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From Semantic Games to Analytic Calculi

DISSERTATION

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Erklärung zur Verfassung der Arbeit

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Hiermit erkläre ich, dass ich diese Arbeit selbständig verfasst habe, dass ich die verwendeten Quellen und Hilfsmittel vollständig angegeben habe und dass ich die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken oder dem Internet im Wortlaut oder dem Sinn nach entnommen sind, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht habe.

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Kurzfassung

Spiele bieten einen natürlichen und fruchtbaren Ansatz für die Logik und schlagen eine Brücke zwischen dem traditionellen semantischen und beweistheoretischen Ansatz. Die vorliegende Arbeit soll diese Verbindung veranschaulichen. Hierzu wird eine Technik untersucht, die semantische Spiele zu Beweisbarkeitsspielen und weiter zu analytischen Kalkülen liftet. Das einfachste Beispiel für diese Technik ist das Liften von Hintikkas Spiel für die klassische Aussagenlogik zu einer Version des Gentzen'schen Sequenzsystems LK.

Wir wenden diese Technik auf die Modallogik an. Um einige konzeptuelle Probleme zu überwinden, müssen wir uns auf die hybride Logik – eine Erweiterung der modalen Logik – zurückgreifen. Deren Sprache erlaubt es uns, im Objektlevel explizit auf Welten Bezug zu nehmen.

In unserer zweiten Fallstudie entwickeln wir ein semantisches Spiel für Choice Logik – ein hybrides Framework für Wahrheit und Präferenzen. Dieses Spiel hat eine reichere Domäne von Auszahlungswerten, die vom Beweisbarkeitsspiel und dem daraus resultierenden Kalkül übernommen wird. Dies führt zu einem gradbasierten Gültigkeitsbegriff in der induzierten Logik.

Unter Verwendung der gesammelten Ideen entwickeln wir ein Framework für das Lifting von allgemeinen semantischen Spielen. Dieses Framework umfasst alle bekannten Fälle in dieser Literatur und dieser Arbeit.

Abstract

Games offer a natural and fruitful approach to logic, bridging the traditional semantic and proof-theoretic approach to logic. The present thesis aims to illustrate this connection by investigating a technique of lifting semantic games to provability games and further to analytic calculi. The simplest example of this technique is lifting Hintikka's game for classical propositional logic to a version of Gentzen's sequent system LK.

We apply this technique to modal logic. To overcome some conceptual issues, we must turn to hybrid logic – an extension of modal logic. The language allows us to refer to worlds within the object language explicitly.

In our second case study, we develop a semantic game for choice logic – a framework for jointly dealing with truth and preferences. This game has a richer domain of payoff values, which is inherited by the provability game and the resulting calculus, giving a degree-based notion of validity in the induced logic.

Using these ideas, we develop a lifting framework to conduct the lifting for general semantic games. This framework encompasses all known cases in the literature and in this thesis.

Hakuna Matata

Contents

CHAPTER 1

Introduction

Games in logic have a long and varied tradition (see, e.g., [41]). Building on the concepts of rational behavior and strategic thinking, they are a powerful tool to bridge the gap between intended and formal semantics. Games thus offer a conceptually more natural approach to logic than the common paradigm of model-theoretic semantics and proof systems. Furthermore, they present a starting point to extend logical reasoning to scenarios with incomplete or imperfect information [45]. Nowadays, many different types of games are employed in various subfields and application areas of logic [55, 1, 40, 42], however, two primary forms of logical games are still central: *semantic games* and *provability games*.

Semantic games (or evaluation games) were developed by Jaako Hintikka [38] as an alternative to the Tarskian-style truth definition. In the version for classical logic, every instance of the game is played over a propositional formula F and an interpretation $\mathcal{I}$ by two players, usually called *I* (or *Me*) and *You*. At each point in the game, one of the players acts as the proponent (**P**), while the other acts as the opponent (**O**) of the current formula. The set of actions at each stage is dictated by the main connective of the current formula: if it is of the form $F_1 \vee F_2$, then the player acting as **P** chooses whether the game continues with F_1 or with F_2. Similarly, **O** chooses between continuing the game with F_1 or F_2 at the formula $F_1 \wedge F_2$. If the current formula is $\neg F$, the game continues with F and a role switch. When the game reaches a propositional variable, **P** wins (and **O** loses) iff $\mathcal{I} \models p$; otherwise, **O** wins (and **P** loses). This game adequately models truth over $\mathcal{I}$ in the following sense: *I* have a winning strategy for the game starting with *Me* as the proponent of the formula F and over the interpretation $\mathcal{I}$ iff $\mathcal{I} \models F$.

In contrast to semantic games, provability games, as developed by Paul Lorenzen[1] [43], do not refer to truth in a particular model but to logical validity. The game is played

[1] We prefer to use the term *provability games* instead of the original *dialogue games*. In the next subsection, we will argue that there is indeed a close connection between proofs and provability games.

by two players, *I* and *You*[2], and consists of attacking assertions of formulas made by the other player and defending against these attacks. The game starts by *Me* asserting a given formula F. The involved formulas dictate the attacks and defenses available to the players. Players take alternating turns, and the player who cannot move at their turn loses the game, while the other wins. Intuitively, this game can be thought of as an exam situation, where *I* am the student and *You* are the examiner [41] who asks *Me* to prove a formula F. If it is of the form $F_1 \vee F_2$, then *You* may ask *Me* to choose and prove one of the two disjuncts. For formulas $F_1 \wedge F_2$, *You* can choose to ask *Me* to prove either one of F_1 and F_2. The case for $F_1 \to F_2$ is more intricate: Here, *I* can agree to prove F_2. But *I* can also turn the table and ask *You* to prove that F_2 is consistent in the first place[3]. Besides the rules connected to logical connectives, procedural rules dictate the overall shape of the dialogue. For example, one rule states that *I* can only assert an atomic formula after it was granted by *You*. Originally, dialogue games were meant to characterize constructive reasoning. Indeed, *I* have a winning strategy for the game over F iff F is valid intuitionstically. However, it was shown [21] that a slight change in the procedural rules results in an adequate game for classical logic.

From the adequacy of both games, we immediately obtain the following consequence: *I* have a winning strategy as the proponent of a fixed formula F in Hintikka's game over the interpretation $\mathcal{I}$, for every $\mathcal{I}$ iff *I* have a winning strategy in Lorenzen's classical game over F. But can this link be made more instructive? In the broadest terms, the central question of interest in this thesis is:

> What are the connections between semantic games and provability games?

More concretely, can a provability game over F be seen as a way of simultaneously playing all possible semantic games over F at once? Since every winning strategy for *Me* in an adequate provability game serves as a token of validity, can every winning strategy for *Me* in the provability game be represented as a derivation in a suitable related proof system? And vice versa, does every derivation in this system represent a winning strategy? Is every winning strategy for *You* in the provability game guided by a winning strategy for *You* in the semantic game over some interpretation $\mathcal{I}$? Finding results concerning the last question promises to shed light on proof search and counter-model extraction in the related proof system.

Answering these questions will be at the core of this thesis. The central idea for tackling these issues is the *lifting technique* that allows us to lift semantic games to provability games and further to useful proof systems.

[2]Originally, the two players are called Proponent and Opponent. For reasons that will become clear in the next paragraph, it is worthwhile to think of the two players in the provability games as being the same as in the semantic game.

[3]Note that in this interpretation the players have slightly asymmetric roles.

1.1 Lifting Semantic Games to Provability Games and Analytic Calculi

In this section, we will discuss the main ideas behind the lifting technique and how it can help to answer the questions raised at the end of the previous section. The lifting was first described in [18, 24] for Giles' game for propositional Łukasiewicz logic [33, 34]. In [22, **?**] it was applied for the truth-degree comparison game for Gödel logic.

To get a good grip on the main ideas, we will illustrate the technique using the example of classical propositional logic and Hintikka's game, lift it to a provability game, and further to a version of Gentzen's sequent calculus **LK**.

To this end, it is helpful to describe Hintikka's game in more detail. Remember that players can be either in the role of the proponent (**P**) or the proponent (**O**), and the current role distribution and formula dictate the players' moves at any moment of the game. We codify the situation where the current formula is G, and I am currently in the role of **P** (correspondingly *You* are **O**) in the *game state* $\mathbf{P} : G$, and similarly for the situation where I am the opponent as $\mathbf{O} : G$. The rules of the game can now be described as follows.

($\mathbf{P}_\vee$) At game states of the form $\mathbf{P} : G_1 \vee G_2$, I choose between the game states $\mathbf{P} : G_1$ and $\mathbf{P} : G_2$ to continue the game.

($\mathbf{O}_\vee$) At $\mathbf{O} : G_1 \vee G_2$, *You* choose between $\mathbf{O} : G_1$ and $\mathbf{O} : G_2$.

($\mathbf{P}_\wedge$) At $\mathbf{P} : G_1 \wedge G_2$, *You* choose between $\mathbf{P} : G_1$ and $\mathbf{P} : G_2$.

($\mathbf{O}_\wedge$) At $\mathbf{O} : G_1 \wedge G_2$, I choose between $\mathbf{O} : G_1$ and $\mathbf{O} : G_2$.

($\mathbf{P}_\neg$) At $\mathbf{P} : \neg G$, the game continues with $\mathbf{O} : G$.

($\mathbf{O}_\neg$) At $\mathbf{O} : \neg G$, the game continues with $\mathbf{P} : G$.

