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Optimal convergence rates in L^2 for a first order system least squares finite element method - part II: Inhomogeneous Robin boundary conditions

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ABSTRACT

We consider divergence-based high order discretizations of an L^2-based first order system least squares formulation of a second order elliptic equation with Robin boundary conditions. For smooth geometries, we show optimal convergence rates in the $L^2(\Omega)$-norm for the scalar variable. Convergence rates for the $L^2(\Omega)$-norm error of the gradient of the scalar variable as well as the vectorial variable are also derived. Numerical examples illustrate the analysis.

1. Introduction

Least Squares Methods and related techniques are an established tool for the numerical treatment of partial differential equations as witnessed by the monographs [5,25], which provide both mathematical analysis and examples of applications in fluid and solid mechanics. Least Squares Methods are successfully used in computational fluid mechanics (see, e.g., [13,25]), solid mechanics (see, e.g., [4,15,23]), electromagnetics (see, e.g., [6,7]), and eigenvalue problems (see, e.g., [1,2,7]). Reasons for the popularity of these methods include their flexibility to deal with a variety of equations and the ease of coupling different equations, the fact that they lead to symmetric positive definite systems by construction, and that they naturally come with error estimators.

For scalar second order problems, an important approach in Least Squares methodologies is to reformulate it as a first order system based on a scalar variable and a vectorial variable and to subsequently minimize the residuum in the L^2-norm. This method, called First Order System Least Squares Method (FOSLS), is computationally attractive and leads to quasi-optimality in a residual norm, [5,12]. Obtaining optimal error estimates in norms other than the natural residual norm, say, L^2 for the scalar variable is the purpose of the present work. Here, optimality refers not only to the optimal achievable convergence rate under the assumption of *sufficient* smoothness of the solution but relates to the fact that the regularity of both the scalar variable and vectorial variable are dictated by the regularity of the data. For example, for data $f \in L^2$, the vectorial variable (later denoted $\boldsymbol{\varphi}$) is merely in $\boldsymbol{H}(\operatorname{div}, \Omega)$; this implies that no rate of convergence is available for the convergence in the residual norm, which measures the error of the vectorial variable in $\boldsymbol{H}(\operatorname{div}, \Omega)$. The tools to overcome this obstacle are duality arguments and approximation operators with suitable orthogonality properties as previously done in [26] and [9]. Under regularity assumptions for the appropriate dual problems, optimal convergence rates can then be established.

In the first part [9] of this series of papers, we analyzed high order finite element discretizations of a first order system least squares (FOSLS) formulation of a Poisson-type second order elliptic problem with homogeneous boundary conditions and obtained optimal error estimates for the L^2-error of the scalar variable. Here, we generalize the approach of [9] to Poisson-type problems with inhomogeneous Robin boundary conditions. Compared to [9] and [26] the presence of the boundary terms, which are L^2-terms, requires additional duality arguments and corresponding approximation results for the operator $\boldsymbol{I}_h^\Gamma$ that effects the required orthogonalities. As an aside, we mention that elliptic problems with Robin boundary conditions arise in applications, for example, in wave propagation problems with impedance boundary conditions. The analysis of the Helmholtz equation with impedance conditions is, however, beyond the scope of the present work as one leaves the realm of strong ellipticity; we refer, however, to [8,14] for analyses of FOSLS discretizations of the Helmholtz equation and also to [21,24,32] for Discontinuous Petrov Galerkin (DPG) discretizations.

The need for rather elaborate duality arguments in the analysis of the FOSLS may be understood as a consequence of the choice of norms in the method, in particular the choice of the computationally convenient L^2-norm. For example, for right-hand sides that are not in L^2, the methodology

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requires some regularization of the right-hand side. We refer to [22] for a method that features a regularization of the input data to make minimal residual methods applicable to problems with low-regularity input data.

The regularity assumptions on the data imposed by the least squares approach in L^2-based spaces can also be relaxed by changing the norms and performing a minimization in a weaker one. This approach also has a substantial history. We refer to the recent [33] for such an approach, where the relevant dual norms are realized computationally.

An important class of minimum residual methods that appeared after the monograph [5] are DPG methods, [11,17–19], which may be understood as minimizing the residual in a norm other than L^2, [20].

Contribution of the present work

Our primary contribution is optimal $L^2(\Omega)$ based convergence results for the least squares approximation of the scalar variable u, the gradient of the scalar variable u and the traces of the scalar and the vector variable. Furthermore, we derive improved $L^2(\Omega)$ estimates for the vector variable $\boldsymbol{\varphi}$. These estimates are explicit in the mesh size h of the quasi-uniform meshes employed and the polynomial degree p utilized, which extends the results of [26].

Outline

In Section 2 we introduce the model problem, its FOSLS formulation and prove a norm equivalence that ensures unique solvability of both the continuous and the discrete least squares formulation. Section 3 provides regularity assertions for the representations in terms of a dual formulation of the scalar variable, its gradient, the vector variable, and the traces. These duality results are given without assuming full elliptic regularity so as to provide the tools for a possible extension to situations without full elliptic shift such as non-convex geometries. In Section 4 we present several error estimates for different quantities of interest, namely, the L^2 and H^1-error of the scalar variable and the L^2 and $\boldsymbol{H}(\mathrm{div})$-error of the vectorial variable. This is obtained in a bootstrapping fashion by systematically improving estimates for these quantities of interest by repeated duality arguments. As a tool for our error analysis, we develop in Lemma 4.4 the constrained approximation operator $\boldsymbol{I}_h^\Gamma$ with certain orthogonality properties that is instrumental for our error analysis. This operator generalizes the corresponding operators $\boldsymbol{I}_h$, $\boldsymbol{I}_h^0$ for problems with homogeneous boundary conditions in [9]. Compared to [9], the bootstrapping argument is more involved since the presence of the boundary terms in the bilinear forms lowers the regularity of some components of dual solutions compared to [9] (see Remark 2.1 for more details). We close the paper in Section 5 with numerical results that showcase the proved convergence rates for the case of solutions with finite (low) Sobolev regularity.

Notation

Throughout this work, Ω denotes a bounded simply connected domain in $\mathbb{R}^d$, $d = 2, 3$, with (piecewise) smooth boundary; the boundary $\Gamma := \partial\Omega$ is assumed connected and $\boldsymbol{n}$ denotes the outward unit normal vector. We flag that the convergence analysis will be performed under the assumption of a full elliptic regularity shift, i.e., Ω is convex or has a smooth boundary Γ. Throughout, we will use rather established notation and refer to Part I of this series, [9] for details. Specifically, we employ Sobolev spaces $H^s(\Omega)$, $\boldsymbol{H}(\mathrm{curl},\Omega)$, $\boldsymbol{H}(\mathrm{div},\Omega)$ as well as the spaces with Dirichlet type boundary conditions $H^s_0(\Omega)$, $\boldsymbol{H}_0(\mathrm{curl},\Omega)$, $\boldsymbol{H}_0(\mathrm{div},\Omega)$. On Γ, we use Sobolev spaces $H^t(\Gamma)$, $t \in [-1, 1]$, defined by local charts (see, e.g., [27]); for $t > 1$, the space $H^t(\Gamma)$ is understood as the trace of $H^{t+1/2}(\Omega)$ endowed with the trace norm. We write $(\cdot,\cdot)_\Omega$ for the $L^2(\Omega)$ inner product and $\langle\cdot,\cdot\rangle_\Gamma$ for the duality pairing that extends the $L^2(\Gamma)$ inner product. We consider regular (i.e., no hanging nodes), shape-regular triangulations $\mathcal{T}_h$ of Ω that satisfy the following additional assumptions on the element maps of the triangulation $\mathcal{T}$:

Assumption 1.1 *(Quasi-uniform regular meshes, [30]).* Let $\widehat{K}$ be the reference simplex. Each element map $F_K : \widehat{K} \to K$ can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy, for constants $C_{\mathrm{affine}}, C_{\mathrm{metric}}, \rho > 0$ independent of K:

$$\left\|A_K'\right\|_{L^\infty(\widehat{K})} \le C_{\mathrm{affine}} h_K, \qquad \left\|(A_K')^{-1}\right\|_{L^\infty(\widehat{K})} \le C_{\mathrm{affine}} h_K^{-1},$$
$$\left\|(R_K')^{-1}\right\|_{L^\infty(\tilde{K})} \le C_{\mathrm{metric}}, \qquad \left\|\nabla^n R_K\right\|_{L^\infty(\tilde{K})} \le C_{\mathrm{metric}} \rho^n n! \qquad \forall n \in \mathbb{N}_0.$$

Here, $\tilde{K} = A_K(\widehat{K})$ and $h_K > 0$ denotes the element diameter.

Remark 1.2. The element maps F_K in Assumption 1.1 are required to be analytic, which is stronger than needed for the ensuing analysis. The structure of the element maps F_K, namely, the fact that F_K is the concatenation of an affine map and a smooth map, provides a simple mechanism for scaling arguments familiar from affine triangulations. The specific form of Assumption 1.1 is taken from [30], where the analyticity of the maps R_K is exploited. □

On the reference simplex $\widehat{K}$ we introduce the scalar polynomial space $\mathcal{P}_p(\widehat{K})$, the Raviart-Thomas $\mathbf{RT}_{p-1}(\widehat{K})$ spaces, the Brezzi-Douglas-Marini spaces $\mathbf{BDM}_p(\widehat{K})$, and the type I and II Nédélec spaces $\mathbf{N}^I_{p-1}(\widehat{K})$, $\mathbf{N}^{II}_{p-1}(\widehat{K})$ in the standard way (see [9, Sec. 1.1.2]); correspondingly, we have on the triangulation $\mathcal{T}$ the following $H^1(\Omega)$-, $\boldsymbol{H}(\mathrm{div},\Omega)$-, and $\boldsymbol{H}(\mathrm{curl},\Omega)$-conforming discrete spaces:

$$S_p(\mathcal{T}_h) := \left\{ u \in H^1(\Omega) : \ u|_K \circ F_K \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\},$$
$$\mathbf{BDM}_p(\mathcal{T}_h) := \left\{ \boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div},\Omega) : \ (\det F_K')(F_K')^{-1}\, \boldsymbol{\varphi}|_K \circ F_K \in \mathbf{BDM}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\},$$
$$\mathbf{RT}_{p-1}(\mathcal{T}_h) := \left\{ \boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div},\Omega) : \ (\det F_K')(F_K')^{-1}\, \boldsymbol{\varphi}|_K \circ F_K \in \mathbf{RT}_{p-1}(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\},$$
$$\mathbf{N}^{I/II}_{p-1}(\mathcal{T}_h) := \left\{ \boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{curl},\Omega) : \ (F_K')^\top\, \boldsymbol{\varphi}|_K \circ F_K \in \mathbf{N}^{I/II}_{p-1}(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \right\}.$$

The first order system formulation of a second order equation requires us to choose two finite element spaces, one for the scalar variable u, i.e., the solution of the second order equation, and one for the vector variable $\boldsymbol{\varphi}$, which we select as $\boldsymbol{\phi} = -\nabla u$. Hence, for the numerical discretization of the first order system we consider the following finite element spaces:

$$S_{p_s}(\mathcal{T}_h) \subseteq H^1(\Omega), \quad S^0_{p_s}(\mathcal{T}_h) \subseteq H^1_0(\Omega),$$
$$\mathbf{V}_{p_v}(\mathcal{T}_h) \subseteq \boldsymbol{H}(\mathrm{div},\Omega), \quad \mathbf{V}^0_{p_v}(\mathcal{T}_h) \subseteq \boldsymbol{H}_0(\mathrm{div},\Omega),$$

where the polynomial approximation of the scalar and vector variable is denoted by $p_s \geq 1$ and $p_v \geq 1$. We set

$$p := \min\{p_s, p_v\}.$$

For brevity of notation, we denote by $\mathbf{V}_{p_v}(\mathcal{T}_h)$ either the Raviart-Thomas space $\mathbf{RT}_{p_v-1}(\mathcal{T}_h)$ or the Brezzi-Douglas-Marini space $\mathbf{BDM}_{p_v}(\mathcal{T}_h)$. The space $\mathbf{V}^0_{p_v}(\mathcal{T}_h)$, which includes the boundary conditions, is understood analogously. Furthermore, depending on the choice of the space $\mathbf{V}_{p_v}(\mathcal{T}_h)$, the Nédélec space $\mathbf{N}_{p_v}(\mathcal{T}_h)$ is either of type I (if $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$) or II (if $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h)$). We apply the same convention to spaces incorporating homogeneous boundary conditions. We refer to [3,16,28,31] for further details.

As in [9], further notational conventions are:

(i) lower case roman letters such as u and v are employed to indicate scalar valued functions;
(ii) lower case boldface greek letters such as $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are used for vector valued functions;
(iii) a subscript h as in u_h and $\boldsymbol{\varphi}_h$ indicates membership in a finite element space;
(iv) if not otherwise stated finite element functions without a $\tilde{\ }$ are in some sense fixed, e.g., they are Galerkin approximations whereas functions with a $\tilde{\ }$ are arbitrary and arise, e.g., in quasi-optimality results;
(v) generic constants are either denoted by $C > 0$ or are hidden inside the symbol $\lesssim$; they are independent of the mesh size h and the polynomial degree p unless otherwise stated. We will not track the parameters γ and α appearing in the model problem (2.1).

2. Model problem with Robin boundary conditions

We assume $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. For fixed γ, $\alpha > 0$ we consider the following model problem:

$$-\Delta u + \gamma u = f \quad \text{in } \Omega, \qquad \partial_n u + \alpha u = g \qquad \text{on } \Gamma. \tag{2.1}$$

As in [9] by setting $\boldsymbol{\varphi} = -\nabla u$ we arrive at the system

$$\nabla \cdot \boldsymbol{\varphi} + \gamma u = f \quad \text{in } \Omega, \qquad \nabla u + \boldsymbol{\varphi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n} - \alpha u = -g \quad \text{on } \Gamma. \tag{2.2}$$

Furthermore, we introduce the Hilbert spaces

$$\boldsymbol{V} := \{\boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div},\Omega) : \boldsymbol{\varphi} \cdot \boldsymbol{n} \in L^2(\Gamma)\} \qquad \text{and} \qquad W := H^1(\Omega),$$

and equip $\boldsymbol{V}$ with the graph norm. The bilinear form b and the linear functional F are given by

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) := (\nabla \cdot \boldsymbol{\varphi} + \gamma u, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega + (\nabla u + \boldsymbol{\varphi}, \nabla v + \boldsymbol{\psi})_\Omega + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n} - \alpha u, \boldsymbol{\psi} \cdot \boldsymbol{n} - \alpha v\rangle_\Gamma,$$
$$F((\boldsymbol{\varphi}, v)) := (f, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega + \langle -g, \boldsymbol{\psi} \cdot \boldsymbol{n} - \alpha v\rangle_\Gamma.$$

Remark 2.1. The boundary terms in the bilinear form b reflect the fact that the Robin boundary conditions, which reduce to Neumann condition in the case $\alpha = 0$, are realized as “natural boundary conditions” in the L^2-minimization process. In comparison, the bilinear form of [9] for the Neumann problem lacks the boundary terms since the Neumann conditions are enforced as “essential boundary conditions” on the vectorial variable. This difference in bilinear form leads to slightly different regularity assertions for the dual problems in Section 3 ahead: e.g., Theorem 3.3 ensures up to $\boldsymbol{H}^2$-regularity for $\boldsymbol{\psi}$ whereas the corresponding [9, Thm. 3.3] even asserts up to $\boldsymbol{H}^3$-regularity. □

We start our analysis with a norm equivalence theorem.

