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# A NOTE ON THE SHIFT THEOREM FOR THE LAPLACIAN IN POLYGONAL DOMAINS

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*in memoriam Ivo M. Babuška (1926–2023)*

*Abstract.* We present a shift theorem for solutions of the Poisson equation in a finite planar cone (and hence also on plane polygons) for Dirichlet, Neumann, and mixed boundary conditions. The range in which the shift theorem holds depends on the angle of the cone. For the right endpoint of the range, the shift theorem is described in terms of Besov spaces rather than Sobolev spaces.

*Keywords*: Besov space; corner domain; corner singularity; Mellin calculus

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## 1. Introduction

The classical shift theorem for second order elliptic boundary value problems expresses the observation that regularity of the solution $u$ is two Sobolev orders better than the right-hand side $f$. For example, for the Laplacian with Dirichlet boundary conditions and smooth domains, this shift theorem takes the form

$$f \in H^{-1+s} \text{ implies } u \in H^{1+s} \tag{1.1}$$

for any $s \geqslant 0$ [19], [22], Chapter 2. In 2D polygonal domains or even Lipschitz domains, the shift theorem (1.1) is still valid, however, for a restricted range of values $s \in [0, s_0)$, where $s_0 = 1/2$ for Lipschitz domains [36] and $s_0 > 1/2$ for

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polygonal $\Omega$ depends on the interior angles of $\Omega$ [22]. While the shift theorem does not hold in the limiting case $s = s_0$ in the scale of Sobolev spaces, we show that it holds in suitable Besov spaces.

We prove the shift theorem in the limiting case $s = s_0$ using the well-known expansion of the solution in terms of singularity functions. For the purpose of exposition, consider a cone $\mathcal{C}$ with apex at the origin and angle $\omega > \pi$. Then, near the origin, a solution $u \in H^1(\mathcal{C})$ of the Dirichlet problem can be written as

$$u = S(f)s^+\chi + u_0, \tag{1.2}$$

where $s^+$ is a known singularity function (see (2.14), where $s^+ = s_1^D$), $\chi$ is a smooth cut-off function with $\chi \equiv 1$ near the origin, $u_0 \in H^2$ for $f \in L^2$, and $f \mapsto S(f)$ is a linear functional. We prove the shift theorem in Besov spaces for the limiting case using three ingredients:

(i) we assert that $s^+ \in B^{\alpha}_{2,\infty}$ for a Besov space $B^{\alpha}_{2,\infty}$;
(ii) we show that $f \mapsto S(f)$ is a linear functional on a Besov space of the type $B^{\alpha'}_{2,1}$;
(iii) we use the Mellin calculus to get a shift theorem for the mapping $f \mapsto u_0$.

Shift theorems involving Besov spaces for the endpoints of the Sobolev scale have been shown to be appropriate in [36], Theorem 2. For Lipschitz domains and Dirichlet conditions, it is shown that the solution $u$ of the Poisson problem $-\nabla \cdot (a\nabla u) = f$ (with sufficiently regular positive definite $a$) satisfies $\|u\|_{B^{1+1/2}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{-1+1/2}_{2,1}(\Omega)}$. (A similar result holds for Neumann boundary conditions.) The proof relies on difference quotient techniques that are adapted to Dirichlet or Neumann conditions; an extension to mixed boundary conditions has to impose convexity conditions on the geometry [18], [17]. The endpoint result of the shift theorem of [36], Theorem 2, implies by interpolation the regularity result for the Poisson problem of [25], Theorems 1.1, 1.3, which was obtained by a completely different method, namely, tools from harmonic analysis, although interpolation spaces are employed *en route*; these tools from harmonic analysis allow one to show shift theorems up to 1/2 in scales of Sobolev spaces for the Dirichlet or Neumann Laplace problem (i.e., homogeneous right-hand side but inhomogeneous boundary conditions) on Lipschitz domains *including* the endpoint 1/2 [23], [24].

Moving from general Lipschitz domains to polygonal (in 2D) or polyhedral (in 3D) gives the solution more structure. A powerful way to describe the solution structure consists in expansions of the form (1.2) and the Mellin calculus to derive these expansions. Expansions in corner domains started with the seminal work on 2D corner domains in [26]; a comprehensive discussion of the 2D case was achieved in [22], [21]. The much more complex higher-dimensional cases and higher order equations and even certain nonlinear equations were addressed in [29], in [15], and in [11], [12], [22], [15], [34], [27], [28], [33], [30]. Formulas for the linear functionals $f \mapsto S(f)$ alluded to above go back to the work by Maz'ya and Plamenevskij [28].

Describing solutions in terms of expansions of the form (1.2) leads to a further possible regularity theory for solutions of elliptic boundary value problems in corner domains, namely, the use of weighted spaces, which has applications to finite element approximation theory on graded meshes [3]. While corner weighted spaces of finite regularity are a natural habitat of solutions and data in the framework of the Mellin calculus, weighted analytic regularity for problems in corner domains was developed by Babuška and Guo in [1], [2] for polygonal domains and by Costabel, Dauge, and Nicaise in [10] for polyhedra.

Elliptic shift theorems in Besov spaces have been derived in [14], [13] with a view to characterize optimal convergence rates for adaptive numerical methods. Our present focus on the limiting case $s = s_0$ is close to the works [5], [7], [6]. Indeed, [6] obtains the same shift theorem as we do but effectively restricts the attention to convex domains with one corner with interior angles between $\pi/2$ and $\pi$; [7] restricts to non-convex domains and right-hand sides $f$ in a Besov space that is the interpolation space between $H^{-1}$ and a subspace of $L^2$ of co-dimension 1.

In the present work, by analyzing the singularity function $s^+$ and the associated linear functional $f \mapsto S(f)$ in (1.2), we are able to lift these restrictions of [7], [6] and show in Theorem 1.1 a local shift theorem near a corner without restrictions on the interior angle in the framework of standard Besov spaces. Additionally, we explicitly consider Dirichlet, Neumann, and mixed boundary conditions.

Our proof of the limiting case of the shift theorem relies on expansions in singularity functions and rather explicit formulas for the stress intensity functions. Extensions to 3D might be possible for geometries with point singularities; the presence of edge singularities would require new tools.

Our main result, Theorem 1.1, is formulated in terms of $L^2$-based Besov spaces. Besov spaces based on $L^p$-spaces can alternatively be considered. In Section 5 we briefly indicate that endpoint results in such $L^p$-based Besov spaces can be obtained by the same approach.

### 1.1. Notation.

**1.1.1. Interpolation spaces.** For Banach spaces $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ with continuous embedding $X_1 \subset X_0$ and $\theta \in (0,1)$, $q \in [1,\infty]$, we define with the so-called "real method"/"$K$-method" the interpolation spaces $X_{\theta,q} := (X_0, X_1)_{\theta,q} :=$

$\{u \in X_0 \mid \|u\|_{(X_0,X_1)_{\theta,q}} < \infty\}$, where the norm $\|u\|_{X_{\theta,q}} := \|u\|_{(X_0,X_1)_{\theta,q}}$ is given by

$$\|u\|_{X_{\theta,q}} := \|u\|_{(X_0,X_1)_{\theta,q}} := \begin{cases} \left(\int_{t=0}^{\infty} (t^{-\theta} K(t,u))^q \frac{\mathrm{d}t}{t}\right)^{1/q}, & q \in [1,\infty), \\ \sup_{t>0} t^{-\theta} K(t,u), & q = \infty \end{cases} \tag{1.3}$$

with the $K$-functional

$$K(t,u) = \inf_{v \in X_1} \|u - v\|_{X_0} + t\|v\|_{X_1}.$$

We refer to [31], [37], [39] for discussions of interpolation spaces. We have the continuous embedding $X_{\theta,q} \subset X_{\theta',q'}$ if $\theta > \theta'$ ($q$, $q'$ arbitrary) or $\theta = \theta'$ and $q \leqslant q'$. We highlight that in the present case of $X_1 \subset X_0$, the integral $\int_0^\infty$ in (1.3) can actually be replaced with the finite integral $\int_0^1$, [16], Chapter 6, Section 7. An important property of interpolation spaces is the Reiteration Theorem [37], Theorem 26.3, which states that for $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and arbitrary $\theta \in (0,1)$, $q_1$, $q_2$, $q \in [1,\infty]$, one has (with norm equivalence) $(X_{\theta_1,q_1}, X_{\theta_2,q_2})_{\theta,q} = X_{\theta_1(1-\theta)+\theta_2\theta,q}$.

**1.1.2. Sobolev and Besov spaces.** For domains $D \subset \mathbb{R}^d$, $d \in \{1,2\}$, we employ the usual Sobolev spaces $H^s(D)$ and $\widetilde{H}^s(D)$ for $s \in \mathbb{R}$ as described in, e.g., [31] or [38]. To be specific and following [31], with the space $\mathcal{S}^\star$ of tempered distributions and the Fourier transformation $\mathcal{F}$, the spaces $H^s(\mathbb{R}^d)$ are given by $H^s(\mathbb{R}^d) = \{u \in \mathcal{S}^\star \mid \|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}u|^2 \,\mathrm{d}\xi < \infty\}$. We set $H^s(D) := \{u \in \mathcal{D}^\star(D) \mid u = U|_D$ for some $U \in H^s(\mathbb{R}^d)\}$ with the norm $\|u\|_{H^s(D)} := \inf\{\|U\|_{H^s(\mathbb{R}^d)} \mid U|_D = u\}$, where $\mathcal{D}^\star(D)$ denotes the space of distributions on $D$. We set $\widetilde{H}^s(D) := \{u \in H^s(\mathbb{R}^d) \mid \operatorname{supp} u \subset \overline{D}\}$ with the norm $\|u\|_{\widetilde{H}^s(D)} = \|u\|_{H^s(\mathbb{R}^d)}$. (The space $\widetilde{H}^s(D)$ is denoted $H^s_{\overline{D}}$ in [31], page 76, but coincides with the space $\widetilde{H}^s(D)$ defined in [31], page 77 by [31], Theorem 3.29.) An important relation of these spaces is the duality relation [31], Theorem 3.30

$$\widetilde{H}^{-s}(D) = (H^s(D))^\star, \quad H^{-s}(D) = (\widetilde{H}^s(D))^\star, \quad s \in \mathbb{R}.$$

Furthermore, one has $H^0(D) = \widetilde{H}^0(D) = L^2(D)$ and, by [22], Corollary 1.4.4.5, for $s \in (0,1/2)$ and by duality for $s \in (-1/2,0)$,

$$H^s(D) = \widetilde{H}^s(D), \quad |s| < 1/2. \tag{1.4}$$

The two scales $H^s(D)$, $\widetilde{H}^s(D)$, $s \in \mathbb{R}$, of Sobolev spaces are scales of interpolation spaces: by [31], Theorems B.8, B.9 we have for $s_1$, $s_2 \in \mathbb{R}$, $\theta \in (0,1)$,

$$(H^{s_1}(D), H^{s_2}(D))_{\theta,2} = H^{(1-\theta)s_1+\theta s_2}(D), \ (\widetilde{H}^{s_1}(D), \widetilde{H}^{s_2}(D))_{\theta,2} = \widetilde{H}^{(1-\theta)s_1+\theta s_2}(D). \tag{1.5}$$

The scales of Besov spaces $B^s_{2,q}(D)$ and $\widetilde{B}^s_{2,q}(D)$ are defined by interpolating between Sobolev spaces: given $s \in \mathbb{R}$, select $s_1 < s < s_2$ and set with $\theta := (s - s_1)/(s_2 - s_1)$

$$B^s_{2,q}(D) := (H^{s_1}(D), H^{s_2}(D))_{\theta,q}, \ \widetilde{B}^s_{2,q}(D) := (\widetilde{H}^{s_1}(D), \widetilde{H}^{s_2}(D))_{\theta,q}. \tag{1.6}$$

The Reiteration Theorem [37], Theorem 26.3, asserts that the precise choice of $s_1$, $s_2$ is immaterial. The formula (1.4) also implies

$$B^s_{2,q}(D) = \widetilde{B}^s_{2,q}(D), \quad |s| < \frac{1}{2}, \quad q \in [1, \infty]. \tag{1.7}$$

**1.2. Setting and main results.** We study the regularity of solutions of elliptic problems in a cone. For an angle $\omega \in (0, 2\pi)$ we therefore introduce in polar coordinates[1] $(r, \varphi)$ the cone $\mathcal{C}$ and the truncated cones $\mathcal{C}_R$ by

$$\mathcal{C} := \{(r\cos\varphi, r\sin\varphi) \mid r > 0, \ \varphi \in G\}, \quad G := (0, \omega), \tag{1.8}$$

$$\mathcal{C}_R := \mathcal{C} \cap B_R(0), \tag{1.9}$$

where $B_r(0) \subset \mathbb{R}^2$ denotes the (open) ball of radius $r > 0$ centered at 0. The two lateral sides of $\mathcal{C}$ are $\Gamma_0 = \{(r, 0) \mid r > 0\}$ and $\Gamma_\omega = \{(r\cos\omega, r\sin\omega) \mid r > 0\}$. The three boundary parts of $\mathcal{C}_R$ are $\Gamma_{0,R} = \Gamma_0 \cap B_R(0)$, $\Gamma_{\omega,R} = \Gamma_\omega \cap B_R(0)$, and $\widetilde{\Gamma}_R := \{(R\cos\varphi, R\sin\varphi) \mid \varphi \in (0, \omega)\}$. We consider $H^1(\mathcal{C}_R)$-functions $u$ that satisfy

$$-\Delta u = f \quad \text{in } \mathcal{C}_R, \tag{1.10a}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{1.10b}$$

$$\partial_n u = 0 \quad \text{on } \Gamma_N. \tag{1.10c}$$

Concerning the boundary conditions, we consider three cases:

- ▷ *Dirichlet case*: $\Gamma_D = \Gamma_{0,R} \cup \Gamma_{\omega,R}$ and $\Gamma_N = \emptyset$;
- ▷ *Neumann case*: $\Gamma_N = \Gamma_{0,R} \cup \Gamma_{\omega,R}$ and $\Gamma_D = \emptyset$;
- ▷ *mixed case*: $\Gamma_D = \Gamma_{0,R}$ and $\Gamma_N = \Gamma_{\omega,R}$.

The equation (1.10) is understood in a weak sense. That is, we define the space $H^1_D(\mathcal{C}_R) := \{v \in H^1(\mathcal{C}_R) \mid v|_{\widetilde{\Gamma}_R \cup \Gamma_D} = 0\}$ and its dual $H^{-1}_D(\mathcal{C}_R) := (H^1_D(\mathcal{C}_R))^\star$. The minimal regularity assumption for (1.10) is $f \in H^{-1}_D(\mathcal{C}_R)$. Then, $u \in H^1(\mathcal{C}_R)$ solves (1.10) if $u|_{\Gamma_D} = 0$ (in the sense of traces) and the equations (1.10a), (1.10c) are satisfied in a weak sense, i.e.,

$$\int_{\mathcal{C}_R} \nabla u \cdot \nabla v = \langle f, v \rangle_{H^{-1}_D(\mathcal{C}_R) \times H^1_D(\mathcal{C}_R)} \quad \forall\, v \in H^1_D(\mathcal{C}_R).$$

[1] Throughout, we freely identify points $\mathbf{x} = (x, y) \in \mathbb{R}^2$ either in Cartesian or polar coordinates $(r, \varphi)$.

For the solutions $u$ of (1.10), we have the following result:

**Theorem 1.1** (shift theorem, Besov spaces). *Let $\omega \in (0, 2\pi)$. Fix $0 < R' < R$. Let $f \in H_D^{-1}(\mathcal{C}_R)$, and let $\chi_R \in C_0^\infty(B_R(0))$ with $\chi_R \equiv 1$ on $B_{R'}(0)$. Then for a solution $u \in H^1(\mathcal{C}_R)$ of (1.10) the following statements hold with implied constants depending only on $\omega$, $R$, $R'$, and $\chi_R$:*

(i) *Dirichlet case: For $\chi_R f \in B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)$ one has $u \in B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_{R'})$ with the estimate*

$$\|u\|_{B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_{R'})} \lesssim \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}. \tag{1.11}$$

(ii) *Neumann case: For $\chi_R f \in B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)$ one has $u \in B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_{R'})$ with the estimate*

$$\|u\|_{B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_{R'})} \lesssim \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}. \tag{1.12}$$

(iii) *Mixed case: If $\chi_R f \in \widetilde{B}_{2,1}^{\pi/(2\omega)-1}(\mathcal{C}_R)$ (if $\omega \geqslant \pi/2$) or $\chi_R f \in B_{2,1}^{\pi/(2\omega)-1}(\mathcal{C}_R)$ (if $\omega < \pi/2$) one has $u \in B_{2,\infty}^{1+\pi/(2\omega)}(\mathcal{C}_{R'})$ with the estimate*

$$\|u\|_{B_{2,\infty}^{1+\pi/(2\omega)}(\mathcal{C}_{R'})} \lesssim \begin{cases} \|\chi_R f\|_{B_{2,1}^{-1+\pi/(2\omega)}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \text{if } \omega < \pi/2, \\ \|\chi_R f\|_{\widetilde{B}_{2,1}^{-1+\pi/(2\omega)}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \text{if } \omega \geqslant \pi/2. \end{cases} \tag{1.13}$$

P r o o f. Item (i) is shown in Section 2, item (ii) is discussed in Section 3, and item (iii) in Section 4. □

R e m a r k 1.2. The cases $\omega = \pi$ for Dirichlet and Neumann boundary conditions can be sharpened. This case corresponds to a smooth geometry so that by standard elliptic regularity theory [19], [20] the solution is as smooth as the right-hand side $f$ permits near the origin, i.e., one has estimates of the form

$$\|u\|_{H^{1+s}(\mathcal{C}_{R'})} \lesssim \|\chi_R f\|_{H^{-1+s}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}$$

for all $s \geqslant 0$. The implied constant additionally depends on $s$. Likewise for mixed boundary conditions in the case $\omega = \pi/2$ and the present setting of homogeneous boundary conditions, the shift theorem holds in a larger range as can be seen from the proof of Theorem 1.1 (iii).

Theorem 1.1 discusses a limiting case of the shift theorem. With similar techniques as those used in the proof of Theorem 1.1, one can show a shift theorem in a range of regularity indices:

**Corollary 1.3.** *Assume the hypotheses and notation of Theorem* 1.1.

(i) *Dirichlet case: A solution* $u \in H^1(\mathcal{C}_R)$ *of* (1.10) *satisfies for* $0 < s < \pi/\omega$ *and* $q \in [1, \infty]$

$$\|u\|_{B^{s+1}_{2,q}(\mathcal{C}_{R'})} \lesssim \|\chi_R f\|_{B^{s-1}_{2,q}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}. \tag{1.14}$$

(ii) *Neumann case: A solution* $u \in H^1(\mathcal{C}_R)$ *of* (1.10) *satisfies for* $0 < s < \pi/\omega$ *and* $q \in [1, \infty]$

$$\|u\|_{B^{s+1}_{2,q}(\mathcal{C}_{R'})} \lesssim \begin{cases} \|\chi_R f\|_{\widetilde{B}^{s-1}_{2,q}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \textit{if } s < 1, \\ \|\chi_R f\|_{B^{s-1}_{2,q}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \textit{if } s \geqslant 1. \end{cases} \tag{1.15}$$

(iii) *Mixed case: A solution* $u \in H^1(\mathcal{C}_R)$ *of* (1.10) *satisfies for* $0 < s < \pi/(2\omega)$ *and* $q \in [1, \infty]$

$$\|u\|_{B^{s+1}_{2,q}(\mathcal{C}_{R'})} \lesssim \begin{cases} \|\chi_R f\|_{\widetilde{B}^{s-1}_{2,q}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \textit{if } s < 1, \\ \|\chi_R f\|_{B^{s-1}_{2,q}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)} & \textit{if } s \geqslant 1. \end{cases} \tag{1.16}$$

P r o o f. The result follows by inspection of the proof of Theorem 1.1. For example, for the case of Dirichlet conditions, the proof rests on two ingredients: (a) the shift theorem for the operator $\widetilde{T}$ of (2.36) and (b) the estimate of the function $\widetilde{f}$ in (2.34). The operator $\widetilde{T}$ of (2.36) is directly amenable to interpolation arguments as it maps $H^{-1} \to H^1$ and, by (2.37), $B^{\pi/\omega-1}_{2,1} \to B^{\pi/\omega+1}_{2,\infty}$. Inspection of the proof of (2.34) leads to having to control $\|\nabla\chi_R \cdot \nabla u\|_{B^{-1+s}_{2,q}}$ and $\|\Delta\chi_R u\|_{B^{-1+s}_{2,q}}$. These terms can be estimated with Lemma A.2.

