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Kurzfassung

Wir beweisen die Existenz von globalen Martingallösungen für eine allgemeine Klasse von Kreuzdiffusionssystemen unter dem Einfluss multiplikativen Rauschens. Die betrachteten Gleichungen beschreiben Mehrkomponentensysteme, welche aus physikalischen oder biologischen Anwendungen stammen. In den meisten Fällen sind die Diffusionsmatrizen dieser Probleme weder symmetrisch noch positiv semi-definit, woran klassische Ansätze scheitern. Viele dieser Systeme besitzen jedoch eine formale Gradientenfluss- oder Entropiestruktur welche wir verwenden, um die Existenz globaler Lösungen zu beweisen. Häufig ist zudem die Gesamtmasse erhalten, was zusammen mit der Entropistruktur, zu fast sicheren obern und unteren Schranken der Lösungsprozesse führt.
Wir präsentieren einen neuen und vereinfachten Ansatz um die Existenz globaler Lösungen stochastischer Kreuzdiffusionssysteme zu zeigen und wenden diesen auf das so-genannte Shigesada Kawasaki Teramoto (SKT) System an. Es modelliert die Entwicklung einer beliebigen Anzahl von Spezies, welche untereinander interagieren. Weiters betrachten wir auch eine generelle Klasse von Modellen, bei denen die Gesamtmasse des Systems erhalten bleibt. Das Existenzresultat gilt unter Annahmen an das Zusammenspiel einer geeigneten Entropiefunktion und dem stochastischen Term.
Zudem untersuchen wir die Äquivalenz von schwachen sowie pfadweise-milden Lösungen von parabolischen, quasilinearen stochastischen partiellen Differenzialgleichungen. Dies erweitert die Äquivalenzergebnisse für nicht-autonome semilineare SPDEs auf den quasilinearen Fall. Das Resultat ist motiviert durch den Existenzbeweis einer lokalen pfadweise-milden Lösung für ein Shigesada Kawasaki Teramoto System in [85] und dem schwachen Lösungsbegriff, den wir in dieser Arbeit verwenden.

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Abstract

We prove the existence of global-in-time martingale solutions to a general class of cross-diffusion systems with multiplicative noise. The equations describe multicomponent systems from physics or biology with volume-filling effects and possess a formal gradient-flow or entropy structure. The diffusion matrix in the studied class of systems is generally neither symmetric nor positive semidefinite, which excludes a direct application of standard methods. Instead, the existence proof is based on the entropy structure of the model. We present a new approach to derive solutions to this type of degenerate system of equations and apply it to the Shigesada Kawasaki Teramoto (SKT) population model, which is one of the more well known examples of cross-diffusion systems. It describes the segregation dynamics of populations with an arbitrary number of species. Additionally, we consider a general class of equations with the property that the starting mass of the system is conserved. Their structure also allows for the derivation of physically relevant lower and upper bounds of the solution processes. The existence result holds under some assumptions with respect to the interplay of entropy density and noise term.
Lastly, the equivalence of the concepts of weak and pathwise mild solutions for parabolic quasilinear stochastic partial differential equations (SPDEs) is proven under suitable conditions. This extends the equivalence results established for nonautonomous semilinear SPDEs to the quasilinear case and was motivated by the analysis performed in [85] where pathwise mild solutions for a particular SKT system were derived.

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Wien, am …

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CHAPTER 1

Introduction

The aim of this thesis is to provide a rigorous theoretical analysis of stochastic cross-diffusion systems which are related to a plethora of applications. The results are mainly based on [51], as well as a forthcoming work on the stichastik SKT system and a correction of the results in [50]. In addition, we present some related supplementary results. In this introductory chapter we acquaint the reader with some of the ideas, results and examples from the study of deterministic cross-diffusion systems which will remain central in the study of their stochastic counterparts.

1.1. Cross-diffusion and entropy

Reaction diffusion phenomena arise in many applications in physics, chemistry and biology, describing the evolution of densities or concentrations in multicomponent systems. An important feature these systems capture is the way how the different components interact with each other. It often happens that the concentration of one component in a system influences the diffusion of the others. This phenomenon is known as cross-diffusion. Such cross-diffusion equations can be generally regarded as strongly coupled parabolic equations and they are of the form

$$\partial_t u = \operatorname{div}(A(u)\nabla u) + f(u) \text{ in } \mathbb{R}^d,\ t > 0, \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^n$ is a vector-valued function, modeling, e.g., population densities or chemical concentrations, $A(u) \in \mathbb{R}^{n\times n}$ is the diffusion matrix and f is a (potentially nonlinear) reaction term. In many systems, motivated by applications, $A(u)$ is neither symmetric nor positive semidefinite such that even the local-in-time existence of solutions to (1.1) may be a nontrivial task. Moreover, due to the strongly coupled interactions, the diffusion matrix is usually not diagonal and tools like maximum principles and parabolic regularity theory cannot be applied in a straight forward way. Moreover it is not clear how to derive lower or upper bounds for the solutions, which may be desired in practical applications. It turns out that some cross-diffusion systems possess an entropy- or gradient flow structure, i.e. a functional which is non-increasing along solutions. In the context of this work, we understand by gradient flow structure the purely algebraic formulation

$$\partial_t u - \operatorname{div}(B\nabla \operatorname{grad} \mathrm{H}[u]) = f(u), \qquad \text{in } \mathbb{R}^d,\ t > 0,$$

where $B \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix and $\operatorname{grad}\mathrm{H}$ is the Fréchet derivative of the so-called entropy functional

$$\mathrm{H}[u] = \int_{\mathbb{R}^d} h(u)\,\mathrm{d}x.$$

One might identify $\operatorname{grad} H[u]$ with $h'(u)$, which motivates the introduction of so called entropy variables $w = h'(u)$. These variables make it possible to express (1.1) in a way which reveals its gradient flow structure. The equation then reads

$$\partial_t u - \operatorname{div}(B\nabla w) = f(u), \qquad \text{where } B = A(u)h''(u)^{-1}, \tag{1.2}$$

where u is now a function of w and given by $u(w) = (h')^{-1}(w)$. This not only brings the equation in a mathematically convenient form, but also introduces a physical aspect as the entropy variables are strongly related to chemical potentials arising in thermodynamics. We refer to the appendix of [77] for a discussion.

The concept of entropy itself originates from Rudolf Clausius and the study of thermodynamics. Ludwig Boltzmann later clarified Clausius' definition by suggesting a statistical way to measure the entropy of an ensemble of ideal gas particles. This notion of entropy is related to the logarithm of the number of micro-states of the gas, see [18]. A key observation was that the Boltzmann equation of kinetic theory possesses a Lyapunov functional related to the entropy. This functional was named the H-function and corresponds to the negative of the physical entropy (see [17]):

$$\mathrm{H}[f] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,v,t)\log f(x,v,t)\,\mathrm{d}x\,\mathrm{d}v,$$

where $f(x,v,t)$ is the distribution function depending on the position x, the velocity v and time $t \geq 0$. The fact that $H[f(t)]$ is non-increasing in time is called the H-theorem which can be interpreted as an exemplification of the second law of thermodynamics, namely that the physical entropy never decreases in closed systems. The H-theorem is one of the central aspects of entropy methods for partial differential equations. For an overview, we refer to [78].

In the mathematical study of cross-diffusion systems, one of the first uses of entropy variables can be found in [48]. Later they were used to analyze population systems in one dimension [59] and multiple space dimensions [35, 36]. Burger and coworkers [31] used entropy variables that allowed for L^∞ bounds without using a maximum principle for systems with volume-filling effects. Volume filling means that there is a physical upper bound on the sum of the components. This idea was then generalized in [77, 132]. Recalling that maximum principles can be difficult to apply to these equations, this strategy provided a natural way to obtain upper bounds and is in a certain sense inherent to the structure of the equation.

Previous results in the literature were concerned with the use of more classical approaches for proving global existence of solutions. In [86, Chap. VII] Ladyženskaya and coauthors reduced the problem of finding a priori estimates of local-in-time solutions u to quasilinear parabolic systems to the problem of deriving L^∞ bounds for u and ∇u. They concluded the global-in-time existence of solutions under some growth conditions on the nonlinearities. In [6], Amann presented a theory for strongly coupled systems. He introduced the concept of $W^{1,p}$ weak solutions and proved their local existence. Under the condition that their $W^{1,p}$ norm, with $p > d$, can be controlled, he even established global existence. Pierre, in [109], showed that global existence follows if the non-negativity of the

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solutions is preserved, the total mass of the components is bounded on finite time intervals, and the reaction term $f(u)$ grows at most linearly in the solution u. This result however only holds true for diagonal diffusion matrices and one may have L^∞ finite-time blow up for super-quadratic reaction terms as shown in [110].

1.2. The SKT model

One of the most well known cross-diffusion systems, which will be investigated in a later chapter, is the population model suggested by Shigesada, Kawasaki and Teramoto [119]. Their base assumption was that the populations of two species prefer the same environment and are influenced by intra- as well as inter-specific population pressures. In other words, the movement of each individual species depends on the density of both species. The evolution of the population densities u_1, u_2 is given by the deterministic system of equations

$$\partial_t u - \operatorname{div}(A(u)\nabla u + D(u)\nabla\phi) = f(u), \quad t > 0, \quad u(0) = u^0 \quad \text{in } \mathcal{O} \tag{1.3}$$

where $\mathcal{O} \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with smooth boundary, $u(t) = (u_1, u_2)^\top (t) : \mathcal{O} \to \mathbb{R}^2$ is a vector-valued function and $A(u) = (A_{ij}(u)) \in \mathbb{R}^{2\times 2}$ is the diffusion matrix given by

$$A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix}. \tag{1.4}$$

$D(u) = (D_{ij}(u)) \in \mathbb{R}^{2\times 2}$ is the drift matrix, whose entries are simply given by $D_{ij}(u) = u_i\delta_{ij}$. $\phi : \mathcal{O} \to \mathbb{R}$ is a potential function which models areas with more or less favorable environmental conditions. Typically the term $f(u)$ is of Lotka-Volterrra type:

$$f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)\, u_i, \quad i = 1, 2, \tag{1.5}$$

where $b_{ij} \geq 0$. The coefficients a_{10}, a_{20} are referred to as diffusion coefficients. a_{11} and a_{22} are usually called self-diffusion coefficients. Self diffusion models the tendency for a species to move along the direction of lower density of their own species. Mathematically self-diffusion can be expressed by a quadratic porous-medium term appearing in the equation governing the evolution of a specific species. The cross-diffusion coefficients a_{12}, a_{21} express flux of one species because of the presence of the other species.
The SKT model (without drift and reaction) can be derived from a lattice random walk in the diffusion limit as illustrated in [77, Appendix B]. A rigorous derivation based on the mean-field(-type) limit of a moderately interacting particle system was recently obtained in [34]. The diffusion matrix A is generally neither symmetric nor positive definite, but has positive eigenvalues. One can use $\min\{0, u_i\}$ as a test function and obtain, by the minimum principle, that the solution is nonnegative, but the derivation of a priori estimates is more complicated. Therefore, the global existence of solutions under general assumptions was an open problem for some decades.

One of the first existence results is due to Kim [80] under the assumptions $a_{11} = a_{22} = 0$ and $(a_{ij} = 1)$ for $i \neq j$. The tridiagonal case $a_{21} = 0$ was investigated by Amann [6], Le [88] and more recently, by Desvillettes and Trescaces [49]. Yagi [128] proved

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an existence theorem under the assumption that the diffusion matrix is positive definite ($a_{12} < 8a_{11}, a_{21} < 8a_{22}, a_{12} = a_{21}$).

The progress on the subject made by Amann [6] was significant as he derived sufficient conditions for the solutions to general quasilinear parabolic systems to exist globally in time. The question whether a local-in-time solution exists globally is reduced to the problem of finding a priori estimates in the $W^{1,p}$ norm with $p > d$. In general, the derivation of such bounds poses a tricky problem. The first global existence result for the SKT model where the only restriction on the coefficients was their positivity was presented in [59] in one space dimension and in [35, 36] in several space dimensions.

From a biological point of view, the SKT model exhibits rather interesting properties. For instance, under certain conditions on the parameters, there exist non-constant steady states which represent pattern formation. It turns out that large diffusion coefficients a_{10}, a_{20} tend to eliminate any pattern, while large cross-diffusion coefficients a_{12}, a_{21} help to create patterns. The interplay between diffusion and cross-diffusion was explored in [92]. For more information on this subject, we refer to [76]. The SKT model can also be formulated for n species, but a detailed-balance-like condition has to be imposed on the coefficients in order to obtain a global solution, see e.g. [37].
There are only very few results for stochastic SKT systems. The first result was presented in [52], where the authors consider an SKT-type system in which the coefficients of the associated diffusion matrix depend quadratically on the densities (and not linearly as in (1.4)). The general structure in [52] allowed to work purely in a Hilbert space framework, leading to a novel result through a standard approach. A different solution concept, which we will encounter in Chapter 3, was used in [85] and led to local-in-time solutions. This approach however requires a positive definite diffusion matrix $A(u)$.

1.3. The Maxwell-Stefan equations

Another important example are the Maxwell Stefan equations, describing the dynamics of a fluid mixture in the diffusive regime. Their applications lie for example in sedimentation, dialysis, respiratory airways, electrolysis, and chemical reactors. The system itself was first suggested by James Maxwell [96] for dilute gases and later by Josef Stefan [121] for fluids. In most cases, a component in a fluid mixture is expected to diffuse down the gradient of its concentration, which is expressed by Fick's law. However, in experiments (see e.g. [55]), it was observed that the diffusion flux of any component is strongly coupled to that of the other components. In other words, cross-diffusion effects appear naturally due to the drag and friction forces between the individual components. The Maxwell Stefan equations take these effects into account and can predict osmotic or uphill diffusion, where the flux goes from low- to high-concentration regions. An example of a Maxwell Stefan system for $n = 3$ components is given by

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u), \quad t > 0, \quad u(0) = u^0 \quad \text{in } \mathcal{O}, \tag{1.6}$$

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with the diffusion matrix

$$A(u) = \frac{1}{a(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix},$$
$$\text{where } a(u) = d_0 d_1 u_1 + d_0 d_2 u_2 + d_1 d_2 u_3.$$

The third component is somewhat hidden, as it is determined by the relation $u_3 = 1 - u_1 - u_2$, which indicates that the mixture is saturated. We will also refer to the property $1 = u_1 + u_2 + u_3$ as volume-filling. The matrix A is generally non-symmetric and not positive definite, but its eigenvalues are positive. This allows for local smooth deterministic solutions; see [7]. The first global existence result for deterministic Maxwell Stefan equations was proved in [65] for initial data around the constant equilibrium state. The existence of local classical solutions was shown in [19]. The entropy structure was revealed in [79], and a general global existence result was proved. Also couplings to Navier-Stokes equations were investigated [30, 38, 95, 100]. For a related system, a so-called Poisson Nernst Planck system, a stochastic forcing based on thermal fluctuations was proposed by physicists in [107]. The Maxwell Stefan system serves as a concrete example of a larger class of cross-diffusion systems incorporating volume-filling effects. We will encounter such systems in Section 2.10. To the best of the author's knowledge, prior to [50], there were no results treating these equations with a stochastic forcing in a mathematically rigorous way.

1.4. The stochastic equations

The main focus of this thesis is to obtain global solutions for stochastic cross-diffusion equations, covering both previously mentioned examples. The stochastic version of equation (1.1) will be given by

$$\partial_t u = \operatorname{div}(A(u)\nabla u) + f(u) + \sigma(u)\dot{W} \text{ in } \mathcal{O} \subset \mathbb{R}^d,\ t > 0. \tag{1.7}$$

More specifically, we will be concerned with the equation in its integral form,

$$u(t) = u(0) + \int_0^t \operatorname{div}\Big(A(u)\nabla u\Big)(s)\,\mathrm{d}s + \int_0^t f(u)(s)\,\mathrm{d}s + \int_0^t \sigma(u)(s)\mathrm{d}W(s),$$

which is formally written as

$$\mathrm{d}u = \operatorname{div}\Big(A(u)\nabla u\Big)\mathrm{d}t + f(u)\,\mathrm{d}t + \sigma(u)\mathrm{d}W(t). \tag{1.8}$$

Here W represents a random noise that can be motivated by uncertainties in the modelling of specific phenomena and may vary in its form depending on the application under consideration. Such systems fall into the larger class of quasilinear stochastic partial differential equations (SPDEs), which received a lot of attention and has been extensively analyzed using the theory of (locally) monotone operators [13, 62, 90, 111] or approximating the corresponding coefficients by locally monotone ones [69]. Recently, there has been a growing interest in developing various solution concepts for quasilinear SPDEs such as: kinetic [47, 57, 63], strong (in the probabilistic sense) and weak (in the PDE

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sense) [45, 42, 69], entropy [41], martingale [47, 52] or (pathwise) mild solutions [85]. We mention that solution concepts for certain quasilinear SPDEs have also been developed via rough paths theory [102], paracontrolled calculus [10, 11, 58], and regularity structures [61, 60]. To the best of the author's knowledge, the techniques employed in above mentioned contexts heavily rely on the diffusion matrix being symmetric and/or positive-(semi)definite [45, 47, 69, 71]. In the context of cross diffusion systems this is however not the case in many applications. Therefore, most of the techniques available in the literature on quasilinear SPDEs do not apply or allow only for local-in-time solutions [71, 85]. A step in the direction of treating systems similar to ours is made in [67], which nicely adapts more complex fixed-point approaches to stochastic equations. This approach is however more related to the deterministic solution theory for cross-diffusion systems which we try to avoid.

Therefore we will adapt the methodology related to the deterministic entropy structure to address the question of global-in-time existence for cross-diffusion systems of the form (1.8). In addition we aim to obtain physically relevant bounds similar to the deterministic ones.

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1.5. Layout of the thesis

The structure of the thesis is as follows.

- In Chapter 2 we discuss the existence of solutions to general stochastic cross-diffusion systems under certain assumptions related to their entropy-structure. We present a new technique for establishing solutions which, after introducing a particular regularization operator and a corresponding "change-of-variables" for the equation, reduces the complexity of the problem significantly and allows for the application of standard techniques. This approach also presents a new alternative in the deterministic setting as it avoids more complicated fixed-point arguments and gives a clearer relation to the techniques used for non-degenerate systems.
  This approach will be applied to the Shigesada-Kawasaki-Teramoto (SKT) population model in Section 2.8 to establish global-in-time existence of solutions under very general conditions. We also study the problem considered in [36], where self-diffusion is absent (see Section 2.9). This setting is mathematically more challenging and requires a careful treatment of the entropy production terms.
  The last application focuses on stochastic cross-diffusion systems exhibiting a so-called volume-filling behaviour, which takes into account that concentrations may saturate. Such effects appear for example in multicomponent fluid models, tumor growth models and models describing the transport of ions in biological cells. The existence of global-in-time bounded martingale solutions to a general class of cross-diffusion systems with multiplicative noise is proven. Due to the volume-filling effects, these equations exhibit a different entropy structure compared to the systems in Chapter 2.8. The strategy of the proofs is similar, but the entropy structure provides more properties of the solutions.

- In Chapter 3, we will directly compare the two solution concepts under which stochastic SKT systems have been studied, namely weak- and pathwise-mild solutions. We provide conditions under which these types of solutions are equivalent for general systems of quasilinear parabolic systems with the specific application to cross-diffusion systems in mind.

- In Chapter 4 we consider (slight) generalizations of known tightness results and extensions to the case where only gradient estimates of nonlinear functions of certain terms are known.

- Chapter 5 covers most, if not all, necessary results to keep the thesis as self-contained as possible. This chapter can be consulted if necessary and can skipped by the advanced reader.

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CHAPTER 2

Stochastic cross-diffusion systems

2.1. Introduction

We study systems of abstract degenerate quasilinear SPDEs in divergence form. The evolution of each component is governed by

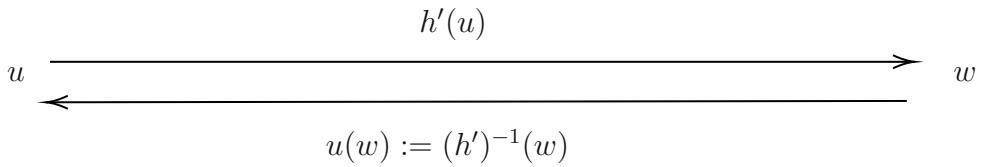
$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u)\,\mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t>0,\ i=1,\dots,n, \tag{2.1}$$

where $\mathcal{O}\subset\mathbb{R}^d$ is a bounded domain. These equations are supplemented with the no-flux boundary and initial conditions,

$$u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \qquad \sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\mathcal{O},\ t>0,\ i=1,\dots,n. \tag{2.2}$$

The precise assumptions on the coefficients and the domain will be stated in the subsequent sections. $u=(u_1,\dots,u_n)$ is a vector of densities, which correspond to population densities or fluid concentrations in the applications considered.
$(W_1,\dots,W_n)$ is an n-dimensional cylindrical Wiener process representing external perturbations or a lack of knowledge of certain physical parameters and σ a matrix of suitable Hilbert-Schmidt operators. The addition of noise to the deterministic dynamics is partially motivated by the fact that the interactions in many models considered actually happen on a mesoscopic scale on which random microscopic fluctuations should still be observable. ν is the exterior unit normal vector to $\partial\mathcal{O}$, and u_i^0 is a possibly random initial datum. In a multitude of applications the matrix A is neither symmetric nor positive semi-definite, hence these systems are usually not within the scope of the classical theory. To ensure a wide applicability of our results, we embrace this structure and only assume that the equation possesses a so-called entropy structure, which is related to a formal gradient flow structure.
In order to analyze the existence of solutions to equation (2.1) (2.2), we go through two steps of approximations. The first step is related to the classical entropy picture:

The entropy picture



At its heart lies a transformation of the equation into so-called entropy variables, which are related to chemical potentials arising in thermodynamics, denoted by w. Expressed in these variables, much of the initial degeneracy within the divergence operator vanishes. Practically, we assume that the solution u_i to be a known, deterministic function of the "true solution" w. This interpretation formally leads to the following equation,

$$du(w)(t) = \operatorname{div}\left(A(u(w))u'(w)\nabla w\right)\,\mathrm{d}t + \sigma(u(w))\,\mathrm{d}W(t). \tag{2.3}$$

The matrix $A(u(w))u'(w)$ will be denoted by $B(w)$ and is usually assumed to be positive semi-definite, which is one of the main benefits of introducing the variables w. We will encounter specific examples of this entropy transformation in a later section. Although well suited for the deterministic setting, which heavily relies on fixed-point arguments, the treatment of implicitly formulated stochastic systems is more delicate. Building on the point of view of "entropy-variables", we introduce another regularization. Interpreting $u(\cdot)$ as an operator, rather than a real valued function, and adding a regularizing term on the left hand side of the equation, leads to the following approximating problem:

$$d\left(u^\varepsilon(w)(t) + \varepsilon L^*Lw(t)\right) = \operatorname{div}\left(A(u)u'(w)\nabla w\right)\,\mathrm{d}t + \sigma(u(w))\,\mathrm{d}W(t). \tag{2.4}$$

In order to formulate this problem as an equation which is explicit on the left-hand-side, we define the process $v(x,t,\omega) := \big(u(w) + \varepsilon L^*Lw\big)(x,t,\omega)$. Assuming the invertibility of the operator $u(\cdot) + \varepsilon L^*L(\cdot)$ mapping a certain Hilbert space $D(L)$ to the Hilbert space $D(L)'$ and denoting its inverse by $R_\varepsilon(\cdot)$, we are able to rewrite equation (2.4) in terms of v,

$$dv(t) = \operatorname{div}\left(\tilde{B}(v)\nabla R_\varepsilon(v)\right)\,\mathrm{d}t + \tilde{\sigma}(v)\,\mathrm{d}W(t), \tag{2.5}$$

leading to an equation which is explicit on the left-hand side. Here $\tilde{B} := A(u(R_\varepsilon(v)))u'(R_\varepsilon(v))$ and $\tilde{\sigma} := \sigma(u(R_\varepsilon(v)))$. Intuitively, the operator L should be thought of as $(1-\Delta)^{m/2}$ and the spaces $D(L)$ and $D(L)'$ as $H^m(\mathcal{O})$ and $(H^m(\mathcal{O}))'$, for m sufficiently large, respectively. We present this Ansatz in pictorial form:

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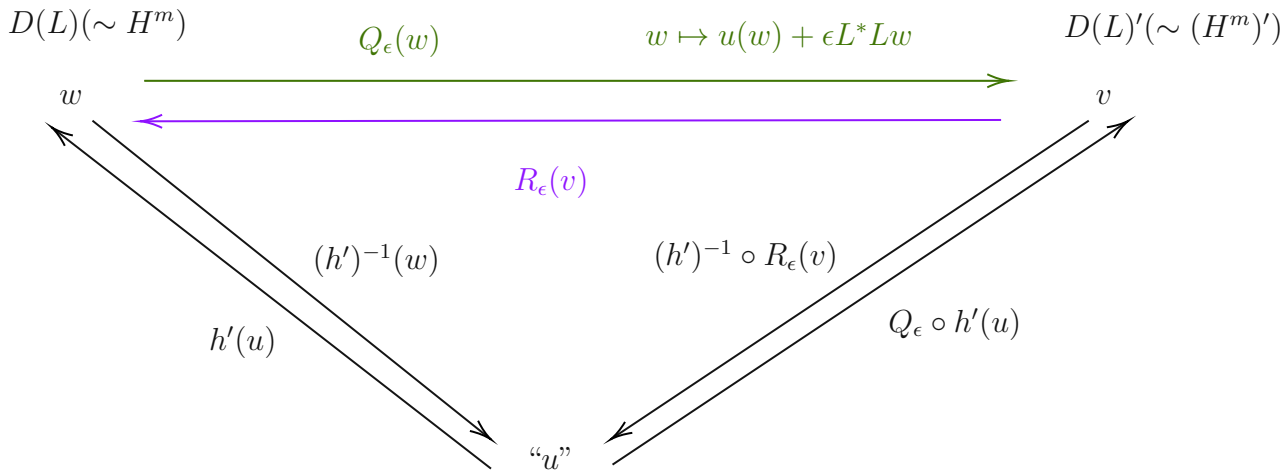
The extended entropy picture



The dynamics of equation (2.5) presents a rather unusual form of evolution, which is driven by a more regular version of the solution itself. This behaviour differentiates it from more commonly used approaches of regularization or mollification and could best be described as regularization by transformation.
The classical entropy method, presented in [77, 31] and illustrated in figure 2.1, motivates the interpretation of the operator R_ε as the result of lifting the function $h'(\cdot) = (u')^{-1}(\cdot)$ onto the level of operators and regularizing it, resulting in a regularized entropy transformation operator $h'_\varepsilon = R_\varepsilon$.
Before we continue, we want to establish, what we understand by a solution to equation (2.1) (2.2). For this, let L be a higher order differential operator whose domain is the separable Hilbert space $D(L)$. $D(L)'$ will be the dual of $D(L)$ with respect to the duality in $L^2(\mathcal{O})$.

DEFINITION 2.1.1. *Let $T > 0$ be fixed, but arbitrary. A (global), probabilistically weak or, martingale solution of (2.1) (2.2) is a collection $\{(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}), \widetilde{\mathbb{F}}, (\widetilde{W}^k)_{k\in\mathbb{N}}, \widetilde{u}\}$, such that $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is a probability space, $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ is a complete filtration, $(\widetilde{W}^k)_{k\in\mathbb{N}}$ is a sequence of independent $\widetilde{\mathbb{F}}$-Wiener processes, $\widetilde{W}$ is a n-dimensional cylindrical Wiener process and $\widetilde{u}$ is a continuous $D(L)'$-valued $\widetilde{\mathbb{F}}$-adapted process, such that for all $\phi \in D(L)$ almost surely the equality*
(2.6)

$$\langle \widetilde{u}_i(t), \phi\rangle = \langle \widetilde{u}_i(0), \phi\rangle + \int_0^t \left\langle \operatorname{div}\left(\sum_{j=1}^n A_{ij}(\widetilde{u})\nabla \widetilde{u}_j\right)(s), \phi \right\rangle \mathrm{d}s + \int_0^t \left\langle \sum_{j=1}^n \sigma_{ij}(\widetilde{u})(s)\,\mathrm{d}\widetilde{W}_j(s), \phi \right\rangle,$$

for $i = 1, \dots, n$, holds for all $t \in [0, T]$, where $\widetilde{u}_0 \overset{d}{=} u_0$. $\langle\cdot,\cdot\rangle$ is to be understood in terms of the duality $\langle\cdot,\cdot\rangle_{D(L)'\times D(L)}$.

REMARK 2.1.2. *We admit to a slight abuse of notation throughout this work: We will use the same notation for a function taking values in a vector space X, and a higher level*

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tensor whose entries take values in the vector space X. This stretch of notation improves the readability without jeopardizing the mathematical accuracy.

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2.2. Notation

In this section in particular, we will adhere to the following notation.

- $\mathbb{N} = \{1, 2, \dots\}$.
- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, \infty)$.
- $\mathcal{O} \subset \mathbb{R}^d$ will be a bounded domain. The dimension of the domain will generally be expressed by d
- The number of components of a system will generally be expressed by $n \in \mathbb{N}$.
- $\mathcal{D} \subset \mathbb{R}^n$ denotes an open set where the solution of our system assumes its values.
- $\mathcal{B}(X)$ denotes the Borel σ algebra of a topological space X.
- $L^0(X)$ denotes the space of measurable functions.
- $L^p(X)$, with $p \in [1, \infty]$ denote the usual Lebesgue spaces of functions u (up to modification on sets of Lebesgue measure 0), for which $\left(\int_X |u(x)|^p \,\mathrm{d}x\right)^{1/p} < \infty$.
- $W^{\alpha,p}(X)$, $\alpha \in \mathbb{R}, p \in [1, \infty]$ denotes the fractional Sobolev (Slobodecki) spaces.
- $H^s(\mathcal{O})$, for $s \notin \mathbb{Z}$, denote the fractional Sobolev spaces obtained by real interpolation. Equivalently one can consider the Bessel-potential/Fourier Sobolev spaces.
- $H^s(\mathcal{O}) = W^{s,2}(\mathcal{O})$, for $s \in \mathbb{R}$.
- $H^m_N(\mathcal{O}) \coloneqq \{w \in H^m(\mathcal{O}) \colon \nabla w \cdot \nu = 0 \text{ on } \partial\mathcal{O}\}$, $m \in \mathbb{N}$.
- $C(X)$, $C^\alpha(X)$ denote the continuous and α Hölder continuous functions on a topological space X.
- $\mathcal{L}_2$ denotes the space of Hilbert-Schmidt operators.
- $\mathcal{L}_2^{n\times n}(X, Y)$: The space of $n \times n$ matrices whose entries are Hilbert-Schmidt operators from X to Y.
- X' (or X^* in section 3) denotes the dual space of a space X.
- $\mathrm{dom}(L)$ denotes the domain of an operator L (set of points).
- $D(L)$ denotes the closure of $\mathrm{dom}(L)$ or a subset thereof, with respect to the norm $\|L \cdot\|$.
- ε denotes a regularization parameter.
- $\delta, \widetilde{\varepsilon}, \widetilde{\widetilde{\varepsilon}}, \overline{\varepsilon}, \hat{\varepsilon}$ denote small parameters.
- C will denote a generic constant, which might change from line to line.
- $\langle \cdot, \cdot \rangle_H$ denotes the inner product on a Hilbert space H. If the context is absolutely clear the subscript will be dropped.
- $\langle \cdot, \cdot \rangle_{X \times X'}$ denotes the dual product between two spaces X, X'. If the context is absolutely clear the subscripts will be dropped.
- $\langle \lambda \rangle = (1 + \lambda)$ for a real number λ.

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2.3. Assumptions

We first collect the necessary assumptions to perform our analysis.
To begin we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and a sequence $\{W_i^k\}_{k\geq 1}$ and $i = 1, \dots, n$ of independent standard $1d$ Brownian motions relative to $\mathcal{F}_t$. To make the distinction clear, we will also use the symbol $\mathbb{F}$ to denote the filtration. In order to avoid unnecessary complications, we may assume that the filtration is complete and right continuous.
For $T > 0$, arbitrary but fixed, we assume $[0, T]$ to be equipped with the Borel sigma-algebra and the Lebesgue measure.
Let $U, \hat{U}$ be separable Hilbert spaces. We will assume, if not stated otherwise, that W is a vector-valued U-cylindrical Wiener process, taking values in $\hat{U}$. In other words, let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U, then $W = (W_1, \dots, W_n)^T$, where

$$W_i = \sum_{k=1}^{\infty} W_i^k e_k,$$

is a continuous $\hat{U}$ valued process.
Next we introduce the assumptions on entropy density h:
Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set. There exists a function $h \in C(\mathcal{D}; [0, \infty))$, such that

(E1) $h \in C^3(\mathcal{D}; [0, \infty))$, h is convex.
(E2) The first derivative, denoted by $h'(\cdot)\colon \mathcal{D} \to \mathbb{R}^n$, is invertible on $\mathbb{R}^n$. We also assume, for every $u \in \mathcal{D}$, that the matrix $h''(u) \in \mathbb{R}^{n\times n}$ is invertible.
(E3) The matrix $B(u) = A(u)(h'')^{-1}(u)$, and hence also the matrix $h''(u)A(u)$, is positive semi-definite for all $u \in \mathcal{D}$.
(E4) The function $u(\cdot) := (h')^{-1}(\cdot)$, from $\mathbb{R}^n$ to $\mathcal{D}$ is growing at most exponentially. The following bound is satisfied. Let $w \in L^\infty(\mathcal{O})$, then

$$\|u(w)\|_{L^\infty(\mathcal{O})} \vee \|u'(w)\|_{L^\infty(\mathcal{O})} \vee \|u''(w)\|_{L^\infty(\mathcal{O})} \leq C \exp\left(C\|w\|_{L^\infty(\mathcal{O})}\right).$$

(E5) There exists a constant $C \geq 0$, such that for every $x \in \overline{\mathcal{D}}$

$$|x| \leq C(1 + h(x)).$$

We now turn to the equation and its coefficients. Let $D(L)$ be a separable Hilbert space of function $w\colon \mathcal{O} \to \mathbb{R}^n$. We assume the following,

(A1) Domain: $\mathcal{O} \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with $\partial\mathcal{O} \in C^\ell$ and $\ell \in \mathbb{N}$ satisfies $\ell > d/2 + 2$.
(A2) Initial datum: Let $p \geq 2$. $u^0 \in L^{p_0}(\Omega; L^p(\mathcal{O}))$, for $p_0 \geq (2/d)\max\{8, d + 2\}$, is $\mathcal{F}_0$-measurable and $u_i^0(x) \in \mathcal{D}$ for a.e. $x \in \mathcal{O}$ $\mathbb{P}$-a.s., $i = 1, \dots, n$.
(A3) The diffusion matrix $A(u)$ is an element of $C^2(\mathcal{D}, \mathbb{R}^n \times \mathbb{R}^n)$ and grows linearly in u.
(A4) $\sigma(u)$ is an $n \times n$ matrix, whose entries are elements of the space of Hilbert Schmidt operators from the Hilbert space U, whose orthonormal basis is given by $(e_k)_{k\in\mathbb{N}}$,

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to $L^2(\mathcal{O})$, which is defined as

$$\mathcal{L}_2(U;L^2(\mathcal{O})) = \left\{ L : U \to L^2(\mathcal{O}) \text{ linear continuous} : \sum_{k=1}^{\infty} \|Le_k\|^2_{L^2(\mathcal{O})} < \infty \right\}$$

and endowed with the norm $\|L\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} = \sum_{k=1}^{\infty} \|Le_k\|^2_{L^2(\mathcal{O})}$. The entries of the matrix $\sigma = (\sigma_{ij})_{i,j=1}^n$ is assumed to be $\mathcal{B}(L^2(\mathcal{O};\mathbb{R}^n))/\mathcal{B}(\mathcal{L}_2(U;L^2(\mathcal{O})))$-measurable and $\mathbb{F}$-adapted.
We will denote the space of $n \times n$ matrices with Hilbert-Schmidt valued entries by $\mathcal{L}_2^{n\times n}$.

(A5) There is a constant $C_\sigma > 0$ such that for any $u_1, u_2 \in L^2(\mathcal{O})$, $\sigma\colon L^2(\mathcal{O};\mathbb{R}^n) \to \mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))$ satisfies

$$\|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \le C_\sigma \|u_1 - u_2\|_{L^2(\mathcal{O})},$$
$$\|\sigma(u)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \le C_\sigma \big(1 + \|u\|_{L^2(\mathcal{O})}\big).$$

(A6) Let h be the entropy density defined previously and $v \in D(L)'$, where $D(L)'$ denotes the dual (in terms of the triple $D(L) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow D(L)'$) of the Hilbert space $D(L)$ introduced in (R2). Recall $u(\cdot)$ from (E3). Then, for $w \in D(L)$,

$$\left(\int_0^t \sum_{k=1}^{\infty} \sum_{i,j=1}^{n} \left(\int_{\mathcal{O}} h'(u(w)))_i \sigma_{ij}(u(w)))_k(s) \,\mathrm{d}x \right)^2 \mathrm{d}s \right)^{1/2}$$
$$\le C\left(1 + \int_0^t \int_{\mathcal{O}} h(u(w))(s) \,\mathrm{d}x \,\mathrm{d}s\right),$$
$$\sum_{k=1}^{\infty} \int_0^t \int_{\mathcal{O}} \mathrm{tr}\Big[(\sigma(u(w))_k(s))^T (u'(w))^{-1} (\sigma(u(w))_k(s)) \Big] \,\mathrm{d}x \,\mathrm{d}s$$
$$\le C\left(1 + \int_0^t \int_{\mathcal{O}} h(u(w))(s) \,\mathrm{d}x \,\mathrm{d}s\right).$$

$\sigma_{ij}(u(w))_k$ denotes the image of $\sigma_{ij}(u(w))(e_k)$, where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of U, $\sigma(u(w))_k$ the image of $\sigma(u(w))(e_k)$ and tr the finite-dimensional matrix trace.

Finally, we address the assumptions needed for our regularization procedure. There exists an operator L such that:

(R1) The operator L is linear and self-adjoint as an unbounded operator on $L^2(\mathcal{O})$.
(R2) The domain of the operator L, denoted by $D(L)$, is a separable Hilbert space.
(R3) The embeddings

$$D(L) \hookrightarrow L^2(\mathcal{O}),\ D(L) \hookrightarrow W^{1,\infty}(\mathcal{O}),$$

are continuous and weak*-dense.
(R4) $0 \in \rho(L)$, or in other words the operator L is invertible.

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REMARK 2.3.1. *(Discussion of the assumptions)*

- *Entropy:*
  Assumptions (E1) (E3) ensure that the equation possesses a suitable entropy structure leading to a so called entropy-estimate of our solution. The entropy-estimate will be crucial, since our Assumptions do not allow for the derivation of classical energy estimates. In the deterministic literature on cross-diffusion systems, such estimates are quite standard [77, 49].
  The regularity assumption imposed on h ensures that the inverse of $h'(\cdot)$, denoted by $(h')^{-1}(\cdot)$ or $u(\cdot)$, is $C^2(\mathbb{R}^n, \mathcal{D})$.
  (E3) also implies that the entropy-density h is in fact strictly convex.
  (E4) should be seen as a "soft" assumption, which is used to establish a quantitative rate of growth in a later section. Incorporating more rapid growth does not pose any mathematical challenges.

- *Equation:*
  Assumption (A3) might seem rather restrictive, and can be relaxed to a weak differentiability and local Lipschitz continuity assumption.
  Let $\mathcal{O} \subset \mathbb{R}^d$ be open and abounded and $u\colon \mathbb{R}^d \to \mathcal{D}$ be an element of $L^1(\mathcal{O})$. Then (E5) guarantees, that there exists a positive constant $C \geq 0$, such that

$$\|u\|_{L^1(\mathcal{O})} \leq C\left(1 + \int_{\mathcal{O}} h(u(x))\,\mathrm{d}x\right).$$

  Assumption (A5) is another Assumption, which can be generalized to local Lipschitz continuity. Such an alteration would only affect the arguments in the limiting procedure.
  (A6) imposes an important interplay between the noise and the entropy-structure. Intuitively, the multiplicative noise needs to compensate for potential singularities of h' and h''. (A6) is mainly chosen to maintain an easy readability of the results. In section 2.11 we will introduce a refined version which alleviates the restrictions on σ significantly and allows for physically relevant structures of σ.
  We want to give an easy example for σ satisfying (A6) in a particular setting which corresponds to an application we will encounter later on. Let $u_i \in L^2(0,T;L^2(\mathcal{O})), u_i > 0$ for $i = 1,\dots,n$ and a.e. (t,x), $(e_k)_{k\in\mathbb{N}}$ be an ONB of U and $(a_k)_{k\in\mathbb{N}}$ be a sequence of real valued numbers such that $(a_k)_{k\in\mathbb{N}} \in l^2(\mathbb{R})$. Further let $\alpha > 0$, and $h(u) = \sum_{i=1}^n u_i \log(u_i) - u_i + 1$, then the operator valued matrix σ with entries

$$\sigma(u)_{ij}(\cdot) = \delta_{ij}\frac{u_i}{1 + (u_i)^{1+\alpha}}\sum_{k=1}^{\infty}\langle a_k e_k, \cdot\rangle_U$$

  satisfies (A4) (A6). It is apparent that this is a rather artificial example which one would not expect to arise in an actual application.

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Under the refined condition (A6)', in the same setting, the components of σ are allowed to be of the form

$$\sigma(u)_{ij}(\cdot) = \delta_{ij} u_i^\alpha \sum_{k=1}^{\infty} \langle a_k e_k, \cdot \rangle_U,$$

with $1/2 \leq \alpha \leq 1$. Such structures are, arguably, more natural, when considering that the noise term should arise in some kind of limiting procedure of a particle system. Although the space U is generic, in most applications we will assume $U = L^2(\mathcal{O})$ or U to be a space such that the noise can be interpreted as a Q-Wiener process on $L^2(\mathcal{O})$. In the second case, (A6)' will allow us to consider

$$\sigma(u)_{ij} = \delta_{ij} u_i^\alpha,$$

with $1/2 \leq \alpha \leq 1$. We recall that such an $L^2(\mathcal{O})$-cylindrical Wiener process actually takes values in $H^\gamma(\mathcal{O})$, for any $\gamma < -\frac{d}{2}$.

- *Regularization:*
  Assumption (R3) allows us to call upon Lemma 2.12.4 and handle terms of the form $\|\nabla \cdot\|_{D(L)'}$.
  (R4) and Lemma 2.12.3 allow us to identify the norm $\|\cdot\|_{D(L)}$ with $\|L \cdot\|_{L^2(\mathcal{O})}$ as well as $\|\cdot\|_{D(L)'}$ with $\|L^{-1} \cdot\|_{L^2(\mathcal{O})}$.
  Assumptions (R1) and (R2) are in place to avoid unnecessary technicalities.

We shall introduce additional notation, which will be used throughout this section.

DEFINITION 2.3.2. *We introduce the $n \times n$ matrix $B(\cdot) \coloneqq A(\cdot)(h'')^{-1}(\cdot)$, as well as $\tilde{B}(\cdot) \coloneqq B(u(R_\varepsilon(\cdot)))$.*
Further we define the mapping $\tilde{\sigma} \colon D(L)' \to \mathcal{L}_2^{n\times n}(U, L^2(\mathcal{O}))$, as $\tilde{\sigma}(\cdot) \coloneqq \sigma(u(R_\varepsilon(\cdot)))$.

REMARK 2.3.3. *We recall that $(h')^{-1}(\cdot)$ and $u(\cdot)$ will essentially denote the same operation. This overload in terms of notation serves the purpose of remaining as consistent as possible with the formulation of our initially posed problem (2.1). It will also help us to keep track of how the final solution enters our estimates.*

REMARK 2.3.4. *When speaking about an unbounded operator L from $L^2(\mathcal{O}) \to L^2(\mathcal{O})$, we imply that $\mathrm{dom}(L)$ and is a subset of $L^2(\mathcal{O})$ and equipped with the topology of $L^2(\mathcal{O})$. The notation $L \colon D(L) \to L^2(\mathcal{O})$ indicates that we consider the operator L on $D(L)$ equipped with its "natural" norm-topology induced by the graph norm (or an equivalent norm).*

Since the existence of an operator L satisfying (R1)-(R4) is important to the construction in the following chapter, we already give a way to construct such an operator, in order to verify that (R1)-(R4) are indeed meaningful.

LEMMA 2.3.5. *Let $D(L)$ and H be given separable Hilbert spaces, such that the embedding $D(L) \hookrightarrow H$ is continuous and dense. Then there exists a self-adjoint, positive operator $L \colon H \to H$, whose domain coincides with $D(L)$.*

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Additionally, the dual space $D(L)'$ is isometric to $D(L^{-1})$, denoting the closure of $\mathrm{dom}(L^{-1})$ *with respect to the norm $\|L^{-1}\cdot\|_H$.*

COROLLARY 2.3.6. *Let $D(L) = H^m(\mathcal{O})$, with $m > d/2+1$ and $H = L^2(\mathcal{O})$, then there exists an operator L, satisfying (R1)-(R4).*

The Lemma and its proof are essentially laid out in [82, p.236 ff.]. For the convenience of the reader, we present an extended version of the arguments.

PROOF. To construct the operator L, we use the properties of the embedding $D(L) \hookrightarrow H$ and the Riesz representation theorem to first construct an auxiliary operator. Let $w \in D(L)$ and $u \in H$, then $w \mapsto \langle w, u\rangle_H$ is clearly a linear bounded operator on $D(L)$. Hence, there exists a unique $\widetilde{u} \in D(L)$, such that

$$\langle w, u\rangle_H = \langle w, \widetilde{u}\rangle_{D(L)}.$$

This relation defines a linear operator $\Lambda^{-1}\colon H \to D(L)$ by

$$\langle w, u\rangle_H = \langle w, \Lambda^{-1}u\rangle_{D(L)}, \text{ for all } w \in D(L),\, u \in H.$$

The operator Λ^{-1} is defined on the whole space H, linear and bounded. Indeed, for $u_1,\, u_2 \in H$,

$$\langle w, \Lambda^{-1}(u_1+u_2)\rangle_{D(L)} = \langle w, u_1+u_2\rangle_H = \langle w, u_1\rangle_H + \langle w, u_2\rangle_H = \langle w, \Lambda^{-1}u_1\rangle_{D(L)} + \langle w, \Lambda^{-1}u_2\rangle_{D(L)}.$$

$$\|\Lambda^{-1}u\|^2_{D(L)} = \langle \Lambda^{-1}u, \Lambda^{-1}u\rangle_{D(L)} = \langle \Lambda^{-1}u, u\rangle_H \le \|\Lambda^{-1}u\|_H\|u\|_H \le \|u\|_H C\|\Lambda^{-1}u\|_{D(L)}$$
$$\Rightarrow \|\Lambda^{-1}u\|_{D(L)} \le C\|u\|_H.$$

The operator is also self-adjoint since it is defined on the whole space H and symmetric. For $u_1, u_2 \in H$,

$$\langle \Lambda^{-1}u_1, u_2\rangle_H = \langle \Lambda^{-1}u_1, \Lambda^{-1}u_2\rangle_{D(L)} = \langle \Lambda^{-1}u_2, \Lambda^{-1}u_1\rangle_{D(L)} = \langle u_1, \Lambda^{-1}u_2\rangle_H,$$

which verifies the symmetry. Setting $u_1 = u_2 = u$, we observe that

$$\langle \Lambda^{-1}u, u\rangle_H = \langle \Lambda^{-1}u, \Lambda^{-1}u\rangle_{D(L)} \ge 0.$$

This yields that the operator is positive. We even claim that it is bijective. Let $u \in H$ be such that $\Lambda^{-1}u = 0$. Then $\langle u, w\rangle_H = 0$ for every $w \in D(L)$. By the density of the embedding, it follows that $u = 0$. We conclude that Λ^{-1} is indeed bijective and hence invertible.
We denote the inverse of Λ^{-1} by $\Lambda\colon \mathrm{dom}(\Lambda) \to H$. It holds that $\mathrm{dom}(\Lambda) = \mathrm{range}(\Lambda^{-1}) \subset D(L)$ and

$$\langle \overline{w}, \Lambda w\rangle_H = \langle \overline{w}, w\rangle_{D(L)},$$

for $\overline{w} \in D(L)$, $w \in \mathrm{dom}(\Lambda) \subset D(L)$.
It also follows directly that Λ is symmetric. Take $w_1, w_2 \in \mathrm{dom}(\Lambda)$,

$$\langle w_1, \Lambda w_2\rangle_H = \langle w_1, w_2\rangle_{D(L)} = \langle w_2, w_1\rangle_{D(L)} = \langle w_2, \Lambda w_1\rangle_H.$$

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Since

$$\mathrm{dom}(\Lambda^*) = \mathrm{range}(((\Lambda)^{-1})^*) = \mathrm{range}((\Lambda)^{-1}) = \mathrm{dom}(\Lambda),$$

we conclude that Λ is self-adjoint. As a consequence of the properties of Λ^{-1}, $\Lambda\colon \mathrm{dom}(\Lambda) \subset H \to H$ is densely defined. As a densely defined, self-adjoint operator, it is closed.
Since

$$\|\Lambda w\|_H \|w\|_H \geq \langle \Lambda w, w\rangle_H = \langle w, w\rangle_{D(L)} \geq c\|w\|_H^2 \geq 0,$$

for every $w \in \mathrm{dom}(\Lambda)$, we conclude that Λ is bounded from below. We are able to introduce the operator $\Lambda^{1/2}\colon \mathrm{dom}(\Lambda^{1/2}) \to H$, which itself is a densely defined, closed operator. Its domain can be obtained by closing the domain of Λ with respect to the norm

$$\|\Lambda^{1/2} w\|_H = \sqrt{\langle \Lambda w, w\rangle_H} = \sqrt{\langle w, w\rangle_{D(L)}} = \|w\|_{D(L)}.$$

We will show that $\mathrm{dom}(\Lambda^{1/2})$ coincides with $D(L)$. To prove this, assume there existed an element $\widetilde{w} \in D(L)$, such that $\langle \widetilde{w}, w\rangle_{D(L)} = 0$ for every $w \in \mathrm{dom}(\Lambda^{1/2})$. Since $\mathrm{dom}(\Lambda) \subset \mathrm{dom}(\Lambda^{1/2})$, this implies in particular that

$$\langle \Lambda w, \widetilde{w}\rangle_H = 0,$$

for $w \in \mathrm{dom}(\Lambda)$. Since $\mathrm{range}(\Lambda) = \mathrm{dom}(\Lambda^{-1}) = H$, we conclude that $\widetilde{w} = 0$. This verifies the claim.
Finally, we define $L \coloneqq \Lambda^{1/2}\colon D(L) \to H$, which is a linear, positive, self-adjoint operator. Choosing the spaces $D(L)$ and H accordingly, once observes that this operator satisfies (R1) (R4).
We can define the inverse of L, given by $L^{-1}\colon \mathrm{dom}(L^{-1}) = \mathrm{range}(L) \subset H \to D(L)$.
It remains to verify that one can identify the dual of $D(L)$ with the space $D(L^{-1})$, which denotes the closure of $\mathrm{dom}(\Lambda)$ with respect to the norm $\|L^{-1}\cdot\|_H$. Let $M = \mathrm{dom}(\Lambda) \subset D(L)$ and F be a linear bounded functional on $D(L^{-1})$, then

$$|F(v)| \leq \|F\|\|v\|_{D(L^{-1})} = \|F\|\|L^{-1}v\|_H,$$

for each $v \in M$. By the invertibility of $L\colon D(L) \to H$, we can perform a change of variables and write $u = L^{-1}v$. Then the functional $F(Lu)$ is bounded with respect to the norm on H, on the set $L^{-1}(M)$, since

$$|F(Lu)| \leq \|F\|\|Lu\|_{D(L^{-1})} = \|F\|\|u\|_H.$$

The density of $L^{-1}(M)$ in H guarantees that $F(Lu)$ admits a unique representation of the form $F(Lu) = \langle u, \widetilde{u}\rangle_H$. Performing the inverse change of variables,

$$F(v) = \langle L^{-1}v, \widetilde{u}\rangle_H = \langle v, L^{-1}\widetilde{u}\rangle_H,$$

where $L^{-1}\widetilde{u} \in D(L)$.
Conversely, for $w \in D(L)$, ever linear functional, on M, of the form $\langle w, v\rangle_H$ is bounded in the norm of $D(L^{-1})$,

$$|\langle w, v\rangle_H| = |\langle Lw, L^{-1}v\rangle_H| \leq \|Lw\|_H \|L^{-1}v\|_H = \|w\|_{D(L)}\|v\|_{D(L^{-1})}.$$

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Therefore, these functionals can be extended by continuity to functionals belonging to $D(L^{-1})'$. It is immediate, that $\|F\|_{D(L^{-1})'} \leq \|w\|_{D(L)}$. We claim that this is indeed an equality. Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of elements in M, such that $w_n \to w$ in $D(L)$. We set $v_n = L^2 w_n$. It follows that

$$\langle v_n, w\rangle_H = \langle L^2 w_n, w\rangle_H = \langle Lw_n, Lw\rangle_H \to \|w\|^2_{D(L)}.$$

For any $\varepsilon > 0$ and sufficiently large n, we have

$$F(v_n) = \langle v_n, w\rangle_H \geq (1-\varepsilon)\|w\|^2_{D(L)} \geq (1-2\varepsilon)\|w_n\|_{D(L)}\|w\|_{D(L)} = (1-2\varepsilon)\|v_n\|_{D(L^{-1})}\|w\|_{D(L)}.$$

Hence we obtain $\|F\|_{D(L^{-1})'} = \|w\|_{D(L)}$. We have shown that the dual space of $D(L^{-1})$, denoted by $D(L^{-1})'$, is isometric to $D(L)$. The isometry can be given explicitly by the equality

$$F(v) = \langle v, w\rangle_H,$$

where $F \in D(L^{-1})', w \in D(L)$ and $v \in M$. Since Hilbert spaces are reflexive, $D(L^{-1})$ is isomorphic to $D(L)'$.

□

Next, we provide a more explicit example of an operator L satisfying our assumptions and which falls under the previous construction. Strictly speaking, this example will not be applicable to our case, as we imposed, so-called, no-flux boundary conditions (see (2.2)). It is nevertheless an illustrative example should provide some intuition regarding the structure of the operator L.

EXAMPLE 2.3.7. *Let us consider the case of homogeneous Dirichlet boundary conditions. Let $L = 1 + (-\Delta)^m$. We consider as its domain the Hilbert space $H_0^{2m}(\mathcal{O})$, where each derivative of order $\leq 2m-1$ vanishes at the boundary. It is immediate that L is bounded from $H_0^{2m}(\mathcal{O})$ to $L^2(\mathcal{O})$ and $H_0^{2m}(\mathcal{O}) \subset L^2(\mathcal{O})$. Hence, we can view it as an unbounded operator on $L^2(\mathcal{O})$. By the density of $C_0^\infty(\mathcal{O})$ in $L^2(\mathcal{O})$, L is densely defined.*
Next, we observe that the spectrum of L is bounded away from 0. For this purpose, let e_i denote a fixed but arbitrary eigenfunction of the negative Laplace operator $-\Delta$.

$$(1 + (-\Delta)^m)e_i = (1 + \lambda_i^m)e_i.$$

By the density of $C_0^\infty(\mathcal{O})$ in $H_0^{2m}(\mathcal{O})$, we see that L is indeed self-adjoint.
Hence $L^*L = (1 + (-\Delta)^m)(1 + (-\Delta)^m)$. L^*L is now viewed as a bounded linear operator from $H_0^{2m}(\mathcal{O})$ into $(H_0^{2m}(\mathcal{O}))'$. To reconcile with the previous notation, $D(L) = H_0^{2m}(\mathcal{O}), D(L)' = (H_0^{2m}(\mathcal{O}))^*, L = L^* = (1 + (-\Delta)^m)$. Choosing $m > d/2 + 1$, we note that, by $H_0^{2m}(\mathcal{O}) \subset H^{2m}(\mathcal{O})$ and the Sobolev embedding theorem, that $H_0^{2m}(\mathcal{O}) \subset W^{1,\infty}(\mathcal{O})$. Since $C_0^\infty(\mathcal{O})$ is dense $H_0^{2m}(\mathcal{O})$ and weak dense in $W^{1,\infty}(\mathcal{O})$, $H_0^{2m}(\mathcal{O})$ is also weak* dense in $W^{1,\infty}(\mathcal{O})$.*

REMARK 2.3.8. *In the following sections, $D(L)$ will be chosen to be the space $H_N^m(\mathcal{O})$ denoting the space of $H^m(\mathcal{O})$ function with homogeneous Neumann boundary condition, with $m > d/2+1$. The reason behind this is that we want to encode the boundary conditions*

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imposed on equation (2.1) (2.2) into the space we are working with.
It is possible to work with the weaker no-flux boundary conditions instead of the homogeneous Neumann boundary conditions, but this would require to alter the formulation of the problem and add additional arguments, due to the non-linearity of the no-flux conditions.

REMARK 2.3.9. *We will not differentiate between the operator L^{-1}, acting on $L^2(\mathcal{O})$ and $D(L)'$, due to the density of $L^2(\mathcal{O})$ in $D(L)'$ and the extension theorem for linear operators.*

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2.4. Strategy of the proof

In order to show the existence of martingale solutions for certain systems of the form (2.1), we will need to introduce part by part the necessary tools. Our strategy for proving this result will be organized into the following steps:

(1) In section 2.5, we fix a separable Hilbert space $D(L)$ which satisfies certain embeddings. A natural choice will be the space $D(L) = H^m_N(\mathcal{O}) := \{w \in H^m(\mathcal{O})\colon \nabla w \cdot \nu = 0 \text{ on } \partial\mathcal{O}\}$, with $m > d/2 + 1$, due to the boundary conditions imposed on the equation. Like the notation $D(L)$ already suggests, we want to construct an operator L, satisfying (R1) (R4), whose domain coincides with $D(L)$. There are several ways to construct such an operator. Our approach is related to the notion of Hilbert-scales and was presented in Lemma 2.3.5. This intermediary operator L will then be used to define a regularization operator given by

$$Q_\varepsilon(\cdot) := u(\cdot) + \varepsilon L^* L \cdot\colon D(L) \to D(L)'. \tag{2.7}$$

The map u, included in this operator is related to the so-called entropy structure of our equation and its key properties are given in (E2), (E3). We prove the invertibility of Q_ε, whose inverse will be denoted by $R_\varepsilon := Q_\varepsilon^{-1}$, as well as the invertibility of its first Fréchet derivative. This will allow us to reformulate equation (2.4) in terms of a $D(L)'$-valued function v and obtain equation (2.5). It also drastically simplifies the problem, since the equation for v is simply a Hilbert space-valued ordinary differential equation.
We also verify that the coefficients of (2.5) exhibit the necessary measurability properties for the equation to be defined.

(2) Next, we want to obtain the, local-in-time, existence of solutions to the regularized equation (2.5). This is done in section 2.6. We first show that the operator

$$v \mapsto \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right),$$

from $D(L)'$ to $D(L)'$ is Fréchet differentiable and its derivative is locally bounded. This allows us to apply standard results from [91], up to a stopping time guaranteeing the that the $D(L)'$ norm of v remains bounded.

(3) In order to show that the solutions exist indeed globally in time, we use Itô's formula to derive a so called entropy-estimate for terms of the form $u(R_\varepsilon(v))$. We use properties of the operator Q_ε to obtain bounds on v. It is in this step, that we return to our original problem (2.4) and define the process u as $u(R_\varepsilon(v))$, which will replace v as our central quantity of interest.

(4) Passing to the limit with respect to the regularization and verifying that $u(R_\varepsilon(v))$ converges (up to a subsequence and change of probability space) to a solution of (2.4) will be shown for each individual example, since the structure of the entropy as well as of the equation itself play an important role in these steps. However the examples treated in section 2.8 and 2.10 already cover two important classes in terms of entropy-structure.

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2.5. The operator setup

In this section, we introduce the operators and functions essential to our regularization procedure and how the functions v (related to formulation (2.5)), w (related to formulation (2.4)) and u (related to formulation (2.1)) are connected.

The goal is to define the components of diagram 2.1 and derive the essential properties of each mapping. We begin by introducing the operator Q_ε, which will be the main tool for our regularization.

DEFINITION 2.5.1. *Let $L\colon L^2(\mathcal{O}) \to L^2(\mathcal{O})$ be an unbounded, linear, self-adjoint operator satisfying (R1)-(R4). For every $\varepsilon > 0$, we define the operator $Q_\varepsilon\colon D(L) \to D(L)'$ by*

$$Q_\varepsilon(\cdot) \coloneqq u(\cdot) + \varepsilon L^* L\,\cdot\,.$$

Next, we establish essential properties of the map Q_ε and its inverse, denoted by R_ε. We will list the properties which will be shown for each operator in the following table to have a concise summary of our results. When reading this section for the first time, the table can be skipped.

(1) $Q_\varepsilon\colon D(L) \to D(L)'$ is
  (a) well defined
  (b) Fréchet differentiable/continuous
  (c) strongly monotone
  (d) coercive

(2) $DQ_{\varepsilon,w}\colon D(L) \to D(L)'$ is
  (a) well defined
  (b) continuous
  (c) strongly monotone
  (d) coercive

(3) $R_\varepsilon\colon D(L)' \to D(L)$ is
  (a) well defined
  (b) Fréchet differentiable/continuous
  (c) strictly monotone
  (d) coercive

(4) $DR_{\varepsilon,w}\colon D(L)' \to D(L)$ is
  (a) well defined
  (b) continuous
  (c) strictly monotone
  (d) coercive

(5) $u(\cdot) = (h')^{-1}(\cdot)$ is
  (a) well defined as an operator which maps $D(L) \to L^\infty(\mathcal{O})$
  (b) Fréchet differentiable/continuous as an operator which maps $D(L) \to L^\infty(\mathcal{O})$
  (c) monotone as an operator which maps $D(L) \to D(L)'$

(6) $h'(\cdot)$ is
  (a) well defined as an operator mapping $\mathrm{dom}(h') \subset L^\infty(\mathcal{O}) \to L^\infty(\mathcal{O})$
  (b) monotone as an operator mapping $\mathrm{dom}(h') \subset D(L) \to D(L)'$

LEMMA 2.5.2. *The non-linear operator $Q_\varepsilon\colon D(L) \to D(L)'$ defined above is well defined, Fréchet differentiable, strongly monotone and coercive. In particular, for any element v in the range of Q_ε, $\exists! w \in D(L)$ such that $Q_\varepsilon(w) = v$.*

For the proof, see the Appendix A.

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LEMMA 2.5.3. *For every $w \in D(L)$, the operator*

$$DQ_{\varepsilon,w}(\cdot) = (u'(w)(\cdot) + \varepsilon L^* L(\cdot))\colon D(L) \to D(L)'$$

is well defined, continuous, strongly monotone and coercive. In particular, for every $\varepsilon > 0$ and $w \in D(L)$, $DQ_{\varepsilon,w}$ is invertible.

For the proof, see the Appendix A.

DEFINITION 2.5.4. *For any $\varepsilon > 0$, we denote the inverse of Q_ε, obtained in Lemma 2.5.2, by $R_\varepsilon\colon D(L)' \to D(L)$.*

LEMMA 2.5.5. *The operator $R_\varepsilon\colon D(L)' \to D(L)$ is Fréchet differentiable, strictly monotone and coercive. In particular it is Lipschitz continuous with Lipschitz constant C/ε.*

For the proof, see the Appendix A.

LEMMA 2.5.6. *For every $w \in D(L)$, the linear operator*

$$DR_{\varepsilon,R_\varepsilon(v)} = DR_{\varepsilon,w} = (DQ_{\varepsilon,w})^{-1}\colon D(L)' \to D(L),$$

is well defined, continuous, strictly monotone and coercive. In particular, for $v, v_1, v_{2 \in D(L)'}$, the following Lipschitz estimate holds,

$$\|(u'(R_\varepsilon(v)) + \varepsilon L^* L)^{-1} v_1 - (u'(R_\varepsilon(v)) + \varepsilon L^* L)^{-1} v_2\|_{D(L)} \leq \frac{1}{C\varepsilon} \|v_1 - v_2\|_{D(L)'}.$$

For the proof, see the Appendix A.

LEMMA 2.5.7. *$u(\cdot)\colon D(L) \to L^\infty(\mathcal{O})$ is Fréchet differentiable and, as a mapping $D(L) \to D(L)'$, it is monotone.*

For the proof, see the Appendix A.

REMARK 2.5.8. *In particular, for $v \in D(L)'$, it holds that*

$$Q_\varepsilon(R_\varepsilon(v)) = u(R_\varepsilon(v)) + \varepsilon L^* L R_\varepsilon(v) = v. \tag{2.8}$$

In anticipation of later proofs, we will state additional properties of the map R_ε. Note that in the following statements, we will identify scalar arguments of the operator R_ε with the respective function taking a constant scalar value.

(1) $R_\varepsilon(u(0)) = 0$,
(2) $R_\varepsilon(0)$ is bounded by a finite constant,
(3) for each $\varepsilon > 0$, R_ε maps $D(L)'$ bounded sets to $D(L)$ bounded sets,
(4) $u(R_\varepsilon(v))$ maps $D(L)'$ bounded sets to $L^\infty(\mathcal{O})$ bounded sets.

LEMMA 2.5.9. *For each $\varepsilon > 0$, $R_\varepsilon(u(0)) = 0$.*

PROOF.

$$0 = R_\varepsilon(Q_\varepsilon(0)) = R_\varepsilon(u(0)).$$

□

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LEMMA 2.5.10. *For each $\varepsilon > 0$, $R_\varepsilon(0)$ is bounded by a finite constant, depending only on u and ε.*

PROOF.

$$\begin{aligned}\|R_\varepsilon(0)\|_{L^\infty(\mathcal{O})} &\leq C\|R_\varepsilon(0)\|_{D(L)} \leq C\|R_\varepsilon(0) - R_\varepsilon(u(0))\|_{D(L)} + C\|R_\varepsilon(u(0))\|_{D(L)} \\ &\leq C\frac{\|u(0)\|_{D(L)'}}{\varepsilon}.\end{aligned}$$

□

LEMMA 2.5.11. *For each $\varepsilon > 0$, R_ε maps $D(L)'$ bounded sets to $D(L)$ bounded sets.*

PROOF. Let $v \in D(L)'$, such that $\|v\|_{D(L)'} \leq R$. By the Lipschitz continuity of R_ε,

$$\begin{aligned}\|R_\varepsilon(v)\|_{D(L)} &\leq \|R_\varepsilon(v) - R_\varepsilon(u(0))\|_{D(L)} \leq C\frac{\|v - u(0)\|_{D(L)'}}{\varepsilon} \\ &\leq C\frac{\|v\|_{D(L)'} + \|u(0)\|_{D(L)'}}{\varepsilon} \leq C\frac{R + \widetilde{C}(u)}{\varepsilon}.\end{aligned}$$

□

LEMMA 2.5.12. *The composition $u(R_\varepsilon(\cdot))$ maps $D(L)'$ bounded sets to $L^\infty(\mathcal{O})$ bounded sets.*

PROOF. First, we note that u, as a real valued function, is defined on the whole space $\mathbb{R}^n$. Hence for any finite value of its argument x, we can find a ball of radius $\tilde{R}_x$, such that $x \in [-\tilde{R}_x, \tilde{R}_x]^n \subset \mathbb{R}^n$. This is a compact set and, by continuity, u maps it to another compact, and hence bounded, set in $\mathbb{R}^n$.
Using the previous Lemma and the embedding $D(L) \hookrightarrow L^\infty(\mathcal{O})$,

$$\|u(R_\varepsilon(v))\|_{L^\infty(\mathcal{O})} \leq C(\tilde{R}_{\|R_\varepsilon(v)\|_{L^\infty(\mathcal{O})}}) \leq C(\tilde{R}_{\frac{R+C(u(0))}{\varepsilon}}).$$

□

2.5.1. Measurability. In order for Equation (2.5) to be meaningful, it is necessary to investigate measurability properties of the previously introduced objects. Therefore, we will show the following:

(1) If $v \in L^0(\Omega, L^0(0,T; D(L)'))$ is progressively measurable, then $R_\varepsilon(v)$, taking values in $D(L)$, is progressively measurable.
(2) If $w \in L^0(\Omega, L^0(0,T; D(L)))$ is progressively measurable, then $u(w)$, taking values in $L^\infty(\mathcal{O})$, is progressively measurable.
(3) If $v \in C([0,T]; D(L)')$, then $R_\varepsilon(v) \in C([0,T]; D(L))$, $u(R_\varepsilon(v)) \in C([0,T]; L^\infty(\mathcal{O}))$. Also $u(R_\varepsilon(v)) \in L^\infty([0,T] \times \mathcal{O})$.

Before we state the first Lemma of this section, we recall the definition of strong measurability,

DEFINITION 2.5.13. *Let $(X, \mathcal{A})$ be a measure space and E be a real or complex Banach space. We call a function $f\colon X \to E$ strongly measurable, if it is Borel measurable and the range $f(X)$ is separably valued.*

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REMARK 2.5.14. *Since simple functions only take finitely many values, hence are separably valued, and measurable functions are closed under pointwise limits (see* [39, Proposition E.1]*), we could rephrase the previous Definition in the following way: We call a function $f\colon X \to E$ strongly measurable, if it is the pointwise limit of a sequence of simple Borel measurable functions.*

LEMMA 2.5.15. *For every fixed $\varepsilon > 0$, the operator R_ε is strongly measurable. If v is progressively measurable, then $R_\varepsilon(v)$ is progressively measurable.*

For the proof, see the Appendix A.

LEMMA 2.5.16. *Let $w \in L^0(\Omega, L^0(0,T;D(L)))$ be a progressively measurable process. Then $u(w)$ is a progressively measurable process taking values in $L^\infty(\mathcal{O})$.*

For the proof, see the Appendix A.

LEMMA 2.5.17. *Let $v \in C([0,T];D(L)')$, then $R_\varepsilon(v) \in C([0,T],D(L))$ and $u(R_\varepsilon(v)) \in L^\infty([0,T]\times\mathcal{O})$.*

For the proof, see the Appendix A.

LEMMA 2.5.18. *Let $1 \le p \le \infty$ and $v \in L^0(\Omega, L^\infty(0,T;D(L)'))$,*

$$\operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right) \in L^0(\Omega, L^p(0,T;D(L)')),$$

for every $1 \le p < \infty$.

For the proof, see the Appendix A.

2.5.2. Relating v and w. Recalling the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, we fixed earlier, measurability in ω will be understood with respect to $\mathcal{F}$ (or $\mathcal{F}_t$) and measurability in t with respect to $([0,T], \mathcal{B}([0,T]), \lambda)$.
We established that $Q_\varepsilon \circ R_\varepsilon = Id$ on $D(L)'$. Later however, we want to apply $Q_\varepsilon \circ R_\varepsilon$ to objects, which are only for a.e. $(\omega,t) \in \Omega\times[0,T]$ in $D(L)'$. This poses the question if, for such a v, $Q_\varepsilon \circ R_\varepsilon(v)$ is still measurable and coincides a.e. with the identity.

LEMMA 2.5.19. *Let $v \in L^0(\Omega\times[0,T];D(L)')$ be the space, or rather the space of equivalence classes, of measurable functions with values in $D(L)'$. Consider the mapping $Q_\varepsilon \circ R_\varepsilon$ from $L^0(\Omega\times[0,T];D(L)') \to L^0(\Omega\times[0,T];D(L)')$, with Q_ε and R_ε defined as before. Then $Q_\varepsilon(R_\varepsilon(v)) \in L^0(\Omega\times[0,T];D(L)')$ and*

$$Q_\varepsilon(R_\varepsilon(v)) = u(R_\varepsilon(v)) + \varepsilon L^* L R_\varepsilon(v) = v. \tag{2.9}$$

For the proof, see the Appendix A.
The previous Lemma enables us to express a $D(L)$ valued process w in terms of a $D(L)'$ valued process and the map R_ε and vice versa, via the map Q_ε. This enables us to rigorously connect (2.4) and (2.5). Equipped with the tools to arrive at a solution for (2.4) by solving (2.5), we will focus on the latter in the next section.

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2.6. Existence of solutions

In this section, we show the existence of solutions for the regularized problem given by (2.10). For this, we use slight modifications of classical results from [91]. Since the regularized equation is still a non-linear equation in v, we will only apply these results locally. We recall the equation, whose solution we are interested in,

$$dv = \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right)\,\mathrm{d}t + \sigma(u(R_\varepsilon(v)))\,\mathrm{d}W(t), \tag{2.10}$$

with initial condition $v_0 = u_0$ and boundary conditions

$$A(u(R_\varepsilon(v)))\nabla u(R_\varepsilon(v))\cdot\nu = B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\cdot\nu = 0,$$

or analogously $\sum_{j=1}^n A_{ij}(u(R_\varepsilon(v)))\nabla u_j(R_\varepsilon(v))\cdot\nu = 0$, on $\partial\mathcal{O}$ for $t>0$ and $i=1,\dots,n$.

Before we state the existence result, we clarify, what we mean by a solution to this equation.

Definition 2.6.1. *Let $T>0$ be fixed, but arbitrary. Let τ be a $\mathbb{F}$ adapted stopping time and v a continuous, $D(L)'$ valued, $\mathbb{F}$ adapted process. A pair (τ, v) is called a local solution of Equation (2.10), if $v(\omega,\cdot,\cdot)\in L^2([0,T\wedge\tau(\omega)); D(L)')\cap C([0,T\wedge\tau(\omega)); D(L)')$,*

$$v(t) = v(0) + \int_0^t \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right)(s)\,\mathrm{d}s + \int_0^t \sigma(u(R_\varepsilon(v)))(s)\,\mathrm{d}W(s), \quad t\in[0,T\wedge\tau), \tag{2.11}$$

and

$$B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\cdot\nu = 0 \quad \text{on } \partial\mathcal{O},$$

$\mathbb{P}$ *-a.s.*

By Lemma 2.5.15, 2.5.16 and 2.5.18, the individual integrals appearing in (2.11) are well defined.

Definition 2.6.2. *A local solution (τ, v) is called a global solution, if it satisfies Definition 2.6.1 and $\mathbb{P}(\tau=\infty)=1$.*

Remark 2.6.3. *Since we arrived at a point where we have to account for the boundary conditions, we will fix $D(L)$ to coincide with the space H^m_N of functions $w\in H^m(\mathcal{O})$, satisfying homogeneous Neumann-boundary conditions, namely*

$$\nabla w\cdot\nu = \nabla R_\varepsilon(v)\cdot\nu = 0, \quad \text{on } \partial\mathcal{O},$$

where ν denotes the unit outer normal vector along $\partial\mathcal{O}$. Even if not specified by a separate notation, these boundary conditions are assumed to hold for functions in $D(L)$. The associated operator L is given by Lemma 2.3.5 with $D(L)=H^m_N$ and $H=L^2(\mathcal{O})$.

In order to prove our main results, we will require the following two Lemmata.

Lemma 2.6.4. *The operator $w\mapsto \operatorname{div}(B(w)\nabla(w))$ from $D(L)$ to $D(L)'$ is Fréchet differentiable.*

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For the proof, see the Appendix A.

LEMMA 2.6.5. *The operator* $v \mapsto \operatorname{div}(B(u(R_\varepsilon(v)))\nabla(R_\varepsilon(v)))$ *from* $D(L)'$ *to* $D(L)'$ *is Fréchet differentiable and its derivative is locally bounded.*

For the proof, see the Appendix A.

THEOREM 2.6.6. *Let Assumptions (A1) (A6), (E1) (E4) and (R1) (R4) be satisfied. Given a process* $v \in L^2(\Omega, C([0,T], D(L)'))$*, we introduce the following sequence of stopping times*

$$\tau_R := \inf\{s \in [0,T] \colon \|v(s)\|_{D(L)'} > R\}.$$

For each $\varepsilon > 0$ *and* $0 < R < \infty$*, Equation 2.10 has a unique local solution, in terms of Definition 2.6.1, denoted by* (τ_R, v^ε)*.*

PROOF. By Lemma 2.6.5, the operator

$$v \mapsto \operatorname{div}(B(u(R_\varepsilon(v)))\nabla(R_\varepsilon(v))) \quad \text{from } D(L)' \text{ to } D(L)', \tag{2.12}$$

is Fréchet differentiable and its derivative is locally bounded. Hence, for $v \in D(L)'$ such that $\|v\|_{D(L)'} \le R$, there exists a constant $c \in [0,\infty)$ such that

$$\langle \operatorname{div}(DB(u(R_\varepsilon(v)))DR_{\varepsilon,v}(\widetilde{v})\nabla R_\varepsilon(v)) + \operatorname{div}(B(u(R_\varepsilon(v)))\nabla DR_\varepsilon(\widetilde{v})) - cI\widetilde{v}, \widetilde{v}\rangle_{D(L)'\times D(L)'} \le -\tilde{c}\|\widetilde{v}\|^2_{D(L)'},$$

for $\widetilde{v} \in D(L)'$. We conclude by [91, Proposition 4.1.4, p.74], that the operator (2.12) satisfies conditions $(H1) - (H3)$ [91, p.70] on a ball $B_R^{D(L)'}$ of radius $0 < R < \infty$ in $D(L)'$. Recalling the notation introduced in Definition 2.3.2, Assumption (A5) together with Lemma 2.5.5 imply that

$$\begin{aligned}
\|\tilde{\sigma}(v_1) - \tilde{\sigma}(v_2)\|_{\mathcal{L}_2^{n\times n}(U,L^2(\mathcal{O}))} &\le C(R)\|u(R_\varepsilon(v_1)) - u(R_\varepsilon(v_2))\|_{L^2(\mathcal{O})} \\
&\le C(R)(\|u'(R_\varepsilon(v_1))\|_{L^\infty(\mathcal{O})}, \|u'(R_\varepsilon(v_2))\|_{L^\infty(\mathcal{O})})\|R_\varepsilon(v_1) - R_\varepsilon(v_2)\|_{D(L)} \\
&\le \frac{C(u',R)}{\varepsilon}\|v_1 - v_2\|_{D(L)'}.
\end{aligned}$$

This fully verifies that $(H1) - (H3)$, as stated in [91, p.70], hold for $v_1, v_2 \in B_R^{D(L)'}$. It remains to verify Assumption $(H4)$ from [91], which follows from a calculation similar to the one in the proof of Lemma 2.6.5 and Lemma 2.5.18.

$$\begin{aligned}
&\|\operatorname{div}(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v))\|_{D(L)'} \\
&\quad\le \|B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\|_{L^1(\mathcal{O})} \\
&\quad\le \|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|\nabla R_\varepsilon(v)\|_{L^1(\mathcal{O})} \\
&\quad\le \|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|R_\varepsilon(v)\|_{D(L)} \\
&\quad\le \|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\left(\frac{C}{\varepsilon}\|v\|_{D(L)'} + \|R_\varepsilon(0)\|_{D(L)}\right) \\
&\quad\le \frac{C(R,u,u',u'')}{\varepsilon}\big(1 + \|v\|_{D(L)'}\big).
\end{aligned}$$

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We can apply [91, Proposition 4.2.4, p.91], by slightly altering its proof as well as some preceding results, up to the stopping time τ_R and obtain a local solution denoted by (τ_R, v^ε). These alterations are sketched in Appendix C. □

REMARK 2.6.7. *Another possibility to obtain a solution up to a small time would be via the Banach fixed-point theorem and a long, but direct calculation involving the operator (2.12). This would also enable us to use the weak formulation and include the no-flux boundary conditions in a natural way. However in this case one needs to take additional care when deriving the entropy estimates.*

2.7. The entropy estimate

The aim of this section is to establish the following, so-called, entropy estimate and extend the local solution we derived in Theorem 2.6.6 to arbitrarily large times.

REMARK 2.7.1. *Lemma 2.12.11, together with the results of Appendix B, allow us to replace all the Gateaux derivatives in the following results by Fréchet derivatives.*

LEMMA 2.7.2. *Let $v \in D(L)'$. We consider the following functional on $D(L)'$,*

$$F(v) := \int_{\mathcal{O}} h(u(R_\varepsilon(v)))\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v)\|^2_{L^2(\mathcal{O})}. \tag{2.13}$$

Its first and second Fréchet derivatives at v, in directions $\xi \in D(L)'$ and $\eta \in D(L)'$, are given by

$$D_\xi F(v) = \langle R_\varepsilon(v), \xi\rangle_{D(L)\times D(L)'} = \langle h'(u(R_\varepsilon(v))), \xi\rangle_{D(L)\times D(L)'},$$
$$D^2_{\xi,\eta} F(v) = \langle (u'(R_\varepsilon(v))(\cdot) + \varepsilon L^* L(\cdot))^{-1}(\eta), \xi\rangle_{D(L)\times D(L)'}.$$

For the proof, see the Appendix A.

Next, we state two technical Lemmas, which will be necessary in order to obtain estimates, which are uniform in ε. Before we begin, we need to recall that, for each $k \in \mathbb{N}$, $\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k$ is in fact a matrix. Therefore the product denoted by $\cdot$ should be interpreted as the Frobenius matrix product, which we will also denote by ":" when we particularly want to emphasize this operation. Hence the matrix trace, denoted by "tr(·)" will enter the following computations and should not be confused with the operator trace which is commonly used to express the Itô correction term (see e.g. [40]) and is explicitly written out in (2.15). In fact, the result of the next Lemma will be applied to the each individual summand in the Itô correction term.