Theorem 2.2 *(Norm equivalence - Robin version of [9, Thm. 2.1]). For all $(\boldsymbol{\varphi}, u) \in \boldsymbol{V} \times W$ there holds*

$$\|u\|^2_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|\boldsymbol{\varphi} \cdot \boldsymbol{n}\|^2_{L^2(\Gamma)} \lesssim b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u)) \lesssim \|u\|^2_{H^1(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|\boldsymbol{\varphi} \cdot \boldsymbol{n}\|^2_{L^2(\Gamma)}.$$

Proof. The upper bound follows directly from the Cauchy-Schwarz inequality. For the lower bound, we proceed similarly to [9, Thm. 2.1]. By definition we have

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\varphi}, u)) = \| \underbrace{\nabla \cdot \boldsymbol{\varphi} + \gamma u}_{=:w} \|^2_{L^2(\Omega)} + \| \underbrace{\nabla u + \boldsymbol{\varphi}}_{=:\boldsymbol{\eta}} \|^2_{L^2(\Omega)} + \| \underbrace{\boldsymbol{\varphi} \cdot \boldsymbol{n} - \alpha u}_{=:\mu} \|^2_{L^2(\Gamma)}.$$

We write $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$ and $u = u_1 + u_2$, where

$$\begin{aligned} \nabla \cdot \boldsymbol{\varphi}_1 + \gamma u_1 &= w && \text{in } \Omega, \\ \nabla u_1 + \boldsymbol{\varphi}_1 &= 0 && \text{in } \Omega, \\ \boldsymbol{\varphi}_1 \cdot \boldsymbol{n} - \alpha u_1 &= 0 && \text{on } \Gamma, \end{aligned} \qquad \begin{aligned} \nabla \cdot \boldsymbol{\varphi}_2 + \gamma u_2 &= 0 && \text{in } \Omega, \\ \nabla u_2 + \boldsymbol{\varphi}_2 &= \boldsymbol{\eta} && \text{in } \Omega, \\ \boldsymbol{\varphi}_2 \cdot \boldsymbol{n} - \alpha u_2 &= \mu && \text{on } \Gamma. \end{aligned}$$

Eliminating $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ leads, in strong form, to

$$-\Delta u_1 + \gamma u_1 = w \quad \text{in } \Omega, \qquad -\Delta u_2 + \gamma u_2 = -\nabla\cdot\boldsymbol{\eta} \quad \text{in } \Omega,$$
$$\partial_n u_1 + \alpha u_1 = 0 \quad \text{on } \Gamma, \qquad \partial_n u_2 + \alpha u_2 = -\mu + \boldsymbol{\eta}\cdot\boldsymbol{n} \quad \text{on } \Gamma.$$

Lax-Milgram provides $\|u_1\|_{H^1(\Omega)} \lesssim \|w\|_{L^2(\Omega)}$ and $\|u_2\|_{H^1(\Omega)} \lesssim \|\boldsymbol{\eta}\|_{L^2(\Omega)} + \|\mu\|_{H^{-1/2}(\Gamma)}$. We set $\boldsymbol{\varphi}_1 = -\nabla u_1$ and $\boldsymbol{\varphi}_2 = \boldsymbol{\eta} - \nabla u_2$ and check that the pairs $(\boldsymbol{\varphi}_1, u_1)$ and $(\boldsymbol{\varphi}_2, u_2)$ satisfy the above two systems and $\|\boldsymbol{\varphi}_1\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim \|w\|_{L^2(\Omega)}$ as well as $\|\boldsymbol{\varphi}_2\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim \|\boldsymbol{\eta}\|_{L^2(\Omega)} + \|\mu\|_{H^{-1/2}(\Gamma)}$. We note $\|\boldsymbol{\varphi}_1\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \|u_1\|_{L^2(\Gamma)}$ and $\|\boldsymbol{\varphi}_2\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \|\mu\|_{L^2(\Gamma)} + \|u_2\|_{L^2(\Gamma)}$. We conclude the proof by observing that $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$ and $u = u_1 + u_2$. □

3. Duality argument

Our error analysis in the following sections relies on several duality arguments for a range of quantities. The following Assumption 3.1 characterizes the range in which the elliptic shift theorem is valid:

Assumption 3.1 *($\hat{s}$ shift property).* Let $\hat{s} \geq -1$. Then for every $f \in H^s(\Omega)$, $g \in H^{s+1/2}(\Gamma)$ and $s \in [-1, \hat{s}]$, the problem

$$-\Delta u + \gamma u = f \quad \text{in } \Omega, \qquad \partial_n u + \alpha u = g \quad \text{on } \Gamma$$

admits the regularity shift $u \in H^{s+2}(\Omega)$ with $\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\Gamma)}$ if $s \geq 0$ and, if $s < 0$, $\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{\widetilde{H}^s(\Omega)} + \|g\|_{H^{s+1/2}(\Gamma)}$. Here, for $s \in (-1, 0)$, we set $H^s(\Omega) = (\widetilde{H}^{-s}(\Omega))'$, $\widetilde{H}^s(\Omega) = (H^{-s}(\Omega))'$ with the Sobolev spaces $H^{-s}(\Omega) = (L^2(\Omega), H^1(\Omega))_{-s,2}$ and $\widetilde{H}^{-s}(\Omega) = (L^2(\Omega), H^1_0(\Omega))_{-s,2}$ defined by the real method of interpolation (see [27] for details).

Remark 3.2. The parameter $\hat{s}$ encodes properties of Γ. For example, by standard elliptic regularity Assumption 3.1 holds for any $\hat{s} \geq 0$ for smooth boundaries Γ. For polygonal/polyhedral domains, $\hat{s}$ is determined by the minimal angle at corners in 2D or corners and edges in 3D. For convex domains Ω, one has $\hat{s} \geq 0$. While $\hat{s}$ is a function of the geometry only, the implied constant in the norm bounds depends additionally on γ and α. □

The proof of the following four duality arguments follows similar patterns. We will therefore present the arguments only for Theorems 3.3 and 3.6 in some detail and refer to [9] for the other cases. We point out that, as mentioned in Remark 2.1, the change in the bilinear form from [9] to the present setting reduces the (provably) achievable regularity of the variable $\boldsymbol{\psi}$ from $\boldsymbol{H}^3(\Omega)$ to $\boldsymbol{H}^2(\Omega)$.

Theorem 3.3 *(Duality argument for the scalar variable — Robin version of [9, Thm. 3.3]). Let Assumption 3.1 be valid for some $\hat{s} \geq -1$. Then, given $(\boldsymbol{\varphi}, w) \in \boldsymbol{V} \times W$ there is a pair $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ with $\|w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s}+1,2)}(\Omega)$, $\nabla\cdot\boldsymbol{\psi} \in H^{\min(\hat{s}+2,2)}(\Omega)$, $\boldsymbol{\psi}\cdot\boldsymbol{n} \in H^{\min(\hat{s}+3/2,3/2)}(\Gamma)$, and $v \in H^{\min(\hat{s}+2,2)}(\Omega)$ with*

$$\|v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} + \|\boldsymbol{\psi}\|_{H^{\min(\hat{s}+1,2)}(\Omega)} + \|\nabla\cdot\boldsymbol{\psi}\|_{H^{\min(\hat{s}+2,2)}(\Omega)} + \|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{H^{\min(\hat{s}+3/2,3/2)}(\Gamma)} \lesssim \|w\|_{L^2(\Omega)}.$$

Proof. By the coercivity result of Theorem 2.2 and Lax-Milgram, there is a unique solution $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ to the following variational problem:

$$(u, w)_\Omega = b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{V} \times W. \tag{3.1}$$

In order to show the regularity assertions about $(\boldsymbol{\psi}, v)$, we introduce the new quantities z, $\boldsymbol{\mu}$, and σ by

$$\nabla\cdot\boldsymbol{\psi} + \gamma v = z \quad \text{in } \Omega, \qquad \nabla v + \boldsymbol{\psi} = \boldsymbol{\mu} \quad \text{in } \Omega, \qquad \boldsymbol{\psi}\cdot\boldsymbol{n} - \alpha v = \sigma \quad \text{on } \Gamma. \tag{3.2}$$

In terms of these quantities, (3.1) reads

$$(u, w)_\Omega = (\nabla u + \boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla\cdot\boldsymbol{\varphi} + \gamma u, z)_\Omega + \langle \boldsymbol{\varphi}\cdot\boldsymbol{n} - \alpha u, \sigma\rangle_\Gamma \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{V} \times W. \tag{3.3}$$

Selecting $u = 0$, we find with an integration by parts

$$0 = (\boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla\cdot\boldsymbol{\varphi}, z)_\Omega + \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, \sigma\rangle_\Gamma = (\boldsymbol{\varphi}, \boldsymbol{\mu} - \nabla z)_\Omega + \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, \sigma + z\rangle_\Gamma,$$

which gives $\boldsymbol{\mu} = \nabla z$ as well as $\sigma = -z|_\Gamma$. Therefore we find by taking $\boldsymbol{\varphi} = 0$ in (3.3)

$$(u, w)_\Omega = (\nabla u, \nabla z)_\Omega + (\gamma u, z)_\Omega + \langle \alpha u, z\rangle_\Gamma \quad \forall u \in H^1(\Omega).$$

That is, z satisfies, in strong form,

$$-\Delta z + \gamma z = w \quad \text{in } \Omega, \qquad \partial_n z + \alpha z = 0 \quad \text{on } \Gamma. \tag{3.4}$$

Assumption 3.1 provides $z \in H^{\min(\hat{s}+2,2)}(\Omega)$ together with the estimate $\|z\|_{H^{\min(\hat{s}+2,2)}(\Omega)} \lesssim \|w\|_{L^2(\Omega)}$. We next proceed as in the proof of [9, Thm. 3.3]. To highlight the fact that $\boldsymbol{\psi}$ is only in $\boldsymbol{H}^{\min(\hat{s}+1,2)}(\Omega)$ compared to [9, Thm. 3.3] we write down the equations for v and $z - v$:

$$-\Delta v + \gamma v = w + (1-\gamma) z \quad \text{in } \Omega, \qquad \partial_n v + \alpha v = (1-\alpha) z \quad \text{on } \Gamma,$$
$$-\Delta(z - v) + \gamma(z - v) = (\gamma - 1) z \quad \text{in } \Omega, \qquad \partial_n (z - v) + \alpha(z - v) = (\alpha - 1) z \quad \text{on } \Gamma.$$

Assumption 3.1 gives $v \in H^{\min(\hat{s}+2,2)}(\Omega)$ since the volume right-hand side is only in $L^2(\Omega)$. The regularity of $z - v$ is limited by the exploitable regularity of the boundary data $(\alpha - 1)z \in H^{\min(\hat{s}+2,2)-1/2}(\Gamma) = H^{\min(\hat{s}+1,1)+1/2}(\Gamma) \subset H^{\min(\min(\hat{s}+1,1),\hat{s})+1/2}(\Gamma) = H^{\min(\hat{s},1)+1/2}(\Gamma)$. Therefore, by Assumption 3.1, we have $z - v \in H^{\min(\hat{s}+2,3)}(\Omega)$ together with the estimate

$$\|z - v\|_{H^{\min(\hat{s}+2,3)}(\Omega)} \lesssim \|w\|_{L^2(\Omega)},$$

and consequently $\boldsymbol{\psi} = \nabla(z - v) \in \boldsymbol{H}^{\min(\hat{s}+1,2)}(\Omega)$. The regularity of $\nabla \cdot \boldsymbol{\psi}$ now follows from the representation $(3.2)_1$ and that of $\boldsymbol{\psi} \cdot \boldsymbol{n}$ from $(3.2)_3$ and $\sigma = -z|_\Gamma$. For this last regularity assertion, we recall that Sobolev spaces $H^s(\Gamma)$ with $s > 1$ are defined in terms of traces of Sobolev functions of $H^{s+1/2}(\Omega)$. □

Theorem 3.4 *(Duality argument for the gradient of the scalar variable - Robin version of [9, Thm. 3.4]). Let Assumption 3.1 be valid for some $\hat{s} \geq -1$. Then, given $(\boldsymbol{\varphi}, w) \in \boldsymbol{V} \times W$ there is a pair $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ with $\|\nabla w\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s}+1,1)}(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$, $\boldsymbol{\psi} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma)$, and $v \in H^1(\Omega)$ with*

$$\|v\|_{H^1(\Omega)} + \|\boldsymbol{\psi}\|_{H^{\min(\hat{s}+1,1)}(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} + \|\boldsymbol{\psi} \cdot \boldsymbol{n}\|_{H^{1/2}(\Gamma)} \lesssim \|\nabla w\|_{L^2(\Omega)}.$$

Proof. Structurally, the proof follows that of Theorem 3.3; details can be found in [10, Thm. 3.4] and the analogous result [9, Thm. 3.4]. We highlight that as in the proof of Theorem 3.3 it is the exploitable regularity of the boundary data that limits the regularity of $\boldsymbol{\psi}$. □

Theorem 3.5 *(Duality argument for the vector valued variable — Robin version of [9, Thm. 3.5]). Let Assumption 3.1 be valid for some $\hat{s} \geq -1$. Then, given $(\boldsymbol{\eta}, u) \in \boldsymbol{V} \times W$ there is a pair $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ with $\|\boldsymbol{\eta}\|^2_{L^2(\Omega)} = b((\boldsymbol{\eta}, u), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^1(\Omega)$, $\boldsymbol{\psi} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma)$, and $v \in H^{\min(\hat{s}+2,2)}(\Omega)$ with*

$$\|v\|_{H^{\min(\hat{s}+2,2)}(\Omega)} + \|\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} + \|\boldsymbol{\psi} \cdot \boldsymbol{n}\|_{H^{1/2}(\Gamma)} \lesssim \|\boldsymbol{\eta}\|_{L^2(\Omega)}.$$

Proof. The proof is structurally similar to that of [9, Thm. 3.5]; details can be found in [10, Thm. 3.5]. □

Theorem 3.6 *(Duality argument for the normal trace of the vector valued variable). Let Assumption 3.1 be valid for some $\hat{s} \geq -1$. Then, given $(\boldsymbol{\eta}, u) \in \boldsymbol{V} \times W$ there is a pair $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ with $\|\boldsymbol{\eta} \cdot \boldsymbol{n}\|^2_{L^2(\Gamma)} = b((\boldsymbol{\eta}, u), (\boldsymbol{\psi}, v))$. Furthermore, $\boldsymbol{\psi} \in \boldsymbol{H}^{\min(\hat{s}+1,1/2)}(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^{\min(\hat{s}+2,3/2)}(\Omega)$, $\boldsymbol{\psi} \cdot \boldsymbol{n} \in L^2(\Gamma)$, and $v \in H^{\min(\hat{s}+2,3/2)}(\Omega)$ with*

$$\|v\|_{H^{\min(\hat{s}+2,3/2)}(\Omega)} + \|\boldsymbol{\psi}\|_{\min(\hat{s}+1,1/2)(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^{\min(\hat{s}+2,3/2)}(\Omega)} + \|\boldsymbol{\psi} \cdot \boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \|\boldsymbol{\eta} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}.$$

Proof. By the coercivity result of Theorem 2.2 and Lax-Milgram, there is a unique $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ satisfying

$$\langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \boldsymbol{\eta} \cdot \boldsymbol{n} \rangle_\Gamma = b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{V} \times W. \tag{3.5}$$

To show the regularity assertions, we introduce the new quantities z, $\boldsymbol{\mu}$, and σ by (3.2) In terms of these quantities, (3.5) reads

$$\langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \boldsymbol{\eta} \cdot \boldsymbol{n} \rangle_\Gamma = (\nabla u + \boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi} + \gamma u, z)_\Omega + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n} - \alpha u, \sigma \rangle_\Gamma \quad \forall (\boldsymbol{\varphi}, u) \in \boldsymbol{V} \times W. \tag{3.6}$$

