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ω-categorical sandwiches for Promise Constraint Satisfaction Problems

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Kurzfassung

Etwa zu der Zeit, als die Dichotomie-Vermutung für *Bedingungserfüllungsprobleme* (engl. Constraint Satisfaction Problems; CSPs) von Bulatov und Zhuk im Jahr 2017 unabhängig bewiesen wurde, zog eine Verallgemeinerung von CSPs, die als *Promise CSPs* (PCSPs) bekannt ist, wachsende Aufmerksamkeit auf sich und entwickelte sich seither zu einem bedeutenden Forschungsthema. Für gegebene endliche Strukturen $\mathbb{A}, \mathbb{B}$ der gleichen relationalen Signatur, so dass $\mathbb{A}$ homomorph auf $\mathbb{B}$ abbildet, kurz $\mathbb{A} \to \mathbb{B}$, beschreibt PCSP$(\mathbb{A}, \mathbb{B})$ die Frage, ob eine gegebene endliche Eingabestruktur homomorph nach $\mathbb{A}$ abbildet oder nicht einmal homomorph nach $\mathbb{B}$ abbildet. Ein vielfältig einsetzbarer Ansatz zur Untersuchung der Komplexität von PCSPs ist die *Sandwiching-Methode*. Wenn es eine Struktur $\mathbb{C}$ gibt, so dass $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$, dann reduziert sich PCSP$(\mathbb{A}, \mathbb{B})$ auf CSP$(\mathbb{C})$. In diesem Fall wird $\mathbb{C}$ eine *Sandwich-Struktur* für $(\mathbb{A}, \mathbb{B})$ genannt.

Barto zeigte, dass ein bestimmtes PCSP von endlichen Strukturen ein unendliches Sandwich mit in polynomieller Zeit lösbarem CSP zulässt, aber mit dieser Methode nicht auf das in Polynomialzeit lösbare CSP einer endlichen Struktur reduziert werden kann, es sei denn P = NP. Kazda, Mayr und Zhuk bewiesen, dass selbst wenn es ein endliches Sandwich mit in polynomieller Zeit lösbarem CSP gibt, die kleinste solche Struktur beliebig groß werden kann. Mottet zeigte, dass ein PCSP, welches in Logik erster Ordnung entscheidbar ist, immer auf das in Polynomialzeit lösbare CSP einer Sandwich-Struktur reduziert werden kann. In seinem Beweis konstruierte er zunächst ein ω*-kategorisches* Sandwich mit in polynomieller Zeit lösbarem CSP und zeigte durch ein Argument aus der *Ramsey*-Theorie, dass ein endlicher Faktor dieser Struktur ebenfalls ein Sandwich mit in polynomieller Zeit lösbarem CSP ist.

In dieser Arbeit wird die Frage untersucht, ob es ein PCSP von endlichen Strukturen gibt, das ein ω-kategorisches Sandwich mit in polynomieller Zeit lösbarem CSP zulässt, jedoch kein endliches Sandwich mit in Polynomialzeit lösbarem CSP hat, um das Gebiet der PCSPs mit Ergebnissen über ω-kategorische CSPs zu verbinden. Während diese Frage offen bleibt, werden wir ein Beispiel für ein Paar $(\mathbb{C}, \mathbb{B})$ aus einer ω-kategorischen Struktur $\mathbb{C}$ mit in polynomieller Zeit lösbarem CSP und einer endlichen Struktur $\mathbb{B}$ mit $\mathbb{C} \to \mathbb{B}$ geben, sodass das CSP jedes endlichen $\mathbb{B}'$ mit $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ NP-vollständig ist. Außerdem werden wir eine Methode beschreiben, um weitere Paare mit dieser Eigenschaft aus bestimmten *homogenen* Ramsey-Strukturen zu konstruieren. Die Konstruktion einer endlichen Struktur $\mathbb{A}$ mit $\mathbb{A} \to \mathbb{C}$, so dass jede endliche Struktur $\mathbb{C}'$ mit $\mathbb{A} \to \mathbb{C}' \to \mathbb{B}$ NP-vollständig ist, würde eine positive Antwort auf die obige Frage geben, angenommen P $\neq$ NP.

Abstract

Around the time when the Dichotomy Conjecture for *Constraint Satisfaction Problems* (CSPs) was independently proved by Bulatov and Zhuk in 2017, a generalisation of CSPs known as *Promise Constraint Satisfaction Problems* (PCSPs) attracted growing attention and has since developed into a significant topic of research. For given finite structures $\mathbb{A}, \mathbb{B}$ of the same relational signature such that $\mathbb{A}$ maps homomorphically to $\mathbb{B}$, denoted $\mathbb{A} \to \mathbb{B}$, $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ asks whether a given finite input structure maps homomorphically to $\mathbb{A}$ or does not even map homomorphically to $\mathbb{B}$. It is promised that all input structures satisfy one of these two cases. A powerful approach to investigating the computational complexity of PCSPs is the *sandwiching method*. If there is some structure $\mathbb{C}$ such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$, then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ reduces to $\mathrm{CSP}(\mathbb{C})$. In this case, $\mathbb{C}$ is called a *sandwich structure* for $(\mathbb{A}, \mathbb{B})$.

Barto showed that a specific finite-domain PCSP admits an infinite polynomial-time tractable sandwich, but does not reduce to a tractable finite-domain CSP via this method, unless $\mathrm{P} = \mathrm{NP}$. Kazda, Mayr, and Zhuk proved that even when there exists a finite tractable sandwich, the smallest such structure may become arbitrarily large. Mottet showed that if a PCSP is first-order definable, it can always be reduced to the CSP of a finite tractable sandwich. In his proof, he first constructed a tractable *ω-categorical* sandwich and used a *Ramsey*-type argument to prove that a finite factor of this structure is also a tractable sandwich.

In this thesis, we will investigate the question whether there is a finite-domain PCSP that admits a tractable ω-categorical sandwich, but no finite tractable one, aiming to connect the PCSP framework to results about ω-categorical CSPs. While this question remains open, we will provide an example of a pair $(\mathbb{C}, \mathbb{B})$ with a tractable ω-categorical structure $\mathbb{C}$ and a finite structure $\mathbb{B}$ with $\mathbb{C} \to \mathbb{B}$ such that the CSP of every finite $\mathbb{B}'$ with $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ is NP-complete. Moreover, we will describe a method to construct more pairs with this property from specific *homogeneous* Ramsey structures. Finding some finite structure $\mathbb{A}$ with $\mathbb{A} \to \mathbb{C}$ such that every finite structure $\mathbb{C}'$ with $\mathbb{A} \to \mathbb{C}' \to \mathbb{B}$ is NP-complete would yield a positive answer to the above question, assuming $\mathrm{P} \neq \mathrm{NP}$.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, im September 2024 [redacted]

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1 Introduction

For many years, computer scientists and mathematicians alike investigated computational problems in order to find the fastest (or at least sufficiently fast) algorithms for each problem. In many cases, algorithms are considered to be sufficiently fast if they can be executed in polynomial time with respect to the input size. This leads directly to one of the most famous problems in mathematics, the P vs NP problem. It is one of the millennium problems, seven well-known complex mathematical problems. The Clay Mathematics Institute selected those problems in the year 2000, promising to award the first correct solution to each problem with 1 million US dollars. The P vs NP problem poses the question whether every problem in NP admits a polynomial-time algorithm that solves the problem. NP is the class of decision problems for which, given a solution for an input, an algorithm can verify its correctness in polynomial time with respect to the size of the input. The P vs NP problem question could be solved by finding a polynomial-time algorithm to an NP-complete problem. An NP-complete problem is a problem that is in NP and every other problem in NP can be reduced to it in polynomial time. Those are the "hardest" problems in NP.
A well-known problem in NP is the *k-colouring problem*. Let $k \in \mathbb{N}$ be fixed. Given a graph $\mathbb{G} = (G, E)$, i.e. a set G and a binary, non-reflexive, symmetric relation E on G, we ask the question whether it is possible to colour the vertices with k colours in a way such that no two adjacent vertices are given the same colour. In terms of computational complexity, for $k \geq 3$, the k-colouring problem is a standard example of an NP-complete problem, whereas the question whether a graph is 2-colourable is the same as asking if the graph is bipartite. This can be solved in linear time.

Many computational problems can be formulated as *Constraint Satisfaction Problems* (CSPs). Let us introduce this notion by reformulating the k-colouring problem. Consider the complete graph $\mathbb{K}_k$ with k vertices. A *graph homomorphism* $h : \mathbb{G} \to \mathbb{H}$ is a function that preserves edges, i.e. $E(x, y) \Rightarrow E(h(x), h(y))$ for all $x, y \in G$. Given a graph $\mathbb{G}$, finding a homomorphism $h : \mathbb{G} \to \mathbb{K}_k$ is equivalent to finding a k-colouring of $\mathbb{G}$. Thus, the k-colouring problem can be formulated as follows.

k-colouring problem
INSTANCE: A finite graph $\mathbb{G}$
QUESTION: Is there a graph homomorphism $h : \mathbb{G} \to \mathbb{K}_k$?

This can be generalised by not only considering $\mathbb{K}_k$, but any graph $\mathbb{H}$. The question whether a given finite graph $\mathbb{G}$ can be mapped homomorphically to $\mathbb{H}$ is called the *$\mathbb{H}$-colouring problem*.

$\mathbb{H}$-colouring problem
INSTANCE: A finite graph $\mathbb{G}$
QUESTION: Is there a graph homomorphism $h : \mathbb{G} \to \mathbb{H}$?

The notion of a CSP further generalises this concept. Formally, consider a relational τ-structure $\mathbb{A}$. Given any finite τ-structure $\mathbb{X}$ (even though the domain of $\mathbb{A}$ could be infinite, we will always restrict the input to finite structures), we can ask whether there exists a homomorphism from $\mathbb{X}$ to $\mathbb{A}$. This problem is then called CSP($\mathbb{A}$). For relational structures $\mathbb{X}, \mathbb{A}$, the existence of a homomorphism $h : \mathbb{X} \to \mathbb{A}$ is denoted by $\mathbb{X} \to \mathbb{A}$.

Constraint Satisfaction Problem of $\mathbb{A}$
INSTANCE: A finite structure $\mathbb{X}$ in the same signature as $\mathbb{A}$
QUESTION: Is there a homomorphism $h : \mathbb{X} \to \mathbb{A}$?

Given a relational τ-structure $\mathbb{A}$, we can analyse the computational complexity of CSP($\mathbb{A}$). Note that for fixed finite $\mathbb{A}$, it can be checked in polynomial time whether a function from a finite structure $\mathbb{X}$ to $\mathbb{A}$ is a homomorphism. Thus, finite-domain CSPs are in NP. We can ask if there is an algorithm that can decide for any τ-structure $\mathbb{X}$ in polynomial time whether $\mathbb{X} \to \mathbb{A}$, i.e. if CSP($\mathbb{A}$) is in P. This has been studied extensively. One of the most influential articles in this area, a paper by Feder and Vardi, was published in 1993 [FV93; FV98]. For the first time, the *dichotomy conjecture* was formulated.

Conjecture 1.1 (Dichotomy Conjecture)**.** *Let $\mathbb{A}$ be a finite relational structure with finite signature. Then either* CSP($\mathbb{A}$) *is solvable in polynomial time, or it is* NP*-complete.*

In the late 1990s, it was discovered that methods from universal algebra offer valuable insights into the complexity classification of finite-domain CSPs [Jea98; JCG97]. These methods played a pivotal role in investigating the dichotomy conjecture, which Bulatov [Bul17] and Zhuk [Zhu17; Zhu20] independently proved it in 2017. A major milestone on the way to achieving this was the understanding of what properties make the CSP of a fixed relational structure $\mathbb{A}$ efficiently solvable or not. It was discovered that hardness comes from the lack of symmetry. However, the objects usually capturing symmetry, automorphisms and endomorphisms, do not provide enough information in this context. It turns out that operations called polymorphisms that can be thought of as multivariate endomorphisms determine the complexity of CSP($\mathbb{A}$) for finite $\mathbb{A}$. For $\mathbb{A} = (A; (R_i)_{i \in I})$ the set Pol($\mathbb{A}$) consisting of all multivariate operations on A that leave all relations R_i for $i \in I$ invariant is called the *polymorphism clone* of $\mathbb{A}$. Elements of Pol($\mathbb{A}$) are called *polymorphisms* of $\mathbb{A}$. Roughly, a rich polymorphism clone yields tractability (solvability in polynomial time), lack of symmetry, conversely, yields hardness. This is the core recognition of the algebraic approach to CSPs and makes it possible to answer questions about relational structures by investigating algebraic structures associated to them.
It is even sufficient to only investigate height-1 identities satisfied by these algebraic structures. *Identities* are equations for operations that hold for all choices of values for the variables. *Height-1* means that each side of the equations has exactly one occurrence of an operation symbol, in particular no nested operations are allowed. Satisfaction of certain

height-1 identities by polymorphisms characterises tractability, whereas the absence of polymorphisms satisfying those identities yields NP-completeness. A k-ary ($k \geq 2$) operation w is called a *weak near-unanimity* (WNU) operation if it satisfies the identities

$$w(x, \dots, x, y) = w(x, \dots, y, x) = \dots = w(y, x, \dots, x).$$

Zhuk proved the following refinement of Conjecture 1.1.

Theorem 1.2 ([Zhu17; Zhu20])**.** *Let $\mathbb{A}$ be a finite-domain relational structure. Then* CSP($\mathbb{A}$) *is tractable if $\mathbb{A}$ has a WNU polymorphism. Otherwise,* CSP($\mathbb{A}$) *is* NP*-complete.*

It was shown that having a WNU polymorphism is equivalent to having other types of polymorphisms, such as a Taylor polymorphism [MM08], a Siggers polymorphism ([Sig10]), and a cyclic polymorphism [BK12], a type of polymorphisms that will play a central role in this thesis.
Driven by the recent success in the CSP setting, it was suggested in [AGH17; BG16] to investigate a more general setting in which problems such as approximate graph colouring and similar ones can be studied. It is a form of approximation of CSPs. Remember that CSP($\mathbb{K}_k$) is the k-colouring problem. Instead of asking if a graph is k-colourable or not we might want to find an m-colouring for a given k-colourable graph and some $m > k$. This problem turns out to be very stubborn. It is believed to be NP-hard for all constants $3 \leq k \leq m$. But despite more than 45 years of studying this question, there was only limited progress. Even for $k = 3$, the best known NP-hardness result was obtained for $m = 5$ [Bar+21], while the best (in terms of m) polynomial-time algorithm for colouring a 3-colourable graph with n vertices uses about $n^{0.199}$ colours [KT17]. The gap between the best known negative and positive results shows just how far we are from understanding the full nature of this problem. The success in the CSP setting was the motivating factor in reformulating this question in the Promise Constraint Satisfaction Problem (PCSP) setting. Let $\mathbb{G}$ be a finite graph. Then the problem above is equivalent to finding a homomorphism $h' : \mathbb{G} \to \mathbb{K}_m$ under the condition that $\mathbb{G} \to \mathbb{K}_k$. This leads naturally to the following more general formulation: Let $\mathbb{A}, \mathbb{B}$ be two structures of the same signature τ such that $\mathbb{A} \to \mathbb{B}$. Then, given a τ-structure $\mathbb{X}$ with $\mathbb{X} \to \mathbb{A}$, find a homomorphism $h : \mathbb{X} \to \mathbb{B}$. Note that a homomorphism to $\mathbb{A}$ (and hence to $\mathbb{B}$) is promised to exist, but not given as an input. This is called the *search version* of the Promise CSP associated with the *template* ($\mathbb{A}, \mathbb{B}$). The condition that $\mathbb{A} \to \mathbb{B}$ is important for the question to make sense. The standard *decision version* of PCSP($\mathbb{A}, \mathbb{B}$) is the task of distinguishing τ-structures $\mathbb{X}$ with $\mathbb{X} \to \mathbb{A}$ from those satisfying $\mathbb{X} \nrightarrow \mathbb{B}$. The fact that $\mathbb{A} \to \mathbb{B}$ guarantees that the two cases are mutually exclusive. However, there could be some structures that do not belong to either case. We can either restrict the input to structures that belong to one of the two cases or allow the algorithm to do anything on all other inputs.

Promise Constraint Satisfaction Problem of ($\mathbb{A}, \mathbb{B}$)
INSTANCE: A structure $\mathbb{X}$ in the same signature as $\mathbb{A}$ and $\mathbb{B}$
QUESTION: Does $\mathbb{X}$ satisfy $\mathbb{X} \to \mathbb{A}$ or not even $\mathbb{X} \to \mathbb{B}$?

We see immediately that the PCSP framework generalises the CSP framework, since for $\mathbb{A} = \mathbb{B}$, we obtain CSP($\mathbb{A}$). Further, the decision version can always be reduced to the

search version by just running the algorithm for the search version, verifying the output and returning "No" if the algorithm fails or gives a wrong output. However, it is still an open question whether the two versions are equivalent for all templates in the sense that there is a polynomial-time reduction in both directions.
Polymorphisms of a finite structure fully determine the complexity of its CSP, hence, it is natural to try and find a similar statement in the PCSP setting. For two structures $\mathbb{A}, \mathbb{B}$ with the same signature, we call homomorphisms from a power of $\mathbb{A}$ to $\mathbb{B}$ *polymorphisms from $\mathbb{A}$ to $\mathbb{B}$* and denote the set of those functions by $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$. There are some methods known to use polymorphisms for investigating the complexity of PCSPs [KO22]:

- If the set of polymorphisms of a PCSP is rich enough, we can use them as a rounding procedure in polynomial-time algorithms allowing to solve a relaxed version of the PCSP and round the relaxed solution by applying appropriate polymorphisms.
- We can use polymorphisms to characterise the existence of specific types of reductions between PCSPs: One PCSP reduces to another by a specific type of reduction if and only if the polymorphisms of the corresponding PCSPs admit certain relations to one another.
- If the polymorphisms for a PCSP are sufficiently limited, we can use this fact directly to prove hardness for the PCSP.

The formulations here are very vague because those approaches will not be of essence in this thesis. We recommend [KO22] for a more thorough discussion and a general introduction to PCSPs.
There are several approaches to solve PCSPs in polynomial time (and characterise the templates solvable by those algorithms) such as local consistency algorithms or relaxing the PCSP to a Linear Programming instance. Another quite natural approach to solving a PCSP is to use the results of the CSP framework. Let $(\mathbb{A}, \mathbb{B})$ be a PCSP template and $\mathbb{C}$ be a relational structure such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$. Then we say that $\mathbb{C}$ is *sandwiched* by $\mathbb{A}$ and $\mathbb{B}$. Note that in this case we can reduce $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ to $\mathrm{CSP}(\mathbb{C})$: Restricting the input to structures either admitting a homomorphism to $\mathbb{A}$ or not even to $\mathbb{B}$, an input structure $\mathbb{X}$ is a yes-instance for $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ if and only if it is a yes-instance of $\mathrm{CSP}(\mathbb{C})$. While this approach is straight forward and might seem very simple, it is so far the most general approach of solving PCSPs via polynomial-time algorithms in the sense that currently, every known tractable PCSP can be reduced to a tractable CSP of a sandwich structure [Mot24; Den+21]. The following meta question arises naturally:

> Given finite $\mathbb{A}, \mathbb{B}$, does there exist some $\mathbb{C}$ sandwiched by $\mathbb{A}, \mathbb{B}$ such that $\mathrm{CSP}(\mathbb{C})$ is tractable?

If we find such a structure for some $\mathbb{A}, \mathbb{B}$, then clearly $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is tractable, whereas a negative answer does not necessarily mean that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is NP-hard. Anyways, it is not even known yet if the meta question is decidable.
For a PCSP template $(\mathbb{A}, \mathbb{B})$ with tractable sandwich $\mathbb{C}$, an interesting property of this PCSP is the minimal size of such a tractable sandwich. It was shown that for every $n > 1$ and prime p there is an example of a PCSP template $(\mathbb{A}, \mathbb{B})$ with $|A| = |B| = n$

admitting a tractable finite sandwich of size p but no smaller one [KMZ22]. So the size of minimal finite sandwiches can grow arbitrarily large even for PCSPs with structures of the same size. Since the complexity of finite-domain CSPs is well-understood, the best case would be to have a finite tractable sandwich. However, it was shown that even if there is a tractable sandwich, it is not guaranteed that there is a finite one as well. Let $\mathbb{A} := (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$ and $\mathbb{B} := (\{0,1\}; \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\})$. Then $\mathbb{A}$ simulates the problem *positive 1-in-3-SAT* (denoted 1-in-3) and $\mathbb{B}$ simulates *positive Not-All-Equal-3-SAT* (denoted NAE-3), which are further explained in Chapter 3. For the template $\mathbb{C} := (\mathbb{Z}; x+y+z=1)$, i.e. the structure with domain $\mathbb{Z}$ and a single ternary relation $R = \{(x,y,z) : x+y+z=1\}$, we have that

$$\text{1-in-3} \xrightarrow{f} (\mathbb{Z}; x+y+z=1) \xrightarrow{g} \text{NAE-3},$$

for the inclusion map f and $g(x) = 0 \Leftrightarrow x \leq 0$. Since $\mathbb{C}$ is known to be tractable [GLS88], it is a tractable sandwich. However, as we will see in Chapter 3, there can be no finite tractable one, unless P = NP.

Theorem 3.1 ([Bar19])**.** *Let* $\mathbb{C}$ *be a finite sandwich for* PCSP(1-in-3, NAE-3)*. Then* CSP($\mathbb{C}$) *is NP-complete.*

Infinite-domain CSPs have been investigated for the past 20 years, using the algebraic approach as well as model theory and Ramsey theory. Many methods and results from finite-domain CSPs were used to obtain results in the infinite-domain cases. However, examples like this suggest that obtaining complexity results for PCSPs requires transfer in the other direction, too. Moreover, the PCSP framework disclosed many interesting aspects not present in the study of CSPs, such as including many interesting and well-known computational problems not expressible as CSPs, revealing connections to many unexpected areas of mathematics such as algebraic topology [Kro+23], matrix analysis [CŽ23], and Boolean function analysis [BGS23], and also the discovery of new algorithms useful for the CSP setting (e.g. [BG21]).
In contrast to Theorem 3.1, there was also a positive result characterising a class of PCSPs that will always allow a finite tractable sandwich structure (Theorem 4.1.2). Atserias showed that first-order definability of CSP($\mathbb{A}$) for a τ-structure $\mathbb{A}$ is equivalent to $\mathbb{A}$ having finite duality [Ats08] (see Chapter 4). An even stronger result (Theorem 4.2.1) was obtained by Rossman [Ros08], which also directly implies $(i) \Leftrightarrow (ii)$ in the following theorem.

Theorem 4.1.2 ([Mot24])**.** *Let* $(\mathbb{A}, \mathbb{B})$ *be a finite PCSP template with finite signature* τ*. The following are equivalent:*

(i) PCSP($\mathbb{A}, \mathbb{B}$) *is first-order definable.*

(ii) $(\mathbb{A}, \mathbb{B})$ *has finite duality.*

(iii) There exists a finite structure $\mathbb{C}$ *with finite duality and such that* $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$.

Although Theorem 4.1.2 is a statement purely about finite structures, the proof of $(ii) \Rightarrow (iii)$ that Mottet provided is an example of applying results from infinite-domain CSPs to

a finite-domain problem. First, an infinite structure $\mathbb{C}$ with the properties of Item (*iii*) is constructed. Using a Ramsey-type argument, it is shown that this structure has a finite factor that also satisfies the required properties.
This approach relies on ω-categoricity of the infinite structure to guarantee finiteness of the factor structure. Considering this result, one might wonder if ω-categoricity itself is enough to "finitize" structures in the sense of the following question asked by Zhuk at the CSP World Congress 2023 in Weissensee, Austria:

Question 1.3. *Let* $(\mathbb{C}, \mathbb{B})$ *be a pair of* τ*-structures, where* $\mathbb{C}$ *is* ω*-categorical with tractable CSP,* $\mathbb{B}$ *is finite and* $\mathbb{C} \to \mathbb{B}$*. Does there exist a finite tractable* $\mathbb{B}'$ *such that* $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$*?*

A positive answer for all such pairs $(\mathbb{C}, \mathbb{B})$ would have immediate negative consequences for the usefulness of ω-categorical structures as sandwich structures for PCSPs: For $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ with tractable ω-categorical $\mathbb{C}$, we would always get a tractable $\mathbb{B}'$ such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}' \to \mathbb{B}$, hence, a finite tractable sandwich $\mathbb{B}'$ for $(\mathbb{A}, \mathbb{B})$. A pair of structures $(\mathbb{A}, \mathbb{B})$ with the same signature that admits such a finite tractable sandwich is called *finitely tractable*. Discussing Question 1.3, Mottet, Pinsker, and Rydval discovered that there is a pair $(\mathbb{C}, \mathbb{B})$ as in the referenced question that is not finitely tractable, unless P = NP, leaving the possibility for tractable ω-categorical structures to reveal results about PCSPs that could not be obtained by the same method (sandwiching a structure) using finite structures. In Chapter 5 we provide a description of the construction of this pair $(\mathbb{C}, \mathbb{B})$ and a formal proof that it has the required properties.
Temporal CSPs, i.e. CSPs with a constraint language that is first-order definable in $(\mathbb{Q}; <)$, were studied extensively [BK10]. The structure $(\mathbb{Q}; <)$ is ω-categorical, its CSP is tractable, and CSP$(\mathbb{Q}; <)$ does not reduce to a CSP of a finite structure by the general reduction method of *canonical functions*. Hence, temporal problems seem like a good choice to use for constructing an example of a pair $(\mathbb{C}, \mathbb{B})$ as above. Adding the relation $I_4(x, y, u, v) \Leftrightarrow (x = y) \Rightarrow (u = v)$ to $(\mathbb{Q}; <)$ (doing so is not necessary for the construction, but convenient), we obtain $(\mathbb{Q}; <, I_4)$, a relational structure in the *Ord-Horn* fragment [NB95], hence, again a tractable structure. Next we construct a structure $\mathbb{C}$ with domain $\mathbb{Q}^3$ whose polymorphism clone consists exactly of the operations obtained by letting polymorphisms of $(\mathbb{Q}; <, I_4)$ act on triples. This structure is again ω-categorical. The complexity of CSPs of ω-categorical structures only depends on the topological polymorphism clone [BP15], i.e. the set of polymorphisms equipped with the *topology of pointwise convergence*. Since the topological polymorphism clone of $\mathbb{C}$ is isomorphic to the topological polymorphism clone of $(\mathbb{Q}; <, I_4)$, CSP$(\mathbb{C})$ is also tractable. The "right side" $\mathbb{B}$ of the example is obtained by factoring $\mathbb{C}$ with respect to orbits of its automorphism group. By ω-categoricity of $\mathbb{C}$, $\mathbb{B}$ is finite. It is known that there can be no pseudo-cyclic polymorphism of $\mathbb{C}$. Further, we will show that any polymorphism of $\mathbb{B}$ can be lifted to a canonical polymorphism of $\mathbb{C}$, i.e. a polymorphism that behaves nicely with respect to orbits of $\mathbb{C}$ under Aut$(\mathbb{C})$. Lifting a cyclic polymorphism would yield a pseudo-cyclic polymorphism on $\mathbb{C}$, hence, $\mathbb{B}$ has no cyclic polymorphism and CSP$(\mathbb{B})$ is NP-complete. Using a Ramsey-type argument, we can show that every $\mathbb{B}'$ with $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ must be homomorphically equivalent to $\mathbb{B}$. Together, we will obtain the following result:

Theorem 5.1. *There exists a pair* $(\mathbb{C}, \mathbb{B})$ *of* τ*-structures with tractable* ω*-categorical* $\mathbb{C}$ *and finite* $\mathbb{B}$ *such that* $\mathbb{C} \to \mathbb{B}$ *and* $(\mathbb{C}, \mathbb{B})$ *is not finitely tractable, unless* $P = NP$.

