DISSERTATION

Analysis of nonlocal cross-diffusion and nonlinear drift-diffussion Systems using Entropy Methods

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Kurzfassung

Kreuzdiffusion ist ein Phänomen in Systemen, bei dem die Diffusion einer Spezies, also der Ausgleich des Konzentrationsunterschiedes innerhalb selbiger, durch den Gradienten anderer im System vorhandener Spezies induziert wird. Aufgrund ihres häufigen Auftretens zum Beispiel in biologischen Systemen oder Prozessen sowie chemischen Reaktionen wurden Kreuzdiffusionssysteme in den letzten Jahrzehnten intensiv untersucht.
Diffusive Systeme mit Drift, in denen die Bewegung von Partikeln nicht nur durch die Änderung von deren Konzentration induziert wird, sondern auch durch eine Kraft, welche mit den Partikeln interagiert, sind wiederum von großem Interesse, da diese mitunter Halbleiter modellieren – ein kleiner jedoch umso essentiellerer Bauteil in den meisten elektrischen Schaltkreisen heutzutage.

Entropiemethoden nutzen die natürliche Präsenz eines Lyapunovfunktionales zur Studie partieller Differentialgleichungen aus und haben sich als mächtiges Werkzeug in der Analyse von Kreuzdiffusions- und Driftdiffusionssystemen erwiesen. Besonders wenn klassische Methoden wie zum Beispiel Maximumprinzipien, (elliptische) Regularitätstheorie, Monotonieargumente, etc. nicht anwendbar sind erweisen sie sich als praktisches Instrument. In der vorliegenden Arbeit illustrieren wir dies anhand unterschiedlichster Anwendungen.

Im ersten Teil der Arbeit betrachten wir Systeme von partiellen Differentialgleichungen, die die Evolution von Populationsdichten oder Zustände von Neuronen beschreiben. Interaktionen zwischen den verschiedenen Spezies führen zu Kreuzdiffusionstermen in den Gleichungen. Weiters erhält man nichtlokale Terme in den Gleichungen, wenn man berücksichtigt, dass Individuen oftmals einen gewissen Interaktionsradius oder eine „Interaktionsdistanz“ haben, beispielsweise deren Sichtfeld oder die nähere Umgebung. Ziel dieses Teiles der Arbeit ist das Erweitern bereits bestehender Resultate für Systeme mit einer beliebigen Anzahl an Gleichungen, wobei wir „minimale“ Forderungen an die Nichtlokalitäten stellen, sowie die Entwicklung eines zuverlässigen numerischen Verfahrens zur approximativen Lösung der Gleichungen.

Die Kreuzdiffusion und die Nichtlokalität stellen eine beträchtliche Herausforderung für die (Existenz-) Analyse dar, denn sie verhindern die Anwendung klassischer Methoden wie etwa Maximumprinzipien oder Regularitätstheorie. Das Ausnutzen der Entropiestruktur des Systems ermöglicht es uns diese Schwierigkeiten zu umgehen. Wir werden die globale Existenz schwacher Lösungen beweisen und zeigen, dass schwache Lösungen und starke Lösungen des Systems übereinstimmen. Hierbei werden wir das Vorhandensein zweier verschiedener Entropiefunktionale im System nutzen, unter der Voraussetzung, dass die Nichtlokalitäten in einem gewissen Sinn positiv semi-definit sind.

Weiters werden wir ein implizites Euler Finite-Volumen Schema entwickeln, welches wichtige strukturelle Eigenschaften des Systems wie etwa dessen Entropiestruktur und die Nichtnegativität von Lösungen auf der diskreten Ebene erhält. Wir werden die Existenz von (diskreten) Lösungen sowie deren Konvergenz gegen schwache Lösungen des Systems bei Verfeinerung des

Gitters beweisen. Wie auch im kontinuierlichen Fall nutzen wir im Existenzbeweis stark die doppelte Entropiestruktur des Systems aus.

Im zweiten Teil dieser Arbeit werden wir uns mit der Analyse eines instationären nichtlinearen Driftdiffusionssystems, welches Memristoren modelliert, beschäftigen. Ein Memristor ist ein nichtlinearer Widerstand mit einem gewissen Gedächtniseffekt, in dem die Elektronen, Löcher und Sauerstoffvakanzen als Ladungsträger fungieren und Drift- sowie Diffusionseffekten ausgesetzt sind. Die nichtlineare Diffusion der Elektronen und Löcher wird mittels Fermi-Dirac Statistik der Ordnung 1/2 modelliert, jene der Sauerstoffvakanzen mit Blakemore Statistik. Jede der Diffusionsgleichungen enthält einen Driftterm, der sich proportional zum Gradienten des elektrischen Potentials des Systems verhält. Das elektrische Potential wiederum ist über eine Poissongleichung mit den vorhandenen Teilchenkonzentrationen gekoppelt. Das Ziel dieses Kapitels ist das Etablieren neuer Erkenntnisse über dieses Modell, da bisher nur wenige Resultate für nichtlineare Driftdiffusionssysteme mit mehr als zwei Spezies existieren.

Die Nichtlinearitäten in der Diffusion, welche nur implizit gegeben sind, stellen eine große Herausforderung für die Existenzanalyse dar. Erschwerend kommen die verschiedenen Randbedingungen der Konzentrationen sowie das Betrachten von drei oder mehr unterschiedlichen Teilchenkonzentrationen hinzu. Der Einsatz klassischer Techniken oder Monotonieargumente wird dadurch unterbunden. Wie auch bei den nichtlokalen Kreuzdiffusionssystemen werden wir diese Schwierigkeiten umgehen, indem wir die Entropiestruktur des Systems ausnutzen. Wir werden die globale Existenz schwacher Lösungen zeigen und, unter geeigneten Regularitätsvoraussetzungen an das elektrische Potential, deren Beschränktheit für alle Zeit. Der Existenzbeweis basiert auf einer Entropiemethode, zusätzlich müssen wir aber feine Abschätzungen für das asymptotische Verhalten der Statistikfunktionen herleiten. Eine weitere Schwierigkeit bereitet die Singularität der Blakemore Statistik, welche wir mit einem Minty-Trick überwinden werden.

Abstract

Systems including cross-diffusion, i.e. the process where the flux of one component or species is induced by the gradient of another component or species, have received increased attention in the PDE community over the last few decades due to the presence of this phenomenon in many real life situations such as chemical reactions or biological processes or systems.
Another important class are diffusive systems containing drift, where the movement of particles is not just induced by the change of their concentration, which is called diffusion, but also by the presence of a force that interacts with the particles. These systems are of high interest since they model, among other things, semiconductors – a small yet essential component of most nowadays electrical circuits.

Entropy methods, where the natural presence of a Lyapunov functional is used to study the behaviour of partial differential equations (PDEs), proved to be a powerful tool in the analysis of cross-diffusion and drift-diffusion systems. These methods are especially useful when standard techniques, for example maximum or comparison principles, (elliptic) regularity theory, monotonicity arguments, etc. cannot be applied. In this thesis we demonstrate the usefulness of these methods in various settings.

In the first part of the thesis we will focus on systems of PDEs that model the evolution of populations of different species or states of neurons. Interactions between the different species lead to the appearance of cross-diffusion terms in the equations. Nonlocal terms enter the equations if we take into account that individuals of a given species usually have some interaction radius or "distance", e.g. their effective area of sight or sense. The aim of this part of the thesis is to extend existing results on systems with an arbitrary number of species by imposing "minimal" conditions on the nonlocalities and to design a reliable numerical approximation to capture the behaviour of solutions.

The main mathematical difficulties come from the presence of cross-diffusion and nonlocalities, preventing the use of standard PDE techniques such as maximum principles and regularity theory. We will show that weak solutions exist globally in time and that a weak-strong uniqueness result holds. The proofs are based on the entropy method, where we exploit the fact that the system we investigate possesses two different entropy functionals, under the assumption that the nonlinearities fulfil a certain positive semi-definiteness condition.

We continue by defining an implicit Euler finite-volume scheme that preserves the important properties of the system such as the nonnegativity of its solutions and its entropy structure on a discrete level. At this point we prove the existence of (discrete) solutions to the scheme and show their convergence to solutions of the continuous scheme when the mesh is refined. As in the continuous case, the existence proof relies heavily on the double entropy structure of the system.

In the second part of the thesis we will analyse an instationary drift-diffusion system, which arises in the modelling of memristors. A memristor is a nonlinear resistor with memory, in

which electrons, holes and oxygen vacancies act as charge carriers experiencing both drift and diffusion phenomena. The nonlinear diffusion of electrons and holes is governed by Fermi-Dirac statistics of order 1/2, while Blakemore statistics are chosen for the oxygen vacancies. Each of the diffusion equations contains a drift term involving the electric potential of the system, which is coupled to the present particle concentrations via a Poisson equation. The main goal of this part is to advance our understanding of this model, as only few results exist for nonlinear drift-diffusion systems having three or more species.

The main challenges we encounter in the analysis of this system come from the nonlinearities in the diffusion, as they are only given implicitly. A misfit of boundary conditions between electrons/holes and oxygen vacancies and the fact that we are dealing with three or more distinct species further complicate the analysis. We will show that weak solutions exist globally in time and that under certain elliptic regularity assumptions they are bounded uniformly in time. The proofs are, as in the case of the nonlocal cross-diffusion systems, based on the entropy method with the additional careful study of the behaviour of the statistics functions. It is worth mentioning that another difficulty arises from the singularity that the Blakemore statistics exhibits, which is overcome by a Minty-type trick.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 3. September 2024

Stefan Portisch

Contents

1 Introduction

The purpose of this thesis is twofold. First,we aim to establish new results in the area of nonlocal cross-diffusion systems for multi-species populations, i.e. systems that involve more than two density functions and hence are comprised of more than two equations describing the evolution of the respective densities. While there exists a plethora of results on the analysis of local cross-diffusion systems (for multiple species), less is known for nonlocal systems of two species, and for nonlocal systems of more than two species analytical results are scarce and usually include some strong requirements on the initial and boundary data or the structure of the equations. Typical examples of these requirements are the differentiability of the interaction kernels or cases where the system can be reduced to two or even one equation. Our aim is to provide a new perspective on the analysis of nonlocal cross-diffusion systems with as little assumptions on the interaction kernels as possible. We especially want to avoid the assumption of the differentiability of the kernel functions.

Second, we aim to advance our knowledge in the analysis of nonlinear drift-diffusion systems arising from semiconductor physics and neuro-computing. Here, only few results exist in the literature for systems that include more than two charge carriers in the semiconductor setting, or species in general, and therefore consist of more than two equations for their respective densities. The scarcity of results in this topic is, in parts, owed to the various mathematical difficulties that have to be overcome in the analysis. Recent developments in the industry (perovskite solar cells) and in neuromorphic computing (artificial synapses), fuelled by the emergence of memristive devices and materials, sparked a new growing interest in such nonlinear systems. In this thesis, we will establish new results in the study of such systems and relate them to existing work such as [71, 77].

At a first glance, it seems that these two topics, nonlocal cross-diffusion systems and nonlinear drift-diffusion systems, have little to do with each other. What unites them in the context of this work is the fact that entropy methods [72, 73] play a crucial role in our approaches. Accordingly, this thesis naturally splits into two main parts with entropy methods acting as an intrinsic underlying thread spanning throughout this work. In the first part of the thesis, which consists of Chapters 2 and 3, we discuss nonlocal cross-diffusion systems modelling multi-species populations or networks. The second part, consisting of Chapter 4, is devoted to the analysis of a nonlinear drift-diffusion system that models memristive devices.

The results in this thesis are based on the publication [74] (Ansgar Jüngel, Stefan Portisch, Antoine Zurek), the publication [75] (Ansgar Jüngel, Stefan Portisch, Antoine Zurek) and the ongoing research collaboration (Maxime Herda, Ansgar Jüngel, Stefan Portisch), for which a manuscript is currently prepared for submission.

1.1 Nonlocal cross-diffusion systems

We start this section by introducing a system of nonlocal partial differential equations, which models multi-species populations or networks. As in [59], we will consider the system on the d-dimensional torus $\mathbb{T}^d$.

1.1.1 Model equations and motivation

We consider the following nonlocal cross-diffusion system:

$$\partial_t u_i - \sigma \Delta u_i = \operatorname{div}(u_i \nabla p_i[u]), \quad t > 0, \quad u_i(0) = u_i^0 \quad \text{in } \mathbb{T}^d, \; i = 1, \ldots, n, \tag{1.1}$$

where $\sigma > 0$ is the diffusion coefficient, $\mathbb{T}^d$ is the d-dimensional torus ($d \geq 1$) and $p_i[u]$ is a nonlocal operator given by

$$p_i[u](x) = \sum_{j=1}^{n} \int_{\mathbb{T}^d} K_{ij}(x-y) u_j(y)\, dy, \quad i = 1, \ldots, n, \tag{1.2}$$

with the kernel functions $K_{ij} : \mathbb{T}^d \to \mathbb{R}$ (extended periodically to $\mathbb{R}^d$), and with the solution vector $u = (u_1, \ldots, u_n)$.

As it was discussed in [59, 87], when the kernel functions are given by $K_{ij} = a_{ij} K$ with numbers $a_{ij} \in \mathbb{R}$ and a nonnegative function K, this model describes the dynamics of a population with n species, where each species can detect other species over a spatial neighborhood by nonlocal sensing, described by the kernel function K. The coefficient a_{ij} is a measure of the strength of attraction (if $a_{ij} < 0$) or repulsion (if $a_{ij} > 0$) of the i-th species to or from the j-th species. A typical choice of K is the characteristic function $\mathbb{1}_B$ of a ball B centered at the origin. The authors of [59] proved the local existence of a unique (strong) solution to system (1.1)–(1.2) for $d \geq 1$ under the condition that K is twice differentiable. Furthermore, they have shown that this solution can be extended globally in one space dimension. However, their assumptions on K exclude the typical case $K = \mathbb{1}_B$ mentioned above. In this thesis we will extend the results of [59] for non-differentiable kernels K_{ij} and prove the global existence of weak solutions to (1.1)–(1.2) as well as a weak-strong uniqueness result in any space dimension. Following [59], we consider (1.1) on the torus; see Remark 9 in Section 2.2 for a discussion about the case where the whole space or a bounded domain is considered.

Another motivation comes from the work [31], where the system (1.1)–(1.2) was rigorously derived from interacting many-particle systems in a mean-field-type limit. As a by-product of this limit, the local existence of smooth solutions to (1.1)–(1.2) has been shown under the assumption that the K_{ij} are smooth. Moreover, under the same assumptions on the kernels, the so-called localization limit was proved, i.e. if K_{ij} converges to the delta distribution times some factor a_{ij}, the solution to the nonlocal system (1.1)–(1.2) converges to a solution to the model (1.1) with

$$p_i[u] = \sum_{j=1}^{n} a_{ij} u_j, \quad i = 1, \ldots, n. \tag{1.3}$$

We note that the local system was first introduced in [56] in the case of two species. In this thesis, we generalize the results of [31] by imposing "minimal" conditions on the initial datum u^0 and the kernels K_{ij}.

A third motivation for our work comes from neuroscience. Indeed, following [11, 58], we see that deterministic nonlocal models of the form (1.1)–(1.2) can be obtained as the mean-field limit of stochastic systems describing the evolution of the states of neurons belonging to different populations. When the number of neurons becomes very large, the solutions of the generalized Hodgkin–Huxley model of [11] can be described in the mean-field limit by a probability distribution u_i for the i-th species, which solves the McKean–Vlasov–Focker–Planck equation of the type

$$\partial_t u_i = \sigma \Delta u_i + \operatorname{div} \sum_{j=1}^{n} \int_{\mathbb{T}^d} M_{ij}(x,y) u_i(x) u_j(y)\, dx\, dy, \quad i = 1, \dots, n, \tag{1.4}$$

where we simplified the diffusion part involving σ. In neural network theory, the kernel function $M_{ij}(x,y)$ describes the weight of a connection between the node x associated to species i and node y associated to species j. In the present work, we simplify the problem further by assuming that the interaction kernels M_{ij} have the special form $M_{ij}(x,y) = \nabla K_{ij}(x-y)$, resulting in (1.1)–(1.2) again.

1.1.2 State of the art

We recall the current state of the art in the study of nonlocal equations and systems, following [74], and mention additional results, which have been found since the publication of this paper.

Most nonlocal models studied in the literature describe a single species. A simple example is the equation

$$\partial_t u = \operatorname{div}(uv),$$

with $v = \nabla(K * u)$. This equation corresponds to the continuity equation for the density u with a nonlocal velocity v. An L^p-theory for this equation was provided in [16], while the Wasserstein gradient-flow structure was explored in [27]. In the context of machine learning, the equation can be seen as the mean-field limit of infinitely many hidden network units [82, 94].

Beyond the study of single-species dynamics one can find some work which deals with the existence of solutions to multi-species nonlocal systems of the form (1.1)–(1.2) or similar to it. For instance, in the case of two species and symmetrizable cross-interaction potentials (meaning $K_{12} = \alpha K_{21}$ for some $\alpha > 0$) without diffusion $\sigma = 0$, a complete existence and uniqueness theory for measure solutions to (1.1)–(1.2) in the whole space with smooth convolution kernels was established in [45] using the Wasserstein gradient-flow theory.

In [56], a nonlocal version of the Shigesada–Kawasaki–Teramoto (SKT) cross-diffusion system, where the diffusion operator is replaced by an integral diffusion operator, was analysed. Similar to our approach, which we will explain shortly, the authors obtain a priori estimates on the solutions via an entropy inequality, which allows them to prove existence of solutions using a compactness argument. In [43], the authors show the existence of weak solutions to a nonlocal version of the SKT system, where the nonlocalities are similar to the ones considered in this work. Assuming some regularity on the convolution kernels, their proof is based on the so-called duality method [42, 80]. They also prove a localization limit result.

In the works [29, 44, 45] the authors analyse nonlocal cross-diffusion systems for two species similar to our model. In [29], the steady states are characterized and numerical simulations are presented, while the works [44, 45] prove the existence of weak measure solutions under a global Lipschitz condition on K_{ij} and ∇K_{ij}. The innovation of the presented work in this thesis is that we impose only integrability conditions on K_{ij} and have (slightly) weakened positive definiteness to detailed balance. The weak-strong uniqueness result and the localization limit are also new in this context.

For the sake of completeness, we mention that a nonlocal system for two species with size exclusion was analysed in [15], using entropy methods. The authors proved the global existence of weak solutions and additionally investigated phase separation effects by means of analytical as well as numerical studies of the energy functional of the system. Contrary to our model, this system has nonlinear diffusion and also takes into account the influence of the total mass density of the species in the equations.

It is worth mentioning that all the aforementioned cited works, except [43], are concerned with two-species models, whereas we will allow for an arbitrary number of species and nondifferentiable kernel functions.

Recently, the authors of [60] showed that for any initial datum u^0 to (1.1) an energy functional of the corresponding solution, which we will touch on shortly, converges to a local minimum. Furthermore, the authors propose a method to find corresponding minimum energy states. In [61] the authors proved the global existence of nonnegative weak solutions to (1.1) in any space dimension, assuming the kernels K_{ij} to be twice differentiable with ∇K_{ij} essentially bounded, but dropping all other assumptions on the kernels. Additionally, they showed that blow-up of solutions in the localized version of (1.1) is possible under certain assumptions on the parameters.

Recently, the authors of [23] derived the local version (1.1) & (1.3) of the system considered in this work from nonlocal interaction systems and used gradient flow techniques to show that solutions of the nonlocal system converge to the local system in the limit of localizing interaction kernels.

1.1.3 Mathematical difficulties and strategy of our proofs

The mathematical difficulties we encounter in the analysis of system (1.1)–(1.2) stem from the cross-diffusion terms and the nonlocality, which prevent the application of standard techniques like maximum principles and regularity theory. For instance, it is well known that nonlocal diffusion operators generally do not possess regularizing effects on the solution [8]. The key ingredient to our analysis lies in the observation that the nonlocal system possesses, like the associated local one, **two** entropies, namely the Shannon-type entropy H_1 [92] and the Rao-type entropy H_2 [89],

$$H_1(u) = \sum_{i=1}^{n} \int_{\mathbb{T}^d} \pi_i u_i (\log u_i - 1)\, dx, \tag{1.5}$$

$$H_2(u) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) u_i(x) u_j(y)\, dx\, dy, \tag{1.6}$$

where $\pi_1, \dots, \pi_n > 0$ are numbers such that

$$\pi_i K_{ij}(x-y) = \pi_j K_{ji}(y-x) \quad \text{for all } i,j = 1,\dots,n,\ x,y \in \mathbb{T}^d \tag{1.7}$$

and

$$\sum_{i,j=1}^{n}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_i K_{ij}(x-y)v_i(x)v_j(y)\,dx\,dy \geq 0 \quad \text{for all } v_i, v_j \in L^2(\mathbb{T}^d). \tag{1.8}$$

A formal computation that is made rigorous in Chapter 2 shows that the following entropy inequalities hold:

$$\frac{dH_1}{dt} + 4\sigma\sum_{i=1}^{n}\int_{\mathbb{T}^d}\pi_i|\nabla\sqrt{u_i}|^2\,dx = -\sum_{i,j=1}^{n}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_i K_{ij}(x-y)\nabla u_i(x)\cdot\nabla u_j(y)\,dx\,dy \leq 0, \tag{1.9}$$

$$\frac{dH_2}{dt} + \sum_{i=1}^{n}\int_{\mathbb{T}^d}\pi_i u_i|\nabla p_i[u]|^2\,dx = -\sigma\sum_{i,j=1}^{n}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_i K_{ij}(x-y)\nabla u_i(x)\cdot\nabla u_j(y)\,dx\,dy \leq 0. \tag{1.10}$$

In particular, the functionals H_1 and H_2 are Lyapunov functionals. We can think of (1.7) as a generalized detailed-balance condition for the Markov chain associated to $(K_{ij}(x-y))$ (for fixed $(x-y)$) with the corresponding reversible measure $(\pi_1,\ldots,\pi_n)$. Condition (1.8), on the other hand, is a generalisation of the usual definition of positive definite kernels in the multi-species case [22]. Examples of kernels that satisfy (1.7) and (1.8) are given in Remark 1 in Section 2.1.

The positive definiteness condition for kernels is essential in reproducing kernel Hilbert theory [86]. From a PDE viewpoint, this condition can be replaced by the smoothness assumption $K_{ij}\in C^1$; see, e.g. [45, 59]. Because of the nonlocality, we cannot conclude estimates in $L^2(\mathbb{T}^d)$ for u_i and ∇u_i from our entropy inequalities like in the local case; see [78] and Section 2.6. Nonetheless, we can deduce bounds for $u_i\log u_i$ in $L^1(\mathbb{T}^d)$ and $\sqrt{u_i}$ in $H^1(\mathbb{T}^d)$ from (1.9).

These bounds are not sufficient to pass to the limit in the approximate problem. In particular, we cannot identify the limit of the product $u_i\nabla p_i[u]$, since u_i and $\nabla p_i[u]$ are elements in spaces larger than $L^2(\mathbb{T}^d)$. We solve this issue by exploiting the uniform $L^2(\mathbb{T}^d)$-bound for $\sqrt{u_i}\nabla p_i[u]$, which follows from (1.10), and prove a "compensated compactness" lemma (see Lemma 13 in Section 2.5): If $u_\varepsilon\to u$ strongly in $L^p(\mathbb{T}^d)$, $v_\varepsilon\rightharpoonup v$ weakly in $L^p(\mathbb{T}^d)$, and $u_\varepsilon v_\varepsilon\rightharpoonup w$ weakly in $L^p(\mathbb{T}^d)$ for some $1<p<2$, then $uv=w$. The estimates from (1.9)–(1.10) are the key for the proof of the global existence of weak solutions to (1.1)–(1.2).

Additionally to the above existence result, we prove the weak-strong uniqueness of solutions, i.e. if u is a weak solution to (1.1)–(1.2) satisfying $u_i\in L^2(0,T;H^1(\mathbb{T}^d))$ and if v is a "strong" solution to this problem with the same initial data, then $u(t)=v(t)$ for a.e. $t\geq 0$. The proof relies on the relative entropy

$$H(u|v) = \sum_{i=1}^{n}\int_{\mathbb{T}^d}\pi_i\big(u_i(\log u_i - 1) - u_i\log v_i + v_i\big)\,dx,$$

a variant of which was used in [52] for reaction-diffusion systems in the context of renormalized solutions and later extended to Shigesada–Kawasaki–Teramoto systems [36]. The recent work [67] generalizes this approach to more general Shigesada–Kawasaki–Teramoto as well as energy-reaction-diffusion systems. To our knowledge, our work is the first one to apply these techniques to nonlocal cross-diffusion systems. The main idea in the proof of the uniqueness is to differentiate $H(u|v)$ and to derive the inequality

$$H(u(t)|v(t)) \le C \sum_{i=1}^n \int_0^t \|u_i - v_i\|^2_{L^1(\mathbb{T}^d)}\, ds \quad \text{for } t > 0,$$

which, together with the Csiszár–Kullback–Pinsker inequality [73, Theorem A.2], allows us to estimate the relative entropy from below by $\|u_i(t) - v_i(t)\|^2_{L^1(\mathbb{T}^d)}$, up to some factor. Grönwall's lemma is then used to conclude the desired result.

It is worth pointing out that our use of this inequality is different from the proofs in [36, 52, 67], where the relative entropy is estimated from below by $|u_i - v_i|^2$ on the set $\{u_i \le K\}$. The difference stems from the nonlocal terms. Indeed, if K_{ij} is bounded,

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)(u_i - v_i)(x)(u_j - v_j)(y)\, dx\, dy \le C \sum_{i=1}^n \left(\int_{\mathbb{T}^d} |u_i - v_i|\, dx \right)^2,$$

leading to an estimate of the difference $|u_i - v_i|$ in $L^1(\mathbb{T}^d)$. In the local case, the associated estimate yields an $L^2(\mathbb{T}^d)$ estimate:

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d} \pi_i a_{ij}(u_i - v_i)(x)(u_j - v_j)(x)\, dx \le C \sum_{i=1}^n \int_{\mathbb{T}^d} |u_i - v_i|^2\, dx.$$

We observe that the uniqueness of weak solutions to cross-diffusion systems is a delicate task, and there are only few results in the literature. Most of the results are based on the fact that the total density $\sum_{i=1}^n u_i$ satisfies a simpler equation for which uniqueness can be shown; see [15, 35].

Contrary to this, in [14], a weak-strong uniqueness result on a cross-diffusion system, based on L^2-estimates and the fact that the studied system could be considered as a perturbation of a system of heat equations, was shown. A so–called duality method was used by the authors of [56] to prove the uniqueness of solutions for a nonlocal version of the Shigesada–Kawasaki–Teramoto system.

To end this short subsection we would like to mention that the bounds obtained in the proof of our existence result are independent of the kernels, thus allowing us to perform the localization limit. To achieve this, we assume that $K_{ij} = B^\varepsilon_{ij} \to a_{ij}\delta_0$ as $\varepsilon \to 0$ in the sense of distributions, where δ_0 is the Dirac delta distribution. Then we show that if u^ε is a weak solution to (1.1)–(1.2), it follows that $u_i^\varepsilon \to u_i$ strongly in $L^1(\mathbb{T}^d \times (0,T))$, and the limit u solves the local system (1.1) and (1.3). As a by-product, we obtain the global existence of weak solutions to this problem; see Section 2.6 for the precise statement.

1.2 A finite-volume scheme for nonlocal cross-diffusion systems

In this section we discuss a slightly modified version of the nonlocal cross-diffusion system presented in the previous section. In particular, we assume local instead of nonlocal self-diffusion. The proof of global existence of weak solutions, which we mentioned earlier, relies heavily on the positive semi-definiteness condition (1.8) of the kernels. While useful in the theoretical setting, checking this condition in practice is rather tedious, which motivates our modification of the system. We will derive an implicit Euler finite-volume scheme (FV-scheme) for the modified system, prove existence of discrete solutions and the convergence of the scheme. Additionally, we investigate numerically the segregation phenomenon such systems exhibit in their localized version [17].

1.2.1 Model equations and connection to previous work

In this part of our work we will design and study a structure-preserving finite-volume discretization of the following one-dimensional nonlocal cross-diffusion initial-value problem:

$$\partial_t u_i = \partial_x(\sigma \partial_x u_i + u_i \partial_x p_i(u)) \quad \text{in } \mathbb{T},\ t > 0, \tag{1.11}$$

$$u_i(\cdot, 0) = u_i^0 \quad \text{in } \mathbb{T},\ i = 1, \ldots, n, \tag{1.12}$$

where $\sigma \geq 0$ is the diffusion coefficient, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus of unit measure, and p_i is the nonlocal operator

$$p_i(u)(x) := a_{ii} u_i(x) + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} (B^{ij} * u_j)(x) = a_{ii} u_i(x) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{\mathbb{T}} a_{ij} B^{ij}(x-y) u_j(y)\, dy, \tag{1.13}$$

where a_{ij} are given constants. The kernel functions $B^{ij} : \mathbb{T} \to \mathbb{R}$ are periodically extended to $\mathbb{R}$ and $u = (u_1, \ldots, u_n)$ is the solution vector. In the case where $B^{ii} = \delta_0$, with $i \in \{1, \ldots, n\}$ and δ_0 being the Dirac measure, we can rewrite p_i as

$$p_i(u) = \sum_{j=1}^{n} a_{ij} (B^{ij} * u_j)(x). \tag{1.14}$$

Equations (1.11) with definition (1.14) and general kernels B^{ij} for $i, j = 1, \ldots, n$ can be derived from stochastic interacting particle systems in the many-particle limit [31].

As mentioned above, we will show that the “full” nonlocal system, that is (1.11) & (1.14), where $B^{ii} \neq \delta_0$ are general kernels, admits global weak solutions, cf. Chapter 2. Our analysis is based on the fact that this system possesses two Lyapunov functionals. More precisely, and as mentioned in Section 1.1, under the assumption of detailed-balance and semi positive-definiteness, i.e. that there exist numbers $\pi_1, \ldots, \pi_n > 0$ such that the kernels B^{ij} satisfy

$$\pi_i a_{ij} B^{ij}(x-y) = \pi_j a_{ji} B^{ji}(y-x) \quad \text{for } i, j = 1, \ldots, n \text{ and a.e. } x, y \in \mathbb{T},$$

and

$$\sum_{i,j=1}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) v_j(y) v_i(x)\, dy\, dx \geq 0 \quad \text{for all } v_i, v_j \in L^2(\mathbb{T}), \tag{1.15}$$

we will show that the Boltzmann-type and Rao-type entropies, respectively,

$$H_B(u) = \sum_{i=1}^{n} \int_{\mathbb{T}} \pi_i u_i (\log u_i - 1)\, dx,$$

$$H_R(u) = \frac{1}{2}\sum_{i,j=1}^n \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y)u_j(y)u_i(x)\,dy\,dx,$$

fulfill the following entropy dissipation inequalities:

$$\frac{dH_B}{dt} + 4\sigma\sum_{i=1}^n \int_{\mathbb{T}} \pi_i |\partial_x \sqrt{u_i}|^2\,dx = -\sum_{i,j=1}^n \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y)\partial_y u_j(y)\partial_x u_i(x)\,dy\,dx \le 0, \tag{1.16}$$

$$\frac{dH_R}{dt} + \sum_{i=1}^n \int_{\mathbb{T}} \pi_i u_i |\partial_x p_i(u)|^2\,dx = -\sigma\sum_{i,j=1}^n \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y)\partial_y u_j(y)\partial_x u_i(x)\,dy\,dx \le 0. \tag{1.17}$$

The Boltzmann entropy is related to the thermodynamic entropy of the system and the Rao entropy is a measure of the functional diversity of the species [89].

While this theoretical framework is suitable to prove the existence of weak solutions, condition (1.15) is cumbersome to check in practice. It is satisfied for smooth kernels such as the Gaussian one, i.e. $B^{ij}(x-y) = \exp(-(x-y)^2/2)$ for $i,j = 1,\dots,n$, but fails to hold for kernels B^{ij} of the type $B^{ij} = \mathbb{1}_K$, where $\mathbb{1}_K$ is the indicator function of some interval K around the origin; see the counterexample in Section 3.6.

The system (1.11) and (1.14), with local or nonlocal self-diffusion terms, describes the dynamics of a population with n species, where the evolution of each species is driven by nonlocal sensing [87]. In other words, each species has the capability to detect other species over a spatial neighborhood, specified by the kernel B^{ij}, and weighted by the strength of attraction ($a_{ij} < 0$) or repulsion ($a_{ij} > 0$). Thus, from a modelling point of view, the case $B^{ij} = \mathbb{1}_K$ is biologically meaningful. To include this case in our analysis (at the continuous or discrete level), we propose to slightly modify the model by considering (1.13) instead of (1.14).

For model (1.11)–(1.13), we impose the following assumptions:

- There exist numbers $\pi_1,\dots,\pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i,j \in \{1,\dots,n\}$.
- $B^{ji}(-x) = B^{ij}(x) \ge 0$ for a.e. $x \in \mathbb{T}$ and all $i,j \in \{1,\dots,n\}$ with $i \ne j$.
- For all $i,j \in \{1,\dots,n\}$ with $i < j$, the matrices
$$M^{ij}(x) := \begin{pmatrix} \pi_i a_{ii} & (n-1)\pi_i a_{ij} B^{ij}(x) \\ (n-1)\pi_j a_{ji} B^{ij}(x) & \pi_j a_{jj} \end{pmatrix} \tag{1.18}$$
are uniformly positive definite for a.e. $x \in \mathbb{T}$. In particular, we could choose some nonpositive off-diagonal coefficients.

The ability to analyse system (1.11)–(1.13) with nonpositive off-diagonal coefficients is a new and meaningful result. However, we notice that under the above assumptions the system is only "weakly" nonlocal in the sense that the self-diffusion coefficients have to dominate the cross-diffusion terms.

We claim that the functionals H_B and H_R are still entropies for system (1.11)–(1.13), where under our asssumptions we now have

$$H_R(u) = \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{T}} \pi_i a_{ii}|u_i(x)|^2\,dx + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^n \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y)u_j(y)u_i(x)\,dy\,dx.$$

Both functionals satisfy some entropy dissipation inequalities similar to (1.16)–(1.17), where, if $i=j$, the terms on the right-hand side are simply given by the square of the $L^2(\mathbb{T})$-norm of $\partial_x u_i$ and we get that the entropy production term

$$Q := \sum_{i=1}^n \int_{\mathbb{T}} \pi_i a_{ii}|\partial_x u_i(x)|^2\,dx + \sum_{\substack{i,j=1\\ i\neq j}}^n \int_{\mathbb{T}}\int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y)\partial_x u_j(y)\partial_x u_i(x)\,dy\,dx \tag{1.19}$$

is nonnegative; see Lemma 31 in Section 3.5. Therefore, at least formally, the functionals H_B and H_R are entropies for system (1.11)–(1.13). In this work, we will translate this property to the discrete level by analysing a two-point flux approximation FV-scheme for (1.11)–(1.13).

1.2.2 Finite-volume methods (in a nutshell)

In this subsection we will give a very brief introduction to finite-volume methods by using the d-dimensional version of equation (1.11) as an example, which then reads as follows:

$$\partial_t u_i = \nabla\cdot(\sigma\nabla u_i + u_i\nabla p_i(u)) \quad \text{in } \mathbb{T}^d,\ t>0,\ i=1,\dots,n. \tag{1.20}$$

Despite the fact that we will study (1.11) in detail in one space dimension and only comment on higher space dimensions, we choose to use the setting of the d-dimensional torus for our presentation in this subsection as it illustrates the concept of finite-volumes better than the one-dimensional case.

Before we start, we want to emphasize that finite-volume methods is a broad research topic and we will barely scratch its surface with this simplified motivation. We therefore refer the interested reader to the foundation [51] for a detailed introduction to finite-volume methods.

To construct a scheme for our system, we start by discretizing the system of equations with an implicit Euler method in time. Consequently, the continuous time derivative in (1.20) is replaced by

$$\partial_t u_i \approx \frac{u_i^k - u_i^{k-1}}{\Delta t},$$

for given $k\in\{1,\dots,N_T\}$, where N_T denotes the number of time steps, Δt is the time step size and u_i^k is the solution at time $t = k\Delta t$.

The idea of finite-volumes is to partition the domain Ω into a set $\mathcal{T}$ of convex, polygonal or polyhedral subsets K, such that $\bigcup_{K\in\mathcal{T}} K = \Omega$. We call $\mathcal{T}$ a mesh and its elements K control volumes or cells. The boundary ∂K of a cell K naturally partitions into a set $\mathcal{E}_K$ of disjoint, convex, $(d-1)$-dimensional sets ς, which we call edges or surfaces of K. Each of these edges ς has associated to it a unique unit normal vector $\nu_{K,\varsigma}$ pointing outwards of K.

Integrating equation (1.20) over a control volume K and (formally) applying the divergence theorem therefore yields

$$\int_K \frac{u_i^k - u_i^{k-1}}{\Delta t}\,dx - \sum_{\varsigma\in\mathcal{E}_K}\int_\varsigma (\sigma\nabla u_i + u_i\nabla p_i(u))\cdot\nu_{K,\varsigma}\,ds = 0. \tag{1.21}$$

The next step is to rewrite equation (1.21) in terms of $u_{i,K}^k$ where $u_{i,K}^k = \mathrm{m}(K)^{-1}\int_K u_i^k(x)\,dx$ is the approximation to the solution u_i^k on the cell K and $\mathrm{m}(K)$ denotes the d-dimensional measure of the cell K. It is rather obvious how to do this for the first term in (1.21), therefore we only sketch it for the integral over the boundary of a cell K. In order to do that, we will need some additional notation.

Each cell K also has an associated point $x_K \in K$ called “middle point”. For this simplified motivation it suffices to think of x_K as the center of mass, although more general conditions are possible, cf. [51]. Let us assume that the edge $\varsigma \in \mathcal{E}_K$ is separating the two cells K and L, we denote this by $\varsigma = K|L$, and that the edge ς is orthogonal to the straight line connecting the two middle points x_K and x_L of the cells K and L, respectively. Furthermore, we define the so-called transmissibility coefficient $\tau_\varsigma := \mathrm{m}(\varsigma)/d_\varsigma$, with $\mathrm{m}(\varsigma)$ the $(d-1)$-dimensional measure of the edge ς and d_ς the Euclidean distance between the two cell centers x_K and x_L. Then

$$-\int_\varsigma (\sigma\nabla u_i + u_i\nabla p_i(u))\cdot\nu_{K,\varsigma}\,ds \approx -\sigma\tau_\varsigma D_{K,\varsigma}u_i^k - \tau_\varsigma u_{i,\varsigma}^k D_{K,\varsigma}p_i^k =: \mathcal{F}_{i,K,\varsigma}^k, \tag{1.22}$$

where $D_{K,\varsigma}v := v_L - v_K$ and the mobilites $u_{i,\varsigma}^k := \widehat{F}(u_{i,K}^k, u_{i,L}^k)$ are an approximation of u_i^k on the edge ς, which we will rigorously define in Chapter 3. Note that we have not given a definition for the approximation p_i^k of $p_i(u)$ either, as this will be done in detail in Chapter 3 as well. The approximation in (1.22) is called two-point flux approximation. Let us point out that the orthogonality assumption for the edges is crucial for the convergence result we will prove. However, that assumption will be trivially satisfied in our case as we have to choose a Cartesian mesh for our finite-volume discretization due to the convolution in the nonlocal operators p_i. In summary, we will obtain the following finite-volume approximation:

$$\frac{\mathrm{m}(K)}{\Delta t}\left(u_{i,K}^k - u_{i,K}^{k-1}\right) + \sum_{\varsigma\in\mathcal{E}_K}\mathcal{F}_{i,K,\varsigma}^k = 0.$$

1.2.3 State of the art

We recall the current state of the art in the study of (1.11)–(1.13) and similar systems. Additional information can be found in [75] and the references therein.

We start by mentioning several works that deal with the design and analysis of numerical schemes for nonlocal cross-diffusion systems. The work [29] studies a positivity-preserving one-dimensional finite-volume scheme for (1.11) with $n = 2$ and additional local cross-diffusion terms. The authors focused on segregated steady states but did not include any numerical analysis. The convergence of this finite-volume scheme was proved in [28], which still focused on the two-species model.

A converging finite-volume scheme for a nonlocal cross-diffusion system modelling either a food chain of three species or an SIR model when the cross-diffusion is dropped, was analysed in [7, 13]. In both works the nonlocality of the system comes from the dependence of the self-diffusion coefficients on the total mass of the corresponding species.

A structure-preserving finite-volume scheme for the nonlocal Shigesada–Kawasaki–Teramoto system was suggested and analysed in [65].

We would also like to mention the paper [26] on a second-order finite-volume scheme for a nonlocal diffusion equation, which preserves the nonnegativity and fulfils a spatially discrete entropy inequality. Related works include a Galerkin scheme for a nonlocal diffusion equation with additive noise [81], a finite-volume discretization of a nonlocal Lévy–Fokker–Planck equation [10], and numerical schemes for nonlocal diffusion equations arising in image processing [55]. To our knowledge, there does not exist any numerical analysis of system (1.11)–(1.13).

In this work we propose a finite-volume scheme which preserves the structure of equations (1.11)–(1.13). Compared to [28], we allow for an arbitrary number of species, include linear diffusion $\sigma \geq 0$, and show that the discrete equivalents of the Boltzmann and Rao entropies are Lyapunov functionals for our scheme. Since we need the positive definiteness of the matrix $M^{ij}(x)$, self-diffusion is needed in our situation. Moreover, in contrast to [28], we impose periodic boundary conditions instead of no-flux conditions. Our proofs rely on the discrete analog of the identity

$$\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j,$$

see (3.8) in Section 3.1.2, which allows us to consider kernels B^{ij} that are not differentiable, whereas in [28] the kernels are required to be in $C_b^2(\mathbb{R})$. Compared to [65], our equations do not have a Laplacian structure, which was used in [65] to define the numerical scheme, and we allow for nonpositive off-diagonal coefficients.

1.3 A charge transport system with Fermi-Dirac statistics for memristors

The last topic we will cover in this thesis is a nonlinear drift-diffusion system modelling memristive devices. Memristors are nonlinear resistors with memory, which show a resistive switching behaviour. They were postulated in the work [39] from 1971.

In neuromorphic computing, memristors can be used to build artificial neurons and synapses [69]. Besides the electrons and holes, which act as charge carriers in general semiconductors, in memristors the oxygen vacancies also act as charge carriers. Applying an electric field in a memristor results in the drift of the oxygen vacancies and changes the boundary between the low- and high-resistance layers. This allows memristors to mimic the conductance response of synapses.

Aside from neuromorphic computing, memristors also play an important role in recent advances in photovoltaic technology, where perovskite solar cells (PSCs) have emerged as a promising technology. In a PSC a perovskite material layer is embedded between two transport layers and exhibits a memristive behaviour [91, 98]. In this application the accumulation of anions plays a fundamental role as well.

Memristive devices can be modelled by a system of drift-diffusion equations for the electrons, holes and oxygen vacancies, coupled to a Poisson equation for the electric potential [63, 84, 97]. Fermi-Dirac statistics of order 1/2 are chosen to govern the nonlinear diffusion of electrons and holes [3, 63, 96, 97]. To correctly model the accumulation of ionic vacancies and take into account the fact that accumulating an excessive number of vacancies is physically unrealistic, the nonlinear diffusion of the oxygen vacancies is governed by Fermi-Dirac statistics of order -1 (also known as Blakemore statistics). This is motivated by the authors of [3] and corresponds to a mean-field ideal lattice gas [12, Eqn. (3.5.1)].

For completeness, let us point out that we consider anorganic oxide-based memristors in this work. In the case of organic semiconductor devices Gauss-Fermi integrals instead of Fermi-Dirac integrals have to be used. This was done by the authors of [62], who showed global existence and boundedness of weak solutions to an instationary drift-diffusion system modelling such an ogranic device with two distinct charge carriers.

Owing to the novelty of the developments in memristor technology and the use of Fermi-Dirac statistics for the nonlinearities, the mathematical analysis of these drift-diffusion systems is rather challenging and new. Only few analytical results for three or more species exist in the literature.

In this thesis we will prove the global existence of weak solutions to this model in up to four space dimensions. Furthermore, in three space dimensions we will show the uniform-in-time boundedness of solutions, given that the electric potential satisfies some elliptic regularity constraint.

1.3.1 Model equations

Let us now present the equations that we will consider in this part.

The Fermi-Dirac integrals of orders 1/2 and -1, defined for all $\eta \in \mathbb{R}$, are given by

$$\begin{aligned} \mathcal{F}_{1/2}(\eta) &:= \frac{1}{\Gamma(1+1/2)} \int_0^\infty \frac{\xi^{\frac{1}{2}}}{1+e^{\xi-\eta}}\, d\xi, \\ \mathcal{F}_{-1}(\eta) &:= \frac{1}{1+e^{-\eta}}, \end{aligned} \tag{1.23}$$

where Γ is the Gamma function

$$\Gamma(z) = \int_0^\infty \frac{t^{z-1}}{e^t}\, dt, \quad z > 0.$$

The corresponding inverse functions to the Fermi-Dirac integrals are given by

$$\begin{aligned} G(z) &:= \mathcal{F}_{1/2}^{-1}(z), \quad z \in (0,\infty), \\ H(z) &:= \mathcal{F}_{-1}^{-1}(z) = \log(z) - \log(1-z), \quad z \in (0,1). \end{aligned} \tag{1.24}$$

We assume that the time evolution of the densities for the electrons n, the holes p, the oxygen vacancies D and the electric potential V is given by

$$\begin{aligned} \partial_t n - \nabla \cdot J_n &= 0, & J_n &= n\nabla G(n) - n\nabla V, \\ \partial_t p + \nabla \cdot J_p &= 0, & J_p &= -(p\nabla G(p) + p\nabla V), \\ \partial_t D + \nabla \cdot J_D &= 0, & J_D &= -(D\nabla H(D) + D\nabla V), \\ \lambda^2 \Delta V &= n - p - D + A, & &\text{in } \Omega,\ t > 0, \end{aligned} \tag{1.25}$$

where J_n, J_p and J_D are the current densities of the electrons, holes and oxygen vacancies, respectively, $\lambda > 0$ is the scaled Debye length and $A(x)$ is the given immobile dopant acceptor density. Following [96] we neglect recombination terms. We use physically motivated initial data and mixed Dirichlet-Neumann boundary conditions:

$$\begin{aligned} n(0,.) = n^I, \quad p(0,.) = p^I, \quad D(0,.) = D^I, &\quad \text{in } \Omega, \\ n = \widetilde{n}, \quad p = \widetilde{p}, \quad V = \widetilde{V}, &\quad \text{on } \Gamma_D,\ t > 0, \\ J_n \cdot \nu = J_p \cdot \nu = \nabla V \cdot \nu = 0, &\quad \text{on } \Gamma_N,\ t > 0, \\ J_D \cdot \nu = 0, &\quad \text{on } \partial\Omega,\ t > 0, \end{aligned} \tag{1.26}$$

where Γ_D denotes the Dirichlet part of the boundary $\partial\Omega$ and Γ_N its Neumann part. The boundary part Γ_D models the Ohmic contacts, at which we prescribe the electron and hole densities as well as the applied voltage, while Γ_N is the union of insulating boundary elements. The no-flux Neumann boundary condition for D reflects the fact that oxygen vacancies cannot pass through the boundary, i.e. they are not supposed to leave the semiconductor domain. These boundary conditions are typically used in the memristor literature [63, 96] and can be seen as a first-order approximation of the densities derived from the semiconductor Boltzmann equation [88].

1.3.2 Mathematical difficulties

The misfit of boundary conditions between the different species gives the main mathematical difficulty in the study of (1.25)–(1.26). Indeed, having mixed Dirichlet-Neumann boundary conditions for n, p, V and no-flux boundary conditions for D creates difficulties in estimates which include mixed terms related to different boundary conditions (e.g. ∇D and ∇V).

Another difficulty is the fact that we consider three species, instead of just two. In a two-species system, the quadratic drift terms can be estimated by exploiting monotonicity properties. For three or more species, this is not possible anymore, cf. [71] and [77].

Further difficulties come from the use of Fermi-Dirac statistics and the fact that the nonlinearities are only given implicitly and behave differently for small and large densities. Fine estimates on the behaviour of the inverse functions of the Fermi-Dirac integrals and their respective derivatives are needed for our analysis, cf. Section 4.4. To illustrate this, let us give the behaviour of the derivative G' close to 0 and at ∞, where G is the inverse of the Fermi-Dirac integral of order $1/2$:

$$G'(z) \sim z^{-1}\mathbb{1}_{(z<\mathcal{F}_{1/2}(0))} + z^{-1/3}\mathbb{1}_{(z>\mathcal{F}_{1/2}(0))},$$

where $\mathbb{1}_M$ stands for the indicator function of the given set M in subscript. The above also illustrates that the nonlinear diffusion in the equation for the electron densities, $n\nabla G(n)$, can be approximated by ∇n in the low density regime and by $\nabla n^{5/3}$ in the high density regime respectively (the same holds for the hole density p). The Blakemore statistics, on the other hand, exhibit a singularity at $D = 1$, bringing additional technical issues, which the authors of [25] dealt with in the case of a one-species equation.

1.3.3 State of the art and strategy of our proofs

To our knowledge, there exist only few analytical results in the literature on drift-diffusion equations for more than two species. The study of the existence of solutions in the low-density regime for the three-species memristor drift-diffusion system was done in [71] using Boltzmann statistics for all three species. The high-density regime was analysed in [77].

In [2] the authors analysed a drift-diffusion model for PSCs similar to (1.25)–(1.26). Fermi-Dirac statistics of order 1/2 are used for the electrons and holes and Blakemore statistics for the oxygen vacancies, but the underlying domain Ω consists of three distinct regions where the electrons and holes can move freely while the oxygen vacancies remain confined to the middle region.

For two space dimensions the authors of [2] proved existence of global (bounded) solutions. Their approach was to truncate the quasi-Fermi potentials and show uniform bounds for the solutions from above and below with iterated L^q estimates. However, this truncation requires the initial data to be pointwise bounded from above as well as bounded away from 0 from below and as a result neither void nor saturation are allowed. An entropy-dissipation inequality for the same model with up to three space dimensions was proven in [1].

In our work we will use a different argument to obtain a global existence result. Based on entropy methods [53, 54, 73] and similar to [71, 77] we will use the free energy of the system to derive a priori estimates. We introduce the energy densities

$$\begin{aligned} \psi_1(n) &:= \int_{\mathcal{F}_{1/2}(0)}^{n} G(z)\,dz, \\ \psi_1(n|\widetilde{n}) &:= \psi_1(n) - \psi_1(\widetilde{n}) - \psi_1'(\widetilde{n})(n-\widetilde{n}) \\ &= \int_{\widetilde{n}}^{n} G(z)\,dz - G(\widetilde{n})(n-\widetilde{n}), \\ \psi_2(D) &:= \int_{\mathcal{F}_{-1}(0)}^{D} H(z)\,dz, \end{aligned} \tag{1.27}$$

and define the free energy functional of the system to be

$$\mathcal{E}[n,p,D,V] := \int_\Omega \psi_1(n|\widetilde{n}) + \psi_1(p|\widetilde{p}) + \psi_2(D) + D\widetilde{V} + \frac{\lambda^2}{2}|\nabla(V-\widetilde{V})|^2\,dx, \tag{1.28}$$

where V solves the Poisson equation in (1.25) with the boundary condition(s) given by (1.26). A formal computation, which will be made rigorous in Section 4.2 for an approximate system, shows that

$$\frac{d\mathcal{E}}{dt}[n,p,D,V] + \int_\Omega \frac{n}{2}|\nabla(G(n)-V)|^2 + \frac{p}{2}|\nabla(G(p)+V)|^2 + D|\nabla(H(D)+V)|^2\,dx \\ \leq C(\widetilde{n},\widetilde{p},\widetilde{V},T), \tag{1.29}$$

which provides a priori estimates for n,p in $L^\infty(0,T;L^{5/3}(\Omega))$, for $\nabla n, \nabla p$ in $L^2(0,T;L^{5/4}(\Omega))$ and for D in $L^2(0,T;H^1(\Omega))$. Improved bounds on the solution are then derived using the Gagliardo-Nirenberg inequality and a bootstrapping argument. Utilizing the asymptotic behaviour of the Fermi-Dirac statistics for large as well as low concentrations of n and p, these a priori estimates will allow us to infer bounds for $\partial_t n$, $\partial_t p$ and $\partial_t D$ in some Sobolev space and to then apply the Aubin–Lions lemma [9, 93] to conclude the compactness of a sequence of approximate solutions. The limit of a subsequence of this sequence is a solution to the original problem (1.25)–(1.26). Identifying the limit of the fluxes in this problem is challenging. Using the derived bounds on G'' allows us to identify J_n and J_p. The flux J_D is a bit more delicate as it exhibits a singularity at $D = 1$ and to find its limit we are going to use a Minty-type trick [49, Lemma D.10].

Additionally, we will also show the existence of bounded weak solutions. The difficulties in achieving this stem from the terms $n\nabla V \cdot \nabla n$ and $p\nabla V \cdot \nabla p$, which we get when testing with n and p in the weak formulation. These terms cannot be bounded easily since we cannot use monotonicity properties. Therefore, as a first step, we will assume that $V \in W^{1,3}(\Omega)$ and improve the regularity of the fluxes by redoing the bootstrapping argument on the level of the approximate system. This will allow us to then use $n^q - \widetilde{n}^q$ and $p^q - \widetilde{p}^q$, $q \in \mathbb{N}$, as test functions in the weak formulation and iteratively derive bounds on n, p in $L^\infty(0,T;L^q(\Omega))$ and for $\nabla n^\alpha, \nabla p^\alpha$ in $L^2(Q_T)$ for all $1 \le \alpha < \infty$. Unfortunately, the bounds will depend on q and therefore might blow up as $q \to \infty$.

To overcome this problem we will need slightly more regularity for the gradient of the potential and will consequently assume that $\nabla V \in L^r(\Omega)$ with $r > 3$. This will allow us to apply an Alikakos-type iteration [5], similar to [71, 77], and obtain estimates for n, p in $L^\infty(0,T;L^{q_k}(\Omega))$ uniformly in $k \in \mathbb{N}$, where q_k is of the order 2^k. By taking the limit $k \to \infty$ we will be able to conclude L^∞-bounds on all densities.

1.4 Outline of the thesis

In this section we give an overview of the structure of the thesis.

In Chapter 2 we will focus on the nonlocal cross-diffusion system (1.1)–(1.2). The main results that we will show are:

- Global existence of weak solutions to the nonlocal system (1.1)–(1.2) for nondifferentiable positive semi-definite kernels in detailed balance;
- Weak-strong uniqueness of solutions to the nonlocal system;
- Localization limit to the local system (1.1) and (1.3).

The chapter is organized as follows: Our hypotheses and main results are made precise in Section 2.1. The global existence of weak solutions to the nonlocal system and some regularity results are proven in Section 2.2. The weak-strong uniqueness result is shown in Section 2.3. In Section 2.4, the localization limit, based on the a priori estimates of Section 2.2, is performed. Finally, we collect some auxiliary lemmas in Section 2.5 and state a global existence result for the local system (1.1) and (1.3) in Section 2.6.

The results in this chapter are based on the research collaboration with Ansgar Jüngel (TU Wien) and Antoine Zurek (UTC) and have been published under the title **Nonlocal cross-diffusion systems for multi-species populations and networks** [74].

In Chapter 3 we present the finite-volume scheme and numerical analysis for the nonlocal cross-diffusion system (1.11)–(1.13). The main results that we will show are as follows (see Section 3.1.3 for details):

- We prove the existence of solutions to the finite-volume scheme, which are nonnegative componentwise, preserve the discrete mass, and satisfy discrete versions of the entropy inequalities (1.16) and (1.17).
- We show that the discrete solutions converge to a weak solution to (1.11)–(1.13) when the mesh size tends to zero. As a by-product, this proves the existence of a weak solution to (1.11)–(1.13).
- We illustrate numerically the rate of convergence of the scheme (in space) in the L^p-norm. Additionally we demonstrate the rate of convergence in different metrics of the solution to the nonlocal system towards the solution of the local one (localization limit). Moreover, we illustrate the segregation phenomenon exhibited by the solutions to (1.11)–(1.13); see also [17].

The chapter is organized as follows. The numerical scheme and our main results are introduced in Section 3.1. We prove the existence of discrete solutions in Section 3.2, while the proof of the convergence of the scheme is presented in Section 3.3. In Section 3.4 numerical experiments are given, Section 3.5 contains some auxiliary results, and we show in Section 3.6 that indicator kernels generally do not fulfill inequality (1.15).

The results of this chapter are based on the research collaboration with Ansgar Jüngel (TU Wien) and Antoine Zurek (UTC) and have been published under the title **A convergent finite-volume scheme for nonlocal cross-diffusion systems for multi-species populations** [75].

Chapter 4 is devoted to the analysis of the memristor model (1.25)–(1.26). The main results that we will show are:

- The existence of global weak solutions;
- The solutions are bounded uniform in time under certain elliptic regularity assumptions.

The chapter is organized as follows. We precisely state our assumptions and main results in Section 4.1. The global existence of weak solutions is proved in Section 4.2 and uniform-in-time bounds are shown in Section 4.3. Lastly, in Section 4.4 we collect and prove necessary properties of the Fermi-Dirac integrals and their respective inverses.

The results in this chapter are based on the research collaboration with Maxime Herda (Inria Lille) and Ansgar Jüngel (TU Wien) and are ongoing work. A manuscript is currently under preparation for submission.

1.5 Note on updated status of references

In our bibliography, we have updated manuscripts that we cited in our papers [74, 75] and which were available as preprint on arXiv or HAL at that time, but have been published in the meantime, to their current published status. This concerns the articles [4, 33, 59, 67, 68].

2 Analysis of nonlocal cross-diffusion systems for multi-species populations and networks

The results in this chapter have been published in [74].

In this chapter we provide the details of the analysis of the nonlocal cross-diffusion system (1.1)–(1.2). We present the main results, i.e. the global existence of weak solutions, the weak-strong uniqueness and the localization limit in Section 2.1. In Section 2.2 we prove the existence Theorem 2 and some improved regularity result for the weak solution, and in Section 2.3 we prove the weak-strong uniqueness of solutions to the nonlocal system. The localization limit is proved in Section 2.4 and some auxiliary results are collected in Section 2.5. An existence result for the local system (1.1) & (1.3) is formulated and proven in Section 2.6.

2.1 Main results

We collect the main theorems, which are proved in the subsequent sections. We impose the following hypotheses:

(H1) Data: Let $d \geq 1$, $T > 0$, $\sigma > 0$, and let $u_i^0 \in L^2(\mathbb{T}^d)$ satisfy $u_i^0 \geq 0$ in $\mathbb{T}^d$, $i = 1, \dots, n$.

(H2) Regularity: $K_{ij} \in L^s(\mathbb{T}^d)$ for $i, j = 1, \dots, n$, where $s = d/2$ if $d > 2$, $s > 1$ if $d = 2$, and $s = 1$ if $d = 1$.

(H3) Detailed balance: There exist $\pi_1, \dots, \pi_n > 0$ such that $\pi_i K_{ij}(x-y) = \pi_j K_{ji}(y-x)$ for all $i, j = 1, \dots, n$ and for a.e. $x, y \in \mathbb{T}^d$.

(H4) Positive definiteness: For all $v_1, \dots, v_n \in L^2(\mathbb{T}^d)$, it holds that

$$\sum_{i,j=1}^n \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) v_i(x) v_j(y)\, dx\, dy \geq 0.$$

We need the same diffusivity σ for all species, since otherwise we cannot prove that the Rao-type functional H_2 is a Lyapunov functional. The reason is the mixing of the species in the definition of H_2, cf. (1.6). The regularity $K_{ij} \in L^s(\mathbb{T}^d)$ of the interaction kernels is required to prove the weak convergence of $(u_i^\delta \nabla p_i[u^\delta])$ in $L^1(\mathbb{T}^d)$, where (u^δ) is an approximate sequence. More precisely, the regularity of K_{ij} implies that $\nabla p_i[u^\delta] \rightharpoonup \nabla p_i[u]$ weakly in $L^d(\mathbb{T}^d)$ (if $d \geq 3$) and, as a consequence of the entropy estimate, we have $u_i^\delta \to u_i$ strongly in $L^r(\mathbb{T}^d)$ for $r < d/(d-2)$. Thus, $u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u]$ weakly in $L^\rho(\mathbb{T}^d)$ for $1 \leq \rho < d/(d-1)$.

Remark 1 (Kernels satisfying Hypotheses (H2)–(H4))**.** Kernels satisfying Hypothesis (H4) with $n = 1$ can be characterized by Mercer's theorem [22, 83].

