.

First, we need to get a representation for the minimizing configuration f^x of the functional $\mathcal{I}_x$. This is done like in Proposition 5.2 in [BFG+19]. The corresponding expansions of the ingredients of the rate function for our setting for $\delta > 0$ are

$$\tilde{E}(f+\delta g) \approx \tilde{E}(f) + 2\delta\langle \dot{f}, \dot{g}\rangle, \tag{3.5.6}$$

$$\tilde{F}(f+\delta g) \approx \tilde{F}(f) + 2\delta\langle (\sigma^2)'(\mathcal{K}(f^2)), \mathcal{K}(fg)\rangle, \tag{3.5.7}$$

$$\tilde{G}(f+\delta g) \approx \tilde{G}(f) + \delta(\langle \sigma(\mathcal{K}(f^2)), \dot{g}\rangle + 2\langle \sigma'(\mathcal{K}(f^2)), \dot{f}\mathcal{K}(fg)\rangle) \tag{3.5.8}$$

Note, that " $\approx$ " is defined in [BFG+19] as

$$A \approx B :\Leftrightarrow A = B + o(\delta), \quad \delta \searrow 0. \tag{3.5.9}$$

If $f = f^x$ is a minimizer then $\delta \mapsto \mathcal{I}_x(f+\delta g)$ has a minimum at $\delta = 0$ for all g. Using (3.5.6), (3.5.7) and (3.5.8) we expand

$$\begin{aligned}
\mathcal{I}_x(f+\delta g) &= \frac{(x-\rho\tilde{G}(f+\delta g))^2}{2\bar{\rho}^2\tilde{F}(f+\delta g)} + \frac{1}{2}\tilde{E}(f+\delta g)\\
&\approx \frac{(x-\rho\tilde{G}(f))^2 - 2\delta\rho(x-\rho\tilde{G}(f))\big(\langle\sigma(\mathcal{K}(f^2)),\dot{g}\rangle + 2\langle\sigma'(\mathcal{K}(f^2)),\dot{f}\mathcal{K}(fg)\rangle\big)}{2\bar{\rho}^2\tilde{F}(f)\big(1+\frac{2\delta}{\tilde{F}(f)}\langle(\sigma^2)'(\mathcal{K}(f^2)),K(fg)\rangle\big)}\\
&\quad + \frac{1}{2}\tilde{E}(f) + \delta\langle\dot{f},\dot{g}\rangle\\
&\approx \frac{(x-\rho\tilde{G}(f))^2 - 2\delta\rho(x-\rho\tilde{G}(f))\big(\langle\sigma(\mathcal{K}(f^2)),\dot{g}\rangle + 2\langle\sigma'(\mathcal{K}(f^2)),\dot{f}\mathcal{K}(fg)\rangle\big)}{2\bar{\rho}^2\tilde{F}(f)}\\
&\quad - \frac{(x-\rho\tilde{G}(f))^2}{2\bar{\rho}^2\tilde{F}(f)}\frac{2\delta}{\tilde{F}(f)}\langle(\sigma^2)'(\mathcal{K}(f^2)),\mathcal{K}(fg)\rangle + \frac{1}{2}\tilde{E}(f) + \delta\langle\dot{f},\dot{g}\rangle.
\end{aligned} \tag{3.5.10}$$

Now, as a consequence, for $f = f^x$ and every $g \in H_0^1[0,1]$,

$$\begin{aligned}
0 = \partial_\delta(\mathcal{I}_x(f+\delta g))_{\delta=0} = &-\frac{2\rho(x-\rho\tilde{G}(f))\big(\langle\sigma(\mathcal{K}(f^2)),\dot{g}\rangle + 2\langle\sigma'(\mathcal{K}(f^2)),\dot{f}\mathcal{K}(fg)\rangle\big)}{2\bar{\rho}^2\tilde{F}(f)}\\
&- \frac{(x-\rho\tilde{G}(f))^2}{2\bar{\rho}^2\tilde{F}^2(f)}2\langle(\sigma^2)'(\mathcal{K}(f^2),\mathcal{K}(fg)\rangle + \langle\dot{f},\dot{g}\rangle.
\end{aligned} \tag{3.5.11}$$

We have $f_0^x = 0$, for any x. We now test with $\dot{g} = \mathbb{1}_{[0,t]}$ for a fixed $t \in [0,1]$ and obtain

$$\begin{aligned}
f_t^x = &\frac{\rho(x-\rho\tilde{G}(f^x))\big(\langle\sigma(\mathcal{K}((f^x)^2)),\mathbb{1}_{[0,t]}\rangle + 2\langle\sigma'(\mathcal{K}((f^x)^2)),\dot{f}^x\mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle\big)}{\bar{\rho}^2\tilde{F}(f^x)}\\
&+ \frac{(x-\rho\tilde{G}(f^x))^2}{2\bar{\rho}^2\tilde{F}^2(f^x)}2\langle(\sigma^2)'(\mathcal{K}((f^x)^2)),\mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle,
\end{aligned} \tag{3.5.12}$$

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where we write

$$\mathrm{id}_{\leq t}(s) = g(s) = \int_0^s \dot{g}(u)\,du = \int_0^s \mathbb{1}_{[0,t]}(u)\,du = \int_0^{s\wedge t} 1\,du = s\wedge t. \tag{3.5.13}$$

Let us recall the ansatz in [BFG+19]. The authors of [BFG+19] choose for fixed x the optimizing function f^x for $\mathcal{I}_x$, i.e. $f^x = \operatorname{argmin}_{f\in H_0^1}\mathcal{I}_x(f)$. Therefore, the first order condition is $\mathcal{I}_x'(f^x) = 0$. The authors of [BFG+19] use the implicit function theorem to show that the minimizing configuration f^x is a smooth function in x (locally around $x=0$). As $\mathcal{I}_x$ is a smooth function, too, this implies the smoothness of $x\mapsto \mathcal{I}_x(f^x) = I(x)$, at least in a neighborhood of 0. Note that for (26) and Lemma 5.3 in [BFG+19], the embedding $\mathcal{K}: H_0^1 \to C$ works, because we have already established that $\mathcal{K}(U\circ f)$ is continuous (see Lemma 3.2.4).

In order to apply the implicit function theorem, the authors of [BFG+19] show that the ingredients of the rate function are Fréchet differentiable by computing their Gateaux derivative. This is more complicated in our case, because of the different integral transform we use. Therefore, we *assume* that the rate function is locally smooth around 0 in Proposition 3.5.1, and, consequently, that Lemma 5.6 in [BFG+19] holds. After establishing that the implicit function theorem can be used, we can proceed as in [BFG+19] up to Theorem 5.12 there.

Next, we will imitate the computations in Theorem 5.12 of [BFG+19] in order to get the expansion of the minimizing configuration in our setting. In fact, if we just want to obtain the second order expansion of the rate function in our setting for Brownian motion squared, it suffices to find the first order expansion of f^x. Assuming the ansatz

$$f_t^x = \alpha_t x + O(x^2), \tag{3.5.14}$$

we get

$$\begin{aligned}
f_t^x &= \alpha_t x + O(x^2),\\
\dot{f}_t^x &= \dot{\alpha}_t x + O(x^2),\\
\sigma(\mathcal{K}((f^x)^2)) &= \sigma_0 + O(x^2),\\
\sigma'(\mathcal{K}((f^x)^2)) &= \sigma_0' + O(x^2),\\
\tilde{F}(f^x) &= \sigma_0^2 + O(x^2),\\
\tilde{G}(f^x) &= \langle \sigma_0, \dot{\alpha}\rangle x + O(x^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \sigma(\mathcal{K}((f^x)^2)), \mathbb{1}_{[0,t]}\rangle &= \sigma_0 t + O(x),\\
2\langle \sigma'(\mathcal{K}((f^x)^2)), \dot{f}^x \mathcal{K}(f^x\,\mathrm{id}_{\leq t})\rangle &= O(x),\\
2\langle (\sigma^2)'(\mathcal{K}((f^x)^2)), \mathcal{K}(f^x\,\mathrm{id}_{\leq t})\rangle &= O(x),\\
x - \rho\tilde{G}(f^x) &= (1-\rho\sigma_0\alpha_1)x + O(x^2),\\
(x-\rho\tilde{G}(f^x))^2 &= O(x^2).
\end{aligned}$$

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We use the previous formulas in (3.5.12) to obtain

$$\begin{aligned} f_t^x &= \frac{\rho((1-\rho\sigma_0\alpha_1)x + O(x^2))(\sigma_0 t + O(x))}{\bar{\rho}^2(\sigma_0^2 + O(x^2))} + \frac{O(x^2)}{2\bar{\rho}^2(\sigma_0^4 + O(x^2))} O(x) \\ &= \frac{\rho(1-\rho\sigma_0\alpha_1)x\sigma_0 t}{\bar{\rho}^2\sigma_0^2} + O(x^2). \end{aligned} \tag{3.5.15}$$

Comparing the coefficients, we get the same result as the authors of [BFG+19] for the first order expansion, i.e.

$$\alpha_t = \frac{\rho(1-\rho\sigma_0\alpha_1)}{\bar{\rho}^2\sigma_0} t. \tag{3.5.16}$$

Setting $t = 1$ and then computing α_1 leads to the formula

$$\alpha_t = \frac{\rho}{\sigma_0} t. \tag{3.5.17}$$

Note that the first order expansion of the minimizing configuration f^x is *exactly* the same as in [BFG+19]. The reason is that the expansions of the ingredients of (3.5.12) are relevant here, and these expansions coincide. For the second order expansion of the rate function, we need second order expansions of its ingredients. These are given in the following formulas, where id^2 denotes the quadractic function $s \mapsto s^2$:

$$\begin{aligned} \frac{1}{2}\tilde{E}(f^x) &= \frac{1}{2}\frac{\rho^2}{\sigma_0^2} x^2 + O(x^3), \\ (x - \rho\tilde{G}(f^x))^2 &= \bar{\rho}^4 x^2 + O(x^3) \\ \tilde{F}(f^x) &= \sigma_0^2 + (\sigma_0^2)' \langle \mathcal{K}(\alpha^2), 1\rangle x^2 + O(x^3) \\ &= \sigma_0^2 + (\sigma_0^2)' \frac{\rho^2}{\sigma_0^2} \langle \mathcal{K}(\mathrm{id}^2), 1\rangle x^2 + O(x^3). \end{aligned}$$

Finally, we get the Taylor expansion of the rate function by taking into account the reasoning above. We insert the expansion

$$f_t^x = \alpha_t x + O(x^2) = \frac{\rho}{\sigma_0} t x + O(x^2) \tag{3.5.18}$$

and the expansions above into Eq. (3.5.12) for the minimizing configuration. Then, we get

$$\begin{aligned} \mathcal{I}_x(f^x) &= \frac{(x - \rho\tilde{G}(f^x))^2}{2\bar{\rho}^2 \tilde{F}(f^x)} + \frac{1}{2}\tilde{E}(f^x) \\ &= \frac{\bar{\rho}^4 x^2 + O(x^3)}{2\bar{\rho}^2\left(\sigma_0^2 + (\sigma_0^2)' \frac{\rho^2}{\sigma_0^2} \langle \mathcal{K}(\mathrm{id}^2), 1\rangle x^2 + O(x^3)\right)} + \frac{1}{2}\frac{\rho^2}{\sigma_0^2} x^2 + O(x^3) \\ &= \frac{\bar{\rho}^2}{2\sigma_0^2} x^2 + O(x^3) + \frac{1}{2}\frac{\rho^2}{\sigma_0^2} x^2 + O(x^3) \\ &= \frac{1}{2\sigma_0^2}(\bar{\rho}^2 + \rho^2) x^2 + O(x^3) \\ &= \frac{1}{2\sigma_0^2} x^2 + O(x^3), \end{aligned} \tag{3.5.19}$$

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and hence the following expansion holds:

$$I(x) = \mathcal{I}_x(f^x) = \frac{1}{2\sigma_0^2}x^2 + O(x^3). \tag{3.5.20}$$

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4. Rough 3/2 – A truncated ansatz

4.1. Introduction – A perspective on 3/2 model(s)

First, as a disclaimer, the ideas presented here are not yet published. Hence, this part of the thesis might contain some open questions and the results obtained up to now have not competed in a peer review process up to now.

In classic stochastic volatility models, several studies [BJO06], [CV03], [Jon03] propose that the so-called 3/2-model is preferable to the Heston model, to avoid the downward sloping volatility of variance smiles there that contradicts empirical findings. Its dynamics is given by

$$dS_t = S_t\sqrt{V_t}\,dB_t, \quad S_0 \geq 1, \tag{4.1.1}$$

$$dV_t = \kappa V_t(\theta - V_t)\,dt + \xi V_t^{3/2}\,dW_t, \quad V_0 = v_0 > 0, \tag{4.1.2}$$

$$d\langle B, W\rangle_t = \rho\,dt. \tag{4.1.3}$$

Here, contrary to the Heston model, the mean reversion speed of the variance is stochastic in the model. Therefore, the variance process reverts more quickly when it is at a high level, thus admitting extreme paths with spikes in instantaneous variance. This means that in periods of market stress with increasing volatility, the vol-of-variance skew steepens, while the opposite would happen in the Heston model. In a classic setting, there are several results about the 3/2-model which include a representation of the cumulant generating function using the confluent hypergeometric function. In this model, the coefficient functions of the integral equation are *not* affine and *not* globally Lipschitz continuous.

Now, in the perspective of rough volatility models, note that there has been an extensive discussion of affine volatility models, e.g. [ER19, ALP19]. In some papers, results for the classic models are transmitted to their rough versions. Considering non-affine models, to our knowledge, there are no papers about a rough variant of the 3/2-model given above. For getting a rough version, we add a fractional kernel to the model in (4.1.1)–(4.1.3) and arrive at

$$dS_t = S_t\sqrt{|V_t|}\,dB_t, \quad S_0 \geq 1, \tag{4.1.4}$$

$$V_t = V_0 + \int_0^t K(t,s)\kappa V_s(\theta - V_s)\,ds + \int_0^t K(t,s)\xi|V_s|^{3/2}\,dW_s, \quad V_0 = v_0 > 0, \tag{4.1.5}$$

$$d\langle B, W\rangle_t = \rho\,dt. \tag{4.1.6}$$

For the kernel take e.g. a Riemann-Liouville kernel $K_\alpha(t,s) = K_\alpha(t-s) \propto (t-s)^{\alpha-1}$ with $\alpha \in (\frac{1}{2}, 1)$ or a suitable kernel for the Fredholm representation of a fractional Brownian motion, i.e. K_H such that

$$B_H(t) = \int_0^t K_H(t,s)\,dB_s = \int_0^T K_H(t,s)\,dB_s,$$

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using the usual convention $K(t,s)=0$ for $s>t$. Introducing the kernel and hence establishing a *stochastic Volterra integral equation* (SVE) brings several problems along considering existence. In the classic 3/2-model (4.1.1)–(4.1.3) existence of a positive solution to the equation (4.1.2) for the instantaneous variance that does not explode is ensured by the Feller condition. For our analysis, we need to introduce an indicator function to truncate the coefficient functions. Doing this, the coefficient functions will be bounded. As a consequence, this will also be the case for the volatility process itself which makes the SVE tractable. We mention at this point that our process V, unlike in the classic 3/2-model, can also become negative. Therefore, we will refer to the process $|V|$ as the instantaneous variance, but the process V will be without any name. As usual, for all our considerations we will use the log-price process X instead of S, i.e. $X_t := \log(S_t)$. Summing up, we introduce the following model.

Definition 4.1.1 (Truncated rough 3/2 model)**.** Let $M>0$ be fixed and very large. The model, given by the SVE

$$dX_t = \sqrt{|V_t|}\,(\bar\rho\, dW_t + \rho\, dB_t) - \frac{1}{2}|V_t|\,dt, \quad X_0 = \log(S_0) \geq 0, \tag{4.1.7}$$

$$V_t = V_0 + \int_0^t b(t,s,V_s)\,ds + \int_0^t \sigma(t,s,V_s)\,dW_s, \quad V_0 = v_0 > 0, \tag{4.1.8}$$

is called the *truncated rough 3/2 model* with truncation size M and parameters $\theta\in\mathbb{R}$, $\kappa>0$ and $\xi>0$. The two Brownian motions B and W are independent. Here, we use $\rho\in[-1,1]$ and $\bar\rho=\sqrt{1-\rho^2}$ to model the correlation between the two equations resp. their driving Brownian motions. The coefficient functions of the one-dimensional SVE (4.1.8) are given by

$$b(t,s,x) = K(t,s)\cdot \bar b(x), \tag{4.1.9}$$

$$\sigma(t,s,x) = K(t,s)\cdot \bar\sigma(x), \tag{4.1.10}$$

with

$$\bar b(x) = \begin{cases} \kappa(-M)(\theta+M), & x<-M,\\ \kappa x(\theta-x), & |x|\leq M,\\ \kappa M(\theta-M), & x>M, \end{cases} \tag{4.1.11}$$

and

$$\bar\sigma(x) = \begin{cases} \xi|x|^{3/2}, & |x|\leq M,\\ \xi M^{3/2}, & |x|>M. \end{cases} \tag{4.1.12}$$

Note, that the only stochastic integral equation in Definition 4.1.1 is (4.1.8) for the process V, because the “equation” (4.1.7) is in fact just an Itô integral, i.e.

$$X_t = X_0 - \int_0^t \frac{1}{2}|V_s|\,ds + \int_0^t \sqrt{|V_s|}(\bar\rho\, dW_s + \rho\, dB_s), \tag{4.1.13}$$

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which is well-defined in our case. Now, we formulate (4.1.7)–(4.1.8) as a two-dimensional system in order to apply large deviation results presented in [JP20]. We can write the model as a two-dimensional process $Y_t = (Y_t^1, Y_t^2)$ satisfying

$$Y_t = Y_0 + \int_0^t B(t,s,Y_s)\,ds + \int_0^t \Sigma(t,s,Y_s)\,dG_s, \quad Y_0 = (X_0, v_0), \tag{4.1.14}$$

where G is a two-dimensional standard Brownian motion, i.e.

$$G_t = \begin{pmatrix} W_t \\ B_t \end{pmatrix},$$

and the coefficient functions B and Σ are given by

$$B(t,s,(y_1,y_2)) = \begin{pmatrix} -\frac{1}{2}|y_2| \\ b(t,s,y_2) \end{pmatrix}, \tag{4.1.15}$$

$$\Sigma(t,s,(y_1,y_2)) = \begin{pmatrix} \bar{\rho}\sqrt{|y_2|} & \rho\sqrt{|y_2|} \\ \sigma(t,s,y_2) & 0 \end{pmatrix}. \tag{4.1.16}$$

Despite the fact that it is not yet clear how to attain appropriate representations on the objects of interest, as characteristic function, pricing formulas, etc., we think it is worth to discuss the problems we are faced with introducing rough volatility to that model. This chapter is organized as follows. In Section 4.2 we will prove that our model equation(s) (4.1.7)–(4.1.8) admit a pathwise unique strong solution on a deterministic interval $[0,T]$ using the ideas of [Zha10]. In Section 4.3, we will show that our specific model meets the requirements needed in [JP20] with convolutional kernels so that we get large and moderate deviation principles. These results will be used in Section 4.3.5 to apply some asymptotic results about the implied volatility and options on realized variance. In Section 4.4, there will be an implementation of our model using a naive Euler approach.

4.2. Existence

In this part, we show that global pathwise unique solutions of our model (4.1.7)–(4.1.8) exist. We use the ideas of [Zha10] and as they do we reduce our analysis to the time range $[0,1]$. In the following, let C be some arbitrary positive constant that may vary from line to line. First, we formulate three very common regularity requirements for the coefficient functions that are needed in [Zha10].

(H1) **Lipschitz**: For all $x, y \in \mathbb{R}^d$ and $s, t \in [0,1]$

$$|b(t,s,x) - b(t,s,y)| \le K_1(t,s)|x-y|, \tag{4.2.1}$$

$$\|\sigma(t,s,x) - \sigma(t,s,y)\|^2 \le K_2(t,s)|x-y|^2, \tag{4.2.2}$$

and

$$\int_0^t (|b(t,s,0)|^\beta + \|\sigma(t,s,0)\|^{2\beta})\,ds \le C, \tag{4.2.3}$$

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where $\beta > 1$, and $K_i(t,s)$, $i = 1, 2$, are two positive functions on $[0,1] \times [0,1]$, and satisfy for some $\alpha > 1$

$$\int_0^t (K_1^\alpha(t,s) + K_2^\alpha(t,s))\,ds \le C, \quad t \in [0,1]. \tag{4.2.4}$$

(H2) **Linear growth**: For all $t, s \in [0,1]$ and $x \in \mathbb{R}^d$

$$|b(t,s,x)| \le \widetilde{K}_1(t,s) \cdot (1 + |x|), \tag{4.2.5}$$

$$\|\sigma(t,s,x)\|^2 \le \widetilde{K}_2(t,s) \cdot (1 + |x|^2), \tag{4.2.6}$$

where $\widetilde{K}_1$ and $\widetilde{K}_2$ satisfy the same integrability condition as K_1 and K_2 above. Furthermore, for all $t, t', s \in [0,1]$ and $x \in \mathbb{R}^d$

$$|b(t',s,x) - b(t,s,x)| \le F_1(t',t,s)(1 + |x|), \tag{4.2.7}$$

$$\|\sigma(t',s,x) - \sigma(t,s,x)\|^2 \le F_2(t',t,s)(1 + |x|^2), \tag{4.2.8}$$

where $F_i(t',t,s)$, $i = 1, 2$, are positive functions on $[0,1] \times [0,1] \times [0,1]$, and satisfy for some $\gamma > 0$

$$\int_0^{t \wedge t'} (F_1(t',t,s) + F_2(t',t,s))\,ds \le C|t - t'|^\gamma. \tag{4.2.9}$$

(H3) **Continuity**: For each $x \in \mathbb{R}^m$ and $t \in [0,1]$, the mappings $(0,t) \ni s \mapsto b(t,s,x) \in \mathbb{R}^d$ and $(0,t) \ni s \mapsto \sigma(t,s,x) \in \mathbb{R}^d \times \mathbb{R}^m$ are continuous.

Note, that for our analysis we can use the fact that we can factorize our coefficient functions into a kernel part and a coefficient part that is only dependent on the state of the process, see (4.1.9) and (4.1.10). Hence, the conditions above can be simplified by formulating assumptions for the kernel part.

Assumption 4.2.1. The Kernel $K(t,s)$ in the coefficient functions (4.1.9) and (4.1.10) of our model (4.1.7)–(4.1.8) satisfies the following conditions:

(i) The kernel is positive $K(t,s) \ge 0$ for all $0 \le s < t \le 1$. Furthermore, we use the convention $K(t,s) = 0$ for $s > t$.

(ii) For fixed t, the kernel $K : (0,t) \to (0,\infty)$ is a continuous function, i.e. the function

$$s \mapsto K(t,s) \tag{4.2.10}$$

is continuous.

