Unterschrift BetreuerIn

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

Ground States in General 2D Dilaton Gravity

ausgeführt am Institut für Theoretische Physik
der Technischen Universität Wien

unter der Anleitung von
Ass.-Prof. Priv.-Doz. Dr. Daniel Grumiller

durch

Martin Laihartinger, BSc.

Wiedner Hauptstrasse 8-10, Turm B,
10. Stock

März 31, 2022

Unterschrift StudentIn

Abstract

General dilaton gravity in 2 dimensions is a useful tool for investigating questions of quantum gravity. It is not only a toy-model for lower dimensional gravity, but can also be derived from compactifications of higher dimensions. We focus on a more general, not power-counting renormalizable action, where we find a 3 parameter family as generalization of the 2 parameter *ab*-family, that comprises spherically reduced gravity from any dimension. We also compute the ground state conditions for Minkowki, Rindler and (A)dS space-times, for the 2 classes of models we investigate.

Contents

1 Introduction

> *" I would rather have questions that can't be answered than answers that can't be questioned. "*
>
> – Richard Feynman

Describing quantum field theory and general relativity with one model is a fundamental problem, even known beyond the area of expertise of physics nowadays. Though there are some famous candidates as solutions like string theory [1], the only self consistent theory that can describe all fundamental forces via vibrating strings in 10 dimensions (superstring theory), there are still many questions which are longing for an answer.

Leaving this theory aside, a complete description for gravity interactions with matter at small scales would be of great importance for calculating and investigating the beginning of the universe [2] and black holes [3], just to state a few examples. For the latter, Hawking radiation [4] gives us a semi-classical description, yet it also raises the question of information loss [5], where a pure quantum state is swallowed and a mixed state is evaporated from the black hole, thus violating unitarity in quantum theory.

The singularities contained in black holes and big bang space-times show the incompleteness of general relativity. In the beginning of the universe as well as the center of a black hole, all matter seems to concentrate at one point, and the curvature of the space-time blows up to infinity. The weak cosmic censorship hypothesis [6] by Roger Penrose states that every curvature singularity is hidden behind a horizon. One may ask now whether a singularity is a technical leftover of the theory or if its notion should be investigated in closer detail. It is usually expected that quantum gravity resolves the singularities arising in general relativity.

As the theories of quantum mechanics and general relativity are both well established by themselves, it seems reasonable to try to apply the laws of quantum mechanics to general relativity in the search of a more conclusive quantum gravity model. In 4 space-time dimensions this gets extremely difficult to do, so a first step in this direction would be to look into simpler models of lower dimensions, e.g. 2 dimensional gravity as a toy model to get better understanding of the theory, or equivalently as a reduction from a 4d space-time with spherical or toroidal symmetry for example, yielding again a 2d gravity model. The literature in lower dimensional gravity is already well established [7, 8]. An interesting result from these developments is that the 2d Einstein-Hilbert action is given by the Gauss-Bonnet term, which is locally trivial. Therefore further structure is then introduced by the dilaton field, which naturally arises in compactifications from higher dimensions [9]. It can also be interpreted as replacing the inverse of the Newton's constant by a scalar field [10], i.e. the dilaton field. The most prominent 2d dilaton model is given by Jackiw and Teitelboim (JT) [11, 12], which was already thoroughly investigated in the 1980's. In the early 1990's, Callan, Giddings, Harvey and Strominger (CGHS) [13] published a paper about a dilaton black hole model, originally inspired by a string theoretical approach. It was also around that time when it was shown that 2d dilaton gravity could be treated as non-linear gauge-theory, i.e. the Poisson σ-model (PSM) [14, 15].

The paper of CGHS also inspired the research for generalized dilaton theories (GDTs). In earlier works it was proposed that all GDTs could be extracted from the dilaton bulk action [16, 17]

$$I[g_{\mu\nu}, X] = -\frac{\kappa}{4\pi}\int d^2x\sqrt{-g}\left[XR - U(X)(\partial X)^2 + V(X)\right] \ , \tag{1}$$

where κ is the gravitational coupling constant, g the determinant of the metric $g_{\mu\nu}$, R the Ricci scalar, $U(X)$ and $V(X)$ arbitrary functions of the dilaton field X. In the first term in the brackets, one could introduce an arbitrary function, but by field redefinitions it is always possible to bring it in the form (1), which we will use throughout the thesis.

Recent work [18] showed that the most general consistent deformation of the JT model to other 2d dilaton gravity models is given by a generally not power-counting renormalizable action

$$I[g_{\mu\nu}, X] = -\frac{\kappa}{4\pi}\int d^2x\sqrt{-g}(XR - 2\mathcal{V}(X, -(\partial X)^2)) \ , \tag{2}$$

instead of the bulk action (1). When we deform a model, we keep the number of field- and gauge-degrees of freedom the same, but modify symmetries, which yields a larger class of models. This is also referred to as "consistent deformation", see [19] for details. The equations of motion that can be deduced from (2) lead to 2 classes of solutions, constant and linear dilaton vacua. Since the constant dilaton vacua are trivial, we shall consider only linear dilaton vacua. By combining the equations of motion (e.o.m.'s) in the right way, one obtains a differential equation including the dilaton field and the function $\mathcal{V}$. Depending on the form of $\mathcal{V}$, the differential equation has several solutions. We will have a look at 2 classes of differential equations resulting from a specific form of $\mathcal{V}$, Bernoulli and exact differential equations. In each case we will calculate the Killing norm and determine specific models that lead to a Minkowski, Rindler or (A)dS ground state. In the case of an exact differential equation, we find a 3 parameter (abc) family of models, that seems to be a generalization of the ab-family, see appendix A in [18], as their results in the special case of (ab) coincide with the ones from the ab-family from previous literature when we compute their ground states.

This thesis is mainly based on the paper [18] and structured as follows. In section 2 we will start with a brief summary on general 2d dilaton gravity and derive it from a PSM, this includes deriving the e.o.m.'s and solve them for linear dilaton vacua, yielding the line element for the dilaton theory. From the line element we deduce the Killing norm, which is needed to compute the ground state conditions on the explicit models for the space-times as mentioned. In subsection 2.3 we also compute the curvature 2-form in the first order formalism, which gives us the conditions on the Killing norm for each space-time we wish to investigate.

In section 3 we will investigate the case of a function $\mathcal{V}$ leading to a Bernoulli differential equation 3.1, where we determine the conditions for Minkowski, Rindler and (A)dS ground states and look into a specific example 3.1.1. In subsection 3.2 we will do the same procedure for the case of an exact diffential equation. We compute conditions on $\mathcal{V}$ for the ground states and apply them on a specific 3 parameter family, which we shall refer to as *abc*-family 3.2.2. We find that the 3 parameter family is pertubatively connected to the 2 parameter family and compare the results in 3.2.3. In section 4 we conclude and give an outlook.