($\mathbf{P}_p$) At $\mathbf{P} : p$, I win and *You* lose if $\mathcal{I} \models p$. Otherwise, I lose and *You* win.

($\mathbf{O}_p$) At $\mathbf{O} : p$, I win and *You* lose if $\mathcal{I} \not\models p$. Otherwise, I lose and *You* win.

The game starts at the formula F with me in the role $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ and over the interpretation $\mathcal{I}$ is denoted $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(F)$.

The vessel for lifting $\mathbf{G}^{\mathrm{CL}}$ to a provability game is the *disjunctive game*. Intuitively, the disjunctive game $\mathbf{DG}^{\mathrm{CL}}(\mathbf{P} : F)$ can be thought of as *Me* and *You* playing all semantic games $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$ over all interpretations $\mathcal{I}$ simultaneously. Note that the rules of this game do not depend on the structure of $\mathcal{I}$ but merely on F. Truth degrees are only needed at the atomic level to determine who wins the particular run of the game. This allows us to require players to play "blindly", i.e., without explicit reference to a model

$\mathcal{I}$. Clearly, if *I* have a winning strategy in such a game, then *I* can win in $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$, for every $\mathcal{I}$, making this strategy an adequate token of logical validity[4].

However, a problem arises already in the simplest cases. Let us consider the example of the disjunctive game starting at $\mathbf{P} : p \vee \neg p$. Clearly, *I* have a winning strategy in the semantic game over every interpretation. However, there is no uniform way of making a good choice in the first turn: No matter whether *I* choose $\mathbf{P} : p$ or $\mathbf{P} : \neg p$, there are still interpretations where *You* win the game eventually. To compensate for this, we allow *Myself* to create "backup copies" and *duplicate* game states. Instead of making a choice in the first turn, *I* duplicate the state and the game continues with the *disjunctive state*[5] $\mathbf{P} : p \vee \neg p \bigvee \mathbf{P} : p \vee \neg p$. Now *I* move to $\mathbf{P} : p$ in the first subgame and to $\mathbf{P} : \neg p$ in the second. Note that this means that in the disjunctive game, *I* additionally take the role of a scheduler, deciding which copy is to be played next. After a role switch in the second subgame, the final state is $\mathbf{P} : p \bigvee \mathbf{O} : p$.

The following winning condition reflects the fact that *My* strategy for the disjunctive game $\mathbf{DG}(\mathbf{P} : p \vee \neg p)$ was successful: *I* win and *You* lose at an elementary disjunctive state iff for every model there is one subgame, where *I* win the corresponding semantic game. Here are the rules of the disjunctive game in detail:

(Dupl) If no game states in D are underlined and D is not terminal, *I* can *duplicate* an $h \in D$ and the game continues with $D \bigvee h$.

(Sched) If no game states in $D = D' \bigvee h$ are underlined and D is not terminal, *I* can *underline* a non-terminal $h \in D$ and the game continues with $D' \bigvee \underline{h}$.

(Move) If $D = D' \bigvee \underline{h}$ then the player who is to move in the semantic game $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(g)$ at the game state g, makes a legal move to the game state g' and the game continues with $D \bigvee g'$. For example, if g is $\mathbf{P} : G_1 \wedge G_2$, then, according to $(\mathbf{P}_\wedge)$, *You* chose a $k \in \{1, 2\}$ and the game continues with $D \bigvee \mathbf{P}, i : G_k$.

(Win) If D consists of game states involving propositional variables only, then *I* win iff *I* win the semantic game in some $g \in D$, and *You* lose. Otherwise, *You* win, and *I* lose.

Note that the rule **(Move)** imports the rules of the semantic game into the disjunctive game. We point out that in the other chapters of this thesis, we have two kinds of disjunctive games: a semantic game $\mathbf{DG}^{\mathrm{CL}}_{\mathcal{I}}$ played over a fixed interpretation $\mathcal{I}$ (in this game, the disjunctive state is winning for *Me* iff some contained game state is winning

[4]The intended use for the disjunctive game might suggest that *conjunctive game* might be a more suitable name for this game. In choosing the name, we are inspired by [24], where the authors introduce *disjunctive strategies*. These strategies are not strategies of any particular game but can be rather seen as *disjunctions* of strategies of the semantic game. In our game-centered approach, we prefer to introduce a new game, where strategies exactly correspond to the disjunctive strategies from [24].

[5]Formally, disjunctive states are multisets of game states of the semantic game.

for *Me* over $\mathcal{I}$), and the provability game, as presented above. This distinction makes the involved proofs easier. For our purposes, it is sufficient to consider the disjunctive game as a provability game.

It can be shown that the disjunctive game is adequate: *I* have a winning strategy σ in $\mathbf{DG}^{\mathrm{CL}}(\mathbf{P} : F)$ iff *I* have winning strategies in all semantic games $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$. Moreover, this lifting answers all the questions we asked at the end of the last section positively. For every model, a winning strategy $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$ can be effectively extracted from σ. Conversely, a countermodel $\mathcal{I}$ and a winning strategy for *You* for $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$ can be computed from *Your* winning strategy for $\mathbf{DG}^{\mathrm{CL}}(\mathbf{P} : F)$.

Furthermore, every winning strategy for *Me* in $\mathbf{DG}^{\mathrm{CL}}$ can be translated into a proof in a weakening-free version of Gentzen's **LK** (for simplicity, we refer to this system as **LK**), as depicted in Table 1.1, and vice versa. To see this, note that there is a 1-1 correspondence between disjunctive states and sequents in **LK**. Every disjunctive state of the form

$$\mathbf{O} : F_1 \bigvee ... \bigvee \mathbf{O} : F_n \bigvee \mathbf{P} : G_1 \bigvee ... \bigvee \mathbf{P} : G_m$$

is directly translated to the sequent

$$F_1, ..., F_n \Rightarrow G_1, ..., G_m.$$

Furthermore, the rules of **LK** can be modeled by strategies for *Me* in the disjunctive game. The logical rules directly correspond to the rules of the underlying semantic game. For example, the rule $(L_\vee)$ corresponds to the rule $(\mathbf{O}_\vee)$, as branching in the proof tree codes branching in *My* strategy, which in turn represents *Your* choice in the semantic game. The rule $(R^1_\vee)$ represents *Me* choosing the left disjunct, while $(R^2_\vee)$ represents *Me* choosing the right disjunct in the game rule $(\mathbf{P}_\vee)$. The contraction rules (L_c) and (R_c) correspond to the duplication rule, **(Dupl)**. The rule **(Sched)** is coded explicitly, by the chosen rule of **LK**. Note that a disjunctive state consisting of game states involving propositional variables only is winning for *Me* iff some variable p appears with both the prefixes **P** and **O**. Hence, the axioms of **LK** exactly correspond to the the winning states of the disjunctive game. Note that these axioms are variants of the usual axioms in **LK** (of the forms $F \Rightarrow F$ and $\bot \Rightarrow$) and are broadly used in weakening-free systems, like **G3K** in [53]. Although we require that all formulas in the axiom sequents consist of propositional variables only, it can be easily shown that disjunctive states corresponding to sequents $\Gamma, p \Rightarrow p, \Delta$ can always be won by *Me*. On the proof-theoretic side, one can show that proofs in systems like **G3K** can always be transformed to make use of propositional axioms only.

Summing up, winning strategies can be seen as proofs in the system **LK**. Counter-model extraction from winning strategies for *You* corresponds to counter-model extraction from failed proof-search in **LK**.

Executing and modifying the described technique for different semantic games will be one of the aims of this thesis.

Table 1.1: Proof system **LK**.

Axioms

$\Gamma, p \Rightarrow p, \Delta$ where Γ, Δ consist of variables only

Structural Rules

$$\frac{\Gamma, F, F \Rightarrow \Delta}{\Gamma, F \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow F, F, \Delta}{\Gamma \Rightarrow \Delta}\ (R_c)$$

Propositional rules

$$\frac{\Gamma, F \Rightarrow \Delta \quad \Gamma, G \Rightarrow \Delta}{\Gamma, F \vee G \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow F, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ (R^1_\vee)$$

$$\frac{\Gamma, F \Rightarrow \Delta}{\Gamma, F \wedge G \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, G \Rightarrow \Delta}{\Gamma, F \wedge G \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta}\ (R_\wedge)$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\Gamma, \neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, F \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta}\ (R_\neg)$$

1.2 Aim of the thesis

We briefly discuss the main goals of this thesis. For a more detailed introduction to the different topics, we forward the reader to the introduction sections of the corresponding chapters. The aim of this thesis is twofold. First, we give a case study of the lifting technique. In Chapter 3, we study the application to modal logic. The challenge, in this case, is conceptual. In modal logic, the game trees of a modal extension of Hintikka's semantic game are not uniform for the underlying model. But this uniformity of game trees is essential for the application of the lifting since it allows the players to play over the model "blindly", as discussed in the previous section. The solution is to turn to hybrid logic, which allows us to refer to the model in the object language explicitly. This extension allows for uniform game trees but comes at the cost of infinite branching, even for finite models, making applying the lifting technique more challenging and forcing us to expand the technique.

In Chapter 4, we consider choice logic, a multi-valued extension of classical logic by preferences. Here, the richer domain of truth values is used to represent how well a given formula is satisfied with respect to the expressed preferences. Consequently, in the corresponding semantic game, the domain of payoff values is richer than the usual win/lose. The induced notion of validity is also many-valued, thus posing a new challenge for the lifting technique.