LEMMA 2.7.3. *Consider the equation*

$$DR_{\varepsilon,w}(a) = (u'(w)(\cdot) + \varepsilon L^* L(\cdot))^{-1} a = b.$$

Let a be an $n\times n$ matrix with $(a_{ij})_{i,j=1,\dots,n} \in L^1(\mathcal{O})$, b be an $n\times n$ matrix with $(b_{ij})_{i,j=1,\dots,n} \in D(L)$ and w be a vector whose entries take values in $D(L)$. For a triple (a,b,w), satisfying the above equation, the following estimates hold,

$$\int_{\mathcal{O}} a : b\,\mathrm{d}x \le \frac{1}{2}\int_{\mathcal{O}} \|\sqrt{u'(w)}b\|^2_F\,\mathrm{d}x + \frac{1}{2}\int \mathrm{tr}\left[a^T (u'(w))^{-1} a\right]\mathrm{d}x,$$

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$$\frac{1}{2}\int_{\mathcal{O}} \|\sqrt{u'(w)}b\|_F^2 \,\mathrm{d}x \leq \frac{1}{2}\int_{\mathcal{O}} \mathrm{tr}\left[a^T(u'(w))^{-1}a\right]\mathrm{d}x = \frac{1}{2}\int_{\mathcal{O}} \|\sqrt{(u'(w))^{-1}}a\|_F^2 \,\mathrm{d}x.$$

$\|\cdot\|_F$ *denotes the Frobenius matrix norm.*

For the proof, see the Appendix A.
The second technical Lemma will be used to prevent an unwanted ε dependence of the term involving the initial datum.

LEMMA 2.7.4. *Assume $v_0 \in L^p(\Omega, L^1(\mathcal{O}))$, for a $p \geq 1$, such that $\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x \leq C$, then, for a.e. ω,*

$$\int_{\mathcal{O}} h(u(R_\varepsilon(v_0)))\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v_0)\|^2_{L^2(\mathcal{O})} \leq \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x. \tag{2.14}$$

For the proof, see the Appendix A.
Equipped with these results, we are ready to prove Theorem 2.7.5.

THEOREM 2.7.5. *Let (τ_R, v^ε) be a local solution of Equation (2.10), obtained from Theorem 2.6.6. For the stopped process $v^{\varepsilon,R}(\cdot) := v^\varepsilon(\omega, \cdot \wedge \tau_R(\omega))$, the following inequality holds*

$$\mathbb{E}\sup_{0\leq t<T\wedge\tau_R}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x + \mathbb{E}\sup_{0\leq t<T\wedge\tau_R}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})} \leq C(v_0,T).$$

The constant C is independent of R and ε.

PROOF. By Lemma 2.7.2, in combination with Lemma 2.12.11 and the results of Appendix B, the first two Fréchet derivatives of the functional F, defined by (2.13), are given by

$$\begin{aligned} D_\xi F(v) &= \langle R_\varepsilon(v), \xi\rangle_{D(L)\times D(L)'} = \langle h'(u(R_\varepsilon(v))), \xi\rangle_{D(L)\times D(L)'},\\ D^2_{\xi,\eta}F(v) &= \langle DR_{\varepsilon,v}(\eta), \xi\rangle = \langle (u'(R_\varepsilon(v))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\eta), \xi\rangle_{D(L)\times D(L)'}, \end{aligned}$$

where $\xi, \eta \in D(L)'$.

Using Itô's formula as stated in [84], see also [104], (the precise assumptions are verified in Appendix B) yields

$$\begin{aligned} \int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(0)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(0)\|^2_{L^2(\mathcal{O})}\\ &+ \sum_{i,j=1}^{n}\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^{\varepsilon,R})))_i\sigma_{ij}(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\ &+ \int_0^t \left\langle h'(u(R_\varepsilon(v^{\varepsilon,R})))(s), \mathrm{div}\left(B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})\right)(s)\right\rangle_{D(L)\times D(L)'}\mathrm{d}s\\ &+ \frac{1}{2}\int_0^t\sum_{k=1}^{\infty}\int_{\mathcal{O}} (u'(R_\varepsilon(v^{\varepsilon,R}))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\cdot\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

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Recall the notation $\sigma_{ij}(u(R_\varepsilon(v)))_k = \sigma_{ij}(u(R_\varepsilon(v)))(e_k)$, introduced in Assumption (A6). Integration by parts (recalling that we equipped $D(L)$ with homogeneous Neumann boundary conditions) and using that $h'(u(R_\varepsilon(v^{\varepsilon,R}))) = h'((h')^{-1}(R_\varepsilon(v^{\varepsilon,R}))) = R_\varepsilon(v^{\varepsilon,R})$, yields

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(0)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(0)\|^2_{L^2(\mathcal{O})}\\
&+ \sum_{i,j=1}^{n}\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^{\varepsilon,R})))_i\sigma_{ij}(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\
&- \int_0^t\int_{\mathcal{O}} \nabla R_\varepsilon(v^{\varepsilon,R})(s)\cdot\left(B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})\right)(s)\,\mathrm{d}x\,\mathrm{d}s\\
&+ \frac{1}{2}\int_0^t\sum_{k=1}^{\infty}\int_{\mathcal{O}} (u'(R_\varepsilon(v^{\varepsilon,R}))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\cdot\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

We see that the terms involving the initial condition and the Itô correction still exhibit an explicit ε dependence which would prevent us from deriving estimates of the solution which are uniform in ε. Therefore we will estimate these two terms further. Applying Lemma 2.7.3 with $a = \sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k$ and $b = (u'(R_\varepsilon(v^{\varepsilon,R}))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k)$, we obtain the following estimate for each $k \in \mathbb{N}$,

$$\begin{aligned}&\int_{\mathcal{O}} (u'(R_\varepsilon(v^{\varepsilon,R}))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k)\cdot\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k\,\mathrm{d}x\\
&\qquad\qquad\leq \int \mathrm{tr}\left[(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k)^T(u'(R_\varepsilon(v^{\varepsilon,R})))^{-1}\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k\right]\mathrm{d}x.\end{aligned}$$

Using this in the Itô correction term, yields

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})} &\leq \int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(0)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(0)\|^2_{L^2(\mathcal{O})}\\
&+ \sum_{i,j=1}^{n}\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^{\varepsilon,R})))_i\sigma_{ij}(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\
&- \int_0^t\int_{\mathcal{O}} \nabla R_\varepsilon(v^{\varepsilon,R})(s)\cdot B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})(s)\,\mathrm{d}x\,\mathrm{d}s\\
&+ \frac{1}{2}\int_0^t\sum_{k=1}^{\infty}\int_{\mathcal{O}} \mathrm{tr}\left[(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))^T(u'(R_\varepsilon(v^{\varepsilon,R})))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\right]\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

"tr" denotes the finite dimensional matrix trace and should not be confused with the operator trace in this term. Next, we turn our attention to the terms involving the initial condition. We apply Lemma 2.7.4, which yields $\mathbb{P}$-a.s. for every $t \in [0, T \wedge \tau_R(\omega))$

$$\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})} \leq \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x$$

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(2.15)
$$
\begin{aligned}
&+\sum_{i,j=1}^n\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^{\varepsilon,R})))_i\sigma_{ij}(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\
&-\int_0^t\int_{\mathcal{O}}\nabla R_\varepsilon(v^{\varepsilon,R})(s)\cdot B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})(s)\,\mathrm{d}x\,\mathrm{d}s\\
&+C\int_0^t\sum_{k=1}^\infty\int_{\mathcal{O}}\mathrm{tr}\bigg[(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))^T(u'(R_\varepsilon(v^{\varepsilon,R})))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\bigg]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

By assumption (E3), the matrix $B(u(R_\varepsilon(v^{\varepsilon,R})))$ is positive semi-definite, which allows us to drop the third term on the right-hand side.
Taking the supremum with respect to time and expectation yields

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x+\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})}\le\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x\\
&+\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\sum_{i,j=1}^n\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^{\varepsilon,R})))_i\sigma_{ij}(u(R_\varepsilon(v^{\varepsilon,R})))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\
&+\frac{1}{2}\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_0^t\sum_{k=1}^\infty\int_{\mathcal{O}}\mathrm{tr}\bigg[(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))^T(u'(R_\varepsilon(v^{\varepsilon,R})))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\bigg]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

An application of the BDG-inequality (see e.g. [94]) leads to

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x+\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})}\le\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x\\
&+C\mathbb{E}\bigg(\int_0^{T\wedge\tau_R}\sum_{k=1}^\infty\sum_{i,j=1}^n\bigg(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s)_{ij}\,\mathrm{d}x\bigg)^2\mathrm{d}s\bigg)^{1/2}\\
&+\frac{1}{2}\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_0^t\sum_{k=1}^\infty\int_{\mathcal{O}}\mathrm{tr}\bigg[(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))^T(u'(R_\varepsilon(v^{\varepsilon,R})))^{-1}(\sigma(u(R_\varepsilon(v^{\varepsilon,R})))_k(s))\bigg]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Using Assumptions (A6), we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x+\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})}\le\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x\\
&+\frac{C}{2}\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\bigg(1+\int_0^t\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(s)\,\mathrm{d}x\,\mathrm{d}s\bigg)\\
&\le\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x+C_2\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_0^t\int_{\mathcal{O}} h(u(R_\varepsilon(v^{\varepsilon,R})))(s)\,\mathrm{d}x\,\mathrm{d}s+C_1.
\end{aligned}
$$

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The positivity of $\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})}$ allows us to drop this term. Recalling the positivity of the entropy density,

$$\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_{\mathcal{O}}h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x\le C_1+\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x + C_2\mathbb{E}\int_0^{T\wedge\tau_R}\sup_{0\le r\le s}\int_{\mathcal{O}}h(u(R_\varepsilon(v^{\varepsilon,R})))(r)\,\mathrm{d}x\,\mathrm{d}s.$$

A simple reformulation, which brings the stopping time inside the integrand and an application of Gronwall's inequality yields a bound which is uniform in ε and R,

$$\mathbb{E}\sup_{0\le t<T\wedge\tau_R}\int_{\mathcal{O}}h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\,\mathrm{d}x\le C(v_0,T\wedge\tau_R)\le C(v_0,T).$$

Similarly,

$$\mathbb{E}\sup_{0\le t<\tau_R\wedge T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^{\varepsilon,R})(t)\|^2_{L^2(\mathcal{O})}\le\mathbb{E}\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x+C(T,v_0)\le C(v_0,T).$$

By Lemma 2.5.17 , the continuity of h and u, we conclude that the supremum holds for every $0<t<\tau_R\wedge T$. □

To conclude this section, we show that the previous estimate already allows us to let the sequence of stopping times τ_R go to ∞.

THEOREM 2.7.6. *Let (τ_R,v^ε) be a local solution of Equation (2.5), obtained via Theorem 2.6.6. Assume that (E5) holds. Then v^ε is a global solution of Equation 2.5.*

PROOF. Combining the results of Theorem 2.7.5 and the representation of v given by Lemma 2.5.19, yields the following estimate, which is uniform in ε,R, (we recall that $v_0=u^0$)

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|v^{\varepsilon,R}(t)\|_{D(L)'}=\mathbb{E}\sup_{0\le t\le T}\|u(R_\varepsilon(v^{\varepsilon,R}))(t)+\varepsilon L^*LR_\varepsilon(v^{\varepsilon,R})(t)\|_{D(L)'}\\
&\le\mathbb{E}\sup_{0\le t\le T}\|u(R_\varepsilon(v^{\varepsilon,R}))(t)\|_{D(L)'}+\mathbb{E}\sup_{0\le t\le T}\varepsilon\|L^*LR_\varepsilon(v^{\varepsilon,R})(t)\|_{D(L)'}\\
&\le C\mathbb{E}\sup_{0\le t\le T}\|u(R_\varepsilon(v^{\varepsilon,R}))(t)\|_{L^1(\mathcal{O})}+\mathbb{E}\sup_{0\le t\le T}\varepsilon\|L^*LR_\varepsilon(v^{\varepsilon,R})(t)\|_{D(L)'}\\
&\le C\mathbb{E}\sup_{0\le t\le T}\|h(u(R_\varepsilon(v^{\varepsilon,R})))(t)\|_{L^1(\mathcal{O})}+C(u)+\mathbb{E}\sup_{0\le t\le T}\varepsilon\|L^*LR_\varepsilon(v^{\varepsilon,R})(t)\|_{D(L)'}\\
&\le C(T,u,v_0).
\end{aligned}$$

This allows us to send $R\to\infty$ and to conclude that we have indeed a solution v^ε for every time T. □

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2.8. Application: SKT with self-diffusion

2.8.1. Setting and Assumptions. In this section, we apply our abstract results to specific examples. We consider the following system of equations

$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^{n} A_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^{n}\sigma_{ij}(u)\,\mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t>0,\ i=1,\dots,n, \tag{2.16}$$

with the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \quad \nabla\left(a_{i0}u_i + \sum_{j=1}^{n} a_{ij}u_iu_j\right)\cdot\nu = 0 \quad \text{on } \partial\mathcal{O},\ t>0,\ i=1,\dots,n. \tag{2.17}$$

The diffusion matrix $A(u) = (A_{ij}(u))$, is given by

$$A_{ij}(u) = \delta_{ij}\left(a_{i0} + \sum_{k=1}^{n} a_{ik}u_k\right) + a_{ij}u_i, \quad i,j = 1,\dots,n. \tag{2.18}$$

Here, $a_{ij} \geq 0$ for $i = 1,\dots,n$ and $j = 0,\dots,n$, $u = (u_1,\dots,u_n)$ is the vector of population densities, $(W_1,\dots,W_n)$ is an n-dimensional cylindrical Wiener process, ν is the exterior unit normal vector to $\partial\mathcal{O}$, and u_i^0 is a possibly random initial datum. We call a_{i0} the diffusion coefficients, a_{ii} the self-diffusion coefficients, and a_{ij} for $i \neq j$ the cross-diffusion coefficients. We say that system (2.16)-(2.17) is *with self-diffusion* if $a_{i0} \geq 0$, $a_{ii} > 0$ for all $i = 1,\dots,n$, and it is *without self-diffusion* if $a_{i0} > 0$, $a_{ii} = 0$ for all $i = 1,\dots,n$.
We impose the following assumptions on the matrix A,

(SKT A1) $a_{ii} > 0, a_{ij} \geq 0$ for $i = 1,\dots,n$, $j = 0,\dots,n$ and the detailed-balance condition is satisfied, i.e., there exist numbers $\pi_1,\dots,\pi_n > 0$ such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i,j = 1,\dots,n. \tag{2.19}$$

Let the entropy density $h\colon \mathcal{D} \to \mathbb{R}$, given by

$$h(u) := \sum_{i=1}^{n} \pi_i u_i \log(u_i) - u_i + 1, \tag{2.20}$$

on the set $\mathcal{D} = (0,\infty)^n$, with suitable constants $\pi_i > 0$, which are determined by (2.19).

THEOREM 2.8.1. *Let $D(L) = H_N^m(\mathcal{O})$, with $m > d/2+1$ and let Assumptions (SKT A1), (A1), (A2), (A4) (A6) be satisfied, , then Equation 2.16 2.17 has a global martingale solution in the sense of Definition 2.1.1.*

First, we verify the assumptions of Theorems 2.6.6 and 2.7.5 for the regularized Equation

$$\mathrm{d}v_i - \operatorname{div}\left(\sum_{j=1}^{n} A_{ij}(u(R_\varepsilon(v)))u'(R_\varepsilon(v))_j\nabla R_\varepsilon(v)_j\right)\mathrm{d}t = \sum_{j=1}^{n}\sigma_{ij}(u(R_\varepsilon(v)))\,\mathrm{d}W_j(t), \tag{2.21}$$

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in $\mathcal{O},\ t>0,\ i=1,\dots,n$.
The function h defined above is twice continuously differentiable and non-negative. (E1) and (E2) are verified directly. So is (E3). Assumption (A3) is satisfied since A only depends on components of the solution in an affine way. (E4) is satisfied, since the inverse of each partial derivative of h is proportional to $\exp(\cdot)$.
For the regularization, we choose the operator L, constructed in Lemma 2.3.5, with $D(L)=H^m_N$ $(m>d/2+1)$ and $H=L^2(\mathcal{O})$, for which we already verified (R1)-(R4).

THEOREM 2.8.2. *The equation*

$$\mathrm{d}v=\operatorname{div}\left(A(u(R_\varepsilon(v)))u'(R_\varepsilon(v))\nabla R_\varepsilon(v)\right)+\sigma(u(R_\varepsilon(v)))\,\mathrm{d}W(t), \tag{2.22}$$

with initial conditions $v_0=u^0$, has a unique global solution $v\in L^2(\Omega,C([0,T],D(L)'))$ in the sense of Definition 2.6.2.

We recall the identity $v=u^\varepsilon(R_\varepsilon(v))+\varepsilon L^*LR_\varepsilon(v)$.

PROOF. This is a direct consequence of Theorem 2.6.6, Theorem 2.7.6 and Lemma 2.5.19. □

2.8.2. Estimates. In order to perform the limit $\varepsilon\to 0$ require uniform estimates. The structure of (2.16) already suggests, that our strategy should be to derive estimates on the quantity $u(R_\varepsilon(v))$ and pass to the limit in terms of u.

LEMMA 2.8.3. *Let v^ε be the solution of (2.22). Then the process $u(R_\varepsilon(v^\varepsilon))$ satisfies the following bounds:*

$$\mathbb{E}\|u(R_\varepsilon(v^\varepsilon))\|_{L^\infty(0,T;L^1(\mathcal{O}))}\le C(u^0,T), \tag{2.23}$$

$$\sum_{i=1}^n\left(a_{i0}\mathbb{E}\|(u^\varepsilon(R_\varepsilon(v^\varepsilon))_i)^{1/2}\|^2_{L^2(0,T;H^1(\mathcal{O}))}+a_{ii}\mathbb{E}\|u(R_\varepsilon(v^\varepsilon))_i\|^2_{L^2(0,T;H^1(\mathcal{O}))}\right)\le C(u^0,T), \tag{2.24}$$

$$\sum_{j\neq i}a_{ij}\mathbb{E}\|\nabla(u(R_\varepsilon(v^\varepsilon))_iu^\varepsilon(R_\varepsilon(v^\varepsilon))_j)^{1/2}\|^2_{L^2(0,T;L^2(\mathcal{O}))}\le C(u^0,T). \tag{2.25}$$

Additionally, we obtain the following estimates,

$$\mathbb{E}\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}\le C(u^0), \tag{2.26}$$

$$\mathbb{E}\sup_{0\le t\le T}\|v^\varepsilon(t)\|^2_{D(L)'}\le C(u^0). \tag{2.27}$$

PROOF. Let v^ε be the solution to (2.22). The starting point of the proof is Theorem 2.7.5,

$$\begin{aligned}\int_{\mathcal{O}}h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x+\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}&\le\int_{\mathcal{O}}h(v_0)\,\mathrm{d}x\\&+\sum_{i,j=1}^n\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W^k_j(s)\end{aligned}$$

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$$- \int_0^t \int_{\mathcal{O}} \nabla h'(u(R_\varepsilon(v^\varepsilon)))(s) \cdot (B(u(R_\varepsilon(v^\varepsilon)))\nabla R_\varepsilon(v^\varepsilon))\,(s)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ C \int_0^t \sum_{k=1}^{\infty} \int_{\mathcal{O}} \mathrm{tr}\left[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v)))^{-1} (\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\right] \mathrm{d}x\,\mathrm{d}s.$$

Note that $h'(u(R_\varepsilon(v))) = R_\varepsilon(v)$. At this point the so-called entropy structure of the equation is crucial. We recall that the matrix $A(u(R_\varepsilon(v)))$ was not symmetric and hence as a bilinear form on $\mathbb{R}^n$ not positive definite, but the matrix $B(u(R_\varepsilon(v))) = A(u(R_\varepsilon(v)))h''(u(R_\varepsilon(v)))^{-1}$ possesses such a positive (semi-) definite structure. Hence the interaction of the differential operator and the entropy, which we also refer to as entropy structure of the equation yields the following estimates (see [37] for more detailed calculations),

$$\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})} + 4\int_0^t \int_{\mathcal{O}} \sum_{i=1}^n \pi_i a_{i0} |\nabla (u(R_\varepsilon(v^\varepsilon))_i)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
(2.28)
$$+ 2\int_0^t \int_{\mathcal{O}} \sum_{i=1}^n \pi_i a_{ii} |\nabla u(R_\varepsilon(v^\varepsilon))_i(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\int_0^t \int_{\mathcal{O}} \sum_{i,j=1}^n \pi_i a_{ij} |\nabla (u(R_\varepsilon(v^\varepsilon))_i u(R_\varepsilon(v^\varepsilon))_j)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + \sum_{k=1}^{\infty}\sum_{i,j=1}^n \int_0^t \int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)$$
(2.29)
$$+ C \int_0^t \sum_{k=1}^{\infty} \int_{\mathcal{O}} \mathrm{tr}\left[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1} (\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\right] \mathrm{d}x\,\mathrm{d}s.$$

We take the expectation of the supremum over $0 < t < T$ and apply the Burkholder Davis Gundy inequality:

$$\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \mathbb{E}\sup_{0\leq t\leq T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}$$
$$+ 4\mathbb{E}\int_0^T \int_{\mathcal{O}} \sum_{i=1}^n \pi_i a_{i0} |\nabla (u(R_\varepsilon(v^\varepsilon))_i)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\mathbb{E}\int_0^T \int_{\mathcal{O}} \sum_{i,j=1}^n \pi_i a_{ij} |\nabla (u(R_\varepsilon(v^\varepsilon)))_i u(R_\varepsilon(v^\varepsilon))_j)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\mathbb{E}\int_0^T \int_{\mathcal{O}} \sum_{i=1}^n \pi_i a_{ii} |\nabla u(R_\varepsilon(v^\varepsilon))_i(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$

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$$\le \mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + \mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_{\mathcal{O}} \log u(R_\varepsilon(v^\varepsilon))_i(s)\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2}$$
$$+ C\mathbb{E}\int_0^T \sum_{k=1}^\infty \int_{\mathcal{O}} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s$$
$$\le \mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + C\mathbb{E}\bigg(1 + \int_0^T \int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(s)\,\mathrm{d}x\,\mathrm{d}s\bigg)$$
$$\le \mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + C\mathbb{E}\bigg(1 + \int_0^T \sup_{0\le r\le s} \int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(r)\,\mathrm{d}x\,\mathrm{d}s\bigg),$$

where we used Assumption (A6) in the last step. By Fubini's theorem and Gronwall's lemma, we conclude that

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x \le C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x,$$

and consequently,

$$\mathbb{E}\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|_{L^2(\mathcal{O})}^2 \le C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x.$$

Lemma 2.5.19 yields,

$$\mathbb{E}\sup_{0\le t\le T}\Big(\|v^\varepsilon(t)\|_{D(L)'}\Big) = \mathbb{E}\sup_{0\le t\le T}\Big(\|u(R_\varepsilon(v^\varepsilon))(t) + \varepsilon L^* LR_\varepsilon(v)(t)\|_{D(L)'}\Big)$$
$$\le C\mathbb{E}\sup_{0\le t\le T}\Big(\|u(R_\varepsilon(v^\varepsilon))(t)\|_{L^1(\mathcal{O})} + \varepsilon\|L^*LR_\varepsilon(v^\varepsilon)(t)\|_{D(L)'}\Big)$$
$$\le C\mathbb{E}\sup_{0\le t\le T}\Big(\|u(R_\varepsilon(v^\varepsilon))(t)\|_{L^1(\mathcal{O})} + \varepsilon\|LR_\varepsilon(v^\varepsilon)(t)\|_{L^2(\mathcal{O})}\Big) \le C(T, v_0).$$

We highlight, that the right hand side is independent of ε. In the last line, we used that

$$\|L^*LR_\varepsilon(v^\varepsilon)(t)\|_{D(L)'} = \sup_{w\in D(L),\|w\|_{D(L)}\le 1} |\langle L^*LR_\varepsilon(v^\varepsilon)(t), w\rangle_{D(L)'\times D(L)}|$$
$$= \sup_{w\in D(L),\|w\|_{D(L)}\le 1} |\langle LR_\varepsilon(v^\varepsilon)(t), Lw\rangle_{L^2(\mathcal{O})}| \le C\|LR_\varepsilon(v^\varepsilon)(t)\|_{L^2(\mathcal{O})}.$$

Estimate (2.24) is obtained from the Poincaré Wirtinger inequality, the previous estimate, and the gradient estimate in (2.29). Moreover, (2.25) also follows from (2.29). □

From now on, we will omit the full structure of the expression $u(R_\varepsilon(v^\varepsilon))$ and will replace it simply by u^ε. This serves the purpose of notational convenience as we will perform the limiting arguments in terms of u^ε.

LEMMA 2.8.4 (Higher-order moments). *Let u^ε be given by $u(R_\varepsilon(v^\varepsilon))$, where v^ε satisfies (2.22) and let $p \le p_0$, with p_0 given in (A2). Then, for any $i = 1,\dots,n$,*

$$\mathbb{E}\|u^\varepsilon\|^p_{L^\infty(0,T;L^1(\mathcal{O}))} \le C(p, v_0, T), \tag{2.30}$$

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$$C(a_{i0})\mathbb{E}\|(u_i^\varepsilon)^{1/2}\|^p_{L^2(0,T;H^1(\mathcal{O}))} + C(a_{ii})\mathbb{E}\|u_i^\varepsilon\|^p_{L^2(0,T;H^1(\mathcal{O}))} \le C(p, v_0, T), \tag{2.31}$$

where $C(p, v_0, T) > 0$ does not depend on ε.
Additionally, we have

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}(t)\bigg)^p \le C(p, v_0, T, L) \tag{2.32}$$

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\|v^\varepsilon(t)\|_{D(L)'}(t)\bigg)^p \le C(p, v_0, T, L), \tag{2.33}$$

where $C(p, v_0, T, L) > 0$ does not depend on ε.

Proof. We raise (2.29) to the power $p \ge 2$. Taking the expectation, applying the Burkholder Davis Gundy inequality and using Assumption (A6), we find that

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le t\le T}\int_\mathcal{O} h(u^\varepsilon)(t)\,\mathrm{d}x + \sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}\bigg)^p + 4^p\mathbb{E}\bigg(\int_0^T\int_\mathcal{O}\sum_{i=1}^n \pi_i a_{i0}|\nabla(u_i^\varepsilon)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^p \\
&\quad + 2^p\mathbb{E}\bigg(\int_0^T\int_\mathcal{O}\sum_{i=1}^n \pi_i a_{ii}|\nabla u_i^\varepsilon(s)|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^p + 2^p\mathbb{E}\bigg(\int_0^T\int_\mathcal{O}\sum_{i,j=1}^n \pi_i a_{ij}|\nabla(u_i^\varepsilon u_j^\varepsilon)^{1/2}(s)|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^p \\
&\le C(p, v_0) + \mathbb{E}\bigg(\int_0^T\sum_{k=1}^\infty\sum_{i,j=1}^n\bigg(\int_\mathcal{O}\log u_i^\varepsilon(s)\sigma_{ij}(u^\varepsilon)_k(s)\,\mathrm{d}x\bigg)^2\mathrm{d}s\bigg)^{p/2} \\
&\quad + C\mathbb{E}\bigg(\int_0^T\sum_{k=1}^\infty\int_\mathcal{O}\mathrm{tr}\Big[(\sigma(u^\varepsilon)_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u^\varepsilon)_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s\bigg)^p \\
&\le C(p, v_0) + C\mathbb{E}\bigg(1 + \int_0^T\int_\mathcal{O} h(u^\varepsilon)(s)\,\mathrm{d}x\,\mathrm{d}s\bigg)^p.
\end{aligned}$$

First we ignore $\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)\|^2_{L^2(\mathcal{O})}$, apply Jensen's and Gronwall's inequality. Using the fact that the entropy dominates the L^1 norm (up to an additive constant), and applying the Poincaré inequality yields the second inequality.
From the first estimate, we obtain

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}\bigg)^p \le C(p, v_0) + C(p,t)\mathbb{E}\int_0^t\bigg(\int_\mathcal{O} h(u^\varepsilon)(s)\,\mathrm{d}x\bigg)^p\mathrm{d}s \le C(p, v_0, t).$$

Combining the previous estimates and using the representation of v given by Lemma 2.5.19,

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{0\le t\le T}\|v^\varepsilon(t)\|_{D(L)'}\bigg)^p &\le \mathbb{E}\bigg(\sup_{0\le t\le T}\|u(R_\varepsilon(v^\varepsilon))(t) + \varepsilon L^* LR_\varepsilon(v^\varepsilon)(t)\|_{D(L)'}(t)\bigg)^p \\
&\le C(p)\bigg(\mathbb{E}\bigg(\sup_{0\le t\le T}\|u(R_\varepsilon(v^\varepsilon))(t)\|_{D(L)'}\bigg)^p + \mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon}{2}\|L^* LR_\varepsilon(v^\varepsilon)(t)\|_{D(L)'}\bigg)^p\bigg) \\
&\le C(p, L)\bigg(\mathbb{E}\bigg(\sup_{0\le t\le T}\|u(R_\varepsilon(v^\varepsilon))(t)\|_{L^2(\mathcal{O})}\bigg)^p + \mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|_{L^2(\mathcal{O})}\bigg)^p\bigg)
\end{aligned}$$

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$$\le C(p,L)\bigg(\mathbb{E}\bigg(\sup_{0\le t\le T}\|u(R_\varepsilon(v^\varepsilon))(t)\|_{L^2(\mathcal{O})}\bigg)^p+\mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon^2}{4}\|LR_\varepsilon(v^\varepsilon)(t)\|_{L^2(\mathcal{O})}^2\bigg)^{p/2}\bigg)$$
$$\le C(p,v_0,L,T).$$

□

We derive further estimates from Lemma 2.8.4.

LEMMA 2.8.5. *Let $a_{ii}>0$ for all $i=1,\dots,n$, $s,r>1$ and let $p\ge 2$, such that $s>\frac{2}{d-1}$. Then for any $i=1,\dots,n$,*

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^{2+2/d}((0,T)\times\mathcal{O})}\le C(p,v_0,T), \tag{2.34}$$
$$\mathbb{E}\|u_i^\varepsilon(t)\|^p_{L^{2+4/d}(0,T;L^2(\mathcal{O}))}\le C(p,v_0,T), \tag{2.35}$$
$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^{r(d+2)/(d(r-1))}(0,T;L^r(\mathcal{O}))}\le C(p,v_0,T), \tag{2.36}$$
$$\mathbb{E}\|u_i^\varepsilon(t)\|^p_{L^s(0,T;L^{sd/(sd-s-2)}(\mathcal{O}))}\le C(p,v_0,T). \tag{2.37}$$

If $s\le\frac{2}{d-1}$, or $d=1$, the norm on the left-hand side of (2.37) can be replaced by $\|u_i^\varepsilon(t)\|_{L^s(0,T;L^q(\mathcal{O}))}$, for any $q<\infty$.

PROOF. Applying the Gagliardo Nirenberg inequality yields,

$$\mathbb{E}\bigg(\int_0^T\|u_i^\varepsilon\|^s_{L^r(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s}\le C\mathbb{E}\bigg(\int_0^T\|u_i^\varepsilon\|^{\theta s}_{H^1(\mathcal{O})}\|u_i^\varepsilon\|^{(1-\theta)s}_{L^1(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s}$$
$$\le C\mathbb{E}\bigg(\|u_i^\varepsilon\|^{(1-\theta)s}_{L^\infty(0,T;L^1(\mathcal{O}))}\int_0^T\|u_i^\varepsilon\|^{\theta s}_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s}$$
$$\le C(T)\Big(\mathbb{E}\|u_i^\varepsilon\|^{(1-\theta)sp_1p/s}_{L^\infty(0,T;L^1(\mathcal{O}))}\Big)^{1/p_1}\bigg\{\mathbb{E}\bigg(\int_0^T\|u_i^\varepsilon\|^{\theta s}_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg)^{p_2p/s}\bigg\}^{1/p_2},$$

where $\frac{1}{p_1}+\frac{1}{p_2}=1$ and

$$\frac{1}{r}=\left(\frac12-\frac1d\right)\theta+1-\theta=1-\frac{d+2}{2d}\theta.$$

Acknowledging the bound (2.24), we set $\theta s=2$, leading to the relation,

$$\frac{s}{r}=\left(\frac12-\frac1d\right)2+s-2=s-\frac{d+2}{d}.$$

Setting $r=s=\frac{2d+2}{d}$ and $p_2=\frac{1}{\theta}=\frac{d+1}{d}$ proves the first estimate.

Rearranging the relation results in

$$s=\frac{1+\frac2d}{1-\frac1r}=\frac{r(d+2)}{d(r-1)},$$

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and

$$r = \frac{sd}{sd - s - 2},$$

we obtain the remaining estimates. □

Next we show bounds for fractional time derivatives of u^ε. This result is used to establish the tightness of the laws of u^ε in certain spaces. We recall the definition of the Sobolev Slobodeckij spaces. Let X be a vector space and let $p \geq 1$, $\alpha \in (0,1)$. Then $W^{\alpha,p}(0,T;X)$ is the set of all functions $v \in L^p(0,T;X)$ for which

$$\|f\|^p_{W^{\alpha,p}(0,T;X)} = \|f\|^p_{L^p(0,T;X)} + \int_0^T \int_0^T \frac{\|f(t) - f(s)\|^p_X}{|t-s|^{1+\alpha p}} \,\mathrm{d}t \,\mathrm{d}s$$

is finite. With this norm, $W^{\alpha,p}(0,T;X)$ becomes a Banach space.
We will also use the notation

$$\|f\|^p_{\dot{W}^{\alpha,p}(0,T;X)} := \int_0^T \int_0^T \frac{\|f(t) - f(s)\|^p_X}{|t-s|^{1+\alpha p}} \,\mathrm{d}t \,\mathrm{d}s$$

for the $W^{\alpha,p}$ semi-norm.

REMARK 2.8.6. *Let $m \in \mathbb{R}$ and $p \geq 1$. If a function f is an element of $W^{m-\delta,p}$ for any $\delta > 0$, we will also adopt the common notation $f \in W^{m-,p}$. The same notation applies to the Hölder spaces $C^{\alpha-}$.*

LEMMA 2.8.7. *[Time regularity with self diffusion] Let $a_{ii} > 0$ for $i = 1, \dots, n$ and $\alpha < 1/2$. Then there exists $C(u^0, T) > 0$ such that, for $p = (2d+4)/d$,*

$$\mathbb{E}\|u^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')} \leq C(v_0, T, L), \tag{2.38}$$

$$\mathbb{E}\|v^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} \leq C(v_0, T, L), \tag{2.39}$$

$$\mathbb{E}\|\varepsilon L^* L R_\varepsilon(v^\varepsilon)\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} \leq C(v_0, T, L). \tag{2.40}$$

REMARK 2.8.8. *As $\alpha p < 1 + \frac{2}{d}$, we can choose α large enough, such that $\alpha p > 1$ holds. By Theorem 5.4.6, we infer that*

$$\mathbb{E}\|u^\varepsilon\|^p_{C^\delta(0,T;D(L)')} \leq C(v_0, T), \tag{2.41}$$

for $\delta < \frac{1}{d+2}$ and uniform in ε.

PROOF. The proof will be split into two steps. First, we derive an estimate of the $W^{\alpha,p}$ semi-norm for the processes v^ε, which we combine with Lemma 2.8.4, in order to obtain the desired result for u^ε. Rewriting Equation (2.22) in terms of u^ε, it remains to show that, for each $i = 1, \dots, n$, the following integral is finite:

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_0^T \frac{\|v_i^\varepsilon(t) - v_i^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+p\alpha}} \,\mathrm{d}t \,\mathrm{d}s \\
&\leq \int_0^T \int_0^T |t-s|^{-1-p\alpha} \mathbb{E} \left\| \int_{s\wedge t}^{t\vee s} \operatorname{div} \sum_{j=1}^n A_{ij}(u^\varepsilon(r)) \nabla u_j^\varepsilon(r) \,\mathrm{d}r \right\|^p_{D(L)'} \mathrm{d}t \,\mathrm{d}s
\end{aligned}$$

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$$+\int_0^T\int_0^T |t-s|^{-1-p\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)\bigg\|_{D(L)'}^p \mathrm{d}t\,\mathrm{d}s$$
$$=: J_1 + J_2.$$

Before we estimate J_1 and J_2, we recall a well-known result for the sake of completeness. Let $g\in L^1(0,T)$ and $\delta<2$, $\delta\neq 1$. We claim that

$$I := \int_0^T\int_0^T |t-s|^{-\delta}\int_{s\wedge t}^{t\vee s} g(r)\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s < \infty. \tag{2.42}$$

Indeed, a change of the integration domain and integration by parts gives

$$\begin{aligned} I &= 2\int_0^T\int_s^T (t-s)^{-\delta}\bigg(\int_s^t g(r)dr\bigg)\,\mathrm{d}t\,\mathrm{d}s \\ &= -\frac{2}{1-\delta}\int_0^T\int_s^T (t-s)^{1-\delta}g(t)\,\mathrm{d}t\,\mathrm{d}s + \frac{2}{1-\delta}\int_0^T (T-s)^{1-\delta}\int_s^t g(r)\,\mathrm{d}r\,\mathrm{d}s, \end{aligned} \tag{2.43}$$

observing that $\lim_{t\to s}(t-s)^{1-\delta}\int_s^t g(r)\,\mathrm{d}r = 0$ for $1-\delta>-1$, since the integrability of g implies that $\lim_{t\to s}(t-s)^{-1}\int_s^t g(r)\,\mathrm{d}r = g(s)$ for a.e. s. The claim follows as the integrals on the right-hand side of (2.43) are finite.

Step 1: We need some preparations before we estimate J_1. We observe that

$$\begin{aligned} \bigg\|\sum_{j=1}^n A_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\bigg\|_{L^1(\mathcal{O})} &= \bigg\|\bigg(a_{i0}+2\sum_{j=1}^n a_{ij}u_j^\varepsilon\bigg)\nabla u_i^\varepsilon + \sum_{j\neq i} a_{ij}u_i^\varepsilon\nabla u_j^\varepsilon\bigg\|_{L^1(\mathcal{O})} \\ &\le C\|\nabla u^\varepsilon\|_{L^2(\mathcal{O})} + C\|u^\varepsilon\|_{L^2(\mathcal{O})}\|\nabla u^\varepsilon\|_{L^2(\mathcal{O})}. \end{aligned}$$

Because of the embedding $L^1(\mathcal{O})\hookrightarrow D(L)'$, it follows that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg\|\int_{s\wedge t}^{t\vee s}\operatorname{div}\sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u_j^\varepsilon(r)\,\mathrm{d}r\bigg\|_{D(L)'}^p \mathrm{d}t\,\mathrm{d}s \\ &\le \mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\bigg\|\operatorname{div}\sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u_j^\varepsilon(r)\bigg\|_{D(L)'}\mathrm{d}r\bigg)^p \mathrm{d}t\,\mathrm{d}s \\ &\le \mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\bigg\|\sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u_j^\varepsilon(r)\bigg\|_{L^1(\mathcal{O})}\mathrm{d}r\bigg)^p \mathrm{d}t\,\mathrm{d}s \\ &\le C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}\,\mathrm{d}r\bigg)^p \\ &\quad + C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}\,\mathrm{d}r\bigg)^p \\ &=: J_{11}+J_{12}. \end{aligned}$$

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We use the Hölder inequality three times to obtain

$$
\begin{aligned}
J_{11} &= C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}\,\mathrm{d}r\bigg)^p\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}|t-s|^{p/2}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{p/2}\,\mathrm{d}t\,\mathrm{d}s.\\
J_{12} &= C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg(\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}\,\mathrm{d}r\bigg)^p\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg\{\bigg(\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{p/2}\times\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{p/2}\bigg\}\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\bigg\{|t-s|^{p/(d+2)}\bigg(\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}^{2(d+2)/d}\,\mathrm{d}r\bigg)^{pd/2(d+2)}\\
&\quad\times\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{p/2}\bigg\}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

Fixing $p=\frac{2(d+2)}{d}$ and estimating the last term from above, we obtain

$$
\begin{aligned}
&\le C\mathbb{E}\bigg(\bigg(\int_0^T\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{p/2}\int_0^T\int_0^T|t-s|^{-1-p\alpha+p/(d+2)}\bigg(\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}^{2(d+2)/d}\,\mathrm{d}r\bigg)\,\mathrm{d}t\,\mathrm{d}s\bigg)\\
&\le C\bigg(\mathbb{E}\bigg(\int_0^T\|\nabla u^\varepsilon(r)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^p\bigg)^{1/2}\\
&\quad\times\bigg(\mathbb{E}\bigg(\int_0^T\int_0^T|t-s|^{-1-p\alpha+p/(d+2)}\int_{s\wedge t}^{t\vee s}\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}^{2(d+2)/d}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s\bigg)^2\bigg)^{1/2}.
\end{aligned}
$$

Thus, in view of estimates (2.42), (2.31) and (2.35), the right hand side of J_{12} is finite, if $p/(d+2)-1-p\alpha>-2$ or $\alpha<(d+2+p)/p(d+2)=1/2$. J_{11} can be estimated directly and the relation $-1-p\alpha+p/2>-1$ suggests again $\alpha<1/2$ for $p=\frac{2(d+2)}{d}$.
To estimate J_2, we use the embedding $L^2(\mathcal{O})\hookrightarrow D(L)'$, the Burkholder-Davis-Gundy inequality, the linear growth of σ, and the Hölder inequality:

$$
\begin{aligned}
J_2 &\le C\int_0^T\int_0^T|t-s|^{-1-p\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n\sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)\bigg\|_{L^2(\mathcal{O})}^p\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_0^T\int_0^T|t-s|^{-1-p\alpha}\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n\sum_{k=1}^\infty\|\sigma_{ij}(u^\varepsilon(r))_k\|_{L^2(\mathcal{O})}^2\,\mathrm{d}r\bigg)^{\frac{p}{2}}\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_0^T\int_0^T|t-s|^{-1-p\alpha+\frac{p}{2}-1}\int_{s\wedge t}^{t\vee s}\mathbb{E}\sum_{j=1}^n\big(1+\|u^\varepsilon(r)\|_{L^2(\mathcal{O})}^p\big)\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

By (2.42) and estimate (2.35), the right-hand side is finite if $-1-p\alpha+\frac{p}{2}-1>-2$ or $\alpha<1/2$. The value of p for this term is dictated by (2.36) and given by $p=2+4/d$.

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We conclude that $v \in L^p(\Omega, W^{\alpha,p}(0,T;D(L)'))$.
Step 2: We recall the a.s. identity

$$v^\varepsilon = u(R_\varepsilon(v^\varepsilon)) + \varepsilon L^* L R_\varepsilon(v^\varepsilon).$$

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} &= \mathbb{E}\|v^\varepsilon - \varepsilon L^* L R_\varepsilon(v^\varepsilon)\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} \\
&\le C(p)\left(\mathbb{E}\|v^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} + \mathbb{E}\|\varepsilon L^* L R_\varepsilon(v^\varepsilon)\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')}\right) \\
&= C(p)\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t) - v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^T\int_0^T \frac{\|\varepsilon L^* L R_\varepsilon(v^\varepsilon)(t) - \varepsilon L^* L R_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg) \\
&= C(p)\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t) - v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p\|L^* L R_\varepsilon(v^\varepsilon)(t) - L^* L R_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg) \\
&\le C(p,L)\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t) - v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p\|R_\varepsilon(v^\varepsilon)(t) - R_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg) \\
&\le C(p,L)\left(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t) - v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s + \mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p}{\varepsilon^p}\frac{\|v^\varepsilon(t) - v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\right) \\
&\le C(p,L)\left(\mathbb{E}\|v^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')}\right).
\end{aligned}$$

We used that R_ε is Lipschitz continuous with Lipschitz constant $\frac{C}{\varepsilon}$ (see Lemma 2.5.5) to obtain the second to last inequality.
The last estimate is obtained in a similar fashion,

$$\begin{aligned}
\mathbb{E}\|\varepsilon L^* L R_\varepsilon(v^\varepsilon)\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} &= \mathbb{E}\|v^\varepsilon - u^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} \\
&\le C(p)\left(\mathbb{E}\|v^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')} + \mathbb{E}\|u^\varepsilon\|^p_{\dot{W}^{\alpha,p}(0,T;D(L)')}\right).
\end{aligned}$$

The estimates on $\mathbb{E}\|u^\varepsilon\|^p_{L^p(0,T;D(L)')}$ are a direct consequence of estimate (2.30). □

By a similar argument, leveraging the definition of Q_ε and the previous estimates, we can actually replace the remaining semi-norms in the statement of Lemma 2.8.7 by the full $W^{\alpha,p}$ norms.
We prove the existence of a martingale solution in the case with self-diffusion.

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2.8.3. Tightness of the laws of (u^ε). We show that the laws of u^ε are tight in a certain sub-Polish space (which is a topological space in which there exists a countable family of continuous functions that separate points [21, Definition 2.1.3]). For this, we introduce the following spaces:

- $C^0([0,T];D(L)')$ is the space of continuous functions $u:[0,T]\to D(L)'$ with the topology $\mathbb{T}_1$ induced by the norm $\|u\|_{C^0([0,T];D(L)')}=\sup_{t\in[0,T]}\|u(t)\|_{D(L)'}$;
- $L^2_w(0,T;H^1(\mathcal{O}))$ is the space $L^2(0,T;H^1(\mathcal{O}))$ with the weak topology $\mathbb{T}_2$;

We define the space

$$\widetilde{Z}_T := C^0([0,T];D(L)')\cap L^2_w(0,T;H^1(\mathcal{O})),$$

endowed with the topology $\widetilde{\mathbb{T}}$ that is the maximum of the topologies of $C^0([0,T];D(L)')$ and $L^2_w(0,T;H^1(\mathcal{O}))$. $\widetilde{Z}_T$ is a sub-Polish space since $C^0([0,T];D(L)')$ is separable and metrizable. For the space $L^2_w\left(0,T;H^1(\mathcal{O})\right)$, it is sufficient to define

$$f_m(u)=\int_0^T (u(t),v_m(t))_{H^1(\mathcal{O})}\,\mathrm{d}t\in\mathbb{R},\qquad \text{where } u\in L^2_w\left(0,T;H^1(\mathcal{O})\right), m\in\mathbb{N}$$

and $(v_m)_{m\in\mathbb{N}}$ is a dense subset of $L^2\left(0,T;H^1(\mathcal{O})\right)$. The functions f_m are a countable family of point separating functionals acting on $L^2\left(0,T;H^1(\mathcal{O})\right)$.

LEMMA 2.8.9. *The set of laws $(\mathcal{L}(u^\varepsilon))_\varepsilon$ is tight in $\widetilde{Z}_T$, as well as $L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$, for $d\geq 3$.*
For $d=1$, tightness is obtained in $L^2(0,T;C(\bar{\mathcal{O}}))$ and for $d=2$ in $L^2(0,T;L^p(\mathcal{O}))$ for any $1\leq p<\infty$.

REMARK 2.8.10. *Recall that the notation $L^{2d/(d-2)-}(\mathcal{O})$ was used to indicate that for a given function f, $f\in L^{2d/(d-2)-\delta}(\mathcal{O})$ for every $\delta>0$. In the following arguments the notation $L^{2d/(d-2)-}(\mathcal{O})$ should be understood in the sense that, assuming $\delta\ll 1$, we choose and fix an arbitrarily small δ and work with the corresponding space.*

PROOF. By (2.41) and Chebyshev's inequality, we conclude that condition $[T]$, given by Definition 5.6.9, is satisfied. Indeed,

$$\begin{aligned}
\sup_\varepsilon \mathbb{P}\left(\sup_{s,t\in[0,T],|t-s|\leq\delta}\|u^\varepsilon(t)-u^\varepsilon(s)\|_{D(L)'}>\eta\right) &\leq \sup_\varepsilon \frac{\mathbb{E}\left(\sup_{s,t\in[0,T],|t-s|\leq\delta}\|u^\varepsilon(t)-u^\varepsilon(s)\|_{D(L)'}\right)}{\eta}\\
&\leq \sup_\varepsilon \frac{\mathbb{E}\left(\delta^\gamma \sup_{s,t\in[0,T],|t-s|\leq\delta}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|_{D(L)'}}{|t-s|^\gamma}\right)}{\eta}\\
&\leq \sup_\varepsilon \frac{\mathbb{E}\left(\delta^\gamma \|u^\varepsilon\|_{C^\gamma([0,T];D(L)')}\right)}{\eta}\\
&\leq C\frac{\delta^\gamma}{\eta},
\end{aligned}$$

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where $0 < \gamma < 1/(d+2)$. Hence, $\forall\, \tilde{\varepsilon} > 0, \forall\, \eta > 0, \exists\, \delta > 0$, such that

$$\sup_{\varepsilon} \mathbb{P}\left(\sup_{s,t\in[0,T],|t-s|\leq\delta} \|u^\varepsilon(t) - u^\varepsilon(s)\|_{D(L)'} > \eta \right) \leq \tilde{\varepsilon}. \tag{2.44}$$

By the entropy estimates and [120, Lemma 5, Theorem 3], with the spaces $X = H^1(\mathcal{O}), B = D(L)'$, we obtain tightness in $C^0([0,T]; D(L)')$ (see also [97, Chapter 2, Section 2]).
Next, we consider the closed ball

$$\overline{B_R^{L_t^2 H_x^1}} = \left\{ f \in\colon \|f\|_{L^2(0,T;H^1(\mathcal{O}))} \leq R \right\}.$$

By (2.24) and Chebyshev's inequality,

$$\mathbb{P}(u^\varepsilon \in \overline{B_R^{L_t^2 H_x^1}}) = \mathbb{P}\left(\omega\colon \|u^\varepsilon\|_{L^2(0,T;H^1(\mathcal{O}))} > R\right) \leq \frac{\mathbb{E}\left(\|u^\varepsilon\|^2_{L^2(0,T;H^1(\mathcal{O}))} \right)}{R^2} \leq \frac{C}{R^2}.$$

By the Banach-Alaoglu Theorem (see 5.1.8), closed balls with respect to the norm topology are weak*-compact. Since $L^2(0,T;H^1(\mathcal{O}))$ is reflexive, it's closed norm balls are weakly compact. Tightness in $L^2_w(0,T;H^1(\mathcal{O}))$ follows directly from (2.24) and the Banach-Alaoglu Theorem (see 5.1.8).

The tightness in $L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$ is a consequence of (2.24), (2.8.7) and Corollary 4.0.9 with $Y = D(L)'$. The case of $d = 1, 2$ can be shown along the same lines. □

For the remainder of this section, we will solely focus on $d \geq 3$, which includes the only physically relevant case, namely $d = 3$.
It follows from the previous lemma that the set of laws $(\mathcal{L}(u^\varepsilon))$ is tight in $Z_T = \widetilde{Z}_T \cap L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$ with the topology $\mathbb{T}$ that is the maximum of $\widetilde{\mathbb{T}}$ and the topology induced by the $L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$ norm.

Lemma 2.8.11. *The set of laws $(\mathcal{L}(\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon)))_\varepsilon$ are tight in $Y_T := L^2_w(0,T;D(L)') \cap L^\infty_{w^*}(0,T;D(L)')$.*

Proof. For the first part of the claim, we consider the closed ball

$$\overline{B_R^{L_t^2 D(L)'_x}} = \left\{ f \in\colon \|f\|_{L^2(0,T;D(L)')} \leq R \right\}.$$

By Estimate (2.26) and Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left(\omega\colon \|\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon)\|_{L^2(0,T;D(L)')} > R\right) &\leq \frac{\mathbb{E}\|\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;D(L)')}}{R^2} \\
&= \frac{\mathbb{E}\varepsilon\|L^*LR_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;D(L)')}}{R^2} \\
&\leq C\frac{\mathbb{E}\varepsilon\|LR_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;L^2(\mathcal{O}))}}{R^2} \leq \frac{C}{R^2}.
\end{aligned}$$

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Since $L^2(0,T;D(L)')$ is reflexive, it's closed norm balls are weakly compact by the Banach-Alaoglu Theorem. Hence, the sequence $(\sqrt{\varepsilon}L^*LR_\varepsilon(v))_\varepsilon$ is tight in $L^2_w(0,T;D(L)')$.
The second part follows from an analogous argument via the estimate

$$\mathbb{P}\left(\omega\colon \|\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon)\|_{L^\infty(0,T;D(L)')} > R\right) \leq \frac{C}{R^2}.$$

□

LEMMA 2.8.12. *The family of measures*

$$\mu_W^\varepsilon = \mathbb{P}(W \in \cdot) \in \Pr\left(C([0,T];\hat{U})\right),$$

is tight.

PROOF. The sequence $(\mu_W^\varepsilon)_\varepsilon$ is constantly equal to one element and consequently weakly compact. By Prokhorov's theorem, it is tight. □

2.8.4. Convergence of (u^ε). We consider the space $Z_T \times Y_T \times C^0([0,T];\hat{U})$, equipped with the probability measure $\mu^\varepsilon := \mu_u^\varepsilon \times \mu_w^\varepsilon \times \mu_W^\varepsilon$, with

$$\mu_u^\varepsilon(\cdot) = \mathbb{P}(u^\varepsilon \in \cdot) \in \Pr\left(C^0([0,T];D(L)') \cap L^2_w(0,T;H^1(\mathcal{O})) \cap L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))\right),$$
$$\mu_w^\varepsilon(\cdot) = \mathbb{P}(\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon) \in \cdot) \in \Pr\left(L^2_w(0,T;D(L)') \cap L^\infty_{w^*}(0,T;D(L)')\right),$$
$$\mu_W^\varepsilon = \mathbb{P}(W \in \cdot) \in \Pr\left(C([0,T];\hat{U})\right).$$

$\Pr(X)$ denotes the space of probability measures on the space X.
$Z_T \times Y_T \times C^0([0,T];\hat{U})$ satisfies the assumptions of the Skorokhod Jakubowski theorem [28, Theorem C1] and the measures μ^ε are tight since set of laws $((\mathcal{L}(u^\varepsilon)), (\mathcal{L}(\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon))), (\mathcal{L}(W))$ are tight in $(Z_T, \mathbb{T}_Z), (Y_T, \mathbb{T}_Y)$ and $C([0,T];\hat{U})$ respectively. The theorem implies the existence of a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and, on this space, $(Z_T \times Y_T \times C^0([0,T];\hat{U}))$-valued random variables $(\widetilde{u}, \widetilde{w}, \widetilde{W})$ and $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ such that $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ has the same law as $(u^\varepsilon, \sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon), W)$ on $\mathcal{B}(Z_T \times Y_T \times C^0([0,T];\hat{U}))$ and, as $\varepsilon \to 0$,

$$(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon) \to (\widetilde{u}, \widetilde{w}, \widetilde{W}) \quad \text{in } Z_T \times Y_T \times C^0([0,T];\hat{U}) \ \widetilde{\mathbb{P}}\text{-a.s.}$$

By the definition of the space Z_T, Y_T, this convergence means $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
\widetilde{u}^\varepsilon &\to \widetilde{u} && \text{in } C^0([0,T];D(L)'),\\
\widetilde{u}^\varepsilon &\rightharpoonup \widetilde{u} && \text{weakly in } L^2(0,T;H^1(\mathcal{O})),\\
\widetilde{u}^\varepsilon &\to \widetilde{u} && \text{in } L^2(0,T;L^{2d/(d-2)-}(\mathcal{O})),\\
\widetilde{w}^\varepsilon &\rightharpoonup \widetilde{w} && \text{weakly in } L^2(0,T;D(L)'),\\
\widetilde{w}^\varepsilon &\rightharpoonup \widetilde{w} && \text{weakly* in } L^\infty(0,T;D(L)'),\\
\widetilde{W}^\varepsilon &\to \widetilde{W} && \text{in } C^0([0,T];\hat{U}).
\end{aligned} \tag{2.45}$$

We derive some regularity properties for the limit $\widetilde{u}$. Note that $\widetilde{u}$ is a Z_T-Borel random variable since $\mathcal{B}(Z_T \times Y_T \times C^0([0,T];\hat{U}))$ is a subset of $\mathcal{B}(Z_T) \times \mathcal{B}(Y_T) \times \mathcal{B}(C^0([0,T];\hat{U}))$.

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We deduce from estimates (2.30) and (2.31) and the fact that u^ε and $\widetilde{u}^\varepsilon$ have the same laws that for any $p \geq 2$,

$$\sup_\varepsilon \widetilde{\mathbb{E}}\bigg(\int_0^t \|\widetilde{u}^\varepsilon(t)\|_{H^1(\mathcal{O})}^2 \,\mathrm{d}t\bigg)^p + \sup_\varepsilon \widetilde{\mathbb{E}}\bigg(\sup_{0\leq t\leq T} \|\widetilde{u}^\varepsilon(t)\|_{D(L)'} \,\mathrm{d}t\bigg)^p \leq C.$$

In view of the embedding $L^1(\mathcal{O}) \hookrightarrow D(L)'$, we infer the existence of a further subsequence of $(\widetilde{u}^\varepsilon)$ (not relabeled) that is weakly converging in $L^p(\widetilde{\Omega}; L^2(0,T;H^1(\mathcal{O})))$ and $L^p(\widetilde{\Omega}; C(0,T;D(L)'))$ as $\varepsilon \to 0$. In view of the convergence of $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we infer that the limit function satisfies

$$\widetilde{\mathbb{E}}\bigg(\int_0^t \|\widetilde{u}(t)\|_{H^1(\mathcal{O})}^2 \,\mathrm{d}t\bigg)^p + \widetilde{\mathbb{E}}\bigg(\sup_{0\leq t\leq T} \|\widetilde{u}(t)\|_{D(L)'}\bigg)^p < \infty.$$

Let $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{F}}^{(\varepsilon)}$ be the filtrations generated by $(\widetilde{u}, \widetilde{w}, \widetilde{W})$ and $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$, respectively. In order to verify that these new random variables induce actual processes, one can follow the arguments in [29, Proposition B4]. The progressive measurability of $\widetilde{u}^\varepsilon$ is a consequence of [25, Appendix B]. Let $\widetilde{B}_j^{\varepsilon,k}(t) := \langle \widetilde{W}_j^\varepsilon(t), e_k\rangle$. It is not too difficult to see, that $\widetilde{B}_j^{\varepsilon,k}$, $k \in \mathbb{N}$, are independent, standard, real-valued $\widetilde{\mathcal{F}}_t$-Wiener Processes. The adaptedness is a direct consequence of their definition and the independence, as well as the independence of the increments $\widetilde{W}^{\varepsilon,k}(t) - \widetilde{W}^{\varepsilon,k}(s)$ with respect to $\widetilde{\mathcal{F}}_s$, is inherited from $B_j^{\varepsilon,k}(\cdot) := \langle W_j(\cdot), e_k\rangle$. Passing to the limit $\varepsilon \to 0$ in the characteristic function, which is done by the dominated convergence theorem, shows that also $\widetilde{W}^k$ are $\widetilde{\mathcal{F}}_t$-martingales with the correct marginal distributions. An application of Lévy's characterization theorem yields that $\widetilde{W}^k$ is indeed a Wiener process. We can conclude that $\widetilde{W}$ is indeed a cylindrical Wiener process.

Since the nonnegativity of u_i^ε for $i = 1, \dots, n$ was an important aspect of our proof, we will obtain the nonnegativity of $\widetilde{u}_i^\varepsilon$ as well as $\widetilde{u}_i$ in the following lemma.

LEMMA 2.8.13 (Nonnegativity). *It holds that $\widetilde{u}_i(x,t) \geq 0$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s. and $i = 1, \dots, n$.*

PROOF. Let $i \in \{1, \dots, n\}$. The transformation into entropy variables $u(R_\varepsilon(v^\varepsilon)) = (h')^{-1}(R_\varepsilon(v^\varepsilon))$, induced by the entropy density (2.20), implies that u_i^ε is nonnegative. It follows that $\mathbb{E}\|(u_i^\varepsilon)^-\|_{L^2(0,T;L^2(\mathcal{O}))} = 0$, where $z^- = \min\{0, z\}$. The function u_i^ε is Z_T-Borel measurable and so is its negative part $(u_i^\varepsilon)^-$. Therefore, using the equivalence of the laws of u_i^ε and $\widetilde{u}_i^\varepsilon$ on Z_T and setting $\mu_{u_i}^\varepsilon := \mathrm{Law}(u_i^\varepsilon) = \mathrm{Law}(\widetilde{u}_i^\varepsilon) =: \widetilde{\mu}_{u_i}^\varepsilon$, we find that

$$\begin{aligned}\widetilde{\mathbb{E}}\|(\widetilde{u}_i^\varepsilon)^-\|_{L^2(0,T;L^2(\mathcal{O}))} &= \int_{L^2(0,T;L^2(\mathcal{O}))} \|y^-\|_{L^2(0,T;L^2(\mathcal{O}))} \,\mathrm{d}\widetilde{\mu}_{u_i}^\varepsilon(y) \\ &= \int_{L^2(0,T;L^2(\mathcal{O}))} \|y^-\|_{L^2(0,T;L^2(\mathcal{O}))} \,\mathrm{d}\mu_{u_i}^\varepsilon(y) = \mathbb{E}\|(u_i^\varepsilon)^-\|_{L^2(0,T;L^2(\mathcal{O}))} = 0.\end{aligned}$$

This shows that $\widetilde{u}_i^\varepsilon \geq 0$ a.e. in $(0,T) \times \mathcal{O}$ $\widetilde{\mathbb{P}}$-a.s. The convergence (up to a subsequence) $\widetilde{u}^\varepsilon \to \widetilde{u}$ a.e. in Z_T $\widetilde{\mathbb{P}}$-a.s. then implies that $\widetilde{u}_i \geq 0$ a.e. in $(0,T) \times \mathcal{O}$ $\widetilde{\mathbb{P}}$-a.s. □

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LEMMA 2.8.14. *It holds for all s, $t \in [0,T]$ with $s \le t$, $i = 1,\dots,n$, and all $\phi_1 \in L^2(\mathcal{O})$, $\phi_2 \in D(L)$ satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \widetilde{u}_i^\varepsilon(t) - \widetilde{u}_i(t), \phi_1 \right\rangle_{L^2} \mathrm{d}t = 0, \tag{2.46}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \left\langle \widetilde{u}_i^\varepsilon(0) - \widetilde{u}_i(0), \phi_2 \right\rangle_{D(L)'\times D(L)} = 0, \tag{2.47}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}t = 0, \tag{2.48}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(0), \phi_2 \right\rangle_{D(L)'\times D(L)} = 0, \tag{2.49}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t = 0, \tag{2.50}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\mathrm{d}\widetilde{W}_j^\varepsilon(s) - \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi_1 \right\rangle_{L^2} \right|^2 \mathrm{d}t = 0. \tag{2.51}$$

PROOF. We know that $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T, $\widetilde{\mathbb{P}}$ -a.s. In particular, $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2([0,T];L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. We conclude that

$$\lim_{\varepsilon\to 0} \int_0^T \langle \widetilde{u}^\varepsilon(t) - \widetilde{u}(t), \phi_1 \rangle_{L^2}^2 \,\mathrm{d}t = 0 \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

By Estimate (2.31), there exists an $r > 1$, such that

$$\widetilde{\mathbb{E}}\left(\left| \int_0^T \|\widetilde{u}^\varepsilon(t) - \widetilde{u}(t)\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}t \right|^r \right) \le C.$$

This bound provides the equi-integrability of $\int_0^T \langle \widetilde{u}^\varepsilon(t) - \widetilde{u}(t), \phi_1 \rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}t$. Vitali's convergence theorem then shows that (2.46) holds.

From Lemma 2.8.9 follows $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $C^0([0,T];D(L)')$ $\widetilde{\mathbb{P}}$ -a.s. and $\widetilde{u}$ is continuous at $t = 0$, we infer that for any ϕ_2,

$$\lim_{\varepsilon\to 0} \langle \widetilde{u}^\varepsilon(0), \phi_2 \rangle_{D(L)'\times D(L)} = \langle \widetilde{u}(0), \phi_2 \rangle_{D(L)'\times D(L)} \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

The claimed convergence in (2.47) follows from Assumption (A2) and Vitali's theorem. (2.48) is a direct consequence of the estimate

$$\widetilde{\mathbb{E}}\left(\int_0^T \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}t \right)^p \le \widetilde{\mathbb{E}}\left(\int_0^T \sqrt{\varepsilon}\|\widetilde{w}_i^\varepsilon(t)\|_{D(L)'}\|\phi_2\|_{D(L)} \,\mathrm{d}t \right)^p \le \sqrt{\varepsilon}^p C(\phi_2, u_0, p).$$

Next, we establish convergence (2.50).

$$\begin{aligned} &\left| \int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \\ &\qquad \le \int_0^t \|A_{ij}(\widetilde{u}^\varepsilon(s)) - A_{ij}(\widetilde{u}(s))\|_{L^2(\mathcal{O})}\|\nabla\widetilde{u}_j^\varepsilon(s)\|_{L^2(\mathcal{O})}\|\nabla\phi_2\|_{L^\infty(\mathcal{O})} \,\mathrm{d}s \end{aligned}$$

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$$
\begin{aligned}
&+\left|\int_0^t A_{ij}(\widetilde{u}(s))\nabla(\widetilde{u}^\varepsilon(s)-\widetilde{u}(s))\cdot\nabla\phi_2\,\mathrm{d}s\right| \\
&\le \|A_{ij}(\widetilde{u}^\varepsilon(s))-A_{ij}(\widetilde{u}(s))\|^2_{L^2(0,T;L^2(\mathcal{O}))}\|\nabla\widetilde{u}_j^\varepsilon(s)\|^2_{L^2(0,T;L^2(\mathcal{O}))}\|\nabla\phi_2\|_{L^\infty(\mathcal{O})} \\
&\quad+\left|\int_0^t A_{ij}(\widetilde{u}(s))\nabla(\widetilde{u}^\varepsilon(s)-\widetilde{u}(s))\cdot\nabla\phi_2\,\mathrm{d}s\right| =: I_1+I_2.
\end{aligned}
$$

By the Lipschitz continuity of A and the uniform bound of $\nabla\widetilde{u}^\varepsilon$, $I_1\to 0$ $\widetilde{\mathbb{P}}$-a.s. For the second integral, we observe that $A_{i,j}(\widetilde{u})\nabla\phi_2\in L^2(0,T;L^2(\mathcal{O}))$ and $\nabla\widetilde{u}_j^\varepsilon\rightharpoonup\nabla\widetilde{u}_j$ weakly in $L^2(0,T;L^2(\mathcal{O}))$. Hence also $I_2\to 0$ $\widetilde{\mathbb{P}}$-a.s. This shows that

$$
\lim_{\varepsilon\to 0}\int_0^t\left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s=\int_0^t\left\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s\quad\widetilde{\mathbb{P}}\text{-a.s.}
$$

for all $\phi_2\in W^{1,\infty}(\mathcal{O})$ satisfying $\nabla\phi_2\cdot\nu=0$ on $\partial\mathcal{O}$. We observe that

$$
\begin{aligned}
\left\|\sum_{j=1}^n A_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\right\|_{L^1(\mathcal{O})}&=\left\|\left(a_{i0}+2\sum_{j=1}^n a_{ij}u_j^\varepsilon\right)\nabla u_i^\varepsilon+\sum_{j\neq i}a_{ij}u_i^\varepsilon\nabla u_j^\varepsilon\right\|_{L^1(\mathcal{O})}\\
&\le C\|\nabla u^\varepsilon\|_{L^2(\mathcal{O})}+C\|u^\varepsilon\|_{L^2(\mathcal{O})}\|\nabla u^\varepsilon\|_{L^2(\mathcal{O})}.
\end{aligned}
$$

Using the embedding $L^1(\mathcal{O})\hookrightarrow L^\infty(\mathcal{O})'$, we estimate

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t\left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s\right|^p\\
&\le\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\left|\int_0^t\left\|\sum_{j=1}^n A_{ij}(\widetilde{u}^\varepsilon)\nabla\widetilde{u}_j^\varepsilon(s)\right\|_{L^1(\mathcal{O})}\mathrm{d}s\right|^p\\
&\le C\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\left|\int_0^t\left\|\left(a_{i0}+2\sum_{j=1}^n a_{ij}\widetilde{u}_j^\varepsilon(s)\right)\nabla\widetilde{u}_i^\varepsilon(s)+\sum_{j\neq i}a_{ij}u_i^\varepsilon(s)\nabla\widetilde{u}_j^\varepsilon(s)\right\|_{L^1(\mathcal{O})}\mathrm{d}s\right|^p\\
&\le C\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\left|\int_0^t\|\nabla\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}+C\|\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}\|\nabla\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s\right|^p\\
&\le C\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\left|\int_0^t\|\nabla\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s+\left(\int_0^t\|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\|\nabla\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right)^{1/2}\right|^p\\
&\le C\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\left|\int_0^t\|\nabla\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s\right|\\
&\quad+C\|\nabla\phi_2\|^p_{L^\infty(\mathcal{O})}\left(\widetilde{\mathbb{E}}\left(\int_0^t\|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right)^p\right)^{1/2}\left(\widetilde{\mathbb{E}}\left(\int_0^t\|\nabla\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right)^p\right)^{1/2}\le C.
\end{aligned}
$$

This bound and the $\widetilde{\mathbb{P}}$-a.s. convergence allow us to apply the Vitali convergence theorem to infer that (2.50) holds.

It remains to prove convergence (2.51). By Lemma 5.3.57, since $\widetilde{W}^\varepsilon\to\widetilde{W}$ in $C^0\left([0,T];\hat{U}\right)$,

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$\widetilde{\mathbb{P}}$ a.s., it is sufficient to show that $\sigma_{ij}(\widetilde{u}^\varepsilon) \to \sigma_{ij}(\widetilde{u})$ in $L^2(0,T;\mathcal{L}_2(U;L^2(\mathcal{O})))\,\widetilde{\mathbb{P}}$ a.s. Indeed, by Assumption (A4)

$$\int_0^t \|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s \le C_\sigma \|\widetilde{u}^\varepsilon - \widetilde{u}\|^2_{L^2(0,T;L^2(\mathcal{O}))}.$$

Since $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2(0,T;L^2(\mathcal{O}))\,\tilde{\mathbb{P}}$ -a.s., we infer that for $t \in [0,T], \omega \in \widetilde{\Omega}$, the right hand side tends to 0. By Lemma 5.3.57, we conclude that $\int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s) \to \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s)$, for $j = 1,\dots,n$, in probability in $L^2(0,T;L^2(\mathcal{O}))$. Furthermore,

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s\right|^p \\
&\quad\le C\widetilde{\mathbb{E}}\left(\int_0^t \left(\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} + \|\sigma_{ij}(\widetilde{u}(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\right)\mathrm{d}s\right)^p \\
&\quad\le C + C\widetilde{\mathbb{E}}\left(\int_0^t \left(\|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})} + \|\widetilde{u}(s)\|^2_{L^2(\mathcal{O})}\right)\mathrm{d}s\right)^p.
\end{aligned}$$

In view of Vitali's convergence theorem, we deduce from this bound and the previous convergence that

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}\int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\mathrm{d}s = 0.$$

We can estimate

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\left\langle\int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s) - \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi_1\right\rangle_{L^2}\right|^2 \\
&\le C\|\phi_1\|^2_{L^2(\mathcal{O})}\widetilde{\mathbb{E}}\int_0^t \|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))} + \|\sigma_{ij}(\widetilde{u}(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s \\
&\le C\|\phi_1\|^2_{L^2(\mathcal{O})} + C\|\phi_1\|^2_{L^2(\mathcal{O})}\widetilde{\mathbb{E}}\left(\int_0^t \|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s + \int_0^t \|\widetilde{u}(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right) \le C.
\end{aligned}$$

This bound allow us to apply the dominated convergence theorem to conclude that for any $\phi_1 \in L^2(\mathcal{O})$,

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}\int_0^T \left|\left\langle\int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s) - \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi_1\right\rangle_{L^2}\right|^2 \mathrm{d}t = 0.$$

This shows (2.51) and finishes the proof. □

Let $\phi \in D(L)$. In order to show that the limit is indeed a solution to the equation, we define, for $t \in [0,T]$ and $i = 1,\dots,n$,

$$\begin{aligned}
\Lambda^\varepsilon_i\big(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon, \phi\big)(t) &:= \big\langle(\widetilde{u}_i(0)), \phi\big\rangle_{L^2} + \big\langle\sqrt{\varepsilon}\widetilde{w}^\varepsilon(0), \phi\big\rangle_{D(L)'\times D(L)} \\
&\quad - \int_0^t \sum_{j=1}^n \big\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}^\varepsilon_j(s), \nabla\phi\big\rangle_{L^2}\,\mathrm{d}s
\end{aligned} \tag{2.52}$$

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$$+\sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\mathrm{d}\widetilde{W}_j^\varepsilon(s), \phi \right\rangle_{L^2},$$

$$\Lambda_i\big(\widetilde{u},\widetilde{w},\widetilde{W},\phi\big)(t) := \langle \widetilde{u}_i(0),\phi\rangle_{L^2} - \sum_{j=1}^n \int_0^t \big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi\big\rangle_{L^2}\,\mathrm{d}s \tag{2.53}$$

$$+\sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi \right\rangle_{L^2}.$$

The following corollary is essentially a consequence of Lemma 2.8.14.

COROLLARY 2.8.15. *It holds for any $\phi_1 \in L^2(\mathcal{O})$ and any $\phi_2 \in D(L)$ satisfying $\nabla\phi_2\cdot\nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{\varepsilon\to 0}\big\|\langle\widetilde{u}_i^\varepsilon,\phi_1\rangle_{L^2} - \langle\widetilde{u}_i,\phi_1\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{\varepsilon\to 0}\big\|\langle\sqrt{\varepsilon}\widetilde{w}_i^\varepsilon,\phi_2\rangle_{D(L)'\times D(L)}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{\varepsilon\to 0}\big\|\langle\widetilde{v}_i^\varepsilon,\phi_2\rangle_{D(L)'\times D(L)} - \langle\widetilde{u}_i,\phi_2\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{\varepsilon\to 0}\big\|\Lambda_i^\varepsilon\big(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon,\phi_2\big) - \Lambda_i\big(\widetilde{u},0,\widetilde{W},\phi_2\big)\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

Since v^ε is a (probabilistically strong) solution to (2.22), and representation (2.8) holds, it satisfies

$$\langle v_i^\varepsilon(t),\phi\rangle_{D(L)'\times D(L)} = \Lambda_i^\varepsilon(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v^\varepsilon),W,\phi)(t) \quad \text{for a.e. } t\in[0,T]\ \mathbb{P}\text{-a.s.},\ i=1,\dots,n,$$

for any $\phi\in D(L)$. Hence,

$$\int_0^T \mathbb{E}\big|\langle v_i^\varepsilon(t),\phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v),W,\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

We deduce from the equivalence of the laws of $(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v),W)$ and $(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon)$ (and an adaptation of the arguments in [14]) that

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle\widetilde{u}_i^\varepsilon(t)+\sqrt{\varepsilon}\widetilde{w}_i^\varepsilon,\phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon,\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

By Corollary 2.8.15, we can pass to the limit $\varepsilon\to 0$ to obtain

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle\widetilde{u}_i(t),\phi\rangle_{L^2} - \Lambda_i(\widetilde{u},0,\widetilde{W},\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

This identity holds for all $\phi\in D(L)$ such that $\nabla\phi\cdot\nu = 0$ on $\partial\mathcal{O}$ and hence, by density, for any $\phi\in D(L)$. This shows that

$$\big|\langle\widetilde{u}_i(t),\phi\rangle_{L^2} - \Lambda_i(\widetilde{u},0,\widetilde{W},\phi)(t)\big| = 0 \quad \text{for a.e. } t\in[0,T]\ \widetilde{\mathbb{P}}\text{-a.s.},\ i=1,\dots,n.$$

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We infer from the definition of Λ_i that

$$\langle \widetilde{u}_i(t), \phi \rangle_{L^2} = \langle \widetilde{u}_i(0), \phi \rangle_{L^2} - \sum_{j=1}^{n} \int_0^t \left\langle A_{ij}(\widetilde{u}(s)) \nabla \widetilde{u}_j(s), \nabla \phi \right\rangle_{L^2} \mathrm{d}s + \sum_{j=1}^{n} \left\langle \int_0^t \sigma_{ij}(\widetilde{u}(s)) \, \mathrm{d}\widetilde{W}_j(s), \phi \right\rangle_{L^2}$$

for a.e. $t \in [0,T]$ $\widetilde{\mathbb{P}}$-a.s. and all $\phi \in D(L)$. Set $\widetilde{U} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$. Then $(\widetilde{u}, 0, \widetilde{W})$, together with $\widetilde{U}$, is a martingale solution to Equation (2.16) (2.17). This finishes the proof of Theorem 2.8.1.

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2.9. Application: SKT without self-diffusion

We again consider equation (2.16) (2.17), but alter the assumptions on A, by setting $a_{ii} = 0$ for $i = 1, \dots, n$. This choice changes the structure of the equation as well as assumptions and complicates the analysis. In order to ensure easier readability of the results, we decided to dedicate a separate section to this case. The system without self diffusion poses the additional problem that our notion of solution needs to be weakened further and is given in the following sense.

DEFINITION 2.9.1. *Let $T > 0$ be fixed, but arbitrary. A (global), probabilistically weak or, martingale solution of (2.16) (2.16), with $a_{ii} = 0$ for $i = 1, \dots, n$, is a collection $\{(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}), \widetilde{\mathbb{F}}, (\widetilde{W}^k)_{k\in\mathbb{N}}, \widetilde{u}, \widetilde{\eta}\}$, such that $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is a probability space, $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ is a complete filtration, $(\widetilde{W}^k)_{k\in\mathbb{N}}$ is a sequence of independent $\widetilde{\mathbb{F}}$-Wiener processes, $\widetilde{W}$ is a n-dimensional cylindrical Wiener process, $\widetilde{u}$ is a continuous $D(L)'$-valued $\widetilde{\mathbb{F}}$-adapted process, $\widetilde{\eta} = (\widetilde{\eta}_{ij})_{i\neq j=1}^n$ is a $L^{(2d+2)/(2d+1)}(0,T; W^{1,(2d+2)/(2d+1)}(\mathcal{O}))$-valued random variable and such that for all $\phi \in D(L)$ almost surely the equality*

(2.54)
$$\langle \widetilde{u}_i(t), \phi\rangle = \langle \widetilde{u}_i(0), \phi\rangle + \int_0^t \left\langle \operatorname{div}\left(\sum_{j=1}^n \widetilde{A}_{ij}(\widetilde{u}, \widetilde{\eta}, \nabla\widetilde{u}_j)\right), \phi\right\rangle \mathrm{d}s + \int_0^t \left\langle \sum_{j=1}^n \sigma_{ij}(\widetilde{u})\,\mathrm{d}\widetilde{W}_j(s), \phi\right\rangle,$$

$i = 1, \dots, n$, holds for all $t \in [0,T]$, where $\widetilde{u}_0 \overset{d}{=} u_0$. $\langle\cdot,\cdot\rangle$ is to be understood as the duality product $\langle\cdot,\cdot\rangle_{D(L)'\times D(L)}$ and the sum of the operators $\widetilde{A}_{ij}$ is given by

$$\sum_{j=1}^n \widetilde{A}_{ij}(\widetilde{u}, \nabla\widetilde{\eta}, \nabla\widetilde{u}_j) := a_{i0}\nabla\widetilde{u}_i + \sum_{j\neq j} a_{ij}\nabla\widetilde{\eta}_{ij}.$$

If the solution is pathwise unique, $\widetilde{\eta}_{ij}$ can be identified with $u_i u_j$. Also in the cases $d = 1, 2$, we are able to identify $\widetilde{\eta}_{i,j}$ with $\widetilde{u}_i\widetilde{u}_j$, for $i \neq j = 1, \dots, n$, since $W^{1,1}(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$.

2.9.1. Assumptions. We impose the following assumptions on the matrix A,

(SKT2 A 1) $a_{ii} = 0, a_{ij} \geq 0$ for $i = 1, \dots, n$, $j = 0, \dots, n$ and the detailed-balance condition is satisfied, i.e., there exist numbers $\pi_1, \dots, \pi_n > 0$ such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \dots, n. \tag{2.55}$$

Let the entropy density $h\colon \mathcal{D} \to \mathbb{R}$, given by

$$h(u) := \sum_{i=1}^n \pi_i u_i \log(u_i) - u_i + 1, \tag{2.56}$$

on the set $\mathcal{D} = (0,\infty)^n$, with suitable constants $\pi_i \geq 0$, which are determined by (2.55). We impose the following assumptions on $\sigma(u)$,

(SKT2 σ1) Let $\gamma \leq \frac{d+2}{(d+1)^2}$. There is a constant $C_\sigma > 0$ such that for any u, $v \in L^2(\mathcal{O})$, $\sigma\colon L^2(\mathcal{O};\mathbb{R}^n) \to \mathcal{L}_2^{n\times n}(U; L^2(\mathcal{O}))$ satisfies

$$\|\sigma(u) - \sigma(v)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \leq C_\sigma \|u - v\|_{L^2(\mathcal{O})},$$

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$$\|\sigma(u)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \le C_\sigma\big(1+\|u\|^\gamma_{L^2(\mathcal{O})}\big) \quad \text{for } d\ge 3,$$
$$\|\sigma(u)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \le C_\sigma\big(1+\|u\|^\rho_{L^2(\mathcal{O})}\big) \quad \text{for } d=2, \text{ where } \rho<1,$$
$$\|\sigma(u)\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))} \le C_\sigma\big(1+\|u\|_{L^2(\mathcal{O})}\big) \quad \text{for } d=1.$$

THEOREM 2.9.2. *Let $d\le 3$ and Assumptions (SKT2 A 1), (A1), (A2) (SKT2 σ1) and (A6) be satisfied, then Equation (2.16) (2.17) has a global martingale solution in the sense of Definition 2.9.1.*

The existence of a global solution to the approximate problem (2.22) follows from Theorems 2.6.6 and 2.7.5 as well as Lemma 2.5.19 as before. Hence we will focus on the estimates, in which the absence of the, so called, self diffusion coefficients manifests.