For the Neumann case (and similarly for the mixed case), the analysis is also reduced to understanding the mapping properties of the corresponding operator $\widetilde{T}$. If $\omega > \pi$ (i.e., $\pi/\omega - 1 \in (-1/2, 0)$) one observes that $\widetilde{B}^{\pi/\omega-1}_{2,1} = B^{\pi/\omega-1}_{2,1}$ so that one has by (3.14) the mapping properties $\widetilde{T}\colon \widetilde{H}^{-1} \to H^1$ and $\widetilde{T}\colon \widetilde{B}^{\pi/\omega-1}_{2,1} \to B^{\pi/\omega+1}_{2,\infty}$. An interpolation argument like in the Dirichlet case concludes the argument. If $\omega < \pi$, one splits the argument into two interpolation steps. First, one observes from Corollary 3.3 for $k = 0$ and $\varepsilon \in (0, 1/2)$ sufficiently small that $\widetilde{T} : H^{\varepsilon} \to H^{2+\varepsilon}$. Hence, $\widetilde{T}\colon \widetilde{H}^{-1} \to H^1$ and $\widetilde{T}\colon \widetilde{H}^{\varepsilon} = H^{\varepsilon} \to H^{2+\varepsilon}$, which provides the desired result for $s \in (0, 1 + \varepsilon)$ by interpolation. For $s \in (1, \pi/\omega)$, one interpolates using the mapping properties $\widetilde{T}\colon H^{\varepsilon} \to H^{2+\varepsilon}$ and $\widetilde{T}\colon B^{-1+\pi/\omega}_{2,1} \to B^{1+\pi/\omega}_{2,\infty}$ provided by (3.14). □

We start with introducing corner-weighted functions that are useful in connection with Mellin transform techniques:

**Definition 2.1** (Weighted spaces). For $s \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$, we put

$$K^s_\gamma(\mathcal{C}) := \Big\{ u \in L^2_{\rm loc}(\mathcal{C}) \mid \|u\|^2_{K^s_\gamma(\mathcal{C})} := \sum_{|\alpha| \leqslant s} \|r^{|\alpha|-s+\gamma} D^\alpha u\|^2_{L^2(\mathcal{C})} < \infty,\ |\alpha| \leqslant s \Big\}.$$

The spaces $K^s_\gamma(\mathcal{C}_R)$ are defined in the same way, just by replacing $\mathcal{C}$ by $\mathcal{C}_R$.

Fractional order Sobolev spaces of functions that are constrained to vanish to a certain order at the origin are shown in the following Lemma 2.2 to be subspaces of suitable weighted Sobolev spaces of the $K^s_\gamma$-type; similar estimates with a focus on integer order Sobolev spaces are well-known in the literature, see, e.g., [27], Chapter 7.1.

**Lemma 2.2.** *Let $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subset B_1(0)$ for some $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$ and assume $\partial^i_x \partial^j_y f(0) = 0$ for $i+j \leqslant k-1$. Then $f \in K^k_{-\varepsilon}(\mathcal{C})$ with the norm estimate*

$$\|f\|_{K^k_{-\varepsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}.$$

P r o o f. See Appendix A. Note that $H^{k+\varepsilon} \subset C^k$ by Sobolev embedding. □

### 2.1. A recap of regularity based on the Mellin calculus.

**2.1.1. Preliminaries.** The following properties of the Mellin transformation are at the heart of the analysis of [26], [15], [27], [28], [33], [30] and are collected in [9], Section 3; we also refer to [35], Chapter 3 for detailed proofs. For a sufficiently regular function $u$ on the cone $\mathcal{C}$, we define its Mellin transform $\mathcal{M}[u]$ by

$$\mathcal{M}[u](\zeta,\varphi) := \frac{1}{\sqrt{2\pi}} \int_{r=0}^{\infty} r^{-{\rm i}\zeta} \widetilde{u}(r,\varphi) \frac{{\rm d}r}{r}, \tag{2.1}$$

where $\widetilde{u}(r,\varphi) = u(r\cos\varphi, r\sin\varphi)$, i.e., the representation of $u$ in polar coordinates. We emphasize, however, that henceforth we will write $u$ for the function both in Cartesian and polar coordinates. The Mellin transformation is connected to the Fourier transformation in that one has with the change of variables $r = {\rm e}^t$

$$\mathcal{M}[u](\zeta,\varphi) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} {\rm e}^{-{\rm i}\zeta t} u({\rm e}^t,\varphi)\,{\rm d}t.$$

This connection with the Fourier transformation is at the heart of the following norm equivalence: if $u(\cdot,\varphi) \in L^2(0,\infty)$, then $\mathcal{M}[u](\cdot,\varphi)$ is in $L^2(\mathbb{R}-\mathrm{i})$ with equivalent norms, and the inverse Mellin transform correspondingly takes the form

$$u = \frac{1}{\sqrt{2\pi}} \int_{\operatorname{Im}\zeta=-\mathrm{i}} r^{\mathrm{i}\zeta} \mathcal{M}[u](\zeta,\varphi)\,\mathrm{d}\zeta.$$

More generally, one has for $k \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ the norm equivalence

$$\|u\|^2_{K^k_\gamma(\mathcal{C})} \sim \int_{\xi\in\mathbb{R}} \|\mathcal{M}[u](\xi - \mathrm{i}\eta)\|^2_{H^k(G;|\xi|)}\,\mathrm{d}\xi,$$
$$\|v\|^2_{H^k(G;|\xi|)} := \sum_{j\leqslant k} (1+|\xi|^2)^{k-j} \|\partial^j_\varphi v\|^2_{L^2(G)}, \quad \eta := k-\gamma-1,$$

where we view the Mellin transformation, which acts only on the variable $r$ (with the dual variable $\zeta$) as a mapping from $K^k_\gamma(\mathcal{C})$ into a space of $H^k(G)$-valued functions. A final important property of the Mellin transformation is that if $u \in K^k_\gamma(\mathcal{C})$ satisfies additionally $\operatorname{supp} u \subset B_1(0)$, then, by a variant of the Paley-Wiener Theorem, $\mathcal{M}[u]$ is actually holomorphic on $\{z \in \mathbb{C} \mid \operatorname{Im} z > -\eta\}$. This property allows one to use the Cauchy integral theorem/residue theorem, whose use leads to expansions in terms of corner singularity functions.

**2.1.2. The isomorphism in weighted spaces and expansion in corner singularity functions.** Let $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$. Consider $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$ and $\partial^i_x \partial^j_y f(0) = 0$ for $i+j \leqslant k-1$. For convenience, we assume $k+1+\varepsilon < 2\pi/\omega$. Note that by Lemma 2.2 the function $f \in K^k_{-\varepsilon}(\mathcal{C})$. Assume that $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ solves the problem

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad u_1 = 0 \quad \text{on } \Gamma_0 \text{ and } \Gamma_\omega. \tag{2.2}$$

Further we pose the auxiliary problem

$$-\Delta u_0 = f \in K^k_{-\varepsilon}(\mathcal{C}), \quad u_0 = 0 \quad \text{on } \Gamma_0 \text{ and } \Gamma_\omega. \tag{2.3}$$

This latter problem admits a unique solution $u_0 \in K^{k+2}_{-\varepsilon}(\mathcal{C})$ by the Mellin calculus going back to [26] with the norm estimate $\|u_0\|_{K^{k+2}_{-\varepsilon}(\mathcal{C})} \lesssim \|f\|_{K^k_{-\varepsilon}(\mathcal{C})}$ (see, e.g., [27], Section 6.1.8, [15], or [35] for more details). By elliptic regularity based on a dyadic decomposition of the cone $\mathcal{C}$ (see Lemma A.3), one actually has $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$ with the estimate $\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C})}$ for each fixed $R > 1$. Following the classical path, we now analyze the relation of the solutions $u_0$ and $u_1$. The Mellin transformation yields

$$\mathcal{L}(\zeta)\mathcal{M}[u_1] = \mathcal{M}[g] \quad \text{on } \{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta > 0\}, \tag{2.4}$$

$$\mathcal{L}(\zeta)\mathcal{M}[u_0] = \mathcal{M}[g] \quad \text{on } \{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta = -1-k-\varepsilon\} \tag{2.5}$$

with the operator $\mathcal{L}(\zeta) := (-\partial_\varphi^2 + \zeta^2)$ and $\mathcal{M}[g]$ being the Mellin transform of the function $g = r^2 f$. Note that $\mathcal{M}[g]$ is holomorphic on $\{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta > -1-k-\varepsilon\}$ with values in $H^k(G)$ and that $\mathcal{M}[u_1]$ is holomorphic on $\{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta > 0\}$ with values in $H^2(G)$. Since the operator $(\mathcal{L}(\zeta))^{-1}$ is meromorphic on $\mathbb{C}$ with poles at the discrete set

$$\pm \mathrm{i}\sigma^D \quad \text{with } \sigma^D := \{\lambda_n^D \mid n \in \mathbb{N}\}, \quad \lambda_n^D := n\frac{\pi}{\omega}, \tag{2.6}$$

we observe that $\mathcal{M}[u_1]$ can be extended meromorphically to $\{\zeta \in \mathbb{C}\colon \operatorname{Im}\zeta > -1-k-\varepsilon\}$ by

$$U(\zeta) := \mathcal{M}[u_1](\zeta) := (\mathcal{L}(\zeta))^{-1}\mathcal{M}[g](\zeta).$$

Let us mention that $U(\zeta)$ and $\mathcal{M}[u_0](\zeta)$ coincide on $\{\zeta \in \mathbb{C} \mid \operatorname{Im}\zeta = -1-k-\varepsilon\}$, as well as $U(\zeta)$ and $\mathcal{M}u_1(\zeta)$ on $\{\operatorname{Im}\zeta = 0\}$. Inverse transformations and the Residue Theorem then lead to

$$u_0 - u_1 = \sum_{\substack{\zeta_0 \in -\mathrm{i}\sigma^D\colon \\ \operatorname{Im}\zeta_0 \in (-1-k-\varepsilon, 0)}} \frac{2\pi\mathrm{i}}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta=\zeta_0}(r^{\mathrm{i}\zeta}(\mathcal{L}(\zeta))^{-1}\mathcal{M}[g](\zeta)). \tag{2.7}$$

Since we assumed $k+1+\varepsilon < 2\pi/\omega$, the sum in (2.7) has at most one term. Determining the residue yields

$$u_1 = \begin{cases} u_0, & \text{if } k+1+\varepsilon < \dfrac{\pi}{\omega}, \\ u_0 - \dfrac{1}{\pi}\Big(\displaystyle\int_{\mathcal{C}} r^{-\lambda_1^D} \sin(\lambda_1\varphi) f(x)\,\mathrm{d}x\Big) r^{\lambda_1^D} \sin(\lambda_1^D\varphi), & \text{if } 1+k+\varepsilon > \dfrac{\pi}{\omega} \end{cases}$$

with $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$ for any chosen $R > 0$. These observations are collected in the following result, cf. [27], Section 6.1.8.

**Proposition 2.3.** *Let $R > 0$. Let $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$ satisfy $k+1+\varepsilon < \lambda_2^D = 2\pi/\omega$ and $k+1+\varepsilon \neq \lambda_1^D = \pi/\omega$. Let $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further assume $\partial_x^i \partial_y^j f(0) = 0$ for $i+j \leqslant k-1$. Then $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ solving*

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad u_1 = 0 \quad \text{for } \varphi \in \{0, \omega\}, \tag{2.8}$$

*has the form*

$$
(2.9)\qquad u_1 = \begin{cases} u_0, & \text{if } \dfrac{\pi}{\omega} = \lambda_1^D > k+\varepsilon+1, \\[2ex] u_0 - \dfrac{1}{\pi}\Big(\displaystyle\int_{\mathcal{C}} r^{-\lambda_1^D}\sin(\lambda_1^D\varphi)f(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big) r^{\lambda_1^D}\sin(\lambda_1^D\varphi), & \\ & \text{if } \dfrac{\pi}{\omega} = \lambda_1^D < k+\varepsilon+1 < \lambda_2^D \end{cases}
$$

*for a* $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$ *with the estimate*

$$
\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}.
$$

The next lemma allows us to remove from Proposition 2.3 the condition that $f$ vanish to order $k-1$ at zero.

**Lemma 2.4.** *Let* $i$, $j$, $k \in \mathbb{N}_0$ *with* $i+j=k$. *Set* $\Sigma_{k+2}^D := \{n \in \{2,\dots,k+2\} \mid n\omega/\pi \in \mathbb{N}\}$ *and* $S_{k+2}^D := \operatorname{span}\{r^n(\ln r\sin(n\varphi) + \varphi\cos(n\varphi)) \mid n \in \Sigma_{k+2}^D\}$. *Then there is a polynomial* $\widetilde{p}_{i,j}$ *of degree* $k+2$ *and a harmonic function* $p'_{i,j} \in S_{k+2}^D$ *such that* $p_{i,j}^D := \widetilde{p}_{i,j} + p'_{i,j}$ *satisfies*

$$
(2.10)\qquad -\Delta p_{i,j}^D = x^i y^j \quad \text{on } \mathcal{C}, \quad p_{i,j}^D|_\Gamma = 0.
$$

*In the special case* $\omega = \pi$, *the contribution* $p'_{i,j}$ *may be taken to be zero.*

Proof. *Step 1:* There is a polynomial $\widehat{p}_{k+2}$ of degree $k+2$ such that $-\Delta\widehat{p}_{k+2} = x^i y^j$. This is shown by induction on $j$: one observes for any $i \in \mathbb{N}_0$ that $\Delta x^{i+2}y^0 = (i+2)(i+1)x^i y^0$ so the case $j=0$ is shown; the formula

$$
\Delta(x^{i+2}y^{j+1}) = (i+2)(i+1)x^i y^{j+1} + (j+1)j x^{i+2}y^{j-1}
$$

provides the induction step from $j$ to $j+1$.

*Step 2:* If $\omega = \pi$, then the boundary condition on the line $\{y=0\}$ can be enforced by subtracting a suitable harmonic polynomial $\operatorname{Re}\sum_{n=0}^{k+2} a_n z^n$ with $z = x + \mathrm{i}y$ and coefficients $a_n \in \mathbb{R}$.

*Step 3:* If $\omega \neq \pi$, then the boundary condition at $\varphi = 0$ is again enforced by subtracting a suitable harmonic polynomial of the form $\operatorname{Re}\sum_{n=0}^{k+2} a_n z^n$. To correct the boundary condition at $\varphi = \omega$, we note (see, e.g., [32], Lemma 6.1.1) that $\operatorname{Im} z^n = r^n\sin(n\varphi)$ and $\operatorname{Im}(z^n\ln z) = r^n(\ln r\sin(n\varphi) + \varphi\cos(n\varphi))$. Both functions vanish on $\varphi = 0$ and are harmonic. Next, we can match a function of the form $r^n$ on the line $\varphi = \omega$ by either $\operatorname{Im} z^n$ if $n\omega/\pi \notin \mathbb{N}$ or by $\operatorname{Im} z^n \ln z$ if $n\omega/\pi \in \mathbb{N}$. The function $\operatorname{Im}(z\ln z)$ is not required in the set $S_{k+2}^D$ since the case $1\cdot\omega/\pi \in \mathbb{N}$ can only arise for $\omega = \pi$. □

**Corollary 2.5.** *Let $R>0$. Let $k\in\mathbb{N}_0$ and $\varepsilon\in(0,1)$ satisfy $k+1+\varepsilon<\lambda_2^D=2\pi/\omega$ and $k+1+\varepsilon\neq\lambda_1^D=\pi/\omega$. Let $f\in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f\subseteq B_1(0)$. Let $\chi\in C_0^\infty(B_1(0))$ with $\chi\equiv 1$ near the origin. Then every function $u_1\in H^1(\mathcal{C})$ with $\operatorname{supp} u_1\subseteq B_1(0)$ solving*

$$-\Delta u_1=f\in H^{k+\varepsilon}(\mathcal{C}),\quad u_1=0\quad \text{for } \varphi\in\{0,\omega\}, \tag{2.11}$$

*has the form $u_1=u_0+\chi P_{k-1}+\delta$ with $u_0\in H^{k+2+\varepsilon}(\mathcal{C}_R)$,*

$$\delta=\begin{cases}0, & \text{if } k+\varepsilon+1<\lambda_1^D=\dfrac{\pi}{\omega},\\[2ex] S_1^D(f)s_1^D, & \text{if } k+\varepsilon+1>\lambda_1^D=\dfrac{\pi}{\omega},\end{cases} \tag{2.12}$$

$$S^D(f):=-\frac{1}{\pi}\bigg(\int_{\mathcal{C}} r^{-\lambda_1^D}\sin(\lambda_1^D\varphi)(f(\mathbf{x})+\Delta(\chi(\mathbf{x})P_{k-1}(\mathbf{x})))\,\mathrm{d}\mathbf{x}\bigg), \tag{2.13}$$

$$s_1^D:=r^{\lambda_1^D}\sin(\lambda_1^D\varphi), \tag{2.14}$$

$$P_{k-1}(\mathbf{x}):=\sum_{i+j\leqslant k-1}\frac{1}{i!j!}p_{i,j}^D(\mathbf{x})(\partial_x^i\partial_y^j f)(0), \tag{2.15}$$

*and $p_{i,j}^D$ are the fixed functions from Lemma 2.4. Furthermore,*

$$\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)}\lesssim\|f\|_{H^{k+\varepsilon}(\mathcal{C})}, \tag{2.16}$$

$$\|P_{k-1}\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)}\lesssim\|f\|_{B_{2,1}^k(\mathcal{C})}\quad \text{if } \Sigma_{k+1}^D=\emptyset, \tag{2.17}$$

$$\|P_{k-1}\|_{B_{2,\infty}^{n^\star+1}(\mathcal{C}_R)}\lesssim\|f\|_{B_{2,1}^k(\mathcal{C})}\quad \text{if } \Sigma_{k+1}^D\neq\emptyset, \tag{2.18}$$

$$n^\star:=\min\Big\{n\in\{2,\dots,k+1\}\mid n\frac{\omega}{\pi}\in\mathbb{N}\Big\},$$

$$\|\Delta(\chi P_{k-1})\|_{H^{k+\varepsilon}(\mathcal{C}_R)}\lesssim\|f\|_{B_{2,1}^k(\mathcal{C})}\lesssim\|f\|_{H^{k+\varepsilon}(\mathcal{C})}. \tag{2.19}$$

*The implied constants depend only on $k$, $\varepsilon$, the angle $\omega$, and the choice of the cut-off function $\chi$.*

Proof. We only consider $k\geqslant 1$, since the claim for $k=0$ is a restatement of Proposition 2.3 and $P_{-1}=0$.