(iii) The kernel is integrable in the sense that there is an $\tilde{\alpha} > 2$ such that

$$\int_0^t K(t,s)^{\tilde{\alpha}}\,ds \le C \tag{4.2.11}$$

for all $t \in [0,1]$.

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(iv) The kernel is L^2-Hölder continuous, i.e. for the modulus of continuity

$$M_K(h) := \sup_{\{t_1,t_2\in[0,T]:\,|t_1-t_2|\le h\}} \int_0^T |K(t_1,s)-K(t_2,s)|^2\,ds, \quad 0\le h\le T, \tag{4.2.12}$$

there exists a constant $C>0$ and $r>0$ such that

$$M_K(h) \le Ch^r. \tag{4.2.13}$$

Remark 4.2.2. An example of a convolutional kernel that satisfies Assumption 4.2.1 is the Riemann-Liouville fractional kernel defined by

$$K_\nu(t,s) = K_\nu(t-s) := \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}, \tag{4.2.14}$$

where $\nu \in \left(\frac{1}{2},1\right)$. Another example is the fractional Brownian motion kernel K_H given by

$$K_H(t,s) := (c_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}}F(t/s))\mathbb{1}_{\{s<t\}}, \quad s,t\in[0,1],$$

where

$$c_H := \left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}\right)^{1/2}$$

and

$$F(u) := c_H\left(\frac{1}{2}-H\right)\int_1^u (r-1)^{H-\frac{3}{2}}(1-r^{H-\frac{1}{2}})\,dr.$$

Note that the fractional Brownian motion with Hurst parameter $H\in(0,1)$ may be defined by

$$B_t := \int_0^t K_H(t,s)\,dW_s.$$

In [Zha08] the author argues why the fractional Brownian motion kernel satisfies the requirements to apply his results (which are essentially a version of those in [Zha10]). For the Riemann-Liouville kernel as defined in (4.2.14), we show in Appendix C that it satisfies Assumption 4.2.1.

In the following we will show that Conditions (H1)–(H3) are satisfied by the process V (4.1.8) given that the Kernel $K(t,s)$ in (4.1.9) and (4.1.10) satisfies Assumption 4.2.1.

Lemma 4.2.3. *Under Assumption 4.2.1 the process V solving (4.1.8) satisfies (H1).*

Proof. Note that $0<s\le t\le 1$ for the whole proof.

Coefficient b. We have for $x,y\in\mathbb{R}$

$$|b(t,s,x)-b(t,s,y)| = K(t,s)\cdot|\bar{b}(x)-\bar{b}(y)|,$$

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where $K(\cdot,\cdot)$ is the kernel function of V and

$$\bar{b}(x) = \begin{cases} -\kappa M(\theta + M), & x < -M, \\ \kappa x(\theta - x), & |x| \leq M, \\ \kappa M(\theta - M), & x > M. \end{cases}$$

Note that on the compact interval $|x| \leq M$ the function $\bar{b}$ is smooth and Lipschitz continuous, since

$$|\partial_x \bar{b}(x)| = |\kappa\theta - 2\kappa x| \leq C_1 + C_2|x| \leq C_1 + C_2 M \leq C$$

on $|x| \leq M$. On $x > M$ ($x < -M$) the function $\bar{b}$ is constant $\bar{b}(M)$ ($\bar{b}(-M)$), and it is continuous on $\mathbb{R}$. Therefore, we have a situation as described in Appendix D.1 resp. (D.1.3). Hence, we can borrow a constant $L_b > 0$ from there and write

$$K(t,s) \cdot |\bar{b}(x) - \bar{b}(y)| \leq K(t,s) \cdot L_b \cdot |x - y|.$$

For Condition (H1) we get

$$K_1(t,s) := L_b \cdot K(t,s).$$

For the origin, we have

$$\int_0^t |b(t,s,0)|^\beta \, ds = \int_0^t K(t,s)^\beta |\bar{b}(0)|^\beta \, ds = 0$$

for every arbitrary $\beta > 1$. For the integrability of the kernel we have

$$\int_0^t K_1^\alpha(t,s) \, ds = \int_0^t L_b^\alpha K(t,s)^\alpha \, ds = L_b^\alpha \int_0^t K(t,s)^\alpha \, ds \leq C,$$

due to (4.2.11), using $\alpha := \tilde{\alpha} > 2$.

Coefficient σ. We have

$$\sigma(t,s,x) = K(t,s)\bar{\sigma}(x) = K(t,s)\xi|x|^{3/2}$$

on $|x| \leq M$. Since

$$\partial_x \bar{\sigma}(x) = \xi \frac{3}{2}\sqrt{|x|} \leq \xi \frac{3}{2}\sqrt{M} \leq C$$

we have the same situation as in (D.1.3). Since the kernel $K(t,s)$ is the same as for b and $\bar{\sigma}(0) = 0$, we have the same reasoning as for b above. However, a difference is that here we get an additional power in the exact formulation of the condition, i.e.

$$K_2(t,s) := L_\sigma^2 \cdot K(t,s)^2.$$

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For the kernel integral, we have

$$\int_0^t K_2(t,s)^\alpha \, ds = \int_0^t L_\sigma^{2\alpha} K(t,s)^{2\alpha} \, ds = L_\sigma^{2\alpha} \int_0^t K(t,s)^{2\alpha} \, ds.$$

Note, that we need some $\alpha > 1$ and hence we set $\alpha = \frac{\tilde{\alpha}}{2} > 1$ and get

$$\int_0^t K(t,s)^{2\alpha} \, ds = \int_0^t K(t,s)^{\tilde{\alpha}} \, ds < C$$

due to (4.2.11). □

Lemma 4.2.4. *Under Assumption 4.2.1 the process V solving* (4.1.8) *satisfies* (H2).

Proof. Note that $0 < s \leq t \leq 1$ for the whole proof.

Coefficient b. We have

$$\begin{aligned} |b(t',s,x) - b(t,s,x)| &= |K(t',s)\bar{b}(x) - K(t,s)\bar{b}(x)| \\ &= |K(t',s) - K(t,s)| \cdot |\bar{b}(x)|. \end{aligned}$$

Note that $\bar{b}$ is a continuous function that is bounded on $x \in [-M, M]$ and constant on $\mathbb{R} \backslash [-M, M]$, hence it is bounded by some constant $B_b > 0$. We can therefore write for any $x \in \mathbb{R}$

$$|\bar{b}(x)| \leq B_b \leq B_b \cdot (1 + |x|).$$

Then, let us define

$$F_1(t',t,s) := B_b \cdot |K(t',s) - K(t,s)|.$$

This is obviously a positive function on $[0,1] \times [0,1] \times [0,1]$. Taking W.L.O.G. $t < t'$, we get the necessary estimate for the integral, i.e.

$$\begin{aligned} \int_0^t F_1(t',t,s) \, ds &= \int_0^t B_b \cdot |K(t',s) - K(t,s)| \, ds \\ &= B_b \cdot \int_0^t |K(t',s) - K(t,s)| \, ds \\ &\leq B_b \cdot \sqrt{\int_0^t |K(t',s) - K(t,s)|^2 \, ds} \\ &\leq C \sqrt{M_k(|t'-t|)} \\ &\leq C |t'-t|^{\frac{r}{2}}, \end{aligned}$$

where we used the kernel assumption (4.2.13) for the last inequality. Hence, we get the necessary estimate, i.e.

$$|b(t',s,x) - b(t,s,x)| \leq F_1(t',t,s) \cdot \big(1 + |x|\big),$$

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where F_1 satisfies the requirements. Clearly, we have

$$|b(t,s,x)| = K(t,s) \cdot |\bar{b}(x)| \leq K(t,s) B_b \big(1+|x|\big),$$

hence

$$\widetilde{K}_1(t,s) := K(t,s) B_b$$

and integrability directly follows from the kernel assumptions as in the proof of Lemma 4.2.3.

Coefficient σ. Here, we have a similar situation as for b. The coefficient factor $\bar{\sigma}$ is bounded by B_σ, so can write

$$\|\sigma(t,s,x)\|^2 \leq K(t,s)^2 B_\sigma^2 (1+|x|^2),$$

hence we set

$$\widetilde{K}_2(t,s) := K(t,s)^2 B_\sigma^2.$$

Integrability directly follows again from the kernel assumptions as in the proof of Lemma 4.2.3. Now, defining

$$F_2(t',t,s) := B_\sigma^2 \cdot |K(t',s) - K(t,s)|^2$$

and using (4.2.13) again, we get the two estimates

$$\int_0^{t\wedge t'} F_1(t',t,s)\, ds \leq C \cdot |t'-t|^{\frac{r}{2}},$$
$$\int_0^{t\wedge t'} F_2(t',t,s)\, ds \leq C \cdot |t'-t|^{r}.$$

So, we have

$$\begin{aligned}\int_0^{t\wedge t'} \left(F_1(t',t,s) + F_2(t',t,s)\right) ds &\leq C_1 \cdot |t'-t|^{\frac{r}{2}} + C_2 \cdot |t'-t|^{r} \\ &\leq C|t'-t|^{\frac{r}{2}\wedge r} \\ &\leq C|t'-t|^{\frac{r}{2}}.\end{aligned}$$

Setting $\gamma = \frac{r}{2}$ we have shown the requirement. □

Clearly, if the kernel satisfies Assumption 4.2.1, it is a continuous function in the second argument. Hence, we can also formulate the following.

Lemma 4.2.5. *Under Assumption 4.2.1 the process V solving (4.1.8) satisfies (H3).*

Proof. The statement directly follows from (4.2.10). □

Now, we have everything we need to formulate an existence result.

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Theorem 4.2.6. *Under Assumption 4.2.1, there exists a pathwise unique continuous adapted solution $V_t(v_0)$ on a deterministic interval $[0,T]$ to (4.1.8). Furthermore, we have the moment estimate*

$$E\left[|V_t|^p\right] \leq C_1 e^{C_2 t}, \tag{4.2.15}$$

for all $t \in [0,T]$ and some constants $C_1, C_2 > 0$ that are only dependent on T and p.

Proof. First, note that due Assumption 4.2.1 together with Lemma 4.2.3, 4.2.4 and 4.2.5 we can use conditions (H1)–(H3) for the coefficient functions of (4.1.8). The existence result then directly follows from Theorem 3.1 in [Zha10]. However, for those who are interested in the detailed proof, the ideas and estimates of [Zha10] have been used to show the existence result in detail. Note, that the proof is a little different from [Zha10], because it is simplified to our state space, i.e. to $\mathbb{R}$. Also the Volterra type Gronwall inequality in [Zha10] has been replaced by Hölder type estimates. For ease of notation we will write $a \preceq_\propto b$ for $a \leq C \cdot b$ and some constant $C > 0$ that may vary from line to line.

Moment estimate. Rebuilding the proof of [Zha10] for our situation, we use Picard's iteration. Let $V_1(t) := v_0$ and define recursively for $n \in \mathbb{N}$,

$$V_{n+1}(t) = v_0 + \int_0^t b(t,s,V_n(s))\,ds + \int_0^t \sigma(t,s,V_n(s))\,dW_s. \tag{4.2.16}$$

To start, let $p \geq 2\alpha^*$ where $\alpha^* = \frac{\alpha}{\alpha-1}$. Note, that α and α^* are conjugated Hölder exponents and as a consequence, $p \geq 2$. The $\alpha > 1$ used here comes from the regularity conditions (H1)–(H2). Now, by the linear growth condition (H2), BDG's inequality (see (1.1.8)) and Hölder's

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inequality we have

$$\begin{aligned}
E\left[|V_{n+1}(t)|^p\right] &\overset{(B.0.3)}{\preceq_\propto} v_0^p + E\left[\left(\int_0^t |b(t,,s,V_n(s))|\,ds\right)^p\right] + E\left[\left|\int_0^t \sigma(t,s,V_n(s))\,dW_s\right|^p\right] \\
&\overset{\text{BDG}}{\preceq_\propto} v_0^p + E\left[\left(\int_0^t |b(t,,s,V_n(s))|\,ds\right)^p\right] + E\left[\left(\int_0^t |\sigma(t,s,V_n(s))|^2\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{(H2)}{\preceq_\propto} v_0^p + E\left[\left(\int_0^t \widetilde{K}_1(t,s)\big(1+|V_n(s)|\big)\,ds\right)^p\right] + E\left[\left(\int_0^t \widetilde{K}_2(t,s)\big(1+|V_n(s)|^2\big)\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{Hölder}}{\preceq_\propto} v_0^p + E\left[\left(\int_0^t \widetilde{K}_1(t,s)^\alpha\right)^{\frac{p}{\alpha}}\left(\int_0^t \big(1+|V_n(s)|\big)^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\right] \\
&\qquad + E\left[\left(\int_0^t \widetilde{K}_2(t,s)^\alpha\,ds\right)^{\frac{p}{2\alpha}}\left(\int_0^t \big(1+|V_n(s)|^2\big)^{\alpha^*}\,ds\right)^{\frac{p}{2\alpha^*}}\right] \\
&\overset{(H2)}{\preceq_\propto} v_0^p + C_{T,p}^1\cdot E\left[\left(\int_0^t \big(1+|V_n(s)|\big)^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\right] \\
&\qquad + C_{T,p}^2\cdot E\left[\left(\int_0^t \big(1+|V_n(s)|^2\big)^{\alpha^*}\,ds\right)^{\frac{p}{2\alpha^*}}\right] \\
&\overset{\text{Jensen}}{\preceq_\propto} v_0^p + C_{T,p}^1\cdot E\left[\int_0^t \big(1+|V_n(s)|\big)^p\,ds\right] + C_{T,p}^2\cdot E\left[\int_0^t \big(1+|V_n(s)|^2\big)^{\frac{p}{2}}\,ds\right] \\
&\overset{(B.0.3)}{\preceq_\propto} v_0^p + C_{T,p}^1\cdot E\left[\int_0^t \big(1+|V_n(s)|^p\big)\,ds\right] + C_{T,p}^2\cdot E\left[\int_0^t \big(1+|V_n(s)|^p\big)\,ds\right] \\
&\overset{\text{Fubini}}{\preceq_\propto} v_0^p + C_{T,p}^1\cdot \int_0^t \big(1+E\left[|V_n(s)|^p\right]\big)\,ds + C_{T,p}^2\cdot \int_0^t \big(1+E\left[|V_n(s)|^p\right]\big)\,ds \\
&= v_0^p + \big(C_{T,p}^1+C_{T,p}^2\big)\cdot\int_0^t \big(1+E\left[|V_n(s)|^p\right]\big)\,ds \\
&\preceq_\propto v_0^p + \big(C_{T,p}^1+C_{T,p}^2\big) + \big(C_{T,p}^1+C_{T,p}^2\big)\cdot\int_0^t E\left[|V_n(s)|^p\right]\,ds \\
&\preceq_\propto v_0^p + C_{T,p} + C_{T,p}\cdot\int_0^t E\left[|V_n(s)|^p\right]\,ds.
\end{aligned}$$

Now, set

$$f_m(t) := \sup_{n=1,\dots,m} E\left[|V_n(t)|^p\right]. \tag{4.2.17}$$

Then, we have

$$f_m(t) \le (v_0^p + C_{T,p}) + C_{T,p}\cdot\int_0^t f_m(s)\,ds,$$

where the constants $C_{T,p}$ are independent of m. Now, using Gronwall's inequality given in

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Theorem 1.1.9, we can estimate

$$\begin{aligned} f_m(t) &\leq v_0^p + C_{T,p} + \int_0^t (v_0^p + C_{T,p}) \cdot C_{T,p} \cdot e^{\int_s^t C_{T,p}\, d\sigma}\, ds \\ &= v_0^p + C_{T,p} + (v_0^p + C_{T,p}) \cdot C_{T,p} \cdot \int_0^t e^{(t-s)C_{T,p}}\, ds \\ &= v_0^p + C_{T,p} + (v_0^p + C_{T,p}) \cdot C_{T,p} \cdot \frac{e^{t\cdot C_{T,p}} - 1}{C_{T,p}} \\ &= v_0^p + C_{T,p} + (v_0^p + C_{T,p})(e^{tC_{T,p}} - 1) \\ &= e^{tC_{T,p}}(v_0^p + C_{T,p}). \end{aligned}$$

The right-hand side is independent of m, hence we can write

$$\sup_{n\in\mathbb{N}} E\left[|V_n(t)|^p\right] \leq e^{tC_{T,p}}(v_0^p + C_{T,p}), \tag{4.2.18}$$

which gives the moment estimate (4.2.15).

Existence. For the existence result we define

$$Z_{n,m}(t) := V_n(t) - V_m(t)$$

and

$$f(t) := \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^2\right]. \tag{4.2.19}$$

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We can write

$$
\begin{aligned}
E\left[|Z_{n+1,m+1}(t)|^p\right] &= E\left[|V_{n+1}(t) - V_{m+1}(t)|^p\right] \\
&= E\left[\left|\int_0^t \left(b(t,s,V_n(s)) - b(t,s,V_m(s))\right) ds + \int_0^t \left(\sigma(t,s,V_n(s)) - \sigma(t,s,V_m(s))\right) dW_s\right|^p\right] \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} E\left[\left(\int_0^t |b(t,s,V_n(s)) - b(t,s,V_m(s))|\, ds\right)^p\right] \\
&\qquad + E\left[\left|\int_0^t \left(\sigma(t,s,V_n(s)) - \sigma(t,s,V_m(s))\right) dW_s\right|^p\right] \\
&\overset{\text{BDG}}{\preceq_\propto} E\left[\left(\int_0^t |b(t,s,V_n(s)) - b(t,s,V_m(s))|\, ds\right)^p\right] \\
&\qquad + E\left[\left(\int_0^t |\sigma(t,s,V_n(s)) - \sigma(t,s,V_m(s))|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{(H1)}}{\preceq_\propto} E\left[\left(\int_0^t K_1(t,s)|V_n(s) - V_m(s)|\, ds\right)^p\right] + E\left[\left(\int_0^t K_2(t,s)|V_n(s) - V_m(s)|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_0^t K_1(t,s)^\alpha\, ds\right)^{\frac{p}{\alpha}} E\left[\left(\int_0^t |V_n(s) - V_m(s)|^{\alpha^*}\, ds\right)^{\frac{p}{\alpha^*}}\right] \\
&\qquad + \left(\int_0^t K_2(t,s)^\alpha\, ds\right)^{\frac{p}{2\alpha}} E\left[\left(\int_0^t |V_n(s) - V_m(s)|^{2\alpha^*}\, ds\right)^{\frac{p}{2\alpha^*}}\right] \\
&\overset{\text{Jensen, (H1)}}{\preceq_\propto} C^1_{T,p} E\left[\int_0^t |V_n(s) - V_m(s)|^p\, ds\right] + C^2_{T,p} E\left[\int_0^t |V_n(s) - V_m(s)|^p\, ds\right] \\
&\overset{\text{Fubini}}{\preceq_\propto} (C^1_{T,p} + C^2_{T,p}) \int_0^t E\left[|V_n(s) - V_m(s)|^p\right] ds \\
&= C_{T,p} \int_0^t E\left[|Z_{m,n}(s)|^p\right] ds.
\end{aligned}
$$

Now, for

$$
g(t) := \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^p\right]
$$

we can write

$$
g(t) \le C_{T,p} \int_0^t g(s)\, ds,
$$

which leads to

$$
\limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^p\right] = 0.
$$

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Note, that $p \geq 2$, hence we can write

$$
\begin{aligned}
0 &= \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^p\right] = \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^{2\frac{p}{2}}\right] \\
&\geq \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^2\right]^{\frac{p}{2}} = \left(\limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^2\right]\right)^{\frac{p}{2}} \\
&\geq 0.
\end{aligned}
$$

Having done this, we finally arive at

$$
f(t) = \limsup_{n,m\to\infty} E\left[|Z_{n,m}(t)|^2\right] = 0. \tag{4.2.20}
$$

The convergence of the Cauchy sequence tells us that there exist an $(\mathcal{F}_t)$-adapted process $V(t)$ such that for almost all $t \in [0,T]$,

$$
\lim_{n\to\infty} E\left[|V_n(t) - V(t)|^2\right] = 0. \tag{4.2.21}
$$

Taking limits in the Picard iteration, we get the stochastic Volterra equation for the limiting process.

Uniqueness. Assume that we have two solutions U and V. Then, the difference can be estimated by

$$
\begin{aligned}
E\left[|U_t - V_t|^p\right] &= E\left[\left|\int_0^t (b(t,s,U_s) - b(t,s,V_s))\,ds + \int_0^t (\sigma(t,s,U_s) - \sigma(t,s,V_s))\,dW_s\right|^p\right] \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} E\left[\left(\int_0^t |b(t,s,U_s) - b(t,s,V_s)|\,ds\right)^p\right] + E\left[\left|\int_0^t (\sigma(t,s,U_s) - \sigma(t,s,V_s))\,dW_s\right|^p\right] \\
&\overset{\text{BDG}}{\preceq_\propto} E\left[\left(\int_0^t |b(t,s,U_s) - b(t,s,V_s)|\,ds\right)^p\right] + E\left[\left(\int_0^t |\sigma(t,s,U_s) - \sigma(t,s,V_s)|^2\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{(H1)}}{\preceq_\propto} E\left[\left(\int_0^t K_1(t,s)|U_s - V_s|\,ds\right)^p\right] + E\left[\left(\int_0^t K_2(t,s)|U_s - V_s|^2\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_0^t K_1(t,s)^\alpha\,ds\right)^{\frac{p}{\alpha}} E\left[\left(\int_0^t |U_s - V_s|^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\right] \\
&\quad + \left(\int_0^t K_2(t,s)^\alpha\,ds\right)^{\frac{p}{2\alpha}} E\left[\left(\int_0^t |U_s - V_s|^{2\alpha^*}\,ds\right)^{\frac{p}{2\alpha^*}}\right] \\
&\overset{\text{Jensen, (H1)}}{\preceq_\propto} (C^1_{T,p} + C^1_{T,p})E\left[\int_0^t |U_s - V_s|^p\,ds\right] \\
&\overset{\text{Fubini}}{\preceq_\propto} C_{T,p}\int_0^t E\left[|U_s - V_s|^p\right]\,ds.
\end{aligned}
$$

From Gronwall's inequality we get that

$$
E\left[|U_t - V_t|^p\right] = 0 \tag{4.2.22}
$$

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for all $t \in [0,1]$. Hence, we have a unique solution for the instantaneous variance V. We even have a unique pathwise solution, because no part of this proof takes advantage of the special form of the filtration for a strong solution, compare to Remark 3.3 in [KS91].