2.9.2. Estimates.

LEMMA 2.9.3 (Entropy inequality). *Let v^ε be the solution to Equation (2.22). The function $u^\varepsilon := u(R_\varepsilon(v^\varepsilon))$ satisfies for $0<t<T$,*

$$\mathbb{E}\|u(R_\varepsilon(v^\varepsilon))\|_{L^\infty(0,T;L^1(\mathcal{O}))} \le C(u^0,T), \tag{2.57}$$

$$\sum_{i=1}^n a_{i0}\mathbb{E}\|(u^\varepsilon(R_\varepsilon(v^\varepsilon))_i)^{1/2}\|^2_{L^2(0,T;H^1(\mathcal{O}))} \le C(u^0,T), \tag{2.58}$$

$$\sum_{j\ne i} a_{ij}\mathbb{E}\|\nabla(u(R_\varepsilon(v^\varepsilon))_i u^\varepsilon(R_\varepsilon(v^\varepsilon))_j)^{1/2}\|^2_{L^2(0,T;L^2(\mathcal{O}))} \le C(u^0,T). \tag{2.59}$$

PROOF. The proof follows from Lemma 2.8.3, with $a_{ii}=0$ for $i=1,\dots,n$. □

LEMMA 2.9.4 (Higher-order moments). *Let u^ε be given by $u(R_\varepsilon(v^\varepsilon))$, where v^ε satisfies Equation (2.22) and let $p\ge 2$. Then, for any $i=1,\dots,n$,*

$$\mathbb{E}\|u^\varepsilon\|^p_{L^\infty(0,T;L^1(\mathcal{O}))} \le C(p,u^0,T), \tag{2.60}$$

$$C(a_{i0})\mathbb{E}\|(u_i^\varepsilon)^{1/2}\|^p_{L^2(0,T;H^1(\mathcal{O}))} \le C(p,u^0,T), \tag{2.61}$$

where $C(p,u^0,T)>0$ does not depend on ε.
Additionally, we have

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}\bigg)^p \le C(p,u^0,T,L) \tag{2.62}$$

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\|v^\varepsilon(t)\|_{D(L)'}\bigg)^p \le C(p,u^0,T,L), \tag{2.63}$$

where $C(p,u^0,T,L)>0$ does not depend on ε.

PROOF. The proof is a direct consequence of Lemma 2.8.4. □

We derive further estimates from Lemma 2.9.4.

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LEMMA 2.9.5. *Let $u^\varepsilon = u(R_\varepsilon(v^\varepsilon))$ and $p \geq 2$, then for any $i = 1, \dots, n$,*

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^2(0,T;W^{1,1}(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.64}$$

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^{1+2/d}((0,T)\times\mathcal{O})} \leq C(p, u^0, T), \tag{2.65}$$

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^{4/d}(0,T;L^2(\mathcal{O}))} \leq C(p, u^0, T) \quad \text{for } d \leq 4, \tag{2.66}$$

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.67}$$

where $\rho_1 = (d+2)/(d+1)$.

PROOF. The identity $\nabla u_i^\varepsilon = 2(u_i^\varepsilon)^{1/2}\nabla(u_i^\varepsilon)^{1/2}$ and the Hölder inequality show that

$$\begin{aligned}
\mathbb{E}\|\nabla u_i^\varepsilon\|^p_{L^2(0,T;L^1(\mathcal{O}))} &\leq C\mathbb{E}\bigg(\int_0^T \|(u_i^\varepsilon)^{1/2}(t)\|^2_{L^2(\mathcal{O})}\|\nabla(u_i^\varepsilon)^{1/2}(t)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/2} \\
&\leq C\mathbb{E}\bigg(\|u_i^\varepsilon\|_{L^\infty(0,T;L^1(\mathcal{O}))}\int_0^T \|\nabla(u_i^\varepsilon)^{1/2}(t)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/2} \\
&\leq C\big(\mathbb{E}\|u_i^\varepsilon\|^p_{L^\infty(0,T;L^1(\mathcal{O}))}\big)^{1/2}\big(\mathbb{E}\|\nabla(u_i^\varepsilon)^{1/2}\|^{2p}_{L^2(0,T;L^2(\mathcal{O}))}\big)^{1/2}.
\end{aligned}$$

Because of (2.60) and (2.61), the right-hand side is bounded. Using (2.60) again, we infer that (2.64) holds. Estimate (2.65) is obtained from the Gagliardo Nirenberg inequality. Let $1 \leq r, s < \infty$. The exact values of r, s will be determined in the following steps. Taking into account estimates (2.60) and (2.61),

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^T \|(u_i^\varepsilon)^{1/2}(t)\|^s_{L^r(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s} \leq C\mathbb{E}\bigg(\int_0^T \|(u_i^\varepsilon)^{1/2}(t)\|^{\theta s}_{H^1(\mathcal{O})}\|(u_i^\varepsilon)^{1/2}(t)\|^{(1-\theta)s}_{L^2(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s} \\
&\leq C(T)\mathbb{E}\bigg(\|u_i^\varepsilon\|^{(1-\theta)s}_{L^\infty(0,T;L^1(\mathcal{O}))}\int_0^T \|(u_i^\varepsilon)^{1/2}(t)\|^{\theta s}_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg)^{p/s} \\
&\leq C(T)\Big(\mathbb{E}\|u_i^\varepsilon\|^{(1-\theta)sp_1}_{L^\infty(0,T;L^1(\mathcal{O}))}\Big)^{1/p_1}\bigg\{\mathbb{E}\bigg(\int_0^T \|(u_i^\varepsilon)^{1/2}(t)\|^{\theta s}_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg)^{p_2 p/s}\bigg\}^{1/p_2},
\end{aligned}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and

$$\frac{1}{r} = \left(\frac{1}{2} - \frac{1}{d}\right)\theta + \frac{1-\theta}{2} = \frac{1}{2} - \frac{\theta}{d}.$$

As in Lemma 2.8.5, we choose $\theta s = 2$, leading to the relation,

$$\frac{s}{r} = \frac{s}{2} - \frac{2}{d} = \frac{sd-4}{2d}. \tag{2.68}$$

(2.65) is obtained by setting $r = s$, $\theta = d/(d+2)$, and $p_2 = \frac{1}{\theta} = \frac{d+2}{d}$. Since we derived the estimate for the square-root, we have to account for the factor $1/2$ in (2.65).
Estimate (2.66) can be shown as in the deterministic case, be reformulating (2.68) and obtaining

$$s = \frac{4r}{d(r-2)},$$

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and

$$r = \frac{2ds}{ds-4}.$$

Accounting again for the fact that we estimated the square root, the claim follows. Finally, estimate (2.67) is a consequence of $\nabla u_i^\varepsilon = 2(u_i^\varepsilon)^{1/2}\nabla(u_i^\varepsilon)^{1/2}$, the Hölder inequality, and estimates (2.61) and (2.65). □

LEMMA 2.9.6. *Let $p \geq 2$. Then for any $i = 1, \ldots, n$ and $j \neq i$,*

$$\mathbb{E}\|(u_i^\varepsilon u_j^\varepsilon)^{1/2}\|^p_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.69}$$

$$\mathbb{E}\|(u_i^\varepsilon u_j^\varepsilon)^{1/2}\|^p_{L^2(0,T;H^1(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.70}$$

$$\mathbb{E}\|(u_i^\varepsilon u_j^\varepsilon)^{1/2}\|^p_{L^{2+2/d}((0,T)\times\mathcal{O})} \leq C(p, u^0, T), \tag{2.71}$$

$$\mathbb{E}\|u_i^\varepsilon u_j^\varepsilon\|^p_{L^{1+2/d}(0,T;L^1(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.72}$$

$$\mathbb{E}\|\nabla(u_i^\varepsilon u_j^\varepsilon)\|^p_{L^{\rho_1}(0,T;L^1(\mathcal{O}))} \leq C(p, u^0, T), \tag{2.73}$$

$$\mathbb{E}\|\nabla(u_i^\varepsilon u_j^\varepsilon)\|^p_{L^{\rho_2}((0,T)\times\mathcal{O})} \leq C(p, u^0, T), \tag{2.74}$$

where $\rho_1 = (d+2)/(d+1)$ and $\rho_2 = (2d+2)/(2d+1)$.

PROOF. The Hölder inequality and estimate (2.60) yield immediately (2.69). By the Poincaré Wirtinger inequality, estimates (2.59) and (2.69) lead to (2.70). Estimates (2.71) and (2.72) follow from the Gagliardo Nirenberg inequality, taking into account estimates (2.69) and (2.70). Indeed,

$$\begin{aligned}
\mathbb{E}\left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}\right\|_{L^r(0,T;L^s(\mathcal{O}))} &\leq C\mathbb{E}\left(\int_0^T \left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{(1-\theta)r}_{L^1(\mathcal{O})} \left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{\theta r}_{H^1(\mathcal{O})} \mathrm{d}t\right)^{\frac{1}{r}} \\
&\leq C\mathbb{E}\sup_{t\in(0,T)}\left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{(1-\theta)}_{L^1(\mathcal{O})} \left(\int_0^T \left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{\theta r}_{H^1(\mathcal{O})} \mathrm{d}t\right)^{\frac{1}{r}} \\
&\leq C\left(\mathbb{E}\sup_{t\in(0,T)}\left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{(1-\theta)r/(r-1)}_{L^1(\mathcal{O})}\right)^{(r-1)/r} \left(\mathbb{E}\left(\int_0^T \left\|\sqrt{u_i^\varepsilon u_j^\varepsilon}(t)\right\|^{\theta r}_{H^1(\mathcal{O})} \mathrm{d}t\right)\right)^{\frac{1}{r}}.
\end{aligned}$$

We proceed as in Lemma 2.8.5.

$$\frac{1}{s} = \left(\frac{1}{2} - \frac{1}{d}\right)\theta + \frac{1-\theta}{1} \iff \frac{1}{s} = 1 - \frac{d+2}{2d}\theta. \tag{2.75}$$

Multiplying both sides by r and choosing $\theta r = 2$ gives

$$\frac{r}{s} = r - \frac{d+2}{2d}r\theta = r - \frac{d+2}{d}.$$

Finally, estimates (2.70), (2.71) and (2.72) imply that

$$\nabla(u_i^\varepsilon u_j^\varepsilon) = 2(u_i^\varepsilon u_j^\varepsilon)^{1/2}\nabla(u_i^\varepsilon u_j^\varepsilon)^{1/2}$$

is bounded in $L^{\rho_1}(0,T;L^1(\mathcal{O}))$ and in $L^{\rho_2}((0,T)\times\mathcal{O})$, verifying (2.73) and (2.74). □

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We show that the fractional time derivative of the function u^ε is uniformly bounded. This result is used to establish the tightness of the laws of u^ε in some Lebesgue spaces. In the present case, without self-diffusion, we assume that $d \leq 4$ in the following calculations. Strictly speaking, we should already restrict ourselves to the case $d \leq 3$, since the estimates we derived previously will only enable us, due to the way we defined σ, to pass to the limit in $d \leq 3$.

LEMMA 2.9.7. *[Time regularity without self-diffusion] Let v^ε be the solution to Equation (2.22), $a_{i0} > 0$ and $a_{ii} = 0$ for all $i = 1, \dots, n$. We obtain the following,*

$$\mathbb{E}\|v^\varepsilon\|^p_{W^{\alpha,p}([0,T];D(L)')} \leq C, \qquad \mathbb{E}\|v^\varepsilon\|^p_{C^\beta([0,T];D(L)')} \leq C,$$
$$\mathbb{E}\|u^\varepsilon\|^p_{W^{\alpha,p}([0,T];D(L)')} \leq C, \qquad \mathbb{E}\|u^\varepsilon\|^p_{C^\beta([0,T];D(L)')} \leq C,$$

for $\alpha < 1/2, p < \min\{(d+2)/(d+1), 4/(d\gamma)\}$, and $\beta < \min\{1/(d+2), 1/2 - (d\gamma)/4\}$.

PROOF. We proceed in a fashion very similar to the previous results and derive an estimate on the expectation of the $W^{\alpha,p}(0,T;X)$ seminorm. Recalling the estimates obtained in Lemma 2.9.4, estimating the seminorm will indeed suffice to show the result.
Define $U^\varepsilon_{1,i} := \int_0^\cdot \operatorname{div} \sum_{j=1}^n A_{ij}(u^\varepsilon)\nabla u^\varepsilon_j \,\mathrm{d}r$.

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T |t-s|^{-1-p\alpha}\left\| \int_{s\wedge t}^{t\vee s} \operatorname{div} \sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u^\varepsilon_j(r)\,\mathrm{d}r\right\|^p_{D(L)'} \mathrm{d}t\,\mathrm{d}s \\
&\leq C\int_0^T\int_0^T |t-s|^{-1-p\alpha}\mathbb{E}\left(\int_{s\wedge t}^{t\vee s} \left\| a_{i0}\nabla u^\varepsilon_i + \sum_{j\neq i} a_{ij}\nabla(u^\varepsilon_i u^\varepsilon_j)\right\|_{L^1(\mathcal{O})} \mathrm{d}r\right)^p \mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

By exchanging the expectation and integration and assuming $\alpha < 1$, this term can be bounded above by

$$\mathbb{E}\|g\|^p_{W^{\alpha,p}(0,T;\mathbb{R})},$$

where $g(t) := \int_0^t \|a_{i0}\nabla u^\varepsilon_i + \sum_{j\neq i} a_{ij}\nabla(u^\varepsilon_i u^\varepsilon_j)\|_{L^1(\mathcal{O})}\,\mathrm{d}r$. We use the simple estimate

$$\begin{aligned}
\mathbb{E}\|g\|^p_{W^{\alpha,p}(0,T;\mathbb{R})} &\leq C\mathbb{E}\|g\|^p_{W^{1,p}(0,T;\mathbb{R})} \\
&\leq \mathbb{E}\left(\left\| \frac{\partial g}{\partial t}\right\|^p_{L^p(0,T;\mathbb{R})} + \|g\|^p_{L^p(0,T;\mathbb{R})}\right) \\
&\leq \mathbb{E}\int_0^T \|a_{i0}\nabla u^\varepsilon_i(t) + \sum_{j\neq i} a_{ij}\nabla(u^\varepsilon_i u^\varepsilon_j)(t)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}t \\
&\quad + \mathbb{E}\int_0^T\int_0^t \|a_{i0}\nabla u^\varepsilon_i(r) + \sum_{j\neq i} a_{ij}\nabla(u^\varepsilon_i u^\varepsilon_j)(r)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

By Estimates (2.67) and (2.73), the terms on the right hand side are finite for $p \leq (d+2)/(d+1)$.

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We define $U_{2,i}^\varepsilon := \int_0^\cdot \sum_{j=1}^n \sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)$, use the embedding $D(L)' \hookrightarrow L^2(\mathcal{O})$ and apply the BDG inequality to $\mathbb{E}\|U_{2,i}^\varepsilon\|^p_{\dot W^{1/2-,p}(0,T;L^2(\mathcal{O}))}$,

$$\begin{aligned}
&\int_0^T\int_0^T |t-s|^{-1-p\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)\bigg\|^p_{L^2(\mathcal{O})}\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_0^T\int_0^T |t-s|^{-1-p\alpha}\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \|\sigma_{ij}(u^\varepsilon(r))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}r\bigg)^{p/2}\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_0^T\int_0^T |t-s|^{-1-p\alpha+p/2-1}\mathbb{E}\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \|\sigma_{ij}(u^\varepsilon(r))\|^p_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_0^T\int_0^T |t-s|^{-1-p\alpha+p/2-1}\mathbb{E}\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n 1+\|u^\varepsilon(r)\|^{\gamma p}_{L^2(\mathcal{O}))}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

In view of the sublinear growth assumption and (2.66), we use integration by parts (or Fubini's theorem) to obtain that the integral is bounded for $\alpha < 1/2$ and $p \le 4/(d\gamma)$. We conclude that $U_{1,i}^\varepsilon \in W^{1-,(d+2)/(d+1)}(0,T;D(L)')$, $U_{2,i}^\varepsilon \in W^{1/2-,4/(d\gamma)}(0,T;D(L)')$ $\mathbb{P}$-a.s. for each $i=1,\dots,n$. By Theorem 5.4.6, we obtain,

$$\begin{aligned}
W^{1-,(d+2)/(d+1)}(0,T;D(L)') &\hookrightarrow C^{\widetilde\alpha-}([0,T];D(L)')\\
W^{1/2-,4/(d\gamma)}(0,T;D(L)') &\hookrightarrow C^{\widetilde\beta-}([0,T]D(L)'),
\end{aligned}$$

with $\widetilde\alpha = 1-(d+1)/(d+2)$ and $\widetilde\beta = 1/2-(d\gamma)/4$. Setting $\beta < \widetilde\alpha\wedge\widetilde\beta$, we conclude

$$\begin{aligned}
\mathbb{E}\|v^\varepsilon\|^p_{C^\beta([0,T];D(L)')} \le{}& C(n)\mathbb{E}\bigg\|\int_0^\cdot \operatorname{div}\sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u_j^\varepsilon(r)\,\mathrm{d}r\bigg\|^p_{C^\beta([0,T];D(L)')}\\
&+C(n)\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)\bigg\|^p_{C^\beta([0,T];D(L)')}\\
\le{}& C(n)\mathbb{E}\bigg\|\int_0^\cdot \operatorname{div}\sum_{j=1}^n A_{ij}(u^\varepsilon(r))\nabla u_j^\varepsilon(r)\,\mathrm{d}r\bigg\|^p_{W^{1-,(d+2)/(d+1)}(0,T;D(L)')}\\
&+C(n)\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \sigma_{ij}(u^\varepsilon(r))\,\mathrm{d}W_j(r)\bigg\|^p_{W^{1/2-,4/(d\gamma)}(0,T;D(L)')} \le C,
\end{aligned}$$

uniform in ε. Using once again 2.5.19 yields,

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon\|^p_{\dot W^{\alpha,p}(0,T;D(L)')} &= \mathbb{E}\|v^\varepsilon - \varepsilon L^*LR_\varepsilon(v^\varepsilon)\|^p_{\dot W^{\alpha,p}(0,T;D(L)')}\\
&\le C(p)\Big(\mathbb{E}\|v^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')} + \mathbb{E}\|\varepsilon L^*LR_\varepsilon(v^\varepsilon)\|^p_{W^{\alpha,p}(0,T;D(L)')}\Big)\\
&= C(p)\bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t)-v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

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$$
\begin{aligned}
&+\mathbb{E}\int_0^T\int_0^T \frac{\|\varepsilon L^*LR_\varepsilon(v^\varepsilon)(t)-\varepsilon L^*LR_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg)\\
&=C\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t)-v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\\
&+\mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p\|L^*LR_\varepsilon(v^\varepsilon)(t)-L^*LR_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg)\\
&\le C(p,L)\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t)-v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\\
&+\mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p\|R_\varepsilon(v^\varepsilon)(t)-R_\varepsilon(v^\varepsilon)(s)\|^p_{D(L)}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg)\\
&\le C(p,L)\Bigg(\mathbb{E}\int_0^T\int_0^T \frac{\|v^\varepsilon(t)-v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\\
&+\mathbb{E}\int_0^T\int_0^T \frac{\varepsilon^p}{\varepsilon^p}\frac{\|v^\varepsilon(t)-v^\varepsilon(s)\|^p_{D(L)'}}{|t-s|^{1+\alpha p}}\,\mathrm{d}t\,\mathrm{d}s\Bigg)\\
&\le C(p,L)\Bigg(\mathbb{E}\|v^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')}\Bigg).
\end{aligned}
$$

An estimate for $\varepsilon L^*LR_\varepsilon(v^\varepsilon)$ is obtained in a similar fashion,

$$
\begin{aligned}
\mathbb{E}\|\varepsilon L^*LR_\varepsilon(v^\varepsilon)\|^p_{W^{\alpha,p}(0,T;D(L)')} &= \mathbb{E}\|v^\varepsilon-u^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')}\\
&\le C(p)\Bigg(\mathbb{E}\|v^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')}+\mathbb{E}\|u^\varepsilon\|^p_{W^{\alpha,p}(0,T;D(L)')}\Bigg).
\end{aligned}
$$

The arguments for the $C^\beta([0,T];D(L)')$ regularity is completely analogous. This concludes the proof. □

2.9.3. Tightness. We show that the laws of u^ε are tight in the sub-Polish space

$$\widetilde{Z}_T := C^0([0,T];D(L)')\cap L^{\rho_1}_w(0,T;W^{1,\rho_1}(\mathcal{O})),$$

endowed with the topology $\widetilde{\mathbb{T}}$ which is the maximum of the topology of $C^0([0,T];D(L)')$ and the weak topology of $L^{\rho_1}_w(0,T;W^{1,\rho_1}(\mathcal{O}))$. Recall that ρ_1 was given in by $\rho_1=\frac{d+2}{d+1}$. From now on, we will assume $d\le 3$.

LEMMA 2.9.8. *Let $d\le 3$. The sequence of laws $(\mathcal{L}(u^\varepsilon))_{N\in\mathbb{N}}$ is tight in*

$$\widetilde{Z}_T,\ L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))\ \text{and}\ L^2(0,T;L^{d/(d-1)}(\mathcal{O})).$$

In the case $d=1$, the space $L^{d/(d-1)}(\mathcal{O})$ can be replaced by $L^q(\mathcal{O})$, for any $1\le q<\infty$ (see [53, Corollary 7.2]).

PROOF. The tightness in $\widetilde{Z}_T$ differs only slightly from Lemma 2.8.9. Indeed one appeals to [120, Lemma 5, Theorem 3] with the spaces $X=W^{1,1}(\mathcal{O})$ and $B=D(L)'$. The

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tightness in $L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))$ follows similarly as in Lemma 2.8.9, observing that the embedding

$$W^{\alpha,q}(0,T;D(L)') \cap L^{(d+2)/(d+1)}(0,T;W^{1,(d+2)/(d+1)}(\mathcal{O})) \hookrightarrow L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))$$

is compact. Here, we use the facts that the embedding $W^{1,(d+2)/(d+1)}(\mathcal{O}) \hookrightarrow L^{d(d+2)/(d^2-2)-}(\mathcal{O})$ is compact and that u_i^ε is bounded in $W^{\alpha,q}(0,T;D(L)')$ and $L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O}))$ due to estimates (2.9.7) and (2.67), respectively. Finally, the last statement is a consequence of the compact embedding

$$W^{\alpha,q}(0,T;D(L))') \cap L^2(0,T;W^{1,1}(\mathcal{O})) \hookrightarrow L^2(0,T;L^{d/(d-1)-}(\mathcal{O}))$$

as well as estimates (2.9.7) and (2.64). In fact, the compactness in $L^2(0,T;L^p(\mathcal{O}))$ is valid up to $p = d/(d-1)$ by taking into account the uniform bound of $u_i^\varepsilon \log u_i^\varepsilon$ in $L^\infty(0,T;L^1(\mathcal{O}))$; see Lemma 4.0.8. □

REMARK 2.9.9. *The embedding $W^{1,(d+2)/(d+1)}(\mathcal{O}) \hookrightarrow L^{d(d+2)/(d^2-2)-}(\mathcal{O})$ is the main technical reason, why we restrict our analysis to $d \le 3$, since the latter space is only continuously embedded in $L^2(\mathcal{O})$ for $d \le 3$.*

The previous lemma shows in particular, that $(\mathcal{L}(u^\varepsilon))$ is tight in

$$Z_T = \widetilde{Z}_T \cap L^2(0,T;L^{d/(d-1)}(\mathcal{O})) \cap L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))$$

with the topology that is the maximum of $\widetilde{\mathbb{T}}$ and the topologies induced by the $L^2(0,T;L^{d/(d-1)}(\mathcal{O}))$ and $L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))$ norms.

LEMMA 2.9.10. *The set of laws $(\mathcal{L}(\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon)))_\varepsilon$ is tight in $Y_T := L^2_w(0,T;D(L)') \cap L^\infty_{w^*}(0,T;D(L)')$.*

The proof is analogous to the one of Lemma 2.8.11.

LEMMA 2.9.11. *The family of measures*

$$\mu_W^\varepsilon = \mathbb{P}(W \in \cdot) \in \Pr\big(C([0,T];\hat{U})\big),$$

is tight.

PROOF. The sequence $(\mu_W^\varepsilon)_\varepsilon$ is constantly equal to one element and consequently weakly compact. By Prokhorov's theorem, it is tight. □

LEMMA 2.9.12. *Let $i,j = 1,\dots,n$ and $i \ne j$. Then the laws of $u_i^\varepsilon u_j^\varepsilon$ are tight in*

$$Z_T^2 := L^{1+2/d}_w(0,T;L^1(\mathcal{O})) \cap L^{(d+2)/(d+1)}_{w*}(0,T;W^{1,1}(\mathcal{O})) \cap L^{(2d+2)/(2d+1)}_w(0,T;W^{1,(2d+2)/(2d+1)}(\mathcal{O})).$$

PROOF. The claim follows from Lemma 2.9.6 and Chebychev's inequality by arguments analogous to the previous proofs. □

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2.9.4. Convergence. We consider the space $Z_T \times Z_T^2 \times Y_T \times C^0([0,T];\hat{U})$, equipped with the probability measure $\mu^\varepsilon := \mu_u^\varepsilon \times \mu_{u^2}^\varepsilon \times \mu_w^\varepsilon \times \mu_W^\varepsilon$, with

$$\begin{aligned}
\mu_u^\varepsilon(\cdot) &= \mathbb{P}(u^\varepsilon \in \cdot) \in \Pr\big(Z_T\big),\\
\mu_u^\varepsilon(\cdot) &= \otimes_{i\neq j=1}^n \mathbb{P}(u_i^\varepsilon u_j^\varepsilon \in \cdot) \in \Pr\big(Z_T^2\big),\\
\mu_w^\varepsilon(\cdot) &= \mathbb{P}(\sqrt{\varepsilon} L^* L R_\varepsilon(v^\varepsilon) \in \cdot) \in \Pr\big(Y_T\big),\\
\mu_W^\varepsilon &= \mathbb{P}(W \in \cdot) \in \Pr\big(C([0,T];\hat{U})\big).
\end{aligned}$$

$Z_T \times Z_T^2 \times Y_T \times C^0([0,T];\hat{U})$ satisfies the assumptions of the Skorokhod Jakubowski theorem [28, Theorem C1] and the measures μ^ε are tight since set of laws $((\mathcal{L}(u^\varepsilon))$, $((\mathcal{L}(u_i^\varepsilon u_j^\varepsilon))_{i\neq j})$, $(\mathcal{L}(\sqrt{\varepsilon} L^* L R_\varepsilon(v^\varepsilon)))$, $(\mathcal{L}(W))$ are tight in $(Z_T, \mathbb{T}_Z)$, $(Z_T^2, \mathbb{T}_{Z^2})$, $(Y_T, \mathbb{T}_Y)$ and $C([0,T];\hat{U})$ respectively. The theorem implies the existence of a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and, on this space, $(Z_T \times Z_T^2 \times Y_T \times C^0([0,T];\hat{U}))$-valued random variables $(\widetilde{u}, (\widetilde{\eta}_{ij})_{i\neq j=1}^n, \widetilde{w}, \widetilde{W})$ and $(\widetilde{u}^\varepsilon, (\widetilde{\eta}_{ij}^\varepsilon)_{i\neq j=1}^n, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ such that $(\widetilde{u}^\varepsilon, (\widetilde{\eta}_{ij}^\varepsilon)_{i\neq j=1}^n, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ has the same law as $(u^\varepsilon, (u_i^\varepsilon u_j^\varepsilon)_{i\neq j=1}^n, \sqrt{\varepsilon} L^* L R_\varepsilon(v^\varepsilon), W)$ on $\mathcal{B}(Z_T \times Z_T^2 \times Y_T \times C^0([0,T];\hat{U}))$ and, as $\varepsilon \to 0$,

$$(\widetilde{u}^\varepsilon, (\widetilde{\eta}_{ij}^\varepsilon)_{i\neq j=1}^n, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon) \to (\widetilde{u}, (\widetilde{\eta}_{ij})_{i\neq j=1}^n, \widetilde{w}, \widetilde{W}) \quad \text{in } Z_T \times Z_T^2 \times Y_T \times C^0([0,T];\hat{U}) \ \widetilde{\mathbb{P}}\text{-a.s.}$$

By the definition of the space Z_T, Y_T, this convergence means $\widetilde{\mathbb{P}}$-a.s. The convergence results (2.45) hold, but only in a weaker sense,

$$\widetilde{u}^\varepsilon \rightharpoonup \widetilde{u} \quad \text{weakly in } L^{(d+2)/(d+1)}(0,T;W^{1,(d+2)/(d+1)}(\mathcal{O})), \tag{2.76}$$

$$\widetilde{u}^\varepsilon \to \widetilde{u} \quad \text{in } L^2(0,T;L^{d/(d-1)}(\mathcal{O})), \tag{2.77}$$

$$\widetilde{u}^\varepsilon \to \widetilde{u} \quad \text{in } L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O})), \tag{2.78}$$

$$\widetilde{\eta}_{ij}^\varepsilon \to \widetilde{\eta}_{ij} \quad \text{weakly* in } L^{(d+2)/(d+1)}(0,T;W^{1,1}(\mathcal{O})), \tag{2.79}$$

$$\widetilde{\eta}_{ij}^\varepsilon \to \widetilde{\eta}_{ij} \quad \text{weakly in } L^{(2d+2)/(2d+1)}(0,T;W^{1,(2d+2)/(2d+1)}(\mathcal{O})), \tag{2.80}$$

$$\widetilde{w}^\varepsilon \rightharpoonup \widetilde{w} \quad \text{weakly in } L^2(0,T;D(L)'), \tag{2.81}$$

$$\widetilde{w}^\varepsilon \rightharpoonup \widetilde{w} \quad \text{weakly* in } L^\infty(0,T;D(L)'), \tag{2.82}$$

$$\widetilde{W}^\varepsilon \to \widetilde{W} \quad \text{in } C^0([0,T];\hat{U}). \tag{2.83}$$

$\widetilde{\mathbb{P}}$-a.s. as $N \to \infty$. Moreover, we deduce from estimates (2.67) and (2.60) that

$$\widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}^\varepsilon(t)\|_{W^{1,(2d+2)/(2d+1)}(\mathcal{O})}^{(2d+2)/(2d+1)} \,\mathrm{d}t\bigg)^p + \widetilde{\mathbb{E}}\bigg(\sup_{0\le t\le T} \|\widetilde{u}^\varepsilon(t)\|_{D(L)'} \,\mathrm{d}t\bigg)^p \le C.$$

By the bound (2.74), and the equivalence of laws, we deduce that

$$\widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{\eta}_{ij}^\varepsilon\|_{W^{1,(2d+2)/(2d+1)}(\mathcal{O})}^{(2d+2)/(2d+1)} \,\mathrm{d}t\bigg)^p \le C. \tag{2.84}$$

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We infer the existence of a subsequence of $(\widetilde{u}^\varepsilon)$ (not relabeled) that is weakly converging in $L^p(\widetilde{\Omega}; L^{(2d+2)/(2d+1)}(0,T; W^{1,(2d+2)/(2d+1)}(\mathcal{O})))$ (and strongly converging in $L^p(\widetilde{\Omega}; C([0,T]; D(L)'))$ (by Vitali's theorem [56, Theorem 6, section III.3.6, p.122])) as $\varepsilon \to 0$. Thus, taking into account the convergence $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we conclude that the limit function satisfies

$$\widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}(t)\|_{W^{1,(d+2)/(d+1)}(\mathcal{O})}^{(d+2)/(d+1)} \,\mathrm{d}t \bigg)^p + \widetilde{\mathbb{E}}\bigg(\sup_{0\le t\le T} \|\widetilde{u}(t)\|_{D(L)'} \bigg)^p < \infty \quad \text{for } p \le p_0.$$

We verify similarly as in Lemma 2.8.13 that $\widetilde{u}_i(x,t) \ge 0$ for a.e. $(x,t) \in \mathcal{O}\times(0,T)$ $\widetilde{\mathbb{P}}$-a.s. and $i = 1,\dots,n$. The only difference to the proof is that we work in the space $L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))$ instead of $L^2(0,T;L^2(\mathcal{O}))$.
In order to verify that these new random variables induce actual processes, one can follow the arguments in [29, Proposition B4]. The progressive measurability of $\widetilde{u}^\varepsilon$ is a consequence of [25, Appendix B]. Let $\widetilde{B}_j^{\varepsilon,k}(t) := \langle \widetilde{W}_j^\varepsilon(t), e_k\rangle$. It is not too difficult to see, that $\widetilde{B}_j^{\varepsilon,k}$, $k \in \mathbb{N}$, are independent, standard, real-valued $\widetilde{\mathcal{F}}_t$-Wiener Processes. The adeptness is a direct consequence of their definition and the independence, as well as the independence of the increments $\widetilde{W}^{\varepsilon,k}(t) - \widetilde{W}^{\varepsilon,k}(s)$ with respect to $\widetilde{\mathcal{F}}_s$, is inherited from $B_j^{\varepsilon,k}(\cdot) := \langle W_j(\cdot), e_k\rangle$. Passing to the limit $\varepsilon \to 0$ in the characteristic function, which is done by the dominated convergence theorem, shows that also $\widetilde{W}^k$ are $\widetilde{\mathcal{F}}_t$-martingales with the correct marginal distributions. An application of Lévy's characterization theorem yields that $\widetilde{W}^k$ is indeed a Wiener process. We can conclude that $\widetilde{W}$ is indeed a cylindrical Wiener process.

The following lemma allows us to identify the quadratic terms.

LEMMA 2.9.13. *Let $(\widetilde{\eta}_{ij}^\varepsilon)$ be the sequence of Z_T^2-valued random variables constructed above. Then it holds for $\phi_2 \in W^{2,2d+1}(\mathcal{O})$ and satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \bigg| \sum_{i,j=1,\, j\ne i}^n \int_0^t \big\langle \nabla\widetilde{\eta}_{ij}^\varepsilon(s) - \nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2 \big\rangle_{L^2} \,\mathrm{d}s \bigg| \,\mathrm{d}t = 0.$$

PROOF. Since ϕ_2 is constant with respect to time, we infer that $\phi_2 \in L^\infty(0,T;W^{2,2d+1}(\mathcal{O}))$. The weak convergence 2.80 implies that, for any $t \le T$,

$$\int_0^t \langle \nabla\widetilde{\eta}_{ij}^\varepsilon(s) - \nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2\rangle_{L^2} \,\mathrm{d}s \to 0 \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

By the equivalence of laws, we infer

$$\widetilde{\mathbb{E}}\bigg(\int_0^t \langle \nabla\widetilde{\eta}_{ij}^\varepsilon(s) - \nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2\rangle_{L^2} \,\mathrm{d}s \bigg)^2 \le C.$$

By Vitali's theorem

$$\widetilde{\mathbb{E}} \bigg| \int_0^t \langle \nabla\widetilde{\eta}_{ij}^\varepsilon(s) - \nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2\rangle_{L^2} \,\mathrm{d}s \bigg| \to 0.$$

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Hence, when considering

$$\widetilde{\mathbb{E}} \int_0^T \left| \sum_{i,j=1,\, j\neq i}^n \int_0^t \left\langle \nabla \widetilde{\eta}_{ij}^\varepsilon(s) - \nabla \widetilde{\eta}_{ij}(s), \nabla \phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t,$$

we use integration by parts (recall that $\nabla \phi_2 \cdot \nu = 0$ on $\partial \mathcal{O}$), which yields

$$\widetilde{\mathbb{E}} \int_0^T \left| \sum_{i,j=1,\, j\neq i}^n \int_0^t \left\langle \widetilde{\eta}_{ij}^\varepsilon(s) - \widetilde{\eta}_{ij}(s), \Delta \phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t.$$

Since ϕ was chosen such that $\Delta \phi_2 \in L^{2d+1}(\mathcal{O})$, the right-hand side converges to 0 by Tonelli's and Vitali's theorem. □

We infer from estimate (2.84) that, by [73, Proposition 1.2.25, p.26] that there exists an $\Omega \times (0,T) \times \mathcal{O}$ strongly measurable version of $\widetilde{\eta}_{ij}^\varepsilon$ as well as $\widetilde{\eta}_{ij}$. In a similar way as for $\widetilde{u}_i^\varepsilon$, one can argue that $\widetilde{\eta}_{ij}^\varepsilon \geq 0$, (ω, t, x)-a.s.

LEMMA 2.9.14. *It holds for all s, $t \in [0,T]$ with $s \leq t$, $i = 1, \dots, n$, and all $\phi_1 \in L^2(\mathcal{O})$ and all $\phi_2 \in D(L)$ and satisfying $\nabla \phi_2 \cdot \nu = 0$ on $\partial \mathcal{O}$ that*

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \widetilde{u}_i^\varepsilon(t) - \widetilde{u}_i(t), \phi_1 \right\rangle_{L^2} \mathrm{d}t = 0, \tag{2.85}$$

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \left\langle \widetilde{u}_i^\varepsilon(0) - \widetilde{u}_i(0), \phi_2 \right\rangle_{D(L)' \times D(L)} = 0, \tag{2.86}$$

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \sqrt{\varepsilon} \widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)' \times D(L)} \mathrm{d}t = 0, \tag{2.87}$$

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \left\langle \sqrt{\varepsilon} \widetilde{w}_i^\varepsilon(0), \phi_2 \right\rangle_{D(L)' \times D(L)} = 0, \tag{2.88}$$

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \int_0^t \left\langle \widetilde{A}_{ij}(\widetilde{u}^\varepsilon, \nabla \widetilde{\eta}_{ij}^\varepsilon, \nabla \widetilde{u}_j^\varepsilon)(s) - \widetilde{A}_{ij}(\widetilde{u}, \nabla \widetilde{\eta}_{ij}, \nabla \widetilde{u}_j)(s), \nabla \phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t = 0, \tag{2.89}$$

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \int_0^t \left\langle \sigma_{ij}(\widetilde{u}^\varepsilon(s)) \mathrm{d}\widetilde{W}_j^{(N)}(s) - \sigma_{ij}(\widetilde{u}(s)) \, \mathrm{d}\widetilde{W}_j(s), \phi_1 \right\rangle_{L^2} \right|^2 \mathrm{d}t = 0. \tag{2.90}$$

PROOF. We know that $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s. In particular, $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2\left([0,T]; L^{(d+2)/(d+1)}(\mathcal{O})\right)$ $\widetilde{\mathbb{P}}$-a.s., which means that for any $t \in [0,T]$. We conclude that

$$\lim_{\varepsilon \to 0} \int_0^T \langle \widetilde{u}^\varepsilon(t) - \widetilde{u}(t), \phi_1 \rangle_{L^2}^{(d+2)/(d+1)} \, \mathrm{d}t = 0 \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

We have for any $r > 1$, by

$$\widetilde{\mathbb{E}} \left(\left| \int_0^T \left\| \widetilde{u}^{(N)}(t) - \widetilde{u}(t) \right\|_{L^2(\mathcal{O})}^{(d+2)/(d+1)} \mathrm{d}t \right|^r \right) \leq C.$$

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This bound provides the equi-integrability of $\int_0^T \left\langle \widetilde{u}^{(N)}(t) - \widetilde{u}(t), \phi_1 \right\rangle_{L^2(\mathcal{O})}^{(d+2)/(d+1)} \mathrm{d}t$. Vitali's convergence theorem then shows that (2.85) holds.
Convergence (2.86) follows $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $C^0\left([0,T]; D(L)'\right)$ $\widetilde{\mathbb{P}}$ -a.s. and $\widetilde{u}$ is continuous at $t=0$, we infer that for any ϕ_2

$$\lim_{\varepsilon\to 0} \langle \widetilde{u}^\varepsilon(0), \phi_2 \rangle_{D(L)'\times D(L)} = \langle \widetilde{u}(0), \phi_2 \rangle_{D(L)'\times D(L)} \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

(2.87) is a direct consequence of the estimate

$$\widetilde{\mathbb{E}}\left(\int_0^T \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}t \right)^p \le \widetilde{\mathbb{E}}\left(\int_0^T \sqrt{\varepsilon} \|\widetilde{w}_i^\varepsilon(t)\|_{D(L)'} \|\phi_2\|_{D(L)} \mathrm{d}t \right)^p \le \sqrt{\varepsilon}^p C(\phi_2, u_0, p).$$

Next, we establish convergence (2.86).

$$\begin{aligned}
&\widetilde{E}\int_0^T \left| \int_0^t \sum_{j=1}^n \left\langle \widetilde{A}_{ij}(\widetilde{u}^\varepsilon, \nabla\widetilde{\eta}_{ij}^\varepsilon, \nabla\widetilde{u}_j^\varepsilon)(s) - \widetilde{A}_{ij}(\widetilde{u}, \nabla\widetilde{\eta}_{ij}, \nabla\widetilde{u}_j)(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t \\
&= \widetilde{E}\int_0^T \left| \int_0^t \left\langle a_{i0}\nabla\widetilde{u}_i^\varepsilon(s) + \sum_{j\neq i} a_{ij}\nabla\widetilde{\eta}_{ij}^\varepsilon(s) - a_{i0}\nabla\widetilde{u}_i(s) - \sum_{j\neq i} a_{ij}\nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t \\
&\le \widetilde{E}\int_0^T \left| \int_0^t \left\langle a_{i0}\nabla\widetilde{u}_i^\varepsilon(s) - a_{i0}\nabla\widetilde{u}_i(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t \\
&\quad + \widetilde{E}\int_0^T \left| \int_0^t \left\langle \sum_{j\neq i} a_{ij}\nabla\widetilde{\eta}_{ij}^\varepsilon(s) - \sum_{j\neq i} a_{ij}\nabla\widetilde{\eta}_{ij}(s), \nabla\phi_2 \right\rangle_{L^2} \mathrm{d}s \right| \mathrm{d}t.
\end{aligned}$$

The convergence of the second term is a direct consequence of Lemma 2.9.13. By Lemma 5.3.57, since $\widetilde{W}^\varepsilon \to \widetilde{W}$ in $C\left([0,T]; \hat{U}\right)$, it is sufficient to show that $\sigma_{ij}\left(\widetilde{u}^{(N)}\right) \to \sigma_{ij}(\widetilde{u})$ in $L^2\left(0,T; \mathcal{L}_2\left(U; L^2(\mathcal{O})\right)\right)$ $\mathbb{P}-$ a.s. Indeed, by Assumption (SKT2 σ1),

$$\begin{aligned}
&\int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))}^2 \mathrm{d}s \\
&\quad = \int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \mathrm{d}s \\
&\quad \le \int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \mathrm{d}s \\
&\qquad + \int_0^t \left\|\sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s)) - \sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U,L^2(\mathcal{O}))} \mathrm{d}s \\
&\quad \le C\int_0^t \left\|\widetilde{u}^\varepsilon(s)\right\|_{L^2(\mathcal{O})}^\gamma \left\|\widetilde{u}^\varepsilon(s) - \widetilde{u}(s)\right\|_{L^2(\mathcal{O})} \mathrm{d}s \\
&\qquad + \int_0^t \left\|\widetilde{u}(s)\right\|_{L^2(\mathcal{O})}^\gamma \left\|\widetilde{u}^\varepsilon(s) - \widetilde{u}(s)\right\|_{L^2(\mathcal{O})} \mathrm{d}s
\end{aligned}$$

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$$\leq C\left(\int_0^t \left\|\widetilde{u}^\varepsilon(s)\right\|_{L^2(\mathcal{O})}^{\gamma p}\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^t \left\|\widetilde{u}^\varepsilon(s)-\widetilde{u}(s)\right\|_{L^2(\mathcal{O})}^{q}\,\mathrm{d}s\right)^{\frac{1}{q}}$$
$$+\left(\int_0^t \left\|\widetilde{u}(s)\right\|_{L^2(\mathcal{O})}^{\gamma p}\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^t \left\|\widetilde{u}^\varepsilon(s)-\widetilde{u}(s)\right\|_{L^2(\mathcal{O})}^{q}\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

We need strong convergence in $L^q(0,T;L^2(\mathcal{O}))$ and boundedness in $L^{\gamma p}(0,T;L^2(\mathcal{O}))$. With γ being less or equal to $(d+2)/(d+1)^2$, we choose $q=\frac{d+2}{d+1}$ and $p=d+1$. By (2.78), the first terms are $\widetilde{\mathbb{P}}$-a.s. finite. Since $\widetilde{u}^\varepsilon\to\widetilde{u}$ in $L^{(d+2)/(d+1)}(0,T;L^2(\mathcal{O}))\,\tilde{\mathbb{P}}$ -a.s., we infer that for $t\in[0,T],\omega\in\widetilde{\Omega}$, the right hand side tends to 0. By Lemma 5.3.57, we conclude that $\int_0^t\sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}_j^\varepsilon(s)\to\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s)$, for $j=1,\dots,n$, in probability in $L^2(0,T;L^2(\mathcal{O}))$. Furthermore,

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t\left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))-\sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\,\mathrm{d}s\right|^p\\
&\quad\leq C\widetilde{\mathbb{E}}\left(\int_0^t\left(\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2+\|\sigma_{ij}(\widetilde{u}(s)\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\right)\mathrm{d}s\right)^p\\
&\quad\leq C+C\widetilde{\mathbb{E}}\left(\int_0^t\left(\|\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}^{2\gamma}+\|\widetilde{u}(s)\|_{L^2(\mathcal{O})}^{2\gamma}\right)\mathrm{d}s\right)^p\\
&\quad\leq C+C\widetilde{\mathbb{E}}\left(\int_0^t\left(\|\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}^{(d+2)/(d+1)}+\|\widetilde{u}(s)\|_{L^2(\mathcal{O})}^{(d+2)/(d+1)}\right)\mathrm{d}s\right)^p.
\end{aligned}$$

In view of Vitali's convergence theorem, we deduce from this bound and the previous convergence that

$$\lim_{\varepsilon\to0}\widetilde{\mathbb{E}}\int_0^t\left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))-\sigma_{ij}(\widetilde{u}(s))\right\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\,\mathrm{d}s=0.$$

We can estimate

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\left\langle\int_0^t\sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}_j^\varepsilon(s)-\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle_{L^2}\right|^2\\
&\quad\leq C\|\phi_1\|_{L^2(\mathcal{O})}^2\widetilde{\mathbb{E}}\int_0^t\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2+\|\sigma_{ij}(\widetilde{u}(s))\|_{\mathcal{L}_2(U;L^2(\mathcal{O}))}^2\,\mathrm{d}s\\
&\quad\leq C\|\phi_1\|_{L^2(\mathcal{O})}^2+C\|\phi_1\|_{L^2(\mathcal{O})}^2\widetilde{\mathbb{E}}\left(\int_0^t\|\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}^{2\gamma}\,\mathrm{d}s+\int_0^T\|\widetilde{u}(s)\|_{L^2(\mathcal{O})}^{2\gamma}\,\mathrm{d}s\right)\leq C.
\end{aligned}$$

This bound allow us to apply the dominated convergence theorem to conclude that for any $\phi_1\in L^2(\mathcal{O})$,

$$\lim_{\varepsilon\to0}\widetilde{\mathbb{E}}\int_0^T\left|\left\langle\int_0^t\sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}_j^\varepsilon(s)-\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle_{L^2}\right|^2\mathrm{d}t=0.$$

This shows (2.90) and finishes the proof. □

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Let $\phi \in D(L)$. Next, we define for $t \in [0,T]$ and $i = 1, \dots, n$,

$$\begin{aligned}\Lambda_i^\varepsilon\big(\widetilde{u}^\varepsilon, \sqrt{\varepsilon}\widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon, \phi\big)(t) &:= \big\langle \widetilde{u}_i(0), \phi\big\rangle_{L^2} + \big\langle \sqrt{\varepsilon}\widetilde{w}^\varepsilon(0), \phi\big\rangle_{D(L)'\times D(L)} \\ &\quad - \int_0^T \sum_{j=1}^n \big\langle \widetilde{A}_{ij}(\widetilde{u}^\varepsilon, \nabla\widetilde{\eta}^\varepsilon, \nabla\widetilde{u}_j^\varepsilon)(s), \nabla\phi\big\rangle_{L^2}\,\mathrm{d}s \\ &\quad + \sum_{j=1}^n \left\langle \int_0^T \sigma_{ij}(\widetilde{u}^\varepsilon(s))\mathrm{d}\widetilde{W}_j^{(N)}(s), \phi\right\rangle_{L^2}, \end{aligned} \tag{2.91}$$

$$\begin{aligned}\Lambda_i\big(\widetilde{u}, 0, \widetilde{W}, \phi\big)(t) &:= \langle \widetilde{u}_i(0), \phi\rangle_{L^2} - \sum_{j=1}^n \int_0^T \big\langle \widetilde{A}_{ij}(\widetilde{u}, \widetilde{\eta}, \nabla\widetilde{u}_j)(s), \nabla\phi\big\rangle_{L^2}\,\mathrm{d}s \\ &\quad + \sum_{j=1}^n \left\langle \int_0^T \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi\right\rangle_{L^2}. \end{aligned} \tag{2.92}$$

The following corollary is essentially a consequence of Lemma 2.9.14.

Corollary 2.9.15. *It holds for any $\phi_1 \in L^2(\mathcal{O})$ and any $\phi_2 \in D(L)$ satisfying $\nabla\phi_2\cdot\nu = 0$ on $\partial\mathcal{O}$ that*

$$\begin{aligned}\lim_{\varepsilon\to 0}\big\|\langle\widetilde{u}_i^\varepsilon, \phi_1\rangle_{L^2} - \langle\widetilde{u}_i, \phi_1\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} &= 0, \\ \lim_{\varepsilon\to 0}\big\|\langle\sqrt{\varepsilon}\widetilde{w}_i^\varepsilon, \phi_2\rangle_{D(L)'\times D(L)}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} &= 0, \\ \lim_{\varepsilon\to 0}\big\|\langle\widetilde{v}_i^\varepsilon, \phi_2\rangle_{D(L)'\times D(L)} - \langle\widetilde{u}_i, \phi_2\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} &= 0, \\ \lim_{\varepsilon\to 0}\big\|\Lambda_i^\varepsilon\big(\widetilde{u}^\varepsilon, \widetilde{\eta}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon, \phi_2\big) - \Lambda_i\big(\widetilde{u}, \widetilde{\eta}, 0, \widetilde{W}, \phi_2\big)\big\|_{L^1(\widetilde{\Omega}\times(0,T))} &= 0. \end{aligned}$$

Since v^ε is a strong solution of Equation (2.22), and representation (2.8) holds, it satisfies

$$\langle v_i^\varepsilon(t), \phi\rangle_{D(L)'\times D(L)} = \Lambda_i^\varepsilon(u^\varepsilon, (u_i^\varepsilon u_j^\varepsilon)_{i\neq j}, \varepsilon L^* LR_\varepsilon(v^\varepsilon), W, \phi)(t) \quad \text{for a.e. } t\in[0,T]\ \mathbb{P}\text{-a.s.},\ i = 1,\dots,n,$$

for any $\phi \in D(L)$. Hence,

$$\int_0^T \mathbb{E}\big|\langle v_i^\varepsilon(t), \phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(u^\varepsilon, (u_i^\varepsilon u_j^\varepsilon)_{i\neq j}, \varepsilon L^* LR_\varepsilon(v), W, \phi)(t)\big|\,\mathrm{d}t = 0, \quad i = 1,\dots,n.$$

We deduce from the equivalence of the laws of $(u^\varepsilon, (u_i^\varepsilon u_j^\varepsilon)_{i\neq j}, \varepsilon L^* LR_\varepsilon(v), W)$ and $(\widetilde{u}^\varepsilon, \widetilde{\eta}, \sqrt{\varepsilon}\widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ (and arguments from [14]) that

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i^\varepsilon(t) + \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon, \phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(\widetilde{u}^\varepsilon, \widetilde{\eta}^\varepsilon, \sqrt{\varepsilon}\widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon, \phi)(t)\big|\,\mathrm{d}t = 0, \quad i = 1,\dots,n.$$

By Corollary 2.9.15, we can pass to the limit $\varepsilon \to 0$ to obtain

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i(t), \phi\rangle_{L^2} - \Lambda_i(\widetilde{u}, \widetilde{\eta}, 0, \widetilde{W}, \phi)(t)\big|\,\mathrm{d}t = 0, \quad i = 1,\dots,n.$$

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This identity holds for all $\phi \in D(L)$ such that $\nabla\phi \cdot \nu = 0$ on $\partial\mathcal{O}$ and hence, by density, for any $\phi \in D(L)$. This shows that

$$\left|\langle \widetilde{u}_i(t), \phi\rangle_{L^2} - \Lambda_i(\widetilde{u}, \widetilde{\eta}, 0, \widetilde{W}, \phi)(t)\right| = 0 \quad \text{for a.e. } t \in [0,T] \ \widetilde{\mathbb{P}}\text{-a.s., } \ i = 1, \dots, n.$$

We infer from the definition of Λ_i that

$$\langle \widetilde{u}_i(t), \phi\rangle_{L^2} = \langle \widetilde{u}_i(0), \phi\rangle_{L^2} - \sum_{j=1}^{n} \int_0^t \left\langle \widetilde{A}_{ij}(\widetilde{u}, \widetilde{\eta}, \nabla\widetilde{u}_j)(s), \nabla\phi \right\rangle_{L^2} \mathrm{d}s + \sum_{j=1}^{n} \left\langle \int_0^t \sigma_{ij}(\widetilde{u}(s))\, \mathrm{d}\widetilde{W}_j(s), \phi \right\rangle_{L^2}$$

for a.e. $t \in [0,T]$ $\widetilde{\mathbb{P}}$-a.s. and all $\phi \in D(L)$. Set $\widetilde{U} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$. Then $\widetilde{U}$ together with $(\widetilde{u}, 0, \widetilde{W})$ is a martingale solution to Equation (2.16) (2.17). This finishes the proof.

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2.10. Application to Volume filling models

In this section, we want to discuss a class of stochastic cross-diffusion equations which covers examples from fluid dynamics of mixtures, electrochemistry, cell biology, and biofilm modeling. In many applications, so-called volume-filling effects need to be taken into account, due to the finite size of the species or components, which means that the unknowns are volume fractions which sum up to one. The dynamics of the concentration (or volume fraction) vector $u = (u_1, \dots, u_n)$ is given by

$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u)\,\mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t>0, \tag{2.93}$$

where $i = 1, \dots, n$ and $\mathcal{O} \subset \mathbb{R}^d$ $(1 \le d)$ is a bounded domain, supplemented with the no-flux boundary and initial conditions,

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ t>0, \quad u_i(0) = u_i^0 \quad \text{in } \mathcal{O},\ i = 1, \dots, n. \tag{2.94}$$

ν denotes again the exterior unit normal vector to $\partial\mathcal{O}$ and u_i^0 is a possibly random initial datum. The concentrations $u_i(\omega, x, t)$ are defined on $\Omega \times \mathcal{O} \times [0, T]$. Together with the solvent concentration u_{n+1}, the concentrations fill up the domain, i.e., $\sum_{i=1}^{n+1} u_i = 1$. We call this assumption *volume filling*. $W = (W_1, \dots, W_n)$ is a U-cylindrical Wiener process.

In contrast to the previous section, the entropy method we employ not only provides a framework for the existence analysis and the proof of positive lower bounds, but also physically relevant upper bounds for the concentrations.

2.10.1. Assumptions and Estimates. In addition to the general Assumptions, we impose a more specific structure on the entropy as well as the matrix appearing in the diffusion operator.

(Vol 1) The set $\mathcal{D}$ is given by $(0,1)^n$. $h\colon \mathcal{D} \to \mathbb{R}$ is C^3, convex and bounded from below.

(Vol 2) h' is invertible as a real function on the whole $\mathbb{R}^n$ and $u(\cdot) = (h')^{-1}(\cdot)\colon \mathbb{R}^n \to \mathcal{D}$ is C^2.

(Vol 3) There exist $c_h > 0$, $0 \le m < 1$ such that for all $u \in \mathcal{D}$, $z \in \mathbb{R}^n$,

$$z^\top h''(u)A(u)z \ge c_h \sum_{i=1}^n \frac{z_i^2}{u_i^{2m}}.$$

In most applications, the entropy density h will be of the form

$$h(u) = \sum_{i=1}^n \left(u_i \log(u_i) - u_i + 1\right) + \left(1 - \sum_{i=1}^n u_i\right)\log\left(1 - \sum_{i=1}^n u_i\right) - \left(1 - \sum_{i=1}^n u_i\right) + 1.$$

THEOREM 2.10.1. *Let Assumptions (Vol 1) (Vol 3), (E4), (A1) (A6), (R1) (R4) be satisfied, then equation 2.93 has a global martingale solution in the sense of Definition 2.1.1.*

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Using Assumptions (Vol 1) (Vol 3), Theorem 2.6.6 and Theorem 2.7.6, we conclude that the regularized equation

$$\mathrm{d}v_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u(R_\varepsilon(v)))u'(R_\varepsilon(v))_j\nabla R_\varepsilon(v)_j\right)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u(R_\varepsilon(v)))\,\mathrm{d}W_j(t), \tag{2.95}$$

with $v_0 = u^0$, has a global solution v^ε, satisfying the following inequality,

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x\\ &+ \sum_{i,j=1}^n\sum_{k=1}^\infty \int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\ &- c_h\int_0^t\int_{\mathcal{O}}\sum_{i=1}^n \frac{(\nabla u(R_\varepsilon(v^\varepsilon)))_i^2}{u(R_\varepsilon(v))_i^{2m}}(s)\,\mathrm{d}x\,\mathrm{d}s\\ &+ C\sum_{k=1}^\infty\int_{\mathcal{O}}\operatorname{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By the chain rule,

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x\\ &+ \sum_{k=1}^\infty\sum_{i,j=1}^n \int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s)\\ &- \frac{c_h}{(1-m)^2}\int_0^t\int_{\mathcal{O}}\sum_{i=1}^n |\nabla(u(R_\varepsilon(v^\varepsilon))_i(s))^{1-m}(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &+ C\sum_{k=1}^\infty\int_0^t\int_{\mathcal{O}}\operatorname{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Taking the supremum with respect to time and the expectation, applying the BDG inequality and Assumption (A6) yields

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \mathbb{E}\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}&\\ + \frac{c_h}{(1-m)^2}\mathbb{E}\int_0^T\int_{\mathcal{O}}\sum_{i=1}^n |\nabla(u(R_\varepsilon(v^\varepsilon))_i(s))^{1-m}(s)|^2\,\mathrm{d}x\,\mathrm{d}s&\\ \le \mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + C_1 + C_2\mathbb{E}\int_0^T\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(s)\,\mathrm{d}x\,\mathrm{d}s.&\end{aligned}$$

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Considering a running supremum in the last double integral, Gronwall's inequality implies,

(2.96)
$$\mathbb{E}\sup_{0\le t\le T}\|u(R_\varepsilon(v^\varepsilon))(t)\|_{L^1(\mathcal{O})}\le C(u^0,T),$$

(2.97)
$$\mathbb{E}\sup_{0\le t\le T}\frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})}\le C(u^0,T),$$

(2.98)
$$\mathbb{E}\int_0^T\int_{\mathcal{O}}\sum_{i=1}^n|\nabla(u(R_\varepsilon(v^\varepsilon))_i(s))^{1-m}|^2\,\mathrm{d}x\,\mathrm{d}s=\|\nabla u(R_\varepsilon(v))^{1-m}\|^2_{L^2(\Omega,L^2(0,T;L^2(\mathcal{O})))}\le C(u^0,T).$$

We recall that these estimates are independent of ε. For notational simplicity, we define

$$u^\varepsilon:=u(R_\varepsilon(v^\varepsilon)).$$

The specific structure of the entropy, imposed in Assumptions (Vol 1) and (Vol 2), implies that $0<u_i^\varepsilon<1$ and therefore results in a uniform bound. This uniform L^∞ bound leads to

$$\mathbb{E}\|u^\varepsilon\|^2_{L^2(0,T;H^1(\mathcal{O}))}\le C(T,u^0), \tag{2.99}$$

where $C(T,u^0)>0$ is independent of ε. Indeed,

$$\begin{aligned}\mathbb{E}\|\nabla u_i^\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))}&=\mathbb{E}\int_0^T\int_{\mathcal{O}}\left|\frac{1}{1-m}(u_i^\varepsilon)^m(t)\nabla(u_i^\varepsilon)^{1-m}(t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\frac{1}{(1-m)^2}\mathbb{E}\int_0^T\int_{\mathcal{O}}|u_i^\varepsilon(t)|^{2m}|\nabla(u_i^\varepsilon)^{1-m}(t)|^2\,\mathrm{d}x\,\mathrm{d}t\\&\le\frac{1}{(1-m)^2}\mathbb{E}\int_0^T\int_{\mathcal{O}}|\nabla(u_i^\varepsilon)^{1-m}(t)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\frac{1}{(1-m)^2}\mathbb{E}\|\nabla(u_i^\varepsilon)^{1-m}\|^2_{L^2(0,T;L^2(\mathcal{O}))}\le C.\end{aligned}$$

Here, we used $|u^\varepsilon(\omega,x,t)|\le 1$ for almost all $(\omega,x,t)\in\Omega\times\mathcal{O}\times[0,T]$ and that $m<1$. A trivial consequence of $|u^\varepsilon(\omega,x,t)|\le 1$ is the estimate

$$\mathbb{E}\|u_i^\varepsilon\|^p_{L^\infty((0,T)\times\mathcal{O})}\le 1, \tag{2.100}$$

for any $p\in[1,\infty)$ and $i=1,\dots,n$.

2.10.2. Tightness of laws. Let v^ε be a solution to (2.95), constructed before. We show that the laws of $u^\varepsilon(=u(R_\varepsilon(v^\varepsilon)))$ are tight in a certain sub-Polish space. Let $d\ge 3$. We introduce the following spaces:

- $C^0([0,T];D(L)')$ equipped with the topology $\mathbb{T}_1$ induced by the norm $\|u\|_{C^0([0,T];D(L)')}=\sup_{t\in(0,T)}\|u(t)\|_{D(L)'}$;
- $L^2_w(0,T;H^1(\mathcal{O}))$ equipped with the weak topology $\mathbb{T}_2$;

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- $L^p(0,T;L^{2d/(d-2)-}(\mathcal{O}))$, with $1 \le p < \infty$, equipped with the topology $\mathbb{T}_3$ induced by the norm $\|\cdot\|_{L^p(0,T;L^{2d/(d-2)-}(\mathcal{O}))}$;
- $C^0([0,T];L^2_w(\mathcal{O}))$, endowed with the weakest topology $\mathbb{T}_4$ such that for all $\psi \in L^2(\mathcal{O})$, the mappings

$$C^0([0,T];L^2_w(\mathcal{O})) \to C^0([0,T];\mathbb{R}), \quad u \mapsto \langle u(\cdot),\psi\rangle,$$

are continuous.

REMARK 2.10.2. *Recall that the notation $L^{2d/(d-2)-}(\mathcal{O})$ indicated that a function was an element of $L^p(0,T;L^{2d/(d-2)-\delta}(\mathcal{O}))$, for every $\delta > 0$. In the current setting we understand, under the assumption that $\delta \ll 1$, we understand by it that fix an arbitrarily small δ and work with the corresponding space.*

We define the space

$$Z_T := C^0([0,T];D(L)') \cap L^2_w(0,T;H^1(\mathcal{O})) \cap L^p(0,T;L^{2d/(d-2)-}(\mathcal{O})) \cap C^0([0,T];L^2_w(\mathcal{O})),$$

where $p < \infty$, endowed with the topology $\mathbb{T}$ that is the maximum of the topologies $\mathbb{T}_i$, $i = 1,2,3,4$, of the corresponding spaces. It is clear that Z_T is a sub-Polish space. Indeed, $C^0([0,T];D(L)')$, $L^2_w(0,T;H^1(\mathcal{O}))$ and $L^p(0,T;L^{2d/(d-2)-}(\mathcal{O}))$ are separable and metrizable for $p > 1$. It remains to consider the space $C^0([0,T];L^2_w(\mathcal{O}))$. Let $(h_m)_{m\in\mathbb{N}}$ be a dense subset of $L^2(0,T;L^2(\mathcal{O}))$ and let $\mathbb{Q}_T$ be the set of rational numbers from the interval $[0,T]$. Then the family $\{f_{m,t} : m \in \mathbb{N}, t \in \mathbb{Q}_T\}$, defined by

$$f_{m,t}(u) = \langle u(t), h_m\rangle_{L^2(\mathcal{O})} \in \mathbb{R}, \qquad \text{where } u \in C^0\left([0,T];L^2_w(\mathcal{O})\right), m \in \mathbb{N}, t \in \mathbb{Q}_T$$

consists of continuous functions separating points in $C^0([0,T];L^2_w(\mathcal{O}))$.

LEMMA 2.10.3. *The set of laws $(\mathcal{L}(u^\varepsilon))_\varepsilon$ is tight in Z_T.*

REMARK 2.10.4. *In caparison to the previous section, we will use the Aldous condition instead of fractional Sobolev estimates. The reason for this is, that we won't require higher moments of the gradients ∇u^ε and the initial condition. Imposing additional assumptions on $\mathbb{E}\big(\int_{\mathcal{O}} h(u(0))\,\mathrm{d}x\big)^p$ would enable us to perform the same estimates as in the previous sections and obtain that $\mathbb{E}\|u^\varepsilon\|^p_{C^\beta([0,T];D(L)')} \le C$ for $\beta < \frac{1}{2}$.*

PROOF. The idea is to apply Lemma 4.0.19 with the spaces $Y = D(L)'$, $\widetilde{Y} = L^2(\mathcal{O})$, $X = H^1(\mathcal{O})$ and $B = L^{2d/(d-2)-}(\mathcal{O})$. Estimates (2.100) and (2.99) imply that conditions (3) and (1) are satisfied. Condition (2) is a consequence of the boundedness. It remains to show that $(u^\varepsilon)_\varepsilon$ satisfies (4), namely the Aldous condition in $D(L)'$. We need to show that for any $\delta > 0$ and $\kappa > 0$, there exists $\theta_0 > 0$ such that for any sequence $(\tau_\varepsilon)_{\varepsilon>0}$ of $\mathbb{F}$-stopping times, it holds that

$$\sup_{\varepsilon>0}\sup_{0<\theta<\theta_0} \mathbb{P}\big\{\|u^\varepsilon(\tau_\varepsilon+\theta) - u^\varepsilon(\tau_\varepsilon)\|_{D(L)'} \ge \kappa\big\} \le \delta.$$

At this point, we need to admit an abuse of notation. In the following, it suffices to work on a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$, which will not be renamed, instead of the full ε-net.

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Let $(\tau_\varepsilon)_{\varepsilon>0}$ be a sequence of $\mathbb{F}$-stopping times such that $0 \le \tau_\varepsilon \le T$ and let $t \in [0,T]$. The solution v^ε to Equation (2.95) satisfies

$$v(t) = v_0 + \int_0^t \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right)(s)\,\mathrm{d}s + \int_0^t \sigma(u(R_\varepsilon(v)))(s)\,\mathrm{d}W(s) \tag{2.101}$$

$$=: J_1^\varepsilon + J_2^\varepsilon(t) + J_3^\varepsilon(t). \tag{2.102}$$

First we the term involving the diffusion operator. Let $\theta > 0$. We use Assumption (A3), the continuous embedding $D(L) \hookrightarrow W^{1,\infty}(\mathcal{O})$, and estimates (2.100) (2.99) to find that

$$\begin{aligned}
&\mathbb{E}\left\| \int_{\tau_\varepsilon}^{\tau_\varepsilon+\theta} \operatorname{div}\left(A(u^\varepsilon)\nabla u^\varepsilon\right)(s)\,\mathrm{d}s\right\|_{D(L)'} \\
&\quad \le C\mathbb{E}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\theta} \left(1+\|u^\varepsilon(s)\|_{L^2(\mathcal{O})}\right)\|\nabla u^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s \\
&\quad \le C\theta^{1/2}\mathbb{E}\left((1+\|u^\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))})\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}\right) \\
&\quad \le C\theta^{1/2}\left\{1+\mathbb{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|_{L^2(\mathcal{O})}^2\right)\right\}^{1/2}\left\{\mathbb{E}\int_0^T \|\nabla u^\varepsilon(s)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s\right\}^{1/2} \\
&\quad \le C(v_0)\theta^{1/2},
\end{aligned}$$

where we applied first the Cauchy Schwarz inequality with respect to time and then with respect to ω.
For the stochastic integral, we take into account Assumption (A5) and the Itô isometry,

$$\begin{aligned}
&\mathbb{E}\left\| \int_{\tau_\varepsilon}^{\tau_\varepsilon+\theta} \sigma_{ij}(u^\varepsilon(s))\,\mathrm{d}W_j(s)\right\|_{D(L)'}^2 \\
&\quad \le \mathbb{E}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\theta} \|\sigma(u^\varepsilon(s))\|_{\mathcal{L}_2^{n\times n}(U;L^2(\mathcal{O}))}^2\,\mathrm{d}s \\
&\quad \le C\mathbb{E}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\theta} 1+\|u^\varepsilon(s)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s \le C(v_0)\theta.
\end{aligned}$$

Let $\kappa > 0$ and $\delta > 0$. In view of the previous estimates and using the Chebyshev inequality, it follows for $i = 2, 3$ that

$$\begin{aligned}
\mathbb{P}\left\{\|J_i^\varepsilon(\tau_\varepsilon+\theta) - J_i^\varepsilon(\tau_\varepsilon)\|_{D(L)'} \ge \kappa\right\} &\le \frac{1}{\kappa}\mathbb{E}\left\|J_i^\varepsilon(\tau_\varepsilon+\theta) - J_i^\varepsilon(\tau_\varepsilon)\right\|_{D(L)'} \\
&\le \frac{C\theta^{1/2}}{\kappa}.
\end{aligned}$$

Thus, choosing $\theta_0 = \min\{1, (\kappa\delta/C)^2\}$, we infer that for $i = 2, 3$,

$$\sup_{\varepsilon>0}\sup_{0<\theta<\theta_0} \mathbb{P}\left\{\|J_i^\varepsilon(\tau_\varepsilon+\theta) - J_i^\varepsilon(\tau_\varepsilon)\|_{D(L)'} \ge \kappa\right\} \le \delta.$$

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This shows that the Aldous condition holds for all three terms J_i^ε $(i = 2, 3)$ and consequently, in view of (2.102), also for $(v^\varepsilon)_\varepsilon$. To obtain this result for the family $(u^\varepsilon)_\varepsilon$, we recall the a.s. identity

$$v^\varepsilon = u(R_\varepsilon)(v^\varepsilon)) + \varepsilon L^* L R_\varepsilon(v^\varepsilon).$$

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon(\tau_N+\theta) - u^\varepsilon(\tau_N)\|_{D(L)'} &= \mathbb{E}\|v^\varepsilon(\tau_N+\theta) - \varepsilon L^* L R_\varepsilon(v^\varepsilon)(\tau_N+\theta) - v^\varepsilon(\tau_N) + \varepsilon L^* L R_\varepsilon(v^\varepsilon)(\tau_N)\|_{D(L)'} \\
&\le \mathbb{E}\|v^\varepsilon(\tau_N+\theta) - v^\varepsilon(\tau_N)\|_{D(L)'} + \varepsilon\mathbb{E}\|L^* L R_\varepsilon(v^\varepsilon)(\tau_N+\theta) - L^* L R_\varepsilon(v^\varepsilon)(\tau_N)\|_{D(L)'} \\
&\le C(u^0)\theta^{1/2} + \varepsilon C\mathbb{E}\|R_\varepsilon(v^\varepsilon)(\tau_N+\theta) - R_\varepsilon(v^\varepsilon)(\tau_N)\|_{D(L)} \\
&\le C(u^0)\theta^{1/2} + \varepsilon\frac{C}{\varepsilon}\mathbb{E}\|v^\varepsilon(\tau_N+\theta) - v^\varepsilon(\tau_N)\|_{D(L)'} \\
&\le C(u^0)\theta^{1/2}.
\end{aligned}$$

Thus, by Lemma 4.0.19, the family of laws of $(u^\varepsilon)_\varepsilon$ is tight in Z_T. □

LEMMA 2.10.5. *The set of laws* $(\mathcal{L}(\sqrt{\varepsilon}L^* L R_\varepsilon(v^\varepsilon)))_\varepsilon$ *is tight in* $Y_T := L^2_w(0,T;D(L)') \cap L^\infty_{w^*}(0,T;D(L)')$.

PROOF. For the first part of the claim, we consider the closed ball

$$\overline{B_R^{L^2_t D(L)'_x}} = \left\{ f \in : \|f\|_{L^2(0,T;D(L)')} \le R \right\}.$$

By Estimate (2.97) and Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left(\omega\colon \|\sqrt{\varepsilon}L^* L R_\varepsilon(v^\varepsilon)\|_{L^2(0,T;D(L)')} > R\right) &\le \frac{\mathbb{E}\|\sqrt{\varepsilon}L^* L R_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;D(L)')}}{R^2} \\
&= \frac{\mathbb{E}\varepsilon\|L^* L R_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;D(L)')}}{R^2} \\
&\le C\frac{\mathbb{E}\varepsilon\|L R_\varepsilon(v^\varepsilon)\|^2_{L^2(0,T;L^2(\mathcal{O}))}}{R^2} \le \frac{C}{R^2}.
\end{aligned}$$

Since $L^2(0,T;H^1(\mathcal{O}))$ is reflexive, it's closed norm balls are weakly compact by the Banach-Alaoglu Theorem. Hence, the sequence $(\sqrt{\varepsilon}L^* L R_\varepsilon(v))_\varepsilon$ is tight in $L^2_w(0,T;D(L)')$.
The second part follows from an identical procedure, where $L^2(\mathcal{O})$ is replaced by $L^\infty(\mathcal{O})$. □

REMARK 2.10.6. *The cases* $d = 1, 2$ *can be treated analogously, with the difference that one obtains tightness in* $L^p(0,T;C(\bar{\mathcal{O}}))$ *and* $L^p(0,T;L^q(\mathcal{O}))$ *for any* $1 \le p, q < \infty$. *These cases however are less relevant in light of the physical applications, which is why we focus on the case* $d \ge 3$.

2.10.3. Convergence. We consider the space $Z_T \times Y_T \times C^0([0,T];\hat{U})$, equipped with the probability measure $\mu^\varepsilon := \mu^\varepsilon_u \times \mu^\varepsilon_w \times \mu^\varepsilon_W$, with

$$\mu^\varepsilon_u(\cdot) = \mathbb{P}(u^\varepsilon \in \cdot) \in \Pr\left(C^0([0,T];D(L)') \cap L^2_w(0,T;H^1(\mathcal{O})) \cap L^p(0,T;L^{2d/(d-2)-}(\mathcal{O})) \cap C^0([0,T];L^2_w(\mathcal{O}))\right),$$
$$\mu^\varepsilon_w(\cdot) = \mathbb{P}(\sqrt{\varepsilon}L^* L R_\varepsilon(v^\varepsilon) \in \cdot) \in \Pr\left(L^2_w(0,T;D(L)') \cap L^\infty_{w^*}(0,T;D(L)')\right),$$

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$\mu_W^\varepsilon = \mathbb{P}(W \in \cdot) \in \Pr\big(C([0,T];\hat{U})\big)$.

$Z_T \times Y_T \times C^0([0,T];\hat{U})$ satisfies the assumptions of the Skorokhod Jakubowski theorem [28, Theorem C1] and the measures μ^ε are tight since set of laws $((\mathcal{L}(u^\varepsilon)), (\mathcal{L}(\sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon))), (\mathcal{L}(W))$ are tight in $(Z_T, \mathbb{T}_Z), (Y_T, \mathbb{T}_Y)$ and $C([0,T];\hat{U})$ respectively. The theorem implies the existence of a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and, on this space, $(Z_T \times Y_T \times C^0([0,T];\hat{U}))$-valued random variables $(\widetilde{u}, \widetilde{w}, \widetilde{W})$ and $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ such that $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$ has the same law as $(u^\varepsilon, \sqrt{\varepsilon}L^*LR_\varepsilon(v^\varepsilon), W)$ on $\mathcal{B}(Z_T \times Y_T \times C^0([0,T];\hat{U}))$ and, as $\varepsilon \to 0$,

$$(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon) \to (\widetilde{u}, \widetilde{w}, \widetilde{W}) \quad \text{in } Z_T \times Y_T \times C^0([0,T];\hat{U}) \ \widetilde{\mathbb{P}}\text{-a.s.}$$

By the definition of the space Z_T, Y_T, this convergence means $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
\widetilde{u}^\varepsilon &\to \widetilde{u} && \text{in } C^0([0,T];D(L)'),\\
\widetilde{u}^\varepsilon &\rightharpoonup \widetilde{u} && \text{weakly in } L^2(0,T;H^1(\mathcal{O})),\\
\widetilde{u}^\varepsilon &\to \widetilde{u} && \text{in } L^2(0,T;L^{2d/(d-2)-}(\mathcal{O})),\\
\widetilde{u}^\varepsilon &\to \widetilde{u} && \text{in } C^0([0,T];L^2_w(\mathcal{O})),\\
\widetilde{w}^\varepsilon &\rightharpoonup \widetilde{w} && \text{weakly in } L^2(0,T;D(L)'),\\
\widetilde{w}^\varepsilon &\rightharpoonup \widetilde{w} && \text{weakly* in } L^\infty(0,T;D(L)'),\\
\widetilde{W}^\varepsilon &\to \widetilde{W} && \text{in } C^0([0,T];\hat{U}).
\end{aligned} \tag{2.103}$$

We derive some regularity properties for the limit $\widetilde{u}$. Note that $\widetilde{u}$ is a Z_T-Borel random variable since $\mathcal{B}(Z_T \times Y_T \times C^0([0,T];\hat{U}))$ is a subset of $\mathcal{B}(Z_T) \times \mathcal{B}(Y_T) \times \mathcal{B}(C^0([0,T];\hat{U}))$. We deduce from estimates (2.30) and (2.31) and the fact that u^ε and $\widetilde{u}^\varepsilon$ have the same laws that,

$$\sup_\varepsilon \widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}^\varepsilon(t)\|^2_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg) + \sup_\varepsilon \widetilde{\mathbb{E}}\bigg(\sup_{0\le t\le T}\|\widetilde{u}^\varepsilon(t)\|_{D(L)'}\,\mathrm{d}t\bigg) \le C.$$

In view of the embedding $L^1(\mathcal{O}) \hookrightarrow D(L)'$, we infer the existence of a further subsequence of $(\widetilde{u}^\varepsilon)$ (not relabeled) that is weakly converging in $L^p(\widetilde{\Omega}; L^2(0,T;H^1(\mathcal{O})))$ and $L^p(\widetilde{\Omega}; C(0,T;D(L)'))$ as $\varepsilon \to 0$. In view of the convergence of $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we infer that the limit function satisfies

$$\widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}(t)\|^2_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg) + \widetilde{\mathbb{E}}\bigg(\sup_{0\le t\le T}\|\widetilde{u}(t)\|_{D(L)'}\bigg) < \infty.$$

Let $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{F}}^{(\varepsilon)}$ be the filtrations generated by $(\widetilde{u}, \widetilde{w}, \widetilde{W})$ and $(\widetilde{u}^\varepsilon, \widetilde{w}^\varepsilon, \widetilde{W}^\varepsilon)$, respectively. In order to verify that these new random variables induce actual processes, one can follow the arguments in [29, Proposition B4]. The progressive measurability of $\widetilde{u}^\varepsilon$ is a consequence of [25, Appendix B]. Let $\widetilde{B}_j^{\varepsilon,k}(t) := \langle \widetilde{W}_j^\varepsilon(t), e_k\rangle$. It is not too difficult to see, that $\widetilde{B}_j^{\varepsilon,k}$, $k \in \mathbb{N}$, are independent, standard, real-valued $\widetilde{\mathcal{F}}_t$-Wiener Processes. The adeptness is a direct consequence of their definition and the independence, as well as the independence of

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the increments $\widetilde{W}^{\varepsilon,k}(t)-\widetilde{W}^{\varepsilon,k}(s)$ with respect to $\widetilde{\mathcal{F}}_s$, is inherited from $B_j^{\varepsilon,k}(\cdot) := \langle W(\cdot), e_k\rangle$. Passing to the limit $\varepsilon \to 0$ in the characteristic function, which is done by the dominated convergence theorem, shows that also $\widetilde{W}^k$ are $\widetilde{\mathcal{F}}_t$-martingales with the correct marginal distributions. An application of Lévy's characterization theorem yields that $\widetilde{W}^k$ is indeed a Wiener process. We can conclude that $\widetilde{W}$ is indeed a cylindrical Wiener process.

Since the upper and lower bound of the solution are , due to the applications, of central importance to us, we verify that this property is still preserved.