*Step 1:* Lemma 2.4 provides functions $p_{i,j}^D$ such that the function

$$P_{k-1}=\sum_{i+j\leqslant k-1}\frac{1}{i!j!}p_{i,j}^D(\partial_x^i\partial_y^j f)(0)$$

solves the problem

$$-\Delta P_{k-1}=\sum_{i+j\leqslant k-1}\frac{1}{i!j!}x^iy^j(\partial_x^i\partial_y^j f)(0),\quad P_{k-1}=0\quad \text{for } \varphi\in\{0,\omega\}. \tag{2.20}$$

We have for $k\geqslant 1$ the embedding $B_{2,1}^k\subset C^{k-1}$, which is asserted in [38], Theorem 2.8.1 (c) and could, alternatively, be obtained by combining the classical

Gagliardo inequality (in 2D) $\|\nabla^j u\|_{L^\infty} \lesssim \|u\|_{H^{k+1}}^\theta \|u\|_{L^2}^{1-\theta}$ with $\theta = (j+1)/(k+1)$ for $0 \leqslant j \leqslant k-1$ with the result [37], Theorem 25.3. In view of (2.15), this embedding leads to the estimates (2.17), (2.18): In the first case, $\Sigma_{k+1}^D = \emptyset$, the functions $p_{i,j}^D$ are polynomials (and hence smooth). In the second case, $\Sigma_{k+1}^D \neq \emptyset$, the functions $p_{i,j}^D$ are sums of polynomials, which are smooth, and functions $p'_{i,j} \in S_{k+1}^D$, which are in $B_{2,\infty}^{n^\star+1}(\mathcal{C}_1)$ by Lemma 2.6 (iv) ahead.

*Step 2:* We now define $\widetilde{u_1} := u_1 - \chi P_{k-1}$, which also has support in $B_1(0)$. Since $u_1$ solves (2.11), the function $\widetilde{u_1} \in H^1(\mathcal{C})$ solves the problem

$$-\Delta \widetilde{u_1} = \widetilde{f} := f + \Delta(\chi P_{k-1}) \in H^{k+\varepsilon}(\mathcal{C}), \quad \widetilde{u_1} = 0 \quad \text{for } \varphi \in \{0, \omega\}. \tag{2.21}$$

By construction of $P_{k-1}$, the right-hand side $\widetilde{f}$ satisfies

$$\partial_x^i \partial_y^j \widetilde{f}(0) = 0 \quad \text{for } i + j \leqslant k-1. \tag{2.22}$$

Thus Proposition 2.3 can be applied to the problem (2.21), and we obtain

$$\widetilde{u_1} = \begin{cases} u_0, & \text{if } k+1+\varepsilon < \lambda_1^D = \dfrac{\pi}{\omega}, \\ u_0 - \dfrac{1}{\pi}\left(\displaystyle\int_{\mathcal{C}} r^{-\lambda_1^D} \sin(\lambda_1^D \varphi) \widetilde{f}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right) r^{\lambda_1^D} \sin(\lambda_1^D \varphi), & \text{if } k+1+\varepsilon > \lambda_1^D = \dfrac{\pi}{\omega} \end{cases}$$

with $\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|\widetilde{f}\|_{H^{k+\varepsilon}(\mathcal{C})}$. To complete the proof, we distinguish the cases $\Sigma_{k+1}^D = \emptyset$ and $\Sigma_{k+1}^D \neq \emptyset$. If $\Sigma_{k+1}^D = \emptyset$, then $P_{k-1}$ is a polynomial of degree $k+1$ and together with (2.17) we get $\|\widetilde{f}\|_{H^{k+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_R)}$. If $\Sigma_{k+1}^D \neq \emptyset$, then $P_{k-1}$ is the sum of a polynomial of degree $k+1$, for which we can argue as in the case $\Sigma_{k+1}^D = \emptyset$, and a harmonic contribution $P'_{k+1} \in S_{k+1}^D$. The function $P'_{k+1}$ is smooth away from the origin and by harmonicity we have $\Delta(\chi P'_{k+1}) = 0$ near the origin so that in total we arrive again at the estimate (2.19) and thus $\|\widetilde{f}\|_{H^{k+\varepsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C})}$. □

**2.2. Regularity of the singularity function and the stress intensity functional.** The following Lemma 2.6 clarifies in which Besov spaces the singularity functions arising in corner domains lie. The proof of Lemma 2.6 (iii), (iv) relies on arguments given in [3] or [4], Theorem 2.1.

**Lemma 2.6.** *The following statements hold.*

(i) *For $\alpha > 1$, $\alpha \notin \mathbb{N}$, set $k := \lfloor \alpha \rfloor - 1$ and let $P_{k-1}$, $\chi$ be as in Corollary 2.5. Then the mapping*

$$f \mapsto S(f) := \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\varphi)(f + \Delta(\chi P_{k-1}))\,\mathrm{d}\mathbf{x}$$

*is bounded and linear on $B_{2,1}^{\alpha-1}(\mathcal{C}_1)$.*

(ii) *Let* $0 < \alpha \leqslant 1$. *Then the mapping*

$$f \mapsto S(f) := \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\varphi) f \, \mathrm{d}\mathbf{x}$$

*is bounded and linear on* $\widetilde{B}^{\alpha-1}_{2,1}(\mathcal{C}_1)$. *For* $\alpha > 1/2$, *it is also bounded and linear on* $B^{\alpha-1}_{2,1}(\mathcal{C}_1) = \widetilde{B}^{\alpha-1}_{2,1}(\mathcal{C}_1)$.

(iii) *For* $\beta \geqslant -1$, *the function* $s^+(r,\varphi) = r^\beta \sin(\beta\varphi)$ *is in the space* $B^{1+\beta}_{2,\infty}(\mathcal{C}_1)$.

(iv) *Let* $\Phi \in C^\infty(\mathbb{R}^2)$ *with* $|\Phi(x,y)| \leqslant Cr^n$ *as* $r \to 0$. *The functions* $v(x,y) = \Phi(x,y)\ln r$ *and* $w(x,y) = \varphi\Phi(x,y)$ *are in the space* $B^{n+1}_{2,\infty}(\mathcal{C}_1)$.

(v) *The statements* (i), (ii), (iii) *remain true if the function* sin *is replaced with* cos.

P r o o f. (i) Choose $0 < \varepsilon \ll 1$ such that $k + 1 - \alpha + \varepsilon < 0$. By the Reiteration Theorem [37], Theorem 26.3,

$$B^{\alpha-1}_{2,1}(\mathcal{C}_1) = (H^{k+\varepsilon}(\mathcal{C}_1), B^{k+1}_{2,1}(\mathcal{C}_1))_{(\alpha-k-1-\varepsilon)/(1-\varepsilon),1}.$$

Now assume $f \in C^\infty(\overline{\mathcal{C}_1})$, $f \neq 0$, the general statement will then follow by density arguments. For

$$\delta := \min\Big\{\|f\|^{1/(1-\varepsilon)}_{H^{k+\varepsilon}(\mathcal{C}_1)}\|f\|^{-1/(1-\varepsilon)}_{B^{k+1}_{2,1}(\mathcal{C}_1)}, \frac{1}{2}\operatorname{diam}\{\mathbf{x} \in \mathcal{C} \colon \chi(\mathbf{x}) = 1\}\Big\},$$

denote by $\chi_\delta$ a smooth cut-off function that equals zero for $r < \delta$ and one for $r > 2\delta$. We have

$$\begin{aligned} S(f) &= \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\varphi)\chi_\delta(f + \Delta(\chi P_{k-1}))\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\mathcal{C}_1} r^{-\alpha}\sin(\alpha\varphi)(1-\chi_\delta)(f + \Delta(\chi P_{k-1}))\,\mathrm{d}\mathbf{x} \\ &=: S_1 + S_2. \end{aligned} \tag{2.23}$$

The first integral, $S_1$, is estimated by

$$\begin{aligned} S_1 &= \bigg|\int_{\mathcal{C}_1} r^{-\alpha+k+\varepsilon}\sin(\alpha\varphi)\chi_\delta\frac{f + \Delta(\chi P_{k-1})}{r^{k+\varepsilon}}\,\mathrm{d}x\bigg| \\ &\lesssim \Big\|\frac{f + \Delta(\chi P_{k-1})}{r^{k+\varepsilon}}\Big\|_{L^2(\mathcal{C}_1)}\bigg|\int_\delta^1 r^{-2\alpha+2k+2\varepsilon+1}\chi_\delta^2\,\mathrm{d}r\bigg|^{1/2} \\ &\lesssim \Big\|\frac{f + \Delta(\chi P_{k-1})}{r^{k+\varepsilon}}\Big\|_{L^2(\mathcal{C}_1)}\delta^{k+1-\alpha+\varepsilon}. \end{aligned}$$

The remaining $L^2$-norm can be handled with Lemma 2.2: We have, since $\partial_x^i \partial_y^j (f + \Delta(\chi P_{k-1}))(0) = 0$ for $i + j \leqslant k - 1$, cf. (2.22),

$$\left\| \frac{f + \Delta(\chi P_{k-1})}{r^{k+\varepsilon}} \right\|_{L^2(\mathcal{C}_1)}^2 \leqslant \|f + \Delta(\chi P_{k-1})\|_{K_{-\varepsilon}^k(\mathcal{C}_1)}^2$$
$$\lesssim \|f + \Delta(\chi P_{k-1})\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^2 \overset{(2.19)}{\lesssim} \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^2.$$

Since the expression $\delta$ involves a minimum, we analyze two cases. If

$$\|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^{1/(1-\varepsilon)} \|f\|_{B_{2,1}^{k+1}(\mathcal{C}_1)}^{-1/(1-\varepsilon)} \leqslant \frac{1}{2} \operatorname{diam}\{\mathbf{x} \in \mathcal{C} \mid \chi(\mathbf{x}) = 1\},$$

we get directly

$$\delta^{k+1-\alpha+\varepsilon} = (\|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^{1/(1-\varepsilon)} \|f\|_{B_{2,1}^{k+1}(\mathcal{C}_1)}^{-1/(1-\varepsilon)})^{k+1-\alpha+\varepsilon},$$

since the exponent $k + 1 - \alpha + \varepsilon < 0$; if, on the other hand,

$$\|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^{1/(1-\varepsilon)} \|f\|_{B_{2,1}^{k+1}(\mathcal{C}_1)}^{-1/(1-\varepsilon)} > \frac{1}{2} \operatorname{diam}\{\mathbf{x} \in \mathcal{C} \mid \chi(\mathbf{x}) = 1\},$$

then the continuous embedding $B_{2,1}^{k+1}(\mathcal{C}_1) \subset H^{k+\varepsilon}(\mathcal{C}_1)$ implies

$$\|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^{1/(1-\varepsilon)} \|f\|_{B_{2,1}^{k+1}(\mathcal{C}_1)}^{-1/(1-\varepsilon)} \lesssim 1,$$

and we arrive at

$$\delta^{k+1-\alpha+\varepsilon} = \left(\frac{1}{2} \operatorname{diam}\{\mathbf{x} \in \mathcal{C} \mid \chi(\mathbf{x}) = 1\}\right)^{k+1-\alpha+\varepsilon}$$
$$\lesssim (\|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}^{1/(1-\varepsilon)} \|f\|_{B_{2,1}^{k+1}(\mathcal{C}_1)}^{-1/(1-\varepsilon)})^{k+1-\alpha+\varepsilon}.$$

For the second integral of (2.23), $S_2$, we obtain

$$S_2 = \left| \int_{\mathcal{C}_1} r^{-\alpha+k} \sin(\alpha\varphi)(1 - \chi_\delta) \frac{f + \Delta(\chi P_{k-1})}{r^k} \, \mathrm{d}x \right|$$
$$\lesssim \left\| \frac{f + \Delta(\chi P_{k-1})}{r^k} \right\|_{L^\infty(\mathcal{C}_{2\delta})} \int_0^{2\delta} r^{-\alpha+k+1} \, \mathrm{d}r.$$

Since

$$\Delta(\chi P_{k-1}) = \Delta P_{k-1} = - \sum_{i+j \leqslant k-1} \frac{1}{i!j!} x^i y^j (\partial_x^i \partial_y^j f)(0)$$

in the region where $\chi \equiv 1$, it follows with the embedding $B^1_{2,1}(\mathcal{C}_1) \subseteq C(\overline{\mathcal{C}_1})$, cf. [38], Theorem 4.6.1, that

$$\begin{aligned} S_2 &\lesssim \left\| \frac{f - \sum_{i+j \leqslant k-1} \frac{1}{i!j!} x^i y^j (\partial_x^i \partial_y^j f)(0)}{r^k} \right\|_{L^\infty(\mathcal{C}_1)} \delta^{k+2-\alpha} \\ &\lesssim \|D^k f\|_{L^\infty(\mathcal{C}_1)} \delta^{k+2-\alpha} \lesssim \|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)} (\|f\|^{1/(1-\varepsilon)}_{H^{k+\varepsilon}(\mathcal{C}_1)} \|f\|^{-1/(1-\varepsilon)}_{B^{k+1}_{2,1}(\mathcal{C}_1)})^{k+2-\alpha}. \end{aligned}$$

In total, we have arrived at

$$|S(f)| \lesssim \|f\|^{(k+2-\alpha)/(1-\varepsilon)}_{H^{k+\varepsilon}(\mathcal{C}_1)} \|f\|^{(\alpha-k-1-\varepsilon)/(1-\varepsilon)}_{B^{k+1}_{2,1}(\mathcal{C}_1)}. \tag{2.24}$$

By [37], Lemma 25.2 the estimate (2.24) implies

$$S \in ((H^{k+\varepsilon}(\mathcal{C}_1), B^{k+1}_{2,1}(\mathcal{C}_1))_{(\alpha-k-1-\varepsilon)/(1-\varepsilon),1})^\star = (B^{\alpha-1}_{2,1}(\mathcal{C}_1))^\star.$$

(ii) By (iii) we have for $\alpha \in (0,1]$ that $s^-(r,\varphi) := r^{-\alpha} \sin(\alpha\varphi) \in B^{-\alpha+1}_{2,\infty}(\mathcal{C}_1)$. From the characterization of dual spaces of interpolation spaces, [37], Lemma 41.3, [38], Theorem 1.11.2, we have for any $\varepsilon \in (0,1/2)$ in view of $-\alpha+1 \in [0,1)$ with $\theta = (-\alpha+1+\varepsilon)/(1+\varepsilon)$

$$B^{-\alpha+1}_{2,\infty} = (H^{-\varepsilon}, H^1)_{\theta,\infty} = (((H^1)^\star, (H^{-\varepsilon})^\star)_{1-\theta,1})^\star = ((\widetilde{H}^{-1}, \widetilde{H}^\varepsilon)_{1-\theta,1})^\star = (\widetilde{B}^{\alpha-1}_{2,1})^\star.$$

For $\alpha > 1/2$, we note that $\alpha - 1 \in (-1/2, 0)$ so that by (1.7) we have $\widetilde{B}^{\alpha-1}_{2,1} = B^{\alpha-1}_{2,1}$.

(iii) *Step 1 ($\beta \in \mathbb{N}_0$):* For $\beta \in \mathbb{N}_0$, the function $(x,y) \mapsto r^\beta \sin(\beta\varphi)$ is a polynomial and thus smooth.

*Step 2 ($\beta > -1$, $\beta \notin \mathbb{N}_0$):* We write the Besov space as the interpolation space

$$B^{1+\beta}_{2,\infty}(\mathcal{C}_1) = (L^2(\mathcal{C}_1), H^{\lfloor\beta\rfloor+2}(\mathcal{C}_1))_{(1+\beta)/(\lfloor\beta\rfloor+2),\infty}.$$

Next we select a smooth cut-off function $\chi_t$ with $\chi_t \equiv 0$ on $B_{t^{1/(\lfloor\beta\rfloor+2)}/2}(0)$ and $\chi_t \equiv 1$ on $B_1(0) \setminus B_{t^{1/(\lfloor\beta\rfloor+2)}}(0)$, and whose derivatives satisfy $\|\nabla^k \chi_t\|_{L^\infty(\mathcal{C}_1)} \lesssim t^{-k/(\lfloor\beta\rfloor+2)}$. We then get

$$\|(1-\chi_t)s^+\|^2_{L^2(\mathcal{C}_1)} \lesssim \int_0^1 (1-\chi_t)^2 r^{2\beta} r \,\mathrm{d}r \lesssim \int_0^{t^{1/(\lfloor\beta\rfloor+2)}} r^{2\beta+1} \,\mathrm{d}r \lesssim t^{2(\beta+1)/(\lfloor\beta\rfloor+2)}. \tag{2.25}$$

For the derivatives we obtain

(2.26)

$$\begin{aligned}
&\|\nabla^{\lfloor\beta\rfloor+2}(\chi_t s^+)\|^2_{L^2(\mathcal{C}_1)}\\
&\quad\lesssim \int_0^1 \sum_{s=0}^{\lfloor\beta\rfloor+2} |\nabla^s\chi_t(r)|^2|\nabla^{\lfloor\beta\rfloor+2-s} r^\beta|^2 r\,\mathrm{d}r\\
&\quad\lesssim \int_{t^{1/(\lfloor\beta\rfloor+2)}/2}^1 r^{2(\beta-\lfloor\beta\rfloor-2)+1}\,\mathrm{d}r + \sum_{s=1}^{\lfloor\beta\rfloor+2} t^{-2s/(\lfloor\beta\rfloor+2)}(t^{1/(\lfloor\beta\rfloor+2)})^{2(\beta+s-\lfloor\beta\rfloor-2)+2}\\
&\quad\lesssim t^{(2\beta-2\lfloor\beta\rfloor-2)/(\lfloor\beta\rfloor+2)} + \sum_{s=1}^{\lfloor\beta\rfloor+2} t^{-2s/(\lfloor\beta\rfloor+2)}(t^{1/(\lfloor\beta\rfloor+2)})^{2(\beta+s-\lfloor\beta\rfloor-2)+2}\\
&\quad\lesssim t^{2(\beta+1)/(\lfloor\beta\rfloor+2)-2}.
\end{aligned}$$

The $L^2$-norm satisfies

$$\|\chi_t s^+\|^2_{L^2(\mathcal{C}_1)} \lesssim \int_{t^{1/(\lfloor\beta\rfloor+2)}/2}^1 r^{2\beta+1}\,\mathrm{d}r \lesssim t^{2(\beta+1)/(\lfloor\beta\rfloor+2)-2}, \tag{2.27}$$

since the integral is bounded and $2(\beta+1)/(\lfloor\beta\rfloor+2)-2<0$. Then (2.25), (2.26), and (2.27) imply $K(t,s^+)\lesssim t^{(1+\beta)/(\lfloor\beta\rfloor+2)}$ and thus $s^+\in B^{1+\beta}_{2,\infty}(\mathcal{C}_1)$.