Selecting $u = 0$, we find after an integration by parts

$$\langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \boldsymbol{\eta} \cdot \boldsymbol{n} \rangle_\Gamma = (\boldsymbol{\varphi}, \boldsymbol{\mu})_\Omega + (\nabla \cdot \boldsymbol{\varphi}, z)_\Omega + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \sigma \rangle_\Gamma = (\boldsymbol{\varphi}, \boldsymbol{\mu} - \nabla z)_\Omega + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \sigma + z \rangle_\Gamma \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{V},$$

which gives $\boldsymbol{\mu} = \nabla z$ as well as $\sigma = \boldsymbol{\eta} \cdot \boldsymbol{n} - z$. Therefore we find with $\boldsymbol{\varphi} = 0$ in (3.6)

$$0 = (\nabla u, \nabla z)_\Omega + (\gamma u, z)_\Omega + \langle \alpha u, z \rangle_\Gamma - \langle \alpha u, \boldsymbol{\eta} \cdot \boldsymbol{n} \rangle_\Gamma \quad \forall u \in H^1(\Omega).$$

That is, z satisfies, in strong form,

$$-\Delta z + \gamma z = 0 \quad \text{in } \Omega, \qquad \partial_n z + \alpha z = \alpha \boldsymbol{\eta} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

Assumption 3.1 provides $z \in H^{\min(\hat{s}+2,3/2)}(\Omega)$ with the estimate $\|z\|_{H^{\min(\hat{s}+2,3/2)}(\Omega)} \lesssim \|\boldsymbol{\eta} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}$. The equations satisfied by v are easily seen to be:

$$-\Delta v + \gamma v = (1 - \gamma) z \quad \text{in } \Omega, \qquad \partial_n v + \alpha v = (1 - \alpha)(z - \boldsymbol{\eta} \cdot \boldsymbol{n}) \quad \text{on } \Gamma.$$

Assumption 3.1 gives $v \in H^{\min(\hat{s}+2,3/2)}(\Omega)$ with the estimate $\|v\|_{H^{\min(\hat{s}+2,3/2)}(\Omega)} \lesssim \|\boldsymbol{\eta} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}$. Finally, we have $\boldsymbol{\psi} = \nabla(z - v) \in \boldsymbol{H}^{\min(\hat{s}+1,1/2)}(\Omega)$. The regularity of $\nabla \cdot \boldsymbol{\psi}$ follows from $(3.2)_1$ and that of $\boldsymbol{\psi} \cdot \boldsymbol{n}$ from $(3.2)_3$. □

Remark 3.7. Usually a duality argument results in a dual solution with higher order Sobolev regularity. This is not the case in Theorem 3.6, where the regularity is *not* improved, since $\boldsymbol{\psi} \cdot \boldsymbol{n}$ is still only in $L^2(\Gamma)$. The sole purpose of this duality argument is to introduce another Galerkin orthogonality that can be utilized in the error analysis. □

4. Error analysis

From here on we will only consider domains Ω satisfying Assumption 3.1 for some $\hat{s} \geq 0$ such as domains with smooth boundary Γ or convex domains. Non-convex polygonal/polyhedral domains would require a more careful analysis.

After recalling results about a commuting diagram operator in Subsection 4.1 we proceed in Subsection 4.2 with introducing and analyzing the operator $\boldsymbol{I}_h^\Gamma$, which features an orthogonality necessary for our analysis. Finally, we prove different error estimates in Subsection 4.4 via a bootstrap argument. We first prove suboptimal estimates for the errors $\|e^u\|_{L^2(\Omega)}$, $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$, and $\|\nabla e^u\|_{L^2(\Omega)}$ in Lemma 4.5, Theorem 4.6, and Lemma 4.9, respectively, where $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$ are the FOSLS errors of the scalar and vectorial variable. We then prove optimal estimates for

$\|\boldsymbol{e}^{\boldsymbol{\varphi}} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}$ and $\|\nabla e^u\|_{L^2(\Omega)}$ in Theorems 4.11 and 4.13. Next, we derive in Theorem 4.15 improved estimates for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ that are numerically seen to be still suboptimal. Finally, we conclude with an optimal estimate for $\|e^u\|_{L^2(\Omega)}$ in Theorem 4.17.

4.1. A commuting diagram operator

In the analysis it is crucial to understand the approximation properties of the vector-valued finite element space in the classical $\boldsymbol{H}(\operatorname{div}, \Omega)$ norm as well as the $L^2(\Gamma)$ norm of the normal trace simultaneously. We are therefore interested in quantifying

$$\inf_{\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)} \|\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{\boldsymbol{H}(\operatorname{div},\Omega)} + \|(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\|_{L^2(\Gamma)}$$

for $\boldsymbol{\psi} \in \boldsymbol{V}$. This infimum can be estimated by specific approximants. For the reader's convenience we briefly summarize some results of [29] concerning the $\boldsymbol{H}(\operatorname{div},\Omega)$-conforming elementwise defined approximation operator $\Pi^{\operatorname{div}}_{p_v} : \boldsymbol{H}^{1/2}(\operatorname{div},\Omega) \to \mathbf{V}_{p_v}(\mathcal{T}_h)$ constructed therein. This operator is defined on the reference element with error estimates that are explicit in the polynomial degree p_v. A simple scaling argument gives the desired h estimates of the global operator:

Proposition 4.1 *(Defs. 2.3, 2.6, Thms. 2.10, 2.13, & Rem. 2.9 in [29]). The global operator $\Pi^{\operatorname{div}}_{p_v}$ satisfies for every $\boldsymbol{\varphi} \in \boldsymbol{H}^{1/2}(\operatorname{div},\Omega)$ and $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$,*

(i) $(\nabla\cdot(\boldsymbol{\varphi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi}), \nabla\cdot\tilde{\boldsymbol{\varphi}}_h)_\Omega = 0$ *and consequently* $\|\nabla\cdot(\boldsymbol{\varphi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi})\|_{L^2(\Omega)} \le \|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}$,
(ii) $\langle(\boldsymbol{\varphi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi})\cdot\boldsymbol{n}, \tilde{\boldsymbol{\varphi}}_h\cdot\boldsymbol{n}\rangle_\Gamma = 0$ *and consequently* $\|(\boldsymbol{\psi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \le \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)}$,
(iii) $\|\boldsymbol{\varphi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi}\|_{\boldsymbol{H}(\operatorname{div},\Omega)} \lesssim \left(\frac{h}{p_v}\right)^{1/2} \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{\boldsymbol{H}^{1/2}(\operatorname{div},\Omega)}$.

Proof. The operator $\Pi^{\operatorname{div}}_{p_v}$ is constructed in [29] and [34]. However, while [29] covered in detail the case of $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}$, the fact that the approximation properties and the commuting diagram properties are also valid for the choice $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}$ is discussed in [34, Sec. 4.8].

For the 2D case, [29] consider the operator curl instead of div. However, since in 2D, the vector-valued curl is just a rotated gradient and the scalar-valued curl a divergence operator applied to the rotated vector, the results of [29] can be reformulated in terms of the operator div.

Property (ii) and (iii) can be found in [29] for the case $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}$. Property (i) follows from the commuting diagram property of $\Pi^{\operatorname{div}}_{p_V}$: for $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$ we calculate with the L^2-projection $\Pi^{L^2}_{p_v} : L^2(\Omega) \to S^{-1}_{p_v} := \{v \in L^2(\Omega) \,|\, v|_K \circ F_K \in \mathcal{P}_{p_v}(\hat{K}) \quad \forall K \in \mathcal{T}\}$

$$(\nabla\cdot(\boldsymbol{\varphi} - \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\varphi}), \nabla\cdot\tilde{\boldsymbol{\varphi}}_h)_\Omega = (\nabla\cdot\boldsymbol{\varphi} - \Pi^{L^2}_{p_v}\nabla\cdot\boldsymbol{\varphi}), \nabla\cdot\tilde{\boldsymbol{\varphi}}_h)_\Omega = 0,$$

where we used $\nabla\cdot\tilde{\boldsymbol{\varphi}}_h \in S^{-1}_h$. □

4.2. The operator $\boldsymbol{I}^\Gamma_h$

We will require an approximation operator with certain orthogonality properties, i.e., an operator similar to $\boldsymbol{I}^0_h$ and $\boldsymbol{I}_h$ constructed in [9, Sec. 4]. Although the operator $\boldsymbol{I}_h$ of [9] is applicable to derive improved convergence results for the present case of Robin boundary conditions, they are only optimal in a pure h-version of the FOSLS method and suboptimal in a p-version context. This is due to the fact that the analysis requires approximation properties of $\boldsymbol{I}_h$ in the $L^2(\Gamma)$ norm for the normal trace, which can only be effected by relying on inverse estimates. Even though, *per se*, these inverse estimates are sharp one loses an order of p when doing so. For optimal p-estimate, it is therefore necessary to define the operator $\boldsymbol{I}^\Gamma_h$ such that the normal trace is appropriately involved.

We introduce the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ and the induced norm $|||\cdot|||$ by

$$\langle\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle\rangle := (\boldsymbol{\varphi},\boldsymbol{\psi})_\Omega + \langle\boldsymbol{\varphi}\cdot\boldsymbol{n},\boldsymbol{\psi}\cdot\boldsymbol{n}\rangle_\Gamma, \qquad |||\boldsymbol{\varphi}||| := \sqrt{\langle\langle\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle\rangle}. \tag{4.1}$$

Construction of $\boldsymbol{I}^\Gamma_h$: We define $\boldsymbol{I}^\Gamma_h$ by a constrained minimization problem:

$$\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi} = \operatorname*{argmin}_{\boldsymbol{\varphi}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)} \frac{1}{2}|||\boldsymbol{\varphi} - \boldsymbol{\varphi}_h|||^2 \qquad \text{s.t.} \qquad (\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}), \nabla\cdot\boldsymbol{\chi}_h)_\Omega = 0 \quad \forall\boldsymbol{\chi}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h).$$

This constrained optimization problem is equivalent to a saddle problem: Find $(\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{V}_{p_v}(\mathcal{T}_h) \times \nabla\cdot\mathbf{V}_{p_v}(\mathcal{T}_h)$ such that

$$\langle\langle\boldsymbol{\varphi}_h,\boldsymbol{\mu}_h\rangle\rangle + (\nabla\cdot\boldsymbol{\mu}_h, \lambda_h)_\Omega = \langle\langle\boldsymbol{\varphi},\boldsymbol{\mu}_h\rangle\rangle \qquad \forall\boldsymbol{\mu}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h), \tag{4.2}$$

$$(\nabla\cdot\boldsymbol{\varphi}_h, \eta_h)_\Omega = (\nabla\cdot\boldsymbol{\varphi}, \eta_h)_\Omega \qquad \forall\eta_h \in \nabla\cdot\mathbf{V}_{p_v}(\mathcal{T}_h). \tag{4.3}$$

Solvability follows in the standard way by asserting an inf-sup condition for the bilinear form $\widetilde{b}(\boldsymbol{\mu},\lambda) := (\nabla\cdot\boldsymbol{\mu},\lambda)_\Omega$ and coercivity of $\langle\langle\cdot,\cdot\rangle\rangle$ on the kernel of $\widetilde{b}$ both on the continuous and the discrete level; the inf-sup condition on the discrete level is inferred from the one on the continuous level using that $\boldsymbol{\Pi}^{\operatorname{div}}_{p_v}$ can be leveraged as a Fortin operator. See [9] or [10, Sec. 4.2] for details. We thus have:

Lemma 4.2. *The operator $\boldsymbol{I}^\Gamma_h$ is well-defined.*

4.3. Helmholtz decompositions

As a tool in the $L^2(\Omega)$ analysis of the operator $\boldsymbol{I}^\Gamma_h$ we need the following decomposition. Compared to [9, Sec. 4] we need a Helmholtz-like decomposition accounting for the regularity of the normal trace:

Lemma 4.3 *(Continuous and discrete Helmholtz-like decomposition - $L^2(\Gamma)$ normal trace). Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$. Let $\boldsymbol{Y} \subset \boldsymbol{H}(\mathrm{curl}, \Omega)$ be given by*

$$\boldsymbol{Y} := \left\{ \boldsymbol{\mu} \in \boldsymbol{H}(\mathrm{curl}, \Omega) : (\nabla \times \boldsymbol{\mu}) \cdot \boldsymbol{n} \in L^2(\Gamma) \right\}.$$

The operators $\boldsymbol{\Pi}^{\mathrm{curl},\Gamma} : \boldsymbol{V} \to \nabla \times \boldsymbol{Y}$ and $\boldsymbol{\Pi}_h^{\mathrm{curl},\Gamma} : \boldsymbol{V} \to \nabla \times \mathbf{N}_{p_v}(\mathcal{T}_h)$ given by

$$\langle\langle \boldsymbol{\Pi}^{\mathrm{curl},\Gamma} \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu} \rangle\rangle = \langle\langle \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu} \rangle\rangle \quad \forall \boldsymbol{\mu} \in \boldsymbol{Y}, \tag{4.4}$$

$$\langle\langle \boldsymbol{\Pi}_h^{\mathrm{curl},\Gamma} \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu}_h \rangle\rangle = \langle\langle \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu}_h \rangle\rangle \quad \forall \boldsymbol{\mu}_h \in \mathbf{N}_{p_v}(\mathcal{T}_h), \tag{4.5}$$

are well-defined. The remainder $\boldsymbol{r}$ in the continuous decomposition $\boldsymbol{\varphi} = \boldsymbol{\Pi}^{\mathrm{curl},\Gamma} \boldsymbol{\varphi} + \boldsymbol{r}$ satisfies $\boldsymbol{r} \in \boldsymbol{H}^1(\Omega)$ with $\|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \|\nabla \cdot \boldsymbol{\varphi}\|_{L^2(\Omega)}$. Additionally the solution $R \in H^2(\Omega)$ of

$$-\Delta R = -\nabla \cdot \boldsymbol{\varphi} \quad \text{in } \Omega, \qquad \partial_n R + R = 0 \quad \text{on } \Gamma,$$

satisfies $\boldsymbol{r} = \nabla R$ together with $\|R\|_{H^2(\Omega)} \lesssim \|\boldsymbol{r}\|_{H^1(\Omega)} \lesssim \|\nabla \cdot \boldsymbol{\varphi}\|_{L^2(\Omega)}$. Furthermore $\boldsymbol{r}$ satisfies

$$\nabla \cdot \boldsymbol{r} = \nabla \cdot \boldsymbol{\varphi} \quad \text{in } \Omega, \qquad \nabla \times \boldsymbol{r} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{r} \cdot \boldsymbol{n} = -R \quad \text{on } \Gamma. \tag{4.6}$$

Proof. The unique solvability of (4.4), (4.5) on the discrete and the continuous level follows immediately from the fact that the variational formulations are just the definition of the orthogonal projections onto $\nabla \times \boldsymbol{Y}$ and $\nabla \times \mathbf{N}_{p_v}(\mathcal{T}_h)$, respectively. For any $\boldsymbol{\mu} \in \boldsymbol{C}_0^\infty(\Omega)$ we find

$$\langle\langle \boldsymbol{r}, \nabla \times \boldsymbol{\mu} \rangle\rangle = (\boldsymbol{r}, \nabla \times \boldsymbol{\mu})_\Omega = 0,$$

which gives $\nabla \times \boldsymbol{r} = 0$. Since $\boldsymbol{\Pi}^{\mathrm{curl},\Gamma} \boldsymbol{\varphi} \in \nabla \times \boldsymbol{Y}$ we conclude $\nabla \cdot \boldsymbol{r} = \nabla \cdot \boldsymbol{\varphi}$. The observation $\nabla \times \boldsymbol{r} = 0$ gives via the exact sequence property