LEMMA 2.10.7. *The limit $\widetilde{u}$ satisfies $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O}\times(0,T)$ $\widetilde{\mathbb{P}}$-a.s.*

PROOF. By Assumptions (Vol 1), (Vol 2), $u^\varepsilon(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O}\times(0,T)$ $\mathbb{P}$-a.s. In particular,

$$\|u^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} := \sum_{i=1}^n \|u_i^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1. \tag{2.104}$$

The set $L^\infty(0,T;L^\infty(\mathcal{O}))$ is continuously embedded in $L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$. Thus, by the Kuratowski theorem (see Theorem 5.7.2), $L^\infty(0,T;L^\infty(\mathcal{O}))$ is a Borel set of $L^p(0,T;L^{2d/(d-2)-}(\mathcal{O}))$. Therefore, by [24, Lemma B.1] (see also Lemma 5.7.3), the set $L^\infty(0,T;\,L^\infty(\mathcal{O}))\cap Z_T$ is a Borel subset of $L^p(0,T;L^{2d/(d-2)-}(\mathcal{O}))\cap Z_T$ which in turn is Z_T. The equivalence of the laws of $\widetilde{u}^\varepsilon$ and u^ε on $\mathcal{B}(Z_T)$ as well as (2.104) then show that

$$\widetilde{\mathbb{P}}\big\{\|\widetilde{u}^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\} = \mathbb{P}\big\{\|u^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\} = 1.$$

By the definition of the norm in (2.104), this means that

$$\sum_{i=1}^n |\widetilde{u}_i^\varepsilon(x,t)| \le 1 \quad \text{for a.e. } (x,t)\in\mathcal{O}\times(0,T)\ \widetilde{\mathbb{P}}\text{-a.s.} \tag{2.105}$$

Next, we show that $\widetilde{u}_i^\varepsilon(x,t) \ge 0$ for a.e. $(x,t)\in\mathcal{O}\times(0,T)$ $\widetilde{\mathbb{P}}$-a.s. Let $v \in L^\infty(0,T;L^\infty(\mathcal{O}))$ and define the closed unit ball with center v by

$$B(v) = \big\{u \in L^\infty(0,T;L^\infty(\mathcal{O})) : \|u-v\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\}.$$

We deduce from (2.105) that

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^\varepsilon \in B(0)\big) = 1, \quad i=1,\ldots,n.$$

Since $0 \le u_i^\varepsilon(x,t) \le 1$ a.e. in $\mathcal{O}\times(0,T)$ $\mathbb{P}$-a.s., we have $\|u_i^\varepsilon - 1\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1$ for all $i=1,\ldots,n$ and consequently, by the equivalence of the laws,

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^\varepsilon \in B(1)\big) = \mathbb{P}\big(u_i^\varepsilon \in B(1)\big) = 1.$$

We infer that

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^\varepsilon \in B(1)\cap B(0)\big) = 1,$$

and this implies that $0 \le \widetilde{u}_i^\varepsilon(x,t) \le 1$ $\widetilde{\mathbb{P}}$-a.s. and, taking into account (2.105), $\sum_{i=1}^n \widetilde{u}_i^\varepsilon(x,t) \le 1$, i.e. $\widetilde{u}^\varepsilon(x,t) \in \overline{\mathcal{D}}$ $\widetilde{\mathbb{P}}$-a.s. Moreover, from (2.103) we know that $\widetilde{u}^\varepsilon$ converges to $\widetilde{u}$

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strongly in $L^2(0,T;L^{2d/(d-2)-}(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and thus we conclude that $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s. □

REMARK 2.10.8. *The following convergence results are very similar to the ones presented in previous sections. They are presented again for completeness and can be skipped.*

LEMMA 2.10.9. *It holds for all s, $t \in [0,T]$ with $s \le t$, $i = 1,\dots,n$, and all $\phi_1 \in L^2(\mathcal{O})$, $\phi_2 \in D(L)$ satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \widetilde{u}_i^\varepsilon(t) - \widetilde{u}_i(t), \phi_1 \right\rangle_{L^2} \mathrm{d}t = 0, \tag{2.106}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \left\langle \widetilde{u}_i^\varepsilon(0) - \widetilde{u}_i(0), \phi_2 \right\rangle_{D(L)'\times D(L)} = 0, \tag{2.107}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}t = 0, \tag{2.108}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(0), \phi_2 \right\rangle_{D(L)'\times D(L)} = 0, \tag{2.109}$$

(2.110)

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}s \right| \mathrm{d}t = 0,$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}} \int_0^T \left| \sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\mathrm{d}\widetilde{W}_j^\varepsilon(s) - \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s), \phi_1 \right\rangle_{L^2} \right|^2 \mathrm{d}t = 0. \tag{2.111}$$

PROOF. We know that $\widetilde{u}^\varepsilon \to \widetilde{u}$ in Z_T, $\widetilde{\mathbb{P}}$-a.s. In particular, $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2\left([0,T];L^{2d/(d-2)-}(\mathcal{O})\right)$ $\widetilde{\mathbb{P}}$-a.s. We conclude that

$$\lim_{\varepsilon\to 0} \int_0^T \langle \widetilde{u}^\varepsilon(t) - \widetilde{u}(t), \phi_1 \rangle_{L^2}^2 \mathrm{d}t = 0 \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

By Estimate (2.100), there exists an $r > 1$, such that

$$\widetilde{\mathbb{E}} \left(\left| \int_0^T \|\widetilde{u}^\varepsilon(t) - \widetilde{u}(t)\|_{L^2(\mathcal{O})}^2 \mathrm{d}t \right|^r \right) \le C.$$

This bound provides the equi-integrability of $\int_0^T \langle \widetilde{u}^\varepsilon(t) - \widetilde{u}(t), \phi_1 \rangle_{L^2(\mathcal{O})}^2 \mathrm{d}t$. Vitali's convergence theorem then shows that (2.106) holds.

From Lemma 2.10.3 follows $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $C^0\left([0,T];D(L)'\right)$ $\widetilde{\mathbb{P}}$-a.s. and $\widetilde{u}$ is continuous at $t = 0$, we infer that for any ϕ_2,

$$\lim_{\varepsilon\to 0} \langle \widetilde{u}^\varepsilon(0), \phi_2 \rangle_{D(L)'\times D(L)} = \langle \widetilde{u}(0), \phi_2 \rangle_{D(L)'\times D(L)} \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

The claimed convergence in (2.107) follows from Assumption (A2) and Vitali's theorem. (2.108) is a direct consequence of the estimate

$$\widetilde{\mathbb{E}} \int_0^T \left\langle \sqrt{\varepsilon}\widetilde{w}_i^\varepsilon(t), \phi_2 \right\rangle_{D(L)'\times D(L)} \mathrm{d}t \le \widetilde{\mathbb{E}} \int_0^T \sqrt{\varepsilon}\|\widetilde{w}_i^\varepsilon(t)\|_{D(L)'}\|\phi_2\|_{D(L)} \mathrm{d}t \le \sqrt{\varepsilon}C(\phi_2, u_0, p).$$

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Next, we establish convergence (2.110).

$$\begin{aligned}
&\left|\int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2\right\rangle_{L^2} \mathrm{d}s\right| \\
&\quad\le \int_0^t \|A_{ij}(\widetilde{u}^\varepsilon(s)) - A_{ij}(\widetilde{u}(s))\|_{L^2(\mathcal{O})}\|\nabla\widetilde{u}_j^\varepsilon(s)\|_{L^2(\mathcal{O})}\|\nabla\phi_2\|_{L^\infty(\mathcal{O})}\,\mathrm{d}s \\
&\qquad + \left|\int_0^t A_{ij}(\widetilde{u}(s))\nabla(\widetilde{u}^\varepsilon(s) - \widetilde{u}(s))\cdot\nabla\phi_2\,\mathrm{d}s\right| \\
&\quad\le \|A_{ij}(\widetilde{u}^\varepsilon(s)) - A_{ij}(\widetilde{u}(s))\|^2_{L^2(0,T;L^2(\mathcal{O}))}\|\nabla\widetilde{u}_j^\varepsilon(s)\|^2_{L^2(0,T;L^2(\mathcal{O}))}\|\nabla\phi_2\|_{L^\infty(\mathcal{O})} \\
&\qquad + \left|\int_0^t A_{ij}(\widetilde{u}(s))\nabla(\widetilde{u}^\varepsilon(s) - \widetilde{u}(s))\cdot\nabla\phi_2\,\mathrm{d}s\right| =: I_1 + I_2.
\end{aligned}$$

By the Lipschitz continuity of A and the uniform bound of $\nabla\widetilde{u}^\varepsilon$, $I_1\to 0$ $\widetilde{\mathbb{P}}$-a.s. For the second integral, we observe that $A_{i,j}(\widetilde{u})\nabla\phi_2\in L^2(0,T;L^2(\mathcal{O}))$ and $\nabla\widetilde{u}_j^\varepsilon\rightharpoonup\nabla\widetilde{u}_j$ weakly in $L^2(0,T;L^2(\mathcal{O}))$. Hence also $I_2\to 0$ $\widetilde{\mathbb{P}}$-a.s. This shows that

$$\lim_{\varepsilon\to 0}\int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s = \int_0^t \left\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

for all $\phi_2\in W^{1,\infty}(\mathcal{O})$ satisfying $\nabla\phi_2\cdot\nu = 0$ on $\partial\mathcal{O}$. Using the embedding $D(L)\hookrightarrow W^{1,\infty}(\mathcal{O})$, we compute

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s\right|^r \\
&\quad\le \|\nabla\phi_2\|^r_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\bigg(\int_0^t\bigg\|\sum_{j=1}^n A_{ij}(\widetilde{u}^\varepsilon)\nabla\widetilde{u}_j^\varepsilon(s)\bigg\|_{L^1(\mathcal{O})}\mathrm{d}s\bigg)^r \\
&\quad\le C\|\nabla\phi_2\|^r_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\bigg(\int_0^t\big(1+\|\widetilde{u}^\varepsilon(s)\|_{L^2(\mathcal{O})}\big)\|\nabla\widetilde{u}_j^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg)^r \\
&\quad\le C\|\nabla\phi_2\|^r_{L^\infty(\mathcal{O})}\widetilde{\mathbb{E}}\bigg[\Big(1+\|\widetilde{u}^\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))}\Big)\int_0^t\|\nabla\widetilde{u}_j^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg]^r \\
&\quad\le C\|\nabla\phi_2\|^r_{L^\infty(\mathcal{O})}\bigg[\widetilde{\mathbb{E}}\Big(1+\|\widetilde{u}^\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))}\Big)^{pr}\bigg]^{1/p}\bigg[\mathbb{E}\bigg(\int_0^t\|\nabla\widetilde{u}_j^\varepsilon(s)\|_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg)^{qr}\bigg]^{1/q},
\end{aligned}$$

where $1/p+1/q=1$. Choosing $r,q>1$, such that $qr=2$, we infer

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t \left\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla\widetilde{u}_j^\varepsilon(s),\nabla\phi_2\right\rangle_{L^2}\mathrm{d}s\right|^r \\
&\quad\le C(t)\|\nabla\phi_2\|^r_{L^\infty(\mathcal{O})}\bigg[\widetilde{\mathbb{E}}\Big(1+\|\widetilde{u}^\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))}\Big)^{pr}\bigg]^{1/p}\bigg[\mathbb{E}\int_0^t\|\nabla\widetilde{u}_j^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg]^{1/q} \\
&\quad\le C(t).
\end{aligned}$$

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This bound and the $\widetilde{\mathbb{P}}$-a.s. convergence allow us to apply the Vitali convergence theorem to infer that (2.110) holds.

It remains to prove convergence (2.111). By Lemma 5.3.57, since $\widetilde{W}^\varepsilon \to \widetilde{W}$ in $C^0\left([0,T];\hat{U}\right)$, $\widetilde{\mathbb{P}}$ -a.s., it is sufficient to show that $\sigma_{ij}\left(\widetilde{u}^\varepsilon\right) \to \sigma_{ij}(\widetilde{u})$ in $L^2\left(0,T;\mathcal{L}_2\left(U;L^2(\mathcal{O})\right)\right)\mathbb{P}-$ a.s. Indeed, by Assumption (A5),

$$\int_0^t \left\|\sigma_{ij}\left(\widetilde{u}^\varepsilon(s)\right)-\sigma_{ij}(\widetilde{u}(s))\right\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s \le C_\sigma \left\|\widetilde{u}^\varepsilon-\widetilde{u}\right\|^2_{L^2(0,T;L^2(\mathcal{O}))}.$$

Since $\widetilde{u}^\varepsilon \to \widetilde{u}$ in $L^2\left(0,T;L^2(\mathcal{O})\right)\tilde{\mathbb{P}}$ -a.s., we infer that for $t\in[0,T], \omega\in\widetilde{\Omega}$, the right hand side tends to 0. By Lemma 5.3.57, we conclude that $\int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s) \to \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s)$, for $j=1,\dots,n$, in probability in $L^2(0,T;L^2(\mathcal{O}))$. Furthermore,

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))-\sigma_{ij}(\widetilde{u}(s))\right\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s\right|^p\\
&\le C\widetilde{\mathbb{E}}\left(\int_0^t\left(\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}+\|\sigma_{ij}(\widetilde{u}(s)\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\right)\mathrm{d}s\right)^p\\
&\le C+C\widetilde{\mathbb{E}}\left(\int_0^t\left(\|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}+\|\widetilde{u}(s)\|^2_{L^2(\mathcal{O})}\right)\mathrm{d}s\right)^p.
\end{aligned}$$

In view of Vitali's convergence theorem, we deduce from this bound and the previous convergence that

$$\lim_{\varepsilon\to0}\widetilde{\mathbb{E}}\int_0^t \left\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))-\sigma_{ij}(\widetilde{u}(s))\right\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\mathrm{d}s=0.$$

We can estimate

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\left\langle\int_0^t\sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s)-\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle_{L^2}\right|^2\\
&\le C\|\phi_1\|^2_{L^2(\mathcal{O})}\widetilde{\mathbb{E}}\int_0^t\|\sigma_{ij}(\widetilde{u}^\varepsilon(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}+\|\sigma_{ij}(\widetilde{u}(s))\|^2_{\mathcal{L}_2(U;L^2(\mathcal{O}))}\,\mathrm{d}s\\
&\le C\|\phi_1\|^2_{L^2(\mathcal{O})}+C\|\phi_1\|^2_{L^2(\mathcal{O})}\widetilde{\mathbb{E}}\left(\int_0^t\|\widetilde{u}^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s+\int_0^t\|\widetilde{u}(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right)\le C.
\end{aligned}$$

This bound allow us to apply the dominated convergence theorem to conclude that for any $\phi_1\in L^2(\mathcal{O})$,

$$\lim_{\varepsilon\to0}\widetilde{\mathbb{E}}\int_0^T\left|\left\langle\int_0^t\sigma_{ij}(\widetilde{u}^\varepsilon(s))\,\mathrm{d}\widetilde{W}^\varepsilon_j(s)-\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle_{L^2}\right|^2\mathrm{d}t=0.$$

This shows (2.111) and finishes the proof. □

Let $\phi\in D(L)$. In order to show that the limit is indeed a solution to the equation, we define, for $t\in[0,T]$ and $i=1,\dots,n$,

$$\Lambda^\varepsilon_i\left(\widetilde{u}^\varepsilon,\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon,\phi\right)(t) := \left\langle\widetilde{u}_i(0),\phi\right\rangle_{L^2} \tag{2.112}$$

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$$- \int_0^t \sum_{j=1}^n \big\langle A_{ij}(\widetilde{u}^\varepsilon(s))\nabla \widetilde{u}_j^\varepsilon(s), \nabla\phi\big\rangle_{L^2}\,\mathrm{d}s + \sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}^\varepsilon(s))\mathrm{d}\widetilde{W}_j^\varepsilon(s), \phi\right\rangle_{L^2},$$

$$\Lambda_i\big(\widetilde{u},\widetilde{w},\widetilde{W},\phi\big)(t) := \langle \widetilde{u}_i(0),\phi\rangle_{L^2} - \sum_{j=1}^n \int_0^t \big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi\big\rangle_{L^2}\,\mathrm{d}s + \sum_{j=1}^n \left\langle \int_0^t \sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi\right\rangle_{L^2}. \tag{2.113}$$

The following corollary is essentially a consequence of Lemma 2.10.9.

COROLLARY 2.10.10. *It holds for any $\phi_1 \in L^2(\mathcal{O})$ and any $\phi_2 \in D(L)$ satisfying $\nabla\phi_2\cdot\nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{\varepsilon\to 0}\big\|\langle\widetilde{u}_i^\varepsilon,\phi_1\rangle_{L^2} - \langle\widetilde{u}_i,\phi_1\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{\varepsilon\to 0}\big\|\langle\widetilde{v}_i^\varepsilon,\phi_2\rangle_{D(L)'\times D(L)} - \langle\widetilde{u}_i,\phi_2\rangle_{L^2}\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$
$$\lim_{\varepsilon\to 0}\big\|\Lambda_i^\varepsilon\big(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon,\phi_2\big) - \Lambda_i\big(\widetilde{u},0,\widetilde{W},\phi_2\big)\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

Since v^ε is a strong solution of (2.95), and representation (2.8) holds, it satisfies

$$\langle v_i^\varepsilon(t),\phi\rangle_{D(L)'\times D(L)} = \Lambda_i^\varepsilon(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v^\varepsilon),W,\phi)(t) \quad \text{for a.e. } t\in[0,T]\ \mathbb{P}\text{-a.s., } i=1,\dots,n,$$

for any $\phi\in D(L)$. Hence,

$$\int_0^T \mathbb{E}\big|\langle v_i^\varepsilon(t),\phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v),W,\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

We deduce from the equivalence of the laws of $(u^\varepsilon,\varepsilon L^*LR_\varepsilon(v),W)$ and $(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon)$ that

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle\widetilde{u}_i^\varepsilon(t)+\sqrt{\varepsilon}\widetilde{w}_i^\varepsilon,\phi\rangle_{D(L)'\times D(L)} - \Lambda_i^\varepsilon(\widetilde{u}^\varepsilon,\sqrt{\varepsilon}\widetilde{w}^\varepsilon,\widetilde{W}^\varepsilon,\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

By Corollary 2.10.10, we can pass to the limit $\varepsilon\to 0$ to obtain

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle\widetilde{u}_i(t),\phi\rangle_{L^2} - \Lambda_i(\widetilde{u},0,\widetilde{W},\phi)(t)\big|\,\mathrm{d}t = 0, \quad i=1,\dots,n.$$

This identity holds for all $\phi\in D(L)$ such that $\nabla\phi\cdot\nu = 0$ on $\partial\mathcal{O}$ and hence, by density, for any $\phi\in D(L)$. This shows that

$$\big|\langle\widetilde{u}_i(t),\phi\rangle_{L^2} - \Lambda_i(\widetilde{u},0,\widetilde{W},\phi)(t)\big| = 0 \quad \text{for a.e. } t\in[0,T]\ \widetilde{\mathbb{P}}\text{-a.s., } i=1,\dots,n.$$

We infer from the definition of Λ_i that

$$\langle\widetilde{u}_i(t),\phi\rangle_{L^2} = \langle\widetilde{u}_i(0),\phi\rangle_{L^2} - \sum_{j=1}^n\int_0^t \big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi\big\rangle_{L^2}\,\mathrm{d}s + \sum_{j=1}^n\left\langle\int_0^t\sigma_{ij}(\widetilde{u}(s))\,\mathrm{d}\widetilde{W}_j(s),\phi\right\rangle_{L^2}$$

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for a.e. $t \in [0,T]$ $\widetilde{\mathbb{P}}$-a.s. and all $\phi \in D(L)$. Set $\widetilde{u} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$. Then $(\widetilde{u}, 0, \widetilde{W})$ is a martingale solution to Equation (2.93) (2.94). This finishes the proof.

2.10.4. Examples. Finally, we present two examples of "Volume filling models".

2.10.5. Maxwell–Stefan systems. Maxwell Stefan equations describe the dynamics of fluid mixtures in the diffusion regime. Applications include membrane electrolysis processes [70], ion transport through nanopores [15], and dynamics of lithium-ion batteries [101]. Here, we consider an uncharged three-species mixture with the concentrations u_1, u_2 and the solvent concentration $u_3 = 1 - u_1 - u_2$. The diffusion matrix is given by

$$A(u) = \frac{1}{a(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix},$$
$$\text{where } a(u) = d_0 d_1 u_1 + d_0 d_2 u_2 + d_1 d_2 u_3,$$

and $d_i > 0$ for $i = 0, 1, 2$ are diffusion coefficients [78, Section 4.1]. The matrix $A(u)$ is Lipschitz continuous on $\mathcal{D}$ since $a(u)$ is strictly positive and bounded from above. The same argument also applies to the linear growth condition.
The entropy density is given by

$$h(u) = \sum_{i=1}^{3} \big(u_i(\log(u_i) - 1) + 1\big).$$

Its derivative $w = h'(u) = (\log(u_1/u_3), \log(u_2/u_3))^\top$ can be explicitly inverted on $\overline{\mathcal{D}}$:

$$u_i = \frac{e^{w_i}}{1 + e^{w_1} + e^{w_2}}, \quad i = 1, 2.$$

We see that (Vol 1), (Vol 2) are satisfied. Moreover, there exists $c > 0$ such that for $z \in \mathbb{R}^n$,

$$z^\top h''(u) A(u) z = \frac{d_2 z_1^2}{u_1 a(u)} + \frac{d_1 z_2^2}{u_2 a(u)} + \frac{d_0 (z_1 + z_2)^2}{u_3 a(u)} \geq c\left(\frac{z_1^2}{u_1} + \frac{z_2^2}{u_2}\right).$$

Thus, Assumption (Vol 3) is satisfied with $m = 1/2$.

2.10.6. Biofilm model. Consider a fluid mixture consisting of n concentrations $u_1, \dots, u_n$ and the solvent concentration u_{n+1} such that $\sum_{i=1}^{n+1} u_i = 1$. We suppose that the concentrations are driven by the partial pressures $p_i = u_i$ $(i = 1, \dots, n)$, while the solvent has the constant partial pressure p_{n+1}. Allowing for the presence of an interphase force and neglecting inertia effects, a volume-filling cross-diffusion model with diffusion matrix $A(u)$, defined by

$$A_{ii}(u) = 1 - u_i, \quad A_{ij}(u) = -u_i \quad \text{for } i \neq j,$$

was formally derived in [78, Example 4.3] from an Euler system with linear friction force. This model can be also used to describe the dynamics of a bacterial biofilm with subpopulations $u_1, \dots, u_n$ and the volume fraction u_{n+1} of "free space", in which the biofilm can

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expand [44]. As in the previous example, we choose the entropy density and the noise term

$$h(u) = \sum_{i=1}^{n+1} \big(u_i(\log u_i - 1) + 1\big), \quad \sigma_{ii}(u) = u_i u_{n+1}, \quad \sigma_{ij}(u) = 0 \quad \text{for } i \neq j.$$

The previous example has shown that Assumptions (Vol 1) and (Vol 2) are satisfied. Assumption (Vol 3) is fulfilled with $m = 1/2$ since for all $u \in \overline{\mathcal{D}}$ and $z \in \mathbb{R}^n$,

$$z^\top h''(u) A(u) z = \sum_{i=1}^n \frac{z_i^2}{u_i}.$$

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2.11. Discussing the noise

2.11.1. Overview and the entropy estimate. In this section, our aim is to discuss a slight generalization of assumption (A6). The rigidity of our assumptions as well as the structure of our entropy estimate prevented us from using interesting and relevant forms of σ and left us with, arguably, more artificial examples for σ.
The proposed alteration of (A6) enables us to provide a wider range of possible structures of our multiplicative noise. First we state the altered condition (A6)', which would replace (A6),

(A6)' Let $\sigma(u(R_\varepsilon(v)))_k = \sigma(u(R_\varepsilon(v)))(e_k)$, where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of U and $v \in D(L)'$, where $D(L)'$ denotes the dual (in terms of the triple $D(L) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow D(L)'$) of $D(L)$ introduced in (R2). For any $\delta > 0$, there exists a constant $C > 0$, such that the following inequality holds,

$$\begin{aligned}
&\left(\int_0^t \sum_{k=1}^{\infty} \sum_{i,j=1}^{n} \left(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x \right)^2 \mathrm{d}s \right)^{1/2} \\
&- \delta \sup_{0\le s\le t} \int_{\mathcal{O}} h(u(R_\varepsilon(v)))\,\mathrm{d}x - \delta \int_0^t \left\langle R_\varepsilon(v^{\varepsilon,R})(s), \operatorname{div}\left(B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})\right)(s) \right\rangle_{D(L)\times D(L)'} \mathrm{d}s \\
&+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathcal{O}} \operatorname{tr}\left[(\sigma(u(R_\varepsilon(v)))_k(s))^T (u'(R_\varepsilon(v)))^{-1} (\sigma(u(R_\varepsilon(v)))_k(s)) \right] \mathrm{d}x\,\mathrm{d}s \\
&\le C\left(1 + \int_0^t \int_{\mathcal{O}} h(u(R_\varepsilon(v)))(s)\,\mathrm{d}x\,\mathrm{d}s\right).
\end{aligned}$$

Although our analysis was performed with a general vector valued U-cylindrical Wiener process, in this section we want to consider a more hands-on setting. Therefore consider a trace-class, positive and symmetric operator Q on $L^2(\mathcal{O})$ and the space $Q^{1/2}(L^2(\mathcal{O}))$, equipped with the norm $\|Q^{-1/2}\cdot\|_{L^2(\mathcal{O})}$. We set $U = Q^{1/2}(L^2(\mathcal{O}))$ and denote by W^Q the $Q^{1/2}(L^2(\mathcal{O}))$ cylindrical Wiener process, we will be working with. This setting is equivalent to considering spatially colored noise on $L^2(\mathcal{O})$ in the form of an (L^2-)Q-Wiener process ($Q \neq \mathrm{Id}$). We argue that this point of view provides more intuitive insight which can be translated again into the setting where $Q = \mathrm{Id}$, by constructing suitable Hilbert-Schmidt operators. In the following, we won't work directly with the operator Q, but rather assume it being defined through its eigenvalues and eigenfunctions.
Denoting the normalized eigenfunctions of the Laplace operator, which form a basis of $L^2(\mathcal{O})$, by $(\eta_k)_{k\in\mathbb{N}}$, considering again a sequence of independent Brownian motions $(W_j^k)_{j=1,\dots,n,k\in\mathbb{N}}$ as well as a sequence $(a_k)_{k\in\mathbb{N}}$ of nonnegative real numbers that converge sufficiently fast to 0 (to be further specified in the examples), we assume the noise to be (formally) of the form $W^Q = (W_1^Q, \dots, W_n^Q)$, where

$$W_j^Q(t) = \sum_{k=1}^{\infty} a_k e_k W_j^k(t). \tag{2.114}$$

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Recall that the terms in (A6) arose from the stochastic integral $\int_0^t \sigma(u(R_\varepsilon(v)(s)))\,\mathrm{d}W_s$. Replacing W with W^Q and shifting our point of view from a U-cylindrical Wiener process to an L^2-Q- Wiener process, we will also slightly adapt (A6)' to this setting, to avoid a potentially confusing abuse of notation.

(A6)'(Q) Let $\sigma(u(R_\varepsilon(v)))_k = \sigma(u(R_\varepsilon(v)))(a_k\eta_k)$, where $(\eta_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$ and $v \in D(L)'$. For any $\delta > 0$, there exists a constant $C > 0$, such that the following inequality holds,

$$\begin{aligned}
&\left(\int_0^t \sum_{k=1}^{\infty}\sum_{i,j=1}^{n}\left(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\right)^2 \mathrm{d}s\right)^{1/2}\\
&-\delta\sup_{0\le s\le t}\int_{\mathcal{O}} h(u(R_\varepsilon(v)))\,\mathrm{d}x - \delta\int_0^t \left\langle R_\varepsilon(v^{\varepsilon,R})(s), \operatorname{div}\left(B(u(R_\varepsilon(v^{\varepsilon,R})))\nabla R_\varepsilon(v^{\varepsilon,R})\right)(s)\right\rangle_{D(L)\times D(L)'}\mathrm{d}s\\
&+\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}} \operatorname{tr}\left[(\sigma(u(R_\varepsilon(v)))_k(s))^T(u'(R_\varepsilon(v)))^{-1}(\sigma(u(R_\varepsilon(v)))_k(s))\right]\mathrm{d}x\,\mathrm{d}s\\
&\le C\left(1+\int_0^t\int_{\mathcal{O}} h(u(R_\varepsilon(v)))(s)\,\mathrm{d}x\,\mathrm{d}s\right).
\end{aligned}$$

If a_k correspond to powers of the eigenvalues of $(1-\Delta)$ and $(\eta_k)_k$ to the normalized eigenfunctions of $(1-\Delta)$, $(e_k)_{k\in\mathbb{N}} = (a_k\eta_k)_{k\in\mathbb{N}}$ can be interpreted as the basis of another Sobolev space.

Before we discuss how our new Assumption improves upon (A6), we make a formal, but rather illustrative consideration.

Assuming that our sequence $(a_k)_{k\in\mathbb{N}}$ has sufficient decay for the stochastic integral with $\sigma(u) \sim u^\alpha$ to be well defined. We want to see which, if any, values $\alpha > 0$ are compatible with (A6). Set $h(x) = x\big(\log(x) - 1\big) + 1$ and $h'(x) = \log(x)$, then

$$\frac{\big(x^\alpha h'(x)\big)^2}{h(x)} = \frac{(x)^{2\alpha}\log(x)^2}{x\big(\log(x)-1\big)+1} \to 0,$$

as $x \to \infty$, if $2\alpha < 1$. This follows from de l'Hospitals rule. To satisfy Assumption (A6) and close the entropy estimate (2.15), we would require $|\sigma(u)| \le C(1+(u)^\alpha)$, with $\alpha < 1/2$. On the other hand the quadratic variation term behaves similar to $\frac{x^{2\alpha}}{x}$. To control this term, we require $\alpha \ge 1/2$. We see that the two terms originating from the noise give contradicting values for powers of u.

The situation is not as hopeless, as it might appear, since Assumption (A6) only relates σ and the entropy h, ignoring all other terms appearing in the entropy estimate (2.15) and consequent estimates. We will show that, by incorporating more of the structure of the equation, leading to (A6)'/(A6)'(Q), more interesting cases of σ can be considered. For this, we recall the entropy inequality derived earlier, but also remind that we are now dealing with a Q Wiener process on $L^2(\mathcal{O})$. Since we did not specify the rate of decay of the sequence $(a_k)_{k\in\mathbb{N}}$, as well as the exact structure of σ, the following calculations should be interpreted as begin purely formal to uncover the form of (A6)'(Q). However, when in

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the setting of the SKT model with self-diffusion, presented earlier with a U cylindrical Wiener process, these calculations don't lack any rigor. For this reason, we leave a bit of ambiguity to the interpretation of $\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k$ as $\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))e_k$ or $\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))\eta_k a_k$. We quickly recall the notation $u(R_\varepsilon(v^\varepsilon)) = u^\varepsilon$.

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(v^\varepsilon)(t)\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x \\ &+ \sum_{k=1}^{\infty}\sum_{i,j=1}^{n}\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s) \\ &- \int_0^t\int_{\mathcal{O}} \nabla R_\varepsilon(v^\varepsilon)(s)\cdot B(u(R_\varepsilon(v^\varepsilon)))(s)\nabla R_\varepsilon(v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \\ &+ C\sum_{k=1}^{\infty}\int_{\mathcal{O}}\mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

For illustration purpose and since it is the case in many of our applications, we assume that the structure of B allows for the following estimate,

$$\begin{aligned}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \sum_{i=1}^{n}\int_0^t \|\nabla u_i^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s &\le \int_{\mathcal{O}} h(v_0)\,\mathrm{d}x \\ &+ \sum_{k=1}^{\infty}\sum_{i,j=1}^{n}\int_0^t\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_i\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\,\mathrm{d}W_j^k(s) \\ &+ C\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{O}}\mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By the BDG inequality,

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} h(u(R_\varepsilon(v^\varepsilon)))(t)\,\mathrm{d}x + \sum_{i=1}^{n}\int_0^T \|\nabla u_i^\varepsilon(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s &\le \mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x \\ &+ C\mathbb{E}\bigg(\int_0^T\sum_{k=1}^{\infty}\sum_{i,j=1}^{n}\bigg(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_j\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2\mathrm{d}s\bigg)^{1/2} \\ &+ \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_0^t\sum_{k=1}^{\infty}\int_{\mathcal{O}}\mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T(u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Instead of immediately resorting to Gronwall's inequality to control the terms on the right-hand side, we consider the combination of the terms on the left hand side. The generalized Poincaré inequality (see [123, Chap. 2, Sect. 1.4]) implies that $\|u^\varepsilon\|_{L^2(0,T;H^1((\mathcal{O}))}$ can be controlled by $\|u^\varepsilon\|_{L^\infty(0,T;L^1((\mathcal{O}))}$ and $\|\nabla u^\varepsilon\|_{L^2(0,T;L^2((\mathcal{O}))}$. An application of Young's inequality and the Sobolev embedding theorem yields,

$$\|u^\varepsilon\|_{L^2(0,T;L^2((\mathcal{O}))} \le C\|u^\varepsilon\|_{L^2(0,T;H^1((\mathcal{O}))} \le C\|u^\varepsilon\|_{L^\infty(0,T;L^1((\mathcal{O}))} + C\|\nabla u^\varepsilon\|^2_{L^2(0,T;L^2((\mathcal{O}))} + C.$$

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Indeed, by the Gagliardo-Nirenberg inequality, one could actually do even better and replace $\|u^\varepsilon\|_{L^2(0,T;L^2((\mathcal{O}))}$ by $\|u^\varepsilon\|_{L^{2(d+1)/d}(0,T;L^{2(d+1)/d}(\mathcal{O}))}$ for $d \geq 3$, but the simpler version of the estimate will suffice for our purposes.
We see, assuming $\|u^\varepsilon\|_{L^\infty(0,T;L^1((\mathcal{O}))} \leq \sup_{0\leq t\leq T}\int_\mathcal{O} h(u(R_\varepsilon(v^\varepsilon)))\,\mathrm{d}x + C$ (see (E5)), that

$$\begin{aligned}
&\mathbb{E}\|u^\varepsilon\|_{L^2(0,T;L^2((\mathcal{O}))} + C \leq C\bigg(1 + \mathbb{E}\int_\mathcal{O} h(v_0)\,\mathrm{d}x \\
&+ \mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_\mathcal{O} h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&+ \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \sum_{k=1}^\infty \int_\mathcal{O} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s\bigg).
\end{aligned}$$

Adding this inequality to the entropy inequality yields

$$\begin{aligned}
&\mathbb{E}\big(\|u^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))} + \|u^\varepsilon\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \sup_{0\leq t\leq T}\int_\mathcal{O} h(u^\varepsilon)\,\mathrm{d}x + \|\nabla u^\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))}\big) \\
&\leq C\bigg(1 + \mathbb{E}\int_\mathcal{O} h(v_0)\,\mathrm{d}x + \mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_\mathcal{O} h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^T \sum_{k=1}^\infty \int_\mathcal{O} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s\bigg).
\end{aligned}$$

By Parseval's identity, and since $L^2(\mathcal{O})$ can be continuously embedded in the closure of $Q^{-1/2}(L^2(\mathcal{O}))$ with respect to the $\|Q^{1/2}\cdot\|_{L^2}$ norm (which itself is a Hilbert space), we obtain

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^T \int_\mathcal{O} \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_\mathcal{O} h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&\quad + \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \sum_{k=1}^\infty \int_\mathcal{O} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\mathbb{E}\bigg(\int_0^T \sum_{i,j=1}^n \int_\mathcal{O} \big(h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))(s)\big)^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2} \\
&\quad + \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \sum_{k=1}^\infty \int_\mathcal{O} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We now apply this observation to explicit examples, which we discussed earlier.

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2.11.2. SKT with self-diffusion. We want to verify that

$$\sigma_{ij}(u) = \delta_{ij}(u_i)^\alpha,$$

for $1/2 \le \alpha \le 1$ and the Q-Wiener process W^Q, whose components are given by

$$W_j^Q = \sum_{k=1}^{\infty} \frac{\eta_k}{\langle \lambda_k \rangle^\rho} W_j^{(k)},$$

where $\rho > (d-1)(d-2)(\alpha - 1/2)/2d + d/2$ and $(\lambda_k)_{k\in\mathbb{N}}$, $(\eta_k)_{k\in\mathbb{N}}$ are the eigenvalues and eigenfunctions of the Laplace operator with Dirichlet or Neumann boundary conditions, are admissible (under Assumption (A6)'(Q)) in the sense that they allow us to close the entropy estimate. To reconcile with the previously used notation $a_k = \frac{1}{\langle \lambda_k \rangle^\rho}$.
The map $u \mapsto \sigma_{ij}(u) = \delta_{ij} u^\alpha$ should be seen as a map $L^2(\mathcal{O}) \to \mathcal{L}_2(H^\beta(\mathcal{O}), L^2(\mathcal{O}))$, where $\beta > \rho$.
We recall the entropy density $h(u^\varepsilon) = \sum_{i=1}^n \pi_i u_i^\varepsilon \log(u_i^\varepsilon) - u_i^\varepsilon + 1$, its derivative $h'(u^\varepsilon)_i = \pi_i \log(u_i^\varepsilon)$ and the entropy inequality

$$\begin{aligned}
&\mathbb{E}\|u^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))} + \mathbb{E}\bigg(\sup_{0\le t\le T} \int_{\mathcal{O}} h(u^\varepsilon)(t)\,\mathrm{d}x + C\sum_{i=1}^n \|\nabla u_i^\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))} + \widetilde{C}\bigg) \\
&\le C\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + \widetilde{C} + \mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&\quad + \frac{1}{2}\mathbb{E}\sup_{0\le t\le T} \int_0^t \sum_{k=1}^\infty \int_{\mathcal{O}} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1} (\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Of the three terms on the right hand side only the last two matter. We will investigate them separately.

LEMMA 2.11.1. *There exist constants $C > 0$ and $0 < \overline{\varepsilon} < 1$, such that*

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i,j=1}^n \bigg(\int_{\mathcal{O}} h'(u(R_\varepsilon(v^\varepsilon)))_j \sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&= \mathbb{E}\bigg(\int_0^T \sum_{k=1}^\infty \sum_{i=1}^n \bigg(\int_{\mathcal{O}} \log(u_i^\varepsilon)(u_i^\varepsilon)^\alpha \frac{\eta_k}{\lambda_k^\rho}\,\mathrm{d}x\bigg)^2 \mathrm{d}s\bigg)^{1/2} \\
&\le C\bigg(1 + \mathbb{E}\int_0^T \int_{\mathcal{O}} h(u_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\bigg) + \overline{\varepsilon}\bigg(\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} h(u_i^\varepsilon)\,\mathrm{d}x + \mathbb{E}\|\nabla u_i^\varepsilon\|^2_{L^2(\mathcal{O})}\bigg).
\end{aligned}$$

PROOF. By the subadditivity of the map $x \mapsto x^{1/2}$ on $\mathbb{R}_+$, we can drop the finite sum and consider each term separately. The calculation will be performed for the case $\alpha = 1$, since the case $\alpha < 1$ follows by even simpler arguments.

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As in the overview, a Parseval argument allows us to estimate

$$\mathbb{E}\bigg(\int_0^T\sum_{k=1}^\infty\bigg(\int_\mathcal{O}h'(u(R_\varepsilon(v^\varepsilon)))_j\sigma_{ij}(u(R_\varepsilon(v^\varepsilon)))_k(s)\,\mathrm{d}x\bigg)^2\,\mathrm{d}s\bigg)^{1/2}$$
$$\le C\mathbb{E}\bigg(\int_0^T\int_\mathcal{O}(u_i^\varepsilon\log(u_i^\varepsilon))^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2}=C\mathbb{E}\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^2(0,T;L^2(\mathcal{O}))}$$
$$\le C\mathbb{E}\bigg(\int_0^T\Big(\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}^\theta\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^p(\mathcal{O})}^{1-\theta}\Big)^2\,\mathrm{d}t\bigg)^{1/2},$$

where $p=2+\delta$ for $\delta>0$, $1/2=\theta+(1-\theta)/p$. By Young's inequality,

$$\le C\mathbb{E}\bigg(\int_0^T\Big(C(\widetilde{\varepsilon})\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+\widetilde{\varepsilon}\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^p(\mathcal{O})}\Big)^2\,\mathrm{d}t\bigg)^{1/2}.$$

We replace $u_i^\varepsilon\log(u_i^\varepsilon)$ by $g(u_i^\varepsilon):=\max\{\overline{C},u_i^\varepsilon\log(u_i^\varepsilon)\}$, with a "large" $\overline{C}>0$.

$$\le C\mathbb{E}\bigg(\int_0^T\Big(C(\widetilde{\varepsilon})\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+\widetilde{\varepsilon}\|g(u_i^\varepsilon)\|_{L^p(\mathcal{O})}\Big)^2\,\mathrm{d}t\bigg)^{1/2}.$$

By the Sobolev embedding theorem,

$$\le C\mathbb{E}\bigg(\int_0^T\Big(C(\widetilde{\varepsilon})\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+\widetilde{\varepsilon}\|g(u_i^\varepsilon)\|_{W^{1,q}(\mathcal{O})}\Big)^2\,\mathrm{d}t\bigg)^{1/2},$$

such that $\frac{1}{q}=\frac{1}{p}-\frac{1}{d}$. We note that $q=\frac{d(2+\delta)}{d+2+\delta}<2$.

$$\le C\mathbb{E}\bigg(\int_0^T\Big(C(\widetilde{\varepsilon})\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+\widetilde{\varepsilon}\big(\|g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}^q+\|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}^q\big)^{1/q}\Big)^2\,\mathrm{d}t\bigg)^{1/2}.$$

Using that for $a,b\ge0$, since $1/q<1$, by subadditivity, $(a^q+b^q)^{1/q}\le a^{q/q}+b^{q/q}$,

$$\le C\mathbb{E}\bigg(\int_0^T\Big(C(\widetilde{\varepsilon})\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+\widetilde{\varepsilon}\big(\|g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}+\|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}\big)\Big)^2\,\mathrm{d}t\bigg)^{1/2}.$$

Next, we estimate the term $\|g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}$. We first use the Gagliardo-Nirenberg-Sobolev inequality,

$$\|g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}\le C\|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}^\theta\|g(u_i^\varepsilon)\|_{L^1(\mathcal{O})}^{1-\theta}+C\|g(u_i^\varepsilon)\|_{L^1(\mathcal{O})},$$

where $0<\theta=\frac{d(q-1)}{d(q-1)+q}<1$. We apply Young's inequality to the first term and use for the second term that, by the boundedness of $x\mapsto x\log(x)$ on bounded intervals, there exists a constant C, such that $\|g(u_i^\varepsilon)\|_{L^1(\mathcal{O})}\le C\big(1+\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}\big)$. Hence

$$\|g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}\le C\|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}+C\|g(u_i^\varepsilon)\|_{L^1(\mathcal{O})}+C\|g(u_i^\varepsilon)\|_{L^1(\mathcal{O})}$$
$$\le C\|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}+C\big(1+\|u_i^\varepsilon\log(u_i^\varepsilon)\|_{L^1(\mathcal{O})}\big).$$

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Using this estimate in our previous inequality, absorbing the constants into those depending on $\widetilde{\varepsilon}$, yields

$$\leq \mathbb{E}\bigg(\int_0^T \Big(C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(\mathcal{O})} + \widetilde{\varepsilon} C\big(1 + \|\nabla g(u_i^\varepsilon)\|_{L^q(\mathcal{O})}\big)\Big)^2 \,\mathrm{d}t \bigg)^{1/2}$$

$$\leq \mathbb{E}\bigg(\int_0^T \Big(C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(\mathcal{O})} + \widetilde{\varepsilon} C\big(1 + \|\chi_{\{u>C\}} \log(u_i^\varepsilon) \nabla u_i^\varepsilon + \nabla u_i^\varepsilon\|_{L^q(\mathcal{O})}\big)\Big)^2 \,\mathrm{d}t \bigg)^{1/2}.$$

Since $\log(x)$ grows slower than any power of x, we estimate for $\overline{C}$ sufficiently large,

$$\leq \mathbb{E}\bigg(\int_0^T \Big(C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(\mathcal{O})} + \widetilde{\varepsilon} C\big(1 + \|\chi_{\{u>C\}} |(u_i^\varepsilon)^\gamma| |\nabla u_i^\varepsilon| + \nabla u_i^\varepsilon\|_{L^q(\mathcal{O})}\big)\Big)^2 \,\mathrm{d}t \bigg)^{1/2}.$$

Choosing $\gamma \leq (2-q)/2q$ and using Hölder's inequality,

$$\leq \mathbb{E}\bigg(\int_0^T \Big(C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(\mathcal{O})} + \widetilde{\varepsilon} C\big(1 + \|(u_i^\varepsilon)^\gamma\|_{L^{2q/(2-q)}(\mathcal{O})} \|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})} + \|\nabla u_i^\varepsilon\|_{L^q(\mathcal{O})}\big)\Big)^2 \,\mathrm{d}t \bigg)^{1/2}$$

$$\leq \mathbb{E} C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^2(0,T;L^1(\mathcal{O}))} + \widetilde{\varepsilon} C \mathbb{E}\bigg[1 + \bigg(\int_0^T \|(u_i^\varepsilon)^\gamma\|_{L^{2q/(2-q)}(\mathcal{O})}^2 \|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}t \bigg)^{1/2} + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^q(\mathcal{O}))} \bigg]$$

$$= \mathbb{E}(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^2(0,T;L^1(\mathcal{O}))} + \widetilde{\varepsilon} C \mathbb{E}\bigg[1 + \bigg(\int_0^T \bigg(\int_{\mathcal{O}} ((u_i^\varepsilon)^{2\gamma q/(2-q)} \,\mathrm{d}x \bigg)^{(2-q)/q} \|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}t \bigg)^{1/2} + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^q(\mathcal{O}))} \bigg]$$

$$\leq \mathbb{E} C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^2(0,T;L^1(\mathcal{O}))} + \widetilde{\varepsilon} C \mathbb{E}\bigg[1 + \sup_{0\leq t\leq T} \|u_i^\varepsilon\|_{L^1(\mathcal{O})}^{(2-q)/2q} \bigg(\int_0^T \|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}t \bigg)^{1/2} + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^q(\mathcal{O}))} \bigg].$$

Applying Young's inequality once again, noting that q can be chosen such that $(2-q)/q < 1$,

$$\leq \mathbb{E} C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^2(0,T;L^1(\mathcal{O}))} + \widetilde{\varepsilon} C \mathbb{E}\big(1 + \sup_{0\leq t\leq T} \|u_i^\varepsilon\|_{L^1(\mathcal{O})}^{(2-q)/q} + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}^2 + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}\big)$$

$$\leq \mathbb{E} C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^2(0,T;L^1(\mathcal{O}))} + \widetilde{\varepsilon} C \mathbb{E}\big(1 + \sup_{0\leq t\leq T} \|u_i^\varepsilon\|_{L^1(\mathcal{O})} + 2\|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}^2 + C\big).$$

Interpolating $L^2(0,T)$ between $L^1(0,T)$ and $L^\infty(0,T)$, yields

$$\leq \mathbb{E} C(\widetilde{\varepsilon}) \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(0,T;L^1(\mathcal{O}))}^{1/2} \|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{1/2} + \widetilde{\varepsilon} C \mathbb{E}\big(1 + \sup_{0\leq t\leq T} \|u_i^\varepsilon\|_{L^1(\mathcal{O})} + \|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}^2\big).$$

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Once again Young's inequality yields

$$\leq \mathbb{E}C(\widetilde{\varepsilon},\widetilde{\widetilde{\varepsilon}})\|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(0,T;L^1(\mathcal{O}))} + \widetilde{\widetilde{\varepsilon}}C(\widetilde{\varepsilon})\mathbb{E}\|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^\infty(0,T;L^1(\mathcal{O}))}$$
$$+ \widetilde{\varepsilon}C\mathbb{E}\big(1 + \sup_{0\leq t\leq T}\|u_i^\varepsilon\|_{L^1(\mathcal{O})} + \|\nabla u_i^\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))}\big).$$

Choosing $\widetilde{\widetilde{\varepsilon}}$ small enough, with respect to the other small parameter $\widetilde{\varepsilon}$, we can absorb all but the first term and a constant C on the left hand side. It remains to treat $\mathbb{E}C(\widetilde{\varepsilon},\widetilde{\widetilde{\varepsilon}})\|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(0,T;L^1(\mathcal{O}))}$.

$$\mathbb{E}\|u_i^\varepsilon \log(u_i^\varepsilon)\|_{L^1(0,T;L^1(\mathcal{O}))} = \mathbb{E}\int_0^T\int_{\mathcal{O}} |u_i^\varepsilon \log(u_i^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \mathbb{E}\int_0^T\int_{\mathcal{O}} |u_i^\varepsilon \log(u_i^\varepsilon) - u_i^\varepsilon + 1| + |u_i^\varepsilon - 1|\,\mathrm{d}x\,\mathrm{d}s.$$

by the positivity of u_i^ε, $u_i^\varepsilon \log(u_i^\varepsilon)$ is bounded from below. Hence we can find a constant, depending on $\mathcal{O}, T$, such that

$$\leq \mathbb{E}\int_0^T\int_{\mathcal{O}} u_i^\varepsilon \log(u_i^\varepsilon) - u_i^\varepsilon + 1 + |u_i^\varepsilon - 1|\,\mathrm{d}x\,\mathrm{d}s + C.$$

Using that $x\log(x) - x + 1$ grows faster than x, a similar argument leads to

$$\leq C\mathbb{E}\int_0^T\int_{\mathcal{O}} u_i^\varepsilon \log(u_i^\varepsilon) - u_i^\varepsilon + 1\,\mathrm{d}x\,\mathrm{d}s + C.$$

□

The structure of the inequality allows for an application of Gronwall's inequality, which finishes the arguments regarding the first term.

REMARK 2.11.2. *To give an answer to the question, why we did not immediately introduce the function* $g(u_i^\varepsilon) := \max\{\overline{C}, u_i^\varepsilon \log(u_i^\varepsilon)\}$, *we highlight that the first steps were necessary to obtain a small constant* $\widetilde{\varepsilon}$. *Without the additional steps this constant would be absent and prevent us from absorbing the correct terms on the left-hand side.*

We turn our attention to the quadratic-variation term,

$$\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \sum_{k=1}^\infty \int_{\mathcal{O}} \mathrm{tr}\Big[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s.$$

LEMMA 2.11.3. *There exist constants* $C > 0$ *and* $0 < \hat{\varepsilon} < 1$, *with* $\overline{\varepsilon} + \hat{\varepsilon} < 1$, *such that*

$$\mathbb{E}\sup_{0\leq t\leq T}\int_0^t \sum_{k=1}^\infty \int_{\mathcal{O}} \mathrm{tr}\Big[(\sigma(u^\varepsilon)_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1}(\sigma(u^\varepsilon)_k(s))\Big]\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq C + \hat{\varepsilon}\Big(\mathbb{E}\|u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))} + \mathbb{E}\|\nabla u_i^\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))}\Big) + C(\hat{\varepsilon})\mathbb{E}\int_0^T\int_{\mathcal{O}} h(u_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.$$

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PROOF. We will use the notation $a_k = \frac{1}{\langle \lambda_k \rangle^\rho}$, in order to highlight the powers of λ_k which appear from the estimates.
For simplicity, we consider each term of the matrix trace separately

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} \int_0^t \sum_{k=1}^\infty \int_{\mathcal{O}} (\sigma(u^\varepsilon)_k(s))^T_{ii} (u'(R_\varepsilon(v^\varepsilon)))^{-1}_{ii} (\sigma(u^\varepsilon)_k(s))_{ii} \,\mathrm{d}x\,\mathrm{d}s \\
&= \mathbb{E} \sup_{0\le t\le T} \int_0^t \sum_{k=1}^\infty a_k^2 \int_{\mathcal{O}} (u_i^\varepsilon)^\alpha \eta_k \frac{1}{u_i^\varepsilon} (u_i^\varepsilon)^\alpha \eta_k \,\mathrm{d}x\,\mathrm{d}s = \mathbb{E} \sup_{0\le t\le T} \int_0^t \sum_{k=1}^\infty a_k^2 \int_{\mathcal{O}} (u_i^\varepsilon)^{2\alpha-1} \eta_k\eta_k \,\mathrm{d}x\,\mathrm{d}s \\
&= \mathbb{E} \sum_{k=1}^\infty a_k^2 \|(u_i^\varepsilon)^{2\alpha-1}\eta_k\eta_k\|_{L^1(0,T;L^1(\mathcal{O}))}.
\end{aligned}$$

We apply Hölder's inequality,

$$\le C(T)\mathbb{E} \sum_{k=1}^\infty a_k^2 \|(u_i^\varepsilon)^{2\alpha-1}\eta_k\|_{L^1(0,T;L^2(\mathcal{O}))}.$$

Since we will need to differentiate between the cases $\alpha < 1$ and $\alpha = 1$, we introduce the real constant $\gamma \ge 2\alpha - 1$ whose value will be determined during the following calculations. Once again, by Hölder's inequality with $1/p + 1/q = \frac{(d-2)\gamma}{2d} + \frac{d-(d-2)\gamma}{2d} = \frac{1}{2}$,

$$\le \mathbb{E} \sum_{k=1}^\infty a_k^2 \int_0^T \|(u_i^\varepsilon)^{2\alpha-1}\|_{L^{2d/((d-2)\gamma)}(\mathcal{O})} \|\eta_k\|_{L^{2d/(d-(d-2)\gamma)}(\mathcal{O})} \,\mathrm{d}t.$$

We interpolate $L^{2d/(d-(d-2)\gamma)}(\mathcal{O})$ in between $L^2(\mathcal{O})$ and $L^\infty(\mathcal{O})$ and obtain

$$\le C\mathbb{E} \sum_{k=1}^\infty a_k^2 \int_0^T \|(u_i^\varepsilon)^{2\alpha-1}\|_{L^{2d/((d-2)\gamma)}(\mathcal{O})} \|\eta_k\|_{L^2(\mathcal{O})}^{(d-(d-2)\gamma)/d} \|\eta_k\|_{L^\infty(\mathcal{O})}^{(d-2)\gamma/d} \,\mathrm{d}t.$$

We recall that $(\eta_k)_{k\in\mathbb{N}}$ were chosen to be an orthonormal basis of $L^2(\mathcal{O})$. Using that $\|\eta_k\|_{L^\infty(\mathcal{O})} \le C(\mathcal{O})\lambda_k^{(d-1)/2}$ (see [66, Sections 2.4, 4, 5]), in the case of Dirichlet as well as Neumann boundary conditions,

$$\le C(\mathcal{O})\mathbb{E} \int_0^T \|(u_i^\varepsilon)^{2\alpha-1}\|_{L^{2d/((d-2)\gamma)}(\mathcal{O})} \,\mathrm{d}t \sum_{k=1}^\infty a_k^2 \lambda_k^{(d-1)(d-2)\gamma/2d}.$$

We now differentiate between the cases $\alpha < 1$ and $\alpha = 1$. In the first case, $\gamma = 2\alpha - 1$ and hence

$$= C(\mathcal{O})\mathbb{E} \int_0^T \|u_i^\varepsilon\|_{L^{2d/((d-2))}(\mathcal{O})}^{2\alpha-1} \,\mathrm{d}t \sum_{k=1}^\infty a_k^2 \lambda_k^{(d-1)(d-2)\gamma/2d}.$$

We apply Young's, as well as Hölder's inequality,

$$\le C(\mathcal{O})\mathbb{E}\left(\int_0^T \widetilde{\varepsilon}\|u_i^\varepsilon\|_{L^{2d/(d-2)}(\mathcal{O})} + C(\widetilde{\varepsilon}) \,\mathrm{d}t \right) \sum_{k=1}^\infty a_k^2 \lambda_k^{(d-1)(d-2)\gamma/2d}$$

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$$\leq C(\mathcal{O})\widetilde{\varepsilon}^{1/2}\mathbb{E}\bigg(\int_0^T \|u_i^\varepsilon\|_{L^{2d/(d-2)}(\mathcal{O})}^2\,\mathrm{d}t\bigg)^{1/2}\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d} + C(\widetilde{\varepsilon})\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}.$$

See [87] for upper and lower bounds of the eigenvalues of the Laplace operator with both, Dirichlet and Neumann boundary conditions. In both cases, we see that for the right hand side to converge, we require $a_k < \lambda_k^{-(d-1)(d-2)(\alpha-1/2)/2d-d/2}$. We see that $W_j^Q = \sum_{k=1}^\infty \frac{e_k}{\langle\lambda_k\rangle^\rho} W_j^{(k)}$, where $\rho > (d-1)(d-2)(\alpha-1/2)/2d+d/2$ and $(\lambda_k)_{k\in\mathbb{N}}$ are the eigenvalues of the Laplace operator, has sufficient decay for the quadratic variation term to converge. Using the (continuous) embedding $H^1(\mathcal{O}) \hookrightarrow L^{2d/(d-2)}(\mathcal{O})$ for $d \geq 2$ yields

$$\leq C(\mathcal{O})\widetilde{\varepsilon}^{1/2}\mathbb{E}\|u_i^\varepsilon\|_{L^2(0,T;H^1(\mathcal{O}))}\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d} + C(\widetilde{\varepsilon})\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}.$$

The estimate $C\widetilde{\varepsilon}^{1/2}\mathbb{E}\|u_i^\varepsilon\|_{L^2(0,T;H^1(\mathcal{O}))} \leq C\widetilde{\varepsilon}^{1/2}\big(1+\mathbb{E}\|u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}+\mathbb{E}\|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}^2\big)$, concludes the argument for $\alpha < 1$.
In the case $\alpha = 1$, we choose $\gamma = 2\alpha - 1 + \delta$, with an arbitrarily small $\delta > 0$. The Gagliardo-Nirenberg inequality yields

$$\begin{aligned}&\mathbb{E}\int_0^T \|u_i^\varepsilon\|_{L^{2d/((d-2)\gamma)}(\mathcal{O})}\,\mathrm{d}t\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}\\&\leq C\mathbb{E}\int_0^T\Big(\|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})}^\theta\|u_i^\varepsilon\|_{L^1(\mathcal{O})}^{1-\theta} + C\|u_i^\varepsilon\|_{L^1(\mathcal{O})}\Big)\,\mathrm{d}t\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d},\end{aligned}$$

where $\theta = \frac{2d+2\gamma-d\gamma}{d+2} < 1$. We use Young's inequality twice,

$$\begin{aligned}&\leq C\mathbb{E}\int_0^T\Big(\widetilde{\varepsilon}\|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})} + C(\widetilde{\varepsilon})\|u_i^\varepsilon\|_{L^1(\mathcal{O})}\Big)\,\mathrm{d}t\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}\\&\leq C\mathbb{E}\int_0^T\Big(\widetilde{\varepsilon}\|\nabla u_i^\varepsilon\|_{L^2(\mathcal{O})}^2 + C + C(\widetilde{\varepsilon})\|u_i^\varepsilon\|_{L^1(\mathcal{O})}\Big)\,\mathrm{d}t\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}\\&\leq\sum_{k=1}^\infty a_k^2\lambda_k^{(d-1)(d-2)\gamma/2d}\bigg(\widetilde{\varepsilon}C\mathbb{E}\|\nabla u_i^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))}^2 + C(\widetilde{\varepsilon})\mathbb{E}\int_0^T\int_{\mathcal{O}} h(u_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + C\bigg).\end{aligned}$$

For the last inequality, we argue as in the proof of Lemma 2.11.1. Since δ was chosen arbitrarily small, the arguments regarding the powers of λ_k essentially don't change and we still require $a_k < \lambda_k^{-(d-1)(d-2)(\alpha-1/2)/2d-d/2}$ for the sum to converge. Choosing $\widetilde{\varepsilon}$ small enough, we are once again in the position to absorb the L^2 norm of the gradient on the left hand side and apply Gronwall's inequality to the remaining terms. This concludes the proof for the case $\alpha = 1$. □

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REMARK 2.11.4. *For $d = 1, 2$, we use the embedding $H^1(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$ for any $1 \leq p < \infty$ and see that*

$$\begin{aligned}
&\mathbb{E}\sum_{k=1}^{\infty} a_k^2 \|(u_i^\varepsilon)^{2\alpha-1} e_k\|_{L^1(0,T;L^2(\mathcal{O}))} \\
&\leq \mathbb{E}\sum_{k=1}^{\infty} a_k^2 \left(\int_0^T \|(u_i^\varepsilon)^{2\alpha-1}\|_{L^{p/\gamma}(\mathcal{O})} \|\eta_k\|_{L^{2p/(p-2\gamma)}(\mathcal{O})} \,\mathrm{d}t \right) \\
&\leq C\mathbb{E}\sum_{k=1}^{\infty} a_k^2 \left(\int_0^T \|u_i^\varepsilon\|_{L^p(\mathcal{O})}^{2\alpha-1} \|\eta_k\|_{L^2(\mathcal{O})}^{(p-2\gamma)/p} \|\eta_k\|_{L^\infty(\mathcal{O})}^{2\gamma/p} \,\mathrm{d}t \right) \\
&\leq C\mathbb{E}\sum_{k=1}^{\infty} a_k^2 \lambda_k^{\gamma(d-1)/p} \left(\int_0^T \|u_i^\varepsilon\|_{L^p(\mathcal{O})}^{2\alpha-1} \,\mathrm{d}t \right).
\end{aligned}$$

As before, Young's inequality, as well as the Gagliardo-Nirenberg inequality again help us to obtain the desired form. The setting $d = 1$ already hints at another special case, namely $\mathcal{O} = \mathbb{R}^d/\mathbb{Z}^d$. Here the eigenfunctions of the Laplace operator (with homogeneous Dirichlet or periodic boundary conditions) are uniformly bounded in $L^\infty(\mathcal{O})$ and one obtains simply $a_k < \lambda_k^{-d/2}$.

The previous two Lemmata verify the claim we made regarding σ and W^Q in the beginning of this subsection. The entropy estimate follows directly by absorbing all terms with a "small" parameter on the left-hand side and applying Gronwall's inequality.

2.11.3. SKT without self-diffusion. Due to the lack of self-diffusion, instead of the $H^1(\mathcal{O})$ estimate, we only obtain an estimate of the $W^{1,1}(\mathcal{O})$ norm of our solution. We need to modify our previous arguments, and distinguish between different dimensions. For $d = 1, 2$, using the positivity of u_i^ε ($u_i^\varepsilon \log(u_i^\varepsilon) = \sqrt{u_i^\varepsilon}\sqrt{u_i^\varepsilon}\log(u_i^\varepsilon)$) and Young's inequality, we obtain

$$\begin{aligned}
&\|u^\varepsilon\|_{L^2(0,T;L^2(\mathcal{O}))} \leq C\|u^\varepsilon\|_{L^2(0,T;W^{1,1}(\mathcal{O}))} \leq C\|u^\varepsilon\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{1/2} \|\nabla(u^\varepsilon)^{1/2}\|_{L^2(0,T;L^2(\mathcal{O}))} \\
&\leq C\left(\|u^\varepsilon\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \|\nabla(u^\varepsilon)^{1/2}\|_{L^2(0,T;L^2(\mathcal{O}))}^2 \right) \\
&\leq C\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x + \mathbb{E}\sum_{i=1}^{n} \pi_i C\widetilde{\varepsilon}\mathbb{E}\left(\int_0^T \int_{\mathcal{O}} (u_i^\varepsilon)^{4\alpha-1} \log\left(u_i^\varepsilon\right)^4 \mathrm{d}x\,\mathrm{d}s \right)^{1/2} \\
&\quad + C(\widetilde{\varepsilon})\mathbb{E}\int_0^T \int_{\mathcal{O}} u_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}s + \widetilde{C}(\widetilde{\varepsilon}) \\
&\quad + \frac{C}{2}\mathbb{E}\int_0^T \sum_{k=1}^{\infty} \int_{\mathcal{O}} \mathrm{tr}\left[(\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s))^T (u'(R_\varepsilon(v^\varepsilon)))^{-1} (\sigma(u(R_\varepsilon(v^\varepsilon)))_k(s)) \right] \mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Hence, we obtain the bound $\alpha < 3/4$ by Hölder's inequality and the fact that the logarithm grows slower than any positive power of its argument. In dimension $d \geq 3$, one is left with $\alpha < 1/2$, which is not admissible since the quadratic variation term is still present. It

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is possible to alter σ in order to account for the quadratic variation term, which would produce a singularity for $u^\varepsilon = 0$. Such a modification would take the form $\sigma_{ij} = \delta_{ij}\frac{(u_i^\varepsilon)^\alpha}{1+(u_i^\varepsilon)^\beta}$, with $\alpha \geq 1/2$ and $\beta \geq \alpha/2$. Although arguably more artificial, this structure prevents a blow-up at 0, while preserving the same asymptotic behaviour of $(u_i^\varepsilon)^\gamma$ for $\gamma \leq 1/2$.
We admit that the analysis of this more complex problem was rather short and more formal. A more detailed discussion might be left for a future point in time, but we hypothesize that there is not much room for improvement.

2.11.4. Volume filling models. A central property of the Volume-filling models was that the solutions u_i^ε remained bounded between 0 and 1. This allows for a rather general structure of σ. Let $m > d/2$. Assuming $D(L) \hookrightarrow H^m(\mathcal{O})$ (continuously and dense), where $H^m(\mathcal{O})$ is assumed to be compatible with homogeneous Dirichlet ($H_0^m(\mathcal{O})$) or Neumann boundary conditions ($H_N^m(\mathcal{O}) := \{w \in H^m(\mathcal{O})\colon \nu \cdot \nabla w = 0 \text{ on } \partial\mathcal{O}\}$, where ν denotes the unit outer normal vector). Let $(\widetilde{h}_n)_{n\in\mathbb{N}}$ be the orthonormal basis of $D(L)'$, given by $(\lambda_n h_n)_{n\in\mathbb{N}}$, where $(h_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$, consisting of eigenfunctions of L. $(\lambda_n)_{n\in\mathbb{N}}$ the corresponding eigenvalues and $(\eta_k)_{k\in\mathbb{N}}$ be another orthonormal basis of $L^2(\mathcal{O})$.
Let us first estimate the integral w.r.t. a cylindrical Wiener process. A rather elementary calculation implies that the stochastic integral will be rather agnostic with respect to the power of its integrand, due to the uniform bound.

$$
\begin{aligned}
\mathbb{E}\Big\| \sum_{k=1}^\infty \int_0^t \sigma(u^\varepsilon)\eta_k \,\mathrm{d}W^k(s)\Big\|_{D(L)'}^2 &= \mathbb{E}\sum_{k=1}^\infty \int_0^t \|\sigma(u^\varepsilon)(s)\eta_k\|_{D(L)'}^2 \,\mathrm{d}s \\
&= \mathbb{E}\sum_{k=1}^\infty \int_0^t \sum_{n\in\mathbb{N}} \langle \sigma(u^\varepsilon)(s)\eta_k, \widetilde{h}_n\rangle_{D(L)'}^2 \,\mathrm{d}s \\
&= \mathbb{E}\sum_{k=1}^\infty \int_0^t \sum_{n\in\mathbb{N}} \langle L^{-1}\sigma(u^\varepsilon)(s)\eta_k, L^{-1}\widetilde{h}_n\rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\
&= \mathbb{E}\sum_{k=1}^\infty \int_0^t \sum_{n\in\mathbb{N}} \langle L^{-1}\sigma(u^\varepsilon)(s)\eta_k, h_n\rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\
&= \mathbb{E}\sum_{k=1}^\infty \int_0^t \sum_{n\in\mathbb{N}} \langle \sigma(u^\varepsilon)(s)\eta_k, L^{-1}h_n\rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\
&= \mathbb{E}\sum_{k=1}^\infty \int_0^t \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^2}\langle \sigma(u^\varepsilon)(s)\eta_k, h_n\rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\
&= \int_0^t \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^2}\mathbb{E}\sum_{k=1}^\infty \langle \sigma(u^\varepsilon)(s)h_n, \eta_k\rangle_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\
&= \mathbb{E}\int_0^t \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^2}\|\sigma(u^\varepsilon)(s)h_n\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}s
\end{aligned}
$$

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$$\leq \mathbb{E}\int_0^t \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^2}\|\sigma(u^\varepsilon)(s)\|^2_{L^\infty(\mathcal{O})}\,\mathrm{d}s$$
$$\leq C\mathbb{E}\int_0^t \|\sigma(u^\varepsilon)(s)\|^2_{L^\infty(\mathcal{O})}\,\mathrm{d}s.$$

For the last line to be finite, it is sufficient, that σ is continuous and defined on the whole space, or simply an $L^\infty(\mathcal{O})$ function. We used that, due to the embedding $D(L) \hookrightarrow H^m(\mathcal{O})$ $(m > d/2)$, one can show that the eigenvalues of L decay , up to a constant, as fast as the ones of the Laplace operator (see (see 2.12.12)).
Here the main limiting factor is again the desired entropy inequality. It allows σ of the form $u_i^\varepsilon(1-\sum_{j=1}^n u_j^\varepsilon)$, which acts in a similar fashion as the scalar noise $u^\varepsilon(1-u^\varepsilon)$, but takes into consideration the densities of the other components in the equation as well. Due to the uniform bounds, it is also possible to consider positive powers (≥ 1) of such expressions. Such examples are tailor made for the specific entropy, since h' exhibits a blow-up when $\sum_{i=1}^n u_i^\varepsilon = 1$. Said blow-up also prevents us from using σ of simple power type.

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2.12. Appendix Chapter 2

2.12.1. Appendix A: proofs.

PROOF. (of Lemma 2.5.2)
Step 1: Q_ε is well defined.
Let $w \in D(L)$.

$$\begin{aligned}\|u(w) + \varepsilon L^* Lw\|_{D(L)'} &\le \|u(w)\|_{D(L)'} + |\varepsilon| \|L^* Lw\|_{D(L)'} \\ &\le \|u(w)\|_{L^2(\mathcal{O})} + |\varepsilon| \|L^* Lw\|_{D(L)'}.\end{aligned}$$

By the embedding of $D(L) \hookrightarrow L^\infty(\mathcal{O})$ and the continuity, on the whole space, of the mapping $u(\cdot) = (h')^{-1}(\cdot)$, the right hand side is finite.
Step 2: Q_ε is Fréchet differentiable.
For $h \in D(L)$, we consider

$$\frac{\|u(w+h) + \varepsilon L^* L(w+h) - u(w) - \varepsilon L^* Lw - Fh\|_{D(L)'}}{\|h\|_{D(L)}}$$
$$\frac{\|u(w+h) - u(w) + \varepsilon L^* Lh - Fh\|_{D(L)'}}{\|h\|_{D(L)}},$$

where $F\colon D(L) \to D(L)'$ is our hypothesized Fréchet derivative. We can apply Taylor's formula to the function u to obtain

$$\begin{aligned}&\frac{\|u(w+h) - u(w) + \varepsilon L^* Lh - Fh\|_{D(L)'}}{\|h\|_{D(L)}} \\ &= \frac{\|u'(w)h + \frac{1}{2}\int_0^1 (1-t)u''(w+th)h^2 \,\mathrm{d}t + \varepsilon L^* Lh - Fh\|_{D(L)'}}{\|h\|_{D(L)}} \\ &\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \|\frac{1}{2}\int_0^1 (1-t)u''(w+th)h^2 \,\mathrm{d}t\|_{D(L)'}}{\|h\|_{D(L)}}\end{aligned}$$

using lemma 2.12.4 with $X = D(L) \hookrightarrow Y' = L^\infty(\mathcal{O})$, $L^1(\mathcal{O}) = Y \hookrightarrow D(L)'$.

$$\begin{aligned}&\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \|\frac{1}{2}\int_0^1 (1-t)u''(w+th)h^2 \,\mathrm{d}t\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}} \\ &\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \|\int_0^1 (1-t)u''(w+th)\,\mathrm{d}t\|_{L^1(\mathcal{O})}\|h^2\|_{L^\infty(\mathcal{O})}}{\|h\|_{D(L)}} \\ &\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \|\int_0^1 (1-t)u''(w+th)\,\mathrm{d}t\|_{L^1(\mathcal{O})}\|h\|^2_{L^\infty(\mathcal{O})}}{\|h\|_{D(L)}} \\ &\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \int_0^1 \|u''(w+th)\|_{L^1(\mathcal{O})}\,\mathrm{d}t\|h\|^2_{D(L)}}{\|h\|_{D(L)}} \\ &\le \frac{\|u'(w)h + \varepsilon L^* Lh - Fh\|_{D(L)'} + \sup_{t\in[0,1]}\|u''(w+th)\|_{L^1(\mathcal{O})}\|h\|^2_{D(L)}}{\|h\|_{D(L)}}.\end{aligned}$$

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Choosing $F = F_w := (u'(w) + \varepsilon L^*L)$, we see that the first term vanishes. We will refer to the operator F_w as $DQ_{\varepsilon,w}$. By the embedding $D(L) \hookrightarrow L^\infty(\mathcal{O})$ and the assumed continuity of u'', it follows that the $\sup_{t\in[0,1]} \|u''(w+th)\|_{L^1(\mathcal{O})}$ is bounded. Hence, letting $\|h\|_{D(L)} \to 0$, we obtain that the Fréchet derivative of Q_ε is in fact given by the operator $DQ_{\varepsilon,w}$. Further properties of this operator will be investigated in Lemma 2.5.3.
The continuity of Q_ε is now a direct consequence of the differentiability.
Step 3: Q_ε is monotone: Let $w_1, w_2 \in D(L)$, then

$$\begin{aligned}\langle Q_\varepsilon(w_1) - Q_\varepsilon(w_2), w_1 - w_2\rangle_{D(L)'\times D(L)} &= \langle u(w_1) + \varepsilon L^*Lw_1 - u(w_2) - \varepsilon L^*Lw_2, w_1 - w_2\rangle \\ &= \langle u(w_1) - u(w_2), w_1 - w_2\rangle + \varepsilon\langle L^*Lw_1 - L^*Lw_2, w_1 - w_2\rangle \\ &\text{using the properties of the operator } L \text{ and the function } u(\cdot), \\ &\text{we can express this duality with respect to the } L^2(\mathcal{O}) \text{ inner product} \\ &= \langle u(w_1) - u(w_2), w_1 - w_2\rangle_{L^2(\mathcal{O})} + \varepsilon\langle L(w_1 - w_2), L(w_1 - w_2)\rangle_{L^2(\mathcal{O})} \\ &= \langle u(w_1) - u(w_2), w_1 - w_2\rangle + \varepsilon\|L(w_1 - w_2)\|^2_{L^2(\mathcal{O})} \\ &= \langle u(w_1) - u(w_2), h'(u(w_1)) - h'(u(w_2))\rangle + \varepsilon\|L(w_1 - w_2)\|^2_{L^2(\mathcal{O})} \geq 0.\end{aligned}$$

Here we used the assumed convexity of h as well as the property that h' is invertible and its inverse is given by $u(\cdot)$. Using Lemma 2.12.3, we note that there exists a constant $c > 0$, such that $\|L(w_1 - w_2)\|_{L^2(\mathcal{O})} \geq c\|w_1 - w_2\|_{D(L)}$. Hence we obtain the strong monotonicity.
Step 4: Coercivity.
We set $w_1 = w$ and $w_2 = 0$ and proceed as in the previous step.

$$\begin{aligned}\langle Q_\varepsilon(w) - Q_\varepsilon(0), w\rangle_{D(L)'\times D(L)} &= \langle u(w) + \varepsilon L^*Lw - u(0), w\rangle \\ &= \langle u(w) + \varepsilon L^*Lw, w\rangle - \langle u(0), w\rangle \geq C(\varepsilon)\|w\|^2_{D(L)}.\end{aligned}$$

Hence,

$$\begin{aligned}\langle Q_\varepsilon(w) - Q_\varepsilon(0), w\rangle_{D(L)'\times D(L)} &= \langle u(w) + \varepsilon L^*Lw - u(0), w\rangle \\ &\geq C(\varepsilon)\|w\|^2_{D(L)} - |\langle u(0), w\rangle| \\ &\geq C(\varepsilon)\|w\|^2_{D(L)} - |u(0)|\|w\|_{L^1(\mathcal{O})} \\ &\geq C(\varepsilon)\|w\|^2_{D(L)} - |u(0)|C(\mathcal{O})\|w\|_{L^2(\mathcal{O})} \\ &\geq C(\varepsilon)\|w\|^2_{D(L)} - \frac{2}{C(\varepsilon)}(|u(0)|C(\mathcal{O}))^2 - \frac{C(\varepsilon)}{2}\|w\|^2_{L^2(\mathcal{O})} \\ &\geq \frac{C(\varepsilon)}{2}\|w\|^2_{D(L)} - \frac{2}{C(\varepsilon)}(|u(0)|C(\mathcal{O}))^2.\end{aligned}$$

Hence, by Theorem 2.12.9, for every $v \in D(L)'$, there exists a $w \in D(L)$, such that $Q_\varepsilon(w) = v$. By the strong monotonicity from step 3, this w is unique. □

Proof. (of Lemma 2.5.3) Step 1: The operator $DQ_{\varepsilon,w}$ is well defined, since for $\widetilde{w} \in D(L)$

$$\|u'(w)\widetilde{w} + \varepsilon L^*L\widetilde{w}\|_{D(L)'} \leq \|u'(w)\widetilde{w}\|_{D(L)'} + |\varepsilon|\|L^*L\widetilde{w}\|_{D(L)'}$$

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$$\leq \|u'(w)\widetilde{w}\|_{L^2(\mathcal{O})} + |\varepsilon|\|L^*L\widetilde{w}\|_{D(L)'}$$
$$\leq \|\widetilde{w}\|_{L^\infty(\mathcal{O})}\|u'(w)\|_{L^2(\mathcal{O})} + |\varepsilon|\|L^*L\widetilde{w}\|_{D(L)'}$$

By the embedding of $D(L) \hookrightarrow L^\infty(\mathcal{O})$ and the continuity of the mapping $u'(\cdot) = ((h')^{-1})'(\cdot)$, the right hand side is finite.
Step 2: The operator $DQ_{\varepsilon,w}$ is continuous. Let $w_1, w_2 \in D(L)$, then

$$\begin{aligned}\|u'(w)(w_1-w_2)+\varepsilon L^*L(w_1-w_2)\|_{D(L)'} &\leq \|u'(w)(w_1-w_2)\|_{D(L)'} + \varepsilon\|L^*L(w_1-w_2)\|_{D(L)'}\\ &\leq \|u'(w)\|_{L^\infty}\|(w_1-w_2)\|_{D(L)} + \varepsilon\|w_1-w_2\|_{D(L)}\\ &\leq (\|u'(w)\|_{L^\infty}+\varepsilon)\|w_1-w_2\|_{D(L)}.\end{aligned}$$

Step 3: The operator $DQ_{\varepsilon,w}$ is strongly monotone.
Let $w_1, w_2 \in D(L)$, then

$$\begin{aligned}\langle DQ_{\varepsilon,w}(w_1) - DQ_{\varepsilon,w}(w_2), w_1-w_2\rangle_{D(L)'\times D(L)} &= \langle u'(w)(w_1-w_2)+\varepsilon L^*L(w_1-w_2), w_1-w_2\rangle\\ &= \langle u'(w)(w_1-w_2), w_1-w_2\rangle + \varepsilon\langle L^*L(w_1-w_2), w_1-w_2\rangle\end{aligned}$$

using the properties of the entropy h, we recall that the matrix $u'(\cdot) = (h'')^{-1}(\cdot)$ is positive semi-definite,

$$\begin{aligned}&= \langle u'(w)(w_1-w_2), w_1-w_2\rangle_{L^2(\mathcal{O})} + \varepsilon\langle L(w_1-w_2), L(w_1-w_2)\rangle_{L^2(\mathcal{O})}\\ &= \int_{\mathcal{O}} (w_1-w_2)^T(h''(w))^{-1})(w_1-w_2)\,\mathrm{d}x + \varepsilon\|L(w_1-w_2)\|^2_{L^2(\mathcal{O})}\\ &\geq \varepsilon\|L(w_1-w_2)\|^2_{L^2(\mathcal{O})} \geq c\varepsilon\|w_1-w_2\|^2_{D(L)}.\end{aligned}$$

We used Lemma 2.12.3 for the last inequality.
Step 4: The operator $DQ_{\varepsilon,w}$ is coercive.
Since the operator $DQ_{\varepsilon,w}$ is linear, it suffices to set $w_2 = 0$ to directly obtain

$$\langle DQ_{\varepsilon,w}(w_1), w_1\rangle_{D(L)'\times D(L)} \geq c\varepsilon\|w_1\|^2_{D(L)}.$$

By Theorem 2.12.9, we conclude that the operator $DQ_{\varepsilon,w}$ is invertible. □

Proof of Lemma 2.5.5. Step 1: The operator R_ε is well defined by construction as a result of the properties of Q_ε.
Step 2: The operator R_ε is Lipschitz continuous.
Let $v_1, v_2 \in D(L)'$, then

$$\begin{aligned}\|R_\varepsilon(v_1)-R_\varepsilon(v_2)\|^2_{D(L)} &= \|R_\varepsilon Q_\varepsilon(w_1) - R_\varepsilon Q_\varepsilon(w_2)\|^2_{D(L)}\\ &= \|w_1-w_2\|^2_{D(L)} \leq C\|L(w_1-w_2)\|^2_{L^2(\mathcal{O})}\\ &\leq \frac{C}{\varepsilon}\langle Q_\varepsilon(w_1)-Q_\varepsilon(w_2), w_1-w_2\rangle = \frac{C}{\varepsilon}\langle v_1-v_2, w_1-w_2\rangle\\ &\leq \frac{C}{\varepsilon}\|v_1-v_2\|_{D(L)'}\|w_1-w_2\|_{D(L)} = \frac{C}{\varepsilon}\|v_1-v_2\|_{D(L)'}\|R_\varepsilon(v_1)-R_\varepsilon(v_2)\|_{D(L)}.\end{aligned}$$

We conclude that

$$\|R_\varepsilon(v_1)-R_\varepsilon(v_2)\|_{D(L)} \leq \frac{C}{\varepsilon}\|v_1-v_2\|_{D(L)'}.$$

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Step 3: The operator R_ε is Fréchet differentiable.
By the inverse function Theorem (Theorem 2.12.8), $DR_{\varepsilon,v} = (DQ_{\varepsilon,R_\varepsilon(v)})^{-1}$, for every $v \in D(L)'$.
Step 4: The operator R_ε is strictly monotone.
Let $v_1, v_2 \in D(L)'$, then

$$\begin{aligned}\langle R_\varepsilon(v_1) - R_\varepsilon(v_2), v_1 - v_2\rangle_{D(L)\times D(L)'} &= \langle R_\varepsilon(v_1) - R_\varepsilon(v_2), Q_\varepsilon(R_\varepsilon(v_1)) - Q_\varepsilon(R_\varepsilon(v_1))\rangle_{D(L)\times D(L)'} \\ &\geq C\|R_\varepsilon(v_1) - R_\varepsilon(v_2)\|^2_{D(L)}.\end{aligned}$$