The reason we use the third power of $\mathbb{Q}$ for the domain of $\mathbb{C}$ is that $(\mathbb{Q};<)$ is finitely bounded with bounds of maximal size 3. This property, homogeneity and the Ramsey property are the core concepts needed for the construction in Chapter 5. This motivates a more general result which we will give in Chapter 6.

Theorem 6.1. *Let $\mathbb{D}'$ be a finitely bounded homogeneous Ramsey structure and $\mathbb{D}$ be a first-order reduct of $\mathbb{D}'$ that is a model-complete core.*
Let $m \in \mathbb{N}$ be greater than all sizes of bounds of $\mathbb{D}'$ and arities of relations of $\mathbb{D}'$ and $\mathbb{D}$.
Then there exists an ω-categorical structure $\mathbb{C}$ with domain $C = D^m$ and a finite structure $\mathbb{B}$ such that $\mathrm{CSP}(\mathbb{B})$ is NP-complete and

- $\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{D}) \curvearrowright D^m$.
- $\mathbb{C} \to \mathbb{B}$.
- *Every finite $\mathbb{B}'$ such that $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ is homomorphically equivalent to $\mathbb{B}$. In particular, if $P \neq NP$, then $(\mathbb{C}, \mathbb{B})$ is not finitely tractable.*

Note that $\mathrm{CSP}(\mathbb{B})$ for $\mathbb{B}$ as above will always be NP-complete, because otherwise we could set $\mathbb{B}' := \mathbb{B}$ to obtain a finite tractable structure with $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$. This result yields a method to construct the "right side" of a potential PCSP with a tractable ω-categorical sandwich from a tractable finitely bounded homogeneous Ramsey structure. An example of a PCSP with tractable ω-categorical sandwich that does not allow a finite tractable sandwich would connect the PCSP framework to a range of results that were already obtained for CSPs of ω-categorical structures (see e.g. [Bod21; Pin22]), proving yet again that results about infinite-domain CSPs can be used to acquire complexity results for finitary problems. Thus, this thesis can be viewed as a step on the way to answer the following open question:

Question 1.4. *Is there a PCSP template $(\mathbb{A}, \mathbb{B})$ with a tractable ω-categorical sandwich $\mathbb{C}$ such that there is no finite tractable sandwich structure?*

A positive answer has been announced by Mottet.

2 Preliminaries

2.1 Notation

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$. For a natural number n we denote the set $\{1, \dots, n\}$ by $[n]$. For a real number $r \in \mathbb{R}$, we define

$$\lfloor r \rfloor := \max\{z \in \mathbb{Z} : z \leq r\},$$
$$\lceil r \rceil := \min\{z \in \mathbb{Z} : z \geq r\}.$$

For a set A, some $n \in \mathbb{N}$ and some element $a \in A^n$, for $i \leq n$ we denote by a_i the i-th entry of a, i.e. $a = (a_1, \dots, a_n) \in A^n$.

2.2 Relational structures and CSPs

A *relational signature* τ is a set $R = \{R_i : i \in I\}$ of relation symbols (with possibly infinite I) each with an associated *arity* $k_i \in \mathbb{N}$. A *relational structure* $\mathbb{A}$ over the signature τ consists of a set A (the *domain*) and relations $R_i^{\mathbb{A}} \subseteq A^{k_i}$ for each $i \in I$. If the domain A is a finite set, we call the structure $\mathbb{A}$ *finite*.
A *homomorphism* from a τ-structure $\mathbb{A}$ with domain A to a τ-structure $\mathbb{B}$ with domain B is a function $h : A \to B$ that preserves all relations. That means that for all $i \in I$, if $(a_1, \dots, a_{k_i}) \in R_i^{\mathbb{A}}$, then $(h(a_1), \dots, h(a_{k_i})) \in R_i^{\mathbb{B}}$. If there is a homomorphism from a structure $\mathbb{A}$ to a structure $\mathbb{B}$, we say that $\mathbb{A}$ *maps homomorphically* to $\mathbb{B}$ and write $\mathbb{A} \to \mathbb{B}$. If $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$, then we say that $\mathbb{A}$ and $\mathbb{B}$ are *homomorphically equivalent*.
An injective map $\phi : \mathbb{A} \to \mathbb{B}$ between τ-structures $\mathbb{A}, \mathbb{B}$ is called an *embedding*, if for all $i \in I$, it holds that $(a_1, \dots, a_{k_i}) \in R_i^{\mathbb{A}} \Leftrightarrow (\phi(a_1), \dots, \phi(a_{k_i})) \in R_i^{\mathbb{B}}$ for all $a_1, \dots, a_{k_i} \in A$. A bijective embedding is called an *isomorphism*. An isomorphism from a structure to itself is called *automorphism*.

In this thesis a *constraint satisfaction problem* (CSP) is a computational problem described by a single relational structure with finite relational signature.

Definition 2.2.1 (CSP($\mathbb{A}$)). Let $\mathbb{A}$ be a (possibly infinite) structure with a finite relational signature τ, also called *template*. Then CSP($\mathbb{A}$) is the computational problem of deciding whether a given finite τ-structure maps homomorphically to $\mathbb{A}$.

Constraint Satisfaction Problem of $\mathbb{A}$
INSTANCE: A finite structure $\mathbb{X}$ in the same signature as $\mathbb{A}$
QUESTION: Is there a homomorphism $h : \mathbb{X} \to \mathbb{A}$?

We can also consider CSP($\mathbb{A}$) as the class of yes-instances of CSP($\mathbb{A}$). We call a homomorphism $h : \mathbb{X} \to \mathbb{A}$ a *solution* of $\mathbb{X}$ for CSP($\mathbb{A}$).
Note that for homomorphically equivalent structures $\mathbb{A}$ and $\mathbb{B}$, a structure $\mathbb{X}$ is a yes-instance of CSP($\mathbb{A}$) if and only if it is a yes-instance of CSP($\mathbb{B}$).

Lemma 2.2.2. *Let $\mathbb{A}, \mathbb{B}$ be homomorphically equivalent. Then*

$$\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{B}).$$

2.3 Actions and orbits

Definition 2.3.1. Let $g : X^m \to X$ be a function. For each $n \in \mathbb{N}$, g also naturally defines a function $g_n : (X^n)^m \to X^n$ by

$$g_n(\begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix}, \dots, \begin{pmatrix} x_1^m \\ \vdots \\ x_n^m \end{pmatrix}) := \begin{pmatrix} g(x_1^1, \dots, x_1^m) \\ \vdots \\ g(x_n^1, \dots, x_n^m) \end{pmatrix}$$

for all $x^1 = (x_1^1, \dots, x_n^1), \dots, x^m = (x_1^m, \dots, x_n^m) \in X^n$. We will denote such g_n by $g \curvearrowright X^n$ and say that g_n is the *action of g on X^n*. For a set G of operations $g : X^{m_g} \to X, m_g \in \mathbb{N}$, we will use the notation

$$G \curvearrowright X^n := \{g \curvearrowright X^n : g \in G\}$$

and we call $G \curvearrowright X^n$ the *action of G on X^n*.

Definition 2.3.2. Let G be a permutation group acting on a set X. For $x \in X^n$ and $g_n := g \curvearrowright X^n$ for all $g \in G$ we call the set

$$\mathcal{O}(x) = \{g_n(x) : g \in G\}$$

the *orbit* of x under G. We call an orbit of an n-tuple an *n-orbit*. We say that G is *oligomorphic* if there are only finitely many n-orbits under G for every $n \in \mathbb{N}$.
For a relational structure $\mathbb{A}$, we denote the orbit of x with respect to Aut($\mathbb{A}$) by $\mathcal{O}^{\mathbb{A}}(x)$.

2.4 Polymorphisms and function clones

Polymorphisms are a powerful tool for analysing the complexity of CSPs. Polymorphisms can be thought of as higher-dimensional endomorphisms.

Definition 2.4.1. Let $\mathbb{A}$ be a relational structure. Then we denote by $\mathbb{A}^n$ the structure with domain A^n and relations defined as follows. For each k-ary relation $R^{\mathbb{A}}$, the relation $R^{\mathbb{A}^n}$ is the set of all tuples $(x^1, \dots, x^k) \in (A^n)^k$ with $(x_i^1, \dots, x_i^k) \in R^{\mathbb{A}}$ for all $i \leq n$, where x_i denotes the i-th entry of x.

Definition 2.4.2 (Pol($\mathbb{A}$))**.** Let $\mathbb{A}$ be a relational structure and $n \geq 1$. Then a *polymorphism* of $\mathbb{A}$ of arity n is a homomorphism from $\mathbb{A}^n \to \mathbb{A}$. We denote the set of n-ary polymorphisms of $\mathbb{A}$ by Pol$(\mathbb{A})^{(n)}$ and the set of all polymorphisms of $\mathbb{A}$ by Pol($\mathbb{A}$).

In other words, an operation $f : A^n \to A$ is a polymorphism of a τ-structure $\mathbb{A}$ if for every $R \in \tau$, f applied component-wise to any n-tuple of elements in $R^{\mathbb{A}} \subseteq A^k$ gives an element of $R^{\mathbb{A}}$, or equivalently, if (a_{ij}) is a $k \times n$ matrix whose columns are tuples in $R^{\mathbb{A}}$, then f applied to the rows yields a k-tuple that is also in $R^{\mathbb{A}}$. In this case, we say that f is *compatible* with the relation R.

For $n \geq 1$ and a set A, denote by $\mathscr{O}_A^{(n)}$ the set of n-ary operations on A. Denote further the set of all operations on A of finite arity by $\mathscr{O}_A := \bigcup_{n \geq 1} \mathscr{O}_A^{(n)}$.

Definition 2.4.3 (Function clone)**.** A *function clone* (over A) is a subset $\mathscr{C} \subseteq \mathscr{O}_A$ satisfying

- $\mathscr{C}$ contains all projections, i.e. for all $1 \leq k \leq n$ it contains $\pi_k^n \in \mathscr{O}_A^{(n)}$ defined by $\pi_k^n(a_1, \ldots, a_n) = a_k$, and
- $\mathscr{C}$ is *closed under composition*, that is, for all $m, n \in \mathbb{N}$, $f \in \mathscr{C} \cap \mathscr{O}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{C} \cap \mathscr{O}_A^{(m)}$ it contains the function $f(g_1, \ldots, g_n) \in \mathscr{O}_A^{(m)}$ defined by

$$(a_1, \ldots, a_m) \mapsto f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)).$$

Consider the set $\mathrm{Pol}(\mathbb{A})$ of polymorphisms of a relational structure $\mathbb{A}$. It is easy to see that all the projections are polymorphisms and that the set of polymorphisms is closed under composition. Thus, from now on we will also call $\mathrm{Pol}(\mathbb{A})$ the *polymorphism clone* of $\mathbb{A}$. Of special interest to us are structures with polymorphism clones that contain a rather large permutation group, an oligomorphic one. A function clone over some set A is called oligomorphic if it contains an oligomorphic permutation group. A classical theorem in model theory, the theorem of Engeler, Svenonius, and Ryll-Nardzewski states that the automorphism group $\mathrm{Aut}(\mathbb{A})$ of a countable structure $\mathbb{A}$ is oligomorphic if and only if $\mathbb{A}$ is *ω-categorical* ([Hod93]), where a relational τ-structure $\mathbb{A}$ is called ω-categorical, if any countable τ-structure $\mathbb{B}$ that satisfies exactly those first-order sentences over τ that are true in $\mathbb{A}$ is isomorphic to $\mathbb{A}$. Since the automorphism group of a relational structure $\mathbb{A}$ is contained in the polymorphism clone of $\mathbb{A}$, it follows that a countable structure $\mathbb{A}$ is ω-categorical if and only if its polymorphism clone is oligomorphic. It was shown that for a finite or countable ω-categorical structure $\mathbb{A}$ the computational complexity of $\mathrm{CSP}(\mathbb{A})$ only depends on the polymorphism clone $\mathrm{Pol}(\mathbb{A})$ [BN06] and later that it only depends on the polymorphism clone viewed as a topological clone [BP15], i.e., viewed as an abstract algebraic structure additionally equipped with the topology of pointwise convergence (see Definition 2.9.2).
Cyclic polymorphisms in particular will play a central role in this thesis. A *cyclic* operation is a k-ary $(k \geq 2)$ operation satisfying the identity $f(x_1, x_2, \ldots, x_k) = f(x_2, \ldots, x_n, x_1)$.

Theorem 2.4.4 ([BK12])**.** *Let $\mathbb{A}$ be a finite relational structure. If $\mathbb{A}$ has no cyclic polymorphism, then CSP($\mathbb{A}$) is NP-complete.*

In fact, assuming P $\neq$ NP, tractability of finite structures can even be characterised by the existence of cyclic polymorphisms, i.e. $\mathrm{CSP}(\mathbb{A})$ is in P if and only if $\mathbb{A}$ has a cyclic polymorphism.

2.5 Primitive positive interpretations

Definition 2.5.1. A first-order τ-formula $\phi(x_1, \dots, x_k)$ is called *primitive positive* if

$$\phi = \exists x_{k+1}, \dots, x_n(\psi_1 \wedge \dots \wedge \psi_m)$$

for *atomic* τ-formulas $\psi_1, \dots, \psi_m$, i.e. ψ_i are of the form $R(y_1, \dots, y_k)$ with $R \in \tau$ and $y_i \in \{x_1, \dots, x_n\}$, of the form $y = y'$ for $y, y' \in \{x_1, \dots, x_n\}$, or $\top$ (true) or $\bot$ (false).

Definition 2.5.2. A σ-structure $\mathbb{B}$ has a *(first-order) interpretation* I in a τ-structure $\mathbb{A}$ if and only if there exists a natural number $d \geq 1$, the *dimension* of I, and

- a τ-formula $B_I(x_1, \dots, x_d)$ (called *domain formula*),
- for each unnested atomic σ-formula $\phi(y_1, \dots, y_k)$ a τ-formula $\phi_I(\overline{x}_1, \dots, \overline{x}_k)$ with disjoint d-tuples $\overline{x}_i$ of distinct variables (called the *defining formulas*),
- a surjective map $h \colon B_I(A^d) \to B$ (called *coordinate map*),

such that for every unnested atomic σ-formula ϕ and all tuples $\overline{a}_i \in B_I(A^d)$

$$\mathbb{B} \models \phi(h(\overline{a}_1), \dots, h(\overline{a}_k)) \Leftrightarrow \mathbb{A} \models \phi_I(\overline{a}_1, \dots, \overline{a}_k) \ .$$

If the formulas B_I and ϕ_I are primitive positive, we say that the interpretation I is *primitive positive*.

In the CSP setting, primitive positive interpretations are used to reduce one problem in polynomial time to another.

Theorem 2.5.3 ([BJK05])**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be structures with finite relational signatures. If $\mathbb{B}$ has a primitive-positive interpretation in $\mathbb{A}$, then there is a polynomial-time reduction from* $\mathrm{CSP}(\mathbb{B})$ *to* $\mathrm{CSP}(\mathbb{A})$.

2.6 Promise CSPs and function minions

Promise CSPs are a natural way to generalise CSPs.

Definition 2.6.1 ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)**.** Let $\mathbb{A}, \mathbb{B}$ be finite relational structures with the same signature τ. Then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is the computational problem of deciding whether a given finite τ-structure $\mathbb{X}$ maps homomorphically to $\mathbb{A}$ or does not map homomorphically to $\mathbb{B}$. If neither is the case, the algorithm can do anything.

Promise Constraint Satisfaction Problem of $(\mathbb{A}, \mathbb{B})$
INSTANCE: A structure $\mathbb{X}$ in the same signature as $\mathbb{A}$ and $\mathbb{B}$
QUESTION: Does $\mathbb{X}$ satisfy $\mathbb{X} \to \mathbb{A}$ or not even $\mathbb{X} \to \mathbb{B}$?

Function minions play a similar role for PCSPs as function clones do for CSPs. First, let us give a more general definition of polymorphisms.

Definition 2.6.2 (Pol($\mathbb{A}, \mathbb{B}$))**.** Let $\mathbb{A}, \mathbb{B}$ be two structures with the same signature. An *n-ary polymorphism from $\mathbb{A}$ to $\mathbb{B}$* is a homomorphism from $\mathbb{A}^n$ to $\mathbb{B}$. We denote the set of all polymorphisms from $\mathbb{A}$ to $\mathbb{B}$ by $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$.

Note that this definition generalises Definition 2.4.2 in the sense that $\mathrm{Pol}(\mathbb{A}, \mathbb{A}) = \mathrm{Pol}(\mathbb{A})$. In this case, $\mathrm{Pol}(\mathbb{A})$ is a function clone. If $\mathbb{A}$ and $\mathbb{B}$ are different, the composition of polymorphisms as in Definition 2.4 is not well-defined. However, $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is always closed under taking minors.

Definition 2.6.3 (Minor)**.** An n-ary function $f : A^n \to B$ is called a *minor* of an m-ary function $g : A^m \to B$ given by a map $\pi : [m] \to [n]$ if

$$f(a_1, \ldots, a_n) = g(a_{\pi(1)}, \ldots, a_{\pi(m)})$$

for all $a_1, \ldots, a_n \in A$.

In other words, f is a minor of g if it is obtained from g by identifying variables, permuting variables, and introducing dummy variables.

Definition 2.6.4 (Function minion)**.** Let $\mathscr{O}(A, B) = \{f : A^n \to B \mid n \geq 1\}$. A *(function) minion* $\mathbb{M}$ on a pair of sets (A, B) is a non-empty subset of $\mathscr{O}(A, B)$ that is closed under taking minors. For fixed $n \geq 1$, let $\mathscr{M}^{(n)}$ denote the set of n-ary functions from $\mathscr{M}$.

For any two relational structures $\mathbb{A}, \mathbb{B}$, it is easy to see that $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is a minion. Thus, we also call $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ the *polymorphism minion* of (the template) $(\mathbb{A}, \mathbb{B})$. Similar to the description of complexity via polymorphisms in the CSP setting it was shown that, up to log-space reductions, the complexity of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ for finite τ-structures $\mathbb{A}, \mathbb{B}$ only depends on the properties of polymorphisms from $\mathbb{A}$ to $\mathbb{B}$, hence, on the polymorphism minion of $(\mathbb{A}, \mathbb{B})$.

2.7 Extreme amenability

Definition 2.7.1 (Topological group)**.** A *topological group* is an (abstract) group $\mathbf{G}$ together with a topology on the domain G such that the function $(g, h) \mapsto gh^{-1}$ from G^2 to G is continuous. In other words, we require the binary group operation and the inverse function to be continuous.

Definition 2.7.2 (Compact space)**.** A topological space X is called *compact* if every open cover has a finite subcover. That is, for every family $(U_i)_{i \in I}$ of open sets with $\bigcup_{i \in I} U_i = X$, there exist $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $\bigcup_{k=1}^{n} U_{i_k} = X$.

Definition 2.7.3 (Continuous group action)**.** Let G be a group with neutral element e and X be a set. A function $\alpha : G \times X \to X$ that satisfies $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$ is called a *group action* of G on X.
If $\mathbf{G}$ is a topological group and X is a topological space, then we call a group action α of $\mathbf{G}$ on X continuous if it is continuous with respect to the product topology on $G \times X$.

Definition 2.7.4 (Hausdorff space)**.** A topological space $(X, \mathcal{T})$ is said to satisfy the separation axiom (T_2), if

$$\forall x, y \in X, x \neq y \ \exists O_x, O_y \in \mathcal{T} : (x \in O_x \wedge y \in O_y) \wedge (O_x \cap O_y = \emptyset).$$

If $(X, \mathcal{T})$ satisfies (T_2), then we call it a *Hausdorff space*.

Definition 2.7.5 (Extreme amenability)**.** A topological group $\mathbf{G}$ is called *extremely amenable* if every continuous action α of $\mathbf{G}$ on a compact Hausdorff space X has a fixed point, i.e. there is $x \in X$ such that $\alpha(g)(x) = x$ for every $g \in G$.

2.8 The Ramsey property

In Chapter 5, we want to show that $\mathrm{Aut}(\mathbb{Q}; <)$ is extremely amenable. This was first proved by Pestov [Pes98]. Afterwards, Kechris, Pestov and Todorčević reproved it in a more general framework, in which the former result can be viewed as a special instance [KPT05]. A crucial factor is to establish a relationship between the concept of *extreme amenability* and the Ramsey property. Let us define the latter in the following.
Let τ be a relational language and $\mathbb{A}, \mathbb{B}$ be τ-structures. We write $\binom{\mathbb{B}}{\mathbb{A}}$ for the set of embeddings of $\mathbb{A}$ into $\mathbb{B}$. Given structures $\mathbb{S}, \mathbb{M}, \mathbb{L}$ and $c \in \mathbb{N}$, we write

$$\mathbb{L} \to (\mathbb{M})^{\mathbb{S}}_c,$$

if for every colouring $\chi : \binom{\mathbb{L}}{\mathbb{S}} \to [c]$ of the embeddings of $\mathbb{S}$ into $\mathbb{L}$ with c colours, there is an embedding $f \in \binom{\mathbb{L}}{\mathbb{M}}$ such that all embeddings $g \in \binom{\mathbb{L}}{\mathbb{S}}$ with $g(S) \subseteq f(M)$ are assigned the same colour. For a τ-structure $\mathbb{A}$, the *age* of $\mathbb{A}$ (denoted by $\mathrm{Age}(\mathbb{A})$) is the class of all finite τ-structures that can be embedded into $\mathbb{A}$.

Definition 2.8.1 (Homogeneous structure)**.** Let τ be a relational signature. A τ-structure $\mathbb{A}$ is called *homogeneous* if every isomorphism between finite substructures of $\mathbb{A}$ can be extended to an automorphism of $\mathbb{A}$.

Definition 2.8.2 (Ramsey property)**.** A class of finite relational structures that is closed under isomorphisms and induced substructures is called *Ramsey*, or is said to have the *Ramsey property*, if for every $\mathbb{S}, \mathbb{M} \in \mathcal{C}$ and for every $c \in \mathbb{N}$ there exists $\mathbb{L} \in \mathcal{C}$ such that $\mathbb{L} \to (\mathbb{M})^{\mathbb{S}}_c$.
A homogeneous structure $\mathbb{C}$ is called *Ramsey* if $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_c$ holds for all $\mathbb{A}, \mathbb{B} \in \mathrm{Age}(\mathbb{C})$ and $c \geq 2$.

For a countable homogeneous structure $\mathbb{A}$, the Ramsey property can be characterised by the topological automorphism group of $\mathbb{A}$: The age of $\mathbb{A}$ is Ramsey if and only if the automorphism group of $\mathbb{A}$ is extremely amenable. This is called the Kechris-Pestov-Todorčević correspondence and a consequence of the following theorem, which was proved in [KPT04].

Theorem 2.8.3 ([KPT05])**.** *Let $\mathbb{A}$ be a countable homogeneous relational structure. Then the following are equivalent.*

(i) $\mathbb{A}$ is Ramsey

(ii) The age of $\mathbb{A}$ has the Ramsey property.

(iii) $\mathrm{Aut}(\mathbb{A})$ *is extremely amenable.*

It is known that for any homogeneous Ramsey structure, there is a linear order that is preserved by all automorphisms.