2 Summary of the PSM and General 2d Dilaton Gravity

In this section we recap and summarize the most important aspects of the PSM and 2d dilaton gravity for our purposes. We will show that the general action (2) is equivalent to a specific type of PSM and derive it's Poisson tensor. From the general action we are going to deduce the e.o.m.'s, with which we can test models with arbitrary functions $\mathcal{V}$ in Minkowski, Rindler and (A)dS space-times. We will derive the conditions for the ground state in each space-time via the first order formalism, where we calculate the curvature 2-form and subsequentially the Ricci scalar.

2.1 Poisson σ-Model

The bulk action for the PSM depending on gauge field 1-forms A_I and target space coordinates X^I is given in [18] by

$$I_{\text{PSM}}[A_I, X^I] = \frac{\kappa}{2\pi}\int (X^I dA_I + \frac{1}{2}P^{IJ}(X^K)A_I \wedge A_J) \ , \tag{3}$$

where κ is a coupling constant and can be interpreted as proportionally inverse to the 2d Newtons constant. We can define the Poisson tensor via the Schouten-Nijenhuis bracket $\{X^I, X^J\} = P^{IJ}$, where one can interpret the scalars X^I as target space coordinates of a Poisson-manifold. The Poisson tensor P^{IJ} is antisymmetric, $P^{IJ} = -P^{JI}$, and satisfies the Jacobi identity

$$P^{IL}\partial_L P^{JK} + P^{JL}\partial_L P^{KI} + P^{KL}\partial_L P^{IJ} = 0 \ . \tag{4}$$

The transformation of the gauge 1-forms A_I and target space coordinates X_I that preserve the action (3) are non-linear in general and given by

$$\delta_\lambda X^I = \lambda_J P^{JI} \qquad \delta_\lambda A_I = d\lambda_I + \partial_I P^{JK} A_J \lambda_K \ . \tag{5}$$

The e.o.m.'s in this case can be obtained by varying the action (3) with respect to the fields A_I and X^I,

$$\frac{\delta I_{\text{PSM}}}{\delta A_I} = \frac{\kappa}{2\pi}\int (-dX^I - \frac{1}{2}P^{IJ}A_J + \frac{1}{2}P^{JI}A_J) = 0$$

$$\frac{\delta I_{\text{PSM}}}{\delta X^I} = \frac{\kappa}{2\pi}\int (dA_I + \frac{1}{2}(\partial_I P^{JK})A_J \wedge A_K) = 0$$

which yields

$$dX^I + P^{IJ}A_J = 0$$
$$dA_I + \frac{1}{2}(\partial_I P^{JK})A_J \wedge A_K = 0 \ . \tag{6}$$

In dilaton gravity the gauge field 1-forms are given by $A_I = (\omega, e_a)$, with spin-connection ω and zweibein e_a, whereas the target space coordinates are composed of a dilaton field X and Lagrange-multipliers for the torsion constraints X^a, i.e. $X^I = (X, X^a)$. Let us for convenience change the index notation to light-cone gauge $X^a = \{X^+, X^-\}$, with Minkowski metric

$$\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

and Levi-Civita symbol

$$\epsilon^{\pm}{}_{\pm} = \pm 1 \ . \tag{8}$$

The action (2) written in first order formalism is given by

$$I_{\text{gen}}[e_a, \omega, X, X^a] = \frac{\kappa}{2\pi}\int \left(X d\omega + X^a(de_a + \epsilon_a{}^b \omega \wedge e_b) + \frac{1}{2}\epsilon^{ab} e_a \wedge e_b \mathcal{V}(X, X^a X_a) \right) \ . \tag{9}$$

When we insert the fields A_I and X_I in the action (3) with a Poisson tensor given by $P^{aX} = \epsilon^a{}_b X^b$, $P^{ab} = \epsilon^{ab}\mathcal{V}(X, X^a X_a)$, i.e.

$$P^{IJ} = \begin{pmatrix} 0 & -X^+ & X^- \\ X^+ & 0 & \mathcal{V}(X, X^a X_a) \\ -X^- & -\mathcal{V}(X, X^a X_a) & 0 \end{pmatrix} \ , \tag{10}$$

we get

$$I_{\text{PSM}}[e_a, \omega, X, X^a] = \frac{\kappa}{2\pi}\int (X d\omega + X^a de_a + P^{aX} e_a \wedge \omega + \frac{1}{2} P^{ab} e_a \wedge e_b) =$$
$$= \frac{\kappa}{2\pi}\int \left(X d\omega + X^a(de_a + \epsilon_a{}^b \omega \wedge e_b) + \frac{1}{2}\epsilon^{ab} e_a \wedge e_b \mathcal{V}(X, X^a X_a) \right) \ , \tag{11}$$

which coincides with (9).

Note that the function $\mathcal{V}(X, X^a X_a)$ only depends on the dilaton field X and the Lorentz invariant product of the Lagrange-multipliers $X^a X_a$, due to the Jacobi identity (4).

2.2 General 2d Dilaton Gravity

The e.o.m.'s for the general 2d dilaton action (9) are

$$\begin{aligned} &dX + X^a \epsilon_a^{\ b} e_b = 0 \\ &dX^a - X_b \epsilon^{ba} \omega + \epsilon^{ab} e_b \mathcal{V} = 0 \\ &d\omega + \frac{1}{2} \epsilon^{ab} e_a \wedge e_b \partial_X \mathcal{V} = 0 \\ &de_a + \epsilon_a^{\ b} \omega \wedge e_b + \frac{1}{2} \epsilon^{cb} e_c \wedge e_b \partial_{X^a} \mathcal{V} = 0 \quad . \end{aligned} \tag{12}$$

The solutions to the e.o.m.'s lead to 2 distinct cases, constant and linear dilaton vacua. We will only consider the second case.