Continuity. The Hölder continuity of the paths can easily be seen in the proof Theorem 3.3 in [Zha10]. For the reader's convenience we show the slightly modified computations for our setup. Using the moment estimate (4.2.18) which was proven above, we get that

$$\sup_{t\in[0,T]} E\left[|V_t|^p\right] < \infty. \tag{4.2.23}$$

Now, set

$$J(t) := \int_0^t \sigma(t,s,V_s)\,dW_s \tag{4.2.24}$$

and write for $0 \le t < t' \le T$,

$$J(t') - J(t) = \int_0^t \left(\sigma(t',s,V_s) - \sigma(t,s,V_s)\right)\,dW_s + \int_t^{t'} \sigma(t',s,V_s)\,dW_s =: J_1(t',t) + J_2(t',t).$$

Here, we want to show that the requirements of the Kolmogorov continuity criterion (see Theorem 1.1.6) are met. Therefore, we *cannot* just use the bound for the kernel integral, but have to subtly apply Hölder twice. For doing this take some $\gamma,\beta > 1$ such that $1 < \gamma\beta < \alpha$. Again, the star superscript indicates the conjugated Hölder exponents and p has to be large enough to apply Jensen, i.e. $p \ge 2\beta^*$ with $\beta^* = \frac{\beta}{\beta-1}$.

We can write

$$\begin{aligned}
E\left[|J_2(t',t)|^p\right] &= E\left[\left|\int_t^{t'} \sigma(t',s,V_s)\,dW_s\right|^p\right] \\
&\overset{\text{BDG}}{\preceq_\propto} E\left[\left(\int_t^{t'} \sigma(t',s,V_s)^2\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{(H2)}}{\preceq_\propto} E\left[\left(\int_t^{t'} \widetilde{K}_2(t',s)\left(1+|V_s|^2\right)\,ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_t^{t'} \widetilde{K}_2(t',s)^\beta\,ds\right)^{\frac{p}{2\beta}} E\left[\left(\int_t^{t'} \left(1+|V_s|^2\right)^{\beta^*}\,ds\right)^{\frac{p}{2\beta^*}}\right] \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} \left(\int_t^{t'} \widetilde{K}_2(t',s)^\beta\,ds\right)^{\frac{p}{2\beta}} E\left[\left(\int_t^{t'} \left(1+|V_s|^{2\beta^*}\right)\,ds\right)^{\frac{p}{2\beta^*}}\right].
\end{aligned}$$

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Now, for the second term we get an estimate of order 1, i.e.

$$
\begin{aligned}
E\left[\left(\int_t^{t'}\left(1+|V_s|^{2\beta^*}\right)ds\right)^{\frac{p}{2\beta^*}}\right] &\overset{\text{Jensen, (B.0.3)}}{\preceq_\propto} E\left[\int_t^{t'}\left(1+|V_s|^p\right)ds\right] \\
&\overset{\text{Fubini}}{\preceq_\propto} \int_t^{t'}\left(1+E[|V_s|^p]\right)ds \\
&\overset{(4.2.23)}{\preceq_\propto} C_{T,p}\int_t^{t'} 1\,ds \\
&= C_{T,p}|t'-t|.
\end{aligned}
$$

For the kernel part we can use Hölder on the kernel and the indicator function, i.e.

$$
\begin{aligned}
\left(\int_t^{t'}\widetilde{K}_2(t',s)^\beta\,ds\right)^{\frac{p}{2\beta}} &\preceq_\propto \left(\int_t^{t'}\widetilde{K}_2(t',s)^{\beta\gamma}\,ds\right)^{\frac{p}{2\beta\gamma}}\left(\int_t^{t'}1\,ds\right)^{\frac{p}{2\beta\gamma^*}} \\
&\overset{(H2)}{\preceq_\propto} C_{T,p}|t'-t|^{\frac{p}{2\beta\gamma^*}}.
\end{aligned}
$$

Note, that $\beta\gamma<\alpha$ ensures integrability, i.e. using Jensen we get

$$
\int_t^{t'}\widetilde{K}_2(t',s)^{\beta\gamma}\,ds=\int_t^{t'}\widetilde{K}_2(t',s)^{\alpha\frac{\beta\gamma}{\alpha}}\,ds\le\left(\int_t^{t'}\widetilde{K}(t',s)^\alpha\right)^{\frac{\beta\gamma}{\alpha}}\le C.
$$

Putting together the computations above, we arrive at

$$
E\left[|J_2(t',t)|^p\right]\preceq_\propto |t'-t|^{1+\frac{p}{2\beta\gamma^*}}. \tag{4.2.25}
$$

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For J_1 we have

$$
\begin{aligned}
E\left[|J_1(t',t)|^p\right] &= E\left[\left|\int_0^t \left(\sigma(t',s,V_s)-\sigma(t,s,V_s)\right)\, dW_s\right|^p\right] \\
&\overset{\text{BDG}}{\preceq_\propto} E\left[\left(\int_0^t |\sigma(t',s,V_s)-\sigma(t,s,V_s)|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{(H2)}}{\preceq_\propto} E\left[\left(\int_0^t F_2(t',t,s)\left(1+|V_s|^2\right) ds\right)^{\frac{p}{2}}\right] \\
&= E\left[\left(\int_0^t \left(F_2(t',t,s)+F_2(t',t,s)\cdot|V_s|^2\right) ds\right)^{\frac{p}{2}}\right] \\
&= E\left[\left(\int_0^t F_2(t',t,s)\, ds + \int_0^t F_2(t',t,s)\cdot|V_s|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} E\left[\left(\int_0^t F_2(t',t,s)\, ds\right)^{\frac{p}{2}} + \left(\int_0^t F_2(t',t,s)\cdot|V_s|^2\, ds\right)^{\frac{p}{2}}\right] \\
&= \left(\int_0^t F_2(t',t,s)\, ds\right)^{\frac{p}{2}} + E\left[\left(\int_0^t F_2(t',t,s)\cdot|V_s|^2\, ds\right)^{\frac{p}{2}}\right] \\
&=: I_1
\end{aligned}
$$

For the second term we need the Minkovski inequality in the version given in Theorem 1.1.10. In (1.1.14) we take the power q on both sides and get

$$
\int_{S_2}\left|\int_{S_1} F(x,y)\mu_1(dx)\right|^q \mu_2(dy) \leq \left[\int_{S_1}\left(\int_{S_2}|F(x,y)|^q\mu_2(dy)\right)^{\frac{1}{q}}\mu_1(dx)\right]^q. \tag{4.2.26}
$$

Then, we apply this inequality using

$$
\begin{aligned}
q &= \frac{p}{2}, \\
\mu_1(ds) &= ds, \\
\mu_2(d\omega) &= P(d\omega), \\
S_1 &= (0,t), \\
S_2 &= \Omega, \\
F(s,\omega) &= F_2(t',t,s)|V_s(\omega)|^2.
\end{aligned}
$$

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Now, for the second term of I_1, we can write

$$E\left[\left(\int_0^t F_2(t',t,s)\cdot |V_s|^2\,ds\right)^{\frac{p}{2}}\right] = \int_\Omega \left|\int_0^t F_2(t',t,s)\cdot |V_s(\omega)|^2\,ds\right|^{\frac{p}{2}} P(d\omega)$$
$$\overset{\text{Minkovski}}{\preceq_\propto} \left(\int_0^t \left(\int_\Omega F_2(t',t,s)^{\frac{p}{2}}\cdot |V_s(\omega)|^{2\frac{p}{2}}\,P(d\omega)\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}$$
$$= \left(\int_0^t E\left[F_2(t',t,s)^{\frac{p}{2}}\cdot |V_s|^p\right]^{\frac{2}{p}} ds\right)^{\frac{p}{2}}$$
$$= \left(\int_0^t \left(F_2(t',t,s)^{\frac{p}{2}} E\left[|V_s|^p\right]\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}$$
$$= \left(\int_0^t F_2(t',t,s)\cdot E\left[|V_s|^p\right]^{\frac{2}{p}}\,ds\right)^{\frac{p}{2}}.$$

Using this we can continue

$$I_1 \preceq_\propto \left(\int_0^t F_2(t',t,s)\,ds\right)^{\frac{p}{2}} + \left(\int_0^t F_2(t',t,s)E[|V_s|^p]^{\frac{2}{p}}\,ds\right)^{\frac{p}{2}}$$
$$\preceq_\propto \left(\int_0^t F_2(t',t,s)\,ds + \int_0^t F_2(t',t,s)E[|V_s|^p]^{\frac{2}{p}}\,ds\right)^{\frac{p}{2}}$$
$$= \left(\int_0^t F_2(t',t,s)(1+E[|V_s|^p]^{\frac{2}{p}})\,ds\right)^{\frac{p}{2}}$$
$$\overset{(4.2.23)}{\preceq_\propto} \left(\int_0^t F_2(t',t,s)\,ds\right)^{\frac{p}{2}}$$
$$\overset{(\text{H2})}{\preceq_\propto} |t'-t|^{\frac{\tilde{\gamma}p}{2}},$$

with $\tilde{\gamma} > 0$ coming from (H2). For any $\tilde{\gamma} > 0$ we can chose p large enough such that the exponent is greater one. Hence, a requirement for the exponent is $p \geq \frac{2}{\tilde{\gamma}}$.

Summing up, the exponent for the Kolmogorov continuity criterion Theorem 1.1.6 is

$$\delta := \left(1 + \frac{p}{2\beta\gamma^*}\right) \wedge \frac{\tilde{\gamma}p}{2}, \tag{4.2.27}$$

where we used $p \geq \frac{2}{\tilde{\gamma}} \vee 2\beta^*$. We have for all $0 \leq t < t' \leq T$ that

$$E\left[|J(t') - J(t)|^p\right] \preceq_\propto |t'-t|^\delta. \tag{4.2.28}$$

Similarly, we can prove for all $0 \leq t < t' \leq T$ that

$$E\left[\left|\int_0^{t'} b(t',s,V_s)\,ds - \int_0^t b(t,s,V_s)\,ds\right|^p\right] \preceq_\propto |t-t'|^{\tilde{\delta}}. \tag{4.2.29}$$

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Using

$$\tilde{J}(t) := \int_0^t b(t, s, V_s)\, ds,$$

we can write

$$\tilde{J}(t') - \tilde{J}(t) = \int_0^t \big(b(t', s, V_s) - b(t, s, V_s)\big)\, ds + \int_t^{t'} b(t', s, V_s)\, ds =: \tilde{J}_1(t', t) + \tilde{J}_2(t', t).$$

As above, we have have for $p \geq \beta^*$

$$\begin{aligned}
E[|\tilde{J}_2(t', t)|^p] &= E\left[\left|\int_t^{t'} b(t', s, V_s)\, ds\right|^p\right] \\
&\leq E\left[\left(\int_t^{t'} \left|b(t', s, V_s)\right|\, ds\right)^p\right] \\
&\overset{\text{(H2)}}{\preceq_\propto} E\left[\left(\int_t^{t'} \widetilde{K}_1(t', s)\big(1 + |V_s|\big)\, ds\right)^p\right] \\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_t^{t'} \widetilde{K}_1(t', s)^\beta\, ds\right)^{\frac{p}{\beta}} E\left[\left(\int_t^{t'} (1 + |V_s|)^{\beta^*}\, ds\right)^{\frac{p}{\beta^*}}\right] \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} \left(\int_t^{t'} \widetilde{K}_1(t', s)^\beta\, ds\right)^{\frac{p}{\beta}} E\left[\left(\int_t^{t'} (1 + |V_s|^{\beta^*})\, ds\right)^{\frac{p}{\beta^*}}\right]
\end{aligned}$$

Again, using Hölder, we get

$$\begin{aligned}
\left(\int_t^{t'} \widetilde{K}_1(t', s)^\beta\, ds\right)^{\frac{p}{\beta}} &\preceq_\propto \left(\int_t^{t'} \widetilde{K}_1(t', s)^{\beta\gamma}\, ds\right)^{\frac{p}{\beta\gamma}} \left(\int_t^{t'} 1\, ds\right)^{\frac{p}{\beta\gamma^*}} \\
&\overset{\text{(H2)}}{\preceq_\propto} C_{T,p}|t' - t|^{\frac{p}{\beta\gamma^*}}.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
E\left[\left(\int_t^{t'} (1 + |V_s|^{\beta^*})\, ds\right)^{\frac{p}{\beta^*}}\right] &\overset{\text{Jensen, (B.0.3)}}{\preceq_\propto} E\left[\int_t^{t'} (1 + |V_s|^p)\, ds\right] \\
&\overset{\text{Fubini}}{\preceq_\propto} \int_t^{t'} \big(1 + E[|V_s|^p]\big)\, ds \\
&\overset{\text{(4.2.23)}}{\preceq_\propto} C_{T,p} \int_t^{t'} 1\, ds \\
&= C_{T,p}|t' - t|.
\end{aligned}$$

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For J_1 we can write

$$\begin{aligned}
E\left[|J_1(t',t)|^p\right] &= E\left[\left|\int_0^t \big(b(t',s,V_s) - b(t,s,V_s)\big)\, ds\right|^p\right] \\
&\preceq_\propto E\left[\left(\int_0^t |b(t',s,V_s) - b(t,s,V_s)|\, ds\right)^p\right] \\
&\overset{\text{(H2)}}{\preceq_\propto} E\left[\left(\int_0^t F_1(t',t,s)\big(1+|V_s|\big)\, ds\right)^p\right]
\end{aligned}$$

Now, using Minkovski's inequality, Theorem 1.1.10, with $F(s,\omega) = F_1(t',t,s)(1+|V_s(\omega)|)$, we can write

$$\begin{aligned}
E\left[\left(\int_0^t F_1(t',t,s)\big(1+|V_s|\big)\, ds\right)^p\right] &= \int_\Omega \left|\int_0^t F_1(t',t,s)\cdot\big(1+|V_s(\omega)|\big)\, ds\right|^p P(d\omega) \\
&\overset{\text{Minkovski}}{\preceq_\propto} \left(\int_0^t \left(\int_\Omega F_1(t',t,s)^p \cdot \big(1+|V_s(\omega)|\big)^p P(d\omega)\right)^{\frac{1}{p}} ds\right)^p \\
&= \left(\int_0^t E\left[F_1(t',t,s)^p \cdot \big(1+|V_s|\big)^p\right]^{\frac{1}{p}} ds\right)^p \\
&= \left(\int_0^t F_1(t',t,s)\cdot E\left[\big(1+|V_s|\big)^p\right]^{\frac{1}{p}} ds\right)^p \\
&\overset{\text{(B.0.3)}}{\preceq_\propto} \left(\int_0^t F_1(t',t,s)\cdot E\left[\big(1+|V_s|^p\big)\right]^{\frac{1}{p}} ds\right)^p \\
&= \left(\int_0^t F_1(t',t,s)\cdot \big(1+E\left[|V_s|^p\right]\big)^{\frac{1}{p}} ds\right)^p \\
&\overset{\text{(4.2.23)}}{\preceq_\propto} \left(\int_0^t F_1(t',t,s)\, ds\right)^p \\
&\overset{\text{(H2)}}{\preceq_\propto} |t'-t|^{\tilde{\gamma}p}.
\end{aligned}$$

As above, we need the exponent to be greater than 1, hence we require $p \geq \frac{1}{\tilde{\gamma}}$. Summing up, we arrive at

$$E[|\tilde{J}(t') - \tilde{J}(t)|^p] \preceq_\propto |t'-t|^{\tilde{\delta}}, \tag{4.2.30}$$

where

$$\tilde{\delta} := \left(1 + \frac{p}{\beta\gamma^*}\right) \wedge \tilde{\gamma}p \tag{4.2.31}$$

and $p \geq \beta^* \vee \frac{1}{\tilde{\gamma}}$.

Now, taking $\delta \wedge \tilde{\delta}$ as exponent and p large enough continuity follows from the Kolmogorov continuity criterion, see Theorem 1.1.6.

QED. Having shown the moment estimate, existence uniqueness and continuity concludes the proof. $\square$

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For the log-price, recall that it is given by

$$X_t = X_0 + \int_0^t \sqrt{|V_s^M|}\,(\bar{\rho}\,dW_s + \rho\,dB_s) - \frac{1}{2}\int_0^t |V_s^M|\,ds.$$

This equation is *not* a Volterra equation, but X is given explicitly by an Itô integral. Hence, it also admits a pathwise unique solution X taking advantage of the fact that V is pathwise unique.

Summing up, we get a pathwise unique solution for the two-dimensional formulation of our model as a stochastic Volterra system.

Corollary 4.2.7. Under Assumption 4.2.1, there exists a pathwise unique continuous adapted solution $Y_t(y_0)$ in a deterministic interval $[0, T]$ to the stochastic Volterra system (4.1.14).

4.3. Large deviations

In this section we establish large deviations results for the model (4.1.7)–(4.1.8). The main reference for this section will be the results of Jacquier and Pannier [JP20]. In the following, we will argue that the two-dimensional formulation (4.1.14) of (4.1.7)–(4.1.8) satisfies the requirements needed to apply their results. In order to do this we rewrite the system (4.1.14) as

$$\begin{aligned}
Y_t &= \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \begin{pmatrix} -\frac{1}{2}|V_s| \\ b(t,s,V_s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} \bar{\rho}\sqrt{|V_s|} & \rho\sqrt{|V_s|} \\ \sigma(t,s,V_s) & 0 \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix} \\
&= \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \begin{pmatrix} -\frac{1}{2}|V_s| \\ K(t,s)\bar{b}(V_s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} \bar{\rho}\sqrt{|V_s|} & \rho\sqrt{|V_s|} \\ K(t,s)\bar{\sigma}(V_s) & 0 \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix} \\
&= \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1\cdot\left(-\frac{1}{2}|V_s|\right) \\ K(t,s)\cdot\bar{b}(V_s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} 1\cdot(\bar{\rho}\sqrt{|V_s|}) & 1\cdot(\rho\sqrt{|V_s|}) \\ K(t,s)\cdot\bar{\sigma}(V_s) & 0\cdot K(t,s) \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix} \\
&= \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}|V_s| \\ \bar{b}(V_s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix} \begin{pmatrix} \bar{\rho}\sqrt{|V_s|} & \rho\sqrt{|V_s|} \\ \bar{\sigma}(V_s) & 0 \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix}.
\end{aligned}$$

Hence, we can set

$$K_{\mathrm{JP}}(t,s) := \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix}, \tag{4.3.1}$$

$$b_{\mathrm{JP}}(s,x) := \begin{pmatrix} -\frac{1}{2}|x| \\ \bar{b}(x) \end{pmatrix}, \tag{4.3.2}$$

$$\sigma_{\mathrm{JP}}(s,x) := \begin{pmatrix} \bar{\rho}\sqrt{|x|} & \rho\sqrt{|x|} \\ \bar{\sigma}(x) & 0 \end{pmatrix}, \tag{4.3.3}$$

which corresponds to the kernel and coefficient functions of (1.1) in [JP20]. Note, that existence and uniqueness is clear from the results in Section 4.2. In [JP20], the authors perform all the proofs for convolutional kernels. In Section 3.5 [JP20] they discuss the extension of their results to non-convolution kernels. However, we will consider the convolutional case to circumvent possible problems with the additional assumptions for the non-convolutional case.

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Assumption 4.3.1. Let K in the model equation (4.1.14) be a convolutional Kernel, i.e. it can be written as function $K : (0, t) \to (0, \infty)$ with

$$K(t, s) := K(t - s). \tag{4.3.4}$$

Remark 4.3.2. Note that we can also use the large deviations result from [Zha10] for the process V. For the log-price, we would have to apply the extended contraction principle, because the two-dimensional formulation (4.1.14) of our problem does *not* satisfy the requirements in [Zha10] in both components. The problem here would be that that the coefficient functions of the log-price do not satisfy the Lipschitz condition that is required in [Zha10], because of the square root in the diffusion part.

4.3.1. Sample path large deviations

For establishing pathwise small-noise large deviations, we consider the scaled equation

$$Y_t^\varepsilon = \begin{pmatrix} X_t^\varepsilon \\ V_t^\varepsilon \end{pmatrix} = \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}|V_s^\varepsilon| \\ \bar{b}(V_s^\varepsilon) \end{pmatrix} ds + \vartheta_\varepsilon \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix} \begin{pmatrix} \bar{\rho}\sqrt{|V_s^\varepsilon|} & \rho\sqrt{|V_s^\varepsilon|} \\ \bar{\sigma}(V_s^\varepsilon) & 0 \end{pmatrix} \begin{pmatrix} dW_s \\ dB_s \end{pmatrix}, \tag{4.3.5}$$

where $\vartheta_\varepsilon \searrow 0$ for $\varepsilon \searrow 0$. We can e.g. take $\vartheta_\varepsilon = \sqrt{\varepsilon}$.

In the following we show that an essential regularity requirement of [JP20] is satisfied by our model.

Assumption 4.3.3 (Assumption 2.3 in [JP20])**.** Under Assumptions 4.2.1 and 4.3.1 the kernel $K_{\mathrm{JP}} : [0, T] \to \mathbb{R}^{d\times d}$ is an upper triangular matrix satisfying the following conditions: $K_{\mathrm{JP}} \in L^2([0, T] : \mathbb{R}^{d\times d})$ and there exists $\gamma \in (0, 2]$ such that, for h small enough,

$$\int_0^h |K_{\mathrm{JP}}(t)|^2\, dt + \int_0^T |K_{\mathrm{JP}}(t + h) - K_{\mathrm{JP}}(t)|^2\, dt = \mathcal{O}(h^\gamma). \tag{4.3.6}$$

Lemma 4.3.4. *Under Assumptions 4.2.1 and 4.3.1 Assumption 4.3.3 holds.*

Proof. Note that due to Assumption 4.3.1 we have

$$K_{\mathrm{JP}}(t, s) = \begin{pmatrix} 1 & 0 \\ 0 & K(t,s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & K(t-s) \end{pmatrix} = K_{\mathrm{JP}}(t - s). \tag{4.3.7}$$

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The integrability can be shown by

$$\begin{aligned}
\int_0^T |K_{\text{JP}}(t)|^2\,dt &= \int_0^T \left(1+|K(t)|^2\right)dt \\
&= T + \int_0^T |K(t)|^2\,dt \\
&= T + \int_0^T |K(t)|^{2\frac{\tilde{\alpha}}{\tilde{\alpha}}}\,dt \\
&= T + \int_0^T (|K(t)|^{\tilde{\alpha}})^{\frac{2}{\tilde{\alpha}}}\,dt \\
&\overset{\text{Jensen}}{\leq} T + \left(\int_0^t |K(t)|^{\tilde{\alpha}}\,dt\right)^{\frac{2}{\tilde{\alpha}}} \\
&\overset{(4.2.11)}{\leq} C.
\end{aligned}$$

Next, we write

$$\begin{aligned}
\int_0^h |K_{\text{JP}}(t)|^2\,dt &= \int_0^h \left(1+|K(t)|^2\right)dt \\
&= h + \int_0^h |K(t)|^2\,dt.
\end{aligned}$$

For the second term choose $\beta := \frac{\tilde{a}}{2} > 1$ and $\beta^* = \frac{\beta}{\beta-1}$. Then, using Hölder inequality on the indicator function we can write

$$\begin{aligned}
\int_0^h |K(t)|^2\,dt &\leq \left(\int_0^h |K(t)|^{2\beta}\,dt\right)^{\frac{1}{\beta}} \left(\int_0^h 1\,dt\right)^{\frac{1}{\beta^*}} \\
&= \left(\int_0^h |K(t)|^{\tilde{\alpha}}\,dt\right)^{\frac{1}{\beta}} h^{\frac{1}{\beta^*}} \\
&\overset{(4.2.11)}{\leq} Ch^{\frac{1}{\beta^*}},
\end{aligned}$$

where $\beta^* > 1$. Summing up, we have

$$\int_0^h |K_{\text{JP}}(t)|^2\,dt \leq h + Ch^{\frac{1}{\beta^*}} \leq Ch^{\frac{1}{\beta^*}},$$

for h small.