Proposition 2.8.4 ([KPT05])**.** *Let $\mathbb{A}$ be a homogeneous Ramsey structure with domain A. Then there exists a linear order on A that is preserved by all automorphisms.*

We can also add this order as a binary relation to the signature of the corresponding structure without losing the Ramsey property. Moreover, this is possible for any relation that is preserved by all automorphisms. It is a well-known fact that the automorphisms of a relational structure $\mathbb{A}$ preserve all relations that are first-order definable in $\mathbb{A}$. A $\tau_{\mathbb{A}}$-structure $\mathbb{A}$ is a *reduct* of a $\tau_{\mathbb{B}}$-structure $\mathbb{B}$ if $\tau_{\mathbb{A}} \subseteq \tau_{\mathbb{B}}$, they have the same domain $A = B$, and for all $R \in \tau_{\mathbb{A}}$ we have $R^{\mathbb{A}} = R^{\mathbb{B}}$. In this case, $\mathbb{A}$ is also called the $\tau_{\mathbb{A}}$*-reduct* of $\mathbb{B}$. Vice versa, $\mathbb{B}$ is called an *expansion* of $\mathbb{A}$. We also say that $\mathbb{B}$ is a $\tau_{\mathbb{B}}$*-expansion* of $\mathbb{A}$. If all relations of $\mathbb{B}$ have a first-order definition in $\mathbb{A}$ without parameters, then $\mathbb{B}$ is called a *first-order expansion* of $\mathbb{A}$. A structure $\mathbb{C}$ is called a *first-order reduct* of $\mathbb{A}$ if it is a reduct of thee xpansion $\mathbb{A}^+$ of $\mathbb{A}$ by all relations that are first-order definable in $\mathbb{A}$.

Proposition 2.8.5. *Let $\mathbb{A}$ be a homogeneous Ramsey structure and $\mathbb{B}$ an expansion of $\mathbb{A}$ with* $\mathrm{Aut}(\mathbb{B}) = \mathrm{Aut}(\mathbb{A})$*. Then $\mathbb{B}$ is also a homogeneous Ramsey structure.*

Proof. Denote the signature of $\mathbb{A}$ by τ and let the signature of $\mathbb{B}$ be $\tau' \supseteq \tau$. To show that $\mathbb{B}$ is still homogeneous, take any isomorphism between finite substructures of $\mathbb{B}$. This is clearly also an isomorphism between the τ-reducts of those structures and therefore extends to a τ-automorphism, which preserves all relations in τ', hence, it is an automorphism of $\mathbb{B}$.
Since $\mathrm{Aut}(\mathbb{A}) = \mathrm{Aut}(\mathbb{B})$, Theorem 2.8.3 implies that $\mathbb{B}$ is Ramsey. □

2.9 Canonisation

Throughout this thesis, we will try to use the finite number of orbits of ω-categorical structures to our advantage. For an ω-categorical structure $\mathbb{A}$, since there are only finitely many n-orbits, factoring $\mathbb{A}^n$ by the equivalence relation $\sim_{\mathrm{Aut}(\mathbb{A})}$ induced by $\mathrm{Aut}(\mathbb{A})$, i.e. $a \sim b$ if there exists $\alpha \in \mathrm{Aut}(\mathbb{A})$ with $\alpha(a) = b$, will yield a finite structure. However, in many cases we want to preserve the property of functions being polymorphisms, or more precisely, we want polymorphisms of $\mathbb{A}$ to induce polymorphisms of the factor structure. This is only possible if the function behaves in a nice way regarding the equivalence classes induced by $\mathrm{Aut}(\mathbb{A})$. The functions that behave nicely in that sense are called canonical functions.

Definition 2.9.1 (Canonical functions)**.** Let $\mathbf{G} \curvearrowright X$ and $\mathbf{H} \curvearrowright Y$ be permutation groups. A function $f : X \to Y$ is called *canonical with respect to* $\mathbf{G}$ *and* $\mathbf{H}$ if for every tuple $t \in X^n$ for some $n \in \mathbb{N}$ and every $\alpha \in \mathbf{G}$ there exists $\beta \in \mathbf{H}$ such that $f\alpha(t) = \beta f(t)$.
For any two relational structures $\mathbb{A}, \mathbb{B}$, a function $f : A \to B$ is called canonical with respect to $\mathbb{A}$ and $\mathbb{B}$ if it is canonical with respect to $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$.

For $f : A^n \to B$, we call f canonical with respect to $\mathbb{A}$ and $\mathbb{B}$ if it is canonical with respect to $\mathrm{Aut}(\mathbb{A})^n \curvearrowright A^n$ and $\mathrm{Aut}(\mathbb{B})$, where $\mathrm{Aut}(\mathbb{A})^n \curvearrowright A^n$ consists of the functions $(a_1, \ldots, a_n) \mapsto (\alpha_1(a_1), \ldots, \alpha_n(a_n))$ for $a_1, \ldots, a_n \in A$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(\mathbb{A})$.

The property of being a homomorphism is a local property and can be checked on finite sets. This is the reason why the closure of a set of homomorphisms with respect to the topology of pointwise convergence also consists only of homomorphisms.

Definition 2.9.2 (Topology of pointwise convergence)**.** For sets X, Y, the topology of pointwise convergence on Y^X is defined as the product topology where Y is taken to be discrete. For $S \subseteq Y^X$, we write $\overline{S}$ for the closure of S in this space.
For $S \subseteq Y^X$ and $f \in Y^X$, we have $f \in \overline{S}$ if and only if for every finite subset $F \subseteq X$, there is some $g_F \in S$ with $f|_F = g_F|_F$.

Theorem 2.9.3 ([BPT13; BP21])**.** *Let* $\mathbf{G} \curvearrowright X$*,* $\mathbf{H} \curvearrowright Y$ *be permutation groups, where* X *is countable,* $\mathbf{G}$ *is extremely amenable, and* $\mathbf{H}$ *is oligomorphic. Let* $f : X \to Y$ *and* Y^X *be equipped with the topology of pointwise convergence. Then*

$$\overline{\mathbf{H}f\mathbf{G}} := \overline{\{\beta f \alpha \mid \alpha \in \mathbf{G}, \beta \in \mathbf{H}\}}$$

contains a canonical function with respect to $\mathbf{G}$ *and* $\mathbf{H}$*.*

Let $\mathbb{A}, \mathbb{B}$ be relational structures and $\mathbf{G} := \mathrm{Aut}(\mathbb{A}), \mathbf{H} := \mathrm{Aut}(\mathbb{B})$. Then given that $\mathrm{Aut}(\mathbb{A})$ is extremely amenable, A is countable, and $\mathrm{Aut}(\mathbb{B})$ is oligomorphic, the existence of a homomorphism $f : \mathbb{A} \to \mathbb{B}$ implies the existence of a canonical homomorphism $f' : \mathbb{A} \to \mathbb{B}$. The previous theorem guarantees that under certain conditions we can use unary functions, in particular homomorphisms, between different structures to generate a canonical one. Sometimes it is useful to do the same for functions of higher arity, e.g. polymorphisms. The next statement is a corollary of the previous one, that allows multivariate functions.

Theorem 2.9.4 ([BPT13],[BP21])**.** *Let* $\mathbb{A}$ *be a homogeneous structure in a finite relational signature with the Ramsey property, let* $\mathbb{B}$ *be a countable* ω*-categorical structure, and let* $f : A^n \to B$ *be an arbitrary function. Then there exists* $g : A^n \to B$ *that is canonical from* $\mathbb{A}$ *to* $\mathbb{B}$ *in the closure of*

$$\Big\{(a_1, \ldots, a_n) \mapsto \beta(f(\alpha_1(a_1), \ldots, \alpha_n(a_n))\ \mid\ \beta \in \mathrm{Aut}(\mathbb{B}), \alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(\mathbb{A})\Big\},$$

with respect to the topology of pointwise convergence. This means that for every $m \in \mathbb{N}$ *and finite subset* S *of* A^m*, there exist* $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(\mathbb{A}), \beta \in \mathrm{Aut}(\mathbb{B})$ *such that*

$$g(a_1, \ldots, a_n) = \beta(f(\alpha_1(a_1), \ldots, \alpha_n(a_n))$$

for all $a_1, \ldots, a_n \in S$*. If* $\mathbb{A} = \mathbb{B}$*, we say that* f *locally interpolates* g *modulo* $\mathrm{Aut}(\mathbb{A})$*.*

2.10 Finitely bounded structures

Let τ be a finite relational signature and $\mathcal{C}$ be a class of τ-structures. Then a finite τ-structure $\mathbb{F}$ is called a *bound* of $\mathcal{C}$ if it does not embed into a structure from $\mathcal{C}$ and it is minimal with this property (with respect to embeddability).

Definition 2.10.1. A class of τ-structures is called *finitely bounded* if it has finitely many bounds up to isomorphism.
A structure $\mathbb{A}$ is called finitely bounded if the age of $\mathbb{A}$ is finitely bounded.

Remark 2.10.2. Note that for a finitely bounded τ-structure $\mathbb{A}$, a finite τ-structure $\mathbb{B}$ embeds into $\mathbb{A}$ if and only if none of the finite bounds of $\mathbb{A}$ embeds into $\mathbb{B}$. To see this, let $\mathcal{F}$ be the set of bounds and note that if $\mathbb{F} \in \mathcal{F}$ embeds into $\mathbb{B}$, an embedding of $\mathbb{B}$ into $\mathbb{A}$ would yield an embedding of $\mathbb{F}$ into $\mathbb{A}$. On the other hand, if no $\mathbb{F} \in \mathcal{F}$ embeds into $\mathbb{B}$ and $\mathbb{B}$ does not embed into $\mathbb{A}$, then $\mathbb{B}$ or a substructure of $\mathbb{B}$ would be a bound that is not in $\mathcal{F}$, a contradiction.

3 Sandwiches for PCSPs

In this chapter, we will investigate sandwiches for PCSPs, in particular, sandwich structures that are minimal in size. In the following section, we state some results about minimal finite sandwiches. In the remaining sections, we explore an example of a PCSP that reduces to an infinite tractable sandwich but not to a finite one, unless P = NP, constructed by Barto [Bar19]. This shows that solving infinite-domain CSPs is necessary in order to prove tractability of PCSPs via reduction to a sandwich structure.

3.1 Minimal finite sandwiches

Recent results show that if there is a finite tractable sandwich, the size of the smallest finite tractable sandwiches can grow even for fixed size of $\mathbb{A}$ and $\mathbb{B}$. In [Den+21], the authors presented an example of a finitely tractable PCSP$(\mathbb{A}, \mathbb{B})$ with $|A| = |B| = 2$ with a tractable sandwich $\mathbb{C}$ for $(\mathbb{A}, \mathbb{B})$ of size $|C| = 3$ such that there is no tractable sandwich of size ≤ 2. The result by [KMZ22], showing that for every prime p and every integer $n > 1$ there are $\mathbb{A}, \mathbb{B}$ of size n such that PCSP$(\mathbb{A}, \mathbb{B})$ admits a tractable sandwich $\mathbb{C}$ of size p but no smaller one, extended this result. Those structures can be chosen to have a single relation of arity n^p. So for fixed size $|A| = |B| = n$, by choosing an appropriate relation, the size of the smallest tractable sandwich can be arbitrarily large.
Even if we restrict the signature to a single ternary relation, it was shown that a similar behaviour can be obtained: For every prime $p \geq 7$, there are $\mathbb{A}, \mathbb{B}$ of size $p - 1$ with a single ternary relation such that PCSP$(\mathbb{A}, \mathbb{B})$ admits a tractable sandwich $\mathbb{C}$ of size p, but there is no tractable sandwich structure of smaller size [KMZ22].

3.2 Infinite sandwiches are necessary

Since the complexity of finite domain CSPs is well-understood, the natural first approach when trying to solve the PCSP for a given template $(\mathbb{A}, \mathbb{B})$ with the sandwiching method is to try to find a finite tractable sandwich. However, in this section we will show that this is not always possible. More precisely, we will show that there are finite $\mathbb{A}, \mathbb{B}$ such that PCSP$(\mathbb{A}, \mathbb{B})$ reduces via $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ to CSP$(\mathbb{C})$ for tractable $\mathbb{C}$ with infinite domain, but there is no finite tractable sandwich for $(\mathbb{A}, \mathbb{B})$, unless P = NP.

The construction and proof follow [Bar19]. The construction uses the templates for the problems *positive 1-in-3-SAT* (denoted 1-in-3) and *positive Not-All-Equal-3-SAT* (denoted NAE-3). The first problem can be formulated as follows. Given a list of triples of variables, decide whether there is a mapping from the set of variables to $\{0, 1\}$ such that exactly one variable in each triple is mapped to 1. For NAE-3, the instances are triples of variables as well, but we only need a mapping to $\{0,1\}$ that does not map all variables in one triple to the same value. Reformulating the respective problems to CSPs, we get the following templates:

$$\begin{aligned} \text{1-in-3}: &\quad (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\}) \\ \text{NAE-3}: &\quad (\{0,1\}; \{0,1\}^3 \backslash \{(0,0,0),(1,1,1)\}). \end{aligned}$$

We will also denote the templates by 1-in-3 and NAE-3 and write PCSP(1-in-3, NAE-3) for the corresponding PCSP. It is well-known that 1-in-3 and NAE-3 are NP-complete problems [Sch78]. However, PCSP(1-in-3, NAE-3) is tractable. To see this, we will reduce the problem to $\mathrm{CSP}(\mathbb{Z}; x+y+z=1)$, i.e. the template with domain $\mathbb{Z}$ and one ternary relation $R = \{(x,y,z) : x+y+z=1\}$, by showing that $(\mathbb{Z}; x+y+z=1)$ is sandwiched by the templates of 1-in-3 and NAE-3. Now $\mathrm{CSP}(\mathbb{Z}; x+y+z=1)$ is the problem of solving a set of linear equations (even of the particular type $x+y+z=1$) over $\mathbb{Z}$. This problem is known to be solvable in polynomial time [GLS88]. Let us define the homomorphisms

$$\text{1-in-3} \xrightarrow{f} (\mathbb{Z}; x+y+z=1) \xrightarrow{g} \text{NAE-3}. \tag{3.1}$$

For f, we take the inclusion map, which is a homomorphism, since for any tuple (x,y,z) with exactly one entry equal to 1 and the others 0, we have $x+y+z=1$. The map g is defined by $g(x)=0 \Leftrightarrow x \leq 0$. Since $x+y+z=1$ implies that neither all of the entries of the triple (x,y,z) are greater than 0, nor all of them are smaller than or equal to 0, tuples in $R^{\mathbb{Z}}$ are mapped to tuples in $R^{\text{NAE-3}}$. Thus, $(\mathbb{Z}; x+y+z=1)$ is a sandwich for PCSP(1-in-3, NAE-3) and by tractability of $(\mathbb{Z}; x+y+z=1)$, also PCSP(1-in-3, NAE-3) is tractable. Hence, we found an infinite tractable sandwich, it remains to show that there can be no finite tractable one. This will make clear that infinite sandwiches are indeed necessary for proving tractability of PCSPs by the sandwiching method. We will show the following Theorem:

Theorem 3.1 ([Bar19]). *Let $\mathbb{C}$ be a finite sandwich for* PCSP(1-in-3, NAE-3)*. Then* $\mathrm{CSP}(\mathbb{C})$ *is NP-complete.*

Let $\mathbb{C} = (C, R)$ be a finite relational structure with $R \subseteq C^3$ such that $f : \text{1-in-3} \to \mathbb{C}$ and $g : \mathbb{C} \to \text{NAE-3}$ are homomorphisms. Since the composition $g \circ f$ is a homomorphism, applied to $(1,0,0) \in R^{\text{1-in-3}}$ component-wise it yields a Not-All-Equal tuple. Hence, $f(0) \neq f(1)$. Renaming the elements of C, we can assume that $\{0,1\} \subseteq C$ and that f is the inclusion map. In particular, we have $\{(1,0,0),(0,1,0),(0,0,1)\} \subseteq R^{\mathbb{C}}$.

If $\mathrm{CSP}(\mathbb{C})$ is tractable, then by Theorem 2.4.4, $\mathbb{C}$ has a cyclic polymorphism of any prime arity $p > |C|$. Let us fix a cyclic polymorphism

$$s \in \mathrm{Pol}(\mathbb{C})^{(p)} \text{ for some prime } p > 60|C| \tag{3.2}$$

and define a function t on C of arity p^2 by

$$\begin{aligned} &t(x_{11}, x_{12}, \ldots, x_{1p}, x_{21}, x_{22}, \ldots, x_{2p}, \ldots, x_{p1}, \ldots, x_{pp}) \\ &\quad := s(s(x_{11}, x_{21} \ldots, x_{p1}), \end{aligned}$$

$$s(x_{12}, x_{22} \ldots, x_{p2}),$$
$$\vdots$$
$$s(x_{1p}, x_{2p} \ldots, x_{pp}))$$

A more convenient way to denote t is to display the arguments in a $p \times p$ matrix whose entry in the i-th row and j-th column is x_{ij}, so we obtain the value

$$t \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} \tag{3.3}$$

by applying s first to the columns and then to the results. In the proof the arguments x_{ij} will mostly be 0's and 1's. Before starting the proof of Theorem 3.1, we introduce some concepts for *zero-one matrices*, i.e. matrices with only 0's or 1's as entries.

Definition 3.2. Let p be the prime from (3.2) and X, Y be $p \times p$ zero-one matrices. The *area of X* is the fraction of ones and is denoted by

$$\lambda(X) = \Big(\sum_{i,j} x_{ij}\Big)/p^2$$

We say that the matrices X, Y are *g-equivalent*, denoted by $X \sim Y$, if $g(t(X)) = g(t(Y))$, where g is the homomorphism defined in 3.1.
The matrix X is called *tame* if the following hold:

(*i*) $\lambda(X) < \frac{1}{3} \;\Rightarrow\; X \sim 0_{p\times p}$, and

(*ii*) $\lambda(X) > \frac{1}{3} \;\Rightarrow\; X \sim 1_{p\times p}$,

where $0_{p\times p}$ and $1_{p\times p}$ denote the $p \times p$ matrices with all entries equal to 0 or 1, respectively.

Note that since p is a prime number greater than 3, the case $\lambda(X) = 1/3$ is not possible. Further, since the equivalence relation $\sim$ is a partition into two blocks, $X \nsim Y \nsim Z$ implies $X \sim Z$.

Definition 3.3. A triple X, Y, Z of $p \times p$ zero-one matrices is called a *cover* if, for every $1 \le i, j \le p$, exactly one of x_{ij}, y_{ij}, z_{ij} is equal to one.

Lemma 3.4. *If X, Y, Z is a cover, then X, Y, Z are not all g-equivalent.*

Proof. By definition of a cover, all tuples (x_{ij}, y_{ij}, z_{ij}) are in $\{(1,0,0),(0,1,0),(0,0,1)\} \subseteq R^{\mathbb{C}}$. Since s preserves $R^{\mathbb{C}}$, so does t. Hence, the triple $(t(X), t(Y), t(Z))$ is in $R^{\mathbb{C}}$ as well and since g is a homomorphism from $\mathbb{C}$ to the template of NAE-3, we know that $g(t(X)), g(t(Y)), g(t(Z))$ can not be all equal, which means that X, Y, Z are not all g-equivalent. □

Let us give an outline of the following proof. First, we will show that certain matrices, called "almost rectangles", are tame. Then we construct two tame matrices X_1, X_2 such that $\lambda(X_1) < 1/3$ and $\lambda(X_2) > 1/3$, but $t(X_1) = t(X_2)$. This will lead to a contradiction, since $0_{p\times p} \nsim 1_{p\times p}$, as we shall see.

3.2.1 Line segments are tame

In this subsection it is mostly convenient to regard the arguments of t as tuples $x = (x_{11}, x_{12}, \dots, x_{pp})$ rather than matrices. However, we will still use the definitions of area, g-equivalence, tameness, and cover by identifying tuples with matrices as in (3.3). We will show that line segments are tame. Line segments are a special case of almost rectangles which we will define in the next section.
By Fermat's Little Theorem, since $p > 3$ is a prime number, there is some $q \in \mathbb{N}$ such that

$$p^2 = 3q + 1. \tag{3.4}$$

In the following, let $\langle i \rangle$ denote the tuple

$$\langle i \rangle = (\underbrace{1, 1, \dots, 1}_{i\text{-times}}, 0, 0, \dots, 0).$$

These tuples are called *line segments*. In this subsection we will prove that line segments are tame. Before we prove that, let us remark that the cyclicity of s is inherited to t.

Lemma 3.5. *The operation t is cyclic.*

Proof. This can be easily seen by using cyclicity of s two times, once the cyclicity of the "outer" and then of the "inner" s in the definition of t. □

Lemma 3.6. *Each line segment $\langle i \rangle$ for $i \in \{0, \dots, p^2\}$, is tame and $\langle 0 \rangle \nsim \langle p^2 \rangle$.*

Proof. First, let us prove by induction that for each $i \in \{0, \dots, q\}$, we have

$$\langle q-i \rangle \sim \langle q-i+1 \rangle \sim \dots \sim \langle q \rangle \nsim \langle q+1 \rangle \sim \dots \sim \langle q+i \rangle \sim \langle q+i+1 \rangle.$$

For the base step $i = 0$, let

$$X = (\underbrace{1, \dots, 1}_{q\text{-times}}, 0, \dots, 0, 0, \dots, 0)$$
$$Y = (0, \dots, 0\, \underbrace{1, \dots, 1}_{q\text{-times}}, 0, \dots, 0)$$
$$Z = (0, \dots, 0, 0, \dots, 0 \quad \underbrace{1, \dots, 1}_{(q+1)\text{-times}} \quad)$$

Then X, Y, Z is a cover, hence, by Lemma 3.4, they are not all g-equivalent. Since t is cyclic, we have $t(\langle q \rangle) = t(X) = t(Y)$ and $t(\langle q+1 \rangle) = t(Z)$. Thus, $\langle q \rangle, \langle q \rangle, \langle q+1 \rangle$ are not all g-equivalent, so $\langle q \rangle \nsim \langle q+1 \rangle$.
Now assume the claim holds for $i-1$. We want to prove it for i. Consider the tuples $\langle q-i \rangle, \langle q+1 \rangle, \langle q+i \rangle$. Since $(q-i)+(q+1)+(q+i) = 3q+1 = p^2$, we can shift the tuples to form a cover and apply the same argument as above to see that $\langle q-i \rangle, \langle q+1 \rangle, \langle q+i \rangle$ are not all g-equivalent. By the induction hypothesis, $\langle q+1 \rangle \sim \langle q+i \rangle$, hence, $\langle q-i \rangle \nsim \langle q+1 \rangle$. Since $\langle q+1 \rangle \nsim \langle q-i+1 \rangle$ (again by the induction hypothesis), we get $\langle q-i \rangle \sim \langle q-i+1 \rangle$. Finally, it remains to show that $\langle q+i \rangle \sim \langle q+i+1 \rangle$,

which can be achieved in a similar fashion by using the triples $\langle q-i\rangle, \langle q\rangle, \langle q+i+1\rangle$.
We have shown

$$\langle 0\rangle \sim \langle 1\rangle \sim \cdots \sim \langle q\rangle \nsim \langle q+1\rangle \sim \cdots \sim \langle 2q\rangle \sim \langle 2q+1\rangle.$$

Using the same arguments again for $\langle 0\rangle, \langle i\rangle, \langle p^2-i\rangle$ with $0 \leq i \leq q$, we obtain $\langle 0\rangle \nsim \langle p^2-i\rangle$. Hence, we get

$$\langle 0\rangle \sim \langle 1\rangle \sim \cdots \sim \langle q\rangle \nsim \langle q+1\rangle \sim \cdots \sim \langle p^2\rangle.$$

Since $\lambda(\langle i\rangle) < 1/3 \Leftrightarrow i \leq q$, all line segments are tame. In particular, we showed that $\langle 0\rangle \nsim \langle p^2\rangle$. □

3.2.2 Almost rectangles are tame

Definition 3.7. Let $1 \leq k_1, \ldots, k_p \leq p$. By

$$[k_1, \ldots, k_p]$$

we denote the matrix whose i-th column begins with k_i ones followed by $p - k_i$ zeros, for each $1 \leq i \leq p$.
An *almost rectangle* is a matrix of the form $[k, k, \ldots, k, l, l, \ldots, l]$ (with arbitrary number of k's, including 0 and p) where $0 \leq k - l \leq 5|C|$. We call the number $k - l$ the *size of the step*.

The remainder of this subsection is devoted to the proof of the following Proposition:

Proposition 3.8. *Each almost rectangle is tame.*

First let us state a fact about almost rectangles of small step size.

Lemma 3.9. *Each almost rectangle of step size ≤ 1 is tame.*

Proof. Note that an almost rectangle of step size 0 or 1 is just a line segment $\langle i\rangle$, thus, this lemma follows from Lemma 3.6. □

To obtain a contradiction, assume that there is a counterexample to Proposition 3.8. In the following, let

$$X = [\underbrace{k, k, \ldots, k}_{m\text{-times}}, l, l, \ldots, l] \tag{3.5}$$

be a minimal counterexample, where minimal means that

(i) X has minimal step size, and

(ii) among the counterexamples with minimal step size, $|\lambda(X) - 1/3|$ is maximal.

We will separate the proof into two cases, $\lambda(X) \geq 5/12$ and $\lambda(X) < 5/12$. In [Bar19] it is remarked that with a sufficiently large p, any number strictly between $1/3$ and $1/2$ would work with the same case distinction and proof idea. The basic idea for both cases is very similar to the approach in the proof of Lemma 3.6.