To solve the e.o.m.'s, we will write them in light-cone gauge $X^a X_a = \eta_{ab} X^a X^b = 2X^+ X^-$,

$$\begin{aligned} &dX + X^- e^+ - X^+ e^- = 0 \\ &(d \pm \omega) X^\pm \pm e^\pm \mathcal{V} = 0 \\ &d\omega + \epsilon \frac{\partial \mathcal{V}}{\partial X} = 0 \\ &(d \pm \omega) e^\pm + \epsilon \frac{\partial \mathcal{V}}{\partial X^\mp} = 0 \quad , \end{aligned} \tag{13}$$

where we introduced the volume form $\epsilon = \frac{1}{2} \epsilon^{ab} e_a \wedge e_b = e^- \wedge e^+$. Next, we will add the second line upper sign multiplied by X^- and the lower sign multiplied by X^+, so we get

$$d(X^+ X^-) - \mathcal{V}(X, 2X^+ X^-) \, dX = 0 \, . \tag{14}$$

This relation implies Casimir conservation $dC = 0$, which can be integrated formally to get the Casimir function $C(X, X^+ X^-)$. In [8] it was shown that the Casimir is related to a conserved mass. This is because the Poisson-tensor along with the generic solutions (26) has rank 2 and therefore a 1-dimensional kernel, which implies the existence of a Casimir function that is constant on-shell and thus leads to a conserved mass. In case of a ground state we have vanishing mass, so the condition for the Casimir function is $C(X, X^+ X^-) = 0$. The equation (14) will become important in the following sections, but first we want to find a solution to the e.o.m.'s. As a first step, let us assume that $X^+ \neq 0$ and introduce the 1-form $Z = \frac{e^+}{X^+}$. From the second line in (13) we then get

$$\omega = -\frac{dX^+}{X^+} - Z \, \mathcal{V} \, . \tag{15}$$

The first line in (13) gives us

$$e^- = \frac{dX}{X^+} + X^- Z \, , \tag{16}$$

so the volume element becomes

$$\epsilon = e^- \wedge e^+ = \frac{dX}{X^+} \wedge e^+ + X^- Z \wedge e^+ = -Z \wedge dX \ . \tag{17}$$

Next, let us divide the fourth line upper sign in (13) by X^+, which yields

$$\frac{de^+}{X^+} + \frac{e^+}{(X^+)^2} \wedge dX^+ = (Z \wedge dX) \frac{1}{X^+} \frac{\partial \mathcal{V}}{\partial X^-} \ . \tag{18}$$

With

$$dZ = d\left(\frac{e^+}{X^+}\right) = \frac{de^+}{X^+} + \frac{e^+}{(X^+)^2} \wedge dX^+ \ , \tag{19}$$

equation (18) becomes

$$dZ = (Z \wedge dX) \frac{1}{X^+} \frac{\partial \mathcal{V}}{\partial X^-} = (Z \wedge dX) \frac{\partial \mathcal{V}}{\partial (X^+ X^-)} \ , \tag{20}$$

where we used

$$\frac{\partial}{\partial X^-} = X^+ \frac{\partial}{\partial (X^+ X^-)} \ . \tag{21}$$

Let us make the Ansatz $Z = dv e^{Q(X)}$ and act with the de-Rahm differential on it

$$dZ = -e^{Q(X)} \frac{dQ}{dX} dv \wedge dX \ . \tag{22}$$

Comparing this with (20), we see that

$$-e^{Q(X)} \frac{dQ}{dX} dv \wedge dX = e^{Q(X)} \frac{\partial \mathcal{V}}{\partial (X^+ X^-)} dv \wedge dX \ ,$$

and therefore

$$\frac{dQ}{dX} = -\frac{\partial \mathcal{V}}{\partial (X^+ X^-)} \ . \tag{23}$$

Following from (14), this can be integrated as

$$Q(X, C) = -\int^X \frac{\partial \mathcal{V}(X, 2X^+ X^-)}{\partial (X^+ X^-)} dX' \ . \tag{24}$$

From the cartan formulation with $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$, the line element is then given by

$$ds^2 = 2e^+ e^- = 2e^Q dv dX + 2e^Q X^+ X^- dv^2 \ . \tag{25}$$

Let us introduce here a radial coordinate $dr = e^Q dX$, which can be integrated to give the dilaton field as a function of the radius $X(r)$. When we substitute the new coordinate in the line element, we get

$$ds^2 = 2e^{2Q} X^+ X^- dv^2 + 2 dv dr \ . \tag{26}$$

2.3 First Order Curvature Formalism

Let us investigete a metric of the form

$$ds^2 = K(r)dv^2 + 2dvdr \ . \tag{27}$$

The Killing vector is determined by the Killing equation

$$\mathcal{L}_\xi(g_{\mu\nu}) = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0 \ , \tag{28}$$

with covariant derivative ∇_μ. As the metric $g_{\mu\nu}$ in (27) does not depend on the coordinate v, one can directly find the Killing vector $\xi = \partial_v$, with Killing norm $K(r) = g_{vv}$. Using the first order formalism, it is easy to compute the curvature scalar for a metric in the form (27). To do so, we make an ansatz for the zweibein

$$e^+ = dv \qquad e^- = \frac{1}{2}K(r)dv + dr \ , \tag{29}$$

which indeed reproduces the line element (27) with $ds^2 = 2e^+e^-$. To obtain the curvature 2-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ , \tag{30}$$

we have to establish the spin-connection $\omega^a{}_b$ first. In 2d gravity, the spin-connection simplifies to $\omega^a{}_b = \epsilon^a{}_b\ \omega$, with the Levi-Civita symbol (8). We can now deduce the spin-connection from the condition of vanishing torsion

$$de^\pm + \epsilon^\pm{}_\pm \omega \wedge e^\pm = 0 \ , \tag{31}$$

which yields the functions

$$\omega \wedge dv = 0 \tag{32}$$

for the upper sign and

$$\frac{1}{2}\partial_r K(r) dr \wedge dv - \omega \wedge \left(\frac{1}{2}K(r)dv + dr\right) = 0 \tag{33}$$

for the lower sign in (31). If we insert the first equation into the second one we get

$$\omega \wedge dr = -\frac{1}{2}\partial_r K(r) dv \wedge dr \ , \tag{34}$$

and the spin-connection reads as

$$\omega = -\frac{1}{2}\partial_r K(r) dv \ . \tag{35}$$

The curvature 2-form (30) simplifies in 2d as well to

$$R^{\pm}{}_{\pm} = \epsilon^{\pm}{}_{\pm} d\omega = \mp \frac{1}{2} \partial_r^2 K(r) dr \wedge dv \ , \tag{36}$$

and the Ricci scalar can be obtained by contracting the curvature 2-form as

$$R = R^{\pm}{}_{\pm vr} e^{\pm r} e^{v}_{\pm} + R^{\pm}{}_{\pm rv} e^{\pm v} e^{r}_{\pm} = \partial_r^2 K(r) \ . \tag{37}$$

For Minkowski space, where we have a vanishing Ricci scalar, $R = 0$, and a constant metric, the Killing norm is constant, $K(r) = \text{const}$.

In Rindler space we have a vanishing Ricci scalar as well, but differently to Minkowski space, the Killing norm is proportional to the radial coordinate $K(r) \propto r$.

For local (A)dS the Ricci scalar is (negative) constant, $R = (-)\text{const}$, so the Killing norm has to be proportional to r^2, i.e. $K(r) \propto r^2$.