The second aim is to consolidate the experience from the different case studies in a general lifting framework. Developing such a general framework is the topic of Chapter 5. This framework encompasses all cases in this thesis and the literature. We will test this framework for the semantic game in [22].

1.3 Publications underlying this thesis

The focus of this thesis is in line with project P32684, funded by the Austrian Science Fund (FWF). The material of this thesis relies on the following publications:

- Chapters 1 and 2 contain no original material.
- Chapter 3 is based on the single-author publications [27, 26].
- Chapter 4 is based on joint work with Michael Bernreiter [29, 31, 30]. In this project, the author of this thesis was responsible for the game- and proof-theoretic results, which are also included in this thesis. Michael Bernreiter contributed results on complexity and his expertise in the area of preference handling in AI. He does not plan to use the material of the above publications in his thesis.
- Chapter 5 consists of unpublished material from the author's collected manuscripts inspired by project meetings with colleagues and his advisor. As an illustration of the general framework, we present an application of the lifting technique to the truth-degree-comparison game for Gödel logic, similar to the shared publication [48]. Since the emphasis and presentation are different, there is no conflict of interest with a chapter of Alexandra Pavlova's thesis building on the same publication.

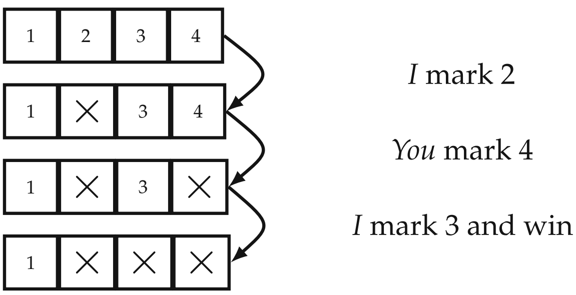
CHAPTER 2

Preliminaries

In this chapter, we provide the game-theoretic and logic background needed in this thesis. We assume that the reader has a strong background in logic, hence we lay our focus on the section on game theory. There is no original research in this chapter.

2.1 Games

In this section, we give a short introduction to some fundamental game-theoretic concepts. For additional material, we refer the reader to [46], or similar introductory books. Our running example in this section will be Hintikka's game, which was already discussed in Chapter 1, and n-Treblecross, a degenerate version of the game Tic-Tac-Toe. In n-Treblecross ($n \geq 3$), the two players, *I* and *You*, take turns marking boxes aligned in a row of length n. The game is slightly asymmetric, as *I* have to take the first turn. Unlike Tic-Tac-Toe, both players mark with the same symbol, an $\times$. The first player to cause three (or more) $\times$s in a row wins, and the other loses. A possible run of 4-Treblecross is depicted below:



Here, *I* start by marking Box 2. Then *You* mark Box 4, after which *I* mark Box 3, and win the game since now three consecutive boxes are marked. If *I* would have marked Box 1 instead, then *You* could have won the game by marking Box 3.

In the simple example of n-Treblecross, we can already identify the characteristics common to all games investigated in this thesis. Our games fall into the following categories:

- Two-player games: there are two players, which we will always call *Me* and *You*.
- Games of perfect information: at each stage, every player is informed of all events that have happened previously in the game.
- Games of complete information: there is common knowledge about the possible moves and payoffs of all players; in our case, who wins and who loses at which stages.
- Sequential games: there are no simultaneous moves of the players.
- Zero-sum games: players' interests are strictly opposed: if *I* win, then *You* lose and vice versa. In our case (in fact, in all games in this thesis), there are no draws.
- Determined games: exactly one of the two players has a winning strategy; we will formally prove this for Treblecross later.

The formal framework below captures all these characteristics. This framework will be used throughout the thesis. For the used notation for sequences, check the appendix.

Definition 2.1.1: Game

Let Stat be a set whose elements are called *(game) states*. A *game* over Stat is a tuple $(\mathrm{Hist}, h^*, \ell, \wp)$, where

- Hist is a subset of sequences over Stat, called *histories* [17] or *runs*.
- h^* is a distinguished element of Hist called the *initial history*. Equipped with $\sqsubseteq$, Hist is a partially ordered set with minimal element h^*. We require that Hist is downwards-closed[a] and every increasing chain in Hist has a supremum in Hist. Maximal elements of Hist are called *terminal histories*, *complete runs*, or *outcomes*. We let Ter be the set of terminal histories of Hist.
- ℓ is a function from non-terminal histories to the set $\{I, Y\}$, called the *labeling function*. If $\ell(h) = I$, we say that h is labeled I, or an I-history, and similarly for Y.
- The *payoff function* $\wp$ maps terminal histories to a linear order $(Z, \preceq)$. Elements of Z are called *values* and $\wp(h)$ is called the *payoff* of h.

[a] In our context, this means that for $h^* \sqsubseteq h_1 \sqsubseteq h_2$ and $h_2 \in H$, we have $h_1 \in H$.

In this thesis, Z is always partitioned into two sets, Z^+ and Z^-, where Z^+ is upwards-closed, Z^- is downwards-closed, and $a \prec b$ for all $a \in Z^-, b \in Z^+$. Z^+ is interpreted as a *winning range*, and payoffs in Z^+ are interpreted as winning for *Me*, while payoffs in the *losing range*, Z^-, are winning for *You*.

Example 2.1.2. The game Treblecross for a n-row, $\mathbf{T}_n$, can be formalized as follows. The initial history is the empty history ϵ. Game states are integers in $\{1, \dots, n\}$. The set of histories, Hist^n, contains ϵ and all sequences $h = \langle h_1, \dots, h_n \rangle$ of different integers in $\{1, \dots, n\}$ such that $\langle h_1, \dots, h_{n-1} \rangle$ does not contain the three consecutive integers $i, i+1, i+2$ for any i. In this thesis, we usually define the set of histories recursively. For $\mathbf{T}_n$, this definition would look as follows:

- The initial history is ϵ.
- Hist^n is the smallest set containing ϵ and closed under the following condition: if h is contained, then $h \smile i$ is contained if $i \notin \text{range}(h)$ and there is no j such that $\{j, j+1, j+2\} \subseteq \text{range}(h)$.
- Non-terminal histories of even length are labeled "I", non-terminal histories of odd length are labeled "Y".
- The payoff function takes values in the domain $Z = \{-1, 1\}$, where $-1 \triangleleft 1$. Terminal histories of even length have the value 1, and terminal histories of odd length have the value -1. Here, the winning range is $\{1\}$, and the losing range is $\{-1\}$.

Example 2.1.3. We have discussed Hintikka's game for classical propositional logic [39] in Chapter 1. We now show how it can be represented in our game format. Let $\mathcal{I}$ be an interpretation. Game states are of the form $\mathbf{Q} : F$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ and F is a propositional formula. Let g be a game state. The evaluation game $\mathbf{G}^{\text{CL}}_{\mathcal{I}}(g)$ is defined as follows:

- The initial history is $\langle g \rangle$.
- If $h = h' \smile \mathbf{Q} : G$ is contained, and G is of the form
  - $G = G_1 \vee G_2$, then also $h \smile \mathbf{Q} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = G_1 \wedge G_2$, then also $h \smile \mathbf{Q} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = G_1 \rightarrow G_2$, then also[1]. $h \smile \bar{\mathbf{Q}} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = \neg G'$, then also $h \smile \bar{\mathbf{Q}} : G'$ is contained
- Non-terminal histories ending in the states of the following forms are labeled:

[1] If $\mathbf{Q} = \mathbf{P}$, then $\bar{\mathbf{Q}} = \mathbf{O}$, and if $\mathbf{Q} = \mathbf{O}$, then $\bar{\mathbf{Q}} = \mathbf{P}$.

labeled "I"	labeled "Y"
$\mathbf{P} : G_1 \vee G_2$	$\mathbf{O} : G_1 \vee G_2$
$\mathbf{O} : G_1 \wedge G_2$	$\mathbf{P} : G_1 \wedge G_2$
$\mathbf{P} : G_1 \to G_2$	$\mathbf{O} : G_1 \to G_2$
$\mathbf{P} : \neg G'$	$\mathbf{O} : \neg G'$

- The payoff function takes values in the domain $Z = \{-1, 1\}$, where $-1 \lhd 1$. Terminal histories ending $\mathbf{P} : a$ have payoff 1 iff $\mathcal{I} \models a$ and -1, otherwise. Terminal histories ending $\mathbf{O} : a$ have payoff 1 iff $\mathcal{I} \not\models a$ and -1, otherwise. Here, the winning range is $\{1\}$ and the losing range is $\{-1\}$.

We will often use the following equivalent definition of a game. Since H is downwards closed, it naturally carries the structure of a tree rooted in the initial history sequence h^*. Therefore, we can think of a game as a tree T whose nodes are labeled "I" or "Y", telling us which of the two players is to move at every non-terminal history. Outcomes are associated with the leaves of the tree or infinite branches. As before, the payoff function $\wp$ maps terminal histories to a linear order.

Example 2.1.4. The game tree for 4-Treblecross is depicted in Figure 2.1. *I* move in all nodes with an even distance to the root $*$, and *You* move in all other nodes. *I* win and *You* lose in leaves with odd distance to the root, and vice versa in all other leaves.