For the kernel differences we can write

$$\begin{aligned}
\int_0^T |K_{\text{JP}}(t+h) - K_{\text{JP}}(t)|^2\,dt &= \int_0^T |K(t+h) - K(t)|^2\,dt \\
&\leq M_K(h) \\
&\overset{(3.1.5)}{\leq} Ch^r,
\end{aligned}$$

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for some $r > 0$. Since h is small, we can put everything together to

$$\int_0^h |K_{\mathrm{JP}}(t)|^2\,dt + \int_0^T |K_{\mathrm{JP}}(t+h) - K_{\mathrm{JP}}(t)|^2\,dt \le C_1 h^{\frac{1}{\beta^*}} + C_2 h^r \le C h^\gamma, \tag{4.3.8}$$

with $\gamma := r \wedge \frac{1}{\beta^*}$. In case $\gamma > 2$ note that for small h we have

$$h^\gamma = h^{\gamma\frac{2}{2}} = h^{2\frac{\gamma}{2}} = \left(h^2\right)^{\frac{\gamma}{2}} \le h^2. \tag{4.3.9}$$

We are only considering an asymptotic result for $h \searrow 0$, hence $\gamma > 0$ is just fine. □

Remark 4.3.5. Note that the Riemann Liouville kernel satisfies this assumption too, because it satisfies Assumption 4.2.1 and is a convolutional kernel.

The authors of [JP20] formulate four conditions that a stochastic Volterra system needs to meet in order to apply their results. They use

(JP1) X_0^ε converges to $x_0 \in \mathbb{R}^d$ as ε tends to zero.

(JP2) For all $\varepsilon > 0$ small enough, the coefficients b_ε and σ_ε are measurable maps on $\mathbb{T} \times \mathbb{R}^d$ and converge pointwise to b and σ as ε goes to zero. Moreover, $b(t,\cdot)$ and $\sigma(t,\cdot)$ are continuous on $\mathbb{R}^d$, uniformly in $t \in \mathbb{T}$.

(JP3) Either **a)** or **b)** holds:

a) For all $\varepsilon > 0$ small enough, b_ε and σ_ε have linear growth unifromly in ε and in $t \in \mathbb{T}$.

b) The process X^ε admits an autonomous $\mathcal{S}_\Upsilon^\Gamma$-subsystem.

(JP4) The small-noise SVE is exact for small enough $\varepsilon > 0$.

Note that these correspond to **H1–H4** of [JP20] and we name them differently to avoid confusions with our conditions (H1)–(H3). It can easily be seen, that our model satisfies these requirements.

Lemma 4.3.6. *Under Assumption 4.2.1 the two-dimensional formulation* (4.1.14) *of our model satisfies* (JP1)–(JP4) *which correspond to* **H1–H4** *of* [JP20].

Proof. **(JP1)** Note that our model starts deterministic with (x_0, v_0).

(JP2) In our small-noise equation (4.3.5), the coefficient functions are still independent of ε. Continuity of the coefficients as defined in (4.1.11) and (4.1.12) can be seen directly. Note that these functions are independent of time, hence continuity holds uniformly in $t \in [0, T]$.

(JP3) We do not need the notion of $\mathcal{S}_\Upsilon^\Gamma$-subsystems here, because we have linear growth for all dimensions.

(JP4) Existence of pathwise unique Hölder continuous solution of the small-noise equation directly follows from Corollary 4.2.7, because the small-noise equation satisfies the needed requirements. □

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Having shown everything above, there is only one detail left to discuss before we can use the large deviations results of [JP20]. The authors of [JP20] address the problem that under weak regularity conditions there may not be a unique solution to the (limiting) control equation, i.e.

$$\varphi_t = x_0 + \int_0^t K_{\mathrm{JP}}(t,s)\left(b_{\mathrm{JP}}(s,\varphi_s) + \sigma_{\mathrm{JP}}(s,\varphi_s)v_s\right) ds, \tag{4.3.10}$$

with $v_s \in L^2[0,T]$ resp. $h(t) := \int_0^t v_s\, ds$ is in the Cameron-Martin space, see Definition 1.2.3. To solve this problem, they build up a framework where they need Assumption 3.1 in [JP20]. However, we can ignore this framework and the assumption, because we have a unique solution to the control equation, see Remark 3.9 in [JP20]. In fact, we can use Lipschitz continuity to show uniqueness of the second component and again, as in Section 4.2, we take advantage of the fact that the first component is just an integral and *not* and actual equation. Despite being redundant, for the convenience of the reader we show the computations below.

Lemma 4.3.7. *Under Assumption 4.2.1 there is a unique solution to the limiting control equation* (4.3.10)*.*

Proof.
Existence. For the second component of (4.3.10) let

$$f_{n,m}(t) := \varphi_2^n(t) - \varphi_2^m(t)$$

and

$$f(t) := \limsup_{n,m\to\infty} |\varphi_2^n(t) - \varphi_2^m(t)|, \tag{4.3.11}$$

where φ_2^n is the Picard iteration for the second component of (4.3.10) given by

$$\varphi_2^{n+1}(t) = v_0 + \int_0^t K(t,s)\cdot\left(\bar{b}(\varphi_2^n(s)) + \bar{\sigma}(\varphi_2(s)^n)v_2(s)\right) ds. \tag{4.3.12}$$

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We can write for $p \geq \tilde{\alpha}^*$ with $\tilde{\alpha}^* = \frac{\tilde{\alpha}}{\tilde{\alpha}-1}$

$$\begin{aligned}
|\varphi_2^n(t) - \varphi_2^m(t)|^p &= \left| \int_0^t K(t,s) \cdot \left(\bar{b}(\varphi_2^n(s)) + \bar{\sigma}(\varphi_2^n(s)) v_2(s) \right) \, ds - \int_0^t K(t,s) \cdot \left(\bar{b}(\varphi_2^m(s)) + \bar{\sigma}(\varphi_2^m(s)) v_2(s) \right) \, ds \right|^p \\
&= \left| \int_0^t K(t,s) \left[\left(\bar{b}(\varphi_2^n(s)) + \bar{\sigma}(\varphi_2^n(s)) v_2(s) \right) - \left(\bar{b}(\varphi_2^m(s)) + \bar{\sigma}(\varphi_2^m(s)) v_2(s) \right) \right] \, ds \right|^p \\
&= \left| \int_0^t K(t,s) \left\{ \left[\bar{b}(\varphi_2^n(s)) - \bar{b}(\varphi_2^m(s)) \right] + v_2(s) \left[\bar{\sigma}(\varphi_2^n(s)) - \bar{\sigma}(\varphi_2^m(s)) \right] \right\} \, ds \right|^p \\
&\leq \left(\int_0^t K(t,s) \left\{ |\bar{b}(\varphi_2^n(s)) - \bar{b}(\varphi_2^m(s))| + |v_2(s)| \cdot |\bar{\sigma}(\varphi_2^n(s)) - \bar{\sigma}(\varphi_2^m(s))| \right\} \, ds \right)^p \\
&\overset{\text{(H1)}}{\preceq_\propto} \left(\int_0^t K(t,s) \left\{ |\varphi_2^n(s) - \varphi_2^m(s)| + |v_2(s)| \cdot |\varphi_2^n(s) - \varphi_2^m(s)| \right\} \, ds \right)^p \\
&= \left(\int_0^t K(t,s)(1 + |v_2(s)|) |\varphi_2^n(s) - \varphi_2^m(s)| \, ds \right)^p \\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_0^t K(t,s)^{\tilde{\alpha}} \, ds \right)^{\frac{p}{\tilde{\alpha}}} \left(\int_0^t (1 + |v_2(s)|)^{\tilde{\alpha}^*} |\varphi_2^n(s) - \varphi_2^m(s)|^{\tilde{\alpha}^*} \, ds \right)^{\frac{p}{\tilde{\alpha}^*}} \\
&\overset{\text{(4.2.11)}}{\preceq_\propto} \left(\int_0^t (1 + |v_2(s)|)^{\tilde{\alpha}^*} |\varphi_2^n(s) - \varphi_2^m(s)|^{\tilde{\alpha}^*} \, ds \right)^{\frac{p}{\tilde{\alpha}^*}} \\
&\overset{\text{Jensen}}{\preceq_\propto} \int_0^t (1 + |v_s(s)|)^p \cdot |\varphi_2^n(s) - \varphi_2^m(s)|^p \, ds.
\end{aligned}$$

Using Gronwall's inequality, we get that

$$\sup_{t \in [0,T]} |f(t)| = 0. \tag{4.3.13}$$

Hence, we have a Cauchy sequence that converges to the second component of the solution of the control equation (4.3.10). The first component trivially exists, because it is given by the integral

$$\varphi_1(t) := x_0 + \int_0^t \left(-\frac{1}{2} |\varphi_2(s)| \right) \, ds + \int_0^t \sqrt{|\varphi_2(s)|} \, (\rho v_1(s) + \bar{\rho} v_2(s)) \, ds. \tag{4.3.14}$$

Uniqueness. Let φ and ψ be two solutions to (4.3.10). Then, for $p \geq 2\alpha^*$ with $\alpha^* = \frac{\alpha}{\alpha-1}$, we

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can write for the second components

$$
\begin{aligned}
|\varphi_2(t)-\psi_2(t)|^p &= \left|\int_0^t K(t,s)\left(\bar b(\varphi_2(s))-\bar b(\psi_2(s))\right)\,ds+\int_0^t K(t,s)\left(\bar\sigma(\varphi_2(s))-\bar\sigma(\psi_2(s))\right)v_2(s)\,ds\right|^p\\
&\preceq_\propto \left(\int_0^t K(t,s)\left|\bar b(\varphi_2(s))-\bar b(\psi_2(s))\right|\,ds\right)^p+\left(\int_0^t K(t,s)\left|\bar\sigma(\varphi_2(s))-\bar\sigma(\psi_2(s))\right|\cdot|v_2(s)|\,ds\right)^p\\
&\overset{\text{(H1)}}{\preceq_\propto} \left(\int_0^t K(t,s)|\varphi_2(s)-\psi_2(s)|\,ds\right)^p+\left(\int_0^t K(t,s)|\varphi_2(s)-\psi_2(s)|\cdot|v_2(s)|\,ds\right)^p\\
&\overset{\text{Hölder}}{\preceq_\propto} \left(\int_0^t K(t,s)^\alpha\,ds\right)^{\frac{p}{\alpha}}\left(\int_0^t|\varphi_2(s)-\psi_2(s)|^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\\
&\qquad+\left(\int_0^t K(t,s)^\alpha\,ds\right)^{\frac{p}{\alpha}}\left(\int_0^t|\varphi_2(s)-\psi_2(s)|^{\alpha^*}|v_2(s)|^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\\
&\overset{\text{(4.2.11)}}{\preceq_\propto} C_{T,p}\left(\int_0^t|\varphi_2(s)-\psi_2(s)|^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}+C_{T,p}\left(\int_0^t|\varphi_2(s)-\psi_2(s)|^{\alpha^*}|v_2(s)|^{\alpha^*}\,ds\right)^{\frac{p}{\alpha^*}}\\
&\overset{\text{Jensen}}{\preceq_\propto} C_{T,p}\int_0^t|\varphi_2(s)-\psi_2(s)|^p\,ds+C_{T,p}\int_0^t|\varphi_2(s)-\psi_2(s)|^p|v_2(s)|^p\,ds\\
&\preceq_\propto C_{T,p}\int_0^t\left(1+|v_2(s)|^p\right)|\varphi_2(s)-\psi_2(s)|^p\,ds.
\end{aligned}
$$

Now, using Gronwall's inequality we get

$$\sup_{t\in[0,T]}|\varphi_2(t)-\psi_2(t)|^p=0. \tag{4.3.15}$$

For getting uniqueness of the first component recall that we have an integral that uses φ_2 as input, i.e.

$$\varphi_1(t)=x_0+\int_0^t\left(-\frac{1}{2}|\varphi_2(s)|\right)ds+\int_0^t\sqrt{|\varphi_2(s)|}\left(\rho v_1(s)+\bar\rho v_2(s)\right)ds. \tag{4.3.16}$$

For two different solutions φ and ψ, we have already shown that $\varphi_2(t)=\psi_2(t)$ for all $t\in[0,T]$. Using that, we directly get

$$\varphi_1(t)-\psi_1(t)=0. \tag{4.3.17}$$

This concludes the proof. □

Finally, we have everything that we need for our large deviations principle.

Theorem 4.3.8. *Under Assumptions 4.2.1 and 4.3.1 the family $\{Y^\varepsilon\}_{\varepsilon>0}$, unique solution of (4.3.5), satisfies a large deviations principle with rate function I and speed $\vartheta_\varepsilon^{-2}$, i.e.*

$$-\inf_{f\in\mathcal{A}^\circ}I(f)\le\liminf_{\varepsilon\searrow0}\vartheta_\varepsilon^2\log P(Y^\varepsilon\in\mathcal{A})\le\limsup_{\varepsilon\searrow0}\vartheta_\varepsilon^2\log P(Y^\varepsilon\in\mathcal{A})\le-\inf_{f\in\bar{\mathcal{A}}}I_T(f), \tag{4.3.18}$$

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for every Borel subset $\mathcal{A}$ of $C[0,T]$. The rate function I is given by

$$I(\varphi) := \inf\left\{\frac{1}{2}\int_0^T |v_s|^2\,ds : v \in L^2,\ \varphi(t) = x_0 + \int_0^t K_{\mathrm{JP}}(t,s)\left(b_{\mathrm{JP}}(s,\varphi_s) + \sigma_{\mathrm{JP}}(s,\varphi_s)v_s\right)ds\right\}$$
$$= \inf_{\{f\in L^2,\,\widetilde{S}(f)=\varphi\}} \frac{1}{2}\int_0^T |f_s|^2\,ds, \tag{4.3.19}$$

whenever $\left\{f \in L^2,\ \widetilde{S}(f) = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\} \neq \emptyset$, and $I(\varphi_1,\varphi_2) = \infty$ otherwise. Here, $\widetilde{S}$ maps $f = (f_1, f_2)$ to the solution of (4.3.10).

Proof. We have proven above that all requirements in Theorem 3.8 in [JP20] hold. The conditions **H1–H4** [JP20] hold due to Lemma 4.3.6. Assumption 2.3 [JP20] holds due to Lemma 4.3.4. Assumption 3.1 [JP20] holds due to Lemma 4.3.7. Assumption 3.6 [JP20] holds automatically, because our two-dimensional formulation satisfies the linear growth condition. For the detailed proof we refer to [JP20] and the computations in their appendix. □

As a consequence, we can formulate large deviation principles for the scaled log-price and the scaled instantaneous variance by using the contraction principle (see Theorem 1.2.5) with the projection to coordinates as continuous function.

For an efficient notation we introduce the convention to write

$$Y^\varepsilon \sim \mathrm{LDP}(I, \vartheta_\varepsilon^{-2}) \tag{4.3.20}$$

if the process $(Y^\varepsilon)_{\varepsilon>0}$ satisfies an LDP with speed $\vartheta_\varepsilon^{-2}$ and rate function I.

Corollary 4.3.9. Under Assumptions 4.2.1 and 4.3.1 the following large deviations principles hold:

(L1) The joint process $Y^\varepsilon = (X^\varepsilon, V^\varepsilon)$ satisfies an LDP, i.e. $Y^\varepsilon \sim \mathrm{LDP}(I, \vartheta_\varepsilon^{-2})$, where the rate function I is given by (4.3.19) resp.

$$I(\varphi_1,\varphi_2) := \inf\left\{\frac{1}{2}\int_0^T \left(f_1^2(s) + f_2^2(s)\right)ds : f_1, f_2 \in L^2[0,T],\ \widetilde{S}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\}, \tag{4.3.21}$$

as long as the set is non-empty and $I(\varphi_1,\varphi_2) = \infty$ else.

(L2) The process V satisfies an LDP, i.e. $V^\varepsilon \sim \mathrm{LDP}(I^V, \vartheta_\varepsilon^{-2})$, where the rate function I^V is given by

$$I^V(\psi_2) := \inf\left\{\frac{1}{2}\int_0^T \left(f_1^2(s) + f_2^2(s)\right) : f_1, f_2 \in L^2[0,T], \widetilde{S}\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \psi_2\end{pmatrix}\right\}, \tag{4.3.22}$$

if this set is non-empty and $I^V(\psi_2) = \infty$ else.

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(L3) The instantaneous variance satisfies an LPD, i.e. $|V^\varepsilon| \sim \mathrm{LDP}(I^{|V|}, \vartheta_\varepsilon^{-2})$, where the rate function $I^{|V|}$ is given by

$$I^{|V|}(\chi_2) := \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T], \widetilde{S}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi_2 \end{pmatrix}, \chi_2 = |\psi_2|\right\}, \tag{4.3.23}$$

if this set is non-empty and $I^{|V|}(\chi_2) = \infty$ else.

(L4) The log-price satisfies an LDP, i.e. $X^\varepsilon \sim \mathrm{LDP}(I^X, \vartheta_\varepsilon^{-2})$, where the rate function I^X is given by

$$I^X(\psi_1) := \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s)) : f_1, f_2 \in L^2[0,T], \widetilde{S}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \varphi_2 \end{pmatrix}\right\}, \tag{4.3.24}$$

if this set is non-empty and $I^X(\psi_1) = \infty$ else.

Proof. We can directly apply the contraction principle in Theorem 1.2.5.

(L2) For V^ε, using $f(x_1, x_2) = x_2$, we can write

$$\begin{aligned} I^V(\psi_2) &= \inf\{I(\varphi_1, \varphi_2) : \psi_2 = f(\varphi_1, \varphi_2)\} \\ &= \inf\{I(\varphi_1, \varphi_2) : \psi_2 = \varphi_2\} \\ &= \inf\{I(\varphi_1, \psi_2)\} \\ &= \inf\left\{\frac{1}{2}\int_0^T \left(f_1^2(s) + f_2^2(s)\right) : f_1, f_2 \in L^2[0,T], \widetilde{S}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi_2 \end{pmatrix}\right\}. \end{aligned}$$

(L3) For $|V^\varepsilon|$, using $f(x) = |x|$, we can write

$$\begin{aligned} I^{|V|}(\chi_2) &= \inf\{I^V(\psi_2) : \chi_2 = f(\psi_2)\} \\ &= \inf\{I^V(\psi_2) : \chi_2 = |\psi_2|\} \\ &= \inf\left\{\frac{1}{2}\int_0^T \left(f_1^2(s) + f_2^2(s)\right) : f_1, f_2 \in L^2[0,T], \tilde{S}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi_2 \end{pmatrix}, \chi_2 = |\psi_2|\right\}. \end{aligned}$$

(L4) Then, for X^ε we project the first component using $f(x_1, x_2) = x_1$ and get

$$\begin{aligned} I^X(\psi_1) &= \inf\{I(\varphi_1, \varphi_2) : \psi_1 = f(\varphi_1, \varphi_2)\} \\ &= \inf\{I(\varphi_1, \varphi_2) : \psi_1 = \varphi_1\} \\ &= \inf\{I(\psi_1, \varphi_2)\} \\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s)) : f_1, f_2 \in L^2[0,T], \widetilde{S}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \varphi_2 \end{pmatrix}\right\}, \end{aligned}$$

which concludes the proof. $\square$

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4.3.2. Sample path moderate deviations

For applying the moderate deviations results of [JP20], let $h_\varepsilon \to \infty$ such that $\vartheta_\varepsilon h_\varepsilon \to 0$ as $\varepsilon \to 0$. Define $\overline{Y}$ to be the limit in law of Y^ε as given in (4.3.5), which is in fact the solution of the Volterra equation

$$\overline{Y}_t = y_0 + \int_0^t K_{\mathrm{JP}}(t,s) b_{\mathrm{JP}}(s, \overline{Y}_s)\, ds. \tag{4.3.25}$$

Then, according to [JP20], the moderate deviations principle for $\{Y^\varepsilon\}_{\varepsilon>0}$ is equivalent to the LDP for the family $\{\eta^\varepsilon\}_{\varepsilon>0}$ defined as

$$\eta^\varepsilon := \frac{Y^\varepsilon - \overline{Y}}{\vartheta_\varepsilon h_\varepsilon}. \tag{4.3.26}$$

Therefore, η^ε satisfies the SVE

$$\eta_t^\varepsilon = \int_0^t K_{\mathrm{JP}}(t,s) \frac{b_{\mathrm{JP}}(s, \overline{Y}_s + \vartheta_\varepsilon h_\varepsilon \eta_s^\varepsilon) - b_{\mathrm{JP}}(s, \overline{Y}_s)}{\vartheta_\varepsilon}\, ds + \int_0^t K_{\mathrm{JP}}(t,s) \frac{\sigma_{\mathrm{JP}}(s, \overline{Y}_s + \vartheta_\varepsilon h_\varepsilon \eta_s^\varepsilon)}{h_\varepsilon}\, dW_s. \tag{4.3.27}$$

for all $\varepsilon > 0$ and is its unique solution if **H4** of [JP20] holds. In the new situation the limiting control equation is given by

$$\psi_t = \int_0^t K_{\mathrm{JP}}(t,s) \left(\nabla b_{\mathrm{JP}}(s, \overline{Y}_s)\psi_s + \sigma_{\mathrm{JP}}(s, \overline{Y}_s) v_s\right)\, ds, \tag{4.3.28}$$

which is way simpler than the control equation for the LDP, because the solution ψ only occurs linearly under the integral. Here, we get an additional requirement on the coefficient functions, i.e. that the function b_{JP} is Lipschitz continuous and differentiable. Recall that in (4.3.2) we do *not* meet this requirement at all. However, for the second component, the equation for the the process V, we can replace $\bar{b}$ by a smooth version $\tilde{b}$ such that we get at least sample path moderate deviations for the process V. How we can attain a smooth version $\tilde{b}$ is shortly discussed in Appendix D.2. In the following, let $\eta^{V,\varepsilon}$ be the second component of η^ε. The second component of the control equation can be written as

$$\overline{V}_t = y_0 + \int_0^t K(t,s)\tilde{b}(\overline{V}_s)\, ds \tag{4.3.29}$$

$$\psi_2(t) = \int_0^t K(t,s) \left(\partial_x \tilde{b}(\overline{V}_s)\psi_2(s) + \bar{\sigma}(\overline{V}_s) v_s\right)\, ds. \tag{4.3.30}$$

Theorem 4.3.10. *Under Assumptions 4.2.1 and 4.3.1 the family $\{\eta^{V,\varepsilon}\}_{\varepsilon>0}$ satisfies a large deviations principle (equivalently $\{V^\varepsilon\}_{\varepsilon>0}$ satisfies a moderate deviations principle) with speed h_ε^2 and rate function*

$$\Lambda(\psi) := \inf_{f \in L^2[0,T],\, \psi = \overline{S}(f)} \frac{1}{2} \int_0^T |f_s|^2\, ds, \tag{4.3.31}$$

where $\overline{S}$ is the solution map of the limiting control equation (4.3.30).