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^1(\Omega) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl}, \Omega) \xrightarrow{\nabla \times} \boldsymbol{H}(\mathrm{div}, \Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \xrightarrow{0} \{0\}$$

the existence of a potential $R \in H^1(\Omega)$ with $\boldsymbol{r} = \nabla R$. The function R is determined up to a constant that we will fix shortly. $\boldsymbol{r} = \nabla R$ implies $-\Delta R = -\nabla \cdot \nabla R = -\nabla \cdot \boldsymbol{r} = -\nabla \cdot \boldsymbol{\varphi}$. To analyze the boundary conditions satisfied by R we insert $\boldsymbol{r} = \nabla R$ into the variational formulation and integrate by parts to get

$$0 = \langle\langle \nabla R, \nabla \times \boldsymbol{\mu} \rangle\rangle = (\nabla R, \nabla \times \boldsymbol{\mu})_\Omega + \langle \partial_n R, (\nabla \times \boldsymbol{\mu}) \cdot \boldsymbol{n} \rangle_\Gamma = \langle R + \partial_n R, (\nabla \times \boldsymbol{\mu}) \cdot \boldsymbol{n} \rangle_\Gamma.$$

Since $(\nabla \times \boldsymbol{\mu}) \cdot \boldsymbol{n} = \nabla_\Gamma \cdot (\boldsymbol{\mu} \times \boldsymbol{n})$ and Γ is connected, we conclude $\partial_n R + R = c$ for some $c \in \mathbb{R}$. Since R is fixed up to a constant, we select it such that $c = 0$. Hence, the function R satisfies the boundary value problem of the statement of the lemma. By Assumption 3.1 and Remark 3.2 we have $\|R\|_{H^2(\Omega)} \lesssim \|\nabla \cdot \boldsymbol{\varphi}\|_{L^2(\Omega)}$. This concludes the proof. ☐

Lemma 4.4. *Let Ω satisfy Assumption 3.1 for some $\hat{s} \geq 0$. The operator $\boldsymbol{I}_h^\Gamma$ satisfies the following estimates for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$:*

$$|||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi}||| \lesssim |||\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h||| + \frac{h}{p_v} \left\| \nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h) \right\|_{L^2(\Omega)}, \tag{4.7}$$

$$\|\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi})\|_{L^2(\Omega)} \leq \left\| \nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h) \right\|_{L^2(\Omega)} \tag{4.8}$$

Proof. The proof parallels the one of [9, Lem. 4.6] by replacing $\|\cdot\|_{L^2(\Omega)}$ with $|||\cdot|||$; we will therefore merely point out the differences; details can be found in [10, Lem. 4.4]. Let $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$ be arbitrary. The orthogonality relations enforced in the construction of the operator $\boldsymbol{I}_h^\Gamma$ readily imply the estimate (4.8). We have with $\boldsymbol{e} = \boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi}$

$$|||\boldsymbol{e}|||^2 = \langle\langle \boldsymbol{e}, \boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h \rangle\rangle + \langle\langle \boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi} \rangle\rangle.$$

Lemma 4.3 allows us to decompose the discrete object $\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi} \in \mathbf{V}_{p_v}(\mathcal{T}_h)$ on a discrete as well as a continuous level:

$$\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi} = \nabla \times \boldsymbol{\mu} + \boldsymbol{r}, \qquad \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi} = \nabla \times \boldsymbol{\mu}_h + \boldsymbol{r}_h$$

for certain $\boldsymbol{\mu} \in \boldsymbol{Y}$, $\boldsymbol{r} \in \boldsymbol{V}$, $\boldsymbol{\mu}_h \in \mathbf{N}_{p_v}(\mathcal{T}_h)$, and $\boldsymbol{r}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$. Since $\nabla \cdot \nabla \times = 0$, property (4.2) of $\boldsymbol{I}_h^\Gamma$ immediately gives

$$\langle\langle \boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi}, \nabla \times \boldsymbol{\mu}_h \rangle\rangle = 0.$$

We therefore have

$$\langle\langle \boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi} \rangle\rangle = \langle\langle \boldsymbol{e}, \nabla \times \boldsymbol{\mu}_h + \boldsymbol{r}_h \rangle\rangle = \langle\langle \boldsymbol{e}, \boldsymbol{r}_h \rangle\rangle = \langle\langle \boldsymbol{e}, \boldsymbol{r}_h - \boldsymbol{r} \rangle\rangle + \langle\langle \boldsymbol{e}, \boldsymbol{r} \rangle\rangle := T_1 + T_2.$$

Before continuing with the treatment of the terms T_1 and T_2, we collect that Lemma 4.3 states that $\boldsymbol{r} = \nabla R$ with

$$\|R\|_{H^2(\Omega)} \lesssim \|\nabla \cdot (\boldsymbol{\varphi}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi})\|_{L^2(\Omega)}, \tag{4.9}$$

$$-\Delta R = -\nabla \cdot (\boldsymbol{\varphi}_h - \boldsymbol{I}_h^\Gamma \boldsymbol{\varphi}) \text{ in } \Omega, \qquad \partial_n R + R = 0 \quad \text{on } \Gamma, \tag{4.10}$$

$$\nabla R = \boldsymbol{r}, \qquad \boldsymbol{r} \cdot \boldsymbol{n} = -R \quad \text{on } \Gamma. \tag{4.11}$$

Treatment of T_1: See [9, Lem. 4.6] for analogous arguments and more details. Proceeding as in [9, Lem. 4.6], one arrives at

$$|||\boldsymbol{r} - \boldsymbol{r}_h||| \leq |||\boldsymbol{r} - \boldsymbol{\Pi}_{p_v}^{\mathrm{div}} \boldsymbol{r}||| \lesssim \|\boldsymbol{r} - \boldsymbol{\Pi}_{p_v}^{\mathrm{div}} \boldsymbol{r}\|_{L^2(\Omega)} + \|(\boldsymbol{r} - \boldsymbol{\Pi}_{p_v}^{\mathrm{div}} \boldsymbol{r}) \cdot \boldsymbol{n}\|_{L^2(\Gamma)}.$$

The volume term is estimated as in [9, Lem. 4.6] using [29, Thm. 2.10 (vi)/Thm. 2.13 (iv)]. To estimate the boundary term we apply Proposition 4.1 to conclude with the face-wise L^2-projection $\Pi_{p_v}^{L^2(\Gamma)}$

$$\|(\boldsymbol{r}-\boldsymbol{\Pi}_{p_v}^{\text{div}}\boldsymbol{r})\cdot\boldsymbol{n}\|_{L^2(\Gamma)} = \|\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{\Pi}_{p_v}^{L^2(\Gamma)}\boldsymbol{r}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \frac{h}{p_v}\|\boldsymbol{r}\cdot\boldsymbol{n}\|_{H^1(\Gamma)} \overset{(4.11)}{\lesssim} \frac{h}{p_v}\|R\|_{H^1(\Gamma)} \lesssim \frac{h}{p_v}\|R\|_{H^2(\Omega)} \overset{(4.9)}{\lesssim} \frac{h}{p_v}\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)}.$$

Summarizing the above development, we have

$$|||\boldsymbol{r}-\boldsymbol{r}_h||| \lesssim \frac{h}{p_v}\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)}. \tag{4.12}$$

Adding and subtracting $\boldsymbol{\varphi}$ and using (4.8) we get

$$T_1 \le |||\boldsymbol{e}|||\cdot|||\boldsymbol{r}-\boldsymbol{r}_h||| \lesssim \frac{h}{p_v}|||\boldsymbol{e}|||\cdot\|\nabla\cdot(\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)} \lesssim \frac{h}{p_v}|||\boldsymbol{e}|||\cdot\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Treatment of T_2: The term T_2 is estimated with a duality argument. Proceeding as in the proof of [9, Lem. 4.6], we get

$$T_2 \lesssim \frac{h}{p_v}\|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}|||\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}|||.$$

Finally we have for any $\tilde{\boldsymbol{\varphi}}_h$

$$\begin{aligned}|||\boldsymbol{e}|||^2 &= \langle\langle\boldsymbol{e},\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\rangle\rangle + \langle\langle\boldsymbol{e},\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}\rangle\rangle = \langle\langle\boldsymbol{e},\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\rangle\rangle + T_1 + T_2\\ &\lesssim |||\boldsymbol{e}|||\cdot|||\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h|||_{L^2(\Omega)} + \frac{h}{p_v}|||\boldsymbol{e}|||\cdot\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} + \frac{h}{p_v}\|\nabla\cdot\boldsymbol{e}\|_{L^2(\Omega)}\cdot|||\tilde{\boldsymbol{\varphi}}_h-\boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}|||.\end{aligned}$$

Adding and subtracting $\boldsymbol{\varphi}$ in the last term and applying estimate (4.8) together with the Young inequality yields the result. □

4.4. Error estimates

Lemma 4.5 *(Suboptimal estimate for $\|e^u\|_{L^2(\Omega)}$ - Robin version of [9, Lem. 4.1]). Let Assumption 3.1 be valid for some $\hat{s}\ge 0$. Let $(\boldsymbol{\varphi}_h,u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi},u)$. Set $e^u = u-u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi}-\boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h\in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$, there holds*

$$\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b \lesssim \frac{h}{p}\|u-\tilde{u}_h\|_{H^1(\Omega)} + \frac{h}{p}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h}{p}\|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. We employ the duality argument of Theorem 3.3 with $w = e^u$. As in [9, Lem. 4.1] we find by Galerkin orthogonality and the Cauchy-Schwarz inequality for any $\tilde{\boldsymbol{\psi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$ and $\tilde{v}_h\in S_{p_s}(\mathcal{T}_h)$

$$\|e^u\|_{L^2(\Omega)}^2 \le \|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b\|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h,v-\tilde{v}_h)\|_b.$$

The norm equivalence in Theorem 2.2 gives

$$\|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h,v-\tilde{v}_h)\|_b \lesssim \|v-\tilde{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\|_{\boldsymbol{H}(\text{div},\Omega)} + \|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)}.$$

Proposition 4.1 and the regularity estimates given by Theorem 3.3 yield the result. □

Theorem 4.6 *(Suboptimal estimate for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ — suboptimal Robin version of [9, Thm. 4.8]). Let Assumption 3.1 be valid for some $\hat{s}\ge 0$. Let $(\boldsymbol{\varphi}_h,u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi},u)$. Set $e^u = u-u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi}-\boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h\in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$,*

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|u-\tilde{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \left(\frac{h}{p}\right)^{1/2}\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. This a somewhat weaker version of [9, Thm. 4.8] that will be improved in Theorem 4.15 below. Let $(\boldsymbol{\psi},v)\in\boldsymbol{V}\times W$ denote the dual solution given by Theorem 3.5 applied to $\boldsymbol{\eta} = \boldsymbol{e}^{\boldsymbol{\varphi}}$. Theorem 3.5 gives $\boldsymbol{\psi}\in\boldsymbol{L}^2(\Omega)$, $\nabla\cdot\boldsymbol{\psi}\in H^1(\Omega)$, $\boldsymbol{\psi}\cdot\boldsymbol{n}\in H^{1/2}(\Gamma)$, and $v\in H^2(\Omega)$, which are controlled by $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. By the Galerkin orthogonality we have for any $(\tilde{\boldsymbol{\psi}}_h,\tilde{v}_h)$

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}^2 = b((\boldsymbol{e}^{\boldsymbol{\varphi}},e^u),(\boldsymbol{\psi},v)) = b((\boldsymbol{e}^{\boldsymbol{\varphi}},e^u),(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h,v-\tilde{v}_h)). \tag{4.13}$$

All volume terms of the right-hand side of (4.13) with the exception of $\langle\langle\boldsymbol{e}^{\boldsymbol{\varphi}},\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\rangle\rangle$ can be estimated as in [9, Proof of Thm. 4.6] (see [10, Thm. 4.6] for details); boundary terms involving e^u are treated with multiplicative trace estimates and the preceding Lemma 4.5 to give

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}^2 \lesssim \|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b\left[\|\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))\|_{L^2(\Omega)} + (h/p)^{1/2}\|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \|v-\tilde{v}_h\|_{H^1(\Omega)}\right] + \langle\langle\boldsymbol{e}^{\boldsymbol{\varphi}},\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\rangle\rangle. \tag{4.14}$$

To analyze the term $\langle\langle\boldsymbol{e}^{\boldsymbol{\varphi}},\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\rangle\rangle$ we follow a similar procedure as in [9, Thm. 4.8] and first perform a Helmholtz decomposition of the vector field $\boldsymbol{\psi}$. Since $\boldsymbol{\psi}\in\boldsymbol{H}(\text{div},\Omega)$ with $\nabla\cdot\boldsymbol{\psi}\in H^1(\Omega)$ and $\boldsymbol{\psi}\cdot\boldsymbol{n}\in H^{1/2}(\Gamma)$ we find $\boldsymbol{\rho}\in\boldsymbol{H}_0(\text{curl},\Omega)$ and $z\in H^2(\Omega)$ such that $\boldsymbol{\psi} = \nabla\times\boldsymbol{\rho}+\nabla z$ in the following way: let $z\in H^1(\Omega)$ with zero average solve

$$-\Delta z = -\nabla\cdot\boldsymbol{\psi} \quad \text{in } \Omega, \qquad \partial_n z = \boldsymbol{\psi}\cdot\boldsymbol{n} \quad \text{on } \Gamma. \tag{4.15}$$

As $\nabla\cdot(\boldsymbol{\psi}-\nabla z) = 0$ and $(\boldsymbol{\psi}-\nabla z)\cdot\boldsymbol{n} = 0$ by construction, the exact sequence property ensures the existence of $\boldsymbol{\rho}\in\boldsymbol{H}_0(\text{curl},\Omega)$ such that $\boldsymbol{\psi}-\nabla z = \nabla\times\boldsymbol{\rho}$. By elliptic regularity (Assumption 3.1) we have $z\in H^2(\Omega)$ together with the estimate

$$\|z\|_{H^2(\Omega)} \lesssim \|\nabla\cdot\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)}.$$

From the weak formulation of (4.15),

$$\forall w \in H^1(\Omega) \quad : \quad (\nabla z, \nabla w)_\Omega = (-\nabla\cdot\boldsymbol{\psi}, w)_\Omega + \langle \boldsymbol{\psi}\cdot\boldsymbol{n}, w\rangle_\Gamma = (\boldsymbol{\psi}, \nabla w)_\Omega,$$

we infer with Lax-Milgram $\|z\|_{H^1(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{L^2(\Omega)}$. Since $\boldsymbol{\rho}\in\boldsymbol{H}_0(\mathrm{curl},\Omega)$ and consequently $\nabla\times\boldsymbol{\rho}\in\boldsymbol{H}_0(\mathrm{div},\Omega)$, we can estimate

$$|||\nabla\times\boldsymbol{\rho}||| = \|\nabla\times\boldsymbol{\rho}\|_{L^2(\Omega)} \le \|\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla z\|_{L^2(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}, \tag{4.16}$$

where we used the estimates of the Helmholtz decomposition as well as the regularity estimates of Lemma 3.5. We now continue estimating (4.14) by applying the Helmholtz decomposition. In essence this is again the procedure of [9, Thm. 4.8] after replacing $\|\cdot\|_{L^2(\Omega)}$ with $|||\cdot|||$. For any $\tilde{\boldsymbol{\psi}}_h^c, \tilde{\boldsymbol{\psi}}_h^g \in \mathbf{V}_{p_v}(\mathcal{T}_h)$ we have with $\tilde{\boldsymbol{\psi}}_h = \tilde{\boldsymbol{\psi}}_h^c + \tilde{\boldsymbol{\psi}}_h^g$

$$\langle\langle \boldsymbol{e}^{\boldsymbol{\varphi}}, \boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\rangle\rangle = \langle\langle \boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla\times\boldsymbol{\rho}-\tilde{\boldsymbol{\psi}}_h^c\rangle\rangle + \langle\langle \boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla z-\tilde{\boldsymbol{\psi}}_h^g\rangle\rangle =: T^c + T^g.$$