Lemma 3.10. *The area $\lambda(X)$ for X defined in 3.5 can not be $\geq 5/12$.*

Proof. Assume that $\lambda(X) \geq 5/12$. Let

$$k_1 = \left\lceil \frac{p-l}{2} \right\rceil, \quad k_2 = \left\lfloor \frac{p-l}{2} \right\rfloor, \qquad l_1 = \left\lceil \frac{p-k}{2} \right\rceil, \quad l_2 = \left\lfloor \frac{p-k}{2} \right\rfloor.$$

Then

$$l_1 + l_2 + k = p = k_1 + k_2 + l, \tag{3.6}$$
$$1 \geq k_1 - k_2 \geq 0, \text{ and } 1 \geq l_1 - l_2 \geq 0. \tag{3.7}$$

Since $k \geq l$, we have $k_1 \geq l_1$ and $k_2 \geq l_2$. Further, by Lemma 3.9, $k - l \geq 2$, hence, both $k_1 - l_1$ and $k_2 - l_2$ are strictly smaller than $k - l$. Consider the matrices

$$Y_i = [\underbrace{l_i, l_i, \ldots, l_i}_{m\text{-times}}, k_i, k_i, \ldots, k_i], \quad i = 1, 2.$$

By shifting the entries in each row of Y_i to the left m times, we get an almost rectangle of smaller step size than $k - l$, which is tame by minimality of X. This changes neither the value of t (by cyclicity of s) nor the area of the matrices. Thus Y_i is g-equivalent to a tame matrix of the same area, hence, tame for $i = 1, 2$.
Let Y_1' (Y_2', resp.) be the matrices obtained from Y_1 (Y_2, resp.) by shifting the first m columns k times ($k+l_1$ times, resp.) down and the remaining columns l times ($l+k_1$ times, resp.) down. Then X, Y_1', Y_2' is a cover (by (3.6)), hence, by Lemma 3.4, X, Y_1', Y_2' are not all g-equivalent. Shifting columns does not change the value of t (by cyclicity of s), thus, also X, Y_1, Y_2 are not all g-equivalent. Since X, Y_1', Y_2' is a cover, we see that

$$1 = \lambda(X) + \lambda(Y_1') + \lambda(Y_2') = \lambda(X) + \lambda(Y_1) + \lambda(Y_2).$$

By (3.7), we get $\lambda(Y_2) \leq \lambda(Y_1)$ and since $k_1 - k_2 \leq 1, l_1 - l_2 \leq 1$, we have $\lambda(Y_1) - \lambda(Y_2) \leq p/p^2 = 1/p$. This yields

$$\lambda(Y_1) = 1 - \lambda(X) - \lambda(Y_2) \leq 1 - \frac{5}{12} - \lambda(Y_1) + \frac{1}{p}.$$

Since $p > 12$ by our choice in 3.2, we get

$$\lambda(Y_2) \leq \lambda(Y_1) < \frac{1}{3}.$$

Tameness of Y_1 and Y_2 yields $Y_1 \sim Y_2 \sim 0_{p\times p}$. By Lemma 3.6, $0_{p\times p} \nsim 1_{p\times p}$, thus, since X, Y_1, Y_2 are not all g-equivalent, we get $X \sim 1_{p\times p}$, which together with $\lambda(X) \geq 5/12 > 1/3$ yields tameness of X, a contradiction. □

It remains to show that $\lambda(X) < 5/12$ also leads to a contradiction, proving that a counterexample X can not exist.

Lemma 3.11. *The area $\lambda(X)$ for X defined in 3.5 can not be smaller than $5/12$.*

Proof. By definition of an almost rectangle, the step size of X is at most $5|C|$, where C is the domain of the sandwich for $(\mathbb{A}, \mathbb{B})$. Together with $p > 60|C|$, we have

$$\frac{5}{12} > \lambda(X) \geq \frac{p(k - 5|C|)}{p^2}$$

which implies

$$k \leq \frac{5p}{12} + 5|C| < \frac{5p}{12} + \frac{p}{12} = \frac{p}{2}.$$

This shows that

$$2l \leq k + l \leq 2k < p.$$

We divide the proof into two cases again. First let us assume that $m < p/2$ (remember that m is the number of rows of X with k entries equal to 1). Define

$$Y = [\underbrace{l, \ldots, l}_{m\text{-times}}, \underbrace{k, \ldots, k}_{m\text{-times}}, l, \ldots, l],$$
$$Z = [\underbrace{p-k-l, \ldots, p-k-l}_{2m\text{-times}}, p-2l, \ldots, p-2l].$$

Similar to the proof of Lemma 3.10, these matrices form a cover with X by shifting columns down (and thereby not changing the value of t). Thus, X, Y, Z are not all g-equivalent and $\lambda(X) + \lambda(Y) + \lambda(Z) = 1$. Furthermore, Y is obtained by shifting all rows of X to the right m times, which neither changes the area nor the value of t, hence, $\lambda(X) = \lambda(Y)$ and $t(X) = t(Y)$. Thus, we have $X \sim Y$ and therefore $X \nsim Z$.
Since $p - 2l \geq p - k - l$ and $(p - 2l) - (p - k - l) = k - l$, by shifting the columns of Z to the left $2m$ times, we get an almost rectangle Z' with the same step size as X. However, $\lambda(Z') = \lambda(Z)$ and $\lambda(X) \neq 1/3$ (since p is prime), thus, we obtain

$$\frac{|\lambda(Z) - 1/3|}{|\lambda(X) - 1/3|} = \frac{|(1 - 2\lambda(X)) - 1/3|}{|\lambda(X) - 1/3|} = \frac{|2(\lambda(X) - 1/3)|}{|\lambda(X) - 1/3|} = 2,$$

hence, $|\lambda(Z) - 1/3| > |\lambda(X) - 1/3|$. By minimality of X, we see that Z' is tame. Thus, Z is tame as well. The calculation also shows, that $\lambda(Z) - 1/3$ and $\lambda(X) - 1/3$ have opposite signs. Now tameness of Z and $X \nsim Z$ yield tameness of X, a contradiction.
It remains to show that $m \geq p/2$ leads to a contradiction as well. In this case, define

$$Y = [l, \ldots, l, \underbrace{k, \ldots, k}_{m\text{-times}}],$$
$$Z = [\underbrace{p-k-l, \ldots, p-k-l}_{(p-m)\text{-times}}, p-2k, \ldots, p-2k, \underbrace{p-k-l, \ldots, p-k-l}_{(p-m)\text{-times}}].$$

Again, we get a cover by shifting columns of Y and Z down. The proof in this case is very similar to the one above. This concludes the proof of the lemma. □

Proof of Proposition 3.8. Assume that the proposition is false. Then take a minimal counterexample X as above. By Lemma 3.10 and Lemma 3.11, the area $\lambda(X)$ of X can be neither $\geq 5/12$ nor smaller than $5/12$, a contradiction. □

3.2.3 The contradiction

Remember that all the proofs in the previous two subsections were based on the assumption that the finite sandwich structure $\mathbb{C}$ for the template $(\mathbb{A}, \mathbb{B})$, where $\mathbb{A}$ is the template for 1-in-3 and $\mathbb{B}$ the template for NAE-3, is tractable. Using a cyclic polymorphism s of $\mathbb{C}$ of arity $p > 60|C|$, which exists due to Theorem 2.4.4, we constructed the function t, which we used to define g-equivalence, where g is the homomorphism $g : \mathbb{C} \to \mathbb{B}$, and tameness. We saw that $0_{p\times p} = \langle 0 \rangle \not\sim \langle p^2 \rangle = 1_{p\times p}$ and almost rectangles are tame. Our aim now is to lead the assumption to a contradiction by constructing two tame matrices X_1, X_2 such that $\lambda(X_1) < {}^1\!/\!_3$ and $\lambda(X_2) > {}^1\!/\!_3$, but $t(X_1) = t(X_2)$.

Proof of Theorem 3.1. Let $\mathbb{C}$ be a tractable finite sandwich and define p, g-equivalence and tameness as in the previous sections. Then $0_{p\times p} \not\sim 1_{p\times p}$. Let

$$m = \frac{p-1}{2}.$$

By the choice of p, we have $p/3 - 2|C| > 0$. Thus, there are $2|C| > |C|$ integers between $p/3 - 2|C| \notin \mathbb{Z}$ and $p/3 - 1$. Since s maps to C, by the pigeonhole principle, we find l_1, l_2 such that

$$\frac{p}{3} - 2|C| < l_1 < l_2 < \frac{p}{3}$$

and

$$s(\underbrace{1,\dots,1}_{l_1\text{-times}},0,\dots,0) = s(\underbrace{1,\dots,1}_{l_2\text{-times}},0,\dots,0).$$

Now define

$$X_i = [\underbrace{k,\dots,k}_{m\text{-times}}, l_i,\dots,l_i], i = 1, 2,$$

where k will be chosen such that $\lambda(X_1) < 1/3 < \lambda(X_2)$ as follows. Note that for $k \leq p/3$, the area of both matrices is smaller than $1/3$. For $k \geq p/3 + 3|C|$, by

$$\lambda(X_i) = \frac{mk + (p-m)l_i}{p^2} \geq \frac{\frac{p-1}{2}(p/3 + 3|C|) + \frac{p+1}{2}(p/3 - 2|C|)}{p^2} = \frac{\frac{p^2}{3} + |C|\frac{(p-5)}{2}}{p^2} > \frac{1}{3},$$

we have that the area of both matrices is greater than $1/3$. Choose k maximal such that $\lambda(X_1) < 1/3$. Since increasing k by one increases the amount of 1's in the matrix by less than increasing l_1 and since we have $l_1 < l_2$, we have that $\lambda(X_2) > 1/3$.
Further, by

$$l_1 < l_2 < \frac{p}{3} < k < \frac{p}{3} + 3|C| \leq l_1 + 5|C| < l_2 + 5|C|,$$

we see that $0 \leq k - l_i \leq 5|C|$ for $i = 1, 2$, and thus, both matrices are almost rectangles. By Proposition 3.8, they are tame, hence, $X_1 \sim 0_{p\times p}$ and $X_2 \sim 1_{p\times p}$.
It remains to show that $t(X_1) = t(X_2)$. To see this, note that the inner s in the definition of t is applied to all columns of the matrices. The first m columns are the same in X_1 and X_2, hence,

so are their images under s. The remaining columns also have the same image under s by our choice of l_1, l_2. Hence, the outer s is applied to the same values for both matrices, which yields $t(X_1) = t(X_2)$, and thus $0_{p\times p} \sim X_1 \sim X_2 \sim 1_{p\times p}$, a contradiction to Lemma 3.6. Since any CSP of a finite relational structure is either in P or NP-complete, CSP($\mathbb{C}$) is NP-complete for any finite sandwich of $(\mathbb{A}, \mathbb{B})$. □

This shows that if we want to reduce some PCSP$(\mathbb{A}, \mathbb{B})$ to the tractable CSP of a sandwich structure $\mathbb{C}$, there are cases in which this sandwich has to be infinite. In particular, the PCSP framework is a proper generalisation of CSPs. On the other hand, in the next section we will see a characterisation of a class of PCSPs that always reduce to a finite tractable sandwich.

4 First-Order definable PCSPs

In this section we will describe a class of PCSPs that guarantees the existence of a finite tractable sandwich structure. We will also examine logical aspects of PCSPs. Some PCSPs can be "solved" by logical sentences. Given a PCSP-template $(\mathbb{A}, \mathbb{B})$ (with signature τ) and a logic $\mathcal{L}$ (e.g. first-order logic), we can ask whether there exists a sentence Φ (in the language of τ) in $\mathcal{L}$ whose class of finite models separates the yes-instances from the no-instances of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$. More precisely, $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is said to be *solvable* or *definable* by a sentence Φ if the following hold for every finite structure $\mathbb{X}$:

(*i*) If $\mathbb{X} \to \mathbb{A}$, then $\mathbb{X} \models \Phi$.

(*ii*) If $\mathbb{X} \nrightarrow \mathbb{B}$, then $\mathbb{X} \not\models \Phi$.

If $\mathcal{L}$ is first-order logic, we say that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is *first-order definable* by Φ.

4.1 Separability problem

Denote by $\mathrm{Mod}(\Phi)$ the class of finite models that satisfy Φ. Let Y be the class of yes-instances and N the class of no-instances of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$, then we have $Y \subseteq \mathrm{Mod}(\Phi)$ and $N \cap \mathrm{Mod}(\Phi) = \emptyset$. In this sense, the yes- and no-instances are separated. Consider the following question in this setting:

Separability problem for $\mathcal{L}$
For which promise constraint satisfaction problems $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ does there exist a sentence $\Phi \in \mathcal{L}$ such that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is solvable by Φ?

This problem is solved for several logics in the context of CSPs (i.e. when $\mathbb{A} = \mathbb{B}$), including first-order logic [Ats08]. In [Mot24], A. Mottet gave an answer to this question in the PCSP setting (see Theorem 4.1.2) in the case that $\mathcal{L}$ is first-order logic. Let us first define a concept that, in the CSP and PCSP context, is strongly connected to first-order definability.

Definition 4.1.1 (Duality)**.** Let $\mathcal{F}$ be a family of finite structures. We say that a structure $\mathbb{X}$ is $\mathcal{F}$*-free*, if there is no $\mathbb{F} \in \mathcal{F}$ such that $\mathbb{F} \to \mathbb{X}$. A PCSP-template $(\mathbb{A}, \mathbb{B})$ is said to have *duality* $\mathcal{F}$ if the following hold for every finite $\mathbb{X}$:

(*i*) If $\mathbb{X} \to \mathbb{A}$, then $\mathbb{X}$ is $\mathcal{F}$-free.

(*ii*) If $\mathbb{X} \nrightarrow \mathbb{B}$, then $\mathbb{X}$ is not $\mathcal{F}$-free.

If $\mathcal{F}$ is a finite set, we say that $(\mathbb{A}, \mathbb{B})$ has *finite duality*.
If any of the definitions above applies to $(\mathbb{A}, \mathbb{A})$, we say that the structure $\mathbb{A}$ has the corresponding property (for example "$\mathbb{A}$ has finite duality").

Now we are ready to state the answer to the Separability problem for first-order logic.

Theorem 4.1.2 ([Mot24]). *Let* $(\mathbb{A}, \mathbb{B})$ *be a finite PCSP template with finite signature* τ. *The following are equivalent:*

(i) $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is first-order definable.*

(ii) $(\mathbb{A}, \mathbb{B})$ *has finite duality.*

(iii) There exists a finite structure $\mathbb{C}$ *with finite duality and such that* $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$.

Note that if some PCSP is first-order definable, then we get a polynomial-time algorithm by checking the truth of the corresponding sentence Φ in a given input structure $\mathbb{X}$. It is known that this can be done in $\mathcal{O}(|X|^{|\Phi|})$.
This gives a sufficient condition for reducibility of a PCSP to a finite tractable sandwich. In the remainder of this chapter, we will show Theorem 4.1.2 by following the proof presented in [Mot24].

4.2 Finite duality and first-order definability

The equivalence of finite duality of a finite structure and its CSP being first-order definable was first proven by A. Atserias in [Ats08]. We will use the following stronger statement by B. Rossman in the proof of Theorem 4.1.2. We say that a sentence Φ is *existential positive* if it does not contain universal quantifiers and negation symbols.

Theorem 4.2.1 (Theorem 4.11 in [Ros08]). *Let* $\mathcal{P} \subseteq \mathcal{Q}$ *be classes of structures, and* Φ *be a first-order sentence such that:*

(i) for all finite $\mathbb{X}, \mathbb{Y}$ *such that* $\mathbb{X} \in \mathcal{P}$ *and* $\mathbb{X} \to \mathbb{Y}$, *we have* $\mathbb{Y} \models \Phi$,

(ii) for all finite $\mathbb{X}, \mathbb{Y}$ *such that* $\mathbb{X} \models \Phi$ *and* $\mathbb{X} \to \mathbb{Y}$, *we have* $\mathbb{Y} \in \mathcal{Q}$.

Then there exists an existential positive sentence Ψ *such that* $\mathcal{P} \subseteq \mathrm{Mod}(\Psi) \subseteq \mathcal{Q}$.

Let us show that this indeed implies Atserias's result.

Theorem 4.2.2. *Let* $\mathbb{A}$ *be a finite relational structure. Then* $\mathbb{A}$ *has finite duality if and only if* $\mathrm{CSP}(\mathbb{A})$ *is first-order definable.*

Proof. Assume first that $\mathbb{A}$ has finite duality $\mathcal{F}$. Identify every $\mathbb{F} = (F, (R_i)_{i=1}^m) \in \mathcal{F}$ with a primitive positive sentence ϕ_F as follows: Enumerate $F = \{f_1, \ldots, f_n\}$ and let $\phi_F = \exists x_1, \ldots, x_n \bigwedge_{i \le m} \phi_i$ where each ϕ_i is a conjunction that encodes all tuples in R_i, i.e. for

$$R_i = \{(f_{1,1}, \ldots, f_{1,k_i}), \ldots, (f_{m_i,1}, \ldots, f_{m_i,k_i})\},$$

we have $\phi_i = \bigwedge_{j \le m_i} R_i(x_{j,1}, \ldots, x_{j,k_i})$. In this case, we call $\mathbb{F}$ the canonical database of ϕ_F. Then for any structure $\mathbb{X}$ and $\mathbb{F} \in \mathcal{F}$, we have $\mathbb{F} \to \mathbb{X}$ if and only if ϕ_F is satisfiable in $\mathbb{X}$, i.e. that there are $x_1, \ldots, x_n \in X$ which satisfy the corresponding conjunction $\bigwedge_{i \le m} \phi_i$. Thus, $\mathrm{CSP}(\mathbb{A})$ is solvable

by $\Phi = \bigwedge_{F \in \mathcal{F}} \neg\phi_F$.
For the other implication assume that CSP($\mathbb{A}$) is solvable by a first-order sentence Φ. Let

$$\mathcal{P} = \mathcal{Q} = \{\mathbb{X} : \mathbb{X} \nrightarrow \mathbb{A}\} = \{\mathbb{X} : \mathbb{X} \models \neg\Phi\}$$

Then for any finite $\mathbb{X}, \mathbb{Y}$ such that $\mathbb{X} \in \mathcal{P}$ and $\mathbb{X} \to \mathbb{Y}$, we have $\mathbb{Y} \nrightarrow \mathbb{A}$ (otherwise $\mathbb{X} \to \mathbb{A}$), thus $\mathbb{Y} \models \neg\Phi$. On the other hand, if $\mathbb{X} \models \neg\Phi$ and $\mathbb{X} \to \mathbb{Y}$, again, $\mathbb{Y} \nrightarrow \mathbb{A}$, hence, $\mathbb{Y} \in \mathcal{Q}$. Applying Theorem 4.2.1 yields an existential positive sentence Ψ such that $\mathcal{P} = \mathrm{Mod}(\Psi) = \{\mathbb{X} : \mathbb{X} \nrightarrow \mathbb{A}\}$. Without loss of generality we can assume that Ψ is of the form $\bigvee_{i \leq k} \exists x_1, \ldots, x_m \psi_i$ where each ψ_i is a conjunction of positive literals that does not contain any equalities (eliminate any equalities by merging the corresponding variables). Reversing the construction from above, each ψ_i encodes a finite structure $\mathbb{F}_i$ and we have that

$$\mathbb{X} \to \mathbb{A} \quad \Leftrightarrow \quad \mathbb{X} \notin \mathcal{P} \quad \Leftrightarrow \quad \mathbb{X} \models \neg\Psi \quad \Leftrightarrow \quad \forall i \leq k : \mathbb{F}_i \nrightarrow \mathbb{X}$$

Hence, $\mathbb{A}$ has finite duality $\mathcal{F} = \{\mathbb{F}_i : i \leq k\}$. □

In the proof of the implication $(ii) \Rightarrow (iii)$ of Theorem 4.1.2, we will, in particular, make use of the finite number of orbits of an ω-categorical structure. The following result, initially by Cherlin, Shelah, and Shi [CSS99], and improved by Hubička and Nešetřil [HN19], yields existence of an ω-categorical structure with duality $\mathcal{F}$ for any finite set $\mathcal{F}$ of finite connected structures. We call a structure *connected* if it is not isomorphic to the disjoint union of two non-empty structures.

Theorem 4.2.3. *Let $\mathcal{F}$ be a finite set of finite connected structures in a finite relational signature. There exists an ω-categorical structure $\mathbb{C}$ such that $\mathbb{C}$ has duality $\mathcal{F}$. Moreover, $\mathbb{C}$ can be chosen to have an expansion $\mathbb{C}^+$ by finitely many relations such that $\mathbb{C}^+$ is homogeneous with the Ramsey property.*

4.3 1-tolerant polymorphisms

For a structure $\mathbb{C}^+$ given by Theorem 4.2.3 and any given $f : C^n \to C$, we can apply Theorem 2.9.4. In particular, if f is a polymorphism, it locally interpolates a polymorphism that is canonical with respect to $\mathrm{Aut}(\mathbb{C}^+)$. Let us now introduce a property of polymorphisms that is preserved under this local interpolation.

Definition 4.3.1 (1-tolerant polymorphism)**.** A polymorphism of a structure $\mathbb{A}$ is called *1-tolerant* if it satisfies that for any m-tuples $\overline{a}_1, \ldots, \overline{a}_n \in A^m$ such that all but at most one of them are in some relation $R^{\mathbb{A}} \subseteq A^m$, we have that $f(\overline{a}_1, \ldots, \overline{a}_n) \in R^{\mathbb{A}}$.

Lemma 4.3.2 ([Mot24])**.** *Let $\mathbb{A}$ be a relational structure. Let f be a 1-tolerant polymorphism of $\mathbb{A}$ and let $\mathbf{G}$ be a subset of $\mathrm{Aut}(\mathbb{A})$. Let g be an operation that is locally interpolated by f modulo $\mathbf{G}$. Then g is a 1-tolerant polymorphism of $\mathbb{A}$.*

Proof. Let g be locally interpolated by f modulo $\mathbf{G}$ and $R \subseteq A^m$ be a relation of $\mathbb{A}$. Let $\overline{a}_1, \ldots, \overline{a}_n \in A^m$ be such that all but at most one of them are in R. Then there exist $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbf{G}$ such that

$$g(\overline{a}_1, \ldots, \overline{a}_n) = \alpha(f(\alpha_1(\overline{a}_1), \ldots, \alpha_n(\overline{a}_n))).$$

Now for all but at most one $i \leq n$, we have that $\alpha_i(\bar{a}_i) \in R$, since all α_i are automorphisms. By 1-tolerance of f, we get that $f(\alpha_1(\bar{a}_1), \ldots, \alpha_n(\bar{a}_n)) \in R$. Finally, α is an automorphism, hence, $\alpha(f(\alpha_1(\bar{a}_1), \ldots, \alpha_n(\bar{a}_n))) = g(\bar{a}_1, \ldots, \bar{a}_n) \in R$. □

Before getting to the proof of Theorem 4.1.2, let us state a result that connects finite duality of finite-domain or ω-categorical structures to 1-tolerant polymorphisms.

Theorem 4.3.3 ([LLT07])**.** *Let $\mathbb{A}$ be a finite or ω-categorical structure. The following are equivalent:*

(i) $\mathbb{A}$ has finite duality.

(ii) $\mathbb{A}$ has a 1-tolerant polymorphism.

4.4 A compactness argument

In the proof of Theorem 4.1.2, we will use a so-called "compactness argument". This notion can be found in various papers. Many times the argument uses König's tree lemma. This lemma states that a rooted tree T with an infinite number of nodes, each with a finite number of children, has a branch of infinite length. The proof of Lemma 4.4.1 is an example of such a compactness argument.

Lemma 4.4.1 ([BD13])**.** *Let τ be any relational signature. Then a countable τ-structure $\mathbb{C}$ admits a homomorphism (an embedding) to a countable ω-categorical τ-structure $\mathbb{B}$ with oligomorphic automorphism group if and only if every finite substructure of $\mathbb{C}$ admits a homomorphism (an embedding) to $\mathbb{B}$.*

Proof. We proof the statement for homomorphisms. The arguments in the other case are analogous. The forward implication is trivial. Let us therefore assume that every finite substructure of $\mathbb{C}$ admits a homomorphism to $\mathbb{B}$. We will use König's tree lemma to show the existence of a homomorphism from $\mathbb{C}$ to $\mathbb{B}$. Let $c_1, c_2, \ldots$ be an enumeration of C. Construct a rooted tree with all nodes being on some level $n \geq 0$. On level n the nodes represent equivalence classes of homomorphisms from the substructure $\mathbb{C}_n$ of $\mathbb{C}$ induced by $C_n := \{c_1, \ldots, c_n\}$ to $\mathbb{B}$. Two such homomorphisms are equivalent, if they map the (ordered) tuple $(c_1, \ldots, c_n)$ to the same n-orbit of $\mathbb{B}$ or, equivalently, if for such homomorphisms f and g, there is some $\alpha \in \mathrm{Aut}(\mathbb{B})$ with $\alpha \circ f = g$. Two equivalence classes of homomorphisms on level n and $n+1$ are adjacent if there are representatives f_n and f_{n+1}, respectively, such that f_n is the restriction of f_{n+1} to $\mathbb{C}_n$. As there are only finitely many n-orbits in $\mathbb{B}$ for each $n \in \mathbb{N}$, each node has finitely many children. For each $n \in \mathbb{N}$, we can map $\mathbb{C}_n$ homomorphically to $\mathbb{B}$, so there are nodes on each level. Thus, we can apply König's lemma and get an infinite branch. We can use this branch to define a homomorphism $h : \mathbb{C} \to \mathbb{B}$ inductively as follows. Denote the equivalence classes of the nodes of this branch by $(H_n)_{n\in\mathbb{N}}$ where H_n is represented by the node on the n-th level. We want to define a sequence $(h_n)_{n\in\mathbb{N}}$ of homomorphisms $h_n : \mathbb{C}_n \to \mathbb{B}$ such that $h_n \in H_n$ and $h_{n+1}|_{C_n} = h_n$ for all $n \in \mathbb{N}$. Let h_0 be the empty function. Then h_0 clearly meets the requirements. Now assume that $h_0, \ldots, h_n$ are already defined. Since the nodes representing H_n and H_{n+1} are adjacent, there is some $h_{n+1} \in H_{n+1}$ such that h_{n+1} extends h_n, hence, this function satisfies the requirements. Thus, inductively defining h_n

for $n \in \mathbb{N}$ in this way, we get a chain of homomorphisms. Hence, by setting $h(c_n) := h_n(c_n)$ for all $n \in \mathbb{N}$, we get a well-defined function $h : C \to B$. Clearly, this is a homomorphism from $\mathbb{C} \to \mathbb{B}$.

□

4.5 A characterisation of first-order solvable finite-domain PCSPs

The proof of Theorem 4.1.2 uses the construction of a finite structure from an infinite one by factoring by an automorphism group. Let us explain this construction in a more general setting.