By Lemma 2.5.2,

$$\|Q_\varepsilon(R_\varepsilon(v_1)) - Q_\varepsilon(R_\varepsilon(v_2))\|_{D(L)'} \leq C(R_\varepsilon(v_1), R_\varepsilon(v_2))\|R_\varepsilon(v_1) - R_\varepsilon(v_2)\|_{D(L)}.$$

Using this inequality, we conclude

$$\begin{aligned}\langle R_\varepsilon(v_1) - R_\varepsilon(v_2), v_1 - v_2\rangle_{D(L)\times D(L)'} &\geq \|R_\varepsilon(v_1) - R_\varepsilon(v_2)\|^2_{D(L)} \\ &\geq \frac{1}{C(R_\varepsilon(v_2), R_\varepsilon(v_2))}\|Q_\varepsilon(R_\varepsilon(v_1)) - Q_\varepsilon(R_\varepsilon(v_2))\|^2_{D(L)'} \\ &= \frac{1}{C(R_\varepsilon(v_2), R_\varepsilon(v_2))}\|v_1 - v_2\|^2_{D(L)'}.\end{aligned}$$

Step 5: The operator R_ε is coercive.
This follows directly by setting $v_1 = v, v_2 = 0$ in the previous calculation:

$$\begin{aligned}\langle R_\varepsilon(v) - R_\varepsilon(0), v\rangle_{D(L)\times D(L)'} &\geq \|R_\varepsilon(v) - R_\varepsilon(0)\|^2_{D(L)} \\ &\geq \frac{1}{C(R_\varepsilon(v), R_\varepsilon(0))}\|Q_\varepsilon(R_\varepsilon(v)) - Q_\varepsilon(R_\varepsilon(0))\|^2_{D(L)'} \\ &= \frac{1}{C(R_\varepsilon(v), R_\varepsilon(0))}\|v\|^2_{D(L)'}.\end{aligned}$$

□

Proof of Lemma 2.5.6. Step 1: The operator $DR_{\varepsilon,w} = (DQ_{\varepsilon,w})^{-1}$ is well defined.
By Lemma 2.5.3 and Theorem 2.12.9, it is clear that the operator is well defined.
Step 2: The operator $DR_{\varepsilon,w} = (DQ_{\varepsilon,w})^{-1}$ is continuous.
The continuity follows directly from the monotonicity of the operator $DQ_{\varepsilon,w}$,

$$\begin{aligned}\|w_1 - w_2\|^2_{D(L)} &\leq \frac{1}{C\varepsilon}\langle u'(w)w_1 - u'(w)w_2, w_1 - w_2\rangle + \varepsilon\langle L^*Lw_1 - L^*Lw_2, w_1 - w_2\rangle \\ &\leq \frac{1}{C\varepsilon}\|(u'(w) + \varepsilon L^*L)w_1 - (u'(w) + \varepsilon L^*L)w_2\|_{D(L)'}\|w_1 - w_2\|_{D(L)}.\end{aligned}$$

Hence

$$\|w_1 - w_2\|_{D(L)} \leq \frac{1}{C\varepsilon}\|(u'(w) + \varepsilon L^*L)w_1 - (u'(w) + \varepsilon L^*L)w_2\|_{D(L)'}.$$

Since the operator is bijective, which follows from Theorem 2.12.9, we can rewrite the expression as

$$\|DQ_\varepsilon(w)^{-1}v_1 - DQ_\varepsilon(w)^{-1}v_2\|_{D(L)} = \|w_1 - w_2\|_{D(L)}$$

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$$\leq \frac{1}{C\varepsilon}\|(u'(w)+\varepsilon L^*L)(DQ_\varepsilon(w)^{-1}v_1)-(u'(w)+\varepsilon L^*L)(DQ_\varepsilon(w))^{-1}v_2)\|_{D(L)'}$$
$$= \frac{1}{C\varepsilon}\|v_1-v_2\|_{D(L)'}.$$

Hence, the operator $(u'(R_\varepsilon(v))+\varepsilon L^*L)^{-1}(\cdot) = (u'(w)+\varepsilon L^*L)^{-1}(\cdot)$ is Lipschitz continuous.
Step 3: The operator $DR_{\varepsilon,w} = (DQ_{\varepsilon,w})^{-1}$ is strictly monotone.
Let $v_1, v_2 \in D(L)'$, then

$$\langle (DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2), v_1-v_2\rangle_{D(L)\times D(L)'}$$
$$= \langle (DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2), DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_1))-DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_1))\rangle_{D(L)\times D(L)'}$$
$$\geq \|(DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2)\|_{D(L)}.$$

By Lemma 2.5.3,

$$\|DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_1))-DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_2))\|_{D(L)'}$$
$$\leq C((DQ_{\varepsilon,w})^{-1}(v_2),(DQ_{\varepsilon,w})^{-1}(v_2))\|(DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2)\|_{D(L)}.$$

Using this inequality, we conclude

$$\langle DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2), v_1-v_2\rangle_{D(L)\times D(L)'} \geq \|(DQ_{\varepsilon,w})^{-1}(v_1)-(DQ_{\varepsilon,w})^{-1}(v_2)\|_{D(L)}$$
$$\geq \frac{1}{C((DQ_{\varepsilon,w})^{-1}(v_2)(DQ_{\varepsilon,w})^{-1}(v_2))}\|DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_1))DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v_2))\|_{D(L)'}$$
$$= \frac{1}{C((DQ_{\varepsilon,w})^{-1}(v_1),(DQ_{\varepsilon,w})^{-1}(v_2))}\|v_1-v_2\|_{D(L)'}.$$

Step 4: The operator $DR_{\varepsilon,w} = (DQ_{\varepsilon,w})^{-1}$ is coercive.
This is again a direct consequence of the monotonicity,

$$\langle (DQ_{\varepsilon,w})^{-1}(v)-(DQ_{\varepsilon,w})^{-1}(0), v\rangle_{D(L)\times D(L)'} \geq \|(DQ_{\varepsilon,w})^{-1}(v)-(DQ_{\varepsilon,w})^{-1}(0)\|^2_{D(L)}$$
$$\geq \frac{1}{C((DQ_{\varepsilon,w})^{-1}(v),(DQ_{\varepsilon,w})^{-1}(0))}\|(DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(v))-(DQ_{\varepsilon,w}((DQ_{\varepsilon,w})^{-1}(0))\|^2_{D(L)'}$$
$$= \frac{1}{C((DQ_{\varepsilon,w})^{-1}(v),(DQ_{\varepsilon,w})^{-1}(0))}\|v\|^2_{D(L)'}.$$

□

Proof of Lemma 2.5.7. Let $w \in D(L)$. By embedding $D(L) \hookrightarrow L^\infty(\mathcal{O})$, there exists a positive constant $C \in \mathbb{R}$ such that for almost every $x \in \mathcal{O}, |w(x)| \leq C$. By assumption, u is a continuous function, defined on the whole $\mathbb{R}^d$. As a continuous function on the reals, it maps the compact set $[-C,C]^d$ to a compact set of $\mathbb{R}^d$, which is in particular bounded. Hence the map $u(\cdot)$ is well defined from $D(L)$ to $L^\infty(\mathcal{O})$.
To verify the differentiablity, let $\phi \in D(L)$. By the mean value Theorem (see Theorem 2.12.6),

$$\frac{\|u(w+\phi)-u(w)-u'(w)\phi\|_{L^\infty(\mathcal{O})}}{\|\phi\|_{D(L)}} \leq \frac{\|\frac{1}{2}\int_0^1(1-t)u''(w+t\phi)\,\mathrm{d}t\phi^2\|_{L^\infty(\mathcal{O})}}{\|\phi\|_{D(L)}}$$

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$$\leq \frac{\|\phi\|^2_{L^\infty(\mathcal{O})}\| \int_0^1 (1-t)u''(w+t\phi)\,\mathrm{d}t\|_{L^\infty(\mathcal{O})}}{\|\phi\|_{D(L)}}$$
$$\leq \|\phi\|_{D(L)}\| \int_0^1 (1-t)u''(w+t\phi)\,\mathrm{d}t\|_{L^\infty(\mathcal{O})}.$$

By assumption, u'' is continuous on $\mathbb{R}^n$ and the integral is bounded by the same arguments used above. We conclude that $u(\cdot)\colon D(L) \to L^\infty(\mathcal{O})$ is Fréchet differentiable.
The monotonicity follows from the calculations performed in the proof of Lemma 2.5.2, with $\varepsilon = 0$. □

Proof of Lemma 2.5.15. Let $v \in L^0(\Omega, L^0(0,T; D(L)'))$, be progressively measurable i.e. $\mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(D(L)')$ measurable, for every $t \in [0,T]$. Let us fix t and $0 \leq s \leq t$. Then v can be approximated by a sequence of simple functions (see [33, Theorem 4.38])

$$S_{n,m}(v)(\omega,s) = \sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\omega,s) v^{ij}, \tag{2.115}$$

where $A_i \in \mathcal{F}_t$, $B_j \in \mathcal{B}([0,t])$, $v^{ij} \in D(L)'$ and χ denotes the indicator function. Before we begin with the proof, we make the following observation,

$$R_\varepsilon(S_{n,m}(v))(\omega,s) = R_\varepsilon\Big(\sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\omega,s) v^{ij}\Big),$$
$$= \sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\cdot,\cdot) R_\varepsilon\Big(\sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\cdot,\cdot) v^{ij}\Big)(\omega,s)$$
$$= \sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\cdot,\cdot) R_\varepsilon\big(v^{ij}\big)(\omega,s).$$

For the second equality, we used that the families of sets $\{A_i\}_{i=1,\dots,n}, \{B_j\}_{j=1,\dots,m}$ partition Ω and $[0,t]$, respectively. We define the function $(\omega,s) \mapsto R_\varepsilon(v(\omega,s,\cdot))$ from $\Omega \times [0,t] \to D(L)$, for every (ω,s) outside a null-set for each representative of the equivalence class. Next, we want to show that there exists a sequence of simple functions $S_{n,m}(R_\varepsilon(v))$ of the form (2.115), approximating $R_\varepsilon(v)$ for almost every $(\omega,s) \in \Omega \times [0,t]$. For this, we define

$$S_{n,m}(R_\varepsilon(v))(\omega,s) = \sum_{i=1}^n \sum_{j=1}^m \chi_{A_i \times B_j}(\omega,s) R_\varepsilon(v^{ij}),$$

where v^{ij} is chosen as in (2.115). We observe that

$$\|R_\varepsilon(v) - S_{n,m}(R_\varepsilon(v))\|_{D(L)} = \|R_\varepsilon(v) - R_\varepsilon(S_{n,m}(v))\|_{D(L)} \leq \frac{C}{\varepsilon}\|v - S_{n,m}(v)\|_{D(L)'}.$$

The right hand side converges in $D(L)'$ for almost every $(\omega,s) \in \Omega \times [0,t]$ and consequently the left hand side converges in $D(L)$ for almost every $(\omega,s) \in \Omega \times [0,t]$. We have found a $\mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(D(L)')$ measurable sequence of simple functions converging to $R_\varepsilon(v)$ for a.e. $(\omega,s) \in \Omega \times [0,t]$. This concludes the proof. □

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PROOF OF LEMMA 2.5.16. We proceed as in the previous proof. Let

$$S_{n,m}(w) = \sum_{i=1}^{n}\sum_{j=1}^{m} \chi_{A_i\times B_j}(\cdot,\cdot)w^{ij}, \tag{2.116}$$

where $A_i \in \mathcal{F}_t$, $B_j \in \mathcal{B}([0,t])$ and $w^{ij} \in D(L)$, such that $S(w) \to w$ in $D(L)$ for almost every (ω, s).

$$\begin{aligned} u(S_{n,m}(w)) &= u\bigg(\sum_{i=1}^{n}\sum_{j=1}^{m} \chi_{A_i\times B_j}(\omega,s)w^{ij}\bigg) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \chi_{A_i\times B_j}(\cdot,\cdot)u\bigg(\sum_{i=1}^{n}\sum_{j=1}^{m} \chi_{A_i\times B_j}(\cdot,\cdot)w^{ij}\bigg)(\omega,s) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \chi_{A_i\times B_j}(\cdot,\cdot)u\big(w^{ij}\big)(\omega,s). \end{aligned}$$

Since $u\colon D(L) \to L^\infty(\mathcal{O})$ is Fréchet differentiable, it is locally Lipschitz continuous. Hence assuming $\|w^{ij}\|_{L^\infty} \le 2\|w\|_{L^\infty(\mathcal{O})}$ for every pair i,j and almost every (ω,s), we obtain

$$\begin{aligned} \|u(w) - S_{n,m}(u(w))\|_{L^\infty(\mathcal{O})} &= \|u(w) - u(S_{n,m}(w))\|_{L^\infty(\mathcal{O})} \\ &\le C(\|u'(2|w|)\|_{L^\infty(\mathcal{O})})\|w - S_{n,m}(w)\|_{L^\infty(\mathcal{O})} \le C(\|u'(2|w|)\|_{L^\infty(\mathcal{O})})\|w - S_{n,m}(w)\|_{D(L)}. \end{aligned}$$

The right hand side converges in $D(L)$ for almost every $(\omega,s) \in \Omega\times[0,t]$ and consequently the left hand side converges in $L^\infty(\mathcal{O})$ for almost every $(\omega,s) \in \Omega\times[0,t]$. This concludes the proof. □

PROOF OF LEMMA 2.5.17. For the first claim, we consider

$$\|R_\varepsilon(v(t)) - R_\varepsilon(v(s))\|_{D(L)} \le \frac{C}{\varepsilon}\|v(t) - v(s)\|_{D(L)'}.$$

By continuity of v, the right hand side approaches 0 as $s \to t$. This verifies the first claim. For the second claim, by virtue of the local Lipschitz continuity of the map $u\colon D(L) \to L^\infty(\mathcal{O})$,

$$\begin{aligned} &\|u(R_\varepsilon(v(t))) - u(R_\varepsilon(v(s)))\|_{L^\infty(\mathcal{O})} \\ &\qquad \le C(\|u(R_\varepsilon(v(t)))\|_{L^\infty(\mathcal{O})}, \|u(R_\varepsilon(v(s)))\|_{L^\infty(\mathcal{O})})\|R_\varepsilon(v(t)) - R_\varepsilon(v(s))\|_{D(L)} \\ &\qquad \le \frac{C(\|u(R_\varepsilon(v(t)))\|_{L^\infty(\mathcal{O})}, \|u(R_\varepsilon(v(s)))\|_{L^\infty(\mathcal{O})})}{\varepsilon}\|v(t) - v(s)\|_{D(L)'}. \end{aligned}$$

We conclude that $u(R_\varepsilon(v)) \in C([0,T]; L^\infty(\mathcal{O}))$.
Hence, for fixed $t \in [0,T]$, $u(R_\varepsilon(v))(t,\cdot)$ is measurable from $\mathcal{O}$ to $\mathbb{R}^n$ and for fixed $x \in \mathcal{O}$, it is continuous from $[0,T]$ to $\mathbb{R}^n$. By [33, Theorem 4.51, p.153], $u(R_\varepsilon(v))(t,x)$ is a jointly measurable Carathéodory function. Considering $u(R_\varepsilon(v)) \in L^\infty(0,T;L^\infty(\mathcal{O}))$ and let N_t be the null-set such that $|u(R_\varepsilon(v))(t,\cdot)| = \infty$. By the joint measurability, we infer that the set $N := \{(t,x) \in [0,T]\times\mathcal{O}\colon x \in N_t\}$ is measurable. A Fubini type argument, see Lemma 2.12.10, shows that $u(R_\varepsilon(v)) \in L^\infty([0,T]\times\mathcal{O})$. □

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PROOF OF LEMMA 2.5.18. Since the operator

$$v \mapsto \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right) \tag{2.117}$$

is separably valued, by Pettis' theorem, it suffices to show that

$$\left\langle \operatorname{div}\left(B(u(R_\varepsilon(v)))\nabla R_\varepsilon(v)\right), g\right\rangle_{L^p(0,T;D(L)')\times L^q(0,T;D(L)')}$$

is measurable. We will verify this result by showing that the operator (2.117) is locally Lipschitz continuous between the spaces in question. Since $L^1(\mathcal{O}) \hookrightarrow D(L)'$,

$$\begin{aligned}
&\int_0^t \left\| \operatorname{div}\left(B(u(R_\varepsilon(v_1)))\nabla R_\varepsilon(v_1)\right)(s) - \operatorname{div}\left(B(u(R_\varepsilon(v_2)))\nabla R_\varepsilon(v_2)\right)(s)\right\|_{D(L)'}^p \mathrm{d}s \\
&\leq \int_0^t \left\|B(u(R_\varepsilon(v_1)))\nabla R_\varepsilon(v_1)(s) - B(u(R_\varepsilon(v_2)))\nabla R_\varepsilon(v_2)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s,
\end{aligned}$$

expanding with $\pm B(u(R_\varepsilon(v_1)))\nabla R_\varepsilon(v_2)$ and using the triangle inequality yields

$$\begin{aligned}
&\leq C(p)\int_0^t \left\|B(u(R_\varepsilon(v_1)))\nabla R_\varepsilon(v_1)(s) - B(u(R_\varepsilon(v_2)))\nabla R_\varepsilon(v_2)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\quad + \int_0^t \left\|B(u(R_\varepsilon(v_1)))(s)\left(\nabla R_\varepsilon(v_1)(s) - \nabla R_\varepsilon(v_2)(s)\right)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\leq C(p)\int_0^t \left\|B(u(R_\varepsilon(v_1)))(s) - B(u(R_\varepsilon(v_2)))(s)\right\|_{L^\infty(\mathcal{O})}^p \left\|\nabla R_\varepsilon(v_1)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\quad + \int_0^t \left\|B(u(R_\varepsilon(v_1)))(s)\right\|_{L^\infty(\mathcal{O})}^p \left\|\nabla R_\varepsilon(v_1)(s) - \nabla R_\varepsilon(v_2)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\leq C(p)\int_0^t \left\|B(u(R_\varepsilon(v_1)))(s) - B(u(R_\varepsilon(v_2)))(s)\right\|_{L^\infty(\mathcal{O})}^p \left\|\nabla R_\varepsilon(v_1)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\quad + \int_0^t \left\|B(u(R_\varepsilon(v_1)))(s)\right\|_{L^\infty(\mathcal{O})}^p \left\|R_\varepsilon(v_1) - R_\varepsilon(v_2)(s)\right\|_{D(L)}^p \mathrm{d}s,
\end{aligned}$$

where we used that $W^{1,1}(\mathcal{O}) \hookrightarrow D(L)$. By the definition of $B(u(\cdot))$, we obtain

$$\begin{aligned}
&\leq C(p)\int_0^t \left\|A(u(\cdot))(h'')^{-1}(u(\cdot))(R_\varepsilon(v_1))(s) - A(u(\cdot))(h'')^{-1}(u(\cdot))(R_\varepsilon(v_2))(s)\right\|_{L^\infty(\mathcal{O})}^p \\
&\qquad\qquad \times \left\|\nabla R_\varepsilon(v_1)(s)\right\|_{L^1(\mathcal{O})}^p \mathrm{d}s \\
&\quad + \frac{C}{\varepsilon}\int_0^t \left\|A(u(\cdot))(h'')^{-1}(u(\cdot))(R_\varepsilon(v_1))(s)\right\|_{L^\infty(\mathcal{O})}^p \left\|v_1(s) - v_2(s)\right\|_{D(L)'}^p \mathrm{d}s.
\end{aligned}$$

Expanding with $\pm A(u(R_\varepsilon(v_1)))(h'')^{-1}(u(R_\varepsilon(v_2)))$ and using the sub-multiplicativity of the $L^\infty(\mathcal{O})$ norm,

$$\leq C(p)\int_0^t \left\|[A(u(R_\varepsilon(v_1)))(s) - A(u(R_\varepsilon(v_2)))(s)](h'')^{-1}(u(R_\varepsilon(v_2)))(s)\right\|_{L^\infty(\mathcal{O})}^p$$

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$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\int_0^t \big\|A(u((R_\varepsilon(v_1)))(s)[(h'')^{-1}(u(R_\varepsilon(v_1)))(s)-(h'')^{-1}(u((R_\varepsilon(v_2))))(s)]\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\frac{C}{\varepsilon}\int_0^t \big\|A(u(\cdot))(h'')^{-1}(u(\cdot))(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\big\|^p_{D(L)'}\,\mathrm{d}s.$$

Using the triangle inequality in the second and third term yields

$$\le C(p)\int_0^t \big\|A(u(R_\varepsilon(v_1)))(s)-A(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\int_0^t \big\|A(u((R_\varepsilon(v_1)))(s)\|^p_{L^\infty(\mathcal{O})}\|(h'')^{-1}(u(R_\varepsilon(v_1)))(s)-(h'')^{-1}(u((R_\varepsilon(v_2))))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\frac{C}{\varepsilon}\int_0^t \big\|A(u(R_\varepsilon(v_1)))(s)\|^p_{L^\infty(\mathcal{O})}\|(h'')^{-1}(u(R_\varepsilon(v_1)))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\,\mathrm{d}s.$$

By the regularity and liner growth assumption on the matrix A, we obtain the following,

$$\le C(p,A(\|u\|_{L^\infty}))\int_0^t \big\|u(R_\varepsilon(v_1))(s)-u(R_\varepsilon(v_2))(s)\|^p_{L^\infty(\mathcal{O})}\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\int_0^t \frac{1}{\varepsilon}(1+\|u((R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1)))(s)-(h'')^{-1}(u((R_\varepsilon(v_2))))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\frac{C}{\varepsilon}\int_0^t (1+\|u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\,\mathrm{d}s.$$

In the following steps, we will use an intermediate value theorem (inequality). Hence, for notational convenience, we replace the factors of the form $\eta R_\varepsilon(v_1)+(1-\eta)R_\varepsilon(v_2)$ by ξ, since their explicit structure won't be relevant.

$$\le C(p,A(\|u\|_{L^\infty}))\int_0^t \big\|u'(\xi_1)(R_\varepsilon(v_1)(s)-R_\varepsilon(v_2))(s)\|^p_{L^\infty(\mathcal{O})}\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$
$$+\int_0^t \frac{1}{\varepsilon}(1+\|u((R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|((h'')^{-1})'(\xi_2)u'(\xi_3)(R_\varepsilon(v_1)(s)-R_\varepsilon(v_2))(s)\|^p_{L^\infty(\mathcal{O})}$$
$$\times \|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s$$

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$$+\frac{C}{\varepsilon}\int_0^t (1+\|u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\,\mathrm{d}s.$$

Once again by the triangle inequality applied to the first term,

$$\begin{aligned}
&\leq C(p,A(\|u\|_{L^\infty}))\int_0^t \|u'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|R_\varepsilon(v_1)(s)-R_\varepsilon(v_2)(s)\|^p_{L^\infty(\mathcal{O})}\\
&\qquad\qquad\times\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}\|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s\\
&+\int_0^t \frac{1}{\varepsilon}(1+\|u((R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|((h'')^{-1})'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|u'(\xi_2)(s)\|^p_{L^\infty(\mathcal{O})}\\
&\qquad\qquad\times\|R_\varepsilon(v_1)(s)-R_\varepsilon(v_2)(s)\|^p_{L^\infty(\mathcal{O})}\|\nabla R_\varepsilon(v_1)(s)\|^p_{L^1(\mathcal{O})}\,\mathrm{d}s\\
&+\frac{C}{\varepsilon}\int_0^t (1+\|u(R_\varepsilon(v_1))\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1))\|^p_{L^\infty(\mathcal{O})}\|v_1-v_2\|^p_{D(L)'}\,\mathrm{d}s.
\end{aligned}$$

Using the continuity of the embedding $L^\infty(\mathcal{O})\hookrightarrow D(L)$,

$$\begin{aligned}
&\leq C(p,A(\|u\|_{L^\infty}))\int_0^t \|u'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|R_\varepsilon(v_1)(s)-R_\varepsilon(v_2)(s)\|^p_{D(L)}\\
&\qquad\qquad\times\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}\|R_\varepsilon(v_1)(s)\|^p_{D(L)}\,\mathrm{d}s\\
&+\int_0^t \frac{1}{\varepsilon}(1+\|u((R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|((h'')^{-1})'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|u'(\xi_2)(s)\|^p_{L^\infty(\mathcal{O})}\\
&\qquad\qquad\times\|R_\varepsilon(v_1)(s)-R_\varepsilon(v_2)(s)\|^p_{D(L)}\|R_\varepsilon(v_1)(s)\|^p_{D(L)}\,\mathrm{d}s\\
&+\frac{C}{\varepsilon}\int_0^t (1+\|u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\,\mathrm{d}s.
\end{aligned}$$

The Lipschitz continuity of $R_\varepsilon\colon D(L)'\to D(L)$ yields,

$$\begin{aligned}
&\leq C(p,A(\|u\|_{L^\infty}),\frac{1}{\varepsilon})\int_0^t \|u'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\|(h'')^{-1}(u(R_\varepsilon(v_2)))(s)\|^p_{L^\infty(\mathcal{O})}\\
&\qquad\qquad\times(C+\|v_1(s)\|^p_{D(L)'})\,\mathrm{d}s\\
&+\int_0^t \frac{1}{\varepsilon}(1+\|u((R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|((h'')^{-1})'(\xi_1)(s)\|^p_{L^\infty(\mathcal{O})}\|u'(\xi_2)(s)\|^p_{L^\infty(\mathcal{O})}\\
&\qquad\qquad\times\|v_1(s)-v_2(s)\|^p_{D(L)'}(C+\|v_1(s)\|^p_{D(L)'})\,\mathrm{d}s\\
&+\frac{C}{\varepsilon}\int_0^t (1+\|u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})})\|(h'')^{-1}(u(R_\varepsilon(v_1))(s)\|^p_{L^\infty(\mathcal{O})}\|v_1(s)-v_2(s)\|^p_{D(L)'}\,\mathrm{d}s.
\end{aligned}$$

By arguments similar to Lemma 2.5.12, we can estimate all terms including an $L^\infty(\mathcal{O})$ norm in $L^\infty(0,T;L^\infty(\mathcal{O}))$ by a constant depending on u,u',u'',ε and v_1,v_2. We conclude that the operator (2.117) is locally Lipschitz continuous, with a constant depending on $p,A,u,u',u'',\frac{1}{\varepsilon}$ and the interval $[0,t]$. □

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PROOF OF LEMMA 2.5.19. By [39, Proposition E1,E2, p.397/398], the strongly measurable function v can (ω,t)-a.e. be approximated by simple functions of the form

$$S_{n,m}(v)=\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\omega,t)v^{ij},$$

where $(A_i)_{i=1,\dots,n}\in\mathcal{F}$, $(B_j)_{j=1,\dots,m}\in\mathcal{B}([0,T])$ and $(v^{ij})_{i=1,\dots,n,j=1,\dots,m}\in D(L)'$. We observe that, since the families of sets $\{A_i\}_{i=1,\dots,n},\{B_j\}_{j=1,\dots,m}$ partition Ω and $[0,T]$ respectively,

$$\begin{aligned}
Q_\varepsilon\Big(R_\varepsilon(S_{n,m}(v))\Big)&=Q_\varepsilon\bigg(R_\varepsilon\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\omega,t)v^{ij}\Big)\bigg)\\
&=Q_\varepsilon\bigg(\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)R_\varepsilon\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)v^{ij}\Big)\Big](\omega,t)\bigg)\\
&=Q_\varepsilon\bigg(\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)R_\varepsilon\big(v^{ij}\big)\Big](\omega,t)\bigg)\\
&=\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)Q_\varepsilon\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)R_\varepsilon\big(v^{ij}\big)\Big)\Big](\omega,t)\\
&=\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)Q_\varepsilon\Big(R_\varepsilon\big(v^{ij}\big)\Big)\Big](\omega,t)\\
&=\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\chi_{A_i\times B_j}(\cdot,\cdot)v^{ij}\Big](\omega,t).
\end{aligned}$$

We are left with a sequence of simple functions in $D(L)'$. If v^{ij} converges at (ω,t) to the function $v(\omega,t,\cdot)$, also $Q_\varepsilon\Big(R_\varepsilon(S_{n,m}(v))\Big)$ converges (ω,t)-a.e. to $v(\omega,t,\cdot)$.

Hence $Q_\varepsilon\Big(R_\varepsilon(v)\Big)$ is a measurable function and coincides with the identity mapping (ω,t)-a.e. (Indeed, in the second to last line, we found a sequence of simple measurable functions approximating $Q_\varepsilon(R_\varepsilon(v))$.) □

PROOF OF LEMMA 2.7.2. For first directional derivative of F in the direction $\xi\in D(L)'$, evaluated at $v\in D(L)'$, we use the definition of the Gateaux

$$D_\xi F(v)=\partial_r\bigg(\int_{\mathcal{O}}h(u(R_\varepsilon(v+r\xi)))\,\mathrm{d}x+\frac{\varepsilon}{2}\langle LR_\varepsilon(v+r\xi),LR_\varepsilon(v+r\xi)\rangle_{L^2(\mathcal{O})}\bigg)\bigg|_{r=0}.$$

By the chain rule, and the fact that $D_\xi L(v)=L\xi$ (by linearity of L),

$$=\int_{\mathcal{O}}h'(u(R_\varepsilon(v)))u'(R_\varepsilon(v))D_\xi R_\varepsilon(v)+\frac{2\varepsilon}{2}LD_\xi R_\varepsilon(v)LR_\varepsilon(v)\,\mathrm{d}x.$$

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We now use the $L^2(\mathcal{O})$ adjoint of the operator L in the second term, which is nothing but the usual $L^2(\mathcal{O})$ inner product. Next we recall that by Assumption (E2), the function $u(\cdot) = (h')^{-1}(\cdot)$ is invertible and we can express $R_\varepsilon(v)$ by $R_\varepsilon(v) = h'((h')^{-1}(R_\varepsilon(v))) = h'(u(R_\varepsilon(v))$.

$$
\begin{aligned}
&= \int_{\mathcal{O}} R_\varepsilon(v) u'(R_\varepsilon(v)) D_\xi R_\varepsilon(v)\,\mathrm{d}x + \varepsilon \left\langle R_\varepsilon(v), \varepsilon L^* L D_\xi R_\varepsilon(v)\right\rangle_{D(L)\times D(L)'} \\
&= \int_{\mathcal{O}} R_\varepsilon(v) u'(R_\varepsilon(v)) D_\xi R_\varepsilon(v)\,\mathrm{d}x + \varepsilon \left\langle R_\varepsilon(v), \varepsilon L^* L D_\xi R_\varepsilon(v)\right\rangle_{D(L)\times D(L)'} \\
&= \left\langle R_\varepsilon(v), u'(R_\varepsilon(v)) D_\xi R_\varepsilon(v) + \varepsilon L^* L D_\xi R_\varepsilon(v)\right\rangle_{D(L)\times D(L)'} \\
&= \left\langle R_\varepsilon(v), (u'(R_\varepsilon(v)) + \varepsilon L^* L) \circ D_\xi R_\varepsilon(v)\right\rangle_{D(L)\times D(L)'}.
\end{aligned}
$$

By Lemma 2.5.6, we obtain

$$= \left\langle R_\varepsilon(v), \xi\right\rangle_{D(L)\times D(L)'}.$$

Once again, we recall that h' and u are inverses of each other,

$$= \left\langle h'(u(R_\varepsilon(v))), \xi\right\rangle_{D(L)\times D(L)'},$$

or formally, if ξ was an $L^2(\mathcal{O})$ function

$$= \int_{\mathcal{O}} h'(u(R_\varepsilon(v)))\xi\,\mathrm{d}x.$$

One could also obtain the result by arguing that the derivative D commutes with the composition of linear operators $L^* \circ L$. Hence,

$$
\begin{aligned}
D_\xi u(R_\varepsilon(v)) + \varepsilon D_\xi L^* L R_\varepsilon(v) &= D_\xi \left(u(R_\varepsilon(v)) + \varepsilon L^* L R_\varepsilon(v)\right) \\
&= D_\xi\big(Q_\varepsilon \circ R_\varepsilon(v)\big) = D_\xi(v) = \xi.
\end{aligned}
$$

Next, we calculate the second Fréchet derivative,

$$
\begin{aligned}
D_{\eta,\xi} F(v) &= D_\eta \langle h'(u(R_\varepsilon(v))), \xi\rangle_{D(L)\times D(L)'} \\
&= D_\eta \langle R_\varepsilon(v), \xi\rangle_{D(L)\times D(L)'} = \langle D_\eta R_\varepsilon(v), \xi\rangle_{D(L)\times D(L)'} = \langle DR_\varepsilon(v)(\eta), \xi\rangle_{D(L)\times D(L)'} \\
&= \langle (u'(R_\varepsilon(v))(\cdot) + \varepsilon L^* L(\cdot))^{-1}(\eta), \xi\rangle_{D(L)\times D(L)'}.
\end{aligned}
$$

We used that, by Lemma 2.5.6,

$$D_\eta R_\varepsilon(v) = (\eta \mapsto u'(R_\varepsilon(v))\eta + \varepsilon L^* L\eta)^{-1} = \big((DQ_\varepsilon)(R_\varepsilon(v)\big)^{-1}(\eta).$$

□

Proof of Lemma 2.7.3. By Lemma 2.5.3, the operator $DQ_{\varepsilon,w}(\cdot) := u'(w)(\cdot) + \varepsilon L^* L(\cdot)$ is strongly monotone, coercive and Lipschitz continuous in its argument from the Hilbert space $D(L)$ into $D(L)'$. Hence, by Theorem 2.12.9, for fixed w, the equation $DQ_{\varepsilon,w} b = a$ has a unique solution.

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Using the invertibility of $DQ_{\varepsilon,w}$, and associating a with an element of $D(L)'$, we can rearrange the terms in the equation,

$$(u'(w)(\cdot) + \varepsilon L^* L(\cdot))^{-1} a = b$$
$$a = (u'(w)(\cdot) + \varepsilon L^* L(\cdot))b.$$

We test the equation with b, with respect to the Frobenius inner product, and integrate with respect to x. In more rigorous terms, we use the duality of $D(L)$ and $D(L)'$ with respect to the $L^2(\mathcal{O})$ inner product.

$$\begin{aligned}\langle a, b\rangle_{D(L)'\times D(L)} &= \int_{\mathcal{O}} u'(w)b : b \,\mathrm{d}x + \langle \varepsilon L^* Lb, b\rangle_{D(L)'\times D(L)} \\ &= \int_{\mathcal{O}} u'(w)b : b \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon Lb : Lb \,\mathrm{d}x.\end{aligned}$$

Since h'' is symmetric, $(h'')^{-1}$ is symmetric as well. By the positive semi-definiteness, it follows that $\sqrt{(h'')}$ as well as $\sqrt{(h'')}^{-1}$ exist and are symmetric.

$$\begin{aligned}&\int_{\mathcal{O}} u'(w)b : b \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \sqrt{u'(w)}\sqrt{u'(w)}b : b \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \mathrm{tr}\left[\left(\sqrt{u'(w)}\sqrt{u'(w)}b\right)^T b\right]\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \mathrm{tr}\left[\left(\sqrt{u'(w)}b\right)^T \sqrt{u'(w))}^T b\right]\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \|\sqrt{u'(w)}b\|_F^2 \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x.\end{aligned}$$

On the other hand, by a direct estimate, we obtain

$$\begin{aligned}\int_{\mathcal{O}} a : b \,\mathrm{d}x &= \int_{\mathcal{O}} a : (\sqrt{u'(w)})^{-1}\sqrt{u'(w)}b \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \mathrm{tr}\left[a^T(\sqrt{u'(w)})^{-1}\sqrt{u'(w)}b\right]\mathrm{d}x \\ &= \int_{\mathcal{O}} \mathrm{tr}\left[\sqrt{u'(w)}ba^T(\sqrt{u'(w)})^{-1}\right]\mathrm{d}x \\ &\le \int_{\mathcal{O}} \sqrt{\mathrm{tr}\left[\left(\sqrt{u'(w)}b\right)^T\sqrt{u'(w)}b\right]\mathrm{tr}\left[\left(a^T(\sqrt{u'(w)})^{-1}\right)^T a^T(\sqrt{u'(w)})^{-1}\right]}\,\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathcal{O}} \mathrm{tr}\left[\left(\sqrt{u'(w)}b\right)^T\sqrt{u'(w)}b\right]\mathrm{d}x + \frac{1}{2}\int \mathrm{tr}\left[\left(a^T(\sqrt{u'(w)})^{-1}\right)^T a^T(\sqrt{u'(w)})^{-1}\right]\mathrm{d}x\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathcal{O}} \operatorname{tr}\left[\left(\sqrt{u'(w)}b\right)^T \sqrt{u'(w)}b\right] \mathrm{d}x + \frac{1}{2}\int \operatorname{tr}\left[a^T(\sqrt{u'(w)})^{-1}\left((\sqrt{u'(w)})^{-1}\right)^T a\right] \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathcal{O}} \operatorname{tr}\left[\left(\sqrt{u'(w)}b\right)^T \sqrt{u'(w)}b\right] \mathrm{d}x + \frac{1}{2}\int \operatorname{tr}\left[a^T(u'(w))^{-1}a\right] \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathcal{O}} \|\sqrt{u'(w)}b\|_F^2 \,\mathrm{d}x + \frac{1}{2}\int \operatorname{tr}\left[a^T(u'(w))^{-1}a\right] \mathrm{d}x.
\end{aligned}
$$

Hence,

$$
\frac{1}{2}\int_{\mathcal{O}} \|\sqrt{u'(w)}b\|_F^2 \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \le \frac{1}{2}\int \operatorname{tr}\left[a^T(u'(w))^{-1}a\right] \mathrm{d}x.
$$

Since $\int_{\mathcal{O}} \varepsilon(Lb : Lb)\,\mathrm{d}x \ge 0$, this proves the claim. □

Proof of Lemma 2.7.4. Let $v_0 \in L^p(\Omega, L^1(\mathcal{O}))$ such that $\mathbb{E}\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x \le C$. We recall that by Assumption (E1) the function h is convex. Hence

$$
h(v_0) \ge h(u(R_\varepsilon(v_0))) + h'(u(R_\varepsilon(v_0)))\big(v_0 - u(R_\varepsilon(v_0))\big).
$$

By Lemma 2.5.19 and the embedding $L^1(\mathcal{O}) \hookrightarrow D(L)'$, we can rewrite v_0 as $Q_\varepsilon(R_\varepsilon(v_0)) = u(R_\varepsilon(v_0)) + \varepsilon L^* L R_\varepsilon(v_0)$. Hence,

$$
\begin{aligned}
h(v_0) &\ge h(u(R_\varepsilon(v_0))) + h'(u(R_\varepsilon(v_0)))\big(u(R_\varepsilon(v_0)) + \varepsilon L^* L R_\varepsilon(v_0) - u(R_\varepsilon(v_0))\big) \\
&= h(u(R_\varepsilon(v_0))) + R_\varepsilon(v_0)\varepsilon L^* L R_\varepsilon(v_0).
\end{aligned}
$$

Integrating both sides with respect to x, we obtain

$$
\begin{aligned}
\int_{\mathcal{O}} h(v_0)\,\mathrm{d}x &\ge \int_{\mathcal{O}} h(u(R_\varepsilon(v_0)))\,\mathrm{d}x + \int_{\mathcal{O}} R_\varepsilon(v_0)\varepsilon L^* L R_\varepsilon(v_0)\,\mathrm{d}x \\
&= \int_{\mathcal{O}} h(u(R_\varepsilon(v_0)))\,\mathrm{d}x + \varepsilon\int_{\mathcal{O}} L R_\varepsilon(v_0) L R_\varepsilon(v_0)\,\mathrm{d}x \\
&= \int_{\mathcal{O}} h(u(R_\varepsilon(v_0)))\,\mathrm{d}x + \varepsilon\|L R_\varepsilon(v_0)\|^2_{L^2(\mathcal{O})} \\
&\ge \int_{\mathcal{O}} h(u(R_\varepsilon(v_0)))\,\mathrm{d}x + \frac{\varepsilon}{2}\|L R_\varepsilon(v_0)\|^2_{L^2(\mathcal{O})}.
\end{aligned}
$$

□

Proof of Lemma 2.6.4. First we formally calculate the Gateaux derivative to get an idea about the potential structure of the Fréchet derivative,

$$
\partial_\varepsilon \operatorname{div}(B(w+\varepsilon\phi)\nabla(w+\varepsilon\phi))|_{\varepsilon=0} = \operatorname{div}(DB(w)\phi\nabla w) + \operatorname{div}(B(w)\nabla\phi).
$$

We will investigate DB separately. Since $w, \phi \in D(L)$, this formally derived operator is indeed well defined as a map from $D(L)$ to $D(L)'$.
For the Fréchet derivative, we consider

$$
\frac{\|\operatorname{div}\left(B(w+h)\nabla(w+h) - B(w)\nabla w - DB(w)h\nabla w - B(w)\nabla h\right)\|_{D(L)'}}{\|h\|_{D(L)}}
$$

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$$\leq C\frac{\|B(w+h)\nabla(w+h)-B(w)\nabla w-DB(w)h\nabla w-B(w)\nabla h\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$\leq C\frac{\|(B(w+h)-B(w)-DB(w)h)\nabla w\|_{L^1(\mathcal{O})}+\|(B(w+h)-B(w))\nabla h\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$\leq C\frac{\|(B(w+h)-B(w)-DB(w)h)\|_{L^\infty(\mathcal{O})}\|\nabla w\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$+C\frac{\|(B(w+h)-B(w))\|_{L^\infty(\mathcal{O})}\|\nabla h\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}.$$

Using the mean-value inequality,

$$\leq C\frac{\|\int_0^1 D^2(1-t)B(w+th)h^2\,\mathrm{d}t\|_{L^\infty(\mathcal{O})}\|\nabla w\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$+C\frac{\|\int_0^1(1-t)DB(w+th)h\,\mathrm{d}t\|_{L^\infty(\mathcal{O})}\|\nabla h\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$\leq C\frac{\sup_{t\in[0,1]}\|D^2B(w+th)\|_{L^\infty(\mathcal{O})}\|h^2\|_{L^\infty(\mathcal{O})}\|\nabla w\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$+C\frac{\sup_{t\in[0,1]}\|DB(w+th)\|_{L^\infty(\mathcal{O})}\|h\|_{D(L)}\|\nabla h\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$\leq C\frac{\sup_{t\in[0,1]}\|D^2B(w+th)\|_{L^\infty(\mathcal{O})}\|h\|_{D(L)}^2\|\nabla w\|_{L^1(\mathcal{O})}}{\|h\|_{D(L)}}$$
$$+C\frac{\sup_{t\in[0,1]}\|DB(w+th)\|_{L^\infty(\mathcal{O})}\|h\|_{D(L)}^2}{\|h\|_{D(L)}}$$
$$\leq C\|h\|_{D(L)}\left(\sup_{t\in[0,1]}\|D^2B(w+th)\|_{L^\infty(\mathcal{O})}\|\nabla w\|_{L^1(\mathcal{O})}+\sup_{t\in[0,1]}\|DB(w+th)\|_{L^\infty(\mathcal{O})}\right).$$

Next, we consider DB.

$$DB=D(A(u(w))(h'')^{-1}(u(w)))=D(A(u(w)))(h'')^{-1}(u(w))+A(u(w))D(h'')^{-1}(u(w)).$$

Explicitly calculating the derivatives, we arrive at

$$D_\phi(A(u(w)))=A'(u(w))u'(w)\phi$$
$$D_\phi(h'')^{-1}(u(w))=((h'')^{-1})'(u(w))u'(w)\phi.$$

Combining these, and slightly abusing the notation,

$$DB(w)(\cdot)=A'(u(w))u'(w)\times(\cdot)(h'')^{-1}(u(w)))+A(u(w))((h'')^{-1})'(u(w))u'(w)\times(\cdot).$$

We proceed to calculating calculate the second Fréchet derivative.

$$D^2B(w)(\cdot_1)(\cdot_2)=A''(u(w))u'(w)u'(w)\times(\cdot_1)(\cdot_2)(h'')^{-1}(u(w)))$$
$$+A'(u(w))u''(w)\times(\cdot_1)(\cdot_2)(h'')^{-1}(u(w)))$$

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$$
\begin{aligned}
&+ A(u(w)) \times (\cdot_1)((h'')^{-1})'(u(w))u'(w)) \times (\cdot_2)\\
&+ A'(u(w))u'(w) \times (\cdot_2)((h'')^{-1})'(u(w))u'(w) \times (\cdot_1)\\
&+ A(u(w))((h'')^{-1})''(u(w))u'(w) \times (\cdot_2)u'(w)) \times (\cdot_1)\\
&+ A(u(w))((h'')^{-1})'(u(w))u''(w) \times (\cdot_2) \times (\cdot_1).
\end{aligned}
$$

By Assumption (R3), $w \in L^\infty(\mathcal{O})$. The matrix A, as a real valued function, is C^2 and defined on the $\mathbb{R}^n$ and so are $u, u' = (h'')^{-1}, u''$. By the same argument as in Lemma 2.5.7, u maps to $L^\infty(\mathcal{O})$ and we conclude that each of the terms $A''(u(w))$, $A'(u(w))$, $A(u(w))$, $u(w)$, $u'(w)$, $u''(w)$ lies in $L^\infty(\mathcal{O})$. Hence also their products, as well as products with other $L^\infty(\mathcal{O})$ functions remain in $L^\infty(\mathcal{O})$. We conclude that $DB(w+th), D^2B(w+th)$ are bounded, for each $t \in [0,1]$. By Assumption (E4) the constant bounding this term is proportional to an exponential times a polynomial of $\|w\|_{L^\infty(\mathcal{O})}$ and $\|h\|_{L^\infty(\mathcal{O})}$. □

Proof of Lemma 2.6.5. We already established in Lemma 2.5.5, that $R_\varepsilon(\cdot)$ is Fréchet differentiable as an operator from $D(L)'$ to $D(L)$. By the chain rule for the Fréchet derivative and Lemma 2.6.4, we conclude that the operator $\operatorname{div}(B(\cdot)\nabla(\cdot)) \circ R_\varepsilon(\cdot)$ given by $\operatorname{div}(B(R_\varepsilon(\cdot))\nabla(R_\varepsilon(\cdot)))$, from $D(L)'$ to $D(L)'$, is Fréchet differentiable with derivative

$$
\operatorname{div}(DB(R_\varepsilon(\cdot_1))DR_{\varepsilon,(\cdot_1)}(\cdot_2)\nabla R_\varepsilon(\cdot_1)) + \operatorname{div}(B(R_\varepsilon(\cdot_1))\nabla DR_{\varepsilon,(\cdot_1)}(\cdot_2)).
$$

Let $v, v_1, v_2 \in D(L)'$ and assume $\|v\|_{D(L)'} \le K$.

$$
\begin{aligned}
&\big|\langle \operatorname{div}(DB(u(R_\varepsilon(v)))DR_{\varepsilon,v}(v_1)\nabla R_\varepsilon(v)) + \operatorname{div}(B(u(R_\varepsilon(v)))\nabla DR_{\varepsilon,v}(v_1)), v_2\rangle_{D(L)'\times D(L)'}\big|\\
&\quad\le \|\operatorname{div}(DB(u(R_\varepsilon(v)))DR_{\varepsilon,v}(v_1)\nabla R_\varepsilon(v)) + \operatorname{div}(B(u(R_\varepsilon(v)))\nabla DR_{\varepsilon,v}(v_1))\|_{D(L)'}\|v_2\|_{D(L)'}\\
&\quad\le \|DB(u(R_\varepsilon(v)))DR_{\varepsilon,v}(v_1)\nabla R_\varepsilon(v) + B(u(R_\varepsilon(v)))\nabla DR_{\varepsilon,v}(v_1)\|_{L^1(\mathcal{O})}\|v_2\|_{D(L)'}\\
&\quad\le \big(\|DB(u(R_\varepsilon(v)))DR_{\varepsilon,v}(v_1)\nabla R_\varepsilon(v)\|_{L^1(\mathcal{O})} + \|B(u(R_\varepsilon(v)))\nabla DR_{\varepsilon,v}(v_1)\|_{L^1(\mathcal{O})}\big)\|v_2\|_{D(L)'}\\
&\quad\le \big(\|DB(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|DR_{\varepsilon,v}(v_1)\|_{L^\infty(\mathcal{O})}\|\nabla R_\varepsilon(v)\|_{L^1(\mathcal{O})}\big)\|v\|_{D(L)'}\\
&\qquad+ \big(\|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|\nabla DR_{\varepsilon,v}(v_1)\|_{L^1(\mathcal{O})}\big)\|v_2\|_{D(L)'}.
\end{aligned}
$$

We can use the Lipschitz continuity of $DR_{\varepsilon,v}(\cdot)$, established in Lemma 2.5.6, to derive

$$
\begin{aligned}
\|DR_{\varepsilon,v}(v_1)\|_{D(L)} &\le \|DR_{\varepsilon,v}(v_1) - DR_{\varepsilon,v}(0)\|_{D(L)} + \|DR_{\varepsilon,v}(0)\|_{D(L)}\\
&\le \frac{c}{\varepsilon}\|v_1\|_{D(L)'} + \|DR_{\varepsilon,v}(0)\|_{D(L)},
\end{aligned}
$$

where the constant c only depends on the spectrum of the operator L, but is independent of everything else. Since $DR_{\varepsilon,v}(\cdot)$ is linear, $\|DR_{\varepsilon,v}(0)\|_{D(L)} = 0$. Hence,

$$
\begin{aligned}
&\big(\|DB(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|DR_{\varepsilon,v}(v_1)\|_{L^\infty(\mathcal{O})}\|\nabla R_\varepsilon(v)\|_{L^1(\mathcal{O})}\big)\|v_2\|_{D(L)'}\\
&\qquad+ \big(\|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|\nabla DR_{\varepsilon,v}(v_1)\|_{L^1(\mathcal{O})}\big)\|v_2\|_{D(L)'}\\
&\quad\le \big(\|DB(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|DR_{\varepsilon,v}(v_1)\|_{D(L)}\|R_\varepsilon(v)\|_{D(L)}\big)\|v_2\|_{D(L)'}\\
&\qquad+ \big(\|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|DR_{\varepsilon,v}(v_1)\|_{D(L)}\big)\|v_2\|_{D(L)'}\\
&\quad\le \frac{C}{\varepsilon^2}\|DB(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|v_1\|_{D(L)'}(\|v\|_{D(L)'} + \|u(0)\|_{D(L)})\|v_2\|_{D(L)'}
\end{aligned}
$$

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$$+\frac{c}{\varepsilon}\|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|v_1\|_{D(L)'}\|v_2\|_{D(L)'}.$$

By the previous arguments,

$$\begin{aligned}
&\frac{C}{\varepsilon^2}\|DB(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|v_1\|_{D(L)'}(\|v_2\|_{D(L)'}+\|u(0)\|_{D(L)})\|v_2\|_{D(L)'}\\
&\qquad+\frac{c}{\varepsilon}\|B(u(R_\varepsilon(v)))\|_{L^\infty(\mathcal{O})}\|v_1\|_{D(L)'}\|v_2\|_{D(L)'}\\
&\quad\le\frac{C(K,B,u,u',u'')}{\varepsilon^2}\|v_1\|_{D(L)'}\|v_2\|_{D(L)'}.
\end{aligned}$$

□

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2.12.2. Appendix B: Itô's formula. In this section, we want to verify the conditions for Itô's formula as presented in [84]. The original statement reads as follows. Let V, H be (separable) Hilbert spaces and assume $V \hookrightarrow H$ is continuous and dense. The inner products and norms on these spaces will be denoted by $(\cdot,\cdot)_V, (\cdot,\cdot)_H$ and $\|\cdot\|_V, \|\cdot\|_H$, respectively. Let $\phi(\eta)$ be a real-valued function on H. Assume that

(Itô 1) For any $\eta, \xi \in H$ the functions $\phi(\eta + t\xi)$ is twice continuously differentiable as a function of t and the functions

$$\phi_{(\xi)}(\eta) := \frac{\partial}{\partial t}\phi(\eta + t\xi)_{t=0}, \quad \phi_{(\xi)(\xi)}(\eta) := \frac{\partial^2}{\partial^2 t}\phi(\eta + t\xi)\Big|_{t=0}$$

are continuous as functions of $(\eta, \xi) \in H \times H$;

(Itô 2) For any $R \in (0, \infty)$ there exists a $K(R)$ such that for all $\eta, \xi \in H$ satisfying $\|\eta\|_H \leq R$ we have

$$\left|\phi_{(\xi)}(\eta)\right| \leq K(R)\|\xi\|_H, \quad \left|\phi_{(\xi)(\xi)}(\eta)\right| \leq K(R)\|\xi\|_H^2.$$

In this situation for any $\eta, \xi, \gamma \in H$ the function $\Phi(t,s) := \phi(\eta + t\xi + s\gamma)$ at all points (t,s) has partial derivatives with respect to t and s which are continuous as functions of (t,s). It follows that $\Phi(t,s)$ is a differentiable function of (t,s) and for any numbers α, β

$$\frac{\partial}{\partial \varepsilon}\Phi(t + \alpha\varepsilon, s + \beta\varepsilon)\Big|_{\varepsilon=0} = \alpha\frac{\partial}{\partial t}\Phi(t,s) + \beta\frac{\partial}{\partial s}\Phi(t,s)$$

For $t = s = 0$ this yields

$$\phi_{(\alpha\xi+\beta\gamma)}(\eta) = \alpha\phi_{(\xi)}(\eta) + \beta\phi_{(\gamma)}(\eta).$$

Hence, the function $\phi_{(\xi)}(\eta)$ as a function of $\xi \in H$ is a continuous linear functional and by Riesz's representation theorem there exists an element $\phi_{(\cdot)}(\eta) \in H$ such that

$$\phi_{(\xi)}(\eta) = \left(\phi_{(\cdot)}(\eta), \xi\right)_H, \quad \left\|\phi_{(\cdot)}(\eta)\right\|_H \leq K\left(\|\eta\|_H\right).$$

(Itô 3) If $\eta \in V$, then $\phi_{(\cdot)}(\eta) \in V$ and

$$\left\|\phi_{(\cdot)}(\eta)\right\|_V \leq K_1\left(1 + \|\eta\|_V\right),$$

where K_1 is a fixed constant:

(Itô 4) For any $v^* \in V$ the function $\left(\phi_{(\cdot)}(v), v^*\right)_V$ is a continuous function on V (in the metric of V).

Let $W_t^1, W_t^2, \ldots$ be a finite or infinite sequence of independent $\{\mathcal{F}_t\}_t$ Wiener processes on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume the following: We are given a sequence of predictable H-valued processes $(\sigma_t^i)_{i\in\mathbb{N}}$, such that for any $T \in (0, \infty)$,

$$\sum_{k=1}^{\infty} \mathbb{E}\int_0^T \|\sigma_t^k\|_H^2 \, \mathrm{d}t < \infty.$$

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Let v_t, v_t^* be V-valued, predictable processes satisfying

$$\mathbb{E}\int_0^T \|v_t\|_V \,\mathrm{d}t < \infty, \qquad \mathbb{E}\int_0^T \|v_t^*\|_V \,\mathrm{d}t < \infty,$$

for any $T \in (0,\infty)$. Finally, suppose that v_0 is an H-valued $\mathcal{F}_0$-measurable random vector and that for every $\varphi \in V$ we have

$$(\varphi, v_t)_H = (\varphi, v_0)_H + \int_0^t (\varphi, v_s^*)_V \,\mathrm{d}s + \sum_k \int_0^t \left(\varphi, \sigma_s^k\right)_H \mathrm{d}W_s^k,$$

for almost all (ω, t).

THEOREM 2.12.1. *There exists a continuous H-valued $\mathcal{F}_t$-adapted process u_t and a set $\Omega' \subset \Omega$ of full measure such that*

(1) $u_t = v_t$ for almost all (ω, t), so that

$$\mathbb{E}\int_0^t \|u_t\|_V^2 \,\mathrm{d}t < \infty,$$

for any $T \in (0,\infty)$.

(2) For all $\omega \in \Omega'$, all $\varphi \in V$ and all $t \geq 0$, we have

$$(\varphi, u_t)_H = (\varphi, v_0)_H + \int_0^t (\varphi, v_s^*)_V \,\mathrm{d}s + \sum_k \int_0^t \left(\varphi, \sigma_s^k\right)_H \mathrm{d}W_s^k. \tag{2.118}$$

(3) If in addition assumptions (Itô 1) (Itô 4) are satisfied, then a.s. for every t,

$$\begin{aligned}\phi(u_t) =& \phi(u_0) + \sum_k \int_0^t \phi_{\left(\sigma_s^k\right)}(u_s) \,\mathrm{d}W_s^k \\ &+ \int_0^t \left[\left(\phi_{(\cdot)}(u_s), v_s^*\right)_V + (1/2)\sum_k \phi_{\left(\sigma_s^k\right)\left(\sigma_s^k\right)}(u_s)\right] \mathrm{d}s.\end{aligned}$$

REMARK 2.12.2. *It should be noted that the inner product in the deterministic integral is the one in V. Hence the function $v^* \in V$ should be viewed as the Riesz-representative of an element of V', denoting the dual of V.*
Theorem 2.12.1 is a combination of the results of [84, Theorem 2.1] *and* [84, Theorem 3.1].

The spaces V and H should be thought of as a part of the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$. The triple in which we solved equation (2.10) and on which we will operate is given by $V = H = V' = D(L)'$. Due to the definition of the functional F (see (2.13)) which incorporates the operator R_ε, another triple enters the calculations, namely $D(L) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow D(L)'$. One should make a mental note to keep these triples separate.
Equation (2.10), explicitly given by

$$v(t) = v(0) + \int_0^t \operatorname{div}\left(B(u(R_\varepsilon(v)))(s)\nabla R_\varepsilon(v)(s)\right) \mathrm{d}s + \int_0^t \sigma(u(R_\varepsilon(v)))(s)\,\mathrm{d}W_s \tag{2.119}$$

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can be expressed in a form consistent with (2.118) as

$$\begin{aligned}\langle \varphi, u_t\rangle_{D(L)'\times D(L)'} &= \langle \phi, v_0\rangle_{D(L)'\times D(L)'} + \int_0^t \langle \varphi, v_s^*\rangle_{D(L)'\times D(L)'}\,\mathrm{d}s + \sum_k \int_0^t \langle \varphi, \sigma_s^k\rangle_{D(L)'\times D(L)'}\,\mathrm{d}W_s^k \\ &= \langle \varphi, v_0\rangle_{D(L)'\times D(L)'} + \int_0^t \langle \varphi, \operatorname{div}\left(B(u(R_\varepsilon(v)))(s)\nabla R_\varepsilon(v)(s)\right)\rangle_{D(L)'\times D(L)'}\,\mathrm{d}s \\ &\quad + \sum_k \int_0^t \langle \varphi, \sigma(u(R_\varepsilon(v)))(s)e_k\rangle_{D(L)'\times D(L)'}\,\mathrm{d}W_s^k.\end{aligned}$$

This formulation is consistent with testing (2.10), with respect to the duality between $D(L)$ and $D(L)'$, with the test functions $\psi := (L^*L)^{-1}\varphi \in D(L)$.
The functionals $\phi, \phi_{(\xi)}$ and $\phi_{(\xi),(\xi)}$ correspond to

$$\begin{aligned}\phi(\eta) &= F(\eta) = \int_{\mathcal{O}} h(u(R_\varepsilon(\eta)))\,\mathrm{d}x + \frac{\varepsilon}{2}\|LR_\varepsilon(\eta)\|^2_{L^2(\mathcal{O})}, \\ \phi_{(\xi)}(\eta) &= D_\xi F(\eta) = \langle R_\varepsilon(\eta), \xi\rangle_{D(L)\times D(L)'} = \langle h'(u(R_\varepsilon(\eta))), \xi\rangle_{D(L)\times D(L)'}, \\ \phi_{(\xi),(\xi)}(\eta) &= D^2_{\xi,\xi}F(\eta) = \langle (u'(R_\varepsilon(\eta))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\xi), \xi\rangle_{D(L)\times D(L)'}.\end{aligned}$$

Since the assumptions require explicit estimates on the Riesz representative, it is fruitful to recall that $D(L)'$ is also part of the triple $D(L) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow D(L)'$. As in Lemma 2.3.5, the operator L can be constructed in such a way that the representative in $D(L)'$ of a functional acting on $D(L)'$ can be expressed via the application of L^*L to an element of $D(L)$, which follows from the definition of the duality between $D(L)$ and $D(L)'$ as the extension of the duality in $L^2(\mathcal{O})$. For $F \in D(L)$ and $\xi \in D(L)'$ this can be expressed through the relation

$$F(\xi) = \langle F, \xi\rangle_{D(L)\times D(L)'} = \langle LF, L^{-1}\xi, \rangle_{L^2(\mathcal{O})} = \langle LF, (LL^{-1})L^{-1}\xi, \rangle_{L^2(\mathcal{O})} \tag{2.120}$$

$$= \langle L^{-1}L^*LF, L^{-1}\xi, \rangle_{L^2(\mathcal{O})} = \langle L^*LF, \xi\rangle_{D(L)'\times D(L)'}, \tag{2.121}$$

where we used the self-adjointedness of L as well as L^{-1}. Starting with the duality in the triple $V = H = V' = D(L)'$ and an element $G \in D(L)'$. The inverse of the operator L^*L allows us to recover (2.120) and an element $F \in D(L)$ via the operation $(L^*L)^{-1}G$. We proceed to verify the assumptions of the theorem. Since $L^2(\mathcal{O}) \hookrightarrow D(L)'$, the conditions will be also satisfied, if we verify the assumptions for elements $\xi \in D(L)'$, which is much more natural, based on our definitions. We begin by verifying (Itô 1). The claimed differentiability is a consequence of the chain rule and the assumed regularity of h (see (E1)). To show that the maps $D_\xi F$ and $D_{\xi,\xi}F$ are indeed continuous let $\eta_1, \eta_2, \xi_1, \xi_2 \in D(L)'$. We begin by verifying the continuity of $D_\xi F$,

$$\begin{aligned}&|D_{\xi_1}F(\eta_1) - D_{\xi_2}F(\eta_2)| = |\langle R_\varepsilon(\eta_1), \xi_1\rangle_{D(L)\times D(L)'} - \langle R_\varepsilon(\eta_2), \xi_2\rangle_{D(L)\times D(L)'}| \\ &\le |\langle R_\varepsilon(\eta_1), \xi_1\rangle_{D(L)\times D(L)'} - \langle R_\varepsilon(\eta_1)\xi_2\rangle_{D(L)\times D(L)'}| \\ &\quad + |\langle R_\varepsilon(\eta_1), \xi_2\rangle_{D(L)\times D(L)'} - \langle R_\varepsilon(\eta_2), \xi_2\rangle_{D(L)\times D(L)'}| \\ &= |\langle R_\varepsilon(\eta_1), \xi_1 - \xi_2\rangle_{D(L)\times D(L)'}| + |\langle R_\varepsilon(\eta_1) - R_\varepsilon(\eta_2), \xi_2\rangle_{D(L)\times D(L)'}|\end{aligned}$$

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$$\leq \|R_\varepsilon(\eta_1)\|_{D(L)}\|\xi_1-\xi_2\|_{D(L)'} + \|R_\varepsilon(\eta_1)-R_\varepsilon(\eta_2)\|_{D(L)}\|\xi_2\|_{D(L)'}$$
$$\leq C(\varepsilon)\bigg((1+\|\eta_1\|_{D(L)'})\|\xi_1-\xi_2\|_{D(L)'} + \|\eta_1-\eta_2\|_{D(L)'}\|\xi_2\|_{D(L)'}\bigg)$$
$$\leq C(\varepsilon)\bigg((1+\|\eta_1\|_{D(L)'})\|\xi_1-\xi_2\|_{D(L)'} + \|\eta_1-\eta_2\|_{D(L)'}\|\xi_2\|_{D(L)'}\bigg).$$

This calculation implies that $D_\xi F$ is jointly locally Lipschitz continuous in ξ and η. For the continuity of $D_{\xi,\xi}F$, we perform an analogous computation,

$$\begin{aligned}
&|D_{\xi_1,\xi_1}F(\eta_1) - D_{\xi_2,\xi_2}F(\eta_2)| \\
&= |\langle (u'(R_\varepsilon(\eta_1))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1),\xi_1\rangle_{D(L)\times D(L)'} - \langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_2),\xi_2\rangle_{D(L)\times D(L)'}| \\
&\leq |\langle (u'(R_\varepsilon(\eta_1))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1),\xi_1\rangle_{D(L)\times D(L)'} - \langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1),\xi_1\rangle_{D(L)\times D(L)'}| \\
&\quad + |\langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1),\xi_1\rangle_{D(L)\times D(L)'} - \langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_2),\xi_1\rangle_{D(L)\times D(L)'}| \\
&\quad + |\langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_2),\xi_1\rangle_{D(L)\times D(L)'} - \langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_2),\xi_2\rangle_{D(L)\times D(L)'}|.
\end{aligned}$$

By linearity of the dual product and the operators $DR_{\varepsilon,\eta_i}(\cdot) = (u'(R_\varepsilon(\eta_1))(\cdot)+\varepsilon L^*L(\cdot))^{-1}$ for $i=1,2$,

$$\begin{aligned}
&\leq |\langle (u'(R_\varepsilon(\eta_1))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1) - (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1),\xi_1\rangle_{D(L)\times D(L)'}| \\
&\quad + |\langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_1-\xi_2),\xi_1\rangle_{D(L)\times D(L)'}| \\
&\quad + |\langle (u'(R_\varepsilon(\eta_2))(\cdot)+\varepsilon L^*L(\cdot))^{-1}(\xi_2),\xi_1-\xi_2\rangle_{D(L)\times D(L)'}| \\
&= I_1+I_2+I_3.
\end{aligned}$$

The continuity of I_2, I_3 follows directly by Lemma 2.5.6. It remains to investigate the continuity, with respect to η, of the first term I_1.

$$\begin{aligned}
&(u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1} - (u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1} \\
&= (u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}\bigg(Id - (u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}\bigg) \\
&= (u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}\bigg(Id - \big(u'(R_\varepsilon(\eta_1))+u'(R_\varepsilon(\eta_2))-u'(R_\varepsilon(\eta_2))+\varepsilon L^*L\big)(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}\bigg) \\
&= (u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}\bigg(Id - \big((u'(R_\varepsilon(\eta_1))-u'(R_\varepsilon(\eta_2))+(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)\big)(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}\bigg) \\
&= (u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}\bigg(Id - (u'(R_\varepsilon(\eta_1))-u'(R_\varepsilon(\eta_2)))(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1} - Id\bigg) \\
&= -(u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}(u'(R_\varepsilon(\eta_1))-u'(R_\varepsilon(\eta_2)))(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}.
\end{aligned}$$

Hence, for $\xi_1 \in D(L)'$.

$$\begin{aligned}
&\|\big((u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1} - (u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}\big)(\xi_1)\|_{D(L)} \\
&\leq \|(u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}(u'(R_\varepsilon(\eta_1))-u'(R_\varepsilon(\eta_2)))(u'(R_\varepsilon(\eta_2))+\varepsilon L^*L)^{-1}(\xi_1)\|_{D(L)} \\
&\leq \|(u'(R_\varepsilon(\eta_1))+\varepsilon L^*L)^{-1}\|_{L(D(L)',D(L))}\|u'(R_\varepsilon(\eta_1))-u'(R_\varepsilon(\eta_2))\|_{L(D(L),D(L)')}
\end{aligned}$$

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$$\times \|(u'(R_\varepsilon(\eta_2)) + \varepsilon L^*L)^{-1}(\xi_1)\|_{D(L)}.$$

We focus on the second term, as the first and third term can be bounded by a constant depending on ε and the norm of ξ_1.

$$\begin{aligned}
&\|u'(R_\varepsilon(\eta_1)) - u'(R_\varepsilon(\eta_2))\|_{L(D(L),D(L)')} = \sup_{\psi\in D(L),\|\psi\|_{D(L)}\le 1} \|(u'(R_\varepsilon(\eta_1)) - u'(R_\varepsilon(\eta_2)))\psi\|_{D(L)'} \\
&\le \sup_{\psi\in D(L),\|\psi\|_{D(L)}\le 1} \|(u'(R_\varepsilon(\eta_1)) - u'(R_\varepsilon(\eta_2)))\psi\|_{L^2(\mathcal{O})} \\
&\le \sup_{\psi\in D(L),\|\psi\|_{D(L)}\le 1} \|u'(R_\varepsilon(\eta_1)) - u'(R_\varepsilon(\eta_2))\|_{L^\infty(\mathcal{O})}\|\psi\|_{L^2(\mathcal{O})} \\
&\le \sup_{\psi\in D(L),\|\psi\|_{D(L)}\le 1} \|(u''(tR_\varepsilon(\eta_1) + (1-t)R_\varepsilon(\eta_2))(R_\varepsilon(\eta_1) - R_\varepsilon(\eta_2))\|_{L^\infty(\mathcal{O})}\|\psi\|_{L^2(\mathcal{O})} \\
&\le \|(u''(tR_\varepsilon(\eta_1) + (1-t)R_\varepsilon(\eta_2))\|_{L^\infty(\mathcal{O})}\|(R_\varepsilon(\eta_1) - R_\varepsilon(\eta_2))\|_{D(L)} \sup_{\psi\in D(L),\|\psi\|_{D(L)}\le 1} \|\psi\|_{L^2(\mathcal{O})} \\
&\le C(u'', \eta_1, \eta_2, \varepsilon)\|\eta_1 - \eta_2\|_{D(L)'(\mathcal{O})}.
\end{aligned}$$

This concludes the first assumption.
Assumption (Itô 2) follows by similar argument. Let $\eta, \xi \in D(L)'$. For the first Gateaux derivative,

$$\begin{aligned}
\big|D_\xi F(\eta)\big| &= \big|\langle h'(u(R_\varepsilon(\eta))), \xi\rangle_{D(L)\times D(L)'}\big| \\
&= \big|\langle R_\varepsilon(\eta), \xi\rangle_{D(L)\times D(L)'}\big| \le \|R_\varepsilon(\eta)\|_{D(L)}\|\xi\|_{D(L)'} \\
&\le C(\varepsilon)\big(1 + \|\eta\|_{D(L)'}\big)\|\xi\|_{D(L)'}.
\end{aligned}$$

The second Gateaux derivative requires additional care.

$$|\langle (u'(R_\varepsilon(\eta))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\xi), \xi\rangle_{D(L)\times D(L)'}| \le \|(u'(R_\varepsilon(\eta))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\xi)\|_{D(L)}\|\xi\|_{D(L)'}.$$

By Lemma 2.5.6, we can bound the first term uniformly in η and obtain

$$\begin{aligned}
&\|(u'(R_\varepsilon(\eta))(\cdot) + \varepsilon L^*L(\cdot))^{-1}(\xi)\|_{D(L)(\mathcal{O})}\|\xi\|_{L^2(\mathcal{O})} \\
&\le C(\varepsilon)\|\xi\|_{D(L)'}\|\xi\|_{D(L)'} \\
&\le C(\varepsilon)\|\xi\|^2_{D(L)'}.
\end{aligned}$$

To verify Assumption (Itô 3), we recall the discussion leading to (2.120) and the relation

(2.122)
$$D_\xi F(\eta) = \langle R_\varepsilon(\eta), \xi\rangle_{D(L)\times D(L)'} = \langle L^{-1}L^*LR_\varepsilon(\eta), L^{-1}\xi, \rangle_{L^2(\mathcal{O})} = \langle L^*LR_\varepsilon(\eta), \xi\rangle_{D(L)'\times D(L)'}.$$

Hence the desired function $\phi_{(\cdot)}(\eta) \in D(L)'$ is given by $L^*LR_\varepsilon(\eta)$.

$$\begin{aligned}
\|L^*LR_\varepsilon(\eta)\|_{D(L)'} &\le C\|R_\varepsilon(\eta)\|_{D(L)} = C\|R_\varepsilon(\eta)\|_{D(L)} \\
&\le C\|R_\varepsilon(\eta) - R_\varepsilon(0)\|_{D(L)} + \|R_\varepsilon(0)\|_{D(L)} \\
&\le C(\varepsilon)\|\eta\|_{D(L)'} + \|R_\varepsilon(0)\|_{D(L)} \\
&\le C(\varepsilon, R_\varepsilon(0))(1 + \|\eta\|_{D(L)'}).
\end{aligned}$$

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Regarding the last Assumption (Itô 4). Let $v^*, v_1, v_2 \in D(L)'$,

$$\begin{aligned}
&\langle L^*LR_\varepsilon(v_1) - L^*LR_\varepsilon(v_2), v^*\rangle_{D(L)'} \\
&= \langle L^*LR_\varepsilon(v_1) - L^*LR_\varepsilon(v_2), v^*\rangle_{D(L)'} \\
&\le \|L^*LR_\varepsilon(v_1) - L^*LR_\varepsilon(v_2)\|_{D(L)'}\|v^*\|_{D(L)'} \\
&\le C\|R_\varepsilon(v_1) - R_\varepsilon(v_2)\|_{D(L)}\|v^*\|_{D(L)'} \\
&\le C_\varepsilon\|v_1 - v_2\|_{D(L)'}\|v^*\|_{D(L)'}.
\end{aligned}$$

This concludes the verification of the necessary assumptions.

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2.12.3. Appendix C: specific lemmata.

LEMMA 2.12.3. *Let , $L\colon D(L) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ be a linear operator and let $\lambda \in \rho(L)$ be given. Then $\|\cdot\| := \|(L-\lambda Id)\cdot\|_{L^2(\mathcal{O})}$ is an equivalent (to the graph norm) norm on $D(L)$.*

PROOF. Since $\lambda \in \rho(L)$, we know that the inverse $(L-\lambda Id)^{-1}\colon L^2(\mathcal{O}) \to L^2(\mathcal{O})(D(L))$ exists and is bounded.

$$\|(L-\lambda Id)w\|_{L^2(\mathcal{O})} \leq \|Lw\|_{L^2(\mathcal{O})} + |\lambda|\|w\|_{L^2(\mathcal{O})} \leq (1+|\lambda|)(\|Lw\|_{L^2(\mathcal{O})} + \|w\|_{L^2(\mathcal{O})}) = C_\lambda \|w\|_{D(L)}.$$

For the other direction,

$$\begin{aligned}\|Lw\|_{L^2(\mathcal{O})} + \|w\|_{L^2(\mathcal{O})} &\leq \|(L-\lambda Id)w\|_{L^2(\mathcal{O})} + (1+|\lambda|)\|w\|_{L^2(\mathcal{O})} \\ &= \|(L-\lambda Id)w\|_{L^2(\mathcal{O})} + (1+|\lambda|)\|(L-\lambda)^{-1}(L-\lambda)w\|_{L^2(\mathcal{O})} \\ &\leq \|(L-\lambda Id)w\|_{L^2(\mathcal{O})} + (1+|\lambda|)\|(L-\lambda)^{-1}\|_{L(L^2,D(L))}\|(L-\lambda)w\|_{L^2(\mathcal{O})} \\ &\leq (1+(1+|\lambda|)\|(L-\lambda)^{-1}\|_{L(L^2,D(L))})\|(L-\lambda)w\|_{L^2(\mathcal{O})}.\end{aligned}$$

□

LEMMA 2.12.4. *Let X, Y be Banach spaces and $\iota : X \to Y'$ denote the continuous embedding of X into Y'. Let $x \in X, y \in Y$, then the mapping $j : Y \to X'$, defined by*

$$\langle j(y), x\rangle_{X'\times X} = \langle \iota(x), y\rangle_{Y'\times Y}$$

is continuous. If $\iota(X)$ is weak-dense in Y' then j is injective, hence j defines a continuous embedding of Y into X'.*

PROOF. Let $x \in X, y \in Y$. By the embedding ι, $\iota(x) \in Y'$, we infer

$$\langle \iota(x), y\rangle_{Y'\times Y} \leq \|\iota(x)\|_{Y'}\|y\|_Y \leq C\|x\|_X\|f\|_Y.$$

Hence the map $e \mapsto \langle \iota(x), y\rangle_{Y'\times Y}$ is an element of X'. We define the map $j\colon Y \to X'$ by

$$y \mapsto \langle \iota(\cdot), y\rangle_{Y'\times Y}.$$

Hence,

$$\|j(y)\|_{X'} \leq \sup_{x\colon \|x\|_X=1} \langle j(y), x\rangle_{X'\times X} \leq \|\iota(x)\|_{Y'}\|y\|_Y \leq \|y\|_Y.$$

The continuity of j is a consequence of the linearity and boundedness.
If $j(f) = 0$, then

$$0 = \langle j(y), x\rangle_{X'\times X} = \langle \iota(x), y\rangle_{Y'\times Y}$$

for every $x \in X$. The weak*-density of $\iota(X)$ in Y' implies that $\langle y', y\rangle = 0$ for any $y' \in Y'$. We conclude that $y = 0$ and consequently that j is a continuous injection. □

THEOREM 2.12.5 (Schauder). *If $L \in \mathcal{K}(X,Y)$, then $L^* \in \mathcal{K}(Y', X')$. And conversely.*

The following theorem has been taken from [54, Theorem 3.2.7, p.129]

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THEOREM 2.12.6. *(Mean Value Theorem). Let X, Y be normed linear spaces and let $f\colon X \to Y$. If for given $a, b \in X$ the directional derivative $\delta f(a+t(b-a); b-a)$ exists for all $t \in [0,1]$, then*

$$\|f(b) - f(a)\|_Y \leq \sup_{t\in[0,1]} \|\delta f(a+t(b-a); b-a)\|_Y$$

and

$$\|f(b) - f(a) - \delta f(a; b-a)\|_Y \leq \sup_{t\in[0,1]} \|\delta f(a+t(b-a); b-a) - \delta f(a; b-a)\|_Y.$$

Moreover, if $Df(a+t(b-a))$ exists for all $t \in [0,1]$, then

$$\|f(b) - f(a)\|_Y \leq \sup_{t\in[0,1]} \|Df(a+t(b-a))\|_{\mathcal{L}(X,Y)} \|b-a\|_X.$$

The following is [54, Theorem 4.2.1, p.157].

THEOREM 2.12.7. *(Implicit Function Theorem). Let X, Y, Z be Banach spaces, $f : X \times Y \to Z$. Let $(a,b) \in X \times Y$ be such a point that*

$$f(a,b) = o.$$

Let $\mathcal{G}$ be an open set in $X \times Y$ containing the point (a,b). Let $f \in C^1(\mathcal{G})$ and let the partial Fréchet derivative $f_2'(a,b)$ be an isomorphism of Y onto Z. Then there are neighborhoods $\mathcal{U}$ of a and $\mathcal{V}$ of b such that for any $x \in \mathcal{U}$ there exists a unique $y \in \mathcal{V}$ for which

$$f(x,y) = o.$$

Denote this y by $\varphi(x)$. Then $\varphi \in C^1(\mathcal{U})$. Moreover, if $f \in C^k(\mathcal{G}), k \in \mathbb{N}$, then $\varphi \in C^k(\mathcal{U})$.

THEOREM 2.12.8. *(Inverse Function Theorem). Suppose X and Y are Banach spaces, $U \subset X$ is open, $f \in C^k(U \to X)$ with $k \geq 1$, $x_0 \in U$ and $Df(x_0)$ is invertible. Then there is a ball $B = B(x_0, r)$ in U centered at x_0 such that*

(1) $V = f(B)$ is open,
(2) $f|_B : B \to V$ is a homeomorphism,
(3) $g \doteq (f|_B)^{-1} \in C^k(V, B)$

and $g'(y) = [f'(g(y))]^{-1}$ for all $y \in V$.

The next Theorem is [54, Theorem 6.1.21, p.375].

THEOREM 2.12.9. *Let X be a reflexive real Banach space. Moreover, let $T : X \to X'$ be an operator satisfying the conditions*

(1) T is bounded;
(2) T is demicontinuous;
(3) T is coercive, i.e.,

$$\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} = +\infty;$$

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(4) T is monotone on the space X, i.e., for all $u, v \in X$ we have

$$\langle T(u) - T(v), u - v\rangle \geq 0,$$

Then the equation

$$T(u) = f^*$$

has at least one solution $u \in X$ for every $f^* \in X'$. If, moreover, $\langle T(u) - T(v), u - v\rangle > 0$ for all $u, v \in X, u \neq v$, then the equation has precisely one solution $u \in X$ for every $f^* \in X'$.