We select $\tilde{\boldsymbol{\psi}}_h^c = \boldsymbol{\Pi}_h^{\mathrm{curl},\Gamma}\nabla\times\boldsymbol{\rho}$ and proceed with the terms T^c, T^g as in the proof of [9, Thm. 4.8] (see [10, Thm. 4.6] for details), to arrive at the analog of [9, (4.22)], which reads

$$\langle\langle \boldsymbol{e}^{\boldsymbol{\varphi}}, \boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\rangle\rangle \lesssim |||\nabla z - \tilde{\boldsymbol{\psi}}_h^g|||\cdot|||\boldsymbol{e}^{\boldsymbol{\varphi}}||| + \left[|||\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h||| + \frac{h}{p_v}\left\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\right\|_{L^2(\Omega)} + \frac{h}{p_v}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b\right]\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}. \tag{4.17}$$

To conclude the proof we estimate the quantities arising in the estimates (4.14) and (4.17). To that end note that $\nabla z\in\boldsymbol{H}^1(\mathrm{div},\Omega)$. Using the estimates of the Helmholtz decomposition, the equation satisfied by z, and the regularity estimates given by Theorem 3.5, we find

$$\|\nabla z\|_{\boldsymbol{H}^1(\mathrm{div},\Omega)} \lesssim \|z\|_{H^2(\Omega)} + \|\Delta z\|_{H^1(\Omega)} \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$
$$\|\nabla z\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} = \|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.$$

Exploiting these regularity estimates and employing the operator of Proposition 4.1 we may find $\tilde{\boldsymbol{\psi}}_h^g\in\mathbf{V}_{p_v}(\mathcal{T}_h)$ with

$$\|\nabla z - \tilde{\boldsymbol{\psi}}_h^g\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim h/p_v\,\|\nabla z\|_{\boldsymbol{H}^1(\mathrm{div},\Omega)} \lesssim h/p_v\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$
$$\|(\nabla z - \tilde{\boldsymbol{\psi}}_h^g)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim (h/p_v)^{1/2}\,\|\nabla z\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} \overset{\text{Prop. 4.1(ii)}}{\lesssim} (h/p_v)^{1/2}\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$
$$|||\nabla z - \tilde{\boldsymbol{\psi}}_h^g||| \lesssim (h/p_v)^{1/2}\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$

where the last one is just a combination of the previous ones. These estimates in turn give (note that $\nabla\cdot\tilde{\boldsymbol{\psi}}_h^c = \nabla\cdot\boldsymbol{\Pi}_h^{\mathrm{curl},\Gamma}\nabla\times\boldsymbol{\rho} = 0$)

$$\|\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))\|_{L^2(\Omega)} = \|\nabla\cdot(\nabla z - \tilde{\boldsymbol{\psi}}_h^g)\|_{L^2(\Omega)} \lesssim h/p_v\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$
$$\|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \le \|(\nabla\times\boldsymbol{\rho} - \boldsymbol{\Pi}_h^{\mathrm{curl},\Gamma}\nabla\times\boldsymbol{\rho})\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \|(\nabla z - \tilde{\boldsymbol{\psi}}_h^g)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \overset{(4.16)}{\lesssim} \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.$$

Furthermore there exists $\tilde{v}_h\in S_{p_s}(\mathcal{T}_h)$ with $\|v-\tilde{v}_h\|_{H^1(\Omega)} \lesssim h/p_s\,\|v\|_{H^2(\Omega)} \lesssim h/p_s\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. We then combine (4.14) and (4.17) to find

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|^2_{L^2(\Omega)} \lesssim (h/p)^{1/2}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} + (h/p)^{1/2}\cdot|||\boldsymbol{e}^{\boldsymbol{\varphi}}|||\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} + \left[|||\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h||| + h/p\left\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\right\|_{L^2(\Omega)} + h/p\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b\right]\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.$$

Canceling one power of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ on both sides, estimating $|||\boldsymbol{e}^{\boldsymbol{\varphi}}|||$ by $\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b$, and collecting the terms, we find

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim (h/p)^{1/2}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b + |||\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h||| + h/p\left\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\right\|_{L^2(\Omega)}.$$

The result follows from the observation that the FOSLS approximation is the orthogonal projection with respect to the b scalar product, the norm equivalence of Theorem 2.2, and collecting terms. □

Remark 4.7. Theorem 4.6 seems suboptimal in the following sense: Given $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Gamma)$, the shift theorem gives $u\in H^2(\Omega)$ and consequently $\boldsymbol{\varphi}\in\boldsymbol{H}^1(\Omega)$. Theorem 4.6 gives

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim h^{3/2}\,\|u\|_{H^2(\Omega)} + h\,\|\boldsymbol{\varphi}\|_{H^1(\Omega)} + h^{1/2}\|\boldsymbol{\varphi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} + h^{1/2}\,\|\nabla\cdot\boldsymbol{\varphi}\|_{L^2(\Omega)} \lesssim h^{1/2}(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}),$$

whereas from a best approximation viewpoint we could hope for $\mathcal{O}(h)$. □

Lemma 4.8 *(Convergence of dual solution for ∇e^u - Robin version of [9, Lem. 4.9]). Let Assumption 3.1 be valid for some $\hat{s}\ge 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Let $(\boldsymbol{\psi}, v)\in\boldsymbol{V}\times W$ be the dual solution given by Theorem 3.4 with $w = e^u$. Let $(\boldsymbol{\psi}_h, v_h)$ be the FOSLS approximation of $(\boldsymbol{\psi}, v)$ and abbreviate $e^v = v - v_h$ and $\boldsymbol{e}^{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then,*

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|\nabla e^u\|_{L^2(\Omega)}, \qquad \|e^v\|_{L^2(\Omega)} \lesssim \frac{h}{p}\,\|\nabla e^u\|_{L^2(\Omega)},$$
$$\|e^v\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\nabla e^u\|_{L^2(\Omega)}, \qquad \|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\nabla e^u\|_{L^2(\Omega)}.$$

Proof. Theorem 3.4 gives $\boldsymbol{\psi}\in\boldsymbol{H}^1(\Omega)$, $\nabla\cdot\boldsymbol{\psi}\in H^1(\Omega)$, and $v\in H^1(\Omega)$ which are controlled by $\|\nabla e^u\|_{L^2(\Omega)}$. Stability gives

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \le \|(\boldsymbol{\psi}, v)\|_b \lesssim \|\nabla e^u\|_{L^2(\Omega)}.$$

Lemma 4.5 gives

$$\|e^v\|_{L^2(\Omega)} \lesssim h/p\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b,$$

which together with the above stability result proves the second estimate. The third one follows by a multiplicative trace inequality together with the second estimate and the norm equivalence theorem in conjunction with the first estimate of the present lemma:

$$\|e^v\|_{L^2(\Gamma)} \lesssim \|e^v\|_{L^2(\Omega)}^{1/2} \|e^v\|_{H^1(\Omega)}^{1/2} \lesssim (h/p)^{1/2}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim (h/p)^{1/2} \|\nabla e^u\|_{L^2(\Omega)}.$$

By Theorem 4.6 we have, for any $\tilde{v}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\psi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$,

$$\|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{1/2} \|v - \tilde{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \left(\frac{h}{p}\right)^{1/2} \|\nabla\cdot(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\|_{L^2(\Omega)}.$$

The regularity of the dual solution together the approximation properties of the pertinent spaces then implies the result. □

Theorem 4.9 *(Suboptimal estimate for $\|\nabla e^u\|_{L^2(\Omega)}$ — suboptimal Robin version of [9, Thm. 4.10]). Let Assumption 3.1 be valid for some $\hat{s} \geq 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$. Then, for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$ there holds*

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \|u - \tilde{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. We proceed as in [9, Thm. 4.10] and denote by $(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)$ the FOSLS approximation of the dual solution given by Theorem 3.4 (duality argument for the gradient of the scalar variable) applied to the right-hand side $w = e^u$. We note that $\|v\|_{H^1(\Omega)}$, $\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)}$, and $\|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)}$ are controlled by $\|\nabla e^u\|_{L^2(\Omega)}$. By Galerkin orthogonality, we have for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$

$$\|e^u\|_{L^2(\Omega)}^2 = b((\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)). \tag{4.18}$$

We specifically choose $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}$. In what follows, we repeatedly use properties of the operator $\boldsymbol{I}_h^\Gamma$ collected in Lemma 4.4. Making use of the regularity properties of the dual solution spelled out in Theorem 3.4 and using Lemma 4.8 we get:

$$\begin{aligned}
(\gamma(u - \tilde{u}_h), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}} + \gamma e^v)_\Omega &\lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)} \|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|u - \tilde{u}_h\|_{H^1(\Omega)} \|\nabla e^u\|_{L^2(\Omega)},\\
(\nabla(u - \tilde{u}_h), \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|\nabla(u - \tilde{u}_h)\|_{L^2(\Omega)} \|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|u - \tilde{u}_h\|_{H^1(\Omega)} \|\nabla e^u\|_{L^2(\Omega)},\\
\langle -\alpha(u - \tilde{u}_h), \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n} - \alpha e^v\rangle_\Gamma &\lesssim \|u - \tilde{u}_h\|_{L^2(\Gamma)} \|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|u - \tilde{u}_h\|_{H^1(\Omega)} \|\nabla e^u\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}, \nabla e^v)_\Omega &= -(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}), e^v)_\Omega + \langle(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\cdot\boldsymbol{n}, e^v\rangle_\Gamma \lesssim \left[h/p\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)} + (h/p)^{1/2}|||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}|||\right] \|\nabla e^u\|_{L^2(\Omega)},\\
(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}), \gamma e^v)_\Omega &\leq \|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)} \|e^v\|_{L^2(\Omega)} \lesssim h/p\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)} \|\nabla e^u\|_{L^2(\Omega)},\\
\langle(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\cdot\boldsymbol{n}, -\alpha e^v\rangle_\Gamma &\leq \|(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \|e^v\|_{L^2(\Gamma)} \lesssim (h/p)^{1/2}|||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}||| \|\nabla e^u\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}, \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}\|_{L^2(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim (h/p)^{1/2}|||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}||| \|\nabla e^u\|_{L^2(\Omega)},\\
(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &= (\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}), \nabla\cdot(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h))_\Omega \lesssim h/p\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)} \|\nabla e^u\|_{L^2(\Omega)},\\
\langle(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\cdot\boldsymbol{n}, \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\rangle_\Gamma &\leq \|(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim |||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}||| \|\nabla e^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.
\end{aligned} \tag{4.19}$$

Inserting these bounds in (4.18), canceling one power of $\|\nabla e^u\|_{L^2(\Omega)}$ on both sides, and collecting the terms yields

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \|u - \tilde{u}_h\|_{H^1(\Omega)} + |||\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi}||| + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^\Gamma\boldsymbol{\varphi})\|_{L^2(\Omega)}.$$

Finally exploiting the estimates of the operator $\boldsymbol{I}_h^\Gamma$ we obtain at the asserted estimate. □

Remark 4.10. Theorem 4.9 seems again suboptimal: Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, the shift theorem gives $u \in H^2(\Omega)$ and consequently $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega)$. Theorem 4.9 gives

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim h\|u\|_{H^2(\Omega)} + h\|\boldsymbol{\varphi}\|_{H^1(\Omega)} + h^{1/2}\|\boldsymbol{\varphi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} + h\|\nabla\cdot\boldsymbol{\varphi}\|_{L^2(\Omega)} \lesssim h^{1/2}(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}),$$

whereas from a best approximation viewpoint we could hope for $\mathcal{O}(h)$. □

Theorem 4.11 *(Optimal estimate for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)}$). Let Assumption 3.1 be valid for some $\hat{s} \geq 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$, there holds*

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{1/2} \|u - \tilde{u}_h\|_{H^1(\Omega)} + \left(\frac{h}{p}\right)^{1/2} \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ denote the dual solution given by Theorem 3.6 with $\boldsymbol{\eta} = \boldsymbol{e}^{\boldsymbol{\varphi}}$. Theorem 3.6 asserts $\boldsymbol{\psi} \in \boldsymbol{H}^{1/2}(\Omega)$, $\nabla\cdot\boldsymbol{\psi} \in H^{3/2}(\Omega)$, $\boldsymbol{\psi}\cdot\boldsymbol{n} \in L^2(\Gamma)$, and $v \in H^{3/2}(\Omega)$, which are controlled by $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)}$. For the analysis we employ the operator $\Pi_{p_v}^{\mathrm{div}}$ from [29] discussed in Proposition 4.1. The main features exploited in the proof are that $\Pi_{p_v}^{\mathrm{div}}$ realizes the L^2 orthogonal projections of the divergence as well as the normal trace. By Galerkin orthogonality we have for any $(\tilde{\boldsymbol{\psi}}_h, \tilde{v}_h)$

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|^2_{L^2(\Gamma)} = b((\boldsymbol{e}^{\boldsymbol{\varphi}},e^u),(\boldsymbol{\psi},v)) = b((\boldsymbol{e}^{\boldsymbol{\varphi}},e^u),(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h, v-\tilde{v}_h)).$$

Choosing $\tilde{\boldsymbol{\psi}}_h = \Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi}$, we estimate

$$(\nabla\cdot\boldsymbol{e}^{\boldsymbol{\varphi}}+\gamma e^u, \nabla\cdot(\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})+\gamma(v-\tilde{v}_h))_\Omega \lesssim \|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b\left[\|\nabla\cdot(\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})\|_{L^2(\Omega)}+\|v-\tilde{v}_h\|_{L^2(\Omega)}\right],$$

$$(\nabla e^u+\boldsymbol{e}^{\boldsymbol{\varphi}}, \nabla(v-\tilde{v}_h)+\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})_\Omega \lesssim \left[\|\nabla e^u\|_{L^2(\Omega)}+\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}\right]\left[\|v-\tilde{v}_h\|_{H^1(\Omega)}+\|\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi}\|_{L^2(\Omega)}\right],$$

$$\langle -\alpha e^u, -\alpha(v-\tilde{v}_h)\rangle_\Gamma \lesssim \|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b\|v-\tilde{v}_h\|_{L^2(\Gamma)},$$

$$\langle \boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}, (\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_\Gamma \overset{\text{orth. of }\Pi^{\operatorname{div}}_{p_v}}{=} \langle(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}, (\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_\Gamma \lesssim \|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)}\|(\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)}.$$

The two missing boundary terms, i.e., $\langle \boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}, -\alpha(v-\tilde{v}_h)\rangle_\Gamma$ and $\langle -\alpha e^u, (\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_\Gamma$, can be written as volume terms by means of partial integration

$$\langle \boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}, -\alpha(v-\tilde{v}_h)\rangle_\Gamma = (\nabla\cdot\boldsymbol{e}^{\boldsymbol{\varphi}}, -\alpha(v-\tilde{v}_h))_\Omega + (\boldsymbol{e}^{\boldsymbol{\varphi}}, -\alpha\nabla(v-\tilde{v}_h))_\Omega,$$

$$\langle -\alpha e^u, (\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_\Gamma = (-\alpha\nabla e^u, (\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))_\Omega + (-\alpha e^u, \nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))_\Omega$$

and can therefore be controlled by the right-hand sides of the first two estimates. We now exploit the regularity estimates given in Theorem 3.6, the properties of $\Pi^{\operatorname{div}}_{p_v}$ given in Proposition 4.1 as well as the approximation properties of the employed spaces to find $\tilde{v}_h$ such that

$$\|\nabla\cdot(\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})\|_{L^2(\Omega)} \lesssim h/p_v\|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)} \lesssim h/p_v\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

$$\|\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi}\|_{L^2(\Omega)} \lesssim (h/p_v)^{1/2}\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{1/2}(\operatorname{div},\Omega)} \lesssim (h/p_v)^{1/2}\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

$$\|(\boldsymbol{\psi}-\Pi^{\operatorname{div}}_{p_v}\boldsymbol{\psi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

$$\|v-\tilde{v}_h\|_{L^2(\Omega)} \lesssim (h/p_s)^{3/2}\,\|v\|_{H^{3/2}(\Omega)} \lesssim (h/p_s)^{3/2}\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

$$\|v-\tilde{v}_h\|_{H^1(\Omega)} \lesssim (h/p_s)^{1/2}\,\|v\|_{H^{3/2}(\Omega)} \lesssim (h/p_s)^{1/2}\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

$$\|v-\tilde{v}_h\|_{L^2(\Gamma)} \lesssim h/p_s\,\|v\|_{H^{3/2}(\Omega)} \lesssim h/p_s\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)},$$

which in turn gives after collecting terms and canceling one power of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)}$ on both sides of the estimate

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim h/p\|(\boldsymbol{e}^{\boldsymbol{\varphi}},e^u)\|_b + (h/p)^{1/2}\left[\|\nabla e^u\|_{L^2(\Omega)}+\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}\right] + \|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)}.$$

Applying Theorems 4.6 and 4.9 to estimate $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ and $\|\nabla e^u\|_{L^2(\Omega)}$ yields the result. □

Remark 4.12. Theorem 4.11 seems optimal in the following sense: Given $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Gamma)$, the shift theorem gives $u\in H^2(\Omega)$ and consequently $\boldsymbol{\varphi}\in\boldsymbol{H}^1(\Omega)$. Theorem 4.11 gives

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim h^{3/2}\,\|u\|_{H^2(\Omega)} + h^{3/2}\,\|\boldsymbol{\varphi}\|_{H^1(\Omega)} + h^{1/2}\|\boldsymbol{\varphi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)} + h\,\|\nabla\cdot\boldsymbol{\varphi}\|_{L^2(\Omega)} \lesssim h^{1/2}(\|f\|_{L^2(\Omega)}+\|g\|_{H^{1/2}(\Gamma)}),$$

which is the rate expected from a best approximation argument. □

We are in position to derive an optimal estimate for $\|\nabla e^u\|_{L^2(\Omega)}$ using the estimate given in Theorem 4.11.