Definition 4.5.1. Let $\mathbb{C}$ be a relational structure and $\mathbf{G} \curvearrowright C$ a permutation group. Then $\mathbf{G}$ induces an equivalence relation $\sim_\mathbf{G}$ by

$$c_1 \sim_\mathbf{G} c_2 \Leftrightarrow \exists g \in \mathbf{G} : g(c_1) = c_2.$$

By $\mathbb{C}/_\mathbf{G}$ we denote the structure whose domain consists of the equivalence classes of $\sim_\mathbf{G}$, and such that for every relatio n $R^\mathbb{C} \subseteq C^k$ in the signature of $\mathbb{C}$, we have $(\mathcal{O}_1, \dots, \mathcal{O}_k) \in R^{\mathbb{C}/\mathbf{G}}$ if, and only if, there exist $c_1 \in \mathcal{O}_1, \dots, c_k \in \mathcal{O}_k$ with $(c_1, \dots, c_k) \in R^\mathbb{C}$.

Let us now prove the central statement of this section:

Proof of Theorem 4.1.2.
$(i) \Rightarrow (ii)$: This will follow from Theorem 4.2.1 in a similar fashion as in the proof of Theorem 4.2.2. Let $\mathcal{P}$ be the class of finite structures $\mathbb{X}$ such that $\mathbb{X} \not\to \mathbb{B}$, and $\mathcal{Q}$ be the class of finite structures $\mathbb{X}$ such that $\mathbb{X} \not\to \mathbb{A}$. Let Φ be a first-order sentence solving PCSP$(\mathbb{A}, \mathbb{B})$. Then we have $\mathcal{P} \subseteq \text{Mod}(\neg\Phi) \subseteq \mathcal{Q}$. Moreover, if $\mathbb{X} \in \mathcal{P}$ and $\mathbb{X} \to \mathbb{Y}$, then $\mathbb{Y} \not\to \mathbb{B}$, since $\mathbb{X} \not\to \mathbb{B}$ and hence, $\mathbb{Y} \models \neg\Phi$. Similarly, if $\mathbb{X} \models \neg\Phi$ and $\mathbb{X} \to \mathbb{Y}$, then $\mathbb{Y} \not\to \mathbb{A}$, since $\mathbb{X} \not\to \mathbb{A}$, hence, $\mathbb{Y} \in \mathcal{Q}$. So we can apply Theorem 4.2.1 to obtain an existential positive sentence Ψ such that $\mathcal{P} \subseteq \text{Mod}(\Psi) \subseteq \mathcal{Q}$. Now Ψ is equivalent to a disjunction $\bigvee_{i \leq m} \Psi_i$ of primitive positive sentences Ψ_i not containing equalities (otherwise we can merge the corresponding variables). Define $\mathbb{F}_{\Psi_i}$ for $i \leq m$ as follows: For $\Psi_i = \exists x_1, \dots, x_n(\psi_1 \wedge \dots \wedge \psi_m)$ with positive atomic formulas ψ_j for $j \leq m$, let $F = \{f_1, \dots, f_n\}$ with distinct elements $f_i \neq f_j$ for $1 \leq i \neq j \leq n$. For each $\psi_j = R(x_{j_1}, \dots, x_{j_k})$ with $R \in \tau$ of arity $k \in \mathbb{N}$, add $(f_{j_1}, \dots, f_{j_k})$ to $R^{\mathbb{F}_{\Psi_i}}$. Note that for any τ-structure $\mathbb{X}$ and any $i \leq m$, we have $\mathbb{X} \models \Psi_i$ if and only if $\mathbb{F}_{\Psi_i} \to \mathbb{X}$. Let $\mathcal{F} := \{\mathbb{F}_{\Psi_i} : i \leq m\}$. Let us now show that $(\mathbb{A}, \mathbb{B})$ has duality $\mathcal{F}$. For any finite $\mathbb{X}$, if there exists $\mathbb{F} \in \mathcal{F}$ such that $\mathbb{F} \to \mathbb{X}$, then $\mathbb{X} \models \Psi$, hence, $\mathbb{X} \in \mathcal{Q}$ and $\mathbb{X} \not\to \mathbb{A}$. On the other hand, if $\mathbb{X} \not\to \mathbb{B}$ then $\mathbb{X} \in \mathcal{P}$, thus, $\mathbb{X} \models \Psi$ and therefore, there is some $\mathbb{F} \in \mathcal{F}$ such that $\mathbb{F} \to \mathbb{X}$. This means that $\mathcal{F}$ forms a duality for $(\mathbb{A}, \mathbb{B})$.
$(ii) \Rightarrow (iii)$: Let $\mathcal{F}$ be a duality for PCSP$(\mathbb{A}, \mathbb{B})$. Let us show that without loss of generality we can assume that $\mathcal{F}$ consists of connected structures. Assume there is some $\mathbb{F} \in \mathcal{F}$ that is isomorphic to the disjoint union of two non-empty structures $\mathbb{F}_1, \mathbb{F}_2$. Since $\mathbb{A}$ is $\mathcal{F}$-free, we have $\mathbb{F} \not\to \mathbb{A}$, hence, either $\mathbb{F}_1 \not\to \mathbb{A}$ or $\mathbb{F}_2 \not\to \mathbb{A}$. Assume without loss of generality that $\mathbb{F}_1 \not\to \mathbb{A}$ and define $\mathcal{F}' := (\mathcal{F} \cup \{\mathbb{F}_1\}) \setminus \{\mathbb{F}\}$. We want to show that $\mathcal{F}'$ is a duality for $(\mathbb{A}, \mathbb{B})$. Therefore, suppose $\mathbb{X} \to \mathbb{A}$. Then $\mathbb{X}$ is $\mathcal{F}$-free and since $\mathbb{F}_1 \not\to \mathbb{A}$, also $\mathbb{F}_1$-free, hence, $\mathcal{F}'$-free. On the other hand, if $\mathbb{X}$ is $\mathcal{F}'$-free, then $\mathbb{X}$ is, in particular, $\mathcal{F}$-free, so $\mathbb{X} \to \mathbb{B}$.

Now by Theorem 4.2.3, there exists an ω-categorical structure $\mathbb{C}$ that has duality $\mathcal{F}$. Since $\mathbb{A}$ is $\mathcal{F}$-free, we have $\mathbb{A} \to \mathbb{C}$. Every finite substructure of $\mathbb{C}$ is $\mathcal{F}$-free and therefore admits a homomorphism to $\mathbb{B}$. By the compactness argument presented in Lemma 4.4.1, we get $\mathbb{C} \to \mathbb{B}$. Together, we have $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$.

Since $\mathbb{C}$ is ω-categorical and has finite duality, Theorem 4.3.3 yields a 1-tolerant polymorphism $f : C^n \to C$. Further, Theorem 4.2.3 yields that there is a homogeneous Ramsey expansion $\mathbb{C}^+$ of $\mathbb{C}$ by finitely many relations. Since $\mathbb{C}^+$ is homogeneous and has finite signature, it is ω-categorical. By Theorem 2.9.4, f locally interpolates an operation $g : C^n \to C$ modulo $\mathrm{Aut}(\mathbb{C}^+)$ which is canonical with respect to $\mathrm{Aut}(\mathbb{C}^+)$. By Lemma 4.3.2, g is a 1-tolerant polymorphism of $\mathbb{C}$.

Consider $\mathbb{C}' := \mathbb{C}/_{\mathrm{Aut}(\mathbb{C}^+)}$. Then $\mathbb{C}'$ is finite. Define $g' : (C')^n \to C'$ as follows: Let $g'(\mathcal{O}_1, \ldots, \mathcal{O}_n)$ be the $\sim_{\mathrm{Aut}(\mathbb{C}^+)}$-class of $g(c_1, \ldots, c_n)$ for any $c_1 \in \mathcal{O}_1, \ldots, c_n \in \mathcal{O}_n$. This definition does not depend on the chosen elements, since g is canonical with respect to $\mathbb{C}^+$. It can be easily seen that g' is a 1-tolerant polymorphism of $\mathbb{C}'$.

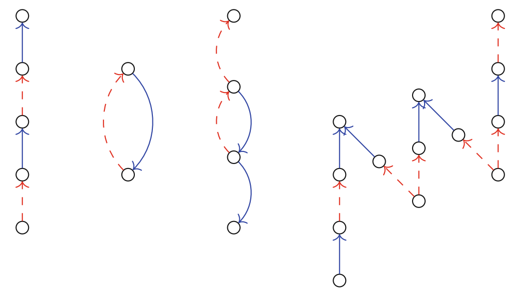
Finally, let $h : \mathbb{C} \to \mathbb{B}$ be a homomorphism. Since $\mathrm{Aut}(\mathbb{C}^+)$ is extremely amenable by Theorem 2.8.3 and $\mathrm{Aut}(\mathbb{B})$ is oligomorphic as an automorphism group of a finite structure, we can apply Theorem 2.9.3 to $\mathrm{Aut}(\mathbb{C}^+)$ and $\mathrm{Aut}(\mathbb{B})$ to obtain a homomorphism h' that is canonical from $\mathbb{C}^+$ to $\mathbb{B}$. Again, it can be easily verified that h' induces a homomorphism $h'' : \mathbb{C}' \to \mathbb{B}$. Hence, we have $\mathbb{A} \to \mathbb{C} \to \mathbb{C}' \to \mathbb{B}$, in particular, $\mathbb{A} \to \mathbb{C}' \to \mathbb{B}$. Since $\mathbb{C}'$ has a 1-tolerant polymorphism g', by Theorem 4.3.3, $\mathbb{C}'$ has finite duality.

$(iii) \Rightarrow (i)$: By Theorem 4.2.2, $\mathrm{CSP}(\mathbb{C})$ is definable by a first-order sentence Φ. This sentence also proves that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is first-order definable. □

We saw a characterisation of first-order solvability of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ by $(\mathbb{A}, \mathbb{B})$ having finite duality. Now let us show that there are proper examples of PCSP templates with finite duality (i.e. templates $(\mathbb{A}, \mathbb{B})$ with finite duality such that neither $\mathbb{A}$ nor $\mathbb{B}$ has finite duality).

Proposition 4.5.2 ([Mot24])**.** *There is a PCSP template* $(\mathbb{A}, \mathbb{B})$ *with finite duality such that neither* $\mathbb{A}$ *nor* $\mathbb{B}$ *has finite duality.*

Proof. Consider the following four structures in relational signature with two binary symbols R and B (denoted by red (dashed) and blue (solid) arrows):



$\mathbb{A}$ $\mathbb{C}$ $\mathbb{B}$ $\mathbb{F}_2$

Then clearly $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$. Moreover, $\mathbb{C}$ has the duality that consists of the following four structures: an R-path of length 2, a B-path of length 2, a graph consisting of 3 nodes x_1, x_2, x_3 with $x_1 B x_2$

and $x_3 R x_2$ (this prevents a vertex with incoming B- and R-edges) and a graph consisting of 3 nodes y_1, y_2, y_3 with $y_2 B y_1$ and $y_2 R y_3$ (this prevents a vertex with outgoing B- and R-edges).
Let us show that neither $\mathbb{A}$ nor $\mathbb{B}$ have finite duality. Let us start with $\mathbb{A}$. Let $\mathbb{S}_1 + \mathbb{S}_2$ ($\mathbb{S}_1 - \mathbb{S}_2$, resp.) for a structure $\mathbb{S}_1$ with a designated endpoint and a directed path $\mathbb{S}_2$ define a concatenation such that the designated endpoint of $\mathbb{S}_1$ and the starting point (endpoint, resp.) of $\mathbb{S}_2$ are merged and the other nodes are added without further merging any nodes. For example, $\mathbb{A} = \mathbb{P} + \mathbb{P}$ for $\mathbb{P}$ being a path consisting of an R-edge followed by a B-edge. Let further $\mathbb{R}$ be a path consisting of a single R-edge, and $\mathbb{B}'$ be a path consisting of a single B-edge. Now define for every $n \geq 1$ the structure

$$\mathbb{F}_n := \mathbb{B}' + \underbrace{\mathbb{P} - \mathbb{P} + \cdots + \mathbb{P} - \mathbb{P}}_{n\text{-times } \mathbb{P}-\mathbb{P}} + \mathbb{P} + \mathbb{R},$$

by inductively defining the endpoints of the partial sums as the last added vertex, such that this definition coincides for $n = 2$ with the structure shown above. We have that $\mathbb{F}_n \nrightarrow \mathbb{A}$ for all $n \geq 1$. However, every $\mathbb{F}_n$ can be mapped to $\mathbb{A}$ homomorphically if we remove one edge. Assume there is some finite duality $\mathcal{F}$ for $\mathbb{A}$. Let $m := \max\{|\mathbb{F}| : \mathbb{F} \in \mathcal{F}\}$. Since $|\mathbb{F}_m| = 4m + 5 > m$, if $\mathbb{F} \to \mathbb{F}_m$ for some $\mathbb{F} \in \mathcal{F}$, then there is at least one edge we can remove and obtain a structure $\mathbb{F}'_m$ such that $\mathbb{F} \to \mathbb{F}'_m$. Thus, also $\mathbb{F} \to \mathbb{A}$, a contradiction. But since $\mathbb{F}_m \nrightarrow \mathbb{A}$, there must be some $\mathbb{F} \in \mathcal{F}$ with $\mathbb{F} \to \mathbb{F}_m$, contradicting the assumption that $\mathbb{A}$ has finite duality.
The proof for $\mathbb{B}$ works similarly by colouring all edges in each $\mathbb{F}_n$ with the same colour. □

5 A pair $(\mathbb{C}, \mathbb{B})$ with ω-categorical $\mathbb{C}$ and finite $\mathbb{B}$ that is not finitely tractable

The first approach when trying to prove tractability for a PCSP via the sandwiching method is to look for a finite tractable sandwich. However, in Chapter 3, we have seen that in some cases there is no finite tractable sandwich structure, but there is an infinite one. Even though oligomorphic clones enjoy many properties of function clones on finite sets, ω-categorical structures encode a much larger class of CSPs than finite structures. So there is hope that even if there is no finite tractable sandwich for a given PCSP template, in some cases we might still be able to find an ω-categorical one. In this chapter we show that under the assumption that P $\neq$ NP, there is an example of a pair $(\mathbb{C}, \mathbb{B})$ as in Question 1.3 that yields a negative answer, i.e. that the pair $(\mathbb{C}, \mathbb{B})$ is not finitely tractable.

Theorem 5.1. *There exists a pair $(\mathbb{C}, \mathbb{B})$ of τ-structures with tractable ω-categorical $\mathbb{C}$ and finite $\mathbb{B}$ such that $\mathbb{C} \to \mathbb{B}$ and $(\mathbb{C}, \mathbb{B})$ is not finitely tractable, unless $P = NP$.*

Note that if we are in the situation of Theorem 5.1 and we would find a finite structure $\mathbb{A}$ with NP-complete CSP such that $\mathbb{A} \to \mathbb{C}$ and such that there is no finite tractable $\mathbb{A}'$ with $\mathbb{A} \to \mathbb{A}' \to \mathbb{B}$, then we would have a PCSP template $(\mathbb{A}, \mathbb{B})$ with a tractable ω-categorical sandwich $\mathbb{C}$ such that $(\mathbb{A}, \mathbb{B})$ is not finitely tractable, answering Question 1.4.

5.1 Some properties of $(\mathbb{Q}; <)$ and $(\mathbb{Q}; <, I_4)$

We will base the construction of a pair $(\mathbb{C}, \mathbb{B})$ as in Theorem 5.1 on a well-known example of an ω-categorical structure, namely $(\mathbb{Q}; <)$. We will show ω-categoricity by proving that $(\mathbb{Q}; <)$ is homogeneous. Then, it will follow from homogeneity in a finite relational language.

Proposition 5.1.1. $(\mathbb{Q}, <)$ *is homogeneous.*

Proof. Let $A, B \subseteq \mathbb{Q}$ be finite subsets of $\mathbb{Q}$ and let $\beta : A \to B$ be an isomorphism. Enumerate $\mathbb{Q} \backslash A := (c_i)_{i \in \mathbb{N}}$ and $\mathbb{Q} \backslash B := (d_j)_{j \in \mathbb{N}}$. Define $\alpha_0 := \beta$. We will use a well-known back-and-forth argument to inductively define partial isomorphisms $\alpha_k : A_k \to B_k$ between finite subsets $A_k, B_k \subseteq \mathbb{Q}$ with $\bigcup_{k \in \mathbb{N}} A_k = \bigcup_{k \in \mathbb{N}} B_k = \mathbb{Q}$ such that for $l < k$ we have $A_l \subseteq A_k$ and $\alpha_k|_{A_l} = \alpha_l$. Suppose $\alpha_0, \ldots, \alpha_m$ for $m \in \mathbb{N}$ are already defined. Define $i_0 := \min\{i \in \mathbb{N} : c_i \notin A_m\}$. Take a partition of A_m into sets $A_<$ of elements smaller than c_{i_0} and $A_>$ of elements greater than c_{i_0}. By density of $\mathbb{Q}$ and since α_m is a partial isomorphism, there exists some $d \in \mathbb{Q} \backslash B_m$ such that $x < d < y$ for all $x \in \alpha_m(A_<), y \in \alpha_m(A_>)$. Let j_0 be the index of d in the enumeration of $\mathbb{Q} \backslash B$. Then, $\alpha_m \cup \{(c_{i_0}, d_{j_0})\}$ is a partial isomorphism extending α_m. Similarly, for $j_1 := \min\{j \in \mathbb{N} : d_j \notin (B_m \cup \{d_{j_0}\})\}$, we find $c_{i_1} \in \mathbb{Q} \backslash (A_m \cup \{c_{i_0}\})$ such that $\alpha_{m+1} := \alpha_m \cup \{(c_{i_0}, d_{j_0}), (c_{i_1}, d_{j_1})\}$ is a partial isomorphism. Now define $\alpha = \bigcup_{k \in \mathbb{N}} \alpha_k$. By construction, α is automorphism extending β. $\square$

Proposition 5.1.2. *A countable homogeneous relational structure $\mathbb{A}$ with finite signature is ω-categorical.*

Proof. We want to show that $\mathbb{A}$ has only finitely many n-orbits for each $n \geq 1$. Let $n \in \mathbb{N}$ and $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n$. If the mapping $a_i \mapsto b_i$ is a partial isomorphism between the substructures induced by $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, then it can be extended to an automorphism by homogeneity. This shows that the orbit of some tuple $(a_1, \ldots, a_n)$ only depends on the relations (including equality) between its entries. Since the signature is finite, there are only finitely many possibilities. Hence, there can only be finitely many n-orbits. □

Corollary 5.1.3. *$(\mathbb{Q}; <)$ is ω-categorical.*

Proof. This follows from Proposition 5.1.1 and Proposition 5.1.3. □

The Ramsey property is a powerful tool for generating canonical functions. We want to show that $(\mathbb{Q}; <)$ is Ramsey by using the finite version of Ramsey's Theorem.

Theorem 5.1.4 (Finite version of Ramsey's theorem, [Ram30])**.** *Let $s, m, c \in \mathbb{N}$ and let $\binom{[l]}{s}$ denote the set of all s-subsets of $[l]$. Then there exists a positive integer l such that for every $\chi : \binom{[l]}{s} \to [c]$ there exists a monochromatic $M \in \binom{[l]}{m}$, i.e. χ is constant on all s-subsets of M.*

Proposition 5.1.5. *The class of finite linear orders is a Ramsey class. Therefore, $(\mathbb{Q}; <)$ is Ramsey and $\mathrm{Aut}(\mathbb{Q}; <)$ is extremely amenable.*

Proof. Note that for any two finite linear orders $\mathbb{L}_1$, $\mathbb{L}_2$ with n_1 and n_2 elements, respectively, $\mathbb{L}_1$ can be embedded into $\mathbb{L}_2$ if and only if $n_1 \leq n_2$ by mapping it to any substructure of $\mathbb{L}_2$ that has n_1 elements. The corresponding embedding must map the smallest element to the smallest and so on, hence, there is a one to one correspondence between subsets of L_2 size n_1 and embeddings of $\mathbb{L}_1$ into $\mathbb{L}_2$. Thus, it is a direct consequence of Theorem 5.1.4 that the class of finite linear orders is a Ramsey class. Since $\mathrm{Age}(\mathbb{Q}; <)$ is precisely the class of finite linear orders, $(\mathbb{Q}; <)$ is Ramsey, and since it is countable and homogeneous, by Theorem 2.8.3, $\mathrm{Aut}(\mathbb{Q}; <)$ is extremely amenable. □

For convenience, we will add another relation to the signature of $(\mathbb{Q}; <)$ and use the obtained structure to construct a pair $(\mathbb{C}, \mathbb{B})$ as in Theorem 5.1.

Definition 5.1.6. The relation I_4 is defined by

$$I_4(x, y, u, v) \Leftrightarrow (x = y) \Rightarrow (u = v).$$

Lemma 5.1.7 ([BCP10])**.** *Let f be a function from A^n to A that depends on all arguments. Then the following are equivalent:*

(i) f is injective.

(ii) f preserves the relation I_4 defined by $x = y \Rightarrow u = v$.

Proof. For the implication from (i) to (ii), suppose that f is injective and let $a, b, c, d \in A^n$ such that $a_i = b_i \Rightarrow c_i \Rightarrow d_i$ for all $i \leq n$. Let $t = (t_1, t_2, t_3, t_4) := (f(a), f(b), f(c), f(d))$. If $a = b$, then $c = d$ and thus, $t_1 = t_2$ and $t_3 = t_4$. By injectivity of f, if $a \neq b$, we have $f(a) \neq f(b)$. In both cases $t_1 = t_2 \Rightarrow t_3 = t_4$ is satisfied.
For the backwards implication, suppose that there are distinct $a, b \in A^n$ with $f(a) = f(b)$. Let J be the set of all $j \in \{1, \ldots, n\}$ such that $a_j \neq b_j$. Since $a \neq b$, J is non-empty. Let $j \in J$ be arbitrary. Since f depends on all arguments, we can find $c, d \in A^n$ with $f(c) \neq f(d)$, such that $c_i = d_i$ for all $i \leq n$ with $i \neq j$, and $c_j \neq d_j$. We claim that (a, b, c, d) shows that f does not preserve I_4. Note that if $a_i = b_i$ then $c_i = d_i$ by construction. Thus, $I_4(a_i, b_i, c_i, d_i)$ for all $i \leq n$, but $f(a) = f(b) \wedge f(c) \neq f(d)$. □

Since I_4 is preserved by any injective function, adding it to the signature of a relational structure will not change the automorphism group. Hence, we obtain the following corollary.

Corollary 5.1.8. $(\mathbb{Q}; <, I_4)$ *is a homogeneous Ramsey structure.*

Proof. Since I_4 is preserved by all injective functions, we have $\mathrm{Aut}(\mathbb{Q}; <) = \mathrm{Aut}(\mathbb{Q}; <, I_4)$. Since $(\mathbb{Q}; <)$ is a homogeneous Ramsey structure, Proposition 2.8.5 yields that $(\mathbb{Q}; <, I_4)$ is, too. □

5.2 Definition and some properties of $\mathbb{C}$

Definition 5.2.1. In the following, let $\mathbb{C}$ be the structure with domain $C := \mathbb{Q}^3$ and the following relations (for $x \in C$ we denote by x_i the i-th entry of x):

- For any pair (i, j) with $1 \leq i \neq j \leq 3$ define the unary relation

$$<_{i,j} (x) \Leftrightarrow x_i < x_j.$$

- For any pair (i, j) with $1 \leq i, j \leq 3$ define the binary relation

$$<'_{i,j} (x, y) \Leftrightarrow x_i < y_j.$$

- For any $k \leq 3$ and k-tuples $(i_1, \ldots, i_k), (j_1, \ldots, j_k)$, each consisting of pairwise different entries ≤ 3, define the binary *compatibility relation*

$$S_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(x, y) \Leftrightarrow \forall l \leq k : x_{i_l} = y_{j_l}$$

  for $x, y \in C$. For $k = 1$, we will also write $S_{i,j}$ instead of $S_{(i),(j)}$.

- For any two pairs $(i_1, i_2), (j_1, j_2)$ with $1 \leq i_1 \neq i_2 \leq 3, 1 \leq j_1 \neq j_2 \leq 3$ define the binary relation

$$I_{(i_1,i_2),(j_1,j_2)}(x, y) \Leftrightarrow (x_{i_1} = x_{i_2}) \Rightarrow (y_{j_1} = y_{j_2})$$

  for $x, y \in C$.

Lemma 5.2.2. *Let $\mathbb{C}$ be the structure from Definition 5.2.1. Then*

$$\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{Q}; <, I_4) \curvearrowright \mathbb{Q}^3.$$

Proof. "$\subseteq$": Let $f : (\mathbb{Q}^3)^n \to \mathbb{Q}^3 \in \mathrm{Pol}(\mathbb{C})$. Define $f' : \mathbb{Q}^n \to \mathbb{Q}$ as follows: For $(q_1, \ldots, q_n) \in \mathbb{Q}^n$, let $x^i := (q_i, q_i, q_i) \in \mathbb{Q}^3$ for $i \leq n$ and define $f'(q_1, \ldots, q_n)$ to be the first entry of $f(x^1, \ldots, x^n)$. To obtain $f = f' \curvearrowright \mathbb{Q}^3$ we will show that if a tuple $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ appears as a row in any n-tuple $(x^1, \ldots, x^n) \in \mathbb{Q}^3$ with $x^j = (x_1^j, x_2^j, x_3^j)$, i.e. there is $i \leq 3$ such that $x_i^j = q_j$ for all $j \leq n$, then the i-th entry of $f(x^1, \ldots, x^n)$ is $f'(q_1, \ldots, q_n)$.
For any $(x^1, \ldots, x^n), (y^1, \ldots, y^n) \in (\mathbb{Q}^3)^n$ with $(x_i^1, \ldots, x_i^n) = (y_j^1, \ldots, y_j^n)$ for some $1 \leq i, j \leq 3$, we have $S_{i,j}(x^l, y^l)$ for all $l \leq n$. Since f is a polymorphism, we get $S_{i,j}(f(x^1, \ldots, x^n), f(y^1, \ldots, y^n))$, which means $f(x^1, \ldots, x^n)_i = f(y^1, \ldots, y^n)_j$.
Thus, we get $f = f' \curvearrowright \mathbb{Q}^3$. The definitions of $<_{i,j}$ and $I_{(i_1,i_2),(j_1,j_2)}$ guarantee $f' \in \mathrm{Pol}(\mathbb{Q}; <, I_4)$.
"$\supseteq$": For every $f \in \mathrm{Pol}(\mathbb{Q}, <, I_4)$, we have that $f \curvearrowright \mathbb{Q}^3 \in \mathrm{Pol}(\mathbb{C})$. This is readily checked. ☐

Later in this chapter we want to obtain a factor structure consisting of all 1-orbits of $\mathbb{C}$. Let us show that this set is finite.