2.1.1 Strategies

Another central notion is that of a *strategy*. We can think of a strategy for a player in a game as a complete game plan, describing a specific action for every possible non-terminal history in which that player is to move. In chess, a strategy for White is a mapping σ from odd-length non-terminal histories h to game states g. We require that

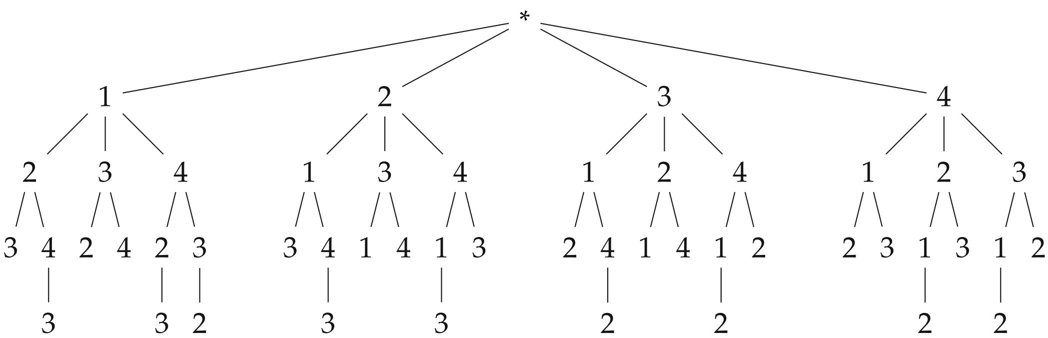


Figure 2.1: The game tree of 4-Treblecross

$\sigma(g)$ is obtained from h by a legal move by White, as given by the rules of chess. Here is the formal definition:

Definition 2.1.5: Strategy

Let H_I be the set of histories labeled "I". A *strategy* σ *for Me* for the game $\mathbf{G} = (H, h^*, \ell, \wp)$ is a function from H_I to H such that $\sigma(h) = h \smile g$, for some game state g. Σ_I denotes the set of strategies for *Me*. Similarly, we define H_Y, strategies for *You*, and Σ_Y.

In our tree view of a game, a strategy for *Me* can be represented as a function mapping each node labeled "I" to one of its children.

We often wish to investigate how a given strategy σ for *Me* performs against other possible strategies by *You*. In the most general setting, we are interested in the possible payoff values if *I* play according to σ. To answer this question, it is unnecessary to know which actions σ prescribes in all possible scenarios. For example, in a game of 4-Treblecross, if σ tells *Me* to mark Box 1 in the first move, it is irrelevant which moves σ prescribes in the history $\langle 2, 3\rangle$. We can thus interpret a strategy σ for *Me* as a subtree of the game tree satisfying the following conditions:

1. The root of T is in σ.
2. For every node of T that occurs in σ and is labeled "Y", all of its children are in σ.
3. For every node of T that occurs in σ and is labeled "I", precisely one of its children is in σ.

Alternatively, σ can be seen as a result of the following process: start with the game tree T and prune away all but one child of every node labeled "I".

Example 2.1.6. The tree in Figure 2.2 depicts a strategy for *Me* in 4-Treblecross.

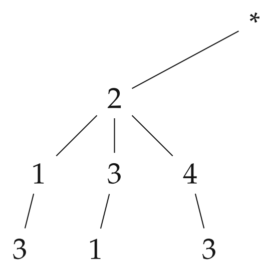


Figure 2.2: Winning strtegy for *Me* in 4-Treblecross

2.1.2 Outcomes

Let σ and μ be strategies for *Me* and *You* in a game $\mathbf{G}$, respectively. If *I* play according to σ and *You* play according to μ, then there is a unique predetermined terminal history that the game will end in. We will call this history $O(\sigma, \mu)$, the *outcome given by* σ *and* μ. We provide a formal definition of $O(\sigma, \mu)$ in terms of histories below. For now, let us take the instructive viewpoint on games as trees and strategies as pruning of the game tree, as described in the previous subsection. Then, the outcome given by σ and μ is a maximal path through the game tree T constructed as follows: first, prune T according to σ, leaving for every I-node all but one of its children. Then, do the same with μ, leaving for every Y-node exactly one child. The result is a unique maximal path through the game tree, $O(\sigma, \mu)$.

Example 2.1.7. Let σ be the strategy depicted in Figure 2.2 for *Me* in 4-Treblecross, and let μ be a strategy for *You* prescribing *You* to move to 4 at the history $h = \langle 2 \rangle$, i.e., $\mu(h) = h \smile 4$. To obtain $O(\sigma, \mu)$, we remove the 1- and 3-branches starting at 2 in σ. Hence, the resulting unique outcome is $\langle 2, 4, 3 \rangle$. This is precisely the run discussed on the first page of this chapter.

We now come to the formal definition of $O(\sigma, \mu)$ in terms of histories. As functions from non-terminal histories to game states, σ and μ have disjoint domains, H_I and H_Y. Let id_{Ter} be the identity function on Ter, the set of terminal histories, i.e., $\mathrm{id}_{Ter}(h) = h$. Therefore, the union $\sigma; \mu := \sigma \cup \mu \cup \mathrm{id}_{Ter}$ is a well-defined function from H to H given by

$$(\sigma; \mu)(h) = \begin{cases} \sigma(h), \text{ if } h \in H_I, \\ \mu(h), \text{ if } h \in H_Y, \\ h, \text{ if } h \in Ter \end{cases}$$

By definition of a strategy, both σ and μ are monotonically increasing functions on H (ordered by the natural ordering of sequences, $\sqsubseteq$) and so is $\sigma; \mu$. Since all chains in H have a supremum, the following notion is well-defined:

Definition 2.1.8: Outcome given by strategies

Let σ and μ be strategies for *Me* and *You* in a game $\mathbf{G} = (H, h^*, \ell, \wp)$. Then, the *outcome* given by σ and μ is defined[a] as

$$O(\sigma, \mu) = \sup\{(\sigma; \mu)^n(h^*) : n \in \mathbb{N}\},$$

where sup is with respect to the usual "is-an-initial-segment-of" ordering $\sqsubseteq$ on sequences.

[a] For a function $f : R \to R$, $f^n(x)$ is the result of applying f to x exactly n times.

We compute the outcome given by σ and μ from Example 2.1.7:

$$(\sigma; \mu)^0(\epsilon) = \epsilon$$

$$(\sigma;\mu)^1(\epsilon) = (\sigma,\mu)(\epsilon) = \sigma(\epsilon) = \langle 2\rangle$$
$$(\sigma;\mu)^2(\epsilon) = (\sigma;\mu)((\sigma,\mu)^1(\epsilon)) = \mu(\langle 2\rangle) = \langle 2,4\rangle$$
$$(\sigma;\mu)^3(\epsilon) = (\sigma,\mu)((\sigma;\mu)^2(\epsilon)) = \sigma(\langle 2,4\rangle) = \langle 2,4,3\rangle$$

Since the history $\langle 2,4,3\rangle$ is terminal, $(\sigma;\mu)^n(\epsilon) = \langle 2,4,3\rangle$, for all $n \geq 3$. Hence, $O(\sigma,\mu) = \langle 2,4,3\rangle$, which coincides with the outcome determined in the example.

2.1.3 Valuating Strategies and Games

As a first metric on how well strategies perform, we introduce the notion of a k-strategy. Intuitively, a k-strategy for *Me* in a game $\mathbf{G}$ ensures that playing according to σ, *My* payoff is at least k irrespective of *Your* actions. Formally, let $k \in Z$, the complete linear order of payoff values of the game $\mathbf{G}$. Then σ is a k-strategy for *Me* if $\inf_{\sigma_Y \in \Sigma_Y}(\wp(\sigma,\sigma_Y)) \succeq k$, where we abbreviate $\wp(O(\sigma_I,\sigma_Y))$ by $\wp(\sigma_I,\sigma_Y)$. Similarly, a strategy μ for *You* is a k-strategy if *My* payoff against μ is at most k, i.e., $\sup_{\sigma_I \in \Sigma_I}(\wp(\sigma_I,\mu)) \preceq k$.

In the tree representation of a strategy σ, the notion of a k-strategy is particularly intuitive: the subtree σ is a k-strategy iff all leaves have payoff $\succeq k$. For example, Figure 2.2 shows a strategy for *Me* in 4-Treblecross where all leaves have payoff 1. Hence, the depicted strategy is a 1-strategy, or, since it guarantees a win for *Me*, a *winning strategy*.

In the general setting, if Z is partitioned into a winning range Z^+ and a losing range Z^-, as described below Definition 2.1.1, then every strategy σ for *Me* with $\inf_{\sigma_Y \in \Sigma_Y}(\wp(\sigma,\sigma_Y)) \in Z^+$ is called a *winning strategy* for *Me*. Every strategy for *You* with $\sup_{\sigma_I \in \Sigma_I}(\wp(\sigma_I,\mu)) \in Z^-$ is called a winning strategy for *You*. The strategy in Figure 2.2 is a 1-strategy, and since 1 is in the winning range of this game, it is a winning strategy for *Me*. On the other hand, one can show that *You* have a winning strategy in 6-Treblecross.

Example 2.1.9. In Hintikka's game for classical propositional logic, *I* have a winning strategy in $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{P} : F)$ iff $\mathcal{I} \models F$, see [39].