An example of a kernel that satisfies Hypotheses (H2)–(H4) is given by the so–called Gaussian kernel $B(|x-y|) = (2\pi)^{-d/2}\exp(-|x-y|^2/2)$. We define for $i, j = 1, \ldots, n$ and $x, y \in \mathbb{R}^d$,

$$K_{ij}(x-y) = B_{ij}^{\varepsilon}(x-y) := \frac{a_{ij}}{(2\pi\varepsilon^2)^{d/2}}\exp\bigg(-\frac{|x-y|^2}{2\varepsilon^2}\bigg),$$

where $\varepsilon > 0$ and $a_{ij} \ge 0$ are such that the matrix $(\pi_i a_{ij})$ is symmetric and positive definite for some $\pi_i > 0$. Thus, Hypothesis (H3) holds. Hypothesis (H4) can be verified as follows. The identity

$$\frac{e^{-|x-y|^2/(2\varepsilon^2)}}{(2\pi\varepsilon^2)^{d/2}} = \int_{\mathbb{R}^d}\frac{e^{-|x-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}}\frac{e^{-|y-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}}\,dz,$$

shows that

$$\begin{aligned}
&\sum_{i,j=1}^n\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_i K_{ij}(x-y)v_i(x)v_j(y)\,dx\,dy = \sum_{i,j=1}^n\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_i a_{ij}\frac{e^{-|x-y|^2/(2\varepsilon^2)}}{(2\pi\varepsilon^2)^{d/2}}v_i(x)v_j(y)\,dx\,dy \\
&= \sum_{i,j=1}^n\pi_i a_{ij}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d}\bigg(\frac{e^{-|x-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}}v_i(x)\bigg)\,dx\int_{\mathbb{T}^d}\bigg(\frac{e^{-|y-z|^2/\varepsilon^2}}{(\pi\varepsilon^2)^{d/2}}v_j(y)\bigg)\,dy\,dz \\
&\ge \frac{\alpha}{(\pi\varepsilon^2)^d}\sum_{i=1}^n\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{T}^d}e^{-|x-z|^2/\varepsilon^2}v_i(x)\,dx\bigg)^2\,dz \ge 0,
\end{aligned}$$

where $\alpha > 0$ is the smallest eigenvalue of $(\pi_i a_{ij})$. This proves the positive definiteness of K_{ij}. Note that $B_{ij}^{\varepsilon} \to a_{ij}\delta_0$ as $\varepsilon \to 0$ in the sense of distributions.

We can construct further examples from the Gaussian kernel. For instance,

$$K_{ij}(x-y) = \frac{a_{ij}}{1+|x-y|^2}, \quad i, j = 1, \ldots, n,\ x, y \in \mathbb{R}^d,$$

satisfies Hypothesis (H4), since

$$\frac{1}{1+|x-y|^2} = \int_0^\infty e^{-s(1+|x-y|^2)}\,ds,$$

and $B(x-y) = \exp(-s(1+|x-y|^2))$ is positive definite. □

We call $u = (u_1, \ldots, u_n)$ a *weak solution* to system (1.1)–(1.2) if it holds for all test functions $\phi_i \in L^{d+2}(0, T; W^{1,d+2}(\mathbb{T}^d))$, $i = 1, \ldots, n$, that

$$\int_0^T\langle\partial_t u_i, \phi_i\rangle\,dt + \sigma\int_0^T\int_{\mathbb{T}^d}\nabla u_i\cdot\nabla\phi_i\,dx\,dt = -\int_0^T\int_{\mathbb{T}^d}u_i\nabla p_i[u]\cdot\nabla\phi_i\,dx\,dt, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $W^{1,d+2}(\mathbb{T}^d)'$ and $W^{1,d+2}(\mathbb{T}^d)$, and the initial datum $u_i(0) = u_i^0$ is satisfied in the sense of $W^{1,d+2}(\mathbb{T}^d)'$.

First, we show the global existence of weak solutions. Let $Q_T = \mathbb{T}^d \times (0, T)$.

Theorem 2 (Global existence)**.** *Let Hypotheses (H1)–(H4) hold. Then there exists a global weak solution $u = (u_1, \ldots, u_n)$ to (1.1)–(1.2) satisfying $u_i \geq 0$ in Q_T and*

$$\begin{aligned} &u_i^{1/2} \in L^2(0,T; H^1(\mathbb{T}^d)), \quad u_i \in L^{1+2/d}(Q_T) \cap L^q(0,T; W^{1,q}(\mathbb{T}^d)), \\ &\partial_t u_i \in L^q(0,T; W^{-1,q}(\mathbb{T}^d)), \quad u_i \nabla p_i[u] \in L^q(Q_T), \end{aligned} \tag{2.2}$$

where $q = (d+2)/(d+1)$ and $i = 1, \ldots, n$. The initial datum in (1.1) is satisfied in the sense of $W^{-1,q}(\mathbb{T}^d) := W^{1,d+2}(\mathbb{T}^d)'$. Moreover, the following entropy inequalities hold:

$$H_1(u(t)) + 4\sigma \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i |\nabla u_i^{1/2}|^2 \, dx \, ds \leq H_1(u^0), \tag{2.3}$$

$$H_2(u(t)) + \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i u_i |\nabla p_i[u]|^2 \, dx \, ds \leq H_2(u^0). \tag{2.4}$$

Unfortunately, we cannot treat vanishing diffusion $\sigma = 0$, since this would not allow us to derive suitable gradient bounds. However, we can allow for arbitrarily small $\sigma > 0$, which means that cross-diffusion may dominate diffusion, or in other words, the cross-diffusion term is generally *not* just a perturbation.

Imposing more regularity on the kernel functions, we can derive $H^1(\mathbb{T}^d)$ regularity for u_i, which is needed for the weak-strong uniqueness result.

Proposition 3 (Regularity)**.** *Let Hypotheses (H1)–(H4) hold and let $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$ for all $i, j = 1, \ldots, n$. Then there exists a weak solution $u = (u_1, \ldots, u_n)$ to system (1.1)–(1.2) satisfying $u_i \geq 0$ in $\mathbb{T}^d$ and*

$$u_i \in L^2(0,T; H^1(\mathbb{T}^d)), \quad \partial_t u_i \in L^2(0,T; H^{-1}(\mathbb{T}^d)), \quad \nabla p_i[u] \in L^\infty(0,T; L^\infty(\mathbb{T}^d)).$$

Moreover, if additionally ∇K_{ij}, $\Delta K_{ij} \in L^\infty(\mathbb{T}^d)$ and $m_0 \leq u_i^0 \leq M_0$ in $\mathbb{T}^d$, then it holds that

$$0 < m_0 e^{-\lambda t} \leq u_i(t) \leq M_0 e^{\lambda t} \text{ in } \mathbb{T}^d \text{ for } t < 0,$$

where $\lambda > 0$ depends on ΔK_{ij} and u^0.

The proof of the $H^1(\mathbb{T}^d)$-regularity is based on standard L^2-estimates if $\nabla K_{ij} \in L^\infty(\mathbb{T}^d)$. The difficulty is the reduced regularity $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$, which requires some care. Indeed, using the test function u_i in the weak formulation of (1.1) leads to a cubic term, which is reduced to a subquadratic term for ∇u_i by combining the Gagliardo–Nirenberg inequality and the uniform $L^1(\mathbb{T}^d)$-bound for u_i.

Similar lower and upper bounds as in Proposition 3 were obtained in [43] with a different proof. Since the L^∞-bounds depend on the derivatives of K_{ij}, they do not carry over in the localization limit to the local system. In fact, it is an open problem whether the local system (1.1) and (1.3) possesses *bounded* weak solutions. The proposition also holds for kernel functions $K_{ij}(x, y)$ that are used in neural network theory; see Remark 10.

Theorem 4 (Weak-strong uniqueness)**.** *Let $K_{ij} \in L^\infty(\mathbb{R}^d)$ and $c \leq u_i^0 \leq C$ in $\mathbb{T}^d$ for all $i, j = 1, \ldots, n$ with constants $0 < c \leq C < \infty$. Let u be a nonnegative weak solution* *to* (1.1)–(1.2) *satisfying* (2.2) *as well as* $u_i \in L^2(0,T;H^1(\mathbb{T}^d)) \cap H^1(0,T;H^{-1}(\mathbb{T}^d))$. *Additionally, let* $v = (v_1,\ldots,v_n)$ *be a "strong" solution to* (1.1)–(1.2)*, i.e. a weak solution to* (1.1)–(1.2) *satisfying*

$$c \le v_i \le C \quad \text{in } Q_T, \quad \partial_t v_i \in L^2(0,T;H^{-1}(\mathbb{T}^d)), \quad v_i \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d)),$$

and having the same initial data as u. *Then* $u(x,t) = v(x,t)$ *for a.e.* $(x,t) \in \mathbb{T}^d \times (0,T)$.

The existence of a strong solution v_i to (1.1)–(1.2) was proved in [31, Prop. 1], but only locally in time and in the whole space setting. While the proof can be adapted to the case of a torus, it is less clear how to extend it globally in time. Theorem 4 cannot be extended in a straightforward way to the whole space case since $v_i \ge c > 0$ would be nonintegrable. In the case of the Maxwell–Stefan cross-diffusion system on a bounded domain $\Omega \subset \mathbb{R}^d$, it is possible to relax the lower bound to $v_i > 0$ a.e. and $\log v_i \in L^2(0,T;H^1(\Omega))$ [68]. The proof could be possibly extended to the whole space, but the computations in [68] are made rigorous by exploiting the specific structure of the Maxwell–Stefan diffusion coefficients.

For the localization limit, we choose the radial kernel

$$K_{ij}^\eta(x-y) = \frac{a_{ij}}{\eta^d} B\left(\frac{|x-y|}{\eta}\right), \quad i,j = 1,\ldots,n, \; x,y \in \mathbb{T}^d, \tag{2.5}$$

where $\eta > 0$, $B \in C^0(\mathbb{R})$, $\operatorname{supp}(B) \subset (-1,1)$, $\int_{-1}^1 B(|z|)\,dz = 1$, and $a_{ij} \ge 0$ is such that $(\pi_i a_{ij})$ is symmetric and positive definite for some $\pi_i > 0$, $i = 1,\ldots,n$.

Theorem 5 (Localization limit)**.** *Let* K_{ij}^η *be given by* (2.5) *and satisfying Hypothesis (H4). Let* u^η *be the weak solution to* (1.1)–(1.2)*, constructed in Theorem* 2*. Then there exists a subsequence of* (u^η) *that is not relabeled such that, as* $\eta \to 0$,

$$u^\eta \to u \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\mathbb{T}^d)),$$

if $d \ge 2$ *and strongly in* $L^2(0,T;L^r(\mathbb{T}^d))$ *for any* $r < \infty$ *if* $d = 1$. *Moreover,* u *is a nonnegative weak solution to* (1.1) *and* (1.3).

The existence of global weak solutions to (1.1) and (1.3) can be proved for any bounded domain Ω with Lipschitz boundary $\partial\Omega$ imposing no-flux boundary conditions; see Section 2.6.

2.2 Global existence for the nonlocal system

We prove the global existence of a nonnegative weak solution u to (1.1)–(1.2) and show the regularity properties of Proposition 3. Since the proof is based on the entropy method similar to [73, Chapter 4], we sketch the standard arguments and focus on the derivation of uniform estimates. We assume throughout this section that Hypotheses (H1)–(H4) hold.

2.2.1 Solution of an approximated system

Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $\delta > 0$, and $m \in \mathbb{N}$ with $m > d/2 + 1$. We proceed by induction over $k \in \mathbb{N}$. To this end, let $u^{k-1} \in L^2(\mathbb{T}^d;\mathbb{R}^n)$ be given, where the superindex k refers to the time step $t_k = k\tau$. Set $u_i(w) = \exp(w_i/\pi_i) > 0$. We wish to find $w^k \in H^m(\mathbb{T}^d;\mathbb{R}^n)$ to the approximated system

$$\begin{aligned}\frac{1}{\tau}\int_{\mathbb{T}^d}(u(w^k)-u^{k-1})\cdot\phi\,dx + \sigma\sum_{i=1}^n\int_{\mathbb{T}^d}\nabla u_i(w^k)\cdot\nabla\phi_i\,dx + \delta b(w^k,\phi)\\ = -\sum_{i=1}^n\int_{\mathbb{T}^d}u_i(w^k)\nabla p_i[u(w^k)]\cdot\nabla\phi_i\,dx,\end{aligned} \tag{2.6}$$

for $\phi = (\phi_1,\ldots,\phi_n) \in H^m(\mathbb{T}^d;\mathbb{R}^n)$. The bilinear form

$$b(w^k,\phi) = \int_{\mathbb{T}^d}\bigg(\sum_{|\alpha|=m}D^\alpha w^k\cdot D^\alpha\phi + w\cdot\phi\bigg)\,dx$$

is coercive on $H^m(\mathbb{T}^d;\mathbb{R}^n)$, i.e. $b(w^k,w^k) \ge C\|w^k\|^2_{H^m(\mathbb{T}^d)}$ for some $C>0$, as a consequence of the generalized Poincaré–Wirtinger inequality (see Lemma 15 in Section 2.5). By a fixed-point argument, which uses a mapping of the type $L^\infty(\mathbb{T}^d) \to H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ and is done in the entropy variable w^k, it is sufficient to derive a uniform bound for all fixed points w_i^k in the space $H^m(\mathbb{T}^d)$ (see [73, Section 4.4] for details). To this end, let $w_i^k \in H^m(\mathbb{T}^d)$ be such a fixed point. We use the admissible test function $\phi_i = w_i^k = \pi_i\log u_i^k$ (with $u_i^k := u_i(w^k)$) in (2.6):

$$\begin{aligned}\sum_{i=1}^n\frac{\pi_i}{\tau}\int_{\mathbb{T}^d}(u_i^k-u_i^{k-1})\cdot\log u_i^k\,dx + 4\sigma\sum_{i=1}^n\pi_i\int_{\mathbb{T}^d}|\nabla(u_i^k)^{1/2}|^2\,dx + \delta b(w^k,w^k)\\ = -\sum_{i=1}^n\int_{\mathbb{T}^d}u_i^k\nabla p_i[u^k]\cdot\nabla w_i^k\,dx = -\sum_{i=1}^n\int_{\mathbb{T}^d}\pi_i\nabla p_i[u^k]\cdot\nabla u_i^k\,dx,\end{aligned}$$

where we used the identity $u_i^k\nabla w_i^k = \pi_i\nabla u_i^k$. An integration by parts gives

$$\int_{\mathbb{T}^d}\nabla K_{ij}(x-y)u_j^k(y)\,dy = \int_{\mathbb{T}^d}\nabla K_{ij}(z)u_j^k(x-z)\,dz = \int_{\mathbb{T}^d}K_{ij}(x-y)\nabla u_j^k(y)\,dy. \tag{2.7}$$

Thus, in view of definition (1.2) of $p_i[u^k]$ and Hypothesis (H4),

$$\begin{aligned}\sum_{i=1}^n\frac{\pi_i}{\tau}\int_{\mathbb{T}^d}(u_i^k-u_i^{k-1})\cdot\log u_i^k\,dx + 4\sigma\sum_{i=1}^n\pi_i\int_{\mathbb{T}^d}|\nabla(u_i^k)^{1/2}|^2\,dx + \delta b(w^k,w^k)\\ = -\sum_{i,j=1}^n\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\pi_iK_{ij}(x-y)\nabla u_j^k(y)\cdot\nabla u_i^k(x)\,dx\,dy \le 0.\end{aligned}$$

The convexity of $f(z) = z(\log z - 1)$ for $z \ge 0$ implies that $f(z) - f(y) \le f'(z)(z-y)$ for all $y,z > 0$. Using this inequality to estimate the first integral on the left-hand side of the displayed inequality and the coercivity of $b(w^k,w^k)$ to estimate the third term, we find that

$$\frac{1}{\tau}\sum_{i=1}^n\int_{\mathbb{T}^d}\pi_i\big(u_i^k(\log u_i^k - 1) - u_i^{k-1}(\log u_i^{k-1}-1)\big)\,dx$$

$$+ 4\sigma \sum_{i=1}^n \pi_i \|\nabla(u_i^k)^{1/2}\|_{L^2(\mathbb{T}^d)}^2 + \delta C \sum_{i=1}^n \|w_i^k\|_{H^m(\mathbb{T}^d)}^2 \le 0, \tag{2.8}$$

where C is the coercivity constant of the bilinear form $b(\cdot,\cdot)$. This provides an estimate uniform in the fixed points w^k in $H^m(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, necessary to conclude the fixed-point argument and giving the existence of a solution $w^k \in H^m(\mathbb{T}^d;\mathbb{R}^n)$ to (2.6). This defines $u^k := u(w^k)$, finishing the induction step.

To derive further uniform estimates, we wish to use $\psi_i = \pi_i p_i[u^k]$ as a test function in (2.6). However, we cannot estimate the term $\delta b(w^k,\psi)$ appropriately. Therefore, we perform the limits $\delta \to 0$ and $\tau \to 0$ separately.

2.2.2 Limit $\delta \to 0$

Let us first list some uniform bounds and convergence results.

Lemma 6. *Let $u^\delta = (u_1^\delta,\ldots,u_n^\delta)$ with $u_i^\delta = u_i(w^k)$ be a solution to (2.6) and let $w_i^\delta = \pi_i \log u_i^\delta$ for $i = 1,\ldots,n$ (slightly abusing the notation). Then there exists a constant $C > 0$, independent of δ, such that*

$$\|(u_i^\delta)^{1/2}\|_{L^{r_1}(\mathbb{T}^d)} + \|\nabla u_i^\delta\|_{L^{r_2}(\mathbb{T}^d)} \le C,$$

where $r_1 = 2d/(d-2)$ and $r_2 = d/(d-1)$ if $d > 2$, $r_1 < \infty$ and $r_2 < 2$ if $d = 2$ and $r_1 = \infty$ and $r_2 = 2$ if $d = 1$. Moreover, it holds, up to a subsequence, as $\delta \to 0$ that

$$\begin{aligned} u_i^\delta &\to u_i \quad \text{strongly in } L^r(\mathbb{T}^d), \quad r < r_1/2, \\ \nabla u_i^\delta &\rightharpoonup \nabla u_i \quad \text{weakly in } L^{r_2}(\mathbb{T}^d), \\ \delta w_i^\delta &\to 0 \quad \text{strongly in } H^m(\mathbb{T}^d). \end{aligned}$$

Proof. Estimate (2.8) and the Poincaré–Wirtinger inequality show that $(u_i^\delta)^{1/2}$ is uniformly bounded in $H^1(\mathbb{T}^d)$ and, by Sobolev's embedding, in $L^{r_1}(\mathbb{T}^d)$. Therefore, we infer that the gradient $\nabla u_i^\delta = 2(u_i^\delta)^{1/2}\nabla(u_i^\delta)^{1/2}$ is uniformly bounded in $L^{r_2}(\mathbb{T}^d)$. By Sobolev's embedding, the sequence (u_i^δ) is relatively compact in $L^r(\mathbb{T}^d)$ for $r < r_1/2$, and there exists a subsequence that is not relabeled such that, as $\delta \to 0$, the claimed convergences of u_i^δ and ∇u_i^δ hold. We deduce from (2.8) that $\sqrt{\delta} w_i^\delta$ is uniformly bounded in $H^m(\mathbb{T}^d)$, hence $\delta w_i^\delta \to 0$ in $H^m(\mathbb{T}^d)$ in the limit $\delta \to 0$. This ends the proof. □

Thanks to Lemma 6, we have, up to a subsequence, $u_i^\delta \to u_i$ a.e. and (u_i^δ) is dominated by some function in $L^r(\mathbb{T}^d)$. By dominated convergence, $p_i[u^\delta] \to p_i[u]$ a.e. and Young's convolution inequality (see Lemma 11 in Section 2.5) shows that, for $d > 2$,

$$\begin{aligned} \|p_i[u^\delta]\|_{L^\infty(\mathbb{T}^d)} &\le \sum_{j=1}^n \left\| \int_{\mathbb{T}^d} K_{ij}(.-y)u_j^\delta(y)\,dy \right\|_{L^\infty(\mathbb{T}^d)} \\ &\le \sum_{j=1}^n \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \|u_j^\delta\|_{L^{d/(d-2)}(\mathbb{T}^d)} \le C. \end{aligned}$$

Here and in the following, $C > 0$ denotes a constant independent of δ with values possibly changing from line to line. In a similar way, we can prove that $(p_i[u^\delta])$ is bounded in $L^r(\mathbb{T}^d)$ for any $r < \infty$ if $d = 2$ and in $L^\infty(\mathbb{T}^d)$ if $d = 1$, assuming that $K_{ij} \in L^1(\mathbb{T}^d)$; see Hypothesis (H2). Lemma 12 in Section 2.5 then implies that $p_i[u^\delta] \to p_i[u]$ strongly in $L^r(\mathbb{T}^d)$ for any $r < \infty$. Furthermore, if $d > 2$, we use again Young's convolution inequality and Lemma 6 to show that

$$\begin{aligned} \|\nabla p_i[u^\delta]\|_{L^{r_3}(\mathbb{T}^d)} &\le \sum_{j=1}^n \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \|\nabla u_j^\delta\|_{L^{d/(d-1)}(\mathbb{T}^d)} \\ &\le \max_{1\le j\le n} \|K_{ij}\|_{L^{d/2}(\mathbb{T}^d)} \sum_{j=1}^n \|\nabla u_j^\delta\|_{L^{d/(d-1)}(\mathbb{T}^d)} \le C, \end{aligned}$$

where $r_3 = d$. Similar computations show that for $d = 2$ we have $\nabla p_i[u^\delta]$ is bounded in $L^{r_3}(\mathbb{T}^d)$ for some $r_3 > 2$ and for $r_3 = 2$ if $d = 1$. Hence, for a subsequence,

$$\nabla p_i[u^\delta] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^{r_3}(\mathbb{T}^d).$$

Combining this with the strong convergence $u_i^\delta \to u_i$ in $L^r(\mathbb{T}^d)$ for $r < r_1/2$, we conclude that the product converges:

$$u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(\mathbb{T}^d).$$

We deduce from the uniform bounds $\|u_i^\delta\|_{L^{r_1/2}(\mathbb{T}^d)} \le C$ and $\|\nabla p_i[u^\delta]\|_{L^{r_3}(\mathbb{T}^d)} \le C$ that the sequence $(u_i^\delta \nabla p_i[u^\delta])$ is bounded in $L^{\min\{2,d/(d-1)\}}(\mathbb{T}^d)$ and

$$u_i^\delta \nabla p_i[u^\delta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^{\min\{2,d/(d-1)\}}(\mathbb{T}^d).$$

Thus, we can pass to the limit $\delta \to 0$ in (2.6) to conclude that $u_i^k := u_i \ge 0$ for $i = 1, \dots, n$ solves

$$\frac{1}{\tau} \int_{\mathbb{T}^d} (u^k - u^{k-1}) \cdot \phi \, dx + \sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \nabla u_i^k \cdot \nabla \phi_i \, dx = - \sum_{i=1}^n \int_{\mathbb{T}^d} u_i^k \nabla p_i[u^k] \cdot \nabla \phi_i \, dx, \tag{2.9}$$

for all test functions $\phi_i \in W^{1,r_3}(\mathbb{T}^d)$. Observe that $p_i[u^k]$ is an element of the space $W^{1,r_3}(\mathbb{T}^d)$ and is an admissible test function; this will be used in the next subsection.

2.2.3 Uniform estimates

We introduce the piecewise constant in time functions $u^{(\tau)}(x,t) = u^k(x)$ for $x \in \mathbb{T}^d$ and for $t \in ((k-1)\tau, k\tau]$, $k = 1, \dots, N$. At time $t = 0$, we set $u_i^{(\tau)}(\cdot, 0) = u_i^0$. Furthermore, we use the time-shift operator $(S_\tau u^{(\tau)})(x,t) = u^{k-1}(x)$ for $x \in \mathbb{T}^d$, $t \in ((k-1)\tau, k\tau]$. Then, summing over k in (2.9), we obtain

$$\begin{aligned} \frac{1}{\tau} \int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi \, dx \, dt + \sigma \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} \nabla u_i^{(\tau)} \cdot \nabla \phi_i \, dx \, dt \\ = - \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \cdot \nabla \phi_i \, dx \, dt, \end{aligned}$$

for piecewise constant functions $\phi : (0,T) \to W^{1,r_3}(\mathbb{T}^d;\mathbb{R}^n)$ and, by a density argument, for all functions $\phi \in L^2(0,T;W^{1,r_3}(\mathbb{T}^d;\mathbb{R}^n))$. Summing the entropy inequality (2.8) over $k = 1,\dots,N$, it follows that

$$H_1(u^{(\tau)}(T)) + 4\sigma \sum_{i=1}^n \int_0^T \pi_i \|\nabla (u_i^{(\tau)})^{1/2}\|^2_{L^2(\mathbb{T}^d)}\,dt \le H_1(u^0). \tag{2.10}$$

These bounds allow us to derive further estimates. Since the $L^1\log L^1$-bound dominates the $L^1(\mathbb{T}^d)$-norm, we deduce from the Poincaré–Wirtinger inequality that

$$\|u_i^{(\tau)} \log u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \|(u_i^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\mathbb{T}^d))} \le C(u^0), \quad i = 1,\dots,n.$$

This implies, by the Gagliardo–Nirenberg inequality with $\theta = d/(d+2)$, that

$$\begin{aligned}
\|u_i^{(\tau)}\|^{1+2/d}_{L^{1+2/d}(Q_T)} &= \int_0^T \|(u_i^{(\tau)})^{1/2}\|^{2+4/d}_{L^{2+4/d}(\mathbb{T}^d)}\,dt \\
&\le C\int_0^T \|(u_i^{(\tau)})^{1/2}\|^{\theta(2d+4)/d}_{H^1(\mathbb{T}^d)} \|(u_i^{(\tau)})^{1/2}\|^{(1-\theta)(2d+4)/d}_{L^2(\mathbb{T}^d)}\,dt \\
&\le C\|u_i^{(\tau)}\|^{2/d}_{L^\infty(0,T;L^1(\mathbb{T}^d))} \int_0^T \|(u_i^{(\tau)})^{1/2}\|^2_{H^1(\mathbb{T}^d)}\,dt \le C(u^0,d),
\end{aligned} \tag{2.11}$$

and by Hölder's inequality with $q = (d+2)/(d+1)$,

$$\begin{aligned}
\|\nabla u_i^{(\tau)}\|_{L^q(Q_T)} &= 2\|(u_i^{(\tau)})^{1/2}\nabla (u_i^{(\tau)})^{1/2}\|_{L^q(Q_T)} \\
&\le 2\|(u_i^{(\tau)})^{1/2}\|_{L^{2+4/d}(Q_T)} \|\nabla (u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C.
\end{aligned} \tag{2.12}$$

These bounds are not sufficient to pass to the limit $\tau \to 0$, since we also need uniform bounds on $u_i^\tau p_i[u^\tau]$ and on the discrete time derivative. To derive further estimates, we use the test function $\phi_i = \pi_i p_i[u^k] \in W^{1,r_3}(\mathbb{T}^d)$ in (2.9):

$$\begin{aligned}
&\frac{1}{\tau}\sum_{i,j=1}^n \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i (u_i^k(x) - u_i^{k-1}(x)) K_{ij}(x-y) u_j^k(y)\,dx\,dy \\
&\quad + \sigma \sum_{i,j=1}^n \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla u_i^k(x)\cdot \nabla u_j^k(y)\,dx\,dy = -\sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i^k |\nabla p_i[u^k]|^2\,dx.
\end{aligned} \tag{2.13}$$

Because of the positive definiteness of $\pi_i K_{ij}$, the second term on the left-hand side is nonnegative. Exploiting the symmetry and positive definiteness of $\pi_i K_{ij}$ (Hypotheses (H3)–(H4)), the first integral can be estimated from below as

$$\begin{aligned}
&\frac{1}{2\tau}\sum_{i,j=1}^n \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)\Big(u_i^k(x)u_j^k(y) - u_i^{k-1}(x)u_j^{k-1}(y) \\
&\qquad + (u_i^k(x) - u_i^{k-1}(x))(u_j^k(y) - u_j^{k-1}(y))\Big)\,dx\,dy \\
&\quad \ge \frac{1}{2\tau}\sum_{i,j=1}^n \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)\big(u_i^k(x)u_j^k(y) - u_i^{k-1}(x)u_j^{k-1}(y)\big)\,dx\,dy
\end{aligned}$$

$$= \frac{1}{\tau}(H_2(u^k) - H_2(u^{k-1})).$$

Therefore, we infer from (2.13) that

$$H_2(u^k) + \tau \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i^k |\nabla p_i[u^k]|^2 \, dx \le H_2(u^{k-1}),$$

and summing this inequality over $k = 1, \dots, N$, we have

$$H_2(u^{(\tau)}(T)) + \sum_{i=1}^n \pi_i \int_0^T \|(u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}]\|^2_{L^2(\mathbb{T}^d)} \, dt \le H_2(u^0). \tag{2.14}$$

The previous bound allows us to derive an estimate for the discrete time derivative.

Lemma 7. *Let $u^{(\tau)}$ be a previously obtained solution to the weak formulation*

$$\begin{aligned} \frac{1}{\tau}\int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi \, dx \, dt + \sigma \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} \nabla u_i^{(\tau)} \cdot \nabla \phi_i \, dx \, dt \\ = - \sum_{i=1}^n \int_0^T \int_{\mathbb{T}^d} u_i^{(\tau)} \nabla p_i[u^{(\tau)}] \cdot \nabla \phi_i \, dx \, dt \end{aligned} \tag{2.15}$$

for all functions $\phi \in L^2(0,T; W^{1,r_3}(\mathbb{T}^d; \mathbb{R}^n))$. Then there exists a constant $C > 0$ independent of τ such that

$$\tau^{-1} \|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^q(0,T;W^{2,(d+2)/2}(\mathbb{T}^d)')} \le C, \tag{2.16}$$

where $q = (d+2)/(d+1)$.

Proof. Estimates (2.11) and (2.14) imply that

$$u_i^{(\tau)} \nabla p_i[u^{(\tau)}] = (u_i^{(\tau)})^{1/2} \cdot (u_i^{(\tau)})^{1/2} \nabla p_i[u^{(\tau)}]$$

is uniformly bounded in $L^q(Q_T)$, where $q = (d+2)/(d+1)$. Let $\phi \in L^{q'}(0,T; W^{2,(d+2)/2}(\mathbb{T}^d))$, where $q' = d+2$. Then $1/q + 1/q' = 1$ and

$$\begin{aligned} &\frac{1}{\tau}\left| \int_0^T \int_{\mathbb{T}^d} (u^{(\tau)} - S_\tau u^{(\tau)}) \cdot \phi \, dx \, dt \right| \\ &\le \sigma \sum_{i=1}^n \|u_i^{(\tau)}\|_{L^{1+2/d}(Q_T)} \|\Delta \phi_i\|_{L^{(d+2)/2}(Q_T)} + \sum_{i=1}^n \|u_i^{(\tau)} \nabla p_i[u^{(\tau)}]\|_{L^q(Q_T)} \|\nabla \phi_i\|_{L^{q'}(Q_T)} \\ &\le C \|\phi\|_{L^{q'}(0,T;W^{2,(d+2)/2}(\mathbb{T}^d))}. \end{aligned}$$

We conclude that

$$\tau^{-1} \|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^q(0,T;W^{2,(d+2)/2}(\mathbb{T}^d)')} \le C,$$

which finishes the proof of the lemma. □

2.2.4 Limit $\tau \to 0$

Estimates (2.10) and (2.16) allow us to apply the Aubin–Lions compactness lemma in the version of [19] to conclude the existence of a subsequence that is not relabeled such that, as $\tau \to 0$,

$$u_i^{(\tau)} \to u_i \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\mathbb{T}^d)),\ i = 1,\ldots,n,$$

if $d \geq 2$ and strongly in $L^2(0,T;L^r(\mathbb{T}^d))$ for any $r < \infty$ if $d = 1$. Strictly speaking, the version of [19] holds for the continuous time derivative, but the technique of [48] shows that the conclusion of [19] also holds for the discrete time derivative. Then, maybe for another subsequence, $u_i^{(\tau)} \to u_i$ a.e. in Q_T, and we deduce from (2.11) that $u_i^{(\tau)} \to u_i$ strongly in $L^r(Q_T)$ for all $r < 1 + 2/d$ (see Lemma 12 in Section 2.5). Furthermore, we obtain from (2.10), (2.12), (2.14), and (2.16) the convergences

$$\begin{aligned} \nabla u_i^{(\tau)} &\rightharpoonup \nabla u_i \quad \text{weakly in } L^q(Q_T),\ i = 1,\ldots,n, \\ \tau^{-1}(u^{(\tau)} - S_\tau u^{(\tau)}) &\rightharpoonup \partial_t u_i \quad \text{weakly in } L^q(0,T;W^{2,(d+2)/2}(\mathbb{T}^d)'), \\ (u_i^{(\tau)})^{1/2}\nabla p_i[u^{(\tau)}] &\rightharpoonup z_i \quad \text{weakly in } L^2(Q_T), \end{aligned} \tag{2.17}$$

where $z_i \in L^2(Q_T)$ and $q = (d+2)/(d+1)$. Since $u_i^{(\tau)} \geq 0$, we infer that $u_i \geq 0$ in Q_T. It remains to identify the limit z_i, which is stated in the following result.

Lemma 8. *Let $u^{(\tau)}$ be the previously obtained solution to (2.15). Then the weak limit z_i in (2.17) can be identified as $z_i = u_i^{1/2}\nabla p_i[u]$, i.e., as $\tau \to 0$,*

$$(u_i^{(\tau)})^{1/2}\nabla p_i[u^{(\tau)}] \rightharpoonup u_i^{1/2}\nabla p_i[u] \quad \textit{weakly in } L^2(Q_T).$$

Proof. We show first that

$$\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

It follows from the strong convergence of $(u_i^{(\tau)})$ that $K_{ij}(x-y)u_j^{(\tau)}(y,t) \to K_{ij}(x-y)u_j(y,t)$ for a.e. $(y,t) \in Q_T$ and for a.e. $x \in \mathbb{T}^d$. Hence, because of the uniform bounds, there holds the convergence $p_i[u^{(\tau)}] \to p_i[u]$ a.e. in Q_T. We deduce from Young's convolution inequality and the uniform bound for $\nabla u_i^{(\tau)}$ in $L^q(Q_T)$ that $\nabla p_i[u^{(\tau)}]$ is uniformly bounded in $L^q(Q_T)$. Therefore,

$$\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

When $d = 2$, we have the convergences of $\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u]$ weakly in $L^{4/3}(Q_T)$ and of $(u_i^{(\tau)})^{1/2} \to u_i^{1/2}$ strongly in $L^4(Q_T)$, which is sufficient to pass to the limit in the product and to identify it with z_i. However, this argument fails when $d > 2$, and we need a more sophisticated proof. The div-curl lemma does not apply, since the exponents of the respective Lebesgue spaces, in which the convergences of $(u_i^{(\tau)})^{1/2}$ and $\nabla p_i[u^{(\tau)}]$ take place, are not conjugate for $d > 2$. Also the generalization [21, Theorem 2.1] to nonconjugate exponents cannot be used for general d.

Our idea is to exploit the fact that the product converges in a space better than L^1. Then Lemma 13 in Section 2.5 immediately implies that

$$(u_i^{(\tau)})^{1/2}\nabla p_i[u^{(\tau)}] \rightharpoonup u_i^{1/2}\nabla p_i[u] \quad \text{weakly in } L^q(Q_T).$$

In fact, estimate (2.14) implies that this convergence holds in $L^2(Q_T)$, which finishes the proof of Lemma 8. □

Combining Lemma 8 and the strong convergence of $(u_i^{(\tau)})^{1/2}$ in $L^2(Q_T)$ gives

$$u_i^{(\tau)}\nabla p_i[u^{(\tau)}] \rightharpoonup u_i\nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

In view of the uniform bounds for $(u_i^{(\tau)})^{1/2}$ in $L^{2+4/d}(Q_T)$ and of $(u_i^{(\tau)})^{1/2}\nabla p_i[u^{(\tau)}]$ in $L^2(Q_T)$, the product $u_i^{(\tau)}\nabla p_i[u^{(\tau)}]$ is uniformly bounded in $L^q(Q_T)$. Thus, the previous weak convergence also holds in $L^q(Q_T)$.

2.2.5 End of the proof

The convergences of the previous subsection allow us to pass to the limit $\tau \to 0$ in (2.9), yielding

$$\int_0^T \langle \partial_t u_i, \phi_i\rangle\, dt + \sigma \int_0^T \int_{\mathbb{T}^d} \nabla u_i \cdot \nabla \phi_i \, dx\, dt = -\int_0^T \int_{\mathbb{T}^d} u_i \nabla p_i[u]\cdot \nabla\phi_i\, dx\, dt,$$

for all smooth test functions. Because of ∇u_i, $u_i\nabla p_i[u] \in L^q(Q_T)$, a density argument shows that the weak formulation holds for all $\phi \in L^{q'}(0,T;W^{1,q'}(\mathbb{T}^d))$, recalling that $q' = d+2$. Then $\partial_t u_i$ lies in the space $L^q(0,T;W^{-1,q}(\mathbb{T}^d))$, where $W^{-1,q}(\mathbb{T}^d) := W^{1,q'}(\mathbb{T}^d)'$. The proof that $u(\cdot,0)$ satisfies the initial datum can be done exactly as in [72, p. 1980]. Finally, using the convexity of H_1 and the lower semi-continuity of convex functions, the entropy inequalities (2.10) and (2.14) become (2.3) and (2.4), respectively, in the limit $\tau \to 0$. This ends the proof of Theorem 2.

Remark 9 (Whole space and bounded domains)**.** We believe that the whole space $\Omega = \mathbb{R}^d$ can be treated by using the techniques of [33], under the assumption that a moment of u^0 is bounded, i.e. $\int_{\mathbb{R}^d} u_i^0(x)(1+|x|^2)^{\alpha/2}\,dx < \infty$ for a suitable $\alpha > 0$. Indeed, standard estimates show that $u_i^\varepsilon(1+|x|^2)^{\alpha/2}$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^d))$, where (u_i^ε) is a sequence of approximating solutions. By the previous proof, $(\sqrt{u_i^\varepsilon})$ is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$, and estimate (2.12) shows that (u_i^ε) is bounded in $L^2(0,T;W^{1,q}(\mathbb{R}^d))$ with $q = (d+2)/(d+1)$. Since $W^{1,q}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d;(1+|x|^2)^{\alpha/2})$ is compactly embedded in $L^r(\mathbb{R}^d)$ for $r < 2q/(2-q)$ and $1 \le q < 2$ (by adapting the proof of [24, Lemma 1]), we can apply the Aubin–Lions lemma, concluding that, up to a subsequence, $u_i^\varepsilon \to u$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$.

The case of bounded domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ seems to be more delicate. We assume no-flux boundary conditions to recover the weak formulation (2.1). The problem comes from the treatment of the boundary integrals when integrating by parts. For instance, we need to integrate by parts in $\nabla p_i[u]$ (see (2.7)) and to control the integral

$$q_i[u](x) := \sum_{j=1}^n \int_{\partial\Omega} K_{ij}(x-y)u_j(y)\nu(y)\,dy,$$

where ν is the exterior unit normal vector of $\partial\Omega$. If $K_{ij} \in L^\infty(\mathbb{R}^d)$, we may estimate this integral by $\|u_i\|_{L^1(\partial\Omega)} \le C\|u_i\|_{W^{1,1}(\Omega)}$. Consequently, $\|q_i[u]\|_{L^\infty(\Omega)} \le C\sum_{j=1}^n \|u_j\|_{W^{1,1}(\Omega)}$. The integral $q_i[u]$ appears in the weak formulation, for instance, as

$$\left|\sigma\int_\Omega \nabla u_i \cdot q_i[u]\,dx\right| \le C\sum_{j=1}^n \|\nabla u_j\|^2_{W^{1,1}(\Omega)} \le 2C\sum_{j=1}^n \|u_j\|_{L^1(\Omega)}\|\nabla u_j^{1/2}\|^2_{L^2(\Omega)},$$

and this integral cannot generally be absorbed by the corresponding term in (2.8) except if $\|u_j^0\|_{L^1(\Omega)}$ is sufficiently small. □

2.2.6 Proof of Proposition 3.

The proof of the $H^1(\mathbb{T}^d)$ regularity requires an approximate scheme that differs from the one used in the proof of Theorem 2. Given $u^{k-1} \in L^2(\mathbb{T}^d;\mathbb{R}^n)$ with $u_i^{k-1} \ge 0$, we wish to find $u^k \in H^1(\mathbb{T}^d;\mathbb{R}^n)$ such that

$$\frac{1}{\tau}\int_{\mathbb{T}^d}(u_i^k - u_i^{k-1})\phi_i\,dx + \sigma\int_{\mathbb{T}^d}\nabla u_i^k \cdot \nabla\phi_i\,dx + \int_{\mathbb{T}^d}\frac{(u_i^k)^+}{1+\delta(u_i^k)^+}\nabla p_i[u^k]\cdot\nabla\phi_i\,dx = 0, \quad (2.18)$$

for $\phi_i \in H^1(\mathbb{T}^d)$, where $\delta > 0$ and $z^+ = \max\{0,z\}$. Since $\nabla K_{ij} \in L^{d+2}(\mathbb{T}^d)$, $\nabla p_i[u^k]$ can be bounded in $L^{d+2}(\mathbb{T}^d)$ in terms of the $L^1(\mathbb{T}^d)$-norm of u^k. Thus, the last term on the left-hand side is well defined. The existence of a solution to this discrete scheme is proved by a fixed-point argument, and the main step is the derivation of uniform estimates. First, we observe that the test function $(u_i^k)^- = \min\{0,u_i^k\}$ yields

$$\begin{aligned}\frac{1}{\tau}\int_{\mathbb{T}^d}(u_i^k - u_i^{k-1})(u_i^k)^-\,dx + \sigma\int_{\mathbb{T}^d}|\nabla(u_i^k)^-|^2\,dx \\ = -\int_{\mathbb{T}^d}\frac{(u_i^k)^+}{1+\delta(u_i^k)^+}\nabla p_i[u^k]\cdot\nabla(u_i^k)^-\,dx = 0,\end{aligned}$$

and consequently, $(u_i^k)^- = 0$ in $\mathbb{T}^d$. Thus, $u_i^k \ge 0$ and we can remove the plus sign in (2.18). Second, we use the test function u_i^k in (2.18) and sum over $i = 1,\ldots,n$:

$$\frac{1}{\tau}\sum_{i=1}^n\int_{\mathbb{T}^d}(u_i^k - u_i^{k-1})u_i^k\,dx + \sigma\sum_{i=1}^n\int_{\mathbb{T}^d}|\nabla u_i^k|^2\,dx = -\sum_{i=1}^n\int_{\mathbb{T}^d}\frac{u_i^k}{1+\delta u_i^k}\nabla p_i[u^k]\cdot\nabla u_i^k\,dx. \quad (2.19)$$

The first integral becomes

$$\sum_{i=1}^n\int_{\mathbb{T}^d}(u_i^k - u_i^{k-1})u_i^k\,dx \ge \frac{1}{2}\sum_{i=1}^n\int_{\mathbb{T}^d}\big((u_i^k)^2 - (u_i^{k-1})^2\big)\,dx.$$

The right-hand side in (2.19) is estimated by Hölder's inequality and Young's convolution inequality:

$$-\sum_{i=1}^n\int_{\mathbb{T}^d}\frac{u_i^k}{1+\delta u_i^k}\nabla p_i[u^k]\cdot\nabla u_i^k\,dx \le \sum_{i=1}^n\|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}\|\nabla p_i[u^k]\|_{L^{d+2}(\mathbb{T}^d)}\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}$$

$$\leq C_K \sum_{i,j=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)} \|u_j^k\|_{L^1(\mathbb{T}^d)} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)},$$

where $C_K > 0$ depends on the $L^{d+2}(\mathbb{T}^d)$-norm of ∇K_{ij} but not on δ. Taking the test function $\phi_i = 1$ in (2.18) shows that $\|u_i^k\|_{L^1(\mathbb{T}^d)} = \|u_i^0\|_{L^1(\mathbb{T}^d)}$ is uniformly bounded. This allows us to reduce the cubic expression on the right-hand side of the previous inequality to a quadratic one. This is the key idea of the proof. Combining the previous arguments, (2.18) becomes

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k)^2\,dx - \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^{k-1})^2\,dx + \tau\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} |\nabla u_i^k|^2\,dx \\ \leq \tau C \sum_{i=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)} \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)} \\ \leq \frac{1}{2}\tau\sigma \sum_{i=1}^n \|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^2 + \tau C \sum_{i=1}^n \|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2.\end{aligned}$$

The Gagliardo–Nirenberg and Poincaré–Wirtinger inequalities (see Lemmas 14 and 15 in Section 2.5) show that

$$\begin{aligned}\|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2 &\leq C\|u_i^k\|_{H^1(\mathbb{T}^d)}^{2\theta} \|u_i^k\|_{L^1(\mathbb{T}^d)}^{2(1-\theta)} \\ &\leq C\big(\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)} + \|u_i^k\|_{L^1(\mathbb{T}^d)}\big)^{2\theta} \|u_i^k\|_{L^1(\mathbb{T}^d)}^{2(1-\theta)} \\ &\leq C(u^0)\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^{2\theta} + C(u^0),\end{aligned}$$

where $\theta = d(d+4)/(d+2)^2 < 1$. We deduce from Young's inequality that for any $\varepsilon > 0$,

$$C(u^0)\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^{2\theta} \leq \theta\varepsilon^{1/\theta}\|\nabla u_i^k\|^2 + (1-\theta)\varepsilon^{-1/(1-\theta)}C(u^0)^{1/(1-\theta)}.$$

After setting $C(\varepsilon) = C(u^0) + \varepsilon^{-1/(1-\theta)}C(u^0)^{1/(1-\theta)}$, we find that

$$\|u_i^k\|_{L^{2+4/d}(\mathbb{T}^d)}^2 \leq \varepsilon^{1/\theta}\|\nabla u_i^k\|_{L^2(\mathbb{T}^d)}^2 + C(\varepsilon).$$

Therefore, choosing $\varepsilon > 0$ sufficiently small, we infer that

$$\frac{1}{2}\sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^k)^2\,dx - \frac{1}{2}\sum_{i=1}^n \int_{\mathbb{T}^d} (u_i^{k-1})^2\,dx + \frac{1}{4}\tau\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} |\nabla u_i^k|^2\,dx \leq \tau C(\varepsilon). \tag{2.20}$$

This provides a uniform $H^1(\mathbb{T}^d)$-estimate for u^k. Defining the fixed-point operator as a mapping from $L^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, the compact embedding $H^1(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$ implies the compactness of this operator (see [73, Chapter 4] for details). This shows that (2.18) possesses a weak solution $u^k \in H^1(\mathbb{T}^d)$.

In order to pass to the limit $(\delta,\tau) \to 0$, we need uniform estimates for the piecewise constant in time functions $u_i^{(\tau)}$, using the same notation as in the proof of Theorem 2. Estimate (2.20) provides uniform bounds for $u_i^{(\tau)}$ in $L^\infty(0,T;L^2(\mathbb{T}^d))$ and $L^2(0,T;H^1(\mathbb{T}^d))$. By the Gagliardo–Nirenberg inequality, $(u_i^{(\tau)})$ is bounded in $L^{2+4/d}(Q_T)$. By Young's convolution inequality,

$$\sup_{t\in(0,T)} \|\nabla p_i[u^{(\tau)}(t)]\|_{L^\infty(\mathbb{T}^d)} \le \sum_{j=1}^n \|\nabla K_{ij}\|_{L^{d+2}(\mathbb{T}^d)} \sup_{t\in(0,T)} \|u_j^{(\tau)}\|_{L^q(\mathbb{T}^d)} \le C,$$

where $q=(d+2)/(d+1)$. Thus, $(\nabla p_i[u^{(\tau)}])$ is bounded in $L^\infty(0,T;L^\infty(\mathbb{T}^d))$. From these estimates, we can derive a uniform bound for the discrete time derivative. Using the uniform estimates $\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\mathbb{T}^d))} \le C$ and $\|\nabla p_i[u^{(\tau)}]\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C$, we can prove in a similar way as in the proof of Theorem 2 that

$$\tau^{-1}\|u^{(\tau)} - S_\tau u^{(\tau)}\|_{L^2(0,T;H^{-1}(\mathbb{T}^d))} \le C.$$

Therefore, by the Aubin–Lions lemma [48], up to a subsequence, as $(\delta,\tau)\to 0$,

$$u_i^{(\tau)} \to u_i \quad \text{strongly in } L^2(Q_T),$$

and this convergence even holds in $L^r(Q_T)$ for any $r < 2+4/d$. We can show as in the proof of Theorem 2 that $p_i[u^{(\tau)}] \to p_i[u]$ a.e. and consequently, for a subsequence, $\nabla p_i[u^{(\tau)}] \rightharpoonup \nabla p_i[u]$ weakly in $L^2(Q_T)$. We infer that

$$u_i^{(\tau)}\nabla p_i[u^{(\tau)}] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

Omitting the details, it follows that $u=(u_1,\ldots,u_n)$ is a weak solution to (1.1)–(1.2) satisfying the regularity $u_i \in L^2(0,T;H^1(\mathbb{T}^d))$ for $i=1,\ldots,n$.

Next, we show the lower and upper bounds for u_i. Define $M(t)=M_0e^{\lambda t}$, where $\lambda>0$ will be specified later. Recall that we assume $\nabla K_{ij}\in L^\infty(\mathbb{T}^d)$, and $0<m_0\le u_i^0\le M_0$ in $\mathbb{T}^d$ (see Proposition 3). Hence, we can apply Young's convolution inequality with $p=1$ and $q,r=\infty$ and estimate

$$\|\nabla p_i[u]\|_{L^\infty(\mathbb{T}^d)} \le \sum_{j=1}^n \|\nabla K_{ij}\|_{L^\infty(\mathbb{T}^d)}\|u_j\|_{L^1(\mathbb{T}^d)} \le C.$$

Then, with the test function $e^{-\lambda t}(u_i-M)^+(t) = e^{-\lambda t}\max\{0,(u_i-M)(t)\}$ in the weak formulation of (2.18), we deduce from

$$\partial_t u_i e^{-\lambda t}(u_i-M)^+ = \frac{1}{2}\partial_t\{e^{-\lambda t}[(u_i-M)^+]^2\} + \frac{\lambda}{2}e^{-\lambda t}[(u_i-M)^+]^2 + \lambda e^{-\lambda t}M(u_i-M)^+$$

that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^d} e^{-\lambda t}(u_i-M)^+(t)^2\,dx + \sigma\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}|\nabla(u_i-M)^+|^2\,dx\,ds \\
&\quad = -\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}(u_i-M)\nabla p_i[u]\cdot\nabla(u_i-M)^+\,dx\,ds - \frac{\lambda}{2}\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}[(u_i-M)^+]^2\,dx\,ds \\
&\qquad - \int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}M\nabla p_i[u]\cdot\nabla(u_i-M)^+\,dx\,ds - \lambda\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}M(u_i-M)^+\,dx\,ds.
\end{aligned}$$

We write $(u_i - M)\nabla(u_i - M)^+ = \frac{1}{2}\nabla[(u_i - M)^+]^2$ and integrate by parts in the first and third terms of the right-hand side:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{T}^d} e^{-\lambda s}(u_i - M)^+(t)^2\,dx + \sigma\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}|\nabla(u_i - M)^+|^2\,dx\,ds \\ &\quad\le \frac{1}{2}\big(\|\Delta p_i[u]\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} - \lambda\big)\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}[(u_i - M)^+]^2\,dx\,ds \\ &\qquad + \big(\|\Delta p_i[u]\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} - \lambda\big)\int_0^t\int_{\mathbb{T}^d} e^{-\lambda s}M(u_i - M)^+\,dx\,ds. \end{aligned}$$

By Young's convolution inequality and the regularity assumptions on K_{ij},

$$\|\Delta p_i[u]\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C\sum_{j=1}^n \|u_j\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \le C_0.$$

Therefore, choosing $\lambda \ge C_0$, it follows that

$$\int_{\mathbb{T}^d} e^{-\lambda t}(u_i - M)^+(t)^2\,dx \le 0,$$

and we infer that $e^{-\lambda t}(u_i - M)^+(t) = 0$ and $u_i(t) \le M(t) = M_0 e^{\lambda t}$ for $t > 0$. The lower bound $u_i(t) \ge m(t) := m_0 e^{-\lambda t}$ is proved in the same way, using the test function $e^{-\lambda t}(u_i - m)^-$, where $(u_i - m)^- = \min\{0, u_i - m\}$.

Remark 10. Proposition 3 holds true for more general kernel functions of the type $K_{ij}(x,y)$. In that case, we need the regularity $\nabla_x K_{ij} \in L_y^\infty L_x^{d+2} \cap L_x^\infty L_y^{d+2}$ to apply the Young inequality for kernels; see [95, Theorem 0.3.1]. For the lower and upper bounds of the solution, we additionally need the regularity $\nabla_x K_{ij}$, $\Delta_x K_{ij} \in L_x^\infty L_y^\infty$. □

2.3 Weak-strong uniqueness for the nonlocal system

In this section, we prove Theorem 4. Let u be a nonnegative weak solution and v be a positive "strong" solution to (1.1)–(1.2), i.e., $v = (v_1, \ldots, v_n)$ is a weak solution to (1.1)–(1.2) satisfying

$$0 < c \le v_i \le C \quad \text{in } Q_T, \quad \partial_t v_i \in L^2(0,T;H^{-1}(\mathbb{T}^d)), \quad v_i \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d)).$$

In particular, we have $\log v_i$, $\nabla \log v_i \in L^\infty(0,T;L^\infty(\mathbb{T}^d))$. Then, for $0 < \varepsilon < 1$, we define the regularized relative entropy density

$$h_\varepsilon(u|v) = \sum_{i=1}^n \pi_i\big((u_i + \varepsilon)(\log(u_i + \varepsilon) - 1) - (u_i + \varepsilon)\log v_i + v_i\big),$$

and the associated relative entropy

$$H_\varepsilon(u|v) = \int_{\mathbb{T}^d} h_\varepsilon(u|v)\,dx.$$

2.3.1 Preparations

We compute

$$\frac{\partial h_\varepsilon}{\partial u_i}(u|v) = \pi_i \log(u_i+\varepsilon) - \pi_i \log v_i, \quad \frac{\partial h_\varepsilon}{\partial v_i}(u|v) = -\pi_i \frac{u_i+\varepsilon}{v_i} + \pi_i.$$

Then we have

$$\nabla \frac{\partial h_\varepsilon}{\partial u_i} = \frac{\pi_i}{u_i+\varepsilon}\nabla u_i - \frac{\pi_i}{v_i}\nabla v_i, \quad \nabla \frac{\partial h_\varepsilon}{\partial v_i} = -\frac{\pi_i}{v_i}\nabla u_i + \frac{\pi_i(u_i+\varepsilon)}{v_i^2}\nabla v_i.$$

The second function is an admissible test function for the weak formulation of (1.1), satisfied by v_i, since $\nabla u_i \in L^2(Q_T)$ and $\nabla v_i \in L^\infty(Q_T)$. Strictly speaking, the first function is not an admissible test function for the weak formulation of (1.1), satisfied by u_i, since it needs test functions in $W^{1,d+2}(\mathbb{T}^d)$. However, the nonlocal term becomes with this test function

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K_{ij}(x-y)\nabla u_j(y)\cdot \frac{\nabla u_i(x)}{u_i(x)+\varepsilon}\,dx\,dy,$$

which is finite since K_{ij} is essentially bounded and $\nabla u_i \cdot \nabla u_j \in L^1(Q_T)$. Thus, by a suitable approximation argument, the following computation can be made rigorous. We find that

$$\begin{aligned}
\frac{d}{dt}H_\varepsilon(u|v) &= \sum_{i=1}^n \left(\left\langle \partial_t u_i, \frac{\partial h_\varepsilon}{\partial u_i}(u|v)\right\rangle + \left\langle \partial_t v_i, \frac{\partial h_\varepsilon}{\partial v_i}(u|v)\right\rangle \right) \\
&= -\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \left(\nabla u_i \cdot \nabla \frac{\partial h_\varepsilon}{\partial u_i}(u|v) + \nabla v_i \cdot \nabla \frac{\partial h_\varepsilon}{\partial v_i}(u|v)\right) dx \\
&\quad - \sum_{i=1}^n \int_{\mathbb{T}^d} \left(u_i \nabla p_i[u]\cdot \nabla \frac{\partial h_\varepsilon}{\partial u_i}(u|v) + v_i \nabla p_i[v] \cdot \nabla \frac{\partial h_\varepsilon}{\partial v_i}(u|v)\right) dx \\
&= -\sigma \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i \left| \frac{\nabla u_i}{\sqrt{u_i+\varepsilon}} - \sqrt{u_i+\varepsilon}\frac{\nabla v_i}{v_i}\right|^2 dx \\
&\quad - \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i \left(\frac{u_i}{u_i+\varepsilon}\nabla p_i[u]\cdot\nabla u_i - \frac{u_i}{v_i}\nabla p_i[u]\cdot \nabla v_i - \nabla p_i[v]\cdot \nabla u_i \right. \\
&\quad \left. + \frac{u_i+\varepsilon}{v_i}\nabla p_i[v]\cdot\nabla v_i \right) dx.
\end{aligned}$$

The first integral is nonpositive. Thus, an integration over $(0,t)$ gives

$$\begin{aligned}
&H_\varepsilon(u(t)|v(t)) - H_\varepsilon(u(0)|v(0)) \\
&\quad \le -\sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i (u_i+\varepsilon)\nabla(p_i[u]-p_i[v])\cdot \nabla \log\frac{u_i+\varepsilon}{v_i}\,dx\,ds \\
&\qquad + \varepsilon \sum_{i=1}^n \int_0^t \int_{\mathbb{T}^d} \pi_i \nabla p_i[u]\cdot\nabla \log\frac{u_i+\varepsilon}{v_i}\,dx\,ds =: I_1 + I_2.
\end{aligned} \tag{2.21}$$

2.3.2 Estimation of I_1 and I_2

Inserting the definition of p_i,

$$\begin{aligned}\nabla(p_i[u]-p_i[v])(x) &= \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}(x-y)\nabla(u_j-v_j)(y)\,dy \\ &= \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}(x-y)\bigg((u_j+\varepsilon)(y)\nabla\log\frac{u_j+\varepsilon}{v_j}(y) \\ &\quad + (u_j-v_j)(y)\nabla\log v_j(y) + \varepsilon\nabla\log v_j(y)\bigg)\,dy,\end{aligned}$$

leads to

$$\begin{aligned}I_1 = &-\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)\bigg((u_i+\varepsilon)(x)(u_j+\varepsilon)(y)\nabla\log\frac{u_j+\varepsilon}{v_j}(y) \\ &\times\nabla\log\frac{u_i+\varepsilon}{v_i}(x) + (u_i+\varepsilon)(x)(u_j-v_j)(y)\nabla\log v_j(y)\cdot\nabla\log\frac{u_i+\varepsilon}{v_i}(x)\bigg)\,dx\,dy\,ds \\ &-\varepsilon\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)(u_i+\varepsilon)(x)\nabla\log v_j(y)\cdot\nabla\log\frac{u_i+\varepsilon}{v_i}(x)\,dx\,dy\,ds \\ =:&\ I_{11}+I_{12}.\end{aligned}$$

Setting

$$U_i = (u_i+\varepsilon)\nabla\log\frac{u_i+\varepsilon}{v_i}, \quad V_i = \frac{1}{2}(u_i-v_i)\nabla\log v_i,$$

we can formulate the first integral as

$$\begin{aligned}I_{11} &= -\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)\big(U_i(x)\cdot U_j(y) + 2U_i(x)\cdot V_j(y)\big)\,dx\,dy\,ds \\ &= -\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)(U_i+V_i)(x)\cdot(U_j+V_j)(y)\,dx\,dy\,ds \\ &\quad + \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)V_i(x)\cdot V_j(y)\,dx\,dy\,ds \\ &\le \frac{1}{4}\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \pi_i K_{ij}(x-y)(u_i-v_i)(x)(u_j-v_j)(y)\nabla\log v_i(x)\cdot\nabla\log v_j(y)\,dx\,dy\,ds \\ &\le \frac{1}{4}\max_{i,j=1,\ldots,n}\|\pi_i K_{ij}\|_{L^\infty(\mathbb{T}^d)}\max_{k=1,\ldots,n}\|\nabla\log v_k\|^2_{L^\infty(Q_T)} \\ &\quad\times\sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d}|(u_i-v_i)(x)|\,dx\int_{\mathbb{T}^d}|(u_j-v_j)(y)|\,dy\,ds \\ &\le C\sum_{i=1}^n \int_0^t \bigg(\int_{\mathbb{T}^d}|u_i-v_i|\,dx\bigg)^2 ds,\end{aligned}$$

using the symmetry and positive definiteness of $\pi_i K_{ij}$ as well as the regularity assumptions on K_{ij} and $\nabla \log v_i$. This estimate is crucial, as it will allow us later to apply Grönwall's lemma to conclude the equality of $u_i(t)$ and $v_i(t)$. The second integral I_{12} is estimated as

$$\begin{aligned} I_{12} &= -\varepsilon \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla \log v_j(y) \cdot \big(\nabla u_i - (u_i+\varepsilon)\nabla \log v_i\big)(x)\, dx\, dy\, ds \\ &\le \varepsilon C \sum_{i,j=1}^n \|\nabla \log v_j\|_{L^\infty(Q_T)} \int_0^t \int_{\mathbb{T}^d} \big(|\nabla u_i| + (u_i+1)|\nabla \log v_i|\big)\, dx\, ds \\ &\le \varepsilon C \sum_{i=1}^n \big(\|\nabla u_i\|_{L^1(Q_T)} + \|u_i\|_{L^1(Q_T)} + 1\big) \le \varepsilon C. \end{aligned}$$

We conclude that

$$I_1 \le C \sum_{i=1}^n \int_0^t \left(\int_{\mathbb{T}^d} |u_i - v_i|\, dx \right)^2 ds + \varepsilon C.$$

It remains to estimate I_2. Here we need the improved regularity $\nabla u_i \in L^2(Q_T)$. Inserting the definition of $p_i[u]$, we have

$$I_2 = \varepsilon \sum_{i,j=1}^n \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \pi_i K_{ij}(x-y) \nabla u_j(y) \cdot \nabla \log \frac{u_i+\varepsilon}{v_i}(x)\, dx\, dy\, ds.$$

Since

$$\varepsilon |\nabla u_j(y) \cdot \nabla \log(u_i+\varepsilon)(x)| = 2\varepsilon \left| \nabla u_j(y) \cdot \frac{\nabla\sqrt{u_i+\varepsilon}}{\sqrt{u_i+\varepsilon}}(x) \right| \le 2\sqrt{\varepsilon} |\nabla u_j(y)| |\nabla \sqrt{u_i(x)}|,$$

we find that

$$I_2 \le C \sum_{i,j=1}^n \big(\varepsilon \|\nabla u_j\|_{L^1(Q_T)} + \sqrt{\varepsilon} \|\nabla u_j\|_{L^2(Q_T)} \|\nabla \sqrt{u_i}\|_{L^2(Q_T)} \big) \le \sqrt{\varepsilon} C.$$

We summarize the estimates for I_1 and I_2 and conclude from (2.21) that

$$H_\varepsilon(u(t)|v(t)) - H_\varepsilon(u(0)|v(0)) \le C \sum_{i=1}^n \int_0^t \left(\int_{\mathbb{T}^d} |u_i - v_i|\, dx \right)^2 ds + \sqrt{\varepsilon} C. \tag{2.22}$$

2.3.3 Limit $\varepsilon \to 0$

We perform first the limit in $H_\varepsilon(u(t)|v(t))$. We claim that for $0 < \varepsilon < 1$,

$$\big|(u_i+\varepsilon)(\log(u_i+\varepsilon) - 1)\big| \le u_i(\log u_i + 1) + 1 + e^{-2}.$$

Indeed, set $f(s) := (s+\varepsilon)(\log(s+\varepsilon) - 1)$ and $g(s) := s(\log s + 1) + 1 + e^{-2}$ for $s \ge 0$. A computation shows that $|f(s)| \le 1$ and $g(s) \ge 1$ on the interval $[0, e-\varepsilon]$. Thus, $|f(s)| \le g(s)$ on this interval. Next, it holds that $0 = f(e-\varepsilon) < g(e-\varepsilon)$ and

$$f'(s) = \log(s+\varepsilon) = \log(1+\varepsilon/s) + \log s < 2 + \log s = g'(s)$$

for $s > e - \varepsilon$. This implies that $f(s) < g(s)$ for $s > e - \varepsilon$ and proves the claim. Due to $u_i \in L^2(0,T;\, H^1(\mathbb{T}^d)) \cap H^1(0,T; H^{-1}(\mathbb{T}^d)) \hookrightarrow C^0([0,T]; L^2(\mathbb{T}^d))$, we infer that

$$\big|(u_i+\varepsilon)(\log(u_i+\varepsilon)-1)\big| \le u_i(\log u_i + 1) + C \in L^\infty(0,T; L^1(\mathbb{T}^d)).$$

Therefore, by dominated convergence, as $\varepsilon \to 0$,

$$\sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i(u_i(t)+\varepsilon)(\log(u_i(t)+\varepsilon)-1)\,dx \to \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i(t)(\log u_i(t)-1)\,dx,$$

and this convergence holds for a.e. $t \in (0,T)$. Furthermore, the assumption $c \le v_i \le C$ in Q_T shows that $\log v_i$ is bounded in Q_T and hence,

$$\sum_{i=1}^n \pi_i(-(u_i+\varepsilon)\log v_i + v_i) \le C(v)\bigg(\sum_{i=1}^n u_i + 1\bigg) \in L^\infty(0,T; L^1(\mathbb{T}^d)).$$

By dominated convergence again,

$$\sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i\big(-(u_i(t)+\varepsilon)\log v_i(t) + v_i(t)\big)\,dx \to \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i\big(-u_i(t)\log v_i(t) + v_i(t)\big)\,dx.$$

This shows that for a.e. $t \in (0,T)$,

$$H_\varepsilon(u(t)|v(t)) \to H(u(t)|v(t)) \quad \text{as } \varepsilon \to 0, \text{ where}$$
$$H(u|v) = \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i\big(u_i(\log u_i - 1) - u_i \log v_i + v_i\big)\,dx,$$

and $H_\varepsilon(u(0)|v(0)) = H_\varepsilon(u^0|u^0) \to 0$. Then we deduce from (2.22) in the limit $\varepsilon \to 0$ that

$$H(u(t)|v(t)) \le C\sum_{i=1}^n \int_0^t \|u_i - v_i\|^2_{L^1(\mathbb{T}^d)}\,ds. \tag{2.23}$$

The constant C depends on the L^∞-norm of $\nabla \log v_i$, which is bounded thanks to our assumptions.