Let us also mention the case of global AdS with $K(r) \propto -(r^2+1)$ and AdS-Rindler where $K(r) \propto -r^2 + Ar + B$. We will only consider the first 3 cases when we investigate the ground state for the dilaton models, so when we write (A)dS, we refer to the local (A)dS space-time.

3 Dilaton Models with Nonlinear Potential

We now study linear dilaton vacua with general action (9) for different classes of functions $\mathcal{V}$. To do so, we first have a look at the form of the e.o.m. (14) and see that it takes on the form of a first order nonlinear ordinary differential equation

$$\frac{dY}{dX} = \mathcal{V}(X, 2Y) \ , \tag{38}$$

where $Y = X^+X^-$. Although there is no general solution to this kind of differential equation, there have been some solutions found for special cases of $\mathcal{V}$. In the following, we have a closer look at 2 of these cases and conclude for which models of $\mathcal{V}$ one gets Minkowski, Rindler or (A)dS ground states.

3.1 Bernoulli Differential Equation

Assuming (38) is of the specific form

$$dY = V_1(X)YdX + V_2(X)Y^n dX \ , \tag{39}$$

with arbitrary functions $V_1(X)$ and $V_2(X)$, the problem is reduced to solving a Bernoulli differential equation. To see this explicitly, as a first step we make the coordinate transformation

$$z = Y^{1-n} \ , \quad Y = z^{\frac{1}{1-n}} \ ,$$

$$dz = (1-n)Y^{-n}dY \ , \quad dY = \frac{1}{1-n} z^{\frac{n}{1-n}} dz \ . \tag{40}$$

Since $n = 1$ is trivial, we assume $n \neq 1$. Inserting (40) in (39) yields

$$dY = \frac{1}{(1-n)Y^{-n}} dz = V_1(X)YdX + V_2(X)Y^n dX \ , \tag{41}$$

$$dz = (1-n)[V_2(X)dX + V_1(X)zdX] \ . \tag{42}$$

This is now similar to the UV-family, a family of linear dilaton models where the function $\mathcal{V}$ takes on the form $\mathcal{V} = V(X) - YU(X)$, see the appendix in [18]. The functions U and V are then identified as $U(X) = -(1-n)V_1(X)$ and $V(X) = (1-n)V_2(X)$. According to [18], we introduce

$$Q(X) := \int^X U(X')dX' = (n-1) \int^X V_1(X')dX' \tag{43}$$

and

$$w(X) := \int^X e^{Q(X')} V(X') dX' = (1-n) \int^X e^{Q(X')} V_2(X') dX' \ . \tag{44}$$

Acting with the de-Rahm differential on the function w gives us

$$dw = (1-n) e^{Q(X)} V_2(X) dX = e^{Q(X)} [dz - (1-n) V_1(X) z dX] =$$

$$= e^{Q(X)} dz + e^{Q(X)} z dQ = d(e^Q z) \ . \tag{45}$$

We can now integrate the expression $d(e^Q z) = dw$,

$$e^Q z = w - C \quad \to \quad z = e^{-Q}(w - C) \ , \tag{46}$$

with the integration constant given by the Casimir C. As the next step we compute the Killing norm. The Killing norm of the metric (25) is given by

$$K = 2e^{2Q} X^+ X^- = 2e^{2Q} Y = 2e^{2Q} z^{\frac{1}{1-n}} = 2e^{Q(2 - \frac{1}{1-n})} (w - C)^{\frac{1}{1-n}} \ . \tag{47}$$

The Casimir function vanishes for the ground state, so the ground state Killing norm becomes

$$K|_{C=0} = 2e^{Q(2 - \frac{1}{1-n})} w^{\frac{1}{1-n}} \ . \tag{48}$$

For a Minkowski ground state, the Killing norm has to be constant, $K|_{C=0} = \text{const}$, which yields the condition

$$w e^{Q(1-n)} = \text{const} \ . \tag{49}$$

Analogously for a Rindler ground state we have

$$w e^{Q(1-n)} \propto r \tag{50}$$

and for an (A)dS ground state

$$w e^{Q(1-n)} \propto r^2 \ , \tag{51}$$

where the radial coordinate is given by

$$r = \int^X e^{Q(X')} dX' \ . \tag{52}$$

3.1.1 Example for a 3 Parameter Family

We consider as an example for the Bernoulli case the functions

$$V_1(X) = \frac{a}{X} \ , \quad V_2(X) = X^b \ , \tag{53}$$

where a, b and n are arbitrary parameters for now, and will be fixed for each ground state later. The function Q from (43) then becomes

$$Q(X) = (n-1) \int^X \frac{a}{X'} \, dX' = (n-1) \ a \ \ln(X) \ , \tag{54}$$

and therefore

$$e^Q = X^{a(n-1)} \ . \tag{55}$$

Similarly, the function w given by (44) is computed as

$$w(X) = (1-n) \int^X e^{Q(X')} X'^b \, dX' = \frac{1-n}{a(n-1)+b+1} X^{a(n-1)+b+1} \ . \tag{56}$$

The Minkowski ground state condition (49) now implies

$$w \ e^{Q(1-n)} = \frac{1-n}{a(n-1)+b+1} X^{a(n-1)+b+1} X^{a(n-1)(1-n)} = \text{const} \ , \tag{57}$$

and leaves us with the result

$$b = (1-n)(2-n)a - 1 \ . \tag{58}$$

The radial coordinate (52) is given by

$$r = \int^X X'^{a(n-1)} dX' = \frac{1}{a(n-1)+1} X^{a(n-1)+1} \ , \tag{59}$$

so our condition (50) for a Rindler ground state is

$$X^{a(n-1)+b+1-a(n-1)^2} \propto X^{a(n-1)+1} \ , \tag{60}$$

which gives us

$$b = (n-1)^2 a \ . \tag{61}$$

For an (A)dS ground state, the relation (51) reads as follows,

$$X^{a(n-1)+b+1-a(n-1)^2} \propto X^{2a(n-1)+2} \ , \tag{62}$$

yielding

$$b = n(n-1)a + 1 \ . \tag{63}$$

3.2 Exact Differential Equation

The second class of differential equations we would like to involve in our investigations are exact differential equations of the form

$$p(X,Y)\ dX + q(X,Y)\ dY = 0\ . \tag{64}$$

If the integrability condition

$$\frac{\partial p}{\partial Y} = \frac{\partial q}{\partial X} \tag{65}$$

holds, there exists a potential $F(X,Y)$, so that $\frac{\partial F}{\partial X} = p$ and $\frac{\partial F}{\partial Y} = q$. Note that the potential is also the Casimir function $F = C$.