Theorem 4.13 *(Optimal estimate for $\|\nabla e^u\|_{L^2(\Omega)}$ — Robin version of [9, Theorem 4.10]). Let Assumption 3.1 be valid for some $\hat{s}\geq 0$. Let $(\boldsymbol{\varphi}_h,u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi},u)$. Set $e^u = u-u_h$. Then, for any $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h\in S_{p_s}(\mathcal{T}_h)$, there holds*

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \|u-\tilde{u}_h\|_{H^1(\Omega)} + \left(\frac{h}{p}\right)^{1/2}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \left(\frac{h}{p}\right)^{1/2}\|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. We refine the proof of Theorem 4.9 making use of Theorem 4.11. To that end, we recall (4.19), which stated

$$\langle(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n}, \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\rangle_\Gamma \leq \|(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)}\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \overset{(*)}{\lesssim} |||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\,\|\nabla e^u\|_{L^2(\Omega)}. \tag{4.20}$$

This estimate can now be improved by refining $\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b$ with the aid of Theorem 4.11, which, together with the available regularity assertions for the dual solution asserted in Theorem 3.4, gives the bound

$$\|\boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|v-\tilde{v}_h\|_{H^1(\Omega)} + \left(\frac{h}{p}\right)^{1/2}\|\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h)\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\nabla e^u\|_{L^2(\Omega)};$$

in turn this enables us to sharpen the bound for $\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b$ and improve $\overset{(*)}{\lesssim}$ in (4.20) to get

$$\langle(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n}, \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\rangle_\Gamma \lesssim (h/p)^{1/2}|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\,\|\nabla e^u\|_{L^2(\Omega)}.$$

All other estimates in the proof of Theorem 4.9 stay the same. Canceling one power of $\|\nabla e^u\|_{L^2(\Omega)}$ on both sides and collecting the terms yields

$$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \|u-\tilde{u}_h\|_{H^1(\Omega)} + \left(\frac{h}{p}\right)^{1/2}|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}||| + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)}.$$

Finally exploiting the estimates of the operator $\boldsymbol{I}^\Gamma_h$ in Lemma 4.4 we arrive at the asserted estimate. □

Before turning to the estimate for $\|e^u\|_{L^2(\Omega)}$ we first derive a slightly better version of Theorem 4.6. To that end we first analyze the convergence of the corresponding dual solution:

Lemma 4.14 *(Convergence of dual solution for $\boldsymbol{e}^{\boldsymbol{\varphi}}$). Let Assumption 3.1 be valid for some $\hat{s} \geq 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ be the dual solution given by Theorem 3.5 with $\boldsymbol{\eta} = \boldsymbol{e}^{\boldsymbol{\varphi}}$. Let $(\boldsymbol{\psi}_h, v_h)$ be the FOSLS approximation of $(\boldsymbol{\psi}, v)$. Denote $e^v = v - v_h$ and $\boldsymbol{e}^{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then,*

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}, \qquad \|e^v\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}, \qquad \|e^v\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},$$
$$\|\nabla e^v\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}, \qquad \|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}, \qquad \|\boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{1/2}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.$$

Proof. Theorem 3.5 gives $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, $\nabla\cdot\boldsymbol{\psi} \in H^1(\Omega)$, $\boldsymbol{\psi}\cdot\boldsymbol{n} \in H^{1/2}(\Omega)$, and $v \in H^2(\Omega)$, which are controlled by $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. Stability of FOSLS gives

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \leq \|(\boldsymbol{\psi}, v)\|_b \lesssim \|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}. \tag{4.21}$$

Lemma 4.5 provides

$$\|e^v\|_{L^2(\Omega)} \lesssim h/p\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b,$$

and together with (4.21) we arrive at the second estimate. The third one follows by a multiplicative trace inequality together with the second estimate and the norm equivalence theorem in conjunction with the first estimate of the present lemma:

$$\|e^v\|_{L^2(\Gamma)} \lesssim \|e^v\|_{L^2(\Omega)}^{1/2}\|e^v\|_{H^1(\Omega)}^{1/2} \lesssim (h/p)^{1/2}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim (h/p)^{1/2}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.$$

Theorems 4.13, 4.6 and 4.11 then yield the remaining three estimates by combining the regularity assertions for the dual solution with the approximation properties of the finite element spaces. □

Theorem 4.15 *(Suboptimal but improved estimate for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ — Robin version of [9, Thm. 4.8]). Let Assumption 3.1 be valid for some $\hat{s} \geq 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for any $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$, there holds*

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)} + \left(\frac{h}{p}\right)^{1/2}\|u - \tilde{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \frac{h}{p}\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. We proceed as in the proof of Theorem 4.9. Let $(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)$ be the FOSLS approximation error of the dual solution $(\boldsymbol{\psi}, v)$ given by Theorem 3.5 (duality argument for the vector variable) corresponding to the right-hand side $\boldsymbol{\eta} = \boldsymbol{e}^{\boldsymbol{\varphi}}$. We have that $\|v\|_{H^2(\Omega)}$, $\|\boldsymbol{\psi}\|_{L^2(\Omega)}$, $\|\nabla\cdot\boldsymbol{\psi}\|_{H^1(\Omega)}$, $\|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)}$ are controlled by $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. As before for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}^2 = b((\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h, u - \tilde{u}_h), (\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)).$$

We again choose $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}$, repeatedly use properties of the operator $\boldsymbol{I}_h^{\Gamma}$ collected in Lemma 4.4, utilize the regularity properties of the dual solution given in Theorem 3.5, and apply Lemma 4.14 to get:

$$\begin{aligned}
(\gamma(u - \tilde{u}_h), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
\langle -\alpha(u - \tilde{u}_h), \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n} - \alpha e^v\rangle_\Gamma &\lesssim \|u - \tilde{u}_h\|_{L^2(\Gamma)}\left[\|\boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \|e^v\|_{L^2(\Gamma)}\right] \lesssim (h/p)^{1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
(\nabla(u - \tilde{u}_h), \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &= -(u - \tilde{u}_h, \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega + \langle u - \tilde{u}_h, \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\rangle_\Gamma \lesssim \left[\|u - \tilde{u}_h\|_{L^2(\Omega)} + (h/p)^{1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)}\right]\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
(\gamma(u - \tilde{u}_h), \gamma e^v)_\Omega &\lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)}\|e^v\|_{L^2(\Omega)} \lesssim h/p\|u - \tilde{u}_h\|_{H^1(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
(\nabla(u - \tilde{u}_h), \nabla e^v)_\Omega &\lesssim \|\nabla(u - \tilde{u}_h)\|_{L^2(\Omega)}\|\nabla e^v\|_{L^2(\Omega)} \lesssim (h/p)^{1/2}\|u - \tilde{u}_h\|_{H^1(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}, \nabla e^v + \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\leq \|\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}\|_{L^2(\Omega)}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim |||\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}|||\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
\langle(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\cdot\boldsymbol{n}, \boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n} - \alpha e^v\rangle_\Gamma &\leq \|(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)}\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim |||\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}|||\,\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)},\\
(\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}), \nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &= (\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}), \nabla\cdot(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h))_\Omega \lesssim h/p\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\|_{L^2(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}.
\end{aligned}$$

Canceling one power of $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ on both sides and summarizing we find

$$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \|u - \tilde{u}_h\|_{L^2(\Omega)} + (h/p)^{1/2}\|u - \tilde{u}_h\|_{L^2(\Gamma)} + (h/p)^{1/2}\|u - \tilde{u}_h\|_{H^1(\Omega)} + |||\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}||| + h/p\|\nabla\cdot(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\|_{L^2(\Omega)}.$$

A trace estimate, using the estimates of the operator $\boldsymbol{I}_h^{\Gamma}$ in Lemma 4.4, and collecting the terms yields the result. □

Lemma 4.16 *(Convergence of dual solution for e^u — Robin version of [9, Lem. 4.11]). Let Assumption 3.1 be valid for some $\hat{s} \geq 0$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Set $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Let $(\boldsymbol{\psi}, v) \in \boldsymbol{V} \times W$ be the dual solution given by Theorem 3.3 with $w = e^u$. Furthermore, let $(\boldsymbol{\psi}_h, v_h)$ be the FOSLS approximation of $(\boldsymbol{\psi}, v)$ and denote $e^v = v - v_h$ and $\boldsymbol{e}^{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$. Then,*

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}}, e^v)\|_b \lesssim \frac{h}{p}\|e^u\|_{L^2(\Omega)}, \qquad \|e^v\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^2\|e^u\|_{L^2(\Omega)}, \qquad \|e^v\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{3/2}\|e^u\|_{L^2(\Omega)},$$

$$\|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim \begin{cases} h\,\|e^u\|_{L^2(\Omega)} & \text{if } \mathbf{V}_{p_v}(\mathcal{T}_h)=\mathbf{RT}_0(\mathcal{T}_h),\\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)}\|e^u\|_{L^2(\Omega)} & \text{else,}\end{cases}$$

$$\|\boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \begin{cases} h\,\|e^u\|_{L^2(\Omega)} & \text{if } \mathbf{V}_{p_v}(\mathcal{T}_h)=\mathbf{RT}_0(\mathcal{T}_h),\\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)}\|e^u\|_{L^2(\Omega)} & \text{else.}\end{cases}$$

Proof. Theorem 3.3 gives $\boldsymbol{\psi}\in\boldsymbol{H}^{\min(\hat{s}+1,2)}(\Omega)$, $\nabla\cdot\boldsymbol{\psi}\in H^2(\Omega)$, $\boldsymbol{\psi}\cdot\boldsymbol{n}\in H^{3/2}(\Gamma)$, and $v\in H^2(\Omega)$, which are controlled by $\|e^u\|_{L^2(\Omega)}$. In view of the optimality of the b-norm we have:

$$\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \lesssim \frac{h}{p}\,\|e^u\|_{L^2(\Omega)}. \tag{4.22}$$

By Lemma 4.5 we have

$$\|e^v\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b,$$

which together with (4.22) gives the second estimate of the lemma. The third estimate of the lemma follows by a multiplicative trace inequality together with the second estimate and the norm equivalence theorem in conjunction with the first estimate of the present lemma:

$$\|e^v\|_{L^2(\Gamma)} \lesssim \|e^v\|^{1/2}_{L^2(\Omega)}\,\|e^v\|^{1/2}_{H^1(\Omega)} \lesssim (h/p)^{3/2}\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \lesssim (h/p)^{3/2}\,\|e^u\|_{L^2(\Omega)}.$$

Theorems 4.6 and 4.11 then yield the remaining estimates by exploiting the regularity of the dual solution and the approximation properties of the employed spaces. ☐

Theorem 4.17 *(Optimal estimate for $\|e^u\|_{L^2(\Omega)}$ — Robin version of [9, Theorem 4.12]). Let Assumption 3.1 be valid for some $\hat{s}\geq 0$. Let $(\boldsymbol{\varphi}_h,u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi},u)$. Furthermore, let $e^u=u-u_h$. Then, for any $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h\in S_{p_s}(\mathcal{T}_h)$, there holds:*
If $\mathbf{V}^0_{p_v}(\mathcal{T}_h)=\mathbf{RT}_0(\mathcal{T}_h)$, then

$$\|e^u\|_{L^2(\Omega)} \lesssim h\,\|u-\tilde{u}_h\|_{H^1(\Omega)} + h\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + h\|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + h\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)};$$

if $\mathbf{V}^0_{p_v}(\mathcal{T}_h)=\mathbf{BDM}_1(\mathcal{T}_h)$, then

$$\|e^u\|_{L^2(\Omega)} \lesssim h\,\|u-\tilde{u}_h\|_{H^1(\Omega)} + h^{\min(\hat{s}+1,3/2)}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + h^{\min(\hat{s}+1,3/2)}\|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + h\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)};$$

if $\mathbf{V}^0_{p_v}(\mathcal{T}_h)\notin\{\mathbf{RT}_0(\mathcal{T}_h),\mathbf{BDM}_1(\mathcal{T}_h)\}$, then

$$\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p}\,\|u-\tilde{u}_h\|_{H^1(\Omega)} + \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)}\|(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} + \left(\frac{h}{p}\right)^2\|\nabla\cdot(\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}.$$

Proof. We proceed as in the proof of Theorem 4.9 with $(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)$ denoting the FOSLS approximation of the dual solution given by Theorem 3.3 (duality argument for the scalar variable) applied to $w=e^u$ As before for any $\tilde{\boldsymbol{\varphi}}_h\in\mathbf{V}_{p_v}(\mathcal{T}_h)$, $\tilde{u}_h\in S_{p_s}(\mathcal{T}_h)$

$$\|e^u\|^2_{L^2(\Omega)} = b((\boldsymbol{\varphi}-\tilde{\boldsymbol{\varphi}}_h,u-\tilde{u}_h),(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)).$$