Taking the test function $\phi_i = 1$ in the weak formulation of (1.1), we directly see conservation of mass, $\int_{\mathbb{T}^d} u_i^0\,dx = \int_{\mathbb{T}^d} u_i(t)\,dx$ for all $t > 0$. Furthermore, since u and v have the same initial data, it follows that $\int_{\mathbb{T}^d} u_i(t)\,dx = \int_{\mathbb{T}^d} v_i(t)\,dx$ for all $t > 0$. Thus, by the classical Csiszár–Kullback–Pinsker inequality [73, Theorem A.2], we have

$$\begin{aligned} H(u|v) &= \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i \log\frac{u_i}{v_i}\,dx + \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i(v_i - u_i)\,dx = \sum_{i=1}^n \int_{\mathbb{T}^d} \pi_i u_i \log\frac{u_i}{v_i}\,dx \\ &\ge 2\bigg(\int_{\mathbb{T}^d} u_i^0\,dx\bigg)^{-1} \sum_{i=1}^n \|u_i - v_i\|^2_{L^1(\mathbb{T}^d)} = C(u^0)\sum_{i=1}^n \|u_i - v_i\|^2_{L^1(\mathbb{T}^d)}. \end{aligned}$$

We infer from (2.23) that

$$\sum_{i=1}^n \|(u_i - v_i)(t)\|^2_{L^1(\mathbb{T}^d)} \le C\int_0^t \sum_{i=1}^n \|u_i - v_i\|^2_{L^1(\mathbb{T}^d)}\,ds.$$

Grönwall's inequality now implies that $\|(u_i - v_i)(t)\|_{L^1(\mathbb{T}^d)} = 0$ and hence $u_i(t) = v_i(t)$ in $\mathbb{T}^d$ for a.e. $t > 0$ and $i = 1, \ldots, n$. $\square$

2.4 Localization limit

We prove Theorem 5. Let u^η be the nonnegative weak solution to (1.1)–(1.2) with kernel (2.5), constructed in Theorem 2. The entropy inequalities (2.3) and (2.4) as well as the proof of Theorem 2 show that all estimates are independent of η. More precisely, the right-hand side of (2.4) depends on K_{ij}^η, but in view of [20, Theorem 4.22], it holds that $K_{ij}^\eta * u_j \to a_{ij}u_j$ as $\eta \to 0$ and therefore, we can bound the right-hand side of (2.4) uniformly in η. Therefore, for $i = 1, \ldots, n$ (see (2.10)–(2.12), (2.14)–(2.16)),

$$\|u_i^\eta \log u_i^\eta\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \|u_i^\eta\|_{L^{1+2/d}(Q_T)} + \|u_i^\eta\|_{L^q(0,T;W^{1,q}(\mathbb{T}^d))} \le C,$$
$$\|(u_i^\eta)^{1/2}\|_{L^2(0,T;H^1(\mathbb{T}^d))} + \|\partial_t u_i^\eta\|_{L^q(0,T;W^{1,d+2}(\mathbb{T}^d)')} + \|(u_i^\eta)^{1/2}\nabla p_i^\eta[u^\eta]\|_{L^2(Q_T)} \le C,$$

where $C > 0$ is independent of η, $q = (d+2)/(d+1)$, and $p_i^\eta[u_i^\eta] = \sum_{j=1}^n \int_{\mathbb{T}^d} K_{ij}^\eta(x-y)u_j^\eta(y)\,dy$. We infer from the Aubin–Lions lemma in the version of [19, 48] that there exists a subsequence (not relabeled) such that, as $\eta \to 0$,

$$u_i^\eta \to u_i \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\mathbb{T}^d)), \quad i = 1, \ldots, n, \tag{2.24}$$

if $d \ge 2$ and strongly in $L^2(0,T;L^r(\mathbb{T}^d))$ for any $r < \infty$ if $d = 1$. Moreover,

$$\nabla u_i^\eta \rightharpoonup \nabla u_i \quad \text{weakly in } L^q(Q_T), \quad i = 1, \ldots, n, \tag{2.25}$$
$$\partial_t u_i^\eta \rightharpoonup \partial_t u_i \quad \text{weakly in } L^q(0,T;W^{1,d+2}(\mathbb{T}^d)'), \tag{2.26}$$
$$(u_i^\eta)^{1/2}\nabla p_i^\eta[u^\eta] \rightharpoonup z_i \quad \text{weakly in } L^2(Q_T), \tag{2.27}$$

where $z_i \in L^2(Q_T)$ for $i = 1, \ldots, n$.

As in Section 2.2, the main difficulty is the identification of z_i with the term $u_i^{1/2}\nabla p_i[u]$, where $p_i[u] := \sum_{j=1}^n a_{ij}\nabla u_j$. Since the kernel functions also depend on η, the proof is different from the one in Section 2.2. We claim that

$$\nabla p_i^\eta[u^\eta] \rightharpoonup \nabla p_i[u] \quad \text{weakly in } L^q(Q_T). \tag{2.28}$$

Indeed, let $\phi \in L^{q'}(Q_T;\mathbb{R}^n)$, where $q' = d+2$ satisfies $1/q + 1/q' = 1$. We compute

$$\begin{aligned}
&\bigg|\int_0^T\int_{\mathbb{T}^d}(\nabla p_i^\eta[u^\eta] - \nabla p_i[u])\cdot\phi\,dx\,dt\bigg| \\
&\quad = \bigg|\sum_{j=1}^n\int_0^T\int_{\mathbb{T}^d}\bigg(\int_{\mathbb{T}^d}K_{ij}^\eta(x-y)\nabla u_j^\eta(y,t)\,dy\bigg)\cdot\phi(x,t)\,dx\,dt \\
&\qquad - \sum_{j=1}^n\int_0^T\int_{\mathbb{T}^d}a_{ij}\nabla u_j(y,t)\cdot\phi(y,t)\,dy\,dt\bigg| \\
&\quad \le \sum_{j=1}^n\bigg|\int_0^T\int_{\mathbb{T}^d}\bigg(\int_{\mathbb{T}^d}K_{ij}^\eta(x-y)\phi(x,t)\,dx - a_{ij}\phi(y,t)\bigg)\cdot\nabla u_j^\eta(y,t)\,dy\,dt\bigg| \\
&\qquad + \sum_{j=1}^n a_{ij}\bigg|\int_0^T\int_{\mathbb{T}^d}\nabla(u_j^\eta - u_j)(y,t)\cdot\phi(y,t)\,dy\,dt\bigg|
\end{aligned}$$

$$\leq \sum_{j=1}^n \left\| \int_{\mathbb{T}^d} K_{ij}^\eta(. - y)\phi(y)\,dy - a_{ij}\phi \right\|_{L^{q'}(Q_T)} \|\nabla u_j^\eta\|_{L^q(Q_T)}$$
$$+ \sum_{j=1}^n a_{ij} \left| \int_0^T \int_{\mathbb{T}^d} \nabla(u_j^\eta - u_j)(y,t) \cdot \phi(y,t)\,dy\,dt \right|.$$

Since B has compact support in $\mathbb{R}$, we can apply Lemma 16 to infer that the first term on the right-hand side, formulated as the convolution $K_{ij}^\eta * \phi - a_{ij}\phi$ (slightly abusing the notation), converges to zero strongly in $L^{q'}(\mathbb{R}^d)$ as $\eta \to 0$. Thus, taking into account the weak convergence (2.25), convergence (2.28) follows.

Because of the convergences (2.24), (2.27), and (2.28), we can apply Lemma 13 from Section 2.5 to infer that $z_i = u_i^{1/2}\nabla p_i[u]$. Therefore,

$$u_i^\eta \nabla p_i^\eta[u^\eta] \rightharpoonup u_i \nabla p_i[u] \quad \text{weakly in } L^1(Q_T).$$

Estimate (2.27) shows that the convergence holds in $L^q(Q_T)$. This convergence as well as (2.25) and (2.26) allow us to perform the limit $\eta \to 0$ in the weak formulation of (1.1), proving that u solves (1.1) and (1.3). □

2.5 Auxiliary results

We recall the Young convolution inequality (the proof in [20, Theorem 4.33] also applies to the torus).

Lemma 11 (Young's convolution inequality)**.** *Let $1 \leq p \leq q \leq \infty$ such that $1+1/q = 1/p+1/r$ and let $K \in L^r(\mathbb{T}^d)$ (extended periodically to $\mathbb{R}^d$). Then for any $v \in L^p(\mathbb{T}^d)$,*

$$\left\| \int_{\mathbb{T}^d} K(\cdot - y)v(y)\,dy \right\|_{L^q(\mathbb{T}^d)} \leq \|K\|_{L^r(\mathbb{T}^d)} \|v\|_{L^p(\mathbb{T}^d)}.$$

The next result is a consequence of Vitali's lemma and is well known. We recall it for the convenience of the reader.

Lemma 12. *Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain, $1 < p < \infty$, and u_ε, $u \in L^1(\Omega)$ be such that (u_ε) is bounded in $L^p(\Omega)$ and $u_\varepsilon \to u$ a.e. in Ω. Then $u_\varepsilon \to u$ strongly in $L^r(\Omega)$ for all exponents $1 \leq r < p$.*

Proof. We have for any $M > 0$,

$$\int_{\{u_\varepsilon \geq M\}} |u_\varepsilon|^r\,dx = \int_{\{u_\varepsilon \geq M\}} |u_\varepsilon|^p |u_\varepsilon|^{-(p-r)}\,dx \leq M^{-(p-r)} \int_\Omega |u_\varepsilon|^p\,dx \leq CM^{-(p-r)} \to 0,$$

as $M \to \infty$. Thus, (u_ε) is uniformly integrable. Since convergence a.e. implies convergence in measure, we conclude with Vitali's convergence theorem. □

The following lemma specifies conditions under which the limit of the product of two converging sequences can be identified.

Lemma 13. *Let $p > 1$ and let $u_\varepsilon \geq 0$, $u_\varepsilon \to u$ strongly in $L^p(\mathbb{T}^d)$, $v_\varepsilon \rightharpoonup v$ weakly in $L^p(\mathbb{T}^d)$, and $u_\varepsilon v_\varepsilon \rightharpoonup w$ weakly in $L^p(\mathbb{T}^d)$ as $\varepsilon \to 0$. Then $w = uv$.*

The lemma is trivial if $p \geq 2$. We apply it in Section 2.2 with $1 < p < 2$. Note that the strong convergence of (u_ε) cannot be replaced by weak convergence. A simple counter-example is given by $u_\varepsilon(x) = \exp(2\pi \mathrm{i} x/\varepsilon) \rightharpoonup 0$ and $v_\varepsilon(x) = \exp(-2\pi \mathrm{i} x/\varepsilon) \rightharpoonup 0$ weakly in $L^2(-1,1)$, but the product fulfils $u_\varepsilon v_\varepsilon \equiv 1 \neq 0 \cdot 0$.

Proof. We define the truncation function $T_1 \in C^2([0,\infty))$ satisfying $T_1(s) = s$ for $0 \leq s \leq 1$, $T_1(s) = 2$ for $s > 3$, and T_1 is nondecreasing and concave in the interval $[1,3]$. Furthermore, we set $T_k(s) = kT_1(s/k)$ for $s \geq 0$ and $k \in \mathbb{N}$. The strong convergence of (u_ε) implies the existence of a not relabelled subsequence such that $u_\varepsilon \to u$ a.e.. Hence, $T_k(u_\varepsilon) \to T_k(u)$ a.e. and since T_k is bounded for fixed $k \in \mathbb{N}$, we conclude by dominated convergence that $T_k(u_\varepsilon) \to T_k(u)$ strongly in $L^r(\mathbb{T}^d)$ for any $r < \infty$. Because of the uniqueness of the limit, the convergence holds for the whole sequence. Thus, $T_k(u_\varepsilon)v_\varepsilon \rightharpoonup T_k(u)v$ weakly in $L^1(\mathbb{T}^d)$. Writing $\overline{z_\varepsilon}$ for the weak limit of a sequence (z_ε) (if it exists), this result means that $\overline{T_k(u_\varepsilon)v_\varepsilon} = T_k(u)v$ and the assumption translates to $\overline{u_\varepsilon v_\varepsilon} = w$. Consequently, $w - T_k(u)v = \overline{(u_\varepsilon - T_k(u_\varepsilon))v_\varepsilon}$. Then we can estimate

$$
\begin{aligned}
\|w - T_k(u)v\|_{L^1(\mathbb{T}^d)} &\leq \sup_{0<\varepsilon<1} \int_{\mathbb{T}^d} |u_\varepsilon - T_k(u_\varepsilon)||v_\varepsilon|\,dx \leq \sup_{0<\varepsilon<1} \int_{\{|u_\varepsilon|>k\}} |u_\varepsilon||v_\varepsilon|\,dx \\
&\leq \frac{1}{k^{p-1}} \sup_{0<\varepsilon<1} \int_{\{|u_\varepsilon|>k\}} |u_\varepsilon|^p |v_\varepsilon|\,dx \leq \frac{1}{k^{p-1}} \sup_{0<\varepsilon<1} \int_{\mathbb{T}^d} |u_\varepsilon|^p (1 + |v_\varepsilon|^p)\,dx \leq \frac{C}{k^{p-1}},
\end{aligned}
$$

where we used the properties $u_\varepsilon = T_k(u_\varepsilon)$ for $u_\varepsilon \leq k$ and $|u_\varepsilon - T_k(u_\varepsilon)| \leq |u_\varepsilon|$ due to $T_k(u_\varepsilon) \leq u_\varepsilon$ for $u_\varepsilon > k$. The constant C depends on the bounds for u_ε in $L^p(\mathbb{T}^d)$ and $u_\varepsilon v_\varepsilon$ in $L^p(\mathbb{T}^d)$. We infer that $T_k(u)v \to w$ strongly in $L^1(\mathbb{T}^d)$ and (for a subsequence) a.e. as $k \to \infty$. Since $T_k(u)v = uv$ in $\{|u| \leq k\}$ for any $k \in \mathbb{N}$ and $\operatorname{meas}\{|u| > k\} \leq \|u\|_{L^1(\mathbb{T}^d)}/k \to 0$, we infer in the limit $k \to \infty$ that $w = uv$ a.e. in $\mathbb{T}^d$. □

For convenience, we recall the Gagliardo–Nirenberg inequality [100, Appendix A, (54a)] and the Poincaré–Wirtinger inequality [50, Sec. 5.8.1, Theorem 1].

Lemma 14 (Gagliardo-Nirenberg inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with Lipschitz boundary. Let $m \in \mathbb{N}$, $\beta \in \mathbb{N}_0^d$ be such that $0 \leq |\beta| \leq m-1$, let $1 \leq p,q,r \leq \infty$, and let $\theta \in [0,1]$ be such that*

$$
\frac{1}{p} = \frac{|\beta|}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\theta + (1-\theta)\frac{1}{q}, \quad \frac{|\beta|}{m} \leq \theta \leq 1.
$$

Then there exists a constant $C > 0$ such that for all $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, it holds that

$$
\|D^\beta u\|_{L^p(\Omega)} \leq C\|u\|_{W^{m,r}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}.
$$

If $1 < r < \infty$ and $m - |\beta| - d/r$ is a nonnegative integer, then the inequality holds only for $|\beta|/m \leq \theta < 1$.

Lemma 15 (Poincaré-Wirtinger inequality). *Let $\Omega \subset \mathbb{R}^d$ $(d \geq q)$ be a bounded domain with Lipschitz boundary and let $1 \leq p \leq \infty$. Then there exists a constant $C > 0$ such that for all $u \in W^{1,p}(\Omega)$,*

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u\, dx \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

In particular, we have

$$\|u\|_{L^p(\Omega)} \leq C(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}).$$

The following lemma states the convergence of a convolution with a sequence of mollifiers, see [20, Theorem 4.22].

Lemma 16. *Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$, and let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of mollifiers, i.e.*

$$\rho_n \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp} \rho_n \subset \overline{B(0,1/n)}, \quad \int \rho_n\, dx = 1, \quad \rho_n \geq 0 \text{ on } \mathbb{R}^d.$$

Then $\rho_n * f \to f$ in $L^p(\mathbb{R}^d)$ as $n \to \infty$.

2.6 Local cross-diffusion system

The existence of global weak solutions to the local system (1.1) and (1.3) in any bounded polygonal domain was shown in [78] by analysing a finite-volume scheme. For existence results on related systems, we refer to, e.g., [4, 6, 32, 38]. For completeness, we state the assumptions and the theorem and indicate how the result can be proved using the techniques of Section 2.2. We assume that $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $T > 0$, and $u^0 \in L^2(\Omega)$ satisfies $u_i^0 \geq 0$ in Ω for $i = 1, \ldots, n$. We set $Q_T = \Omega \times (0,T)$.

Theorem 17 (Existence for the local system). *Let $\sigma > 0$, $a_{ij} \geq 0$, and let the matrix $(u_i a_{ij})$ be positively stable for all $u_i > 0$, $i = 1, \ldots, n$. Assume that there exist $\pi_1, \ldots, \pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i,j = 1, \ldots, n$. Then there exists a global weak solution to (1.1) and (1.3) with no-flux boundary conditions, satisfying $u_i \geq 0$ in Q_T and*

$$u_i \in L^2(0,T; H^1(\Omega)) \cap L^{2+4/d}(Q_T), \quad \partial_t u_i \in L^q(0,T; W^{-1,q}(\Omega)),$$

for $i = 1, \ldots, n$, where $q = (d+2)/(d+1)$. The initial datum in (1.1) is satisfied in the sense of $W^{-1,q}(\Omega)$. Moreover, the following entropy inequalities are satisfied:

$$\begin{aligned} &\frac{dH_1}{dt} + 4\sigma \sum_{i=1}^n \int_\Omega \pi_i |\nabla \sqrt{u_i}|^2\, dx + \alpha \sum_{i=1}^n \int_\Omega |\nabla u_i|^2\, dx \leq 0, \\ &\frac{dH_2^0}{dt} + \sum_{i=1}^n \int_\Omega \pi_i u_i |\nabla p_i[u]|^2\, dx + \alpha\sigma \sum_{i=1}^n \int_\Omega |\nabla u_i|^2\, dx \leq 0, \end{aligned} \tag{2.29}$$

where $\alpha > 0$ is the smallest eigenvalue of $(\pi_i a_{ij})$ and $H_2^0(u) := \frac{1}{2}\sum_{i,j=1}^n \int_\Omega \pi_i a_{ij} u_i u_j\, dx \geq 0$.

We call a matrix *positively stable* if all eigenvalues have a positive real part. This condition means that (1.1) is parabolic in the sense of Petrovskii, which is a minimal condition to ensure the local solvability [6]. Inequalities (1.9)–(1.10) and (2.29) reveal a link between the entropy structures of the nonlocal and local systems. This link was explored recently in detail for related systems in [43].

Proof. If Ω is the torus, the theorem is a consequence of the localization limit (Theorem 5). If Ω is a bounded domain, the result can be proved by using the techniques of the proof of Theorem 2. In fact, the proof is simpler since the problem is local. The entropy identities are (formally)

$$\begin{aligned} \frac{dH_1}{dt} + 4\sigma\sum_{i=1}^n\int_\Omega \pi_i|\nabla\sqrt{u_i}|^2\,dx &= -\sum_{i,j=1}^n\int_\Omega \pi_i a_{ij}\nabla u_i\cdot\nabla u_j\,dx, \\ \frac{dH_2^0}{dt} + \sum_{i=1}^n\int_\Omega \pi_i u_i|\nabla p_i[u]|^2\,dx &= -\sigma\sum_{i,j=1}^n\int_\Omega \pi_i a_{ij}\nabla u_i\cdot\nabla u_j\,dx. \end{aligned} \tag{2.30}$$

We claim that the matrix $(\pi_i a_{ij})$ is positive definite. Let $A_1 := \operatorname{diag}(u_i/\pi_i)$ and $A_2 := (\pi_i a_{ij})$. Then A_1 is symmetric and positive definite by our assumptions, A_2 is symmetric and the product $A_1A_2 = (u_i a_{ij})$ is positively stable. Therefore, by [37, Prop. 3], A_2 is positive definite. We infer that the right-hand sides in (2.30) are nonpositive, and we derive estimates for an approximate family of u_i in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$. By the Gagliardo–Nirenberg inequality, this yields bounds for u_i in $L^{2+4/d}(Q_T)$. Consequently, $u_i\nabla p_i[u]$ is bounded in $L^q(Q_T)$, where $q = (d+2)/(d+1)$ (we can even choose $q = 4(d+2)/(3d+4)$), and the time derivative $\partial_t u_i$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$. These estimates are sufficient to deduce from the Aubin–Lions lemma the relative compactness for the approximate family of u_i in $L^2(Q_T)$. The limit in the approximate problem, similar to (2.6), shows that the limit satisfies (1.1) and (1.3). Finally, using the lower semicontinuity of convex functions and the norm, the weak limit in the entropy inequalities leads to (2.29). □

3 A convergent finite-volume scheme for nonlocal cross-diffusion systems for multi-species populations

The results in this chapter have been published in [75].

This chapter is devoted to the design and analysis of a finite-volume scheme for the nonlocal cross-diffusion system (1.11)–(1.12) & (1.13). We start by introducing some necessary notation in Section 3.1.1. Then we introduce the scheme and its properties in Section 3.1.2 and state our main results, the existence of discrete solutions and the convergence of the scheme, in Section 3.1.3. The proof of the existence Theorem 22 is detailed in Section 3.2 and the convergence Theorem 23 is proved in Section 3.3. Finally, we present numerical experiments in Section 3.4, where we investigate the convergence rate of our scheme in space as well as the rate of convergence of the localization limit and we illustrate the segregation pattern of the system due to the nonlocal kernels. Some auxiliary results are collected in Section 3.5 and in Section 3.6 we show that indicator functions of a ball with a radius $r > 0$ in general do not fulfil the positive semi-definiteness condition (1.8).

3.1 Notation and numerical scheme

3.1.1 Notation

A uniform mesh $\mathcal{T}$ of the torus $\mathbb{T}$ consists of N intervals (or cells) K_ℓ of length $\Delta x = 1/N$, given by $K_\ell = (x_{\ell-1/2}, x_{\ell+1/2})$ with end points $x_{\ell\pm1/2} = (\ell \pm 1/2)\Delta x$ and centers $x_\ell = \ell\Delta x$ for $\ell \in G = \mathbb{Z} \setminus N\mathbb{Z}$. For given end time $T > 0$, let $N_T \in \mathbb{N}$ and define the time step size $\Delta t = T/N_T$ and the time steps $t_k = k\Delta t$. A space-time discretization of $Q_T := \mathbb{T}\times(0,T)$ is denoted by $\mathcal{D}$; it consists of the space discretization $\mathcal{T}$ of $\mathbb{T}$ and the time discretization $(N_T, \Delta t)$ of $(0,T)$.

We introduce some function spaces. The space of piecewise constant (in space) functions is given by

$$\mathcal{V}_\mathcal{T} = \left\{ v : \mathbb{T} \to \mathbb{R} : \exists (v_\ell)_{\ell\in G} \subset \mathbb{R},\ v(x) = \sum_{\ell\in G} v_\ell \mathbb{1}_{K_\ell}(x) \right\},$$

where $\mathbb{1}_{K_\ell}$ is the indicator function of K_ℓ. We identify the function $v \in \mathcal{V}_\mathcal{T}$ and the numbers $(v_\ell)_{\ell\in G}$ by writing $v = (v_\ell)_{\ell\in G}$. For $q \in [1,\infty)$ and $v \in \mathcal{V}_\mathcal{T}$, we introduce the $L^q(\mathbb{T})$-norm, the discrete $W^{1,q}(\mathbb{T})$-seminorm, and the discrete $W^{1,q}(\mathbb{T})$-norm by, respectively,

$$\|v\|_{0,q,\mathcal{T}}^q = \sum_{\ell\in G} \Delta x |v_\ell|^q, \quad |v|_{1,q,\mathcal{T}}^q = \sum_{\ell\in G} \Delta x \left| \frac{v_{\ell+1} - v_\ell}{\Delta x} \right|^q,$$

$$\|v\|_{1,q,\mathcal{T}}^q = |v|_{1,q,\mathcal{T}}^q + \|v\|_{0,q,\mathcal{T}}^q.$$

We also define the discrete $L^\infty(\mathbb{T})$-norm by $\|v\|_{0,\infty,\mathcal{T}} = \max_{\ell\in G}|v_\ell|$. Let us point out that for functions $v \in \mathcal{V}_\mathcal{T}$ it holds that $\|v\|_{0,q,\mathcal{T}} = \|v\|_{L^q(\mathbb{T})}$. We set

$$\mathrm{D}_\ell v := \frac{v_{\ell+1}-v_\ell}{\Delta x} \quad \text{and} \quad \mathrm{D}v := (\mathrm{D}_\ell v)_{\ell\in G}.$$

We briefly recall the definition of the space $\mathrm{BV}(\mathbb{T})$ of functions of bounded variation. A function $v \in L^1(\mathbb{T})$ belongs to $\mathrm{BV}(\mathbb{T})$ if its total variation $\mathrm{TV}(v)$, given by

$$\mathrm{TV}(v) = \sup\left\{ \int_\mathbb{T} v(x)\partial_x\phi(x)\,dx : \ \phi \in C_0^1(\mathbb{T}),\ |\phi(x)| \le 1 \quad \text{for all } x \in \mathbb{T} \right\},$$

is finite. We endow the space $\mathrm{BV}(\mathbb{T})$ with the norm

$$\|v\|_{BV(\mathbb{T})} = \|v\|_{L^1(\mathbb{T})} + \mathrm{TV}(v) \quad \text{for all } v \in \mathrm{BV}(\mathbb{T}).$$

In particular, it holds that $\|v\|_{BV(\mathbb{T})} = \|v\|_{1,1,\mathcal{T}}$ for any $v \in \mathcal{V}_\mathcal{T} \cap \mathrm{BV}(\mathbb{T})$.

For any given $q \in [1,\infty)$, we associate to these norms a dual norm with respect to the $L^2(\mathbb{T})$-inner product by

$$\|v\|_{-1,q',\mathcal{T}} = \sup\left\{ \left| \int_\mathbb{T} vw\,dx \right| : w \in \mathcal{V}_\mathcal{T},\ \|w\|_{1,q,\mathcal{T}} = 1 \right\},$$

where $1/q + 1/q' = 1$. Then, the following estimate holds for all v, $w \in \mathcal{V}_\mathcal{T}$:

$$\left| \int_\mathbb{T} vw\,dx \right| \le \|v\|_{-1,q',\mathcal{T}} \|w\|_{1,q,\mathcal{T}}.$$

We also need the space of piecewise constant (in time) functions taking values in $\mathcal{V}_\mathcal{T}$:

$$\mathcal{V}_\mathcal{D} = \left\{ v : \mathbb{T}\times(0,T] \to \mathbb{R} : \exists (v^k)_{k=1,\ldots,N_T},\ v(x,t) = \sum_{k=1}^{N_T} \mathbb{1}_{(t_{k-1},t_k]}(t) v^k(x) \right\},$$

and the discrete $L^p(0,T;W^{1,q}(\mathbb{T}))$-norm

$$\left(\sum_{k=1}^{N_T} \Delta t \|v^k\|_{1,q,\mathcal{T}}^p \right)^{1/p}, \quad \text{where } 1 \le p,q < \infty,\ v \in \mathcal{V}_\mathcal{D}.$$

3.1.2 Numerical scheme

The initial datum (1.12) is approximated by

$$u_{i,\ell}^0 = \frac{1}{\Delta x} \int_{K_\ell} u_i^0(x)\,dx \quad \text{for } \ell \in G,\ i = 1,\ldots,n. \tag{3.1}$$

For given $k \in \{1,\ldots,N_T\}$ and $u^{k-1} \in \mathcal{V}_\mathcal{T}^n$, the values $u^k = (u_{i,\ell}^k)_{i=1,\ldots,n,\,\ell\in G}$ are determined by the implicit Euler finite-volume scheme

$$\frac{\Delta x}{\Delta t}(u_{i,\ell}^k - u_{i,\ell}^{k-1}) + \mathcal{F}_{i,\ell+1/2}^k - \mathcal{F}_{i,\ell-1/2}^k = 0, \quad i = 1,\ldots,n,\ \ell \in G, \tag{3.2}$$

with the numerical fluxes

$$\mathcal{F}^k_{i,\ell+1/2} = -\frac{\sigma}{\Delta x}(u^k_{i,\ell+1} - u^k_{i,\ell}) - \frac{u^k_{i,\ell+1/2}}{\Delta x}(p^k_{i,\ell+1} - p^k_{i,\ell}), \tag{3.3}$$

where the discrete nonlocal operators are given by

$$p^k_{i,\ell} = a_{ii}u^k_{i,\ell} + \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\ell' \in G} \Delta x a_{ij} B^{ij}_{\ell-\ell'} u^k_{j,\ell'}, \quad B^{ij}_{\ell-\ell'} = \frac{1}{\Delta x}\int_{K_{\ell-\ell'}} B^{ij}(y)\,dy, \tag{3.4}$$

for all $i, j = 1, \ldots, n$ and ℓ, $\ell' \in G$. We will show in the proof of Lemma 27 that the identity $p^k_{i,\ell} = a_{ii}u^k_i(x_\ell) + \sum_{j\neq i} a_{ij}(B^{ij} * u^k_j)(x_\ell)$ holds for $\ell \in G$, verifying the consistency of the discretization of $p^k_{i,\ell}$. The mobility $u^k_{i,\ell+1/2} = \widehat{F}(u^k_{i,\ell}, u^k_{i,\ell+1})$ is assumed to satisfy the following properties for all $u_{i,\ell}$, $u_{i,\ell+1}$:

- The function $\widehat{F} : [0,\infty)^2 \to [0,\infty)$ is continuous and satisfies $\widehat{F}(u_{i,\ell}, u_{i,\ell}) = u_{i,\ell}$ as well as $\min\{u_{i,\ell}, u_{i,\ell+1}\} \leq \widehat{F}(u_{i,\ell}, u_{i,\ell+1}) \leq \max\{u_{i,\ell}, u_{i,\ell+1}\}$.
- There exists $c_0 > 0$ such that the following discrete chain rule holds:

$$u_{i,\ell+1/2}(p_{i,\ell+1} - p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \geq c_0(p_{i,\ell+1} - p_{i,\ell})(u_{i,\ell+1} - u_{i,\ell}). \tag{3.5}$$

Remark 18 (Examples for mobilities)**.** Property (3.5) is satisfied if $u_{i,\ell}$ (we omit the superindex k) is defined by the upwind approximation

$$u_{i,\ell+1/2} = \begin{cases} u_{i,\ell+1} & \text{if } p_{i,\ell+1} - p_{i,\ell} \geq 0, \\ u_{i,\ell} & \text{if } p_{i,\ell+1} - p_{i,\ell} < 0, \end{cases} \tag{3.6}$$

or by the logarithmic mean

$$u_{i,\ell+1/2} = \begin{cases} \dfrac{u_{i,\ell+1} - u_{i,\ell}}{\log u_{i,\ell+1} - \log u_{i,\ell}} & \text{if } u_{i,\ell+1} > 0,\ u_{i,\ell} > 0, \text{ and } u_{i,\ell+1} \neq u_{i,\ell}, \\ u_{i,\ell} & \text{if } u_{i,\ell+1} = u_{i,\ell} > 0, \\ 0 & \text{else.} \end{cases} \tag{3.7}$$

We refer to Lemma 32 in Section 3.5 for a proof. □

Remark 19 (Symmetry of discrete kernels)**.** Definition (3.4) of $B^{ij}_{\ell-\ell'}$ is consistent with the discrete analog of $B^{ji}(-x) = B^{ij}(x)$. Indeed, with the change of variables $y \mapsto -y$,

$$B^{ji}_{-\ell'} = \frac{1}{\Delta x}\int_{K_{-\ell'}} B^{ji}(y)\,dy = \frac{1}{\Delta x}\int_{K_{\ell'}} B^{ji}(-y)\,dy = \frac{1}{\Delta x}\int_{K_{\ell'}} B^{ij}(y)\,dy = B^{ij}_{\ell'}.$$

Remark 20 (Discrete derivative of the convolution)**.** A shift of Δx in definition (3.4) of $B^{ij}_{\ell-\ell'}$ shows that $B^{ij}_{\ell-\ell'} = B^{ij}_{(\ell+1)-(\ell'+1)}$, which leads to

$$\sum_{\ell' \in G}(B^{ij}_{(\ell+1)-\ell'} - B^{ij}_{\ell-\ell'})\,u_{j,\ell'} = \sum_{\ell' \in G}\left(B^{ij}_{(\ell+1)-(\ell'+1)}u_{j,\ell'+1} - B^{ij}_{\ell-\ell'}u_{j,\ell'}\right) \tag{3.8}$$

$$= \sum_{\ell' \in G} B^{ij}_{\ell-\ell'}(u_{j,\ell'+1} - u_{j,\ell'})$$

for all $\ell \in G$, $i, j = 1, \dots, n$. This is the discrete analog of the rule $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$. □

Remark 21 (Asymptotic-preserving scheme)**.** For $j \neq i$, consider kernels B^{ij}_ε for some parameter $\varepsilon > 0$ and $B^{ij}_\varepsilon \to \delta_0$ in the sense of distributions as $\varepsilon \to 0$. Let $p^{k,\varepsilon}_{i,\ell}$ be defined as in (3.4) with the kernels $B^{ij}(y)$ replaced by $B^{ij}_\varepsilon(y)$. Then, as $\varepsilon \to 0$,

$$p^{k,\varepsilon}_{i,\ell} \to \sum_{j=1}^n a_{ij} \left(\delta_0 * u_j\right)(x_\ell) = \sum_{j=1}^n a_{ij} u_{j,\ell}.$$

Thus, our numerical scheme is asymptotic-preserving in the sense that the method converges to a finite-volume scheme for the local system, which also preserves the nonnegativity, conserves the mass, and dissipates the Boltzmann and Rao entropies. □

3.1.3 Main results

We impose the following hypotheses:

(H1) Domain and parameters: $\mathbb{T}$ is a one-dimensional torus, i.e. we impose periodic boundary conditions, $T > 0$, $\sigma \geq 0$, and $Q_T := \mathbb{T} \times (0, T)$.

(H2) Initial datum: $u^0 = (u^0_1, \dots, u^0_n) \in L^2(\mathbb{T}; \mathbb{R}^n)$ satisfies $u^0_i \geq 0$ in $\mathbb{T}$.

(H3) Kernels: Let $B^{ij} \in L^\infty(\mathbb{T})$ for $j \neq i$ be nonnegative functions satisfying the symmetry condition $B^{ji}(x) = B^{ij}(-x)$ for a.e. $x \in \mathbb{T}$. There exist numbers $\pi_1, \dots, \pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ (detailed-balance condition), and the matrices M^{ij}, defined in (1.18), are uniformly positive definite for a.e. $x \in \mathbb{T}$.

We consider the one-dimensional equations mainly for notational simplicity. In several space dimensions $d > 1$, we infer uniform estimates in spaces with weaker integrability than in one space dimension, because of Sobolev embeddings. Thanks to the positive definiteness condition on $M^{ij}_{\ell-\ell'}$, we obtain a bound for u_i in the discrete $L^2(0, T; H^1(\mathbb{T}))$-norm, which allows us to conclude, together with the Rao entropy estimate, by the discrete Gagliardo–Nirenberg inequality, a bound for u_i in $L^2(Q_T)$, which is sufficient to estimate the product $u_i \partial_x p_i(u)$. In the one-dimensional situation, this procedure simplifies; see Lemma 28. We discuss the multidimensional case in Remark 30.

Our results also hold if $\sigma = 0$, since the condition $\sigma > 0$ provides an estimate for u_i in the discrete norm of $L^2(0, T; W^{1,1}(\mathbb{T}))$, while the positive definiteness condition on $M^{ij}_{\ell-\ell'}$ allows us to conclude a stronger bound in the discrete norm of $L^2(0, T; H^1(\mathbb{T}))$. Notice that kernels of the type $B^{ij} = \mathbb{1}_K$ satisfy Hypothesis (H3) (for suitable π_i and a_{ij}).

Condition $u^0 \in L^2(\mathbb{T}; \mathbb{R}^n)$ in Hypothesis (H2) is needed to obtain a finite initial Rao entropy $H_R(u^0)$. For the existence result, the assumption on the kernels can be weakened to $B^{ij} \in L^1(\mathbb{T})$. The boundedness condition on B^{ij} in Hypothesis (H3) is needed in the proof of the convergence of the scheme.

We introduce for a given nonnegative function $u \in \mathcal{V}_\mathcal{T}^n$ the discrete entropies

$$\mathcal{H}_B(u) = \sum_{i=1}^n \sum_{\ell \in G} \Delta x \pi_i h(u_{i,\ell}), \quad h(s) = s(\log s - 1), \tag{3.9}$$
$$\mathcal{H}_R(u) = \frac{1}{2} \sum_{i=1}^n \sum_{\ell \in G} \Delta x \pi_i a_{ii} |u_{i,\ell}|^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \sum_{\ell,\ell' \in G} (\Delta x)^2 \pi_i a_{ij} B^{ij}_{\ell-\ell'} u_{j,\ell'} u_{i,\ell},$$

and the matrices

$$M^{ij}_{\ell-\ell'}(x) := \begin{pmatrix} \pi_i a_{ii} & (n-1)\pi_i a_{ij} B^{ij}_{\ell-\ell'}(x) \\ (n-1)\pi_j a_{ji} B^{ij}_{\ell-\ell'}(x) & \pi_j a_{jj} \end{pmatrix} \text{ for } i<j, \ \ell,\ell' \in G. \tag{3.10}$$

In view of Hypothesis (H3), they are symmetric and positive definite uniformly in $\ell, \ell' \in G$ and $x \in \mathbb{T}$, i.e. $z^\top M^{ij}_{\ell-\ell'}(x) z \geq c_M |z|^2$ for $z \in \mathbb{R}^2$, $x \in \mathbb{T}$ and some $c_M > 0$.

Our first main result is the existence of discrete solutions.

Theorem 22 (Existence of discrete solutions)**.** *Let Hypotheses (H1)–(H3) hold. Then there exists a solution $u^k \in \mathcal{V}_\mathcal{T}^n$ to system (3.1)–(3.4) for all $k = 1, \ldots, N_T$, satisfying $u^k_{i,\ell} \geq 0$ for all $i = 1, \ldots, n$, $\ell \in G$ and the discrete entropy inequalities*

$$\mathcal{H}_B(u^k) + \frac{c_0 \Delta t}{n-1} \sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell' \in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \tag{3.11}$$
$$+ 4\sigma \Delta t \sum_{i=1}^n \pi_i |(u_i^k)^{1/2}|^2_{1,2,\mathcal{T}} \leq \mathcal{H}_B(u^{k-1}),$$
$$\mathcal{H}_R(u^k) + \Delta t \sum_{i=1}^n \sum_{\ell \in G} \Delta x \pi_i u^k_{i,\ell+1/2} \left(\frac{p^k_{i,\ell+1} - p^k_{i,\ell}}{\Delta x} \right)^2 \tag{3.12}$$
$$+ \frac{\sigma \Delta t}{(n-1)} \sum_{\substack{i,j=1 \\ i<j}}^n \sum_{\ell,\ell' \in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \leq \mathcal{H}_R(u^{k-1}).$$

Furthermore, the solution conserves the mass, $\sum_{\ell \in G} \Delta x u^k_{i,\ell} = \int_\mathbb{T} u_i^0(x)\,dx$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, N_T$.

This theorem is proved by solving a fixed-point problem based on a topological degree argument, similar as in [79]. For this step, we formulate (3.2) in terms of the entropy variable $w_i = \pi_i \log u_i$ and regularize the equations by adding the discrete analog of $-\varepsilon \Delta w_i + \varepsilon w_i$. The regularization ensures the coercivity in the variable w_i. After transforming back to the original variable $u_i = \exp(w_i/\pi_i)$, we obtain automatically the positivity of u_i (and nonnegativity after passing to the limit $\varepsilon \to 0$). Like on the continuous level, the derivation of the discrete entropy inequalities (3.11) and (3.12) relies on the detailed-balance condition $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j = 1, \ldots, n$.

For our second main result, we need to introduce some notation. We define the "diamond" cell of the dual mesh $T_{\ell+1/2} = (x_\ell, x_{\ell+1})$ with center $x_{\ell+1/2}$. These cells define another partition of $\mathbb{T}$. The gradient of $v \in \mathcal{V}_\mathcal{D}$ is then defined by

$$\partial_x^\mathcal{D} v(x,t) = \mathrm{D}_\ell v^k = \frac{v_{\ell+1}^k - v_\ell^k}{\Delta x} \quad \text{for } x \in T_{\ell+1/2},\ t \in (t_{k-1}, t_k].$$

We also introduce a sequence of space-time discretizations $(\mathcal{D}_m)_{m\in\mathbb{N}}$ indexed by the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$ satisfying $\eta_m \to 0$ as $m \to \infty$. The corresponding spatial mesh is denoted by $\mathcal{T}_m$ with $G_m = \mathbb{Z} \setminus N_m\mathbb{Z}$ and the number of time steps by N_T^m. Finally, to simplify the notation, we set $\partial_x^m := \partial_x^{\mathcal{D}_m}$.

Theorem 23 (Convergence of the scheme)**.** *Let Hypotheses (H1)–(H3) hold and let $(\mathcal{D}_m)$ be a sequence of uniform space-time discretizations satisfying $\eta_m \to 0$ as $m \to \infty$. Let (u_m) be the solutions to (3.1)–(3.4) constructed in Theorem 22. Then there exists $u = (u_1, \ldots, u_n)$ satisfying $u_i \geq 0$ in Q_T and, up to a subsequence, as $m \to \infty$,*

$$u_{i,m} \to u_i \quad \text{strongly in } L^2(Q_T),$$
$$\partial_x^m u_{i,m} \rightharpoonup \partial_x u_i \quad \text{weakly in } L^2(Q_T),$$

and u is a weak solution to (1.11)–(1.12), i.e., it holds for all $\psi_i \in C_0^\infty(\mathbb{T} \times [0,T))$ and for all $i = 1, \ldots, n$ that

$$\int_0^T \int_\mathbb{T} u_i \partial_t \psi_i \, dx\, dt + \int_\mathbb{T} u_i^0 \psi_i(\cdot, 0)\, dx = \int_0^T \int_\mathbb{T} (\sigma \partial_x u_i + u_i \partial_x p_i(u)) \partial_x \psi_i \, dx\, dt.$$

The proof of Theorem 23 is based on suitable estimates uniform with respect to Δx_m and Δt_m, derived from the discrete entropy inequalites. A discrete version of the Aubin–Lions lemma from [57] yields the strong convergence of a subsequence of solutions (u_m) to (3.2)–(3.4). The most technical part is the identification of the limit function as a weak solution to (1.11)–(1.12).

3.2 Proof of Theorem 22

Theorem 22 is proved by induction over $k = 1, \ldots, N_T$. We first regularize the problem and prove the existence of an approximate solution by using a topological degree argument for the fixed-point problem. The discrete entropy inequalities yield a priori estimates independent of the approximation parameter. The de-regularization limit is performed thanks to the Bolzano–Weierstraß theorem.

Let $k \in \{1, \ldots, N_T\}$ and $u^{k-1} \in \mathcal{V}_\mathcal{T}^n$ satisfying $u_{i,\ell}^{k-1} \geq 0$ for $i = 1, \ldots, n$, $\ell \in G$ be given.

3.2.1 Solution to a linearized regularized scheme

We prove the existence of a unique solution to a linearized regularized problem, which allows us to define the fixed-point operator. Let $R > 0$, $\varepsilon > 0$ and define

$$Z_R := \big\{ w = (w_1, \ldots, w_n) \in \mathcal{V}_\mathcal{T}^n : \|w_i\|_{1,2,\mathcal{T}} < R \text{ for } i = 1, \ldots, n \big\}.$$

We introduce the mapping $F : Z_R \to \mathbb{R}^{nN}$, $w \mapsto w^\varepsilon$, where w^ε is the solution to the linear regularized problem

$$-\varepsilon \frac{w^\varepsilon_{i,\ell+1} - 2w^\varepsilon_{i,\ell} + w^\varepsilon_{i,\ell-1}}{\Delta x} + \varepsilon \Delta x w^\varepsilon_{i,\ell} = -\Delta x \frac{u_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} - (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}), \tag{3.13}$$

where $i = 1, \dots, n$, $\ell \in G$, $u_{i,\ell}$ is defined by $u_{i,\ell} = \exp(w_{i,\ell}/\pi_i)$, $\mathcal{F}_{i,\ell\pm1/2}$ is defined as in (3.3) with u^k_i replaced by u_i and $p^k_{i,\ell}$ replaced by

$$p_{i,\ell} = a_{ii} u_{i,\ell} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\ell' \in G} \Delta x a_{ij} B^{ij}_{\ell-\ell'} u_{j,\ell'}.$$

We claim that F is well defined. For this, we write (3.13) in the form

$$Mw^\varepsilon = v, \quad \text{where } v_{i,\ell} = -\Delta x \frac{u_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} - (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}).$$

The matrix $M \in \mathbb{R}^{nN \times nN}$ is a block diagonal matrix with entries $M' \in \mathbb{R}^{N\times N}$, which are tridiagonal matrices such that $M'_{\ell,\ell} = \varepsilon\Delta x + 2\varepsilon/\Delta x$, $M'_{\ell+1,\ell} = M'_{\ell,\ell+1} = -\varepsilon/\Delta x$. We can decompose the full system $Mw^\varepsilon = v$ into the subsystems $M'w^\varepsilon_i = v_i$ for $i = 1, \dots, n$. Since M' is strictly diagonally dominant, there exists a unique solution to $M'w^\varepsilon_i = v_i$ and consequently for $Mw^\varepsilon = v$ by setting $w^\varepsilon = (w^\varepsilon_1, \dots, w^\varepsilon_n)$. We infer that the mapping F is well defined.

3.2.2 Continuity of F

We fix $i \in \{1, \dots, n\}$, multiply (3.13) by $w^\varepsilon_{i,\ell}$, and sum over $\ell \in G$:

$$\begin{aligned} \varepsilon \sum_{\ell\in G} \bigg(-\frac{w^\varepsilon_{i,\ell+1} - 2w^\varepsilon_{i,\ell} + w^\varepsilon_{i,\ell-1}}{\Delta x} + \Delta x w^\varepsilon_{i,\ell}\bigg) w^\varepsilon_{i,\ell} \\ = -\sum_{\ell\in G} \Delta x \frac{u_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} w^\varepsilon_{i,\ell} - \sum_{\ell \in G} (\mathcal{F}_{i,\ell+1/2} - \mathcal{F}_{i,\ell-1/2}) w^\varepsilon_{i,\ell}. \end{aligned} \tag{3.14}$$

The left-hand side can be rewritten by using discrete integration by parts (or summation by parts):

$$\begin{aligned} \varepsilon \sum_{\ell\in G} \bigg(-\frac{(w^\varepsilon_{i,\ell+1} - w^\varepsilon_{i,\ell}) - (w^\varepsilon_{i,\ell} - w^\varepsilon_{i,\ell-1})}{\Delta x} w^\varepsilon_{i,\ell} + \Delta x (w^\varepsilon_{i,\ell})^2 \bigg) \\ = \varepsilon \sum_{\ell\in G} \frac{(w^\varepsilon_{i,\ell+1} - w^\varepsilon_{i,\ell})^2}{\Delta x} + \varepsilon \sum_{\ell\in G} \Delta x (w^\varepsilon_{i,\ell})^2 = \varepsilon \|w^\varepsilon_i\|^2_{1,2,\mathcal{T}}. \end{aligned} \tag{3.15}$$

The first term on the right-hand side of (3.14) is estimated by the Cauchy–Schwarz inequality, where we take into account that $w \in Z_R$, which in turn implies a finite discrete $L^2(\mathbb{T})$-norm for $u_{i,\ell} = \exp(w_{i,\ell}/\pi_i)$:

$$\bigg| -\sum_{\ell\in G} \Delta x \frac{u_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} w^\varepsilon_{i,\ell} \bigg| \le C(\Delta t) \|u_i - u^{k-1}_i\|_{0,2,\mathcal{T}} \|w^\varepsilon_i\|_{0,2,\mathcal{T}} \le C(\Delta t, R) \|w^\varepsilon_i\|_{1,2,\mathcal{T}},$$

where here and in the following $C > 0$, $C(\Delta t, R) > 0$, etc. are generic constants with values changing from line to line. We split the second term on the right-hand side of (3.14) into two parts:

$$\begin{aligned}
&-\sum_{\ell\in G}(\mathcal{F}_{i,\ell+1/2}-\mathcal{F}_{i,\ell-1/2})w^\varepsilon_{i,\ell} = I_1 + I_2, \quad \text{where} \\
&I_1 = \sigma\sum_{\ell\in G}\bigg(\frac{u_{i,\ell+1}-u_{i,\ell}}{\Delta x} - \frac{u_{i,\ell}-u_{i,\ell-1}}{\Delta x}\bigg)w^\varepsilon_{i,\ell}, \\
&I_2 = \sum_{\ell\in G}\bigg(u_{i,\ell+1/2}\frac{p_{i,\ell+1}-p_{i,\ell}}{\Delta x} - u_{i,\ell-1/2}\frac{p_{i,\ell}-p_{i,\ell-1}}{\Delta x}\bigg)w^\varepsilon_{i,\ell}.
\end{aligned}$$

For I_1, we use discrete integration by parts, the Cauchy–Schwarz inequality, and the fact that $w \in Z_R$:

$$\begin{aligned}
|I_1| &= \bigg| -\sigma\sum_{\ell\in G}\Delta x\frac{u_{i,\ell+1}-u_{i,\ell}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \\
&\le \sigma\bigg(\sum_{\ell\in G}\Delta x\bigg|\frac{u_{i,\ell+1}-u_{i,\ell}}{\Delta x}\bigg|^2\bigg)^{1/2}\bigg(\sum_{\ell\in G}\Delta x\bigg|\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg|^2\bigg)^{1/2} \\
&= \sigma|u_i|_{1,2,\mathcal{T}}|w^\varepsilon_i|_{1,2,\mathcal{T}} \le C(R)\|w^\varepsilon_i\|_{1,2,\mathcal{T}}.
\end{aligned}$$

Using discrete integration by parts, and definition (3.4) of $p_{i,\ell}$, we obtain

$$\begin{aligned}
|I_2| &= \bigg| -\sum_{\ell\in G}\Delta x u_{i,\ell+1/2}\frac{p_{i,\ell+1}-p_{i,\ell}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \le I_{21}+I_{22}, \quad \text{where} \\
I_{21} &= \bigg|\sum_{\ell\in G}\Delta x u_{i,\ell+1/2}a_{ii}\frac{(u_{i,\ell+1}-u_{i,\ell})}{\Delta x}\frac{(w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell})}{\Delta x}\bigg|, \\
I_{22} &= \bigg|\sum_{\substack{j=1\\j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2u_{i,\ell+1/2}a_{ij}\frac{B^{ij}_{\ell+1-\ell'}-B^{ij}_{\ell-\ell'}}{\Delta x}u_{j,\ell'}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg|.
\end{aligned}$$

For I_{21}, because of the bound in Z_R, we can estimate $u_{i,\ell+1/2} \le \max\{u_{i,\ell+1}, u_{i,\ell}\} \le C(R)$. Then, thanks to the Cauchy–Schwarz inequality, we obtain

$$I_{21} \le C(R)a_{ii}\,|u_i|_{1,2,\mathcal{T}}\,|w^\varepsilon_i|_{1,2,\mathcal{T}} \le C(R)\,\|w^\varepsilon_i\|_{1,2,\mathcal{T}}.$$

For I_{22}, applying the discrete analog (3.8) of the rule $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$,

$$\begin{aligned}
I_{22} &= \bigg|\sum_{\substack{j=1\\j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2u_{i,\ell+1/2}a_{ij}B^{ij}_{\ell-\ell'}\frac{u_{j,\ell'+1}-u_{j,\ell'}}{\Delta x}\frac{w^\varepsilon_{i,\ell+1}-w^\varepsilon_{i,\ell}}{\Delta x}\bigg| \\
&= \bigg|\sum_{\substack{j=1\\j\neq i}}^n\sum_{\ell,\ell'\in G}(\Delta x)^2u_{i,\ell+1/2}a_{ij}B^{ij}_{\ell-\ell'}(\mathrm{D}_{\ell'}u_j)(\mathrm{D}_\ell w_i)\bigg|,
\end{aligned}$$

where we used the notation of Section 3.1.1. Similarly to I_{21}, we infer that

$$I_{22} \leq C(R) \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \sum_{\ell \in G} \Delta x \Big(\sum_{\ell' \in G} \Delta x B^{ij}_{\ell-\ell'} \mathrm{D}_{\ell'} u_j \Big) \mathrm{D}_\ell w_i.$$

Then, by the Cauchy–Schwarz inequality and the discrete convolution inequality cf. Lemma 33 in Section 3.5,

$$\begin{aligned} I_{22} &\leq C(R) \sum_{\substack{j=1 \\ j \neq i}}^{n} \Big\{ \sum_{\ell \in G} \Delta x \Big(\sum_{\ell' \in G} \Delta x B^{ij}_{\ell-\ell'} \mathrm{D}_{\ell'} u_j \Big)^2 \Big\}^{1/2} |w_i|_{1,2,\mathcal{T}} \\ &\leq C(R) \sum_{\substack{j=1 \\ j \neq i}}^{n} \|B^{ij}\|_{L^1(\mathbb{T})} |u_j|_{1,2,\mathcal{T}} |w_i|_{1,2,\mathcal{T}} \leq C(R) \|w_i\|_{1,2,\mathcal{T}}. \end{aligned}$$

Combining these estimates, we deduce from (3.14) that $\varepsilon \|w_i^\varepsilon\|_{1,2,\mathcal{T}} \leq C(\Delta t, R)$.

We can proceed to show the continuity of F. Let $(w^k)_{k \in \mathbb{N}}$ be such that $w^k \to w \in Z_R$ as $k \to \infty$ and set $w^{\varepsilon,k} := F(w^k)$. We have just proved that $(w^{\varepsilon,k})_{k \in \mathbb{N}}$ is bounded with respect to the $\| \cdot \|_{1,2,\mathcal{T}}$-norm. By the Bolzano–Weierstraß theorem, there exists a subsequence (not relabeled) such that $w^{\varepsilon,k} \to w^\varepsilon$ in Z_R as $k \to \infty$. Performing the limit $k \to \infty$ in (3.14), satisfied for $w^{\varepsilon,k}$, shows that w^ε solves the scheme (3.14) with $u_{i,\ell} = \exp(w_i^\varepsilon / \pi_i)$. This means that $w^\varepsilon = F(w)$, which proves the continuity of F.

3.2.3 Existence of a fixed point

We show that $F : Z_R \to \mathbb{R}^{nN}$ admits a fixed point by using a topological degree argument. We recall that the Brouwer topological degree is a mapping $\deg : M \to \mathbb{Z}$, where

$$M = \big\{ (f, Z, y) : f \in C^0(\mathbb{T}),\ Z \text{ is open, bounded},\ y \notin f(\partial Z) \big\};$$

see [41, Chap. 1, Theorem 3.1] for details and properties.