Definition 2.1.10: Maximin and Minimax values

Given a game $\mathbf{G}$, with payoff function taking values in a complete linear order, the *maximin value* is defined as

$$v_I(\mathbf{G}) = \sup_{\sigma_I \in \Sigma_I} \inf_{\sigma_Y \in \Sigma_Y} \wp(\sigma_I,\sigma_Y).$$

Any strategy witnessing the supremum (i.e. any $v_I(\mathbf{G})$-strategy) is called a *maximin strategy*.
The *minimax value* is defined as

$$v_Y(\mathbf{G}) = \inf_{\sigma_Y \in \Sigma_Y} \sup_{\sigma_I \in \Sigma_I} \wp(\sigma_I,\sigma_Y).$$

Any strategy witnessing the infimum is called a *minimax strategy*. If $v_I(\mathbf{G}) = v_Y(\mathbf{G})$ we call this value the *value of* $\mathbf{G}$ and denote it by $v(\mathbf{G})$. If the value of $\mathbf{G}$ exists, we call this game *determined*.

Since *I* have a winning strategy in 4-Treblecross, the value of this game is 1. Similarly, the value of 6-Treblecross is -1, since it can be shown that *You* have a winning strategy. Note that, in general, not every game is determined. For example, there are games, where neither *I* nor *You* have winning strategies. The important class of finite-valued games, however, is determined. A game is called *finite-valued* if the pointwise image of Ter under $\wp$ is a finite set, i.e., $|\wp(Ter)| < \infty$. For example, n-Treblecross and every finite game (games where the set of histories is finite) is finite-valued.

Theorem 2.1.11: Finite-valued games are determined

Finite-valued games are determined. Moreover, maximin and minimax strategies exist in these games.

Proof. If $\mathbf{G}$ is finite-valued, then the infima and suprema in the definitions of v_I and v_Y are over finite sets and thus witnessed. Let σ_I^* and σ_Y^* be strategies such that $v_I = \inf_{\sigma_Y \in \Sigma_Y} \wp(\sigma_I^*, \sigma_Y)$ and $v_Y = \sup_{\sigma_I \in \Sigma_I} \wp(\sigma_I, \sigma_Y^*)$. Assume, towards a contradiction that $v_Y \prec v_I$. But then

$$\wp(\sigma_I^*, \sigma_Y^*) \preceq \sup_{\sigma_I \in \Sigma_I} \wp(\sigma_I, \sigma_Y^*) = v_Y \prec v_I = \inf_{\sigma_Y \in \Sigma_Y} \wp(\sigma_I^*, \sigma_Y) \preceq \wp(\sigma_I^*, \sigma_Y^*),$$

which is impossible. We thus have $v_I \preceq v_Y$. The converse inequality holds for all games in general, see [46]. □

As an application of this theorem, we get that for every $n \geq 3$, the finite-valued game n-Treblecross is determined, i.e., exactly one of the players has a winning strategy. We conclude this subsection with the following notion:

Definition 2.1.12: Strategic equivalence of games

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ where the payoff functions take values in the same domain Z. We call the games *strategically equivalent* and write $\mathbf{G}_1 \cong \mathbf{G}_2$ iff they have the same value.

Note that this notion is very general. Two games can be strategically equivalent even if they do not share any of their structure. All that is required is that they have the same game value. For example, *I* have 1-strategies in the games 3-Treblecross, 4-Treblecross, and $\mathrm{GCL}_{\{b\}}(F)$ with $F = ((a \vee b) \vee c) \wedge \neg(a \vee d)$, making them strategically equivalent.

2.1.4 Subgames

In this section, we recall the important notion of subgames and derive a useful lemma that will be useful for later proofs.

Definition 2.1.13: Subgame

Let $\mathbf{G} = (H, h^*, \ell, \wp)$ be a game and h a history. Then the game $\mathbf{G}@h = (H_h, h, \ell_h, \wp_h)$, called the *subgame of* $\mathbf{G}$ *starting at* h, is defined as follows:

- The initial history is h.
- H_h is the set of histories in H extending h.
- ℓ_h is ℓ restricted to H_h
- $\wp_h$ is $\wp$ restricted to the terminal histories in H_h.

$\mathbf{G}'$ is an *immediate subgame* of $\mathbf{G}$ if $\mathbf{G}' = \mathbf{G}@(h^* \smile g)$ and $h^* \smile g \in H$.

We easily see that $\mathbf{G}_h$ is again a game in terms of Definition 2.1.1. To motivate the following lemma, let us say in a game $\mathbf{G}$, it is *My* turn at the initial history h^* , and let us suppose *I* have a strategy σ guaranteeing a payoff of at least z in one of the immediate subgames, $\mathbf{G}'$, of $\mathbf{G}$. Then *I* have a z-strategy in $\mathbf{G}$, too: simply go to $\mathbf{G}'$ in the first turn and then play according to σ. Similarly, if initially it is *Your* turn and *I* have z-strategies σ_i in all immediate subgames $\mathbf{G}_i$ of $\mathbf{G}$, then *I* have a z-strategy for $\mathbf{G}$. If *You* go to $\mathbf{G}_i$ in the first turn, then *I* play according to σ_i and the outcome will have a payoff of at least z. Let us prove these observations formally.

Lemma 2.1.14: Subgame-to-game

Let $\mathbf{G}$ be a game with initial history h^* and let $h_i = h^* \smile g_i$ such that $(h_i)_{i\in I}$ are all the minimal histories extending h^*.

1. If $\ell(h^*) = I$, then *I* have a z-strategy in $\mathbf{G}$ iff there is some j such that *I* have a z-strategy in $\mathbf{G}@h_j$.
2. If $\ell(h^*) = Y$, then *I* have a z-strategy in $\mathbf{G}$ iff for every i, *I* have a z-strategy in $\mathbf{G}@h_i$.

A symmetric result holds for *Your* strategies.

Proof. Note that the histories H in $\mathbf{G}$ are precisely the histories in $\bigcup_{i\in I} H_{h_i}$ plus the initial history h^*.

For 1, let σ be *My* z-strategy σ in $\mathbf{G}$. Let $\sigma(h^*) = h_j$, let μ_j be a strategy for *You* in $\mathbf{G}@h_j$ and let μ be an arbitrary strategy agreeing with μ on $H^Y_{h_j}$. Let σ_j be σ restricted to H_{h_j}.

We have

$$\begin{aligned}
\wp_{h_j}(\sigma_j, \mu_j) &= \wp_{h_j}(O(\sigma_j, \mu_j)) \\
&= \wp_{h_j}(\sup\{(\sigma_j; \mu_j)^n(h_i) : n \in \mathbb{N}\}) \\
&= \wp(\sup\{(\sigma; \mu)^n(h^*) : n \in \mathbb{N}\}) \\
&= \wp(\sigma, \mu) \succeq z,
\end{aligned}$$

where the third equality follows from the fact $(\sigma; \mu)(h^*) = \sigma(h^*) = h_j$ and the facts that σ agrees with σ_j on $H^I_{h_j}$ and μ agrees with μ_j on $H^Y_{h_j}$. The inequality follows from the assumption on σ. Since *Your* strategy μ_j was arbitrary, we get $\inf_{\mu \in \Sigma_Y} \wp_{h_j}(\sigma_j, \mu_j) \succeq z$, i.e., σ_j is a z-strategy for *Me*.

For the other direction, suppose *I* have a z-strategy σ_j in $\mathbf{G}@h_j$. Let σ be a strategy for *Me* in $\mathbf{G}$ such that $\sigma(h^*) = h_j$, σ agrees with σ_j on $H^I_{h_j}$ and is arbitrary everywhere else. Let μ be a strategy for *You* in $\mathbf{G}$ and let μ_j be its restriction to $H^Y_{h_j}$. Similar to before, we have

$$\begin{aligned}
\wp(\sigma, \mu) &= \wp(O(\sigma, \mu)) \\
&= \wp(\sup\{(\sigma; \mu)^n(h^*) : n \in \mathbb{N}\}) \\
&= \wp_{h_j}(\sup\{(\sigma_j; \mu_j)^n(h_j) : n \in \mathbb{N}\}) \\
&= \wp_{h_j}(\sigma_j, \mu_j) \succeq z.
\end{aligned}$$

For 2, let σ be *My* z-strategy σ in $\mathbf{G}$ and fix the subgame h_j. Let σ_j be σ restricted to $H^I_{h_j}$. Let μ_j be a strategy for *You* in $\mathbf{G}@h_j$ and define μ such that $\mu(h^*) = h_j$, μ agrees with μ_j on $H^Y_{h_j}$ and is arbitrary everywhere else. Then

$$\begin{aligned}
\wp_{h_j}(\sigma_j, \mu_j) &= \wp_{h_j}(O(\sigma', \mu)) \\
&= \wp_{h_j}(\sup\{(\sigma'; \mu)^n(h_i) : n \in \mathbb{N}\}) \\
&= \wp(\sup\{(\sigma; \mu)^n(h^*) : n \in \mathbb{N}\}) \\
&= \wp(\sigma, \mu) \succeq z,
\end{aligned}$$

For the other direction, suppose *I* have a z-strategy σ_i for each $\mathbf{G}@h_i$. We set $\sigma = \bigcup_{i \in I} \sigma_i$, i.e., σ agrees with every σ_i on $H^I_{h_i}$. To show that σ is a z-strategy, note that all strategies for *You* in $\mathbf{G}$ can be written as $(h^*, h_j) \cup \bigcup_{i \in I} \mu_i$, where each μ_i is a strategy for *You* in $\mathbf{G}@h_i$ and h_j is *Your* choice at the initial history. Let μ be such a strategy. We have

$$\begin{aligned}
\wp(\sigma, \mu) &= \wp(O(\sigma, \mu)) \\
&= \wp(\sup\{(\sigma, \mu)^n(h^*) : n \in \mathbb{N}\}) \\
&= \wp_{h_j}(\sup\{(\sigma_i, \mu_i)^n(h_i) : n \in \mathbb{N}\}) \\
&= \wp_{h_j}(\sigma_j, \mu) \succeq z,
\end{aligned}$$

where we used $(\sigma; \mu)(h^*) = \mu(h^*) = h_i$ and the facts that σ agrees with σ_j on $H^I_{h_j}$ and μ agrees with μ_j on $H^Y_{h_j}$. The inequality follows from the assumption that σ_j is a z-strategy. Since μ was arbitrary, we conclude that σ is a z-strategy. $\square$

We point out that for finite games, this lemma justifies the backward-induction technique for computing a game's value. This technique works as follows: start with the tree representation of a finite game $\mathbf{G}$ and label every leaf with its corresponding payoff. Then, move to the nodes that have labeled successors. For each I-node, label this node with the maximum of all of its connected leaves. For each Y-node, label it with the minimum. Move one level higher and repeat the procedure. When the root is labeled, the lemma guarantees that this label coincides with the value of the game.