As in the previous section, we would like to compute the Killing norm as a first step. To do so let us start with the integrating factor

$$Q(X,C) = -\int^X \frac{\partial \mathcal{V}(X',2Y)}{\partial Y} dX'\ . \tag{66}$$

From (64), one can see that the function $\mathcal{V}$ is given by $\mathcal{V} = -\frac{p}{q}$, so we get

$$Q(X,C) = \int^X \frac{\partial}{\partial Y}\left(\frac{p}{q}\right) dX' = \int^X \left(\frac{1}{q}\frac{\partial p}{\partial Y} - \frac{p}{q}\,\frac{1}{q}\frac{\partial q}{\partial Y}\right) dX'\ .$$

The first term in the integral on the right can be rewritten with the relation (65), which yields

$$Q(X,C) = \int^X \left(\frac{1}{q}\frac{\partial q}{\partial X'} - \frac{p}{q}\,\frac{1}{q}\frac{\partial q}{\partial Y}\right) dX' = \int^X \frac{\partial}{\partial X'}\ln(q)\ dX' + \int^X \mathcal{V}\,\frac{\partial}{\partial Y}\ln(q)\ dX'\ .$$

To simplify this a little further, let us change the integration variable in the second integral to $dX = \mathcal{V}^{-1}dY$, and we obtain

$$Q(X,C) = \int^X \frac{\partial}{\partial X'}\ln(q)\ dX' + \int^Y \frac{\partial}{\partial Y'}\ln(q)\ dY'\ . \tag{67}$$

We can use the fundamental theorem of calculus for both integrals on the right hand side. Doing so we get a constant of integration depending on Y for the first and one depending on X for the second integral. To avoid this we rewrite the second integrand as follows

$$\frac{1}{q}\frac{\partial q}{\partial Y}\ dY = \frac{1}{q}dq - \frac{1}{q}\frac{\partial q}{\partial X}\ dX\ , \tag{68}$$

where we used

$$dq(X,Y) = \frac{\partial q}{\partial Y}\ dY + \frac{\partial q}{\partial X}\ dX\ . \tag{69}$$

Inserting this in (67) yields

$$Q(X,C) = \int \frac{1}{q}\, dq = \ln\ q + c\ , \tag{70}$$

where c is a constant that does not depend on Y or X. For the computation of the Killing norm and the ground state, the constant c is irrelevant, so we set $c = 0$, i.e. $e^{2Q} = q^2$. The Killing norm now has a rather simple form

$$K = 2e^{2Q} X^+ X^- = 2q^2 Y\ . \tag{71}$$

The Killing horizon is located at zeros of the Killing norm $K = 0$. If we exclude the option $q \to 0$, which would imply $Q \to -\infty$, there is an unique Killing horizon for every zero of Y in the range of definition of X. Further motivation for this choice will be given in the next section with the definition of the radial coordinate.

3.2.1 Quadratic Potential

The literature about potentials linear in Y, which yield power counting renormalizable models, is already well established. Therefore we are including terms of quadratic order of Y in the potential, leading new but also not power-counting renormalizable models. The potential is given by

$$F(X,Y) = V_0(X) + V_1(X)Y + V_2(X)Y^2\ , \tag{72}$$

where the $V_i(X)$ are as before arbitrary functions in X. For the ground state we have $F(X,Y) = 0$, which can be used to solve for Y as

$$Y_\pm = \frac{-V_1 \pm \sqrt{V_1^2 - 4V_2V_0}}{2V_2}\ . \tag{73}$$

The function $q(X,Y)$ then becomes

$$q(X,Y_\pm) = V_1 + 2V_2Y_\pm = \pm\sqrt{V_1^2 - 4V_2V_0}\ , \tag{74}$$

and so the Killing norm is given by

$$K^\pm|_{C=0} = (V_1^2 - 4V_2V_0)\frac{-V_1 \pm \sqrt{V_1^2 - 4V_2V_0}}{V_2}\ . \tag{75}$$

In case of a Minkowski ground state this expression should be constant, which yields the conditions on the functions $V_i(X)$ for this model

$$(V_1^2 - 4V_2V_0)\frac{-V_1 \pm \sqrt{V_1^2 - 4V_2V_0}}{V_2} = \text{const}\ . \tag{76}$$

The same procedure applies for a Rindler ground state as

$$(V_1^2 - 4V_2V_0)\frac{-V_1 \pm \sqrt{V_1^2 - 4V_2V_0}}{V_2} \propto r \ , \tag{77}$$

and an (A)dS ground state with

$$(V_1^2 - 4V_2V_0)\frac{-V_1 \pm \sqrt{V_1^2 - 4V_2V_0}}{V_2} \propto r^2 \ , \tag{78}$$

where the radial coordinate is given by

$$r = \int^X q(X', Y)dX' \ . \tag{79}$$

We can see from (79) that a positive radial coordinate implies positivity of the function q, and the assumption in the previous subsection where we exclude the case $q \to 0$ seems to be a reasonable choice.

The 2 branches obtained from the quadratic solution of $Y^{\pm}$ can be understood as follows. As we will see in the upcoming subsection of pertubative continuation, the lower sign Y^- leads to the same result for the Killing norm as in the linear case. This lower sign solution for quadratic potentials can be seen as the direct generalization of the linear theory, i.e. the ab-family. The 2 branches can lead to different dilaton models. While the lower sign has a straightforward interpretation, the upper sign is more peculiar and does not have a well defined limit for $V_2(X) \to 0$.

3.2.2 *abc*-Family

A 3 parameter family of 2d dilaton models with a potential obeying (64) can be given by

$$V_0(X) = A\ X^a \ , \quad V_1(X) = B\ X^b \ , \quad V_2(X) = D\ X^c \ , \tag{80}$$

which we shall refer to as the abc-family according to the parameters, and with constants A, B and D. When we insert the functions (80) in (75), the Killing norm reads as

$$K^{\pm}|_{C=0} = (B^2X^{2b} - 4AD\ X^{a+c})\left(-\frac{B}{D}X^{b-c} \pm \sqrt{\frac{B^2}{D^2}X^{2b-2c} - 4\frac{A}{D}X^{a-c}}\right) \ . \tag{81}$$

Minkowski ground state:

Let us begin our ground state investigations for the *abc*-family in Minkowski space, where $K^{\pm}|_{C=0} = \text{const}$. To do so, let us rewrite the expression (81) as

$$K^{\pm}|_{C=0} = (B^2 - 4AD\ X^{a+c-2b})X^{3b-c}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a+c-2b}}\right) = \text{const} \ , \quad (82)$$

where we pulled out a factor of X^{2b} from the first parentheses in (81), and a factor X^{b-c} from the second parentheses. We can now argue that the left hand side of (82) can only be constant in the dilaton field X for arbitrary values of the coefficients A, B and D, if each term in the product is constant, i.e.