If we show that any solution $(w^\varepsilon, \rho) \in \overline{Z}_R \times [0,1]$ to the fixed-point equation$w^\varepsilon = \rho F(w^\varepsilon)$ satisfies $(w^\varepsilon, \rho) \notin \partial Z_R \times [0,1]$ for sufficiently large values of $R > 0$, then we deduce from the invariance by homotopy that $\deg(I - \rho F, Z_R, 0)$ is invariant in ρ. Then, choosing $\rho = 0$, it follows that $\deg(I, Z_R, 0) = 1$ and, if $\rho = 1$, $\deg(I - F, Z_R, 0) = \deg(I, Z_R, 0) = 1$. This implies that there exists $w^\varepsilon \in Z_R$ such that $(I - F)(w^\varepsilon) = 0$, which is the desired fixed point.

Let (w^ε, ρ) be a fixed point of $w^\varepsilon = \rho F(w^\varepsilon)$. If $\rho = 0$, there is nothing to show. Therefore, let $\rho > 0$. Then w_i^ε solves

$$-\varepsilon \frac{w^\varepsilon_{i,\ell+1} - 2w^\varepsilon_{i,\ell} + w^\varepsilon_{i,\ell-1}}{\Delta x} + \varepsilon \Delta x w^\varepsilon_{i,\ell} = -\rho \Big(\Delta x \frac{u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} + \mathcal{F}^\varepsilon_{i,\ell+1/2} - \mathcal{F}^\varepsilon_{i,\ell-1/2} \Big) \tag{3.16}$$

for all $\ell \in G$ and $i = 1, \ldots, n$, where $u^\varepsilon_{i,\ell} = \exp(w^\varepsilon_{i,\ell} / \pi_i)$, and the fluxes $\mathcal{F}^\varepsilon_{i,\ell \pm 1/2}$ are defined as in (3.3) with $u^k_{i,\ell}$ replaced by $u^\varepsilon_{i,\ell}$. We multiply the previous equation by $\Delta t w^\varepsilon_{i,\ell}$, sum over $\ell \in G$ and $i = 1, \ldots, n$ and use discrete integration by parts as in (3.15):

$$\varepsilon \Delta t \sum_{i=1}^{n} \|w_i^\varepsilon\|^2_{1,2,\mathcal{T}} = -\rho \sum_{i=1}^{n} \sum_{\ell \in G} \big(\Delta x (u^\varepsilon_{i,\ell} - u^{k-1}_{i,\ell}) w^\varepsilon_{i,\ell} + \Delta t (\mathcal{F}^\varepsilon_{i,\ell+1/2} - \mathcal{F}^\varepsilon_{i,\ell-1/2}) w^\varepsilon_{i,\ell} \big). \tag{3.17}$$

For the first term on the right-hand side, we use $w_{i,\ell}^\varepsilon = \pi_i \log u_{i,\ell}^\varepsilon$ and the convexity of the function $h(s) = s(\log s - 1)$:

$$(u_{i,\ell}^\varepsilon - u_{i,\ell}^{k-1})\pi_i \log u_{i,\ell}^\varepsilon \geq \pi_i\big(h(u_{i,\ell}^\varepsilon) - h(u_{i,\ell}^{k-1})\big).$$

Recalling definition (3.9) of $\mathcal{H}_B$, this shows that

$$-\rho\sum_{i=1}^n\sum_{\ell\in G}\Delta x(u_{i,\ell}^\varepsilon - u_{i,\ell}^{k-1})w_{i,\ell}^\varepsilon \leq -\rho\big(\mathcal{H}_B(u^\varepsilon) - \mathcal{H}_B(u^{k-1})\big).$$

Like in Section 3.2.2, we split the second term in (3.17) into two parts:

$$-\rho\Delta t\sum_{i=1}^n\sum_{\ell\in G}(\mathcal{F}_{i,\ell+1/2}^\varepsilon - \mathcal{F}_{i,\ell-1/2}^\varepsilon)w_{i,\ell}^\varepsilon = I_3 + I_4, \quad \text{where} \tag{3.18}$$

$$I_3 = \rho\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\bigg(\frac{u_{i,\ell+1}^\varepsilon - u_{i,\ell}^\varepsilon}{\Delta x} - \frac{u_{i,\ell}^\varepsilon - u_{i,\ell-1}^\varepsilon}{\Delta x}\bigg)w_{i,\ell}^\varepsilon,$$

$$I_4 = \rho\Delta t\sum_{i=1}^n\sum_{\ell\in G}\bigg(u_{i,\ell+1/2}^\varepsilon\frac{p_{i,\ell+1}^\varepsilon - p_{i,\ell}^\varepsilon}{\Delta x} - u_{i,\ell-1/2}^\varepsilon\frac{p_{i,\ell}^\varepsilon - p_{i,\ell-1}^\varepsilon}{\Delta x}\bigg)w_{i,\ell}^\varepsilon.$$

We use discrete integration by parts, the definition $w_{i,\ell}^\varepsilon = \pi_i \log u_{i,\ell}^\varepsilon$, and the elementary inequality $(a-b)(\log a - \log b) \geq 4(\sqrt{a} - \sqrt{b})^2$ for a, $b > 0$ to estimate the first term:

$$\begin{aligned}
I_3 &= -\rho\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\frac{u_{i,\ell+1}^\varepsilon - u_{i,\ell}^\varepsilon}{\Delta x}(w_{i,\ell+1}^\varepsilon - w_{i,\ell}^\varepsilon) \\
&\leq -4\rho\sigma\Delta t\sum_{i=1}^n\sum_{\ell\in G}\frac{\pi_i}{\Delta x}\big((u_{i,\ell+1}^\varepsilon)^{1/2} - (u_{i,\ell}^\varepsilon)^{1/2}\big)^2 = -4\rho\sigma\Delta t\sum_{i=1}^n\pi_i|(u_i^\varepsilon)^{1/2}|_{1,2,\mathcal{T}}^2.
\end{aligned}$$

For the second term I_4, we use discrete integration by parts and $w_{i,\ell}^\varepsilon = \pi_i \log u_{i,\ell}^\varepsilon$ again as well as property (3.5) (discrete chain rule):

$$\begin{aligned}
I_4 &= -\rho\frac{\Delta t}{\Delta x}\sum_{i=1}^n\sum_{\ell\in G}\pi_i u_{i,\ell+1/2}^\varepsilon(p_{i,\ell+1}^\varepsilon - p_{i,\ell}^\varepsilon)(\log u_{i,\ell+1}^\varepsilon - \log u_{i,\ell}^\varepsilon) \\
&\leq -\rho c_0\frac{\Delta t}{\Delta x}\sum_{i=1}^n\sum_{\ell\in G}\pi_i(p_{i,\ell+1}^\varepsilon - p_{i,\ell}^\varepsilon)(u_{i,\ell+1}^\varepsilon - u_{i,\ell}^\varepsilon).
\end{aligned}$$

Then, inserting definition (1.13) of $p_{i,\ell}^\varepsilon$ and using the discrete analog (3.8) of the derivation of a convolution, $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$,

$$I_4 \leq -\rho c_0\frac{\Delta t}{\Delta x}(I_{41} + I_{42}), \quad \text{where}$$

$$I_{41} = \sum_{i=1}^n\sum_{\ell\in G}\pi_i a_{ii}(u_{i,\ell+1}^\varepsilon - u_{i,\ell}^\varepsilon)^2,$$

$$I_{42} = \sum_{\substack{i,j=1\\ j\neq i}}^{n} \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ij} B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell}).$$

We insert $(n-1)^{-1}\sum_{j\neq i} 1 = 1$ and $\sum_{\ell'\in G}\Delta x = 1$ (note that $\mathrm{m}(\mathbb{T}) = 1$) in I_{41} and split the resulting sum into two parts:

$$I_{41} = \frac{1}{n-1}\sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ii}(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell})^2 + \frac{1}{n-1}\sum_{\substack{i,j=1\\ i>j}}^{n} \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ii}(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell})^2.$$

We exchange i and j as well as ℓ and ℓ' in the second term, which leads to

$$I_{41} = \frac{1}{n-1}\sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x\big[\pi_i a_{ii}(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell})^2 + \pi_j a_{jj}(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'})^2\big].$$

Similarly, we distinguish between $i<j$ and $i>j$ in I_{42} and exchange i and j as well as ℓ and ℓ' in the sum over $i>j$, leading to

$$\begin{aligned} I_{42} &= \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x \pi_i a_{ij} B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell}) \\ &\quad + \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x \pi_j a_{ji} B^{ji}_{\ell'-\ell}(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell})(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'}). \end{aligned}$$

By Remark 19, we have $B^{ji}_{\ell'-\ell} = B^{ij}_{\ell-\ell'}$. Therefore,

$$I_{42} = \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} \Delta x (\pi_i a_{ij} + \pi_j a_{ji}) B^{ij}_{\ell-\ell'}(u^\varepsilon_{j,\ell'+1} - u^\varepsilon_{j,\ell'})(u^\varepsilon_{i,\ell+1} - u^\varepsilon_{i,\ell}).$$

The sum of I_{41} and I_{42} can be written as a quadratic form in $\mathrm{D}_\ell u^\varepsilon_i$ and $\mathrm{D}_{\ell'} u^\varepsilon_j$ with the matrix $M^{ij}_{\ell-\ell'}$, defined in (3.10). This shows that

$$I_4 \le -\frac{\rho c_0 \Delta t}{(n-1)} \sum_{\substack{i,j=1\\ i<j}}^{n} \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix}^\top M^{ij}_{\ell-\ell'} \begin{pmatrix} \mathrm{D}_\ell u^\varepsilon_i \\ \mathrm{D}_{\ell'} u^\varepsilon_j \end{pmatrix} \le 0.$$

Collecting the estimates for I_3 and I_4 in (3.18), we deduce from (3.17) the following regularized discrete entropy inequality:

$$\rho \mathcal{H}_B(u^\varepsilon) + \varepsilon \Delta t \sum_{i=1}^{n} \|w^\varepsilon_i\|^2_{1,2,\mathcal{T}} + 4\rho\sigma\Delta t \sum_{i=1}^{n} \pi_i |(u^\varepsilon_i)^{1/2}|^2_{1,2,\mathcal{T}} \tag{3.19}$$

$$+\frac{\rho c_0\Delta t}{(n-1)}\sum_{\substack{i,j=1\\ i<j}}^{n}\sum_{\ell,\ell'\in G}(\Delta x)^2\begin{pmatrix}\mathrm{D}_\ell u_i^\varepsilon\\ \mathrm{D}_{\ell'}u_j^\varepsilon\end{pmatrix}^\top M_{\ell-\ell'}^{ij}\begin{pmatrix}\mathrm{D}_\ell u_i^\varepsilon\\ \mathrm{D}_{\ell'}u_j^\varepsilon\end{pmatrix}\le\rho\mathcal{H}_B(u^{k-1}).$$

We proceed with the topological degree argument. We set $R=1+(\mathcal{H}_B(u^{k-1})/(\varepsilon\Delta t))^{1/2}$. Then (3.19) implies that

$$\varepsilon\Delta t\sum_{i=1}^{n}\|w_i^\varepsilon\|_{1,2,\mathcal{T}}^2\le\rho\mathcal{H}_B(u^{k-1})\le\mathcal{H}_B(u^{k-1})=\varepsilon\Delta t(R-1)^2<\varepsilon\Delta tR^2,$$

and hence $w^\varepsilon\notin\partial Z_R$. We infer that $\deg(I-F,Z_R,0)=1$ and consequently, F admits a fixed point. Note that we did not use the estimate for u_i^ε in the seminorm $|\cdot|_{1,2,\mathcal{T}}$ at this point, such that $\sigma=0$ is admissible here (and also in the following two subsections).

3.2.4 Limit $\varepsilon\to 0$

There exists a constant $C>0$ such that $C(s-1)\le h(s)$ for all $s\ge 0$. Hence,

$$C\pi_i\Delta x(u_{i,\ell}^\varepsilon-1)\le\pi_i\Delta xh(u_{i,\ell}^\varepsilon)\le\mathcal{H}_B(u^\varepsilon)\le\mathcal{H}_B(u^{k-1}),$$

for all $\ell\in G$, $i=1,\ldots,n$. Thus, $(u_{i,\ell}^\varepsilon)$ is bounded in ε and the Bolzano–Weierstraß theorem implies the existence of a subsequence (not relabeled) such that $u_{i,\ell}^\varepsilon\to u_{i,\ell}^k\ge 0$ as $\varepsilon\to 0$. It follows from (3.19) that $\varepsilon w_{i,\ell}^\varepsilon\to 0$. Thus, the limit $\varepsilon\to 0$ in (3.16) shows that u^k is a solution to the numerical scheme (3.2)–(3.4). Moreover, the limit $\varepsilon\to 0$ in (3.19) leads to the discrete entropy inequality (3.11).

3.2.5 Discrete Rao entropy inequality

We prove inequality (3.12). To this end, we multiply (3.2) by $\Delta t\pi_ip_{i,\ell}^k$ and sum over $\ell\in G$, $i=1,\ldots,n$:

$$\sum_{i=1}^{n}\sum_{\ell\in G}\Delta x\pi_i(u_{i,\ell}^k-u_{i,\ell}^{k-1})p_{i,\ell}^k+\sum_{i=1}^{n}\sum_{\ell\in G}\Delta t\pi_i(\mathcal{F}_{i,\ell+1/2}^k-\mathcal{F}_{i,\ell-1/2}^k)p_{i,\ell}^k=0. \tag{3.20}$$

For the first term in (3.20), we use the definition of $p_{i,\ell}^k$:

$$\sum_{i=1}^{n}\sum_{\ell\in G}\Delta x\pi_i(u_{i,\ell}^k-u_{i,\ell}^{k-1})p_{i,\ell}^k=I_5+I_6,\quad\text{where}$$

$$I_5=\sum_{i=1}^{n}\sum_{\ell\in G}\Delta x\pi_ia_{ii}(u_{i,\ell}^k-u_{i,\ell}^{k-1})u_{i,\ell}^k,$$

$$I_6=\sum_{\substack{i,j=1\\ j\ne i}}^{n}\sum_{\ell,\ell'\in G}(\Delta x)^2\pi_ia_{ij}B_{\ell-\ell'}^{ij}(u_{i,\ell}^k-u_{i,\ell}^{k-1})u_{j,\ell'}^k.$$

We rewrite I_5 and I_6 according to

$$I_5 = \frac{1}{2}\sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i a_{ii}\big((u_{i,\ell}^k)^2 - (u_{i,\ell}^{k-1})^2\big) + \frac{1}{2}\sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i a_{ii}\big(u_{i,\ell}^k - u_{i,\ell}^{k-1}\big)^2,$$

$$I_6 = \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \pi_i a_{ij} B_{\ell-\ell'}^{ij}(u_{i,\ell}^k u_{j,\ell'}^k - u_{i,\ell}^{k-1} u_{j,\ell'}^{k-1})$$
$$+ \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \pi_i a_{ij} B_{\ell-\ell'}^{ij}(u_{i,\ell}^k - u_{i,\ell}^{k-1})(u_{j,\ell'}^k - u_{j,\ell'}^{k-1}).$$

Combining the second terms in I_5 and I_6, using similar computations as for I_4 in Section 3.2.3, and applying Hypothesis (H3) shows that the second term of $I_5 + I_6$ is nonnegative leading to

$$I_5 + I_6 \geq \frac{1}{2}\sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i a_{ii}\big((u_{i,\ell}^k)^2 - (u_{i,\ell}^{k-1})^2\big)$$
$$+ \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \pi_i a_{ij} B_{\ell-\ell'}^{ij}(u_{i,\ell}^k u_{j,\ell'}^k - u_{i,\ell}^{k-1} u_{j,\ell'}^{k-1}).$$

Then it holds that

$$\sum_{i=1}^n \sum_{\ell\in G} \Delta x \pi_i (u_{i,\ell}^k - u_{i,\ell}^{k-1}) p_{i,\ell}^k \geq \mathcal{H}_R(u^k) - \mathcal{H}_R(u^{k-1}).$$

Now, we split the second term in (3.20) again into two parts:

$$\sum_{i=1}^n \sum_{\ell\in G} \Delta t \pi_i (\mathcal{F}_{i,\ell+1/2}^k - \mathcal{F}_{i,\ell-1/2}^k) p_{i,\ell}^k = I_7 + I_8, \quad \text{where}$$

$$I_7 = -\sigma \Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i \left(\frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x} - \frac{u_{i,\ell}^k - u_{i,\ell-1}^k}{\Delta x} \right) p_{i,\ell}^k,$$

$$I_8 = -\Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i \left(u_{i,\ell+1/2}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x} - u_{i,\ell-1/2}^k \frac{p_{i,\ell}^k - p_{i,\ell-1}^k}{\Delta x} \right) p_{i,\ell}^k.$$

We reformulate I_7 by using discrete integration by parts:

$$I_7 = \sigma \Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x} (p_{i,\ell+1}^k - p_{i,\ell}^k).$$

Then, with similar computations as for I_4 in Section 3.2.3, we obtain

$$I_7 = \frac{\sigma \Delta t}{(n-1)} \sum_{\substack{i,j=1\\ i<j}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \geq 0.$$

Finally, the term I_8 can be rewritten as

$$I_8 = \Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i u_{i,\ell+1/2}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x}(p_{i,\ell+1}^k - p_{i,\ell}^k) = \Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i \Delta x \big|(u_{i,\ell+1/2}^k)^{1/2} \mathrm{D}_\ell p_i^k\big|^2.$$

Hence, we infer from (3.20) that

$$\begin{aligned}\mathcal{H}_R(u^k) &+ \Delta t \sum_{i=1}^n \sum_{\ell\in G} \pi_i \Delta x \big|(u_{i,\ell+1/2}^k)^{1/2} \mathrm{D}_\ell p_i^k\big|^2 \\ &+ \frac{\sigma \Delta t}{(n-1)} \sum_{\substack{i,j=1\\ i<j}}^n \sum_{\ell,\ell'\in G} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \le \mathcal{H}_R(u^{k-1}),\end{aligned}$$

which proves (3.12).

Finally, conservation of mass follows from summing (3.2) over $\ell \in G$ and observing that the sum over the numerical fluxes vanishes. This ends the proof of Theorem 22.

3.3 Proof of Theorem 23

To prove the convergence of the scheme, we first derive some uniform estimates and then apply a discrete Aubin–Lions compactness lemma.

3.3.1 Uniform estimates

Let $(u_m)_{m\in\mathbb{N}}$ be a sequence of finite-volume solutions to (3.2)–(3.4) associated to the mesh $\mathcal{D}_m$ and constructed in Theorem 22. The conservation of mass and the discrete entropy inequalities (3.11) and (3.12) show that, after summing over $k = 1, \ldots, N_T^m$,

$$\max_{k=1,\ldots,N_T^m} \|u_i^k\|_{0,2,\mathcal{T}_m}^2 + \sum_{k=1}^{N_T^m} \Delta t_m \|(u_i^k)^{1/2}\|_{1,2,\mathcal{T}_m}^2 \le C, \quad i = 1, \ldots, n, \tag{3.21}$$

where $C > 0$ denotes here and in the following a constant that is independent of the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$, but possibly depending on u^0 and T. Because of the positive definiteness of $M_{\ell-\ell'}^{ij}$, we conclude a bound for u_i^k in the norm $\|\cdot\|_{1,2,\mathcal{T}_m}$.

Lemma 24. *Let the assumptions of Theorem 23 hold. Then there exists $C > 0$ independent of η_m (but depending on the positive definiteness constant c_M) such that for all $m \in \mathbb{N}$ and all $i = 1, \ldots, n$,*

$$\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|_{1,2,\mathcal{T}_m}^2 \le C. \tag{3.22}$$

Proof. We infer from (3.11) that

$$\frac{c_0}{n-1} \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\substack{i,j=1\\ i<j}}^n \sum_{\ell,\ell'\in G_m} (\Delta x)^2 \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix}^\top M_{\ell-\ell'}^{ij} \begin{pmatrix} \mathrm{D}_\ell u_i^k \\ \mathrm{D}_{\ell'} u_j^k \end{pmatrix} \le \mathcal{H}_B(u^0).$$

Since $M^{ij}_{\ell-\ell'}$ is uniformly positive definite with constant $c_M > 0$,

$$\begin{aligned}
&\frac{c_0}{n-1}\sum_{\substack{i,j=1\\i<j}}^{n}\sum_{\ell,\ell'\in G_m}(\Delta x)^2\begin{pmatrix}\mathrm{D}_\ell u_i^k\\ \mathrm{D}_{\ell'}u_j^k\end{pmatrix}^\top M^{ij}_{\ell-\ell'}\begin{pmatrix}\mathrm{D}_\ell u_i^k\\ \mathrm{D}_{\ell'}u_j^k\end{pmatrix}\\
&\quad\geq \frac{c_M c_0}{n-1}\sum_{\substack{i,j=1\\i<j}}^{n}\sum_{\ell,\ell'\in G_m}(\Delta x)^2\big(|\mathrm{D}_\ell u_i^k|^2+|\mathrm{D}_{\ell'}u_j^k|^2\big)\\
&\quad= c_M c_0\sum_{i=1}^{n}\sum_{\ell\in G_m}\Delta x|\mathrm{D}_\ell u_i^k|^2 + c_M c_0\sum_{j=1}^{n}\sum_{\ell'\in G_m}\Delta x|\mathrm{D}_{\ell'}u_j^k|^2\\
&\quad= 2c_M c_0\sum_{i=1}^{n}\sum_{\ell\in G_m}\Delta x|\mathrm{D}_\ell u_i^k|^2.
\end{aligned}$$

Together with the first bound in (3.21), this finishes the proof. □

Lemma 25. *Let the assumptions of Theorem 23 hold. Then there exists a constant $C > 0$ independent of η_m (but depending on σ) such that for all $m \in \mathbb{N}$, $i = 1, \ldots, n$,*

$$\sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{1,1,\mathcal{T}_m} + \sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{0,\infty,\mathcal{T}_m} \leq C.$$

Moreover, there exists another constant, still denoted by $C > 0$ and independent of η_m, such that

$$\sum_{k=1}^{N_T^m}\Delta t_m|p_i^k|^2_{1,2,\mathcal{T}_m} \leq C. \tag{3.23}$$

Proof. As $\mathrm{m}(\mathbb{T}) = 1$, thanks to the Cauchy–Schwarz inequality,

$$|u_i^k|_{1,1,\mathcal{T}_m} = \sum_{\ell\in G_m}|u_{i,\ell+1}^k - u_{i,\ell}^k| \leq |u_i^k|_{1,2,\mathcal{T}_m}.$$

Using (3.22), this shows that

$$\begin{aligned}
\sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{1,1,\mathcal{T}_m} &\leq 2\sum_{k=1}^{N_T^m}\Delta t_m\big(\|u_i^k\|^2_{0,1,\mathcal{T}_m} + |u_i^k|^2_{1,1,\mathcal{T}_m}\big)\\
&\leq 2T\max_{k=1,\ldots,N_T^m}\|u_i^k\|^2_{0,1,\mathcal{T}_m} + 2\sum_{k=1}^{N_T^m}\Delta t_m|u_i^k|^2_{1,2,\mathcal{T}_m} \leq C(u^0,T).
\end{aligned}$$

To show the discrete $L^\infty(\mathbb{T})$-bound, we apply the continuity of the embedding $\mathrm{BV}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ (in one space dimension). We conclude that, for $i = 1, \ldots, n$,

$$\sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{0,\infty,\mathcal{T}_m} \leq C\sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{BV(\mathbb{T})} = C\sum_{k=1}^{N_T^m}\Delta t_m\|u_i^k\|^2_{1,1,\mathcal{T}_m} \leq C(u^0,T).$$

For the last part, we estimate as follows:

$$\begin{aligned}|p_i^k|_{1,2,\mathcal{T}_m}^2 &= \sum_{\ell\in G_m} \Delta x_m \bigg|\frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m}\bigg|^2 \\ &\le C a_{ii}^2 |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \bigg| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} \frac{B_{\ell+1-\ell'}^{ij} - B_{\ell-\ell'}^{ij}}{\Delta x_m} u_{j,\ell'}^k \bigg|^2 \\ &\le C |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \bigg| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \frac{u_{j,\ell'+1}^k - u_{j,\ell'}^k}{\Delta x_m} \bigg|^2 \\ &\le C |u_i^k|_{1,2,\mathcal{T}_m}^2 + C \sum_{\ell\in G_m} \Delta x_m \bigg| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \bigg|^2.\end{aligned}$$

Then we deduce from the elementary inequality $(\sum_{j=1,\,j\neq i}^n a_j)^2 \le (n-1)\sum_{j=1,\,j\neq i}^n a_j^2$ for $a_j\in\mathbb{R}$ and the discrete Young convolution inequality in Lemma 33 that

$$\begin{aligned}&\sum_{\ell\in G_m} \Delta x_m \bigg| \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \bigg|^2 \\ &\le (n-1) \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell\in G_m} \Delta x_m \bigg(\sum_{\ell'\in G_m} \Delta x_m a_{ij} B_{\ell-\ell'}^{ij} \mathrm{D}_{\ell'} u_j^k \bigg)^2 \le C \sum_{\substack{j=1\\ j\neq i}}^n \|B^{ij}\|_{L^2(\mathbb{T})}^2 |u_j^k|_{1,1,\mathcal{T}_m}^2.\end{aligned}$$

Summing over k, we infer that

$$\sum_{k=1}^{N_T^m} \Delta t_m |p_i^k|_{1,2,\mathcal{T}_m}^2 \le C \bigg\{ \sum_{i=1}^n \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 + \sum_{\substack{j=1\\ j\neq i}}^n \bigg(\|B^{ij}\|_{L^2(\mathbb{T})}^2 \sum_{k=1}^{N_T^m} \Delta t_m |u_j^k|_{1,1,\mathcal{T}_m}^2 \bigg) \bigg\} \le C,$$

where we used Lemma 25 for the last inequality. It is at this point, where we need the discrete $L^2(0,T;H^1(\mathbb{T}))$-bound of $(u_{m,i})$. This ends the proof. □

Next, we show a uniform bound for the discrete time derivative.

Lemma 26. *Let the assumptions of Theorem 23 hold. Then there exists $C>0$ independent of η_m such that for all $m\in\mathbb{N}$, $i=1,\ldots,n$,*

$$\sum_{k=1}^{N_T^m} \Delta t_m \bigg\| \frac{u_i^k - u_i^{k-1}}{\Delta t_m} \bigg\|_{-1,2,\mathcal{T}_m}^{4/3} \le C.$$

Proof. Let $\phi = (\phi_\ell)_{\ell\in G_m} \in \mathcal{V}_{\mathcal{T}_m}$ be such that $\|\phi\|_{1,2,\mathcal{T}_m} = 1$. We multiply (3.2) by ϕ_ℓ, sum over $\ell\in G_m$, and use discrete integration by parts:

$$\sum_{\ell\in G_m} \Delta x_m \frac{u_{i,\ell}^k - u_{i,\ell}^{k-1}}{\Delta t_m} \phi_\ell = \sigma \sum_{\ell\in G_m} \bigg(\frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} - \frac{u_{i,\ell}^k - u_{i,\ell-1}^k}{\Delta x_m} \bigg) \phi_\ell \tag{3.24}$$

$$+\sum_{\ell\in G_m}\left(u^k_{i,\ell+1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}-u^k_{i,\ell-1/2}\frac{p^k_{i,\ell}-p^k_{i,\ell-1}}{\Delta x_m}\right)\phi_\ell$$
$$=-\sigma\sum_{\ell\in G_m}\Delta x_m\frac{u^k_{i,\ell+1}-u^k_{i,\ell}}{\Delta x_m}\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}-\sum_{\ell\in G_m}\Delta x_m u^k_{i,\ell+1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}$$
$$=: I_9+I_{10}.$$

By the Cauchy–Schwarz inequality,

$$|I_9|\le\sigma\sum_{\ell\in G_m}\Delta x_m\big((u^k_{i,\ell+1})^{1/2}+(u^k_{i,\ell})^{1/2}\big)\left|\frac{(u^k_{i,\ell+1})^{1/2}-(u^k_{i,\ell})^{1/2}}{\Delta x_m}\right|\left|\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}\right|$$
$$\le 2\sigma\|(u^k_i)^{1/2}\|_{0,\infty,\mathcal{T}_m}|(u^k_i)^{1/2}|_{1,2,\mathcal{T}_m}|\phi|_{1,2,\mathcal{T}_m}.$$

Furthermore, using $(u^k_{i,\ell+1/2})^{1/2}\le\max\{(u^k_{i,\ell})^{1/2},(u^k_{i,\ell+1})^{1/2}\}\le\|(u^k_i)^{1/2}\|_{0,\infty,\mathcal{T}_m}$,

$$|I_{10}|\le\sum_{\ell\in G_m}\Delta x_m\big|(u^k_{i,\ell+1/2})^{1/2}\big|\left|(u^k_{i,\ell+1/2})^{1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}\right|\left|\frac{\phi_{\ell+1}-\phi_\ell}{\Delta x_m}\right|$$
$$\le\|(u^k_i)^{1/2}\|_{0,\infty,\mathcal{T}_m}\left(\sum_{\ell\in G_m}\Delta x_m\left|(u^k_{i,\ell+1/2})^{1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}\right|^2\right)^{1/2}|\phi|_{1,2,\mathcal{T}_m}.$$

Applying the elementary inequality $(a+b)^r\le C(a^r+b^r)$ for all $a,b\ge 0$ and $r>1$, inserting the previous estimates into (3.24), and using Hölder's inequality, we find that

$$\sum_{k=1}^{N_T^m}\Delta t_m\left\|\frac{u^k_i-u^{k-1}_i}{\Delta t_m}\right\|^{4/3}_{-1,2,\mathcal{T}_m}=\sum_{k=1}^{N_T^m}\Delta t_m\sup_{\|\phi\|_{1,2,\mathcal{T}_m}=1}\left|\sum_{\ell\in G_m}\Delta x_m\frac{u^k_{i,\ell}-u^{k-1}_{i,\ell}}{\Delta t_m}\phi_\ell\right|^{4/3}$$
$$\le C\sum_{k=1}^{N_T^m}\Delta t_m\|(u^k_i)^{1/2}\|^{4/3}_{0,\infty,\mathcal{T}_m}|(u^k_i)^{1/2}|^{4/3}_{1,2,\mathcal{T}_m}$$
$$+C\sum_{k=1}^{N_T^m}\Delta t_m\|(u^k_i)^{1/2}\|^{4/3}_{0,\infty,\mathcal{T}_m}\left(\sum_{\ell\in G_m}\Delta x_m\left|(u^k_{i,\ell+1/2})^{1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}\right|^2\right)^{2/3}$$
$$\le C\left(\sum_{k=1}^{N_T^m}\Delta t_m\|(u^k_i)^{1/2}\|^4_{0,\infty,\mathcal{T}_m}\right)^{1/3}\left(\sum_{k=1}^{N_T^m}\Delta t_m|(u^k_i)^{1/2}|^2_{1,2,\mathcal{T}_m}\right)^{2/3}$$
$$+C\left(\sum_{k=1}^{N_T^m}\Delta t_m\|(u^k_i)^{1/2}\|^4_{0,\infty,\mathcal{T}_m}\right)^{1/3}\left(\sum_{k=1}^{N_T^m}\Delta t_m\sum_{\ell\in G_m}\Delta x_m\left|(u^k_{i,\ell+1/2})^{1/2}\frac{p^k_{i,\ell+1}-p^k_{i,\ell}}{\Delta x_m}\right|^2\right)^{2/3}$$
$$\le C(u^0,T),$$

and the last bound follows from Lemma 25 and the discrete Rao entropy inequality (3.12). □

3.3.2 Compactness

We claim that the estimates from Lemmas 25 and 26 are sufficient to conclude the relative compactness of $(u_m)_{m\in\mathbb{N}}$. In fact, the result follows from the discrete Aubin–Lions lemma in the version of [57, Theorem 3.4] if the following two properties are satisfied:

- For any $(v_m)_{m\in\mathbb{N}} \subset \mathcal{V}_{\mathcal{T}_m}$ such that $\sup_{m\in\mathbb{N}} \|v_m\|_{1,2,\mathcal{T}_m} \le C$ for some $C > 0$, there exists a function $v \in L^2(\mathbb{T})$ satisfying, up to a subsequence, $v_m \to v$ strongly in $L^2(\mathbb{T})$. This property follows from [51, Theorem 14.1].
- If $v_m \to v$ strongly in $L^2(\mathbb{T})$ and $\|v_m\|_{-1,2,\mathcal{T}_m} \to 0$ as $m \to \infty$, then $v = 0$. This property can be replaced by the condition that $\|\cdot\|_{1,2,\mathcal{T}_m}$ and $\|\cdot\|_{-1,2,\mathcal{T}_m}$ are dual norms with respect to the $L^2(\mathbb{T})$-norm, which is the case [57, Remark 6]. A more detailed proof can be found in [79, Prop. 10].

Hence, it follows from [57, Theorem 3.4] that there exists a subsequence, which is not relabeled, such that

$$u_{m,i} \to u_i \quad \text{strongly in } L^1(0,T;L^2(\mathbb{T})) \text{ as } m \to \infty.$$

Let us now adapt the Gagliardo–Nirenberg inequality to our situation. Let $k = 1,\dots,N_T^m$ be fixed. We first apply Lemma 34 with $s = p = 2$:

$$\|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m} \le C\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^{1/2}\|u_{m,i}^k\|_{0,2,\mathcal{T}}^{1/2}.$$

Then, it follows from the Hölder inequality

$$\|u_{m,i}^k\|_{0,6,\mathcal{T}_m} \le \|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m}^{2/3}\|(u_{m,i}^k)^{1/3}\|_{0,6,\mathcal{T}_m} = \|u_{m,i}^k\|_{0,\infty,\mathcal{T}_m}^{2/3}\|u_{m,i}^k\|_{0,2,\mathcal{T}_m}^{1/3}$$

that

$$\|u_{m,i}^k\|_{0,6,\mathcal{T}_m} \le C\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^{1/3}\,\|u_{m,i}^k\|_{0,2,\mathcal{T}}^{2/3}.$$

Therefore,

$$\sum_{k=1}^{N_T}\Delta t_m\|u_{m,i}^k\|_{0,6,\mathcal{T}_m}^6 \le C\max_{k=1,\dots,N_T}\|u_{m,i}\|_{0,2,\mathcal{T}_m}^4\sum_{k=1}^{N_T}\Delta t_m\,\|u_{m,i}^k\|_{1,2,\mathcal{T}_m}^2.$$

Recalling estimates (3.21) and (3.22), we conclude that the sequence $(u_{m,i})_{m\in\mathbb{N}}$ is uniformly bounded in $L^6(\mathbb{T})$. The convergence dominated theorem implies that, up to a subsequence, for every $p < 6$,

$$u_{m,i} \to u_i \quad \text{strongly in } L^p(Q_T) \text{ as } m \to \infty.$$

Lemma 25 implies that the sequence of discrete derivatives $(\partial_x^m u_{m,i})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Thus, there exists a subsequence (not relabeled) such that $\partial_x^m u_{m,i} \rightharpoonup v_i$ weakly in $L^2(Q_T)$, and the proof of [30, Lemma 4.4] allows us to identify $v_i = \partial_x u_i$.

Lemma 27. *The following convergences hold, up to subsequences, as $m \to \infty$:*

$$p_{m,i} \to p_i(u) \quad \textit{strongly in } L^2(Q_T),$$
$$\partial_x p_{m,i} \rightharpoonup \partial_x p_i(u) \quad \textit{weakly in } L^2(Q_T), \quad i = 1,\dots,n.$$

Proof. We follow the strategy of [65, Corollary 14]. First, we rewrite $p^k_{i,\ell}$ defined in (3.4). By a change of variables, we have

$$\begin{aligned} p^k_{i,\ell} &= a_{ii}u^k_{m,i,\ell} + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} a_{ij}\Big(\int_{K_{\ell-\ell'}} B^{ij}(y)\,dy\Big)u^k_{m,j,\ell'} \\ &= a_{ii}u^k_{m,i,\ell} + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G_m} a_{ij}\int_{K_{\ell'}} B^{ij}(x_\ell - z)u^k_{m,j}(z)dz \\ &= a_{ii}u^k_{m,i}(x_\ell) + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(B^{ij} * u^k_{m,j})(x_\ell). \end{aligned}$$

We introduce the piecewise constant function Q^{ij}_m by setting $Q^{ij}_m := (B^{ij} * u_{m,j})(x_\ell)$ in K_ℓ for $\ell \in G_m$. Then

$$p_i(u) - p_{m,i} = a_{ii}(u_i - u_{m,i}) + \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(B^{ij} * u_j - Q^{ij}_m).$$

Since we know that $u_i - u_{m,i} \to 0$ strongly in $L^2(Q_T)$, it is sufficient to prove the following convergence $B^{ij} * u_j - Q^{ij}_m \to 0$ strongly in $L^2(Q_T)$. For this, we write

$$(B^{ij} * u_j - Q^{ij}_m)(x,t) = B^{ij} * (u_j - u_{m,j})(x,t) + \int_{\mathbb{T}} (B^{ij}(x-y) - B^{ij}(x_\ell - y))u_{m,j}(y,t)\,dy.$$

By Young's convolution inequality, we have

$$\|B^{ij} * (u_j - u_{m,j})\|_{L^2(Q_T)} \le \|B^{ij}\|_{L^1(\mathbb{T})}\|u_j - u_{m,j}\|_{L^2(Q_T)} \to 0.$$

Setting $\xi(x,y) = B^{ij}(x-y) - B^{ij}(x_\ell - y)$ for $x \in K_\ell$ and $y \in \mathbb{T}$, we estimate

$$\begin{aligned} \Big\| \int_{\mathbb{T}} \xi(\cdot,y)u_{m,j}(y,t)\,dy\Big\|^2_{L^2(Q_T)} &\le \int_{\mathbb{T}} \|\xi(x,\cdot)\|^2_{L^2(\mathbb{T})}\,dx\|u_{m,j}\|^2_{L^2(Q_T)} \\ &\le \sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}\|^2_{L^2(\mathbb{T})}\|u_{m,j}\|^2_{L^2(Q_T)}. \end{aligned}$$

Since $(u_{m,j})$ is bounded in $L^2(Q_T)$, it remains to verify that the first factor converges to zero as $\Delta x_m \to 0$. This follows from the density of continuous functions in $L^2(\mathbb{T})$. Indeed, let $\varepsilon > 0$ and B^{ij}_ε be continuous such that $\|B^{ij}_\varepsilon - B^{ij}\|_{L^2(\mathbb{T})} \le \varepsilon$. Then

$$\begin{aligned} \sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}\|_{L^2(\mathbb{T})} &\le \sup_{|z|\le\Delta x_m} \|B^{ij}(z+\cdot) - B^{ij}_\varepsilon(z+\cdot)\|_{L^2(\mathbb{T})} \\ &\quad + \sup_{|z|\le\Delta x_m} \|B^{ij}_\varepsilon(z+\cdot) - B^{ij}_\varepsilon\|_{L^2(\mathbb{T})} + \|B^{ij}_\varepsilon - B^{ij}\|_{L^2(\mathbb{T})} \\ &\le 2\varepsilon + \sup_{|z|\le\Delta x_m} \|B^{ij}_\varepsilon(z+\cdot) - B^{ij}_\varepsilon\|_{L^2(\mathbb{T})}. \end{aligned}$$

The last term is smaller than ε if we choose Δx_m sufficiently small. Consequently, we have shown that $\sup_{|z|\le\Delta x_m}\|B^{ij}(z+\cdot)-B^{ij}\|^2_{L^2(\mathbb{T})}\to 0$ as $m\to\infty$ and $B^{ij}*u_j-Q^{ij}_m\to 0$ strongly in $L^2(Q_T)$. This proves the first part of the lemma.

Thanks to (3.23), we have shown that $(\partial_x^m p_{m,i})_{m\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Hence, up to a subsequence, $\partial_x^m p_{m,i}\rightharpoonup z$ weakly in $L^2(Q_T)$. The first part of the proof shows that $z=\partial_x p_i(u)$, finishing the proof. □

3.3.3 Convergence of the scheme

We show that the limit $u=(u_1,\ldots,u_n)$ of the finite-volume solutions is a weak solution to (1.11)–(1.12). Let $i\in\{1,\ldots,n\}$ be fixed, let $\psi_i\in C_0^\infty(\mathbb{T}\times[0,T))$ be given, and denote the mesh size by $\eta_m=\max\{\Delta x_m,\Delta t_m\}$. We set $\psi_{i,\ell}^k:=\psi_i(x_\ell,t_k)$, multiply (3.2) by $\Delta t_m\psi_{i,\ell}^{k-1}$ and sum over $\ell\in G_m$, $k=1,\ldots,N_T^m$. This yields $F_1^m+F_2^m+F_3^m=0$, where

$$
\begin{aligned}
F_1^m &= \sum_{k=1}^{N_T^m}\sum_{\ell\in G_m}\Delta x_m(u_{i,\ell}^k-u_{i,\ell}^{k-1})\psi_{i,\ell}^{k-1},\\
F_2^m &= -\sigma\sum_{k=1}^{N_T^m}\Delta t_m\sum_{\ell\in G_m}\left(\frac{u_{i,\ell+1}^k-u_{i,\ell}^k}{\Delta x_m}-\frac{u_{i,\ell}^k-u_{i,\ell-1}^k}{\Delta x_m}\right)\psi_{i,\ell}^{k-1},\\
F_3^m &= -\sum_{k=1}^{N_T^m}\Delta t_m\sum_{\ell\in G_m}\left(u_{i,\ell+1/2}^k\frac{p_{i,\ell+1}^k-p_{i,\ell}^k}{\Delta x_m}-u_{i,\ell-1/2}^k\frac{p_{i,\ell}^k-p_{i,\ell-1}^k}{\Delta x_m}\right)\psi_{i,\ell}^{k-1}.
\end{aligned}
$$

Furthermore, we introduce the terms

$$
\begin{aligned}
F_{10}^m &= -\int_0^T\int_{\mathbb{T}}u_{m,i}\partial_t\psi_i\,dx\,dt-\int_{\mathbb{T}}u_{m,i}(x,0)\psi_i(x,0)\,dx,\\
F_{20}^m &= \sigma\int_0^T\int_{\mathbb{T}}\partial_x^m u_{m,i}\partial_x\psi_i\,dx\,dt,\\
F_{30}^m &= \int_0^T\int_{\mathbb{T}}u_{m,i}\partial_x^m p_{m,i}\partial_x\psi_i\,dx\,dt.
\end{aligned}
$$

Lemma 28. *Let the assumptions of Theorem 23 hold. Then it holds that, as $m\to\infty$,*

$$F_{10}^m\to-\int_0^T\int_{\mathbb{T}}u_i\partial_t\psi_i\,dx\,dt-\int_{\mathbb{T}}u_i^0(x)\psi_i(x,0)\,dx, \tag{3.25}$$

$$F_{20}^m\to\sigma\int_0^T\int_{\mathbb{T}}\partial_x u_i\partial_x\psi_i\,dx\,dt, \tag{3.26}$$

$$F_{30}^m\to\int_0^T\int_{\mathbb{T}}u_i\partial_x p_i(u)\partial_x\psi_i\,dx\,dt. \tag{3.27}$$

Proof. The strong convergence of $(u_{m,i})_{m\in\mathbb{N}}$ as well as the weak convergence of $(\partial_x^m u_{m,i})_{m\in\mathbb{N}}$ in $L^2(Q_T)$ together with the fact that $u_{m,i}(x,0)=(\Delta x_m)^{-1}\int_{K_\ell}u_i^0(z)dz$ for $x\in K_\ell$ and $\ell\in G$ immediately show convergences (3.25) and (3.26). It remains to verify (3.27). We know from

Lemma 27 that $\partial_x^m p_{m,i} \rightharpoonup \partial_x p_i(u)$ weakly in $L^2(Q_T)$. Since $u_{m,i} \to u_i$ strongly in $L^2(Q_T)$, this implies that

$$u_{m,i}\partial_x^m p_{m,i} \rightharpoonup u_i \partial_x p_i(u) \quad \text{weakly in } L^1(Q_T).$$

In fact, since $u_{m,i}^{1/2}\partial_x^m p_{m,i}$ is uniformly bounded in $L^2(Q_T)$ and $u_{m,i}^{1/2}$ is uniformly bounded in $L^\infty(0,T;L^4(\mathbb{T}))$, this weak convergence even holds in $L^2(0,T;L^{4/3}(\mathbb{T}))$. This proves (3.27) and ends the proof. □

Lemma 29. *Let the assumptions of Theorem 23 hold. Then it holds that, as $m \to \infty$,*

$$F_{10}^m - F_1^m \to 0, \quad F_{20}^m - F_2^m \to 0, \quad F_{30}^m - F_3^m \to 0.$$

The lemma implies that

$$\begin{aligned} F_{10}^m + F_{20}^m + F_{30}^m &= (F_{10}^m - F_1^m) + (F_{20}^m - F_2^m) + (F_{30}^m - F_3^m) + (F_1^m + F_2^m + F_3^m) \\ &= (F_{10}^m - F_1^m) + (F_{20}^m - F_2^m) + (F_{30}^m - F_3^m) \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Therefore, thanks to Lemma 28, we conclude that $u = (u_1, \ldots, u_n)$ is a weak solution to system (1.11)–(1.12). This finishes the proof of Theorem 23, once Lemma 29 is proved.

Proof of Lemma 29. The limit $F_{10}^m - F_1^m \to 0$ is shown in [30, Theorem 5.2]. For the convergence of $F_{20}^m - F_2^m$, we use discrete integration by parts:

$$\begin{aligned} F_2^m &= \sigma \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell \in G_m} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m}(\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}) \\ &= \sigma \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} \int_{x_\ell}^{x_{\ell+1}} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} \int_{t_{k-1}}^{t_k} \frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{\Delta x_m}\, dx\, dt, \\ F_{20}^m &= \sigma \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} \int_{t_{k-1}}^{t_k} \int_{x_\ell}^{x_{\ell+1}} \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m} \partial_x \psi_i \, dx\, dt. \end{aligned}$$

By the mean-value theorem,

$$\left| \int_{t_{k-1}}^{t_k} \frac{1}{\Delta x_m} \int_{x_\ell}^{x_{\ell+1}} \left(\frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{\Delta x_m} - \partial_x \psi_i \right) dx\, dt \right| \le C \Delta t_m \eta_m.$$

This shows that, as $m \to \infty$,

$$\begin{aligned} |F_2^m - F_{20}^m| &\le \sigma \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} \left| \int_{t_{k-1}}^{t_k} \int_{x_\ell}^{x_{\ell+1}} \left(\frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{\Delta x_m} - \partial_x \psi_i \right) \frac{u_{i,\ell+1}^k - u_{i,\ell}^k}{\Delta x_m}\, dx\, dt \right| \\ &\le C\eta_m \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell \in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k| = C\eta_m \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,1,\mathcal{T}_m} \to 0, \end{aligned}$$

where we used the uniform discrete $L^2(0,T;W^{1,1}(\mathbb{T}))$-bound from Lemma 25.

It remains to prove that $|F_{30}^m - F_3^m| \to 0$. First, using discrete integration by parts, we rewrite F_3^m as well as F_{30}^m as

$$F_3^m = \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} \int_{t_{k-1}}^{t_k} u_{i,\ell+1/2}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} (\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1})\, dt,$$

$$F_{30}^m = \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} \int_{t_{k-1}}^{t_k} \bigg(\int_{x_\ell}^{x_{\ell+1/2}} u_{i,\ell}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} \partial_x \psi_i \, dx + \int_{x_{\ell+1/2}}^{x_{\ell+1}} u_{i,\ell+1}^k \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} \partial_x \psi_i \, dx \bigg).$$

Then we find that

$$\begin{aligned} |F_3^m - F_{30}^m| = \bigg| &\sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} (u_{i,\ell+1/2}^k - u_{i,\ell}^k) \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} \\ &\times \int_{t_{k-1}}^{t_k} \bigg(\frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{2} - \int_{x_\ell}^{x_{\ell+1/2}} \partial_x \psi_i(x)\, dx \bigg) dt \\ &+ \sum_{k=1}^{N_T^m} \sum_{\ell \in G_m} (u_{i,\ell+1/2}^k - u_{i,\ell+1}^k) \frac{p_{i,\ell+1}^k - p_{i,\ell}^k}{\Delta x_m} \\ &\times \int_{t_{k-1}}^{t_k} \bigg(\frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{2} - \int_{x_{\ell+1/2}}^{x_{\ell+1}} \partial_x \psi_i(x)\, dx \bigg) dt \bigg|. \end{aligned}$$

Thanks to the regularity of ψ_i, there exists a constant C independent of η_m such that

$$\bigg| \int_{t_{k-1}}^{t_k} \bigg(\frac{\psi_{i,\ell+1}^{k-1} - \psi_{i,\ell}^{k-1}}{2} - \int_{x_\ell}^{x_{\ell+1/2}} \partial_x \psi_i(x)\, dx \bigg) dt \bigg| \le C \eta_m \Delta t_m.$$

We obtain a similar expression if we integrate $\partial_x \psi_i$ over $(x_{\ell+1/2}, x_{\ell+1})$. Thus, since

$$\begin{aligned} |u_{i,\ell+1/2}^k - u_{i,\ell}^k| &\le |u_{i,\ell+1}^k - u_{i,\ell}^k| \quad \text{and} \\ |u_{i,\ell+1/2}^k - u_{i,\ell+1}^k| &\le |u_{i,\ell}^k - u_{i,\ell+1}^k|, \end{aligned}$$

we have

$$\begin{aligned} |F_3^m - F_{30}^m| &\le 2C\eta_m \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell \in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k| |\mathrm{D}_\ell\, p_i^k| \\ &\le 2C\eta_m \bigg(\sum_{i=1}^n a_{ii} \sum_{k=1}^{N_T^m} \Delta t_m |u_i^k|_{1,2,\mathcal{T}_m}^2 \\ &\quad + \sum_{\substack{j=1 \\ j\ne i}}^n \sum_{k=1}^{N_T^m} \Delta t_m \sum_{\ell,\ell' \in G_m} |u_{i,\ell+1}^k - u_{i,\ell}^k| |a_{ij} (B_{\ell+1-\ell'}^{ij} - B_{\ell-\ell'}^{ij}) u_{j,\ell'}^k| \bigg). \end{aligned}$$

It follows for $j \in \{1,\dots,n\}$ with $j \neq i$, using the discrete analog (3.8) of $\partial_x B^{ij} * u_j = B^{ij} * \partial_x u_j$, that

$$\max_{\ell \in G_m} \bigg(\sum_{\ell' \in G_m} |a_{ij}(B^{ij}_{\ell+1-\ell'} - B^{ij}_{\ell-\ell'})u^k_{j,\ell'}| \bigg) = \max_{\ell \in G_m} \bigg(\sum_{\ell' \in G_m} \Delta x_m |a_{ij}||B^{ij}_{\ell-\ell'}||\mathrm{D}_{\ell'} u^k_j| \bigg)$$
$$\leq |a_{ij}| \|B^{ij}\|_{L^\infty(\mathbb{T})} |u^k_j|_{1,1,\mathcal{T}_m}.$$

At this point, we need the regularity condition $B^{ij} \in L^\infty(\mathbb{T})$ from Hypothesis (H3). Hence, it holds that

$$|F^m_3 - F^m_{30}| \leq 2C\eta_m \bigg(\sum_{i=1}^n \sum_{k=1}^{N^m_T} \Delta t_m |u^k_i|^2_{1,2,\mathcal{T}_m} + \sum_{k=1}^{N^m_T} \Delta t_m |u^k_i|_{1,1,\mathcal{T}_m} \sum_{\substack{j=1 \\ j\neq i}}^n |u^k_j|_{1,1,\mathcal{T}_m} \bigg).$$

It remains to apply the Cauchy–Schwarz inequality to conclude that

$$|F^m_3 - F^m_{30}| \leq 2C\eta_m \bigg\{ \sum_{i=1}^n \sum_{k=1}^{N^m_T} \Delta t_m |u^k_i|^2_{1,2,\mathcal{T}_m}$$
$$+ \sum_{\substack{j=1 \\ j\neq i}}^n \bigg(\sum_{k=1}^{N^m_T} \Delta t_m |u^k_i|^2_{1,1,\mathcal{T}_m} \bigg)^{1/2} \bigg(\sum_{k=1}^{N^m_T} \Delta t_m |u^k_j|^2_{1,1,\mathcal{T}_m} \bigg)^{1/2} \bigg\}.$$

Finally, we infer from Lemma 25 that $|F^m_3 - F^m_{30}| \to 0$ as $m \to \infty$. Here, we need the discrete $L^2(0,T;H^1(\mathbb{T}))$-bound for u_i, which follows if $a_{ii} > 0$. This concludes the proof of Lemma 29. □

Remark 30 (Multidimensional case)**.** Theorems 22 and 23 also hold in the multidimensional situation. The proof of Theorem 22 does not change, but the Sobolev embeddings in the proof of Theorem 23 change because of their dependence on the space dimension. We only sketch the changes. We consider a uniform mesh on $\mathbb{T}^d$ by taking the tensor product of the mesh $\mathcal{T}$ introduced in Section 3.1.1. The cells K_ℓ are then d-dimensional cubes with cell centers $\ell = (\ell_1,\dots,\ell_d)$ and measure $\mathrm{m}(K_\ell) = (\Delta x)^d$. We write $\varsigma = K_\ell|K_{\ell'}$ for the edge (or hyper-face) ς between the neighboring cells K_ℓ and $K_{\ell'}$, and $\mathcal{E}_\ell$ for the set of edges of the cell K_ℓ. Finally, for every $\varsigma = K_\ell|K_{\ell'}$, we define the transmissibility coefficient $\tau_\varsigma := \mathrm{m}(\varsigma)/\mathrm{d}_\varsigma$ with $\mathrm{m}(\varsigma) = (\Delta x)^{d-1}$ and d_ς being the Euclidean distance between the cell centers. The numerical scheme (3.2)–(3.3) changes to

$$\mathrm{m}(K_\ell)\frac{u^k_{i,\ell} - u^{k-1}_{i,\ell}}{\Delta t} + \sum_{\varsigma \in \mathcal{E}_\ell} \mathcal{F}^k_{i,\ell,\varsigma} = 0, \quad i = 1,\dots,n,\ \ell \in G^d, \tag{3.28}$$

$$\mathcal{F}^k_{i,\ell,\varsigma} = -\sigma\,\tau_\varsigma\,\mathrm{D}_{\ell,\varsigma} u^k_i - \tau_\varsigma\, u^k_{i,\varsigma}\,\mathrm{D}_{\ell,\varsigma} p^k_i, \tag{3.29}$$

where we have set $\mathrm{D}_{\ell,\varsigma} v := v_{\ell'} - v_\ell$ for an edge $\varsigma := K_\ell|K_{\ell'}$, the mobilities are defined by $u^k_{i,\varsigma} = \widehat{F}(u^k_{i,\ell}, u^k_{i,\ell'})$ with $\widehat{F}$ as in Section 3.1.2, and the discrete nonlocal operators are given by

$$p_{i,\ell}^k = a_{ii}u_{i,\ell}^k + \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\ell'\in G^d} \mathrm{m}(K_{\ell'})a_{ij}B_{\ell,\ell'}^{ij}u_{j,\ell'}^k, \quad B_{\ell,\ell'}^{ij} = \frac{1}{\mathrm{m}(K_{\ell-\ell'})}\int_{K_{\ell-\ell'}} B^{ij}(y)\,dy. \tag{3.30}$$

Let u_m be a solution to (3.28)–(3.30) associated to some space-time discretization indexed by the mesh size $\eta_m = \max\{\Delta x_m, \Delta t_m\}$ satisfying $\eta_m \to 0$ as $m \to \infty$. The corresponding spatial mesh is denoted by $\mathcal{T}_m^d$ and the number of time steps by N_T^m. The uniform estimates (3.21) and (3.22) also hold for $d \geq 2$, but the regularity obtained in Lemma 25 is slightly weaker. Indeed, the embedding $\mathrm{BV}(\mathbb{T}^d) \hookrightarrow L^{d/(d-1)}(\mathbb{T}^d)$ (with $d/(d-1) = \infty$ if $d = 1$) yields

$$\sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|_{1,1,\mathcal{T}_m^d}^2 + \sum_{k=1}^{N_T^m} \Delta t_m \|u_i^k\|_{0,d/(d-1),\mathcal{T}_m^d}^2 \leq C,$$

see for instance [18, 79] for the definitions of the discrete norms. Then from Hölder's inequality $\|v\|_{0,2d/(2d-1),\mathcal{T}_m^d} \leq \|v^{1/2}\|_{0,2d/(d-1),\mathcal{T}_m^d}\|v^{1/2}\|_{0,2,\mathcal{T}_m^d}$ for $v \in \mathcal{V}_{\mathcal{T}_m^d}$ we get the following bound on the discrete time derivative (replacing the estimate in Lemma 26):

$$\sum_{k=1}^{N_M^T} \Delta t_m \left\| \frac{u_i^k - u_i^{k-1}}{\Delta t_m} \right\|_{-1,2d/(2d-1),\mathcal{T}_m^d}^{4/3} \leq C.$$

Similarly as in the one-dimensional case, we conclude from [57, Theorem 3.4] the existence of a subsequence (which is not relabeled) such that $u_{m,i} \to u_i$ strongly in $L^1(0,T; L^{2d/(2d-1)}(\mathbb{T}^d))$ as $m \to \infty$. We deduce from the discrete Gagliardo–Nirenberg inequality [18, Lemma 3.1]

$$\|u_i^k\|_{0,2d/(d-1),\mathcal{T}_m^d} \leq C\|u_i^k\|_{1,2,\mathcal{T}_m^d}^{1/2}\|u_i^k\|_{0,2,\mathcal{T}_m^d}^{1/2},$$

that the strong convergence $u_{m,i} \to u_i$ holds in $L^p(Q_T)$ for every $p < 2d/(d-1)$ (instead of $p < 6$ in the one-dimensional case) and in particular in $L^2(Q_T)$. Thus, the statement of Lemma 27 holds, and we have $\nabla^m p_{m,i} \rightharpoonup \nabla p_i(u)$ weakly in $L^2(Q_T)$, where ∇^m denotes the discrete gradient. In particular, $u_{m,i}\nabla^m p_{m,i} \rightharpoonup u_i \nabla p_i(u)$ weakly in $L^{4/3}(Q_T)$ as in the one-dimensional case. From this point on, the convergence of the scheme follows the lines of Section 3.3.3. □

3.4 Numerical experiments

In this section, we present several numerical experiments to illustrate the behavior of the scheme. The scheme was implemented in one space dimension using Matlab. In all the subsequent numerical tests, we choose the upwind mobility (3.6). The code is available at https://gitlab.tuwien.ac.at/asc/nonlocal-crossdiff. Our code is an adaptation of the one developed in [65] for the approximation of the nonlocal SKT system. We refer the reader to [65, Section 6.1] for a complete presentation of the different methods used to implement the scheme.

3.4.1 Test case 1. Rate of convergence in space for various L^p-norms, convolution kernels, and initial data

We investigate the rate of convergence in space of the scheme at final time $T = 1$. In all test cases of this section, we consider $n = 2$ species, $\sigma = 10^{-4}$, the coefficient matrix $A = (a_{ij})_{1\leq i,j\leq 2}$ given by

$$A = \begin{pmatrix} 0.1251 & 0.25 \\ 1 & 2 \end{pmatrix},$$

and $\pi_1 = 4$, $\pi_2 = 1$. We consider various initial data and kernels. More precisely, we choose

$$u_1^0(x) = \mathbb{1}_{[1/4,3/4]}(x), \quad u_2^0(x) = \mathbb{1}_{[0,1/4]}(x) + \mathbb{1}_{[3/4,1]}(x), \tag{3.31}$$

$$u_1^0(x) = \cos(2\pi x) + 1, \quad u_2^0(x) = \sin(2\pi x - \pi/2) + 1, \tag{3.32}$$

$$u_1^0(x) = \max(1 - |1 - 2x|, 0), \quad u_2^0(x) = \max(1 - 2|x|, 0) \tag{3.33}$$

and the kernels

$$B^{ij}(z) = \mathbb{1}_{[-0.3,0.3]}(z), \tag{3.34}$$

$$B^{ij}(z) = 2\max(1 - |z|/0.3, 0), \tag{3.35}$$

$$B^{ij}(z) = \exp\left(-|z|^2/2\varepsilon^2\right)/\sqrt{2\pi\varepsilon^2}, \ \varepsilon = 10^{-3}. \tag{3.36}$$

First, we consider a mesh of $N_{init} = 32$ cells and the time step size $\Delta t_{init} = 1/64$. Then, starting from this initial mesh, we refine the mesh in space by doubling the number of cells and halving the time step size, i.e. $N_{new} = 2N_{old}$ and $\Delta t_{new} = \Delta t_{old}/2$. This refinement of the meshes is in agreement with the first-order convergence rate of the Euler discretization in time and the expected first-order convergence rate in space of the scheme, due to the choice of the upwind mobility in the numerical fluxes. As exact solutions to system (1.11)–(1.13) are not explicitly known, we refine the mesh in space and time until $N_{end} = 2048$ and $\Delta t_{end} = 1/4096$, and we consider the solutions of the scheme obtained for N_{end} and Δt_{end} as reference solutions. The error is computed between the reference solutions and the solutions obtained for $N = 1024$ cells and $\Delta t = 1/2048$ at final time $T = 1$. Finally, using linear regression in logarithmic scale, we present in Table 3.1 the experimental order of convergence in the L^1- and L^∞-norms. As expected, we observe a rate of convergence around one. In Table 3.1, the numbers in bold letters denote the number of the test case available in our code (see the file loadTestcase.m).