2.2 Logics

We assume that the reader has a firm background in logic and, therefore, only point out some conceptual differences between the standard logic literature and this thesis.

Most notably, we adopt multi-valued versions of the central logical concepts of truth, validity, proof, and provability. We acknowledge that this approach has interesting and far-reaching consequences in philosophy and logic. However, this discussion lies beyond the scope of this thesis. Instead, we merely point out that this choice is a natural consequence of the intuitive ideas underlying the games presented in Chapters 4 and Chapters 5. For a detailed account, we refer the interested reader to [36, 37].

This multi-valued approach to logic stems from a degree-based notion of truth. A degree-based semantics for a logic L is a a class of models, $\mathfrak{I}$, together with a mapping where each $\mathcal{I} \in \mathfrak{I}$ is assigned a degree-function $d_{\mathcal{I}}$ mapping to a linear order $(Z, \preceq)$. The value $d_{\mathcal{I}}(F)$ is interpreted as the *truth degree* of the formula F under the model $\mathcal{I}$. This induces a degree-based notion of validity: the *validity degree* of a formula F is defined as $d(F) = \inf_{\mathcal{I} \in \mathrm{I}} d_{\mathcal{I}}(F)$, where we assume that this infimum always exists.

We will be concerned with logic games, i.e., games capturing the above degrees. On the semantic side, a *semantic game* $\mathbf{G}^{\mathrm{L}}$ for a logic L is a collection of games where every pair of formula F and model $\mathcal{I}$ get assigned a game $\mathbf{G}^{\mathrm{L}}_{\mathcal{I}}(F)$. Then $\mathbf{G}^{\mathrm{L}}$ is called *adequate* (with respect to L) if the value of $\mathbf{G}^{\mathrm{L}}_{\mathcal{I}}(F)$ (see Definition 2.1.10) is $d_{\mathcal{I}}(F)$ for each $\mathcal{I} \in \mathfrak{I}$, $F \in \mathcal{L}$. For example, Hintikka's game is adequate with respect to classical logic, as discussed in Examples 2.1.3 and 2.1.9.

On the validity side, a *provability game* $\mathbf{DG}^{\mathrm{L}}$ for a logic L is a collection of games where every formula F gets assigned a game $\mathbf{DG}^{\mathrm{L}}(F)$. Then $\mathbf{DG}^{\mathrm{L}}$ is called *adequate* if the value of $\mathbf{DG}^{\mathrm{L}}_{\mathcal{I}}(F)$ is $d(F)$, for each $F \in \mathcal{L}$. An example of an adequate game for classical logic is a version of Lorenzen's dialogue game [43, 21, **?**].

One of the main themes of this thesis is the lifting of adequate semantic games to adequate provability games.

CHAPTER 3

Hybrid Logic

3.1 Introduction

In this chapter, we will apply the lifting technique to modal logic. Modal logic is an extension of the language of propositional logic by the modality $\Box$. This simple yet expressive language is used as a tool for formalizing different areas of logic investigation: epistemology ($\Box F$ is read as "(the agent) knows that F"), deontic logic ($\Box F$ stands for "it is obligatory that F"), temporal logics ($\Box F$ means "F always holds"), and many others.

The usual semantics for modal logics are relational structures. A model is a tuple $\mathcal{M} = (\mathsf{W}, \mathsf{R}, \mathsf{V})$, where W is a set, whose elements are called worlds. R is an accessibility relation between worlds and V maps every propositional variable to the set of worlds where p is true. Consequently, formulas are evaluated at a world w in a model $\mathcal{M}$. Depending on the area of investigation, worlds and the accessibility relation have different interpretations in terms of knowledge states and epistemological indistinguishability, deontic states and the betterness-relations, points in time and the is-in-the-future-relation, etc.

There is a straightforward extension of Hintikka's game to modal logic. In addition to the current roles of the players, the current formula F, we must also keep track of the current world w in the model. In the world w, the rules for the connectives $\wedge, \vee, \rightarrow, \neg$ are as usual. For example, if F is of the form $F_1 \vee F_2$, then the proponent chooses F_i and the game continues with F_i in the same world w. The extension concerns the treatment of the modal operator: if $F = \Box G$, then the opponent chooses a world v that is R-accessible from w and the game continues with G at v. If there is no R-accessible world, the proponent wins and the opponent loses. The game ends if a propositional variable p is reached at a world u. The proponent wins and the opponent loses if p is true at u, otherwise, the opponent wins and the proponent loses.

We see from the informal description of this semantic game that its game tree depends not only on the syntax of the involved formula but also on the relational structure of

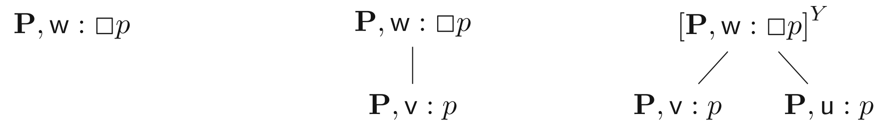


Figure 3.1: Three game trees for $\Box p$.

the model. In Figure 3.1 we see three game trees over different models, but for the same situation where *I* am the proponent of the formula $\Box p$ at the world w. In the underlying model of the first game tree, there are no R-accessible worlds and *I* win the game immediately. In the second model, w has one R-accessible world, v, and *I* must be prepared to defend p at v. In the third model, w has two R-successors, v and u, and *You* may choose between them.

This new behavior is in stark contrast to the evaluation game for propositional logic where semantic information is needed only in the final stage of the game to decide who wins. This poses a conceptual problem for the lifting technique as described in Chapter 1. Both our intuitions behind the resulting provability game – a simultaneous play over all models, and a play of the evaluation game without explicit reference to a particular model – require that the game tree of the evaluation game is uniform over all models.

We overcome this problem by turning to *hybrid logic* – an extension of modal logic that allows for explicit reference to worlds and the accessibility relation within the object language. The language contains an infinite collection of *nominals* $i, j, \ldots$ and a binary relation symbol R. Nominals are used as names for worlds of the model, while the formula $R(i, j)$ is read as "the world with name j is R-accessible from the world with name j".

It is worth noting that hybrid logic substantially increases the expressivity of the language. It enables us to grasp many frame properties that are provably not expressible in "orthodox" modal logic while keeping the same computational complexity[1] [11]. Apart from this, using nominals can be an advantage for modeling in temporal logic [8] as well as in making the link between modal logic and description logics more explicit [12].

What is important for us here is that by using nominals, the rule for $\Box$ in Hintikka's game can be reformulated in the following way: in the world with name i, if the current formula is $\Box G$, then the opponent *chooses a nominal* j and the game continues at the world with name j and with the formula $R(i, j) \to G$. Note that the proponent can immediately win the game over this formula if j is not R-accessible from i in the model: according to the rule for "$\to$", they can choose to continue the game with $R(i, j)$ and a role switch. As the new opponent, they win the false atomic formula $R(i, j)$. The important advantage of the new rule is that the game tree is now independent of the accessibility relation R of the underlying model, allowing us to apply the lifting technique.

[1] That is true in the basic hybrid logic. The logic containing quantification over nominal is known to be undecidable.

The coherence of hybrid logic with games was first demonstrated by Patrick Blackburn, who designed a Lorenzen-style [43] dialogue game for hybrid logic [10]. Sara Negri presented the labeled proof-system **G3K** that resembles the semantics of modal logic [47]. Our approach combines the best of two worlds: it supplements the clear semantic motivation of **G3K** with an accessible game-theoretic viewpoint of Blackburn's dialogue game. Similar to **G3K**, a failed search for a winning strategy in the disjunctive game directly gives rise to a countermodel. Although the number of players' possible choices may be infinite in the disjunctive game, we show that winning strategies can be represented as finitely branching trees. This representation allows us to interpret winning strategies as derivations in a proof calculus, similar to other sequent systems for hybrid logic [9, 13].