We again choose $\tilde{\boldsymbol{\varphi}}_h=\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}$, utilize properties of the operator $\boldsymbol{I}^\Gamma_h$ collected in Lemma 4.4, make use of the regularity assertions for the dual solution of Theorem 3.3, and apply Lemma 4.16:

$$\begin{aligned}
(\gamma(u-\tilde{u}_h),\nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}}+\gamma e^v)_\Omega &\lesssim \|u-\tilde{u}_h\|_{L^2(\Omega)}\,\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \lesssim h/p\,\|u-\tilde{u}_h\|_{H^1(\Omega)}\,\|e^u\|_{L^2(\Omega)},\\
(\nabla(u-\tilde{u}_h),\nabla e^v+\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega &\lesssim \|\nabla(u-\tilde{u}_h)\|_{L^2(\Omega)}\,\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \lesssim h/p\,\|u-\tilde{u}_h\|_{H^1(\Omega)}\,\|e^u\|_{L^2(\Omega)},\\
\langle-\alpha(u-\tilde{u}_h),\boldsymbol{e}^{\boldsymbol{\psi}}\cdot\boldsymbol{n}-\alpha e^v\rangle_\Gamma &\lesssim \|u-\tilde{u}_h\|_{L^2(\Gamma)}\,\|(\boldsymbol{e}^{\boldsymbol{\psi}},e^v)\|_b \lesssim h/p\,\|u-\tilde{u}_h\|_{H^1(\Omega)}\,\|e^u\|_{L^2(\Omega)},\\
(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi},\nabla e^v)_\Omega &= -(\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}),e^v)_\Omega + \langle(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n},e^v\rangle_\Gamma \lesssim \left[(h/p)^2\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)} + (h/p)^{3/2}|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\right]\|e^u\|_{L^2(\Omega)},\\
(\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}),\gamma e^v)_\Omega &\leq \|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)}\,\|e^v\|_{L^2(\Omega)} \lesssim (h/p)^2\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)}\,\|e^u\|_{L^2(\Omega)},\\
\langle(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n},-\alpha e^v\rangle_\Gamma &\leq \|(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\cdot\boldsymbol{n}\|_{L^2(\Gamma)}\,\|e^v\|_{L^2(\Gamma)} \lesssim (h/p)^{3/2}|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\,\|e^u\|_{L^2(\Omega)},
\end{aligned}$$

$$(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi},\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega \lesssim \|\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}\|_{L^2(\Omega)}\|\boldsymbol{e}^{\boldsymbol{\psi}}\|_{L^2(\Omega)} \lesssim \begin{cases} h|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\,\|e^u\|_{L^2(\Omega)} & \text{if } \mathbf{V}_{p_v}(\mathcal{T}_h)=\mathbf{RT}_0(\mathcal{T}_h),\\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)}|||\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}|||\,\|e^u\|_{L^2(\Omega)} & \text{else,}\end{cases}$$

$$(\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}),\nabla\cdot\boldsymbol{e}^{\boldsymbol{\psi}})_\Omega = (\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi}),\nabla\cdot(\boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}_h))_\Omega \lesssim \begin{cases} h\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)}\,\|e^u\|_{L^2(\Omega)} & \text{if } p_v=1,\\ \left(\frac{h}{p}\right)^2\|\nabla\cdot(\boldsymbol{\varphi}-\boldsymbol{I}^\Gamma_h\boldsymbol{\varphi})\|_{L^2(\Omega)}\,\|e^u\|_{L^2(\Omega)} & \text{else,}\end{cases}$$

$$\langle (\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\cdot \boldsymbol{n}, \boldsymbol{e}^{\boldsymbol{\Psi}}\cdot \boldsymbol{n}\rangle_\Gamma \le \|(\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi})\cdot \boldsymbol{n}\|_{L^2(\Gamma)} \|\boldsymbol{e}^{\boldsymbol{\Psi}}\cdot \boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \begin{cases} h|||\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}|||\,\|e^u\|_{L^2(\Omega)} & \text{if } \mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_0(\mathcal{T}_h), \\ \left(\frac{h}{p}\right)^{\min(\hat{s}+1,3/2)} |||\boldsymbol{\varphi} - \boldsymbol{I}_h^{\Gamma}\boldsymbol{\varphi}|||\,\|e^u\|_{L^2(\Omega)} & \text{else.} \end{cases}$$

Canceling one power of $\|e^u\|_{L^2(\Omega)}$ on both sides, using the estimates of the operator $\boldsymbol{I}_h^{\Gamma}$ and collecting the terms yields the result. □

Corollary 4.18. *Let Γ be smooth, $f \in H^s(\Omega)$ and $g \in H^{s+1/2}(\Gamma)$ for some $s \ge 0$, and denote $C_{f,g} := \|f\|_{H^s(\Omega)} + \|g\|_{H^{s+1/2}(\Gamma)}$. Then the solution to (2.2) satisfies $u \in H^{s+2}(\Omega)$, $\boldsymbol{\varphi} \in \boldsymbol{H}^{s+1}(\Omega)$, $\boldsymbol{\varphi}\cdot\boldsymbol{n} \in \boldsymbol{H}^{s+1/2}(\Gamma)$, and $\nabla\cdot\boldsymbol{\varphi} \in H^s(\Omega)$. Let $(\boldsymbol{\varphi}_h, u_h)$ be the FOSLS approximation of $(\boldsymbol{\varphi}, u)$. Let $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\phi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$. Then, for the lowest order case $p_v = 1$,*

$$\|e^u\|_{L^2(\Omega)} \lesssim h^{\min(s+1,2)} \|f\|_{H^s(\Omega)}.$$

For $p_v > 1$ there holds

$\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$	$\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h)$
$\|e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1,p_s,p_v+1/2)+1} C_{f,g}$	$\|e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1,p_s,p_v+1)+1} C_{f,g}$
$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1,p_s,p_v+1/2)} C_{f,g}$	$\|\nabla e^u\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1,p_s,p_v+1)} C_{f,g}$
$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1/2,p_s+1/2,p_v)} C_{f,g}$	$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1/2,p_s+1/2,p_v+1)} C_{f,g}.$
$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1/2,p_s+1/2,p_v)} C_{f,g}$	$\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)} \lesssim \left(\frac{h}{p}\right)^{\min(s+1/2,p_s+1/2,p_v+1)} C_{f,g}.$

Proof. By the smoothness of Γ, Assumption 3.1 holds for any $\hat{s}$. The regularity of $\boldsymbol{\varphi}$ follows from $\boldsymbol{\varphi} = -\nabla u$. We next inspect the quantities in the estimates of Theorems 4.11, 4.13, 4.15, and 4.17:

$$\begin{aligned}
\|u - \tilde{u}_h\|_{L^2(\Omega)} &\lesssim (h/p)^{\min(s+1,p_s)+1} \|u\|_{H^{s+2}(\Omega)} \lesssim (h/p)^{\min(s+1,p_s)+1} C_{f,g}, \\
\|u - \tilde{u}_h\|_{H^1(\Omega)} &\lesssim (h/p)^{\min(s+1,p_s)} \|u\|_{H^{s+2}(\Omega)} \lesssim (h/p)^{\min(s+1,p_s)} C_{f,g}, \\
\|u - \tilde{u}_h\|_{L^2(\Gamma)} &\lesssim (h/p)^{\min(s+1,p_s)+1/2} \|u\|_{H^{s+2}(\Omega)} \lesssim (h/p)^{\min(s+1,p_s)+1/2} C_{f,g}, \\
\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} &\lesssim \begin{cases} (h/p)^{\min(s+1,p_v)} \|\boldsymbol{\varphi}\|_{H^{s+1}(\Omega)} \lesssim (h/p)^{\min(s+1,p_v)} C_{f,g} & \mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h), \\ (h/p)^{\min(s+1,p_v+1)} \|\boldsymbol{\varphi}\|_{H^{s+1}(\Omega)} \lesssim (h/p)^{\min(s+1,p_v+1)} C_{f,g} & \mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h), \end{cases} \\
\|(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\cdot\boldsymbol{n}\|_{L^2(\Gamma)} &\lesssim \begin{cases} (h/p)^{\min(s+1/2,p_v)} \|\boldsymbol{\varphi}\|_{H^{s+1}(\Omega)} \lesssim (h/p)^{\min(s+1/2,p_v)} C_{f,g} & \mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h), \\ (h/p)^{\min(s+1/2,p_v+1)} \|\boldsymbol{\varphi}\|_{H^{s+1}(\Omega)} \lesssim (h/p)^{\min(s+1/2,p_v+1)} C_{f,g} & \mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h), \end{cases} \\
\|\nabla\cdot(\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)} &\lesssim (h/p)^{\min(s,p_v)} \|\nabla\cdot\boldsymbol{\varphi}\|_{H^s(\Omega)} \lesssim h^{\min(s,p_v)} C_{f,g}.
\end{aligned}$$

The bounds in Theorems 4.11, 4.13, 4.15 and 4.17 together with the above estimates give the asserted rates. □

Remark 4.19. Note that Corollary 4.18 predicts the same rates for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ and $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\cdot\boldsymbol{n}\|_{L^2(\Gamma)}$. This again suggests the suboptimality of the estimate for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. □

5. Numerical examples

Our numerical examples are obtained with hp-FEM code NETGEN/NGSOLVE by J. Schöberl, [35,36]. In Example 5.1 we consider a right-hand side $f \in \cap_{\varepsilon>0} \left(H^{1/2-\varepsilon}(\Omega)\right) \setminus H^{1/2}(\Omega)$ so that $u \in \cap_{\varepsilon>0} H^{5/2-\varepsilon}(\Omega)$ and $\boldsymbol{\varphi} \in \cap_{\varepsilon>0} \boldsymbol{H}^{3/2-\varepsilon}(\Omega)$. In all graphs presented, we plot the actual numerical results (red dots), the rate that is guaranteed by Corollary 4.18 (in black with the number written out near the bottom right), and a reference line for the best rate possible with the employed space $S_{p_s}(\mathcal{T}_h)$ or $\mathbf{V}_{p_v}(\mathcal{T}_h)$ given the Sobolev regularity of the solution u (in blue with the number written out near the top left).

Example 5.1. Our computational domain Ω is the unit sphere in $\mathbb{R}^2$, and we take $f(x,y) = \mathbb{1}_{[0,1/2]}(\sqrt{x^2+y^2})$, which is a step function supported by a disk of radius $1/2$. In (2.1) we set $\gamma = 2$ and $\alpha = 1$. The exact solution u is determined by the condition $\partial_n u = 0$ on Γ. The right-hand side boundary data g is calculated according to the choice $\alpha = 1$. The solution has finite regularity $u \in H^{5/2-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. We perform both the h-version and the p-version of the FOSLS method. The solution u is in fact piecewise smooth, but the meshes employed for both the h-version and the p-version are not aligned with the regions of smoothness of u, but rather such that the meshes do not resolve the circle of radius $1/2$, where the solution u has limited regularity. Regarding the h-version, we employ every combination of $\mathbf{V}_{p_v}(\mathcal{T}_h)$ and $S_{p_s}(\mathcal{T}_h)$ for $p_v, p_s \in \{1,2,3,4,5\}$. For the p-version, we select a fixed mesh with mesh size $h \approx 0.6$, chose $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$ and $p_v = p_s = p$, for $p = 1,\dots,27$. The numerical results for the h-version are plotted in Figs. 1 and 2 for $\|e^u\|_{L^2(\Omega)}$, in Figs. 3 and 4 for $\|\nabla e^u\|_{L^2(\Omega)}$, and in Figs. 5 and 6 for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. The numerical results for the p-version are plotted in Fig. 7. We make the following observations:

- In Figs. 1 and 2 for $\|e^u\|_{L^2(\Omega)}$, we observe that the convergence rates asserted in Corollary 4.18 are attained. However, for the lowest order case $p_v = 1$, the theoretical results seem suboptimal.

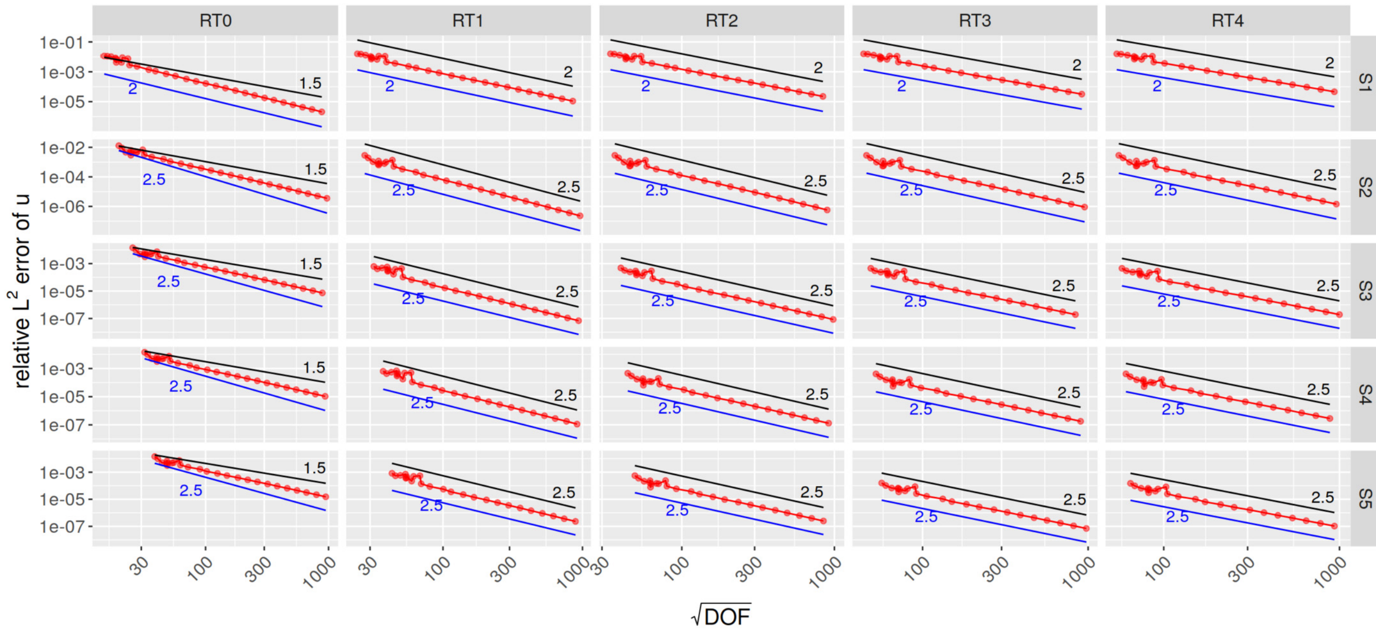

Fig. 1. h-convergence of $\|e^u\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$, see Example 5.1.

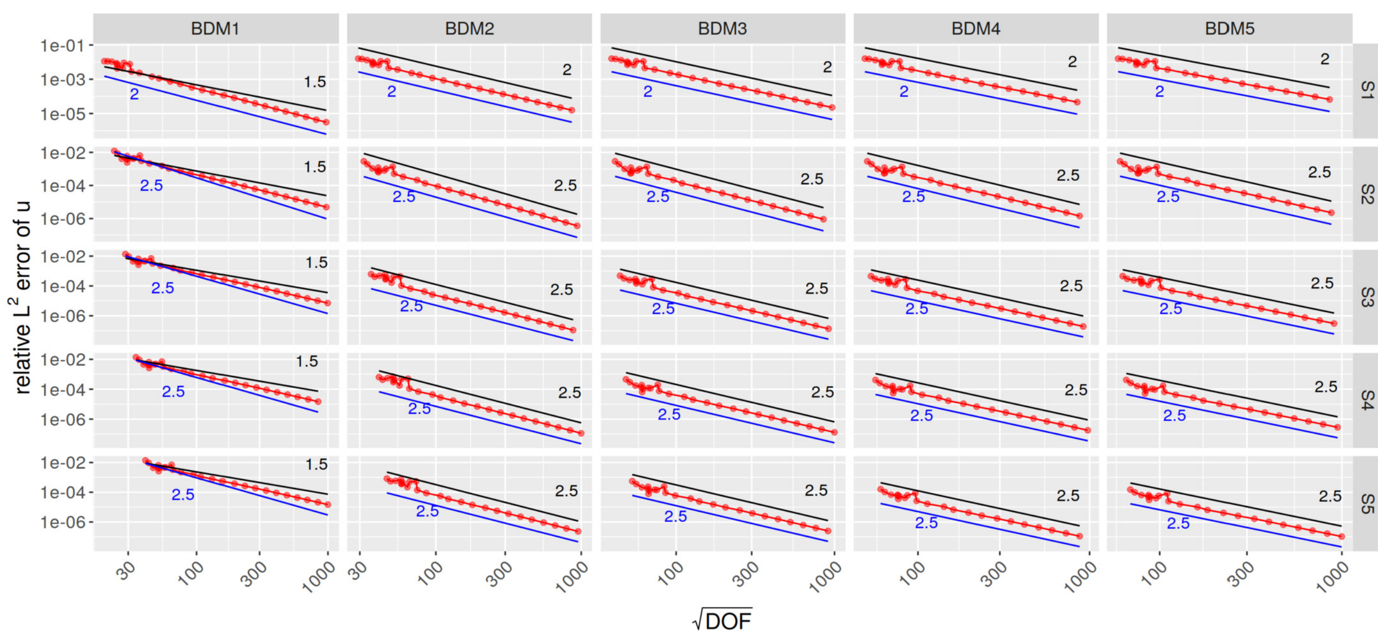

Fig. 2. h-convergence of $\|e^u\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h)$, see Example 5.1.