3.4.2 Test case 2. Rate of convergence of the localization limit in various metrics

In the second test case, following [65], we evaluate numerically the rate of convergence of the localization limit. More precisely, for some sequences of kernels converging towards the Dirac measure δ_0, we compute the rate of convergence in different metrics of the solutions to scheme (3.1)–(3.4) towards its local version, i.e. $B^{ij} = \delta_0$ for all $i, j = 1, \ldots, n$. At the continuous level, one can show, by adapting the approach of [74], that the localization limit holds thanks to a compactness method; see also [43] for the SKT system. However, so far no explicit rate of convergence is available. The goal of this numerical test is to obtain a better insight into this rate of convergence. Besides, it also illustrates Remark 21.

Kernel → / Initial Data ↓	Indicator (3.34)	Triangle (3.35)	Gaussian (3.36)
(3.31)	**Testcase 13** L^1-order: 1.1741 L^1-error: $9.76 \cdot 10^{-4}$ L^∞-order: 1.14 L^∞-error: $1.49 \cdot 10^{-3}$	**Testcase 16** L^1-order: 1.1741 L^1-error: $9.76 \cdot 10^{-4}$ L^∞-order: 1.1331 L^∞-error: $1.68 \cdot 10^{-3}$	**Testcase 19** L^1-order: 1.0109 L^1-error: $3.20 \cdot 10^{-3}$ L^∞-order: 0.98437 L^∞-error: $2.45 \cdot 10^{-2}$
(3.32)	**Testcase 14** L^1-order: 1.0948 L^1-error: $1.81 \cdot 10^{-5}$ L^∞-order: 1.0486 L^∞-error: $4.73 \cdot 10^{-5}$	**Testcase 17** L^1-order: 1.0336 L^1-error: $2.78 \cdot 10^{-5}$ L^∞-order: 1.0092 L^∞-error: $8.57 \cdot 10^{-5}$	**Testcase 20** L^1-order: 0.93381 L^1-error: $2.35 \cdot 10^{-3}$ L^∞-order: 0.91831 L^∞-error: $8.87 \cdot 10^{-3}$
(3.33)	**Testcase 15** L^1-order: 0.97752 L^1-error: $6.39 \cdot 10^{-5}$ L^∞-order: 0.99787 L^∞-error: $1.74 \cdot 10^{-4}$	**Testcase 18** L^1-order: 0.97495 L^1-error: $5.35 \cdot 10^{-5}$ L^∞-order: 0.99741 L^∞-error: $11.48 \cdot 10^{-4}$	**Testcase 21** L^1-order: 0.9611 L^1-error: $9.27 \cdot 10^{-4}$ L^∞-order: 0.9761 L^∞-error: $3.69 \cdot 10^{-3}$

Table 3.1: Orders of convergence in the L^1- and L^∞-norms in space at final time $T = 1$ for different kernels and initial data.

We consider the following parameters (for all 6 test cases of this section): $n = 3$ species, diffusion parameter $\sigma = 10^{-4}$, coefficient matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.125 \\ 0.4 & 1 & 0.2 \\ 0.25 & 0.2 & 1 \end{pmatrix},$$

and $\pi_1 = 4$, $\pi_2 = 2$, $\pi_3 = 2$. We choose the final time $T = 1$, a mesh of $N = 512$ cells, and the time step size $\Delta t = 10^{-3}$. Furthermore, we take the nonsmooth initial data

$$u_1^0(x) = \mathbb{1}_{[3/6,5/6]}(x), \quad u_2^0(x) = \mathbb{1}_{[0,1/6]}(x) + \mathbb{1}_{[5/6,1]}(x), \quad u_3^0(x) = \mathbb{1}_{[1/6,3/6]}(x), \tag{3.37}$$

and the smooth initial data

$$u_1^0(x) = \cos(2\pi x) + 1, \quad u_2^0(x) = \sin(2\pi x) + 1, \tag{3.38}$$
$$u_3^0(x) = (\cos(2\pi x) + \sin(2\pi x) + 2)/2.$$

The kernels are chosen according to

$$B_\alpha^{ij}(z) = \mathbb{1}_{[-\alpha,\alpha]}(z)/2\alpha, \tag{3.39}$$
$$B_\alpha^{ij}(z) = \max(1 - |z|/\alpha, 0)/\alpha, \tag{3.40}$$
$$B_\alpha^{ij}(z) = \exp\left(-|z|^2/2\alpha^2\right)/\sqrt{2\pi\alpha^2}. \tag{3.41}$$

In our experiments, starting from $\alpha_{init} = 2^7\Delta x$, we successively halve α until we arrive at $\alpha = \Delta x$. For each value of α, we compute the solutions to the nonlocal scheme (3.1)–(3.4) at final time. We evaluate the L^1, L^∞, and Wasserstein distance W_1 between the solution to the nonlocal scheme and the solution to the local one (for this, it is enough to set $\alpha = 0$ in our code). Since we work in one space dimension, we can explicitly compute the Wasserstein distance W_1; see [90, Chapter 2]. The rates of convergence are estimated by linear regression (in log scale) and the results are presented in Table 3.2. Surprisingly, we observe a slightly better rate of convergence in the case of nonsmooth initial data. As before, the names in bold letters in Table 3.2 denote the name of the test case available in our code (see also the file loadTestcase.m).

Kernel → / Initial Data ↓	(3.39)		(3.40)		(3.41)	
nonsmooth (3.37)	**Testcase NLTL2**		**Testcase NLTL4**		**Testcase NLTL6**	
	L^1-order:	1.8280	L^1-order:	1.8709	L^1-order:	1.7386
	L^∞-order:	1.8271	L^∞-order:	1.8698	L^∞-order:	1.7379
	W_1-order:	1.8306	W_1-order:	1.8724	W_1-order:	1.7426
smooth (3.38)	**Testcase NLTL3**		**Testcase NLTL5**		**Testcase NLTL7**	
	L^1-order:	1.7430	L^1-order:	1.8240	L^1-order:	1.5991
	L^∞-order:	1.7462	L^∞-order:	1.8261	L^∞-order:	1.6038
	W_1-order:	1.7451	W_1-order:	1.8252	W_1-order:	1.6023

Table 3.2: Rates of convergence of the localization limit in the L^1-, L^∞- and W_1-metric for different initial data and kernels.

3.4.3 Test case 3. Segregation phenomenon

In this numerical experiment, we set $\sigma = 0$. Under the assumptions of $n = 2$ species, $a_{ij} = 1$, and $B^{ij} = \delta_0$ for $i, j = 1, 2$, it has been shown in [17] that if the initial data are segregated (initial data with disjoint supports) then the solutions remain segregated (i.e., they have disjoint supports) for all time. The main goal of this subsection is to illustrate the segregation pattern due to the nonlocal terms, i.e. $B^{ij} \neq \delta_0$. We expect that the solutions to the nonlocal model, given segregated initial data, are completely segregated, and that there exists a small region, i.e. a "gap" between the supports of the species, with a size that is related to the radius of the interaction kernels. Let us notice that in the subsequent test cases, Hypothesis (H3) is never satisfied. However, we did not encounter any numerical issues with our code.

We launched the code for a mesh of 512 cells and the time step size $\Delta t = 10^{-4}$. In the case of $n = 2$ species, we considered the initial data

$$u_1^0(x) = \mathbb{1}_{[0.1,0.4]}(x), \quad u_2^0(x) = \mathbb{1}_{[0.6,0.8]}(x),$$

while for $n = 3$ species, we have taken

$$u_1^0(x) = \mathbb{1}_{[0.5,0.6]}(x), \quad u_2^0(x) = \mathbb{1}_{[0.8,0.9]}(x), \quad u_3^0(x) = \mathbb{1}_{[0.1,0.2]}(x).$$

In both cases, we set $a_{ij} = 1$ for all $i, j = 1, \ldots, n$.

In Figures 3.1 and 3.2, we present the segregation pattern at times $t = 0.02$ and $t = 0.2$ obtained for the local model, $B^{ij} = \delta_0$, and the nonlocal model with

$$B^{ij}(z) = 100 \cdot \mathbb{1}_{[-0.1,0.1]}(z).$$

For small times, the support of the species extends until reaching the support of another species. In the local model, the species slightly mix (due to numerical diffusion), while we observe a “gap” between the supports of the solutions in the nonlocal model. This “gap” is of order 0.1 which is the size of the radius of the kernels B^{ij}. Similar numerical results have been observed in [28, Section 6] but using different kernel functions and two species only.

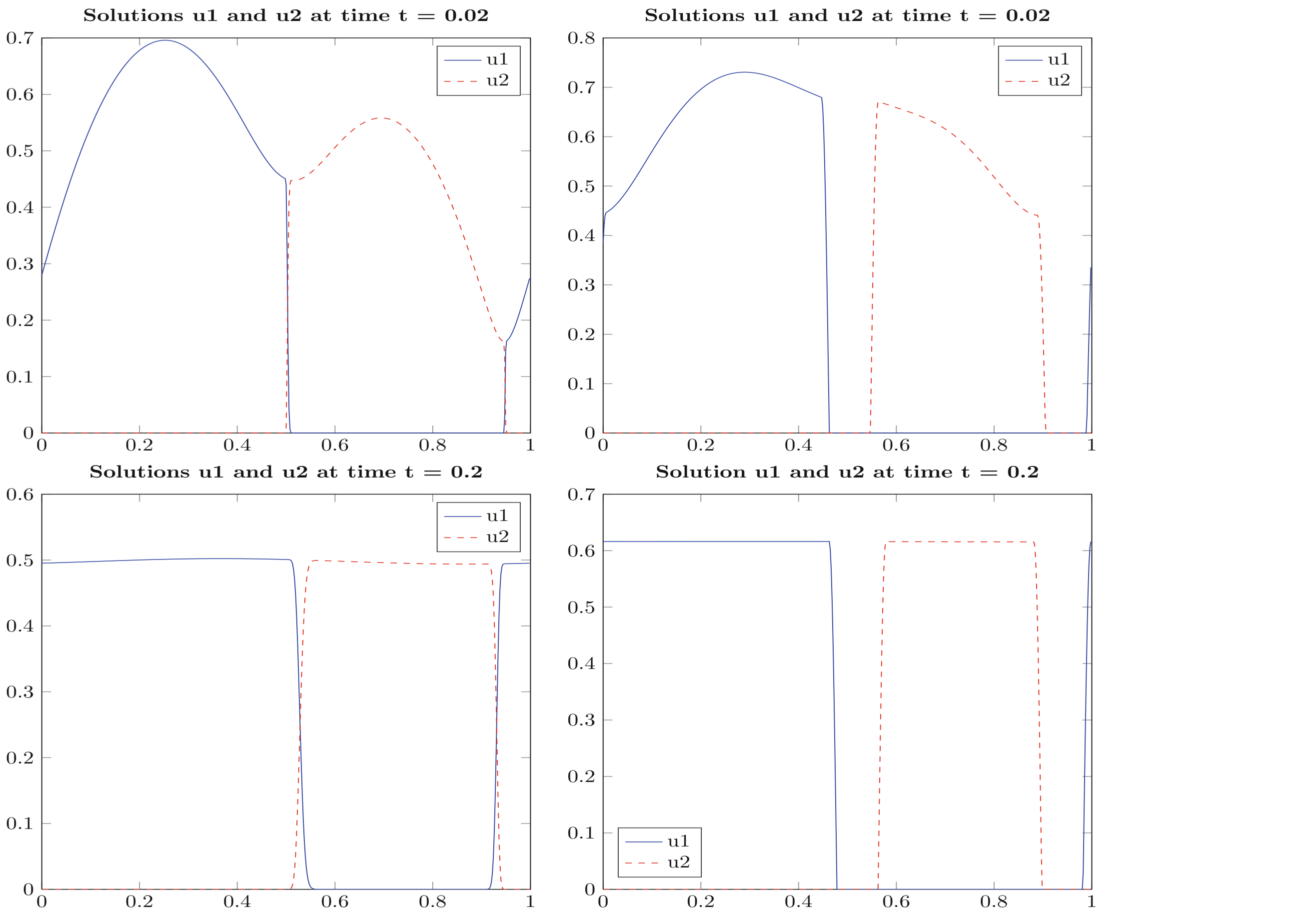

Figure 3.1: Comparison of the segregation pattern for *two* species at times $t = 0.02$ (top) and $t = 0.2$ (bottom) obtained from the local model (left) and nonlocal model (right). The solutions are almost in the steady state at $t = 0.2$.

3.4.4 Test case 4. Dissipation of entropy

In the last numerical experiment, we plot the two entropies $\mathcal{H}_B(u(t))$ and $\mathcal{H}_R(u(t))$ over time in semi-logarithmic scale to illustrate the entropy production as proved in Theorem 22. We

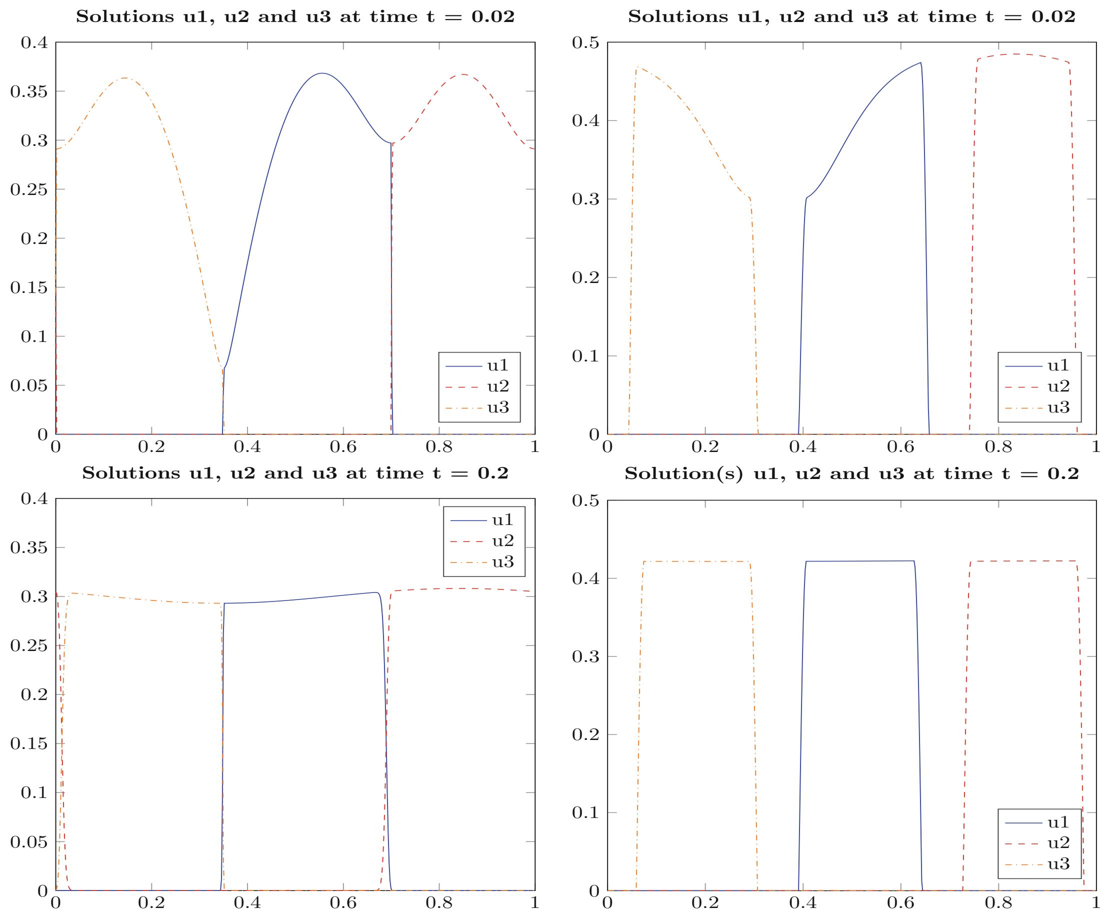

Figure 3.2: Comparison of the segregation patterns for *three* species at times $t = 0.02$ (top) and $t = 0.2$ (bottom) obtained from the local model (left) and nonlocal model (right). The solutions are almost in the steady state at $t = 0.2$.

set the final time $T = 1.5$, the time step size $\Delta t = 10^{-4}$, use a mesh of $N = 512$ cells, and choose $n = 2$ species. The remaining parameters are taken as in Section 3.4.1; see Table 3.1 and the test cases therein. As expected, the entropies are decreasing functions of time. The Rao entropy decays first quickly but then stabilizes slowly, while the Boltzmann entropy takes more time to stabilize.

3.5 Some auxiliary results

Lemma 31. *Under Hypothesis (H3), the entropy dissipation Q, defined in* (1.19)*, is nonnegative.*

Proof. We follow the approach of [43] and write $Q = Q_1 + \cdots + Q_3$, where

$$Q_1 = \frac{1}{n-1} \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \pi_i a_{ii} |\partial_x u_i(x)|^2 \, dx + \frac{1}{n-1} \sum_{i,j=1,\, i>j}^{n} \int_{\mathbb{T}} \pi_i a_{ii} |\partial_x u_i(y)|^2 \, dy,$$

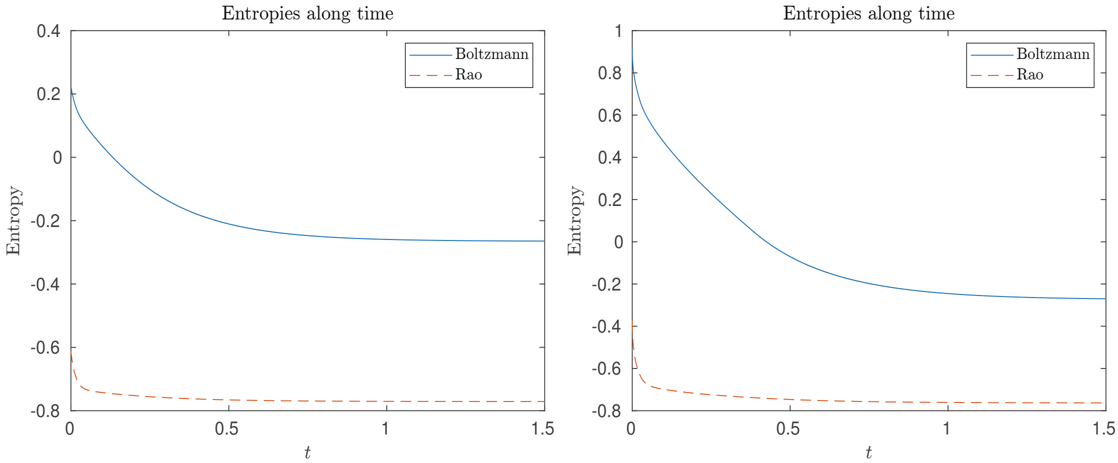

Figure 3.3: Temporal decay of the Boltzmann and Rao entropies for test cases 15 (left) and 16 (right) in semi-logarithmic scale.

$$Q_2 = \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x)\, dy\, dx,$$
$$Q_3 = \sum_{i,j=1,\, i>j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x)\, dy\, dx.$$

Exchanging i and j in the second integral of Q_1 and using $\mathrm{m}(\mathbb{T}) = 1$, we have

$$Q_1 = \frac{1}{n-1} \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \left(\pi_i a_{ii} |\partial_x u_i(x)|^2 + \pi_j a_{jj} |\partial_x u_j(y)|^2 \right) dy\, dx.$$

Exchanging i and j as well as x and y in Q_3 gives

$$\begin{aligned} Q_3 &= \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_j a_{ji} B^{ji}(y-x) \partial_x u_j(y) \partial_x u_i(x)\, dy\, dx \\ &= \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_j a_{ji} B^{ij}(x-y) \partial_x u_j(y) \partial_x u_i(x)\, dy\, dx. \end{aligned}$$

We collect these expressions to obtain

$$Q = \frac{1}{(n-1)} \sum_{i,j=1,\, i<j}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \begin{pmatrix} \partial_x u_i(x) \\ \partial_x u_j(y) \end{pmatrix}^{\top} M^{ij}(x-y) \begin{pmatrix} \partial_x u_i(x) \\ \partial_x u_j(y) \end{pmatrix} dy\, dx \geq 0,$$

where M^{ij} is defined in (1.18), and the last inequality follows from Hypothesis (H3). □

Lemma 32. *The upwind approximation* (3.6) *and the logarithmic mean* (3.7) *satisfy property* (3.5) *of the mobilities* $u_{i,\sigma}$.

Proof. The proof is based on the following inequalities for the logarithmic mean:

$$\min\{a,b\} \le \frac{a-b}{\log a - \log b} \le \max\{a,b\} \quad \text{for all } a,b>0. \tag{3.42}$$

They imply the linear growth $u_{i,\ell+1/2} \le \max\{u_{i,\ell}, u_{i,\ell+1}\}$ for the logarithmic mean, which also holds, by definition, for the upwind approximation. We show that property (3.5) is satisfied for the upwind approximation (3.6). Let $p_{i,\ell+1} - p_{i,\ell} \ge 0$. Then, by (3.42),

$$\begin{aligned} u_{i,\ell+1/2}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) &= u_{i,\ell+1}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \\ &\ge (p_{i,\ell+1}-p_{i,\ell})(u_{i,\ell+1}-u_{i,\ell}). \end{aligned}$$

On the other hand, if $p_{i,\ell+1} - p_{i,\ell} < 0$, again by (3.42),

$$\begin{aligned} u_{i,\ell+1/2}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) &= u_{i,\ell}(p_{i,\ell+1}-p_{i,\ell})(\log u_{i,\ell+1} - \log u_{i,\ell}) \\ &\ge (p_{i,\ell+1}-p_{i,\ell})(u_{i,\ell+1}-u_{i,\ell}). \end{aligned}$$

Property (3.5) follows immediately after inserting definition (3.7) of the logarithmic mean. This ends the proof. □

Lemma 33 (Discrete Young convolution inequality)**.** *Let $1 \le p,q \le \infty$ and $1 \le r \le \infty$ be such that $1 + 1/r = 1/p + 1/q$ and let $B \in L^p(\mathbb{T})$ and $v = (v_\ell)_{\ell\in G} \in \mathcal{V}_\mathcal{T}$. Furthermore, define $B_{\ell-\ell'} := (\Delta x)^{-1}\int_{K_{\ell-\ell'}} B(y)\,dy$ for every ℓ and $\ell' \in G$. Then*

$$\left(\sum_{\ell\in G}\Delta x\left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'}\, v_{\ell'}\right|^r\right)^{1/r} \le \|B\|_{L^p(\mathbb{T})}\|v\|_{0,q,\mathcal{T}}.$$

Proof. First, let $\ell \in G$ be fixed. Then

$$\left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'} v_{\ell'}\right| \le \sum_{\ell'\in G}\Delta x\big(|B_{\ell-\ell'}|^p|v_{\ell'}|^q\big)^{1/r}|B_{\ell-\ell'}|^{(r-p)/r}|v_{\ell'}|^{(r-q)/r}.$$

Thanks to the assumption $1 = 1/p + 1/q - 1/r$, we can apply Hölder's inequality with the exponents r, $pr/(r-p)$, and $qr/(r-q)$ to obtain

$$\begin{aligned} \left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'} v_{\ell'}\right| &\le \left(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\right)^{1/r}\left(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p\right)^{(r-p)/pr} \\ &\quad\times\left(\sum_{\ell'\in G}\Delta x|v_{\ell'}|^q\right)^{(r-q)/qr} \\ &= \left(\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\right)^{1/r}\|B\|_{0,p,\mathcal{T}}^{(r-p)/r}\|v\|_{0,q,\mathcal{T}}^{(r-q)/r}. \end{aligned}$$

Then, taking the exponent r and summing over $\ell \in G$,

$$\sum_{\ell\in G}\Delta x\left|\sum_{\ell'\in G}\Delta x B_{\ell-\ell'}v_{\ell'}\right|^r \le \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\left(\sum_{\ell\in G}\Delta x\sum_{\ell'\in G}\Delta x|B_{\ell-\ell'}|^p|v_{\ell'}|^q\right)$$

$$\leq \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\bigg(\sum_{\ell'\in G}\Delta x|v_{\ell'}|^q\sum_{\ell\in G}\Delta x|B_{\ell-\ell'}|^p\bigg)$$
$$\leq \|B\|_{0,p,\mathcal{T}}^{r-p}\|v\|_{0,q,\mathcal{T}}^{r-q}\|v\|_{0,q,\mathcal{T}}^{q}\|B\|_{0,p,\mathcal{T}}^{p} = \|B\|_{0,p,\mathcal{T}}^{r}\|v\|_{0,q,\mathcal{T}}^{r}.$$

Finally, it holds that

$$\begin{aligned}\|B\|_{0,p,\mathcal{T}}^p &\leq \sum_{\ell\in G}\Delta x\bigg|\frac{1}{\Delta x}\int_{K_\ell}B(y)\,dy\bigg|^p \leq \sum_{\ell\in G}\bigg(\int_{K_\ell}|B(y)|^p\,dy\bigg)\bigg(\int_{K_\ell}\frac{dx}{\Delta x}\bigg)^{p-1}\\ &\leq \sum_{\ell\in G}\int_{K_\ell}|B(y)|^p\,dy = \|B\|_{L^p(\mathbb{T})}^p,\end{aligned}$$

which concludes the proof. □

Lemma 34. *Let $s>1$ and $p>1$. Then there exists a constant $C>0$ only depending on s such that for any sequence $u=(u_\ell)_{\ell\in G}$ it holds that*

$$\|u\|_{0,\infty,\mathcal{T}} \leq C\|u\|_{1,p,\mathcal{T}}^{1/s}\|u\|_{0,(s-1)p/(p-1),\mathcal{T}}^{1-1/s}.$$

Proof. We adapt the proof of [18, Lemma 4.1] to the one-dimensional case. Due to the embedding $\mathrm{BV}(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ applied to the sequence $(|u_\ell|^s)_{\ell\in G}$,

$$\|u\|_{0,\infty,\mathcal{T}}^s \leq C\bigg(\|u\|_{0,s,\mathcal{T}}^s + \sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big|\bigg). \tag{3.43}$$

Since $s>1$, we have

$$\sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big| \leq s\sum_{\ell\in G}\big(|u_\ell|^{s-1}+|u_{\ell+1}|^{s-1}\big)|u_\ell - u_{\ell+1}|.$$

We apply Hölder's inequality with exponents p and $p/(p-1)$:

$$\sum_{\ell\in G}\big||u_\ell|^s - |u_{\ell+1}|^s\big| \leq 2s\bigg(\sum_{\ell\in G}\frac{|u_\ell-u_{\ell+1}|^p}{\Delta x^{p-1}}\bigg)^{1/p}\bigg(\sum_{\ell\in G}\Delta x|u_\ell|^{\frac{(s-1)p}{p-1}}\bigg)^{(p-1)/p}.$$

Besides, using again Hölder's inequality (with the same exponents), we find that

$$\|u\|_{0,s,\mathcal{T}} = \bigg(\sum_{\ell\in G}\Delta x|u_\ell||u_\ell|^{s-1}\bigg)^{1/s} \leq \|u\|_{0,p,\mathcal{T}_m}^{1/s}\|u\|_{0,(s-1)p/(p-1),\mathcal{T}}^{(s-1)/s}.$$

Then, inserting the last two inequalities into (3.43) yields the desired result. This concludes the proof of Lemma 34. □

3.6 Counter-example

We claim that there exist kernels B^{ij}, being indicator functions, and piecewise constant functions $u_1, \dots, u_n$ such that the positive semi-definiteness condition

$$J := \sum_{i,j=1}^{n} \int_{\mathbb{T}} \int_{\mathbb{T}} \pi_i a_{ij} B^{ij}(x-y) u_j(y) u_i(x) \, dy \, dx \geq 0,$$

is *not* satisfied. For this statement, we assume that the matrix $(\pi_i a_{ij}) \in \mathbb{R}^{n \times n}$ is (symmetric and) positive definite. With the notation of Section 3.1.1, we set $\Delta x = 1/N$ for some even number $N > 5$ and choose $r = 3\Delta x/2$ as well as the kernels

$$B^{ij}(x) = \mathbb{1}_{(-r,r)}(x) \quad \text{for } x \in \mathbb{T}.$$

Let $u_i = (u_{i,\ell})_{\ell \in G} \in \mathcal{V}_{\mathcal{T}}$ for $i = 1, \dots, n$. Then we can write J as

$$J = \sum_{i,j=1}^{n} \sum_{\ell,\ell' \in G} \pi_i a_{ij} \widehat{M}^{ij}_{\ell,\ell'} u_{j,\ell'} u_{i,\ell}, \quad \text{where } \widehat{M}^{ij}_{\ell,\ell'} = \int_{K_\ell} \int_{K_{\ell'}} B^{ij}(x-y) \, dy \, dx. \tag{3.44}$$

A straightforward, but tedious computation shows that the matrix $\widehat{M}^{ij} = (\widehat{M}^{ij}_{\ell,\ell'})_{\ell,\ell' \in G} \in \mathbb{R}^{N \times N}$ is pentadiagonal with entries

$$M^{ij}_{\ell,\ell'} = (\Delta x)^2, \quad M^{ij}_{\ell,\ell \pm 1} = \frac{7}{8}(\Delta x)^2, \quad M^{ij}_{\ell,\ell \pm 2} = \frac{1}{8}(\Delta x)^2.$$

This matrix possesses the eigenvector $w \in \mathbb{R}^N$, defined by $w_\ell = 1$ for odd ℓ and $w_\ell = -1$ for even ℓ, associated with the negative eigenvalue $\lambda = -4(\Delta x)^2$.

Let $v_1, \dots, v_n \in \mathbb{R}^n$ be the eigenvectors of the symmetric matrix $(\pi_i a_{ij})_{i,j=1,\dots,n}$ associated with the eigenvalues $0 < \nu_1 \leq \dots \leq \nu_n$, respectively. We define the $nN \times nN$ matrix $\widehat{M} = (\pi_i a_{ij} \widehat{M}^{ij})$ consisting of the $N \times N$ blocks $\pi_i a_{ij} \widehat{M}^{ij}$. It can be verified that this matrix $\widehat{M}$ possesses the eigenvector $z = (z_1, \dots, z_n) \in \mathbb{R}^{nN}$ with $z_i = v_{n,i} w \in \mathbb{R}^N$ for $i = 1, \dots, n$ associated with the eigenvalue $\lambda \nu_n = -4(\Delta x)^2 \nu_n$. Then, choosing $u_i = z_i$ in (3.44), we find that

$$J = \sum_{i,j=1}^{n} \pi_i a_{ij} z_i^\top \widehat{M}^{ij} z_j = -4(\Delta x)^2 \nu_n \sum_{i=1}^{n} |z_i|^2 < 0.$$

This provides the desired counter-example.

4 Analysis of a charge transport system with Fermi-Dirac statistics for memristive devices

The results in this chapter are from an ongoing research collaboration with Maxime Herda (Inria Lille) and Ansgar Jüngel (TU Wien). A manuscript for submission is currently in preparation.

In this chapter we analyse an instationary nonlinear drift-diffusion system that models memristive devices. We present our hypothesis and main results, i.e. the global existence of weak solutions and the uniform-in-time boundedness of weak solutions in Section 4.1. In Section 4.2 we prove the existence Theorem 35 and Section 4.3 is concerned with the proof of bounded solutions, cf. Theorem 36. The necessary estimates on the statistics functions, their inverses and the corresponding derivatives, are collected and proved in Section 4.4.

4.1 Main results

We will impose the following assumptions.

(A1) Domain: $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\mu(\Gamma_D) > 0$, where μ is the $(d-1)$-dimensional Lebesgue measure, Γ_N is relatively open in $\partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

(A2) Data: $T > 0$, $\lambda > 0$, $A \in L^\infty(\Omega)$.

(A3) Boundary data: $\widetilde{n}, \widetilde{p}, \widetilde{V} \in W^{1,\infty}(\Omega)$ with $\widetilde{n}, \widetilde{p} > 0$ in Ω.

(A4) Initial data: $n^I, p^I, D^I \in L^2(\Omega)$ satisfy $n^I, p^I, D^I \geq 0$ in Ω and such that

$$\mathcal{E}[n^I, p^I, D^I, V^I] < \infty.$$

Furthermore, we assume that $\overline{D^I} := \operatorname{ess\,sup}_{x\in\Omega} D^I(x) \leq 1$ and $D^I_\Omega < 1$, where

$$D^I_\Omega := \frac{1}{\mathrm{m}(\Omega)} \int_\Omega D^I \, dx \tag{4.1}$$

and $\mathrm{m}(\Omega)$ is the d-dimensional Lebesgue measure of Ω.

(A5) Elliptic Regularity: There exists some $r \geq 3$ such that for $C > 0$ and all $f \in L^{3r/(r+3)}(\Omega)$ the weak solution V of the Poisson problem

$$\Delta V = f \text{ in } \Omega, \quad V = \widetilde{V} \text{ on } \Gamma_D, \quad \nabla V \cdot \nu = 0 \text{ on } \Gamma_N, \tag{4.2}$$

satisfies the estimate

$$\|V\|_{W^{1,r}(\Omega)} \leq C\|f\|_{L^{3r/(r+3)}(\Omega)} + C. \tag{4.3}$$

Additionally, we set $Q_T := (0,T)\times\Omega$, $W^{1,q}_D(\Omega) := \{u \in W^{1,q}(\Omega)\colon u = 0 \text{ on } \Gamma_D\}$ and introduce the initial electric potential $V^I - \widetilde{V} \in H^1_D(\Omega)$ as the unique solution to

$$\begin{aligned} \lambda^2\Delta V^I &= n^I - p^I - D^I + A, \text{ in } \Omega,\\ V^I &= \widetilde{V} \text{ on } \Gamma_D, \quad \nabla V^I\cdot\nu = 0 \text{ on } \Gamma_N. \end{aligned} \tag{4.4}$$

Constants $C > 0$ in the following computations may change their value from line to line.

Let us discuss our assumptions. The boundary data in (A3) are supposed to be time independent to simplify the computations. Assumption (A4) marks the biggest difference of our work to [2]. While the authors of [2] have to assume pointwise positive bounds on the initial data from below and above and far from saturation for ion vacancies, we allow for void as well as saturation. We only suppress $D^I_\Omega = 1$, which would be physically unrealistic. The most restrictive assumption is (A5). In general, for the solution to elliptic problems (4.2) with mixed boundary conditions one can only expect $V \in W^{1,r}(\Omega)$ for some $r > 2$, cf. [64]. Under certain geometric conditions to the Dirichlet and Neumann boundary part of $\partial\Omega$, this regularity is improved to $r > 3$, as is shown in [47]. In particular, it is necessary that Γ_D and Γ_N meet at an angle not larger than π. The authors of [47] also argued that, while it is a restrictive condition, it is satisfied in most applications. Assumption (A5) was also used by the authors of [77] to prove existence of bounded weak solutions to the degenerate drift-diffusion system, which is an approximation to (1.25) in the high density regime of the species n and p.

Our main result is the global existence of weak solutions in space dimension $d \le 4$.

Theorem 35 (Global Existence). *Let the assumptions (A1)–(A4) hold and assume that $d \le 4$ for the spatial dimension. Then there exists a weak solution (n,p,D,V) to system (1.25)–(1.26) with (4.4) satisfying*

$$\begin{aligned} &n,p \in L^\infty(0,T;L^{5/3}(\Omega)) \cap L^2(0,T;W^{1,\kappa}(\Omega)),\\ &D,\sqrt{D} \in L^\infty(Q_T)\cap L^2(0,T;H^1(\Omega)), \quad V \in L^\infty(0,T;H^1(\Omega)),\\ &\left.\begin{aligned} &2nG'(n)\nabla\sqrt{n} - \sqrt{n}\nabla V,\ 2pG'(p)\nabla\sqrt{p} + \sqrt{p}\nabla V\\ &2\nabla\tanh^{-1}(\sqrt{D}) + \sqrt{D}\nabla V \end{aligned}\right\} \in L^2(Q_T),\\ &\partial_t n, \partial_t p \in L^1(0,T;W^{1,\kappa'}_D(\Omega)') \cap L^2(0,T;W^{1,5}_D(\Omega)'), \quad \partial_t D \in L^2(0,T;H^1(\Omega)'), \end{aligned} \tag{4.5}$$

where $1/\kappa' = 1 - 1/\kappa$ and κ depends on the spatial dimension d as follows

$$\kappa \begin{cases} = 2, & d = 1,\\ < 2, & d = 2,\\ < \frac{8}{5}, & d = 3,\\ < \frac{16}{11}, & d = 4. \end{cases}$$

The fluxes have to be understood in the senses

$$\begin{aligned} J_n &= nG'(n)\nabla n - n\nabla V \in L^1(0,T;L^\kappa(\Omega)) \cap L^2(0,T;L^{5/4}(\Omega)),\\ -J_p &= pG'(p)\nabla p + p\nabla V \in L^1(0,T;L^\kappa(\Omega)) \cap L^2(0,T;L^{5/4}(\Omega)),\\ -J_D &= -\nabla\log(1-D) + D\nabla V \in L^2(Q_T). \end{aligned}$$

The Dirichlet boundary conditions in (1.26), $n = \tilde{n},\ p = \tilde{p}$ *on* $\Gamma_D, t > 0$, *are satisfied in the sense of traces and the initial condition in* (1.26) *holds in the sense*

$$n(t,.) \to n^I,\ p(t,.) \to p^I \quad \text{strongly in } W_D^{1,\kappa'}(\Omega)' \cap W_D^{1,5}(\Omega)',$$
$$D(t,.) \to D^I \quad \text{strongly in } H^1(\Omega)',$$

as $t \to 0$. *The solution satisfies the free energy inequality*

$$\mathcal{E}[n,p,D,V](\tau) + \frac{1}{2}\int_0^\tau \int_\Omega |2nG'(n)\nabla\sqrt{n} - \sqrt{n}\nabla V|^2 + |2pG'(p)\nabla\sqrt{p} + \sqrt{p}\nabla V|^2\,dx\,dt$$
$$+ \frac{1}{2}\int_0^\tau \int_\Omega |2\nabla \tanh^{-1}(\sqrt{D}) + \sqrt{D}\nabla V|^2\,dx\,dt \le \mathcal{E}^I + C(\mathcal{E}^I, \Lambda, T), \quad (4.6)$$

for all $\tau \in (0,T]$, *where the initial free energy* $\mathcal{E}^I := \mathcal{E}[n^I, p^I, D^I, V^I]$ *is defined in* (1.28),

$$\Lambda := 2\left(\|\nabla(G(\widetilde{n}) - \widetilde{V})\|^2_{L^\infty(Q_T)} + \|\nabla(G(\widetilde{p}) + \widetilde{V})\|^2_{L^\infty(Q_T)}\right), \quad (4.7)$$

and it holds that $C(\mathcal{E}^I, \Lambda, T) = 0$ *if* $\Lambda = 0$.

The property $\Lambda = 0$ means that the boundary conditions are in thermal equilibrium and the free energy then is a nonincreasing function in time.

As already mentioned in Section 1.3.3, we will approximate (1.25) – (1.26) by truncating both the drift and the diffusion term (more precisely we cut off the densities, but leave the potentials alone) and prove the existence of a solution (n_k, p_k, D_k, V_k) to the approximate problem. Estimates uniform in the approximation parameter k, obtained via an approximate free energy inequality, will allow us to then take the limit $k \to \infty$ and thus prove existence of a solution (n, p, D, V) to (1.25) – (1.26).

As a second result we prove the boundedness of solutions under slightly stricter assumptions.

Theorem 36 (Bounded Solutions)**.** *Let assumptions (A1)–(A4) (assumptions of Theorem 35) and (A5) hold with* $r = 3$, *let* $d = 3$ *and assume that* $n^I, p^I, D^I \in L^\infty(\Omega)$. *Then the weak solution constructed in Theorem 35 fulfills*

$$n,p,D \in L^\infty(0,T;L^q(\Omega)), \quad \text{for all } 1 \le q < \infty, \qquad V \in L^\infty(0,T;W^{1,3}(\Omega)),$$
$$\nabla n^\alpha, \nabla p^\alpha \in L^2(Q_T), \quad \text{for all } 1 \le \alpha < \infty. \quad (4.8)$$

If additionally $r > 3$ *in assumption (A5), there holds the improved regularity*

$$n,p,D \in L^\infty(0,T;L^\infty(\Omega)), \qquad V \in L^\infty(0,T;W^{1,r}(\Omega)). \quad (4.9)$$

As stated in Section 1.3.3, the theorem is proved by an Alikakos-type iteration method. The restriction to three space dimensions comes from the regularity assumption (4.3).

Remark 37. Our results hold for an arbitrary number of charged particles, since we use the Poisson equation only through the norm estimates on V and ∇V in the various L^q-spaces. The system of equations for the charge densities would then read as

$$\partial_t u_i = \nabla \cdot (u_i \nabla G(u_i) + z_i u_i \nabla V), \quad i \in I,$$
$$\partial_t u_i = \nabla \cdot (u_i \nabla H(u_i) + z_i u_i \nabla V), \quad i \in I_0,$$
$$\lambda^2 \Delta V = -\sum_{i \in I \cup I_0} z_i u_i + A(x),$$

where $z_i \in \mathbb{R}$ corresponds to the charge number of the species u_i, I and I_0 are the respective sets of indices, and initial and mixed boundary conditions are chosen according to (1.26).

Remark 38. Let us point out that our results also hold when using Fermi-Dirac statistics of order 1/2 instead of Blakemore statistics for the oxygen vacancies. The proofs for D are then analogue to the proofs for p, with the exception that vanishing boundary terms due to the no-flux boundary condition simplify the computations a bit and conservation of mass (for D) stays upright. The regularity results will be the same for all species in that case, cf. [71, 77].

Let us also introduce the following notation, which will be used throughout the entire chapter.

Notation 1. *Given terms A and B, we write $A \lesssim B$ if there exists a constant $C > 0$, such that it holds that $A \le CB$. If $A \lesssim B \lesssim A$ holds, we write $A \sim B$. Furthermore, if there exist two constants $C_1, C_2 > 0$, such that $A \le C_1 B + C_2$ is true, we write $A \lesssim B + 1$.*

4.2 Proof of Theorem 35

In this section we prove the global existence of weak solutions to (1.25)–(1.26). We first show the existence of solutions to an approximate problem, followed by deriving uniform estimates and then pass to the limit to show that the original problem has weak solutions.

4.2.1 Approximate problem to (1.25)–(1.26)

The approximate problem is defined by cutting off the nonlinearities. To this end we introduce for $z \in \mathbb{R}$ and $k \in \mathbb{N}$ the truncations

$$T_k(z) := \max(0, \min(k, z)),$$

$$S_k^1(z) := \begin{cases} 1, & z \le 0, \\ zG'(z), & 0 < z \le k, \\ k^{2/3} z^{1/3} G'(z), & k < z, \end{cases} \qquad S_k^2(z) := \begin{cases} 1, & z \le 0, \\ zH'(z), & 0 < z \le \frac{k}{k+1}, \\ 1 + k, & \frac{k}{k+1} < z. \end{cases} \tag{4.10}$$

Note that it holds that $zH'(z) = \frac{1}{1-z}$ and, thanks to Lemma 73, $zG'(z) \sim 1 + z^{\frac{2}{3}}$. This allows us to define the approximate system as follows:

$$\left.\begin{aligned} \partial_t n_k - \nabla \cdot \left(S_k^1(n_k)\nabla n_k - T_k(n_k)\nabla V_k\right) &= 0, \\ \partial_t p_k - \nabla \cdot \left(S_k^1(p_k)\nabla p_k + T_k(p_k)\nabla V_k\right) &= 0, \\ \partial_t D_k - \nabla \cdot \left(S_k^2(D_k)\nabla D_k + T_{\frac{k}{k+1}}(D_k)\nabla V_k\right) &= 0, \\ \lambda^2 \Delta V_k - n_k + p_k + D_k + A &= 0, \end{aligned}\right\} \quad \text{in } \Omega,\ t > 0, \tag{4.11}$$

supplemented with initial and boundary conditions

$$\begin{aligned} n_k(0,.) = n^I, \quad p_k(0,.) = p^I, \quad D_k(0,.) = D^I, &\quad \text{in } \Omega, \\ n_k = \widetilde{n}, \quad p_k = \widetilde{p}, \quad V_k = \widetilde{V}, &\quad \text{on } \Gamma_D,\ t > 0, \\ \nabla n_k \cdot \nu = \nabla p_k \cdot \nu = \nabla V_k \cdot \nu = 0, &\quad \text{on } \Gamma_N,\ t > 0, \\ \left(S_k^2(D_k)\nabla D_k + T_{\frac{k}{k+1}}(D_k)\nabla V_k\right) \cdot \nu = 0, &\quad \text{on } \partial\Omega,\ t > 0. \end{aligned} \tag{4.12}$$

4.2.2 Existence of solutions to the approximate problem (4.11)–(4.12)

We prove that the approximate problem has a weak solution. This can be done by a Schaefer fixed point argument, which requires us to first linearize the approximate problem. To this end, let $(n^*, p^*, D^*) \in L^2(Q_T; \mathbb{R}^3)$ and $\sigma \in [0,1]$. Then we define the linearized problem

$$\left.\begin{aligned} \partial_t n - \nabla \cdot \big(S_k^1(n^*)\nabla n - \sigma T_k(n^*)\nabla V\big) &= 0,\\ \partial_t p - \nabla \cdot \big(S_k^1(p^*)\nabla p + \sigma T_k(p^*)\nabla V\big) &= 0,\\ \partial_t D - \nabla \cdot \Big(S_k^2(D^*)\nabla D + \sigma T_{\frac{k}{k+1}}(D^*)\nabla V\Big) &= 0,\\ \lambda^2 \Delta V - n + p + D + \sigma A &= 0, \end{aligned}\right\} \quad \text{in } \Omega,\ t>0, \tag{4.13}$$

together with initial and boundary conditions

$$\begin{aligned} n(0,.) = \sigma n^I, \quad p(0,.) = \sigma p^I, \quad D(0,.) = \sigma D^I, &\quad \text{in } \Omega,\\ n = \sigma\widetilde{n}, \quad p = \sigma\widetilde{p}, \quad V = \sigma\widetilde{V}, &\quad \text{on } \Gamma_D,\ t>0,\\ \nabla n\cdot\nu = \nabla p\cdot\nu = \nabla V\cdot\nu = 0, &\quad \text{on } \Gamma_N,\ t>0,\\ \Big(S_k^2(D^*)\nabla D + \sigma T_{\frac{k}{k+1}}(D^*)\nabla V\Big)\cdot\nu = 0, &\quad \text{on } \partial\Omega,\ t>0, \end{aligned} \tag{4.14}$$

which has a unique weak solution $(n,p,D,V) \in L^2(Q_T;\mathbb{R}^4)$, cf. [99, Theorem 23.A].

This allows us to define the fixed-point operator F and prove the existence of a solution to the approximate system (4.11)–(4.12)

$$F\colon \begin{cases} L^2(Q_T;\mathbb{R}^3)\times[0,1] \to L^2(Q_T;\mathbb{R}^3)\\ (n^*,p^*,D^*,\sigma) \mapsto (n,p,D) \text{ solution to (4.13)-(4.14)}. \end{cases} \tag{4.15}$$

Lemma 39. *Let assumptions (A1)–(A4) hold. Then, for $\sigma = 1$, the operator $F(.,1)$ mapping $L^2(Q_T;\mathbb{R}^3) \to L^2(Q_T;\mathbb{R}^3)$ defined by (4.15) has a fixed point $(n,p,D) \in L^2(Q_T)^3$, i.e. it solves the equation $F(n,p,D,1) = (n,p,D)$.*

Proof. We will use the Leray-Schauder fixed point theorem to establish the existence of a fixed point. It is straightforward to check that, for $\sigma = 0$, the unique solution to system (4.13)–(4.14) is $(n,p,D,V) = (0,0,0,0)$, hence $F(n^*,p^*,D^*,0) = 0$ for all $(n^*,p^*,D^*) \in L^2(Q_T;\mathbb{R}^3)$.

Next, we show that F is continuous. To this end, let $(n_m^*, p_m^*, D_m^*, \sigma_m) \to (n^*, p^*, D^*, \sigma)$ in $L^2(Q_T;\mathbb{R}^3)\times[0,1]$ and set $(n_m,p_m,D_m) := F(n_m^*,p_m^*,D_m^*,\sigma_m)$, $(n,p,D) := F(n^*,p^*,D^*,\sigma)$, respectively. We have to show that $(n_m,p_m,D_m) \to (n,p,D)$ as $m\to\infty$. We use the test functions $n_m - \sigma_m\widetilde{n}$, $p_m - \sigma_m\widetilde{p}$, D_m and $V_m - \sigma_m\widetilde{V}$ in the weak formulation of (4.13). Starting with $V_m - \sigma_m \widetilde{V}$ in the weak formulation of the Poisson equation we obtain

$$\begin{aligned}\lambda^2 \int_\Omega \left|\nabla(V_m - \sigma_m \widetilde{V})\right|^2 dx &= -\lambda^2 \int_\Omega \sigma_m \nabla\widetilde{V} \cdot \nabla(V_m - \sigma_m \widetilde{V})\, dx \\ &\quad - \int_\Omega (n_m - p_m - D_m - \sigma_m A)(V_m - \sigma_m \widetilde{V})\, dx \\ &\le \frac{\lambda^2}{2} \int_\Omega \sigma_m^2 \left|\nabla\widetilde{V}\right|^2 dx + \frac{\lambda^2}{2} \int_\Omega \left|\nabla(V_m - \sigma_m \widetilde{V})\right|^2 dx \\ &\quad + \frac{1}{2\delta} \int_\Omega (n_m - p_m - D_m - \sigma_m A)^2\, dx \\ &\quad + \frac{\delta C_P^2}{2} \int_\Omega \left|\nabla(V_m - \sigma_m \widetilde{V})\right|^2 dx,\end{aligned}$$

where we have used Young's inequality for products and the Poincaré inequality. Choosing δ sufficiently small and rearranging terms, we obtain

$$\frac{\lambda^2 - \delta C_P^2}{2} \int_\Omega \left|\nabla(V_m - \sigma_m \widetilde{V})\right|^2 dx \le \frac{\lambda^2 \sigma_m^2}{2} \int_\Omega \left|\nabla\widetilde{V}\right|^2 dx + \frac{1}{2\delta} \int_\Omega (n_m - p_m - D_m - \sigma_m A)^2\, dx.$$

Computing the square on the left-hand side, dropping the nonnegative part with $|\nabla\widetilde{V}|^2$, rearranging terms and using Young's inequality again for the product, we get

$$\begin{aligned}\frac{\lambda^2 - \delta C_P^2}{2} \int_\Omega |\nabla V_m|^2\, dx &\le \frac{\varepsilon}{2} \int_\Omega |\nabla V_m|^2\, dx + \frac{\lambda^2 \sigma_m^2 \varepsilon + (\lambda^2 - \delta C_P^2)^2 \sigma_m^2}{2\varepsilon} \int_\Omega |\nabla\widetilde{V}|^2\, dx \\ &\quad + \frac{4}{2\delta} \int_\Omega n_m^2 + p_m^2 + D_m^2 + \sigma_m^2 A^2\, dx.\end{aligned}$$

Choosing ε sufficiently small, we can absorb the first term on the right-hand side into the left-hand side and thus, after integrating over $t \in (0, \tau)$ get the estimate

$$\int_0^\tau \int_\Omega |\nabla V_m|^2\, dx\, dt \le C + C \int_0^\tau \int_\Omega n_m^2 + p_m^2 + D_m^2\, dx\, dt.$$

We proceed with the test function $n_m - \sigma_m \widetilde{n}$ in the equation for n_m and get

$$\begin{aligned}&\int_0^\tau \langle \partial_t (n_m - \sigma_m \widetilde{n}), n_m - \sigma_m \widetilde{n}\rangle\, dt \\ &\quad + \int_0^\tau \int_\Omega (S_k^1(n_m^*) \nabla(n_m - \sigma_m \widetilde{n}) - \sigma_m T_k(n_m^*) \nabla(V_m - \sigma_m \widetilde{V})) \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt \\ &\quad + \int_0^\tau \int_\Omega \sigma_m (S_k^1(n_m^*) \nabla\widetilde{n} - T_k(n_m^*) \sigma_m \nabla\widetilde{V}) \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt = 0.\end{aligned}$$

Simplifying terms, we obtain

$$\begin{aligned}&\frac{1}{2} \int_\Omega (n_m(\tau) - \sigma_m \widetilde{n})^2\, dx - \frac{1}{2} \int_\Omega (n_m(0) - \sigma_m \widetilde{n})^2\, dx \\ &\qquad + \int_0^\tau \int_\Omega S_k^1(n_m^*) \nabla n_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt\end{aligned}$$

$$= \sigma_m \int_0^\tau \int_\Omega T_k(n_m^*) \nabla V_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt.$$

Now we estimate term by term. Using $n_m(0) = \sigma_m n^I$ together with Young's inequality for products and $0 \leq \sigma_m \leq 1$ we get

$$\frac{1}{2} \int_\Omega (n_m(\tau) - \sigma_m \widetilde{n})^2 - (n_m(0) - \sigma_m \widetilde{n})^2\, dx \geq \frac{1}{4} \int_\Omega n_m(\tau)^2\, dx - C_1.$$

To estimate the last integral on the left-hand side we split the domain Ω into the disjoint union of the sets $\Omega = \{n_m^* \leq 0\} \cup \{0 < n_m^* \leq k\} \cup \{k < n_m^*\}$ to use the definition of $S_k^1(n_m^*)$ and combine this with the estimates from Corollary 74 and Young's inequality for products to obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega S_k^1(n_m^*) \nabla n_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt \\
&\quad = \int_0^\tau \int_{\{n_m^* \leq 0\}} \nabla n_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt \\
&\qquad + \int_0^\tau \int_{\{0 < n_m^* \leq k\}} n_m^* G'(n_m^*) \nabla n_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt \\
&\qquad + \int_0^\tau \int_{\{k < n_m^*\}} k^{2/3} (n_m^*)^{1/3} G'(n_m^*) \nabla n_m \cdot \nabla(n_m - \sigma_m \widetilde{n})\, dx\, dt \\
&\quad \geq \int_0^\tau \int_{\{n_m^* \leq 0\}} \frac{1}{2} |\nabla n_m|^2 - \frac{\sigma_m^2}{2} |\nabla \widetilde{n}|^2\, dx\, dt \\
&\qquad + \int_0^\tau \int_{\{0 < n_m^* \leq k\}} \frac{C}{2} |\nabla n_m|^2 - \frac{\tilde{C}^2 (1 + k^{2/3})^2 \sigma_m^2}{2C} |\nabla \widetilde{n}|^2\, dx\, dt \\
&\qquad + \int_0^\tau \int_{\{k < n_m^*\}} \frac{C}{2} |\nabla n_m|^2 - \frac{\tilde{C}^2 (1 + k^{2/3})^2 \sigma_m^2}{2C} |\nabla \widetilde{n}|^2\, dx\, dt \\
&\quad \geq \frac{\min(1, C)}{2} \int_0^\tau \int_\Omega |\nabla n_m|^2\, dx\, dt \\
&\qquad - \frac{(1 + k^{2/3})^2 \max(C, \tilde{C}^2)}{2C} \int_0^\tau \int_\Omega |\nabla \widetilde{n}|^2\, dx\, dt.
\end{aligned}$$

We estimate the right-hand side using Young's inequality for products, together with the two inequalities $T_k(n_m^*) \leq k$, $\sigma_m \leq 1$ and the estimate previously obtained for ∇V_m and obtain

$$\begin{aligned}
\sigma_m \int_0^\tau \int_\Omega T_k(n_m^*) \nabla V_m \cdot \nabla(n_m - \sigma_n \widetilde{n})\, dx\, dt &\leq \frac{\delta}{2} \int_0^\tau \int_\Omega |\nabla(n_m - \sigma_m \widetilde{n})|^2\, dx\, dt \\
&\quad + \frac{k^2 \sigma_m^2}{2\delta} \int_0^\tau \int_\Omega |\nabla V_m|^2\, dx\, dt \\
&\leq \delta \int_0^\tau \int_\Omega |\nabla n_m|^2 + |\nabla \widetilde{n}|^2\, dx\, dt \\
&\quad + \frac{k^2}{2\delta} \left(C + C \int_0^\tau \int_\Omega n_m^2 + p_m^2 + D_m^2\, dx\, dt \right).
\end{aligned}$$

Combining all these estimates, choosing $\delta = \frac{\min(1,C)}{4}$ and rearranging terms, as well as multiplying or dividing by constants, we get

$$\int_\Omega n_m^2(\tau)\,dx + \int_0^\tau\int_\Omega |\nabla n_m|^2\,dx\,dt \le C(k) + C(k)\int_0^\tau\int_\Omega n_m^2 + p_m^2 + D_m^2\,dx\,dt.$$

The computations when using $p_m - \sigma_m\widetilde{p}$ as a test function are analogue to the previous computations with $n_m - \sigma_m\widetilde{n_m}$ and the final estimate is identical, hence we do not write them down explicitly.

Finally, we use D_m as a test function in the equation for D_m. Rewriting and simplifying terms analogous to the case of n_m yields

$$\begin{aligned}\frac{1}{2}\int_\Omega D_m^2(\tau)\,dx - \frac{1}{2}\int_\Omega D_m^2(0)\,dx \\ + \int_0^\tau\int_\Omega S_k^2(D_m^*)|\nabla D_m|^2\,dx\,dt \\ = -\sigma_m\int_0^\tau\int_\Omega T_{\frac{k}{k+1}}(D_m^*)\nabla V_m\cdot\nabla D_m\,dx\,dt.\end{aligned}$$

The second term in the first line is a positive constant, so there is nothing to do about this. To deal with the term in the second line we note that $S_k^2(D_m^*) \ge 1$ by definition, hence

$$\int_0^\tau\int_\Omega S_k^2(D_m^*)|\nabla D_m|^2\,dx\,dt \ge \int_0^\tau\int_\Omega |\nabla D_m|^2\,dx\,dt.$$

To estimate the integral in the third line, we use that $T_{\frac{k}{k+1}}(D_m^*) \le 1$, and together with Young's inequality for products and $\sigma_m \le 1$ we obtain

$$\begin{aligned}-\sigma_m\int_0^\tau\int_\Omega T_{\frac{k}{k+1}}(D_m^*)\nabla V_m\cdot\nabla D_m\,dx\,dt &\le \frac{1}{2}\int_0^\tau\int_\Omega |\nabla D_m|^2\,dx\,dt \\ &\quad + \frac{1}{2}\left(C + C\int_0^\tau\int_\Omega n_m^2 + p_m^2 + D_m^2\,dx\,dt\right).\end{aligned}$$

We put everything together, rearrange terms and end up with

$$\int_\Omega D_m^2(\tau)\,dx + \int_0^\tau\int_\Omega |\nabla D_m|^2\,dx\,dt \le C + C\int_0^\tau\int_\Omega n_m^2 + p_m^2 + D_m^2\,dx\,dt.$$

Hence, by adding all three inequalities, we get

$$\begin{aligned}\int_\Omega n_m^2(\tau) + p_m^2(\tau) + D_m^2(\tau)\,dx + \int_0^\tau\int_\Omega |\nabla n_m|^2 + |\nabla p_m|^2 + |\nabla D_m|^2\,dx\,dt \\ \le C(k) + C(k)\int_0^\tau\int_\Omega n_m^2 + p_m^2 + D_m^2\,dx\,dt.\end{aligned}$$

Let us note that the bound on the right-hand side is independent of σ_m. By Grönwall's inequality we now obtain that

$$\|n_m(\tau)\|_{L^2(\Omega)}^2 + \|p_m(\tau)\|_{L^2(\Omega)}^2 + \|D_m(\tau)\|_{L^2(\Omega)}^2 \le C(k)e^{C(k)\tau},$$

which yields uniform bounds on n_m, p_m, D_m in $L^2(Q_T)$, independent of $(n_m^*, p_m^*, D_m^*, \sigma_m)$. Plugging this into the combined inequality which we used for the Grönwall argument, we also see that we have a uniform bound on the gradients, namely

$$\int_0^\tau \|\nabla n_m\|_{L^2(\Omega)}^2 + \|\nabla p_m\|_{L^2(\Omega)}^2 + \|\nabla D_m\|_{L^2(\Omega)}^2 \, dt \le C(k) + C(k)(e^{C(k)\tau} - 1) = C(k)e^{C(k)\tau}.$$

Thus, we have a bound on n_m, p_m, D_m in $L^2(0,T;H^1(\Omega))$, uniform in $(n_m^*, p_m^*, D_m^*, \sigma_m)$. We will derive uniform bounds on the time derivatives in $L^2(0,T;H_D^1(\Omega)')$ and $L^2(0,T;H^1(\Omega)')$ as well. To this end, let $\varphi \in L^2(0,T;H_D^1(\Omega))$ such that $\|\varphi\|_{L^2(0,T;H_D^1(\Omega))} = 1$. Using it as a test function in the weak formulation for n_m (where we have already dropped cancelling terms) and applying Hölder's inequality, together with the bounds $S_k^1(z) \le C(k)$, $T_k(z) \le k$ and the estimate on ∇V_m in $L^2(Q_T)$, we compute

$$\begin{aligned}
\left| \int_0^T \int_\Omega \langle \partial_t (n_m - \sigma_m \widetilde{n}), \varphi \rangle \, dt \right| &= \left| \int_0^T \int_\Omega (S_k^1(n_m^*) \nabla n_m - \sigma_m T_k(n_m^*) \nabla V_m) \cdot \nabla \varphi \, dx \, dt \right| \\
&\le \int_0^T \int_\Omega C(k) |\nabla n_m| |\nabla \varphi| + k |\nabla V_m| |\nabla \varphi| \, dx \, dt \\
&\le C(k) \int_0^T \|n_m\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} + \|\nabla V_m\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)} \, dt \\
&\le C(k) \|n_m\|_{L^2(0,T;H^1(\Omega))} + C(k) \|\nabla V_m\|_{L^2(Q_T)} \\
&\le C(k).
\end{aligned}$$

Note that $\sigma_m \widetilde{n}$ is independent of t, hence we obtained a uniform estimate on $\partial_t n_m$ in $L^2(0,T;H_D^1(\Omega)')$. The estimates for the two remaining time derivatives are computed in the same way. Since the computations for $\partial_t p_m$ are identical to the previous computations, we do not write them down and only detail those for $\partial_t D_m$.