This chapter is structured as follows: Section 3.2 is a recap of hybrid logic. In Section 3.3, we present the evaluation game and recall the central notion of a strategy. The disjunctive game over a model is introduced in Section 3.4 and the general disjunctive game in Section 3.5. This section also contains the main result. In Section 3.6, we use this result to directly derive games that adequately model validity and entailment. Finally, we show how winning strategies can be finitized and formulated as proofs in a sequent calculus presented in Section 3.7.

3.2 Basic notions in Hybrid Logic

In this section, we recall some basic notions and results in hybrid logic. For more information, the reader may wish to consult books like [11].

The language of hybrid modal logic is as follows: We start from two disjoint, countably infinite sets N (set of nominals) and P (set of propositional variables). Nominals are usually called "i, j, k, ..." propositional variables are called "p, q, ...". Formulas F are built according to the following grammar:

$$F ::= \bot \mid p \mid i \mid R(i,j) \mid F \wedge F \mid F \vee F \mid F \to F \mid \neg F \mid @_i F \mid \forall i.F \mid \Box F \mid \Diamond F$$

Formulas of the form $\bot$, p, i, and $R(i,j)$ are called *elementary*. Intuitively, we can think of the nominal i as the name of a particular world in a model. Hence, i is true in exactly one world. The formula $@_i F$ stands for the fact that F is true in the world with the name i. The relational claim $R(i,j)$ says that the world with name j is accessible from the world with name i. Though usually defined as $@_i \Diamond j$, we include it as an elementary formula to prevent circular definitions in the description of the game rules in the following sections. $\forall i$ allows universal quantification over nominals. As is common practice, we require that in the formula $\forall i.F$, the nominal i does not appear bound by another quantifier in F. In other words, there is no occurrence of $\forall i$ in F. Using this restriction, we can simply define substitution $F[i/j]$ as the result of replacing every occurrence of i in F by j.

We now define the semantics formally: A model $\mathcal{M}$ for hybrid modal logic is a tuple $(\mathsf{W}, \mathsf{R}, \mathsf{V}, \mathsf{g})$, where

1. W is a non-empty set. Its elements are called *worlds*.
2. $R \subseteq W \times W$ is called *accessibility relation*. As usual, we write wRv instead of $(w, v) \in R$. The set of *accessible worlds from* w is $wR := \{v \in W : wRv\}$.
3. $V : P \to \mathcal{P}(W)$[2] is called *valuation function*.
4. $g : N \to W$ is called *assignment*. If $g(i) = w$, we say that i is a *name* of w, or simply that w has a name. If g is surjective, i.e. every world has a name, we call $\mathcal{M}$ *named*.

The pair (W, R) is called a *frame* and $\mathcal{M}$ is said to be *based on* this frame. Truth of formulas in the world w of W is defined recursively:

$\mathcal{M}, w \models \bot$ never,
$\mathcal{M}, w \models p$, iff $w \in V(p)$,
$\mathcal{M}, w \models i$, iff $g(i) = w$,
$\mathcal{M}, w \models R(i, j)$, iff $g(i) R g(j)$,
$\mathcal{M}, w \models F \wedge G$, iff $\mathcal{M}, w \models F$ and $\mathcal{M}, w \models G$,
$\mathcal{M}, w \models F \vee G$, iff $\mathcal{M}, w \models F$ or $\mathcal{M}, w \models G$,
$\mathcal{M}, w \models F \to G$, iff $\mathcal{M}, w \not\models F$ or[3] $\mathcal{M}, w \models G$,
$\mathcal{M}, w \models \neg F$, iff $\mathcal{M}, w \not\models F$,
$\mathcal{M}, w \models @_i F$, iff $\mathcal{M}, g(i) \models F$,
$\mathcal{M}, w \models \forall i.F$, iff for all[4] $j \in N$, $\mathcal{M}, w \models F[i/j]$,
$\mathcal{M}, w \models \Box F$, iff for all[5] $v \in W$, $(w, v) \notin R$ or $\mathcal{M}, v \models F$,
$\mathcal{M}, w \models \Diamond F$, iff for some $v \in W$, wRv and $\mathcal{M}, v \models F$.

Example 3.2.1. We invite the reader to check that $\mathcal{M}, w_1 \models \Box(j \vee \neg\Box p)$, where $\mathcal{M}$ is as depicted in Figure 3.2

We say that a formula F is true in the model $\mathcal{M}$ and write $\mathcal{M} \models F$ iff $\mathcal{M}, w \models F$ for every world w. For a class of models $\mathfrak{M}$, we write $\models_{\mathfrak{M}} F$ and say that F is *valid over* $\mathfrak{M}$ iff for all $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M} \models F$. We say that F is *valid* (we write $\models F$) iff F is valid over the class of all models. For a set of formulas $\mathcal{T}$, we write $\mathcal{M}, w \models \mathcal{T}$ iff $\mathcal{M}, w \models T$ for every $T \in \mathcal{T}$. We say that $\mathcal{T}$ *(locally) entails* F and write $\mathcal{T} \models F$ iff $\mathcal{M}, w \models \mathcal{T}$ implies $\mathcal{M}, w \models F$, for every model $\mathcal{M}$ and world w. Similarly, we define $\mathcal{T} \models_{\mathfrak{M}} F$. We say that

[2] $\mathcal{P}(W)$ denotes the power set of W.
[3] This is equivalent to the usual "if $\mathcal{M}, w \models F$, then $\mathcal{M}, w \models G$".
[4] This is different from the definition in the literature but equivalent for the class of named models, which will be used for the largest part of this chapter. We chose this formulation since it fits our game format better.
[5] This is equivalent to the usual "For all $v \in W$, if wRv, then $\mathcal{M}, v \models F$". Our formulation is more easily representable as a game rule.

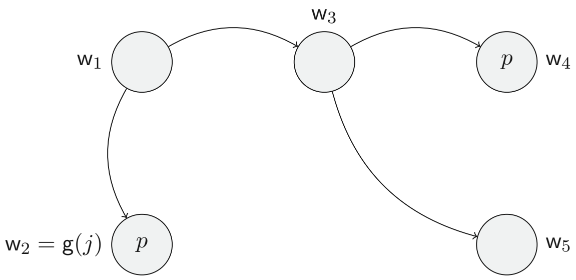


Figure 3.2: Model with worlds $\mathsf{w}_1, ..., \mathsf{w}_5$. Arrows represent the accessibility relation. We write p inside the circle representing a world w iff $\mathsf{w} \in \mathsf{V}(p)$

$\mathfrak{M}$ is *characterized* by a set of formulas $\mathcal{T}$ iff $\mathcal{M} \in \mathfrak{M} \Leftrightarrow \mathcal{M} \models \mathcal{T}$, for every model $\mathcal{M}$. If such a $\mathcal{T}$ exists, we say that $\mathfrak{M}$ is *characterizable*.

For a frame (W, R), we say that G is *valid* over (W, R) and write $(\mathsf{W}, \mathsf{R}) \models G$ iff $\mathcal{M} \models G$, for every model $\mathcal{M}$ based on (W, R). A set of formulas $\mathcal{T}$ is *valid* over (W, R) iff $(\mathsf{W}, \mathsf{R}) \models T$ for every $T \in \mathcal{T}$. A class of frames $\mathfrak{F}$ is *characterized* by a set of formulas $\mathcal{F}$ iff $(\mathsf{W}, \mathsf{R}) \in \mathfrak{F} \Leftrightarrow (\mathsf{W}, \mathsf{R}) \models \mathcal{F}$. In this paper, we will be interested in finite sets $\mathcal{F}$ consisting of *pure* formulas, i.e., formulas not containing propositional variables where every nominal is bound. In this case, one can show that $\mathcal{M} \models \mathcal{F}$ iff $\mathcal{M}$ is based on a frame from $\mathfrak{F}$. Therefore, we will from now on assume that *characterizable* for a class of frames $\mathfrak{F}$ means characterizable by such an $\mathcal{F}$, and conveniently identify $\mathfrak{F}$ with the class of all models based on a frame from $\mathfrak{F}$ (and use the $\models_{\mathfrak{F}}$-notation accordingly).

For example, the class of irreflexive models is characterized by the formula $\forall i.\neg R(i, i)$. The class of frames with two worlds is characterized by $\neg\forall i.\forall j.\neg\forall k.(k = i \vee k = j)$. We point out that the expression of frame properties using the $\forall$-quantifier is an elegant way of reducing questions of validity over classes of frames to questions of entailment. However, including this quantifier in our language comes at a cost, as the validity problem becomes undecidable in general. However, if we restrict the use of $\forall$ to instances of many frame properties, the satisfiability problem is indeed decidable. We thus get both the mentioned reduction and decidability.

The most important class for what follows is the class of named models $\mathfrak{N}$. The accessibility relation in such named models is completely determined by the truth values of the relational formulas. For these models, we can thus state the following semantic facts about the modal operators without explicitly referring to the semantics of the accessibility relation. Let $\mathsf{w} = \mathsf{g}(i)$, then

$$\mathcal{M}, \mathsf{w} \models \Box F, \text{ iff for all } j \in N, \mathcal{M}, \mathsf{g}(j) \models R(i, j) \rightarrow F,$$
$$\mathcal{M}, \mathsf{w} \models \Diamond F, \text{ iff for some } j \in N, \mathcal{M}, \mathsf{g}(j) \models R(i, j) \wedge F.$$
$$\mathcal{M} \models F, \text{ iff for all } i \in N, \mathcal{M}, \mathsf{g}(i) \models F.$$

As noted in the introduction to this chapter, this semantic fact will play an important role in defining the evaluation game in such a way that the lifting technique can be applied. Furthermore, we will make use of the fact that questions of validity and entailment can be reduced to named models:

Lemma 3.2.2: Lemma 2.7 in [13]

Let $\mathcal{T}$ be a finite set of formulas and $\mathfrak{F}$ a characterizable class of frames. Then for all F,

$$\mathcal{T} \models_{\mathfrak{F}} F \iff \mathcal{T} \models_{\mathfrak{F}\cap\mathfrak{N}} F.$$

3.3 A Game for Truth

We are now ready to introduce the evaluation game for hybrid logic. We give both an intuitive and formal definition and recall the notion of a (winning) strategy in Subsection 3.3.1. The adequacy of the game is proved in Subsection 3.3.2, and Subsection 3.3.3 shows how to internalize nominals into the game.