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Proof. We need to show that **H2–H8**, Assumptions 2.3 and 3.6 of [JP20] are satisfied.

H2–H4. This was already shown in the proof of the large deviations result, Theorem 4.3.8.

H5. The Lipschitz condition for $\tilde{b}$ can be argued in the same way as for $\bar{b}$ in the proof of Lemma 4.2.3. The modified version $\tilde{b}$ is differentiable, see Appendix D.2.

H6. The coefficient function $\bar{\sigma}$ is locally Hölder continuous, since it is Lipschitz continuous according to the computations in the proof of Lemma 4.2.3.

H7. This is an initial condition that holds trivially, because we start deterministic.

H8. Trivially holds, because we have $b = b_\varepsilon$.

Assumption 2.3 of [JP20]. Holds due to Lemma 4.3.4.

Assumption 3.6 of [JP20]. Is satisfied automatically, because our equation satisfies the linear growth condition. □

4.3.3. Small-time large deviations

In this part, we want to establish a small-time large deviation principle. Therefore, we need an additional kernel assumption for being able to use the small-noise LDP for proving a small-time LDP.

Assumption 4.3.11. The kernel of our model (4.1.7)–(4.1.8) is homogeneous of degree $\zeta \in \mathbb{R}$ in the sense that there is a $\zeta \in \mathbb{R}$ such that

$$K(t\varepsilon, s\varepsilon) = \varepsilon^\zeta K(t, s) \tag{4.3.32}$$

for all $t, s \in [0, T]$ and $\varepsilon > 0$. Furthermore, let ζ be such that $\zeta + 1 > 0$ and $\zeta + \frac{1}{2} > 0$, i.e.

$$\zeta > -\frac{1}{2}.$$

Remark 4.3.12. Note that we will stick to Assumption 4.3.1 to use convolutional kernels. However, our notation uses general kernels to let space for further adaptions later.

Now, define

$$\widetilde{X}_t^\varepsilon := \varepsilon^\zeta X_{t\varepsilon}, \tag{4.3.33}$$

$$\widetilde{V}_t^\varepsilon := V_{t\varepsilon} \tag{4.3.34}$$

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Using the homogeneity property from Assumption 4.3.11, we can write

$$\begin{aligned}\widetilde{V}_t^\varepsilon = V_{t\varepsilon} &= V_0 + \int_0^{t\varepsilon} b(t\varepsilon, s, V_s)\, ds + \int_0^{t\varepsilon} \sigma(t\varepsilon, s, V_s)\, dW_s \\ &= V_0 + \int_0^{t\varepsilon} K(t\varepsilon, s)\bar{b}(V_s)\, ds + \int_0^{t\varepsilon} K(t\varepsilon, s)\bar{\sigma}(V_s)\, dW_s \\ &= V_0 + \varepsilon \int_0^{t} K(t\varepsilon, u\varepsilon)\bar{b}(V_{u\varepsilon})\, du + \int_0^{t} K(t\varepsilon, u\varepsilon)\bar{\sigma}(V_{u\varepsilon})\, dW_{u\varepsilon} \\ &\overset{(4.3.32)}{=} V_0 + \varepsilon \cdot \varepsilon^\zeta \int_0^{t} K(t, u)\bar{b}(V_{u\varepsilon})\, du + \varepsilon^\zeta \int_0^{t} K(t, u)\bar{\sigma}(V_{u\varepsilon})\, dW_{u\varepsilon} \\ &\overset{d}{=} V_0 + \varepsilon^{\zeta+1} \int_0^{t} K(t, u)\bar{b}(V_{u\varepsilon})\, du + \varepsilon^{\zeta+\frac{1}{2}} \int_0^{t} K(t, u)\bar{\sigma}(V_{u\varepsilon})\, dW_u,\end{aligned}$$

where we used the Brownian scaling property

$$\frac{1}{\sqrt{\varepsilon}} W_{u\varepsilon} \overset{d}{=} W_u$$

for the last equation. For the scaled log-price we have

$$\begin{aligned}\widetilde{X}_t^\varepsilon = \varepsilon^\zeta X_{t\varepsilon} &= \varepsilon^\zeta X_0 - \varepsilon^\zeta \int_0^{t\varepsilon} \frac{1}{2}|V_s|\, ds + \varepsilon^\zeta \int_0^{t\varepsilon} \sqrt{|V_s|}\,(\bar{\rho}\, dW_s + \rho\, dB_s) \\ &= \varepsilon^\zeta X_0 - \varepsilon^{\zeta+1} \int_0^{t} \frac{1}{2}|V_{u\varepsilon}|\, du + \varepsilon^\zeta \int_0^{t} \sqrt{|V_{u\varepsilon}|}\,(\bar{\rho}\, dW_{u\varepsilon} + \rho\, dB_{u\varepsilon}) \\ &\overset{d}{=} \varepsilon^\zeta X_0 - \varepsilon^{\zeta+1} \int_0^{t} \frac{1}{2}|V_{u\varepsilon}|\, du + \varepsilon^{\zeta+\frac{1}{2}} \int_0^{t} \sqrt{|V_{u\varepsilon}|}\,(\bar{\rho}\, dW_u + \rho\, dB_u)\,.\end{aligned}$$

Now, using $\vartheta_\varepsilon := \varepsilon^{\zeta+\frac{1}{2}}$, we can write

$$\begin{pmatrix}\widetilde{X}_t^\varepsilon \\ \widetilde{V}_t^\varepsilon\end{pmatrix} = \begin{pmatrix}\varepsilon^\zeta X_0 \\ V_0\end{pmatrix} + \int_0^t \begin{pmatrix}1 & 0 \\ 0 & K(t,u)\end{pmatrix}\begin{pmatrix}-\varepsilon^{\zeta+1}\frac{1}{2}|\widetilde{V}_u^\varepsilon| \\ \varepsilon^{\zeta+1}\bar{b}(\widetilde{V}_u^\varepsilon)\end{pmatrix} ds + \vartheta_\varepsilon \cdot \int_0^t \begin{pmatrix}1 & 0 \\ 0 & K(t,u)\end{pmatrix}\begin{pmatrix}\bar{\rho}\sqrt{|\widetilde{V}_u^\varepsilon|} & \rho\sqrt{|\widetilde{V}_u^\varepsilon|} \\ \bar{\sigma}(\widetilde{V}_u^\varepsilon) & 0\end{pmatrix}\begin{pmatrix}dB_s \\ dW_s\end{pmatrix} \tag{4.3.35}$$

Here, we have a framework as in [JP20], this time with ε-dependent drift coefficient

$$b_\varepsilon(s,(x,y)) = \begin{pmatrix}-\varepsilon^{\zeta+1}\frac{1}{2}|y| \\ \varepsilon^{\zeta+1}\bar{b}(y)\end{pmatrix}. \tag{4.3.36}$$

This drift coefficient satisfies all the regularity requirements needed in [JP20] and we have $b_\varepsilon \to b = 0$ for $\varepsilon \to 0$. Additionally, the differentiability and Lipschitz condition of the limiting control equation in the large deviations section of [JP20] are trivially met. Therefore, we can also apply moderate deviations in the small-time case. The volatility coefficient is the same as in the previous setting. Now, consider the controlled equation to $\widetilde{V}^\varepsilon$, i.e.

$$\widetilde{V}_t^{\varepsilon,v} = v_0 + \varepsilon^{\zeta+1}\int_0^t K(t,u)\bar{b}(\widetilde{V}_u^{\varepsilon,v})\, du + \varepsilon^{\zeta+\frac{1}{2}}\int_0^t K(t,u)\bar{\sigma}(V_u^{\varepsilon,v})\, dW_u + \int_0^t K(t,u)\bar{\sigma}(V_u^{\varepsilon,v})v_u\, du, \tag{4.3.37}$$

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and the limiting control equation

$$\varphi(t) = v_0 + \int_0^t K(t,s)\bar{\sigma}(\varphi(s))v_s\,ds. \tag{4.3.38}$$

The control of the log-price is given by

$$\widetilde{X}_t^{\varepsilon,w} = \varepsilon^{\zeta} X_0 - \varepsilon^{\zeta+1}\int_0^t \frac{1}{2}|\widetilde{V}_u^{\varepsilon,v}|\,du + \varepsilon^{\zeta+\frac{1}{2}}\int_0^t \sqrt{|V_u^{\varepsilon,v}|}\,(\bar{\rho}\,dW_u + \rho\,dB_u) + \int_0^t \sqrt{|V_u^{\varepsilon,v}|}(\bar{\rho}v_u + \rho w_u)\,du, \tag{4.3.39}$$

with limiting equation

$$\psi(t) = \int_0^t \sqrt{|\psi(s)|}(v_s + w_s)\,ds. \tag{4.3.40}$$

Now, we have again everything we need to formulate a large deviations result. In the following, we will use the shorthand from (4.3.20) to formulate a large deviations result.

Corollary 4.3.13. Under Assumptions 4.2.1, 4.3.1 and 4.3.11 the following hold:

(L1) The joint scaled process $(\widetilde{X}^{\varepsilon}, \widetilde{V}^{\varepsilon})$ satisfies an LDP, i.e. $(\widetilde{X}^{\varepsilon}, \widetilde{V}^{\varepsilon}) \sim \mathrm{LDP}(I, \varepsilon^{-(2\zeta+1)})$, where the rate function I is given by

$$I(\varphi_1,\varphi_2) := \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\, S\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\right\}, \tag{4.3.41}$$

where S is the solution map of the control equation, i.e. for $f_1, f_2 \in L^2[0,T]$ the first and second components of $S\begin{pmatrix} f_1 \\ f_2\end{pmatrix}$ solve the control equations (4.3.40) and (4.3.38).

(L2) The scaled log-price $\widetilde{X}^{\varepsilon}$ satisfies an LDP, i.e. $\widetilde{X}^{\varepsilon} \sim \mathrm{LDP}(I^X, \varepsilon^{-(2\zeta+1)})$, where the rate function I^X is given by

$$I^X(\psi) = \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\, S\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\psi \\ \varphi_2\end{pmatrix}\right\}, \tag{4.3.42}$$

whenever this set is non-empty and $I(\varphi_1,\varphi_2) = \infty$ else.

(L3) The log-price satisfies a small-time LDP, i.e. $\varepsilon^{\zeta}X_{\varepsilon} \sim \mathrm{LDP}(I_1^X, \varepsilon^{-(2\zeta+1)})$, where the rate function I_1^X is given by

$$I_1^X(x) := \inf\left\{I^X(\psi) : x = \psi(1)\right\}, \tag{4.3.43}$$

whenever this set is non-empty and $I_1^X(x) = \infty$ else.

(L4) The scaled process $\widetilde{V}^{\varepsilon}$ satisfies an LDP, i.e. $\widetilde{V}^{\varepsilon} \sim \mathrm{LDP}(I^V, \varepsilon^{-(2\zeta+1)})$, where the rate function I^V is given by

$$I^V(\psi) := \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\, S\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1 \\ \psi\end{pmatrix}\right\}, \tag{4.3.44}$$

whenever this set is non-empty and $I^V(\psi) = \infty$ else.

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(L5) The process V satisfies a small-time LDP, i.e. $V_\varepsilon \sim \mathrm{LDP}(I_1^V, \varepsilon^{-(2\zeta+1)})$, where the rate function I_1^V is given by

$$I_1^V(y) := \inf\left\{I^V(\psi) : y = \psi(1)\right\}, \tag{4.3.45}$$

whenever this set is non-empty and $I_1^V(y) = \infty$ else.

(L6) The instantaneous variance satisfies a small-time LDP, i.e. $|V_\varepsilon| \sim \mathrm{LDP}(I_1^{|V|}, \varepsilon^{-(2\zeta+1)})$, where the rate function $I_1^{|V|}$ is given by

$$I_1^{|V|} := \inf\left\{I_1^V(y) : z = |y|\right\}, \tag{4.3.46}$$

whenever this set is non-empty and $I_1^V(y) = \infty$ else.

Proof. This is actually Proposition 4.3 in [JP20]. For (L1) the reasoning is essentially the same as in the proof of Theorem 4.3.8. The only difference here is that we have a drift coefficient that is dependent of ε and tends to zero as ε tends to zero. For the others we can directly apply the contraction principle Theorem 1.2.5.

(L2)

$$\begin{aligned} I^X(\psi) &= \inf\{I(\varphi_1,\varphi_2) : \psi = f(\varphi_1,\varphi_2) = \varphi_1\} \\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\ S\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \psi = \varphi_1\right\} \\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\ S\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \psi \\ \varphi_2 \end{pmatrix}\right\}. \end{aligned}$$

(L3)

$$\begin{aligned} I_1^X(x) &:= \inf\left\{I^X(\psi) : x = f(\psi),\right\} \\ &= \inf\left\{I^X(\psi) : x = \psi(1)\right\}. \end{aligned}$$

(L4)

$$\begin{aligned} I^V(\psi) &= \inf\{I(\varphi_1,\varphi_2) : \psi = f(\varphi_1,\varphi_2) = \varphi_2\} \\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\ S\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \psi = \varphi_2\right\} \\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s))\,ds : f_1, f_2 \in L^2[0,T],\ S\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi \end{pmatrix}\right\}. \end{aligned}$$

(L5)

$$\begin{aligned} I_1^V(y) &:= \inf\left\{I^V(\psi) : y = f(\psi),\right\} \\ &= \inf\left\{I^V(\psi) : y = \psi(1)\right\}. \end{aligned}$$

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(L6)

$$
\begin{aligned}
I_1^{|V|}(z) &:= \inf\left\{I_1^V(y) : z = f(y)\right\} \\
&= \inf\left\{I_1^V(y) : z = |y|\right\}.
\end{aligned}
$$

□

4.3.4. Small-time moderate deviations

For small-time moderate deviations let $h_\varepsilon = \varepsilon^{-\beta}$ for any $\beta \in (0, \zeta + \frac{1}{2})$, and define the two-dimensional process

$$
\eta^\varepsilon = \frac{1}{\vartheta_\varepsilon h_\varepsilon}(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon - v_0) = \frac{1}{\varepsilon^{\zeta+\frac{1}{2}-\beta}}(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon - v_0). \tag{4.3.47}
$$

According to [JP20], the case $\beta = 0$ corresponds to the Central Limit Theorem, whereas $\beta = \zeta + \frac{1}{2}$ is the LDP regime, so that MDP precisely corresponds to some interpolation between the two. The authors of [JP20] show that all requirements that are needed are satisfied given the additional Assumption 4.5 in [JP20]. This assumption is about the homogeneity and local Hölder continuity of the coefficient functions. It is met for our model, hence we can cite their result.

Corollary 4.3.14 (Proposition 4.6 [JP20])**.** Under Assumptions 4.2.1, 4.3.1 and 4.3.11 the following hold:

(M1) The joint process $(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon)$ satisfies an MDP, i.e. $(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon) \sim \text{MDP}(\Lambda, \varepsilon^{-2\beta})$, where the rate function Λ is given by

$$
\Lambda(\varphi_1, \varphi_2) := \inf\left\{\frac{1}{2}\int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds : f_1, f_2 \in L^2[0,T], \Gamma\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right\}, \tag{4.3.48}
$$

if this set is non-empty and $\Lambda(\varphi_1, \varphi_2) = \infty$ otherwise. Here, Γ maps (f_1, f_2) to the solution to the control equation of the process η^ε.

(M2) The scaled log-price satisfies an MDP, i.e. $\widetilde{X}^\varepsilon \sim \text{MDP}(\Lambda^X, \varepsilon^{-2\beta})$, where the rate function Λ^x is given by

$$
\begin{aligned}
\Lambda^X(\psi) &:= \inf\{\Lambda(\varphi_1, \varphi_2) : \psi = \varphi_1\} \\
&= \inf\left\{\frac{1}{2}\int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds : f_1, f_2 \in L^2[0,T], \Gamma\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \psi \\ \varphi_2 \end{pmatrix}\right\},
\end{aligned} \tag{4.3.49}
$$

wherever this set is non-empty and $\Lambda(\psi) = \infty$ otherwise.

(M3) The log-price satisfies a small-time MDP, i.e. $\varepsilon^\zeta X_\varepsilon \sim \text{MDP}(\Lambda_1^X, \varepsilon^{-2\beta})$, where the rate function Λ_1^X is given by

$$
\Lambda_1^X(x) := \inf\{\Lambda^X(\psi) : x = \psi(1)\}, \tag{4.3.50}
$$

whenever this set is non-empty and $\Lambda_1^X(x) = \infty$ otherwise.

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(M4) The scaled process $\widetilde{V}^\varepsilon$ satisfies an MDP, i.e. $\widetilde{V}^\varepsilon \sim \mathrm{MDP}(\Lambda^V, \varepsilon^{-2\beta})$, where the rate function Λ^V is given by

$$\begin{aligned}\Lambda^V(\psi) &:= \inf\{\Lambda(\varphi_1,\varphi_2) : \psi = \varphi_2\} \\ &= \inf\left\{\frac{1}{2}\int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds : f_1, f_2 \in L^2[0,T], \Gamma\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_2 \\ \psi\end{pmatrix}\right\},\end{aligned} \tag{4.3.51}$$

whenever this set is non-empty and $\Lambda^V(\psi) = \infty$ otherwise.

(M5) The process V satisfies a small-time MDP, i.e. $V^\varepsilon \sim \mathrm{MDP}(\Lambda_1^V, \varepsilon^{-2\beta})$, where the rate function Λ_1^V is given by

$$\Lambda_1^V(y) := \inf\{\Lambda^V(\psi) : y = \psi(1)\}, \tag{4.3.52}$$

whenever this set is non-empty and $\Lambda_1^V(y) = \infty$ otherwise.

4.3.5. Applications

In this section, we cite some results to motivate why we want to have large (and moderate) deviations for asymptotic results.

Implied volatility asymptotics

The results on small-time large and moderate deviations in Sections 4.3.3 and 4.3.4 can be used to obtain asymptotics for the implied volatility, see Section 4.1.3 in [JP20]. In the following, we cite their results.

Following [JP20] we say that for each maturity $t \geq 0$ and log-moneyness $k \in \mathbb{R}$, the implied volatility $\hat{\sigma}(t,k)$ is the unique non-negative solution to $C_{\mathrm{BS}}(t,k,\hat{\sigma}(t,k)) = C(t,k)$, where C_{BS} corresponds to the price of a European call option under the Black-Scholes model, and C a given Call option price. According to [JP20], this notion is only well-defined if the underlying stock price is a true martingale, which has not been assumed so far. Therefore, the authors of [JP20] add this requirement as an assumption.

Assumption 4.3.15 (Assumption 4.7 [JP20])**.** The process $\exp(\widetilde{X}^\varepsilon)$ in (4.3.35) is a true martingale for small enough $\varepsilon > 0$.

Corollary 4.3.16 (Corollary 4.8 [JP20])**.** Let Assumption 4.3.15 hold.

(LDP) Under the same assumptions as in Corollary 4.3.13,

$$\lim_{t\searrow 0} \widehat{\sigma}\left(t, kt^{-\zeta}\right)^2 = \begin{cases} \frac{k^2}{2\inf_{x\geq k} I_1^X(x)}, & \text{if } k > 0, \\ \frac{k^2}{2\inf_{x\leq k} I_1^X(x)}, & \text{if } k < 0. \end{cases} \tag{4.3.53}$$

(MDP) Under the same assumptions as in Corollary 4.3.14 and for any $\beta \in (0, \zeta + \frac{1}{2})$, $k \neq 0$, we have

$$\lim_{t\searrow 0} \widehat{\sigma}\left(t, kt^{\frac{1}{2}-\beta}\right)^2 = \begin{cases} \frac{k^2}{2\inf_{x\geq k} \Lambda_1^X(x)}, & \text{if } k > 0, \\ \frac{k^2}{2\inf_{x\leq k} \Lambda_1^X(x)}, & \text{if } k < 0. \end{cases} \tag{4.3.54}$$

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Realized variance

Motivated by [LMS21], a transformation of the instantaneous variance that might be interesting is the integrated variance, also known as realized variance (RV), which is given by

$$\frac{1}{T}\int_0^T d\langle X\rangle_t = \frac{1}{T}\int_0^T \sqrt{|V_t|}^2\, dt = \frac{1}{T}\int_0^T |V_t|\, dt. \tag{4.3.55}$$

Note that we need to take the absolute value of the process V, because in fact only the process $|V|$ is the instantaneous variance. Now, define the $C[0,T]$-operator

$$\mathrm{RV}(f)(\cdot) : f \mapsto \frac{1}{\cdot}\int_0^{\cdot} f(s)\, ds, \quad \mathrm{RV}(f)(0) := f(0). \tag{4.3.56}$$

Lemma 4.3.17. *The operator* RV *given in* (4.3.56) *is continuous with respect to the* sup *norm on $C[0,T]$.*

Proof. As in the proof of Corollary 3.4 [LMS21] let $\delta \in C[0,T]$ be a small perturbation which allows us to write

$$\begin{aligned}
\|\mathrm{RV}(f+\delta) - \mathrm{RV}(f)\|_\infty &= \sup_{t\in[0,T]} |\mathrm{RV}(f+\delta)(t) - \mathrm{RV}(f)(t)| \\
&= \sup_{t\in[0,T]} \left|\frac{1}{t}\int_0^t (f(s)+\delta(s))\, ds - \frac{1}{t}\int_0^t f(s)\, ds\right| \\
&= \sup_{t\in[0,T]} \left|\frac{1}{t}\int_0^t \delta(s)\, ds\right| \\
&\le \sup_{t\in[0,T]} \frac{1}{t}\int_0^t |\delta(s)|\, ds \\
&\le \frac{1}{t}\int_0^t M_\delta\, ds \\
&= M_\delta,
\end{aligned}$$

where $M_\delta := \sup_{t\in[0,T]} |\delta(t)|$, which is finite as $\delta \in C[0,T]$. Clearly, M_δ tends to zero as δ tends to zero, and hence the operator RV is continuous with respect to the sup norm on $C[0,T]$. □

Recall that from Corollary 4.3.13 we know that $(V_t)_{t>0}$ satisfies a large deviations principle as t tends to zero with speed $t^{-(2\zeta+1)}$ and rate function I_1^V given by (4.3.45). Then, using the continuity of the realized variance operator RV we get a large deviations result for the realized variance.