*Step 3* ($\beta=-1$)*:* Fix $\varepsilon\in(0,1/2)$. We assert $s^+\in B^0_{2,1}(\mathcal{C}_1)=(H^{-\varepsilon}(\mathcal{C}_1),H^\varepsilon(\mathcal{C}_1))_{1/2,1}$ using the "Babuška trick" [22], Theorem 1.4.5.3. Let $\chi_t$ be the cut-off function of Step 2 (with $\beta=-1$) and split $s^+=r^{-1}\sin(\varphi)$ as $s^+=(1-\chi_t)s^++\chi_t s^+$. We estimate with the Cauchy-Schwarz inequality and Lemma 2.2 to get

$$\begin{aligned}
&\|(1-\chi_t)s^+\|_{H^{-\varepsilon}(\mathcal{C}_1)}\\
&\quad=\sup_{v\in\widetilde{H}^\varepsilon(\mathcal{C}_1)\,|\,\|v\|_{\widetilde{H}^\varepsilon}=1}\int_{\mathcal{C}_1}(1-\chi_t)r^\varepsilon s^+ r^{-\varepsilon}v\,\mathrm{d}\mathbf{x} \overset{\text{C.S.}}{\lesssim} t^\varepsilon \sup_{\|v\|_{\widetilde{H}^\varepsilon=1}}\|r^{-\varepsilon}v\|_{L^2(\mathcal{C}_1)} \overset{\text{Lemma 2.2}}{\lesssim} t^\varepsilon.
\end{aligned}$$

For the term $\|\chi_t s^+\|_{H^\varepsilon(\mathcal{C}_1)}$, we employ the "Babuška trick", i.e., we use the Sobolev embedding $W^{1,p}\subset H^\varepsilon$ for $1/p=1-\varepsilon/2$, [22], Theorem 1.4.5.2, [38], Theorem 2.8.1, to conveniently estimate the norm $\|\cdot\|_{H^\varepsilon}$. A calculation shows for $t<1$

$$\|\chi_t s^+\|_{H^\varepsilon(\mathcal{C}_1)}\lesssim\|\chi_t s^+\|_{W^{1,p}(\mathcal{C}_1)}\lesssim t^{-\varepsilon}.$$

In conclusion, the $K$-functional of $s^+$ for the pair $(H^{-\varepsilon}(\mathcal{C}_1),H^\varepsilon(\mathcal{C}_1))$ satisfies $K(\tau,s^+)\lesssim t^\varepsilon+\tau t^{-\varepsilon}$. Selecting $t=\tau^{1/(2\varepsilon)}$ shows $K(\tau,s^+)\lesssim\tau^{1/2}$ so that $s^+\in(H^{-\varepsilon}(\mathcal{C}_1),H^\varepsilon(\mathcal{C}_1))_{1/2,\infty}=B^0_{2,\infty}(\mathcal{C}_1)=\widetilde{B}^0_{2,\infty}(\mathcal{C}_1)$.

(iv) First we consider $v=\Phi(x,y)\ln r$ with $\Phi=O(r^n)$ as $r\to0$. We define for every $0<t<1$ the function

$$v_t:=\Phi(x,y)\int_r^1-\chi_t(\tau)\tau^{-1}\,\mathrm{d}\tau,$$

where $\chi_t$ is a smooth cut-off function with the properties $\chi_t(\tau) = 1$ for $\tau > t$ and $\chi_t(\tau) = 0$ for $\tau < t/2$ and $|\nabla^j \chi| \lesssim t^{-j}$. Note that $v_t \in C^\infty(\mathcal{C}_1)$ and $v_t = v$ for $r > t$. We obtain

$$\|v - v_t\|^2_{L^2(\mathcal{C})} = \int_{\varphi=0}^{\omega} \int_0^t \Phi^2 \left| \int_r^1 \frac{1}{\tau}(\chi_t(\tau) - 1)\,\mathrm{d}\tau \right|^2 r\,\mathrm{d}r \lesssim \int_0^t r^{2n+1} \left| \int_r^t \frac{1}{\tau}\,\mathrm{d}\tau \right|^2 \mathrm{d}r$$
$$\lesssim \int_0^t r^{2n} t \frac{r}{t} \ln^2 \frac{t}{r}\,\mathrm{d}r = t^{2n+2} \int_0^1 r^{2n+1} \ln^2 r\,\mathrm{d}r \lesssim t^{2n+2}.$$

By the smoothness of $\Phi$, we have $|\nabla^j \Phi| = O(r^{n-j})$ for $j \in \{0, \ldots, n\}$ and $|\nabla^j \Phi| = O(1)$ for $j > n$. The product rule then implies (using that $r \sim t$ on $\operatorname{supp} \nabla \chi_t \subset \overline{B}_t(0) \setminus B_{t/2}(0)$)

$$\begin{aligned}
|\nabla^{n+2} v_t| &\lesssim \sum_{\mu=0}^{n} |\nabla^\mu \Phi| \left| \nabla^{n+1-\mu} \frac{\chi_t}{r} \right| + |r^{-1}\chi_t| + \left| \int_r^1 \chi_t \tau^{-1}\,\mathrm{d}\tau \right| \\
&\lesssim \sum_{\mu=0}^{n} r^{n-\mu} r^{-(n+2-\mu)} \chi_{\overline{B}_t(0)\setminus B_{t/2}(0))} + r^{-1}\chi_{r>t/2} + |\ln r| \\
&\lesssim r^{-2} \chi_{\overline{B}_t(0)\setminus B_{t/2}(0)} + r^{-1}\chi_{r>t/2} + |\ln r|.
\end{aligned}$$

Consequently,

$$\|\nabla^{n+2} v_t\|^2_{L^2(\mathcal{C})} = \int_{\varphi=0}^{\omega} \int_{r=0}^{1} |\nabla^{n+2} v_t|^2 r\,\mathrm{d}r\,\mathrm{d}\varphi \lesssim t^{-2}.$$

Since the above calculations hold for every $t \in (0, 1)$, we choose $t = \tau^{1/(n+2)}$ and get for the $K$-functional

$$K(\tau, v)^2 \lesssim \|v - v_t\|^2_{L^2(\mathcal{C}_1)} + \tau^2 \|v_t\|^2_{H^{n+2}(\mathcal{C}_1)} \lesssim \tau^{2(n+1)/(n+2)},$$

which shows $v \in (L^2(\mathcal{C}_1), H^{n+2}(\mathcal{C}_1))_{(n+1)/(n+2),\infty} = B^{n+1}_{2,\infty}(\mathcal{C}_1)$.

The proof that the function $w = \Phi(x, y)\varphi$ is in $B^{n+1}_{2,\infty}(\mathcal{C}_1)$ follows similar lines using the function $v_t := \Phi(x, y)\varphi\chi_t$. □

The core of proof of shift Theorem 1.1 is the following abstract result:

**Lemma 2.7.** *Let $(X_1, \|\cdot\|_{X_1}) \subset (X_0, \|\cdot\|_{X_0})$ and $(Y_1, \|\cdot\|_{Y_1}) \subset (Y_0, \|\cdot\|_{Y_0})$ be Banach spaces with continuous embeddings. Let $q_j$, $p_j \in [1, \infty]$, $\theta_j \in (0, 1)$, $j = 1, \ldots, J$, with $0 < \theta_1 < \theta_2 < \ldots < \theta_J < 1$. Let $\widetilde{T}\colon X_0 \to Y_0$, $S_1 : X_1 \to Y_1$, $S_{\theta_j}\colon X_{\theta,p_j} \to Y_{\theta,q_j}$, $j \in \{1, \ldots, J\}$, be bounded linear. Assume $\widetilde{T} f = S_1(f) + \sum_{j=1}^{J} S_{\theta_j}(f)$ for all $f \in X_1$. Then, $\widetilde{T}\colon (X_0, X_1)_{\theta_1,p_1} \to (Y_0, Y_1)_{\theta_1,\infty}$ is bounded linear.*

Proof. For $t > 0$, decompose $f \in X_{\theta_1,p_1}$ as $f = f_0 + f_1$ with $f_0 \in X_0$, $f_1 \in X_1$, and

$$\|f_0\|_{X_0} + t\|f_1\|_{X_1} \leqslant 2K(t,f) \lesssim t^{\theta_1}\|f\|_{X_{\theta_1,\infty}} \lesssim t^{\theta_1}\|f\|_{X_{\theta_1,p_1}}. \tag{2.28}$$

By [8], Lemma and the interpolation inequality, we have additionally

$$\|f_1\|_{X_{\theta_1,p_1}} \overset{[8],\text{ Lemma}}{\leqslant} 3\|f\|_{X_{\theta_1,p_1}}, \tag{2.29}$$

$$\begin{aligned}\|f_1\|_{X_{\theta_j,p_j}} &\lesssim \|f_1\|_{X_{\theta_1,p_1}}^{(1-\theta_j)/(1-\theta_1)}\|f_1\|_{X_1}^{(\theta_j-\theta_1)/(1-\theta_1)} \\ &\overset{(2.28)}{\lesssim} t^{-(\theta_j-\theta_1)}\|f_1\|_{X_{\theta_1,p_1}}, \quad j = 2,\dots,J.\end{aligned} \tag{2.30}$$

We write $\widetilde{T}f = \widetilde{T}f_0 + \widetilde{T}f_1 = \widetilde{T}f_0 + S_1(f_1) + \sum_{j=1}^{J} S_{\theta_j}(f_1)$. For $t > 0$ and $j \in \{1,\dots,J\}$ decompose $S_{\theta_j}(f_1) = s_{j,0}(f_1) + s_{j,1}(f_1)$ with $s_{j,0}(f_1) \in Y_0$, $s_{j,1}(f_1) \in Y_1$ and

$$\begin{aligned}&\|s_{1,0}(f_1)\|_{Y_0} + t\|s_{1,1}(f_1)\|_{Y_1} \\ &\qquad\leqslant 2K(t,S_{\theta_1}(f_1)) \lesssim t^{\theta_1}\|S_{\theta_1}(f_1)\|_{Y_{\theta_1,q_1}} \lesssim t^{\theta_1}\|f_1\|_{X_{\theta_1,p_1}} \overset{(2.29)}{\lesssim} t^{\theta_1}\|f\|_{X_{\theta_1,p_1}}, \\ &\|s_{j,0}(f_1)\|_{Y_0} + t\|s_{j,1}(f_1)\|_{Y_1} \\ &\qquad\leqslant 2K(t,S_{\theta_j}(f_1)) \lesssim t^{\theta_j}\|S_{\theta_j}(f_1)\|_{Y_{\theta_j,q_j}} \lesssim t^{\theta_j}\|f_1\|_{X_{\theta_j,p_j}} \overset{(2.30)}{\lesssim} t^{\theta_1}\|f\|_{X_{\theta_1,p_1}}.\end{aligned}$$

This implies the decomposition

$$\widetilde{T}f = \left(\widetilde{T}f_0 + \sum_{j=1}^{J} s_{j,0}(f_1)\right) + \left(S_1(f_1) + \sum_{j=1}^{J} s_{j,1}(f_1)\right) =: y_0 + y_1$$

with

$$\begin{aligned}\|y_0\|_{Y_0} + t\|y_1\|_{Y_1} &\lesssim \|f_0\|_{X_0} + t^{\theta_1}\|f\|_{X_{\theta_1,p_1}} + t(\|f_1\|_{X_1} + t^{\theta_1-1}\|f\|_{X_{\theta_1,p_1}}) \\ &\overset{(2.28)}{\lesssim} t^{\theta_1}\|f\|_{X_{\theta_1,p_1}}.\end{aligned}$$

Hence, $\widetilde{T}f \in X_{\theta_1,\infty}$ with $\|\widetilde{T}f\|_{X_{\theta_1,\infty}} \lesssim \|f\|_{X_{\theta_1,p_1}}$. □

We are now in position to prove shift Theorem 1.1 for Dirichlet boundary conditions.

P r o o f of Theorem 1.1 (i) (Dirichlet conditions). We denote by $\chi_r$, $r > 0$, a smooth cutoff function with $\operatorname{supp}\chi_r \subset B_r(0)$ and $\chi_r \equiv 1$ near 0. We assume for simplicity $R < 1$. By Remark 1.2 a stronger shift theorem holds for $\omega = \pi$ so that we will assume $\omega \neq \pi$.

*Step 0 (local regularity):* Since the two lines $\{\varphi = 0\}$ and $\{\varphi = \omega\}$ are smooth, local elliptic regularity gives for any $0 < R_1 < R_2 < R_3 < R$ that for any $s \in \mathbb{N}_0$ (see, e.g., [19], Section 6; this is even true for any $s \geqslant 0$, see Lemma A.2 for details)

$$\|u\|_{H^{s+2}(\mathcal{C}_{R_2}\setminus\mathcal{C}_{R_1})} \lesssim \|f\|_{H^s(\mathcal{C}_{R_3})} + \|u\|_{H^1(\mathcal{C}_{R_3})}. \tag{2.31}$$

*Step 1 (localized equation):* Since $R' < R$ and we are interested in the regularity of $u$ in $\mathcal{C}_{R'}$, we fix a smooth cut-off function $\chi_{\widetilde{R}} \in C_0^\infty(B_{\widetilde{R}}(0))$ with $\chi_{\widetilde{R}} \equiv 1$ on $\mathcal{C}_{R'}$, where $R_2 < R' < \widetilde{R} < R_3 < R < 1$ are such that $\chi_R \equiv 1$ on $\mathcal{C}_{R_3}$. We set $\widetilde{u} := \chi_{\widetilde{R}} u$, and we note that $\widetilde{u}$ satisfies

$$\begin{aligned} -\Delta\widetilde{u} &= -\chi_{\widetilde{R}}\Delta u - 2\nabla\chi_{\widetilde{R}}\cdot\nabla u - \Delta\chi_{\widetilde{R}} u \\ &= \chi_{\widetilde{R}} f - 2\nabla\chi_{\widetilde{R}}\cdot\nabla u - \Delta\chi_{\widetilde{R}} u =: \widetilde{f} \quad \text{in } \mathcal{C}_R, \end{aligned} \tag{2.32}$$

$$\widetilde{u} = 0 \quad \text{on } \Gamma_D \cup \widetilde{\Gamma}_R = \partial\mathcal{C}_R. \tag{2.33}$$

*Claim:*

$$\|\widetilde{f}\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} \lesssim \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}. \tag{2.34}$$

To see this, we consider the cases $\omega < \pi$ and $\omega \geqslant \pi$ separately.

*Case 1:* $\omega \leqslant \pi$. Let $s := \lfloor \lambda_1^D \rfloor = \lfloor \pi/\omega \rfloor \geqslant 1$ and note the continuous embedding $H^{s+1} \subset B_{2,1}^{\pi/\omega}$ so that together with the support properties of $\nabla\chi_{\widetilde{R}}$

$$\begin{aligned} \|\widetilde{f}\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} &\overset{H^{s+1}\subset B_{2,1}^{\pi/\omega}}{\lesssim} \|\chi_{\widetilde{R}} f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|u\|_{H^{s+1}(\mathcal{C}_{\widetilde{R}}\setminus\mathcal{C}_{R'})} \\ &\overset{(2.31)}{\lesssim} \|\chi_{\widetilde{R}} f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|f\|_{H^{s-1}(\mathcal{C}_{R_3}\setminus\mathcal{C}_{R_2})} + \|u\|_{H^1(\mathcal{C}_R)} \\ &\lesssim \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} + \|u\|_{H^1(\mathcal{C}_R)}, \end{aligned} \tag{2.35}$$

where, in the last step, we used $\chi_{\widetilde{R}}\chi_R = \chi_{\widetilde{R}}$ and the continuity of the multiplication with smooth functions in Sobolev (and hence, by interpolation, in Besov spaces) so that $\|\chi_{\widetilde{R}} f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} = \|\chi_{\widetilde{R}}\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)} \lesssim \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)}$ and $\|f\|_{H^{s-1}(\mathcal{C}_{R_3}\setminus\mathcal{C}_{R_2})} \leqslant \|f\|_{H^{s-1}(\mathcal{C}_{R_3})} \leqslant \|\chi_R f\|_{H^{s-1}(\mathcal{C}_R)} \leqslant \|\chi_R f\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)}$ as $s - 1 \in \mathbb{N}_0$.

*Case 2:* $\omega > \pi$. Again, set $s := \lfloor \lambda_1^D \rfloor = \lfloor \pi/\omega \rfloor = 0$ and note the continuous embedding $L^2 \subset B_{2,1}^{\pi/\omega - 1}$. One may then argue as in the first line of (2.35), which gives the result.

*Step 2:* In order to analyze the regularity of $\widetilde{u}$ on $\mathcal{C}_{R'}$, we select $\overline{R} < R$ and $\chi_{\overline{R}} \in C_0^\infty(B_{\overline{R}}(0))$ with $\chi_{\overline{R}} \equiv 1$ on $\operatorname{supp} \chi_R \subset B_R(0)$ and introduce the operators $T$, $\widetilde{T}$ by

$$T\colon \begin{cases} H^{-1}(\mathcal{C}_R) \to H_0^1(\mathcal{C}_R), \\ f \mapsto v, \end{cases} \qquad \widetilde{T}\colon \begin{cases} H^{-1}(\mathcal{C}_R) \to H_0^1(\mathcal{C}_R), \\ f \mapsto \chi_{R'} T \chi_{\overline{R}} f, \end{cases} \tag{2.36}$$

where $v = Tf$ solves

$$-\Delta v = f \text{ in } \mathcal{C}_R, \quad v = 0 \text{ on } \Gamma_D \cup \widetilde{\Gamma}_R = \partial \mathcal{C}_R.$$

Then, since $\operatorname{supp} \widetilde{f} \subset B_{\widetilde{R}}(0)$, we have $\chi_{\overline{R}} \widetilde{f} = \widetilde{f}$ and therefore $\widetilde{u} = \widetilde{T}\widetilde{f}$. The proof of Theorem 1.1 (i) is complete once we ascertain

$$\|\widetilde{T}\widetilde{f}\|_{B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_{R'})} \lesssim \|\widetilde{f}\|_{B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R)}, \tag{2.37}$$

which we will do in ensuing Steps 3–5.

*Step 3:* We show (2.37) for the case $\lambda_1^D = \pi/\omega \notin \mathbb{N}$ and $\omega < \pi$ using Lemma 2.7 and Corollary 2.5. Since $\omega < \pi$ and $\lambda_1^D \notin \mathbb{N}$, we can find $(k, \varepsilon) \in \mathbb{N}_0 \times (0,1)$ such that $\lambda_1^D < k + \varepsilon + 1 < \lambda_2^D$ together with $\lfloor k + \varepsilon + 1 \rfloor = \lfloor \lambda_1^D \rfloor \geqslant 1$. Set $\theta_1 := \pi/(\omega(k + \varepsilon + 1)) \in (0,1)$ as well as

$$X_0 := H^{-1}(\mathcal{C}_R), \quad X_1 = H^{k+\varepsilon}(\mathcal{C}_R), \quad Y_0 := H_0^1(\mathcal{C}_R), \quad Y_1 = H^{2+k+\varepsilon}(\mathcal{C}_R). \tag{2.38}$$

We have $B_{2,1}^{\pi/\omega-1}(\mathcal{C}_R) = (X_0, X_1)_{\theta_1,1}$ and $B_{2,\infty}^{\pi/\omega+1}(\mathcal{C}_R) = (Y_0, Y_1)_{\theta_1,\infty}$. By the Lax-Milgram theorem, we have that $\widetilde{T}\colon X_0 \to Y_0$ is bounded, linear. With $u_0$, $S_1^D$, $s_1^D$, $\chi P_{k-1}$ given by Corollary 2.5, we set $S_1(f) = u_0(f) + \chi P_{k-1}$, $S_{\theta_1}(f) = S^D(f) s_1^D$. We note that

$$\Sigma_{k+1}^D = \left\{ n \in \left\{2, \dots, \left\lfloor \frac{\pi}{\omega} \right\rfloor \right\} \;\middle|\; n\frac{\omega}{\pi} \in \mathbb{N} \right\} = \emptyset$$

in view of

$$\left\lfloor \frac{\pi}{\omega} \right\rfloor \frac{\omega}{\pi} < 1.$$

Corollary 2.5 shows that $S_1\colon X_1 \to Y_1$ is bounded, linear; Lemma 2.6 (i) asserts that $S_1^D \in ((X_0, X_1)_{\theta_1,1})^\star$ and $s_1^D \in (X_0, X_1)_{\theta,\infty}$. The desired assertion (2.37) now follows from Lemma 2.7.