Therefore, let $\varphi \in L^2(0,T;H^1(\Omega))$ with $\|\varphi\|_{L^2(0,T;H^1(\Omega))} = 1$. Then for $\partial_t D_m$ we obtain, using that $S_k^2(z) \le C(k)$,

$$\left| \int_0^T \int_\Omega \langle \partial_t D_m, \varphi \rangle \, dt \right| \le C(k) \|D_m\|_{L^2(0,T;H^1(\Omega))} + C(k) \|\nabla V_m\|_{L^2(Q_T)} \le C(k).$$

This shows that the time derivatives $\partial_t n_m, \partial_t p_m$ are bounded in $L^2(0,T;H_D^1(\Omega)')$ and $\partial_t D_m$ is bounded in $L^2(0,T;H^1(\Omega)')$, uniformly in $(n_m^*, p_m^*, D_m^*, \sigma_m)$. Due to Aubin-Lions lemma [9, 93] there exists a subsequence $(n_{m'}, p_{m'}, D_{m'})$ and $\zeta = (\zeta_n, \zeta_p, \zeta_D)$ such that

$$\begin{aligned}
n_{m'} \to \zeta_n, \quad p_{m'} \to \zeta_p, \quad D_{m'} \to \zeta_D, \quad &\text{strongly in } L^2(Q_T), \\
\partial_t n_{m'} \rightharpoonup \partial_t \zeta_n, \quad \partial_t p_{m'} \rightharpoonup \partial_t \zeta_p, \quad &\text{weakly in } L^2(0,T;H_D^1(\Omega)'), \\
\partial_t D_{m'} \rightharpoonup \partial_t \zeta_D, \quad &\text{weakly in } L^2(0,T;H^1(\Omega)').
\end{aligned}$$

These convergences are good enough to take the limit in the weak formulation, which shows that $(\zeta_n, \zeta_p, \zeta_D)$ is a solution to the system for (n^*, p^*, D^*, σ), i.e. $\zeta = F(n^*, p^*, D^*, \sigma)$. Since the solution to the system is unique, it follows that $(n, p, D) = (\zeta_n, \zeta_p, \zeta_D)$, and furthermore we get that the entire sequence (n_m, p_m, D_m) converges to (n, p, D). In conclusion, F is continuous.

Next, we show that all fixed points of $F(.,\sigma)$ are uniformly bounded in $L^2(Q_T)$, independent of σ. For this we recall that, thanks to Grönwall's inequality, we had the bound

$$\|n_m(\tau)\|^2_{L^2(\Omega)} + \|p_m(\tau)\|^2_{L^2(\Omega)} + \|D_m(\tau)\|^2_{L^2(\Omega)} \le C(k)e^{C(k)\tau},$$

where the constant $C(k)$ was independent of the input $(n^*_m, p^*_m, D^*_m, \sigma_m)$. This yields the uniform bound for the fixed points of $F(.,\sigma)$, independent of σ.

It remains to show compactness of F. Again, we will use the already derived uniform bounds to conclude this. To this end, let $(n^*_m, p^*_m, D^*_m, \sigma_m)$ be a bounded sequence in $L^2(Q_T)^3 \times [0,1]$. Again, denote by $(n_m, p_m, D_m) = F(n^*_m, p^*_m, D^*_m, \sigma_m)$. The bounds derived above allow us to conclude that $(n_m, p_m, D_m)_{m\in\mathbb{N}}$ is uniformly bounded in the right spaces to apply Aubin-Lions lemma, thus there exists a subsequence $(n_{m'}, p_{m'}, D_{m'})_{m'\in\mathbb{N}}$, which is strongly converging in $L^2(Q_T)^3$, hence F is compact.

By the Leray-Schauder fixed-point theorem, it follows that $F(.,1)$ has (at least) one fixed point $(n,p,D) \in L^2(Q_T)^3$, which concludes the proof. ☐

To close this subsection, we will show that the obtained fixed point is nonnegative and that we have conservation of mass for D_k. More precisely, the following statement holds.

Lemma 40. *Let the assumptions (A1)-(A4) hold and let (n_k, p_k, D_k) be the fixed point obtained in Lemma 39, i.e. (n_k, p_k, D_k, V_k) is a weak solution to (4.11)-(4.12). Then it holds that the solution is nonnegative, i.e. $n_k, p_k, D_k \ge 0$.*

Proof. We will prove the nonnegativity of n_k. The strategy will be to test with the negative part of the solution and then conclude that it has to be 0. Let $n_k^- = \min(0, n_k)$ and note that we have to use $n_k^- - \widetilde{n}^-$ as a test function in order to respect the Dirichlet boundary condition, but since $\widetilde{n} > 0$ in Ω, integrals involving the negative part of $\widetilde{n}$ vanish. By definition of n_k^-, it holds that $T_k(n_k)\mathbb{1}_{(n_k\le 0)} = 0$. Since $\nabla n_k^- = \mathbb{1}_{(n_k\le 0)}\nabla n_k$, the weak formulation reduces to

$$\int_0^\tau \langle \partial_t n_k, n_k^-\rangle\, dt + \int_0^\tau \int_\Omega S_k^1(n_k)\nabla n_k \cdot \nabla n_k^-\, dx\, dt = 0.$$

We reformulate the first integral and see

$$\begin{aligned}\int_0^\tau \langle \partial_t n_k, n_k^-\rangle\, dt &= \frac{1}{2}\int_0^\tau \frac{d}{dt}\int_\Omega (n_k^-)^2\, dx\, dt \\ &= \frac{1}{2}\int_\Omega (n_k^-)^2(\tau)\, dx - \frac{1}{2}\int_\Omega (n_k^-)^2(0)\, dx \\ &= \frac{1}{2}\int_\Omega (n_k^-)^2(\tau)\, dx,\end{aligned}$$

since $n_k^-(0) = \min(0, n^I)$ and $n^I \ge 0$ by assumption. To compute the second integral, we note that $S_k^1(n_k)\mathbb{1}_{(n_k\le 0)} = 1$, therefore we have

$$\begin{aligned}\int_0^\tau \int_\Omega S_k^1(n_k)\nabla n_k \cdot \nabla n_k^-\, dx\, dt &= \int_0^\tau \int_\Omega S_k^1(n_k)\mathbb{1}_{(n_k\le 0)}\nabla n_k \cdot \nabla n_k\, dx\, dt \\ &= \int_0^\tau \int_\Omega \mathbb{1}_{(n_k\le 0)}\nabla n_k \cdot \nabla n_k\, dx\, dt \\ &= \int_0^\tau \int_\Omega |\nabla n_k^-|^2\, dx\, dt.\end{aligned}$$

Combining our computations we obtain

$$\frac{1}{2}\int_\Omega (n_k^-)^2(\tau)\,dx + \int_0^\tau \int_\Omega |\nabla n_k^-|^2\,dx\,dt = 0,$$

hence $n_k^- = 0$ a.e in Q_T, since both integrands are nonnegative. This shows that $n_k \geq 0$. The computations to show that $p_k \geq 0$ and $D_k \geq 0$ are completely identical, thus we do not write them down.

□

Lemma 41. *Let the assumptions (A1)–(A4) hold and let (n_k, p_k, D_k) be the fixed point obtained in Lemma 39, i.e. (n_k, p_k, V_k) is a weak solution to (4.11)–(4.12). Then it holds that the mass of D_k is conserved for all $\tau \in [0,T]$,*

$$\int_\Omega D_k(\tau)\,dx = \int_\Omega D^I\,dx, \quad \text{for all } \tau \in [0,T]. \tag{4.16}$$

Proof. We use 1 as a test function in the weak formulation for D_k and directly obtain

$$0 = \int_0^\tau \langle \partial_t D_k, 1\rangle\,dt = \int_0^\tau \frac{d}{dt}\int_\Omega D_k\,dx\,dt = \int_\Omega D_k(\tau)\,dx - \int_\Omega D^I\,dx.$$

□

4.2.3 Uniform estimates

The next step is to derive k-uniform estimates. Since the densities are only nonnegative and G, H are singular at 0 we cannot use $G(n_k) - G(\widetilde{n}) - (V_k - \widetilde{V})$, $G(p_k) - G(\widetilde{p}) + V_k - \widetilde{V}$ and $H(D_k) + V_k$ directly as a test function and thus need to regularize. Let us recall the cut-offs

$$T_k(z) = \max(0, \min(k, z)),$$

$$S_k^1(z) = \begin{cases} 1, & z \leq 0, \\ zG'(z), & 0 < z \leq k, , \\ k^{2/3}z^{1/3}G'(z), & k < z, \end{cases} \qquad S_k^2(z) = \begin{cases} 1, & z \leq 0, \\ zH'(z), & 0 < z \leq \frac{k}{k+1}, \\ 1+k, & \frac{k}{k+1} < z. \end{cases}$$

We now define the regularizations via the following desired relations:

$$\begin{aligned}
\sqrt{T_k(n_k)+\delta}\,\nabla g'_{k,\delta}(n_k) &= \nabla \widetilde{g}_{k,\delta}(n_k), \\
\sqrt{T_k(n_k)+\delta}\,\nabla \widetilde{g}_{k,\delta}(n_k) &= S_k^1(n_k)\nabla n_k, \\
\sqrt{T_k(p_k)+\delta}\,\nabla g'_{k,\delta}(p_k) &= \nabla \widetilde{g}_{k,\delta}(p_k), \\
\sqrt{T_k(p_k)+\delta}\,\nabla \widetilde{g}_{k,\delta}(p_k) &= S_k^1(p_k)\nabla p_k, \\
\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla h'_{k,\delta}(D_k) &= \nabla \widetilde{h}_{k,\delta}(D_k), \\
\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla \widetilde{h}_{k,\delta}(D_k) &= S_k^2(D_k)\nabla D_k,
\end{aligned}$$

which yields

$$\begin{aligned} g''_{k,\delta}(s) &= \frac{S^1_k(s)}{T_k(s)+\delta} = \begin{cases} \frac{sG'(s)}{s+\delta}, & 0 < s \le k, \\ \frac{k^{2/3}s^{1/3}G'(s)}{k+\delta}, & k < s, \end{cases} \\ \widetilde{g}'_{k,\delta}(s) &= \sqrt{T_k(s)+\delta}\, g''_{k,\delta}(s) = \begin{cases} \frac{sG'(s)}{\sqrt{s+\delta}}, & 0 < s \le k, \\ \frac{k^{2/3}s^{1/3}G'(s)}{\sqrt{k+\delta}}, & k < s, \end{cases} \\ h''_{k,\delta}(s) &= \frac{S^2_k(s)}{T_{\frac{k}{k+1}}(s)+\delta} = \begin{cases} \frac{1}{1-s}\frac{1}{s+\delta}, & 0 \le s \le \frac{k}{k+1}, \\ \frac{1+k}{\frac{k}{k+1}+\delta}, & \frac{k}{k+1} < s, \end{cases} \\ \widetilde{h}'_{k,\delta}(s) &= \sqrt{T_{\frac{k}{k+1}}(s)+\delta}\, h''_{k,\delta}(s) = \begin{cases} \frac{1}{1-s}\frac{1}{\sqrt{s+\delta}}, & 0 \le s \le \frac{k}{k+1}, \\ \frac{1+k}{\sqrt{\frac{k}{k+1}+\delta}}, & \frac{k}{k+1} < s, \end{cases} \end{aligned} \tag{4.17}$$

with the respective anti-derivatives

$$\begin{aligned} g_{k,\delta}(s) &= \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{S^1_k(z)}{T_k(z)+\delta}\, dz\, dy, \\ \widetilde{g}_{k,\delta}(s) &= \int_0^s \frac{S^1_k(y)}{\sqrt{T_k(y)+\delta}}\, dy, \\ h_{k,\delta}(s) &= \int_{\mathcal{F}_{-1}(0)}^{s} \int_{\mathcal{F}_{-1}(0)}^{y} \frac{S^2_k(z)}{T_{\frac{k}{k+1}}(z)+\delta}\, dz\, dy, \\ \widetilde{h}_{k,\delta}(s) &= \int_0^s \frac{S^2_k(y)}{\sqrt{T_{\frac{k}{k+1}}(y)+\delta}}\, dy. \end{aligned} \tag{4.18}$$

We also define

$$\begin{aligned} G_{k,\delta}(s|\widetilde{s}) &:= g_{k,\delta}(s) - g_{k,\delta}(\widetilde{s}) - g'_{k,\delta}(\widetilde{s})(s-\widetilde{s}), \\ H_{k,\delta}(s) &:= h_{k,\delta}(s) + s\widetilde{V}, \end{aligned} \tag{4.19}$$

and the regularized entropy

$$\begin{aligned} \mathcal{E}_{k,\delta}[n_k,p_k,D_k,V_k](t) &:= \int_\Omega G_{k,\delta}(n_k|\widetilde{n}) + G_{k,\delta}(p_k|\widetilde{p}) + H_{k,\delta}(D_k) + \frac{\lambda^2}{2}|\nabla(V-\widetilde{V})|^2\, dx, \\ \mathcal{E}_{k,\delta}[n_k,p_k,D_k,V_k](0) &:= \mathcal{E}_{k,\delta}[n^I,p^I,D^I,V^I] = \mathcal{E}^I_{k,\delta}, \\ \Lambda_{k,\delta} &:= 2\left(\left\| \nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V}) \right\|^2_{L^\infty(Q_T)} + \left\| \nabla(g'_{k,\delta}(\widetilde{p}) + \widetilde{V}) \right\|^2_{L^\infty(Q_T)} \right). \end{aligned} \tag{4.20}$$

The following estimate holds.

Lemma 42. *There exist constants $C_1, C_2 > 0$ such that for any k and δ satisfying the inequalities $0 < \delta < \mathcal{F}_{1/2}(0) < k$ and $s > 0$ the following estimate holds:*

$$T_k(s)^{5/3} \le C_1 g_{k,\delta}(s) + C_2. \tag{4.21}$$

Proof. Observe that since $g_{k,\delta}$ is nonnegative, if $0 < s \leq \mathcal{F}_{1/2}(0)$ then by assumption $s < k$ and $T_k(s)^{5/3} = s^{\frac{5}{3}} \leq \mathcal{F}_{1/2}(0)^{\frac{5}{3}}$. If $\mathcal{F}_{1/2}(0) < s \leq k$, we can infer from the definition of $g_{k,\delta}$ and from Corollary 74 that

$$
\begin{aligned}
g_{k,\delta}(s) &= \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{zG'(z)}{z+\delta}\,dz\,dy \geq C \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{1+z^{2/3}}{z+\delta}\,dz\,dy \\
&\geq C \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{z^{2/3}}{z+\delta}\,dz\,dy \geq \frac{C}{2} \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} z^{-1/3}\,dz\,dy \\
&= \frac{3C}{4}\left(\frac{3}{5}s^{5/3} - \mathcal{F}_{1/2}(0)^{2/3}s + \frac{2}{5}\mathcal{F}_{1/2}(0)^{5/3}\right),
\end{aligned}
$$

where we used $\delta < \mathcal{F}_{1/2}(0)$ for the second inequality in the second line. This shows the claim for arguments $s \in (\mathcal{F}_{1/2}(0), k]$. Finally if $s > k$, then $T_k(s)^{5/3} = T_k(k)^{5/3}$ and $g_{k,\delta}(k) \leq g_{k,\delta}(s)$, so inequality (4.21) also holds on this interval. □

To take the limit $\delta \to 0$ we now derive δ-uniform bounds.

Lemma 43. *Let the assumptions (A1)–(A4) hold and let (n_k, p_k, D_k, V_k) be a weak solution to (4.11)–(4.12) as obtained in Lemma 39. Then for all $\delta > 0$ and all $0 < \tau < T$ there holds the following regularized energy inequality:*

$$
\begin{aligned}
&\mathcal{E}_{k,\delta}[n_k, p_k, D_k, V_k](t) \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(p_k) + \sqrt{T_k(p_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k|^2\,dx\,dt \leq \mathcal{E}^I_{k,\delta} + C(\mathcal{E}^I_{k,\delta}, \widetilde{V}, \delta, \Lambda_{k,\delta}, T).
\end{aligned}
\tag{4.22}
$$

The constant $C(\mathcal{E}^I_{k,\delta}, \widetilde{V}, \delta, \Lambda_{k,\delta}, T)$ vanishes if $\Lambda_{k,\delta} = 0$ and $\delta = 0$.

Proof. We use $g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V})$, $g'_{k,\delta}(p_k) - g'_{k,\delta}(\widetilde{p}) + V_k - \widetilde{V}$ and $h'_{k,\delta}(D_k) + V_k$ as test functions in the weak formulations and add the equations. For the left-hand side this yields

$$
\begin{aligned}
&\int_0^\tau \langle \partial_t n_k, g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V})\rangle\,dt \\
&\qquad + \int_0^\tau \langle \partial_t p_k, g'_{k,\delta}(p_k) - g'_{k,\delta}(\widetilde{p}) + (V_k - \widetilde{V})\rangle\,dt \\
&\qquad\qquad + \int_0^\tau \langle \partial_t D_k, h'_{k,\delta}(D_k) + V_k\rangle\,dt.
\end{aligned}
$$

We rewrite this expression term by term, use that $\widetilde{n}, \widetilde{p}, \widetilde{D}$ are independent of t, and get

$$\begin{aligned}
\langle \partial_t n_k, g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n})\rangle &= \frac{d}{dt}\int_\Omega (g_{k,\delta}(n_k) - g_{k,\delta}(\widetilde{n})) - g'_{k,\delta}(\widetilde{n})(n_k - \widetilde{n})\,dx \\
\langle \partial_t p_k, g'_{k,\delta}(p_k) - g'_{k,\delta}(\widetilde{p})\rangle &= \frac{d}{dt}\int_\Omega (g_{k,\delta}(p_k) - g_{k,\delta}(\widetilde{p})) - g'_{k,\delta}(\widetilde{p})(p_k - \widetilde{p})\,dx \\
\langle \partial_t D_k, h'_{k,\delta}(D_k) + \widetilde{V}\rangle &= \frac{d}{dt}\int_\Omega h_{k,\delta}(D_k) + D_k\widetilde{V}\,dx, \\
-\langle \partial_t(n_k - p_k - D_k), V_k - \widetilde{V}\rangle &= \frac{\lambda^2}{2}\frac{d}{dt}\int_\Omega |\nabla(V_k - \widetilde{V})|^2\,dx.
\end{aligned}$$

By (4.20), this is exactly

$$\mathcal{E}_{k,\delta}[n_k, p_k, D_k, V_k](\tau) - \mathcal{E}^I_{k,\delta}.$$

Next, we compute and estimate the right-hand sides of the weak formulation. We start with the equation for n_k,

$$\begin{aligned}
&\int_0^\tau \langle \partial_t n_k, g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V})\rangle\,dt \\
&\qquad = -\int_0^\tau \int_\Omega (S^1_k(n_k)\nabla n_k - T_k(n_k)\nabla V_k)\cdot\nabla(g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V}))\,dx\,dt.
\end{aligned}$$

Rewriting the first term of the product using the definition of $\widetilde{g}'_{k,\delta}(n_k)$ yields

$$\begin{aligned}
S^1_k(n_k)\nabla n_k - T_k(n_k)\nabla V_k &= \sqrt{T_k(n_k)+\delta}\left(\frac{S^1_k(n_k)}{\sqrt{T_k(n_k)+\delta}}\nabla n_k - \sqrt{T_k(n_k)+\delta}\,\nabla V_k\right) \\
&\quad + \delta\nabla V_k \\
&= \sqrt{T_k(n_k)+\delta}\,(\nabla\widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k) + \delta\nabla V_k.
\end{aligned}$$

We also rewrite the first part of the test function,

$$\nabla(g'_{k,\delta}(n_k) - V_k) = g''_{k,\delta}(n_k)\nabla n_k - \nabla V_k = \frac{\nabla\widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k}{\sqrt{T_k(n_k)+\delta}}.$$

Plugging these into the integral and further using Young's inequality we obtain

$$
\begin{aligned}
&-\int_0^\tau \int_\Omega (S_k^1(n_k)\nabla n_k - T_k(n_k)\nabla V_k)\cdot \nabla(g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V}))\,dx\,dt \\
&\quad = -\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad - \int_0^\tau \int_\Omega \frac{\delta}{\sqrt{T_k(n_k)+\delta}} \nabla V_k \cdot (\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k)\,dx\,dt \\
&\qquad + \int_0^\tau \int_\Omega \sqrt{T_k(n_k)+\delta}\,(\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k)\cdot \nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})\,dx\,dt \\
&\qquad + \int_0^\tau \int_\Omega \delta \nabla V_k \cdot \nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})\,dx\,dt \\
&\quad \le -\frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad + \int_0^\tau \int_\Omega \frac{\delta^2}{T_k(n_k)+\delta}|\nabla V_k|^2\,dx\,dt + \int_0^\tau \int_\Omega (T_k(n_k)+\delta)|\nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})|^2\,dx\,dt \\
&\qquad + \frac{\delta}{2}\int_0^\tau \int_\Omega |\nabla V_k|^2\,dx\,dt + \frac{\delta}{2}\int_0^\tau \int_\Omega |\nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})|^2\,dx\,dt \\
&\quad \le -\frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad + 2\delta \int_0^\tau \int_\Omega |\nabla V_k|^2\,dx\,dt + 2\|\nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})\|^2_{L^\infty(Q_T)} \int_0^\tau \int_\Omega T_k(n_k) + \delta\,dx\,dt.
\end{aligned}
$$

We estimate the first integral in the very last line by

$$
\begin{aligned}
2\delta \int_0^\tau \int_\Omega |\nabla V_k|^2\,dx\,dt &= 2\delta \int_0^\tau \int_\Omega |\nabla(V_k - \widetilde{V}) + \nabla \widetilde{V}|^2\,dx\,dt \\
&\le 4\delta \int_0^\tau \int_\Omega |\nabla(V_k - \widetilde{V})|^2\,dx\,dt + 4\delta \int_0^\tau \int_\Omega |\nabla \widetilde{V}|^2\,dx\,dt.
\end{aligned}
$$

The term $T_k(n_k)$ in the last integral can be estimated by $G_{k,\delta}(n_k)$, which we will do once we add the estimates for all species n_k, p_k and D_k.

Combining the last estimates, we find that

$$
\begin{aligned}
&-\int_0^\tau \int_\Omega (S_k^1(n_k)\nabla n_k - T_k(n_k)\nabla V_k)\cdot \nabla(g'_{k,\delta}(n_k) - g'_{k,\delta}(\widetilde{n}) - (V_k - \widetilde{V}))\,dx\,dt \\
&\quad \le -\frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad + 4\delta \int_0^\tau \int_\Omega |\nabla(V_k - \widetilde{V})|^2\,dx\,dt + 4\delta \int_0^\tau \int_\Omega |\nabla \widetilde{V}|^2\,dx\,dt \\
&\qquad + 2\|\nabla(g'_{k,\delta}(\widetilde{n}) - \widetilde{V})\|^2_{L^\infty(Q_T)} \int_0^\tau \int_\Omega T_k(n_k) + \delta\,dx\,dt.
\end{aligned}
$$

Note that we did not plug in the estimate for $T_k(n_k)$ yet, since we will use it after finally adding all the inequalities.

Computing and estimating the right-hand side of the weak formulation for p_k is analogue to the computations for n_k, hence we omit the details and only state the final estimate

$$
\begin{aligned}
&-\int_0^\tau\int_\Omega (S_k^1(p_k)\nabla p_k + T_k(p_k)\nabla V_k)\cdot\nabla(g'_{k,\delta}(p_k) - g'_{k,\delta}(\widetilde{p}) + (V_k - \widetilde{V}))\,dx\,dt \\
&\qquad\le -\frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{g}_{k,\delta}(p_k) + \sqrt{T_k(p_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad\quad + 4\delta\int_0^\tau\int_\Omega |\nabla(V_k-\widetilde{V})|^2\,dx\,dt + 4\delta\int_0^\tau\int_\Omega |\nabla\widetilde{V}|^2\,dx\,dt \\
&\qquad\quad + 2\|\nabla(g'_{k,\delta}(\widetilde{p}) + \widetilde{V})\|^2_{L^\infty(Q_T)}\int_0^\tau\int_\Omega T_k(p_k)+\delta\,dx\,dt.
\end{aligned}
$$

And lastly, we estimate the right-hand side of the weak formulation for D_k:

$$
\begin{aligned}
&\int_0^\tau \langle\partial_t D_k, h'_{k,\delta}(D_k) + V_k\rangle\,dt \\
&\qquad\qquad = -\int_0^\tau\int_\Omega (S_k^2(D_k)\nabla D_k + T_{\frac{k}{k+1}}(D_k)\nabla V_k)\cdot\nabla(h'_{k,\delta}(D_k) + V_k)\,dx\,dt.
\end{aligned}
$$

We rewrite the first factor on the right-hand side to

$$
\begin{aligned}
&S_k^2(D_k)\nabla D_k + T_{\frac{k}{k+1}}(D_k)\nabla V_k \\
&\qquad = \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\left(\frac{S_k^2(D_k)}{\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}}\nabla D_k + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k\right) \\
&\qquad\quad - \delta\nabla V_k \\
&\qquad = \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,(\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k) - \delta\nabla V_k,
\end{aligned}
$$

and reformulate the gradient of the test function to

$$
\nabla(h'_{k,\delta}(D_k) + V_k) = h''_{k,\delta}(D_k)\nabla D_k + \nabla V_k = \frac{\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k}{\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}}.
$$

Plugging these computations into the integral and using Young's inequality we obtain

$$
\begin{aligned}
&-\int_0^\tau\int_\Omega (S_k^2(D_k)\nabla D_k + T_{\frac{k}{k+1}}(D_k)\nabla V_k)\cdot\nabla(h'_{k,\delta}(D_k)+V_k)\,dx\,dt \\
&\quad= -\int_0^\tau\int_\Omega |\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad+\int_0^\tau\int_\Omega \frac{\delta}{\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}}\nabla V_k\cdot(\nabla\widetilde{h}_{k,\delta}(D_k)+\sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k)\,dx\,dt \\
&\quad\le -\frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad+\frac{\delta}{2}\int_0^\tau\int_\Omega |\nabla V_k|^2\,dx\,dt \\
&\quad\le -\frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\qquad+\delta\int_0^\tau\int_\Omega |\nabla(V_k-\widetilde{V})|^2\,dx\,dt + \delta\int_0^\tau\int_\Omega|\nabla\widetilde{V}|^2\,dx\,dt.
\end{aligned}
$$

Now, we add all estimates, recall $\Lambda_{k,\delta} = 2(\|\nabla(g'_{k,\delta}(\widetilde{n})-\widetilde{V})\|^2_{L^\infty(Q_T)}+\|\nabla(g'_{k,\delta}(\widetilde{p})+\widetilde{V})\|^2_{L^\infty(Q_T)})$, and obtain

$$
\begin{aligned}
\mathcal{E}_{k,\delta}[n_k,p_k,D_k,V_k](\tau) &+ \frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&+ \frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{g}_{k,\delta}(p_k) + \sqrt{T_k(p_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&+ \frac{1}{2}\int_0^\tau\int_\Omega |\nabla\widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)+\delta}\,\nabla V_k|^2\,dx\,dt \\
&\le \mathcal{E}^I_{k,\delta} + \Lambda_{k,\delta}\int_0^\tau\int_\Omega T_k(n_k)+T_k(p_k)+\delta\,dx\,dt \\
&\quad + 9\delta\int_0^\tau\int_\Omega |\nabla(V_k-\widetilde{V})|^2\,dx\,dt + 9\delta\int_0^\tau\int_\Omega |\nabla\widetilde{V}|^2\,dx\,dt \\
&\le \mathcal{E}^I_{k,\delta} + C(\delta,\widetilde{n},\widetilde{p},\widetilde{V}) + C\int_0^\tau \mathcal{E}_{k,\delta}[n_k,p_k,D_k,V_k]\,dt.
\end{aligned}
$$

By Grönwall's inequality we conclude that

$$
\mathcal{E}_{k,\delta}[n_k,p_k,D_k,V_k](\tau) \le (\mathcal{E}^I_{k,\delta} + C(\delta,\widetilde{n},\widetilde{p},\widetilde{V}))e^{C\tau}.
$$

Using this information in the previous inequality shows (4.22). □

The next step is taking the limit $\delta\to 0$. To this end, we first define the following auxiliary functions:

$$
\begin{aligned}
g_k''(s) &:= \frac{S_k^1(s)}{T_k(s)} = \begin{cases} G'(s), & 0 < s \le k, \\ k^{-1/3} s^{1/3} G'(s), & k < s, \end{cases} \\
\widetilde{g}_k'(s) &:= \sqrt{T_k(s)} g_k''(s) = \begin{cases} \sqrt{s} G'(s), & 0 < s \le k, \\ k^{1/6} s^{1/3} G'(s), & k < s, \end{cases} \\
h_k''(s) &:= \frac{S_k^2(s)}{T_{\frac{k}{k+1}}(s)} = \begin{cases} \frac{1}{1-s}\frac{1}{s}, & 0 < s \le \frac{k}{k+1}, \\ \frac{k+1}{(\frac{k}{k+1})} = (k+1)\frac{k+1}{k}, & \frac{k}{k+1} < s, \end{cases} \\
\widetilde{h}_k'(s) &:= \sqrt{T_{\frac{k}{k+1}}(s)} h_k''(s) = \begin{cases} \frac{1}{1-s}\frac{1}{\sqrt{s}}, & 0 < s \le \frac{k}{k+1}, \\ \frac{k+1}{\sqrt{\frac{k}{k+1}}} = (k+1)\sqrt{\frac{k+1}{k}}, & \frac{k}{k+1} < s. \end{cases}
\end{aligned}
\tag{4.23}
$$

The respective anti-derivatives are

$$
\begin{aligned}
g_k(s) &= \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{S_k^1(z)}{T_k(z)}\, dz\, dy, \\
\widetilde{g}_k(s) &= \int_0^s \frac{S_k^1(y)}{\sqrt{T_k(y)}}\, dy, \\
h_k(s) &= \int_{\mathcal{F}_{-1}(0)}^{s} \int_{\mathcal{F}_{-1}(0)}^{y} \frac{S_k^2(z)}{T_{\frac{k}{k+1}}(z)}\, dz\, dy, \\
\widetilde{h}_k(s) &= \int_0^s \frac{S_k^2(y)}{\sqrt{T_{\frac{k}{k+1}}(y)}}\, dy.
\end{aligned}
\tag{4.24}
$$

Furthermore, we define

$$
\begin{aligned}
G_k(s|\widetilde{s}) &:= g_k(s) - g_k(\widetilde{s}) - g_k'(\widetilde{s})(s - \widetilde{s}), \\
H_k(s) &:= h_k(s) + s\widetilde{V},
\end{aligned}
\tag{4.25}
$$

and the approximate entropy

$$
\begin{aligned}
\mathcal{E}_k[n_k, p_k, D_k, V_k](t) &:= \int_\Omega G_k(n_k|\widetilde{n}) + G_k(p_k|\widetilde{p}) + H_k(D_k) + \frac{\lambda^2}{2}|\nabla(V_k - \widetilde{V})|^2\, dx, \\
\mathcal{E}_k[n_k, p_k, D_k, V_k](0) &:= \mathcal{E}_k[n^I, p^I, D^I, V^I] = \mathcal{E}_k^I, \\
\Lambda_k &:= 2\left(\|\nabla(g_k'(\widetilde{n}) - \widetilde{V})\|^2_{L^\infty(Q_T)} + \|\nabla(g_k'(\widetilde{p}) + \widetilde{V})\|^2_{L^\infty(Q_T)} \right).
\end{aligned}
\tag{4.26}
$$

Now the following estimate holds:

Lemma 44. *Let the assumptions (A1)–(A4) hold and let (n_k, p_k, D_k, V_k) be a weak solution to (4.11)-(4.12). Then there exists a constant $C(\mathcal{E}_k^I, \Lambda_k, T) > 0$ such that for all $0 < \tau < T$ it* *holds that*

$$
\begin{aligned}
&\mathcal{E}_k[n_k, p_k, D_k, V_k](\tau) \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k|^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{g}_k(p_k) + \sqrt{T_k(p_k)}\,\nabla V_k|^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_\Omega |\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k|^2\,dx\,dt \le \mathcal{E}_k^I + C(\mathcal{E}_k^I, \Lambda_k, T).
\end{aligned} \tag{4.27}
$$

The constant $C(\mathcal{E}_k^I, \Lambda_k, T)$ vanishes if $\Lambda_k = 0$. Furthermore, there exists a constant $C > 0$ independent of k, such that the following bounds hold for all $k \in \mathbb{N}$:

$$
\begin{aligned}
\|g_k(n_k)\|_{L^\infty(0,T;L^1(\Omega))} + \|g_k(p_k)\|_{L^\infty(0,T;L^1(\Omega))} + \|h_k(D_k)\|_{L^\infty(0,T;L^1(\Omega))} &\le C, \\
\left\|\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k\right\|_{L^2(Q_T)} + \left\|\nabla \widetilde{g}_k(p_k) + \sqrt{T_k(p_k)}\,\nabla V_k\right\|_{L^2(Q_T)} &\le C, \\
\|\nabla V_k\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k\right\|_{L^2(Q_T)} &\le C.
\end{aligned} \tag{4.28}
$$

Proof. Since the computations for n_k and p_k are analogue, we will only explicitly write them down for n_k. We have to show the following convergences as $\delta \to 0$:

$$
\begin{aligned}
\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k) + \delta}\,\nabla V_k &\rightharpoonup \nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k, &&\text{weakly in } L^2(Q_T), \\
\nabla \widetilde{h}_{k,\delta}(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k) + \delta}\,\nabla V_k &\rightharpoonup \nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k, &&\text{weakly in } L^2(Q_T), \\
g_{k,\delta}(n_k) &\to g_k(n_k), &&\text{strongly in } L^1(Q_T), \\
h_{k,\delta}(D_k) &\to h_k(D_k), &&\text{strongly in } L^1(Q_T).
\end{aligned}
$$

Using the following estimates andconvergences as $\delta \to 0$,

$$
\begin{aligned}
|\sqrt{T_k(n_k) + \delta} - \sqrt{T_k(n_k)}| = \frac{\delta}{|\sqrt{T_k(n_k) + \delta} + \sqrt{T_k(n_k)}|} &< \sqrt{\delta} \to 0, \\
|\sqrt{T_{\frac{k}{k+1}}(D_k) + \delta} - \sqrt{T_{\frac{k}{k+1}}(D_k)}| = \frac{\delta}{|\sqrt{T_{\frac{k}{k+1}}(D_k) + \delta} + \sqrt{T_{\frac{k}{k+1}}(D_k)}|} &< \sqrt{\delta} \to 0, \\
|T_k(n_k) + \delta - T_k(n_k)| = \delta &\to 0, \\
|T_{\frac{k}{k+1}}(D_k) + \delta - T_{\frac{k}{k+1}}(D_k)| = \delta &\to 0,
\end{aligned}
$$

we can conclude by the monotone convergence theorem that

$$
\begin{aligned}
g_{k,\delta}(n_k) &= \int_{\mathcal{F}_{1/2}(0)}^{n_k} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{T_k^1(z)}{T_k(z) + \delta}\,dz\,dy \\
&\to \int_{\mathcal{F}_{1/2}(0)}^{n_k} \int_{\mathcal{F}_{1/2}(0)}^{y} \frac{T_k^1(z)}{T_k(z)}\,dz\,dy = g_k(n_k), \quad \text{a.e. in } Q_T,
\end{aligned}
$$

and similarly this convergence holds for $\widetilde{g}_{k,\delta}(n_k) \to \widetilde{g}_k(n_k)$, $g_{k,\delta}(p_k) \to g_k(p_k)$, $\widetilde{g}_{k,\delta}(p_k) \to \widetilde{g}_k(p_k)$ and $h_{k,\delta}(D_k) \to h_k(D_k)$, $\widetilde{h}_{k,\delta}(D_k) \to \widetilde{h}_k(D_k)$ almost everywhere in Q_T.

Next, we derive uniform in δ bounds on $g_k(n_k), \widetilde{g}_k(n_k)$, $g_k(p_k)$, $\widetilde{g}_k(p_k)$, $h_k(D_k)$, $\widetilde{h}_k(D_k)$, which will allow us to apply the dominated convergence theorem and obtain the desired strong and weak convergences. We start with $\widetilde{g}_{k,\delta}(s)$. Assuming that $s > k$, we have

$$\begin{aligned}\widetilde{g}_k(s) = \int_0^s \widetilde{g}_k'(y)\,dy = \int_0^k \sqrt{y}G'(y)\,dy + \int_k^s k^{1/6}y^{1/3}G'(y)\,dy \\ \leq C\int_0^k y^{1/6} + y^{-1/2}\,dy + Ck^{1/6}\int_k^s 1 + y^{-2/3}\,dy \\ \leq C(k)(s+1),\end{aligned}$$

and the same inequality holds for $s \leq k$, since the constant is allowed to depend on k. Hence, we conclude that

$$\widetilde{g}_{k,\delta}(n_k) \leq \widetilde{g}_k(n_k) \leq C(k)(n_k+1).$$

Next, we estimate $\widetilde{h}_{k,\delta}(s)$. Assuming $s > \frac{k}{k+1}$, we get

$$\widetilde{h}_k(s) = \int_0^s \widetilde{h}_k'(y)\,dy = \int_0^{\frac{k}{k+1}} \frac{1}{1-y}\frac{1}{\sqrt{y}}\,dy + \int_{\frac{k}{k+1}}^s (k+1)\sqrt{\frac{k+1}{k}}\,dy \leq C(k)(s+1).$$

Note that again for $s \leq \frac{k}{k+1}$ this bound trivially holds as well, hence we have

$$\widetilde{h}_{k,\delta}(D_k) \leq \widetilde{h}_k(D_k) \leq C(k)(D_k+1).$$

Next, we give a bound on $g_{k,\delta}(s)$. Assume that $s > k$ and w.l.o.g. $k > 1$, then we can compute

$$\begin{aligned}g_k(s) &= \int_{\mathcal{F}_{1/2}(0)}^k \int_{\mathcal{F}_{1/2}(0)}^y G'(z)\,dz\,dy + \int_k^s \int_{\mathcal{F}_{1/2}(0)}^k G'(z)\,dz\,dy + \int_k^s\int_k^y k^{-1/3}z^{1/3}G'(z)\,dz\,dy \\ &\leq C\int_{\mathcal{F}_{1/2}(0)}^k y^{2/3}\,dy + G(k)(s-k) + Ck^{-1/3}\int_k^s y + 3y^{1/3} - k - 3k^{1/3}\,dy \\ &\leq C(k)(s^2+1),\end{aligned}$$

where we used Lemma 77 to estimate $G(y) \leq Cy^{2/3}$. Again, if $s \leq k$, the above bound holds trivially. This allows us to conclude

$$g_{k,\delta}(n_k) \leq g_k(n_k) \leq C(k)(n_k^2+1).$$

Finally, we estimate $h_{k,\delta}(s)$. Assume that $s > \frac{k}{k+1}$, then we can compute (cf. $\mathcal{F}_{-1}(0) = 1/2$)

$$\begin{aligned}h_k(s) &= \int_{\mathcal{F}_{-1}(0)}^{\frac{k}{k+1}} \int_{\mathcal{F}_{-1}(0)}^y \frac{1}{1-z}\frac{1}{z}\,dz\,dy \\ &\quad + \int_{\frac{k}{k+1}}^s \int_{\mathcal{F}_{-1}(0)}^{\frac{k}{k+1}} \frac{1}{1-z}\frac{1}{z}\,dz\,dy + \int_{\frac{k}{k+1}}^s \int_{\frac{k}{k+1}}^y (k+1)\frac{k+1}{k}\,dz\,dy \\ &\leq C(k) + C(k)(s+1) + C(k)(s^2+1) \leq C(k)(s^2+1).\end{aligned}$$

Again, if $s \le \frac{k}{k+1}$, the above bound holds trivially and we conclude

$$h_{k,\delta}(D_k) \le h_k(D_k) \le C(k)(D_k^2 + 1).$$

Using the derived uniform in δ bounds together with the Q_T-a.e. convergences and the regularities $n_k, p_k, D_k \in L^2(Q_T)$, we can conclude that

$$\begin{aligned}
\widetilde{g}_{k,\delta}(n_k) &\to \widetilde{g}_k(n_k), \quad \text{strongly in } L^2(Q_T),\\
g_{k,\delta}(n_k) &\to g_k(n_k), \quad \text{strongly in } L^1(Q_T),\\
\widetilde{h}_{k,\delta}(D_k) &\to \widetilde{h}_k(D_k), \quad \text{strongly in } L^2(Q_T),\\
h_{k,\delta}(D_k) &\to h_k(D_k), \quad \text{strongly in } L^1(Q_T).
\end{aligned}$$

From (4.22) we infer that there exists a constant $C > 0$ independent of δ such that

$$\begin{aligned}
\left\| \nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k) + \delta}\, \nabla V_k \right\|_{L^2(Q_T)} + \left\| \nabla \widetilde{g}_{k,\delta}(p_k) + \sqrt{T_k(p_k) + \delta}\, \nabla V_k \right\|_{L^2(Q_T)}&\\
+ \left\| \nabla \widetilde{h}_{k,\delta}(D_k) + \sqrt{T_k^3(D_k) + \delta}\, \nabla V_k \right\|_{L^2(Q_T)} &\le C,
\end{aligned}$$

hence there exists a subsequence that converges weakly in $L^2(Q_T)$. By the previous arguments we can identify the weak limit and obtain

$$\begin{aligned}
\nabla \widetilde{g}_{k,\delta}(n_k) - \sqrt{T_k(n_k) + \delta}\, \nabla V_k &\rightharpoonup \nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\, \nabla V_k\\
\nabla \widetilde{g}_{k,\delta}(p_k) + \sqrt{T_k(p_k) + \delta}\, \nabla V_k &\rightharpoonup \nabla \widetilde{g}_k(p_k) + \sqrt{T_k(p_k)}\, \nabla V_k\\
\nabla \widetilde{h}_{k,\delta}(D_k) + \sqrt{T_k^3(D_k) + \delta}\, \nabla V_k &\rightharpoonup \nabla \widetilde{h}_k(D_k) + \sqrt{T_k^3(D_k)}\, \nabla V_k,
\end{aligned}$$

all weakly in $L^2(Q_T)$. Furthermore, thanks to the derived convergences, we can take the limit $\delta \to 0$ in (4.22), which shows (4.27). The uniform bounds in (4.28) are now a direct consequence of (4.27), which finishes the proof. □

In the next lemma we collect some estimates, which are needed to improve the regularity of the solution.

Lemma 45. *For $s > 0$ and $k > 1$ it holds that*

$$\begin{aligned}
G'(s) &\le g_k''(s),\\
s^{5/3} &\lesssim g_k(s) + 1,\\
T_k(s)^{7/6} &\lesssim \widetilde{g}_k(s),\\
\widetilde{g}_k(s)^{10/7} &\lesssim g_k(s) + 1,\\
T_k(s)^{5/3} &\lesssim g_k(s) + 1.
\end{aligned} \tag{4.29}$$

Proof. The first inequality is a direct consequence of the definition of $g_k''(s)$ in (4.23). To show the second estimate, we use the first inequality together with the estimate on G' from Lemma 73 and compute

$$g_k(s) \ge C \int_{\mathcal{F}_{1/2}(0)}^{s} \int_{\mathcal{F}_{1/2}(0)}^{y} z^{-1} + z^{-1/3}\, dz\, dy \ge \widetilde{C}_1 s^{5/3} - \widetilde{C}_2.$$

To show the third estimate, let $0 < s \leq k$. By definition we then have $T_k(s) = s$ and, thanks to Corollary 74, $s^{1/6} \leq Cs^{1/2}G'(s)$. Combining this with the definition of $\widetilde{g}_k'(s)$ we obtain

$$T_k'(s)T_k(s)^{1/6} \leq Cs^{1/2}G'(s) = C\widetilde{g}_k'(s).$$

Integrating this inequality from 0 to s yields

$$T_k(s)^{7/6} \leq C\widetilde{g}_k(s).$$

If $s > k$, the same inequality holds since T_k is constant and $\widetilde{g}_k$ is nondecreasing. To show the fourth estimate, let again $0 < s \leq k$. Then, from the definition of $\widetilde{g}_k(s)$ and by Corollary 74 we have

$$\widetilde{g}_k(s) = \int_0^s y^{1/2}G'(y)\,dy \leq C\int_0^s y^{1/6} + y^{-1/2}\,dy = C\left(\frac{6}{7}s^{7/6} + 2s^{1/2}\right).$$

It directly follows that

$$\widetilde{g}_k(s)^{10/7} \leq C\left(s^{5/3} + 1\right).$$

If $s > k$, we can use this estimate together with Corollary 74 and compute

$$\begin{aligned}\widetilde{g}_k(s) &= \int_0^k y^{1/2}G'(y)\,dy + \int_k^s k^{1/6}y^{1/3}G'(y)\,dy \\ &\leq C\left(k^{7/6} + k^{1/2}\right) + Ck^{1/6}\left(s + s^{1/3}\right) \leq Cs^{7/6}.\end{aligned}$$

Together with the estimate on $s \in (0, k]$ we can conclude that

$$\widetilde{g}_k(s)^{10/7} \leq C\left(s^{5/3} + 1\right) \leq C_1 g_k(s) + C_2, \quad \forall\ s > 0.$$

The last inequality is a direct consequence of the previously proven inequalities. This finishes the proof. □

With the estimates from Lemma 44 and Lemma 45 we can derive the following uniform bounds.

Lemma 46. *Let the assumptions (A1)–(A4) hold and let $(n_k, p_k, D_k, V_k)_{k\in\mathbb{N}}$ be the sequence of solutions to (4.11)–(4.12). Then there exists a constant $C > 0$ independent of $k \in \mathbb{N}$, such* *that*

$$
\begin{aligned}
\left\|\sqrt{T_k(n_k)}\right\|_{L^\infty(0,T;L^{10/3}(\Omega))} + \left\|\sqrt{T_k(p_k)}\right\|_{L^\infty(0,T;L^{10/3}(\Omega))} &\le C,\\
\|\widetilde{g}_k(n_k)\|_{L^\infty(0,T;L^{10/7}(\Omega))} + \|\widetilde{g}_k(p_k)\|_{L^\infty(0,T;L^{10/7}(\Omega))} &\le C,\\
\|\nabla\widetilde{g}_k(n_k)\|_{L^2(0,T;L^{5/4}(\Omega))} + \|\nabla\widetilde{g}_k(p_k)\|_{L^2(0,T;L^{5/4}(\Omega))} &\le C,\\
\left\|\sqrt{T_k(n_k)}\,\nabla V_k\right\|_{L^\infty(0,T;L^{5/4}(\Omega))} + \left\|\sqrt{T_k(p_k)}\,\nabla V_k\right\|_{L^\infty(0,T;L^{5/4}(\Omega))} &\le C,\\
\|n_k\|_{L^\infty(0,T;L^{5/3}(\Omega))} + \|\sqrt{n_k}\|_{L^\infty(0,T;L^{10/3}(\Omega))} &\le C,\\
\|p_k\|_{L^\infty(0,T;L^{5/3}(\Omega))} + \|\sqrt{p_k}\|_{L^\infty(0,T;L^{10/3}(\Omega))} &\le C,\\
\|\nabla n_k\|_{L^2(0,T;L^{5/4}(\Omega))} + \|\nabla p_k\|_{L^2(0,T;L^{5/4}(\Omega))} &\le C,\\
\left\|\nabla\widetilde{h}_k(D_k)\right\|_{L^2(Q_T)} + \left\|\sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k\right\|_{L^\infty(0,T;L^2(\Omega))} &\le C,\\
\|\nabla D_k\|_{L^2(Q_T)} + \left\|\nabla\sqrt{D_k}\right\|_{L^2(Q_T)} &\le C.
\end{aligned}
\tag{4.30}
$$

Proof. All estimates except those for $\nabla n_k, \nabla p_k, \nabla D_k$ and $\nabla\sqrt{D_k}$ follow directly from Lemma 44 and Lemma 45 by simply combining the results. Let us note that in order to obtain the estimates on $\nabla\widetilde{h}_k(D_k)$ as well as $\sqrt{T_{\frac{k}{k+1}}(D_k)}\nabla V_k$ one additionally has to use that $T_{\frac{k}{k+1}}(D_k) < 1$ for all $k \in \mathbb{N}$. To show the bound on ∇n_k and ∇p_k, we write

$$|\nabla\widetilde{g}_k(n_k)| = |\widetilde{g}_k'(n_k)\nabla n_k|.$$

By definition of $\widetilde{g}_k'$, together with Corollary 74, we have

$$\widetilde{g}_k'(s) = \begin{cases} s^{1/2}G'(s) \sim s^{1/6} + s^{-1/2}, & s \in (0,k),\\ k^{1/6}s^{1/3}G'(s) \sim k^{1/6}(1 + s^{-2/3}), & s \ge k, \end{cases}$$

hence there exists a constant $C > 0$ independent of s and k such that

$$\widetilde{g}_k'(s) \ge C.$$

This shows that

$$|\nabla\widetilde{g}_k(n_k)| = |\widetilde{g}_k'(n_k)\nabla n_k| \ge C|\nabla n_k|,$$

which proves the bound for ∇n_k and ∇p_k. In order to show the bound for ∇D_k and $\nabla\sqrt{D_k}$ we remark that $\widetilde{h}_k'(s) > 1$ for all $s > 0$. Moreover, from the definition of $\widetilde{h}_k'$ we also readily see that it holds that $\widetilde{h}_k'(s) > \frac{1}{\sqrt{s}}$ for all $s > 0$. Together, this shows that

$$
\begin{aligned}
|\nabla\widetilde{h}_k(D_k)| &= |\widetilde{h}_k'(D_k)\nabla D_k| > |\nabla D_k|\\
|\nabla\widetilde{h}_k(D_k)| &= |\widetilde{h}_k'(D_k)\nabla D_k| > |\frac{1}{\sqrt{D_k}}\nabla D_k| = 2|\nabla\sqrt{D_k}|,
\end{aligned}
$$

which shows the claim and thus finishes the proof. □

The next step is to improve the spatial regularity of some of the terms by using the Gagliardo-Nirenberg inequality and a bootstrapping argument.

Lemma 47. *Let $d \leq 9$, then there exists a constant $C > 0$ independent of k such that the following holds:*

$$\begin{aligned} \left\|\sqrt{T_k(n_k)}\right\|_{L^2(0,T;L^q(\Omega))} + \left\|\sqrt{T_k(p_k)}\right\|_{L^2(0,T;L^q(\Omega))} &\leq C, \\ \left\|\sqrt{T_k(n_k)}\,\nabla V_k\right\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} + \left\|\sqrt{T_k(p_k)}\,\nabla V_k\right\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} &\leq C, \\ \|\nabla\widetilde{g}_k(n_k)\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} + \|\nabla\widetilde{g}_k(p_k)\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} &\leq C, \\ \|\widetilde{g}_k(n_k)\|_{L^2(0,T;L^{2qd/[(q+2)d-2q]}(\Omega))} + \|\widetilde{g}_k(p_k)\|_{L^2(0,T;L^{2qd/[(q+2)d-2q]}(\Omega))} &\leq C, \\ \|\nabla n_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} + \|\nabla p_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} &\leq C, \end{aligned} \tag{4.31}$$

where the range of the exponent q depends on the dimension d as follows,

$$q \in \begin{cases} [2,\infty], & d = 1, \\ [2,\infty), & d = 2, \\ [2, \frac{8d}{3(d-2)}), & d \in [3,9]. \end{cases} \tag{4.32}$$

Observe that the lower bound on q is due to the condition $2q/(q+2) \geq 1$.

Proof. Let us define $q_0 := 10/3$, which is the regularity exponent of $\sqrt{T_k(n_k)}$ in space as obtained in Lemma 46. With this notation, we can rewrite the spatial exponents from Lemma 46 as follows, where a doublearrow $\longleftrightarrow$ means that the term on the right-hand side is the spatial exponent of the term on the left-hand side:

$$\begin{aligned} \sqrt{T_k(n_k)} &\longleftrightarrow q_0, \\ \sqrt{T_k(n_k)}\nabla V_k &\longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{2}\right)^{-1}, \\ \nabla\widetilde{g}_k(n_k) &\longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{2}\right)^{-1}. \end{aligned}$$

By the Gagliardo-Nirenberg inequality we can use $\nabla\widetilde{g}_k(n_k)$ to improve the regularity of $\widetilde{g}_k(n_k)$. With $\theta = 1$ we obtain the following spatial exponent:

$$\widetilde{g}_k(n_k) \longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{2} - \frac{1}{d}\right)^{-1}.$$

By the estimate from (4.29), we obtain the following improved spatial exponent for $\sqrt{T_k(n_k)}$:

$$\sqrt{T_k(n_k)} \longleftrightarrow \frac{7}{3}\left(\frac{1}{q_0} + \frac{1}{2} - \frac{1}{d}\right)^{-1} =: q_1.$$

This allows us to derive the following recursion for the spatial exponent of $\sqrt{T_k(n_k)}$:

$$\frac{1}{q_{m+1}} = \frac{3}{7}\left(\frac{1}{q_m} + \frac{1}{2} - \frac{1}{d}\right).$$

Solving this recursion for its limit ℓ we get

$$\frac{1}{\ell} = \frac{3}{7}\left(\frac{1}{\ell} + \frac{1}{2} - \frac{1}{d}\right) \Rightarrow \ell = \frac{8d}{3(d-2)}.$$

From $d = 10$ on we would not gain any regularity. Furthermore, since we are using the Gagliardo-Nirenberg inequality, the constants in the bounds might blow up, hence we can only do finitely many steps in the recursion and thus cannot reach the limit ℓ. For the case $d = 1$, however, the exponent $q = \infty$ is due to Sobolev embeddings and no bootstrapping argument is needed. The arguments for p_k are exactly the same, hence we will omit them. This concludes the proof. □

4.2.4 Aubin-Lions lemma & identification of limits

Next, we derive uniform bounds for n_k, p_k, D_k as well as $\partial_t n_k, \partial_t p_k, \partial_t D_k$ in order to apply the Aubin-Lions lemma and extract a converging subsequence.

To this end, let us recall the equations in our system:

$$\begin{aligned}
\partial_t n_k &= \nabla \cdot \left(\sqrt{T_k(n_k)}\,\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\sqrt{T_k(n_k)}\,\nabla V_k\right), \\
\partial_t p_k &= \nabla \cdot \left(\sqrt{T_k(p_k)}\,\nabla \widetilde{g}_k(p_k) + \sqrt{T_k(p_k)}\,\sqrt{T_k(p_k)}\nabla V_k\right), \\
\partial_t D_k &= \nabla \cdot \left(\sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k\right).
\end{aligned}$$

We also note that in the equations for n_k and p_k we will have to use test functions that have a vanishing trace on the Dirichlet boundary Γ_D of Ω.

Lemma 48. *Let the assumptions (A1)–(A4) hold, let $d \le 5$ and let $(n_k, p_k, D_k, V_k)_{k\in\mathbb{N}}$ be the sequence of solutions to (4.11)–(4.12). Then there exists a constant $C > 0$ independent of k such that*

$$\begin{aligned}
\|\partial_t n_k\|_{L^1(0,T;W_D^{1,2q/(q-4)}(\Omega)')} + \|\partial_t p_k\|_{L^1(0,T;W_D^{1,2q/(q-4)}(\Omega)')} + \|\partial_t D_k\|_{L^2(0,T;H^1(\Omega)')} &\le C, \\
\|n_k\|_{L^2(0,T;W^{1,2q/(q+2)}(\Omega))} + \|p_k\|_{L^2(0,T;W^{1,2q/(q+2)}(\Omega))} + \|D_k\|_{L^2(0,T;H^1(\Omega))} &\le C.
\end{aligned} \tag{4.33}$$

Proof. We start with the bounds for D_k and $\partial_t D_k$. Using the Poincaré-Wirtinger inequality, we can estimate

$$\|D_k\|_{L^2(Q_T)} \le \|D_k - D_{k,\Omega}\|_{L^2(Q_T)} + \|D_{k,\Omega}\|_{L^2(Q_T)} \le C_P \|\nabla D_k\|_{L^2(Q_T)} + \|D_{k,\Omega}\|_{L^2(Q_T)},$$

where

$$D_{k,\Omega}(t) = \frac{1}{\mathrm{m}(\Omega)} \int_\Omega D_k(t)\,dx.$$

By Lemma 41 there holds conservation of mass, i.e.

$$\int_\Omega D_k(t)\,dx = \int_\Omega D^I\,dx,$$

and together with the uniform bound on ∇D_k from (4.30) we obtain a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\|D_k\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

To derive a bound for $\partial_t D_k$, we recall that $\sqrt{T_{\frac{k}{k+1}}(D_k)}$ is uniformly bounded in $L^\infty(Q_T)$ and by (4.28) we have that $\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k$ is uniformly bounded in $L^2(Q_T)$. Combining this yields that there exists a constant $C > 0$ independent of k such that

$$\left\| \sqrt{T_{\frac{k}{k+1}}(D_k)}\, \nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\sqrt{T_{\frac{k}{k+1}}(D_k)}\, \nabla V_k \right\|_{L^2(Q_T)} \leq C.$$

This shows that $\partial_t D_k$ is uniformly bounded in $L^2(0,T;H^1(\Omega)')$.

Next, we show the uniform bounds for n_k, p_k and $\nabla n_k, \nabla p_k$. From (4.31) we have that the sequences ∇n_k and ∇p_k are uniformly bounded in $L^2(0,T;L^{2q/(q+2)}(\Omega))$. By the Poincaré inequality we therefore obtain that

$$\begin{aligned}\|n_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} &\leq C\,\|\nabla n_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))}\\ &\quad + C(T)\left(\|\widetilde{n}\|_{L^{2q/(q+2)}(\Omega)} + \|\nabla\widetilde{n}\|_{L^{2q/(q+2)}(\Omega)}\right) \leq C.\end{aligned}$$

To derive the uniform bound on $\partial_t n_k$ and $\partial_t p_k$ we recall that $\sqrt{T_k(n_k)}$ is uniformly bounded in $L^2(0,T;L^q(\Omega))$ and $\nabla \widetilde{g}_k(n_k) + \sqrt{T_k(n_k)}\nabla V_k$ is uniformly bounded in $L^2(0,T;L^{2q/(q+2)}(\Omega))$ thanks to (4.31). Combining these estimates yields the existence of a constant $C > 0$ independent of k such that

$$\left\| \sqrt{T_k(n_k)}\, \nabla \widetilde{g}_k(n_k) + \sqrt{T_k(n_k)}\sqrt{T_k(n_k)}\, \nabla V_k \right\|_{L^1(0,T;L^{2q/(q+4)}(\Omega))} \leq C.$$

This shows that $\partial_t n_k$ and $\partial_t p_k$ are uniformly bounded in $L^1(0,T;W_D^{1,2q/(q-4)}(\Omega)')$, if $q > 4$, which is the case if $d < 6$. □

Having collected all necessary uniform bounds, we can now state the convergence result.

Lemma 49. *Let the assumptions (A1)–(A4) hold and let $(n_k, p_k, D_k)_{k\in\mathbb{N}}$ be the sequence of solutions to (4.11)–(4.12). Then there exists a subsequence (which we have not relabelled) and functions $n, p \in L^2(0,T;L^{2q/(q+2)}(\Omega))$ and $D \in L^2(Q_T)$, such that in the limit $k \to \infty$ and for space dimension $d \leq 5$ the following convergences hold:*

$$\begin{aligned} n_k &\to n, \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)),\\ p_k &\to p, \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)),\\ D_k &\to D, \quad \text{strongly in } L^2(Q_T), \end{aligned} \tag{4.34}$$

where a Lebesgue space with exponent $r-$ means that the statement holds for all spaces $L^s(\Omega)$ with $1 \leq s < r$.