Corollary 4.3.18. The integrated variance process $(\mathrm{RV}(|V|)(t))_{t\in[0,T]}$ satisfies a large deviations principle on $\mathbb{R}_+^*$ as t tends to zero, with speed $t^{-(2\zeta+1)}$ and rate function $\hat{\Lambda}$ defined as

$$\hat{\Lambda}(y) := \inf\{I_1^V(\psi) : y = \mathrm{RV}(|\psi|)(1)\}, \tag{4.3.57}$$

where $\hat{\Lambda}(v_0) = 0$.

Remark 4.3.19. The rate function (4.3.57) can be written as

$$\begin{aligned}\hat{\Lambda}(y) &= \inf\{I_1^V(\psi) : y = \mathrm{RV}(|\psi|)(1)\}\\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s)\, ds : f_1, f_2 \in L^2[0,T],\, S\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix},\, \varphi_2 = \psi,\, \psi(1) = x,\, y = \mathrm{RV}(x)(1)\right\}\\ &= \inf\left\{\frac{1}{2}\int_0^T (f_1^2(s) + f_2^2(s)\, ds : f_1, f_2 \in L^2[0,T],\, S\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \begin{pmatrix}\varphi_1 \\ \psi\end{pmatrix},\, y = \mathrm{RV}(|\psi|)(1)\right\}.\end{aligned}$$

Proof of Corollary 4.3.18. Recall that we have an LDP on $C[0,T]$ for the process $(V_{\varepsilon\cdot})_\varepsilon$ due to Corollary 4.3.13. Using the contraction principle, Theorem 1.2.5, for the continuous function $f(x) = |x|$ we get the same for $(|V_{\varepsilon\cdot}|)_\varepsilon$. Then, we use that RV is a continuous operator, hence we again apply the contraction principle and get an LPD on $C[0,T]$ for $(\mathrm{RV}(|V_{\varepsilon\cdot}|))_\varepsilon$. Then, for all $t \in [0,T]$, we can write

$$\mathrm{RV}(|V_{\varepsilon\cdot}|)(t) = \mathrm{RV}(|V_\cdot|)(\varepsilon t). \tag{4.3.58}$$

Hence, we have this LDP on $C[0,T]$ for the process $(\mathrm{RV}(|V|)(\varepsilon\cdot))_\varepsilon$. Evaluating at $t = 1$ we use the contraction principle to finally arrive at an LDP in $\mathbb{R}$ for $(\mathrm{RV}(|V|)(\varepsilon))_\varepsilon$ and speed $\varepsilon^{-(2\zeta+1)}$. Mapping ε to t yields the result. □

The authors of [LMS21] argue that if the rate function of the input process V to the realized variance operator RV is continuous, so is the rate function of $\mathrm{RV}(|v|)$. Since we have not yet discussed the continuity of the rate function, we will make a continuity assumption for citing their result.

Assumption 4.3.20. The rate function I_1^V given by (4.3.45) is continuous.

Corollary 4.3.21 (Corollary 3.6 [LMS21])**.** Under Assumption 4.3.20 the rate function $\hat{\Lambda}$ given by (4.3.57) is continuous.

Corollary 4.3.22 (Corollary 3.7 [LMS21])**.** Let Assumption 4.3.20 hold, $\beta := 2\zeta + 1$ and assume that the rate function can be represented as $\hat{\Lambda}(e^{\cdot})$. Then, for log moneyness $k := \log\left(\frac{K}{\mathrm{RV}(v)(0)}\right) \neq 0$, the following equality holds true for Call options on integrated variance:

$$\lim_{t\searrow 0} t^\beta \log E\left[\left(\mathrm{RV}(|V|)(t) - e^k\right)^+\right] = -I(k), \tag{4.3.59}$$

where I is defined as

$$I(x) := \begin{cases} \inf_{y>x} \hat{\Lambda}(e^y), & x > 0, \\ \int_{y<x} \hat{\Lambda}(e^y), & y < 0. \end{cases} \tag{4.3.60}$$

Similarly, for log moneyness $k := \log\left(\frac{K}{\sqrt{\mathrm{RV}(v)(0)}}\right) \neq 0$,

$$\lim_{t\searrow 0} t^\beta \log E\left[\left(\sqrt{\mathrm{RV}(v)(t)} - e^k\right)^+\right] = -\bar{I}(k), \tag{4.3.61}$$

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where $\bar{I}$ is defined analogously as

$$\bar{I}(x) := \begin{cases} \inf_{y>x} \hat{\Lambda}(e^{2y}), & x > 0 \\ \inf_{y<x} \hat{\Lambda}(e^{2y}), & x < 0. \end{cases} \tag{4.3.62}$$

Proof. The proof can be found in Appendix B.2 [LMS21]. □

In [LMS21], the authors also consider implied volatility asymptotics. Note, that for the Call price of the realized variance the implied volatility $\hat{\sigma}(T, k)$ to be the solution to

$$E\left[\left(\mathrm{RV}(v)(T) - e^k\right)^+\right] = C_{\mathrm{BS}}(\mathrm{RV}(v)(0), k, T, \hat{\sigma}(T, k)), \tag{4.3.63}$$

where C_{BS} denotes the Call price in the Black-Scholes model. Using Corollary 4.3.22, we can, according to [LMS21], deduce the small-time behavior of the implied volatility $\hat{\sigma}$, as defined in (4.3.63).

Corollary 4.3.23 (Corollary 3.8 [LMS21])**.** Let Assumption 4.3.20 hold and $\beta := 2\zeta + 1$. Then, the small-time asymptotic behavior of the implied volatility is given by the following limit, for a log moneyness $k \neq 0$:

$$\lim_{t \searrow 0} t^{1-\beta} \hat{\sigma}^2(t, k) =: \hat{\sigma}^2(k) = \frac{k^2}{2I(k)}. \tag{4.3.64}$$

4.4. Simulation

In this section, we want to implement the model (4.1.7)–(4.1.8). As a disclaimer we note that the naive Euler scheme used here is just an easy-to-understand proposal for simulating the paths of our model without any error analysis, etc. For sure, there are more sophisticated and efficient implementations possible and extensive error analysis would be interesting.

4.4.1. The Algorithm

Recall our model equation

$$X_t = X_0 - \int_0^t \frac{1}{2}|V_s|\, ds + \int_0^t \sqrt{|V_s|}(\bar{\rho}\, dW_s + \rho\, dB_s), \quad X_0 = x_0 > 0,$$
$$V_t = V_0 + \int_0^t K(t,s)\bar{b}(V_s)\, ds + \int_0^t K(t,s)\bar{\sigma}(V_s)\, dW_s, \quad V_0 = v_0 > 0.$$

Remark 4.4.1. Note that the process V can also have negative values. Therefore, if we want to analyze the properties of the instantaneous variance, we have to take

$$\mathcal{V}_t := |V_t|.$$

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For the simulation of the paths we used a naive Euler scheme for the process V and computed the Itô integral for the log-price X the same way. A difference to Itô diffusions we have to take into account for the equation of V is that the kernel forces us to use all the previous values of V_t instead of just one time-step before. However, we do not mind, because we use matrix arithmetics to avoid a further loop handling this as can be seen below. We define a grid

$$\mathbb{T} := \{t_0, t_1, \ldots, t_N\},$$

where

$$h := t_{j+1} - t_j = \frac{T}{N}.$$

Then, we can write

$$\begin{aligned}
V_{t_{j+1}} &= V_0 + \int_0^{t_{j+1}} K(t_{j+1}, s)\bar{b}(V_s)\,ds + \int_0^{t_{j+1}} K(t_{j+1}, s)\bar{\sigma}(V_s)\,dW_s \\
&= V_0 + \sum_{k=0}^{j} \left(\int_{t_k}^{t_{k+1}} K(t_{j+1}, s)\bar{b}(V_s)\,ds + \int_{t_k}^{t_{k+1}} K(t_{j+1}, s)\bar{\sigma}(V_s)\,dW_s \right) \\
&\approx V_0 + \sum_{k=0}^{j} \left(K(t_{j+1}, t_k)\bar{b}(V_{t_k})(t_{k+1} - t_k) + K(t_{j+1}, t_k)\bar{\sigma}(V_{t_k})(W_{t_{k+1}} - W_{t_k}) \right) \\
&= V_0 + \sum_{k=0}^{j} K(t_{j+1}, t_k) \left\{ \bar{b}(V_{t_k})(t_{k+1} - t_k) + \bar{\sigma}(V_{t_k})(W_{t_{k+1}} - W_{t_k}) \right\} \\
&= V_0 + \sum_{k=0}^{j} K(t_{j+1}, t_k) \left\{ \bar{b}(V_{t_k}) \cdot h + \bar{\sigma}(V_{t_k}) \cdot \sqrt{h} \cdot G_k \right\},
\end{aligned}$$

where $G_k \sim \mathcal{N}(0,1)$ i.i.d. For an efficient implementation we will do some linear algebra. Therefore, we define

$$D_k := \bar{b}(V_{t_k}) \cdot h + \bar{\sigma}(V_{t_k}) \cdot \sqrt{h} \cdot G_k, \tag{4.4.1}$$

which allows us to write

$$\begin{aligned}
V_{t_{j+1}} &= V_0 + \sum_{k=0}^{j} K(t_{j+1}, t_k) \left\{ \bar{b}(V_{t_k}) \cdot h + \bar{\sigma}(V_{t_k}) \cdot \sqrt{h} \cdot G_k \right\} \\
&= V_0 + \sum_{k=0}^{j} K(t_{j+1}, t_k) \cdot D_k \\
&= V_0 + K(t_{j+1}, t_0) \cdot D_0 + K(t_{j+1}, t_1) \cdot D_1 + \cdots + K(t_{j+1}, t_j) \cdot D_j \\
&= V_0 + \begin{pmatrix} K(t_{j+1}, t_0) & K(t_{j+1}, t_1) & \cdots & K(t_{j+1}, t_j) \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_1 \\ \vdots \\ D_j \end{pmatrix}.
\end{aligned}$$

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For handling different paths denote by $D_k^(i)$ the D_k coming from a simulation $W^{(i)}$ of Brownian motion for $i = 1, \dots, m$. In fact, we have the factor $\sqrt{h}$ extra, so we draw a Gaussian $G^{(i)}$. Using the superscript for the path, we can write

$$V_{t_{j+1}}^{(i)} = V_0 + \begin{pmatrix} K(t_{j+1}, t_0) & K(t_{j+1}, t_1) & \dots & K(t_{j+1}, t_j) \end{pmatrix} \cdot \begin{pmatrix} D_0^{(i)} \\ D_1^{(i)} \\ \vdots \\ D_j^{(i)} \end{pmatrix}. \tag{4.4.2}$$

Using a matrix, we can write all paths at once, i.e.

$$\left(V_{t_{j+1}}^{(i)}\right)_{i=1,\dots,m} = V_0 + \begin{pmatrix} K(t_{j+1}, t_0) & K(t_{j+1}, t_1) & \dots & K(t_{j+1}, t_j) \end{pmatrix} \cdot \begin{pmatrix} D_0^{(1)} & D_0^{(2)} & \cdots & D_0^{(m)} \\ D_1^{(1)} & D_1^{(2)} & \cdots & D_1^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ D_j^{(1)} & D_j^{(2)} & \cdots & D_j^{(m)} \end{pmatrix},$$

and including all time steps we arrive at

$$\left(V_{t_j}^{(i)}\right)_{j=0,\dots,N}^{i=1,\dots,m} = V_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ K(t_1, t_0) & 0 & \cdots & 0 \\ K(t_2, t_0) & K(t_2, t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K(t_N, t_0) & K(t_N, t_1) & \dots & K(t_N, t_{N-1}) \end{pmatrix} \cdot \begin{pmatrix} D_0^{(1)} & D_0^{(2)} & \cdots & D_0^{(m)} \\ D_1^{(1)} & D_1^{(2)} & \cdots & D_1^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ D_{N-1}^{(1)} & D_{N-1}^{(2)} & \cdots & D_{N-1}^{(m)} \end{pmatrix}.$$

Despite having a nice representation here, we emphasize, that for each row, we need to know the entries of all rows before. Hence, our implementation cannot compute everything at once, but will need an appropriate updating algorithm. We can do this the following way. For the D_k we have

$$D_k^{(i)} = \bar{b}\big(V_{t_k}^{(i)}\big) \cdot h + \bar{\sigma}\big(V_{t_k}^{(i)}\big) \cdot \sqrt{h} \cdot G_k^{(i)}, \tag{4.4.3}$$

for $k = 0, \dots N-1$ and $i = 1, \dots, m$. If we want to compute this simultaneously for all paths let G be a Gaussian sample matrix of size $N \times m$. For ease of notation, we use $(a : b)$ in the matrix indices as a shorthand for $i = a, \dots, b$. Using pointwise multiplication we get

$$\begin{aligned} \left(D_k^{(i)}\right)_{i=1,\dots,m} &= \left(D_k^{(1)}, \quad D_k^{(2)}, \quad \cdots, \quad D_k^{(m)}\right) \\ &= D_k^{(1:m)} \\ &= \left(\bar{b}(V_{t_k}^{(1)}) \cdot h + \bar{\sigma}(V_{t_k}^{(1)}) \cdot \sqrt{h} \cdot G_k^{(1)}, \quad \cdots, \quad \bar{b}(V_{t_k}^{(m)}) \cdot h + \bar{\sigma}(V_{t_k}^{(m)}) \cdot \sqrt{h} \cdot G_k^{(m)}\right) \\ &= \left(\bar{b}(V_{t_k}^{(1:m)}) \cdot h\right) + \left(\bar{\sigma}(V_{t_k}^{(1:m)})\sqrt{h}\right) \cdot \begin{pmatrix} G_k^{(1)} & G_k^{(2)} & \cdots & G_k^{(m)} \end{pmatrix} \\ &= \bar{b}(V_{t_k}^{(1:m)}) \cdot h + \bar{\sigma}(V_{t_k}^{(1:m)}) \cdot \sqrt{h} \cdot G_k^{(1:m)}, \end{aligned}$$

where, with a slightly abuse of notation, the coefficient functions $\bar{b}$ and $\bar{\sigma}$ are evaluated pointwise. Now, we can formulate an algorithm that only adds a row to the matrix containing the D_k values.

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Algorithm 4.4.2. *We can perform the following updating algorithm:*

- *Set a starting value*

$$V_{t_0}^{(1:m)} = \begin{pmatrix} V_0 & V_0 & \cdots & V_0 \end{pmatrix}, \tag{4.4.4}$$

which is an $1 \times m$-matrix containing the initial value.

- *For the iteration computing $V_{t_{j+1}}$, we need the matrix $D_{0:j}^{(1:m)}$. For the previous iteration for V_{t_j}, we had the matrix $D_{0:(j-1)}^{(1:m)}$, hence we compute just the row*

$$D_j^{(1:m)} = \bar{b}\big(V_{t_j}^{(1:m)}\big) \cdot h + \bar{\sigma}\big(V_{t_j}^{(1:m)}\big)\sqrt{h} \cdot G_j^{(1:m)}. \tag{4.4.5}$$

Then, we stack this to the already computed matrix from the previous time iteration, i.e.

$$D_{0:j}^{(1:m)} = \begin{pmatrix} D_{0:(j-1)}^{(1:m)} \\ D_j^{(1:m)} \end{pmatrix} = \begin{pmatrix} D_0^{(1)} & D_0^{(2)} & \cdots & D_0^{(m)} \\ D_1^{(1)} & D_1^{(2)} & \cdots & D_1^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ D_j^{(1)} & D_j^{(2)} & \cdots & D_j^{(m)} \end{pmatrix}. \tag{4.4.6}$$

- *Now, we can perform the vector-matrix multiplication from (4.4.2) simultaneously for all paths, i.e.*

$$V_{t_{j+1}}^{(1:m)} = \begin{pmatrix} K(t_{j+1}, t_0) & K(t_{j+1}, t_1) & \ldots & K(t_{j+1}, t_j) \end{pmatrix} \cdot \begin{pmatrix} D_0^1 & D_0^2 & \cdots & D_0^m \\ D_1^1 & D_1^2 & \cdots & D_1^m \\ \vdots & \vdots & \ddots & \vdots \\ D_j^1 & D_j^2 & \cdots & D_j^m \end{pmatrix}. \tag{4.4.7}$$

Note that a $1 \times (j+1)$ matrix is multiplied with a $(j+1) \times m$ matrix. The resulting $1 \times m$ matrix resp. vector $V_{t_{j+1}}^{(1:m)}$ contains the value of all paths at a certain time point.

For computing the log-price take the same grid as above. Note that the log-price is, in fact, not given as the solution to an equation, but just an Itô integral. We can write

$$\begin{aligned}
X_{t_j} &= X_0 - \int_0^{t_j} \frac{1}{2}|V_s|\,ds + \int_0^{t_j} \sqrt{|V_s|}(\bar{\rho}\,dW_s + \rho\,dB_s) \\
&= X_0 + \sum_{k=0}^{j-1} \left(\int_{t_k}^{t_{k+1}} -\frac{1}{2}|V_s|\,ds + \int_{t_k}^{t_{k+1}} \sqrt{|V_s|}(\bar{\rho}\,dW_s + \rho\,dB_s) \right) \\
&\approx X_0 + \sum_{k=0}^{j-1} \left(-\frac{1}{2}|V_{t_k}|(t_{k+1} - t_k) + \sqrt{|V_{t_k}|}\left(\bar{\rho}(W_{t_{k+1}} - W_{t_k}) + \rho(B_{t_{k+1}} - B_{t_k})\right) \right) \\
&= X_0 + \sum_{k=0}^{j-1} \left[-\frac{1}{2}|V_{t_k}|h + \sqrt{|V_{t_k}|}\left(\bar{\rho}\Delta_h W + \rho\Delta_h B\right) \right] \\
&= X_0 + \sum_{k=0}^{j-1} \left[-\frac{1}{2}|V_{t_k}|h + \sqrt{|V_{t_k}|}\sqrt{h}\left(\bar{\rho}G_k + \rho\widetilde{G}_k\right) \right],
\end{aligned}$$

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where $G_k, \widetilde{G}_k \sim \mathcal{N}(0,1)$ i.i.d. for $k = 0, \ldots, j-1$. Performing everything columnwise to resolve the different parts, we write for $j \geq 1$

$$X_{t_j}^{(i)} = X_0 + \sum_{k=0}^{j-1} \left[-\frac{1}{2} |V_{t_k}^{(i)}| h + \sqrt{|V_{t_k}^{(i)}|} \sqrt{h} \left(\bar{\rho} G_k^{(i)} + \rho \widetilde{G}_k^{(i)} \right) \right], \tag{4.4.8}$$

where the Gaussian matrix G was used to simulate the driving Brownian motion of the process V. The log-price does not have an integral kernel and it is not an actual equation. Therefore, it is possible to compute (4.4.8) directly with matrix arithmetics without any loop. In the following, we derive the exact computation. To get a proper representation of the sum as matrix multiplication define for $j \geq 1$ the $N \times N$ matrix

$$\tilde{A}_{j,k} := \delta(k \leq j) := \begin{cases} 1, & k \leq j, \\ 0, & \text{else}. \end{cases} \tag{4.4.9}$$

Hence, we can write

$$\begin{aligned}
X_{t_j}^{(i)} &= X_0 + \sum_{k=0}^{j-1} \left[-\frac{1}{2} |V_{t_k}^{(i)}| h + \sqrt{|V_{t_k}^{(i)}|} \sqrt{h} \left(\bar{\rho} G^{(i)_k} + \rho \widetilde{G}_k^{(i)} \right) \right] \\
&= X_0 - \frac{1}{2} \cdot h \cdot \left(|V_{t_0}^{(i)}| + |V_{t_1}^{(i)}| + \ldots + |V_{t_{j-1}}^{(i)}| \right) \\
&\quad + \sqrt{h} \cdot \left(\sqrt{|V_{t_0}^{(i)}|} \left(\bar{\rho} G_0^{(i)} + \rho \widetilde{G}_0^{(i)} \right) + \ldots + \sqrt{|V_{t_{j-1}}^{(i)}|} \left(\bar{\rho} G_{j-1}^{(i)} + \rho \widetilde{G}_{j-1}^{(i)} \right) \right) \\
&= X_0 - \frac{1}{2} \cdot h \cdot \tilde{A}_{j,(1:N)} \cdot \begin{pmatrix} |V_{t_0}^{(i)}| \\ |V_{t_1}^{(i)}| \\ \vdots \\ |V_{t_{N-1}}^{(i)}| \end{pmatrix} + \sqrt{h} \cdot \tilde{A}_{j,(1:N)} \cdot \begin{pmatrix} \sqrt{|V_{t_0}^{(i)}|} \left(\bar{\rho} G_0^{(i)} + \rho \widetilde{G}_0^{(i)} \right) \\ \sqrt{|V_{t_1}^{(i)}|} \left(\bar{\rho} G_1^{(i)} + \rho \widetilde{G}_1^{(i)} \right) \\ \vdots \\ \sqrt{|V_{t_{N-1}}^{(i)}|} \left(\bar{\rho} G_{N-1}^{(i)} + \rho \widetilde{G}_{N-1}^{(i)} \right) \end{pmatrix}.
\end{aligned}$$

For all time steps simultaneously we then get

$$\begin{pmatrix} X_{t_1}^{(i)} \\ X_{t_2}^{(i)} \\ \vdots \\ X_{t_N}^{(i)} \end{pmatrix} = X_0 - \frac{1}{2} \cdot h \cdot \tilde{A}_{(1:N),(1:N)} \cdot \begin{pmatrix} |V_{t_0}^{(i)}| \\ |V_{t_1}^{(i)}| \\ \vdots \\ |V_{t_{N-1}}^{(i)}| \end{pmatrix} + \sqrt{h} \cdot \tilde{A}_{(1:N),(1:N)} \cdot \begin{pmatrix} \sqrt{|V_{t_0}^{(i)}|} \left(\bar{\rho} G_0^{(i)} + \rho \widetilde{G}_0^{(i)} \right) \\ \sqrt{|V_{t_1}^{(i)}|} \left(\bar{\rho} G_1^{(i)} + \rho \widetilde{G}_1^{(i)} \right) \\ \vdots \\ \sqrt{|V_{t_{N-1}}^{(i)}|} \left(\bar{\rho} G_{N-1}^{(i)} + \rho \widetilde{G}_{N-1}^{(i)} \right) \end{pmatrix}.$$

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For all time steps and paths simultaneously we get

$$\begin{pmatrix} X_{t_1}^{(1)} & \cdots & X_{t_1}^{(m)} \\ X_{t_2}^{(1)} & \cdots & X_{t_2}^{(m)} \\ \vdots & \ddots & \vdots \\ X_{t_N}^{(1)} & \cdots & X_{t_N}^{(m)} \end{pmatrix} = X_0 - \frac{1}{2} \cdot h \cdot \tilde{A}_{(1:N),(1:N)} \cdot \begin{pmatrix} |V_{t_0}^{(1)}| & \cdots & |V_{t_0}^{(m)}| \\ |V_{t_1}^{(1)}| & \cdots & |V_{t_1}^{(m)}| \\ \vdots & \ddots & \vdots \\ |V_{t_{N-1}}^{(1)}| & \cdots & |V_{t_{N-1}}^{(m)}| \end{pmatrix}$$

$$+\sqrt{h} \cdot \tilde{A}_{(1:N),(1:N)} \cdot \begin{pmatrix} \sqrt{|V_{t_0}^{(1)}|}\left(\bar{\rho} G_0^{(1)} + \rho \widetilde{G}_0^{(1)}\right) & \cdots & \sqrt{|V_{t_0}^{(m)}|}\left(\bar{\rho} G_0^{(m)} + \rho \widetilde{G}_0^{(m)}\right) \\ \sqrt{|V_{t_1}^{(1)}|}\left(\bar{\rho} G_1^{(1)} + \rho \widetilde{G}_1^{(1)}\right) & \cdots & \sqrt{|V_{t_1}^{(m)}|}\left(\bar{\rho} G_1^{(m)} + \rho \widetilde{G}_1^{(m)}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{|V_{t_{N-1}}^{(1)}|}\left(\bar{\rho} G_{N-1}^{(1)} + \rho \widetilde{G}_{N-1}^{(1)}\right) & \cdots & \sqrt{|V_{t_{N-1}}^{(m)}|}\left(\bar{\rho} G_{N-1}^{(m)} + \rho \widetilde{G}_{N-1}^{(m)}\right) \end{pmatrix}.$$

Using these computations, we see that a simple matrix multiplication computes all values of the log-price at once.