*Step 4:* We show (2.37) for the case $\lambda_1^D = \pi/\omega \in \mathbb{N}$ and $\omega < \pi$ using Lemma 2.7 and Corollary 2.5. As in Step 3, choose $(k, \varepsilon) \in \mathbb{N}_0 \times (0,1)$ with $\lambda_1^D = \pi/\omega < k + \varepsilon + 1 < \lambda_1^D + 1 < \lambda_2^D$ such that $\lfloor k + \varepsilon + 1 \rfloor = \lambda_1^D$ and take with this choice of $k$, $\varepsilon$ the spaces $X_0$, $X_1$, $Y_0$, $Y_1$ as in (2.2) and set $\theta_1 = \pi/(\omega(k + \varepsilon + 1))$ so that $B^{\pi/\omega-1}_{2,1}(\mathcal{C}_R) = (X_0, X_1)_{\theta_1,1}$ and $B^{\pi/\omega+1}_{2,1}(\mathcal{C}_R) = (Y_0, Y_1)_{\theta_1,\infty}$. In Corollary 2.5, our choice of $k$ corresponds to $\Sigma^D_{k+1} = \{\pi/\omega\}$ and $n^\star = \pi/\omega = \lambda^D_1$. Again, $\widetilde{T}\colon X_0 \to Y_0$ is bounded by Lax-Milgram. With the functions $u_0$, $S^D$, $s^D_1$, $P_{k-1}$ from Corollary 2.5, Corollary 2.5 gives the decomposition $\widetilde{T}f = (u_0(f) + S^D(f)s^D_1) + \chi P_{k-1} =: S_1(f) + S_{\theta_1}(f)$. Since $s^D_1$ is a polynomial (and thus smooth), Corollary 2.5 asserts the boundedness of $S_1\colon X_1 \to Y_1$ and $S_{\theta_1}\colon X_{\theta_1,1} \to Y_{\theta_1,\infty}$. Lemma 2.7 then implies (2.37).

*Step 5:* We show (2.37) for the case $\omega > \pi$. The procedure is similar to that of preceding Steps 3, 4. Since $\omega \in (\pi, 2\pi)$, we have $\lambda^D_1 = \pi/\omega < 1 < 2\pi/\omega = \lambda^D_2$ so that we may select $k = 0$ and $\varepsilon \in (0, 1)$ such that $\lambda^D_1 < 1 < k + \varepsilon + 1 < \lambda^D_2$. Then $\Sigma^D_{k+1} = \emptyset$, $P_{k-1} \equiv 0$ in Corollary 2.5, and we may argue as in Step 3 with the choice $X_0$, $X_1$, $Y_0$, $Y_1$ from (2.2) and $\theta_1 = \pi/(\omega(k + \varepsilon + 1))$; note that $1/2 < \pi/\omega < 1$, so that Lemma 2.6 (ii) implies $S^D_1 \in (B^{\pi/\omega-1}_{2,1}(\mathcal{C}_R))^\star = ((X_0, X_1)_{\theta_1,1})^\star$. □

## 3. Neumann boundary conditions

The case of Neumann boundary conditions is handled in a way similar to the Dirichlet case of Section 2.1.2. We put

$$\sigma^N := \{\lambda^N_n \mid n = 0, 1, \ldots\} \quad \text{with } \lambda^N_n := n\frac{\pi}{\omega}. \tag{3.1}$$

Analogous to Proposition 2.3, we have:

**Proposition 3.1.** *Let $R > 0$. Let $k \in \mathbb{N}_0$ and $\varepsilon \in (0, 1)$ satisfy $k + 1 + \varepsilon < \lambda^N_2 = 2\pi/\omega$, $k + 1 + \varepsilon \neq \lambda^N_1 = \pi/\omega$, and let $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further assume $\partial^i_x\partial^j_y f(0) = 0$ for $i + j \leqslant k - 1$. Then every function $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$, solving*

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad \partial_n u_1 = 0 \quad \text{for } \varphi \in \{0, \omega\} \tag{3.2}$$

*has the form*

$$u_1 = \begin{cases} u_0, & \text{if } \dfrac{\pi}{\omega} = \lambda^N_1 > k + \varepsilon + 1, \\ u_0 - \dfrac{1}{\pi}\left(\displaystyle\int_{\mathcal{C}} r^{-\lambda^N_1} \cos(\lambda^N_1\varphi) f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right) r^{\lambda^N_1} \cos(\lambda^N_1\varphi), & \\ & \text{if } \dfrac{\pi}{\omega} = \lambda^N_1 < k + \varepsilon + 1 < \lambda^N_2 \end{cases} \tag{3.3}$$

*for a $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$ with the estimate*

$$\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}.$$

P r o o f. The procedure is as in the Dirichlet case of Section 2.1.2: The Mellin transformation yields the equations (2.4), (2.5) for $\mathcal{M}[u_1]$ and $\mathcal{M}[u_0]$ together with the Neumann boundary conditions $\partial_\varphi \mathcal{M}[u_1] = \partial_\varphi \mathcal{M}[u_0] = 0$ on $\{\varphi = 0\}$ and $\{\varphi = \omega\}$. The operator $\mathcal{L}(\zeta)$ is meromorphic on $\mathbb{C}$ with poles at $\pm \mathrm{i}\sigma^N$. As in the Dirichlet case, $u_0 \in K^{k+2}_{-\varepsilon}(\mathcal{C})$ so that $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$. Since $0 \in \pm\mathrm{i}\sigma^N$, the inverse Mellin transformation cannot be performed on the line $\{\operatorname{Im}\zeta = 0\}$. In contrast to the Dirichlet case, where $u_1 \in K^1_0(\Gamma)$ due to the vanishing of $u_1$ on $\{\varphi = 0\}$ and $\{\varphi = \omega\}$ we only have $u_1 \in K^1_\delta(\mathcal{C})$, $\delta > 0$ arbitrary, in the case of Neumann boundary conditions. This implies that the inverse Mellin transformation has to be done on a line $\{\operatorname{Im}\zeta = \delta\}$ for chosen $\delta > 0$. The Cauchy integral formula relating $u_0$ and $u_1$ now uses the lines $\{\operatorname{Im}\zeta = \delta\}$ and $\{\operatorname{Im}\zeta = -(k+1+\varepsilon)\}$ and leads to

$$u_0 - u_1 = \sum_{\substack{\zeta_0 \in -\mathrm{i}\sigma^N \\ \operatorname{Im}\zeta_0 \in (-1-k-\varepsilon, \delta)}} \frac{2\pi\mathrm{i}}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta=\zeta_0} (r^{\mathrm{i}\zeta}(\mathcal{L}(\zeta))^{-1}\mathcal{M}g(\zeta)).$$

For the evaluation of residues, we note that the double pole of $\mathcal{L}^{-1}$ at $\zeta_0 = 0$ leads to two contributions to the sum; if $k + 1 + \varepsilon > \pi/\omega$, then a third contribution arises in the sum. The residues can be evaluated explicitly:

$$\begin{aligned} &\text{at } \zeta_0 = 0: && \left(\frac{\mathrm{i}}{\omega}\frac{1}{\sqrt{2\pi}}\int_{\mathcal{C}} f\,\mathrm{d}\mathbf{x}\right)\ln r + \left(\frac{\mathrm{i}}{\omega}\frac{1}{\sqrt{2\pi}}\int_{\mathcal{C}} \ln r f\,\mathrm{d}\mathbf{x}\right)1 =: s_0 + s_1, \\ &\text{at } \zeta_0 = -\mathrm{i}\lambda_1^N: && -\left(\frac{1}{\pi}\int_{\mathcal{C}} r^{-\lambda_1^N}\cos(\lambda_1^N\varphi) f\,\mathrm{d}\mathbf{x}\right) r^{\lambda_1^N}\cos(\lambda_1^N\varphi). \end{aligned}$$

By assumption $u_1 \in H^1(\mathcal{C})$ so that the contribution $s_0$ has to vanish. The contribution $s_1$ is a constant function and hence smooth. Additionally, the function $(x,y) \mapsto \ln r$ is in $B^1_{2,\infty}(\mathcal{C}_1)$ by Lemma 2.6 so that $f \mapsto \int_{\mathcal{C}} \ln r f$ is a bounded linear functional on $(B^1_{2,\infty}(\mathcal{C}_1))^\star \supset H^{k+\varepsilon}(\mathcal{C}_1)$ and thus the sum $u_0 + s_1$ is in $H^{k+2+\varepsilon}(\mathcal{C}_1)$ with the stated estimate. $\square$

The Neumann analog of Lemma 2.4 is:

**Lemma 3.2.** *Let $i$, $j$, $k \in \mathbb{N}_0$ with $i + j = k$. Set $\Sigma^N_{k+2} := \{n \in \{2, \ldots, k+2\} \mid n\omega/\pi \in \mathbb{N}\}$ and $S^N_{k+2} := \operatorname{span}\{r^n(\ln r\cos(n\varphi) - \varphi\sin(n\varphi)) \mid n \in \Sigma^N_{k+2}\}$. Then there is a polynomial $\widetilde{p}_{i,j}$ of degree $k+2$ and a harmonic function $p'_{i,j} \in S^N_{k+2}$ such that $p^N_{i,j} := \widetilde{p}_{i,j} + p'_{i,j}$ satisfies*

$$-\Delta p^N_{i,j} = x^i y^j \quad \text{on } \mathcal{C}, \quad \partial_n p^N_{i,j}|_\Gamma = 0. \tag{3.4}$$

*In the special case $\omega = \pi$, the contribution $p'_{i,j}$ may be taken to be zero.*

P r o o f. The proof is very similar to that of Lemma 2.4. The correction on $\{\varphi = 0\}$ is now done with polynomials of the form $\operatorname{Im} \sum_{n=0}^{k+2} a_n z^n$, $a_n \in \mathbb{R}$. For the correction on $\{\varphi = \omega\}$ one uses the functions $\operatorname{Re} z^n = r^n \cos(n\varphi)$ if $n\omega/\pi \notin \mathbb{N}_0$ and the function $\operatorname{Re}(z^n \ln z) = r^n(\ln r \cos(n\varphi) - \varphi \sin(n\varphi))$ if $n\omega/\pi \in \mathbb{N}_0$. □

**Corollary 3.3.** *Let $R > 0$. Let $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$ satisfy $k+1+\varepsilon < 2\pi/\omega = \lambda_2^N$, $k+1+\varepsilon \neq \pi/\omega = \lambda_1^N$, and $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Let $\chi \in C_0^\infty(B_1(0))$ with $\chi \equiv 1$ near the origin. Then every function $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$, solving*

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad \partial_n u_1 = 0 \quad \textit{for } \varphi \in \{0,\omega\} \tag{3.5}$$

*has the form $u_1 = u_0 + \chi P_{k-1} + \delta$ with $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$,*

$$\delta = \begin{cases} 0, & \textit{if } \dfrac{\pi}{\omega} = \lambda_1^N > k+\varepsilon+1, \\ S^N(f) s_1^N, & \textit{if } \dfrac{\pi}{\omega} = \lambda_1^N < k+\varepsilon+1 < \lambda_2^N, \end{cases} \tag{3.6}$$

$$S^N(f) := -\frac{1}{\pi}\Big(\int_{\mathcal{C}} r^{-\lambda_1^N} \cos(\lambda_1^N \varphi)(f(\mathbf{x}) + \Delta(\chi(\mathbf{x}) P_{k-1}(\mathbf{x})))\,\mathrm{d}\mathbf{x}\Big), \tag{3.7}$$

$$s_1^N := r^{\lambda_1^N} \cos(\lambda_1^N \varphi), \tag{3.8}$$

$$P_{k-1}(\mathbf{x}) := \sum_{i+j \leqslant k-1} \frac{1}{i!j!} p_{i,j}^N(\mathbf{x})(\partial_x^i \partial_y^j f)(0), \tag{3.9}$$

*and $p_{i,j}^N$ are the fixed functions from Lemma 3.2. Furthermore, the following estimates hold:*

$$\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C})}, \tag{3.10}$$

$$\|P_{k-1}\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C})} \quad \textit{if } \Sigma^N_{k+1} = \emptyset, \tag{3.11}$$

$$\|P_{k-1}\|_{B^{n^\star+1}_{2,\infty}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C})} \quad \textit{if } \Sigma^N_{k+1} \neq \emptyset, \tag{3.12}$$

$$n^\star := \min\Big\{n \in \{2,\ldots,k+1\} \mid n\frac{\omega}{\pi} \in \mathbb{N}\Big\},$$

$$\|\Delta(\chi P_{k-1})\|_{H^{k+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C})} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C})}. \tag{3.13}$$

*The implied constants depend only $k$, $\varepsilon$, the angle $\omega$, and the choice of the cut-off function $\chi$.*

P r o o f. Follows in the same way as that of Corollary 2.5. □

P r o o f of Theorem 1.1 (ii). The proof follows the strategy developed for the case of Dirichlet boundary conditions; the main difference lies in the fact that the basic stability estimate is now $\widetilde{T}\colon \widetilde{H}^{-1}(\mathcal{C}_R) \to H^1(\mathcal{C}_R)$ instead of $\widetilde{T}\colon H^{-1}(\mathcal{C}_R) \to H^1(\mathcal{C}_R)$ for the Dirichlet case.

The regularity assertions of Step 0 still hold. The mapping $T$ has to be replaced with $T\colon \widetilde{H}^{-1}(\mathcal{C}_R) \to H^1_D(\mathcal{C}_R)$, where $Tf$ solves

$$-\Delta Tf = f \quad \text{in } \mathcal{C}_R, \quad \partial_n(Tf) = 0 \quad \text{on } \Gamma_N, \quad Tf = 0 \quad \text{on } \widetilde{\Gamma}_R,$$

and $\Gamma_N = \{\varphi = 0\} \cup \{\varphi = \omega\}$. We set $\widetilde{T} := (\chi_{R'} T \chi_{\overline{R}})$ as in Step 2 of the Dirichlet case. The modified right-hand side $\widetilde{f}$ is defined as in (2.32), and it suffices to ascertain

$$\|\widetilde{T}\widetilde{f}\|_{B^{\pi/\omega+1}_{2,\infty}(\mathcal{C}_{R'})} \lesssim \|\widetilde{f}\|_{B^{\pi/\omega-1}_{2,1}(\mathcal{C}_R)}. \tag{3.14}$$

As in the proof of Theorem 1.1 (i) for the Dirichlet case, we distinguish between the cases $\omega < \pi$ and $\omega > \pi$, the case $\omega = \pi$ having already been discussed in Remark 1.2.

*Step 1 (preliminaries):* Let $\omega < \pi$. Then, $\lambda_1^N > 1$. Select $k = 0$, $\varepsilon \in (0,1)$ such that $1 < s' := k+\varepsilon+1 < \lambda_1^N$. Corollary 3.3 asserts that $\widetilde{T}\colon H^{s'-1}(\mathcal{C}_R) \to H^{s'+1}(\mathcal{C}_R)$ is a bounded operator.

*Step 2:* Let $\omega < \pi$ and $\pi/\omega \notin \mathbb{N}$. Select $(k,\varepsilon) \in \mathbb{N}_0 \times (0,1)$ with $\lambda_1^N < k+\varepsilon+1 < \lambda_2^N$ such that $\lfloor k+\varepsilon+1 \rfloor = \lfloor \lambda_1^N \rfloor \geqslant 1$. Take with $s'$ from Step 1

$$X_0 := H^{s'-1}(\mathcal{C}_R),\ X_1 = H^{k+\varepsilon}(\mathcal{C}_R),\ Y_0 := H^{s'+1}(\mathcal{C}_R),\ Y_1 = H^{2+k+\varepsilon}(\mathcal{C}_R). \tag{3.15}$$

Note that with $\theta_1 := (\pi/\omega - 1 - (s'-1))/(k+\varepsilon-(s'-1)) \in (0,1)$, we have $B^{\pi/\omega-1}_{2,1}(\mathcal{C}_R) = (X_0, X_1)_{\theta_1,1}$ and $B^{\pi/\omega+1}_{2,\infty}(\mathcal{C}_R) = (Y_0,Y_1)_{\theta_1,\infty}$. Then, $\widetilde{T}$ satisfies the assumptions of Lemma 2.7: the mapping properties $X_0 \to Y_0$ are given by the above Step 1, and the mapping properties of the decomposition of $\widetilde{T}f = (u_0(f) + \chi P_{k-1}) + S_1^N(f)s_1^N =: S_1(f) + S_{\theta_1}(f)$ for arguments $f \in X_1$ is provided by Corollary 3.3 in conjunction with Lemma 2.6 (i).

*Step 3:* Let $\omega < \pi$ and $\pi/\omega \in \mathbb{N}$. Choose $k$, $\varepsilon$, $X_0$, $X_1$, $Y_0$, $Y_1$ as in Step 2 above. As in the Dirichlet case, choose the decomposition $\widetilde{T}f = (u_0(f) + S_1^N(f)s_1^N) + \chi P_{k-1} =: S_1(f) + S_{\theta_1}(f)$; Corollary 3.3 provides that $S_1$, $S_{\theta_1}$ satisfy the assumptions of Lemma 2.7.