LEMMA 2.12.10. *Let $(X, \mathcal{F}, \mu_X)$ and $(Y, \mathcal{G}, \mu_Y)$ be σ-finite measure spaces and $(X \times Y, \mathcal{F} \otimes \mathcal{G}, \mu)$ be the product measure space. Consider a product measurable function $f : X \times Y \to Z$, where Z is a Banach space. It holds that*

$$\operatorname{esssup}_{(x,y)\in X\times Y} \|f(x,y)\|_Z = \operatorname{esssup}_{x\in X} \operatorname{esssup}_{y\in Y} \|f(x,y)\|_Z = \operatorname{esssup}_{y\in Y} \operatorname{esssup}_{x\in X} \|f(x,y)\|_Z.$$

PROOF. We recall the definition of the L^∞ norm as $\|f\|_\infty = \inf\{M > 0 : \mu(\{x\colon \|f(x)\| > M\}) = 0\}$.
By the definition of $\|f\|_{L^\infty(X\times Y;Z)}$, one has $\|f(x,y)\|_Z \leq \|f\|_{L^\infty(X\times Y;Z)}$ for $\mu_X \otimes \mu_Y$ -almost every $(x, y) \in X \times Y$. For a measurable set $A \subset X \times Y$, the product measure $\mu_X \otimes \nu_Y = 0$ if and only if $\mu_Y(A_x) = 0$, where $A_x := \{y \in Y\colon (x, y) \in A\}$, for almost every $x \in X$. It follows that for μ_X -almost every $x \in X$, it holds that $\|f(x,y)\|_Z \leq \|f\|_{L^\infty(X\times Y;Z)}$ for μ_Y- almost every $y \in Y$, i.e. $\|f_x\|_{L^\infty(Y;Z)} \leq \|f\|_{L^\infty(X\times Y;Z)}$ for μ_X -almost very $x \in X$. This means that $\operatorname{essup}_{x\in X} \|f_x\|_{L^\infty(Y;Z)} \leq \|f\|_{L^\infty(X\times Y;Z)}$. Conversely, we have $\|f_x\|_{L^\infty(Y;Z)} \leq \operatorname{essup}_{x\in X} \|f_x\|_{L^\infty(Y;Z)}$ for μ_X -almost every $x \in X$. In other words, it holds that $\|f_x(y)\|_Z \leq \operatorname{essup}_{x\in X} \|f_x\|_{L^\infty(Y;Z)}$, for μ_Y -almost every $y \in Y$, for μ_X -almost every $x \in X$. By the observation regarding the null-sets of the product measure, this means that $\|f(x,y)\|_Z = \|f_x(y)\|_Z \leq \operatorname{essup}_{x\in X} \|f_x\|_{L^\infty(Y;Z)}$ for $\mu_X \otimes \mu_Y$ -almost every $(x, y) \in X \times Y$, and hence that $\|f\|_{L^\infty(X\times Y;Z)} \leq \operatorname{essup}_{x\in X} \|f_x\|_{L^\infty(Y;Z)}$.
Exchanging the roles of X and Y concludes the proof. □

LEMMA 2.12.11. [54, Proposition 3.2.15] *If $f : X \to Y$ is a map of linear normed spaces such that in a certain open neighbourhood O_{x_0} of $x_0 \in X$ the Gateaux differential $\delta f(x)$ exists and is linear and continuous for all $x \in O_{x_0}$ and the map*

$$x \mapsto \delta f(x)$$

from X into $L(X, Y)$ is continuous at x_0, then f is Fréchet differentiable at x_0.

LEMMA 2.12.12. *Let $s > d/2$ and the embedding $D(L) \hookrightarrow H^s(\mathcal{O})$ be continuous and dense. We denote by $(\lambda_k^L)_{k\in\mathbb{N}}$ the eigenvalues of the operator L and by $(\lambda_k^\Delta)_{k\in\mathbb{N}}$ the eigenvalues of the operator $1 - \Delta$. Then there exists a constant $C > 1$, depending on the operators L, $1 - \Delta$, as well as on s, such that*

$$(\lambda_i^L)^{-2} \leq C(L, 1 - \Delta, s)(\lambda_i^\Delta)^{-s}, \tag{2.123}$$

for every $i \in \mathbb{N}$.

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PROOF. Before we begin, we note that for $s_1 \leq s_2 \in \mathbb{N}$ with continuous and dense embeddings $H^{s_2}(\mathcal{O}) \hookrightarrow D(L) \hookrightarrow H^{s_1}(\mathcal{O})$ the following inequalities hold,

$$\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2} \leq c\sum_{n\in\mathbb{N}}(\lambda_n^L)^{-2} \leq C\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}. \tag{2.124}$$

Indeed, let $(h_n)_{n\in\mathbb{N}}$ denote the eigenfunctions of the operator L, then for every $n \in \mathbb{N}$,

$$\|h_n\|^2_{H^{s_2}(\mathcal{O})'} \leq c(L)\|L^{-1}h_n\|^2_{L^2(\mathcal{O})} = c(L)(\lambda_n^L)^{-2}\|h_n\|^2_{L^2(\mathcal{O})} \leq C(L)\|h_n\|^2_{H^{s_1}(\mathcal{O})'}$$

$$\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2}\langle h_n, e_k\rangle^2 \leq c(L)(\lambda_n^L)^{-2} \leq C(L)\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}\langle h_n, e_k\rangle^2.$$

Summing over n,

$$\sum_{n\in\mathbb{N}}\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2}\langle h_n, e_k\rangle^2 \leq c(L)\sum_{n\in\mathbb{N}}(\lambda_n^L)^{-2} \leq C(L)\sum_{n\in\mathbb{N}}\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}\langle h_n, e_k\rangle^2$$

$$\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2}\sum_{n\in\mathbb{N}}\langle h_n, e_k\rangle^2 \leq c(L)\sum_{n\in\mathbb{N}}(\lambda_n^L)^{-2} \leq C(L)\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}\sum_{n\in\mathbb{N}}\langle h_n, e_k\rangle^2$$

$$\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2}\|e_k\|^2_{L^2(\mathcal{O})} \leq c(L)\sum_{n\in\mathbb{N}}(\lambda_n^L)^{-2} \leq C(L)\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}\|e_k\|^2_{L^2(\mathcal{O})}$$

$$\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_2} \leq c(L)\sum_{n\in\mathbb{N}}(\lambda_n^L)^{-2} \leq C(L)\sum_{k=1}^{\infty}(\lambda_k^{\Delta})^{-s_1}.$$

We will now use (2.124) to prove (2.123). If the claim was not true, then for every constant $C > 1$, there would be at least one index i, such that

$$(\lambda_i^L)^{-2} > C(\lambda_i^{\Delta})^{-s}.$$

We define the constants $C_n := n^2\sum_{k=1}^{\infty}(\lambda_k^L)^{-2}$ and denote the subset of indices j, where $(\lambda_j^L)^{-2} \leq C_n(\lambda_j^{\Delta})^{-s}$, by J_n. Since the set J_n is non-empty by assumption, we can sum over its components,

$$\infty > \sum_{k=1}^{\infty}(\lambda_k^L)^{-2} \geq \sum_{j\in J_n}(\lambda_j^L)^{-2} > n^2\sum_{k=1}^{\infty}(\lambda_k^L)^{-2}\sum_{j\in J_n}(\lambda_i^{\Delta})^{-s}.$$

Hence

$$\frac{1}{n^2} \geq \sum_{j\in J_n}(\lambda_i^{\Delta})^{-s}.$$

We observe that $\bigcup_{n=1}^{\infty} J_n \supseteq \bigcup_{n=2}^{\infty} J_n \supseteq \cdots \supseteq \bigcup_{N+1=2}^{\infty} J_n \supseteq \cdots \supseteq \limsup_{n\to\infty} J_n$. Dividing both sides of the inequality by $\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}$, yields

$$\frac{1}{\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}}\frac{1}{n^2} \geq \frac{1}{\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}}\sum_{j\in J_n}(\lambda_i^{\Delta})^{-s}.$$

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The right hand side can now be interpreted as a probability measure on $\mathbb{N}$. Since

$$\infty > \frac{1}{\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}} \sum_{n=1}^{\infty} \frac{1}{n^2} \geq \frac{1}{\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}} \sum_{n=1}^{\infty} \sum_{j \in J_n} (\lambda_i^{\Delta})^{-s},$$

an application of the Borel-Cantelli Lemma, denoting $\widetilde{J} = \limsup_{n\to\infty} J_n$, yields

$$\frac{1}{\sum_{k=1}^{\infty}(\lambda_i^{\Delta})^{-s}} \sum_{j \in \widetilde{J}} (\lambda_i^{\Delta})^{-s} = 0.$$

We conclude that only for finitely many $n \in \mathbb{N}$, there exists a non-empty set of indices J_n such that $(\lambda_i^L)^{-2} > n(\lambda_i^{\Delta})^{-s}$. Denoting this finite set of natural numbers by I, there exists a constant $\widetilde{C}$, depending on the supremum of I, for which $(\lambda_i^L)^{-2} > \widetilde{C}(\lambda_i^{\Delta})^{-s}$ is not satisfied, leading to a contradiction. □

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2.12.4. Appendix D: local existence. In this section, we sketch the alterations which need to be applied to [91, Proposition 4.2.4, p.91], as well as the preceding results [91, Theorem 3.1.1, p.56] and [91, Proposition 3.1.4, p.57] in order to obtain a local existence result. We recall that the goal is to obtain a (local) solution of the equation

$$X(t) = X(0) + \int_0^t A(s, X(s))\,\mathrm{d}s + \int_0^t B(s, X(s))\,\mathrm{d}W_s,$$

in the sense of [91, Definition 4.2.1, p.90]. Since this section is focused on the quoted result, we will adapt their notation. Recall that the equation is posed on the Gelfand-Triple $V \hookrightarrow H \hookrightarrow V'$.
Since the proof of [91, Proposition 4.2.4, p.91] is based on a Galerkin projection in the Hilbert space H, we consider the finite dimensional equation

$$\mathrm{d}X^{(n)}(t) = P_n A(t, X^{(n)})\,\mathrm{d}t + P_n B(t, X^{(n)})\,\mathrm{d}W_t^{(n)}, \tag{2.125}$$

which puts us in the setting of [91, Theorem 3.1.1, p.56]. The main changes need to be made at this stage. We sketch the proof, with an additional minor alteration which allows us to ignore some additional Lemmas used in [91] and leave this section more connected. We begin by introducing the following Euler-type discretization with step size $1/m$ by

(2.126)
$$X^{(n),m}(t) = X^{(n),m}(0) + \int_{\lfloor tm\rfloor/m}^t P_n A(s, X^{(n),m}(\lfloor tm\rfloor/m))\,\mathrm{d}t + \int_{\lfloor tm\rfloor/m}^t P_n B(s, X^{(n),m}(\lfloor tm\rfloor/m))\,\mathrm{d}W_s^{(n)}.$$

An equivalent way to write this is

(2.127)
$$X^{(n),m}(t) = X^{(n),m}(0) + \int_0^t P_n A(s, X^{(n),m}(\kappa(m,s)))\,\mathrm{d}t + \int_0^t P_n B(s, X^{(n),m}(\kappa(m,s))\,\mathrm{d}W_s^{(n)},$$

where $\kappa(m,t) := \lfloor tm\rfloor/m (\leq t)$. The process $t \mapsto X^{(n),m}(t)$ is adapted and continuous, hence

$$\begin{aligned} t \mapsto p^{(n),m}(t) := X^{(n),m}(\kappa(m,t)) - X^{(n),m}(t) = &- \int_{\kappa(m,t)}^t P_n A(s, X^{(n),m}(\kappa(m,s)))\,\mathrm{d}t \\ &- \int_{\kappa(m,t)}^t P_n B(s, X^{(n),m}(\kappa(m,s))\,\mathrm{d}W_s^{(n)} \end{aligned}$$

defines an adapted cadlag process. Hence, we obtained another equivalent formulation of (2.127), given by

$$\begin{aligned} X^{(n),m}(t) = X^{(n),m}(0) &+ \int_0^t P_n A(s, X^{(n),m}(s) + p^{(n),m}(s))\,\mathrm{d}t \\ &+ \int_0^t P_n B(s, X^{(n),m}(s) + p^{(n),m}(s))\,\mathrm{d}W_s^{(n)}. \end{aligned}$$

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Next, we fix $R \geq 0$ and define

$$\tau^{(n),m}(R) := \inf\left\{t \geq 0 \colon |X^{(n),m}(t)| > \frac{R}{3}\right\}$$

and

$$r(R) := \frac{R}{4}.$$

Then, since $\kappa(m,t) \coloneqq \lfloor tm \rfloor / m \leq t$,

$$|p^{(n),m}(t)| \leq \frac{2R}{3} \text{ and } |X^{(n),m}(t)| \leqslant \frac{R}{3} \text{ if } t \in (0, \tau^{(n),m}(R)].$$

Fix m_1, m_2 and let $\tau \geq 0$ be a finite stopping time such that $\tau \leq \tau^{(n),m_1}(R) \wedge \tau^{(n),m_2}(R) \wedge T$. Then by Itô's formula,

$$\begin{aligned}
&\|X^{(n),m_1}(\tau) - X^{(n),m_2}(\tau)\|^2 \\
&\quad = 2\int_0^\tau \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\,\mathrm{d}W_s^n\rangle \\
&\qquad + 2\int_0^\tau \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nA(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nA(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\rangle\,\mathrm{d}s \\
&\qquad + \int_0^\tau \|P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s)) + p^{(n),m_2}(s))\|^2\,\mathrm{d}s.
\end{aligned}$$

By rewriting $X^{(n),m_1}(s) - X^{(n),m_2} = X^{(n),m_1}(s) + p^{(n),m_1}(s) - (X^{(n),m_2} + p^{(n),m_2}(s)) - p^{(n),m_1}(s) + p^{(n),m_2}(s)$ and the (local) monotonicity assumption,

$$\begin{aligned}
&\|X^{(n),m_1}(\tau) - X^{(n),m_2}(\tau)\|^2 \\
&\quad \leq 2\int_0^\tau \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\,\mathrm{d}W_s^n\rangle \\
&\qquad + 2\int_0^\tau \langle p^{(n),m_2}(s) - p^{(n),m_1}(s), P_nA(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nA(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\rangle\,\mathrm{d}s \\
&\qquad + C(R)\int_0^\tau \|X^{(n),m_1}(s) + p^{(n),m_1}(s) - X^{(n),m_2}(s) - p^{(n),m_2}(s)\|^2\,\mathrm{d}s.
\end{aligned}$$

By the continuity of A and the choice of stopping time,

$$\begin{aligned}
&\leq 2\int_0^\tau \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\,\mathrm{d}W_s^n\rangle \\
&\quad + \int_0^\tau 4C(R)\big(\|p^{(n),m_2}(s)\|^2 + \|p^{(n),m_1}(s)\|^2\big) + 2C(R)\|X^{(n),m_1}(s) - X^{(n),m_2}(s)\|^2\,\mathrm{d}s \\
&\leq 2\int_0^\tau \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\,\mathrm{d}W_s^n\rangle \\
&\quad + \int_0^\tau 4C(R)\big(\|p^{(n),m_2}(s)\|^2 + \|p^{(n),m_1}(s)\|^2\big) + 2C(R)\sup_{r\in[0,s]}\|X^{(n),m_1}(r) - X^{(n),m_2}(r)\|^2\,\mathrm{d}s.
\end{aligned}$$

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We want to apply the stochastic Gronwall inequality [118] with the terms

$$M(t) :=$$
$$2\int_0^{t\wedge\tau} \langle X^{(n),m_1}(s) - X^{(n),m_2}(s), P_nB(s, X^{(n),m_1}(s) + p^{(n),m_1}(s)) - P_nB(s, X^{(n),m_2}(s) + p^{(n),m_2}(s))\,\mathrm{d}W_s^n\rangle,$$
$$H(t) := \int_0^{t\wedge\tau} C(R)\big(\|p^{(n),m_2}(s)\|^2 + \|p^{(n),m_1}(s)\|^2\big)\,\mathrm{d}s.$$

In order to obtain a convergence result, we have to show that $H^*(T) := \sup_{0\le s\le T} H(s)$ converges to 0 as $m_1, m_2 \to \infty$. This can be verified as follows:

$$\|p(m_i,t)\| \le \sup_{\lfloor m_i t\rfloor/m_i \le s\le t} \left\| \int_{\kappa(m,t)}^{s} P_nA(r, X^{(n),m}(\kappa(m,r)))\,\mathrm{d}r\right\| + \sup_{\lfloor m_i t\rfloor/m_i \le s\le t} \left\| \int_{\kappa(m,t)}^{s} P_nB(r, X^{(n),m}(\kappa(m,r))\,\mathrm{d}W_r^{(n)}\right\|,$$

and since A, B are locally bounded,

$$\mathbb{E}\mathbf{1}_{\{\tau^{(n),m}(R)\ge s\}}\|p(m_i,t)\| \to 0, \tag{2.128}$$

as $m\to\infty$, uniformly in $[0,T]$. From (2.128) and Chebyshev's inequality it follows that

$$\lim_{m_1,m_2\to\infty} \mathbb{P}\left(\sup_{t\in[0,\tau^{(n),m_1}(R)\wedge\tau^{(n),m_2}(R)\wedge T} \|X^{(n),m_1}(t) - X^{(n),m_2}(t)\| \ge \varepsilon\right) \to 0.$$

Define

$$\tau^{(n)}(R) := \liminf_{m\to\infty} \tau^{(n),m}(R).$$

Since the space of continuous processes is complete with respect to the locally (in t) uniform convergence in probability, there exists a continuous ($\mathcal{F}_t$ adapted, $\mathbb{R}^d$ valued) process $X^{(n)}(t)$, defined on $[0,\tau^{(n)}(R))$ to which $X^{(n),m}$ converges $\mathbb{P}$-a.s. locally in $C([0,\tau^{(n)}(R));\mathbb{R}^d)$. The positivity of the stopping time $\tau^{(n)}(R)$ follows from standard arguments (see for example [127]).

To argue the convergence of the right hand side of (2.126), one observes that $\overline{B(R)}^{\mathbb{R}^d} := \{x\in\mathbb{R}^d\colon |x|\le R\}$ is compact and the continuous functions P_nA, P_nB are uniformly continuous on $\overline{B(R)}^{\mathbb{R}^d}$. Hence

$$\lim_{m_1,m_2\to\infty} \mathbb{P}\left(\sup_{t\in[0,\tau^{(n),m_1}(R)\wedge\tau^{(n),m_2}(R)\wedge T} \|P_nA(X^{(n),m_1}(t)) - P_nA(X^{(n),m_2}(t))\| \ge \varepsilon\right) \to 0$$
$$\lim_{m_1,m_2\to\infty} \mathbb{P}\left(\sup_{t\in[0,\tau^{(n),m_1}(R)\wedge\tau^{(n),m_2}(R)\wedge T} \|P_nB(X^{(n),m_1}(t)) - P_nB(X^{(n),m_2}(t))\| \ge \varepsilon\right) \to 0.$$

Together with the continuity of P_nA, P_nB, this yields that $X^{(N)}$ is a solution to equation (2.125) on $[0,\tau^{(n)}(R))$.

The uniqueness is again a direct consequence of Itô's formula.

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Since the stopping time $\tau^{(n)}(R)$ is not the most convenient when passing to the limit in n, we want to relate it to the stopping time $\widetilde{\tau}^{(n)}(R) := \inf\{t \geq 0 \colon \|X^{(n)}(t)\| > R\}$. Since $\tau^{(n)}(R)$ is monotone in R, we denote by $\tau_\infty^{(n)} = \lim_{R\to\infty} \tau^{(n)}(R)$. We infer that on the set $\{\tau_\infty^{(n)} < \infty\}$,

$$\lim_{t\to\tau_\infty^{(n)}} \|X^{(n)}(t,\omega)\| = \lim_{R\to\infty} \|X^{(n)}(\tau^{(n)}(R),\omega)\| = \infty.$$

Our goal is to show that $\mathbb{P}(\widetilde{\tau}^{(n)}(R) < \tau_\infty^{(n)})$. Since $\widetilde{\tau}^{(n)}(R)$ is monotone increasing in R, we will show the even stronger claim

$$\mathbb{P}\left(\lim_{R\to\infty} \widetilde{\tau}^{(n)}(R) < \tau_\infty^{(n)}\right) = 0.$$

Assume the contrary, namely that $\mathbb{P}\left(\lim_{R\to\infty} \widetilde{\tau}^{(n)}(R) = \widetilde{\tau}_\infty^{(n)} > \tau_\infty^{(n)}\right) > 0$. Since $X^{(n)}$ is a local solution of (2.125), there is an increasing sequence of stopping times η_k, such that $X^{(n)}$ satisfies (2.125) on $[0,\eta_k)$. Define a new family of stopping times by

$$\widetilde{\eta}_{R,k} := \widetilde{\tau}^{(n)}(R) \wedge \eta_k; \quad \widetilde{\eta}_k := \sup_R \widetilde{\eta}_{R,k}.$$

Since $\widetilde{\eta}_{R,k} \leq \widetilde{\tau}^{(n)}(R) \leq \widetilde{\tau}_\infty^{(n)}$, it follows that $\widetilde{\eta}_k \leq \widetilde{\tau}_\infty^{(n)}$. We also note that $\lim_k \widetilde{\eta}_k = \widetilde{\tau}_\infty^{(n)}$. Since $\widetilde{\eta}_k \nearrow \tilde{\tau}_\infty^{(n)}$ and $\mathbb{P}(\widetilde{\tau}_\infty^{(n)} > \tau_\infty^{(n)}) > 0$, there exists a number k such that $\mathbb{P}\left(\widetilde{\eta}_k > \tau_\infty^{(n)}\right) > 0$. Hence, we have $\|X^{(n)}(t,\omega)\| \leq k$ for $t \in \left[\tau_\infty^{(n)}(\omega), \widetilde{\eta}_k(\omega)\right)$ leading to a contradiction.
We conclude that, for each fixed $n \in \mathbb{N}$ and $R > 0$, we can solve equation (2.125) up to the stopping time $\tau^{(n)}(R)$.
The passage to the limit $n \to \infty$, which is the subject of [91, Proposition 4.2.4, p.91] and [91, Lemma 4.2.9, p.91], needs to be altered in a similar fashion. One defines again a stopping time by $\tau(R) := \liminf_{n\to\infty} \tau^{(n)}(R)$ and argues along the same lines as in the previous arguments, without the superscript (n).

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2.12.5. Appendix E: Positivity of the stopping times.

DEFINITION 2.12.13. *For a solution v of equation (2.10), we call the stopping time $\tau := \inf\{t > 0 \colon \|v(t)\|_{D(L)'} = \infty\}$ an explosion time and say that v explodes, if $\mathbb{P}(\tau(\omega) < \infty) = 1$.*

We want to show that the stopping time $\tau_R(\omega) := \inf\{t \in [0,T] \colon \|v^\varepsilon(t)\|_{D(L)'} > R\}$, used in the proof of Theorem 2.6.6, is in indeed strictly positive (with positive probability) and that the explosion time is strictly positive with probability 1. Let $\delta > 0$. We note that

$$\{\omega \colon \sup_{0\le t\le\delta} \|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'} < R\} \subset \{\omega \colon \tau_R(\omega) > \delta\}.$$

We proceed by applying Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}(\tau_R > \delta) &\ge \mathbb{P}(\sup_{t\in[0,\delta]} \|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'} \le R) \ge 1 - \frac{\mathbb{E}(\sup_{t\in[0,\delta]} \|v^\varepsilon(t\wedge\tau_R)\|^2_{D(L)'})}{R^2} \\
&\ge 1 - \frac{\mathbb{E}\sup_{t\in[0,\delta]} \|u(0) + \int_0^{t\wedge\tau_R} \operatorname{div}\left(B(u(R_\varepsilon(v^\varepsilon)))\nabla R_\varepsilon(v^\varepsilon)\right)(s)\,\mathrm{d}s + \int_0^{t\wedge\tau_R} \sigma(u(R_\varepsilon(v^\varepsilon)))(s)\,\mathrm{d}W(s)\|^2_{D(L)'}}{R^2} \\
&\ge 1 - C\frac{\mathbb{E}\|v_0\|^2_{D(L)'}}{R^2} \\
&\quad - C\frac{\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]} \|\operatorname{div}\left(B(u(R_\varepsilon(v^\varepsilon)))\nabla R_\varepsilon(v^\varepsilon)\right)(s)\|^2_{D(L)'}\,\mathrm{d}s}{R^2} \\
&\quad - C\frac{\mathbb{E}\sup_{t\in[0,\delta]} \|\int_0^{t\wedge\tau_R} \sigma(u(R_\varepsilon(v^\varepsilon)))(s)\,\mathrm{d}W(s)\|^2_{D(L)'}}{R^2} \\
&= 1 - I_1 - I_2 - I_3.
\end{aligned}$$

First, we estimate I_2.

$$\begin{aligned}
I_2 &= \int_0^\delta \chi_{[0,\tau_R]} \|\operatorname{div}\left(B(u(R_\varepsilon(v^\varepsilon)))\nabla R_\varepsilon(v^\varepsilon)\right)(s)\|^2_{D(L)'}\,\mathrm{d}s \\
&\le \int_0^\delta \chi_{[0,\tau_R]} \|B(u(R_\varepsilon(v^\varepsilon)))\nabla R_\varepsilon(v^\varepsilon)(s)\|^2_{L^1(\mathcal{O})}\,\mathrm{d}s \\
&\le \int_0^\delta \chi_{[0,\tau_R]} \|B(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^p(\mathcal{O})} \|\nabla R_\varepsilon(v^\varepsilon)(s)\|^2_{L^q(\mathcal{O})}\,\mathrm{d}s \\
&\le \int_0^\delta \chi_{[0,\tau_R]} \|B(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^p(\mathcal{O})} \|R_\varepsilon(v^\varepsilon)(s)\|^2_{D(L)}\,\mathrm{d}s \\
&\le \frac{C}{\varepsilon}\int_0^\delta \chi_{[0,\tau_R]} \|B(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^p(\mathcal{O})} (\|v^\varepsilon(s)\|^2_{D(L)'} + \|R_\varepsilon(0)\|^2_{D(L)})\,\mathrm{d}s \\
&\le \frac{C}{\varepsilon}\int_0^\delta \chi_{[0,\tau_R]} \|A(u(R_\varepsilon(v^\varepsilon)))(h'')^{-1}(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^p(\mathcal{O})} (\|v^\varepsilon(s)\|^2_{D(L)'} + \|R_\varepsilon(0)\|^2_{D(L)})\,\mathrm{d}s \\
&\le \frac{C}{\varepsilon}\int_0^\delta \chi_{[0,\tau_R]} \|A(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^\infty(\mathcal{O})} \|(h'')^{-1}(u(R_\varepsilon(v^\varepsilon)))(s)\|^2_{L^p(\mathcal{O})} (\|v^\varepsilon(s)\|^2_{D(L)'} + \|R_\varepsilon(0)\|^2_{D(L)})\,\mathrm{d}s
\end{aligned}$$

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by the growth assumption on A,

$$\leq \frac{C(A)}{\varepsilon}\int_0^\delta \chi_{[0,\tau_R]}\big(1+\|u(R_\varepsilon(v^\varepsilon))(s)\|_{L^\infty(\mathcal{O})}\big)^2\|(h'')^{-1}(u(R_\varepsilon(v^\varepsilon)))(s)\|_{L^p(\mathcal{O})}^2 \times (\|v^\varepsilon(s)\|_{D(L)'}^2+\|R_\varepsilon(0)\|_{D(L)}^2)\,\mathrm{d}s.$$

By the presence of the indicator function, continuity of $(h'')^{-1}$ and the arguments used in Lemma 2.5.12, we conclude that the first two factors can be bounded by a constant depending on R.
To obtain an estimate on I_3, we use the BDG inequality,

$$I_3 = \mathbb{E}\sup_{t\in[0,\delta]}\|\int_0^t \chi_{[0,\tau_R]}\sigma(u(R_\varepsilon(v^\varepsilon)))(s)\,\mathrm{d}W(s)\|_{D(L)'}^2$$
$$\leq \mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}\|\sigma(u(R_\varepsilon(v^\varepsilon)))(s)\|_{\mathcal{L}_2^{n\times n}(U,L^2(\mathcal{O}))}^2\,\mathrm{d}s$$

by the linear growth assumption on σ, we obtain

$$\leq C\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}C(1+\|u(R_\varepsilon(v^\varepsilon))(s)\|_{L^2(\mathcal{O})})^2\,\mathrm{d}s$$
$$\leq C\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}C(1+\|u(R_\varepsilon(v^\varepsilon))(s)-u(R_\varepsilon(0))\|_{L^2(\mathcal{O})}+\|u(R_\varepsilon(0))\|_{L^2(\mathcal{O})})^2\,\mathrm{d}s$$
$$\leq C\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}(1+\|u'(\eta R_\varepsilon(v^\varepsilon)(s))\|_{L^2(\mathcal{O})}^2\|R_\varepsilon(v^\varepsilon)(s)\|_{L^\infty(\mathcal{O})}^2+\|u(0)\|_{L^2(\mathcal{O})}^2)\,\mathrm{d}s$$
$$\leq C\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}\big(1+\|u'(\eta R_\varepsilon(v^\varepsilon)(s))\|_{L^2(\mathcal{O})}^2\big(\|R_\varepsilon(v^\varepsilon)(s)-R_\varepsilon(0)\|_{L^\infty(\mathcal{O})}^2+\|R_\varepsilon(0)\|_{L^\infty(\mathcal{O})}^2\big)\big)\,\mathrm{d}s$$
$$+C\mathbb{E}\int_0^\delta \|u(0)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s$$
$$\leq C\mathbb{E}\int_0^\delta \chi_{[0,\tau_R]}\left(1+\|u'(\eta R_\varepsilon(v^\varepsilon)(s))\|_{L^2(\mathcal{O})}^2\left(\frac{C}{\varepsilon}\|v^\varepsilon(s)\|_{D(L)'}^2+\|R_\varepsilon(0)\|_{L^\infty(\mathcal{O})}^2\right)+\|u(0)\|_{L^2(\mathcal{O})}^2\right)\mathrm{d}s.$$

By Assumption (E4), we can quantify the dependence on the radius R,

$$\frac{\mathbb{E}\sup_{t\in[0,\delta]}\|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'}^2}{R^2}$$
$$\leq \frac{C(R_\varepsilon(0))}{R^2}\mathbb{E}\|v_0\|_{D(L)'}^2$$
$$+\mathbb{E}\sup_{t\in[0,\delta]}\frac{\delta}{R^2}\frac{C(A,R_\varepsilon(0))}{\varepsilon}\big(1+\|R_\varepsilon(v^\varepsilon)(t\wedge\tau_R)\|_{L^\infty(\mathcal{O})}^4\big)\exp\big(2\|R_\varepsilon(v^\varepsilon)(t\wedge\tau_R)\|_{L^\infty(\mathcal{O})}\big)\times\big(1+\|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'}^2\big)$$
$$+\mathbb{E}\sup_{t\in[0,\delta]}\frac{\delta}{R^2}\frac{C(\sigma,R_\varepsilon(0))}{\varepsilon}\big(1+\|R_\varepsilon(v^\varepsilon)(t\wedge\tau_R)\|_{L^\infty(\mathcal{O})}^2\big)\exp\big(\|R_\varepsilon(v^\varepsilon)(t\wedge\tau_R)\|_{L^\infty(\mathcal{O})}\big)$$

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$$\times \left(1+\|v^\varepsilon(t\wedge\tau_R)\|^2_{D(L)'}\right)$$

using the inequality $\|R_\varepsilon(v^\varepsilon)(r)\|_{L^\infty(\mathcal{O})} \le \frac{C}{\varepsilon}\|v^\varepsilon(r)\|_{D(L)'} + \|R_\varepsilon(0)\|_{L^\infty(\mathcal{O})}$,

$$\le \frac{C(R_\varepsilon(0))}{R^2}\mathbb{E}\|v_0\|^2_{D(L)'}$$
$$+ \mathbb{E}\sup_{t\in[0,\delta]}\frac{\delta}{R^2}\frac{C(A,R_\varepsilon(0))}{\varepsilon}\left(1+\|v^\varepsilon(t\wedge\tau_R)\|^4_{D(L)'}\right)\exp\left(2\|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'}\right)$$
$$\times \left(1+\|v^\varepsilon(t\wedge\tau_R)\|^2_{D(L)'}\right)$$
$$+ \mathbb{E}\sup_{t\in[0,\delta]}\frac{\delta}{R^2}\frac{C(\sigma,R_\varepsilon(0))}{\varepsilon}\left(1+\|v^\varepsilon(t\wedge\tau_R)\|^2_{D(L)'}\right)\exp\left(\|v^\varepsilon(t\wedge\tau_R)\|_{D(L)'}\right)$$
$$\times \left(1+\|v^\varepsilon(t\wedge\tau_R)\|^2_{D(L)'}\right)$$
$$\le \frac{C(R_\varepsilon(0))}{R^2}\mathbb{E}\|v_0\|^2_{D(L)'} + \frac{\delta}{R^2}\frac{C(A,\sigma,R_\varepsilon(0))}{\varepsilon}(1+R^2)^4\exp(2R).$$

For each $\delta>0$, we can find a radius R, such that

$$R \le \log\left(\frac{1}{\delta^{1/3}}\right) < R+1.$$

Hence the term above can be estimated by

$$\le \frac{C(R_\varepsilon(0))}{R^2}\mathbb{E}\|v_0\|^2_{D(L)'} + \frac{\delta}{R^2}\frac{C(A,\sigma,R_\varepsilon(0))}{\varepsilon}\left(1+\log\left(\delta^{1/3}\right)^2\right)^4\left(\frac{1}{\delta^{1/3}}\right)^2$$
$$\le \frac{C}{\log\left(\delta^{1/3}\right)^2}\left(\mathbb{E}\|v_0\|^2_{D(L)'} + \delta^{1/3}(1+\log\left(\delta^{1/3}\right))^8\right).$$

According to the definition of τ_R,

$$\left\{\sup_{0\le t\le\delta}\|v^\varepsilon(t\wedge\tau_R)\| < R\right\} \subset \{\tau_R>\delta\} \subset \{\lim_{R\to\infty}\tau_R>\delta\}.$$

Let $\tau := \lim_{R\to\infty}\tau_R$, then it follows that

$$\mathbb{P}(\tau>\delta) \ge 1 - \frac{C}{\log\left(\delta^{1/3}\right)^2}\left(\mathbb{E}\|v_0\|^2_{D(L)'} + \delta^{1/3}(1+\log\left(\delta^{1/3}\right))^8\right).$$

Since the terms $\delta^{1/3}(1+\log\left(\delta^{1/3}\right))^8$, $\frac{C}{\log\left(\delta^{1/3}\right)^2}$ approach 0 as $\delta\to 0$, and

$$\mathbb{P}(\tau=0) = \mathbb{P}\big(\cap_{\delta>0}\{\tau<\delta\}\big) = \lim_{\delta\to 0}\mathbb{P}(\tau<\delta),$$

$$\mathbb{P}(\tau=0) \le \lim_{\delta\to 0}\frac{C}{\log\left(\delta^{1/3}\right)^2}\left(\mathbb{E}\|v_0\|^2_{D(L)'} + \delta^{1/3}(1+\log\left(\delta^{1/3}\right))^8\right) = 0.$$

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2.12.6. Appendix F: Basic properties of the Frobenius product. Some results regarding the Frobenius inner product.

$$\langle A, B\rangle_F = \mathrm{Tr}(A^T B) = \sum_{i,j=1}^{d} A_{ij} B_{ij}.$$

For $m \times n$ matrices A, B, we have $\|AB\|_F^2 \coloneqq \mathrm{Tr}((AB)^T AB)$.

$$\|AB\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{p}\left|\sum_{k=1}^{n} a_{ik}b_{kj}\right|^2 \le \sum_{i=1}^{m}\sum_{j=1}^{p}\left(\sum_{k=1}^{n}|a_{ik}|^2\sum_{k=1}^{n}|b_{kj}|^2\right) = \sum_{i=1}^{m}\sum_{j=1}^{p}\left(\sum_{k,l=1}^{n}|a_{ik}|^2|b_{lj}|^2\right)$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n}|a_{ik}|^2\sum_{l=1}^{n}\sum_{j=1}^{p}|b_{lj}|^2 = \|A\|_F^2\|B\|_F^2.$$

Let V be an orthogonal matrix,

$$\|A\|_F^2 = \mathrm{tr}\left(A^T A\right) = \mathrm{tr}\left(A^T AVV^T\right) = \mathrm{tr}\left(V^T A^T AV\right) = \|AV\|_F^2.$$

In particular, if v_k form an orthonormal basis, then $\|A\|_F^2 = \sum_{k=1}^{\infty}\|Av_k\|_2^2$. It is straightforward that $\|A\|_2 \le \|A\|_F$. Then we have

$$\|AB\|_F^2 = \sum_{k=1}^{\infty}\|ABv_k\|_2^2 \le \sum_{k=1}^{\infty}\|A\|_2^2\|Bv_k\|_2^2 \le \|A\|_F^2\sum_{k=1}^{\infty}\|Bv_k\|_2^2 = \|A\|_F^2\|B\|_F^2.$$

LEMMA 2.12.14. *Let B a positive (semi-)definite $n \times n$ matrix, then*

$$\lambda_{\max}(B)\|A\|_F^2 \ge \mathrm{tr}(A^T BA) \ge \lambda_{\min}(B)\|A\|_F^2.$$

PROOF. For the first direction,

$$\mathrm{tr}\left(A^T BA\right) = \mathrm{tr}\left(A^T Q\Lambda Q^T A\right) = \mathrm{tr}\left(\Lambda Q^T AA^T Q\right)$$
$$\ge \lambda_{\min}(B)\,\mathrm{tr}\left(Q^T AA^T Q\right) = \lambda_{\min}(B)\,\mathrm{tr}\left(AA^T\right) = \lambda_{\min}(B)\|A\|_F^2.$$

The second direction works analogously,

$$\mathrm{tr}\left(A^T BA\right) = \mathrm{tr}\left(A^T Q\Lambda Q^T A\right) = \mathrm{tr}\left(\Lambda Q^T AA^T Q\right)$$
$$\le \lambda_{\max}(B)\,\mathrm{tr}\left(Q^T AA^T Q\right) = \lambda_{\max}(B)\,\mathrm{tr}\left(AA^T\right) = \lambda_{\max}(B)\|A\|_F^2.$$

□

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CHAPTER 3

Equivalence of solution concepts

We indicated earlier that previous to the results presented in Section 2.8, that local-in-time existence of solutions to a two-species SKT system was obtained in [85]. In their paper, they used the concept of so-called pathwise mild solution. This solution concept, for semilinear parabolic SPDEs with nonautonomous random generators, was introduced by Pronk and Veraar in [113]. They bypassed the issue of having non adapted integrands appearing in the classical definition of mild solutions by the use of integration by parts. This solution concept was then extended to the case of quasilinear parabolic SPDEs, in the previously mentioned work [85] where they proved the existence of a unique local-in-time pathwise mild solution for the abstract Cauchy problem

$$\begin{aligned} &\mathrm{d}u(t) = [A(u(t))u(t) + F(t,u(t))]\,\mathrm{d}t + \sigma(t,u(t))\,\mathrm{d}W(t), \quad t \in (0,\infty), u(t) \in \mathbb{R}^n, \\ &u(0) = u_0. \end{aligned}$$

This formulation covers the structure motivated by the previous chapters,

$$\begin{cases} \mathrm{d}u = \operatorname{div}\left(\hat{A}(u)\nabla u\right)\,\mathrm{d}t + \sigma(u)\,\mathrm{d}W_t, & t > 0, \\ u(0) = u^0, \end{cases} \tag{3.1}$$

on an open, bounded domain $\mathcal{O} \subset \mathbb{R}^d$ $(d \geq 1)$, with smooth boundary $\partial\mathcal{O}$. Here $\hat{A} = (\hat{A}_{ij})$ is an $n \times n$ diffusion matrix, σ is a nonlinear term and $W = (W_1, \ldots, W_n)$ is a cylindrical Wiener process. The precise assumptions on the coefficients will be stated in Section 3.1 and 3.2. The previous system rewrites componentwise as

$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^{n} \hat{A}_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^{n} \sigma_{ij}(u)\mathrm{d}W_t^j, \quad t > 0, \tag{3.2}$$

with $u_i(0) = u_i^0$ in $\mathcal{O}$, $i = 1, \ldots, n$, and is augmented by either no-flux boundary conditions

$$\sum_{j=1}^{n} \hat{A}_{ij}(u)\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ i = 1, \ldots, n,\ t > 0, \tag{3.3}$$

or homogeneous Dirichlet boundary conditions

$$u_i(t,x) = 0 \quad \text{on } \partial\mathcal{O},\ i = 1, \ldots, n,\ t > 0. \tag{3.4}$$

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In order to investigate mild solutions for (3.2), we write it as an abstract quasilinear Cauchy problem

$$\begin{cases} \mathrm{d}u = A_u u \, \mathrm{d}t + \sigma(u) \, \mathrm{d}W_t \\ u(0) = u^0, \end{cases} \tag{3.5}$$

where the linear operator A_u is given by $A_u v \coloneqq \operatorname{div}(\hat{A}(u)\nabla v)$.

REMARK 3.0.1. *We took the liberty of changing the notation in this section, compared to the previous sections, in order to prevent a clash with commonly used notation.*

- *The diffusion matrix will be denoted by $\hat{A}$. Previously it was denoted by A, which we reserve for the non-linear operator* $\operatorname{div}(\hat{A}(u)\nabla u)$.
- *The dual space of X will be denoted by X^*. One reason for changing to this notation is, that it improves the readability of the notation which will be introduced in Definition 3.1.7. In this section we reserve the notation $\cdot'$ for the one dimensional derivative.*

A stochastic process u is called a pathwise mild solution of (3.2) if

$$u(t) = U^u(t,0)u^0 - \int_0^t U^u(t,s)A_u(s) \int_s^t \sigma(u(\tau)) \, \mathrm{d}W_\tau \, \mathrm{d}s + U^u(t,0) \int_0^t \sigma(u(s)) \, \mathrm{d}W_s, \tag{3.6}$$

where $U^u(\cdot,\cdot)$, is the random evolution family generated by A_u, see Section 5.2 for further details. This formula can be motivated using integration by parts and overcomes the nonadaptedness of the random evolution family $U^u(\cdot,\cdot)$. More precisely, as observed in [113], the mapping $\omega \mapsto U^u(t,s,\omega)$ is only $\mathcal{F}_t$ -measurable. However, in order to define the stochastic convolution $\int_0^t U^u(t,s,\omega) \, \mathrm{d}W_s$ as an Itô integral, this mapping should be $\mathcal{F}_s$ -measurable. In order to solve this issue, one uses the integration by parts formula to obtain (3.6). Pathwise mild solutions for hyperbolic SPDEs with additive noise were analyzed in [99].

The non-adaptedness of the integrand in the definition of a stochastic integral was firstly discussed by Alós, León and Nualart in [4, 89] using the Skorokhod- and the Russo-Vallois [116] forward integral. Similar to [113], in [89] such problems arise for semilinear SPDEs with random, nonautonomous generators. Furthermore, in [4, 89] it was shown that a Skorokhod-mild solution for such SPDEs does not satisfy the weak formulation, whereas the forward mild (based on the Russo-Vallois integral) does. In [113], the authors showed that the pathwise mild solution is equivalent to the forward mild one.

In this section we follow the results obtained in [51] and relate two solution concepts for quasilinear SPDEs, namely weak and pathwise mild, with a particular emphasis on cross-diffusion systems. Equivalence results for weak, pathwise- and forward-mild solutions for semilinear SPDEs obtained in [113] and the existence of pathwise mild solutions for quasilinear SPDEs obtained in [85], raise a natural question regarding the equivalence of these solution concepts in the quasilinear case. Such an aspect is also important to study from a numerical point of view, since numerical schemes are aligned to the solution concept at

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hand. The same holds true for dynamical systems. In fact, many results regarding dynamics and asymptotic behavior of semilinear SPDEs, rely on a semigroup approach. If we take the SKT system as a motivation, then there are deterministic results regarding the existence of attractors using weak [108] as well as mild [129] solution concepts. We emphasize that for semilinear PDEs and SPDEs numerous results regarding the equivalence of weak and mild solutions are well-known, see e.g. [8, 9] and [12] for PDEs and e.g. [40] and [125] for SPDEs. On the other hand, for quasilinear PDEs and SPDEs the literature discussing the equivalence of various solution concepts is very scarce. Results regarding the existence of strong solutions for quasilinear PDEs are available in [5], respectively for SPDEs in [72] under rather restrictive assumptions on the coefficients. Due to regularity reasons, one can not expect to obtain strong solutions for equations our analysis is motivated by. For certain elliptic-parabolic PDEs, using accretive operators and nonlinear semigroups, assertions regarding the equivalence of weak and mild solutions have been derived in [64, 103] and the references specified therein. For nonlinear degenerate problems, the concept of entropy solution was introduced by Carrillo [32]. In [81], the equivalence between weak and entropy solutions was established for an elliptic-parabolic-hyperbolic degenerate PDE. However, to the best of our knowledge there are no other works in the literature discussing the equivalence of various solution concepts for quasilinear SPDEs such as (3.2).

3.1. Preliminaries

Let $T > 0$ be arbitrary but fixed and $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$ be a filtered stochastic basis. Let X $(\|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$, Y $(\|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ and Z $(\|\cdot\|_Z, \langle\cdot,\cdot\rangle_Z)$ be separable Hilbert spaces such that the embeddings

$$Z \hookrightarrow Y \hookrightarrow X$$

are continuous. The choice of these Hilbert spaces depends on the corresponding quasilinear problem, see Section 3.3 for concrete examples. We identify X with its topological dual X^*. Let H denote another separable Hilbert space with orthonormal basis $\{\eta_n\}_{n\in\mathbb{N}}$ and $(W_t)_{t\in[0,T]}$ is a cylindrical Wiener process over X, taking values in H. The cylindrical Wiener process $(W_t)_{t\geq 0}$ can be written as the series

$$W_t = \sum_{n=1}^{\infty} e_n \beta_n(t), \tag{3.7}$$

where $\{\beta_n(\cdot)\}_{n\in\mathbb{N}}$ are mutually independent real valued standard Brownian motions and $\{e_n\}_{n\in\mathbb{N}}$ denotes an orthonormal basis of X and the sequence (3.7) converges in H $\mathbb{P}$-a.s. The space of Hilbert-Schmidt operators from H to X will be denoted by $\mathcal{L}_2(H;X)$ and will be endowed with the norm

$$\|L\|^2_{\mathcal{L}_2(H;X)} \coloneqq \sum_{k=1}^{\infty} \|L\eta_k\|^2_X .$$

We recall some auxiliary results related to the regularity of the stochastic integral with respect to a cylindrical Wiener process.

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PROPOSITION 3.1.1. [112, Prop. 4.4] *Let $p \in [2,\infty)$, $0 < \alpha < 1/2$ and σ be a strongly measurable adapted process belonging to $L^0(\Omega; L^p(0,T;\mathcal{L}_2(H,X)))$. Then, the stochastic integral*

$$\int_0^{\cdot} \sigma(r)\, \mathrm{d}W_r \in L^0(\Omega; W^{\alpha,p}(0,T;X)).$$

The continuous embedding $W^{\alpha,p}(0,T;X) \hookrightarrow C^{\alpha-1/p}(0,T;X)$, for $1/p < \alpha < 1/2$, provides Hölder regularity of the stochastic integral, see [113, Prop. 4.1] for the full generality of the statement.

PROPOSITION 3.1.2. *Let $p \in [2,\infty)$, $1/p < \alpha < 1/2$ and $\sigma \in L^0(\Omega; L^p(0,T;\mathcal{L}_2(H,X)))$ be a strongly measurable adapted process. Then, there exists a positive constant C_T independent of σ, converging to 0 for $T \searrow 0$, such that*

$$\left\| \int_0^t \sigma(r)\ \mathrm{d}W_r \right\|_{L^p(\Omega, C^{\alpha-1/p}(0,T;X))} \leq C_T \left\|\sigma\right\|_{L^0(\Omega;L^p(0,T;\mathcal{L}_2(H,X)))}.$$

We let $\mu > 0$ and introduce the following function space

$$\mathcal{Z} := L^0(\Omega; L^\infty([0,T];Z) \cap C^\mu([0,T];Y)).$$

Throughout this section, we choose u, $v \in \mathcal{Z}$ be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes.

Recalling (3.2), we write

$$u \mapsto A_v u = \operatorname{div}\left(\hat{A}(v)\nabla u\right). \tag{3.8}$$

Therefore, A_v is a linear time-dependent random operator. In order to highlight this dependence we use the notation

$$A_v(t,\omega) := A_{v(t,\omega)}.$$

To simplify the notation, we drop the parameter ω and simply write $A_v(t)$. We now collect essential results regarding evolution families. These are extracted from [113]. For further details regarding evolution systems for nonautonomous operators, we refer to the monographs by Pazy [106] and Yagi [130] as well as to [2].

In order to deal with the time and ω-dependence of A_v we impose, as in [113], that the Acquistapace-Terreni conditions hold for every $\omega \in \Omega$. These were introduced in [2] for time-dependent generators and involve a sectoriality condition on A_v together with a suitable Hölder-regularity. Since we are dealing with nonlinear generators we additionally impose a certain Lipschitz continuity assumption [130, 85].

ASSUMPTION 3.1.3. (Assumptions generators) *For $\vartheta \in (\frac{\pi}{2},\pi)$, let Σ_ϑ be an open sectorial domain, i.e.*

$$\Sigma_\vartheta := \{\lambda \in \mathbb{C} : |\arg\lambda| < \vartheta\}. \tag{3.9}$$

(A1) *A_v is a sectorial operator on X, i.e., there exists a $\vartheta \in (\frac{\pi}{2},\pi)$, such that for every $(t,\omega) \in [0,T]\times\Omega$,*

$$\Sigma_\vartheta \cup \{0\} \subset \rho(A_v(t,\omega)).$$

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(A2) *The resolvent operator* $(\lambda \mathrm{Id} - A_v)^{-1}$ *satisfies the Hile-Yosida condition, i.e., there exists a constant* $M \geq 1$ *such that for every* $(t, \omega) \in [0, T] \times \Omega$,

$$\left\|(\lambda \mathrm{Id} - A_v(t,\omega))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda| + 1}, \quad \text{for } \lambda \in \rho(A_v(t,\omega)).$$

(A3) *There exist two exponents* $\nu, \delta \in (0, 1]$ *with* $\nu + \delta > 1$ *such that for every* $\omega \in \Omega$ *there exists a constant* $L(\omega) \geq 0$ *such that for all* $s, t \in [0, T]$

$$\left\|(A_v(t,\omega))^{\nu}\left(A_v(t,\omega)^{-1} - A_v(s,\omega)^{-1}\right)\right\|_{\mathcal{L}(X)} \leq L(\omega)|t - s|^{\delta}.$$

(A4) *Let* $0 < \nu \leq 1$ *be fixed. Then, for every* $\omega \in \Omega$ *there exists a constant* $L(\omega) > 0$ *such that*

$$\left\|(A_u(t,\omega))^{\nu}\left(A_u(t,\omega)^{-1} - A_v(t,\omega)^{-1}\right)\right\|_{\mathcal{L}(X)} \leq L(\omega)\|u(t,\omega) - v(t,\omega)\|_Y.$$

The conditions (A1) (A3) will be referred to as the (AT) conditions.

Since we aim to relate mild and weak solutions for (3.2), we impose similar assumptions on the adjoint A_v^* defined on X^* of the operator A_v with parameters ϑ^*, M^*, ν^* and δ^*. Recall that X is a Hilbert space and we identified it with its dual X^*.

Assumption 3.1.4. (Assumptions adjoint operators)

(A1*) A_v^* *is a sectorial operator, i.e., there exists a* $\vartheta^* \in (\frac{\pi}{2}, \pi)$, *such that for all* $(t, \omega) \in [0, T] \times \Omega$,

$$\Sigma_{\vartheta^*} \cup \{0\} \subset \rho(A_v^*(t,\omega)),$$

where Σ_{ϑ^*} *is defined as in* (3.9).

(A2*) *There exists a constant* $M^* \geq 1$ *such that for every* $(t, \omega) \in [0, T] \times \Omega$,

$$\left\|(\lambda \mathrm{Id} - A_v^*(t,\omega))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M^*}{|\lambda| + 1}, \quad \text{for } \lambda \in \rho(A_v^*(t,\omega)).$$

(A3*) *There exist two exponents* $\nu^*, \delta^* \in (0, 1]$ *with* $\nu^* + \delta^* > 1$ *such that for every* $\omega \in \Omega$ *there exists a constant* $L^*(\omega) \geq 0$ *such that for all* $s, t \in [0, T]$

$$\left\|(A_v^*(t,\omega))^{\nu^*}\left(A_v^*(t,\omega)^{-1} - A_v^*(s,\omega)^{-1}\right)\right\|_{\mathcal{L}(X)} \leq L^*(\omega)|t - s|^{\delta^*}.$$

(A4*) *Let* $0 < \nu^* \leq 1$ *be fixed. Then, for every* $\omega \in \Omega$ *there exists a constant* $L^*(\omega) > 0$ *such that*

$$\left\|(A_u^*(t,\omega))^{\nu^*}\left(A_u^*(t,\omega)^{-1} - A_v^*(t,\omega)^{-1}\right)\right\|_{\mathcal{L}(X)} \leq L^*(\omega)\|u(t,\omega) - v(t,\omega)\|_Y.$$

We refer the reader to Section 5.2 for details on the fractional power of the operator A_u and its adjoint.

Remark 3.1.5. (1) *We assume that the random variables* $L(\omega)$, $L^*(\omega)$ *are uniformly bounded with respect to* ω. *This assumption can be dropped by a suitable localization argument, see* [113, Section 5.3].

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(2) *The Assumption (A1) implies that $-A_v$ is a sectorial operator. Alternatively, one can assume as in [130], that the spectrum of $A_v(t,\omega)$ is contained in an open sectorial domain with angle $0 < \varphi < \frac{\pi}{2}$ for every $t \in [0,T]$ and $\omega \in \Omega$, i.e.*

$$\sigma(A_v(t,\omega)) \subset \Sigma_\varphi := \{\lambda \in \mathbb{C} : |\arg \lambda| < \varphi\},$$

which would imply that A_v is a sectorial operator.

(3) *Examples of operators satisfying **Assumptions 3.1.3 and 3.1.4** are given in Section 3.3.*

ASSUMPTION 3.1.6. (Constant domains) *For simplicity we assume that the domains of A_v and A_v^* are constant, i.e. (A3) and (A3*) are satisfied for $\nu = 1$ and $\nu^* = 1$, respectively. These conditions are called in literature the Kato Tanabe assumptions [2, 113].*

DEFINITION 3.1.7. *To emphasize the fact that we are working with constant domains, we introduce the following notations for $t \in [0,T]$ and $\omega \in \Omega$:*

$$\mathcal{D}_v := D(-A_v(t,\omega)), \qquad \mathcal{D}_v^\alpha := D\left((-A_v(t,\omega))^\alpha\right),$$
$$\mathcal{D}_v^* := D\left((-A_v(t,\omega))^*\right), \qquad \mathcal{D}_v^{\alpha*} := D\left(((-A_v(t,\omega))^\alpha)^*\right).$$

In general, conditions (A1) and (A2) can be difficult to verify for a given system, but in the Hilbert space framework, it suffices to apply the following criterion. According to [130, Chapter 2.1], we can associate to $-A_v$ a bilinear form $a(v;\cdot,\cdot)$ on a separable Hilbert space V which is densely and continuously embedded in X. More precisely, we set

$$a(v;w_1,w_2) := \langle -A_v w_1, w_2\rangle_X, \qquad \text{for all } w_1, w_2 \in V. \tag{3.10}$$

In this case, in order to verify (A1) and (A2) it suffices to show that [130, Chapter 2.1.1]

$$a(v;w,w) \geq \kappa \|w\|_V^2, \qquad \forall\, w \in V, \tag{3.11}$$

$$|a(v;w_1,w_2)| \leq M\|w_1\|_V\|w_2\|_V, \qquad \forall\, w_1, w_2 \in V, \tag{3.12}$$

for some constants $\kappa > 0$ and $M > 0$.

The assumptions (A1) (A3) allow us to apply pathwise the deterministic results from [2] for the generation of an evolution family for nonautonomous operators and obtain as in [113, Theorem 2.2] the following statement.

THEOREM 3.1.8. *Let $\Delta := \{(s,t) \in [0,T]^2 : s \leq t\}$. Assume that (A1) (A3) hold true for the linear nonautonomous operator $A_v(t,\omega)$. Then, there exists a unique map $U^v : \Delta \times \Omega \to \mathcal{L}(X)$ such that*

(T1) *for all $t \in [0,T]$, $U^v(t,t) = \mathrm{Id}$;*
(T2) *for all $r \leq s \leq t$, $U^v(t,s)U^v(s,r) = U^v(t,r)$;*
(T3) *for every $\omega \in \Omega$, the map $U(\cdot,\cdot,\omega) : \Delta \to \mathcal{L}(X)$ is strongly continuous;*
(T4) *there exists a mapping $C : \Omega \to \mathbb{R}_+$, such that for all $s \leq t$, one has*

$$\|U^v(t,s)\|_{\mathcal{L}(X)} \leq C;$$

(T5) *for every $s < t$ it holds pointwise in Ω that*

$$\frac{\partial}{\partial t} U^v(t,s) = A_v(t)U^v(t,s). \tag{3.13}$$

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Moreover, there exists a mapping $C : \Omega \to \mathbb{R}_+$ *such that*

$$\|A_v(t)U^v(t,s)\|_{\mathcal{L}(X)} \leq C(t-s)^{-1}.$$

In [113, Prop. 2.4] the following measurability result for the evolution family U^v was established. This fact prevents us from defining the stochastic convolution as an Itô-integral.

PROPOSITION 3.1.9. *The evolution system* $U^v : \Delta \times \Omega \to \mathcal{L}(X)$ *is strongly measurable in the uniform operator topology. Moreover, for each* $t \geq s$*, the mapping* $\omega \mapsto U^v(t,s,\omega) \in \mathcal{L}(X)$ *is strongly* $\mathcal{F}_t$*-measurable in the uniform operator topology.*

REMARK 3.1.10. *If one replaces* (A1) *by* (3.10)*, then* (3.13) *becomes*

$$\frac{\partial}{\partial t}U^v(t,s) = -A_v(t)U^v(t,s).$$

In the following, we point out spatial- and time-regularity results of the evolution family U^v, *cf.* [113, Lemma 2.6]. We provide a brief overview on fractional powers of sectorial operators in Section 5.2.

LEMMA 3.1.11. *Let conditions* (A1) (A3) *be satisfied by the linear operator* $A_v(t,\omega)$*. Then, there exists a mapping* $C\colon \Omega \to \mathbb{R}_+$ *such that for all* $0 \leq s < t \leq T$*,* $\theta \in [0,1]$*,* $\lambda \in (0,1)$ *and* $\gamma \in [0,\delta)$*, the following estimates are valid*

$$\|U^v(t,s)\,(-A_v(s))^{\gamma}\,x\|_X \leq C\frac{\|x\|_X}{(t-s)^{\gamma}}, \qquad x \in \mathcal{D}_v^{\gamma}, \tag{3.14}$$

$$\left\|(-A_v(t))^{\theta}U^v(t,s)(-A_v(s))^{-\theta}\right\|_{\mathcal{L}(X)} \leq C. \tag{3.15}$$

Moreover, for $(s,t) \in \Delta$*, the map*

$$(s,t) \mapsto (-A_v(t))^{\theta}U^v(t,s)(-A_v(s))^{-\theta}$$

is strongly continuous.

The following result, see [113, Lemma 2.7] for the proof, allows one to improve the regularity of the evolution family U^v, provided that the adjoint operator A_v^* satisfies the assumptions (A1*) (A3*).

LEMMA 3.1.12. *Let* $A_v(t,\omega)$ *and* $A_v^*(t,\omega)$ *satisfy* (A1) (A3) *and* (A1*) (A3*)*, respectively. Then, for every* $t \in (0,T]$*, the map* $s \mapsto U^v(t,s)$ *belongs to* $C^1([0,t),\mathcal{L}(X))$*, and for every* $x \in \mathcal{D}_v$ *and* $\omega \in \Omega$*, it holds that*

$$\frac{\partial}{\partial s}U^v(t,s)x = -U^v(t,s)A_v(s)x. \tag{3.16}$$

Moreover, for $\beta \in [0,1]$*,* $0 < \gamma < \delta^*$*,* $0 \leq \theta < \delta^*$*,* $\mu \in (0,1)$*,* $\lambda \in (0,1)$*, the following inequalities hold:*

$$\left\|U^v(t,s)(-A_v(s))^{\beta}x\right\|_X \leq C\frac{\|x\|_X}{(t-s)^{\beta}}, \qquad x \in \mathcal{D}_v^{\beta}, \tag{3.17}$$

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$$\left\|U^v(t,s)(-A_v(s))^{1+\theta}x\right\|_X \leq C\frac{\|x\|_X}{(t-s)^{1+\theta}}, \qquad x\in\mathcal{D}_v^{1+\theta}, \tag{3.18}$$

$$\left\|(-A_v(t))^{-\lambda}U^v(t,s)(-A_v(s))^{1+\gamma}x\right\|_X \leq C\frac{\|x\|_X}{(t-s)^{1+\gamma-\lambda}}, \quad x\in\mathcal{D}_v^{1+\gamma}. \tag{3.19}$$

REMARK 3.1.13. *Note that the estimates* (3.18) *and* (3.19) *hold true only if the adjoint operator* $(A_v(t,\omega))^*$ *satisfies* (A1*) (A3*).

REMARK 3.1.14. *Lemma 3.1.11 and Lemma 3.1.12 remain valid if the roles of* $A_v(t)$ *and* $(A_v(t))^*$ *are interchanged. This is due to the fact that we identified* X *with its dual* X^*.

For the sake of completeness, we point out the following statement on the adjoint of an evolution family U^v. Regarding **Assumption 3.1.4** we conclude by Theorem 3.1.8 that for every $(t,\omega)\in(0,T]\times\Omega$ the nonautonomous linear operators $(A_v^*(t-\tau,\omega))_{\tau\in[0,t]}$ generate an evolution family $V^v(t;\tau,s)_{0\leq s\leq\tau\leq t}$. Due to [1, Proposition 2.9]

$$(U^v(t,s))^* = V(t;t-s,0) \quad \text{ for } s,t\in\Delta.$$

Based on Lemma 3.1.12, we motivate a very useful identity resembling the fundamental theorem of calculus. This will be extensively used in Section 3.2. For further details we refer to [113].

LEMMA 3.1.15. *Let* (A1) (A3) *and* (A1*) (A3*) *be satisfied by* $A_v(t,\omega)$ *and* $A_v^*(t,\omega)$, *respectively. For* $\lambda>\delta^*$, $x^*\in\mathcal{D}_v^{\lambda*}$, *the following identity holds true for every* $\omega\in\Omega$ *and* $x\in X$:

$$\int_0^t \langle U^v(t,s)A_v(s)x,x^*\rangle\,\mathrm{d}s = \langle U^v(t,0)x,x^*\rangle - \langle x,x^*\rangle\,, \tag{3.20}$$

where $\langle\cdot,\cdot\rangle$ *denotes the inner product in* X.

SKETCH OF THE PROOF. Since $A_v(t,\omega)$ satisfies (A1) (A3), by Theorem 3.1.8 there exists an evolution family $U^v:\Delta\times\Omega\to\mathcal{L}(X)$. In particular, (3.13) holds. Assume $x\in\mathcal{D}_v$, then integrating (3.13) with respect to time leads to

$$U^v(t,t)x - U^v(t,0)x = -\int_0^t U^v(t,s)A_v(s)x\,\mathrm{d}s.$$

Testing the above identity by $x^*\in X^*$, we obtain

$$\int_0^t \langle U^v(t,s)A_v(s)x,x^*\rangle\,\mathrm{d}s = \langle U^v(t,0)x,x^*\rangle - \langle x,x^*\rangle\,. \tag{3.21}$$

Next we verify that (3.21) holds for $x^*\in\mathcal{D}_v^*$ and $x\in X$. This is justified by (3.19), which implies that for every $s<t$, and fixed v, the operator $A_v^{-\lambda}(t)U^v(t,s)A_v^{1+\gamma}(s)$, for $\lambda>\delta^*$ and $\gamma<\delta^*$, can be uniquely extended to a bounded linear operator on X. The claim follows regarding that

$$\int_0^t \langle U^v(t,s)A_v(s)x,x^*\rangle\,\mathrm{d}s$$

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$$= \int_0^t \left\langle (-A_v(t))^{-\lambda} U^v(t,s)(-1)(-A_v(s))^{1+\gamma}(-A_v(s))^{-\gamma}x, ((-A_v(t))^{\lambda})^* x^* \right\rangle \, \mathrm{d}s$$
$$\leq \int_0^t \frac{\|(-A_v(s))^{-\gamma}x\|_X}{(t-s)^{1+\gamma-\lambda}} \|((-A_v(t))^{\lambda})^* x^*\|_{X^*} \, \mathrm{d}s,$$

where we used (3.19) in the last step. □

3.2. The main result

In this section we introduce two solution concepts: pathwise mild and weak, for the quasilinear SPDE

$$\begin{cases} \mathrm{d}u = A_u u \, \mathrm{d}t + \sigma(u) \, \mathrm{d}W_t \\ u(0) = u^0. \end{cases} \tag{3.22}$$

We are interested in showing the equivalence of these solution concepts. As already mentioned, Pronk and Veraar introduced in [113] the concept of pathwise mild solution for semilinear SPDEs with random nonautonomous generators, which was later extended in [85] to quasilinear problems. The precise definition of the pathwise mild solution is given below.

Definition 3.2.1 (Local pathwise mild solution). *A local pathwise mild solution for (3.22) is a pair (u,τ) where τ is a strictly positive stopping time and $\{u(t) : t \in [0,\tau)\}$ is a $\mathcal{Z}$-valued $(\mathcal{F}_t)_{t\in[0,\tau)}$-adapted stochastic process which satisfies for every $t \in [0,\tau)$, $\mathbb{P}$-a.s.*

$$u(t) = U^u(t,0)u_0 + U^u(t,0)\int_0^t \sigma(u(s)) \, \mathrm{d}W_s - \int_0^t U^u(t,s)A_u(s)\int_s^t \sigma(u(\tau)) \, \mathrm{d}W_\tau \, \mathrm{d}s, \tag{3.23}$$

where U^u is the evolution family generated by the operator A_u.

Definition 3.2.2 (Maximal solution). *We call $\{u(t) : t \in [0,\tau)\}$ a maximal local pathwise mild solution of (3.22) if for any other local pathwise mild solution $\{\widetilde{u}(t) \colon t \in [0,\widetilde{\tau})\}$ satisfying $\tilde{\tau} \geq \tau$ a.s. and $\tilde{u}|_{[0,\tau)}$ is equivalent to u, one has $\tilde{\tau} = \tau$ a.s. If $\{u(t) \colon t \in [0,\tau)\}$ is a maximal local pathwise mild solution of (3.22), then the stopping time τ is called its lifetime.*

In [85], the existence of a maximal local pathwise mild solution was established under additional regularity assumptions on the nonlinear term σ. These were necessary for the fixed point argument. For the convenience of the reader, we recall these assumptions.

Assumption 3.2.3 (Existence of pathwise mild solution).

(1) **Assumption 3.1.3** *holds.*
(2) *The mapping $\sigma : \Omega \times [0,T] \times X \to \mathcal{L}_2\left(H, \mathcal{D}_u^{2\beta}\right)$ is locally Lipschitz continuous, meaning that there exist constants $L_\sigma = L_\sigma(u,v) > 0$, $l_\sigma = l_\sigma(u) > 0$ such that*

$$\|\sigma(u) - \sigma(v)\|_{\mathcal{L}_2\left(H,\mathcal{D}_u^{2\beta}\right)} \leq L_\sigma \|u - v\|_X, \qquad u, v \in Z,$$
$$\|\sigma(u)\|_{\mathcal{L}_2\left(H,\mathcal{D}_u^{2\beta}\right)} \leq l_\sigma \left(1 + \|u\|_X\right), \qquad u \in Z.$$

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Under the above assumptions, the following existence result for (3.22) holds true [85, Theorem 3.11].

THEOREM 3.2.4. *Let* **Assumption 3.2.3** *hold true. Then, there exists a unique maximal local pathwise mild solution u for (3.22) such that $u \in L^0\left(\Omega; \mathcal{B}\left([0,\tau_\infty); \mathcal{D}_u^\beta\right)\right) \cap L^0\left(\Omega; C^\delta\left([0,\tau_\infty); \mathcal{D}_u^\alpha\right)\right)$, where $\alpha \in [0,1)$, $\beta \in (1/2,1)$, $\beta > \alpha$ and $\delta \in (0, \beta-\alpha)$.*

We now give the definition of a weak solution of (3.22).

DEFINITION 3.2.5. *We call an $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathcal{Z}$-valued process u a weak solution of (3.22), if the following identity holds $\mathbb{P}$-a.s.*

$$\begin{aligned}\langle u(t), \varphi(t)\rangle = \langle u^0, \varphi(0)\rangle - \int_0^t a\left(u(s); u(s), \varphi(s)\right)\,\mathrm{d}s + \int_0^t \langle u(s), \varphi'(s)\rangle\,\mathrm{d}s \\ - \int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle\,\mathrm{d}s + \left\langle \int_0^t \sigma(u(s))\,\mathrm{d}W_s, \varphi(t)\right\rangle,\end{aligned} \tag{3.24}$$

for every $\varphi \in L^0\left(\Omega; C^1\left([0,T]; \mathcal{D}_u^*\right)\right)$, where $a(u;\cdot,\cdot)\colon V \times V \to \mathbb{R}$ is the bilinear form corresponding to the operator A_u, as introduced in (3.10).

REMARK 3.2.6. *Note that the test functions φ are allowed to depend on time and on ω. We do not make any assumption regarding the adaptedness of these test functions, but only on their spatial and temporal regularity.*

REMARK 3.2.7. *In comparison to the standard weak formulation of (3.22), the weak formulation (3.24) contains two additional terms*

$$\begin{aligned} L_1 &:= \int_0^t \langle u(s), \varphi'(s)\rangle\,\mathrm{d}s, \\ L_2 &:= -\int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle\,\mathrm{d}s. \end{aligned}$$

The term L_1 appears quite naturally when we test (3.22) with a time-dependent function φ, apply the product rule, i.e.

$$\left\langle \frac{d}{ds}u(s), \varphi(s)\right\rangle = \frac{d}{ds}\langle u(s), \varphi(s)\rangle - \langle u(s), \varphi'(s)\rangle$$

and integrate over time. The term L_2 can be again justified by the product rule, more precisely

$$\langle \sigma(u(s))\,\mathrm{d}W_s, \varphi(s)\rangle = \frac{d}{ds}\left\langle \int_0^s \sigma(u(r))\,\mathrm{d}W_r, \varphi(s)\right\rangle - \left\langle \int_0^s \sigma(u(r))\,\mathrm{d}W_r, \varphi'(s)\right\rangle.$$

Integrating this relation with respect to the time variable, and interchanging the stochastic integral and the inner product gives the representation in Definition 3.2.5.

By Definition 3.2.5, the weak formulation of equation (3.1) is given by

$$\langle u(t), \varphi(t)\rangle = \langle u^0, \varphi(0)\rangle - \int_0^t \left\langle \nabla u(s), B^T(u)\nabla\varphi(s)\right\rangle\,\mathrm{d}s + \int_0^t \langle u(s), \varphi'(s)\rangle\,\mathrm{d}s \tag{3.25}$$

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$$- \int_0^t \left\langle \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau, \varphi'(s) \right\rangle \, \mathrm{d}s + \left\langle \int_0^t \sigma(u(s)) \, \mathrm{d}W_s, \varphi(t) \right\rangle.$$

The key idea. We describe the intuition behind the approach we use in order to prove the equivalence of pathwise mild and weak solutions for (3.22). Such an idea is standard in the context of quasilinear problems. Similar to the proof of existence of solutions [130, 85], we firstly work with nonautonomous equations with random coefficients. More precisely, instead of treating (3.22) we consider the linear equation

$$\begin{cases} \mathrm{d}u = A_v u \, \mathrm{d}t + \sigma(u) \, \mathrm{d}W_t \\ u(0) = u^0, \end{cases} \tag{3.26}$$

where $A_v u = \operatorname{div}(B(v)\nabla u)$ as specified in (3.8). This is a linear nonautonomous Cauchy problem, which due to [113], has a unique pathwise mild solution

$$u(t) = U^v(t,0)u^0 - \int_0^t U^v(t,s)A_v(s) \int_s^t \sigma(u(\tau))\, \mathrm{d}W_\tau \, \mathrm{d}s + U^v(t,0) \int_0^t \sigma(u(s))\, \mathrm{d}W_s. \tag{3.27}$$

Its weak formulation reads as

$$\begin{aligned} \langle u(t), \varphi(t) \rangle = \langle u^0, \varphi(0) \rangle + \int_0^t \langle A_v u, \varphi(s) \rangle \, \mathrm{d}s + \int_0^t \langle u(s), \varphi'(s) \rangle \, \mathrm{d}s \\ - \int_0^t \left\langle \int_0^s \sigma(v(\tau))\, \mathrm{d}W_\tau, \varphi'(s) \right\rangle \, \mathrm{d}s + \left\langle \int_0^t \sigma(v(s)) \, \mathrm{d}W_s, \varphi(t) \right\rangle, \end{aligned} \tag{3.28}$$

for every test function $\varphi \in L^0\left(\Omega; C^1\left([0,T]; \mathcal{D}^*_u\right)\right)$. According to [113, Section 4.4], the weak solution (3.28) is equivalent with the pathwise mild solution (3.27). Returning to the quasilinear problem (3.22), one can show by means of fixed point arguments as in [85], that under **Assumption 3.2.3**

$$u(t) = U^u(t,0)u^0 - \int_0^t U^u(t,s)A_u(s) \int_s^t \sigma(u(\tau))\, \mathrm{d}W_\tau \, \mathrm{d}s + U^u(t,0) \int_0^t \sigma(u(s))\, \mathrm{d}W_s,$$

is the pathwise mild solution of (3.22). In conclusion, we approach the quasilinear SPDE (3.22) as a semilinear SPDE where the nonlinear map A_u is viewed as a linear nonautonomous map for a fixed u, which is the pathwise mild solution of the quasilinear SPDE (3.22). In this case, the tools developed in [113, Section 4.4] can be employed in order to establish the equivalence of weak (3.24) and pathwise mild (3.23) solutions in the quasilinear case.

In order to prove our main result, we make the following assumptions.

Assumption 3.2.8.

(1) *For simplicity, we assume that the initial condition* $u^0 = 0$.
(2) **Assumption 3.1.3, 3.1.4** *and* **3.1.6** *are satisfied.*
(3) *The diffusion coefficient* $\sigma : \Omega \times [0,T] \times X \to \mathcal{L}_2(H,X)$ *is a strongly measurable adapted process. Furthermore* $\sigma(u) \in L^0(\Omega, L^p(0,T;\mathcal{L}_2(H,X)))$ *for* $p \in (2,\infty)$.

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Since we deal with a nonautonomous evolution equation, we introduce an appropriate space that incorporates both time and space regularity of the test functions similar to [113].

DEFINITION 3.2.9. *For $t \in [0,T]$ and $\beta \geq 0$, we let $\Gamma^u_{t,\beta}$ be the subspace of all test functions $\varphi \in L^0(\Omega; C^1([0,t]; X^*))$, such that*

(1) *for all $s \in [0,t)$ and $\omega \in \Omega$, we have $\varphi(s) \in \mathcal{D}_u^{(\beta+1)*}$ and $\varphi'(s) \in \mathcal{D}_u^{\beta*}$.*
(2) *the process $s \mapsto A_u^*(s)\varphi(s)$ belongs to $L^0(\Omega; C([0,t]; X^*))$.*
(3) *There is a mapping $C : \Omega \to \mathbb{R}_+$ and $\varepsilon > 0$ such that for all $s \in [0,t)$*

$$\left\| \left((-A_u(s))^{1+\beta}\right)^* \varphi(s) \right\|_{X^*} + \left\| \left((-A_u(s))^{\beta}\right)^* \varphi'(s) \right\|_{X^*} \leq C(t-s)^{-1+\varepsilon}.$$

In the following, we use test functions of the form $\varphi(s) = U(t,s)^*x^*$ for $x^* \in \mathcal{D}_u^*$. Thus, we need to show that such a $\varphi \in \Gamma^u_{t,\beta}$ for some $\beta > 0$. This is established in the next lemma. We recall that the parameters δ respectively δ^* stand for the Hölder exponents in the (AT) conditions for A_u and A_u^* as specified in (A3) and (A3*), respectively.

LEMMA 3.2.10. *Let $x^* \in \mathcal{D}_u^*$ and $t \in [0,T]$. For $\beta \in [0,\delta^*)$, the process $\varphi : [0,t] \times \Omega \to X^*$ defined as $\varphi(s) := U^u(t,s)^*x^*$ belongs to $\Gamma^u_{t,\beta}$.*

PROOF. In order to show that $\varphi(s) \in \Gamma^u_{t,\beta}$ we verify the three conditions of Definition 3.2.9. For (1), using the definition of the norm on $\mathcal{D}_u^{(\beta+1)*}$, we have by (3.19) and Remark 3.1.14

$$\begin{aligned}
\|\varphi(s)\|_{\mathcal{D}_u^{(\beta+1)*}} &= \| \left((-A_u(s))^{1+\beta}\right)^* (U^u(t,s))^*x^* \|_{X^*} \\
&= \| \left((-A_u(s))^{\beta+1}\right)^* (U^u(t,s))^* ((-A_u(t))^{-\lambda})^* ((-A_u(t))^{\lambda})^* x^* \|_{X^*} \\
&\leq \| \left((-A_u(s))^{\beta+1}\right)^* (U^u(t,s))^* ((-A_u(t))^{-\lambda})^* \|_{\mathcal{L}(X^*)} \|((-A_u(t))^{\lambda})^* x^* \|_{X^*} \\
&\leq \frac{C}{(t-s)^{1+\beta-\lambda}} \|((-A_u(t))^{\lambda})^* x^* \|_{X^*} < \infty,
\end{aligned}$$

where $\lambda \in (\beta, \delta^*)$.
Next we verify condition (2). Using [1, Prop. 2.9], we obtain

$$\varphi'(s) = \frac{d}{ds}\varphi(s) = -(A_u(s))^*(U^u(t,s))^*x^*.$$

By Lemma 3.1.11, we immediately see that φ is continuously differentiable for $s < t$. For the case $s = t$, we refer to [2, Theorem 6.5]. Now from the above identity, we have

$$\begin{aligned}
\|\varphi'(s)\|_{\mathcal{D}_u^{\beta*}} &= \| \left((-A_u(s))^{\beta}\right)^* \varphi'(s) \|_{X^*} \\
&= \| \left((-A_u(s))^{1+\beta}\right)^* (U^u(t,s))^*x^* \|_{X^*} = \|\varphi(s)\|_{\mathcal{D}_u^{(\beta+1)*}} < \infty,
\end{aligned}$$

using (1). Condition (3) of Definition 3.2.9 immediately follows from the previous two estimates, choosing $\varepsilon := \lambda - \beta$. □

Due to **Assumption 3.2.8** and regarding the regularity of the stochastic integral (recall Proposition 3.1.1 an 3.1.2), the terms appearing in (3.24) are well-defined for $\varphi(s) = U(t,s)^*x^* \in \Gamma^u_{t,\beta}$, where $x \in \mathcal{D}_u^*$.

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LEMMA 3.2.11. *Let* **Assumption 3.2.8** *hold and let* $\beta \in [0, \delta^*)$. *Then the following mappings*

$$u \mapsto \int_0^t a(u; u, \varphi(s))\, \mathrm{d}s,$$
$$u \mapsto \int_0^t \left\langle \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau, \varphi'(s) \right\rangle \mathrm{d}s,$$
$$u \mapsto \int_0^t \left\langle U^u(t,s) A_u(s) \int_s^t \sigma(u(\tau))\, \mathrm{d}W_\tau, x^* \right\rangle \mathrm{d}s$$

are well-defined from $\mathcal{Z}$ *to* $L^0(\Omega; \mathbb{R})$.

PROOF. For the first term we obviously have

$$|a(u; u, \varphi(t))| = |\langle -A_u u, \varphi(t)\rangle| = |\langle u, (-A_u)^* \varphi(t)\rangle| \le C\|u\|_X \|(-A_u)^* \varphi\|_X.$$

For the second integral, we obtain

$$\int_0^t \left|\left\langle \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau, \varphi'(s) \right\rangle\right| \mathrm{d}s = \int_0^t \left|\left\langle (-A_u)^{-\beta} \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau, ((-A_u)^\beta)^* \varphi'(s) \right\rangle\right| \mathrm{d}s$$
$$\le \int_0^t \left(\left\| (-A_u)^{-\beta} \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau \right\|_X \left\| ((-A_u)^\beta)^* \varphi'(s) \right\|_{X^*} \right) \mathrm{d}s$$
$$\le C \sup_{s\in[0,t]} \left\| (-A_u)^{-\beta} \int_0^s \sigma(u(\tau))\, \mathrm{d}W_\tau \right\|_X \int_0^t \frac{1}{(t-s)^{1-\varepsilon}}\, \mathrm{d}s,$$

where we use property (3) of Definition 3.2.9.

For the third term, setting $\varphi(s) := U^u(t,s)^* x^*$ for $x^* \in \mathcal{D}_u^*$, we have

$$\int_0^s \left|\left\langle U^u(s,r) A_u(r) \int_r^s \sigma(u(\tau))\, \mathrm{d}W_\tau, x^* \right\rangle\right| \mathrm{d}r$$
$$= \int_0^s \left|\left\langle (-A_u(s))^{-\lambda} U^u(s,r) (A_u(r)) (-A_u(r))^\beta (-A_u(r))^{-\beta} \int_r^s \sigma(u(\tau))\, \mathrm{d}W_\tau, \left((-A_u(s))^\lambda\right)^* x^* \right\rangle\right| \mathrm{d}r$$
$$= \int_0^s \left|\left\langle (-A_u(r))^{-\beta} \int_r^s \sigma(u(\tau))\, \mathrm{d}W_\tau, ((-A_u(r))^{1+\beta})^* U^u(s,r)^* \left((-A_u(s))^{-\lambda}\right)^* \left((-A_u(s))^\lambda\right)^* x^* \right\rangle\right| \mathrm{d}r$$
$$\le \int_0^s \Bigg(\left\| (-A_u(r))^{-\beta} \int_r^s \sigma(u(\tau))\, \mathrm{d}W_\tau \right\|_X$$
$$\left\| ((-A_u(r))^{1+\beta})^* U^u(s,r)^* \left((-A_u(s))^{-\lambda}\right)^* \left((-A_u(s))^\lambda\right)^* x^* \right\|_{X^*} \Bigg) \mathrm{d}r$$
$$\le C_\sigma \int_0^s (s-r)^{-1-\beta+\lambda} \left\| \left((-A_u(s))^\lambda\right)^* x^* \right\|_{X^*} \mathrm{d}r \le C_\sigma s^{\lambda-\beta} \left\| \left((-A_u(s))^\lambda\right)^* x^* \right\|_{X^*}$$
$$\le C'_\sigma \left\| \left((-A_u(s))^\lambda\right)^* x^* \right\|_{X^*},$$

where we used (3.19) for $A_u^*(\cdot)$. The last term is bounded for $\lambda \in (\beta, \delta^*)$. □

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Collecting all the previous deliberations, we now state the main result of this paper.