Lemma 5.2.3. *Let $\mathbb{C}$ be the structure from Definition 5.2.1. Then $\mathbb{C}$ is homogeneous. In particular, $\mathbb{C}$ is ω-categorical.*

Proof. Let $\mathbb{C}_1, \mathbb{C}_2$ be finite substructures of $\mathbb{C}$ and $\alpha_0 : C_1 \to C_2$ a partial isomorphism. Define D_i for $i \leq 2$ as the set of entries appearing in some $c \in C_i$ and $\mathbb{D}_1$ and $\mathbb{D}_2$ as the substructures of $(\mathbb{Q}; <, I_4)$ induced by D_1 and D_2. Then if some element $d \in D_1$ appears as an entry of two elements of C_1, i.e. $d = c_i = c'_j$ for some $c = (c_1, c_2, c_3), c' = (c'_1, c'_2, c'_3) \in C_1$ and $1 \leq i, j \leq 3$, the relation $S_{i,j}$ guarantees that $\alpha_0(c)_i = \alpha_0(c')_j$, hence, α_0 induces a function $\beta_0 : D_1 \to D_2$. By the relations $<'_{i,j}$, and since α_0 is a partial isomorphism, we have that β_0 and its inverse preserve $<$. Hence, β_0 and its inverse are injective and thus preserve I_4. So β_0 is a partial isomorphism. By homogeneity of $(\mathbb{Q}; <, I_4)$, we can extend β_0 to an automorphism β of $(\mathbb{Q}; <, I_4)$. Now $\alpha := \beta \curvearrowright \mathbb{Q}^3$ is an automorphism of $\mathbb{C}$ extending α_0. This yields homogeneity of $\mathbb{C}$. As a homogeneous structure in a finite relational language, $\mathbb{C}$ is ω-categorical. ☐

The relation I_4 is an *Ord-Horn* relation (see [NB95] for an introduction of Ord-Horn), hence, $(\mathbb{Q}; <, I_4)$ belongs to the well-known class of structures with Ord-Horn constraint languages. Thus, it is tractable [NB95]. Since $\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{Q}; <, I_4) \curvearrowright \mathbb{Q}^3$, by the function $\phi : f \mapsto f \curvearrowright \mathbb{Q}^3$, we find a bijection from $\mathrm{Pol}(\mathbb{Q}; <, I_4)$ to $\mathrm{Pol}(\mathbb{C})$. Equip $\mathrm{Pol}(\mathbb{Q}; <, I_4)$ and $\mathrm{Pol}(\mathbb{C})$ with the topology of pointwise convergence. Fixing the behaviour of functions in $\mathrm{Pol}(\mathbb{Q}; <, I_4)$ on a finite subset fixes the behaviour of functions in $\mathrm{Pol}(\mathbb{C})$ on the finitely many induced triples and vice versa. Since the behaviour of functions on finite subsets determines the topology of pointwise convergence, ϕ and its inverse are continuous as functions between the topological clones. Hence, the polymorphism clones are topologically isomorphic. Since the complexity of an ω-categorical structure depends only on its topological polymorphism clone and $(\mathbb{Q}; <, I_4)$ is tractable, $\mathbb{C}$ is tractable as well [BP15]. Note that tractability also follows from $\mathbb{C}$ having a primitive positive interpretation in $(\mathbb{Q}; <, I_4)$. This yields the following corollary.

Corollary 5.2.4. *Let $\mathbb{C}$ be the structure from Definition 5.2.1. Then $\mathbb{C}$ is tractable.*

Definition 5.2.5. Let A be a set and $\mathbf{G} \curvearrowright A$ a permutation group. Then $f : A^n \to A$ with $n \geq 2$ is called *pseudo-cyclic with respect to* $\mathbf{G}$ if there exist $g_1, g_2 \in \overline{\mathbf{G}}$ such that

$$g_1 \circ f(a_1, \dots, a_n) = g_2 \circ f(a_2, \dots, a_n, a_1),$$

for all $a_1, \dots, a_n \in A$.
Let $\mathbb{A}$ be a relational structure, then we call $f : A^n \to A$ with $n \geq 2$ pseudo-cyclic if it is pseudo-cyclic with respect to $\mathrm{Aut}(\mathbb{A})$.

We will later use a Ramsey argument to show that any polymorphism of the factor structure $\mathbb{C}/_{\mathrm{Aut}(\mathbb{C})}$ (see Definition 5.3.1) can be lifted to a canonical polymorphism on $\mathbb{C}$. Moreover, we will show that cyclic polymorphisms are liftable to pseudo-cyclic polymorphisms. To prove NP-completeness of $\mathrm{CSP}(\mathbb{C}/_{\mathrm{Aut}(\mathbb{C})})$, we will use the fact that there can be no pseudo-cyclic polymorphism on $\mathbb{C}$.

Lemma 5.2.6. *Let $\mathbb{C}$ be the structure from Definition 5.2.1. Then* $\mathrm{Pol}(\mathbb{C})$ *does not contain a pseudo-cyclic polymorphism.*

Proof. First, note that all polymorphisms of $\mathbb{C}$ that depend on all arguments are injective. To see this, suppose that we have $f : C^n \to C \in \mathrm{Pol}(\mathbb{C})$ that depends on all arguments. By Lemma 5.2.2 we have $f' \in \mathrm{Pol}(\mathbb{Q}; <, I_4)$ such that $f = f' \curvearrowright \mathbb{Q}^3$. Since f' preserves I_4 and also depends on all arguments, it is injective by Lemma 5.1.7, and thus, f is injective, too.
Suppose we have a pseudo-cyclic polymorphism $f : C^m \to C$ that does not depend on all arguments. Assume without loss of generality it depends only on $x_1, \dots, x_n$ with $n < m$. Then by defining $f'(x_1, \dots, x_n) := f(x_1, \dots, x_n, x_n, \dots, x_n)$, we get a pseudo-cyclic polymorphism that does depend on all arguments. Hence, without loss of generality let us assume that we have a pseudo-cyclic polymorphism $f : C^n \to C$ that depends on all arguments. Then $f = f' \curvearrowright \mathbb{Q}^3$ for some pseudo-cyclic $f' \in \mathrm{Pol}(\mathbb{Q}; <, I_4)$. Consider

$$d_1 := f'(1, 0, \dots, 0), d_2 := f'(0, 1, 0, \dots, 0), \dots, d_n := f'(0, \dots, 0, 1)$$

Suppose that there are $\alpha_1, \alpha_2 \in \overline{\mathrm{Aut}(\mathbb{Q}; <, I_4)}$ such that $\alpha_1(d_i) = \alpha_2(d_{i+1})$ for all $i \leq n-1$ and $\alpha_1(d_n) = \alpha_2(d_1)$. Since f' is injective, we have $d_1 < d_2$ or $d_2 < d_1$. Assume without loss of generality $d_1 < d_2$ (the other case is analogous). Since α_1, α_2 preserve $<$, we have $\alpha_j(d_{i_1}) < \alpha_j(d_{i_2})$ if and only if $d_{i_1} < d_{i_2}$ for $j \leq 2$ and $i_1, i_2 \leq n$. Thus,

$$\begin{aligned} d_1 < d_2 &\Leftrightarrow \alpha_1(d_1) < \alpha_1(d_2) \\ &\Leftrightarrow \alpha_2(d_2) < \alpha_2(d_3) \\ &\Leftrightarrow d_2 < d_3 \\ &\dots \\ &\Leftrightarrow d_{n-1} < d_n \\ &\Leftrightarrow \alpha_1(d_{n-1}) < \alpha_1(d_n) \\ &\Leftrightarrow \alpha_2(d_n) < \alpha_2(d_1) \\ &\Leftrightarrow d_n < d_1. \end{aligned}$$

This yields $d_1 < d_2 < \dots < d_n < d_1$, a contradiction. ☐

Finally, we show that $\mathbb{C}$ is Ramsey. This will prove useful when lifting polymorphisms from the finite factor structure to polymorphisms on $\mathbb{C}$.

Proposition 5.2.7. *$\mathbb{C}$ is a homogeneous Ramsey structure.*

Proof. By Lemma 5.2.3, $\mathbb{C}$ is homogeneous. Since $\mathbb{C}$ is countable and homogeneous, by Theorem 2.8.3, being Ramsey is equivalent to $\mathrm{Aut}(\mathbb{C})$ being extremely amenable. With respect to the topology of pointwise convergence, $\mathrm{Aut}(\mathbb{Q}; <, I_4)$ is topologically isomorphic to $\mathrm{Aut}(\mathbb{C})$ by $f \mapsto f \curvearrowright \mathbb{Q}^3$. Since $\mathrm{Aut}(\mathbb{Q}; <, I_4)$ is extremely amenable, so is $\mathrm{Aut}(\mathbb{C})$, hence, $\mathbb{C}$ is Ramsey. $\square$

5.3 Definition and some properties of $\mathbb{B}$

Since $\mathbb{C}$ is ω-categorical, in particular it only has finitely many 1-orbits. Factoring $\mathbb{C}$ by its automorphism group is thus a natural choice to obtain a finite homomorphic image of $\mathbb{C}$. This approach will prove sufficient to obtain a finite structure $\mathbb{B}$ with the same signature as $\mathbb{C}$ such that $(\mathbb{C}, \mathbb{B})$ satisfies the properties in Theorem 5.1.

Definition 5.3.1. In the following, let $\mathbb{B} := \mathbb{C}/_{\mathrm{Aut}(\mathbb{C})}$, i.e. B is the set of 1-orbits of C under $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aut}(\mathbb{Q}; <, I_4) \curvearrowright \mathbb{Q}^3$ (since $\mathbb{C}$ is ω-categorical, this set is finite) and the relations are defined as follows: For any relation $R^{\mathbb{C}}$ on $\mathbb{C}$ with arity k, we have

$$R^{\mathbb{B}}(\mathcal{O}_1, \ldots, \mathcal{O}_k) \Leftrightarrow \exists c_1 \in \mathcal{O}_1, \ \ldots \ , c_k \in \mathcal{O}_k : R^{\mathbb{C}}(c_1, \ldots, c_k)$$

for all $\mathcal{O}_1, \ldots, \mathcal{O}_k \in B$.

Note that $\mathbb{C} \to \mathbb{B}$, since relations of $\mathbb{B}$ are precisely the images of those of $\mathbb{C}$ under the factor map. Let us now show that we can lift polymorphisms from $\mathbb{B}$ to canonical polymorphisms in $\mathbb{C}$.

Proposition 5.3.2. *Let $\mathbb{C}$ and $\mathbb{B}$ be the structures from Definition 5.2.1 and 5.3.1, respectively. Let $f \in \mathrm{Pol}(\mathbb{B})^{(n)}$. Then there exists $f'' \in \mathrm{Pol}(\mathbb{Q}; <, I_4)^{(n)}$ that is canonical with respect to $\mathrm{Aut}(\mathbb{Q}; <, I_4)$, and such that $f'' \curvearrowright B = f$.*

Proof. For any $x, y, z \in \mathbb{Q}^n$ with $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}}\big((x_i, y_i, z_i)\big)$ for $1 \leq i \leq n$, define

$$\hat{f}(x, y, z) := f(\mathcal{O}_1, \ldots, \mathcal{O}_n).$$

We can interpret $\hat{f}(x, y, z)$ as an element in B as well as a $\mathbb{C}$-orbit (a set). We want to define a function $f'' : \mathbb{Q}^n \to \mathbb{Q}$ inductively such that for all $x, y \in \mathbb{Q}^n$ we have that the order relation between $f''(x)$ and $f''(y)$ is the same as the order relation between q_1 and q_2 for any $(q_1, q_2, q_3) \in \hat{f}(x, y, z)$ with any $z \in \mathbb{Q}^n$. This is necessary for f'' to act on orbits in the same way as f does. Let us first observe that the required order relation between $f''(x)$ and $f''(y)$ does not depend on z. We even obtain the stronger statement that whenever x and y appear in the input of $\hat{f}$, the order relation between the corresponding entries of a tuple in the image under $\hat{f}$ (interpreted as a $\mathbb{C}$-orbit) will always be the same, e.g. if we have $(q_1, q_2, q_3) \in \hat{f}(x, y, z_1)$ with $q_1 < q_2$, then $(p_1, p_2, p_3) \in \hat{f}(z_2, y, x)$ satisfies $p_3 < p_2$ for any $z_1, z_2 \in \mathbb{Q}^n$.
Let $x^1, x^2, x^3, y^1, y^2, y^3 \in \mathbb{Q}^n$ and $x^{i_1} = y^{j_1}, x^{i_2} = y^{j_2}$ with $1 \leq i_1 \neq i_2 \leq 3$, $1 \leq j_1 \neq j_2 \leq 3$. For

$\mathcal{O}_i := \mathcal{O}^{\mathbb{C}}\big((x_i^1, x_i^2, x_i^3)\big)$, $\mathcal{O}'_i := \mathcal{O}^{\mathbb{C}}\big((y_i^1, y_i^2, y_i^3)\big), i \leq n$, we have $S^{\mathbb{B}}_{(i_1,i_2),(j_1,j_2)}\big(\mathcal{O}_i, \mathcal{O}'_i\big)$ for all $i \leq n$. Thus, since f is a polymorphism, we get

$$S^{\mathbb{B}}_{(i_1,i_2),(j_1,j_2)}\big(\hat{f}(x^1, x^2, x^3), \hat{f}(y^1, y^2, y^3)\big),$$

hence, there are tuples $(p_1, p_2, p_3) \in \hat{f}(x^1, x^2, x^3), (q_1, q_2, q_3) \in \hat{f}(y^1, y^2, y^3)$ with $p_{i_1} = q_{j_1}$ and $p_{i_2} = q_{j_2}$. In particular, the order relation between the i_1-th and the i_2-th entry of a tuple in $\hat{f}(x^1, x^2, x^3)$ coincide with the order relation between the j_1-th and the j_2-th entry of a tuple in $\hat{f}(y^1, y^2, y^3)$.

This shows that if we want to define f'' as mentioned above, then $\hat{f}$ determines a unique order relation between $f''(x)$ and $f''(y)$ that has to be satisfied. In particular, for $x = y$ this relation is equality.

Let $(x^k)_{k\in\mathbb{N}}$ be an enumeration of elements in $\mathbb{Q}^n$ and let $G_k := \{x^0, \ldots, x^k\}$ for $k \in \mathbb{N}$. Choose any $q_0 \in \mathbb{Q}$ and let $g_0 : G_0 \to \mathbb{Q}$ be the function mapping x^0 to q_0. Assume $g_k : G_k \to \mathbb{Q}$ is already defined such that for all $x, y \in G_k$ the order relation between $g_k(x)$ and $g_k(y)$ coincides with the relation between q_1 and q_2 for any $(q_1, q_2, q_3) \in \hat{f}(x, y, z)$ and any $z \in \mathbb{Q}^n$. Note that this is true for g_0.

For $x^{k+1} \in \mathbb{Q}^n$, we want to find $q_{k+1} \in \mathbb{Q}$ such that for all $i \leq k$ and any $z \in \mathbb{Q}^n$

$$\begin{aligned} q_{k+1} < g_k(x^i) &\Leftrightarrow \exists (q_1, q_2, q_3) \in \hat{f}(x^{k+1}, x^i, z) : q_1 < q_2, \\ q_{k+1} = g_k(x^i) &\Leftrightarrow \exists (q_1, q_2, q_3) \in \hat{f}(x^{k+1}, x^i, z) : q_1 = q_2, \text{ and} \\ q_{k+1} > g_k(x^i) &\Leftrightarrow \exists (q_1, q_2, q_3) \in \hat{f}(x^{k+1}, x^i, z) : q_1 > q_2. \end{aligned} \tag{5.1}$$

This is possible, if we can extend the linear on $\{g_k(x^1), \ldots, g_k(x^k)\}$ as a substructure of $(\mathbb{Q}; <, I_4)$ to the set $\{g_k(x^1), \ldots, g_k(x^k), q_{k+1}\}$, where q_{k+1} is a variable distinct from all $g_k(x^i)$'s. Assume that the binary relation extending $<$ defined as in (5.1) is not a linear order. Since all axioms of a linear order use at most three variables, we could obtain a violation on a triple. Assume that the definition yields for example $q_{k+1} < g_{k+1}(x^i)$, $q_{k+1} = g_{k+1}(x^j)$ and $g_{k+1}(x^i) < g_{k+1}(x^j)$. Then by the arguments in the beginning of the proof, this would yield a triple $(q_1, q_2, q_3) \in \hat{f}(x^{k+1}, x^i, x^j)$ with $q_1 < q_2$, $q_1 = q_3$ and $q_2 < q_3$, which is not possible. All other cases can be excluded analogously. Thus, we find $q_{k+1} \in \mathbb{Q}$ satisfying (5.1). Now define $g_{k+1} := g_k \cup \{(x^{k+1}, q_{k+1})\}$. By construction, g_{k+1} satisfies the induction hypothesis and extends g_k. Finally, set $f'' := \bigcup_{k\in\mathbb{N}} g_k$. Then for all $x, y \in \mathbb{Q}^n$, we have that the order relation between $f''(x)$ and $f''(y)$ is the same as the order relation between q_1 and q_2 for any $(q_1, q_2, q_3) \in \hat{f}(x, y, z)$ with any $z \in \mathbb{Q}^n$. In particular, since the orbit of an element in $\mathbb{C}$ is fully determined by the order relations between its entries, for $f' := f'' \curvearrowright \mathbb{Q}^3$ and for all $x_1, \ldots, x_n \in \mathbb{Q}^3$ with $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}}(x_i) \in B$ we get

$$f(\mathcal{O}_1, \ldots, \mathcal{O}_n) = \mathcal{O} \Rightarrow f'(x_1, \ldots, x_n) \in \mathcal{O}.$$

Thus, $f'' \curvearrowright B = f$. This together with the definitions of $<^{\mathbb{B}}_{i,j}$ and $I^{\mathbb{B}}_{(i_1,i_2),(j_1,j_2)}$ yields that f' and f'' are polymorphisms. For demonstration, we show that $<_{i,j}$ is preserved by f'. Let $x_1, \ldots, x_n \in \mathbb{Q}^3$ with $<_{i,j} (x_k)$ for all $k \leq n$ and $\mathcal{O}_k := \mathcal{O}^{\mathbb{C}}(x_k) \in B$. Then we get $<^{\mathbb{B}}_{i,j} (\mathcal{O}_k)$ for all $k \leq n$ and since f is a polymorphism, also $<^{\mathbb{B}}_{i,j} \big(f(\mathcal{O}_1, \ldots, \mathcal{O}_n)\big)$. Thus, by $f'(x_1, \ldots, x_n) \in f(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, we get $<_{i,j} \big(f'(x_1, \ldots, x_n)\big)$.

Finally, for any m-tuples $\overline{q}_1, \ldots, \overline{q}_n \in \mathbb{Q}^m$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(\mathbb{Q}; <, I_4)$, the order relations between entries of $f''(\overline{q}_1, \ldots, \overline{q}_n)$ and $f''(\alpha_1(\overline{q}_1), \ldots, \alpha_n(\overline{q}_n))$ coincide on all sub-triples, hence, the two tuples are ordered identically. Thus,

$$\mathcal{O}^{(\mathbb{Q};<,I_4)}\big(f''(\overline{q}_1, \ldots, \overline{q}_n)\big) = \mathcal{O}^{(\mathbb{Q};<,I_4)}\big(f''(\alpha_1(\overline{q}_1), \ldots, \alpha_n(\overline{q}_n))\big),$$

which means that there exists $\beta \in \operatorname{Aut}(\mathbb{Q};<,I_4)$ such that

$$f''(\overline{a}_1, \ldots, \overline{a}_n) = \beta(f''(\alpha_1(\overline{a}_1), \ldots, \alpha_n(\overline{a}_n))).$$

Hence, f'' is canonical with respect to $\operatorname{Aut}(\mathbb{Q};<,I_4)$. □

Now let us show that lifting a cyclic polymorphism on $\mathbb{B}$ yields a pseudo-cyclic polymorphism on $\mathbb{C}$.

Lemma 5.3.3. *Let $\mathbb{B}$ be the structure from Definition 5.3.1. Let $f \in \operatorname{Pol}(\mathbb{B})^{(n)}$ be a cyclic polymorphism. Then any $f' \in \operatorname{Pol}(\mathbb{Q};<,I_4)^{(n)}$ that is canonical with respect to $\operatorname{Aut}(\mathbb{Q};<,I_4)$ and such that $f' \curvearrowright B = f$ is pseudo-cyclic.*

Proof. Claim 1: Let $f_1', f_2' \in \operatorname{Pol}(\mathbb{Q};<,I_4)^{(n)}$ be canonical with respect to $\operatorname{Aut}(\mathbb{Q};<,I_4)$ and such that $f_1' \curvearrowright B = f_2' \curvearrowright B$. Then there are $e_1, e_2 \in \overline{\operatorname{Aut}(\mathbb{Q};<,I_4)}$ such that $e_1 \circ f_1' = e_2 \circ f_2'$.
Proof of Claim 1: Consider any m-tuples $s^1, \ldots, s^n \in \mathbb{Q}^m$ for $m \geq 3$. Then since $f_1' \curvearrowright B = f_2' \curvearrowright B$, the order relations between the entries of $f_1'(s^1, \ldots, s^n)$ coincide with those of $f_2'(s^1, \ldots, s^n)$ on all sub-triples, hence, the tuples are ordered identically. This yields $\mathcal{O}(f_1'(s^1, \ldots, s^n)) = \mathcal{O}(f_2'(s^1, \ldots, s^n))$. Hence, for any finite subset S and tuples $u^1, \ldots, u^n \in \mathbb{Q}^{|S|^n}$ such that $\forall \overline{s} \in S^n \exists i : \overline{s} = (u_i^1, \ldots, u_i^n)$, we get $\alpha_1, \alpha_2 \in \operatorname{Aut}(\mathbb{Q};<,I_4)$ with

$$\alpha_1(f_1'(u^1, \ldots, u^n)) = \alpha_2(f_2'(u^1, \ldots, u^n)),$$

which is equivalent to

$$\alpha_1(f_1'(s_1, \ldots, s_n)) = \alpha_2(f_2'(s_1, \ldots, s_n)) \text{ for all } s_1, \ldots, s_n \in S.$$

We can now use a standard compactness argument presented e.g. in [BPP21]. Take an increasing sequence of finite sets $(S_k)_{k\in\mathbb{N}}$ such that $\bigcup_{k\in\mathbb{N}} S_k = \mathbb{Q}$ and assign α_1^k, α_2^k as above for each S_k. Consider the set

$$\{(\gamma \circ \alpha_1^k, \gamma \circ \alpha_2^k) : \ k \in \mathbb{N} \text{ and } \gamma \in \operatorname{Aut}(\mathbb{Q};<,I_4)\},$$

which is a subset of $\overline{\operatorname{Aut}(\mathbb{Q};<,I_4)}^2$. It follows from the arguments in [BP15] that for an ω-categorical structure $\mathbb{A}$ and all $k \geq 1$, the space $\overline{\operatorname{Aut}(\mathbb{A})}^k$ factored by the equivalence relation where $(\delta_1, \ldots, \delta_k)$ and $(\delta_1', \ldots, \delta_k')$ are identified if and only if there exists $\gamma \in \operatorname{Aut}(\mathbb{A})$ such that $(\delta_1, \ldots, \delta_k) = (\gamma \circ \delta_1', \ldots, \gamma \circ \delta_k')$ is compact. Hence, there is an accumulation point (e_1, e_2) of the above set in $\overline{\operatorname{Aut}(\mathbb{Q};<,I_4)}^2$. Thus, we have $e_1, e_2 \in \overline{\operatorname{Aut}(\mathbb{Q};<,I_4)}$ with

$$e_1 \circ f_1' = e_2 \circ f_2',$$

which proves Claim 1.
Now let $f \in \operatorname{Pol}(\mathbb{B})^{(n)}$ be a cyclic polymorphism and $f' \in \operatorname{Pol}(\mathbb{Q};<,I_4)^{(n)}$ be canonical with respect to $\operatorname{Aut}(\mathbb{Q};<,I_4)$ such that $f' \curvearrowright B = f$. Let $f_1' := f'$ and f_2' be defined by $f_2'(x_1, \ldots, x_n) = f'(x_2, \ldots, x_n, x_1)$ for $x_1 \ldots, x_n \in \mathbb{Q}$. Then f_1' and f_2' satisfy the hypothesis of Claim 1, hence there are $e_1, e_2 \in \overline{\operatorname{Aut}(\mathbb{Q};<,I_4)}$ such that $e_1 \circ f_1' = e_2 \circ f_2'$, which shows that f' is pseudo-cyclic. □

Corollary 5.3.4. *Let $\mathbb{B}$ be the structure from Definition 5.3.1. Then CSP($\mathbb{B}$) is NP-complete.*

Proof. Assume that there is a cyclic polymorphism $f \in \text{Pol}(\mathbb{B})$. Then by Proposition 5.3.2, there exists $f'' \in \text{Pol}(\mathbb{Q}; <, I_4)^{(n)}$ that is canonical with respect to $\text{Aut}(\mathbb{Q}; <, I_4)$, and such that $f'' \curvearrowright B = f$. By Lemma 5.3.3, f'' is pseudo-cyclic, hence, $f' \curvearrowright \mathbb{Q}^3 \in \text{Pol}(\mathbb{C})$ is pseudo-cyclic, too, a contradiction to Lemma 5.2.6. Thus, by Theorem 2.4.4, CSP($\mathbb{B}$) is NP-complete. □

5.4 $(\mathbb{C}, \mathbb{B})$ is not finitely tractable

We are now ready to prove the main theorem of this chapter.