$$\begin{aligned} X^{3b-c} &= \text{const} \ , \\ B^2 - 4AD\ X^{a+c-2b} &= \text{const} \ , \\ -\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a+c-2b}} &= \text{const} \ . \end{aligned} \quad (83)$$

From these relations we can deduce 2 independent algebraic equations for the parameters a, b and c,

$$\begin{aligned} a + c - 2b &= 0 \ , \\ 3b - c &= 0 \ . \end{aligned} \quad (84)$$

Solving the system yields the relations of the parameters for a Minkowski ground state,

$$a = -b \ , \ c = 3b \ , \ \text{ and } b \text{ arbitrary.} \quad (85)$$

Additional to the case of generic A, B and D, one gets isolated solutions for $A = 0$ or $B = 0$. Note that the case $D = 0$ leads to potentials linear in Y, and is therefore not in our interest to study here.

In the special case of $A = 0$ and $B \neq 0$, the Killing norm for the Minkowski ground state (82) becomes

$$K^{\pm}|_{C=0} = B^2 X^{3b-c}\left(-\frac{B}{D} \pm \frac{B}{D}\right) = \text{const} \ . \quad (86)$$

One can see that the 2 branches of the solution lead to 2 different models. For the lower sign we get the ground state condition

$$c = 3b \ , \quad (87)$$

with no additional restrictions. For the upper sign, the Killing norm is always zero, $K^{+}|_{C=0} \equiv 0$, with no restrictions on the parameters. The condition for a Minkowski ground state is therefore always met for $A = 0$ and the upper sign branch of the solution.

For $B = 0$ and $A \neq 0$, (82) reads as follows

$$K^{\pm}|_{C=0} = \mp 4AD\ X^{a+b}\sqrt{-4\frac{A}{D}X^{a+c-2b}} = \text{const} \ , \tag{88}$$

which is solved for

$$c = -3a \ . \tag{89}$$

Rindler ground state:

Next, we determine the parameters a, b and c for Rindler space with arbitrary coefficients A, B and D. First, we have to calculate the radial coordinate (79) for the abc-family,

$$r = \int^{X} (BX'^{b} + 2YDX'^{c})dX' = \frac{B}{b+1}X^{b+1} + \frac{2D}{c+1}X^{c+1}Y \ . \tag{90}$$

The condition for a Rindler ground state (77) then becomes

$$K^{\pm}|_{C=0} = (B^2X^{2b} - 4AD\ X^{a+c})\left(-\frac{B}{D}X^{b-c} \pm \sqrt{\frac{B^2}{D^2}X^{2b-2c} - 4\frac{A}{D}X^{a-c}}\right) \propto$$
$$\frac{B}{b+1}X^{b+1} + \frac{2D}{c+1}X^{b+1}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right) \ . \tag{91}$$

We can now again rewrite the right hand side in the first line by pulling out a factor of X^{3b-c} analogous to (82). Afterwards we divide the whole equation by X^{b+1} and bring the second term in the second line on the other side, which then reads as

$$\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right)\left(-\frac{2D}{c+1} + X^{2b-c-1}(B^2 - 4AD\ X^{a-2b+c})\right) \propto \frac{B}{b+1} \ . \tag{92}$$

The right hand side of (92) is just a constant, so we can argue again that both terms on the right have to be constant for arbitrary A, B and D, to satisfy the condition. This can only be the case when

$$X^{a-2b+c} = \text{const} \ , \qquad X^{2b-c-1} = \text{const} \tag{93}$$

and therefore

$$a - 2b + c = 0 \ , \tag{94}$$

$$2b - c - 1 = 0 \ . \tag{95}$$

The solution yields the relations on the parameters for a Rindler ground state and is given by

$$a = 1 \ , \ c = 2b - 1 \text{ and } b \text{ arbitrary.} \tag{96}$$

When we set the coefficient $A = 0$, we again get different models for the 2 branches of $K^{\pm}|_{C=0}$. The relation (92) in this case reads as

$$\left(-\frac{B}{D} \pm \frac{B}{D}\right)\left(-\frac{2D}{c+1} + B^2 X^{2b-c-1}\right) \propto \frac{B}{b+1} \ . \tag{97}$$

The lower sign $K^{-}|_{C=0}$ leads to a Rindler ground state with the condition on the parameters given by

$$c = 2b - 1 \ . \tag{98}$$

The upper sign $K^{+}|_{C=0}$ only obeys the relation (97) for the trivial case of $B = 0$. Therefore, if $A = 0$ and $B \neq 0$, the branch of $K^{+}|_{C=0}$ does not yield a ground state in Rindler space.

Analogous for $B = 0$, relation (92) becomes

$$4AD \ X^{a-1} = -\frac{2D}{c+1} \ , \tag{99}$$

with the solution

$$a = 1 \ . \tag{100}$$

(A)dS ground state:

Our final ground state for the abc-family will be in (A)dS space. The condition (78) for the abc-family (80) with generic A, B and D becomes after some simplification

$$K^{\pm}|_{C=0} = (B^2 - 4AD\ X^{a-2b+c})X^{3b-c}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right) \propto$$

$$r^2 = X^{2b+2}\left[\left(\frac{B}{b+1}\right)^2 + \frac{4BD}{(b+1)(c+1)}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right) + \right.$$

$$\left. + \left(\frac{2D}{c+1}\right)^2\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right)^2\right] . \tag{101}$$

Dividing each side by X^{2b+2} yields

$$(B^2 - 4AD\ X^{a-2b+c})X^{b-c-2}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right) \propto$$

$$\left[\left(\frac{B}{b+1}\right)^2 + \frac{4BD}{(b+1)(c+1)}\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right) + \right.$$

$$\left. + \left(\frac{2D}{c+1}\right)^2\left(-\frac{B}{D} \pm \sqrt{\frac{B^2}{D^2} - 4\frac{A}{D}X^{a-2b+c}}\right)^2\right] . \tag{102}$$

For arbitrary constants A, B and D the equation holds true if

$$X^{a+c-2b} = \text{const} \quad \text{and} \quad X^{b-c-2} = \text{const} . \tag{103}$$

The algebraic equations for the parameters a, b and c are thus given by

$$a + c - 2b = 0 ,$$

$$b - c - 2 = 0 , \tag{104}$$

with the result for the (A)dS ground state relations

$$a = b + 2 , \ c = b - 2 \text{ and } b \text{ arbitrary.} \tag{105}$$

In the case of $A = 0$, we again have 2 solutions depending on the branch of $K^{\pm}|_{C=0}$. Relation (102) becomes for $K^{-}|_{C=0}$

$$X^{b-c-2} = \text{const} \ , \tag{106}$$

yielding

$$c = b - 2 \ . \tag{107}$$

The upper sign on the other hand leads to $K^{+}|_{C=0} \equiv 0$, independent of the parameters. This implies that only the trivial case $A = B = 0$ for the positive branch has an (A)dS ground state.