THEOREM 3.2.12. *Let* **Assumption 3.2.8** *be satisfied and let* $\beta \in (0, \delta^*)$. *Then the following assertions are valid.*

1.) *If there exists a pathwise mild solution* u *for* (3.22) *on the interval* $[0,T]$. *Then,* u *satisfies* (3.24) *for all* $\varphi \in \Gamma^u_{t,\beta}$, $t \in [0,T]$, $\mathbb{P}$*-a.s.*
2.) *If there exists a weak solution* u *for* (3.22) *satisfying* (3.24) *for all* $\varphi \in \Gamma^u_{t,\beta}$, $t \in [0,T]$, $\mathbb{P}$*-a.s.. Then,* u *satisfies* (3.23) *for* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

PROOF. 1.) We start by showing that a pathwise mild solution of (3.22) is also a weak solution. Assume that (3.23) holds and fix $t \in [0,T]$. Let $\lambda \in (\beta, \delta^*)$ and u be the mild solution of (3.22) with zero initial condition, i.e.

$$u(t) = -\int_0^t U^u(t,r)A_u(r)\int_r^t \sigma(u(\tau))\,\mathrm{d}W_\tau\,\mathrm{d}r + U^u(t,0)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau. \tag{3.29}$$

Applying $x^* \in \mathcal{D}_u^{\lambda *}$ to (3.29) and using that $\int_r^t \sigma(u(\tau))\mathrm{d}W_\tau = \int_0^t \sigma(u(\tau))\mathrm{d}W_\tau - \int_0^r \sigma(u(\tau))\mathrm{d}W_\tau$ to obtain

$$\begin{aligned}\int_0^t \left\langle U^u(t,r)A_u(r)\int_r^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r &= \int_0^t \left\langle U^u(t,r)A_u(r)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r \\ &\quad - \int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r,\end{aligned}$$

further leads to

$$\begin{aligned}\langle u(t), x^*\rangle &= -\int_0^t \left\langle U^u(t,r)A_u(r)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r + \left\langle U^u(t,0)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \\ &\quad + \int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r.\end{aligned} \tag{3.30}$$

We use the identity (3.20) for $x = \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau$ and rewrite the first term on the right-hand-side as

$$\begin{aligned}&\int_0^t \left\langle U^u(t,r)A_u(r)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r \\ &\quad = \left\langle U^u(t,0)\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle - \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle.\end{aligned} \tag{3.31}$$

Using (3.31) in (3.30), we obtain

$$\langle u(t), x^*\rangle = \int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}r + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle. \tag{3.32}$$

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In order to show the equivalence with the weak solution, we need to use test functions $\varphi \in \Gamma^u_{t,\beta}$. To this aim we choose $x^* = (-A_u(s))^* \varphi(s)$ in (3.32) and integrate over time to obtain

$$\begin{aligned}(3.33)\qquad &\int_0^t \langle u(s), (-A_u(s))^* \varphi(s)\rangle \,\mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^* \varphi(s)\right\rangle \mathrm{d}s \\ &= \int_0^t \int_0^s \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^* \varphi(s)\right\rangle \mathrm{d}r\,\mathrm{d}s \\ &= \int_0^t \int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^* \varphi(s)\right\rangle \mathrm{d}s\,\mathrm{d}r,\end{aligned}$$

where we used Fubini's theorem in the last step. Using (3.13), we obtain for all $x \in X$ and $0 \le r \le t \le T$,

$$\begin{aligned}&\int_r^t \langle U^u(s,r)A_u(r)x, (-A_u(s))^* \varphi(s)\rangle \,\mathrm{d}s = -\int_r^t \langle A_u(s)U^u(s,r)A_u(r)x, \varphi(s)\rangle \,\mathrm{d}s \\ &= -\int_r^t \left\langle \frac{d}{\mathrm{d}s}U^u(s,r)A_u(r)x, \varphi(s)\right\rangle \mathrm{d}s \\ &= -\int_r^t \frac{d}{\mathrm{d}s}\langle U^u(s,r)A_u(r)x, \varphi(s)\rangle \,\mathrm{d}s + \int_r^t \langle U^u(s,r)A_u(r)x, \varphi'(s)\rangle \,\mathrm{d}s \\ &= -\langle U^u(t,r)A_u(r)x, \varphi(t)\rangle + \langle U^u(r,r)A_u(r)x, \varphi(r)\rangle + \int_r^t \langle U^u(s,r)A_u(r)x, \varphi'(s)\rangle \,\mathrm{d}s.\end{aligned}$$

Furthermore

$$\begin{aligned}(3.34)\qquad &\langle U^u(t,r)A_u(r)x, \varphi(t)\rangle + \langle x, (-A_u(r))^* \varphi(r)\rangle \\ &\qquad = -\int_r^t \langle U^u(s,r)A_u(r)x, (-A_u(s))^* \varphi(s)\rangle \,\mathrm{d}s + \int_r^t \langle U^u(s,r)A_u(r)x, \varphi'(s)\rangle \,\mathrm{d}s.\end{aligned}$$

Note that the expressions above are well-defined. Indeed, for $\varphi \in \Gamma^u_{t,\beta}$ and $x \in X$, choosing $1 > \lambda > \theta > 0$ and applying(3.19), we infer that

$$\begin{aligned}|\langle U^u(s,r)A_u(r)x, (-A_u(s))^* \varphi(s)\rangle| &= \left|\left\langle (-A_u(s))^{-\lambda}U^u(s,r)A_u(r)x, \left((-A_u(s))^{1+\lambda}\right)^* \varphi(s)\right\rangle\right| \\ &\le \|(-A_u(s))^{-\lambda}U^u(s,r)(-A_u(r))^{1+\theta}(-A_u(r))^{-\theta}x\|_X \| \left((-A_u(s))^{1+\lambda}\right)^* \varphi(s)\|_{X^*} \\ &\le C(s-r)^{-1+\lambda-\theta}(t-s)^{-1+\varepsilon}\|(-A_u(r))^{-\theta}x\|_X \\ &\le C(s-r)^{-1+\lambda-\theta}(t-s)^{-1+\varepsilon}\|x\|_X.\end{aligned}$$

Setting $x = \int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau$ in (3.34) further leads to

$$\begin{aligned}(3.35)\qquad &\left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle + \left\langle \int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(r))^* \varphi(r)\right\rangle \\ &\qquad = -\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^* \varphi(s)\right\rangle \mathrm{d}s\end{aligned}$$

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$$+\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle \mathrm{d}s.$$

Next we use (3.35) to deal with the right-hand-side of (3.33). From (3.33) we get

(3.36)
$$\begin{aligned}\int_0^t \langle u(s), (-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s &= \int_0^t\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^*\varphi(s)\right\rangle \mathrm{d}s\,\mathrm{d}r\\ &\quad+\int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^*\varphi(s)\right\rangle \mathrm{d}s.\end{aligned}$$

Thus, using (3.35) in (3.36) results in

$$\begin{aligned}\int_0^t \langle u(s), (-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s &= -\int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle \mathrm{d}r\\ &\quad-\int_0^t \left\langle \int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(r))^*\varphi(r)\right\rangle \mathrm{d}r\\ &\quad+\int_0^t\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle \mathrm{d}s\,\mathrm{d}r\\ &\quad+\int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^*\varphi(s)\right\rangle \mathrm{d}s.\end{aligned}$$

The above expression on simplification results in

(3.37)
$$\begin{aligned}\int_0^t \langle u(s), (-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s &= -\int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle \mathrm{d}r\\ &\quad+\int_0^t\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle \mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

Moreover, choosing $x^* = \varphi(t)$ in (3.32), we get

$$\begin{aligned}\langle u(t), \varphi(t)\rangle &= \int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle \mathrm{d}r\\ &\quad+\left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle.\end{aligned}\tag{3.38}$$

Using (3.38) and (3.37), we obtain

(3.39)
$$\begin{aligned}\int_0^t \langle u(s), (-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s &= -\langle u(t), \varphi(t)\rangle + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi(t)\right\rangle\\ &\quad+\int_0^t\int_r^t \left\langle U^u(s,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(s)\right\rangle \mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

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Further, choosing $x^* = \varphi'(t)$ in (3.32), we can express the integrand of the last term in the right-hand-side of (3.39) as

(3.40)
$$\int_0^t \left\langle U^u(t,r)A_u(r)\int_0^r \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(t)\right\rangle \mathrm{d}r = \langle u(t),\varphi'(t)\rangle - \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, \varphi'(t)\right\rangle.$$

Using Fubini's theorem and plugging in the relation (3.40) in (3.39), we infer that

$$\begin{aligned}\int_0^t \langle u(s), (-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s = &-\langle u(t),\varphi(t)\rangle + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau,\varphi(t)\right\rangle \\ &+\int_0^t \langle u(s),\varphi'(s)\rangle\,\mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau,\varphi'(s)\right\rangle \mathrm{d}s.\end{aligned}$$

From the previous expression we conclude that u satisfies (3.24) and is therefore a weak solution of (3.22).

2.) Let u be a weak solution of (3.22). Then $u \in \mathcal{Z}$ and satisfies the weak formulation (3.24), namely

$$\begin{aligned}\langle u(t),\varphi(t)\rangle = &-\int_0^t a\,(u(s);u(s),\varphi(s))\,\mathrm{d}s + \int_0^t \langle u(s),\varphi'(s)\rangle\,\mathrm{d}s \\ &-\int_0^t \left\langle \int_0^s \sigma(u(\tau))\;\mathrm{d}W_\tau,\varphi'(s)\right\rangle \mathrm{d}s + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau,\varphi(t)\right\rangle.\end{aligned}$$

In particular, by Theorem 3.1.8 there exists an evolution family $U^u\colon \Delta\times\Omega \to \mathcal{L}(X)$ generated by the operator A_u. Using the relation between $a(u;\cdot,\cdot)$ and A_u, recall (3.10), the previous expression rewrites as

$$\begin{aligned}\langle u(t),\varphi(t)\rangle = &-\int_0^t \langle u(s),(-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s + \int_0^t \langle u(s),\varphi'(s)\rangle\,\mathrm{d}s \\ &-\int_0^t \left\langle \int_0^s \sigma(u(\tau))\;\mathrm{d}W_\tau,\varphi'(s)\right\rangle \mathrm{d}s + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau,\varphi(t)\right\rangle.\end{aligned} \tag{3.41}$$

From Lemma 3.2.10, for a fixed $t\in[0,T]$ and $x^*\in\mathcal{D}_u^*$, $\varphi(s) = U^u(t,s)^*x^*$ belongs to $\Gamma^u_{t,\beta}$. Now with this choice of test functions and using that $\varphi'(s) = (-A_u(s))^*\varphi(s)$, we obtain from (3.41) that

$$\begin{aligned}\langle u(t),U^u(t,t)^*x^*\rangle = &-\int_0^t \langle u(s),(-A_u(s))^*U^u(t,s)^*x^*\rangle\,\mathrm{d}s \\ &+\int_0^t \langle u(s),(-A_u(s))^*\varphi(s)\rangle\,\mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau))\;\mathrm{d}W_\tau,(-A_u(s))^*\varphi(s)\right\rangle \mathrm{d}s \\ &+\left\langle \int_0^t \sigma(u(\tau))\;\mathrm{d}W_\tau,U^u(t,t)^*x^*\right\rangle,\end{aligned}$$

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which simplifies further to

$$\begin{aligned}\langle u(t), x^*\rangle &= -\int_0^t \langle u(s), (-A_u(s))^*\, U^u(t,s)^* x^*\rangle \,\mathrm{d}s \\ &\quad + \int_0^t \langle u(s), (-A_u(s))^* U^u(t,s)^* x^*\rangle \,\mathrm{d}s + \left\langle \int_0^t \sigma(u(\tau)) \,\mathrm{d}W_\tau, x^*\right\rangle \\ &\quad - \int_0^t \left\langle \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, (-A_u(s))^*\, U^u(t,s)^* x^*\right\rangle \mathrm{d}s \\ &= \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle - \int_0^t \left\langle U^u(t,s)\,(-A_u(s)) \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s.\end{aligned}$$

All in all, we obtained that

$$\langle u(t), x^*\rangle = \int_0^t \left\langle U^u(t,s) A_u(s) \int_0^s \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle.$$

Splitting the stochastic integral in the first term on the right-hand-side, into two stochastic integrals results in

$$\begin{aligned}\langle u(t), x^*\rangle &= \int_0^t \left\langle U^u(t,s) A_u(s) \left(\int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau - \int_s^t \sigma(u(\tau))\,\mathrm{d}W_\tau\right), x^*\right\rangle \mathrm{d}s \\ &\quad + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \\ &= \int_0^t \left\langle U^u(t,s) A_u(s) \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s + \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \\ &\quad - \int_0^t \left\langle U^u(t,s) A_u(s) \int_s^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s. \end{aligned} \tag{3.42}$$

Using the identity (3.20) with $x = \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau$, the first term on the right-hand-side of (3.42) can be written as

$$\begin{aligned}&\int_0^t \left\langle U^u(t,s) A_u(s) \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s \\ &\quad = \left\langle U^u(t,0) \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle - \left\langle \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle.\end{aligned}$$

Using the above identity in (3.42) entails

$$\langle u(t), x^*\rangle = \left\langle U^u(t,0) \int_0^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle - \int_0^t \left\langle U^u(t,s) A_u(s) \int_s^t \sigma(u(\tau))\,\mathrm{d}W_\tau, x^*\right\rangle \mathrm{d}s.$$

In the previous deliberations $x^* \in \mathcal{D}_u^*$. Since $\mathcal{D}_u^*$ is dense in X, the result can be extended to every $x^* \in X$ by the Hahn-Banach theorem. This justifies that the weak solution of (3.22) satisfies the mild formulation (3.23). □

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For time-independent test functions, we recover the equivalence of the pathwise mild solution with the following standard weak formulation

$$\langle u(t), x^* \rangle = -\int_0^t a\left(u(s); u(s), x^*\right) \, \mathrm{d}s + \left\langle \int_0^t \sigma(u(s)) \, \mathrm{d}W_s, x^* \right\rangle, \tag{3.43}$$

for $x^* \in \mathcal{D}_u^*$.

THEOREM 3.2.13. *Let* **Assumption 3.2.8** *be satisfied. Then the following assertions are valid.*

1.) *If there exists a pathwise mild solution u for (3.22) on the interval $[0,T]$. Then, u satisfies (3.43) for all $x^* \in \mathcal{D}_u^*$, $t \in [0,T]$, $\mathbb{P}$-a.s.*
2.) *If there exists a weak solution u for (3.22) satisfying (3.43) for all $x^* \in \mathcal{D}_u^*$, $t \in [0,T]$, $\mathbb{P}$-a.s. Then, u satisfies (3.23) for $t \in [0,T]$, $\mathbb{P}$-a.s.*

PROOF. 1.) Let $t \in [0,T]$. We consider only the case $\sigma(u(t)) \in \mathcal{D}_u$ such that the process $t \mapsto A_u(t)\sigma(u(t))$ is adapted and belongs to $L^0(\Omega; L^p(0,T; \mathcal{L}_2(H,X)))$.

For $x^* \in \mathcal{D}_u^*$, we simply set $\varphi(t) := x^*$. However, such a test function does not belong to $\Gamma_{t,\beta}^u$, since $\varphi \notin \mathcal{D}_u^{1+\beta}$. Though, the additional spatial regularity of σ enables us to perform the same proof as before. From (3.33), we obtain for the test function x^* instead of $A_u(s)^*\varphi(s)$

$$\begin{aligned} &\int_0^t \langle u(s), x^* \rangle \, \mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) \, \mathrm{d}W_\tau, x^* \right\rangle \mathrm{d}s \\ &\qquad = \int_0^t \int_r^t \left\langle U^u(s,r) A_u(r) \int_0^r \sigma(u(\tau)) \, \mathrm{d}W_\tau, x^* \right\rangle \mathrm{d}s \, \mathrm{d}r. \end{aligned}$$

Testing with $(-A_u(s))^* x^*$ instead of $(-A_u(s))^*\varphi(s)$ entails

$$\begin{aligned} \int_r^t \langle U^u(s,r) A_u(r) x, A_u(s)^* x^* \rangle \, \mathrm{d}s &= \int_r^t \frac{d}{ds} \langle U^u(s,r) A_u(r) x, x^* \rangle \, \mathrm{d}s \\ &= \langle U^u(t,r) A_u(r) x, x^* \rangle - \langle A_u(r) x, x^* \rangle . \end{aligned}$$

Analogously to the proof of Theorem 3.2.12, the result now follows for $\sigma(\cdot) \in \mathcal{D}_u$. The general case follows from a suitable approximation argument for $\sigma(\cdot)$ as in [113, Theorem 4.9].

2.) We show now that a solution satisfying (3.43), also verifies (3.23). To this aim, we fix $t \in [0,T]$, and let $f \in C^1([0,t])$, $x^* \in \mathcal{D}_u^*$. Using the density of $\mathcal{D}_u^*$ in X, it suffices to consider test functions of the form $\varphi(t) = f(t) \otimes x^*$. Using such test functions in (3.43), we infer that

$$\langle u(t), \varphi(t) \rangle - \left\langle \int_0^t \sigma(u(s)) \, \mathrm{d}W_s, \varphi(t) \right\rangle = -\int_0^t \langle u(s), (-A_u(s))^* x^* \rangle \, \mathrm{d}s f(t). \tag{3.44}$$

The integration by parts formula results in

$$\int_0^t \int_0^s \langle u(r), (-A_u(r))^* \, x^* \rangle \, \mathrm{d}r f'(s) \, \mathrm{d}s$$

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$$= \int_0^t \langle u(s), (-A_u(s))^* x^* \rangle \, \mathrm{d}s f(t) - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle \, \mathrm{d}s.$$

The above identity and (3.43) further entail that

$$\begin{aligned}
&\langle u(t), \varphi(t) \rangle - \left\langle \int_0^t \sigma(u(s)) \, \mathrm{d}W_s, \varphi(t) \right\rangle \\
&= - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle \, \mathrm{d}s + \int_0^t \langle u(s), x^* \rangle f'(s) \, \mathrm{d}s \\
&\quad - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) \, \mathrm{d}W_\tau, x^* \right\rangle f'(s) \, \mathrm{d}s \\
&= - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle \, \mathrm{d}s + \int_0^t \langle u(s), \varphi'(s) \rangle \, \mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) \, \mathrm{d}W_\tau, \varphi'(s) \right\rangle \mathrm{d}s.
\end{aligned}$$

Recalling $a(u; u, \varphi) = -\langle A_u u, \varphi \rangle$, we obtain from the previous expression the weak formulation (3.24), for test functions having the structure $\varphi(t) = f(t) \otimes x^*$. To extend the previous identity to test functions belonging to $\Gamma^u_{\beta,t}$, we firstly extend it to simple functions $\varphi \colon \Omega \to C^1([0,t]; X^*) \cap C([0,t]; \mathcal{D}^*_u)$. By an approximation argument, this can be further extended to any function $\varphi \in L^0(\Omega; C^1([0,t]; X) \cap C([0,t]; \mathcal{D}^*_u))$. Choosing an arbitrary $x^* \in \mathcal{D}^*_u$ and setting $\varphi(s) := U^u(t,s)^* x^*$, by Lemma 3.2.10, we have that $\varphi(s) \in L^0(\Omega; C^1([0,t]; X) \cap C([0,t]; \mathcal{D}^*_u))$. This proves the statement □

3.3. Examples

In this section, we present two examples of parabolic quasilinear SPDEs, to which the theory developed in this paper applies. The existence theory for pathwise mild-, martingale- and weak solutions for these problems is well-known, see [85, 52, 69]. After introducing these SPDEs and recalling the corresponding existence results, we show that they satisfy our assumptions. Therefore, the pathwise mild and weak solution concepts are equivalent in these cases.

EXAMPLE 3.3.1. **(The stochastic SKT population model)** *Let $\mathcal{O} \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary. We fix parameters $\alpha_1, \alpha_2, \delta_{11}, \delta_{21} > 0$. We are interested in studying a cross-diffusion SPDE, which was originally introduced by Shigesada, Kawasaki and Teramato [119] in the deterministic setting, to analyze population segregation by induced cross-diffusion in a two-species model. Note that the nonlinear drift term correspond to those arising in the classical Lokta-Volterra competition model. The stochastic SKT system is given by*

$$\begin{aligned}
\mathrm{d}u_1 &= \left(\Delta\left(\alpha_1 u_1 + \gamma_1 u_1 u_2 + \beta_1 u_1^2\right) + \delta_{11} u_1 - \theta_{11} u_1^2 - \theta_{12} u_1 u_2\right) \mathrm{d}t + \sigma_1(u_1, u_2) \, \mathrm{d}W_t^1, \\
\mathrm{d}u_2 &= \left(\Delta\left(\alpha_2 u_2 + \gamma_2 u_1 u_2 + \beta_2 u_2^2\right) + \delta_{21} u_2 - \theta_{21} u_1 u_2 - \theta_{22} u_2^2\right) \mathrm{d}t + \sigma_2(u_1, u_2) \, \mathrm{d}W_t^2,
\end{aligned} \tag{3.45}$$

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for $t \in [0,T]$ and $x \in \mathcal{O}$ and is supplemented with the following boundary and initial conditions:

$$\begin{aligned} &\frac{\partial}{\partial n}u_1(t,x) = \frac{\partial}{\partial n}u_2(t,x) = 0, && t>0,\ x\in\partial\mathcal{O},\\ &u_1(x,0) = u_1^0(x) \geq 0, \quad u_2(x,0) = u_2^0(x) \geq 0, && x\in\mathcal{O}. \end{aligned}$$

$W = (W^1, W^2)$ is an H-valued cylindrical Wiener process. The solution $u := (u_1, u_2)$, where $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ denote the densities of two competing species S_1 and S_2 at certain location $x \in \mathcal{O}$, at time t. The coefficients $\theta_{11}, \theta_{22} > 0$ denote the intraspecies competition rates in S_1, respectively in S_2 and $\theta_{12}, \theta_{21} > 0$ stand for the interspecies competition rates between S_1 and S_2. Furthermore, the terms $\Delta(\beta_1 u_1^2)$ and $\Delta(\beta_2 u_2^2)$ represent the self-diffusions of S_1 and S_2 with rates $\beta_1, \beta_2 \geq 0$, and $\Delta(\gamma_1 u_1 u_2)$, $\Delta(\gamma_2 u_1 u_2)$ represent the cross-diffusions of S_1 and S_2 with rates $\gamma_1, \gamma_2 \geq 0$.

The SKT system (3.45) can be rewritten as an abstract quasilinear SPDE:

$$\begin{cases} \mathrm{d}u = [A_u u + F(u)]\ \mathrm{d}t + \sigma(u)\ \mathrm{d}W_t, \quad t \in [0,T] \\ u(0) = u^0, \end{cases} \tag{3.46}$$

where

$$A_u u := \operatorname{div}(\hat{A}(u)\nabla u) - \Gamma u,$$

with

$$\hat{A}(u) = \begin{pmatrix} \alpha_1 + 2\beta_1 u_1 + \gamma_1 u_2 & \gamma_1 u_1 \\ \gamma_2 u_2 & \alpha_2 + 2\beta_2 u_2 + \gamma_2 u_1 \end{pmatrix}, \qquad \Gamma(u) = \begin{pmatrix} \delta_{11} & 0 \\ 0 & \delta_{21} \end{pmatrix}.$$

The nonlinear term F corresponds to the Lotka-Volterra type competition model

$$F(u) = \begin{pmatrix} 2\delta_{11}u_1 - \theta_{11}u_1^2 - \theta_{12}u_1u_2 \\ 2\delta_{21}u_2 - \theta_{21}u_1u_2 - \theta_{22}u_2^2 \end{pmatrix}.$$

In order to ensure the positive definiteness of the matrix B, the following restriction on the parameters is necessary:

$$\gamma_1^2 < 8\alpha_1\beta_1 \quad \text{and} \quad \gamma_2^2 < 8\alpha_2\beta_2. \tag{3.47}$$

This assumption is required in order to show that A_u generates a parabolic evolution system U^u for $u \in Z$, for a natural choice of Z, see below. The (AT) conditions (A1) (A3) are satisfied for a standard choice of Hilbert spaces $Z \hookrightarrow Y \hookrightarrow X$, where $X := L^2(\mathcal{O})$, $Z := H^{1+\varepsilon}(\mathcal{O})$ and $Y := H^{1+\varepsilon_0}(\mathcal{O})$ for $0 < \varepsilon_0 < \varepsilon$, see [130, Chapter 15.2.2]. Moreover, according to [130, Chapter 1.8.2], (A1*) (A3*) are also satisfied by the adjoint operator $(A_u)^*$.

REMARK 3.3.2. *The choice of the space $Z = H^{1+\varepsilon}(\mathcal{O})$ is natural. Using the Sobolev embedding $W^{k,p}(\mathcal{O}) \hookrightarrow C(\bar{\mathcal{O}})$ for $kp > d$, we observe that for fixed $t \in [0,T]$ and $d = 2$, the choice of $Z = H^{1+\varepsilon}(\mathcal{O})$ ensures that (3.12) is satisfied.*

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We also emphasize that the domains of the fractional powers of A_u, for $u \in Z$, can be identified with Sobolev spaces, see [130, Prop. 15.3]. More precisely, we have

$$\begin{cases} \mathcal{D}_u^\theta = H^{2\theta}(\mathcal{O}), & \text{for } 0 \leq \theta < \frac{3}{4} \\ \mathcal{D}_u^\theta = H_N^{2\theta}(\mathcal{O}), & \text{for } \frac{3}{4} < \theta \leq 1, \end{cases}$$

where $H_N^{2\theta}(\mathcal{O})$ incorporates the Neumann boundary conditions. Furthermore, the nonlinear drift term is locally Lipschitz continuous on X. Letting $u^0 \in Z$ a.s. and assuming a local Lipschitz continuity on σ (recall **Assumption 3.2.3**), [85, Theorem 4.3] provides the existence of a local-in-time pathwise mild solution u of (3.46) such that $u \in L^0\left(\Omega; \mathcal{B}\left([0,\tau)\,; Z\right)\right) \cap L^0\left(\Omega; C^\delta\left([0,\tau)\,; Y\right)\right)$, where $\delta \in (0, \frac{\varepsilon-\varepsilon_0}{2})$.

The main result establishes the equivalence of this pathwise mild solution with the weak solution. This statement is a direct consequence of Theorem 3.2.12 and Theorem 3.2.13. Note that the results in Theorem 3.2.12 and Theorem 3.2.13 hold if an additional drift term F is incorporated.

THEOREM 3.3.3. *The local pathwise mild solution (u,τ) of (3.46) is also a weak solution. More precisely, for $t \in [0,\tau)$, the following relation holds $\mathbb{P}$-a.s.*

$$\begin{aligned} \langle u(t), x^* \rangle = \langle u^0, x^* \rangle &- \int_0^t \langle \hat{A}(u(s))\nabla u(s), \nabla x^* \rangle \,\mathrm{d}s - \int_0^t \langle \Gamma(u(s)), x^* \rangle \,\mathrm{d}s \\ &+ \int_0^t \langle F(u(s)), x^* \rangle \,\mathrm{d}s + \left\langle \int_0^t \sigma(u(s)) \,\mathrm{d}W_s, x^* \right\rangle, \end{aligned} \tag{3.48}$$

where $x^* \in \mathcal{D}_u^*$.

PROOF. By Theorem 3.2.12 part (1), we infer that the pathwise mild solution $u = (u_1, u_2)$ satisfies the weak formulation for $t \in [0,\tau)$

$$\begin{aligned} \langle u(t), \varphi(t) \rangle = \langle u^0, \varphi(0) \rangle &- \int_0^t \langle \hat{A}(u(s))\nabla u(s), \nabla \varphi(s) \rangle \,\mathrm{d}s - \int_0^t \langle \Gamma(u(s)), \varphi(s) \rangle \,\mathrm{d}s \\ &+ \int_0^t \langle u(s), \varphi'(s) \rangle \,\mathrm{d}s - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) \,\mathrm{d}W_\tau, \varphi'(s) \right\rangle \,\mathrm{d}s \\ &+ \int_0^t \langle F(u(s)), \varphi(s) \rangle \,\mathrm{d}s + \left\langle \int_0^t \sigma(u(s)) \,\mathrm{d}W_s, \varphi(t) \right\rangle, \end{aligned} \tag{3.49}$$

for every time-dependent test function $\varphi \in L^0\left(\Omega; C^1\left([0,t]; \mathcal{D}_u^*\right)\right)$. Now, Theorem 3.2.13 part (1) entails the usual weak formulation (3.49). □

REMARK 3.3.4. *Theorem 3.3.3 also provides a regularity result for the weak solution, i.e. $u \in L^0(\Omega; C^\delta([0,\tau); Y))$.*

EXAMPLE 3.3.5. *We let $d \geq 1$ and consider a quasilinear parabolic stochastic partial differential equation on a d-dimensional domain $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ of the*

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form

$$\begin{cases} \mathrm{d}u = \operatorname{div}(\hat{A}(u)\nabla u)\,\mathrm{d}t + \sigma(u)\mathrm{d}W_t, & \text{in } (0,T)\times\mathcal{O} \\ u = 0 & \text{on } [0,T]\times\partial\mathcal{O} \\ u(0,x) = u^0(x), & x\in\mathcal{O}, \end{cases} \tag{3.50}$$

where $(W_t)_{t\in[0,T]}$ *is a cylindrical Wiener process taking values in a Hilbert space* $H \supset X := L^2(\mathcal{O})$.

Such equations have been extensively studied in the literature, see [45, 47, 69], under the following assumptions on the coefficients B and σ:

ASSUMPTION 3.3.6. (1) *The coefficients* $\hat{A}:\mathbb{R}\to\mathbb{R}^{d\times d}$ *are nonlinear functions, such that the diffusion matrix* $\hat{A} = \left(\hat{A}_{ij}\right)_{i,j=1}^{d}$ *is of class* C_b^1*, symmetric, uniformly positive definite and bounded, i.e. there exist constants* $\kappa, C > 0$ *such that*

$$\kappa \mathrm{I} \le \hat{A} \le C\mathrm{I}. \tag{3.51}$$

(2) *For each* $u\in X$ *we consider a mapping* $\sigma(u): H\to X$ *defined by*

$$\sigma(u)e_k = \sigma_k(\cdot, u(\cdot)),$$

where $\sigma_k \in C(\mathcal{O}\times\mathbb{R})$*. We further suppose that* σ *satisfies usual Lipschitz and linear growth conditions, i.e.*

$$\sum_{k\in\mathbb{N}} |\sigma_k(x,\xi_1) - \sigma_k(x,\xi_2)|^2 \le C|\xi_1-\xi_2|^2, \qquad \forall\, x\in\mathcal{O}, \xi_1, \xi_2\in\mathbb{R},$$

$$\sum_{k\in\mathbb{N}} |\sigma_k(x,\xi)|^2 \le C\left(1+|\xi|^2\right), \qquad \forall\, x\in\mathcal{O}, \xi\in\mathbb{R}.$$

In particular, these assumptions imply that σ maps X to $\mathcal{L}_2(H,X)$. Thus, given a predictable process u that belongs to $L^2(\Omega, L^2(0,T;X))$, the stochastic integral is a well-defined X-valued process.

The previous assumptions on σ can be relaxed and an additional regular drift term can be incorporated [45, 47, 69]. An example of such a drift term is given by $\operatorname{div}(F(u))$, where

$$F = (F_1,\dots,F_d): \mathbb{R} \longrightarrow \mathbb{R}^d$$

is continuously differentiable with bounded derivatives.

REMARK 3.3.7. *In contrast to Example 3.45, the diffusion matrix* B *additionally satisfies the boundedness assumption* (3.51).

Next, we give the definition of a weak solution of (3.50).

DEFINITION 3.3.8. *An* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted,* X*-valued continuous process* $(u(t))_{t\in[0,T]}$ *is called a weak solution for* (3.50) *if for any* $\varphi\in C_0^\infty(\mathcal{O})$*, the following identity holds for* $t\in[0,T]$*,* $\mathbb{P}$*-a.s.*

$$\langle u(t),\varphi\rangle = \langle u^0,\varphi\rangle - \int_0^t \langle \hat{A}(u(s))\nabla u(s), \nabla\varphi\rangle\,\mathrm{d}s + \int_0^t \langle \sigma(u(s))\,\mathrm{d}W_s, \varphi\rangle.$$

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Under the previous assumptions on $\hat{A}$ and σ, together with suitable regularity conditions on the initial data, the existence of a weak solution was established in [45, 47].

Moreover, assuming higher spatial regularity on σ, the regularity of this weak solution can be improved [45, Theorem 2.6]. For the convenience of the reader we indicate this statement. To this aim we let $\eta > 0$, set $D_T := [0,T] \times \mathcal{O}$ and consider the Hölder space $C^{\eta/2,\eta}(D_T)$ with different time and space regularity, endowed with the norm

$$||f||_{C^{\eta/2,\eta}(D_T)} = \sup_{(t,x)\in D_T} |f(t,x)| + \sup_{(t,x)\neq(s,y)\in D_T} \frac{|f(t,x)-f(s,y)|}{\max\{|t-s|^{\eta/2}+|x-y|^{\eta}\}}.$$

THEOREM 3.3.9. *Assume that*

- $u^0 \in L^m\left(\Omega; C^{\iota}(\bar{\mathcal{O}})\right)$ *for some* $\iota > 0$ *and all* $m \in [2,\infty)$, *and* $u^0 = 0$ *on* $\partial\mathcal{O}$ *a.s.*
- $\|\sigma(u)\|_{\mathcal{L}_2\left(H, H_0^1(\mathcal{O})\right)} \leq C\left(1 + \|u\|_{H_0^1(\mathcal{O})}\right)$.

Then the weak solution u *of* (3.50) *belongs to* $L^m\left(\Omega; C^{\eta/2,\eta}(\overline{D_T})\right)$, *for all* $m \in [2,\infty)$.

Now, we are ready to state the main result for this example, based on Theorem 3.3.9.

THEOREM 3.3.10. *Let assumptions of Theorem 3.3.9 and* ***Assumption 3.3.6*** *hold. Then, there exists a pathwise mild solution of* (3.50). *More precisely, there exists an* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted process* u *such that for* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

$$u(t) = U^u(t,0)u^0 + U^u(t,0)\int_0^t \sigma(u(s))\,\mathrm{d}W_s - \int_0^t U^u(t,s)A_u(s)\int_s^t \sigma(u(\tau))\,\mathrm{d}W_\tau\,\mathrm{d}s, \tag{3.52}$$

where $A_u u := \operatorname{div}(B(u)\nabla u)$ *and* $U^u(\cdot,\cdot)$ *is the evolution family generated by* A_u. *Moreover,* $u \in L^0(\Omega; \mathcal{B}([0,T]; H^1(\mathcal{O}))) \cap L^0(\Omega; C^{\delta}([0,T]; L^2(\mathcal{O})))$.

PROOF. Under **Assumption 3.3.6**, the existence of a weak solution u

$$u \in L^2\left(\Omega; C\left([0,T]; L^2(\mathcal{O})\right) \cap L^2\left(0,T; H^1(\mathcal{O})\right)\right)$$

was established in [45, 47]. The bilinear form a for $u \in H^1(\mathcal{O})$ given by

$$a(u; \mathrm{v}, \mathrm{w}) := \langle \hat{A}(u)\nabla \mathrm{v}, \nabla \mathrm{w}\rangle, \qquad \mathrm{v}, \mathrm{w} \in H^1(\mathcal{O})$$

satisfy conditions (3.11) and (3.12) with $Z := H^1(\mathcal{O})$, $X = Y := L^2(\mathcal{O})$, thanks to (3.51). Therefore, (A1) and (A2) from **Assumption 3.1.3** hold. The Lipschitz assumption on B entails that (A3) holds too. Similarly, it can be shown that the conditions (A1*) (A3*) of **Assumption 3.1.4** are satisfied by the adjoint A_u^*, see [130, Chapter 1.8.2] for details. We are left to verify (A4). To this aim, we require to show that $u \in C^{\delta}([0,T]; L^2(\mathcal{O}))$ for some $\delta > 0$. The weak solution $u \in L^m\left(\Omega; C^{\eta/2,\eta}(\overline{D_T})\right)$, for all $m \in [2,\infty)$, by Theorem 3.3.9. Regarding this along with the following equivalent norm [83] on $C^{\eta/2,\eta}(D_T)$

$$\|u\|_{\eta/2,\eta,D_T} = \sup_{(t,x),(s,x)\in D_T} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\eta/2}} + \sup_{(t,x),(t,y)\in D_T} \frac{|u(t,x)-u(t,y)|}{|x-y|^{\eta}}$$

and boundedness of the domain $\mathcal{O}$, we conclude that $u \in C^{\delta}([0,T]; L^2(\mathcal{O}))$ for $\delta = \eta/2$. Hence, we can infer from Theorem 3.1.8 that the operator $A_u u = \operatorname{div}(\hat{A}(u)\nabla u)$ generates an evolution family U^u. According to Theorem 3.2.13 part (2), this evolution family along

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with the weak solution u satisfies (3.52). Consequently, we obtain that u is a pathwise mild solution of (3.50). □

REMARK 3.3.11. *Note that the regularity on σ assumed in Theorem 3.3.9 is necessary in order to obtain the Hölder regularity of the solution.*

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CHAPTER 4

Nonlinear deterministic and stochastic compactness results

The goal of this section is to obtain a general version of the non-linear compactness results presented in [132]. With such a result at our disposal, we will state a tightness criterion which relates to the problems in Chapter 2 where we consider the stochastic SKT model and a general class of volume filling cross-diffusion equations. In [68], Hittmeir and Jüngel stated the following convergence result for a sequence satisfying some a priori estimates, see [68, Proposition 2.1] for more details. Let $\mathcal{O} \subset \mathbb{R}^d (d \geq 2)$ be a bounded domain with $\partial\mathcal{O} \in C^{0,1}, T > 0$ and $s \geq 0$. Furthermore, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative, real-valued functions satisfying

$$\|\sqrt{u_n}\|_{L^2(0,T;H^1(\mathcal{O}))} + \|u_n \log u_n\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \|\partial_t u_n\|_{L^1(0,T;(H^s(\mathcal{O}))')} \leq C$$

for some $C > 0$ independent of n. Then, up to a subsequence, as $n \to \infty$,

$$u_n \to u \text{ strongly in } L^2\left(0,T;L^{d/d-1}(\mathcal{O})\right).$$

A correct(ed) proof of the above proposition was recently given in [20], where the authors first show that a subsequence $(u_n)_{n\in\mathbb{N}}$, which is not relabelled, converges strongly in $L^2(0,T;L^1(\mathcal{O}))$ and then use the boundedness of $\|u_n \log u_n\|_{L^\infty(0,T;L^1(\mathcal{O}))}$ to show that the subsequence indeed converges in $L^2\left(0,T;L^{d/(d-1)}(\mathcal{O})\right)$. We will use the ideas from their proof to prove our main result, which is a slight generalization of their statement.

LEMMA 4.0.1. *Let $\mathcal{O} \subset \mathbb{R}^d$, $\partial\mathcal{O} \in C^{0,1}, T > 0$, $p_1, p_2, q_1, q_2, q, p \geq 1$, which satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}.$$

Let $D_f \subset \mathbb{R}$ be open and $f : D_f \to \mathbb{R}$ be C^1 with invertible derivative. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, taking values in D_f, satisfying

$$\|\nabla f(u_n)\|_{L^{p_1}(0,T;L^{q_1}(\mathcal{O}))} + \left\|(f^{-1})'(f(u_n))\right\|_{L^{p_2}(0,T;L^{q_2}(\mathcal{O}))} \leq C_0, \tag{4.1}$$

and is uniformly bounded in $L^1(0,T;L^1(\mathcal{O}))$. Then u_n is uniformly bounded in $L^p(0,T;W^{1,q}(\mathcal{O}))$.

PROOF. For $p_1, p_2, q_1, q_2 < \infty$, the proof is a simple application of Hölder's inequality.

$$\left(\int_0^T \left(\int_\mathcal{O} |\nabla u_n|^q \, \mathrm{d}x\right)^{\frac{p}{q}} \, \mathrm{d}t\right)^{\frac{1}{q}} = \left(\int_0^T \left(\int_\mathcal{O} |(f^{-1})'(f(u_n))\nabla f(u_n)|^q \, \mathrm{d}x\right)^{\frac{p}{q}} \, \mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^T \left(\int_\mathcal{O} |(f^{-1})'(f(u_n))|^{q_2} \, \mathrm{d}x\right)^{\frac{p}{q_2}} \left(\int_\mathcal{O} |\nabla f(u_n)|^{q_1} \, \mathrm{d}x\right)^{\frac{p}{q_1}} \, \mathrm{d}t\right)^{\frac{1}{p}}$$

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$$\leq \left(\int_0^T \left(\int_{\mathcal{O}} |(f^{-1})'(f(u_n))|^{q_2} \, \mathrm{d}x \right)^{\frac{p_2}{q_2}} \mathrm{d}t \right)^{\frac{1}{p_2}} \left(\int_0^T \left(\int_{\mathcal{O}} |\nabla f(u_n)|^{q_1} \, \mathrm{d}x \right)^{\frac{p_1}{q_1}} \mathrm{d}t \right)^{\frac{1}{p_1}}.$$

The condition

$$\|u_n\|_{L^1(0,T;L^1(\mathcal{O}))} \leq C$$

and the generalized Poincaré inequality (5.4.10) result in

$$\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq C.$$

In the case that one or more of the parameters p_1, p_2, q_1, q_2 are ∞, the proof is modified in the obvious way. □

NOTATION 4.0.2. *We say that a function $u \in L^{p-}(\mathcal{O})$, if $u \in L^q(\mathcal{O})$ for every $q < p$. The same notation will be used with respect to other parameters and always means that a certain property holds true up to, but not including, a certain value of the parameter.*

The next lemma is a straight forward combination of known compactness results related to Sobolev spaces. It will prove useful for later results.

LEMMA 4.0.3. *Let $\mathcal{O} \subset \mathbb{R}^d$, $\partial\mathcal{O} \in C^{0,1}$, $T > 0$, $p, q, k, l \geq 1$. We assume $0 < \alpha < 1$ if $r \geq p \geq 1$ and $\alpha > \frac{1}{r} - \frac{1}{p}$, if $1 \leq r < p$.*
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, satisfying

$$\|\nabla u_n\|_{L^p(0,T;L^q(\mathcal{O}))} + \|u_n\|_{L^k\left(0,T;L^l(\mathcal{O})\right)} + \|u_n\|_{W^{\alpha,r}(0,T;Y)} \leq C_0, \tag{4.2}$$

uniformly in n. Let Y be a Banach space such that the embedding

$$\begin{cases} L^{\frac{qd}{d-q}-}(\mathcal{O}) \hookrightarrow Y & \text{if } d > q, \\ L^p(\mathcal{O}) \hookrightarrow Y & \text{for some } p < \infty \text{ if } d = q, \\ C(\mathcal{O}) \hookrightarrow Y & \text{if } d < q, \end{cases}$$

is continuous. Then there exists a strongly convergent subsequence $(u_{n_m})_{n_m}$,

- *in $L^p\left(0,T;L^{\frac{dq}{d-q}-}(\mathcal{O})\right)$, if $q < d$,*
- *in $L^p\left(0,T;L^k(\mathcal{O})\right)$, for any $k < \infty$, if $q = d$,*
- *in $L^p\left(0,T;C(\overline{\mathcal{O}})\right)$, (more precisely in $C^{(1-\frac{d}{q})-}$, such that $(1-\frac{d}{q})- > 0$), if $q > d$.*

If $p = \infty$, the space $L^p(0,T,\cdot)$ is replaced by the space of continuous functions in time $C([0,T];\cdot)$.

PROOF. By the generalized Poincaré inequality, we conclude that

$$\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq C,$$

uniformly in n.
We first consider the case $q < d$. Observe that the embedding $W^{1,q}(\mathcal{O}) \hookrightarrow L^s(\mathcal{O})$ is compact for all $1 \leq s < \frac{dq}{d-q}$. By Corollary 5.5.10, with the spaces $Y := Y$, $B := L^s(\mathcal{O})$ and $X := W^{1,q}(\mathcal{O})$, a bounded set in $L^p(0,T,W^{1,q}(\mathcal{O})) \cap W^{\alpha,r}(0,T;Y)$ is compact in $L^p(0,T,L^s(\mathcal{O}))$ ($C([0,T];L^s(\mathcal{O}))$ for $p = \infty$).

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For the second case $q = d$ we observe that, by the boundedness of the domain the embedding $L^q(\mathcal{O}) \hookrightarrow L^{q-\delta}(\mathcal{O})$ is continuous for any $\delta > 0$ such that $q > q - \delta > 0$. We are back in the setting of case 1 and $W^{1,q-\delta}(\mathcal{O}) \hookrightarrow L^s(\mathcal{O})$ is compact for all $1 \leq s < \frac{d(q-\delta)}{d-(q-\delta)}$.
The last case is treated analogous to the first one and is also a direct consequence of the Sobolev embedding theorem (see 5.4.3). □

Next, we incorporate non-linear potentially non-linear functions into the previous result.

THEOREM 4.0.4. *Let $\mathcal{O} \subset \mathbb{R}^d$, $\partial\mathcal{O} \in C^{0,1}, T > 0$, $p_1, p_2, q_1, q_2, q, p \geq 1$, which satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}.$$

We assume $0 < \alpha < 1$ if $r \geq p \geq 1$ and $\alpha > \frac{1}{r} - \frac{1}{p}$, if $1 \leq r < p$.
Let $D_f \subset \mathbb{R}$ be open and $f : D_f \to \mathbb{R}$ be C^1 with invertible derivative. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, taking values in D_f, satisfying

$$\|\nabla f(u_n)\|_{L^{p_1}(0,T;L^{q_1}(\mathcal{O}))} + \left\|(f^{-1})'(f(u_n))\right\|_{L^{p_2}(0,T;L^{q_2}(\mathcal{O}))} + \|u_n\|^r_{W^{\alpha,r}(0,T;Y)} \leq C_0, \tag{4.3}$$

as well as

$$\|u_n\|_{L^1(0,T;L^1(\mathcal{O}))} \leq C,$$

uniformly in n. Y is assumed to be Banach space such that the embedding

$$\begin{cases} L^{\frac{qd}{d-q}-}(\mathcal{O}) \hookrightarrow Y & \text{for any } \varepsilon > 0 \text{ if } d > q, \\ L^p(\mathcal{O}) \hookrightarrow Y & \text{for some } p < \infty \text{ if } d = q, \\ C(\mathcal{O}) \hookrightarrow Y & \text{if } d < q, \end{cases}$$

is continuous. Under these conditions, we obtain the following:

- *If $q < d$, then there exists a strongly convergent subsequence $(u_{n_m})_{n_m}$ in $L^p\left(0,T;L^{\frac{dq}{d-q}-}(\mathcal{O})\right)$.*
- *If $q = d$, then there exists a strongly convergent subsequence $(u_{n_m})_{n_m}$ in $L^p\left(0,T;L^k(\mathcal{O})\right)$, for any $k < \infty$.*
- *If $q > d$, then there exists a strongly convergent subsequence $(u_{n_m})_{n_m}$ in $L^p\left(0,T;C(\overline{\mathcal{O}})\right)$, (more precisely in $C^{(1-\frac{d}{q})-}$, such that $(1-\frac{d}{q})- > 0$).*

If $p = \infty$, the space $L^p(0,T,\cdot)$ is replaced by the space of continuous functions in time $C([0,T];\cdot)$.

PROOF. The proof follows from Lemma 4.0.1 and Lemma 4.0.3. □

REMARK 4.0.5. *If f is the identity, $D_f = \mathbb{R}$ and one recovers a standard Gradient estimate in $L^p(0,T;L^q(\mathcal{O}))$.*
In the setting of [68], by the positivity of the functions u_n, the set D_f is given by $(0,\infty)$. It should be remarked that in our applications, we will usually deal with the set $\overline{\mathcal{D}}_f$ instead of $\mathcal{D}_f$, which will either be $[0,\infty)$ or $[0,1]$. The statement and proof of the theorem can easily adapted by including the assumption for the one-sided derivatives at the boundary of $\mathcal{D}_f$.

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REMARK 4.0.6. *By Corollary 5.5.12 the above result can easily be extended to weaker integrability assumptions in form of* $\nabla u_n \in L^1_{\text{loc}}(0,T;L^q(\mathcal{O})), u_n \in W^{s,1}_{\text{loc}}(0,T;Y)$ *and for some* $p,r>1$, $\|u_n\|_{L^p(0,T;L^r(\mathcal{O}))} \leq \|(f^{-1})'(f(u_n))\|_{L^{p_2}(0,T;L^{q_2}(\mathcal{O}))}$. *Here we obtain relative compactness in* $L^{p-\varepsilon}(0,T;L^{\frac{qd}{d-q}-\delta}(\mathcal{O}))$ *for any* $\varepsilon,\delta>0$ *and* $q<d$.

Another direct, but noteworthy consequence is the following.

REMARK 4.0.7. *The previous results also holds true if* $\|u_n\|_{W^{\alpha,r}(0,T;Y)}$ *is replaced by either an estimate on the derivative of the form* $\left\|\frac{d}{dt}u_n\right\|_{L^r(0,T;Y)}$ *or the condition,*

$$\|\tau_h u_n - u_n\|_{L^r(0,T-h;Y)} \to 0 \text{ as } h \to 0,$$

uniformly in n. *For details see* [120].

In the version presented in [20, 68] there is one estimate we didn't account for yet, namely $\|u_n \log u_n\|_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C$, which is a consequence of their entropy estimate and allows us to drop the "$-$" with respect to the space integrability. The assumption of convexity is deeply rooted in the boundedness by entropy method, see [77], so it becomes natural to substitute the specific function $x\log(x)$ for a general function ϕ from a class of functions sharing similar properties. A good choice are functions which act as the so-called principle part of an N-function. N-functions play an important role in the study of Orlicz spaces and uniform integrability. We refer to [3, 117, 115] for more information.

LEMMA 4.0.8. *Let us fix* $s=\frac{qd}{d-q}$. *Let the sequence* $(u_n)_{n\in\mathbb{N}}$ *satisfy the assumptions of Theorem 4.0.4 in the case where* $q<d$ *with the restriction that* $q_2<\infty$. *Let* $\phi\colon\mathbb{R}\to\mathbb{R}$ *be a convex function such that*

$$\lim_{x\to\infty}\frac{\phi(x)}{x^q}=\infty, \qquad \lim_{x\to\infty}\frac{\phi(x)}{|((f^{-1})'(f(x)))|^{q_2}}=\infty. \tag{4.4}$$

Moreover, we assume that

$$\|\phi(u_n)\|_{L^{\frac{p_2}{q_2}}(0,T;L^1(\mathcal{O}))}\leq C. \tag{4.5}$$

Then the sequence (u_n) *converges strongly in* $L^p(0,T;L^s(\mathcal{O}))$.

PROOF. We prove the result for $p_2<\infty$. The proof can be adapted in an obvious fashion when $p=\infty$. First we note that there exists a subsequence of (u_n), which is not relabeled, such that $u_n\to u$ strongly in $L^1(0,T;L^1(\mathcal{O}))$ and also a.e. in $\mathcal{O}\times(0,T)$. We proceed as in [20]. Let $L>1$ be given, and set $v_n^L=\min\{u_n,L\}$ and $w_n^L=\max\{u_n-L,0\}$. Then $u_n=v_n^L+w_n^L$. It holds that $v_n^L\to v^L=\min\{u,L\}$ a.e. and $w_n^L\to w^L=\max\{u-L,0\}$ a.e. with $u=v^L+w^L$. Then

$$\|u_n-u\|_{L^p(0,T;L^s(\mathcal{O}))}\leq \left\|v_n^L-v^L\right\|_{L^p(0,T;L^s(\mathcal{O}))}+\left\|w_n^L\right\|_{L^p(0,T;L^s(\mathcal{O}))} + \left\|w^L\right\|_{L^p(0,T;L^s(\mathcal{O}))}$$

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$$=: I_1 + I_2 + I_3.$$

Let's estimate the terms $I_2,\ I_3$ first. We want to use continuous embedding of $W^{1,q}(\mathcal{O}) \hookrightarrow L^s(\mathcal{O})$ to estimate these terms. As in the proof of Lemma 4.0.1, we apply Hölders inequality, this gives

$$\begin{aligned}
\left\|\nabla w_n^L\right\|_{L^p(0,T;L^q(\mathcal{O}))} &= \left(\int_0^T \left(\int_{\{u_n>L\}} |\nabla u_n|^q \ \mathrm{d}x\right)^{\frac{p}{q}} \ \mathrm{d}t\right)^{\frac{1}{p}} \\
\left\|\nabla w_n^L\right\|_{L^p(0,T;L^q(\mathcal{O}))} &= \left(\int_0^T \left(\int_{\{u_n>L\}} \left(|(f^{-1})'(f(u_n))||\nabla f(u_n)|\right)^q \ \mathrm{d}x\right)^{\frac{p}{q}} \ \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \|\nabla f(u_n)\|_{L^{p_1}(0,T;L^{q_1}(\mathcal{O}))} \left(\int_0^T \left(\int_{\{u_n>L\}} |(f^{-1})'(f(u_n))|^{q_2} \ \mathrm{d}x\right)^{\frac{p_2}{q_2}} \ \mathrm{d}t\right)^{\frac{1}{p_2}} \\
&\leq C \left(\int_0^T \left(\int_{\{u_n>L\}} \delta\phi(u_n) \ \mathrm{d}x\right)^{\frac{p_2}{q_2}} \ \mathrm{d}t\right)^{1/p_2} \\
&\leq C\delta^{\frac{1}{q_2}}.
\end{aligned}$$

In the second to last line we used that, by the properties of φ, there exists an L_0 and $\delta > 0$ such that $\frac{\phi(x)}{|(f^{-1})'(f(x))|^q} > \frac{1}{\delta}$ for all $x > L_0$. Hence, if $u \in L^\phi(\mathcal{O})$ then, there exists $L_0 > 0$ and $\delta > 0$ such that $|(f^{-1})'(f(u))|^q \leq \delta\varphi(|u|)$, for $|u| > L_0$. Further,

$$\begin{aligned}
\left\|w_n^L\right\|_{L^p(0,T;L^q(\mathcal{O}))} &= \left(\int_0^T \left(\int_{\{u_\varepsilon>L\}} (u_n - L)^q \ \mathrm{d}x\right)^{\frac{p}{q}} \ \mathrm{d}t\right)^{\frac{1}{p}} \leq \left(\int_0^T \left(\int_{\{u_\varepsilon>L\}} |u_n|^q \ \mathrm{d}x\right)^{\frac{p}{q}} \ \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^T \left(\int_{\mathcal{O}} \delta\phi(u_n) \ \mathrm{d}x\right)^{\frac{p}{q}} \ \mathrm{d}t\right)^{\frac{1}{q}} \leq C\delta^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$\left\|w_n^L\right\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq C(\delta^{\frac{1}{q}} + \delta^{\frac{1}{q_2}}).$$

By lower semicontinuity of the norm, we obtain

$$\left\|w^L\right\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq C(\delta^{\frac{1}{q}} + \delta^{\frac{1}{q_2}}).$$

From the Sobolev embedding, we conclude that

$$I_2 + I_3 \leq C(\delta^{\frac{1}{q}} + \delta^{\frac{1}{q_2}}).$$

Observing that $|v_n^L(x,t) - v^L(x,t)|^s \leq (2L)^s$ and $|v_n^L(x,t) - v^L(x,t)|^s \to 0$ a.e. in $\mathcal{O}\times(0,T)$, the dominated convergence theorem (see Lemma 5.3.11) implies that $\left(\int_{\mathcal{O}} |v_n^L(x,t) - v^L(x,t)|^s \ \mathrm{d}x\right)^{\frac{p}{s}} \to$

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0 a.e. in $(0,T)$ and hence $\|v_n^L(x,t) - v^L(x,t)\|_{L^s(\mathcal{O})} \to 0$ a.e. in $(0,T)$. Moreover,

$$\left(\int_0^T \|v_n^L(x,t) - v^L(x,t)\|_{L^s(\mathcal{O})}^p \, \mathrm{d}t\right)^{\frac{1}{p}} \leq 2L|\mathcal{O}|^s T^{\frac{1}{p}} =: C(L).$$

Set $f_n^L(t) := \|v_n^L(x,t) - v^L(x,t)\|_{L^s(\mathcal{O})}^p$ and recall that $f_n^L(t) \to 0$ a.e. in $(0,T)$ as well as $\|f_n^L(t)\|_{L^p(0,T)} \leq C(L)$. Again by the dominated convegence theorem, we conclude that

$$\lim_{n\to\infty} \left(\int_0^T \|v_n^L(x,t) - v^L(x,t)\|_{L^s(\mathcal{O})}^p \, \mathrm{d}t\right)^{\frac{1}{p}} \leq 2L|\mathcal{O}|^s T^{\frac{1}{p}} = 0.$$

Thus, there exists $n_0 > 0$ such that for all $0 < n_0 < n$ we have $I_1 = \left\|v_n^L - v^L\right\|_{L^p(0,T;L^s(\mathcal{O}))} \leq \delta(L)$. Therefore,

$$\|u_n - u\|_{L^p(0,T;L^s(\mathcal{O}))} \leq C\delta(n,L) + C(\delta(L)^{\frac{1}{p}} + \delta(L)^{\frac{1}{p_1}}).$$

□

We are now equipped to state our first tightness conditions. We begin with a rather standard Lemma, which is quite useful in applications.

LEMMA 4.0.9. *Let $\mathcal{O} \subset \mathbb{R}^d$, $\partial\mathcal{O} \in C^{0,1}, T > 0$, $p,q,k,l \geq 1$. We assume $0 < \alpha < 1$ if $r \geq p \geq 1$ and $\alpha > \frac{1}{r} - \frac{1}{p}$, if $1 \leq r < p$.*
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, satisfying

$$\mathbb{E}\left[\|\nabla u_n\|_{L^p(0,T;L^q(\mathcal{O}))}\right] + \mathbb{E}\left[\|u_n\|_{L^k\left(0,T;L^l(\mathcal{O})\right)}\right] + \mathbb{E}\left[\|u_n\|_{W^{\alpha,r}(0,T;Y)}\right] \leq C_0, \tag{4.6}$$

uniformly in n. Y is assumed to be Banach space such that the embedding

$$\begin{cases} L^{\frac{qd}{d-q}-}(\mathcal{O}) \hookrightarrow Y & \text{for any } \varepsilon > 0 \text{ if } d > q, \\ L^p(\mathcal{O}) \hookrightarrow Y & \text{for some } p < \infty \text{ if } d = q, \\ C(\mathcal{O}) \hookrightarrow Y & \text{if } d < q, \end{cases}$$

is continuous. Let $1 \leq s < \frac{dq}{d-q}$ and $\mathbb{P}_n$ denote the law of u_n on the space

$$Z = \begin{cases} L^p\left(0,T;L^s(\mathcal{O})\right), & \text{if } q < d, \\ L^p\left(0,T;L^k(\mathcal{O})\right), \text{for some } k < \infty, & \text{if } d = q, \\ L^p\left(0,T;C(\overline{\mathcal{O}})\right), & \text{if } d < q. \end{cases}$$

Then for every $\varepsilon > 0$, there exists a compact subset K_ε of Z, such that

$$\mathbb{P}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

If $p = \infty$, the space $L^p(0,T,\cdot)$ is replaced by the space of continuous functions in time $C([0,T];\cdot)$.

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PROOF. Let

$$B_r^1 := \{v \in L^p(0,T;W^{1,q}(\mathcal{O})) \,|\, \|v\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq R\},$$
$$B_r^2 := \{v \in W^{\alpha,r}(0,T;Y) \,|\, \|v\|_{W^{\alpha,r}(0,T;Y)} \leq R\}.$$

By Lemma 4.0.3, we know that the intersection $B_{r_1}^1 \cap B_{r_2}^2$ is relatively compact in Z. We choose $r_1 = r_2 = \widetilde{r} = \frac{2C}{\varepsilon}$ for $\varepsilon > 0$. By the Chebyshev inequality,

$$\mathbb{P}_n\left(\left(B_{\widetilde{r}}^1 \cap B_{\widetilde{r}}^2\right)^c\right) = \mathbb{P}_n\left(\left(B_{\widetilde{r}}^1\right)^c \cup \left(B_{\widetilde{r}}^2\right)^c\right) = \mathbb{P}\left(\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} > \widetilde{r} \vee \|u_n\|_{W^{\alpha,r}(0,T;Y)} > \widetilde{r}\right)$$
$$\leq \mathbb{P}\left(\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} > \widetilde{r}\right) + \mathbb{P}\left(\|u_n\|_{W^{\alpha,r}(0,T;Y)} > \widetilde{r}\right)$$
$$\leq \frac{2}{\widetilde{r}}\mathbb{E}\left[\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} + \|u_n\|_{W^{\alpha,r}(0,T;Y)}\right] \leq \frac{2C}{\widetilde{r}} \leq \varepsilon.$$

Choosing K_ε as the closure, in Z, of the set $\{v \in Z \,|\, v \in B^1_{\frac{2C}{\varepsilon}} \cap B^2_{\frac{2C}{\varepsilon}}\}$ concludes the proof. □

As in the deterministic compactness results, we now generalize the previous Lemma by introducing potentially non-linear functions to our estimates.

THEOREM 4.0.10. *Let $\mathcal{O} \subset \mathbb{R}^d$, $\partial\mathcal{O} \in C^{0,1}, T > 0$, $q \geq 1$ $0 < \alpha < 1$ if $r \geq p \geq 1$ and $\alpha > \frac{1}{r} - \frac{1}{p}$, if $1 \leq r < p$.*
Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 with invertible derivative. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-adapted processes satisfying, uniformly in n,
(4.7)
$$\mathbb{E}\left[\|\nabla f(u_n)\|^{k_1}_{L^{p_1}(0,T;L^{q_1}(\mathcal{O}))}\right] + \mathbb{E}\left[\left\|(f^{-1})'(f(u_n))\right\|^{k_2}_{L^{p_2}(0,T;L^{q_2}(\mathcal{O}))}\right] + \mathbb{E}\left[\|u_n\|^r_{W^{\alpha,r}(0,T;Y)}\right] \leq C_0,$$
where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and $\frac{1}{k_1} + \frac{1}{k_2} \leq 1$. Further we assume that

$$\mathbb{E}\left[\|u_n\|_{L^1(0,T;L^1(\mathcal{O}))}\right] \leq C_0,$$

uniformly in n. Y is assumed to be Banach space such that the embedding

$$\begin{cases} L^{\frac{qd}{d-q}-}(\mathcal{O}) \hookrightarrow Y & if d > q, \\ L^p(\mathcal{O}) \hookrightarrow Y \text{ for some } p < \infty & if d = q, \\ C(\overline{\mathcal{O}}) \hookrightarrow Y & if d < q, \end{cases}$$

is continuous. Let $1 \leq s < \frac{dq}{d-q}$. Let $\mathbb{P}_n$ denote the law of u_n on the space

$$Z = \begin{cases} L^p\left(0,T;L^s(\mathcal{O})\right), & \text{if } q < d, \\ L^p\left(0,T;L^k(\mathcal{O})\right), \text{for some } k < \infty, & \text{if } d = q, \\ L^p\left(0,T;C(\overline{\mathcal{O}})\right), & \text{if } d < q. \end{cases}$$

Then for every $\varepsilon > 0$, there exists a compact subset K_ε of Z, such that

$$\mathbb{P}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

If $p = \infty$, the space $L^p(0,T,\cdot)$ is replaced by the space of continuous functions in time $C([0,T];\cdot)$.

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We remind the reader that the space $C(\overline{\mathcal{O}})$ is in fact separable, since $\overline{\mathcal{O}}$ is a compact set in $\mathbb{R}^d$.

PROOF. The proof follows from Lemma 4.0.1, Hölder's inequality and Lemma 4.0.9. □

COROLLARY 4.0.11. *Let the conditions of Theorem 4.0.10 be satisfied with $d > q$, with the restriction that $q_2 < \infty$. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a convex function satisfying (4.4). Moreover, we assume that*

$$\mathbb{E}\left[\|\phi(u_n)\|_{L^{\frac{p_2}{q_2}}(0,T;L^1(\mathcal{O}))}\right] \leq C. \tag{4.8}$$

Then the laws of the sequence $(u_n)_{n\in\mathbb{N}}$ are tight in $L^p(0,T;L^{\frac{qd}{d-q}}(\mathcal{O}))$.

PROOF. We recall that by Lemma 4.0.8, a strongly convergent sequence in $L^p(0,T;L^{\frac{qd}{d-q}-}(\mathcal{O}))$ for which the uniform bound $\|\phi(u_n)\|_{L^p(0,T;L^1(\mathcal{O}))} \leq C$ holds, has a convergent subsequence in $L^p(0,T;L^{\frac{qd}{d-q}}(\mathcal{O}))$. By the considerations of Lemma 4.0.4 and Corollary 5.5.10, the set $B_r^1 \cap B_r^2 \cap B_r^3$, where

$$\begin{aligned}
B_r^1 &:= \{v \in L^p(0,T;W^{1,q}(\mathcal{O})) \,|\, \|v\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq R\},\\
B_r^2 &:= \{v \in W^{\alpha,r}(0,T;Y) \,|\, \|v\|_{W^{\alpha,r}(0,T;Y)} \leq R\},\\
B_r^3 &:= \{v \in L^p(0,T;L^1(\mathcal{O})) \,|\, \|\phi(v)\|_{L^{\frac{p_2}{q_2}}(0,T;L^1(\mathcal{O}))} \leq R\},
\end{aligned}$$

is relatively compact in $L^p(0,T;L^{\frac{qd}{d-q}}(\mathcal{O}))$. Choose $r_1 = r_2 = r_3 = \widetilde{r} = \frac{3C}{\varepsilon}$ for $\varepsilon > 0$. By the Chebyshev inequality,

$$\begin{aligned}
&\mathbb{P}_n\left(\left(B_{\widetilde{r}}^1 \cap B_{\widetilde{r}}^2 \cap B_{\widetilde{r}}^3\right)^c\right)\\
&\quad\leq \frac{3}{\widetilde{r}}\mathbb{E}\left[\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} + \|u_n\|_{W^{\alpha,r}(0,T;Y)} + \|\phi(u_n)\|_{L^{\frac{p_2}{q_2}}(0,T;L^1(\mathcal{O}))}\right] \leq \frac{3C}{\widetilde{r}} \leq \varepsilon.
\end{aligned}$$

Choosing K_ε as the closure, in $L^p(0,T;L^{\frac{qd}{d-q}}(\mathcal{O}))$, of the set $\{v \in L^p(0,T;L^{\frac{qd}{d-q}}(\mathcal{O})) \,|\, v \in B^1_{\frac{3C}{\varepsilon}} \cap B^2_{\frac{3C}{\varepsilon}} \cap B^3_{\frac{3C}{\varepsilon}}\}$ concludes the proof since we obtained a compact set for which $\mathbb{P}_n(K_\varepsilon) \geq 1-\varepsilon$ is satisfied. □

COROLLARY 4.0.12. *Let the conditions of Lemma 4.0.9 (Theorem 4.0.10) be satisfied. If $\alpha \leq 1/r$ then the laws of $(u_n)_{n\in\mathbb{N}}$ are tight in $L^{\widetilde{p}}(0,T;Y), \forall\, \widetilde{p} < r_* = r/(1-\alpha r)$. If $\alpha > 1/r$ then the laws of $(u_n)_{n\in\mathbb{N}}$ are tight in $C(0,T;Y)$.*

EXAMPLE 4.0.13. *We assume a setting comparable to [68]. Let T be finite, $\mathcal{O} \subset \mathbb{R}^d$ be bounded and $(u_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative, real-valued functions satisfying*

$$\mathbb{E}\,\|\sqrt{u_n}\|_{L^2(0,T;H^1(\mathcal{O}))} + \mathbb{E}\,\|u_n \log u_n\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \mathbb{E}\,\|u_n\|_{W^{\alpha,r}(0,T;(H^s(\mathcal{O}))'} \leq C,$$

for some $C > 0$ independent of n. We assume $\alpha > 0, r > 1, s > d/2$. First we note that $f(u_n) = \sqrt{u_n}$ and $(f^{-1})'(f(u_n)) = \sqrt{u_n}$. By standard calculus, we conclude

$$\mathbb{E}\,\|\sqrt{u_n}\|_{L^\infty(0,T;L^2(\mathcal{O}))} = \mathbb{E}\,\|u_n\|^{1/2}_{L^\infty(0,T;L^1(\mathcal{O}))}$$

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$$\leq C + \mathbb{E}\,\|u_n \log u_n\|^{1/2}_{L^\infty(0,T;L^1(\mathcal{O}))}.$$

We are now ready to apply Theorem 4.0.10 with

$$f(x) = \sqrt{x}, p_1 = 2, q_1 = 2, p_2 = \infty, q_2 = 2, k_1 = k_2 = 2, r \geq 2, \alpha > 0, Y = H^s(\mathcal{O})'.$$

We obtain tightness of the laws of (u_n) in $L^2(0,T;L^{d/(d-1)-}(\mathcal{O}))$. Next, we apply Lemma 4.0.11 with $\phi(x) = x\log(x)$. Since $q = 1$ and $q_2 = 2$, we conclude

$$\lim_{x\to\infty} \frac{x\log(x)}{\sqrt{x}^2} = \lim_{x\to\infty} \frac{x\log(x)}{x} = \infty.$$

We obtain tightness in $L^2(0,T;L^{d/(d-1)}(\mathcal{O}))$. Finally, if $\alpha r > 1$, Corollary 4.0.12 implies also tightness in $C(0,T;H^s(\mathcal{O})')$.

Next we want to formulate another tightness result in the spirit of Brzeźniak and coworkers, as it was presented in [26], and are based on [98]. We begin by stating the following deterministic compactness result.

THEOREM 4.0.14. *Let $X, B, \widetilde{Y}, Y^\theta, Y$ be Banach spaces such that the embedding $X \hookrightarrow B$ as well as $\widetilde{Y} \hookrightarrow Y$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Further there exists a real constant $M > 0$ and a parameter $\theta \in (0,1)$ such that*

$$\|u_n\|_{Y^\theta} \leq M\|u_n\|^\theta_Y \|u_n\|^{1-\theta}_{\widetilde{Y}}. \tag{4.9}$$

Let $1 \leq p < q \leq \infty$ or $1 < p \leq q \leq \infty$ and

$$Z := C([0,T];Y) \cap L^{q-}(0,T;B) \cap L^p(0,T;B) \cap C([0,T];Y^\theta),$$

equipped with the supremum of the corresponding topologies, $\mathcal{T}$. If a subset $\mathcal{K} \subset Z$ satisfies

(1) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(0,T;X)} \leq C$, ,
(2) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^q(0,T;B)} \leq C$,
(3) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;\widetilde{Y})} \leq C$,
(4) $\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \sup_{|t-s|\leq\delta, s,t\leq T} \|u_n(t) - u_n(s)\|_Y = 0$,

then it is $\mathcal{T}$ compact. If $p = \infty$, the set $\mathcal{K}$ is compact in $Z := C([0,T];Y) \cap C([0,T];B) \cap C([0,T];Y^\theta)$ with respect to the supremum of the topologies.
In particular,

$$\begin{aligned} (1), (2), (4) &\Rightarrow \text{compactness in } L^{q-}(0,T;B) \cap L^p(0,T;B), \\ (3), (4) &\Rightarrow \text{compactness in } C([0,T];Y), \\ (3), (4), (4.9) &\Rightarrow \text{compactness in } C([0,T];Y^\theta). \end{aligned}$$

PROOF. (1), (2), (4) satisfy the assumptions of Theorem 5.5.9 as well as 5.5.11. Hence we obtain compactness in $L^{q-}(0,T;B) \cap L^p(0,T;B)$. The compactness in $C([0,T];Y)$ is a consequence of (3), (4), the compactness of the embedding $\widetilde{Y} \hookrightarrow Y$ and the Arzela-Ascoli theorem in Banach spaces (see Theorem 5.5.3).The compactness in $C([0,T];Y^\theta)$ does not follow directly by one of the theorems but was already proven implicitly in Theorem 5.5.15.

$$\|\tau_h u_n - u_n\|_{L^\infty(0,T-h;Y^\theta)} \leq M\,\|\tau_h u_n - u_n\|^{1-\theta}_{L^\infty(0,T-h;\widetilde{Y})}\,\|\tau_h u_n - u_n\|^\theta_{L^\infty(0,T-h;Y)} \leq 2C^{1-\theta}\epsilon^\theta_h.$$

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$\epsilon_h^\theta \to 0$ by Assumption (4). It remains to verify that the embedding $\widetilde{Y} \hookrightarrow Y^\theta$ is compact. Taking a bounded sequence $v_m \in Y$, by the compact embedding $\widetilde{Y} \hookrightarrow Y$, there exists a convergent subsequence in Y. By assumption,

$$\|v_m - v\|_{Y^\theta} \leq M \|v_m - v\|_{\widetilde{Y}}^{1-\theta} \|v_m - v\|_Y^\theta \to 0.$$

An application of the Arzela-Ascoli theorem concludes the arguement. □

COROLLARY 4.0.15. *Let $X, B, \widetilde{Y}, Y^\theta, Y, Z$ be as in Lemma 4.0.14. Additionally, assume that X has a pre-dual space and let $p > 1$, then $\mathcal{K}$ is compact in*

$$Z \cap L^p_{w^*}(0,T;X).$$

If $p = 1$, conditions (2), (3), (4), as well as

(4') $\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(0,T;X)} \leq C$,

(4') *the family $(u_n)_{n\in\mathbb{N}}$ is uniformly integrable in $L^1(0,T;X)$,*

then $\mathcal{K}$ is compact in

$$Z \cap L^1_w(0,T;X).$$

PROOF. The proof of the first statement is a direct consequence of the Banach-Alaoglu theorem. The second statement follows from the well known Theorem of Dunford and Pettis, see also [23, 126]. □

Next we state a generalized version of [26, Lemma 3.2]. First we state a related result and introduce some notation. We define $C\left([0,T];\widetilde{Y}_{w^*}\right)$ as the space of weakly* continuous functions $u : [0,T] \to \widetilde{Y}$ endowed with the topology, such that for all ϕ the pre-dual of $\widetilde{Y}$ the mappings

$$C\left([0,T];\widetilde{Y}_{w^*}\right) \ni u \mapsto \langle u(\cdot), \phi\rangle \in C([0,T];\mathbb{R}),$$

are continuous. In particular, $u_n \to u$ in $C\left([0,T];\widetilde{Y}_{w^*}\right)$ iff for all ϕ in the pre-dual of $\widetilde{Y}$:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle u_n(t) - u(t), \phi\rangle| = |\phi(u_n(t) - u(t))| = 0.$$

We write $C\left([0,T];\widetilde{Y}_w\right)$ for the same space of weakly continuous functions.

Consider the ball

$$\mathbb{B}^r := \left\{ y \in \widetilde{Y}; \quad \|y\|_{\widetilde{Y}} \leq r \right\}.$$

If $\widetilde{Y}$ is separable, the weak* topology induced on $\mathbb{B}^r$ is metrizable. Let q denote the metric compatible with the weak topology on $\mathbb{B}^r$. Let us consider the following space $C\left([0,T];\mathbb{B}^r_{w^*}\right)$ = the space of weakly continuous functions $u : [0,T] \to \widetilde{Y}$ and such that $\sup_{t\in[0,T]} |u(t)|_{\widetilde{Y}} \leq r$. The space $C\left([0,T];\mathbb{B}^r_{w^*}\right)$ is metrizable with

$$\varrho(u,v) = \sup_{t\in[0,T]} q(u(t),v(t)).$$

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REMARK 4.0.16. *By Theorem 5.1.11, the unit-ball on $L^p(0,T;X)$ is w^*-metrizable if and only if $L^p(0,T;X)$ is separable, in which case compactness and sequential compactness coincide.*

LEMMA 4.0.17. *Let ${}'Y$ be a Banach space and $\widetilde{Y}$ a separable Banach space. We denote their dual spaces by $\widetilde{Y}', Y$. Further assume that the embeddings ${}'Y \hookrightarrow \widetilde{Y} \hookrightarrow \widetilde{Y}' \hookrightarrow Y$ are continuous and ${}'Y \subset \widetilde{Y}'$ is dense. Let $\widetilde{\mathbb{B}}^r_w$ denote the ball of radius r in $\widetilde{Y}$, equipped with the weak topology. Assume that the following conditions are satisfied,*

(1) *$u_n \to u$ in $C([0,T];Y)$,*
(2) *$\sup_{t\in[0,T]} \|u_n\|_{\widetilde{Y}} \leq r$.*

Then $u, u_n \in C\left([0,T];\widetilde{\mathbb{B}}^r_w\right)$ and $u_n \to u$ in $C\left([0,T];\widetilde{\mathbb{B}}^r_w\right)$ as $n \to \infty$.

PROOF. We claim that

$$u_n \to u \quad \text{in} \quad C\left([0,T];\widetilde{\mathbb{B}}^r_w\right) \quad \text{as} \quad n \to \infty$$

i.e. that for all $\phi \in \widetilde{Y}'$

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left|\langle u_n(s) - u(s), \phi\rangle_{\widetilde{Y}\times\widetilde{Y}'}\right| = 0.$$

To verify this claim, we fix $\phi \in \widetilde{Y}'$ and $\varepsilon > 0$. Since ${}'Y$ is dense in $\widetilde{Y}'$, there exists $\phi_\varepsilon \in {}'Y$ such that $|\phi - \phi_\varepsilon|_{\widetilde{Y}'} \leq \varepsilon$. Using (2), we infer that for all $s \in [0,T]$ the following estimates hold

$$\begin{aligned}
\left|\langle u_n(s) - u(s), \phi\rangle_{\widetilde{Y}\times\widetilde{Y}'}\right| &\leq \left|\langle u_n(s) - u(s), \phi - \phi_\varepsilon\rangle_{\widetilde{Y}\times\widetilde{Y}'}\right| + \left|\langle u_n(s) - u(s), \phi_\varepsilon\rangle_{Y\times{}'Y}\right| \\
&\leq \|u_n(s) - u(s)\|_{\widetilde{Y}} \|\phi - \phi_\varepsilon\|_{\widetilde{Y}'} + \left|\langle u_n(s) - u(s), \phi_\varepsilon\rangle_{Y\times{}'Y}\right| \\
&\leq \varepsilon \cdot \|u_n - u\|_{L^\infty(0,T;\widetilde{Y})} + \left|\langle u_n(s) - u(s), \phi_\varepsilon\rangle_{Y\times{}'Y}\right| \\
&\leq 2\varepsilon \cdot \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;\widetilde{Y})} + \left|\langle u_n(s) - u(s), \phi_\varepsilon\rangle_{Y\times{}'Y}\right| \\
&\leq 2\varepsilon r + \sup_{s\in[0,T]} \left|\langle u_n(s) - u(s), \phi_\varepsilon\rangle_{Y\times{}'Y}\right| \\
&\leq 2\varepsilon r + \sup_{s\in[0,T]} \|u_n(s) - u(s)\|_Y \|\phi_\varepsilon\|_{{}'Y}.
\end{aligned}$$

Passing to the limit $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \sup_{s\in[0,T]} \left|\langle u_n(s) - u(s), \phi\rangle_{\widetilde{Y}\times\widetilde{Y}'}\right| \leq 2r\varepsilon.$$

Since ε is arbitrary,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left|\langle u_n(s) - u(s), \phi\rangle_{\widetilde{Y}\times\widetilde{Y}'}\right| = 0,$$

for every $\phi \in \widetilde{Y}'$. Since $C\left([0,T];\widetilde{\mathbb{B}}^r_w\right)$ is a complete metric space, we infer that $u \in C\left([0,T];\widetilde{\mathbb{B}}^r_w\right)$ as well. This completes the proof. □

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REMARK 4.0.18. *This result is consistent with lemma 3.2 in* [26]*, where* $\widetilde{Y}$ *is a separable Hilbert space and the embedding* $'Y \hookrightarrow \widetilde{Y} \eqsim \widetilde{Y}'$ *is continuous and dense.*

We will need the previous results in the case where $\widetilde{Y}$ is a separable Hilbert space in a later chapter. Hence we will collect them in the following lemma.

LEMMA 4.0.19. *Let* $B, 'Y, Y$ *be separable Banach spaces,* X *be a reflexive separable Banach space,* $\widetilde{Y}$ *a separable Hilbert space such that* $'Y \hookrightarrow \widetilde{Y} \hookrightarrow Y$ *with compact and dense embeddings and the conditions from Theorem 4.0.14 be satisfied with* $1 < p < \infty$*. Let*

$$Z \coloneqq C([0,T];Y) \cap L^{q-}(0,T;B) \cap L^p(0,T;B) \cap L^p_w(0,T;X) \cap C([0,T];\widetilde{Y}_w).$$

Then a subset $\mathcal{K} \subset \widetilde{Z}$ *satisfying*

(1) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(0,T;X)} \leq C$,
(2) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^q(0,T;B)} \leq C$,
(3) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;\widetilde{Y})} \leq C$,
(4) $\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \sup_{|t-s|\leq\delta, s,t\leq T} \|u_n(t) - u_n(s)\|_Y = 0$,

is compact with respect to the supremum of the individual topologies.