Proof of Theorem 5.1.
Let $\mathbb{C}$ and $\mathbb{B}$ be defined as in Definition 5.2.1 and Definition 5.3.1, respectively. Then $\mathbb{B}$ is finite and by Lemma 5.2.3, $\mathbb{C}$ is ω-categorical. By construction, $\mathbb{C} \to \mathbb{B}$. It remains to prove that $(\mathbb{C}, \mathbb{B})$ is not finitely tractable, unless P = NP.
Let $\mathbb{B}'$ be a finite tractable structure such that $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ and let $h : \mathbb{C} \to \mathbb{B}'$ be a homomorphism. Define $\mathbf{H} := \{\text{id}_{B'}\}$. Since any automorphism group on a finite set is oligomorphic, so is $\mathbf{H}$. Since $\mathbb{C}$ is Ramsey, $\text{Aut}(\mathbb{C})$ is extremely amenable. Applying Theorem 2.9.3 to $\text{Aut}(\mathbb{C})$ and $\mathbf{H}$, we get $h' \in \overline{\{h\alpha | \alpha \in \text{Aut}(\mathbb{Q}; <, I_4)\}}$ that is canonical with respect to $\text{Aut}(\mathbb{C})$ and $\mathbf{H}$. The closure with respect to the topology of pointwise convergence of a set of homomorphisms consists only of homomorphisms, hence, h' is a homomorphism from $\mathbb{C}$ to $\mathbb{B}'$. The orbits of B' under $\mathbf{H}$ are singletons, so all elements in a $\mathbb{C}$-orbit must be mapped to the same element. Let $\mathcal{O} \in B$. Then all $c \in \mathcal{O}$ are mapped to the same element $b_{\mathcal{O}} \in B'$ by h'. Consider the function $f : B \to B'$ that maps each $\mathcal{O} \in B$ to $b_{\mathcal{O}} \in B'$. Let $R^{\mathbb{B}}$ be any relation on $\mathbb{B}$ of arity k. Then by definition,

$$R^{\mathbb{B}}(\mathcal{O}_1, \dots, \mathcal{O}_k) \Leftrightarrow \exists c_1 \in \mathcal{O}_1, \ \dots, c_k \in \mathcal{O}_k : R^{\mathbb{C}}(c_1, \dots, c_k).$$

Since $h'(c_i) = b_{\mathcal{O}_i}$ for all $i \leq k$, we also have $R^{\mathbb{B}'}(b_{\mathcal{O}_1}, \dots, b_{\mathcal{O}_k})$, hence, f is a homomorphism. Thus, $\mathbb{B}$ and $\mathbb{B}'$ are homomorphically equivalent. By Lemma 2.2.2, CSP($\mathbb{B}'$) is NP-complete, hence, if P $\neq$ NP, CSP($\mathbb{B}'$) $\notin$ P. □

In fact, due to Proposition 5.2.7, using the same arguments as in the previous proof, the following stronger version of Theorem 5.1 is obtained.

Corollary 5.4.1. *There exists a pair $(\mathbb{C}, \mathbb{B})$ of structures in a finite relational language with a tractable homogeneous Ramsey structure $\mathbb{C}$ and finite $\mathbb{B}$ such that $\mathbb{C} \to \mathbb{B}$ and $(\mathbb{C}, \mathbb{B})$ is not finitely tractable, unless P = NP.*

6 More pairs that are not finitely tractable

In the previous chapter we have seen an example of a pair $(\mathbb{C}, \mathbb{B})$ with tractable ω-categorical structure $\mathbb{C}$ and finite $\mathbb{B}$ such that $\mathbb{C} \to \mathbb{B}$ and $(\mathbb{C}, \mathbb{B})$ is not finitely tractable. In this chapter we will use a similar approach to obtain a generalised method to construct such examples. We used $(\mathbb{Q}; <, I_4)$ in the previous chapter to construct $\mathbb{C}$, in particular, we used homogeneity, ω-categoricity and the Ramsey property of $(\mathbb{Q}; <, I_4)$. Even though it was convenient to add I_4, this relation was not essential for the construction, which can be seen by the construction in this chapter. We only need to use properties that $(\mathbb{Q}; <)$ has.
One essential property of $(\mathbb{Q}; <)$ that was somewhat hiddenly used is that it is finitely bounded. This follows from the fact that checking whether a binary relation $<^X$ on a set X is a linear order can be done by checking only triples in X. That is because the conditions of a linear order are for the relation to be irreflexive, antisymmetric, total, and transitive, hence, they all use at most three variables. If $<^X$ satisfies all those conditions and X is countable, then we can embed $\mathbb{X} := (X; <^X)$ into $(\mathbb{Q}; <)$. Thus, some finite (or countably infinite) $\mathbb{X} = (X; <^X)$ can be embedded into $(\mathbb{Q}; <)$ if and only if it does not contain any subset S of size 3 or smaller such that the relation $<^X$ is no linear order on the substructure of $\mathbb{X}$ induced by S. This means that we only need to forbid substructures with maximal size 3, hence, finitely many. That is the reason why we used the third power of $\mathbb{Q}$ as the domain for the structure $\mathbb{C}$. In this chapter we will again define appropriate relations on a power of the domain of a finitely bounded homogeneous Ramsey structure $\mathbb{D}'$, where the dimension depends on the bounds of $\mathbb{D}'$.
Whereas in the previous chapter we concluded that there is no pseudo-cyclic polymorphism on $(\mathbb{Q}; <)$ by showing that it would create a cycle in the order relation, in this chapter we will use model-completeness to obtain a contradiction to a pseudo-cyclic polymorphism of a certain structure. An ω-categorical structure $\mathbb{D}$ is called a *model-complete core* if for any unary polymorphism f of $\mathbb{D}$ and every finite subset $S \subseteq D$ there is some $\alpha \in \mathrm{Aut}(\mathbb{D})$ such that $f|_S = \alpha|_S$.

Theorem 6.1. *Let $\mathbb{D}'$ be a finitely bounded homogeneous Ramsey structure and $\mathbb{D}$ be a first-order reduct of $\mathbb{D}'$ that is a model-complete core.*
Let $m \in \mathbb{N}$ be greater than all sizes of bounds of $\mathbb{D}'$ and arities of relations of $\mathbb{D}'$ and $\mathbb{D}$.
Then there exists an ω-categorical structure $\mathbb{C}$ with domain $C = D^m$ and a finite structure $\mathbb{B}$ such that CSP$(\mathbb{B})$ *is NP-complete and*

- $\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{D}) \curvearrowright D^m$.
- $\mathbb{C} \to \mathbb{B}$.
- *Every finite $\mathbb{B}'$ such that $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ is homomorphically equivalent to $\mathbb{B}$. In particular, if $P \neq NP$, then $(\mathbb{C}, \mathbb{B})$ is not finitely tractable.*

6.1 Definition and some properties of $\mathbb{C}$

Definition 6.1.1. Let $\mathbb{D}, \mathbb{D}'$ and $m \in \mathbb{N}$ be as in Theorem 6.1 and denote the signatures of $\mathbb{D}$ and $\mathbb{D}'$ by τ and τ', respectively. Let $\mathbb{C}$ ($\mathbb{C}'$, respectively) be the structure with domain $C = D^m$ and the signature consisting of the following relations (we denote by x_i the i-th entry of x for $x \in C$):

- For any relation symbol $R \in \tau$ (τ', respectively) of arity $k \leq m$ and k-tuple $(i_1, \dots, i_k) \in [m]^k$ with pairwise different entries, define the unary relation
$$R_{(i_1,\dots,i_k)}(x) \Leftrightarrow R(x_{i_1}, \dots, x_{i_k}).$$
- For any $k \leq m$ and k-tuples $(i_1, \dots, i_k), (j_1, \dots, j_k)$, each consisting of pairwise different entries $i_l, j_l \leq m$ for $l \leq k$, define for $x, y \in C$ with $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m)$ the binary *compatibility relation*
$$S_{(i_1,\dots,i_k),(j_1,\dots,j_k)}(x, y) \quad \Leftrightarrow \quad \forall l \leq k : x_{i_l} = y_{j_l}.$$
For $k = 1$, we will also write $S_{i,j}$ instead of $S_{(i),(j)}$.

Further, we define some relations on $\mathbb{C}'$ to obtain a homogeneous structure.

- For any relation symbol $R \in \tau'$ of arity $k \leq m$ and k-tuple $(i_1, \dots, i_k) \in [m]^k$, define the k-ary relation
$$R'_{(i_1,\dots,i_k)}(x^1, \dots, x^k) \Leftrightarrow R(x^1_{i_1}, \dots, x^k_{i_k}).$$

Lemma 6.1.2. *Let $\mathbb{D}$ be as in Theorem 6.1 and $\mathbb{C}, \mathbb{C}'$ be the structures from Definition 6.1.1. Then*
$$\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{D}) \curvearrowright D^m \text{ and}$$
$$\mathrm{Pol}(\mathbb{C}') = \mathrm{Pol}(\mathbb{D}') \curvearrowright D^m$$

Proof. "$\subseteq$": Let $f : C^n \to C \in \mathrm{Pol}(\mathbb{C})$. Define $f' : D^n \to D$ as follows: For $(d_1, \dots, d_n) \in D^n$, let $x^i := (d_i, \dots, d_i) \in D^m$ for $i \leq n$ and define $f'(d_1, \dots, d_n)$ to be the first entry of $f(x^1, \dots, x^n)$. Then, similarly to the proof of Lemma 5.2.2, by the definition of $S_{i,j}$ for $i, j \leq m$, we have
$$f(x^1, \dots, x^n) = \begin{pmatrix} f'(x^1_1, \dots, x^n_1) \\ f'(x^1_2, \dots, x^n_2) \\ \vdots \\ f'(x^1_m, \dots, x^n_m) \end{pmatrix},$$
i.e. $f = f' \curvearrowright D^m$. By the definitions of $R_{(i_1,\dots,i_k)}$ for $R \in \tau$, we get that $f' \in \mathrm{Pol}(\mathbb{D})$.
"$\supseteq$": For every $f' \in \mathrm{Pol}(\mathbb{D})$, we have that $f' \curvearrowright D^m \in \mathrm{Pol}(\mathbb{C})$. This is readily checked.
The proof for $\mathbb{C}'$ is analogous. □

Lemma 6.1.3. *Let $\mathbb{C}, \mathbb{C}'$ be the structures from Definition 6.1.1. Then $\mathbb{C}$ and $\mathbb{C}'$ are ω-categorical, $\mathbb{C}$ is a model-complete core, and $\mathbb{C}'$ is homogeneous.*

Proof. Since $\mathrm{Aut}(\mathbb{C}') = \mathrm{Aut}(\mathbb{D}') \curvearrowright D^m$, $\mathbb{D}'$ is ω-categorical, and every n-orbit of $\mathbb{C}'$ for $n \in \mathbb{N}$ corresponds to an mn-orbit of $\mathbb{D}'$, there are finitely many n-orbits of $\mathbb{C}'$ for all $n \in \mathbb{N}$. Hence, $\mathbb{C}'$ is ω-categorical. Since $\mathbb{D}$ is a first-order reduct of $\mathbb{D}'$, we have

$$\mathrm{Aut}(\mathbb{C}) = \mathrm{Aut}(\mathbb{D}) \curvearrowright D^m \supseteq \mathrm{Aut}(\mathbb{D}') \curvearrowright D^m = \mathrm{Aut}(\mathbb{C}').$$

Thus, $\mathrm{Aut}(\mathbb{C}')$-orbits are subsets of $\mathrm{Aut}(\mathbb{C})$-orbits, which yields that $\mathbb{C}$ also has finitely many n-orbits for each $n \in \mathbb{N}$. Thus, $\mathbb{C}$ is ω-categorical, too.
Let f be a unary polymorphism of $\mathbb{C}$ and $S \subseteq C$ be a finite subset. Let $T \subseteq D$ be the set consisting of all entries of tuples in $S \subseteq D^m$ and let $f' \in \mathrm{Pol}(\mathbb{D})$ such that $f = f' \curvearrowright D^m$. Since T is finite and $\mathbb{D}$ is a model-complete core, there is some $\alpha' \in \mathrm{Aut}(\mathbb{D})$ such that $f'|_T = \alpha'|_T$. Hence, for $\alpha := \alpha' \curvearrowright D^m$, we have $\alpha \in \mathrm{Aut}(\mathbb{C})$ and $f|_S = \alpha|_S$. thus, $\mathbb{C}$ is a model-complete core.
Homogeneity of $\mathbb{C}'$ can be shown in a similar fashion as in Lemma 5.2.3 using the relations R' for $R \in \tau'$. Finally, since $\mathbb{C}'$ is a countable homogeneous structure and due to the fact that $\mathrm{Aut}(\mathbb{D}')$ and $\mathrm{Aut}(\mathbb{C}')$ are topologically isomorphic by $f \mapsto f \curvearrowright D^m$, the Ramsey property of $\mathbb{C}'$ follows from $\mathbb{D}'$ being Ramsey by Theorem 2.8.3. □

In Chapter 5 we proved that there is no pseudo-cyclic polymorphism on the infinite structure by using the linear order and the relation I_4. We can not do this here. However, we can show that for a pseudo-cyclic polymorphism f of $\mathbb{C}$ that is canonical with respect to $\mathrm{Aut}(\mathbb{C}')$ and whose action on $\mathbb{C}/_{\mathrm{Aut}(\mathbb{C}')}$ is cyclic there is an infinite set on which f is constant. This will yield a contradiction to model-completeness of $\mathbb{C}$. We can find such an infinite set using the infinite version of the Ramsey theorem. The proof of Lemma 6.1.5 was outlined in [Pin22].

Theorem 6.1.4 (Ramsey)**.** *Let A be a countably infinite set and let $n, k \in \mathbb{N}$. Let $\binom{A}{n}$ denote the set of all n-subsets of A. Then for every colouring $\chi : \binom{A}{n} \to [k]$ there exists an infinite monochromatic set $S \subseteq A$, i.e. χ is constant on all n-subsets of S.*

Lemma 6.1.5. *Let $\mathbb{C}$ be the structure from Definition 6.1.1. Then $\mathrm{Pol}(\mathbb{C})$ does not contain a pseudo-cyclic polymorphism f that is canonical with respect to $\mathrm{Aut}(\mathbb{C}')$ and such that $f \curvearrowright \mathbb{C}/_{\mathrm{Aut}(\mathbb{C}')}$ is cyclic.*

Proof. Let $f \in \mathrm{Pol}(\mathbb{C})^{(n)}$ be as above. Since $\mathbb{C}'$ is a homogeneous Ramsey structure, by Proposition 2.8.4, there is a linear order $<$ on C that is preserved by all automorphisms of $\mathbb{C}'$. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the n-orbits of C with respect to $\mathrm{Aut}(\mathbb{C}')$ and assign to each finite subset $\{c_1, \dots, c_n\} \subseteq C$ of size n the number $i \leq k$ such that $(c_1, \dots, c_n) \in \mathcal{O}_i$ where the elements c_j are sorted in ascending order. Since each n-subset of C is assigned a unique number $i \leq k$, by Ramsey's theorem, we get an infinite monochromatic subset S, that means that two k-tuples on S with $k \leq n$ belong to the same orbit if and only if the order $<$ agrees on them. Take any $s_1, \dots, s_n \in S$. We want to show that $f(s_1, \dots, s_n) = f(s_2, \dots, s_n, s_1)$. Assume for a contradiction $f(s_1, \dots, s_n) < f(s_2, \dots, s_n, s_1)$ (the other case is analogous). Then by cyclicity of $f \curvearrowright \mathbb{C}/_{\mathrm{Aut}(\mathbb{C}')}$, the pairs

$$(f(s_1, \dots, s_n), f(s_2, \dots, s_n, s_1)) \text{ and } (f(s_2, \dots, s_n, s_1), f(s_3, \dots, s_1, s_2))$$

must be in the same orbit, in particular, $f(s_2, \ldots, s_n, s_1) < f(s_3, \ldots, s_1, s_2)$. Applying this argument inductively yields

$$f(s_1, \ldots, s_n) < f(s_2, \ldots, s_n, s_1) < \cdots < f(s_n, \ldots, s_{n-1}) < f(s_1, \ldots, s_n),$$

a contradiction.
In particular, $f(s_1, \ldots, s_n) = f(s_2, \ldots, s_n, s_1)$ holds for $s_1 < s_2 < \cdots < s_n \in S$. By canonicity of f with respect to $\mathrm{Aut}(\mathbb{C}')$ we get that $f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$ for all $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ with $s_i < t_i$ for $i \leq n-1$ and $t_n < s_n$. But for any $s_1, \ldots, s_n, s'_1, \ldots, s'_n \in S$ we find $t_1, \ldots, t_n \in S$ with $t_i > \max(s_i, s'_i)$ for $i \leq n-1$ and $t_n < \min(s_n, s'_n)$. This yields

$$f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) = f(s'_1, \ldots, s'_n)$$

Thus, f is constant on S. But since $\mathbb{C}$ is a model-complete core, $\tilde{f}(x) := f(x, \ldots, x)$ must act like an automorphism on every finite subset of C, a contradiction. □

6.2 Definition and some properties of $\mathbb{B}$

In order to prove Theorem 6.1, we need to construct a finite structure $\mathbb{B}$ such that $\mathbb{C} \to \mathbb{B}$ and $(\mathbb{C}, \mathbb{B})$ is not finitely tractable. Again, we will use a similar construction as in Chapter 5. The domain of $\mathbb{B}$ again is the set of 1-orbits of C, yet this time not under $\mathrm{Aut}(\mathbb{C})$, but under $\mathrm{Aut}(\mathbb{C}')$.

Definition 6.2.1. In the following, let $\mathbb{B} := \mathbb{C}/_{\mathrm{Aut}(\mathbb{C}')}$, i.e. B is the set of 1-orbits of C under $\mathrm{Aut}(\mathbb{C}') = \mathrm{Aut}(\mathbb{D}') \curvearrowright D^m$ (since $\mathbb{C}'$ is ω-categorical, this set is finite) and the relations are defined as follows. For any relation $R^{\mathbb{C}}$ on $\mathbb{C}$ with arity k, we have

$$R^{\mathbb{B}}(\mathcal{O}_1, \ldots, \mathcal{O}_k) \Leftrightarrow \exists c_1 \in \mathcal{O}_1, \ \ldots \ , c_k \in \mathcal{O}_k : R^{\mathbb{C}}(c_1, \ldots, c_k)$$

for all $\mathcal{O}_1, \ldots, \mathcal{O}_k \in B$.

By construction, we have $\mathbb{C} \to \mathbb{B}$. Let us now show that we can lift polymorphisms of $\mathbb{B}$ to polymorphisms of $\mathbb{C}$. That is where finite boundedness of $\mathbb{D}'$ comes into play.

Proposition 6.2.2. *Let $\mathbb{B}$ be the structure from Definition 6.2.1. Let $f \in \mathrm{Pol}(\mathbb{B})^{(n)}$. Then there is $f'' \in \mathrm{Pol}(\mathbb{D})^{(n)}$ that is canonical with respect to $\mathrm{Aut}(\mathbb{D}')$ and such that $f'' \curvearrowright B = f$.*

Proof. The idea of the proof is to define a structure $\mathbb{D}'_n$ with domain D^n such that a homomorphism from this structure to $\mathbb{D}'$ can be used to define a polymorphism on $\mathbb{D}$ that satisfies the required conditions.
For $x^1, \ldots, x^m \in D^n$ and $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}'}(x^1_i, \ldots, x^m_i)$ for $i \leq n$, define

$$\hat{f}(x^1, \ldots, x^m) := f(\mathcal{O}_1, \ldots, \mathcal{O}_n).$$

Let $\mathbb{D}'_n$ be the structure with domain D^n and the following relations:

- For every relation R of $\mathbb{D}'$ of arity k and $x^1, \ldots, x^k \in D^n$, let $R^{\mathbb{D}'_n}(x^1, \ldots, x^k)$ if and only if there exist $x^{k+1}, \ldots, x^m \in D^n$ and $(a_1, \ldots, a_m) \in \hat{f}(x^1, \ldots, x^m)$ with $R(a_1, \ldots, a_k)$. Note that by definition $k < m$.

Observe that the definition of the relation does not depend on the tuples $x^{k+1}, \ldots, x^m \in D^n$. To see this, let R be a relation of $\mathbb{D}'$ of arity $k \in \mathbb{N}$ and $x^1, \ldots, x^m, y^{k+1}, \ldots, y^m \in D^n$. Then for all $i \leq n$, we have

$$S_{(1,\ldots,k),(1,\ldots,k)}\big(\, (x_i^1, \ldots, x_i^k, x_i^{k+1}, \ldots, x_i^m)\,,\, (x_i^1, \ldots, x_i^k, y_i^{k+1}, \ldots, y_i^m)\,\big)$$

Thus, for $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}'}(x_i^1, \ldots, x_i^m)$ and $\mathcal{O}_i' := \mathcal{O}^{\mathbb{C}'}(x_i^1, \ldots, x_i^k, y_i^{k+1}, \ldots, y_i^m)$, we get

$$S^{\mathbb{B}}_{(1,\ldots,k),(1,\ldots,k)}(\mathcal{O}_i, \mathcal{O}_i'),$$

and since f is a polymorphism, for $\mathcal{O} := f(\mathcal{O}_1, \ldots, \mathcal{O}_n) = \hat{f}(x^1, \ldots, x^m)$, $\mathcal{O}' := f(\mathcal{O}_1', \ldots, \mathcal{O}_n') = \hat{f}(x^1, \ldots, x^k, y^{k+1}, \ldots, y^m)$, this yields

$$S^{\mathbb{B}}_{(1,\ldots,k),(1,\ldots,k)}(\mathcal{O}, \mathcal{O}').$$

This, however, means that there are tuples $(a_1, \ldots, a_m) \in \mathcal{O}, (b_1, \ldots, b_m) \in \mathcal{O}'$ with $a_i = b_i$ for $i \leq k$. Hence, for any tuples $(a_1', \ldots, a_m') \in \mathcal{O}, (b_1', \ldots, b_m') \in \mathcal{O}'$, we have $R(a_1', \ldots, a_k') \Leftrightarrow R(b_1', \ldots, b_k')$.
Next, define the binary relation $\sim$ on D^n.

- For $x^1, x^2 \in D^n$, let $x^1 \sim x^2$ if and only if there exist $x^3, \ldots, x^m \in D^n$ and $(a_1, \ldots, a_m) \in \hat{f}(x^1, \ldots, x^m)$ with $a_1 = a_2$.