Proof. The uniform bounds from Lemma 48 allow us to apply the Aubin-Lions lemma and thus obtain a subsequence $(n_k, p_k, D_k)_{k\in\mathbb{N}}$, which is not relabelled, together with some functions $n, p \in L^2(0,T; L^{2q/(q+2)}(\Omega))$ and $D \in L^2(Q_T)$, such that

$$\begin{aligned} n_k &\to n, \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)), \\ p_k &\to p, \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)), \\ D_k &\to D, \quad \text{strongly in } L^2(Q_T). \end{aligned}$$

Observe that we have used the following chain of embeddings when applying the Aubin-Lions lemma for n_k and p_k:

$$W_D^{1,2q/(q+2)}(\Omega) \hookrightarrow L^{2q/(q+2)}(\Omega) \subset L^{2q/(q+4)}(\Omega) \subset W_D^{1,2q/(q-4)}(\Omega)'.$$

The improved regularity of the limit and convergences for n_k and p_k in the respective spaces are now a consequence of (4.30).

□

It remains to identify the limits of all terms in the system. We start with the limits in the equations for n_k and p_k.

Lemma 50. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49 and let (n, p, D) be the respective limit. Then it holds that*

$$\begin{aligned} T_k(n_k) &\to n, \quad \textit{strongly in } L^1(0,T;L^{q/2-}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)), \\ T_k(p_k) &\to p, \quad \textit{strongly in } L^1(0,T;L^{q/2-}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)), \\ \sqrt{T_k(n_k)} &\to \sqrt{n}, \quad \textit{strongly in } L^2(0,T;L^{q-}(\Omega)) \cap L^{\infty-}(0,T;L^{10/3-}(\Omega)), \\ \sqrt{T_k(p_k)} &\to \sqrt{p}, \quad \textit{strongly in } L^2(0,T;L^{q-}(\Omega)) \cap L^{\infty-}(0,T;L^{10/3-}(\Omega)). \end{aligned} \tag{4.35}$$

Proof. We start with the convergence of $T_k(n_k)$ and $T_k(p_k)$. For $r > 1$ we compute

$$|n_k - k|\mathbb{1}_{(n_k\geq k)} \leq |n_k|\mathbb{1}_{(n_k\geq k)} \leq n_k\left(\frac{n_k}{k}\right)^{r-1}\mathbb{1}_{(n_k\geq k)} \leq \frac{n_k^r}{k^{r-1}}\mathbb{1}_{(n_k\geq k)}.$$

Using this computation together with the uniform bound of n_k in $L^2(0,T;L^{2q/(q+2)}(\Omega))$, we obtain for the limit $k\to\infty$

$$\begin{aligned} \|T_k(n_k) - n_k\|_{L^1(Q_T)} &= \int_0^T \int_{\{n_k\geq k\}} |k - n_k|\,dx\,dt \\ &\leq \int_0^T \int_\Omega \frac{n_k^r}{k^{r-1}}\,dx\,dt = \frac{\|n_k\|^r_{L^r(Q_T)}}{k^{r-1}} \leq \frac{C}{k^{r-1}} \to 0. \end{aligned}$$

This shows the strong $L^1(Q_T)$-convergence of $T_k(n_k)$ and $T_k(p_k)$ and hence the strong $L^2(Q_T)$-convergence of $\sqrt{T_k(n_k)}$ and $\sqrt{T_k(p_k)}$. From the uniform bounds in (4.30) and (4.31), we can conclude the desired strong convergence of $T_k(n_k)$, $T_k(p_k)$, $\sqrt{T_k(n_k)}$ and $\sqrt{T_k(p_k)}$. This finishes the proof. □

Lemma 51. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49 and let (n, p, D) be the respective limit. Then it holds that*

$$\begin{aligned} \sqrt{T_k(n_k)}\,\nabla V_k &\rightharpoonup \sqrt{n}\,\nabla V, \quad \text{weakly in } L^2(0,T;L^{2q/(q+2)}(\Omega)), \\ \sqrt{T_k(p_k)}\,\nabla V_k &\rightharpoonup \sqrt{p}\,\nabla V, \quad \text{weakly in } L^2(0,T;L^{2q/(q+2)}(\Omega)). \end{aligned} \tag{4.36}$$

Proof. From (4.28) we infer that

$$\nabla V_k \rightharpoonup \nabla V, \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)).$$

Combining this with the strong convergence $\sqrt{T_k(n_k)} \to \sqrt{n}$ in $L^2(0,T;L^{q-}(\Omega))$ we obtain

$$\sqrt{T_k(n_k)}\,\nabla V_k \rightharpoonup \sqrt{n}\,\nabla V, \quad \text{weakly in } L^2(0,T;L^{2q/(q+2)-}(\Omega)),$$

and together with the uniform bound from (4.31) we conclude

$$\sqrt{T_k(n_k)}\,\nabla V_k \rightharpoonup \sqrt{n}\,\nabla V, \quad \text{weakly in } L^2(0,T;L^{2q/(q+2)}(\Omega)).$$

The same holds for $\sqrt{T_k(p_k)}\,\nabla V_k$, which finishes the proof. □

Next, we will show the convergence of $\sqrt{T_k(n_k)}\,\nabla\widetilde{g}_k(n_k)$ and $\sqrt{T_k(p_k)}\,\nabla\widetilde{g}_k(p_k)$. For this we need an intermediate result.

Lemma 52. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49 and let (n, p, D) be the respective limit. Then it holds that*

$$\begin{aligned} \sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) &\to nG'(n), \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)), \\ \sqrt{T_k(p_k)}\,\widetilde{g}_k'(p_k) &\to pG'(p), \quad \text{strongly in } L^2(0,T;L^{2q/(q+2)}(\Omega)) \cap L^{\infty-}(0,T;L^{5/3-}(\Omega)). \end{aligned} \tag{4.37}$$

Proof. We prove the convergence in $L^{\infty-}(0,T;L^{5/3-}(\Omega))$. To this end, let $1 \le r < \infty$ and let $1 \le s < 5/3$. Then we compute

$$\begin{aligned} &\left\|\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) - nG'(n)\right\|_{L^r(0,T;L^s(\Omega))} \\ &\quad \le \left\|\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) - n_kG'(n_k)\right\|_{L^r(0,T;L^s(\Omega))} \\ &\qquad + \left\|n_kG'(n_k) - nG'(n)\right\|_{L^r(0,T;L^s(\Omega))} \\ &\quad = \left(\int_0^T \left(\int_\Omega \left|\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) - n_kG'(n_k)\right|^s dx\right)^{r/s} dt\right)^{1/r} \\ &\qquad + \left(\int_0^T \left(\int_\Omega \left|n_kG'(n_k) - nG'(n)\right|^s dx\right)^{r/s} dt\right)^{1/r}. \end{aligned}$$

Using the definitions of $\widetilde{g}_k'$ and T_k and combining them with the asymptotics from Corollary 74, we can estimate the first integrand (note that on $\{n_k \le k\}$ it vanishes)

$$\begin{aligned}\left|\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) - n_k G'(n_k)\right|^s \mathbb{1}_{(n_k\ge k)} &= \left|n_k^{1/3} G'(n_k)\left(n_k^{2/3} - k^{2/3}\right)\right|^s \mathbb{1}_{(n_k\ge k)}\\ &\le \left|n_k G'(n_k)\right|^s \mathbb{1}_{(n_k\ge k)} \le \left|\frac{n_k^{1+1/3} G'(n_k)}{k^{1/3}}\right|^s \mathbb{1}_{(n_k\ge k)}\\ &\lesssim \left|\frac{n_k + n_k^{1/3}}{k^{1/3}}\right|^s \mathbb{1}_{(n_k\ge k)} \lesssim \left|\frac{n_k}{k^{1/3}}\right|^s \mathbb{1}_{(n_k\ge k)}\\ &= \frac{|n_k|^s}{k^{s/3}} \mathbb{1}_{(n_k\ge k)}.\end{aligned}$$

We use Lemma 78 to estimate the second integrand

$$\left|n_k G'(n_k) - nG'(n)\right|^s = \left|\int_{n_k}^{n} (zG'(z))'\,dz\right|^s \le \left|\int_{n_k}^{n} |(zG'(z))'|\,dz\right|^s \lesssim \left|\int_{n_k}^{n} 1\,dz\right|^s = |n - n_k|^s.$$

We combine both estimates and, using that n_k is uniformly bounded in $L^\infty(0,T;L^{5/3}(\Omega))$ due to (4.30) together with the strong convergence $n_k \to n$ in $L^{\infty-}(0,T;L^{5/3-}(\Omega))$ from (4.34), we obtain for the limit $k \to \infty$

$$\begin{aligned}\left\|\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) - nG'(n)\right\|_{L^r(0,T;L^s(\Omega))} &\le \left(\int_0^T \left(\int_\Omega \frac{|n_k|^s}{k^{s/3}}\,dx\right)^{r/s} dt\right)^{1/r}\\ &\quad + \left(\int_0^T \left(\int_\Omega |n_k - n|^s\,dx\right)^{r/s} dt\right)^{1/r}\\ &= k^{-1/3}\,\|n_k\|_{L^r(0,T;L^s(\Omega))} + \|n_k - n\|_{L^r(0,T;L^s(\Omega))} \to 0.\end{aligned}$$

The strong convergence in $L^2(0,T;L^{2q/(q+2)}(\Omega))$ is proven in exactly the same way. The computations to show the bounds for p_k are identical, hence we skip them. This concludes the proof. □

We will now show that the terms $nG'(n)\nabla n$ and $pG'(p)\nabla p$ are well defined and identify them as the limits of $\sqrt{T_k(n_k)}\,\nabla\widetilde{g}_k(n_k)$ and $\sqrt{T_k(p_k)}\,\nabla\widetilde{g}_k(p_k)$. To do so, we still need to improve on the regularity of $nG'(n)$ and $pG'(p)$.

Lemma 53. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49, let (n, p, D) be the respective limit and let $d \le 4$. Then there hold the convergences*

$$\begin{aligned}\sqrt{T_k(n_k)}\left(\nabla\widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k\right) &\rightharpoonup nG'(n)\nabla n - n\nabla V,\\ \sqrt{T_k(p_k)}\left(\nabla\widetilde{g}_k(p_k) + \sqrt{T_k(p_k)}\,\nabla V_k\right) &\rightharpoonup pG'(p)\nabla p + p\nabla V,\end{aligned} \tag{4.38}$$

both weakly in $L^1(0,T;L^{2q/(q+2)}(\Omega)) \cap L^2(0,T;L^{5/4}(\Omega))$.

Proof. First, we have to improve the regularity of $nG'(n)$. From the definition of $\widetilde{g}_k'$ and by Corollary 74 we directly see that

$$\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) \leq n_k G'(n_k) \lesssim 1 + n_k^{2/3}.$$

Using the Gagliardo-Nirenberg inequality together with the uniform bound on ∇n_k from (4.31), which also holds for ∇n, we obtain

$$\|n_k\|_{L^2(0,T;L^r(\Omega))} \leq C\,\|\nabla n_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} + C\,\|n_k\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} \leq C,$$

where $r = \left(\frac{1}{2} + \frac{1}{q} - \frac{1}{d}\right)^{-1}$. The same estimate also holds for n, which yields

$$\left\|n_k^{2/3}\right\|_{L^3(0,T;L^{3r/2}(\Omega))} + \left\|n^{2/3}\right\|_{L^3(0,T;L^{3r/2}(\Omega))} \leq C.$$

Together with (4.37) we conclude that

$$\sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k) \to nG'(n), \quad \text{strongly in } L^2(0,T;L^{3r/2-}(\Omega)).$$

Combining this with the weak convergence of $\nabla n_k \rightharpoonup \nabla n$ in $L^2(0,T;L^{2q/(q+2)}(\Omega))$ shows that

$$\sqrt{T_k(n_k)}\,\nabla\widetilde{g}_k(n_k) = \sqrt{T_k(n_k)}\,\widetilde{g}_k'(n_k)\nabla n_k \rightharpoonup nG'(n)\nabla n, \quad \text{weakly in } L^1(Q_T).$$

Observe that this requires

$$\frac{1}{q} + \frac{1}{2} + \frac{2}{3}\left(\frac{1}{q} + \frac{1}{2} - \frac{1}{d}\right) < 1,$$

which is the case for $d \leq 4$. From (4.35) and (4.36) we get

$$\sqrt{T_k(n_k)}\sqrt{T_k(n_k)}\,\nabla V_k \rightharpoonup \sqrt{n}\sqrt{n}\,\nabla V, \quad \text{weakly in } L^1(0,T;L^{q/2-}(\Omega)),$$

and we have that

$$\sqrt{T_k(n_k)}\left(\nabla\widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k\right) \rightharpoonup nG'(n)\nabla n - n\nabla V, \quad \text{weakly in } L^1(Q_T).$$

From (4.28), (4.30) and (4.31) it follows that $\sqrt{T_k(n_k)}(\nabla\widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\,\nabla V_k)$ is uniformly bounded in $L^1(0,T;L^{2q/(q+2)}(\Omega)) \cap L^2(0,T;L^{5/4}(\Omega))$, which finishes the proof. □

The next lemma deals with the limit of the terms $\nabla\widetilde{g}_k(n_k), \nabla\widetilde{g}_k(p_k)$ in the energy inequality (4.27).

Lemma 54. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49, let (n, p, D) be the limit and $d \leq 4$. Then there hold the weak convergences*

$$\begin{aligned} \nabla\sqrt{n_k} &\rightharpoonup \nabla\sqrt{n}, \quad \textit{weakly in } L^2(0,T;L^{2q/(q+2)}(\Omega)),\\ \nabla\sqrt{p_k} &\rightharpoonup \nabla\sqrt{p}, \quad \textit{weakly in } L^2(0,T;L^{2q/(q+2)}(\Omega)), \end{aligned} \tag{4.39}$$

as well as

$$\begin{aligned} \nabla\widetilde{g}_k(n_k) &\rightharpoonup 2nG'(n)\nabla\sqrt{n}, \quad \textit{weakly in } L^1(Q_T),\\ \nabla\widetilde{g}_k(p_k) &\rightharpoonup 2pG'(p)\nabla\sqrt{p}, \quad \textit{weakly in } L^1(Q_T), \end{aligned} \tag{4.40}$$

for $k \to \infty$.

Proof. From the definition of $\widetilde{g}_k'(s)$, (4.23), together with Corollary 74 we have that

$$\widetilde{g}_k'(s) \gtrsim s^{-1/2},$$

and from the uniform bounds in (4.31) we readily see that

$$\|\nabla\sqrt{n_k}\|_{L^2(0,T;L^{2q/(q+2)}(\Omega))} \le C.$$

This shows the weak convergence in (4.39). Repeating the computations from the proofs of Lemma 52 and Lemma 53, but with $\sqrt{n_k}$ instead of $\sqrt{T_k(n_k)}$ shows that

$$\sqrt{n_k}\,\widetilde{g}_k'(n_k) \to nG'(n), \quad \text{strongly in } L^2(0,T;L^{3r/2-}(\Omega)).$$

We combine this with the weak convergence of $\nabla\sqrt{n_k} \rightharpoonup \nabla\sqrt{n}$ and obtain

$$\nabla\widetilde{g}_k(n_k) = 2\sqrt{n_k}\,\widetilde{g}_k'(n_k)\nabla\sqrt{n_k} \rightharpoonup 2nG'(n)\nabla\sqrt{n}, \quad \text{weakly in } L^1(Q_T),$$

which finishes the proof. □

It remains to show the convergence of the terms in the equation for D_k. More specifically, we have to prove the convergence of $\sqrt{T_{\frac{k}{k+1}}(D_k)}\,\widetilde{h}_k'(D_k)\nabla D_k$ and of $T_{\frac{k}{k+1}}(D_k)\nabla V_k$ and show that $DH'(D)\nabla D$ is well defined. Showing the convergence of $\sqrt{T_{\frac{k}{k+1}}(D_k)}\,\widetilde{h}_k'(D_k)\nabla D_k$ requires some intermediate results. To this end, let us define

$$\begin{aligned} L(s) &:= -\log(1-s), \\ L_k(s) &:= \begin{cases} -\log(1-s), & 0 \le s \le \frac{k}{k+1}, \\ (k+1)s - k + \log(k+1), & \frac{k}{k+1} < s, \end{cases} \\ L_k'(s) &= \begin{cases} \frac{1}{1-s}, & 0 \le s \le \frac{k}{k+1}, \\ k+1, & \frac{k}{k+1} < s. \end{cases} \end{aligned} \tag{4.41}$$

Observe that the functions L_k are designed in such a way that they are continuous and that it holds that

$$L_k'(s) = \sqrt{T_{\frac{k}{k+1}}(s)}\,\widetilde{h}_k'(s).$$

Furthermore, $L_k(s) \le L_{k+1}(s)$ for all $s \in [0,1)$, L_k is monotonically increasing for all $k \in \mathbb{N}$ and $L_k \to L$ locally uniformly on $[0,1)$, hence $L_k \nearrow L$. We proceed by deriving uniform bounds for $L_k(D_k)$.

Lemma 55. *Let the assumptions (A1)–(A4) hold and let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49. Then there exists a constant $C > 0$ independent of $k \in \mathbb{N}$, such that*

$$\|L_k(D_k)\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{4.42}$$

Proof. We first show the uniform bound on $\nabla L_k(D_k)$. To this end, we note that $s \le \frac{k}{k+1}$ implies $\frac{1}{\sqrt{s}} > 1$ and we have

$$L_k'(s) < \widetilde{h}_k'(s), \quad \text{for } 0 < s \le \frac{k}{k+1}.$$

From $\sqrt{\frac{k+1}{k}} > 1$ it also directly follows that

$$L_k'(s) < \widetilde{h}_k'(s), \quad \text{for } \frac{k}{k+1} < s,$$

and together this shows that

$$|\nabla L_k(D_k)| = \left|L_k'(D_k)\nabla D_k\right| < \left|\widetilde{h}_k'(D_k)\nabla D_k\right| = \left|\nabla \widetilde{h}_k(D_k)\right|.$$

From the uniform bound in (4.30) we deduce

$$\|\nabla L_k(D_k)\|_{L^2(Q_T)} \leq \left\|\nabla \widetilde{h}_k(D_k)\right\|_{L^2(Q_T)} \leq C.$$

To show the uniform bound on $L_k(D_k)$ in $L^2(Q_T)$, we follow the proof of [25, Lemma 4.1]. We define

$$\hat{D} := \frac{1 + D_\Omega^I}{2} > D_\Omega^I,$$

and note that it holds that

$$L_k(D_k) \leq (L_k(D_k) - L_k(\hat{D}))^+ + L_k(\hat{D}),$$

where $(.)^+$ denotes the positive part. Using the elemental inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we estimate

$$\int_0^T \int_\Omega |L_k(D_k)|^2 \, dx \, dt \leq 2\int_0^T \int_\Omega |(L_k(D_k) - L_k(\hat{D}))^+|^2 \, dx \, dt + 2\int_0^T \int_\Omega |L_k(\hat{D})|^2 \, dx \, dt.$$

Since $\hat{D} < 1$ is constant, we can directly estimate the second integral by

$$2\int_0^T \int_\Omega |L_k(\hat{D})|^2 \, dx \, dt \leq 2T \, \mathrm{m}(\Omega) L_k(\hat{D})^2 \leq 2T \, \mathrm{m}(\Omega) \log(1-\hat{D})^2 < +\infty.$$

To estimate the first integral, we define

$$a_k := (L_k(D_k) - L_k(\hat{D}))^+,$$

and show that it is uniformly bounded in $L^2(Q_T)$ with respect to k. Note that for better readability we dropped the dependence on time t, i.e. $a_k = a_k(t)$. With

$$a_{k,\Omega} := \frac{1}{\mathrm{m}(\Omega)} \int_\Omega a_k \, dx,$$

we compute

$$\int_\Omega |a_k - a_{k,\Omega}|^2 \, dx = \int_{\{a_k=0\}} |a_{k,\Omega}|^2 \, dx + \int_{\Omega \setminus \{a_k=0\}} |a_k - a_{k,\Omega}|^2 \, dx \geq \mathrm{m}(\{a_k = 0\}) |a_{k,\Omega}|^2.$$

By the Poincaré-Wirtinger inequality, we obtain

$$\int_\Omega |a_k - a_{k,\Omega}|^2 \, dx \leq C_P^2 \|\nabla a_k\|_{L^2(\Omega)}^2 \leq C_P^2 \|\nabla L_k(D_k)\|_{L^2(\Omega)}^2.$$

If we can derive a lower bound on $\mathrm{m}(\{a_k = 0\})$, the previous two inequalities yield an upper bound on $a_{k,\Omega}$. By the definition of a_k, we have that $a_k = 0$ if and only if $D_k \leq \hat{D}$. Therefore, and by conservation of mass (4.16),

$$\hat{D}\left(\mathrm{m}(\Omega) - \mathrm{m}(\{a_k = 0\})\right) = \int_{\{D_k > \hat{D}\}} \hat{D}\,dx \leq \int_\Omega D_k\,dx = \int_\Omega D^I\,dx = \mathrm{m}(\Omega) D^I_\Omega.$$

Replacing $\hat{D}$ by its definition, we see that

$$\frac{1 + D^I_\Omega}{2}\left(\mathrm{m}(\Omega) - \mathrm{m}(\{a_k = 0\})\right) \leq \mathrm{m}(\Omega) D^I_\Omega,$$

and rearranging terms yields

$$\mathrm{m}(\Omega)\frac{1 - D^I_\Omega}{1 + D^I_\Omega} \leq \mathrm{m}(\{a_k = 0\}).$$

Combining the derived bounds we compute

$$\int_\Omega |a_k|^2\,dx \leq 2\int_\Omega |a_k - a_{k,\Omega}|^2\,dx + 2\int_\Omega |a_{k,\Omega}|^2\,dx \leq C\left\|\nabla L_k(D_k)\right\|^2_{L^2(\Omega)}.$$

Integrating over time, combining the estimates and recalling that $a_k = (L_k(D_k) - L_k(\hat{D}))^+$ this yields

$$\int_0^T\int_\Omega |L_k(D_k)|^2\,dx\,dt \leq C\int_0^T\int_\Omega |\nabla L_k(D_k)|^2\,dx\,dt + C \leq \tilde{C},$$

where $\tilde{C} > 0$ is a constant independent of k. Thus we have shown that $L_k(D_k)$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$, which finishes the proof. □

The uniform bound in Lemma 55 shows that there exists $L^* \in L^2(Q_T)$ such that

$$L_k(D_k) \rightharpoonup L^*, \quad \text{weakly in } L^2(Q_T). \tag{4.43}$$

The next lemmata are concerned with the identification of $L^* = L(D)$, where $D = \lim_{k\in\mathbb{N}} D_k$ is the limit obtained in Lemma 49.

Lemma 56. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49 and (n,p,D) its limit. Then it holds that $L(D) \in L^2(Q_T)$ and in particular $D < 1$ almost everywhere in Q_T.*

Proof. For $k \in \mathbb{N}$ fixed we have that L_k is continuous on $[0,+\infty)$ and monotonically increasing (by definition), hence

$$L_k(D) = \lim_{\ell\to\infty} L_k(D_\ell), \quad \text{at least } Q_T - a.e.$$

Using that $(L_k^2)_{k\in\mathbb{N}}$ is an increasing, convex, nonnegative sequence of functions and applying Fatou's lemma [49, Lemma D.11] we estimate

$$\int_0^T \int_\Omega L_k(D)^2\,dx\,dt = \int_0^T \int_\Omega \lim_{\ell\to\infty} L_k(D_\ell)^2\,dx\,dt = \int_0^T \int_\Omega \liminf_{\ell\to\infty} L_k(D_\ell)^2\,dx\,dt$$
$$\leq \int_0^T \int_\Omega \liminf_{\ell\to\infty} L_\ell(D_\ell)^2\,dx\,dt \leq \liminf_{\ell\to\infty} \int_0^T \int_\Omega L_\ell(D_\ell)^2\,dx\,dt \leq C,$$

where the first inequality is due to $L_k \leq L_{k+1}$ pointwise, the second inequality comes from Fatou's lemma and the constant $C > 0$ does not depend on k (thanks to the uniform bound from Lemma 55). Again using Fatou's lemma, the above inequality allows us to conclude that

$$\int_0^T \int_\Omega L(D)^2\,dx\,dt \leq \liminf_{k\to\infty} \int_0^T \int_\Omega L_k(D)^2\,dx\,dt \leq C < +\infty,$$

hence $L(D) \in L^2(Q_T)$ and $D < 1$ almost everywhere in Q_T. This finishes the proof. □

Next, we define for $0 < \eta < 1$ and for all $k \in \mathbb{N}$

$$\begin{aligned} D_\eta &:= (1-\eta)D + \eta, \\ D_{k,\eta} &:= (1-\eta)D_k + \eta. \end{aligned} \tag{4.44}$$

Observe that $D_{k,\eta} \to D_\eta$ strongly in $L^2(Q_T)$ for all $0 < \eta < 1$.

Lemma 57. *Let the assumptions (A1)–(A4) hold and let (n_k, p_k, D_k) be the convergent subsequence from Lemma 49. Then there exists a constant $C > 0$ that is independent of $k \in \mathbb{N}$ and $0 < \eta < \frac{1}{4}$ such that*

$$\|L_k(D_{k,\eta})\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{4.45}$$

Proof. The proof is analogue to the proof of Lemma 55. □

Let us remark that the uniform bound in Lemma 57 yields for all $0 < \eta < \frac{1}{4}$ the existence of $L^*_\eta \in L^2(Q_T)$ such that

$$L_k(D_{k,\eta}) \rightharpoonup L^*_\eta, \quad \text{weakly in } L^2(Q_T). \tag{4.46}$$

Next, we show some regularity for $L(D_\eta)$.

Lemma 58. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49 and let (n, p, D) be its limit. Then it holds that $L(D_\eta) \in L^2(Q_T)$ for all $0 < \eta < \frac{1}{4}$ and in particular $D_\eta < 1$ almost everywhere in Q_T. Furthermore, there exists a constant $C > 0$ independent of η such that*

$$\|L(D_\eta)\|_{L^2(Q_T)} \leq C. \tag{4.47}$$

Proof. The proof is identical to the proof of Lemma 56. □

Now we are ready to identify $L^* = L(D)$.

Lemma 59 (Minty-type Trick)**.** *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence from Lemma 49 and let (n, p, D) its limit. Then it holds that $L(D) = L^*$ almost everywhere in Q_T and for $k \to \infty$ we have the convergences*

$$\begin{aligned} L_k(D_k) &\rightharpoonup L(D), \quad \text{weakly in } L^2(Q_T), \\ \nabla L_k(D_k) &\rightharpoonup \nabla L(D), \quad \text{weakly in } L^2(Q_T). \end{aligned} \tag{4.48}$$

In particular, this means that $DH'(D)\nabla D \in L^2(Q_T)$ and

$$\sqrt{T_{\frac{k}{k+1}}(D_k)}\, \nabla \widetilde{h}_k(D_k) \rightharpoonup -\nabla \log(1 - D), \quad \text{weakly in } L^2(Q_T). \tag{4.49}$$

Proof. Due to the monotonicity of L_k we have

$$\int_0^T \int_\Omega (L_k(D_\eta) - L_k(D_k))(D_\eta - D_k)\, dx\, dt \geq 0,$$

for all $k \in \mathbb{N}$ and $0 < \eta < \frac{1}{4}$. Now we take the limit $k \to \infty$ by expanding the expression on the left-hand side and inspecting each of the four terms. Then, a combination of monotone convergence, weak convergence of $L_k(D_k) \rightharpoonup L^*$ in $L^2(Q_T)$ and strong convergence of $D_k \to D$ from (4.34) shows that

$$\int_0^T \int_\Omega (L(D_\eta) - L^*)(D_\eta - D)\, dx\, dt \geq 0.$$

Using the definition of $D_\eta = (1 - \eta)D + \eta$, we get that for all $0 < \eta < \frac{1}{4}$

$$\int_0^T \int_\Omega (L(D_\eta) - L^*)(1 - D)\, dx\, dt \geq 0.$$

Taking the limit $\eta \to 0$ (dominated convergence) we get that

$$\int_0^T \int_\Omega (L(D) - L^*)(1 - D)\, dx\, dt \geq 0.$$

Next, we show the inverse inequality. Using the monotonicity of the L_k, i.e. $L_k \leq L_{k+1}$ pointwise, the weak convergence of $L_k(D_k)$ and Fatou's lemma, we estimate

$$\begin{aligned} \int_0^T \int_\Omega (1 - D) L_k(D)\, dx\, dt &= \int_0^T \int_\Omega (1 - D) \lim_{\ell \to \infty} L_k(D_\ell)\, dx\, dt \\ &\leq \int_0^T \int_\Omega (1 - D) \liminf_{\ell \to \infty} L_\ell(D_\ell)\, dx\, dt \\ &\leq \liminf_{\ell \to \infty} \int_0^T \int_\Omega (1 - D) L_\ell(D_\ell)\, dx\, dt \\ &\leq \int_0^T \int_\Omega (1 - D) L^*\, dx\, dt < +\infty. \end{aligned}$$

Again, by dominated convergence, we can take the limit $k \to \infty$ to see

$$\int_0^T \int_\Omega (1-D)L(D)\,dx\,dt \le \int_0^T \int_\Omega (1-D)L^*\,dx\,dt,$$

or equivalently

$$\int_0^T \int_\Omega (L(D)-L^*)(1-D)\,dx\,dt \le 0.$$

Combining these two inequalities, together with the fact that $D < 1$ almost everywhere in Q_T yields the desired identification of L^*,

$$L^* = L(D), \quad \text{almost everywhere in } Q_T.$$

By (4.43) this means

$$L_k(D_k) \rightharpoonup L(D), \quad \text{weakly in } L^2(Q_T),$$
$$\nabla L_k(D_k) \rightharpoonup \nabla L(D), \quad \text{weakly in } L^2(Q_T),$$

and consequently, since

$$\nabla L_k(D_k) = \sqrt{T_{\frac{k}{k+1}}(D_k)}\nabla \widetilde{h}_k(D_k)$$

and

$$DH'(D)\nabla D = -\nabla \log(1-D) = \nabla L(D),$$

we have shown that $DH'(D)\nabla D \in L^2(Q_T)$ and identified the limit

$$\sqrt{T_{\frac{k}{k+1}}(D_k)}\nabla \widetilde{h}_k(D_k) \rightharpoonup -\nabla \log(1-D), \quad \text{weakly in } L^2(Q_T),$$

which finishes the proof. □

It remains to show the convergence of $T_{\frac{k}{k+1}}(D_k)\nabla V_k$ and to identify its limit.

Lemma 60. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence from Lemma 49 and (n,p,D) its limit. Then it holds that*

$$T_{\frac{k}{k+1}}(D_k)\nabla V_k \rightharpoonup D\nabla V, \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)). \tag{4.50}$$

Moreover, there holds the weak convergence

$$\sqrt{T_{\frac{k}{k+1}}(D_k)}\left(\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)}\,\nabla V_k\right) \rightharpoonup -\nabla \log(1-D) + D\nabla V, \quad \textit{weakly in } L^2(Q_T). \tag{4.51}$$

Proof. We recall the definition of $T_{\frac{k}{k+1}}(s)$ and define $T(s)$:

$$T_{\frac{k}{k+1}}(s) = \begin{cases} s, & 0 \le s \le \frac{k}{k+1}, \\ \frac{k}{k+1}, & \frac{k}{k+1} < s \end{cases}$$
$$T(s) := \max(0, \min(s,1)).$$

From the definition of $T(s)$ we readily see that $T_{\frac{k}{k+1}} \to T$ uniformly on $[0, +\infty)$, due to

$$|T(s) - T_k(s)| \le \frac{1}{k+1}, \quad \forall\, s \ge 0.$$

This allows us to compute

$$\begin{aligned}|T_{\frac{k}{k+1}}(D_k) - T(D)| &= |T_{\frac{k}{k+1}}(D_k) - T_{\frac{k}{k+1}}(D) + T_{\frac{k}{k+1}}(D) - T(D)| \\ &\le |T_{\frac{k}{k+1}}(D_k) - T_{\frac{k}{k+1}}(D)| + |T_{\frac{k}{k+1}}(D) - T(D)| \le |D_k - D| + \frac{1}{k+1}.\end{aligned}$$

By the strong convergence $D_k \to D$ in $L^2(Q_T)$ we can conclude that

$$\int_0^T \int_\Omega |T_{\frac{k}{k+1}}(D_k) - T(D)|^2 \, dx \, dt \le 2 \int_0^T \int_\Omega |D_k - D|^2 \, dx \, dt + \frac{2T \, \mathrm{m}(\Omega)}{(k+1)^2} \to 0, \text{ as } k \to \infty.$$

Thanks to Lemma 56 we have that $D < 1$ Q_T-a.e., which implies that $T(D) = D$ almost everywhere in Q_T, and consequently

$$T_{\frac{k}{k+1}}(D_k) \to D, \quad \text{strongly in } L^2(Q_T),$$

and due to the uniform bounds on $T_{\frac{k}{k+1}}(D_k)$ in $L^\infty(Q_T)$, we even have that

$$T_{\frac{k}{k+1}}(D_k) \to D, \quad \text{strongly in } L^r(Q_T), \ \forall\, r < \infty.$$

We combine this with the convergence $\nabla V_k \rightharpoonup \nabla V$ weakly* in $L^\infty(0, T; L^2(\Omega))$, which is due to (4.28), and obtain that

$$T_{\frac{k}{k+1}}(D_k) \nabla V_k \rightharpoonup D \nabla V, \quad \text{weakly in } L^r(0, T; L^{2r/(r+2)}(\Omega)), \ \forall\, r < \infty.$$

Thanks to the uniform bounds, we even get that

$$T_{\frac{k}{k+1}}(D_k) \nabla V_k \rightharpoonup D \nabla V, \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)).$$

Combining this convergence with (4.49) yields the desired result

$$\sqrt{T_{\frac{k}{k+1}}(D_k)} \left(\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)} \, \nabla V_k \right) \rightharpoonup -\nabla \log(1 - D) + D \nabla V, \quad \text{weakly in } L^2(Q_T),$$

which finishes the proof. □

The next lemma is concerned with the identification of the limit of $\nabla \widetilde{h}_k(D_k)$ in the energy inequality (4.27).

Lemma 61. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence obtained in Lemma 49, let (n, p, D) be the limit and $d \le 4$. Then there holds the weak convergence*

$$\nabla \widetilde{h}_k(D_k) \rightharpoonup 2 \nabla \tanh^{-1}(\sqrt{D}), \quad \textit{weakly in } L^2(Q_T). \tag{4.52}$$

Proof. We will follow the proofs of Lemmas 55, 56, 57, 58 and 59. Since the computations are almost identical, we will only sketch the parts where they differ. We define $\widetilde{h}$ and recall the definition of $\widetilde{h}_k$:

$$\widetilde{h}(s) := 2\tanh^{-1}(\sqrt{s}),$$
$$\widetilde{h}_k(s) = \begin{cases} 2\tanh^{-1}(s), & 0 \le s \le \frac{k}{k+1}, \\ (k+1)\sqrt{\frac{k+1}{k}}s - \sqrt{k(k+1)} + 2\tanh^{-1}(\sqrt{k/(k+1)}), & \frac{k}{k+1} < s. \end{cases} \tag{4.53}$$

From (4.30) we obtain that $\|\nabla\widetilde{h}_k(D_k)\|_{L^2(Q_T)} \le C$ and repeating the proof of Lemma 55 shows that $\|\widetilde{h}_k(D_k)\|_{L^2(Q_T)} \le C$. Together, this shows that $\|\widetilde{h}_k(D_k)\|_{L^2(0,T;H^1(\Omega))} \le C$ and following the proof of Lemma 56 shows $\widetilde{h}(D) \in L^2(Q_T)$.

Defining

$$D_\eta := (1-\eta)D + \eta,$$
$$D_{k,\eta} := (1-\eta)D_k + \eta,$$

we follow the proof of Lemma 57 to obtain a uniform bound on $\nabla\widetilde{h}_k(D_{k,\eta})$. A straightforward computation shows that

$$\nabla\widetilde{h}_k(D_{k,\eta}) = (1-\eta)\widetilde{h}_k'(D_{k,\eta})\nabla D_k = \begin{cases} \frac{1-\eta}{(1-D_{k,\eta})\sqrt{D_{k,\eta}}}\nabla D_k, & D_{k,\eta} \le \frac{k}{k+1}, \\ (1-\eta)(k+1)\sqrt{\frac{k+1}{k}}\nabla D_k, & \frac{k}{k+1} < D_{k,\eta}. \end{cases}$$

Observe that $D_k \le D_{k,\eta}$ if $D_{k,\eta} \le 1$ and therefore $D_{k,\eta} \le k/(k+1)$ implies $1/\sqrt{D_{k,\eta}} \le 1/\sqrt{D_k}$. Using the definition of $D_{k,\eta}$ and $\sqrt{(k+1)/k} \le \sqrt{2}$ shows

$$|\nabla\widetilde{h}_k(D_{k,\eta})| \begin{cases} \le |\nabla\widetilde{h}_k(D_k)|, & D_{k,\eta} \le \frac{k}{k+1}, \\ < \sqrt{2}|\nabla L_k(D_{k,\eta})|, & \frac{k}{k+1} < D_{k,\eta}. \end{cases}$$

This allows us to conclude that $\|\nabla\widetilde{h}_k(D_{k,\eta})\|_{L^2(Q_T)} \le C$. Following the proof of Lemma 57 we can show that $\|\widetilde{h}_k(D_{k,\eta})\|_{L^2(Q_T)} \le C$. Together, this proves that $\|\widetilde{h}_k(D_{k,\eta})\|_{L^2(0,T;H^1(\Omega))} \le C$, and as in Lemma 58 we get that $\|\widetilde{h}(D_\eta)\|_{L^2(Q_T)} \le C$. Lastly, proceeding as in the proof of Lemma 59 (Minty-type Trick) shows the weak convergences

$$\widetilde{h}_k(D_k) \rightharpoonup 2\tanh^{-1}(\sqrt{D}), \quad \text{weakly in } L^2(Q_T),$$
$$\nabla\widetilde{h}_k(D_k) \rightharpoonup 2\nabla\tanh^{-1}(\sqrt{D}), \quad \text{weakly in } L^2(Q_T),$$

which finishes the proof. □

We are now ready to identify the limits of $\partial_t n_k, \partial_t p_k, \partial_t D_k$. This is done in the following result.

Lemma 62. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence from Lemma 49 and (n, p, D) its limit. Then it holds for $k \to \infty$ that*

$$\begin{aligned} \partial_t n_k &\rightharpoonup \partial_t n, \quad \text{weakly in } L^1(0,T;W_D^{1,2q/(q-2)}(\Omega)') \cap L^2(0,T;W_D^{1,5}(\Omega)'), \\ \partial_t p_k &\rightharpoonup \partial_t p, \quad \text{weakly in } L^1(0,T;W_D^{1,2q/(q-2)}(\Omega)') \cap L^2(0,T;W_D^{1,5}(\Omega)'), \\ \partial_t D_k &\rightharpoonup \partial_t D, \quad \text{weakly in } L^2(0,T;H^1(\Omega)'). \end{aligned} \tag{4.54}$$

Proof. We start by proving the convergence of $\partial_t n_k$ and $\partial_t p_k$. The arguments are exactly the same, so we only show this for $\partial_t n_k$. Due to the convergence in (4.38) we have that for all test functions $\varphi \in L^\infty(0,T;W_D^{1,2q/(q-2)}(\Omega)) \cup L^2(0,T;W_D^{1,5}(\Omega))$ it holds that

$$\int_0^T \langle \partial_t n_k, \varphi \rangle \, dt = -\int_0^T \int_\Omega \sqrt{T_k(n_k)} \left(\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)} \, \nabla V_k \right) \cdot \nabla \varphi \, dx \, dt$$
$$\to -\int_0^T \int_\Omega \left(nG'(n) \nabla n - n \nabla V \right) \cdot \nabla \varphi \, dx \, dt.$$

Since $W_D^{1,2q/(q-2)}(\Omega)$ and $W_D^{1,5}(\Omega)$ are reflexive spaces, so are their dual spaces and thus we can apply [34, Lemma 3.2] to conclude that there holds the convergence of $\partial_t n_k \rightharpoonup \xi$ weakly in $L^1(0,T;W_D^{1,2q/(q-2)}(\Omega)') \cap L^2(0,T;W_D^{1,5}(\Omega)')$. The strong convergence of $n_k \to n$ in at leat $L^1(Q_T)$ allows us to identify $\xi = \partial_t n$ and we obtain that $\partial_t n_k \rightharpoonup \partial_t n$ converges weakly in $L^1(0,T;W_D^{1,2q/(q-2)}(\Omega)') \cap L^2(0,T;W_D^{1,5}(\Omega)')$. Taking the limit $k \to \infty$ in the weak formulation

$$\int_0^T \langle \partial_t n_k, \varphi \rangle \, dt + \int_0^T \int_\Omega \sqrt{T_k(n_k)} \left(\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)} \, \nabla V_k \right) \cdot \nabla \varphi \, dx \, dt = 0,$$

now leads to

$$\int_0^T \langle \partial_t n, \varphi \rangle \, dt + \int_0^T \int_\Omega \left(nG'(n) \nabla n - n \nabla V \right) \cdot \nabla \varphi \, dx \, dt = 0,$$

for all $\varphi \in L^\infty(0,T;W_D^{1,2q/(q-2)}(\Omega)) \cup L^2(0,T;W_D^{1,5}(\Omega))$. The convergence of $\partial_t D_k$ is essentially the same. Thanks to (4.49) and Lemma 60, we have that for all $\varphi \in L^2(0,T;H^1(\Omega))$ it holds that

$$\int_0^T \langle \partial_t D_k, \varphi \rangle \, dt = -\int_0^T \int_\Omega \sqrt{T_{\frac{k}{k+1}}(D_k)} \left(\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)} \, \nabla V_k \right) \cdot \nabla \varphi \, dx \, dt$$
$$\to -\int_0^T \int_\Omega (-\nabla \log(1-D) + D \nabla V) \cdot \nabla \varphi \, dx \, dt.$$

Since $H^1(\Omega)$ is reflexive, we obtain that $\partial_t D_k \rightharpoonup \zeta$ weakly in $L^2(0,T;H^1(\Omega)')$. The strong convergence $D_k \to D$ in $L^2(Q_T)$ now allows us to identify $\zeta = \partial_t D$ and we see that $\partial_t D_k \rightharpoonup \partial_t D$ weakly in $L^2(0,T;H^1(\Omega)')$. Again taking the limit in the weak formulation

$$\int_0^T \langle \partial_t D_k, \varphi \rangle \, dx \, dt + \int_0^T \int_\Omega \sqrt{T_{\frac{k}{k+1}}(D_k)} \left(\nabla \widetilde{h}_k(D_k) + \sqrt{T_{\frac{k}{k+1}}(D_k)} \, \nabla V_k \right) \cdot \nabla \varphi \, dx \, dt = 0,$$

now leads to

$$\int_0^T \langle \partial_t D, \varphi \rangle \, dx \, dt + \int_0^T \int_\Omega (-\nabla \log(1-D) + D \nabla V) \cdot \nabla \varphi \, dx \, dt = 0,$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$. This shows (4.54) and finishes the proof. □

It remains to show in which sense the intial data and the boundary conditions are fulfilled.

Lemma 63. *Let the assumptions (A1)–(A4) hold, let (n_k, p_k, D_k) be the convergent subsequence from Lemma 49 and (n, p, D) its limit. Then, for $t > 0$ and in the sense of the trace operator* $\mathrm{tr}\colon W^{1,2q/(q+2)}(\Omega) \to L^{2q/(q+2)}(\partial\Omega)$*, it holds that $n = \widetilde{n}$ and $p = \widetilde{p}$ on Γ_D for $t > 0$. The initial data are fulfilled in the sense*

$$\begin{aligned} n(t,.) \to n^I,\ p(t,.) \to p^I, &\quad \textit{strongly in } W_D^{1,2q/(q-2)}(\Omega)' \cap W_D^{1,5}(\Omega)', \\ D(t,.) \to D^I, &\quad \textit{strongly in } H^1(\Omega)', \end{aligned} \tag{4.55}$$

as $t \to 0$.

Proof. Since $n, p \in L^2(0,T; W^{1,2q/(q+2)}(\Omega))$ we directly see that $n = \widetilde{n}$ and $p = \widetilde{p}$ on Γ_D holds for $t > 0$ in the claimed sense. The regularity of $n, p, \partial_t n, \partial_t p$ shows that

$$\begin{aligned} n, p \in W^{1,1}(0,T; W_D^{1,2q/(q-2)}(\Omega)') &\hookrightarrow C([0,T]; W_D^{1,2q/(q-2)}(\Omega)'), \\ n, p \in W^{1,2}(0,T; W_D^{1,5}(\Omega)') &\hookrightarrow C([0,T]; \cap W_D^{1,5}(\Omega)'). \end{aligned}$$

Hence, we have that

$$n(t,.) \to n^I,\ p(t,.) \to p^I \quad \text{strongly in } W_D^{1,2q/(q-2)}(\Omega)' \cap W_D^{1,5}(\Omega)',\ \text{as } t \to 0.$$

In the same way one shows that

$$D(t,.) \to D^I, \quad \text{strongly in } H^1(\Omega)',\ \text{as } t \to 0.$$

This finishes the proof and also concludes the proof of Theorem 35. □

4.3 Proof of Theorem 36

In this section we prove the boundedness of weak solutions to (1.25)–(1.26) & (4.4). First, we improve the regularity on the level of the approximate system by redoing the bootstrapping argument from the proof of Lemma 47. This will allow us, after passing to the limit $k \to \infty$, to use the solution $n - \widetilde{n}$ and $p - \widetilde{p}$ as a test function in the original system and following up with another bootstrapping argument we prove bounds for n and p in $L^\infty(0,T; L^q(\Omega))$ for all $1 \le q < \infty$. These bounds will depend on q, so as a final step we will then do an Alikakos-type iteration to establish the second statement of Theorem 36.

Since the computations in this section are identical for n and p, we will formulate all statements for both n and p, but do the computations only for n.

4.3.1 Improved regularity of solutions to the approximate system (4.11)–(4.12)

We start by improving the regularity of the uniform bounds for the solutions n_k and p_k to the approximate system.

Lemma 64. *Let the assumptions (A1)–(A5) hold with $r = 3$, let $d = 3$ and (n_k, p_k, D_k, V_k) be the solution to (4.11)–(4.12). Then there exists a constant $C > 0$ independent of $k \in \mathbb{N}$ such* *that the following improved regularity holds:*

$$
\begin{aligned}
\left\|\sqrt{T_k(n_k)}\right\|_{L^{14/3}(0,T;L^{14}(\Omega))} + \left\|\sqrt{T_k(p_k)}\right\|_{L^{14/3}(0,T;L^{14}(\Omega))} &\leq C,\\
\|T_k(n_k)\|_{L^{7/3}(0,T;L^7(\Omega))} + \|T_k(p_k)\|_{L^{7/3}(0,T;L^7(\Omega))} &\leq C,\\
\|\nabla \widetilde{g}_k(n_k)\|_{L^2(Q_T)} + \|\nabla \widetilde{g}_k(p_k)\|_{L^2(Q_T)} &\leq C,\\
\|\widetilde{g}_k(n_k)\|_{L^2(0,T;L^6(\Omega))} + \|\widetilde{g}_k(p_k)\|_{L^2(0,T;L^6(\Omega))} &\leq C,\\
\|\nabla n_k\|_{L^2(Q_T)} + \|\nabla p_k\|_{L^2(Q_T)} &\leq C,\\
\|n_k\|_{L^2(0,T;L^6(\Omega))} + \|p_k\|_{L^2(0,T;L^6(\Omega))} &\leq C.
\end{aligned}
\tag{4.56}
$$

Proof. Let us recall the uniform bound from (4.30),

$$\|n_k\|_{L^\infty(0,T;L^{5/3}(\Omega))} \leq C,$$

which by assumption (A5) allows us to directly conclude that

$$\|V_k\|_{L^\infty(0,T;W^{1,3}(\Omega))} \leq C, \quad \text{for all } k \in \mathbb{N}.$$

With this improved regularity we redo the bootstrapping from the proof of Lemma 47. For the convenience of the reader, let us also recall that

$$\nabla \widetilde{g}_k(n_k) = \nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\nabla V_k + \sqrt{T_k(n_k)}\nabla V_k,$$

and thanks to (4.27) and (4.30) we have the bounds

$$
\begin{aligned}
\left\|\nabla \widetilde{g}_k(n_k) - \sqrt{T_k(n_k)}\nabla V_k\right\|_{L^2(Q_T)} + \left\|\sqrt{T_k(n_k)}\nabla V_k\right\|_{L^\infty(0,T;L^{5/4}(\Omega))} &\leq C,\\
\left\|\sqrt{T_k(n_k)}\right\|_{L^\infty(0,T;L^{10/3}(\Omega))} &\leq C.
\end{aligned}
$$

Denoting the spatial regularity exponent of $\sqrt{T_k(n_k)}$ by $q_0 := 10/3$, we obtain the following spatial exponents:

$$
\begin{aligned}
\sqrt{T_k(n_k)} &\longleftrightarrow q_0,\\
\sqrt{T_k(n_k)}\nabla V_k &\longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{3}\right)^{-1},\\
\nabla \widetilde{g}_k(n_k) &\longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{3}\right)^{-1}.
\end{aligned}
$$

Using the Gagliardo-Nirenberg inequality with $\theta = 1$, we can improve the spatial regularity of $\widetilde{g}_k(n_k)$ to

$$\widetilde{g}_k(n_k) \longleftrightarrow \left(\frac{1}{q_0} + \frac{1}{3} - \frac{1}{3}\right)^{-1} = q_0$$

and by (4.29) we obtain as improved spatial exponent for $\sqrt{T_k(n_k)}$

$$\sqrt{T_k(n_k)} \longleftrightarrow \frac{7}{3}q_0 = 70/9 =: q_1.$$

Iterating this argument would yield that $q_k = (7/3)^k q_0$ and consequently $\sqrt{T_k(n_k)} \in L^q(\Omega)$ for all $1 \le q < \infty$. However, we cannot improve the spatial regularity of $\nabla \widetilde{g}_k(n_k)$ beyond $L^2(\Omega)$. Therefore, plugging in $q = 6$ as spatial regularity for $\sqrt{T_k(n_k)}$ we obtain

$$\frac{1}{6} + \frac{1}{3} = \frac{1}{2} \qquad \Longrightarrow \|\nabla \widetilde{g}_k(n_k)\|_{L^2(Q_T)} \le C.$$

Consequently,

$$\|\widetilde{g}_k(n_k)\|_{L^2(0,T;L^6(\Omega))} \le C,$$

and by (4.29) we conclude that

$$\left\| T_k(n_k)^{7/6} \right\|_{L^6(\Omega)}^2 \le C \|\widetilde{g}_k(n_k)\|_{L^6(\Omega)}^2 .$$

Thanks to

$$\left\| T_k(n_k)^{7/6} \right\|_{L^6(\Omega)}^2 = \|T_k(n_k)\|_{L^7(\Omega)}^{7/3}, \qquad \left\| \sqrt{T_k(n_k)}^{7/3} \right\|_{L^6(\Omega)}^2 = \left\| \sqrt{T_k(n_k)} \right\|_{L^{14}(\Omega)}^{14/3},$$

we obtain

$$\left\| \sqrt{T_k(n_k)} \right\|_{L^{14/3}(0,T;L^{14}(\Omega))} + \|T_k(n_k)\|_{L^{7/3}(0,T;L^7(\Omega))} \le C.$$

Lastly, as $|\nabla \widetilde{g}_k(n_k)| \ge C |\nabla n_k|$ we get

$$\|\nabla n_k\|_{L^2(Q_T)} \le C,$$

and again by the Gagliardo-Nirenberg inequality

$$\|n_k\|_{L^2(0,T;L^6(\Omega))} \le C,$$

which finishes the proof. □

As a direct result, we obtain the following improved regularity of the solution n and p.

Corollary 65. *Let the assumptions (A1)–(A5) hold with $r = 3$, let $d = 3$ and (n, p, D, V) be the obtained weak solution to (1.25)–(1.26) & (4.4). Then it holds that*

$$\begin{aligned} n, p &\in L^{7/3}(0,T; L^7(\Omega)), \\ \nabla n, \nabla p &\in L^2(Q_T), \\ V &\in L^\infty(0,T; W^{1,3}(\Omega)). \end{aligned} \tag{4.57}$$

4.3.2 Proving bounds in $L^\infty(0,T;L^q(\Omega))$ for $1 \le q < \infty$

Having obtained enough regularity to use $n - \widetilde{n}$ as a test function in the weak formulation, we will prove the first part of Theorem 36 in this subsection, i.e. we will show (4.8). The crucial part will be estimating the terms $n \nabla V \cdot \nabla n$, which requires the improved regularity of n and p. We start with an auxiliary result.

Lemma 66. *Let the assumptions from Theorem 36 hold with $r = 3$. Then it holds that*

$$\begin{aligned} &n, p \in L^\infty(0,T;L^2(\Omega)),\\ &\nabla n^{4/3}, \nabla p^{4/3} \in L^2(Q_T),\\ &n, p \in L^{8/3}(0,T;L^8(\Omega)). \end{aligned} \tag{4.58}$$

Proof. We use $n - \widetilde{n}$ as a test function in the weak formulation and obtain

$$\int_0^T \langle \partial_t n, n - \widetilde{n}\rangle\,dt + \int_0^T \int_\Omega nG'(n)\nabla n \cdot \nabla(n - \widetilde{n})\,dx\,dt = \int_0^T \int_\Omega n\nabla V \cdot \nabla(n-\widetilde{n})\,dx\,dt.$$

Since $\widetilde{n}$ is independent of time t, we can rewrite the first integral as

$$\int_0^T \langle \partial_t n, n - \widetilde{n}\rangle\,dt = \int_0^T \frac{1}{2}\frac{d}{dt}\|n - \widetilde{n}\|^2_{L^2(\Omega)}\,dt = \frac{1}{2}\|n(T) - \widetilde{n}\|^2_{L^2(\Omega)} - \frac{1}{2}\|n^I - \widetilde{n}\|^2_{L^2(\Omega)}.$$

We split the second integral into two parts

$$\int_0^T \int_\Omega nG'(n)\nabla n \cdot \nabla(n - \widetilde{n})\,dx\,dt = I_1 - I_2,$$

with

$$\begin{aligned} I_1 &:= \int_0^T \int_\Omega nG'(n)\nabla n \cdot \nabla n\,dx\,dt,\\ I_2 &:= \int_0^T \int_\Omega nG'(n)\nabla n \cdot \nabla \widetilde{n}\,dx\,dt. \end{aligned}$$

Using $nG'(n) \sim 1 + n^{2/3}$ from Corollary 74, we can estimate as follows:

$$\begin{aligned} I_1 &\geq C\int_0^T \int_\Omega n^{2/3}|\nabla n|^2\,dx\,dt = C\left(\frac{3}{4}\right)^2 \int_0^T \int_\Omega |\nabla n^{4/3}|^2\,dx\,dt,\\ I_2 &\leq C\int_0^T \int_\Omega (1 + n^{2/3})|\nabla n||\nabla \widetilde{n}|\,dx\,dt\\ &\leq C\|\nabla n\|_{L^2(Q_T)}\|\nabla \widetilde{n}\|_{L^2(Q_T)} + C\|n^{2/3}\|_{L^{7/2}(0,T;L^{21/2}(\Omega))}\|\nabla n\|_{L^2(Q_T)}\|\nabla \widetilde{n}\|_{L^\infty(Q_T)} < +\infty. \end{aligned}$$

The last integral is estimated as

$$\int_0^T \int_\Omega n\nabla V \cdot \nabla(n - \widetilde{n})\,dx\,dt \leq C\|n\|_{L^{7/3}(0,T;L^7(\Omega))}\|\nabla V\|_{L^\infty(0,T;L^3(\Omega))}\|\nabla(n-\widetilde{n})\|_{L^2(Q_T)} < +\infty.$$

Putting the estimates together shows that

$$\begin{aligned} \frac{1}{2}\|n(T) - \widetilde{n}\|^2_{L^2(\Omega)} &+ C\left(\frac{3}{4}\right)^2 \int_0^T \int_\Omega |\nabla n^{4/3}|^2\,dx\,dt\\ &\leq \frac{1}{2}\|n^I - \widetilde{n}\|^2_{L^2(\Omega)} + C\|\nabla n\|_{L^2(Q_T)}\|\nabla \widetilde{n}\|_{L^2(Q_T)}\\ &\quad + C\|n^{2/3}\|_{L^{7/2}(0,T;L^{21/2}(\Omega))}\|\nabla n\|_{L^2(Q_T)}\|\nabla \widetilde{n}\|_{L^\infty(Q_T)}\\ &\quad + C\|n\|_{L^{7/3}(0,T;L^7(\Omega))}\|\nabla V\|_{L^\infty(0,T;L^3(\Omega))}\|\nabla(n-\widetilde{n})\|_{L^2(Q_T)}, \end{aligned}$$

and we obtain that $(n - \widetilde{n}) \in L^\infty(0,T;L^2(\Omega))$ and $\nabla n^{4/3} \in L^2(Q_T)$. Since $\widetilde{n} \in L^\infty(\Omega)$, we directly see that $n \in L^\infty(0,T;L^2(\Omega))$. Using the Gagliardo-Nirenberg inequality (with $\theta = 1$), we estimate

$$\|n\|_{L^8(\Omega)}^{8/3} = \|n^{4/3}\|_{L^6(\Omega)}^2 \le C\|\nabla n^{4/3}\|_{L^2(\Omega)}^2 + C\|n^{4/3}\|_{L^1(\Omega)}^2.$$

Thanks to the regularities $\nabla n^{4/3} \in L^2(Q_T)$ and $n \in L^\infty(0,T;L^2(\Omega))$, we are able to conclude that $n \in L^{8/3}(0,T;L^8(\Omega))$, which finishes the proof. □

Lemma 67. *Let the assumptions from Theorem 36 hold with $r = 3$. Then for all $q \in \mathbb{N}_{\ge 2}$ it holds that*

$$\begin{aligned} n, p &\in L^\infty(0,T;L^q(\Omega)), \\ \nabla n^\alpha, \nabla p^\alpha &\in L^2(Q_T), \quad \textit{for all } 1 \le \alpha < \infty. \end{aligned} \tag{4.59}$$

In particular, this means that (4.8) holds.