- In Figs. 3 and 4 for $\|\nabla e^u\|_{L^2(\Omega)}$, the convergence rates guaranteed by Corollary 4.18 are achieved and are optimal in terms of the regularity of the data.
- In Figs. 5 and 6 for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$, we observe better convergence than ensured by Corollary 4.18, which is in agreement with our comments in Remarks 4.7 and 4.19.
- In Fig. 7 for the p-version of the FOSLS method, we observe the convergence rate predicted by Corollary 4.18 and that it is optimal for $\|e^u\|_{L^2(\Omega)}$ and $\|\nabla e^u\|_{L^2(\Omega)}$. We again note the suboptimality of our estimates for $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$. Finally, for $\|\boldsymbol{e}^{\boldsymbol{\varphi}} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}$ we remark convergence $\mathcal{O}(p^{-2.5})$ (with a green reference line), whereas Corollary 4.18 merely guarantees convergence $\mathcal{O}(p^{-1})$. Since the solution $(\boldsymbol{\varphi}, u)$ is smooth near Γ a local error analysis near Γ could possibly explain this super-convergence behavior.

Data availability

Data will be made available on request.

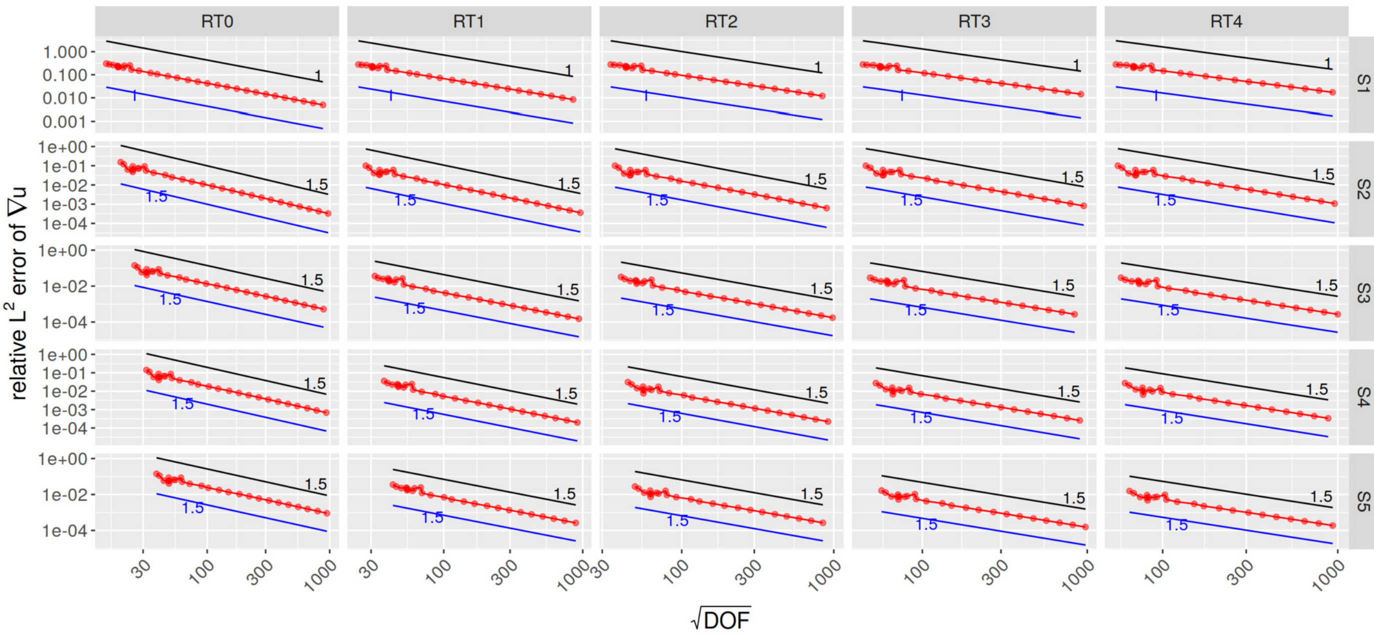

Fig. 3. h-convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$, see Example 5.1.

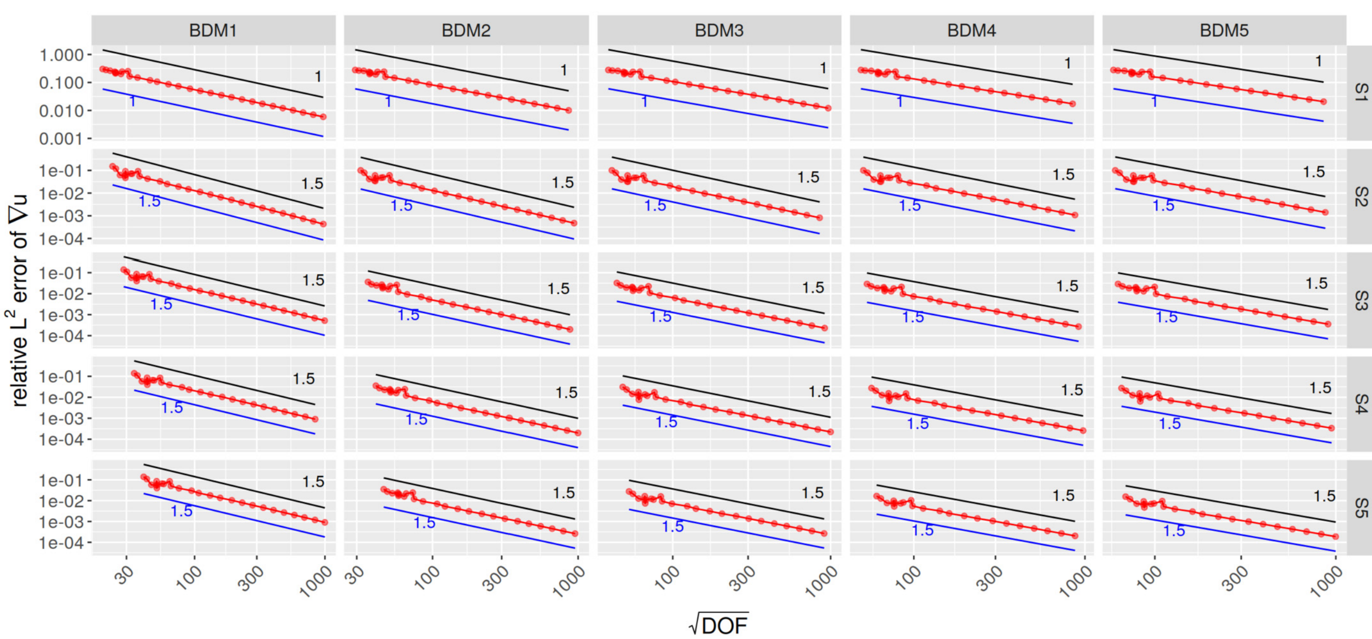

Fig. 4. h-convergence of $\|\nabla e^u\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h)$, see Example 5.1.

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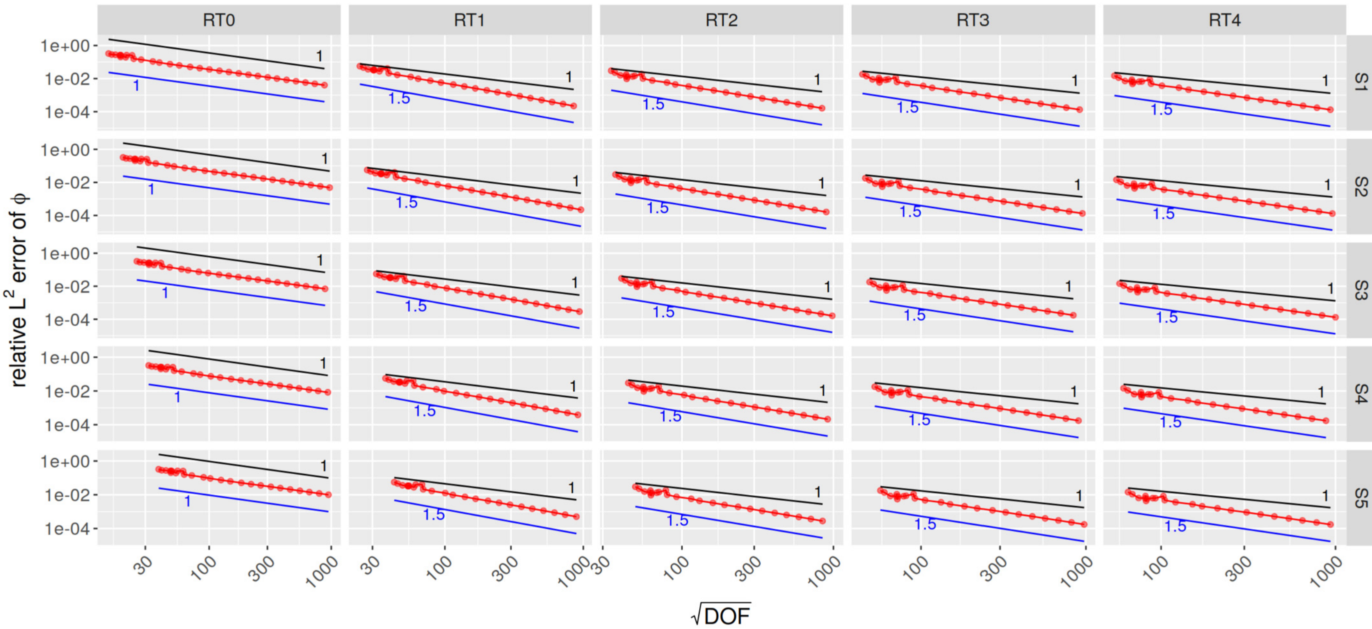

Fig. 5. h-convergence of $\|e^{\varphi}\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$, see Example 5.1.

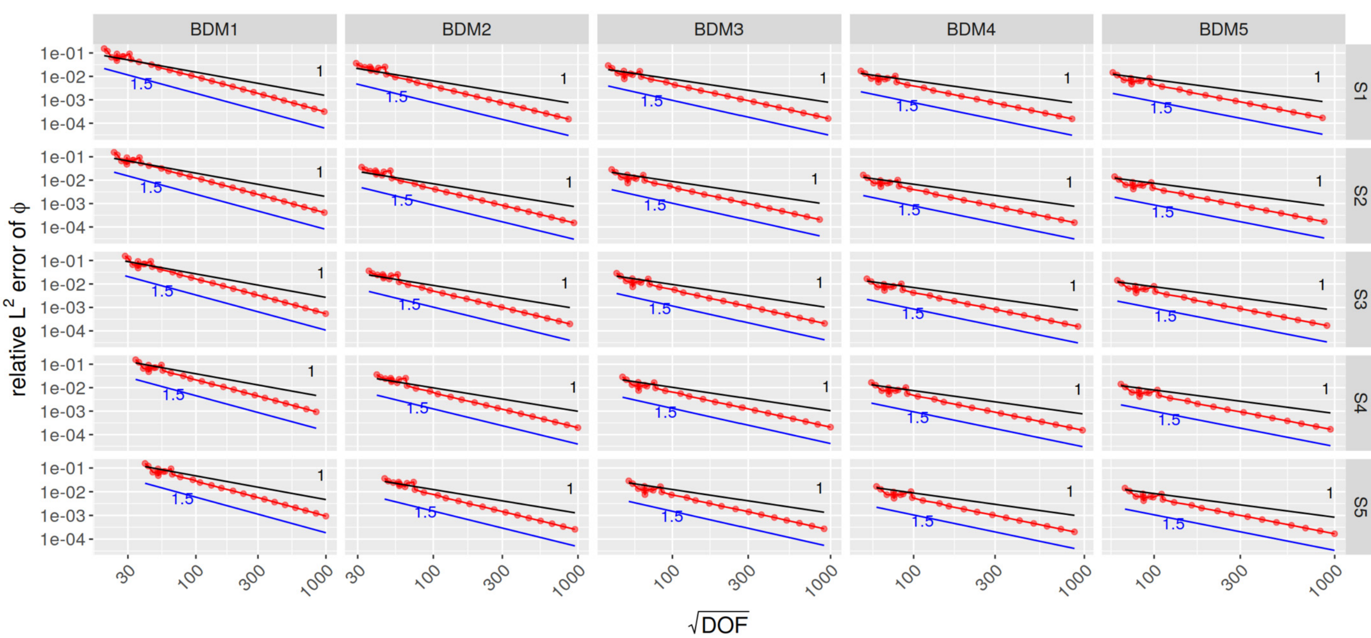

Fig. 6. h-convergence of $\|e^{\varphi}\|_{L^2(\Omega)}$ using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{BDM}_{p_v}(\mathcal{T}_h)$, see Example 5.1.

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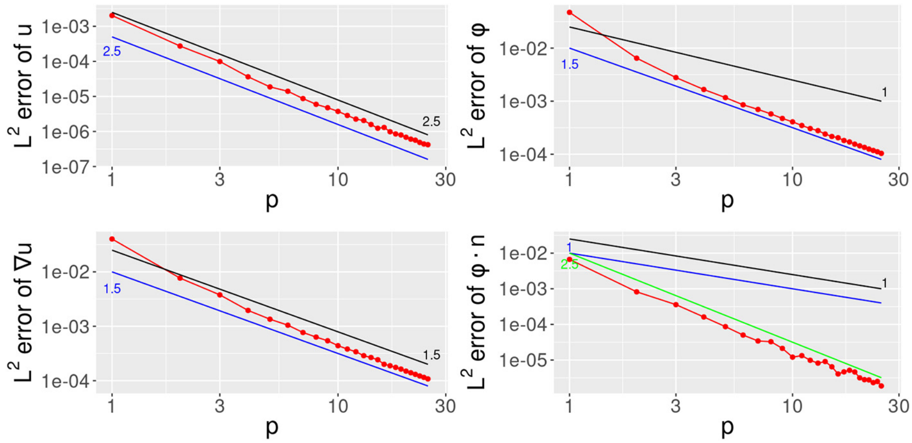

Fig. 7. p-convergence of $\|e^u\|_{L^2(\Omega)}$ (top left), $\|\nabla e^u\|_{L^2(\Omega)}$ (bottom left), $\|\boldsymbol{e}^{\boldsymbol{\varphi}}\|_{L^2(\Omega)}$ (top right), $\|\boldsymbol{e}^{\boldsymbol{\varphi}} \cdot \boldsymbol{n}\|_{L^2(\Gamma)}$ (bottom right) using $\mathbf{V}_{p_v}(\mathcal{T}_h) = \mathbf{RT}_{p_v-1}(\mathcal{T}_h)$ and $S_{p_s}(\mathcal{T}_h)$ with $p_v = p_s$, see Example 5.1.

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