It can be shown using the relation $S^{\mathbb{B}}_{(1,2),(2,1)}$ that $\sim$ is an equivalence relation. It can be shown in a similar fashion as above that for $x^1 \sim x^2$, the relations $S_{(1,2),(i_1,i_2)}$ and $S^{\mathbb{B}}_{(1,2),(i_1,i_2)}$ guarantee that if a tuple $(d^1, \ldots, d^m) \in (D^n)^m$ satisfies $x^1 = d^{i_1}, x^2 = d^{i_2}$, then any $(a_1, \ldots, a_m) \in \hat{f}(d^1, \ldots, d^m)$ satisfies $a_{i_1} = a_{i_2}$.
We will show that if $x^1 \sim x^2$, they admit the same relations to other elements in $\mathbb{D}_n'$, i.e. for any relation $R^{\mathbb{D}_n'}$ of arity $k \in \mathbb{N}$, $y^2, \ldots, y^k \in D^n$ and $1 \leq i \leq k$, we have

$$R^{\mathbb{D}_n'}(y^2, \ldots, y^i, x^1, y^{i+1}, \ldots, y^k) \Leftrightarrow R^{\mathbb{D}_n'}(y^2, \ldots, y^i, x^2, y^{i+1}, \ldots, y^k).$$

Assume without loss of generality $R^{\mathbb{D}_n'}(y^2, \ldots, y^i, x^1, y^{i+1}, \ldots, y^k)$ (the reverse implication is analogous). Then we have

$$(a_1, \ldots, a_m) \in \hat{f}(y^2, \ldots, y^i, x^1, y^{i+1}, \ldots, y^k, x^2, y^0, \ldots, y^0)$$

with $R(a_1, \ldots, a_k)$. By swapping x^1 and x^2 and applying the compatibility relation

$$S_{(1,\ldots,k+1),(1,\ldots,i-1,k+1,i+1,\ldots,k,i)}$$

(and the corresponding relation on $\mathbb{B}$), which states that the first $k+1$ entries of two tuples are the same except for swapping the i-th and $(k+1)$-th entry, we get

$$(b_1, \ldots, b_m) \in \hat{f}(y^2, \ldots, y^i, x^2, y^{i+1}, \ldots, y^k, x^1, y^0, \ldots, y^0)$$

with $R(b_1, \ldots, b_{i-1}, b_{k+1}, b_{i+1}, \ldots, b_k)$. Since $b_{k+1} = b_i$, we get $R(b_1, \ldots, b_k)$, hence,

$$R^{\mathbb{D}_n'}(y^2, \ldots, y^i, x^2, y^{i+1}, \ldots, y^k).$$

We want to find a homomorphism $h : \mathbb{D}'_n \to \mathbb{D}'$. By the reasoning above, it is sufficient to find a homomorphism from $\mathbb{D}'_n/_\sim$ to $\mathbb{D}'$. Our aim is to find an embedding $\phi : \mathbb{D}'_n/_\sim \to \mathbb{D}'$. By Lemma 4.4.1, it is sufficient to show that every finite substructure of $\mathbb{D}'_n/_\sim$ can be embedded into $\mathbb{D}'$. Assume there is some finite substructure $\mathbb{S}$ that does not embed into $\mathbb{D}'$. Then there is some bound $\mathbb{F}$ of $\mathbb{D}'$ that can be embedded into $\mathbb{S}$. Without loss of generality let $F \subseteq S$ and let $F = \{[d^1]_\sim, \ldots, [d^k]_\sim\}$. Recall that $k < m$, since $m > |F|$ for every bound $\mathbb{F}$ of $\mathbb{D}'$. Take some $[d^{k+1}]_\sim, \ldots, [d^m]_\sim \in \mathbb{D}'_n/_\sim$ and let $(e_1, \ldots, e_m) \in \mathcal{O} := \hat{f}(d^1, \ldots, d^m)$. Then by the definition of the relations on $\mathbb{D}'_n$, it holds that the mapping defined by $[d^i]_\sim \mapsto e_i$ for $1 \leq i \leq k$ is an embedding. This, however, yields an embedding of the bound $\mathbb{F}$ into $\mathbb{D}'$, a contradiction. Thus, we can embed every finite substructure of $\mathbb{D}'_n/_\sim$ into $\mathbb{D}'$ and hence by Lemma 4.4.1, there is an embedding $\phi : \mathbb{D}'_n/_\sim \to \mathbb{D}'$. By setting $f''(x) := \phi([x]_\sim)$, we get a homomorphism $f'' : \mathbb{D}'_n \to \mathbb{D}'$.
Let us interpret this homomorphism as an operation $f'' : D^n \to D$ by identifying elements in $\mathbb{D}'_n$ with n-tuples of elements in D.
We want to show that this operation is canonical with respect to $\mathrm{Aut}(\mathbb{D}')$. Take any $x^1, \ldots, x^n \in D^l$ for some $l \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(\mathbb{D}')$. Then by construction, the relations between the entries of $f''(x^1, \ldots, x^n)$ as elements of $\mathbb{D}'$ and those between the entries of $f''(\alpha_1(x^1), \ldots, \alpha_n(x^n))$ coincide. Hence, by homogeneity of $\mathbb{D}'$, there is some $\beta \in \mathrm{Aut}(\mathbb{D}')$ such that

$$f''(x^1, \ldots, x^n) = \beta(f''(\alpha_1(x^1), \ldots, \alpha_n(x^n))).$$

Let $f' := f'' \curvearrowright D^m$. For any $x^1, \ldots, x^n \in D^m$ with $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}'}(x^i)$, the unary relations of $f'(x^1, \ldots, x^n)$ as an element of $\mathbb{C}'$ can be obtained from the relations between the entries of $f'(x^1, \ldots, x^n)$ as elements in $\mathbb{D}'$ as in Definition 6.1.1. By construction, these coincide with the relations between the entries of any $z \in \mathcal{O} := f(\mathcal{O}_1, \ldots, \mathcal{O}_n)$ as elements in $\mathbb{D}'$. Thus, as an element of $\mathbb{C}'$, $f'(x^1, \ldots, x^n)$ admits the same unary relations as any $z \in \mathcal{O}$ and by homogeneity of $\mathbb{C}'$, we have $f'(x^1, \ldots, x^n) \in \mathcal{O}$. Thus, $f'' \curvearrowright B = f$.
Finally, we show that f'' is a polymorphism of $\mathbb{D}$. To see this, let $x^1, \ldots, x^n \in R \subseteq D^k$ for some relation R of $\mathbb{D}$ with $\mathrm{ar}(R) = k$. We want to show that for $z^i := (x^1_i, \ldots, x^n_i)$ for $i \leq k$, we have $R(f''(z^1), \ldots, f''(z^k))$. For $k \leq m$, define $y^i := (x^i_1, \ldots, x^i_k, x_0, \ldots, x_0) \in D^m$ for $i \leq n$ and some $x_0 \in D$. Then we have $R_{(1,\ldots,k)}(y^i)$ for all i, hence, for $\mathcal{O}_i := \mathcal{O}^{\mathbb{C}'}(y^i)$ we get $R^{\mathbb{B}}_{(1,\ldots,k)}(\mathcal{O}_i)$ for all $i \leq n$. Since f is a polymorphism, for $\mathcal{O} := f(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, this yields $R^{\mathbb{B}}_{(1,\ldots,k)}(\mathcal{O})$. Now let $z^{k+1} := \cdots := z^m := (x_0, \ldots, x_0) \in D^n$. By construction, $(f''(z^1), \ldots, f''(z^m)) \in \mathcal{O}$. Since R is first-order definable in τ' and therefore preserved by $\mathrm{Aut}(\mathbb{D}')$, we have $R(a_1, \ldots, a_k)$ for all $(a_1, \ldots, a_m) \in \mathcal{O}$, hence, $R(f''(z^1), \ldots, f''(z^k))$. □

Lemma 6.2.3. *Let $\mathbb{B}$ be the structure from Definition 6.2.1 and $\mathbb{C}, \mathbb{C}'$ be the structures from Definition 6.1.1. Let $f \in \mathrm{Pol}(\mathbb{B})^{(n)}$ be a cyclic polymorphism. Then any $f' \in \mathrm{Pol}(\mathbb{D})^{(n)}$ that is canonical with respect to $\mathrm{Aut}(\mathbb{D}')$ and such that $f' \curvearrowright B = f$ is pseudo-cyclic.*

Proof. Claim 1: Let $f'_1, f'_2 \in \mathrm{Pol}(\mathbb{D})^{(n)}$ be canonical with respect to $\mathrm{Aut}(\mathbb{D}')$ such that $f'_1 \curvearrowright B = f'_2 \curvearrowright B$. Then there are $e_1, e_2 \in \overline{\mathrm{Aut}(\mathbb{D})}$ such that $e_1 \circ f'_1 = e_2 \circ f'_2$.
Proof of Claim 1: Consider any k-tuples $s^1, \ldots, s^n \in \mathbb{Q}^k$ for $k \geq m$. Then since $f'_1 \curvearrowright B = f'_2 \curvearrowright B$, the τ'-relations between the entries of $f'_1(s^1, \ldots, s^n)$ as elements of $\mathbb{D}'$ coincide with those of $f'_2(s^1, \ldots, s^n)$ on all m-tuples, hence, by homogeneity, $\mathcal{O}^{\mathbb{D}'}(f'_1(s^1, \ldots, s^n)) = \mathcal{O}^{\mathbb{D}'}(f'_2(s^1, \ldots, s^n))$.

By similar arguments as in Lemma 5.3.3, for any finite subset S, we get $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{D}') \subseteq \mathrm{Aut}(\mathbb{D})$ with

$$\alpha_1(f'_1(s_1, \ldots, s_n)) = \alpha_2(f'_2(s_1, \ldots, s_n)) \text{ for all } s_1, \ldots, s_n \in S.$$

By a compactness argument presented in [BPP21] (which is also described in the proof of Lemma 5.3.3), there are $e_1, e_2 \in \overline{\mathrm{Aut}(\mathbb{D})}$ with

$$e_1 \circ f'_1 = e_2 \circ f'_2,$$

which proves Claim 1.

Now let $f \in \mathrm{Pol}(\mathbb{B})^{(n)}$ be a cyclic polymorphism and $f' \in \mathrm{Pol}(\mathbb{D})^{(n)}$ be canonical with respect to $\mathrm{Aut}(\mathbb{D}')$, and such that $f' \curvearrowright B = f$. Let $f'_1 := f'$ and f'_2 be defined by $f'_2(x_1, \ldots, x_n) = f'(x_2, \ldots, x_n, x_1)$ for $x_1 \ldots, x_n \in D$. Then f'_1 and f'_2 satisfy the properties of Claim 1, hence there are $e_1, e_2 \in \overline{\mathrm{Aut}(\mathbb{D})}$ such that $e_1 \circ f'_1 = e_2 \circ f'_2$, which shows that f' is pseudo-cyclic. □

Corollary 6.2.4. *Let $\mathbb{B}$ be the structure from Definition 6.2.1. Then CSP($\mathbb{B}$) is NP-complete.*

Proof. If there is a cyclic polymorphism $f \in \mathrm{Pol}(\mathbb{B})$, then by Proposition 6.2.2, there is $f'' \in \mathrm{Pol}(\mathbb{D})$ that is canonical with respect to $\mathrm{Aut}(\mathbb{D}')$ and such that $f'' \curvearrowright B = f$. By Lemma 6.2.3, this polymorphism is pseudo-cyclic, hence, also $f' := f'' \curvearrowright D^m \in \mathrm{Pol}(\mathbb{C})$ is pseudo-cyclic and by canonicity of f'' with respect to $\mathrm{Aut}(\mathbb{D}')$, f' is canonical with respect to $\mathrm{Aut}(\mathbb{C}')$. Finally, $f' \curvearrowright B = f$. By Lemma 6.1.5, such f' can not exist. Hence, there is no cyclic polymorphism of $\mathbb{B}$ and by Theorem 2.4.4, CSP($\mathbb{B}$) is NP-complete. □

6.3 $(\mathbb{C}, \mathbb{B})$ is not finitely tractable

Proposition 6.3.1. *Assume $P \neq NP$. Let $\mathbb{B}$ be the structure from Definition 6.2.1 and $\mathbb{C}$ from 6.1.1. Then $(\mathbb{C}, \mathbb{B})$ is not finitely tractable.*

Proof. Let $\mathbb{B}'$ be a finite tractable structure such that $\mathbb{C} \to \mathbb{B}' \to \mathbb{B}$ and let $h : \mathbb{C} \to \mathbb{B}'$ be a homomorphism. We will use a similar approach as in the proof of Theorem 5.1. Define $\mathbf{H} := \{\mathrm{id}_{B'}\}$, which is oligomorphic since B' is finite. Since $\mathbb{C}'$ is Ramsey, $\mathrm{Aut}(\mathbb{C}')$ is extremely amenable. Applying Theorem 2.9.3 to $\mathrm{Aut}(\mathbb{C}')$ and $\mathbf{H}$ yields $h' \in \overline{\{h\alpha | \alpha \in \mathrm{Aut}(\mathbb{C}')\}}$ that is canonical with respect to $\mathrm{Aut}(\mathbb{C}')$ and $\mathbf{H}$. Since $\mathrm{Aut}(\mathbb{C}') \subseteq \mathrm{Aut}(\mathbb{C})$, this is a homomorphism from $\mathbb{C}$ to $\mathbb{B}'$. The orbits of B' under $\{\mathrm{id}_{B'}\}$ are singletons, so all elements in a $\mathbb{C}'$-orbit must be mapped to the same element. Let $\mathcal{O} \in B$. Then all $c \in \mathcal{O}$ are mapped to the same element $b_{\mathcal{O}} \in B'$ by h'. As in the proof of Theorem 5.1, the function $f : B \to B'$ that maps each $\mathcal{O} \in B$ to $b_{\mathcal{O}}$ is a homomorphism. Since $\mathbb{B}$ and $\mathbb{B}'$ are homomorphically equivalent, CSP($\mathbb{B}'$) is NP-complete, a contradiction to P ≠ NP. □

Now let us proof the main theorem of this chapter.

Proof of Theorem 6.1. Let $\mathbb{C}$ and $\mathbb{B}$ be defined as in Definition 6.1.1 and Definition 6.2.1, respectively. Then $\mathbb{B}$ is finite and by Lemma 6.1.3, $\mathbb{C}$ is ω-categorical. Further, by Lemma 6.1.2, we have $\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{D}) \curvearrowright D^m$. By construction, $\mathbb{C} \to \mathbb{B}$, and by Proposition 6.3.1, the pair $(\mathbb{C}, \mathbb{B})$ is not finitely tractable, unless P = NP. □

References

[AGH17] Per Austrin, Venkatesan Guruswami, and Johan Håstad. "$(2+\varepsilon)$-Sat Is NP-hard". In: *SIAM Journal on Computing* 46.5 (2017), pp. 1554–1573. DOI: 10.1137/15M1006507. eprint: https://doi.org/10.1137/15M1006507. URL: https://doi.org/10.1137/15M1006507.

[Ats08] Albert Atserias. "On digraph coloring problems and treewidth duality". In: *European Journal of Combinatorics* 29.4 (2008). Homomorphisms: Structure and Highlights, pp. 796–820. ISSN: 0195-6698. DOI: https://doi.org/10.1016/j.ejc.2007.11.004. URL: https://www.sciencedirect.com/science/article/pii/S0195669807002004.

[Bar+21] Libor Barto, Jakub Bulín, Andrei Krokhin, and Jakub Opršal. "Algebraic Approach to Promise Constraint Satisfaction". In: *J. ACM* 68.4 (July 2021). ISSN: 0004-5411. DOI: 10.1145/3457606. URL: https://doi.org/10.1145/3457606.

[Bar19] Libor Barto. "Promises Make Finite (Constraint Satisfaction) Problems Infinitary". In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, June 2019. DOI: 10.1109/lics.2019.8785671. URL: http://dx.doi.org/10.1109/LICS.2019.8785671.

[BCP10] Manuel Bodirsky, Hubie Chen, and Michael Pinsker. "The reducts of equality up to primitive positive interdefinability". In: *The Journal of Symbolic Logic* 75.4 (2010), pp. 1249–1292. DOI: 10.2178/jsl/1286198146.

[BD13] Manuel Bodirsky and Víctor Dalmau. "Datalog and constraint satisfaction with infinite templates". In: *Journal of Computer and System Sciences* 79.1 (2013), pp. 79–100. ISSN: 0022-0000. DOI: https://doi.org/10.1016/j.jcss.2012.05.012. URL: https://www.sciencedirect.com/science/article/pii/S0022000012001213.

[BG16] Joshua Brakensiek and Venkatesan Guruswami. "New hardness results for graph and hypergraph colorings". In: *Proceedings of the 31st Conference on Computational Complexity*. CCC '16. Tokyo, Japan: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016. ISBN: 9783959770088.

[BG21] Joshua Brakensiek and Venkatesan Guruswami. *Promise Constraint Satisfaction: Algebraic Structure and a Symmetric Boolean Dichotomy*. 2021. arXiv: 1704.01937.

[BGS23] Joshua Brakensiek, Venkatesan Guruswami, and Sai Sandeep. "Conditional Dichotomy of Boolean Ordered Promise CSPs". In: *TheoretiCS* Volume 2 (Jan. 2023). ISSN: 2751-4838. DOI: 10.46298/theoretics.23.2. URL: http://dx.doi.org/10.46298/theoretics.23.2.

[BJK05] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. "Classifying the Complexity of Constraints Using Finite Algebras". In: *SIAM Journal on Computing* 34.3 (2005), pp. 720–742. DOI: 10.1137/S0097539700376676. eprint: https://doi.org/10.1137/S0097539700376676. URL: https://doi.org/10.1137/S0097539700376676.

[BK10] Manuel Bodirsky and Jan Kára. "The complexity of temporal constraint satisfaction problems". In: *Journal of the ACM* 57.2 (Feb. 2010). ISSN: 0004-5411. DOI: 10.1145/1667053.1667058. URL: https://doi.org/10.1145/1667053.1667058.

[BK12] Libor Barto and Marcin Kozik. "Absorbing Subalgebras, Cyclic Terms, and the Constraint Satisfaction Problem". In: *Logical Methods in Computer Science* Volume 8, Issue 1 (Feb. 2012). ISSN: 1860-5974. DOI: `10.2168/lmcs-8(1:7)2012`. URL: `http://dx.doi.org/10.2168/LMCS-8(1:7)2012`.

[BN06] Manuel Bodirsky and Jaroslav Nešetřil. "Constraint Satisfaction with Countable Homogeneous Templates". In: *Journal of Logic and Computation* 16.3 (June 2006), pp. 359–373. ISSN: 0955-792X. DOI: `10.1093/logcom/exi083`. eprint: `https://academic.oup.com/logcom/article-pdf/16/3/359/3384005/exi083.pdf`. URL: `https://doi.org/10.1093/logcom/exi083`.

[Bod21] Manuel Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction.* Lecture Notes in Logic. Cambridge University Press, 2021.

[BP15] Manuel Bodirsky and Michael Pinsker. "Topological Birkhoff". In: *Transactions of the American Mathematical Society* 367.4 (2015), pp. 2527–2549. ISSN: 00029947. URL: `http://www.jstor.org/stable/24513079` (visited on 09/07/2024).

[BP21] Manuel Bodirsky and Michael Pinsker. "Canonical Functions: a proof via topological dynamics". In: *Contributions to discrete Mathematics* 16.2 (2021), pp. 36–45. ISSN: 1715-0868. DOI: `https://doi.org/10.11575/cdm.v16i2.71724`. URL: `http://hdl.handle.net/20.500.12708/138246`.

[BPP21] Manuel Bodirsky, Michael Pinsker, and András Pongrácz. "Projective Clone Homomorphisms". In: *The Journal of Symbolic Logic* 86.1 (2021), pp. 148–161. DOI: `10.1017/jsl.2019.23`.

[BPT13] Manuel Bodirsky, Michael Pinsker, and Todor Tsankov. "Decidability of Definability". In: *The Journal of Symbolic Logic* 78.4 (2013), pp. 1036–1054. ISSN: 00224812, 19435886. URL: `http://www.jstor.org/stable/43303697` (visited on 09/07/2024).

[Bul17] Andrei A. Bulatov. "A dichotomy theorem for nonuniform CSPs". In: *Proceedings of the 58th IEEE Annual Symposium on Foundations of Computer Science (FOCS'17), Berkeley, CA, USA, October 15-17, 2017.* 2017, pp. 319–330.

[CSS99] Gregory Cherlin, Saharon Shelah, and Niandong Shi. "Universal Graphs with Forbidden Subgraphs and Algebraic Closure". In: *Advances in Applied Mathematics* 22.4 (1999), pp. 454–491. ISSN: 0196-8858. DOI: `https://doi.org/10.1006/aama.1998.0641`. URL: `https://www.sciencedirect.com/science/article/pii/S0196885898906418`.

[CŽ23] Lorenzo Ciardo and Stanislav Živný. "CLAP: A New Algorithm for Promise CSPs". In: *SIAM Journal on Computing* 52.1 (Jan. 2023), pp. 1–37. ISSN: 1095-7111. DOI: `10.1137/22m1476435`. URL: `http://dx.doi.org/10.1137/22M1476435`.

[Den+21] Guofeng Deng, Ezzeddine El Sai, Trevor Manders, Peter Mayr, Poramate Nakkirt, and Athena Sparks. "Sandwiches for promise constraint satisfaction". In: *Algebra Universalis* 82.1 (2021), p. 15. ISSN: 1420-8911. DOI: `10.1007/s00012-020-00702-5`. URL: `https://doi.org/10.1007/s00012-020-00702-5`.

[FV93] Tomás Feder and Moshe Y. Vardi. "Monotone monadic SNP and constraint satisfaction". In: *Proceedings of the 25th Annual ACM Symposium on Theory of Computing (STOC'93), San Diego, CA, USA, May 16 - 18, 1993.* 1993, pp. 612–622.

[FV98] Tomás Feder and Moshe Y. Vardi. "The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory". In: *SIAM Journal on Computing* 28.1 (1998), pp. 57–104.

[GLS88] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization.* eng. Berlin [u.a.]: Springer, 1988. URL: `http://eudml.org/doc/204222`.

[HN19] Jan Hubička and Jaroslav Nešetřil. "All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)". In: *Advances in Mathematics* 356 (Nov. 2019), p. 106791. ISSN: 0001-8708. DOI: 10.1016/j.aim.2019.106791. URL: http://dx.doi.org/10.1016/j.aim.2019.106791.

[Hod93] Wilfrid Hodges. *Model Theory*. Cambridge University Press, 1993.

[JCG97] Peter Jeavons, David Cohen, and Marc Gyssens. "Closure properties of constraints". In: *J. ACM* 44.4 (July 1997), pp. 527–548. ISSN: 0004-5411. DOI: 10.1145/263867.263489. URL: https://doi.org/10.1145/263867.263489.

[Jea98] Peter Jeavons. "On the algebraic structure of combinatorial problems". In: *Theoretical Computer Science* 200.1 (1998), pp. 185–204. ISSN: 0304-3975. DOI: https://doi.org/10.1016/S0304-3975(97)00230-2. URL: https://www.sciencedirect.com/science/article/pii/S0304397597002302.

[KMZ22] Alexandr Kazda, Peter Mayr, and Dmitriy Zhuk. "Small Promise CSPs that reduce to large CSPs". In: *Logical Methods in Computer Science* Volume 18, Issue 3 (Aug. 2022). ISSN: 1860-5974. DOI: 10.46298/lmcs-18(3:25)2022. URL: http://dx.doi.org/10.46298/lmcs-18(3:25)2022.

[KO22] Andrei Krokhin and Jakub Opršal. "An invitation to the promise constraint satisfaction problem". In: *ACM SIGLOG News* 9.3 (Aug. 2022), pp. 30–59. DOI: 10.1145/3559736.3559740. URL: https://doi.org/10.1145/3559736.3559740.

[KPT05] Alexander Sotirios Kechris, Vladimir Pestov, and Stevo Todorčević. "Fraïssé Limits, Ramsey Theory, and Topological Dynamics of Automorphism Groups". In: *Geometric & Functional Analysis GAFA* 15.1 (2005), pp. 106–189. ISSN: 1420-8970. DOI: 10.1007/s00039-005-0503-1. URL: https://doi.org/10.1007/s00039-005-0503-1.

[Kro+23] Andrei Krokhin, Jakub Opršal, Marcin Wrochna, and Stanislav Živný. "Topology and Adjunction in Promise Constraint Satisfaction". In: *SIAM Journal on Computing* 52.1 (Feb. 2023), pp. 38–79. ISSN: 1095-7111. DOI: 10.1137/20m1378223. URL: http://dx.doi.org/10.1137/20M1378223.

[KT17] Ken-ichi Kawarabayashi and Mikkel Thorup. "Coloring 3-Colorable Graphs with Less than n 1/5 Colors". In: *Journal of the ACM* 64 (Mar. 2017), pp. 1–23. DOI: 10.1145/3001582.

[LLT07] Benoit Larose, Cynthia Loten, and Claude Tardif. "A Characterisation of First-Order Constraint Satisfaction Problems". In: *Logical Methods in Computer Science* Volume 3, Issue 4 (Nov. 2007). ISSN: 1860-5974. DOI: 10.2168/lmcs-3(4:6)2007. URL: http://dx.doi.org/10.2168/LMCS-3(4:6)2007.

[MM08] Miklós Maróti and Ralph Mckenzie. "Existence theorems for weakly symmetric operations". In: *Algebra Universalis* 59 (Dec. 2008), pp. 463–489. DOI: 10.1007/s00012-008-2122-9.

[Mot24] Antoine Mottet. "Promise and Infinite-Domain Constraint Satisfaction". In: *Annual Conference for Computer Science Logic*. 2024. URL: https://api.semanticscholar.org/CorpusID:267523787.

[NB95] Bernhard Nebel and Hans-Jürgen Bürckert. "Reasoning about temporal relations: a maximal tractable subclass of Allen's interval algebra". In: *Journal of the ACM* 42.1 (Jan. 1995), pp. 43–66. ISSN: 0004-5411. DOI: 10.1145/200836.200848. URL: https://doi.org/10.1145/200836.200848.

[Pes98] Vladimir G. Pestov. "ON FREE ACTIONS, MINIMAL FLOWS, AND A PROBLEM BY ELLIS". In: *Transactions of the American Mathematical Society* 350 (1998), pp. 4149–4165. URL: https://api.semanticscholar.org/CorpusID:117351761.

[Pin22] M. Pinsker. "Current Challenges in Infinite-Domain Constraint Satisfaction: Dilemmas of the Infinite Sheep". In: *2022 IEEE 52nd International Symposium on Multiple-Valued Logic (ISMVL)*. Los Alamitos, CA, USA: IEEE Computer Society, May 2022, pp. 80–87. DOI: 10.1109/ISMVL52857.2022.00019. URL: https://doi.ieeecomputersociety.org/10.1109/ISMVL52857.2022.00019.

[Ram30] Frank Plumpton Ramsey. "On a Problem of Formal Logic". In: *Proceedings of the London Mathematical Society* s2-30.1 (1930), pp. 264–286. DOI: https://doi.org/10.1112/plms/s2-30.1.264. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s2-30.1.264. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s2-30.1.264.

[Ros08] Benjamin Rossman. "Homomorphism preservation theorems". In: *Journal of the ACM* 55.3 (Aug. 2008). ISSN: 0004-5411. DOI: 10.1145/1379759.1379763. URL: https://doi.org/10.1145/1379759.1379763.

[Sch78] Thomas J. Schaefer. "The complexity of satisfiability problems". In: *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing*. STOC '78. San Diego, California, USA: Association for Computing Machinery, 1978, pp. 216–226. ISBN: 9781450374378. DOI: 10.1145/800133.804350. URL: https://doi.org/10.1145/800133.804350.

[Sig10] Mark Siggers. "A strong Mal'cev condition for locally finite varieties omitting the unary type". In: *Algebra Universalis* 64 (Oct. 2010), pp. 15–20. DOI: 10.1007/s00012-010-0082-3.

[Zhu17] Dmitriy Zhuk. "A proof of CSP dichotomy conjecture". In: *Proceedings of the 58th IEEE Annual Symposium on Foundations of Computer Science (FOCS'17), Berkeley, CA, USA, October 15-17, 2017*. 2017, pp. 331–342.

[Zhu20] Dmitriy Zhuk. "A proof of the CSP dichotomy conjecture". In: *Journal of the ACM* 67.5 (2020), pp. 30/1–78.