Proof. The proof is done by induction. Motivated by the result of Lemma 66, let us assume that $n \in L^\infty(0,T;L^q(\Omega)) \cap L^{(3q+2)/3}(0,T;L^{3q+2}(\Omega))$ and $\nabla n^{(3q+2)/6} \in L^2(Q_T)$. For $q = 2$ this holds thanks to Lemma 66. Using $n^q - \widetilde{n}^q$ as a test function we have

$$\int_0^T \langle \partial_t n, n^q - \widetilde{n}^q\rangle\, dt + \int_0^T\int_\Omega nG'(n)\nabla n \cdot \nabla(n^q - \widetilde{n}^q)\, dx\, dt = \int_0^T\int_\Omega n\nabla V \cdot \nabla(n^q - \widetilde{n}^q)\, dx\, dt.$$

To estimate the integrals we introduce the following notation:

$$\begin{aligned} I_1 &:= \int_0^T \langle \partial_t n, n^q\rangle\, dt, & \widetilde{I}_1 &:= \int_0^T \langle \partial_t n, \widetilde{n}^q\rangle\, dt, \\ I_2 &:= \int_0^T\int_\Omega nG'(n)\nabla n \cdot \nabla n^q\, dx\, dt, & \widetilde{I}_2 &:= \int_0^T\int_\Omega nG'(n)\nabla n \cdot \nabla \widetilde{n}^q\, dx\, dt, \\ I_3 &:= \int_0^T\int_\Omega n\nabla V \cdot \nabla n^q\, dx\, dt, & \widetilde{I}_3 &:= \int_0^T\int_\Omega n\nabla V \cdot \nabla \widetilde{n}^q\, dx\, dt. \end{aligned}$$

Let us start by estimating the integrals $\widetilde{I}_1$, $\widetilde{I}_2$ and $\widetilde{I}_3$. The first integral gives

$$\widetilde{I}_1 = \int_0^T \langle \partial_t n, \widetilde{n}^q\rangle\, dt \le C\|\partial_t n\|_{L^{6/5}(0,T;(W^{1,6}(\Omega)\cap H_D^1(\Omega))')}\|\widetilde{n}\|_{L^\infty(\Omega)}^q \le C(q,\widetilde{n},T).$$

For the second integral, using Corollary 74, we compute

$$\begin{aligned} \widetilde{I}_2 = \int_0^T\int_\Omega nG'(n)\nabla n \cdot \nabla\widetilde{n}^q\, dx\, dt &= q\int_0^T\int_\Omega nG'(n)\widetilde{n}^{q-1}\nabla n \cdot \nabla\widetilde{n}\, dx\, dt \\ &\le Cq\|\widetilde{n}\|_{L^\infty(\Omega)}^{q-1}\|\nabla\widetilde{n}\|_{L^\infty(\Omega)}\int_0^T\int_\Omega (1 + n^{2/3})|\nabla n|\, dx\, dt \\ &\le Cq\|\widetilde{n}\|_{L^\infty(\Omega)}^{q-1}\|\nabla\widetilde{n}\|_{L^\infty(\Omega)}\|\nabla n\|_{L^2(Q_T)}\|1 + n^{2/3}\|_{L^2(Q_T)} \\ &\le C(q,T,\widetilde{n}). \end{aligned}$$

Estimating $\widetilde{I}_3$ is straightforward:

$$\begin{aligned}\widetilde{I}_3 = \int_0^T\int_\Omega n\nabla V\cdot\nabla\widetilde{n}^q\,dx\,dt &= q\int_0^T\int_\Omega n\widetilde{n}^{q-1}\nabla V\cdot\nabla\widetilde{n}\,dx\,dt\\ &\le Cq\|\widetilde{n}\|_{L^\infty(\Omega)}^{q-1}\|\nabla\widetilde{n}\|_{L^\infty(\Omega)}\|\nabla V\|_{L^\infty(0,T;L^3(\Omega))}\|n\|_{L^\infty(0,T;L^{5/3}(\Omega))}\\ &\le C(q,T,\widetilde{n}).\end{aligned}$$

Let us now turn to the integrals I_1, I_2 and I_3. I_1 can be rewritten as

$$I_1 = \int_0^T\langle\partial_t n, n^q\rangle\,dt = \frac{1}{q+1}\int_0^T\frac{d}{dt}\|n\|_{L^{q+1}(\Omega)}^{q+1}\,dt = \frac{1}{q+1}\|n(T)\|_{L^{q+1}(\Omega)}^{q+1} - \frac{1}{q+1}\|n^I\|_{L^{q+1}(\Omega)}^{q+1}.$$

To estimate I_2 we again use Corollary 74 and get

$$\begin{aligned}I_2 = \int_0^T\int_\Omega nG'(n)\nabla n\cdot\nabla n^q\,dx\,dt &= q\int_0^T\int_\Omega nG'(n)n^{q-1}\nabla n\cdot\nabla n\,dx\,dt\\ &\ge Cq\int_0^T\int_\Omega n^{2/3}n^{q-1}|\nabla n|^2\,dx\,dt\\ &= Cq\int_0^T\int_\Omega |n^{(q-1)/2+1/3}\nabla n|^2\,dx\,dt\\ &= Cq\left(\frac{6}{3q+5}\right)^2\int_0^T\int_\Omega\left|\nabla n^{(3q+5)/6}\right|^2\,dx\,dt.\end{aligned}$$

We rewrite I_3 and use Young's inequality to estimate

$$\begin{aligned}I_3 = \int_0^T\int_\Omega n\nabla V\cdot\nabla n^q\,dx\,dt &= q\int_0^T\int_\Omega n\nabla V\cdot n^{q-1}\nabla n\,dx\,dt\\ &= q\int_0^T\int_\Omega n\nabla V\cdot n^{(3q-5)/6}n^{(3q-1)/6}\nabla n\,dx\,dt\\ &= q\frac{6}{3q+5}\int_0^T\int_\Omega n^{1+(3q-5)/6}\nabla V\cdot\nabla n^{(3q+5)/6}\,dx\,dt\\ &\le \frac{\delta}{2}q\left(\frac{6}{3q+5}\right)^2\int_0^T\int_\Omega\left|\nabla n^{(3q+5)/6}\right|^2\,dx\,dt\\ &\quad+\frac{q}{2\delta}\int_0^T\int_\Omega n^{2+(3q-5)/3}|\nabla V|^2\,dx\,dt.\end{aligned}$$

Choosing $\delta = C$, where $C > 0$ is the constant from the estimate of I_2, we can absorb the first term into I_2. It remains to estimate the second term. By our induction assumption, we have

$$n^{2+(3q-5)/3} = n^{(3q+1)/3}\in L^{(3q+2)/(3q+1)}(0,T;L^{3(3q+2)/(3q+1)}(\Omega)).$$

Using Hölder's inequality we get

$$\frac{q}{2C}\int_0^T\int_\Omega n^{2+(3q-5)/3}|\nabla V|^2\,dx\,dt$$

$$\leq C(q,T,G')\|n^{(3q+1)/3}\|_{L^{\frac{3q+2}{3q+1}}(0,T;L^{\frac{3(3q+2)}{3q+1}}(\Omega))}\|\nabla V\|_{L^\infty(0,T;L^3(\Omega))} < +\infty.$$

Combining all the estimates, we obtain

$$\begin{aligned}\frac{1}{q+1}\|n(T)\|^{q+1}_{L^{q+1}(\Omega)} + \frac{Cq}{2}\left(\frac{6}{3q+5}\right)^2 \int_0^T \int_\Omega \left|\nabla n^{(3q+5)/6}\right|^2 \,dx\,dt \\ \leq \frac{1}{q+1}\|n^I\|^{q+1}_{L^{q+1}(\Omega)} + C(q,T,\widetilde{n}) \\ + C(q,T,G')\|n^{(3q+1)/3}\|_{L^{\frac{3q+2}{3q+1}}(0,T;L^{\frac{3(3q+2)}{3q+1}}(\Omega))}\|\nabla V\|_{L^\infty(0,T;L^3(\Omega))} \\ < +\infty.\end{aligned}$$

This shows that

$$n \in L^\infty(0,T;L^{q+1}(\Omega)),$$
$$\nabla n^{(3(q+1)+2)/6} \in L^2(Q_T),$$

and by the Gagliardo-Nirenberg inequality

$$n \in L^{(3(q+1)+2)/3}(0,T;L^{3(q+1)+2}(\Omega)).$$

To finalize the proof, we note that for $\alpha \in [1,4/3]$ there holds

$$|\nabla n^\alpha| \leq |\nabla n| + |\nabla n^{4/3}|$$

and for $\alpha > 4/3$ there exists a unique $q \in \mathbb{N}_{\geq 2}$ such that $3q+2 \leq 6\alpha \leq 3(q+1)+2$. As a result $n^{\alpha-1} \leq n^{(3q+2)/6-1} + n^{(3(q+1)+2)/6-1}$ and consequently

$$|\nabla n^\alpha| \leq |\nabla n^{(3q+2)/6}| + |\nabla n^{(3(q+1)+2)/6}|,$$

which finishes the proof. □

4.3.3 Bounded solutions

Let us remark that the constants from the bounds obtained in Lemma 67 depend on q and hence might blow up as $q \to \infty$. The final step therefore is to derive bounds on the solution in the spaces $L^\infty(0,T;L^q(\Omega))$, which are uniform in $q \in \mathbb{N}$. This will then allow us to take the limit $q \to \infty$. We follow the proof of [77, Lemma 16], where an Alikakos-type iteration was detailed for pure Neumann boundary conditions to simplify the presentation. Since we have the case of mixed Dirichlet-Neumann boundary conditions, we will nonetheless detail the rather technical computations.

Lemma 68. *Let the assumptions from Theorem 36 hold with some $r > 3$. Then it holds that*

$$n,p,D \in L^\infty(0,T;L^\infty(\Omega)), \quad V \in L^\infty(0,T;W^{1,r}(\Omega)). \tag{4.60}$$

Proof. Thanks to assumption (A5) and the regularity $n,p \in L^\infty(0,T;L^{5/3}(\Omega))$ we have that

$$\|V\|_{L^\infty(0,T;W^{1,3+\varrho}(\Omega))} \leq C\|n-p-D+A\|_{L^\infty(0,T;L^{5/3}(\Omega))} + C < \infty,$$

for some $0 < \varrho \leq 3/4$. We use $n^q - \widetilde{n}^q$ for some $q > 1$ as a test function in the weak formulation, but do not integrate over time yet. Repeating the estimates from the proof of Lemma 67 and using Corollary 74 yields

$$\frac{1}{q+1}\frac{d}{dt}\|n\|_{L^{q+1}(\Omega)}^{q+1} + C(G')q\left(\frac{6}{3q+5}\right)^2 \|\nabla n^{(3q+5)/6}\|_{L^2(\Omega)}^2$$
$$\leq q\int_\Omega n\nabla V \cdot n^{q-1}\nabla n\, dx + C\|\widetilde{n}\|_{L^\infty(\Omega)}^q + Cq\|\widetilde{n}\|_{L^\infty(\Omega)}^{q-1}.$$

We rewrite

$$\frac{3q+5}{6} = \frac{q+\frac{5}{3}}{2}$$

and estimate the integral on the right-hand side with Hölder's inequality

$$\begin{aligned} q\int_\Omega n\nabla V \cdot n^{q-1}\nabla n\, dx &= q\int_\Omega n^{(q+1/3)/2}\nabla V \cdot n^{(q-1/3)/2}\nabla n\, dt \\ &= \frac{2q}{q+\frac{5}{3}}\int_\Omega n^{(q+1/3)/2}\nabla V \cdot \nabla n^{(q+5/3)/2}\, dx \\ &\leq 2\|n^{(q+1/3)/2}\|_{L^{6-\beta}(\Omega)}\|\nabla V\|_{L^{3+\varrho}(\Omega)}\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}, \end{aligned}$$

where $\beta = \frac{4\varrho}{1+\varrho}$. Plugging this estimate in, we obtain

$$\frac{1}{q+1}\frac{d}{dt}\|n\|_{L^{q+1}(\Omega)}^{q+1} + \frac{4qC(G')}{(q+5/3)^2}\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}^2$$
$$\leq C(q+1)(\|\widetilde{n}\|_{L^\infty(\Omega)}^q + 1) + C\|n^{(q+1/3)/2}\|_{L^{6-\beta}(\Omega)}\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}. \tag{4.61}$$

Using $(q+1/3)/(q+5/3) < 1$, we estimate the first factor in the second term on the right-hand side as

$$\begin{aligned} \|n^{(q+1/3)/2}\|_{L^{6-\beta}(\Omega)} &= \|n^{(q+5/3)/2}\|_{L^{(6-\beta)\frac{q+1/3}{q+5/3}}(\Omega)}^{\frac{q+1/3}{q+5/3}} \\ &\leq C(\Omega)\|n^{(q+5/3)/2}\|_{L^{6-\beta}(\Omega)}^{\frac{q+1/3}{q+5/3}} \leq C(1+\|n^{(q+5/3)/2}\|_{L^{6-\beta}(\Omega)}). \end{aligned}$$

We use the Gagliardo-Nirenberg inequality to estimate $\|n^{(q+5/3)/2}\|_{L^{6-\beta}(\Omega)}$,

$$\|n^{(q+5/3)/2}\|_{L^{6-\beta}(\Omega)} \leq C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}^\theta \|n^{(q+5/3)/2}\|_{L^1(\Omega)}^{1-\theta} + C\|n^{(q+5/3)/2}\|_{L^1(\Omega)},$$

where $\theta = \frac{30-6\beta}{30-5\beta} \in (0,1)$. At this point we need that $\varrho > 0$, as we require $\theta \in (0,1)$. Otherwise we will not be able to absorb the gradient term(s) into the left-hand side of (4.61) later.

Combining the estimates, we bound the term on the right-hand side of (4.61) by

$$\begin{aligned}C\|n^{(q+1/3)/2}\|_{L^{6-\beta}(\Omega)}\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}&\\ \le C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}&\\ \times\Big(1+\|\nabla n^{(q+5/3)/2}\|^{\theta}_{L^2(\Omega)}\|n^{(q+5/3)/2}\|^{1-\theta}_{L^1(\Omega)}+\|n^{(q+5/3)/2}\|_{L^1(\Omega)}\Big)&\\ \le C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}&\\ +C\|\nabla n^{(q+5/3)/2}\|^{1+\theta}_{L^2(\Omega)}\|n^{(q+5/3)/2}\|^{1-\theta}_{L^1(\Omega)}&\\ +C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}\|n^{(q+5/3)/2}\|_{L^1(\Omega)}.&\end{aligned}$$

The goal is to absorb all gradient terms into the left-hand side of (4.61), hence we use Young's inequality in the following two forms (for all $a, b \ge 0$):

$$ab \le \delta a^r + \frac{b^s}{s r^{s/r}\delta^{s/r}}, \tag{4.62}$$

$$ab \le \delta a^2 + \frac{b^2}{4\delta}, \tag{4.63}$$

with

$$\delta = \frac{1}{3}\frac{4qC(G')}{(q+5/3)^2}.$$

Using (4.63), we estimate the first term

$$\begin{aligned}C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)} &\le \frac{4qC(G')}{3(q+5/3)^2}\|\nabla n^{(q+5/3)/2}\|^2_{L^2(\Omega)}+\frac{C^2}{4}\frac{3(q+5/3)^2}{4qC(G')}\\ &\le \frac{4qC(G')}{3(q+5/3)^2}\|\nabla n^{(q+5/3)/2}\|^2_{L^2(\Omega)}+Cq.\end{aligned}$$

Using (4.62) with $r = 2/(1+\theta)$ and $s = 2/(1-\theta)$, we can estimate the second term

$$\begin{aligned}C\|\nabla n^{(q+5/3)/2}\|^{1+\theta}_{L^2(\Omega)}\|n^{(q+5/3)/2}\|^{1-\theta}_{L^1(\Omega)} &\le \frac{4qC(G')}{3(q+5/3)^2}\|\nabla n^{(q+5/3)/2}\|^2_{L^2(\Omega)}\\ &\quad + C^{2/(1-\theta)}\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)}\\ &\quad\times\frac{1-\theta}{2}\left(\frac{1+\theta}{2}\right)^{\frac{1+\theta}{1-\theta}}\left(\frac{3(q+5/3)^2}{4qC(G')}\right)^{\frac{1+\theta}{1-\theta}}.\end{aligned}$$

Again, using (4.63), we estimate the third term

$$\begin{aligned}C\|\nabla n^{(q+5/3)/2}\|_{L^2(\Omega)}\|n^{(q+5/3)/2}\|_{L^1(\Omega)} &\le \frac{4qC(G')}{3(q+5/3)^2}\|\nabla n^{(q+5/3)/2}\|^2_{L^2(\Omega)}\\ &\quad+\frac{C^2}{4}\frac{3(q+5/3)^2}{4qC(G')}\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)}.\end{aligned}$$

We combine all estimates so far and plug them into (4.61), absorb the gradient terms into the left-hand side and obtain

$$\begin{aligned}\frac{1}{q+1}\frac{d}{dt}\|n\|^{q+1}_{L^{q+1}(\Omega)} \le C(q+1)(\|\widetilde{n}\|^q_{L^\infty(\Omega)}+1)\\ + \frac{C^2}{4}\frac{3(q+5/3)^2}{4qC(G')} + \frac{C^2}{4}\frac{3(q+5/3)^2}{4qC(G')}\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)}\\ + C^{\frac{2}{1-\theta}}\frac{1-\theta}{2}\left(\frac{1+\theta}{2}\right)^{\frac{1+\theta}{1-\theta}}\left(\frac{3(q+5/3)^2}{4qC(G')}\right)^{\frac{1+\theta}{1-\theta}}\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)}.\end{aligned}$$

Now there exists a constant $C(\varrho, G') > 0$ independent of q such that

$$\frac{3C^2(q+5/3)^2}{16qC(G')} \le C(\varrho, G')q$$

and together with

$$\frac{1+\theta}{1-\theta} = \frac{60-11\beta}{\beta}$$

we see that there also exists a constant $C(\varrho, G') > 0$ such that

$$C^{\frac{2}{1-\theta}}\frac{1-\theta}{2}\left(\frac{1+\theta}{2}\right)^{\frac{1+\theta}{1-\theta}}\left(\frac{3(q+5/3)^2}{4qC(G')}\right)^{\frac{1+\theta}{1-\theta}} \le C(\varrho, G')q^{\frac{60-11\beta}{\beta}}.$$

As a result we obtain

$$\frac{1}{q+1}\frac{d}{dt}\|n\|^{q+1}_{L^{q+1}(\Omega)} \le C(q+1)\|\widetilde{n}\|^q_{L^\infty(\Omega)} + C(\varrho, G')(q^{\frac{60-11\beta}{\beta}}+q)(1+\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)}).$$

Let us note that

$$\|n^{(q+5/3)/2}\|^2_{L^1(\Omega)} = \|n\|^{q+5/3}_{L^{(q+5/3)/2}(\Omega)},$$

so together with $(60-11\beta)/\beta > 1$ (note that $\beta < 4$) we rewrite our estimate to

$$\frac{1}{q+1}\frac{d}{dt}\|n\|^{q+1}_{L^{q+1}(\Omega)} \le C(q+1)\|\widetilde{n}\|^q_{L^\infty(\Omega)} + C(\varrho, G')q^{\frac{60-11\beta}{\beta}}(1+\|n\|^{q+5/3}_{L^{(q+5/3)/2}(\Omega)}).$$

We integrate this inequality over $t \in (0,\tau)$, set $\gamma := (60-11\beta)/\beta + 1$ and get

$$\begin{aligned}\|n(\tau)\|^{q+1}_{L^{q+1}(\Omega)} &\le \|n^I\|^{q+1}_{L^{q+1}(\Omega)} + C\tau(q+1)^2\|\widetilde{n}\|^q_{L^\infty(\Omega)} + C(\varrho, G')q^\gamma\int_0^\tau 1+\|n\|^{q+5/3}_{L^{(q+5/3)/2}(\Omega)}\,dt\\ &\le C(\Omega)Tq^2(\|n^I\|^{q+1}_{L^\infty(\Omega)}+\|\widetilde{n}\|^{q+1}_{L^\infty(\Omega)})\\ &\quad + C(\varrho, G', T)q^\gamma(1+\|n\|^{q+5/3}_{L^\infty(0,T;L^{(q+5/3)/2}(\Omega))}).\end{aligned}$$

Taking the supremum over $\tau \in (0,T)$ yields

$$\|n\|^{q+1}_{L^\infty(0,T;L^{q+1}(\Omega))} \le Cq^2(\|n^I\|^{q+1}_{L^\infty(\Omega)}+\|\widetilde{n}\|^{q+1}_{L^\infty(\Omega)}) + Cq^\gamma(1+\|n\|^{q+5/3}_{L^\infty(0,T;L^{(q+5/3)/2}(\Omega))}).$$

Since this inequality holds for arbitrary $q > 1$, we can set

$$q_k := q + 1; \quad q_{k-1} := \frac{q + 5/3}{2},$$

which defines the recursion

$$q_{k-1} = \frac{q_k + 2/3}{2}, \quad \text{or } q_k = 2(q_{k-1} - 1/3).$$

Solving this explicitly we obtain

$$q_k = 2^k(q_0 - 2/3) + 2/3, \quad k \in \mathbb{N},$$

where $q_0 > 1$ can be arbitrary, but fixed. Furthermore, we define

$$b_k := \|n\|^{q_k}_{L^\infty(0,T;L^{q_k}(\Omega))} + \|n^I\|^{q_k}_{L^\infty(\Omega)} + \|\widetilde{n}\|^{q_k}_{L^\infty(\Omega)} + 1.$$

Thanks to the estimates above and using $q_k \le 2q_{k-1}$, we deduce

$$\begin{aligned}
b_k &\le (Cq^2 + 1)(\|n^I\|^{q_k}_{L^\infty(\Omega)} + \|\widetilde{n}\|^{q_k}_{L^\infty(\Omega)}) + C(q_k - 1)^\gamma (1 + \|n\|^{2q_{k-1}}_{L^\infty(0,T;L^{q_{k-1}}(\Omega))}) + 1 \\
&\le Cq_k^\gamma \left(\|n^I\|^{2q_{k-1}}_{L^\infty(\Omega)} + \|\widetilde{n}\|^{2q_{k-1}}_{L^\infty(\Omega)} + \|n\|^{2q_{k-1}}_{L^\infty(0,T;L^{q_{k-1}}(\Omega))} + 1 \right) \\
&\le Cq_k^\gamma b_{k-1}^2 \\
&\le C^k q_k^\gamma b_{k-1}^2.
\end{aligned}$$

Since $q_k \le 3^k$ for k sufficiently large, and setting $M := C3^\gamma$, we get the recursive estimate

$$b_k \le C^k 3^{k\gamma} b_{k-1}^2 = M^k b_{k-1}^2.$$

To solve this, we introduce $c_k := M^{k+2} b_k$ and get

$$c_k \le M^{2(k+1)} b_{k-1}^2 = (M^{k+1} b_{k-1})^2 = c_{k-1}^2,$$

hence

$$c_k \le c_0^{2^k}.$$

Thus, we obtain for b_k

$$b_k = M^{-(k+2)} c_k \le M^{-(k+2)} c_0^{2^k} = M^{-(k+2)} (M^2 b_0)^{2^k} = M^{2^{k+1} - (k+2)} b_0^{2^k}.$$

We recall the definition of b_k and see

$$\|n\|^{q_k}_{L^\infty(0,T;L^{q_k}(\Omega))} \le b_k \le M^{2^{k+1}-(k+2)} \left(\|n\|^{q_0}_{L^\infty(0,T;L^{q_0}(\Omega))} + \|n^I\|^{q_0}_{L^\infty(\Omega)} + \|\widetilde{n}\|^{q_0}_{L^\infty(\Omega)} + 1 \right)^{2^k},$$

or, after taking the q_k-th root

$$\|n\|_{L^\infty(0,T;L^{q_k}(\Omega))} \le M^{(2^{k+1}-(k+2))/q_k} \left(\|n\|^{q_0}_{L^\infty(0,T;L^{q_0}(\Omega))} + \|n^I\|^{q_0}_{L^\infty(\Omega)} + \|\widetilde{n}\|^{q_0}_{L^\infty(\Omega)} + 1 \right)^{2^k/q_k}.$$

It remains to bound the exponents independent of $k \in \mathbb{N}$:

$$\begin{aligned}
\frac{2^{k+1} - (k+2)}{q_k} &= \frac{2^{k+1} - (k+2)}{2^k(q_0 - 2/3) + 2/3} \le \frac{2^{k+1}}{2^k(q_0 - 2/3)} = \frac{2}{q_0 - 2/3}, \\
\frac{2^k}{q_k} &= \frac{2^k}{2^k(q_0 - 2/3) + 2/3} \le \frac{2^k}{2^k(q_0 - 2/3)} = \frac{1}{q_0 - 2/3}.
\end{aligned}$$

These bounds allow us to take the limit $k \to \infty$ in the above estimate. Hence, we conclude that $n \in L^\infty(0,T;L^\infty(\Omega))$, which finishes the proof of Lemma 68 and proves Theorem 36. □

4.4 Auxiliary results on Fermi-Dirac statistics

In this section we recall the definition of the Fermi-Dirac integral $\mathcal{F}_j$ of order $j > -1$. We will then derive important properties, including asymptotic behaviour of $\mathcal{F}_j$ and G, which are needed in the proofs throughout this chapter. The order $j > -1$ is kept general because we need estimates on $\mathcal{F}'_{1/2} = \mathcal{F}_{-1/2}$.

We recall that

$$\mathcal{F}_j(\eta) = \frac{1}{\Gamma(1+j)} \int_0^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi, \quad \text{for } j > -1,$$
$$G(z) = \mathcal{F}_{1/2}^{-1}(z),$$

and that there holds the well known identity

$$\mathcal{F}'_j = \mathcal{F}_{j-1}, \quad \text{for all } j > 0. \tag{4.64}$$

Let us also recall the following notation, which we introduced in Section 4.1:

Notation. *Given terms A and B, we write $A \lesssim B$ if there exists a constant $C > 0$, such that it holds that $A \leq CB$. If $A \lesssim B \lesssim A$ holds, we write $A \sim B$. Furthermore, if there exist two constants $C_1, C_2 > 0$, such that $A \leq C_1 B + C_2$ is true, we write $A \lesssim B + 1$.*

First, we give a bound on the Fermi-Dirac integral.

Lemma 69. *For all $j > -1$ and all $\eta \in \mathbb{R}$ it holds that*

$$\mathcal{F}_j(\eta) \sim e^\eta \mathbb{1}_{(\eta \leq 0)} + \left(\eta^{j+1} + 1\right) \mathbb{1}_{(\eta > 0)}. \tag{4.65}$$

More specifically, for all $\eta \leq 0$, there hold the bounds

$$\frac{1}{2} e^\eta \leq \mathcal{F}_j(\eta) \leq e^\eta. \tag{4.66}$$

Proof. We first prove (4.66), which also shows (4.65) for $\eta \leq 0$. To this end let $\eta \leq 0$, then for all $\xi \geq 0$ it holds that

$$e^{\xi-\eta} \leq 1 + e^{\xi-\eta} \leq 2e^{\xi-\eta},$$

and reordering terms shows

$$\frac{1}{2e^{\xi-\eta}} \leq \frac{1}{1+e^{\xi-\eta}} \leq \frac{1}{e^{\xi-\eta}}.$$

This allows to directly compute

$$\frac{e^\eta}{2} = \frac{e^\eta}{2\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi = \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{2e^{\xi-\eta}}\, d\xi \leq \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi = \mathcal{F}_j(\eta),$$

and

$$\mathcal{F}_j(\eta) = \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi \leq \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^{\xi-\eta}}\, d\xi = \frac{e^\eta}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi = e^\eta,$$

which shows (4.66).

Proving the inequalities for $\eta > 0$ is a bit more involved. We first split $\mathcal{F}_j$ into two parts as follows:

$$\mathcal{F}_j(\eta) = \frac{1}{\Gamma(j+1)} \left(\int_0^\eta + \int_\eta^\infty \right) \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi = I_{F1} + I_{F2}.$$

Now we estimate each term separately. Using $e^{\xi-\eta} \le 1$ for $\xi \le \eta$ we get

$$I_{F1} = \frac{1}{\Gamma(j+1)} \int_0^\eta \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi \ge \frac{1}{2\Gamma(j+1)} \int_0^\eta \xi^j\, d\xi = \frac{\eta^{j+1}}{2(j+1)\Gamma(j+1)} = \frac{\eta^{j+1}}{2\Gamma(j+2)},$$

and using $1 + e^{\xi-\eta} \ge 1$ for all ξ, η we get

$$I_{F1} = \frac{1}{\Gamma(j+1)} \int_0^\eta \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi \le \frac{1}{\Gamma(j+1)} \int_0^\eta \xi^j\, d\xi = \frac{\eta^{j+1}}{(j+1)\Gamma(j+1)} = \frac{\eta^{j+1}}{\Gamma(j+2)}.$$

It remains to estimate I_{F2}. We first prove the bounds for $j \ge 0$ and afterwards for $j < 0$. Using the transformation $\xi \mapsto \xi + \eta$, together with the estimates $(\xi+\eta)^j \ge \xi^j$ for all $\eta, \xi \ge 0$ and $1 + e^\xi \le 2e^\xi$ for all $\xi \ge 0$, we obtain

$$\begin{aligned} I_{F2} = \frac{1}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi &= \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{(\xi+\eta)^j}{1+e^\xi}\, d\xi \\ &\ge \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{1+e^\xi}\, d\xi \ge \frac{1}{2\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi = \frac{\Gamma(j+1)}{2\Gamma(j+1)} = \frac{1}{2}. \end{aligned}$$

For the upper bound on I_{F2} we again use the transformation $\xi \mapsto \xi + \eta$, together with the inequalities $(\xi+\eta)^j \le (2\eta)^j$ for $\xi \le \eta$ and $(\xi+\eta)^j \le (2\xi)^j$ for $\eta \le \xi$ as well as $1 \le 1 + e^\xi$ and $e^\xi \le 1 + e^\xi$ for all $\xi \in \mathbb{R}$, and obtain

$$\begin{aligned} I_{F2} = \frac{1}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi &= \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{(\xi+\eta)^j}{1+e^\xi}\, d\xi \\ &= \frac{1}{\Gamma(j+1)} \left(\int_0^\eta + \int_\eta^\infty \right) \frac{(\xi+\eta)^j}{1+e^\xi}\, d\xi \\ &\le \frac{(2\eta)^j}{\Gamma(j+1)} \int_0^\eta \frac{1}{1+e^\xi}\, d\xi + \frac{2^j}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{1+e^\xi}\, d\xi \\ &\le \frac{(2\eta)^j}{\Gamma(j+1)} \int_0^\eta e^{-\xi}\, d\xi + \frac{2^j}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{e^\xi}\, d\xi \\ &\le \frac{(2\eta)^j(1-e^{-\eta})}{\Gamma(j+1)} + \frac{2^j}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi \\ &= \frac{(2\eta)^j(1-e^{-\eta})}{\Gamma(j+1)} + 2^j \\ &\le \frac{(2\eta)^j}{\Gamma(j+1)} + 2^j. \end{aligned}$$

This shows that

$$\mathcal{F}_j(\eta) \le \frac{\eta^{j+1}}{\Gamma(j+2)} + \frac{(2\eta)^j}{\Gamma(j+1)} + 2^j,$$

$$\mathcal{F}_j(\eta) \geq \frac{\eta^{j+1}}{2\Gamma(j+2)} + \frac{1}{2},$$

which shows that for $j \geq 0$ it holds that

$$\mathcal{F}_j(\eta)\mathbb{1}_{(\eta>0)} \sim (\eta^{j+1} + 1)\mathbb{1}_{(\eta>0)}.$$

It now remains to prove the claim for $j < 0$. To derive the estimate from above, we again use the transformation $\xi \mapsto \xi + \eta$, together with the estimates $(\xi + \eta)^j \leq \xi^j$ and $1 + e^\xi \geq \xi$, and obtain

$$\begin{aligned} I_{F2} = \frac{1}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi &= \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{(\xi+\eta)^j}{1+e^\xi}\, d\xi \\ &\leq \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi \\ &= \frac{\Gamma(j+1)}{\Gamma(j+1)} \\ &= 1. \end{aligned}$$

This shows the estimate from above

$$\mathcal{F}_j(\eta) \leq \frac{\eta^{j+1}}{\Gamma(j+2)} + 1.$$

To obtain the estimate from below, we need a more careful estimate on I_{F1} first. Let us assume that $\eta \geq 1$. Then it holds that

$$\begin{aligned} \frac{\xi^j}{1+e^{\xi-\eta}} &\geq \frac{\xi^j}{2}, && \text{for all } \xi \leq \eta, \\ \xi^j &\geq \frac{1}{2}\left(\xi^j + \eta^{-1}\right), && \text{for all } \xi \in [1, \eta]. \end{aligned}$$

Using this, we estimate

$$\begin{aligned} I_{F1} &= \frac{1}{\Gamma(j+1)} \int_0^\eta \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi = \frac{1}{\Gamma(j+1)} \left(\int_0^1 + \int_1^\eta \right) \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi \\ &\geq \frac{1}{2\Gamma(j+1)} \int_0^1 \xi^j\, d\xi + \frac{1}{4\Gamma(j+1)} \int_1^\eta \xi^j + \eta^{-1}\, d\xi \\ &= \frac{1}{4\Gamma(j+1)} \int_0^\eta \xi^j\, d\xi + \frac{1}{4\Gamma(j+1)} \int_0^1 \xi^j\, d\xi + \frac{1}{4\Gamma(j+1)} \int_1^\eta \eta^{-1}\, d\xi \\ &= \frac{\eta^{j+1}}{4(j+1)\Gamma(j+1)} + \frac{1}{4\Gamma(j+1)} \left(\frac{1}{j+1} + 1 - \eta^{-1} \right) \\ &> \frac{\eta^{j+1}}{4\Gamma(j+2)} + \frac{1}{4\Gamma(j+1)}. \end{aligned}$$

Estimating I_{F2} is now simple and straightforward:

$$I_{F2} = \frac{1}{\Gamma(j+1)} \int_\eta^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi \geq 0.$$

Thus, we have shown that

$$\mathcal{F}_j(\eta) \gtrsim \eta^{j+1} + 1, \quad \text{for all } \eta \geq 1.$$

Now assume that $\eta \in [0, 1]$. Since $\mathcal{F}_j(\eta)$ is continuous and strictly monotonically increasing, we have that $F_j(0)$ is its minimum on $[0, 1]$ and we estimate

$$\mathcal{F}_j(0) = \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{1+e^\xi}\, d\xi > \frac{1}{2\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi = \frac{1}{2}.$$

Hence, there exists a constant $C > 0$, in fact $C = \frac{1}{4}$ suffices, such that

$$\mathcal{F}_j(\eta) \geq C(\eta^{j+1} + 1), \quad \text{for all } \eta \in [0, 1].$$

This shows all necessary estimates on $\mathcal{F}_j(\eta)$ and we conclude that for $j < 0$ it also holds that

$$\mathcal{F}_j(\eta)\mathbb{1}_{(\eta>0)} \sim (\eta^{j+1} + 1)\mathbb{1}_{(\eta>0)},$$

which finishes the proof. □

The next corollary is a direct consequence of Lemma 69.

Corollary 70. *For $j > 0$ it holds that*

$$\begin{aligned} \mathcal{F}_j'(\eta) &= \mathcal{F}_{j-1}(\eta) \sim \mathcal{F}_j(\eta)\mathbb{1}_{(\eta\leq 0)} + \mathcal{F}_j(\eta)^{\frac{j}{j+1}}\mathbb{1}_{(\eta>0)}, \\ \mathcal{F}_{\frac{1}{2}}'(\eta) &= \mathcal{F}_{-\frac{1}{2}}(\eta) \sim \mathcal{F}_{\frac{1}{2}}(\eta)\mathbb{1}_{(\eta\leq 0)} + \mathcal{F}_{\frac{1}{2}}(\eta)^{\frac{1}{3}}\mathbb{1}_{(\eta>0)}, \end{aligned} \tag{4.67}$$

where the second line is the special case $j = \frac{1}{2}$.

Next, we improve the lower bound on $\mathcal{F}_j(\eta)$ for $\eta \leq \eta_0 < 0$ to obtain bounds on its inverse.

Lemma 71. *Let $j > -1$ and let $\eta_0 < 0$ be fixed. Then it holds for all $\eta \leq \eta_0$ and all $\xi \geq 0$ that*

$$\frac{e^\eta}{1+e^{\eta_0}} \leq \mathcal{F}_j(\eta) \leq e^\eta. \tag{4.68}$$

Proof. The upper bound is a consequence of Lemma 69. For the lower bound we compute

$$1 + e^{\xi-\eta} \leq e^\xi e^{\eta_0-\eta} + e^{\xi-\eta} = (1 + e^{\eta_0})e^{\xi-\eta},$$

from which we conclude

$$\begin{aligned} \frac{e^\eta}{1+e^{\eta_0}} = \frac{e^\eta}{1+e^{\eta_0}} \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{e^\xi}\, d\xi &= \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{(1+e^{\eta_0})} e^{\xi-\eta}\, d\xi \\ &\leq \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{\xi^j}{1+e^{\xi-\eta}}\, d\xi = \mathcal{F}_j(\eta). \end{aligned}$$

This finishes the proof. □

As a direct consequence we can give bounds on the inverse of $\mathcal{F}_j$.

Corollary 72. *Let $j > -1$ and let $z_0 > 0$ such that $\mathcal{F}_j^{-1}(z_0) < 0$. Then for all $0 < z \leq z_0$ it holds that*

$$\log z \leq \mathcal{F}_j^{-1}(z) \leq \log z + \log\left(1 + e^{\mathcal{F}_j^{-1}(z_0)}\right). \tag{4.69}$$

Proof. We define $\eta_0 := \mathcal{F}_j^{-1}(z_0)$ and $\eta := \mathcal{F}_j^{-1}(z)$. By applying the logarithm to (4.68) we obtain

$$\log\left(\frac{e^\eta}{1 + e^{\eta_0}}\right) \leq \log \mathcal{F}_j(\eta) \leq \eta.$$

Simplifying this expression leads to

$$\eta - \log(1 + e^{\eta_0}) \leq \log \mathcal{F}_j(\eta) \leq \eta,$$

and rewriting $\eta = \mathcal{F}_j^{-1}(z)$ yields

$$\mathcal{F}_j^{-1}(z) - \log(1 + e^{\mathcal{F}_j^{-1}(z_0)}) \leq \log z \leq \mathcal{F}_j^{-1}(z).$$

Rearranging terms proves (4.69). □

Let us recall the definition of G and compute its derivative:

$$G(z) = \mathcal{F}_{1/2}^{-1}(z),$$

$$G'(z) = \frac{1}{\mathcal{F}'_{1/2}(\mathcal{F}_{1/2}^{-1}(z))} = \frac{1}{\mathcal{F}_{-1/2}(\mathcal{F}_{1/2}^{-1}(z))} = \frac{1}{\mathcal{F}_{-1/2}(G(z))}.$$

The next lemma gives an estimate on the behaviour of G'.

Lemma 73. *For $z \in (0, \infty)$ it holds that*

$$G'(z) \sim z^{-1} + z^{-1/3}. \tag{4.70}$$

Proof. From the computation of G' and by Corollary 70 we directly obtain

$$\begin{aligned} G'(z) = \left(\mathcal{F}_{-1/2}(\mathcal{F}_{1/2}^{-1}(z))\right)^{-1} &\sim \left(\mathcal{F}_{1/2}(\mathcal{F}_{1/2}^{-1}(z))\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)\leq 0)} + \mathcal{F}_{1/2}(\mathcal{F}_{1/2}^{-1}(z))^{1/3}\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)>0)}\right)^{-1} \\ &= \left(z\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)\leq 0)} + z^{1/3}\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)>0)}\right)^{-1} \\ &= z^{-1}\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)\leq 0)} + z^{-1/3}\mathbb{1}_{(\mathcal{F}_{1/2}^{-1}(z)>0)}. \end{aligned}$$

This clearly shows the upper bound

$$G'(z) \lesssim z^{-1} + z^{-1/3}, \quad z \in (0, \infty).$$

For the lower bound we distinct the two cases $z \in (0, \mathcal{F}_{1/2}(0)]$ and $z > \mathcal{F}_{1/2}(0)$. In the case of $z \leq \mathcal{F}_{1/2}(0) < 1$ it immediately follows that $z^{-1} < z^{-1/3}$, and therefore

$$z^{-1} > \frac{1}{2}(z^{-1} + z^{-1/3}).$$

In the case $z > \mathcal{F}_{1/2}(0)$ it holds that $z^{-1/3} \geq z^{-1}$ for $z \geq 1$, and for $z \in (\mathcal{F}_{1/2}(0), 1)$ it is easy to see that $z^{-1/3} > \mathcal{F}_{1/2}(0)z^{-1}$, which shows

$$z^{-1/3} > \frac{\mathcal{F}_{1/2}(0)}{2}(z^{-1} + z^{-1/3}).$$

Combining the cases shows the estimate

$$G'(z) \gtrsim z^{-1} + z^{-1/3},$$

which finishes the proof. □

The next corollary is a direct consequence of Lemma 73

Corollary 74. *For $z \in (0, \infty)$ it holds that*

$$\begin{aligned} z^{1/3}G'(z) &\sim 1 + z^{-2/3}, \\ z^{1/2}G'(z) &\sim z^{1/6} + z^{-1/2}, \\ zG'(z) &\sim 1 + z^{2/3}. \end{aligned} \tag{4.71}$$

The next lemma is a preparation to compute the limit of $zG'(z)$ as $z \to 0^+$.

Lemma 75. *For all $0 < z_0 < \mathcal{F}_{1/2}(0)$ it holds that*

$$\left(1 + e^{G(z_0)}\right)^{-1} \leq \varliminf_{z \to 0^+} zG'(z) \leq \varlimsup_{z \to 0^+} zG'(z) \leq 1 + e^{G(z_0)}. \tag{4.72}$$

Proof. We define $\eta := G(z)$ and $\eta_0 := G(z_0)$. Now let $0 < z_1 \leq z_0$ such that

$$\log z + \log\left(1 + e^{G(z_0)}\right) \leq G(z_0), \ \forall z \leq z_1$$

holds. Applying $\mathcal{F}'_{1/2}$ to (4.69) yields

$$\mathcal{F}_{-1/2}(\log z) = \mathcal{F}'_{1/2}(\log z) \leq \mathcal{F}'_{1/2}(G(z)) \leq \mathcal{F}_{-1/2}(\log z + \log(1 + e^{G(z_0)}))$$

and using the estimate from (4.68) we get

$$\frac{e^{\log z}}{1 + e^{\eta_0}} \leq \mathcal{F}'_{1/2}(G(z)) \leq e^{\log z} e^{\log(1+e^{G(z_0)})}.$$

Simplifying terms, recalling that $G'(z) = (\mathcal{F}'_{1/2}(G(z)))^{-1}$ and rewriting $\eta_0 = G(z_0)$ we obtain

$$(1 + e^{G(z_0)})^{-1} \leq zG'(z) \leq 1 + e^{G(z_0)}.$$

Since this inequality holds for all $0 < z \leq z_1 < z_0$, we have proven (4.72). □

As a direct consequence we can compute the limit of $zG'(z)$.

Corollary 76. *The limit of $zG'(z)$ for $z \to 0^+$ exists and is*

$$\lim_{z\to 0^+} zG'(z) = 1. \tag{4.73}$$

Proof. We note that

$$\lim_{z_0\to 0} G(z_0) = \lim_{z_0\to 0} \mathcal{F}_{1/2}^{-1}(z_0) = -\infty,$$

therefore

$$\lim_{z_0\to 0^+} 1 + e^{G(z_0)} = 1.$$

Hence, by (4.72) we have

$$1 \leq \varliminf_{z\to 0^+} zG'(z) \leq \varlimsup_{z\to 0^+} zG'(z) \leq 1.$$

□

We also need a reformulation of Lemma 69 in order to obtain estimates on $G = \mathcal{F}_{1/2}^{-1}$.

Lemma 77. *For $z \in (0, \infty)$ it holds that*

$$\begin{aligned} \mathcal{F}_{1/2}^{-1}(z) &\lesssim (\log z + \log 2)\mathbb{1}_{(z\leq \mathcal{F}_{1/2}(0))} + (z-1)^{2/3}\mathbb{1}_{(z>\mathcal{F}_{1/2}(0))}, \\ \mathcal{F}_{1/2}^{-1}(z) &\gtrsim (\log z)\mathbb{1}_{(z\leq \mathcal{F}_{1/2}(0))} + (z-1)^{2/3}\mathbb{1}_{(z>\mathcal{F}_{1/2}(0))}. \end{aligned} \tag{4.74}$$

For $z \leq \mathcal{F}_{1/2}(0)$ the estimates hold with constants $C = 1$.

Proof. The result is a direct consequence of Lemma 69. Setting $z = \mathcal{F}_{1/2}(\eta)$, restricting $\eta \leq 0$ and taking the logarithm in (4.66) (with $j = 1/2$), we immediately obtain

$$\begin{aligned} \log z &\leq \mathcal{F}_{1/2}^{-1}(z), \\ \mathcal{F}_{1/2}^{-1}(z) &\leq \log z + \log 2. \end{aligned}$$

To show the second part, we take (4.65) with $\eta > 0$, which gives

$$C_1\eta^{3/2} + C_2 \leq \mathcal{F}_{1/2}(\eta) \leq C_3\eta^{3/2} + C_4,$$

for some constants $C_1, C_2, C_3, C_4 > 0$. Again, setting $z = \mathcal{F}_{1/2}(\eta)$, a direct computation shows that

$$\left(C_3^{-1}(z - C_4)\right)^{2/3} \leq \mathcal{F}_{1/2}^{-1}(z) \leq \left(C_1^{-1}(z - C_2)\right)^{2/3}.$$

Reformulating this, we get

$$(z-1)^{2/3} \lesssim \mathcal{F}_{1/2}^{-1}(z) \lesssim (z-1)^{2/3} \Leftrightarrow \mathcal{F}_{1/2}^{-1}(z) \sim (z-1)^{2/3}.$$

This shows the claim and finishes the proof. □

Additionally to Corollary 74 we also need an estimate on $(zG'(z))'$.

Lemma 78. *For $z \in (0,\infty)$ it holds that*

$$(zG'(z))' \lesssim \mathbb{1}_{(z<\mathcal{F}_{1/2}(0))} + z^{-1/3}\mathbb{1}_{(z\geq\mathcal{F}_{1/2}(0))}. \tag{4.75}$$

Proof. We first prove the part for $z < \mathcal{F}_{1/2}(0)$. Using the chain rule we compute

$$\begin{aligned}(zG'(z))' &= G'(z) + zG''(z) \\ &= \frac{1}{\mathcal{F}'_{1/2}(\mathcal{F}^{-1}_{1/2}(z))} - \frac{z\mathcal{F}''_{1/2}(\mathcal{F}^{-1}_{1/2}(z))}{\mathcal{F}'_{1/2}(\mathcal{F}^{-1}_{1/2}(z))^3} \\ &= \frac{\mathcal{F}'_{1/2}(\mathcal{F}^{-1}_{1/2}z)^2 - z\mathcal{F}''_{1/2}(\mathcal{F}^{-1}_{1/2}(z))}{\mathcal{F}'_{1/2}(\mathcal{F}^{-1}_{1/2}(z))^3}.\end{aligned}$$

We again write $\eta = \mathcal{F}^{-1}_{1/2}(z)$ and define the auxiliary functions

$$\begin{aligned}f(\eta) &:= \frac{\mathcal{F}'_{1/2}(\eta)^2 - \mathcal{F}_{1/2}(\eta)\mathcal{F}''_{1/2}(\eta)}{\mathcal{F}'_{1/2}(\eta)^3}, \\ g(\eta) &:= \mathcal{F}'_{1/2}(\eta)^2 - \mathcal{F}_{1/2}(\eta)\mathcal{F}''_{1/2}(\eta).\end{aligned}$$

To prove the asymptotic behaviour for $z < \mathcal{F}_{1/2}(0)$ or as $z \to 0$, we equivalently prove the behaviour for $\eta < 0$ or as $\eta \to -\infty$. To this end, we compute the derivative of $\mathcal{F}_{1/2}(\eta)$ by exchanging the derivative (with respect to η) with the integral (with respect to ξ) in the definition of $\mathcal{F}_{1/2}(\eta)$ and obtain (setting $C_{1/2} := \Gamma(1+1/2)^{-1}$)

$$\begin{aligned}\mathcal{F}_{1/2}(\eta) &= C_{1/2}\int_0^\infty \frac{\sqrt{\xi}}{1+e^{\xi-\eta}}\,d\xi, \\ \mathcal{F}'_{1/2}(\eta) &= C_{1/2}\int_0^\infty \frac{\sqrt{\xi}e^{\xi-\eta}}{(1+e^{\xi-\eta})^2}\,d\xi, \\ \mathcal{F}''_{1/2}(\eta) &= C_{1/2}\int_0^\infty \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi.\end{aligned}$$

Using these integral representations with different integration variables and by applying Fubini's theorem, we compute the auxiliary function as follows:

$$
\begin{aligned}
g(\eta) &= C_{1/2}^2 \int_0^\infty \int_0^\infty \frac{\sqrt{\xi\nu} e^{\xi-\eta} e^{\nu-\eta}}{(1+e^{\xi-\eta})^2(1+e^{\nu-\eta})^2}\, d\xi\, d\nu \\
&\quad - C_{1/2}^2 \int_0^\infty \int_0^\infty \frac{\sqrt{\xi\nu} e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3(1+e^{\nu-\eta})}\, d\xi\, d\nu \\
&= C_{1/2}^2 \int_0^\infty \int_0^\infty \sqrt{\xi\nu} \frac{e^{\xi-\eta}e^{\nu-\eta}(1+e^{\xi-\eta}) - e^{\xi-\eta}(e^{\xi-\eta}-1)(1+e^{\nu-\eta})}{(1+e^{\xi-\eta})^3(1+e^{\nu-\eta})^2}\, d\xi\, d\nu \\
&= C_{1/2}^2 \int_0^\infty \int_0^\infty \sqrt{\xi\nu} e^{\xi-\eta} \frac{e^{\nu-\eta} + e^{\nu-\eta}e^{\xi-\eta} - e^{\xi-\eta} - e^{\nu-\eta}e^{\xi-\eta} + 1 + e^{\nu-\eta}}{(1+e^{\xi-\eta})^3(1+e^{\nu-\eta})^2}\, d\xi\, d\nu \\
&= C_{1/2}^2 \int_0^\infty \int_0^\infty \sqrt{\xi\nu} e^{\xi-\eta} \frac{2e^{\nu-\eta} - e^{\xi-\eta} + 1}{(1+e^{\xi-\eta})^3(1+e^{\nu-\eta})^2}\, d\xi\, d\nu \\
&= C_{1/2}^2 \int_0^\infty \int_0^\infty e^{\xi-\eta}\sqrt{\xi\nu} \left(\frac{2}{(1+e^{\xi-\eta})^3(1+e^{\nu-\eta})} - \frac{1}{(1+e^{\xi-\eta})^2(1+e^{\nu-\eta})^2} \right) d\xi\, d\nu \\
&= 2\mathcal{F}_{1/2}(\eta) C_{1/2} \int_0^\infty \frac{\sqrt{\xi} e^{\xi-\eta}}{(1+e^{\xi-\eta})^3}\, d\xi - \mathcal{F}_{1/2}'(\eta) C_{1/2} \int_0^\infty \frac{\sqrt{\nu}}{(1+e^{\nu-\eta})^2}\, d\nu.
\end{aligned}
$$

We estimate the two remaining integrals. Using

$$
\frac{1}{1+e^{\xi-\eta}} \le e^{\eta}, \text{ for } \xi \in (0,\infty),
$$

we can estimate

$$
C_{1/2} \int_0^\infty \frac{\sqrt{\xi} e^{\xi-\eta}}{(1+e^{\xi-\eta})^3}\, d\xi \le e^{\eta} C_{1/2} \int_0^\infty \frac{\sqrt{\xi} e^{\xi-\eta}}{(1+e^{\xi-\eta})^2}\, d\xi = e^{\eta} \mathcal{F}_{1/2}'(\eta)
$$

and in the same way

$$
C_{1/2} \int_0^\infty \frac{\sqrt{\nu}}{(1+e^{\nu-\eta})^2}\, d\nu \le e^{\eta} \mathcal{F}_{1/2}(\eta).
$$

Combining these estimates with the previous computations and using (4.65) we obtain

$$
g(\eta) \le 2e^{\eta}\mathcal{F}_{1/2}(\eta)\mathcal{F}_{1/2}'(\eta) + e^{\eta}\mathcal{F}_{1/2}(\eta)\mathcal{F}_{1/2}'(\eta) = 3e^{\eta}\mathcal{F}_{1/2}(\eta)\mathcal{F}_{1/2}'(\eta) \lesssim e^{3\eta}
$$

and hence, by again using (4.65)

$$
f(\eta) = \frac{g(\eta)}{\mathcal{F}_{1/2}'(\eta)^3} \lesssim 1, \text{ for } \eta < 0.
$$

This proves the first part of (4.75), i.e.

$$
(zG'(z))' \lesssim 1, \text{ for } z < \mathcal{F}_{1/2}(0).
$$

To show the second part, let us quickly recall that

$$
(zG'(z))' = G'(z) + zG''(z), \text{ and } G'(z) \sim z^{-1} + z^{-1/3}.
$$

Therefore, we only have to show that

$$zG''(z) = -\frac{z\mathcal{F}''_{1/2}(\mathcal{F}^{-1}_{1/2}(z))}{\mathcal{F}'_{1/2}(\mathcal{F}^{-1}_{1/2}(z))^3} \lesssim z^{-1/3}, \text{ for } z \geq \mathcal{F}_{1/2}(0).$$

Again, we use the representation

$$\mathcal{F}''_{1/2}(\eta) = C_{1/2}\int_0^\infty \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi.$$

Now let $\eta_0 \gg 0$ and $0 < \varepsilon < \eta_0$ arbitrary. For $\eta > \eta_0$ we split the integral of $\mathcal{F}''_{1/2}$ at ε and obtain

$$-\mathcal{F}''_{1/2}(\eta) = -C_{1/2}\int_0^\varepsilon \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi - C_{1/2}\int_\varepsilon^\infty \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi =: I_1 + I_2.$$

Estimating I_1 is straightforward:

$$\begin{aligned}
I_1 &= -C_{1/2}\int_0^\varepsilon \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi \\
&= C_{1/2}\int_0^\varepsilon \frac{\sqrt{\xi}e^{\xi-\eta}(1-e^{\xi-\eta})}{(1+e^{\xi-\eta})^3}\,d\xi \\
&\leq C_{1/2}e^{\varepsilon-\eta}\int_0^\varepsilon \sqrt{\xi}\,d\xi \\
&= \frac{2C_{1/2}}{3}e^{\varepsilon-\eta}\varepsilon^{3/2}.
\end{aligned}$$

Next, we want to integrate by parts twice in I_2, therefore we first compute

$$\begin{aligned}
\partial_\xi \frac{1}{1+e^{\xi-\eta}} &= -\frac{e^{\xi-\eta}}{(1+e^{\xi-\eta})^2} \\
\partial_\xi^2 \frac{1}{1+e^{\xi-\eta}} &= \frac{e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}.
\end{aligned}$$

Thus, integrating by parts twice in I_2 and splitting the resulting integral then at η yields

$$\begin{aligned}
I_2 &= -C_{1/2}\int_\varepsilon^\infty \frac{\sqrt{\xi}e^{\xi-\eta}(e^{\xi-\eta}-1)}{(1+e^{\xi-\eta})^3}\,d\xi \\
&= -C_{1/2}\left(-\sqrt{\xi}\frac{e^{\xi-\eta}}{(1+e^{\xi-\eta})^2}\bigg|_\varepsilon^\infty - \int_\varepsilon^\infty \frac{-\xi^{-1/2}e^{\xi-\eta}}{2(1+e^{\xi-\eta})^2}\,d\xi\right) \\
&= -C_{1/2}\left(\frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2} - \left(\frac{\xi^{-1/2}}{2(1+e^{\xi-\eta})}\bigg|_\varepsilon^\infty - \int_\varepsilon^\infty \frac{-\xi^{-3/2}}{4(1+e^{\xi-\eta})}\,d\xi\right)\right) \\
&= -C_{1/2}\left(\frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2} - \left(-\frac{\varepsilon^{-1/2}}{2(1+e^{\xi-\eta})} + \int_\varepsilon^\infty \frac{\xi^{-3/2}}{4(1+e^{\xi-\eta})}\,d\xi\right)\right) \\
&= -C_{1/2}\left(\frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2} + \frac{\varepsilon^{-1/2}}{2(1+e^{\xi-\eta})}\right) + C_{1/2}\left(\int_\varepsilon^\eta + \int_\eta^\infty\right)\frac{\xi^{-3/2}}{4(1+e^{\xi-\eta})}\,d\xi \\
&= -C_{1/2}\left(\frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2} + \frac{\varepsilon^{-1/2}}{2(1+e^{\xi-\eta})}\right) + C_{1/2}(I_3 + I_4).
\end{aligned}$$

It is now straightforward to estimate I_3,

$$I_3 = \int_\varepsilon^\eta \frac{\xi^{-3/2}}{4(1+e^{\xi-\eta})}\,d\xi \leq \frac{1}{4(1+e^{\varepsilon-\eta})}\int_\varepsilon^\eta \xi^{-3/2}\,d\xi = -\frac{2\xi^{-1/2}}{4(1+e^{\varepsilon-\eta})}\bigg|_\varepsilon^\eta = \frac{\varepsilon^{-1/2}-\eta^{-1/2}}{2(1+e^{\varepsilon-\eta})},$$

and the term I_4,

$$I_4 = \int_\eta^\infty \frac{\xi^{-3/2}}{4(1+e^{\xi-\eta})}\,d\xi \leq \frac{2}{4}\int_\eta^\infty \xi^{-3/2}\,d\xi = -\xi^{-1/2}\bigg|_\eta^\infty = \eta^{-1/2}.$$

Collecting all estimates we find that

$$\begin{aligned}
-\mathcal{F}''_{1/2}(\eta) &\leq C_{1/2}\left(\frac{2\varepsilon^{3/2}e^{\varepsilon-\eta}}{3} - \frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2} - \frac{\varepsilon^{-1/2}}{2(1+e^{\varepsilon-\eta})} + \frac{\varepsilon^{-1/2}-\eta^{-1/2}}{2(1+e^{\varepsilon-\eta})} + \eta^{-1/2}\right)\\
&= C_{1/2}\left(\frac{1+2e^{\varepsilon-\eta}}{2(1+e^{\varepsilon-\eta})}\eta^{-1/2} + \frac{2\varepsilon^{3/2}e^{\varepsilon-\eta}}{3} - \frac{\sqrt{\varepsilon}e^{\varepsilon-\eta}}{(1+e^{\varepsilon-\eta})^2}\right).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we find in the limit $\varepsilon \to 0$ that

$$-\mathcal{F}''_{1/2}(\eta) \leq C_{1/2}\frac{1+2e^{-\eta}}{2(1+e^{-\eta})}\eta^{-1/2} \leq \frac{3}{4}\eta^{-1/2}, \quad \text{for all } \eta \geq 0.$$

Using (4.67) to estimate the denominator and (4.74) to estimate $\mathcal{F}_{1/2}^{-1}(z)$, this allows us to provide the needed estimate

$$-\frac{z\mathcal{F}''_{1/2}(\mathcal{F}_{1/2}^{-1}(z))}{\mathcal{F}'_{1/2}(\mathcal{F}_{1/2}^{-1}(z))} \lesssim \frac{z\mathcal{F}_{1/2}^{-1}(z)^{-1/2}}{\mathcal{F}_{1/2}(\mathcal{F}_{1/2}^{-1}(z))^{3/3}} = \mathcal{F}_{1/2}^{-1}(z)^{-1/2} \lesssim (z-1)^{(-1/2)(2/3)} = (z-1)^{-1/3} \lesssim z^{-1/3}.$$

Hence, we obtain

$$(zG'(z))' \lesssim z^{-1/3} + z^{-1} \lesssim z^{-1/3}, \text{ for } z \geq \mathcal{F}_{1/2}(0).$$

This shows the second part of (4.75) and finishes the proof. □

5 Discussion and outlook

We briefly summarise our results and give an outlook over possible extensions and open problems connected to this thesis.

5.1 Nonlocal cross-diffusion systems

We have proven the global existence of weak solutions for a class of nonlocal cross-diffusion systems, imposing only "minimal" conditions on the interaction kernels K_{ij}. Furthermore, we showed a weak-strong uniqueness result for the solutions, the boundedness of solutions given that the kernels are twice differentiable with bounded second derivatives, and the localization limit of the system.

While these results extend the existing literature, there is of course room for further research. The existence of bounded solutions to the local system remains an open question. The authors of [61] gave a partial answer to that question, listing conditions for the parameters a_{ij} which will lead to a blow-up of solutions. However, these conditions do not cover our assumptions of a detailed balance and positive stability, cf. Theorem 17.

Furthermore, the case of bounded domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ remains open. It seems that for essentially bounded interaction kernels K_{ij} and sufficiently small initial conditions u_j^0 one might be able to obtain a result (cf. Remark 9), but this leaves room for improvement.

Last but not least, we observe that the uniqueness of solutions is a delicate topic in cross-diffusion systems. We have proven weak-strong uniqueness of solutions under the assumption that the strong solution is bounded away from zero as well as from above. A next step could be to improve this result to nonnegative bounded strong solutions, since the positive lower bound is due to our use of a relative entropy method and therefore seems to be a technical assumption.

5.2 A finite-volume scheme for nonlocal cross-diffusion systems

We have designed and analysed an implicit Euler finite-volume scheme for a class of nonlocal cross-diffusion systems and proved existence of discrete solutions as well as their convergence to weak solutions of the system when the mesh size is refined.

Additionally, we did some numerical experiments and observed segregation of species, given that the initial data are segregated. For the local system and two species, this was proven in [17]. However, for more than two species and for nonlocal cross-diffusion systems in general, this remains an open question. Our observations hint that a segregation result is also plausible in this more general setting but to our knowledge no analytical result exists yet.

Furthermore, the data we obtained in our numerical experiments also suggest that solutions stay bounded. Due to the convergence of the scheme, obtaining bounds on the solutions, which for example only depend on the L^1-norms of the kernels K_{ij}, could be a way to also prove the existence of bounded solutions to the continuous system studied in Chapter 2.

5.3 A charge transport system with Fermi-Dirac statistics for memristors

We have shown the global existence and, under additional assumptions, the uniform boundedness in time of weak solutions to a nonlinear drift-diffusion system modelling memristive devices. Fermi-Dirac statistics of order 1/2 and Blakemore statistics were used to govern the nonlinear diffusion of the different charge carriers.

An interesting question to investigate would be the existence of periodic solutions. Neurons exhibit a switch-like behaviour followed by a refractory period. Combining one or more resistive switching random access memory devices together with a parallel capacitor in a circuit, one can model these relaxation oscillations in a controlled fashion, see [69] and the references therein. However, the need for the use of an integrated capacitor limits the scalability of such devices, hence studying memristor models for the existence of relaxation oscillations provides an interesting question.

Moreover, the results in [71] and [77] suggest that a weak-strong uniqueness result should also hold for the system we studied in Chapter 4, but this is still subject to investigation.

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