Algorithm 4.4.3. *Having already computed the process V resp. the values $V_{t_j}^{(i)}$ for $j = 0, \dots, N$ and $i = 1, \dots m$, we can compute the log-price as follows:*

- *Take the Gaussian matrix G which was used to simulate the driving Brownian motion of the process V. Compute another Gaussian matrix $\widetilde{G}$ of the same dimension $(N+1) \times m$.*
- *Create an $(N+1) \times N$ matrix A, where*

$$A_{1,(1:N)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4.4.10}$$

and the quadratic submatrix $A_{(2:(N+1)),1:N}$ is a lower triangular matrix with ones, i.e. the whole matrix is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{(N+1)\times N}. \tag{4.4.11}$$

- *Resolving the sum in (4.4.8), the matrix multiplication, we have to perform is given by*

$$\left(X_{t_j}^{(i)}\right)_{j=0,\dots,N}^{i=1,\dots,m} = X_0 - \frac{1}{2} \cdot h \cdot A \cdot \left(|V_{t_j}^{(i)}|\right)_{j=0,\dots,N}^{i=1,\dots,m} + \sqrt{h} \cdot A \cdot \left(\sqrt{|V_{t_j}^{(i)}|} \cdot (\bar{\rho} \cdot G_j^{(i)} + \rho \widetilde{G}_j^{(i)})\right)_{j=0,\dots,N}^{i=1,\dots,m}. \tag{4.4.12}$$

Here, the indices indicate that the absolute value, the square root as well as the multiplication with the Gaussian paths are performed pointwise. The actual matrix multiplication is performed with A.

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Remark 4.4.4. Using a programming language that has a good syntax for pointwise operations, e.g. R, is convenient here. Using the dot for pointwise operations and $*$ for matrix multiplication the computation in (4.4.12) would look like

$$X\texttt{[0:N]} = X_0 - \frac{1}{2} \cdot h \cdot A\texttt{[0:N,1:N]} * |V\texttt{[1:N,1:m]}| \\ + \sqrt{h} \cdot A\texttt{[0:N,1:N]} * \left(\sqrt{|V\texttt{[1:N,1:m]}|} \cdot \left(\bar{\rho} \cdot G\texttt{[1:N,1:m]} + \rho \cdot \widetilde{G}\texttt{[1:N,1:m]} \right) \right),$$

where again the absolute value and the square root are taken as pointwise operations.

4.4.2. The source code

To conclude this part, we show an implementation example in R and some plots of the paths of the log-price and the process V resp. the instantaneous variance $|V|$. For those who want to reproduce the implementation, the source code is provided. For nice plots, we need the `ggplot` package, and we load the `tidyverse` package to have a convenient syntax for preparing the plots.

```
# libraries ########################################################

library(tidyverse)
```

Listing 4.1: Load libraries

We, for now arbitrarily, choose the model parameters.

```
# model parameters #################################################

t0 = 0
T = 1.3
V0 = 1.1
X0 = 1.3
rho = 0.3
rho_bar = sqrt(1 - rho^2)

kappa = 2.3
theta = 0.3
xi = 1.35
truncation_size = M = 10^13
```

Listing 4.2: Set model parameters

We initialize the grid for the simulation. Note that using `m_max` precomputes a Gaussian matrix that may be bigger than the number of paths we want. For a lower number of paths we just take a sub-matrix.

```
# simulation parameters ############################################

m_max = 20000
h = 0.001
N = T/h
t_grid = seq(t0, T, by = h)

# check grid
length(t_grid) == N + 1
```

Listing 4.3: Set simulation parameters

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Next, we draw the two Gaussian sample matrices G and $\widetilde{G}$.

```
# draw Gaussian sample as a matrix ####################################

G_1 = rnorm(m_max * N)
G_1 = matrix(G_1, nrow = N)

G_2 = rnorm(m_max * N)
G_2 = matrix(G_2, nrow = N)
```

Listing 4.4: Draw Gaussian sample

Next we implemented a vector valued version of the Riemann-Liouville fractional kernel given by (4.2.14).

```
# Riemann-Liouville kernel function definition
    #########################################

alpha = 1   # can be changed before function call

K_scal = function(t, s){
  if(min(s, t) < 0){
    stop("s and t must be >= 0!")
  }

  if(s >= t){
    res = 0
  }else{
    res = (t - s)^(alpha - 1) / gamma(alpha)
  }

  return(res)
}



K_vec = function(t, s){

  n_t = length(t)

  if(n_t != length(s)){
    stop("t and s must be of same length!")
  }

  res = rep(0, n_t)
  for(j in (1:n_t)){
    res[j] = K_scal(t[j], s[j])
  }

  return(res)
}

K = K_vec
```

Listing 4.5: Function definition Riemann-Liouville kernel

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Next, we defined the coefficient functions $\bar{b}$ and $\bar{\sigma}$, again implemented as a vector-valued version.

```r
# integrand coefficient functions ############################################

# REMARK: The trunacation size can be changed before the function call

sigma_bar = function(x) {
  res = rep(0, length(x))

  index_ok = which(abs(x) <= truncation_size)
  index_modified =  which(abs(x) > truncation_size)

  res[index_ok] = xi * abs(x[index_ok])^(3/2)
  res[index_modified] = xi * rep(truncation_size, length(index_modified))^(3/2)

  return(res)
}

b_bar = function(x) {
  res = rep(0, length(x))

  index_ok = which(abs(x) <= truncation_size)
  index_modified_plus = which(x > truncation_size)
  index_modified_minus = which(x < -truncation_size)

  res[index_ok] = kappa * x[index_ok] * (theta - x[index_ok])
  res[index_modified_plus] = kappa * rep(truncation_size, length(index_modified_plus))
      * (theta - rep(truncation_size, length(index_modified_plus)))
  res[index_modified_minus] = kappa * (-1) * rep(truncation_size, length(index_modified
      _minus)) * (theta + rep(truncation_size, length(index_modified_minus)))

  return(res)
}


# Check truncation
plot(sigma_bar, xlim = c(-truncation_size * 10, truncation_size * 10))
plot(b_bar, xlim = c(-truncation_size * 10, truncation_size * 10))
```

Listing 4.6: Coefficient functions $\bar{b}$ and $\bar{\sigma}$

Now, we are able to simulate the paths of V using Algorithm 4.4.2.

```r
# Simulation process V ############################################

alpha = 0.6

n_sim = 2000
n_sim <= m_max

V = matrix(NA, nrow = N+1, ncol = n_sim)
V[1, ] = rep(V0, n_sim)

D = NULL
for(j in 0:(N-1)){

  j_tilde = j + 1

  # kernel
  kernel_vec = K(rep(t_grid[(j_tilde + 1)], times = j_tilde), t_grid[1:j_tilde])
  vec_1 = kernel_vec

  # D
```


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```
  D = rbind(D, b_bar(V[j_tilde, ]) * h + sigma_bar(V[j_tilde, ]) * sqrt(h) * G_1[j_tilde, (1:n_sim)])
  vec_2 = D

  # Matrix multiplication for V_{t_{j + 1}} = V[j_tilde + 1, ]
  V[(j_tilde + 1), ] = V[1, ] + vec_1 %*% vec_2

}
```

Listing 4.7: Simulation of the process V

Next, we use a small script using `ggplot()` to sample some paths of the instantaneous variance $|V|$. Note that it is just an example, and we will give plots for different $\alpha \in (\frac{1}{2}, 1)$ below.

```
# Plot sample paths of the instantaneous variance |V| ############################################

choice = sample(1:n_sim, size = 3)

df = NULL
for(j in (1:length(choice))){

  df = rbind(df, tibble(t = t_grid, V = abs(V[, choice[j]]), path = as.factor(choice[j])))

}

ggplot(df, aes(x = t, y = V, col = path)) +
  geom_line() +
  geom_hline(yintercept = 0, col = "blue") +
  geom_hline(yintercept = 5, col = "red")
```

Listing 4.8: Plot random choice of paths of the instantaneous variance $|V|$

For the simulation of X we use Algorithm 4.4.3.

```
# Simulation log-price X ############################################

# The process V is assumed to be already simulated
# The driving Brownian motion of V has been drawn using G_1
# --> G_1 is multiplied by rho_bar and G_2  by rho

m = n_sim

tmp_mat = matrix(1, N, N)
ones_triangular = lower.tri(tmp_mat, diag = TRUE) * tmp_mat
A = rbind(rep(0, times = N), ones_triangular)

X = X0 - 1/2 * h * A %*% abs(V[1:N, ]) + sqrt(h) * A %*% (  sqrt(abs(V[1:N, ])) * (rho_bar * G_1[1:N, 1:m] + rho * G_2[1:N, 1:m])  )

# Check result
all.equal(dim(X), c(N+1, m))
```

Listing 4.9: Simulation of the log-price X

An example of plotting the log-price works the same way as in Listing 4.8.

```
# Plot sample paths of the log-price X ############################################

choice = sample(1:n_sim, size = 3)

df = NULL
```


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```
for(j in (1:length(choice))){

  df = rbind(df, tibble(t = t_grid, X= X[, choice[j]], path = as.factor(choice[j])))

}

ggplot(df, aes(x = t, y = X, col = path)) +
  geom_line() +
  geom_hline(yintercept = 0, col = "blue") +
  geom_hline(yintercept = 5, col = "red")
```

Listing 4.10: Plot random choice of paths of the log-price X

4.4.3. The plots

In this section, we shortly discuss the numeric results. The model parameters are chosen as in Listings 4.2 and 4.3, i.e.

$$\begin{aligned} t &\in [0, 1.3], \\ v_0 &= 1.1, \\ x_0 &= 1.3, \\ \rho &= 0.3, \\ \bar{\rho} &= \sqrt{1-\rho^2}, \\ \kappa &= 2.3, \\ \theta &= 0.3, \\ \xi &= 1.35, \\ M &= 10^{13} \end{aligned}$$

and

$$\begin{aligned} h &= 0.001, \\ N &= \frac{T}{h}. \end{aligned}$$

First, in Figure 4.1–4.5 we plot paths for different values of α. Here, α denotes the parameter of the Riemann-Liouville kernel, i.e. $\nu = \alpha$ in the definition of (4.2.14). We see that the paths get smoother as α tends to one which corresponds to the case of the classic 3/2-model. In fact, the solution to our equation (4.1.8) cannot explode, because we truncated the coefficient functions in the definition of $\bar{b}$ and $\bar{\sigma}$, see (4.1.11) and (4.1.12). Despite this truncation, which allows us to mathematically handle the model, there are also paths that tend to the boundary created by the truncation. This suggests that if there was no truncation the solution would explode. Paths like this can bee seen in Figure 4.1 and 4.6. The latter one uses $\alpha = 0.6$, where we have already seen "good" paths in Figure 4.2. Note that in the classic 3/2-model there are no explosions due to the Feller condition. We observe, that for one simulation of the Gaussian matrices $G, \widetilde{G}$ resp. G_1 and G_2 we get a threshold α_0, such that there seem to be no (pseudo-)explosions for $\alpha > 0$ in the truncated model and hence in the non-truncated one, see Figure 4.7–4.8. Interesting would be if it is possible to find a threshold that is independent of

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the Gaussian sample matrices and is somehow connected to the choice of the model parameters. This is postponed for future work.

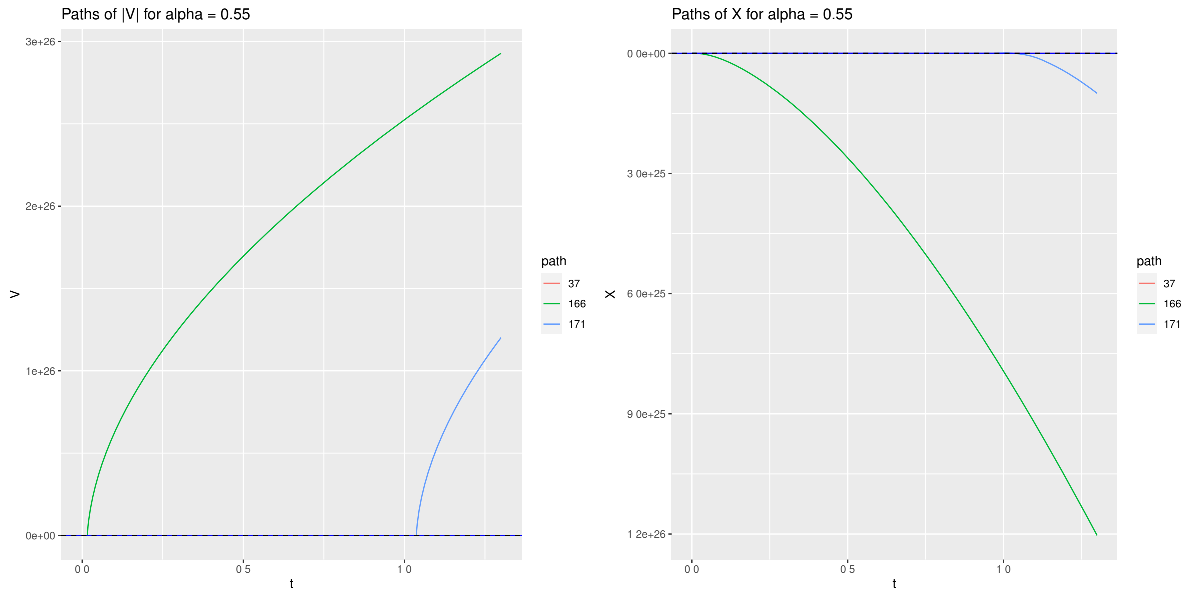

Figure 4.1.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.55$.

Figure 4.2.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.6$.

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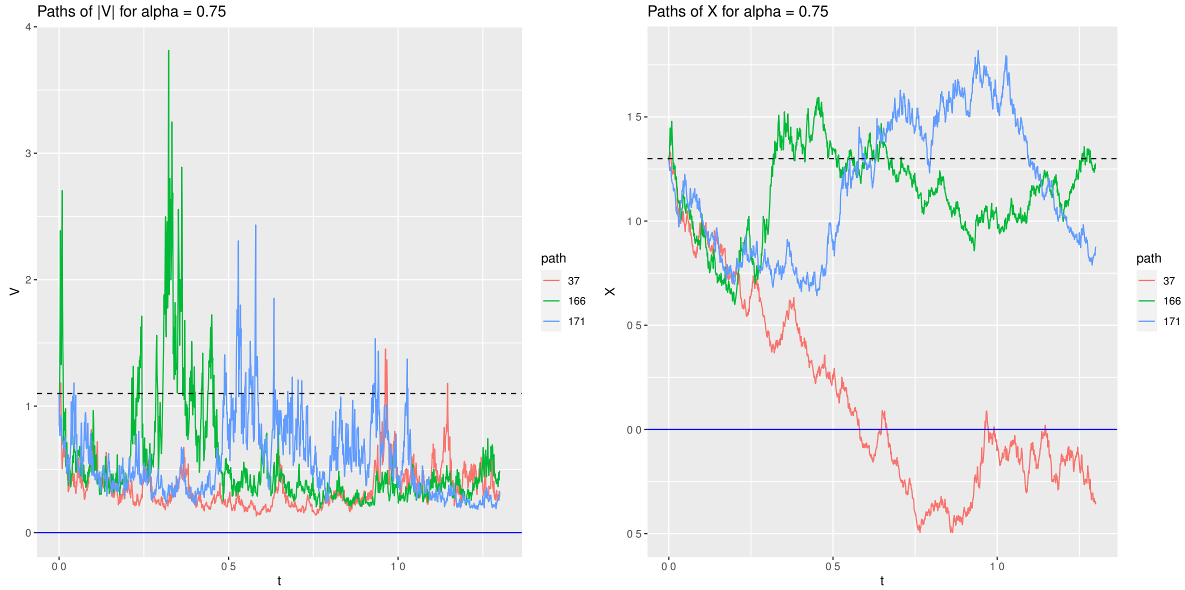

Figure 4.3.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.75$.

Figure 4.4.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.9$.

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Figure 4.5.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 1$.

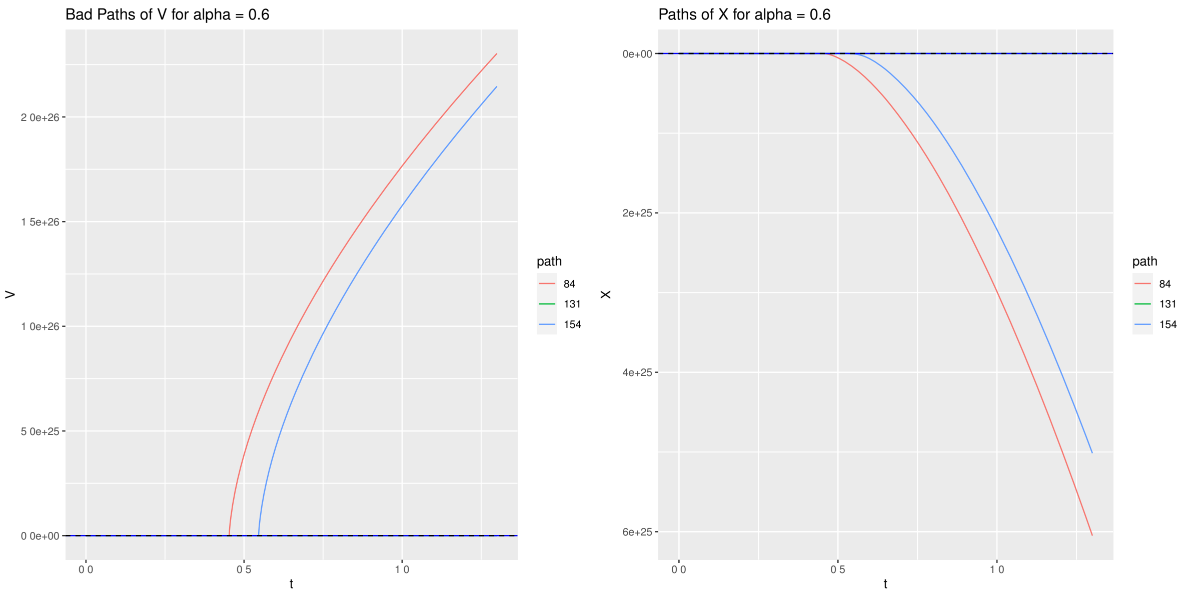

Figure 4.6.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.6$ with paths that do not behave well.

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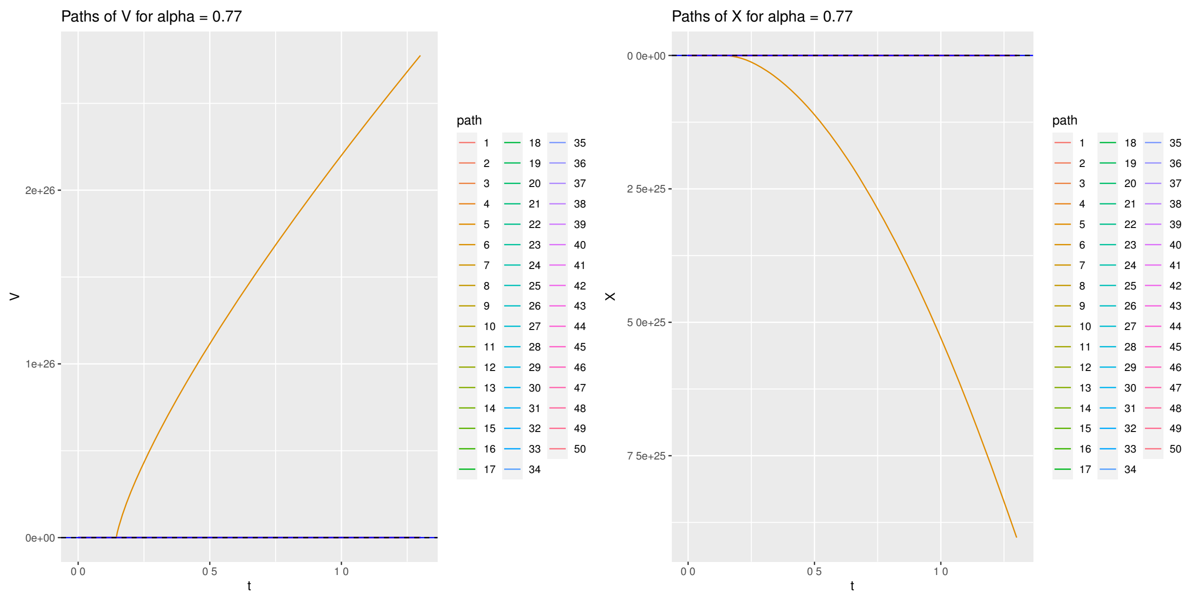

Figure 4.7.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.77$ with a large number of paths plotted.