For $B = 0$, relation (102) reads

$$X^{3b-c} X^{(a+b)/2-b} \propto \text{const} \ , \tag{108}$$

which can be solved for the parameters as follows

$$c = a + 4b \ . \tag{109}$$

The solutions for the ground state conditions only depended on the branches of $K^{\pm}|_{C=0}$ for the special case $A = 0$, for all 3 space-times we investigated. While we had a physical interpretation for the lower sign $K^{-}|_{C=0}$, the upper sign $K^{+}|_{C=0}$ seems to lead to trivial solutions for the dilaton models.

Pertubative continuation:

We would like to check that the *abc*-family devolves into the special case of the *ab*-family for $D \to 0$. The function $q(X, Y_{\pm})$ given by the first term in (81) simplifies for $D \to 0$ to

$$q(X, Y_{\pm}) = \pm\sqrt{B^2 X^{2b} - 4AD \ X^{a+c}} \quad \to \quad q_{\text{lin}} = \pm B \ X^b \ . \tag{110}$$

The redefined product of the Lagrangian multipliers $Y_{\pm}$ is given by the second term in (81)

$$Y_{\pm} = -\frac{B}{2D} X^{b-c} \mp \sqrt{\frac{B^2}{4D^2} X^{2b-2c} - \frac{A}{D} X^{a-c}} \ . \tag{111}$$

For small values of D this expression seems to diverge, so we rewrite it as a first step to

$$Y_{\pm} = -\frac{B}{2D} X^{b-c} \mp \frac{1}{D} \sqrt{\frac{B^2}{4} X^{2b-2c} - AD \ X^{a-c}} \ . \tag{112}$$

Now one can Taylor-expand the square root at the point $D = 0$,

$$\sqrt{\frac{B^2}{4}X^{2b-2c} - AD\ X^{a-c}} \sim \frac{B}{2}X^{b-c} - \frac{A\ X^{a-c}}{\sqrt{B^2\ X^{2b-2c}}}D + \mathcal{O}(D^2)\ . \tag{113}$$

We drop the terms containing higher powers of D and insert it into (112),

$$Y_\pm \sim -\frac{B}{2D}X^{b-c} \mp \frac{B}{2D}X^{b-c} \pm \frac{A}{B}X^{a-b}\ . \tag{114}$$

Choosing the lower sign in $Y_\pm$, we receive the solution of a potential $F(X,Y)$ linear in Y,

$$Y_- \sim Y_{\text{lin}} = -\frac{A}{B}X^{a-b}\ . \tag{115}$$

The branch with the upper sign in (114) diverges for the limit $D \to 0$ and therefore has no correspondence to the linear solution, as we have mentioned in subsection 3.2.1. The Killing norm $K = 2q^2Y$ with the results (110) and (115) reads then as follows,

$$K_{\text{lin}}|_{C=0} = -2AB\ X^{a+b}\ . \tag{116}$$

When we compare this result with (120) in the next section, we see that one does indeed recover the ab-family for small values of D.

3.2.3 ab-Family

In the case of $D = 0$ in (80), we recover the ab-family, similar to [18]

$$F(X,Y) = AX^a + BX^b\ Y\ . \tag{117}$$

With (71), the conditions for the ground states are easily recovered. The function $q(X,Y)$ is given by

$$q = V_1(X) = BX^b\ , \tag{118}$$

and with $F(X,Y) = 0$ we get

$$Y = -\frac{V_0(X)}{V_1(X)} = -\frac{A}{B}X^{a-b}\ . \tag{119}$$

The Killing norm now reads as follows,

$$K|_{C=0} = 2q^2Y = -2AB\ X^{a+b}\ . \tag{120}$$

In the case of a Minkowski ground state $K|_{C=0} = \text{const}$, we get $a = -b$.

For a Rindler ground state we have $K|_{C=0} \propto r$ with

$$r = \frac{B}{b+1} X^{b+1} \ . \tag{121}$$

This yields the relation

$$X^{a+b} \propto X^{b+1} \tag{122}$$

and the result $a = 1$ with b arbitrary.

The (A)dS ground state condition states $K|_{C=0} \propto r^2$. Inserting the Killing norm (120) and the radial coordinate (121) leaves us with

$$X^{a+b} \propto X^{2b+2} \ . \tag{123}$$

So for the ab-family defined as special case (117), the relation for the exponents is $a = b + 2$ in case of an (A)dS ground state.

3.2.4 Comparing Results for the ab-Family

Let us check if the results above coincide with the ones for the $a'b'$-family from previous literature, where we write $a'b'$ for the model from [18] instead of ab, to avoid confusion. According to [18], the function $\mathcal{V}$ is given by

$$\mathcal{V}_{a'b'}(X, Y) = V(X) - U(X)\, Y \ , \tag{124}$$

with

$$V(X) \propto X^{a'+b'} \quad \text{and} \quad U(X) = -\frac{a'}{X} \ . \tag{125}$$

Now, one defines

$$\tilde{Q} := \int^X U(X')dX' = -\int^X \frac{a'}{X'} dX' = \ln X^{-a'} \tag{126}$$

and

$$\tilde{w} := \int^X e^{\tilde{Q}(X')} V(X')dX' = \int^X X'^{b'} dX' = \frac{1}{b'+1} X^{b'+1} \ , \tag{127}$$

where we denoted $\tilde{Q}$ and $\tilde{w}$ instead of Q and w to avoid confusion with the previous section. Acting with the de-Rahm differential on $\tilde{w}$ yields

$$d\tilde{w} = e^{\tilde{Q}} V dX = e^{\tilde{Q}}(YUdX + dY) = (Y de^{\tilde{Q}} + e^{\tilde{Q}} dY) = d(Y e^{\tilde{Q}}) \ , \tag{128}$$

which can be integrated to

$$Y = e^{-\tilde{Q}}(\tilde{w} - \tilde{C}) \ , \tag{129}$$

with Casimir function $\tilde{C}$. For the ground state, $\tilde{C} = 0$, we then get

$$Y = e^{-\tilde{Q}}\tilde{w} \; . \tag{130}$$

The Killing norm is the same as before, so the final result for the Minkowski ground state is

$$K|_{\tilde{C}=0} = 2e^{2\tilde{Q}}Y = \frac{2}{b'+1}X^{-a'+b'+1} = \text{const} \quad \rightarrow \quad a' = 1 + b' \; . \tag{131}$$