*Step 4:* Let $\omega > \pi$. Since $\omega \in (\pi, 2\pi)$, we have $\lambda_1^N = \pi/\omega < 1 < 2\pi/\omega = \lambda_2^N$ and may therefore select $k = 0$ and $\varepsilon \in (0,1/2)$ such that $\lambda_1^N < 1 < k+\varepsilon+1 < \lambda_2^N$. Set

$$\begin{aligned} X_0 &:= \widetilde{H}^{-1}(\mathcal{C}_R),\ X_1 = H^{k+\varepsilon}(\mathcal{C}_R) = \widetilde{H}^{k+\varepsilon}(\mathcal{C}_R),\\ Y_0 &:= H^1(\mathcal{C}_R),\ Y_1 = H^{2+k+\varepsilon}(\mathcal{C}_R), \end{aligned} \tag{3.16}$$

and $\theta_1 = (\pi/\omega - 1 + 1)/(k + \varepsilon + 1) \in (0, 1)$ so that $(X_0, X_1)_{\theta_1,1} = \widetilde{B}^{\pi/\omega-1}_{2,1}(\mathcal{C}_R)$ and $(Y_0, Y_1)_{\theta_1,1} = B^{\pi/\omega+1}_{2,1}(\mathcal{C}_R)$. As in the Dirichlet case, we note $\Sigma^N_{k+1} = \emptyset$ and that Corollary 3.3 provides the assumptions of Lemma 2.7 so that $\widetilde{T}\colon \widetilde{B}^{\pi/\omega-1}_{2,1}(\mathcal{C}_R) \to B^{\pi/\omega+1}_{2,\infty}(\mathcal{C}_R)$. Since $\omega \in (\pi, 2\pi)$, we have $\pi/\omega - 1 \in (-1/2, 0)$ and thus $\widetilde{B}^{\pi/\omega-1}_{2,1}(\mathcal{C}_R) = B^{\pi/\omega-1}_{2,1}(\mathcal{C}_R)$, cf. (1.7). □

## 4. Mixed boundary conditions

The case of mixed boundary conditions is similar to the Neumann case. We recall that $\Gamma_D = \{\varphi = 0\}$ and $\Gamma_N = \{\varphi = \omega\}$. Set

$$\sigma^M := \{\lambda^M_n \mid n \in \mathbb{N}\} \quad \text{with } \lambda^M_n := \Big(n - \frac{1}{2}\Big)\frac{\pi}{\omega}. \tag{4.1}$$

The operator $(\mathcal{L}(\zeta))^{-1}$ arising in the case of mixed boundary conditions is meromorphic on $\mathbb{C}$ with poles at $\pm i\sigma^M$. With similar arguments as in Proposition 2.3 one obtains:

**Proposition 4.1.** *Let $R > 0$. For $k \in \mathbb{N}_0$ and $\varepsilon \in (0, 1)$, let $k + 1 + \varepsilon < \lambda^M_3$, $k + 1 + \varepsilon \notin \{\lambda^M_1, \lambda^M_2\} = \{\pi/(2\omega), 3\pi/(2\omega)\}$, and $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further assume $\partial^i_x \partial^j_y f(0) = 0$ for $i + j \leqslant k - 1$. Then $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$, solving*

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad u_1 = 0 \quad \text{on } \{\varphi = 0\}, \quad \partial_n u_1 = 0 \quad \text{on } \{\varphi = \omega\}, \tag{4.2}$$

*has the form*

$$u_1 = \begin{cases} u_0 & \text{if } k + \varepsilon + 1 < \dfrac{\pi}{2\omega} = \lambda^M_1, \\ u_0 + S^M_1(f) s^M_1, & \text{if } \lambda^M_1 = \dfrac{\pi}{2\omega} < k + \varepsilon + 1 < \dfrac{3\pi}{2\omega} = \lambda^M_2, \\ u_0 + S^M_1(f) s^M_1 + S^M_2(f) s^M_2, & \text{if } \dfrac{3\pi}{2\omega} = \lambda^M_2 < k + \varepsilon + 1 < \lambda^M_3, \end{cases} \tag{4.3}$$

*where*

$$S^M_1(f) = -\frac{1}{\pi}\Big(\int_{\mathcal{C}} r^{-\pi/(2\omega)} \sin\big(\frac{\pi}{2\omega}\varphi\big) f(x)\,\mathrm{d}x\Big), \; s^M_1 = r^{\pi/(2\omega)} \sin\big(\frac{\pi}{2\omega}\varphi\big), \tag{4.4a}$$

$$S^M_2(f) = -\frac{1}{\pi}\Big(\int_{\mathcal{C}} r^{-3\pi/(2\omega)} \sin\big(\frac{3\pi}{2\omega}\varphi\big) f(x)\,\mathrm{d}x\Big), \; s^M_2 = r^{3\pi/(2\omega)} \sin\Big(\frac{3\pi}{2\omega}\varphi\Big) \tag{4.4b}$$

*together with the estimate*

$$\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}.$$

**Lemma 4.2.** *Let $i$, $j$, $k \in \mathbb{N}_0$ with $i+j=k$. Set $\Sigma^M_{k+2} := \{n \in \{1,\dots,k+2\} \mid n\omega/\pi + 1/2 \in \mathbb{N}\}$ and $S^M_{k+2} := \operatorname{span}\{r^n(\ln r \sin(n\varphi) + \varphi\cos(n\varphi)) \mid n \in \Sigma^M_{k+2}\}$. Then there is a polynomial $\widetilde{p}_{i,j}$ of degree $k+2$ and a harmonic function $p'_{i,j} \in S^M_{k+2}$ such that $p^M_{i,j} := \widetilde{p}_{i,j} + p'_{i,j}$ satisfies*

$$-\Delta p^M_{i,j} = x^i y^j \quad \text{on } \mathcal{C}, \quad p^M_{i,j} = 0 \text{ on } \{\varphi = 0\}, \quad \partial_n p^M_{i,j} = 0 \text{ on } \{\varphi = \omega\}. \tag{4.5}$$

Proof. One proceeds as in the proof of Lemma 2.4, the only difference being the correction on the line $\{\varphi = \omega\}$. For that, one uses, for $n \geqslant 1$, the functions $\operatorname{Im} z^n$ if $n\omega/\pi + 1/2 \notin \mathbb{N}$ and $\operatorname{Im}(z^n \ln z)$ if $n\omega/\pi + 1/2 \in \mathbb{N}$. □

**Corollary 4.3.** *Let $R > 0$. Let $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$ satisfy $k+1+\varepsilon < \lambda^M_3 = 5\pi/(2\omega)$ and $k+1+\varepsilon \notin \{\lambda^M_1, \lambda^M_2\} = \{\pi/(2\omega), 3\pi/(2\omega)\}$. Let $f \in H^{k+\varepsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Let $\chi \in C^\infty_0(B_1(0))$ with $\chi \equiv 1$ near the origin. Then every function $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$, solving*

$$-\Delta u_1 = f \in H^{k+\varepsilon}(\mathcal{C}), \quad u_1 = 0 \quad \text{on } \{\varphi = 0\}, \quad \partial_n u_1 = 0 \quad \text{on } \{\varphi = \omega\}, \tag{4.6}$$

*has the form $u_1 = u_0 + \chi P_{k-1} + \delta$ with $u_0 \in H^{k+2+\varepsilon}(\mathcal{C}_R)$,*

$$\delta = \begin{cases} 0, & \text{if } k+\varepsilon+1 < \lambda^M_1 = \dfrac{\pi}{2\omega}, \\ S^M_1(f + \Delta(\chi P_{k-1}))s^M_1, & \text{if } \lambda^M_1 < k+\varepsilon+1 < \lambda^M_2, \\ S^M_1(f + \Delta(\chi P_{k-1}))s^M_1 + S^M_2(f + \Delta(\chi P_{k-1}))s^M_2, & \\ & \text{if } \lambda^M_2 < k+\varepsilon+1 < \lambda^M_3, \end{cases} \tag{4.7}$$

$$P_{k-1}(\mathbf{x}) := \sum_{i+j \leqslant k-1} \frac{1}{i!j!} p^M_{i,j}(\mathbf{x})(\partial^i_x \partial^j_y f)(0), \tag{4.8}$$

*where $S^M_1$, $S^M_2$, $s^M_1$, $s^M_2$ are given in (4.4), and $p^M_{i,j}$ are the fixed functions from Lemma 4.2. Furthermore,*

$$\|u_0\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}, \tag{4.9}$$

$$\|P_{k-1}\|_{H^{k+2+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C}_1)} \quad \text{if } \Sigma^M_{k+1} = \emptyset, \tag{4.10}$$

$$\|P_{k-1}\|_{B^{n^\star+1}_{2,\infty}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C}_1)} \quad \text{if } \Sigma^M_{k+1} \neq \emptyset, \tag{4.11}$$

$$n^\star := \min\Big\{n \in \{1,\dots,k+1\} \mid n\frac{\omega}{\pi} + \frac{1}{2} \in \mathbb{N}\Big\},$$

$$\|\Delta(\chi P_{k-1})\|_{H^{k+\varepsilon}(\mathcal{C}_R)} \lesssim \|f\|_{B^k_{2,1}(\mathcal{C}_1)} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C})}. \tag{4.12}$$

*The implied constants depend only $k$, $\varepsilon$, the angle $\omega$, and the choice of the cut-off function $\chi$.*

Proof. Follows as in the Dirichlet case. □

As in the Neumann case, the proof of Theorem 1.1 (iii) comes down to showing the mapping property $\widetilde{T}\colon B^{\pi/(2\omega)-1}_{2,1}(\mathcal{C}_R) \to B^{\pi/(2\omega)+1}_{2,\infty}(\mathcal{C}_R)$ for $\pi/(2\omega)-1 > -1/2$ and $\widetilde{T}\colon \widetilde{B}^{\pi/(2\omega)-1}_{2,1}(\mathcal{C}_R) \to B^{\pi/(2\omega)+1}_{2,\infty}(\mathcal{C}_R)$ for $\pi/(2\omega)-1 \leqslant -1/2$, where $\widetilde{T} = \chi_{R'} T \chi_{\overline{R}}$ with $Tf$ being the solution of

$$-\Delta Tf = 0 \text{ in } \mathcal{C}_R, \quad Tf = 0 \text{ on } \Gamma_{0,R} \cup \widetilde{\Gamma}_R, \quad \partial_n Tf = 0 \text{ on } \Gamma_{\omega,R}.$$

Proof of Theorem 1.1 (iii). *Case 1:* $\omega < \pi/2$. Since $\omega < \pi/2$, we may select $(k,\varepsilon) \in \mathbb{N}_0 \times (0,1)$ so that $1 < \lambda_1^M < k+1+\varepsilon < \lambda_2^M$ and $\lfloor k+\varepsilon+1 \rfloor = \lfloor \lambda_1^M \rfloor$. This is the setting, where exactly one singularity function appears in the representation of the solution in Corollary 4.3 and the arguments of the Neumann case (Steps 2, 3 of the proof of Theorem 1.1 (ii)) apply.

*Case 2:* $\omega = \pi/2$. Select $k = 0$, $\varepsilon \in (0,1/2)$ such that $1 = \lambda_1^M < k+\varepsilon+1 < \lambda_2^M = 3$. In Corollary 4.3, this corresponds to the case of one "singularity" function $S_1^M(f)s_1^M$, which is in fact a polynomial and thus smooth, and $P_{k-1} = 0$. Lemma 2.6 (ii) provides $S_1^M \in (\widetilde{B}^0_{2,1}(\mathcal{C}_R))^\star$. Set $X_0 = \widetilde{H}^{-1}(\mathcal{C}_R)$, $X_1 = \widetilde{H}^{\varepsilon}(\mathcal{C}_R)$, $Y_0 = H^1(\mathcal{C}_R)$, $Y_1 = H^{2+\varepsilon}(\mathcal{C}_R)$, $\theta = 1/(1+\varepsilon)$. From Lemma 2.7 and the solution representation of Corollary 4.3 we infer $\widetilde{T}\colon \widetilde{B}^{\pi/(2\omega)-1}_{2,1} \to B^{\pi/(2\omega)+1}_{2,\infty}$.

*Case 3:* $\omega \in (\pi/2, 3\pi/2)$. In this case, one can select $k = 0$ and $\varepsilon \in (0,1)$ such that $\lambda_1^M < 1 < k+\varepsilon+1 < \lambda_2^M$ so that in the application of Corollary 4.3 a single singularity function arises. This is handled as in the Neumann case with $\omega > \pi$ there.

*Case 4:* $\omega \in (3\pi/2, 2\pi)$. We have $\lambda_1^M < \lambda_2^M < 1 < \frac{5}{4} < \lambda_3^M$. Select $k = 0$ and $\varepsilon \in (0,1/2)$ such that $\lambda_2^M < 1 < k+\varepsilon+1 < \lambda_3^M$. Set $X_0 = \widetilde{H}^{-1}(\mathcal{C}_R)$, $X_1 = \widetilde{H}^{\varepsilon}(\mathcal{C}_R)$, $Y_0 = H^1(\mathcal{C}_R)$, $Y_1 = H^{2+k+\varepsilon}(\mathcal{C}_R)$, $\theta_j = \lambda_j^M/(1+k+\varepsilon)$, $j \in \{1,2\}$, so that $X_{\theta_j,1} = \widetilde{B}^{\lambda_j^M-1}_{2,1}(\mathcal{C}_R)$ and $Y_{\theta_j,\infty} = B^{\lambda_j^M+1}_{2,\infty}(\mathcal{C}_R)$. Note $P_{k-1} \equiv 0$ in Corollary 4.3. Corollary 4.3 yields for $f \in X_1$ the decomposition $\widetilde{T}f = u_0(f) + S_1^M(f)s_1^M + S_2^M(f)s_2^M =: S_1(f) + S_{\theta_1}(f) + S_{\theta_2}(f)$ with $S_{\theta_j}\colon X_{\theta_j,1} \to Y_{\theta_j,\infty}$ (cf. Lemma 2.6). The desired result now follows from Lemma 2.7.

*Case 5:* $\omega = 3\pi/2$. In contrast to the preceding case $\omega > 3\pi/2$, we have $\lambda_1^M < \lambda_2^M = 1 < 5/4 < \lambda_3^M$. We take $(k,\varepsilon) = (0,\varepsilon)$ as in the preceding case. The function $s_2^M$ is a polynomial and hence smooth. Corollary 4.3 yields for $f \in X_1$ the decomposition $\widetilde{T}f = (u_0(f) + S_2^M(f)s_2^M) + S_1^M(f)s_1^M =: S_1(f) + S_{\theta_1}(f)$. Lemma 2.6 shows that the operators $S_1$ and $S_{\theta_1}$ have the mapping properties required by Lemma 2.7. This concludes the proof. □

## 5. An extension to $L^p$-based Besov spaces

The Besov spaces $B^s_{2,q}$ considered so far are based on $L^2$-spaces. Several results obtained in the framework of these Besov spaces can be generalized to $L^p$-based Besov spaces. We illustrate this in the present section for the simplest case, that of the Dirichlet boundary condition and the assumption of $W^{2,p}$-regularity.

For $k \in \mathbb{N}_0$, $p \in (1,\infty)$, and bounded domains $\Omega$, we introduce the spaces $W^{k,p}(\Omega)$ in the usual way by requiring that derivatives up to order $k$ be in $L^p(\Omega)$, [22]. For $s \geqslant 0$ with $s \notin \mathbb{N}_0$ one puts $W^{s,p}(\Omega) := (W^{\lfloor s\rfloor,p}(\Omega), W^{\lceil s\rceil,p})_{s-\lfloor s\rfloor,p}$. The norm $\|\cdot\|_{W^{s,p}(\Omega)}$ is in fact equivalent to the Aronstein-Slobodeckij norm [37], Chapter 36. A second important fact is that the Reiteration Theorem [37], Chapter 26 allows one to show that for $k$, $m \in \mathbb{N}_0$ and $k < s < m$ with $s \notin \mathbb{N}_0$ one has $W^{s,p}(\Omega) = (W^{k,p}(\Omega), W^{m,p}(\Omega))_{(s-k)/(m-k),p}$ [37], Chapter 34. Analogous to the case of Sobolev spaces, one defines for $s \geqslant 0$, $s \notin \mathbb{N}_0$, Besov spaces $B^s_{p,q}(\Omega)$ by interpolation

$$B^s_{p,q}(\Omega) = (W^{\lfloor s\rfloor,p}(\Omega), W^{\lceil s\rceil,p}(\Omega))_{s-\lfloor s\rfloor,q}$$

and notes that the Reiteration theorem would allow us to represent these spaces by interpolating between the spaces $W^{s_1,p}(\Omega)$, $W^{s_2,p}(\Omega)$ with $0 \leqslant s_1 < s < s_2$, [37], Chapter 24, [38], Section 4.3.2. As is customary in connection with $L^p$ spaces, we denote by $p' = p/(p-1)$ the conjugate exponent.

Analogous to the spaces $K^s_\gamma(\mathcal{C})$, it is convenient to define $K^{s,p}_\gamma(\mathcal{C})$ by the norm

$$\|u\|^p_{K^{s,p}_\gamma(\mathcal{C})} := \sum_{|\alpha|\leqslant s} \|r^{|\alpha|-s+\gamma} D^\alpha u\|^p_{L^p(\mathcal{C})}. \tag{5.1}$$

We consider solutions $u \in W^{1,p}(\mathcal{C})$ of

$$-\Delta u = f \quad \text{in } \mathcal{C}, \quad u = 0 \text{ on } \varphi \in \{0,\omega\}. \tag{5.2}$$

**5.1. Regularity of the singularity functions and stress intensity functionals.** Let $k \in \mathbb{N}_0$. If $k > 2/p$, then by [38], Theorem 4.6.1 we have the embedding $W^{k,p}(\mathcal{C}_R) \subset C^{\lfloor k-2/p\rfloor}(\mathcal{C}_R)$ so that one may define as in (2.15) the polynomial $P_{\lfloor k-2/p\rfloor}$. We have:

**Lemma 5.1.** *For $k \in \mathbb{N}_0$, $p \in (1,2)\cup(2,\infty)$, $f \in W^{k,p}(\mathcal{C}_1)$ let $P_{\lfloor k-2/p\rfloor}$ be given by (2.15), $\chi \in C^\infty_0(B_1(0))$ with $\chi \equiv 1$ near the origin. Then, $f - \Delta(\chi P_{\lfloor k-2/p\rfloor}) \in K^{k,p}_0(\mathcal{C}_1)$.*

P r o o f. The proof follows structurally that of Lemma A.1. Note that $k - 2/p$ is not an integer, which allows one to avoid the introduction of $\varepsilon > 0$ like in the proof of Lemma A.1. □

In the following Lemma 5.2, which allows us to ascertain the regularity of the singularity functions and the stress intensity functional, the functions $P_{\lfloor\alpha-2\rfloor}$ of (2.15) arise. These functions vanish for $\alpha-2<0$ and are well-defined for $f\in B_{p,1}^{\lfloor\alpha-2\rfloor+2/p}\subset C^{\lfloor\alpha-2\rfloor}$ in view of [38], Theorem 4.6.1.

**Lemma 5.2.** *Let $p\in(1,2)\cup(2,\infty)$.*

(i) *For $\beta+2/p>0$ the function $s^+(r,\varphi)=r^\beta\sin(\beta\varphi)$ is in the space $B_{p,\infty}^{\beta+2/p}(\mathcal{C}_1)$.*

(ii) *Let $\Phi\in C^\infty(\mathbb{R}^2)$ with $|\Phi(x,y)|\leqslant Cr^n$ as $r\to 0$ for some $n\in\mathbb{N}_0$. Then the functions $v(x,y)=\Phi(x,y)\ln r$ and $w(x,y)=\varphi\Phi(x,y)$ are in the space $B_{p,\infty}^{n+2/p}(\mathcal{C}_1)$.*

(iii) *Let $\alpha-2/p'>0$ with $\alpha-2\notin\mathbb{N}_0$. Let $P_{\lfloor\alpha-2\rfloor}$ be given by (2.15). Then*

$$f\mapsto S(f):=\int_{\mathcal{C}_1} r^{-\alpha}\sin(\alpha\varphi)(f+\Delta(\chi P_{\lfloor\alpha-2\rfloor}))$$

*is bounded linear on $B_{p,1}^{\alpha-2/p'}(\mathcal{C}_1)$.*

P r o o f. (i) is shown similarly to the proof of Lemma 2.6 (iii) by estimating the $K$-functional through the splitting $r^\beta=r^\beta\chi_t+r^\beta(1-\chi_t)$ for a suitable $t$-dependent cut-off function $\chi_t$.

(ii) is shown by appropriately modifying the proof of Lemma 2.6 (iv).

(iii) The proof follows that of Lemma 2.6 (i) in that $B_{p,1}^{\alpha-2/p'}$ is suitably written as an interpolation space. We distinguish the cases $0<\alpha<2$ and $\alpha>2$.