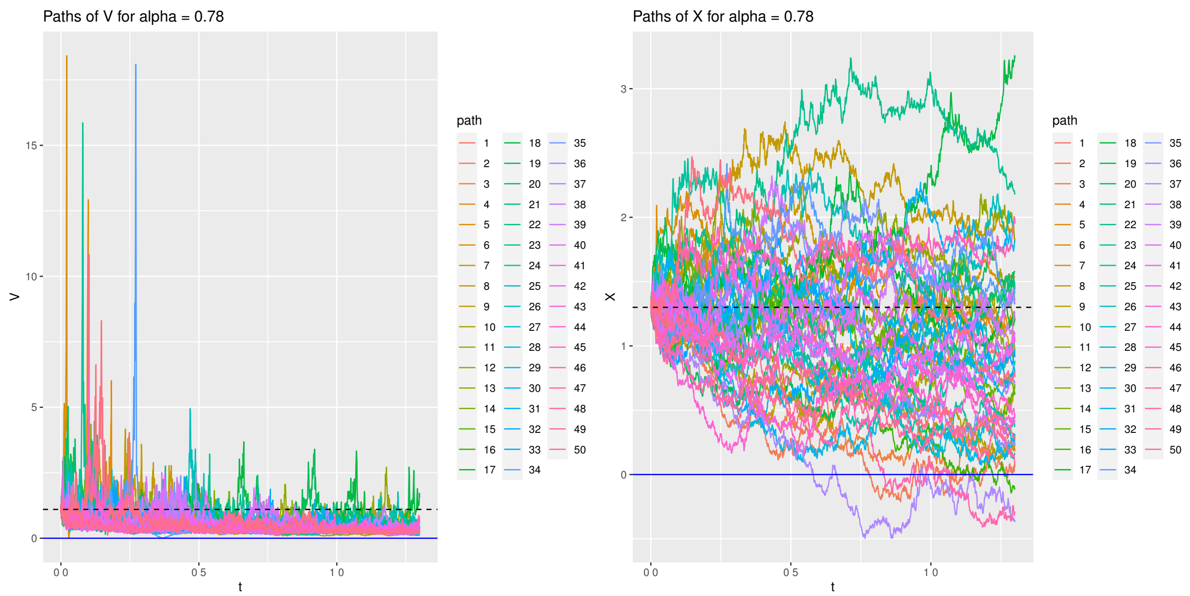

Figure 4.8.: Simulation of $|V|$ and X with Riemann Liouville parameter $\alpha = 0.78$ with a large number of paths plotted.

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A. Asymptotics

In this part we recall some notation for asymptotic results. The formulations are directly taken from [Ger18].

Definition A.0.1 ([FS09]). Let $\mathbb{S}$ be a set and $s_0 \in \mathbb{S}$. We assume a notion of neighbourhood to exist on $\mathbb{S}$, such that $s_0 \in \overline{\mathbb{S}}$ is possible, e.g. $\mathbb{S} = \mathbb{R}$ and $s_0 = +\infty$. Two functions $f, g : \mathbb{S}\backslash\{s_0\} \to \mathbb{R}$ ($\mathbb{C}$) are given.

(i) Write

$$f(s) = o(g(s)), \quad s \to s_0, \tag{A.0.1}$$

if

$$\lim_{s \to s_0} \left| \frac{f(s)}{g(s)} \right| = 0. \tag{A.0.2}$$

In other words, for any (arbitrary small) $\varepsilon > 0$, there exists a neighbourhood $\mathcal{V}_\varepsilon$ of s_0 (depending on ε), such that

$$|f(s)| \leq \varepsilon |g(s)|, \quad s \in \mathcal{V}_\varepsilon, \quad s \neq s_0.$$

We say *"f is of order smaller than g"*, or *"f is little-oh of g"*, or *"f is asymptotically dominated by g"* (as s tends to s_0).

(ii) Write

$$f(s) = \mathcal{O}(g(s)), \quad s \to s_0, \tag{A.0.3}$$

if

$$\limsup_{s \to s_0} \left| \frac{f(s)}{g(s)} \right| < \infty. \tag{A.0.4}$$

In other words, there exists a neighborhood $\mathcal{V}$ of s_0 and a constant $C > 0$ such that

$$|f(s)| \leq C|g(s)|, \quad s \in \mathcal{V}, \quad s \neq s_0.$$

One also says that *"f is of order at most g"*, or *"f is big–Oh of g"*, or *"f is bounded from above by g (up to a constant factor) asymptotically"* (as s tends to s_0).

(iii) Write

$$f(s) \sim g(s), \quad s \to s_0, \tag{A.0.5}$$

if

$$\lim_{s \to s_0} \frac{f(s)}{g(s)} = 1. \tag{A.0.6}$$

One also says that *"f and g are asymptotically equivalent"* (as s tends to s_0).

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Remark A.0.2. Note that very often the term $s \to s_0$ is omitted, since it can mostly be identified by the context.

Remark A.0.3. Note that $f(s) = \mathcal{O}(1)$ resp. $f \in \mathcal{O}(1)$ (for $s \to s_0$) means that f is bounded for $s \to s_0$, since this is equivalent to

$$\limsup_{s\to s_0} \frac{|f(s)|}{1} < \infty.$$

Lemma A.0.4. *Some properties of the Landau-Notation are the following:*

(i) ***Product**: For $f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$ we have $f_1 f_2 = \mathcal{O}(g_1 g_2)$. Especially we have $f\mathcal{O}(g) = \mathcal{O}(fg)$.*

(ii) ***Sum**: For $f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$ we have $f_1 + f_2 = \mathcal{O}(|g_1| + |g_2|)$. This implies that for $f_1, f_2 \in \mathcal{O}(g)$ we get $f_1 + f_2 \in \mathcal{O}(g)$ which means that $\mathcal{O}(g)$ is a convex cone. If f and g are positive functions, we get $\mathcal{O}(f) + \mathcal{O}(g) = \mathcal{O}(f + g)$.*

(iii) ***Constant multiplication**: Let k be a constant. Then we have $\mathcal{O}(kg) = \mathcal{O}(g)$ if k is nonzero. From $f = \mathcal{O}(g)$ we get $kf = \mathcal{O}(g)$.*

Proof. The properties above can be computed straight forward using the definition of the big-$\mathcal{O}$-notation.

Ad (i). Here we get with suppression of the argument for ease of notation

$$\limsup_{s\to s_0} \left|\frac{f_1 f_2}{g_1 g_2}\right| \leq \limsup_{s\to s_0} \left(\left|\frac{f_1}{g_1}\right| \left|\frac{f_1}{g_1}\right|\right) \leq \left(\limsup_{s\to s_0} \left|\frac{f_1}{g_1}\right|\right) \left(\limsup_{s\to s_0} \left|\frac{f_2}{g_2}\right|\right) < \infty.$$

Ad (ii). We have for $|f_1/g_1|, |f_2/g_2| < C$ that

$$\frac{|f_1 + f_2|}{|g_1| + |g_2|} \leq \frac{|f_1|}{|g_1| + |g_2|} + \frac{|f_2|}{|g_1| + |g_2|} \leq \frac{|f_1|}{|g_1|} + \frac{|f_2|}{|g_2|} < 2C$$

which leads to $f_1 + f_2 = \mathcal{O}(|g_1| + |g_2|)$.

Ad (iii). For $|f/g| < C$ we get

$$\left|\frac{kf}{g}\right| < |k|C.$$

$\square$

Remark A.0.5. Note that the notation can be combined with other arithmetic operators, e.g. $g(s) = h(s) + \mathcal{O}(f(s))$ expresses the same as $g(s) - h(s) = \mathcal{O}(f(s))$.

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B. Inequalities

Proposition B.0.1. *The square root function is sub-additive on* $\mathbb{R}_+$ *in the sense that*

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \tag{B.0.1}$$

for every $x, y \geq 0$. *Actually this holds for fractional powers, i.e.*

$$(a+b)^\delta \leq a^\delta + b^\delta \tag{B.0.2}$$

for $a, b > 0$ *and* $\delta \in (0,1)$.

Proposition B.0.2. *For* $p \geq 1$ *and* $a, b \in \mathbb{R}$, *we have*

$$(a+b)^p \leq 2^{p-1}(|a|^p + |b|^p). \tag{B.0.3}$$

Proposition B.0.3. *The power function with fractional power, i.e. exponent in* $(0,1)$, *is Hölder continuous with constant* 1 *and the Hölder exponent is the same as the given fractional exponent. Hence, for* $x, y > 0$ *and* $\alpha \in (0,1)$ *we have*

$$|x^\alpha - y^\alpha| \leq |x-y|^\alpha. \tag{B.0.4}$$

Corollary B.0.4. Taking $\alpha = \frac{1}{2}$ in Proposition B.0.3, we get for any $x, y > 0$ that

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}. \tag{B.0.5}$$

Lemma B.0.5 (Lemma A.1 [Zha08]). *For any* $\alpha \in (0,1)$, *we have*

$$\int_0^t \frac{(t'-t)^\alpha}{(t-s)^\alpha (t'-s)^\alpha}\, ds \leq C(t'-t)^{\alpha \wedge (\frac{1-\alpha}{2})}, \tag{B.0.6}$$

for all $0 < t < t' \leq 1$.

Proof. The proof can be found on p. 2244 [Zha08]. By substitution, we have

$$\int_0^t \frac{(t'-t)^\alpha}{(t-s)^\alpha (t'-s)^\alpha}\, ds = (t'-t)^{1-\alpha} \int_0^{\frac{t}{t'-t}} \frac{1}{u^\alpha (1+u)^\alpha}\, du. \tag{B.0.7}$$

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If $\alpha \in (0, \frac{1}{2})$, the right-hand side of (B.0.7) is less than

$$(t'-t)^{1-\alpha} \int_0^{\frac{t}{t'-t}} \frac{1}{u^{2\alpha}} \leq C(t'-t)^{\alpha}.$$

If $\alpha \in [\frac{1}{2}, 1)$, by Young's inequality, the right-hand side of (B.0.7) is less than

$$\begin{aligned}
&(t'-t)^{1-\alpha} \int_0^{\frac{t}{t'-t}} \left[\frac{2\alpha}{(\alpha+1)u^{\frac{\alpha+1}{2}}} + \frac{1-\alpha}{(1+\alpha)(1+u)^{\frac{\alpha(1+\alpha)}{1-\alpha}}} \right] du \\
&= \frac{4\alpha t^{\frac{1-\alpha}{2}} (t'-t)^{\frac{1-\alpha}{2}}}{1-\alpha^2} + \frac{(t'-t)^{1-\alpha}}{(1+\alpha)(\alpha^2+2\alpha-1)} \left[1 - \left(\frac{t}{t'-t} \right)^{\frac{1-2\alpha-\alpha^2}{1-\alpha}} \right] \\
&\leq C(t'-t)^{\frac{1-\alpha}{2}} + C(t'-t)^{1-\alpha} \\
&\leq C(t'-t)^{\frac{1-\alpha}{2}}.
\end{aligned}$$

$\square$

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C. Computations for the Riemann-Liouville kernel

Proposition C.0.1. *The Riemann-Liouville fractional kernel given by* (4.2.14) *satisfies Assumption 4.2.1.*

Proof. Let $\nu \in (\frac{1}{2}, 1)$. For the sake of simplicity, we omit the normalizing constant $\Gamma(\nu)$ and write

$$K_\nu(t,s) = K_\nu(t-s) := (t-s)^{\nu-1}. \tag{C.0.1}$$

In the following, we have to prove (4.2.10), (4.2.11) and (4.2.13).

Assumption (4.2.10) The function f given by

$$f : \begin{cases} (0,t) \to \mathbb{R}, \\ s \mapsto K_\nu(t,s) = \frac{1}{(t-s)^{1-\nu}}, \end{cases} \tag{C.0.2}$$

is continuous clearly continuous.

Assumption (4.2.11) We have

$$\int_0^t K_\nu(t,s)^{\tilde{\alpha}}\, ds = \int_0^t (t-s)^{(\nu-1)\tilde{\alpha}}\, ds, \tag{C.0.3}$$

which is integrable if and only if

$$(\nu - 1)\tilde{\alpha} > -1.$$

This is equivalent to $\tilde{\alpha} < \frac{1}{1-\nu}$, because $\nu - 1$ is negative for $\nu \in \left(\frac{1}{2}, 1\right)$. Hence, we need

$$\tilde{\alpha} \in \left(1, \frac{1}{1-\nu}\right).$$

For $\nu \in \left(\frac{1}{2}, 1\right)$ we get that

$$\frac{1}{1-\nu} \in (2, \infty).$$

Hence, we can find an $\tilde{\alpha}$ such that

$$\tilde{\alpha} \in \left(2, \frac{1}{1-\nu}\right), \tag{C.0.4}$$

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so $\tilde{\alpha} > 2$. Using $\kappa := (\nu - 1)\tilde{\alpha} \in (-1, 0)$ we get

$$\int_0^t K_\nu(t,s)^{\tilde{\alpha}}\, ds = \int_0^t (t-s)^\kappa\, ds = \int_0^t u^\kappa\, ds = \frac{t^{\kappa+1}}{\kappa+1} \leq \frac{T^{\kappa+1}}{\kappa+1}.$$

Hence, we get a uniform bound for the integral.

Assumption (4.2.13) We have

$$M_{K_\nu}(h) := \sup_{\{t_1,t_2\in[0,T]:\,|t_1-t_2|\leq h\}} \int_0^T |K_\nu(t_1,s) - K_\nu(t_2,s)|^2\, ds. \tag{C.0.5}$$

Using $t_1 < t_2$, we can write

$$\begin{aligned}
\int_0^{t_1} |K_\nu(t_1,s) - K_\nu(t_2,s)|^2\, ds &= \int_0^{t_1} |K_\nu(t_1-s) - K_\nu(t_2-s)|^2\, ds \\
&= \int_0^{t_1} |(t_1-s)^{\nu-1} - (t_2-s)^{\nu-1}|^2\, ds \\
&= \int_0^{t_1} \left| \frac{1}{(t_1-s)^{1-\nu}} - \frac{1}{(t_2-s)^{1-\nu}} \right|^2 ds \\
&= \int_0^{t_1} \left| \left(\frac{1}{t_1-s}\right)^{1-\nu} - \left(\frac{1}{t_2-s}\right)^{1-\nu} \right|^2 ds \\
&\overset{(B.0.4)}{\leq} \int_0^{t_1} \left| \left(\frac{1}{t_1-s}\right) - \left(\frac{1}{t_2-s}\right) \right|^{2(1-\nu)} ds \\
&= \int_0^{t_1} \left| \left(\frac{(t_2-s)-(t_1-s)}{(t_1-s)(t_2-s)}\right) \right|^{2(1-\nu)} ds \\
&= \int_0^{t_1} \left| \left(\frac{t_2-t_1}{(t_1-s)(t_2-s)}\right) \right|^{2(1-\nu)} ds \\
&= \int_0^{t} \frac{(t_2-t_1)^{2(1-\nu)}}{(t_1-s)^{2(1-\nu)}(t_2-s)^{2(1-\nu)}}\, ds.
\end{aligned}$$

Now, using $\gamma := 2(1-\nu) \in (0,1)$ in Lemma B.0.5 we can write

$$\int_0^t \frac{(t_2-t_1)^{2(1-\nu)}}{(t_1-s)^{2(1-\nu)}(t_2-s)^{2(1-\nu)}}\, ds \leq C(t_2-t_1)^{\gamma\wedge\frac{1-\gamma}{2}} = C(t_2-t_1)^{2(1-\nu)\wedge(\nu-\frac{1}{2})}.$$

Next, we need to estimate

$$\begin{aligned}
\int_{t_1}^{t_2} K_\nu(t_2,s)^2\, ds &\leq \left(\int_{t_1}^{t_2} K_\nu(t_2,s)^{2\beta}\, ds\right)^{\frac{1}{\beta}} \left(\int_{t_1}^{t_2} 1\, ds\right)^{\frac{1}{\beta^*}} \\
&\leq C(t_2-t_1)^{\frac{1}{\beta^*}}
\end{aligned}$$

using Hölder inequality. The Hölder exponent $\beta > 1$ is chosen such that $1 < 2\beta < \tilde{\alpha}$, which is possible, because of (C.0.4). The conjugated Hölder exponent then is $\beta^* := \frac{\beta}{\beta-1}$. Putting

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everything together we arrive at

$$\begin{aligned}\int_0^T |K_\nu(t_1,s)-K_\nu(t_2,s)|^2\,ds &= \int_0^{t_1} |K_\nu(t_1,s)-K_\nu(t_2,s)|^2\,ds + \int_{t_1}^{t_2} |K_\nu(t_2,s)|^2\,ds + \int_{t_2}^T 0\,ds\\ &\leq C_1(t_2-t_1)^{2(1-\nu)\wedge(\nu-\frac{1}{2})} + C_2(t_2-t_1)^{\frac{1}{\beta^*}}\\ &\leq C(t_2-t_1)^r,\end{aligned}$$

where

$$r := 2(1-\nu)\wedge\left(\nu-\frac{1}{2}\right)\wedge\frac{1}{\beta^*}. \tag{C.0.6}$$

Finally, we arrive at

$$M_{K_\nu}(h) \leq Ch^r, \tag{C.0.7}$$

for some $r>0$ resp. r given in (C.0.6). $\square$

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D. Regularity requirements

D.1. A short note on Lipschitz continuity

For this section, we use for a continuous function $f : D \subset \mathbb{R} \to \mathbb{R}$ the convention

$$\|f\|_\infty := \sup_{x \in D} |f(x)|. \tag{D.1.1}$$

Definition D.1.1. A function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}$ is called *Lipschitz continuous* if for all x, $y \in D$

$$|f(x) - f(y)| \leq L \cdot |x - y|. \tag{D.1.2}$$

Remark D.1.2. The following (rather obvious) observations should be mentioned, because we want to properly get the Lipschitz constant in our application.

(i) If a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is smooth, it is Lipschitz continuous if and only if the first derivative is bounded on D.

(ii) If a continuous function $f : D \subset \mathbb{R} \to \mathbb{R}$ is piecewise smooth, it is Lipschitz continuous if and only if the first derivative is bounded on any compact interval containing only points of smoothness. In this case, the Lipschitz constant is the maximum of all Lipschitz constants of compact intervals containing only points of smoothness.

To see this, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, that is smooth on $|x| \leq M$ and constant on $|x| > M$, i.e.

$$f(x) = \begin{cases} \tilde{f}(-M), & x < -M \\ \tilde{f}(x), & x \in [-M, M], \\ \tilde{f}(M), & x > M. \end{cases} \tag{D.1.3}$$

Here, $\tilde{f} : [-M, M] \to \mathbb{R}$ is a smooth Lipschitz-continuous function with Lipschitz constant $L_{\tilde{f}}$. Note that the function f is bounded. In the following, we want to get a Lipschitz constant that works uniformly for the whole domain of f.

Case $|x - y| > 1$. We can write

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2 \cdot \|f\|_\infty \leq 2 \cdot \|f\|_\infty \cdot |x - y|.$$

Hence, we have

$$L_1 := 2 \cdot \|f\|_\infty.$$

Case $|x-y| \leq 1$. We let $M \geq 1$, because we use it as a very large boundary for the instantaneous variance in our model. Using W.L.O.G. $x < y$, we can distinguish between the following cases:

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- $x < y < -M$: We have $|f(x) - f(y)| = |\tilde{f}(-M) - \tilde{f}(-M)| = 0$.
- $x < -M < y$: We have
$$|f(x) - f(y)| = |\tilde{f}(-M) - \tilde{f}(y)| \leq L_{\tilde{f}} \cdot |(-M) - y| \leq L_{\tilde{f}} \cdot |x - y|.$$
- $-M < x < y < M$: We have Lipschitz continuity on $[-M, M]$, hence
$$|f(x) - f(y)| = |\tilde{f}(x) - \tilde{f}(y)| \leq L_{\tilde{f}} \cdot |x - y|.$$
- $x < M < y$: We have
$$|f(x) - f(y)| = |\tilde{f}(x) - \tilde{f}(M)| \leq L_{\tilde{f}} \cdot |x - M| \leq L_{\tilde{f}} \cdot |x - y|.$$
- $M < x < y$: We have $|f(x) - f(y)| = |\tilde{f}(M) - \tilde{f}(M)| = 0$.

Summing up, we can keep the original Lipschitz constant $L_{\tilde{f}}$ for the case $|x - y| \leq 1$.

Putting the cases together, we get that f is Lipschitz continuous on $(-\infty, \infty)$ with Lipschitz constant $L_f := \max\{L_{\tilde{f}}, 2 \cdot \|f\|_\infty\}$.

D.2. A smooth version of $\bar{b}$

In this section, we want to sketch how the coefficients function $\bar{b}$ of (4.1.8) can be W.L.O.G. replaced by a smooth version. Recall that we have

$$\bar{b}(x) = \begin{cases} \kappa(-M)(\theta + M), & x \in (-\infty, -M), \\ \kappa x(\theta - x), & x \in [-M, M], \\ \kappa M(\theta - M), & x \in (M, \infty). \end{cases} \tag{D.2.1}$$

A smooth version of $\bar{b}$ needs to be of the form

$$\tilde{b}(x) = \begin{cases} u(-M - \epsilon), & x \in (-\infty, -M - \epsilon) \\ u(x), & x \in [-M - \epsilon, -M] \\ \bar{b}(x), & x \in [-M, M], \\ g(x), & x \in [M, M + \epsilon], \\ g(M + \epsilon), & x \in (M + \epsilon, \infty). \end{cases} \tag{D.2.2}$$

Here, the corresponding functions need to match the function values and the derivatives for meeting the smooth pasting condition. We sketch the interpolation around M. First note that

$$\bar{b}(x) = -\kappa x^2 + \kappa\theta x, \tag{D.2.3}$$
$$\partial_x \bar{b}(x) = -2\kappa x + \kappa\theta. \tag{D.2.4}$$

For g we need the conditions

$$\begin{aligned} g(M) &= \bar{b}(M), \\ g'(M) &= \bar{b}'(M), \\ g'(M + \epsilon) &= 0. \end{aligned}$$

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Hence, we have three conditions to match and can choose g to be a polynom, i.e.

$$\begin{aligned} g(x) &= ax^2 + bx + c, \\ g'(x) &= 2ax + b. \end{aligned}$$

The matching conditions lead to the equations

$$\begin{aligned} M^2 \cdot a + M \cdot b + c &= \kappa M(\theta - M), \\ 2M \cdot a + \quad b \quad &= \kappa(\theta - 2M), \\ 2(M+\epsilon) \cdot a + \quad b \quad &= 0, \end{aligned}$$

which can be written as linear system, i.e.

$$\begin{pmatrix} M^2 & M & 1 \\ 2M & 1 & 0 \\ 2(M+\epsilon) & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \kappa M(\theta - M) \\ \kappa(\theta - 2M) \\ 0 \end{pmatrix}. \tag{D.2.5}$$

The rank on the left-hand side is full and hence our interpolation function has a unique solution given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} M^2 & M & 1 \\ 2M & 1 & 0 \\ 2(M+\epsilon) & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \kappa M(\theta - M) \\ \kappa(\theta - 2M) \\ 0 \end{pmatrix}. \tag{D.2.6}$$

The same can be done around $-M$.

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