PROOF. Except for the last space in the intersection forming Z, the proof follows from Theorem 4.0.14. We take a sequence in the intersection without $C([0,T];\widetilde{Y}_w)$, use (3) and Lemma 4.0.17 to conclude the proof. □

We now proceed to incorporate the previous results in a statement regarding the tightness of laws of a sequence of stochastic processes, similar to Theorem 4.0.10. We will omit the results obtained in Lemma 4.0.17, as their inclusion is straight forward, but decreases the readability of the following result.

THEOREM 4.0.20. *Let* $\mathcal{O} \subset \mathbb{R}^d$*,* $T > 0$ *and* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of continuous,* $\mathbb{F}$*-adapted,* Y *-valued processes satisfying the following conditions*

(1) $$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\nabla f(u_n)\|^{k_1}_{L^{p_1}(0,T;L^{q_1}(\mathcal{O}))}\right] + \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left\|(f^{-1})'(f(u_n))\right\|^{k_2}_{L^{p_2}(0,T;L^{q_2}(\mathcal{O}))}\right] \leq C,$$
$$\text{for } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \text{ and } \frac{1}{k_1} + \frac{1}{k_2} \leq 1, \text{ where } p, q > 1.$$

(2) $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|u_n\|_{L^1(0,T;L^1(\mathcal{O}))}\right] \leq C.$

(3) $(u_n)_{n\in\mathbb{N}}$ *satisfies Aldous condition* (5.12) *in* Y*.*

We define the space $(Z, \mathcal{T})$ *by*

$$Z \coloneqq \begin{cases} C([0,T];Y) \cap L^p(0,T;L^s(\mathcal{O})) \cap L^p_{w^*}(0,T;W^{1,q}(\mathcal{O})), & \text{if } q < d, \\ C([0,T];Y) \cap L^p(0,T;L^k(\mathcal{O})) \cap L^p_{w^*}(0,T;W^{1,q}(\mathcal{O})), \text{ for some } k < \infty, & \text{if } q = d, \\ C([0,T];Y) \cap L^p(0,T;C(\overline{\mathcal{O}})) \cap L^p_{w^*}(0,T;W^{1,q}(\mathcal{O})), & \text{if } q > d, \end{cases}$$

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where $\mathcal{T}$ corresponds to the supremum of the respective topologies and $s < qd/(d-q)$. Let $\mathbb{P}_n$ be the law of u_n on Z. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of Z such that

$$\mathbb{P}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

If $p = \infty$, the second space in the intersection, $L^p(0,T,\cdot)$, is replaced by the space of continuous functions in time $C([0,T];\cdot)$.

PROOF. By condition (1) and Lemma 4.0.1, we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|\nabla u_n\|_{L^p(0,T;L^q(\mathcal{O}))}\right] \leq C.$$

Condition (2) and the generalized Poincaré inequality 5.4.10 result in

$$\mathbb{E}\left[\|u_n\|_{L^p(0,T;W^{1,q}(\mathcal{O}))}\right] \leq C.$$

Let

$$B^1_{R_1} = \left\{v \in Z : \|v\|_{L^p(0,T;W^{1,q}(\mathcal{O}))} \leq R_1\right\},$$

for $R_1 > 0$ with $C/R_1 \leq \frac{\varepsilon}{2}$.

Furthermore, condition (3) and Lemma 5.6.10 imply that there exists a set $A_{\frac{\varepsilon}{2}} \subset C([0,T];Y)$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n\left(A_{\frac{\varepsilon}{2}}\right) \geq 1 - \frac{\varepsilon}{2}$$

where $\mathbb{P}_n$ is the law of u_n and

$$\lim_{\delta\to 0} \sup_{u_n\in A_{\frac{\varepsilon}{2}}} \sup_{s,t\in[0,T],|t-s|\leq\delta} \|u_n(t) - u_n(s)\|_Y = 0.$$

Let us define

$$\mathcal{K}_\varepsilon := \overline{B^1_{\frac{2C}{\varepsilon}} \cap A_{\frac{\varepsilon}{2}}}.$$

Exactly as in Theorem 4.0.10 one can verify that

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(\mathcal{K}_\varepsilon) \geq 1 - \varepsilon.$$

Next, we want to show that $\mathcal{K}_\varepsilon$ is relatively compact in Z.
We first consider the case $q < d$.
We recall that by the Sobolev Embedding Theorem, $W^{1,q}(\mathcal{O})$ is compactly embedded in $L^s(\mathcal{O})$ for every $s \in \left[1, \frac{qd}{d-q}\right)$. Then, by Theorem 4.0.14, $\mathcal{K}_\varepsilon$ is relatively compact in $C([0,T];Y) \cap L^p_{w^*}(0,T;W^{1,q}(\mathcal{O})) \cap L^p(0,T;L^s(\mathcal{O}))$ for $s \in \left[1, \frac{qd}{d-q}\right)$. Consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}_\varepsilon$. Then we infer that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$u_{n_k} \underset{k\to\infty}{\longrightarrow} u \text{ strongly in } C([0,T];Y) \cap L^p(0,T;L^s(\mathcal{O})) \cap L^p_{w^*}(0,T;W^{1,q}(\mathcal{O})), s \in \left[1, \frac{qd}{d-q}\right).$$

The cases when $q = d$ and $q > d$ follow in an analogous way. The only aspect one needs to recall is that continuous functions on compact sets are separable, in order to conclude that the notions of compactness and sequential compactness coincide. □

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REMARK 4.0.21. *If one wants to treat the case where $p = 1$ and $q \geq 1$, it is possible to do so by dropping the space $L^p_{w^*}(0,T;W^{1,q}(\mathcal{O}))$ completely or replacing (2) by an estimate similar to (4.5) of the form $\left\|\phi(\|u_n\|_{W^{1,q}(\mathcal{O})})\right\|_{L^p(0,T)}$, providing a required equi-integrability.*

In the setting $1 < q < d$, we could incorporate the estimate (4.5) to obtain again a slightly better convergence. The arguments are similar to Corollary 4.0.11.

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CHAPTER 5

Auxiliary results from functional and stochastic analysis

5.1. Functional analysis

Throughout this section, let H be a (real or complex) Hilbert space with the norm $\|\cdot\|_H$ induced by the inner product $\langle\cdot,\cdot\rangle_H$ and E a (real or complex) Banach space with the norm $\|\cdot\|_E$. The results stated in this section are mainly based on [22, 131, 130].

DEFINITION 5.1.1. *We denote by E^* the dual space of E, that is, the space of all continuous linear functionals on E. The (dual) norm on E^* is defined by*

$$\|L\|_{E^*} = \sup_{\substack{\|x\|\leq 1\\ x\in E}} |L(x)| = \sup_{\substack{\|x\|\leq 1\\ x\in E}} L(x).$$

Given $L \in E^*$ and $x \in E$ we shall often write $\langle L, x\rangle$ instead of $L(x)$; we say that $\langle\cdot,\cdot\rangle$ is the scalar product for the duality E^*, E.

THEOREM 5.1.2. *Let $G \subset E$ be a linear subspace. If $f : G \to \mathbb{R}$ is a continuous linear functional, then there exists $g \in E^*$ that extends g and such that*

$$\|f\|_{E^*} = \sup_{\substack{x\in G\\ \|x\|\leq 1}} |g(x)| = \|g\|_{G'}.$$

REMARK 5.1.3. *G is dense in E if and only if the extension is unique.*

DEFINITION 5.1.4. *A sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$, $x_n \to x$, if*

$$L(x_n) \to L(x), \qquad \text{for all } L \in E^*,$$

where E^* denotes the topological dual space of E.

The bidual E^{**} is the dual of E^* with norm

$$\|\xi\|_{E^{**}} = \sup_{\substack{L\in E^*\\ \|L\|_{E^*}\leq 1}} |\langle \xi, L\rangle| \quad (\xi \in E^{**}).$$

There is a canonical injection $J : E \to E^{**}$ defined as follows: given $x \in E$, the map $L \mapsto \langle L, x\rangle$ is a continuous linear functional on E^*. Thus it is an element of E^{**}, which we denote by Jx. We have

$$\langle Jx, L\rangle_{E^{**},E^*} = \langle L, x\rangle_{E^*,E} \quad \forall\, x \in E,\ \forall\, L \in E^*.$$

If J turns out to be surjective then one says that E is reflexive, and E^{**} is identified with E.

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DEFINITION 5.1.5. *We denote the topology of weak topology by $\sigma(E, E^*)$ on E. It is the coarsest topology such that each $L \in E^*$ is a continuous map from E to $\mathbb{R}(\mathbb{C})$.*

PROPOSITION 5.1.6. *Let (x_n) be a sequence in E. Then*

- *$x_n \to x$ weakly in $\sigma(E, E^*) \Leftrightarrow \langle f, x_n\rangle \to \langle f, x\rangle,\ \forall f \in E^*$.*
- *If $x_n \to x$ strongly, then $x_n \to x$ weakly in $\sigma(E, E^*)$.*
- *If $x_n \to x$ weakly in $\sigma(E, E^*)$, then $(\|x_n\|_E)$ is bounded and $\|x\|_E \leq \liminf \|x_n\|_E$.*
- *If $x_n \to x$ weakly in $\sigma(E, E^*)$ and if $f_n \to f$ strongly in E^* (i.e., $\|f_n - f\|_{E^*} \to 0$), then $\langle f_n, x_n\rangle \to \langle f, x\rangle$*

PROPOSITION 5.1.7. *Let (x_n) be a sequence in E^*. Then*

- *$f_n \xrightarrow{\star} f$ in $\sigma(E^*, E) \Leftrightarrow \langle f_n, x\rangle \to \langle f, x\rangle,\ \forall x \in E$.*
- *If $f_n \to f$ strongly, then $f_n \to f$ in $\sigma(E^*, E^{**})$.*
- *If $f_n \to f$ in $\sigma(E^*, E^{**})$, then $f_n \xrightarrow{\star} f$ in $\sigma(E^*, E)$.*
- *If $f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$ then $(\|f_n\|_{E^*})$ is bounded and $\|f\|_{E^*} \leq \liminf \|f_n\|_{E^*}$.*
- *If $f_n \xrightarrow{\star} f$ in $\sigma(E^*, E)$ and if $x_n \to x$ strongly in E, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.*

THEOREM 5.1.8 (Banach-Alaoglu-Bourbaki). *The closed unit ball*

$$B_{E^*} = \{f \in E^*; \|f\|_{E^*} \leq 1\}$$

is compact in the weak* *topology $\sigma(E^*, E)$*

DEFINITION 5.1.9. *We say that a topological space E is separable if there exists a subset $D \subset E$ that is countable and dense.*

Separability properties are closely related to the metrizability of the weak topologies. A topological space X is said to be metrizable if there is a metric on X that induces the topology of X.

THEOREM 5.1.10. *Let E be a separable Banach space. Then B_{E^*} is metrizable in the weak * topology $\sigma(E^*, E)$. Conversely, if B_{E^*} is metrizable in $\sigma(E^*, E)$, then E is separable.*

THEOREM 5.1.11. *Let E be a Banach space such that E^* is separable. Then B_E is metrizable in the weak topology $\sigma(E, E^*)$. Conversely, if B_E is metrizable in $\sigma(E, E^*)$, then E^* is separable.*

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. If $E = F$, we will write $\mathcal{L}(E)$.

THEOREM 5.1.12. *Let F be a Banach space and E a normed space. Then, $\mathcal{L}(E, F)$, with the norm*

$$\|L\|_{\mathcal{L}(E,F)} := \sup_{x \in E, \|x\|_E = 1} \|Lx\|_F, \quad L \in \mathcal{L}(E, F)$$

is a Banach space.

DEFINITION 5.1.13. *A sequence of operators $(L_n)_{n \in N} \subset \mathcal{L}(E, F)$ is said to:*

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- *converge in operator norm to* $L \in \mathcal{L}(E,F)$, *if*
$$\|L_n - L\|_{\mathcal{L}(E,F)} \to 0 \text{ as } n \to \infty,$$
- *strongly converge to* $L \in \mathcal{L}(E,F)$, *iff*
$$L_n x \to Lx,$$
strongly in Y *for each* x *in* E.

THEOREM 5.1.14 (Banach-Steinhaus). *Let* E *and* F *be two Banach spaces and let* $(L_i)_{i\in I}$ *be a family (not necessarily countable) of continuous linear operators from* E *into* F. *Assume that*
$$\sup_{i\in I} \|L_i x\|_F < \infty \quad \forall\, x \in E.$$
Then
$$\sup_{i\in I} \|L_i\|_{\mathcal{L}(E,F)} < \infty.$$

COROLLARY 5.1.15. *Let* E *and* F *be Banach spaces. Let* (L_n) *be a sequence of continuous linear operators from* E *into* F *such that for every* $x \in E, L_n x$ *converges (as* $n \to \infty$ *) to a limit denoted by* Lx. *Then we have*

- $\sup_n \|L_n\|_{\mathcal{L}(E,F)} < \infty$,
- $L \in \mathcal{L}(E,F)$,
- $\|L\|_{\mathcal{L}(E,F)} \le \liminf_{n\to\infty} \|L_n\|_{\mathcal{L}(E,F)}$.

DEFINITION 5.1.16. *Let* E *and* F *be two Banach spaces. An unbounded linear operator from* E *into* F *is a linear map* $A : D(A) \subset E \to F$ *defined on a linear subspace* $D(A) \subset E$ *with values in* F. *The set* $D(A)$ *is called the domain of* A

One says that A is bounded (or continuous) if $D(A) = E$ and if there is a constant $c \ge 0$ such that
$$\|Au\|_F \le c\|u\|_E, \quad \forall\, u \in E.$$
Here are some important definitions and further notation:

- Graph of $A = G(A) = \{[u, Au]; u \in D(A)\} \subset E \times F$,
- Range of $A = R(A) = \{Au; u \in D(A)\} \subset F$,
- Kernel of $A = N(A) = \{u \in D(A); Au = 0\} \subset E$.

DEFINITION 5.1.17. *A map* A *is said to be closed if* $G(A)$ *is closed in* $E \times F$.

Let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined. We shall introduce an unbounded operator $A^* : D(A^*) \subset F^* \to E^*$ as follows. First, one defines its domain:
$$D(A^*) = \{v \in F^*;\ \exists\, c \ge 0 \text{ such that } |\langle v, Au\rangle| \le c\|u\|_E \quad \forall\, u \in D(A)\}.$$
From the Uniform Boundedness Principle and the Closed Graph Theorem, it is clear that $D(A^*)$ is a linear subspace of F^*. We shall now define A^*v. Given $v \in D(A^*)$, consider the map $g : D(A) \to \mathbb{R}$ defined by
$$g(u) = \langle v, Au\rangle \quad \forall\, u \in D(A).$$

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We have

$$|g(u)| \leq c\|u\|_E \quad \forall\, u \in D(A).$$

By Hahn-Banach (analytic form; see Theorem 5.1.2) there exists a linear map $f : E \to \mathbb{R}$ that extends g and such that

$$|f(u)| \leq c\|u\|_E \quad \forall\, u \in E.$$

It follows that $f \in E^*$. Note that the extension of g is unique, since $D(A)$ is dense in E. Set

$$A^* v = f.$$

DEFINITION 5.1.18. *The unbounded linear operator $A^* : D(A^*) \subset F^* \to E^*$ is called the adjoint of A. In brief, the fundamental relation between A and A^* is given by*

$$\langle v, Au\rangle_{F^*,F} = \langle A^* v, u\rangle_{E^*,E} \quad \forall\, u \in D(A), \quad \forall\, v \in D(A^*).$$

THEOREM 5.1.19. *Let E and F be two reflexive Banach spaces. Let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined and closed. Then $D(A^*)$ is dense in F^*. Thus A^{**} is well defined $(A^{**} : D(A^{**}) \subset E^{**} \to F^{**})$ and it may also be viewed as an unbounded operator from E into F. Then we have*

$$A^{**} = A.$$

DEFINITION 5.1.20. *A bounded operator $L \in \mathcal{L}(E, F)$ is said to be compact if $L(B_E)$ has compact closure in F (in the strong topology).*

DEFINITION 5.1.21. *Let E and F be topological spaces. A linear operator $L : E \to F$ is called (sequentially) compact, if for each bounded sequence $(x_n)_{n\in\mathbb{N}} \subset E$, the sequence $(Lx_n)_{n\in\mathrm{N}}$ has a convergent subsequence in F.*

REMARK 5.1.22. *For metric spaces, the notions of compactness and sequential compactness coincide.*

REMARK 5.1.23. *If E is separable, then B_{E^*} (as defined in 5.1.8) is* weak**-metrizable and hence also* weak**-sequentially compact.*

LEMMA 5.1.24. *Let E and F be Banach spaces, then $\mathcal{K}(E, F)$, set of all compact operators from E into F is a closed (and hence complete) subspace of $\mathcal{L}(E, F)$ with respect to the operator norm.*

DEFINITION 5.1.25. *Let H be a separable Hilbert space. A bounded operator $L \in \mathcal{L}(H)$ is called a Hilbert-Schmidt operator if there is a Hilbert basis (e_n) in H such that*

$$\|L\|_{HS}^2 = \sum \|Le_n\|_H^2 < \infty.$$

One can prove that this definition is independent of the basis and that $\|\cdot\|_{HS}$ is a norm. Every Hilbert-Schmidt operator is compact. An alternative notation for the space of Hilbert-Schmidt operators is $\mathcal{L}_2$.

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LEMMA 5.1.26. *Let E, F be Banach spaces and $\iota\colon E \to F'$ denote the continuous embedding of E into F'. Then the mapping $j\colon F \to E'$, defined by*

$$\langle j(y), x\rangle = \langle y, \iota(x)\rangle$$

is continuous. If $\iota(X)$ is weak-dense in F' then j is injective, hence j defines a continuous embedding of F into E'.*

DEFINITION 5.1.27. *An unbounded linear operator $A : D(A) \subset H \to H$ is said to be monotone if it satisfies*

$$(Av, v) \geq 0 \quad \forall\, v \in D(A).$$

It is called maximal monotone if, in addition, $R(I + A) = H$, i.e.,

$$\forall\, f \in H \quad \exists\, u \in D(A) \text{ such that } u + Au = f.$$

PROPOSITION 5.1.28. *Let A be a maximal monotone operator. Then*

- *$D(A)$ is dense in H,*
- *A is a closed operator,*
- *For every $\lambda > 0, (I + \lambda A)$ is bijective from $D(A)$ onto $H, (I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.*

DEFINITION 5.1.29. *Let $A : D(A) \subset H \to H$ be an unbounded linear operator with $\overline{D(A)} = H$ Identifying H^* with H, we may view A^* as an unbounded linear operator in H. One says that*
A is symmetric if $(Au, v) = (u, Av)\ \forall\, u, v \in D(A)$,
A is self-adjoint if $D(A^*) = D(A)$ and $A^* = A$.

PROPOSITION 5.1.30. *Let A be a maximal monotone symmetric operator. Then A is self-adjoint.*

DEFINITION 5.1.31. *Let $L \in \mathcal{L}(E)$. The resolvent set, denoted by $\rho(L)$, is defined by*

$$\rho(L) = \{\lambda \in \mathbb{R}; (L - \lambda I) \text{ is bijective from } E \text{ onto } E\}.$$

The spectrum, denoted by $\sigma(L)$, is the complement of the resolvent set, i.e., $\sigma(T) = \mathbb{R}\backslash\rho(L)$. A real number λ is said to be an eigenvalue of L if it lies in the spectrum. In other words, $\lambda \in \sigma(L)$ it holds that

$$N(L - \lambda I) \neq \{0\},$$

and $N(L - \lambda I)$ is the corresponding eigenspace to the eigenvalue λ.

It is useful to keep in mind that if $\lambda \in \rho(L)$ then $(L - \lambda I)^{-1} \in \mathcal{L}(E)$.

LEMMA 5.1.32. *If $L \in \mathcal{L}(H)$ is self-adjoint, then the eigenvalues of L are real. Moreover, the eigenspaces corresponding to two different eigenvalues are orthogonal.*

DEFINITION 5.1.33. *Let X, Y be two Banach spaces. A sesquilinear form on $X \times Y$ is called a duality product if it satisfies*

$$|\langle u, v\rangle| \leq \|u\|_X \|v\|_Y, \quad u \in X,\ v \in Y,$$
$$\|u\|_X = \sup_{\|v\|_Y \leq 1} |\langle u, v\rangle|, \quad u \in X,$$

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$$\|v\|_Y = \sup_{\|u\|_X \le 1} |\langle u, v\rangle|, \quad v \in Y.$$

When there exists such a duality product between two Banach spaces X, Y, we call Y the adjoint space. Since this relation in symmetric in terms of the duality $\langle v, u\rangle_{Y\times X} = \overline{\langle u, v\rangle}_{X\times Y}$, we call $\{X, Y\}$ an adjoint pair.

REMARK 5.1.34. *As it can be seen in the case of nonreflexive Banach spaces, the adjoint space might not be unique. In the following we will restrict ourselves to reflexive Banach spaces. The results still remain true for nonreflexive spaces, but X^* has to be interpreted as an unspecified adjoint space.*

For the next theorem, we will be inconsistent in our notation, denoting the dual space of X by X' and the adjoint space by X^*.

THEOREM 5.1.35. *Let X be a reflexive Banach space, and let $\{X, X^*\}$ be an adjoint pair with duality product $\langle\cdot,\cdot\rangle$. Then, the isometry $J : X^* \to X'$ defined by*

$$(Jv)(u) = \langle u, v\rangle, \quad u \in X,\ v \in X^*$$

is onto and hence an isomorphism. Consequently, for any $\Phi \in X'$, there exists a unique vector $G \in X^*$ such that $\|\Phi\|_{X'} = \|v\|_{X^*}$ and $\Phi(u) = \langle u, v\rangle$ for all $u \in X$.

Let $Z \subset X \subset Z^*$ be a triplet of spaces. By definition, $\{Z, Z^*\}$ is an adjoint pair. We consider a sesquilinear form $a(u, v)$ defined on $Z \times Z$, or shortly on Z. More precisely, let $a(u, v)$ be a complex-valued function defined for $(u, v) \in Z \times Z$ satisfying

$$\begin{cases} a(\alpha u + \beta\tilde{u}, v) = \alpha a(u, v) + \beta a(\tilde{u}, v), & \alpha, \beta \in \mathbb{C}, u, \tilde{u}, v \in \mathrm{Z} \\ a(u, \alpha v + \beta\hat{v}) = \bar{\alpha} a(u, v) + \bar{\beta} a(u, \hat{v}), & \alpha, \beta \in \mathbb{C}, u, v, \hat{v} \in \mathrm{Z} \end{cases}$$

When $a(u, v)$ satisfies the condition

$$|a(u, v)| \le M\|u\|_Z\|v\|_Z, \quad u, v \in Z, \tag{5.1}$$

with some constant M, we say that $a(u, v)$ is a continuous form. Obviously, (5.1) implies that $a(u_n, v_n) \to a(u, v)$ if $u_n \to u$ and $v_n \to v$ in Z, respectively. Consider a continuous sesquilinear form $a(u, v)$ on Z. For each $u \in Z$, $\overline{a(u,\cdot)}$ is a continuous linear functional on Z.

Therefore, by Theorem 5.1.35, there exists a unique $\phi \in Z^*$ such that $\overline{a(u, v)} = \langle v, \Phi\rangle_{Z\times Z^*}$, for all $v \in Z$, i.e., $a(u, v) = (\Phi, v)$ for all $v \in Z$. The correspondence $A : u \mapsto \Phi$ is obviously a linear operator from Z into Z^*. This operator A is called the linear operator associated with the form $a(u, v)$. Therefore,

$$a(u, v) = \langle Au, v\rangle \quad \text{for all } u \in Z,\ v \in Z.$$

From this we observe that A is a bounded operator satisfying

$$\|Au\|_{Z^*} = \sup_{\|v\|\le 1} |\langle Au, v\rangle| \le M\|u\|_Z, \quad u \in Z.$$

Hence, $\|A\|_{\mathcal{L}(Z,Z^*)} \le M$. When $a(u, v)$ satisfies the condition

$$\operatorname{Re} a(u, u) \ge \delta\|u\|_Z^2, \quad u \in Z, \tag{5.2}$$

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with some positive constant $\delta > 0$, we say that $a(u, v)$ is a coercive form. Clearly, (5.2) implies that $u = 0$ if $a(u, u) = 0$.

The following theorem is well known as the Lax-Milgram theorem. For details and its proof, see [131, Theorem, p. 92] or [43, Theorem 7, p. 368].

THEOREM 5.1.36. *Let $a(u, v)$ be a continuous and coercive sesquilinear form on Z (i.e., (5.1) and (5.2)). Then, for any $\psi \in Z^*$, there exists a unique element $v \in Z$ such that $\psi(u) = a(u, v)$ for $v \in Z$.*

THEOREM 5.1.37. *Let $a(u, v)$ satisfy (5.1) and (5.2) and let A be a linear operator associated with $a(u, v)$. Then, A is an isomorphism from Z onto Z^* with $\delta\|u\|_Z \leq \|Au\|_{Z^*} \leq M\|u\|_Z$ as well and is a densely defined, closed linear operator in Z^*.*

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5.2. Fractional powers of sectorial operators

We now collect further properties of sectorial operators, which can be found in [106, 130].

Let X be a Banach space with norm $\|\cdot\|_X$ and A be a linear sectorial operator of X with angle $0 \le \vartheta_A < \pi$. As before an open sectorial domain Σ_ϑ for $\vartheta_A < \vartheta < \pi$ is given by

$$\Sigma_\vartheta = \{\lambda \in \mathbb{C};\, |\arg \lambda| < \vartheta\}, \quad \vartheta_A < \vartheta < \pi.$$

We define, for each complex number z with $\operatorname{Re} z > 0$, the bounded linear operator

$$A^{-z} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-z} (\lambda - A)^{-1} \mathrm{d}\lambda,$$

using the Dunford integral in $\mathcal{L}(X)$, where Γ is the contour surrounding the spectrum $\sigma(A)$, running counterclockwise in $\mathbb{C} \setminus (\infty, 0] \cap \rho(A)$. If $z = n \in \mathbb{N}$ it can be shown, [130, Chapter 2.7.1], that this definition coincides with the standard definition of $A^{-n} = (A^n)^{-1}$.

A^{-z} is an analytic function for $\operatorname{Re} z > 0$ with values in $\mathcal{L}(X)$. The following theorem [130, Theorem 2.21] is concerned with the convergence of A^{-z} as $z \to 0$.

THEOREM 5.2.1. *For any $0 < \vartheta < \frac{\pi}{2}$, as $z \to 0$ with $z \in \overline{\Sigma_\vartheta} \setminus \{0\}$, A^{-z} converges to* Id *strongly on X.*

It also holds that A^{-z} satisfies the law of exponent, i.e.

$$A^{-z} A^{-z'} = A^{-(z+z')}, \quad \operatorname{Re} z > 0,\ \operatorname{Re} z' > 0,$$

which leads to the following theorem, see [130, Theorem 2.22].

THEOREM 5.2.2. *The $\mathcal{L}(X)$-valued function A^{-z} is an analytic semigroup defined in the half-plane $\{z \in \mathbb{C}; \operatorname{Re} z > 0\}$.*

The fractional power A^α, for every real number $-\infty < \alpha < \infty$ is defined by $A^\alpha := (A^{-\alpha})^{-1}$*, see [130, Chapter 2.7.2] for details. The following theorem lists some of the properties of the fractional power A^α, *cf.* [130, Theorem 2.22, 2.23].

THEOREM 5.2.3. *Let A be a sectorial operator on X with angle ϑ_A. Then*

1. *for $-\infty < \alpha < 0$, A^α are bounded operators on X. $A^0 = \mathrm{Id}$ on X and A^α are densely defined, closed linear operators of X for $\alpha > 0$.*
2. *Let $0 \le \alpha_1 < \alpha_2 < \infty$, then $D(A^{\alpha_2}) \subset D(A^{\alpha_1})$.*
3. *A^α satisfies the law of exponent, i.e.*
$$A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}, \quad -\infty < \alpha, \beta < \infty.$$
4. *For $0 < \alpha < 1$, A^α is a sectorial operator on X with angle $\le \alpha \vartheta_A$.*

ASSUMPTION 5.2.4. *For $\vartheta \in (\frac{\pi}{2}, \pi)$, let Σ_ϑ be an open sectorial domain, i.e.*

$$\Sigma_\vartheta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \vartheta\}. \tag{5.3}$$

(A1) *A is a sectorial operator on X, i.e., there exists a $\vartheta \in (\frac{\pi}{2}, \pi)$, such that for every $t \in [0, T]$,*
$$\Sigma_\vartheta \cup \{0\} \subset \rho(A).$$

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(A2) *The resolvent operator* $(\lambda \mathrm{Id} - A)^{-1}$ *satisfies the Hille-Yoshida condition, i.e., there exists a constant* $M \geq 1$ *such that for every* $(t, \omega) \in [0, T] \times \Omega$,

$$\left\|(\lambda \mathrm{Id} - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda| + 1}, \quad \text{for } \lambda \in \rho(A).$$

Let $0 < \alpha < 1$ and A^α be the fractional powers of A. Let M_π be the constant appearing in Assumption (A2) with angle $\vartheta = \pi$. Let us introduce the spaces

$$D_\alpha(A) = \left\{ \mathrm{v} \in X \colon \sup_{0<\rho<\infty} \rho^\alpha \|A(\rho + A)^{-1}\mathrm{v}\|_X < \infty \right\}, \quad 0 \leq \alpha \leq 1.$$

$D_\alpha(A)$ are normed spaces, equipped with the norms

$$\|\mathrm{v}\|_{D_\alpha(A)} = \sup_{0<\rho<\infty} \rho^\alpha \|A(\rho + A)^{-1}\mathrm{v}\|_X.$$

In the following we compare the domain of fractional powers of the operator A to $D_\alpha(A)$ [130, Theorem 2.24].

THEOREM 5.2.5. *For any* $0 < \alpha < 1, D(A^\alpha) \subset D_\alpha(A)$, *and the estimate*

$$\|\mathrm{v}\|_{D_\alpha(A)} \leq C\left(1 + M_\pi\right)^2 \|A^\alpha \mathrm{v}\|_X, \quad \mathrm{v} \in D(A^\alpha)$$

holds true. Conversely, for any $0 < \alpha < \alpha' < 1, D_{\alpha'}(A) \subset D(A^\alpha)$, *and the estimate*

$$\|A^\alpha \mathrm{v}\|_X \leq C_{\alpha,\alpha'} \left[\|\mathrm{v}\|_{D_{\alpha'}(A)} + (1 + M_\pi)\|\mathrm{v}\|_X\right], \quad \mathrm{v} \in D_{\alpha'}(A),$$

holds true.

Next, we compare domains of fractional powers of two sectorial operators A and B of X for which $D(A) \subset D(B)$ continuously, i.e. there exists a constant $C > 0$ such that

$$\|B\mathrm{v}\|_X \leq C\|A\mathrm{v}\|_X, \quad \mathrm{v} \in D(A).$$

THEOREM 5.2.6. [130, Theorem 2.25] *Let* A *and* B *be two sectorial operators of* X *satisfying the above relationship between their domains, as well as assumptions* (A1) *and* (A2). *Then, for any* $0 < \alpha < \alpha' < 1$, $D(A^{\alpha'}) \subset D(B^\alpha)$ *and the estimate*

$$\|B^\alpha \mathrm{v}\|_X \leq C_{\alpha',\alpha} \|A^{\alpha'} \mathrm{v}\|_X, \quad \mathrm{v} \in D(A^{\alpha'})$$

holds true, where $C_{\alpha,\alpha'} > 0$ *is determined by* α, α', M_π *and* C.

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5.3. Stochastic preliminaries

This sections results can be found in [16, 40, 93, 111, 114].
We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for this section and assume that H is a separable Hilbert space with the norm $\|\cdot\|_H$ and E be a separable Banach space with norm $\|\cdot\|_E$.

DEFINITION 5.3.1. *The pair $(\Omega, \mathcal{F})$ is called a measurable space.*

DEFINITION 5.3.2. *Let $(\Omega_1, \mathcal{F})$ and $(\Omega_2, \mathcal{G})$ be two measurable spaces. A map $X : \Omega_1 \to \Omega_2$ is said to be measurable if for every $A \in \mathcal{G}$*

$$X^{-1}(A) = \{\omega \in \Omega_1 : X(\omega) \in A\} \in \mathcal{F}.$$

We call such a function X, a random variable on Ω_1.

DEFINITION 5.3.3. *The smallest σ -algebra containing all open (or closed) subsets of X is called the Borel σ-algebra of X and is denoted by $\mathcal{B}(X)$.*

PROPOSITION 5.3.4. *$\mathcal{B}(E)$ is the smallest σ-algebra of subsets of E containing all sets of the form*

$$\{x \in E : \varphi(x) \leq \alpha\}, \quad \varphi \in E^*, \ \alpha \in \mathbb{R}.$$

REMARK 5.3.5. *The separability of E was important in the previous proposition. Without it, one only has the inclusion $\mathcal{B}([E]_w) \subset \mathcal{B}(E)$, where $\mathcal{B}([E]_w)$ is the σ-algebra generated by $\{x \in E : \varphi(x) \leq \alpha\}$, $\varphi \in E^*$, $\alpha \in \mathbb{R}$.*

DEFINITION 5.3.6. *Let $(\Omega, \mathcal{F})$ be a measurable space. A mapping $X : \Omega \to E$ is said to be Borel measurable if for each $A \in \mathcal{B}(E), X^{-1}(A) \in \mathcal{F}$. Such a Borel measurable map is called an E-valued random variable on Ω.*

DEFINITION 5.3.7. *Let $(\Omega, \mathcal{F})$ be a measurable space and X be an E-valued random variable (on Ω). The smallest σ-algebra $\sigma(X)$ containing all sets $X^{-1}(A), A \in \mathcal{B}(X)$, is called the σ-algebra generated by X.*

LEMMA 5.3.8. *Let $(\Omega, \mathcal{F})$ be a measurable space. Assume that X and Y are E-valued random variables (on Ω). Then*

- *for $\alpha, \beta \in \mathbb{R}, \alpha X + \beta Y$ is an E-valued random variable on Ω,*
- *the mapping $\Omega \ni \omega \mapsto \|X(\omega)\|_E$ is a real-valued random variable on Ω.*

THEOREM 5.3.9 (Lebesgue's Monotone Convergence Theorem). *Let $(f_n)_{n\in\mathrm{N}}$ be a sequence of real valued measurable functions on Ω, satisfying*

- *$0 \leq f_1 \leq f_2 \leq \ldots \leq \infty$,*
- *$f_n \to f$ for almost every ω, as $n \to \infty$.*

Then f is measurable and

$$\int_\Omega f_n \, \mathrm{d}\mu \to \int_\Omega f \, \mathrm{d}\mu \text{ as } n \to \infty.$$

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THEOREM 5.3.10 (Fatous's Lemma). *If $f_n : \Omega \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$, then*

$$\int_\Omega \left(\liminf_{n\to\infty} f_n \right) \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu,$$
$$\limsup_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu \leq \int_\Omega \left(\limsup_{n\to\infty} f_n \right) \mathrm{d}\mu.$$

LEMMA 5.3.11 (Lebesgue's dominated convergence theorem for L^p spaces). *Let $1 \leq p < \infty$ and μ be a finite measure. Suppose $(f_n)_{n\in\mathbb{N}}$ be the sequence of real valued measurable functions on Ω such that $f_n \to f$ in measure as $n \to \infty$. If there exists a function $g \in L^p(\mu)$ such that $|f_n(\omega)| \leq g(\omega)$ for every $\omega \in \Omega$ then $f \in L^p(\mu)$ and*

$$\lim_{n\to\infty} \left(\int_\Omega |f_n - f|^p \, \mathrm{d}\mu \right)^{1/p} = 0.$$

DEFINITION 5.3.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a positive measure space. A set $\Phi \subset L^1(\mu)$ is called uniformly integrable, if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that*

$$\int_U |f| \, \mathrm{d}\mu < \varepsilon,$$

whenever $f \in \Phi$ and $\mu(U) < \delta$.

THEOREM 5.3.13 (Vitali Convergence Theorem). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega, \mathcal{F}, \mu), f \in L^p(\Omega, \mathcal{F}, \mu)$, with $1 \leq p < \infty$. Then, $f_n \to f$ in L^p if and only if*

(1) *f_n converge in measure to f.*
(2) *For every $\varepsilon > 0$, there exists a measurable set U_ε with $\mu(U_\varepsilon) < \infty$ such that for every $V \in \mathcal{F}$ disjoint from U_ε we have, for every $n \in \mathbb{N}$ $\int_V |f_n|^p \, \mathrm{d}\mu < \varepsilon^p$.*
(3) *For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, if $U \in \mathcal{F}$ and $\mu(U) < \delta(\varepsilon)$ then, for every $n \in \mathbb{N}$ we have $\int_U |f_n|^p \, \mathrm{d}\mu < \varepsilon^p$.*

If $\mu(\Omega) < \infty$, the second condition is trivially satisfied. Condition (1) and (3) imply uniform integrability and uniform integrability by itself already implies condition (3).

DEFINITION 5.3.14. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be an E-valued random variable on Ω. Then a mapping $\mathcal{L}(X) : \mathcal{B}(E) \to [0, 1]$ defined by*

$$\mathcal{L}(X)(A) = \mathbb{P}\left(X^{-1}(A)\right) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}), \quad A \in \mathcal{B}(E),$$

is called the law (or the distribution) of X.

A proof and further discussion can be found in [105].

PROPOSITION 5.3.15 (Change of variables). *Let X be a random variable with values in a separable Banach space E and denote it's law by $\mu = \mathcal{L}(X)$. Additionally, let F be continuous mapping between separable Banach spaces E and G (equipped with their Borel*

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sigma-algebras) which is Bochner integrable with respect to μ. Then $F(X)$ is Bochner integrable with respect to $\mathbb{P}$ and the following formula holds

$$\int_\Omega F(X)\,\mathrm{d}\mathbb{P} = \int_E F(u)\mu(\,\mathrm{d}u),$$

where both integrals are interpreted as Bochner integrals.

DEFINITION 5.3.16. *An E-valued continuous-time stochastic process $(X_t)_{t\in\mathbb{T}}$ is a family of E-valued random variables indexed by time t. $\mathbb{T}$ will either denote $\mathbb{T} := [0,T]$ or $\mathbb{T} := [0,\infty)$. For each $\omega \in \Omega$, the map*

$$X(\omega) : t \to X_t(\omega) \in E$$

is called a path (or trajectory) of the process X.

We will use the notation X instead of $\{X(t)\}_{t\in\mathbb{T}}$ for simplicity. And throughout this section we will assume that X is an E-valued stochastic process on $\mathbb{T}$ unless specified otherwise.

DEFINITION 5.3.17. *A process X is called continuous if $\mathbb{P}$ -a.s. the trajectories of X are continuous on $\mathbb{T}$, i.e. there exists $\bar{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for each $\omega \in \Omega$, the mapping $\mathbb{T} \ni t \to X(t,\omega) \in E$ is continuous.*

DEFINITION 5.3.18. *An E-valued stochastic process $\widetilde{X}$ on T is called a modification (or version) of the process X if*

$$\mathbb{P}\left(\{\omega \in \Omega\colon X(t,\omega) \neq \widetilde{X}(t,\omega)\}\right) = 0.$$

DEFINITION 5.3.19. *A process X is called stochastically continuous or continuous in probability at $t_0 \in [0,T]$ if for every $\varepsilon, \delta > 0$, there exists a $\rho > 0$ such that for each $t \in [t_0 - \rho, t_0 + \rho] \cap [0,T]$,*

$$\mathbb{P}\left(\omega \in \Omega : \|X(t,\omega) - X(t_0,\omega)\|_E \geq \epsilon\right) \leq \delta.$$

THEOREM 5.3.20 (Kolmogorov continuity criterion). *Let $X(t), t \in [0,T]$, be a stochastic process with values in a complete metric space (E,ρ) such that for some constants $C > 0, \varepsilon > 0, \delta > 1$ and all $t,s \in [0,T]$*

$$\mathbb{E}\left((\rho(X(t),X(s)))^\delta\right) \leq C|t-s|^{1+\varepsilon}.$$

Then there exists a version of X with $\mathbb{P}$ -almost all trajectories being Hölder continuous functions with an arbitrary exponent smaller than $\frac{\varepsilon}{\delta}$. In particular X has a continuous version.

THEOREM 5.3.21. *Assume that E is a separable Banach space, $\mathcal{O} \in \mathbb{R}^d$ is regular and that for some constants $C > 0, \varepsilon > 0, \delta > 1$ and all $\xi, \eta \in \mathcal{O}$*

$$\mathbb{E}\left(\|X(\xi) - X(\eta)\|_E^\delta\right) \leq C|\xi - \eta|^{d+\varepsilon}.$$

Then, there exists a modification of X which has Hölder continuous realizations with an arbitrary exponent smaller than $\frac{\varepsilon}{\delta}$.

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DEFINITION 5.3.22. *An increasing family of σ -algebras $\{\mathcal{F}_t\}, t \in \mathbb{T}$, such that for all t, $\mathcal{F}_t \subset \mathcal{F}$ is called a filtration, if $\mathcal{F}_s \subset \mathcal{F}_t$ for any $0 \leq s \leq t < \infty$.*
Denote by $\mathcal{F}_{t+}$ the intersection of all $\mathcal{F}_s$ where $s > t$.
The filtration is said to be normal if

- *$\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$,*
- *$\mathcal{F}_t = \mathcal{F}_t+$ for all $t \in T$.*

We will denote the family $(\mathcal{F}_t)_{t\in\mathbb{T}}$ by $\mathbb{F}$.

DEFINITION 5.3.23. *If for arbitrary $t \in \mathbb{T}$ the random variable $X(t)$ is $\mathcal{F}_t$ -measurable then the process X is said to be adapted (to the family $\mathcal{F}_t$).*

LEMMA 5.3.24. *Limits of progressively measurable processes are progressively measurable.*

DEFINITION 5.3.25. *A random variable $\tau : \Omega \to [0, \infty]$ i.e. a random time, is a stopping time (w.r.t filtration $\mathbb{F}$) if for all $t \in \mathbb{T}$*

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

REMARK 5.3.26. *One can easily see that if τ and σ are two stopping times then $\tau \wedge \sigma, \tau \vee \sigma$ and $\tau + \sigma$ are also stopping times.*

REMARK 5.3.27. *Every stopping time τ is $\mathcal{F}_\tau$ -measurable, where*

$$\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } t \in \mathbb{T}\}$$

is a σ-algebra.

From now on, we will use the notation $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ to describe a filtration.

DEFINITION 5.3.28. *An H -valued process X is called an $\mathbb{F}$ -martingale if*

(1) *X is adapted to $\mathbb{F}$,*
(2) *for each $t \in [0, T]$, $\mathbb{E}(\|X(t)\|_H) < \infty$,*
(3) *for each $t, s \in [0, T]$ with $t \geq s$,*

$$\mathbb{E}(X(t)|\mathcal{F}_s) = X(s), \text{ i.e. } \forall A \in \mathcal{F}_s \quad \int_A X(t)\, \mathrm{d}\mathbb{P} = \int_A X(s)\, \mathrm{d}\mathbb{P}.$$

LEMMA 5.3.29. *Let $\mathcal{M}_T^2$ be the space of all H-valued, continuous and square integrable martingales X on $[0, T]$. Then $\mathcal{M}_T^2$ is a Banach space with respect to the following norm*

$$\|X\|_{\mathcal{M}_T^2} = \left(\mathbb{E} \sup_{t\in[0,T]} \|X(t)\|_H^2\right)^{\frac{1}{2}}, \quad X \in \mathcal{M}_T^2.$$

THEOREM 5.3.30 (Burkholder-Davis-Gundy (BDG) inequality). *Let $1 \leq p < \infty$, then for all H-valued continuous martingales M with $M_0 = 0$ and stopping time τ, there exist positive constants c_p and C_p such that*

$$\mathbb{E}\left[\langle M(\tau)\rangle^{p/2}\right] \leq \mathbb{E}\left(\sup_{0\leq t\leq\tau} \|M_t\|\right)^p \leq C_p \mathbb{E}\left[\langle M(\tau)\rangle^{p/2}\right], \tag{5.4}$$

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where $\langle M \rangle$ denotes the quadratic variation of M.

DEFINITION 5.3.31. *The product (Borel) sigma-algebra on $E^{\mathbb{T}}$ is denoted by $\mathcal{B}\left(E^{\mathbb{T}}\right)$, and is the one generated by the coordinate maps*

$$X \mapsto X(t) \text{ for fixed } t \in \mathbb{T},$$

via

$$\sigma(\{X : X(t) \in U\} : t \in \mathbb{T},\ U \subset \mathcal{B}(E)).$$

It is referred to as the cylindrical σ-algebra.
We call sets of the form $\{X : (X(t_1), \ldots, X(t_n)) \in A\}$ for $t_i \in \mathbb{T}$ and $A \subset \mathcal{B}(E)$ cylinder or cylindrical sets.

DEFINITION 5.3.32. *The probability measure $\mathbb{Q} := \mathbb{P} \circ X^{-1} \in \mathcal{P}\left(E^{\mathbb{T}}\right)$ is called distribution of X.*

DEFINITION 5.3.33. *On the separable Banach space E, we introduce the cylindrical σ -algebra $\mathcal{E}(E)$ as the σ -algebra generated by the cylindrical sets C, i.e*

$$C = \{x \in X : (x_1^*(x), \ldots, x_n^*(x)) \in A\},$$

where $x_1^*, \ldots, x_n^* \in E^*$ and $A \in \mathcal{B}(\mathbb{R}^n)$, called a base of C. We will also use the notation $\mathcal{E}(E) = \mathcal{E}(E, E^*)$.

THEOREM 5.3.34. *If F is a separable Fréchet space, then $\mathcal{E}(F) = \mathcal{B}(F)$. Moreover, there is a countable family $U \subset F^*$ separating the points in F (i.e., such that for every pair of points $x \neq y \in F$ there is $u \in U$ such that $u(x) \neq u(y)$) such that $\mathcal{E}(F) = \mathcal{E}(F, U)$.*

DEFINITION 5.3.35. *Let E be a Banach space. A probability measure μ on $(E, \mathcal{B}(E))$ is said to be Gaussian if $\mu \circ f^{-1}$ is a Gaussian measure in $\mathbb{R}$ for every $f \in E^*$. The measure μ is called centred (or symmetric) if all the measures $\mu \circ f^{-1}$ are centred and it is called nondegenerate if for any $f \neq 0$ the measure $\mu \circ f^{-1}$ is nondegenerate.*

REMARK 5.3.36. *Notice that if $f \in E^*$ then $f \in L^p(E, \mu)$ for every $p \geq 1$. Indeed, the integral*

$$\int_E |f(x)|^p \mu(\mathrm{d}x) = \int_{\mathbb{R}} |y|^p \left(\mu \circ f^{-1}\right)(\mathrm{d}y)$$

is finite since $\mu \circ f^{-1}$ is Gaussian in $\mathbb{R}$.

THEOREM 5.3.37. *A Borel probability measure μ on E is Gaussian if and only if its characteristic function is given by*

$$\hat{\mu}(f) = \int_E \exp(if(x))\mu(\mathrm{d}x) = \exp\left\{im(f) - \frac{1}{2}C(f, f)\right\}, \quad f \in E^*,$$

where m is a linear functional on E^* and C is a nonnegative symmetric bilinear form on E^*.

DEFINITION 5.3.38. *Let H be a separable Hilbert space. An H-valued stochastic process W on $[0, \infty)$ is said to be Gaussian if, for arbitrary $n \in \mathbb{N}$ and arbitrary positive numbers $t_1, t_2, \ldots, t_n$, the H^n-valued random variable $(W(t_1), W(t_2), \ldots, W(t_n))$ is Gaussian.*

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DEFINITION 5.3.39. *A real valued Wiener process (Brownian motion) is an $\mathbb{R}$-valued stochastic process defined for $t \in [0, \infty)$ such that*

- *$W(0) = 0$ a.s.,*
- *the sample paths $t \mapsto W(t)$ are almost surely continuous,*
- *for $0 \leq s \leq t < \infty$, $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$,*
- *for $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$, the increments*

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$$

are independent.

THEOREM 5.3.40 (Levy characterization). *An adapted $\mathbb{R}$-valued stochastic process $(W(t))_{t \geq 0}$ is a Brownian motion, if and only if $W(0) = 0$ a.s. and W is a continuous martingale with quadratic variation $\langle W \rangle_t = t$.*

DEFINITION 5.3.41. *We call $W(t) = \left(W^1(t), W^2(t), \cdots, W^d(t)\right)$, a d-dimensional Wiener process if $W^1(t), \cdots, W^d(t)$ are independent $\mathbb{R}$ -valued Wiener processes.*

Let U be a Hilbert space with norm $\|\cdot\|_U$ and $Q \in \mathcal{L}(U)$ be a symmetric non-negative operator. We also assume that $\operatorname{Tr} Q < \infty$. Then there exists a complete orthonormal basis $\{e_k\}$ in U, and a bounded sequence of non-negative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \ldots.$$

Without loss of generality, we may assume that Q is injective.

DEFINITION 5.3.42. *A U-valued stochastic process $W(t), t \geq 0$ is called a Q -Wiener process if*

- *$W(0) = 0$ a.s.*
- *W has a.s. continuous trajectories,*
- *W has independent increments,*
- *$\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t - s)Q), 0 \leq s \leq t$.*

If a process $W(t), t \in [0, T]$ satisfies these properties for $t, s \in [0, T]$ then we say that W is a Q -Wiener process on $[0, T]$.

LEMMA 5.3.43. *Let W be a Q-Wiener process on U with $\operatorname{Tr}(Q) < \infty$. Then the following statements hold:*

- *W is a Gaussian process on U and*

$$\mathbb{E}(W(t)) = 0, \quad \operatorname{Cov}(W(t)) = tQ, \quad t \geq 0.$$

- *For arbitrary t,*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j \quad a.e.$$

where

$$\beta_j(t) = \frac{1}{\lambda_j} \langle W(t), e_j \rangle_{H_0}, \quad j = 1, 2, \ldots$$

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are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ *and the series is convergent in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

PROPOSITION 5.3.44. *For any trace class symmetric non-negative operator* Q *on a separable Hilbert space* U*, there exists a* Q*-Wiener process* $(W(t))_{t\geq 0}$.

THEOREM 5.3.45. *A martingale* $M \in \mathcal{M}_T^2(U)$ *with* $M(0) = 0$*, is a* Q*-Wiener process on* $[0, T]$ *adapted to the filtration* $\mathcal{F}_t$, $t \geq 0$ *and with increments* $M(t) - M(s)$, $t \geq s$*, independent of* $\mathcal{F}_s$, $s \geq 0$*, if and only if the quadratic variation satisfies* $\langle M\rangle_t = tQ$, $t \geq 0$.

PROPOSITION 5.3.46. *Let* $(e_i)_{i\in\mathbb{N}}$ *be an orthonormal basis of* U*. Let* $\hat{U}$ *be a Hilbert space such that* $U_0 = Q^{1/2}(U)$ *is embedded into* $\hat{U}$ *with a Hilbert-Schmidt embedding* J*. Then the formula*

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j \beta_j(t), \quad t \geq 0$$

defines a $\hat{U}$ *-valued Wiener process. Moreover, if* Q_1 *is the covariance of* W *then the spaces* $Q_1^{1/2}\left(\hat{U}\right)$ *and* $Q^{1/2}(U)$ *are identical.*

This way of embedding U into a larger space $\hat{U}$ in a Hilbert-Schmidt way can also be used to define the notion of a process which is white-in-time and white in the Hilbert space $\hat{U}$ and which is called a cylindrical Wiener process.

REMARK 5.3.47. *What one actually does here is to find a space* $\hat{U}$*, such that* U *acts as the Cameron-Martin space for some Gaussian measure on* $\hat{U}$*. For a definition of the Cameron-Martin space and further discussion we refer to* [16, 40, 114].

DEFINITION 5.3.48. *With* U *and* $\hat{U}$ *defined as above with Hilbert-Schmidt embedding* J*. Then a* JJ^* *- Wiener process in* $\hat{U}, W = \{W(t) : t \in [0, 1]\}$ *is called a cylindrical Wiener process on* U *and satisfies for* h_1, h_2 *in* $\hat{U}$ *and* $s, t \in [0, 1]$

$$\mathbf{E}\left\langle W(s), h_1\right\rangle_{\hat{U}} \left\langle W(t), h_2\right\rangle_{\hat{U}} = \min\{s, t\}\left\langle J^* h_1, J^* h_2\right\rangle_U.$$

It is important to remark that a cylindrical Wiener process W on U does not actually take values in U. This is the reason for embedding U into a larger Hilbert space $\hat{U}$. However in many ways the statistics of $W(t)$ don't depend on the space $\hat{U}$. Indeed we can write

$$W(t) = \sum_{k\in\mathbb{N}} e_k W_k(t),$$

where $\{W_k(t)\}_{k\in\mathbb{N}}$ are a sequence of iid Wiener processes on $\mathbb{R}$ and $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for U and the convergence happens in $\hat{U}$. However, the formula does not explicitly depend on $\hat{U}$ and the convergence clearly holds in any other Hilbert space U'' such that U embeds into U'' via a Hilbert-Schmidt inclusion.

DEFINITION 5.3.49. *A process* $\Phi = \{\Phi(t) : t \in \mathbb{R}_+\}$ *with values in* $\mathcal{L}_2(U_0, H)$ *is called an elementary process if there exists a sequence of times* $\{t_i\}_{i=1}^n \subset \mathbb{R}_+$ *with* $t_0 < \ldots < t_n$*, and*

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a collection of $\mathcal{L}_2(U_0,H)$ -valued random variables $\{\Phi_k\}_{k=0}^{n-1}$ with Φ_k being $\mathcal{F}_{t_k}$ measurable and $\mathbf{E}\,\|\Phi_k\|^2_{\mathcal{L}_2(U_0,H)} < \infty$ such that Φ can be written as

$$\Phi(t) = \sum_{k=1}^{n} \Phi_k \mathbb{1}_{(t_{k-1},t_k]}(t).$$

DEFINITION 5.3.50. *For elementary processes Φ we define the stochastic integral by the formula*

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) = \sum_{m=0}^{k-1} \Phi_m \left(W_{t_{m+1}\wedge t} - W_{t_m\wedge t}\right),$$

and denote it by $\Phi\cdot W(t), t\in[0,T]$.

DEFINITION 5.3.51. *Let $\Phi(t), t\in[0,T]$, be a measurable $\mathcal{L}_2(U_0,H)$-valued process; we define the norms*

$$\begin{aligned}\|\Phi\|_t &= \left[\mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}s\right]^{1/2}\\ &= \left[\mathbb{E}\int_0^t \mathrm{Tr}\left[\left(\Phi(s)Q^{1/2}\right)\left(\Phi(s)Q^{1/2}\right)^*\right]\,\mathrm{d}s\right]^{1/2},\end{aligned}$$

for $t\in[0,T]$.

PROPOSITION 5.3.52. *If a process Φ is elementary and $\|\Phi\|_T < \infty$ then the process $\Phi\cdot W$ is a continuous, square integrable H -valued martingale on $[0,T]$ and*

$$\mathbb{E}|\Phi\cdot W(t)|^2 = \|\Phi\|_t^2, \quad 0\le t\le T.$$

PROPOSITION 5.3.53. *The following statements hold.*

- *If a mapping Φ from $[0,T]\times\Omega$, into $\mathcal{L}(U,H)$ is $\mathcal{L}(U,H)$ -predictable then Φ is also $\mathcal{L}_2(U_0,H)$ -predictable. In particular, elementary processes are $\mathcal{L}_2(U_0,H)$ -predictable.*
- *If Φ is an $\mathcal{L}_2(U_0,H)$ -predictable process such that $\|\Phi\|_T<\infty$ then there exists a sequence $\{\Phi_n\}$ of elementary processes such that $\|\Phi-\Phi_n\|_T\to 0$ as $n\to\infty$.*

REMARK 5.3.54. *We are able now to extend the definition of stochastic integral to all $\mathcal{L}_2$-predictable processes Φ such that $\|\Phi\|_T < +\infty$. These processes form a Hilbert space denoted by $\mathcal{N}_W^2(0,T;\mathcal{L}_2(U_0,H))$, more simply $\mathcal{N}_W^2(0,T)$ or $\mathcal{N}_W^2$, and, by the previous proposition, elementary processes form a dense set in $\mathcal{N}_W^2(0,T)$.*

THEOREM 5.3.55. *The stochastic integral is defined for all $\Phi\in\mathcal{N}_W(0,T;\mathcal{L}_2(U_0,H))$.*

THEOREM 5.3.56. *The quadratic covariation of a stochastic integral is given by*

$$\left\langle \int_0 X_s\,\mathrm{d}W_s\right\rangle_t = \int_0^t \|X_s\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}s.$$

We will need the following Lemma from [46].

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LEMMA 5.3.57. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space, H a separable Hilbert space. Consider a sequence of stochastic bases $\mathcal{S}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_t > 0, \mathbb{P}, W^n)$, that is a sequence so that each W^n is cylindrical Brownian motion (over U) with respect to $\mathcal{F}_t^n$. Assume that $\{\sigma^n\}_{n\geq 1}$ are a collection of H -valued $\mathcal{F}_t^n$ predictable processes such that $\sigma^n \in L^2([0,T], L_2(U,H))$ a.s. Finally consider $\mathcal{S} = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W\right)$ and $G \in L^2([0,T], L_2(U,H))$, which is $\mathcal{F}_t$ predictable. If*

$$W^n \to W \quad \text{in probability in } C\left([0,T], \hat{U}\right),$$
$$\sigma^n \to \sigma \quad \text{in probability in } L^2([0,T]; L_2(U,H)),$$

then

$$\int_0^t \sigma^n \, \mathrm{d}W^n \to \int_0^t \sigma \, \mathrm{d}W \quad \text{in probability in } L^2([0,T]; H).$$

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5.4. Inequalities and embedding results

DEFINITION 5.4.1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$, $p \geq 1$, and let $s = k + \sigma$, where $k = [s]$ is an integer and $0 < \sigma < 1$. Then we can define the fractional Sobolev spaces $W^{s,p}(\mathcal{O})(= B^s_{p,p}(\mathcal{O}))$ as $\{f \in W^{k,p}(\mathcal{O})\colon \sup_{|\alpha|=k} \|D^\alpha f\|_{\dot{W}^{s,p}(\mathcal{O})} < \infty\}$, where $\|f\|_{\dot{W}^{s,p}(\mathcal{O})}$ is defined by*

$$\|u\|_{\dot{W}^{s,p}(\mathcal{O})} = \left[\iint_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p\sigma}}\,\mathrm{d}x\,\mathrm{d}y\right]^{\frac{1}{p}}.$$

REMARK 5.4.2. *For $0 < s_0 < s_1 < \infty$, $W^{s_1,p}(\mathcal{O}) \subset W^{s_0,p}(\mathcal{O}) \subset L^p(\mathcal{O})$ with continuous embeddings.*

According to [133, Theorem 2.5.1, Remark 2.5.2], (see also [124, Theorem 2.8 .1 / Remark 2 and Theorem 4.6 .1]), we get the following embedding theorem.

THEOREM 5.4.3. *Let $\mathcal{O}$ be $\mathbb{R}^d, \mathbb{R}^d_+$, or a bounded domain with Lipschitz boundary. Let $1 < p < \infty$ and $0 \leq s < \infty$.*

- *If $0 \leq s < \frac{d}{p}$, then $W^{s,p}(\mathcal{O}) \subset L^r(\mathcal{O})$ with continuous embedding where $r \leq \frac{pd}{d-ps}$. Moreover the embedding is compact if $s > 0$ and $r < \frac{pd}{d-ps}$.*
- *If $s = \frac{d}{p}$, then $W^{\frac{d}{p},p}(\mathcal{O}) \subset L^r(\mathcal{O})$ with compact embedding where r is any finite number such that $p \leq r < \infty$.*
- *If $s > \frac{d}{p}$, then $W^{s,p}(\mathcal{O})$ is continuously embedded in $C^{s-\lfloor d/p\rfloor-1,\nu}(\overline{\mathcal{O}})$, where $\lfloor\cdot\rfloor$ denotes the integer part and*

$$\nu = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p} \text{ if } \frac{d}{p} \notin \mathbb{Z} \\ \text{arbitrary positive number in } (0,1) \text{ if } \frac{d}{p} \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < \left[\frac{d}{p}\right] + 1 - \frac{d}{p}$.

THEOREM 5.4.4 (Morrey's inequality). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ open, bounded and suppose $\partial\mathcal{O}$ is C^1. Assume $u \in W^{1,p}(U)$ $n < p \leq \infty$. Then there exists a function $u^* \in C^{0,\gamma}(\bar{\mathcal{O}})$ for $\gamma = 1 - \frac{d}{p}$, such that $u = u^*$ for a.e. $x \in \mathcal{O}$ and the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{\mathcal{O}})} \leq C\|u\|_{W^{1,p}(\mathcal{O})},$$

holds.

THEOREM 5.4.5. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain. Let $k > 0$ and $q < \infty$ satisfy*

$$q > \frac{p^*}{p^*-1}, \text{ where } p^* = \frac{dp}{d-kp} \text{ if } kp < d,$$
$$q > 1 \text{ for } kp = d,$$
$$q \geq 1 \text{ if } kp > d.$$

Then the space $L^q(\mathcal{O})$ is compactly embedded into the space $W^{-k,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

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THEOREM 5.4.6. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ open, bounded and have Lipschitz boundary $\partial\mathcal{O}$ ($\in C^{0,1}$). Let $p \in [1, +\infty), s \in (0,1)$ such that $sp > d$. Then, there exists $C > 0$, depending on n, s, p and $\mathcal{O}$, such that*

$$\|u\|_{C^{0,\alpha}(\mathcal{O})} \leq C \left(\|f\|^p_{L^p(\mathcal{O})} + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}},$$

for any $u \in L^p(\mathcal{O})$, with $\alpha := (sp - d)/p$.

The following estimates are known as Gagliardo-Nirenberg's inequality. When $\mathcal{O} = \mathbb{R}^d$, the proof is given in [122]. Thanks to the extension theorem, the proofs for other cases are immediately reduced to this case.

THEOREM 5.4.7. *Let $\mathcal{O}$ be $\mathbb{R}^d_+$ or a bounded domain with Lipschitz boundary. There exists a linear operator $\mathfrak{E}$ mapping functions in $\mathcal{O}$ to functions in $\mathbb{R}^d$ with the following properties:*

(1) $(\mathfrak{E}u)\,|_{\mathcal{O}} = u$.
(2) *$\mathfrak{E}$ is a continuous operator from $W^{k,p}(\mathcal{O})$ to $W^{k,p}\left(\mathbb{R}^d\right)$ for any $1 \leq p \leq \infty$ and any $k = 0, 1, 2, \ldots$, namely,*

$$\|\mathfrak{E}u\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{p,k} \|u\|_{W^{k,p}(\mathcal{O})}, \quad u \in W^{k,p}(\mathcal{O})$$

with some constant $C_{p,k} > 0$ depending on p and k.

THEOREM 5.4.8 (Gagliardo-Nirenberg's inequality). *Let $\mathcal{O}$ be $\mathbb{R}^d, \mathbb{R}^d_+$, or a bounded domain with Lipschitz boundary, and let $\theta \in [0,1]$, $1 \leq q \leq p \leq \infty$*

- *If $1 \leq p < n$, then*

$$W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}) \subset L^r(\mathcal{O}),$$

with the estimate

$$\|u\|_{L^r(\mathcal{O})} \leq C_{p,q,r} \|u\|^{\theta}_{W^{1,p}(\mathcal{O})} \|u\|^{1-\theta}_{L^q(\mathcal{O})}, \quad u \in W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}),$$

where $q \leq r \leq \frac{np}{n-p}$, and a is given by

$$\frac{1}{r} = \theta \left(\frac{1}{p} - \frac{1}{n} \right) + \frac{1-\theta}{q}.$$

- *If $p = n$, then*

$$W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}) \subset L^r(\mathcal{O}),$$

with the estimate

$$\|u\|_{L^r(\mathcal{O})} \leq C_{p,q,r} \|u\|^{\theta}_{W^{1,d}(\mathcal{O})} \|u\|^{1-\theta}_{L^q(\mathcal{O})}, \quad u \in W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}),$$

where r is any finite number such that $q \leq r < \infty$, and a is given by

$$\frac{1}{r} = \frac{1-\theta}{q}.$$

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- *If $n < p \leq \infty$, then*

$$W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}) \subset L^r(\mathcal{O}),$$

with the estimate

$$\|u\|_{L^r(\mathcal{O})} \leq C_{p,q,r} \|u\|^{\theta}_{W^{1,p}(\mathcal{O})} \|u\|^{1-\theta}_{L^q(\mathcal{O})}, \quad u \in W^{1,p}(\mathcal{O}) \cap L^q(\mathcal{O}),$$

where $q \leq r \leq \infty$, and a is given by

$$\frac{1}{r} = \theta \left(\frac{1}{p} - \frac{1}{n} \right) + \frac{1-\theta}{q}.$$

We finally present the compact embedding theorem. The proof is given in [124, Theorem 4.10.1].

THEOREM 5.4.9. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. For $1 < p < \infty$ and $s > 0$, the embedding $i : W^{s,p}(\mathcal{O}) \to L^p(\mathcal{O})$ is compact.*

THEOREM 5.4.10 (Generalized Poincaré inequality). *Let $\mathcal{O}$ be a bounded set in $\mathbb{R}^d$ with Lipschitz boundary and let p be a continuous seminorm on $H^m(\mathcal{O})$ such that $p(u) = 0$ for any polynomial u of maximal degree $m-1$ implies that $u = 0$. Then there exists $C_P > 0$ such that for all $u \in H^m(\mathcal{O})$,*

$$\|u\|_{H^m(\mathcal{O})} \leq C_P \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\mathcal{O})} + p(u) \right).$$

This result is stated in [123, Chap. 2, Sect. 1.4]. *Examples for p are $p(u) = \|u\|_{L^2(\mathcal{O})}$, $p(u) = \|u\|_{L^2(\partial\mathcal{O})}$, or, if $m = 1$, $p(u) = \left| \int_{\mathcal{O}} u \, \mathrm{d}x \right|$.*

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5.5. Compactness and convergence results

The results in this section follow mainly the work [120]. Let (X, ρ) be a metric space. Consider the set,

$$C(X) := \{f : X \to \mathbb{R} \text{ is continuous }\},$$

then $C(X)$ is a Banach space endowed with the norm

$$\|f\|_{C(X)} := \sup_{x \in X} |f(x)|.$$

We say that a sequence (f_n) converges to f in $C(X)$ if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0,$$

ie. (f_n) converges uniformly to f in X.

DEFINITION 5.5.1. *Let (X, ρ) be a metric space. A family of functions $\Lambda \subset C(X)$ is called equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \Lambda$, $|f(u) - f(\mathrm{v})| < \varepsilon$, for all $u, \mathrm{v} \in X$ satisfying $\rho(u, \mathrm{v}) < \delta$ The family Λ is called uniformly bounded if there exists a constant $C > 0$ such that $|f(u)| \leq C$, for all $f \in \Lambda$ and for all $u \in X$.*

DEFINITION 5.5.2. *If (X, ρ) be a metric space, a subset $K \subset X$ is called precompact (or relatively compact) if closure $\bar{K}$ of K is compact in X.*

THEOREM 5.5.3. *[Arzela-Ascoli Theorem]* [75, Theorem 5, p.201] *Let (X, ρ) be a compact metric space For a family of functions $\Lambda \subset C(X)$ following conditions are equivalent:*

- *Λ is relatively compact,*
- *Λ is equicontinuous and uniformly bounded.*

Following is an immediate consequence of the Arzelà-Ascoli theorem.

COROLLARY 5.5.4. *Let (X, ρ) be a compact metric space. If a sequence of functions $(f_n) \subset C(X)$ is equicontinuous and uniformly bounded then it contains a uniformly convergent subsequence.*

THEOREM 5.5.5. *Let B be a Banach space and $F \subset L^p(0, T; B)$. F is relatively compact in $L^p(0, T; B)$ for $1 \leq p < \infty$, or in $C(0, T, B)$ for $p = \infty$ if and only if:*

$$\left\{ \int_{t_1}^{t_2} f(t)\, \mathrm{d}t : f \in F \right\} \quad \text{is relatively compact in } B, \quad \forall\, 0 < t_1 < t_2 < T, \tag{5.5}$$

$$\|\tau_h f - f\|_{L^p(0, T-h; B)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F. \tag{5.6}$$

LEMMA 5.5.6. *Let $f \in W^{\alpha, r}(0, T; B), 0 < \alpha < 1, 1 \leq r \leq \infty$ and let p be such that*

$$p \leq \infty \text{ if } \alpha > \frac{1}{r},\ p < \infty \text{ if } \alpha = \frac{1}{r},\ p < r_* = \frac{r}{1 - \alpha r} \text{ if } \alpha < \frac{1}{r}.$$

Then $f \in L^p(0, T; B)$ and there exists a C independent of f such that, $\forall\, h > 0$

$$\|\tau_h f - f\|_{L^p(0, T-h, B)} \leq \begin{cases} C h^{\alpha + 1/p - 1/r} \|f\|_{W^{\alpha, r}(0, T; B)} & \text{if } r \leq p \leq \infty \\ C h^{\alpha} T^{1/p - 1/r} \|f\|_{W^{\alpha, r}(0, T; B)} & \text{if } 1 \leq r \leq p. \end{cases}$$

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COROLLARY 5.5.7. *Let X, B be Banach spaces and the embedding $X \hookrightarrow B$ be compact. Further let F be bounded in $W^{s_0,r_0}_{\mathrm{loc}}(0,T;X) \cap \dot{W}^{s,r}(0,T;B)$[1] where $s > 0, 1 \le r \le \infty$ and s_0 is real, $1 \le r_0 \le \infty$.*
If $s \le 1/r$ then F is relatively compact in $L^p(0,T;B),\ \forall\, p < r_* = r/(1-sr)$.
If $s > 1/\gamma$ then F is relatively compact in $C(0,T;B)$.

THEOREM 5.5.8. *Let X, B be Banach spaces and the embedding $X \hookrightarrow B$ be compact. Let F be bounded in $L^q(0,T;B) \cap W^{-m,1}_{\mathrm{loc}}(0,T;X) \cap \cap W^{s,1}_{\mathrm{loc}}(0,T;B)$, where $1 < q \le \infty, m$ is integer, $s > 0$. Then $\bar{F}$ is relatively compact in $L^p(0,T;B),\ \forall\, p < q$.*

In the following results, we will often consider the scenario where X, B, Y are Banach spaces such that

$$X \subset B \subset Y \text{ with compact embedding } X \to B. \tag{5.7}$$

THEOREM 5.5.9. *Assume (5.7), $1 \le p \le \infty$ and*
(1) *F is bounded in $L^p(0,T;X)$,*
(2) *$\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$.*
Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p = \infty$).

THEOREM 5.5.10. *Assume (5.7) and $1 \le p < \infty, 1 \le r \le \infty$. Let F be bounded in $L^p(0,T;X) \cap W^{s,r}(0,T;Y)$ where $s > 0$ if $r \ge p$ and where $s > 1/r - 1/p$ if $r \le p$. Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p = \infty$).*

THEOREM 5.5.11. *Assume (5.7). Let $1 < q \le \infty$ and F is bounded in $L^q(0,T;B) \cap L^1_{\mathrm{loc}}(0,T;X)$, $\forall\, 0 < t_1 < t_2 < T,\ \|\tau_h f - f\|_{L^1(t_1,t_2;Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$. Then F is relatively compact in $L^p(0,T;B), \forall\, p < q$.*

COROLLARY 5.5.12. *Let F be bounded in $L^1_{\mathrm{loc}}(0,T;X) \cap L^q(0,T;B) \cap W^{s,1}_{\mathrm{loc}}(0,T;Y)$ where $1 < q \le \infty,\ s > 0$. Then F is relatively compact in $L^p(0,T;B)\ \forall\, p < q$.*

For the next results, the following assumptions will be needed.

ASSUMPTION 5.5.13. *Let X, B, Y be Banach spaces, such that*

$$\|v\|_B < M \|v\|_X^{1-\theta} \|v\|_Y^{\theta}, \quad \forall\, v \in X \cap Y, \qquad \text{where } 0 < \theta < 1. \tag{5.8}$$

$$X \subset B \subset Y \text{ with compact embedding } X \to Y. \tag{5.9}$$

THEOREM 5.5.14. *Assume (5.8), (5.9), $1 \le p_i \le \infty$ and F is bounded in $L^{p_0}(0,T;X)$. $\|\tau_h f - f\|_{L^{p_1}(0,T-h;Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$. Then $\bar{E}$ is relatively compact in $L^{p_0}(0,T;B)$ where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. The order p_θ is intermediate between p_0 and p_1, then this result is fruitful only when $p_1 > p_0$.*

COROLLARY 5.5.15. *Assume (5.8),(5.9) and $1 \le p_0 \le \infty,\ 1 \le r_1 \le \infty$. Let F be bounded in $L^{p_0}(0,T;X)$ and $\partial F/\partial t$ be bounded in $L^{r_1}(0,T;Y)$. If $\theta\,(1 - 1/r_1) \le (1-\theta)/p_0$ then F is relatively compact in $L^p(0,T;B), \forall\, p < p_*$ where $1/p_* = (1-\theta)/p_0 - \theta\,(1-1/r_1)$. If $\theta\,(1-1/r_1) > (1-\theta)/p_0$ then F is relatively compact in $C(0,T;B)$.*

[1] $\dot{W}^{s,r}$ denotes the $W^{s,r}$ seminorm.

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Denote

$$\begin{cases} W = W^{s_0,r_0}(0,T;X) \cap W^{s_1,r_1}(0,T;Y), \quad \text{where} \quad s_i \in \mathbb{R}, 1 \le r_i \le \infty, \\ s_\theta = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad s_* = s_\theta - \frac{1}{r_\theta}. \end{cases} \tag{5.10}$$

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5.6. Weak convergence of measures and thightness

In general we are interested in convergence properties of measures (or more generally compactness properties of the space $\mathcal{P}(U)$).

DEFINITION 5.6.1. *Let $\mathcal{P}(U)$ denote the space of probability measures on U.*

DEFINITION 5.6.2. *A sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(U)$ is said to converge weakly to a probability measure $\mu \in \mathcal{P}(U)$ if for every bounded continuous function $\varphi \in C_b(U)$ we have*

$$\lim_{n\to\infty} \int_U \varphi \, \mathrm{d}\mu_n = \int_U \varphi \, \mathrm{d}\mu.$$

PROPOSITION 5.6.3. *A probability measure $\mu \in \mathcal{P}(U)$ is a regular measure. That is, for every $A \in \mathcal{B}(U)$ and $\epsilon > 0$, there is a compact set K_ϵ such that*

$$\mu\left(A \backslash K_\epsilon\right) \leq \epsilon.$$

DEFINITION 5.6.4. *We say family $M \subset \mathcal{P}(U)$ of probability measures is called tight (or uniformly tight) if for each $\epsilon > 0$ there exists compact set $K_\epsilon \subset U$ such that*

$$\sup_{\mu\in M} \mu\left(U \backslash K_\epsilon\right) < \epsilon.$$

Such a condition prevents the measures $\{\mu_n\}$ from sending mass off to infinity, since they must keep most of their mass concentrated on a compact set.

THEOREM 5.6.5 (Prokhorov). *On a metric space $\mathcal{S}$,*

(i) *if a collection of probability measures $\{\mathbb{P}_\alpha\}_{\alpha\in\mathcal{A}}$ is tight, then it is relatively compact.*
(ii) *Suppose S is also separable and complete. If $\{\mathbb{P}_\alpha\}_{\alpha\in\mathcal{A}}$ is relatively compact then it is tight.*

THEOREM 5.6.6 (Skorohod). *Let $\{\mu_n\} \subset \mathcal{P}(U)$ converge weakly to $\mu \in \mathcal{P}(U)$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of random variables, $\{X_n\}$ and X with $\mathcal{L}\left(X_n\right) = \mu_n$ and $\mathcal{L}(X) = \mu$ such that $X_n \to X$, $\mathbb{P}$-almost surely.*

We will need the following Jakubowski's generalisation of the Skorohod Theorem, in the form given by Brzezniak and Ondreját [28, Theorem C.1], see also [74], as we deal with non-metric spaces.

THEOREM 5.6.7. *Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let us denote by $\mathcal{S}$ the σ -algebra generated by the maps $\{f_m\}$. Then*

- *every compact subset of $\mathcal{X}$ is metrizable,*
- *if $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on $(\mathcal{X}, \mathcal{S})$, then there exists a subsequence $(m_k)_{k\in\mathbb{N}}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{X}$-valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω.*

Let $(\mathcal{S}, \varrho)$ be a separable and complete metric space.

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DEFINITION 5.6.8. *Let $u \in C([0,T];\mathcal{S})$. The modulus of continuity of u on $[0,T]$ is defined by*

$$m(u,\delta) := \sup_{s,t\in[0,T],|t-s|\leq\delta} \varrho(u(t),u(s)), \quad \delta > 0.$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$ -adapted $\mathcal{S}$-valued processes.

DEFINITION 5.6.9. *We say that the sequence $(X_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$ -valued random variables satisfies condition [T] iff $\forall\,\varepsilon>0, \forall\,\eta>0, \exists\,\delta>0$*

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left\{m\left(X_n,\delta\right) > \eta\right\} \leq \varepsilon. \tag{5.11}$$

LEMMA 5.6.10. [26, Lemma 2.4] *Assume that $(X_n)_{n\in\mathbb{N}}$ satisfies condition* [T]. *Let $\mathbb{P}_n$ be the law of X_n on $C([0,T];\mathcal{S}), n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a subset $A_\varepsilon \subset C([0,T];\mathcal{S})$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n\left(A_\varepsilon\right) \geq 1-\varepsilon$$

and

$$\lim_{\delta\to 0} \sup_{u\in A_\varepsilon} m(u,\delta) = 0.$$

Now we recall the Aldous condition [A], which is connected with condition [T]. This condition allows us to investigate the modulus of continuity for the sequence of stochastic processes by means of stopped processes.

DEFINITION 5.6.11 (Aldous condition). *A sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition* [A] *iff $\forall\,\varepsilon>0\ \forall\,\eta>0, \exists\,\delta>0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$ -stopping times with $\tau_n \leq T$ one has*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta} \mathbb{P}\left\{\rho\left(X_n\left(\tau_n+\theta\right), X_n\left(\tau_n\right)\right) \geq \eta\right\} \leq \varepsilon. \tag{5.12}$$

LEMMA 5.6.12. [97, Theorem 3.2] *Conditions [A] and [T] are equivalent.*

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5.7. Kuratowski's Theorem and related results

DEFINITION 5.7.1. *A topological space X is completely metrizable if there is a metric d inducing the topology of X such that (X,d) is complete. A separable completely metrizable space is called Polish.*

THEOREM 5.7.2. *Assume that X_1, X_2 are Polish spaces with their Borel σ -fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. If $\varphi : X_1 \to X_2$ is an injective Borel measurable map then for any $E_1 \in \mathcal{B}(X_1), E_2 := \varphi(E_1) \in \mathcal{B}(X_2)$*

LEMMA 5.7.3. *Let X_1, X_2 and Z be topological spaces such that X_1 is a Borel subset of X_2. Then $X_1 \cap Z$ is a Borel subset of $X_2 \cap Z$, where $X_2 \cap Z$ is a topological space too, with the topology given by*

$$\tau(X_2 \cap Z) = \{A \cap B : A \in \tau(X_2), B \in \tau(Z)\}.$$

LEMMA 5.7.4. *Let X_1, X_2, Y be topological spaces such that $X_1 \subset X_2, X_1$ has trace topology from X_2 and $X_1 \cap Y = X_2 \cap Y$ then*

$$\tau(X_1 \cap Y) = \tau(X_2 \cap Y).$$

PROPOSITION 5.7.5. [27, Proposition C3] *Suppose that X_1, X_2 are two topological spaces with their Borel σ -fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. Suppose that $\phi : X_1 \longrightarrow X_2$ is an injective Borel measurable map such that for any $E_1 \in \mathcal{B}(X_1)$ $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$. Then if $g : X_1 \to \mathbb{R}$ is a Borel measurable map then a function $f : X_2 \to \mathbb{R}$ defined by*

$$f(x_2) = \begin{cases} g(\phi^{-1}(x_2)), & \text{if } x_2 \in \phi(X_1) \\ \infty, & \text{if } x_2 \in X_2 \backslash \phi(X_1) \end{cases}$$

is also Borel measurable.

THEOREM 5.7.6. [27, Theorem C3] *Let X_1 and X_2 be a topological spaces such that for each $i = 1, 2$ there exists a sequence $\{f_{i,m}\}$ of continuous functions $f_{i,m} : X_i \to \mathbb{R}$ that separate points of X_i. Let us denote by $\mathcal{S}_i$ the σ -algebra generated by the maps $\{f_{i,m}\}$. If $\phi : X_1 \longrightarrow X_2$ is an injective measurable map, then for any $E_1 \in \mathcal{S}_1$ $E_2 := \phi(E_1) \in \mathcal{S}_2$.*

PROPOSITION 5.7.7. *Assume that E is a separable Banach space, then for arbitrary $T > 0$ and $r \geq p \geq 1$ the spaces $L^r(0,T;E)$ and $C([0,T];E)$ are Borel subsets of $L^p(0,T;E)$.*

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