*The case $\alpha<2$:* Then $P_{\lfloor\alpha-2\rfloor}\equiv 0$. We write for $\varepsilon>0$ so small that $\alpha+\varepsilon<2$

$$B_{p,1}^{\alpha-2/p'}(\mathcal{C}_1)=(L^p(\mathcal{C}_1),B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1))_{\theta,1},\quad \theta=\frac{\alpha-2/p'}{\alpha-2/p'+\varepsilon}.$$

As in the proof Lemma 2.6 (i), one splits for $\delta>0$ the expression $S(f)=S_1+S_2$ as in (2.23). For $S_1$, the Hölder inequality yields

$$|S_1|\leqslant\|f\|_{L^p(\mathcal{C}_1)}\|\chi_\delta r^{-\alpha}\|_{L^{p'}(\mathcal{C}_1)}\lesssim\delta^{-\alpha+2/p'}\|f\|_{L^p(\mathcal{C}_1)}.$$

Select $q>1$ by the condition $2/q=2/p-(\alpha-2/p'+\varepsilon)=2-\alpha-\varepsilon\in(0,2)$. By [38], Theorem 4.6.1 (c) we then have $B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)\subset B_{p,q}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)\subset L^q(\mathcal{C}_1)$ so that

$$|S_2|\leqslant\|f\|_{L^q(\mathcal{C}_1)}\|(1-\chi_\delta)r^{-\alpha}\|_{L^{q'}(\mathcal{C}_1)}\lesssim\delta^{-\alpha+2/q'}\|f\|_{B_{2,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)}\lesssim\delta^\varepsilon\|f\|_{B_{2,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)}.$$

As in the proof Lemma 2.6 (i), we select

(5.3)
$$\delta=\min\Big\{\frac{1}{2}\operatorname{diam}\{\mathbf{x}\in\mathcal{C}\mid\chi(\mathbf{x})=1\},(\|f\|_{L^p(\mathcal{C}_1)}\|f\|^{-1}_{B_{2,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)})^{1/(\alpha-2/p'+\varepsilon)}\Big\}.$$

For brevity, we only consider the case when the minimum in (5.3) is given by the second term. Then

$$S_1 + S_2 \lesssim \|f\|_{L^p(\mathcal{C}_1)}^{1-\theta} \|f\|_{B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)}^{\theta}.$$

An appeal to [37], Lemma 25.2 concludes the proof.

*The case* $\alpha > 2$*:* Select $\varepsilon > 0$ such that $\alpha - 2 - \lfloor \alpha - 2 \rfloor + \varepsilon < 1$. Since $\lfloor \alpha - 2 \rfloor + 2/p < \alpha - 2/p'$, we write

$$B_{p,1}^{\alpha-2/p'}(\mathcal{C}_1) = (B_{p,1}^{\lfloor \alpha-2 \rfloor + 2/p}(\mathcal{C}_1), B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1))_{\theta,1}, \quad \theta = \frac{\alpha - 2 - \lfloor \alpha - 2 \rfloor}{\alpha - 2 + \varepsilon - \lfloor \alpha - 2 \rfloor}.$$

As in the proof of Lemma 2.6 (i), one splits for $\delta > 0$ the expression $S(f) = S_1 + S_2$ as in (2.23). For $S_1$, we note that $B_{p,1}^{\lfloor \alpha-2 \rfloor + 2/p}(\mathcal{C}_1) \subset C^{\lfloor \alpha - 2 \rfloor}(\overline{\mathcal{C}_1})$ by [38], Theorem 4.6.1 (f). Hence,

$$\begin{aligned} |S_1| &\leqslant \|r^{-\lfloor \alpha-2 \rfloor}(f + \chi \Delta P_{\lfloor \alpha - 2 \rfloor})\|_{L^\infty(\mathcal{C}_1)} \|\chi_\delta r^{-\alpha + \lfloor \alpha - 2 \rfloor}\|_{L^1(\mathcal{C}_1)} \\ &\lesssim \delta^{-(\alpha-2)+\lfloor \alpha - 2 \rfloor} \|f\|_{B_{2,1}^{\lfloor \alpha-2 \rfloor + 2/p}(\mathcal{C}_1)}. \end{aligned}$$

For $S_2$, we have by [38], Theorem 4.6.1 (f), that $B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1) = B_{p,1}^{\alpha-2+2/p+\varepsilon}(\mathcal{C}_1) \subset C^{\alpha-2+\varepsilon}(\overline{\mathcal{C}_1})$ so that in view of the fact that $-\Delta P_{\lfloor \alpha - 2 \rfloor}$ is the Taylor expansion of $f$ at the origin of order $\lfloor \alpha - 2 \rfloor$ and $0 < \alpha - 2 - \lfloor \alpha - 2 \rfloor + \varepsilon < 1$, we get

$$\begin{aligned} |S_2| &\leqslant \|r^{-(\alpha-2+\varepsilon)}(f + \chi \Delta P_{\lfloor \alpha - 2 \rfloor})\|_{L^\infty(\mathcal{C}_1)} \|(1 - \chi_\delta) r^{-\alpha + (\alpha-2)+\varepsilon}\|_{L^1(\mathcal{C}_1)} \\ &\lesssim \delta^{\varepsilon} \|f\|_{B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)}. \end{aligned}$$

We select $\delta$ as in (5.3) with the exponent replaced with $1/(\alpha - 2 + \varepsilon - \lfloor \alpha - 2 \rfloor)$. Then, as in the proof of Lemma 2.6 (i), one arrives at

$$S_1 + S_2 \lesssim \|f\|_{B_{p,1}^{\lfloor \alpha-2 \rfloor + 2/p}(\mathcal{C}_1)}^{1-\theta} \|f\|_{B_{p,1}^{\alpha-2/p'+\varepsilon}(\mathcal{C}_1)}^{\theta}.$$

An appeal to [37], Lemma 25.2 concludes the proof. □

**5.2. Expansion in the corner singularity functions.** The $L^p$-theory for elliptic problems in domains with conical points that was developed by Maz'ya and Plamenevskij leads to the following Proposition 5.3.

**Proposition 5.3.** *Let* $R > 0$*. Let* $p \in (1, 2) \cup (2, \infty)$*,* $k \in \mathbb{N}_0$*, and assume* $k + 2 - 2/p = k + 2/p' \notin \pm\sigma^D$*. Let* $f \in W^{k,p}(\mathcal{C})$ *satisfy* $\partial_x^i \partial_y^j f(0) = 0$ *for* $i + j < k - 2/p$*. Then, a solution* $u_1 \in W^{1,p}(\mathcal{C})$ *with* $\operatorname{supp} u_1 \subset B_1(0)$ *solving*

(5.4) $$-\Delta u_1 = f, \quad u_1 = 0 \quad \textit{for } \varphi \in \{0, \omega\}$$

*has the form*

$$u_1 = u_0 - \frac{1}{\pi} \sum_{j\colon \lambda_j^D < k+2-2/p} \int_{\mathcal{C}} r^{-\lambda_j^D} \sin(\lambda_j^D \varphi) f(x) \, \mathrm{d}\mathbf{x} \; r^{\lambda_j^D} \sin(\lambda_j^D \varphi)$$

*with*

$$\|u_0\|_{K_0^{k+2,p}(\mathcal{C}_R)} \lesssim \|f\|_{K_0^{k,p}(\mathcal{C})} \overset{\text{L. 5.1}}{\lesssim} \|f\|_{W^{k,p}(\mathcal{C})}. \tag{5.5}$$

P r o o f. The procedure is similar to that in the $L^2$-based setting in Section 2.1. Note that $\operatorname{supp} f \subset B_1(0)$. By density, one may assume $f \in W^{k,p}(\mathcal{C}) \cap C^\infty(\overline{\mathcal{C}})$ so that the formulas of the $L^2$-based setting are applicable. Since $\operatorname{supp} f \subset B_1(0)$, the Mellin transform $\mathcal{M}[g] = \mathcal{M}[r^2 f]$ is holomorphic in the strip $\{\zeta \in \mathbb{C} \mid \operatorname{Im} > -k-2+2/p\}$. As in Section 2.1, the function $u_0$ is defined by an appropriate inverse Mellin transformation on the line $\operatorname{Im} = -k-2+2/p$, and the residue theorem yields

$$u_0 - u_1 = \sum_{\zeta_0 \in -\mathrm{i}\sigma^D\colon \operatorname{Im}\zeta_0 \in (-k-2+2/p,0)} \operatorname*{Res}_{\zeta=\zeta_0} \big(r^{\mathrm{i}\zeta} (\mathcal{L}(\zeta))^{-1} \mathcal{M}[g](\zeta)\big). \tag{5.6}$$

Evaluating the residue yields the claimed representation. The estimate for $u_0$ is taken from [33], Proposition 2.3, but goes back at least to [29], Theorem 4.1. □

As in Section 2.1, the condition that $f$ vanishes to sufficient order can be removed by adding additional polynomials or logarithmic singularities:

**Corollary 5.4.** *Let $R > 0$. Let $p \in (1,2) \cup (2,\infty)$, $k \in \mathbb{N}_0$, and assume $k+2-2/p = k+2/p' \notin \pm\sigma^D$. Let $f \in W^{k,p}(\mathcal{C})$. Then, a solution $u_1 \in W^{1,p}(\mathcal{C})$ with $\operatorname{supp} u_1 \subset B_1(0)$ solving* (5.4) *can be represented as $u_1 = u_0 + \chi P_{\lfloor k-2/p \rfloor} + S$ with*

$$S = -\frac{1}{\pi} \sum_{j\colon \lambda_j^D < k+2-2/p} \int_{\mathcal{C}} r^{-\lambda_j^D} \sin(\lambda_j^D \varphi) f + \Delta(\chi P_{\lfloor k-2/p \rfloor}) \, \mathrm{d}\mathbf{x} \; r^{\lambda_j^D} \sin(\lambda_j^D \varphi),$$

$$P_{\lfloor k-2/p \rfloor} = \sum_{i+j<k-2/p} \frac{1}{i!j!} p_{i,j}^D(\mathbf{x}) \partial_x^i \partial_y^j f(0),$$

*where the functions $p_{i,j}^D$ are from Lemma 2.4 and*

$$\|u_0\|_{W^{k+2,p}(\mathcal{C}_R)} \lesssim \|f\|_{W^{k,p}(\mathcal{C}_1)}.$$

*The function $P_{\lfloor k-2/p \rfloor} \equiv 0$ if $k-2/p < 0$ and satisfies, for $k-2/p > 0$, the estimates*

$$\|P_{\lfloor k-2/p \rfloor}\|_{W^{k+2,p}(\mathcal{C}_R)} \lesssim \|f\|_{B_{p,1}^{\lfloor k-2/p \rfloor + 2/p}(\mathcal{C}_1)} \quad \text{if } \Sigma^D_{\lfloor k-2/p \rfloor+2} = \emptyset,$$

$$\|P_{\lfloor k-2/p \rfloor}\|_{B_{p,\infty}^{n^*+2/p}(\mathcal{C}_R)} \lesssim \|f\|_{B_{p,1}^{\lfloor k-2/p \rfloor + 2/p}(\mathcal{C}_1)},$$

$$n^* := \min\Big\{n \in \Sigma^D_{\lfloor k-2/p \rfloor+2} \mid n\frac{\omega}{\pi} \in \mathbb{N}\Big\} \quad \text{if } \Sigma^D_{\lfloor k-2/p \rfloor+2} \neq \emptyset.$$

P r o o f. Follows as in Corollary 2.5. □

**5.3. A shift theorem.** The representation formula of Corollary 5.4 allows one to infer a shift theorem in Besov spaces as in the $L^2$-case:

**Theorem 5.5.** *Let $R > 0$. Let $p \in (1,2) \cup (2,\infty)$ and $\chi \in C_0^\infty(B_1(0))$ with $\chi \equiv 1$ near 0. Let $k := \min\{n \in \mathbb{N} \mid \lambda_1^D + 2/p < n + 2\}$. Assume that one of the following two conditions holds:*

(i) $2 < \lambda_1^D + 2/p < k + 2 < \lambda_2^D + 2/p$.

(ii) $k < 2/p$ *and* $2 < \lambda_1^D + 2/p < k + 2$.

*Assume furthermore that $k + 2/p' \notin \pm\sigma^D$. Then for $f \in B_{2,1}^{\lambda_1^D + 2/p - 2}(\mathcal{C}_1)$ a solution $u$ of* (5.2) *satisfies*

$$\|u\|_{B_{2,\infty}^{\lambda_1^D + 2/p}(\mathcal{C}_R)} \lesssim \|\chi f\|_{B_{2,1}^{\lambda_1^D + 2/p - 2}(\mathcal{C}_1)} + \|u\|_{W^{1,p}(\mathcal{C}_1)}.$$

P r o o f. In the following, we assume $\omega \neq \pi$ since in the case $\omega = \pi$ one has a full shift theorem analogous to the case discussed in Remark 1.2, see [20], Section 9.

The two conditions (i), (ii) are such that the procedure already used in the $L^2$ setting is applicable. Inspection of the proof of Theorem 1.1 (i) shows that it relies on the following ingredients (A)–(D):

(A) Local regularity assertions as in Steps 0–1 of that proof that underlie the estimate (2.34). The local regularity in $L^p$-spaces is available, e.g., [20], Section 9.

(B) Solution operators $T$ and $\widetilde{T}$ for the Dirichlet problem as in (2.36). In contrast to (2.38), where $X_0 = H^{-1}(\mathcal{C}_R)$, $Y_0 = H_0^1(\mathcal{C}_R)$, we view $T\colon L^p \to W^{2,p} \cap W_0^{1,p}$ and select

$$X_0 = L^p(\mathcal{C}_R), \quad X_1 = W^{k,p}(\mathcal{C}_R), \quad Y_0 = W^{2,p}(\mathcal{C}_R), \quad Y_1 = W^{k+2,p}(\mathcal{C}_R), \tag{5.7}$$

where we recall that $k$ is taken as the smallest integer with $k + 2 > \lambda_1^D + 2/p$. One then has $B_{p,1}^{\lambda_1^D - 2/p'} = (X_0, X_1)_{\theta,1}$ and $B_{p,\infty}^{\lambda_1^D + 2/p} = (Y_0, Y_1)_{\theta,\infty}$ for $\theta = (\lambda_1^D - 2/p')/k$.

The mapping property $\widetilde{T}\colon X_0 \to Y_0$ follows from the assumption $2 < \lambda_1^D + 2/p$ since this implies that the sum $S$ in the solution representation in Corollary 5.4 is empty for $k = 0$ and that the function $P_{\lfloor k-2/p \rfloor}$ vanishes for $k = 0$ so that $u_1 = u_0$.

(C) The representation formula for the solution for data from $X_1$. This is provided by Corollary 5.4.

(D) The interpolation argument of Lemma 2.7. In the $L^2$-setting, it was applied in two situations:

(a) The sum $S$ contains exactly one singularity function.

(b) The sum $S$ contains several singularity functions but $P_{k-1} \equiv 0$; this was the case in Theorem 1.1 (iii) for $\omega \in (3\pi/2, 2\pi)$.

These two cases correspond to the conditions (i) and (ii), respectively. In the remainder of the proof, we discuss in more detail the application of the interpolation argument of Lemma 2.7.

*Proof under condition (i) and* $\lambda_1^D \notin \mathbb{N}$*:* The definition of $k$ reads

$$k + 1 - 2/p \leqslant \lambda_1^D < k + 2 - 2/p, \tag{5.8}$$

which gives

$$\lfloor k + 2 - 2/p \rfloor \geqslant \lfloor \lambda_1^D \rfloor \geqslant \lfloor k + 1 - 2/p \rfloor = \lfloor k + 2 - 2/p \rfloor - 1. \tag{5.9}$$

We claim that $\Sigma^D_{\lfloor k-2/p\rfloor+2} = \emptyset$. To see this, note that (5.9) implies $\lfloor k - 2/p \rfloor + 2 = \lfloor k + 2 - 2/p \rfloor \leqslant \lfloor \lambda_1^D \rfloor + 1$ so that for any $n \in \mathbb{N}$ with $n \leqslant \lfloor k - 2/p \rfloor + 2$ we have

$$\frac{n}{\lambda_1^D} \leqslant \frac{1 + \lfloor \lambda_1^D \rfloor}{\lambda_1^D} \leqslant 1 + \frac{1}{\lambda_1^D} < 3.$$

For $n \in \Sigma^D_{\lfloor k-2/p\rfloor+2}$, one has $n/\lambda_1^D \in \mathbb{N}$, leading to the possible cases $n = \lambda_1^D$ and $n = 2\lambda_1^D$. The first case is excluded by $\lambda_1^D \notin \mathbb{N}$. The second case is also excluded since otherwise $2\lambda_1^D = n \leqslant \lfloor k - 2/p \rfloor + 2 \leqslant 1 + \lfloor \lambda_1^D \rfloor \leqslant 1 + \lambda_1^D$, which implies $\lambda_1^D \leqslant 1$, and then $2\lambda_1^D = n \in \mathbb{N}$ leads to $\lambda_1^D \in \{1/2, 1\}$, which is not possible due to $\omega \in (0, 2\pi)$ and $\omega \neq \pi$. Since $\Sigma^D_{\lfloor k-2/p\rfloor+2} = \emptyset$, the interpolation argument based on Lemma 2.7 can be done as in the proof of Theorem 1.1 (i) for the case $\lambda_1^D \notin \mathbb{N}$ there.

*Proof under condition (i) and* $\lambda_1^D \in \mathbb{N}$. The value $k$ satisfying (5.8) is given by

$$k = \begin{cases} \lambda_1^D & \text{if } p \in (1,2) \\ \lambda_1^D - 1 & \text{if } p \in (2,\infty) \end{cases} \quad \text{leading to} \quad \lfloor k - 2/p \rfloor + 2 = \begin{cases} \lambda_1^D & \text{if } p \in (1,2), \\ \lambda_1^D & \text{if } p \in (2,\infty). \end{cases}$$

Hence, $\Sigma^D_{\lfloor k-2/p\rfloor+2} = \{\lambda_1^D\}$ and $n^\star = \lambda_1^D$ in the estimate for $P_{\lfloor k-2/p\rfloor}$ in Corollary 5.4. That is, Corollary 5.4 ascertains $\|P_{\lfloor k-2/p\rfloor}\|_{B^{\lambda_1^D+2/p}_{p,\infty}} \lesssim \|f\|_{B^{\lambda_1^D+2/p-2}_{p,1}}$ as used in the proof of Theorem 1.1 (i) for the case $\lambda_1^D \in \mathbb{N}$ there.

*Proof under condition (ii).* Multiple singularity functions in the representation of Corollary 5.4 are allowed. However, the condition $k < 2/p$ ensures that $P_{\lfloor k-2/p\rfloor} = 0$ so that as in the proof of Theorem 1.1 (iii) for $\omega \in (3\pi/2, 2\pi)$ one can use Lemma 2.7 since the linear functionals $f \mapsto \int_C r^{-\lambda_j^D} \sin(\lambda_j^D \varphi) f \,\mathrm{d}\mathbf{x}$ are bounded, linear on $B^{\lambda_j^D - 2/p'}_{2,1}$ for $j \in \mathbb{N}$ with $\lambda_j^D + 2/p < k + 2$. □

R e m a r k 5.6. Theorem 5.5 is based on the assumption that the solution operator for the Dirichlet problem maps $L^p \to W^{2,p}$, which is expressed by the condition $2 < \lambda_1^D + 2/p$. This restriction is due to our working with positive order Besov spaces $B^s_{p,q}$, $s > 0$. To remove this restriction, one would have to use negative order Besov spaces similarly to the way it is done in the $L^2$-setting.