Similarly we get for a Rindler ground state

$$K|_{\tilde{C}=0} \propto r = \int^X e^{\tilde{Q}(X')}dX' = \frac{1}{1-a'}X^{1-a'} \tag{132}$$

and therefore

$$X^{-a'+b'+1} \propto X^{1-a'} \quad \rightarrow \quad b' = 0 \; . \tag{133}$$

The (A)dS ground state condition leads to

$$X^{-a'+b'+1} \propto X^{2-2a'} \quad \rightarrow \quad a' = 1 - b' \; . \tag{134}$$

Let us now compare our results for the ab-family as special case $D = 0$ from the abc-family with the results for the $a'b'$-family from [18]. To do so, let us first compare the functions

$$\mathcal{V}_{ab} = -\frac{\frac{\partial F}{\partial X}}{\frac{\partial F}{\partial Y}} = -\frac{Aa}{B}X^{a-b-1} - \frac{b}{X}Y \; , \tag{135}$$

from the case of an exact differential equation determining the function $\mathcal{V}$, and

$$\mathcal{V}_{a'b'} = \tilde{c}X^{a'+b'} + \frac{a'}{X}Y \; , \tag{136}$$

from the $a'b'$-family in [18], with $\tilde{c}$ being an arbitrary constant. Demanding $\mathcal{V}_{ab} = \mathcal{V}_{a'b'}$, we see that $a' = -b$ and so $b' = a - 1$. The Minkowski, Rindler and (A)dS ground state conditions (131), (133) and (134) can then be rewritten as follows

$$\text{Minkowski}: \; a' = 1 + b' \quad \hat{=} \quad -b = 1 + a - 1 \quad \rightarrow \quad a = -b \; , \tag{137}$$

$$\text{Rindler}: \; b' = 0 \quad \hat{=} \quad a - 1 = 0 \quad \rightarrow \quad a = 1 \; , \tag{138}$$

$$\text{(A)dS}: \; a' = 1 - b' \quad \hat{=} \quad -b = 1 - a + 1 \quad \rightarrow \quad a = b + 2 \; . \tag{139}$$

When we compare these results with the ones in section 3.2.3, we see that they coincide perfectly, which is an affirmation that our generalization of the ab-family for linear dilaton vacua is correct.

4 Conclusion and Outlook

We found that for a general dilaton action (2) the e.o.m. for the dilaton field depending on the Lagrange multipliers results in a first-order non-linear ordinary differential equation with function $\mathcal{V}$ determining the specific kind of equation. Though there is no general solution for this problem, we could investigate 2 specific solution-methods for general 2d dilaton models and parameter families thereof.
In the case of a Bernoulli differential equation, we computed the general conditions for Minkowki, Rindler and (A)dS ground states via the Killing norm and made further investigations for a 3 parameter family, where we found a model for each space-time.
For the exact differential equation-form of the e.o.m. (38), we computed the function $Q(X, C)$ in (70) and found that it only depends on the logarithm of the function $q(X, Y)$ from (64). This yielded us the simple result for the Killing norm (71), which only depends on the function $q(X, Y)^2$ and Y. Because of the degeneracy of the Killing horizon at $q = 0$, we could conclude that the only non-extremal Killing horizon for this, still very general class of models, is only located at values of $Y = 0$.
By fixing the potential $F(X, Y)$ from the exact differential equation in quadratic order of Y, we found a 3 parameter family of 2d dilaton models, that seems to be a generalization of the 2 parameter *ab*-family from [18], as the results of the ground states in the case $D = 0$ for each Minkowski, Rindler and (A)dS coincide with the ones for the *ab*-family from previous literature, which was also confirmed by the pertubative continuation.
Another interesting result of the generalized 3 parameter *abc*-family is, that beside the similarity of the conditions on the parameters a and b for the *abc*- and *ab*-family, the parameter c is also depending on one of the other parameters, and thus does not yield another degree of freedom.
As an outlook, one could investigate further solution-methods for the differential equation (38) like seperable equations with $\mathcal{V}(X, Y) = V_1(X) \cdot V_2(Y)$, homogeneous equations with $\mathcal{V}(X, Y) = \mathcal{V}(\lambda X, \lambda Y)$ where λ is a constant parameter leaving the function $\mathcal{V}$ unchanged, or perhaps even new solutions methods for the differential equation by exploiting the first order formulation to some degree.
Also, we have just considered potentials for the exact differential equations that are of quadratic order in Y. Looking into higher orders of Y would be interesting as it could lead to further generalizations of families of dilaton models. Given the rather simple form of the general Killing norm (71), a ground state examination of models with potentials $F(X, Y)$ with third or even fourth order in Y can be done analytically, as cubic and quartic polynomials have a general analytical solution.

References

[1] J.G. Polchinski, *String theory, volume I: An introduction to the bosonic string* (Cambridge university press Cambridge, 1998)

[2] S. Hawking, W. Israel, General Relativity: an Einstein Centenary Survey (2010)

[3] R.M. Wald, *General relativity* (University of Chicago press, 2010)

[4] D.N. Page, New Journal of Physics **7**, 203 (2005)

[5] S.W. Hawking, arXiv preprint arXiv:1509.01147 (2015)

[6] K. Landsman, Foundations of Physics **51**, 1 (2021)

[7] J.D. Brown, *Lower dimensional gravity* (World Scientific, 1988)

[8] D. Grumiller, W. Kummer, D. Vassilevich, Physics Reports **369**, 327 (2002)

[9] D. Bailin, A. Love, Reports on Progress in Physics **50**, 1087 (1987)

[10] S. Hawking, Communications in Mathematical Physics **25**, 167 (1972)

[11] R. Jackiw, Nuclear Physics B **252**, 343 (1985)

[12] C. Teitelboim, Physics Letters B **126**, 41 (1983)

[13] C.G. Callan Jr, S.B. Giddings, J.A. Harvey, A. Strominger, Physical Review D **45**, R1005 (1992)

[14] P. Schaller, T. Strobl, Modern Physics Letters A **9**, 3129 (1994)

[15] T. Strobl, Arxiv preprint hep-th/0011248 (2000)

[16] J.G. Russo, A.A. Tseytlin, Nuclear Physics B **382**, 259 (1992)

[17] S. Odintsov, I. Shapiro, Physics Letters B **263**, 183 (1991)

[18] D. Grumiller, R. Ruzziconi, C. Zwikel, SciPost Phys. **12**, 032 (2022), 2109.03266

[19] G. Barnich, F. Brandt, M. Henneaux, Physics Reports **338**, 439 (2000)