## Appendix A. Weighted spaces and elliptic regularity in weighted spaces

We introduce for $0 < \varrho < \sigma$ the annuli

$$A(\varrho,\sigma) := \{x \in \mathcal{C}\colon\ \varrho < |x| < \sigma\}. \tag{A.1}$$

**Lemma A.1.** *Let* $f \in H^{k+\varepsilon}(\mathcal{C})$ *with* $\operatorname{supp} f \subset B_1(0)$ *for some* $k \in \mathbb{N}_0$ *and* $\varepsilon \in (0,1)$, *and assume* $\partial_x^i \partial_y^j f(0) = 0$ *for* $i + j \leqslant k-1$. *Then* $f \in K^k_{-\varepsilon}(\mathcal{C})$ *with the norm estimate* $\|f\|_{K^k_{-\varepsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\varepsilon}(\mathcal{C}_1)}$.

Proof of Lemma A.1/Lemma 2.2. We show the result for the cases $\varepsilon \in (0,1/2)$ and $\varepsilon \in (1/2,1)$ separately. The limiting case $\varepsilon = 1/2$ is then obtained by an interpolation argument (cf. [37], Chapter 23 for the interpolation of $L^2$-based spaces with weights). We also flag that we may assume $\omega \neq \pi$ as in the case $\omega = \pi$ the cone $\mathcal{C}$ can be split into 2 cones with apertures $\neq \pi$ and each cone is considered separately.

*Step 1. Claim:* For $\omega \neq \pi$ there holds

(A.2)

$$\|f\|_{H^{k+\varepsilon}(A(1,2))} \leqslant C|\nabla^k f|_{H^\varepsilon(A(1,2))} + \begin{cases} \displaystyle\sum_{l=1}^{2}\sum_{j=0}^{k-1} \|\nabla^j f\|_{L^2(\Gamma_l^1)}, & k \geqslant 1, \\ \displaystyle\sum_{l=1}^{2} \|f\|_{L^2(\Gamma_l^1)}, & k \geqslant 0, \text{ and } \varepsilon > 1/2, \end{cases}$$

where $\Gamma_l^1$, $l \in \{1,2\}$, are the two straight parts of $\partial A(1,2)$. This estimate is a variant of a classical Poincaré inequality. By the Deny-Lions lemma and the connectedness of $A(1,2)$, the full norm $\|f\|_{H^{k+\varepsilon}}$ and the seminorm $|\nabla^f f|_{H^\varepsilon}$ are equivalent up the polynomials of degree $k$. To see that, for $k \geqslant 1$, the map $f \mapsto \|f\|_* := \sum\limits_{l=1}^{2}\sum\limits_{j=0}^{k-1} \|\nabla^j f\|_{L^2(\Gamma_l^1)}$ is a norm on $\mathcal{P}_k$, the space of polynomials of degree $k$, we have to show that $\pi \in \mathcal{P}_k$ and $\|\pi\|_* = 0$ implies $\pi = 0$. Assuming, as we may, that $\Gamma_1^1 \subset \mathbb{R} \times \{0\}$ and writing $\pi(x,y) = \sum\limits_{i+j\leqslant k} a_{ij}x^i y^j$, we see that $a_{ij} = 0$ for all $i$ and $j = 0,\ldots,k-1$ so that $\pi(x,y) = a_{0k}y^k$. Since the line $\Gamma_2^1$ is not colinear with $\Gamma_1^1$ due to $\omega \neq \pi$, we finally conclude $\pi = 0$. The case $k = 0$ is easy.

*Step 2.* $k \geqslant 1$ *and* $\varepsilon \neq 1/2$*:* For $d > 0$ consider $A(d,2d)$, denote $\Gamma_l^d$, $l \in \{1,2\}$, the two straight parts of $\partial A(d,2d)$, and write $\widehat{f}$ for the function $f$ scaled to the reference element $A(1,2)$. Scaling to $A(1,2)$, using the norm equivalence (A.2), and scaling back yields

$$\text{(A.3)} \int_{A(d,2d)} r^{-2k-2\varepsilon}|f|^2 \lesssim d^{-2k-2\varepsilon+2}\|\widehat{f}\|^2_{L^2(A(1,2))} \lesssim d^{-2k-2\varepsilon+2}\|\widehat{f}\|^2_{H^{k+\varepsilon}(A(1,2))}$$

$$\lesssim d^{-2k-2\varepsilon+2}\left(|D^k\widehat{f}|^2_{H^\varepsilon(A(1,2))} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|\nabla^j\widehat{f}\|^2_{L^2(\Gamma_l^1)}\right)$$

$$\lesssim |D^k f|^2_{H^\varepsilon(A(d,2d))} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|r^{-(k+\varepsilon-(1/2+j))}\nabla^j f\|^2_{L^2(\Gamma_l^d)},$$

where we used $d \sim r$ on $A(d,2d)$. Covering $\mathcal{C}$ by annuli of the form $A(d,2d)$, we obtain

$$\int_{\mathcal{C}} r^{-2k-2\varepsilon}|f|^2 \lesssim |D^k f|^2_{H^\varepsilon(\mathcal{C}_1)} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|r^{-(k+\varepsilon-(1/2+j))}\nabla^j f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})},$$

where $\Gamma_l^{\mathcal{C}_1}$, $l = 1$, $2$, denote the straight-lined parts of $\partial\mathcal{C}_1$. As $f \in H^{k+\varepsilon}(\mathcal{C}_1)$, the trace theorem gives $\nabla^j f|_{\Gamma_l^{\mathcal{C}_1}} \in H^{k-j+\varepsilon-1/2}(\Gamma_l^{\mathcal{C}_1})$ for $j \in \{0,\dots,k-1\}$. Since $\partial_x^i\partial_y^{j'} f(0) = 0$ for $i + j' \leqslant k-1$, we even have $\chi(\nabla^j f)|_{\Gamma_l^{\mathcal{C}_1}} \in H_0^{k-j+\varepsilon-1/2}(\Gamma_l^{\mathcal{C}_1})$, cf. [31], Theorem 3.40, for smooth cut-off functions $\chi$ with $\chi \equiv 1$ near the origin. (The cut-off function is merely introduced for notational convenience to localize near the origin.) It follows by [22], Theorem 1.4.4.4

$$\text{(A.4)} \qquad \sum_{j=0}^{k-1}\|r^{-(k+\varepsilon-(1/2+j))}\nabla^j f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} \lesssim \|f\|^2_{H^{k+\varepsilon-1/2}(\Gamma_l^{\mathcal{C}_1})} \lesssim \|f\|^2_{H^{k+\varepsilon}(\mathcal{C}_1)}.$$

The higher derivatives of $f$ appearing in the norm $\|\cdot\|_{K^k_{-\varepsilon}(\mathcal{C})}$ are treated by a similar argument.

*Step 3. $k = 0$ and $\varepsilon \neq 1/2$:* The result in the case $k = 0$ and $\varepsilon \in (0,1/2)$ is found in [22], Theorem 1.4.4.3. For $\varepsilon \in (1/2,1)$, we proceed as in Step 2, replacing the sum $\sum_{j=0}^{k-1}$ with the sum $\sum_{l=1}^{2}\|f\|_{L^2(\Gamma_l^1)}$. The key estimate [22], Theorem 1.4.4.4 is again applicable, which yields the result. □

**Lemma A.2.** *Let $\varrho_1 < \varrho_2 < \varrho_3 < \varrho_4$ and $\widehat{A}_2 := A(\varrho_1,\varrho_4)$, $\widehat{A}_2 := A(\varrho_2,\varrho_3)$. Let $k \in \mathbb{N}_0$, $\varepsilon \in (0,1)$, $f \in H^{k+\varepsilon}(\widehat{A}_2)$ and $u \in H^1(\widehat{A}_2)$ satisfy $-\Delta u = f$ on $\widehat{A}_2$ with additionally either homogeneous Dirichlet conditions, homogeneous Neumann conditions, or homogeneous mixed boundary conditions on the two straight parts of $\partial\widehat{A}_2$. Then, there is $C > 0$, depending only on $\omega$ and $\varrho_i$, $i = 1,\dots,4$, such that $|\nabla^{k+2}u|_{H^\varepsilon(\widehat{A}_1)} \leqslant C(\|f\|_{H^{k+\varepsilon}(\widehat{A}_2)} + \|u\|_{H^1(\widehat{A}_2)})$. More generally, for any $q \in [1,\infty]$ and $s > 1/2$, one has $\|u\|_{B^{s+2}_{2,q}(\widehat{A}_1)} \leqslant C(\|f\|_{B^{-1+s}_{2,q}(\widehat{A}_2)} + \|u\|_{H^1(\widehat{A}_2)})$. For Dirichlet boundary conditions, this estimate actually holds for $s > 0$.*

P r o o f. We restrict to $s > 1/2$ in the second part of the lemma in order to achieve a unified notation since the spaces $B^{-1+s}_{2,q}$ and $\widetilde{B}^{-1+s}_{2,q}$ differ for $s \leqslant 1/2$.

This is a rather standard elliptic regularity theorem. We sketch the proof to illuminate the point that also regularity assertions in Besov spaces are possible. We restrict to Dirichlet boundary conditions at one straight edge $\Gamma_1$ of $\widehat{A}_2$ and merely consider a local situation of the three half-balls $H_\varrho := B_\varrho(x_0) \cap \widehat{A}_2$, $H_{\varrho'} := B_{\varrho'}(x_0) \cap \widehat{A}_2$, $H_{\varrho''} := B_{\varrho''}(x_0) \cap \widehat{A}_2$ with $x_0 \in \Gamma_1$ and $0 < \varrho'' < \varrho' < \varrho$. Consider $\chi_\varrho \in C^\infty_0(B_\varrho(x_0))$, $\chi_{\varrho'} \in C^\infty_0(B_{\varrho'}(x_0))$ with $\chi_\varrho \equiv 1$ on $H_{\varrho'}$ and $\chi_{\varrho'} \equiv 1$ on $H_{\varrho''}$. Let $T$ be the solution operator for the Poisson problem on $H_\varrho$ with Dirichlet boundary conditions on $\partial H_\varrho$. Then, by elliptic regularity, the map $f \mapsto \chi_{\varrho'} T \chi_\varrho$ is bounded $H^s(H_\varrho) \to H^{s+2}(H_{\varrho'})$ for any $s \in \mathbb{N}_0 \cup \{-1\}$. By interpolation, the map $f \mapsto \chi_{\varrho'} T \chi_\varrho$ is bounded $B^s_{2,q}(H_\varrho) \to B^{s+2}_{2,q}(B_{\varrho'})$ for any $s > -1$, $q \in [1,\infty]$. The difference $u_0 := u - \chi_{\varrho'} T \chi_\varrho f$ satisfies homogeneous Dirichlet conditions on $\Gamma_1$ and is harmonic on $H_{\varrho'}$. Hence, $u_0$ is smooth (up to $\Gamma_1$) on $H_{\varrho'}$ and the interior regularity provides $\|u_0\|_{H^{s'+2}(H_{\varrho''})} \lesssim \|u_0\|_{H^1(H_{\varrho'})}$ for any $s' \geqslant 0$. Finally, $\|u_0\|_{H^1(H_{\varrho'})} \leqslant \|u\|_{H^1(H_{\varrho'})} + \|\chi_{\varrho'} T \chi_\varrho f\|_{H^1(H_{\varrho'})} \lesssim \|u\|_{H^1(H_{\varrho'})} + \|\chi f\|_{H^{-1}(\widehat{A}_2)} \lesssim \|u\|_{H^1(H_\varrho)} + \|f\|_{H^s_{2,q}(\widehat{A}_2)}$, where $\chi$ is yet another smooth cut-off function with $\chi \equiv 1$ on $B_\varrho(x_0)$ and supported by a sufficiently small neighborhood of $B_\varrho(x_0)$.

The local estimates can be combined into a global one by a smooth partition of unity to result in the desired estimates for the sets $\widehat{A}_1$ and $\widehat{A}_2$. □

The following lemma demonstrates a type of scaling arguments employed in connection with weighted spaces.

**Lemma A.3.** *Let $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, and $\varepsilon \in (0,1)$. Further let $f \in L^2_{\mathrm{loc}}(\mathcal{C})$ and $u \in H^1_{\mathrm{loc}}(\mathcal{C})$ satisfy $-\Delta u = f$ with either homogeneous Dirichlet boundary conditions, homogeneous Neumann boundary conditions, or homogeneous mixed boundary conditions. Then:*

(i) *For $f \in K^k_\gamma(\mathcal{C})$, $u \in K^0_{\gamma-k-2}(\mathcal{C})$ and $\beta \in (\mathbb{N}_0)^2$, $|\beta| \in [0, k+2]$, it holds*

$$\|r^{-|\beta|+\gamma} D^{k+2-|\beta|} u\|_{L^2(\mathcal{C})} \lesssim \sum_{|\alpha| \leqslant k} \|r^{|\alpha|-k+\gamma} D^\alpha f\|_{L^2(\mathcal{C})} + \|r^{-k-2+\gamma} u\|_{L^2(\mathcal{C})}. \tag{A.5}$$

(ii) *Let $f \in H^{k+\varepsilon}(\mathcal{C}_1)$ with $\operatorname{supp} f \subset B_1(0)$. Further assume that there exists $R > 1$ such that $u \in K^{k+2}_{-\varepsilon}(\mathcal{C}_R)$. Then*

$$|D^{k+2} u|_{H^\varepsilon(\mathcal{C}_1)} \lesssim |D^k f|_{H^\varepsilon(\mathcal{C}_1)} + \sum_{|\alpha| \leqslant k} \|r^{|\alpha|-k-\varepsilon} D^\alpha f\|_{L^2(\mathcal{C}_1)} + \sum_{|\alpha| \leqslant k+2} \|r^{|\alpha|-k-2-\varepsilon} D^\alpha u\|_{L^2(\mathcal{C}_R)}.$$

P r o o f. Introduce the annuli $\widehat{A_1} := A(1/2,2)$ and $\widehat{A_2} := A(1/4,4)$, cf. (A.1). For $\varrho > 0$ scaling yields for $A_{i,\varrho} := \varrho\widehat{A_i}$, $i = 1,2$, $\widehat{u}(\xi) := u(\varrho\xi)$ and $\widehat{f}(\xi) := f(\varrho\xi)$ that $-\Delta\widehat{u} = \varrho^2\widehat{f}$ on $\widehat{A_2}$.

We start the proof of (i) by noting the elliptic regularity estimate

$$\|\widehat{u}\|_{H^{k+2}(\widehat{A_1})} \lesssim \varrho^2\|\widehat{f}\|_{H^k(\widehat{A_2})} + \|\widehat{u}\|_{L^2(\widehat{A_2})}. \tag{A.6}$$

We multiply (A.6) by $\varrho^{\gamma-k-2}$ and obtain after scaling

$$\begin{aligned}\sum_{|\alpha|\leqslant k+2} &\varrho^{2|\alpha|+2\gamma-2k-4}\|D^\alpha u\|^2_{L^2(A_{1,\varrho})}\\ &\lesssim \sum_{|\alpha|\leqslant k} \varrho^{2|\alpha|+2\gamma-2k}\|D^\alpha f\|^2_{L^2(A_{2,\varrho})} + \varrho^{2\gamma-2k-4}\|u\|^2_{L^2(A_{2,\varrho})}.\end{aligned}$$

The definition of the annuli implies $2^{-i}\varrho < r < 2^i\varrho$, $i = 1,\ 2$, on $A_{i,\varrho}$. Thus we get further

$$\sum_{|\alpha|\leqslant k+2} \|r^{|\alpha|+\gamma-k-2}D^\alpha u\|^2_{L^2(A_{1,\varrho})} \lesssim \sum_{|\alpha|\leqslant k} \|r^{|\alpha|+\gamma-k}D^\alpha f\|^2_{L^2(A_{2,\varrho})} + \|r^{\gamma-k-2}u\|^2_{L^2(A_{2,\varrho})}.$$

We now cover $\mathcal{C}$ by annuli $A_{1,2^{-j}}$, $j \in \mathbb{Z}$. Since they have only a finite overlap, we obtain

$$\|r^{-|\beta|+\gamma}D^{k+2-|\beta|}u\|^2_{L^2(\mathcal{C})} \lesssim \sum_{|\alpha|\leqslant k} \|r^{|\alpha|+\gamma-k}D^\alpha f\|^2_{L^2(\mathcal{C})} + \|r^{\gamma-k-2}u\|^2_{L^2(\mathcal{C})}$$

for $|\beta| \in [0, k+2]$, whereupon the result follows.

The proof of (ii) follows in a similar way. We prove it assuming $R > 2$. We have

$$|D^{k+2}\widehat{u}|_{H^\varepsilon(\widehat{A_1})} \lesssim \varrho^2\|\widehat{f}\|_{H^{k+\varepsilon}(\widehat{A_2})} + \|\widehat{u}\|_{H^{k+2}(\widehat{A_2})}, \tag{A.7}$$

which follows from Lemma A.2 after scaling with $\varrho$. Then the usual scaling arguments yield

$$\begin{aligned}&|D^{k+2}u|^2_{H^\varepsilon(A_{1,\varrho})}\\ &\quad\lesssim \varrho^{2-2(k+2+\varepsilon)}|D^{k+2}\widehat{u}|^2_{H^\varepsilon(\widehat{A_1})}\\ &\quad\lesssim \varrho^{2-2(k+2+\varepsilon)}\Bigg(\varrho^4\sum_{|\alpha|\leqslant k}\|D^\alpha\widehat{f}\|^2_{L^2(\widehat{A_2})} + \varrho^4|D^k\widehat{f}|^2_{H^\varepsilon(\widehat{A_2})} + \sum_{|\alpha|\leqslant k+2}\|D^\alpha\widehat{u}\|^2_{L^2(\widehat{A_2})}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \varrho^{2-2(k+2+\varepsilon)}\Big(\varrho^4 \sum_{|\alpha|\leqslant k} \varrho^{-2+2|\alpha|}\|D^\alpha f\|^2_{L^2(A_{2,\varrho})} + \varrho^4\varrho^{-2+2(k+\varepsilon)}|D^k f|^2_{H^\varepsilon(A_{2,\varrho})} \\
&\quad + \sum_{|\alpha|\leqslant k+2} \varrho^{-2+2|\alpha|}\|D^\alpha u\|^2_{L^2(A_{2,\varrho})}\Big) \\
&\lesssim \sum_{|\alpha|\leqslant k} \varrho^{-2k-2\varepsilon+2|\alpha|}\|D^\alpha f\|^2_{L^2(A_{2,\varrho})} + |D^k f|^2_{H^\varepsilon(A_{2,\varrho})} \\
&\quad + \sum_{|\alpha|\leqslant k+2} \varrho^{-2k-2\varepsilon-4+2|\alpha|}\|D^\alpha u\|^2_{L^2(A_{2,\varrho})}.
\end{aligned}
$$

As in (i) we obtain by covering arguments

$$
\begin{aligned}
|D^{k+2}u|^2_{H^\varepsilon(\mathcal{C}_1)} \lesssim &\sum_{|\alpha|\leqslant k} \|r^{-k-\varepsilon+|\alpha|}D^\alpha f\|^2_{L^2(\mathcal{C}_1)} + |D^k f|^2_{H^\varepsilon(\mathcal{C}_1)} \\
&+ \sum_{|\alpha|\leqslant k+2} \|r^{-k-\varepsilon-2+|\alpha|}D^\alpha u\|^2_{L^2(\mathcal{C}_R)}.
\end{aligned}
$$

□



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