As in the case for propositional logic, the *evaluation game* for hybrid logic is played over a model $\mathcal{M} = (\mathsf{W}, \mathsf{R}, \mathsf{V}, \mathsf{g})$ by two players, *Me* and *You*, who argue about the truth of a formula F at a world w. At each stage of the game, one player acts as the proponent, while the other acts as the opponent of the claim that a formula F is true at the world w. We represent the situation where *I* am the proponent (and *You* are the opponent) by the *game state* $\mathbf{P}, \mathsf{w} : F$, and the situation where *I* am the opponent (and *You* are the proponent) by $\mathbf{O}, \mathsf{w} : F$. We add another kind of game state of the form $\mathbf{P} : F$ or $\mathbf{O} : F$, representing the claim that F is true in the whole model. We call a game state *elementary* if its involved formula is elementary. To get a firm understanding of the evaluation game, we start with a semi-formal description. Let g be a game state. The game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ starts at the game state g and proceeds according to the following rules[6]:

($\mathbf{P}_\vee$) At game states of the form $\mathbf{P}, \mathsf{w} : G_1 \vee G_2$, *I* choose between the game states $\mathbf{P}, \mathsf{w} : G_1$ and $\mathbf{P}, \mathsf{w} : G_2$ to continue the game.

($\mathbf{O}_\vee$) At $\mathbf{O}, \mathsf{w} : G_1 \vee G_2$, *You* choose between $\mathbf{O}, \mathsf{w} : G_1$ and $\mathbf{O}, \mathsf{w} : G_2$.

($\mathbf{P}_\wedge$) At $\mathbf{P}, \mathsf{w} : G_1 \wedge G_2$, *You* choose between $\mathbf{P}, \mathsf{w} : G_1$ or with $\mathbf{P}, \mathsf{w} : G_2$.

($\mathbf{O}_\wedge$) At $\mathbf{O}, \mathsf{w} : G_1 \wedge G_2$, *I* choose between $\mathbf{O}, \mathsf{w} : G_1$ and $\mathbf{O}, \mathsf{w} : G_2$.

($\mathbf{P}_\to$) At $\mathbf{P}, \mathsf{w} : G_1 \to G_2$, *I* choose between $\mathbf{O}, \mathsf{w} : G_1$ and $\mathbf{P}, \mathsf{w} : G_2$.

($\mathbf{O}_\to$) At $\mathbf{O}, \mathsf{w} : G_1 \to G_2$, *You* choose between $\mathbf{P}, \mathsf{w} : G_1$ and $\mathbf{O}, \mathsf{w} : G_2$.

($\mathbf{P}_\neg$) At $\mathbf{P}, \mathsf{w} : \neg G$, the game continues with $\mathbf{O}, \mathsf{w} : G$.

($\mathbf{O}_\neg$) At $\mathbf{O}, \mathsf{w} : \neg G$, the game continues with $\mathbf{P}, \mathsf{w} : G$.

[6] In Section 3.3.3, the rules for $\Box$ and $\Diamond$ are altered. The version from that section will be used for the largest part of the chapter.

($\mathbf{P}_@$) At $\mathbf{P}, \mathsf{w} : @_i G$, the game continues with $\mathbf{O}, \mathsf{g}(i) : G$.

($\mathbf{O}_@$) At $\mathbf{O}, \mathsf{w} : @_i G$, the game continues with $\mathbf{O}, \mathsf{g}(i) : G$.

($\mathbf{P}_\forall$) At $\mathbf{P}, \mathsf{w} : \forall i.G$, *You* choose a nominal j, and the game continues with $\mathbf{P}, \mathsf{w} : G[i/j]$.

($\mathbf{O}_\forall$) At $\mathbf{O}, \mathsf{w} : \forall i.G$, *I* choose a nominal j, and the game continues with $\mathbf{O}, \mathsf{w} : G[i/j]$.

($\mathbf{P}_\Box$) At $\mathbf{P}, \mathsf{w} : \Box G$, *I* win if $\mathsf{wR} = \emptyset$. Otherwise, *You* choose an R-successor v and the game continues with $\mathbf{P}, \mathsf{v} : G$.

($\mathbf{O}_\Box$) At $\mathbf{O}, \mathsf{w} : \Box G$, *You* win if $\mathsf{wR} = \emptyset$. Otherwise, *I* choose an R-successor v and the game continues with $\mathbf{P}, \mathsf{v} : G$.

($\mathbf{P}_\Diamond$) At $\mathbf{P}, \mathsf{w} : \Diamond G$, *You* win if $\mathsf{wR} = \emptyset$. Otherwise, *I* choose an R-successor v and the game continues with $\mathbf{P}, \mathsf{v} : G$.

($\mathbf{O}_\Diamond$) At $\mathbf{O}, \mathsf{w} : \Diamond G$, *I* win if $\mathsf{wR} = \emptyset$. Otherwise, *You* choose an R-successor v and the game continues with $\mathbf{P}, \mathsf{v} : G$.

($\mathbf{P}_U$) At $\mathbf{P} : G$, *You* choose a world w and the game continues with $\mathbf{P}, \mathsf{w} : G$.

($\mathbf{O}_U$) At $\mathbf{O} : G$, *I* choose a world w and the game continues with $\mathbf{O}, \mathsf{w} : G$.

($\mathbf{P}_{el}$) Let el be an elementary formula. *I* win and *You* lose at $\mathbf{P}, \mathsf{w} : el$ iff $\mathcal{M}, \mathsf{w} \models el$. Otherwise, *You* win and *I* lose.

($\mathbf{O}_{el}$) At $\mathbf{O}, \mathsf{w} : el$, *I* win and *You* lose iff $\mathcal{M}, \mathsf{w} \not\models el$. Otherwise, *You* win and *I* lose.

Note the symmetry between the **P**- and **O**-versions of the rules. The last rule represents the winning conditions of the game. Interestingly, also the rules for $\Box$ and $\Diamond$ contain winning conditions, if the current world has no successors. We point out that the condition does not depend on the whole history, but only on its final game state.

Example 3.3.1. Consider the model $\mathcal{M}$ in Figure 3.2 and the following run of the game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$. First, *You* must choose a neighboring world. Since *You* know that *I* can defend j at w_2, let us say that *You* choose w_3 and *I* must then defend $j \vee \neg\Box p$ at w_3. Clearly, *I* will choose the second disjunct. According to the rule of negation, a role switch occurs: *I* am now the opponent and *You* the proponent of $\Box p$ at w_3. Hence, *I* must choose a neighboring world and *You* must defend p there. As *My* choice is between the p-world w_4 and the non-p-world w_5, *I* will choose w_5 and win the game. This run of the game is depicted in Figure 3.3

We now formally define the evaluation game in terms of Definition 2.1.1. Game states are of the form $\mathbf{Q} : F$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ and F is a formula, or of the form $\mathbf{Q}, \mathsf{w} : F$ where, additionally, w is a world in a model $\mathcal{M}$. The set of game states is denoted $\mathrm{Stat}^{\mathrm{Hyb}}_{\mathcal{M}}$.

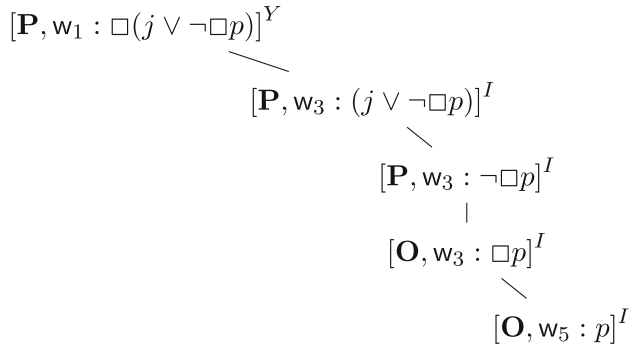


Figure 3.3: A run of the game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$

Definition 3.3.2: Evaluation game for hybrid logic[a]

[a] For the version of the game that is used for the most part of this chapter, see Section 3.3.3

Let $\mathcal{M}$ be a model and g a game state in $\text{Stat}^{\text{Hyb}}_{\mathcal{M}}$. The evaluation game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(g)$ is defined as follows:

- The initial history is $\langle g \rangle$.
- The set of histories $\text{Hist}^{\text{Hyb}}_g$ is the minimal set containing $\langle g \rangle$ and satisfying the following conditions: if $h = \langle \mathbf{Q} : G \rangle$ is contained, then so is $h \smile \mathbf{Q}, \mathsf{u} : G$, for every world u. If $h = h' \smile \mathbf{Q}, \mathsf{v} : G$ is contained, and G is of the form
  - $G = G_1 \vee G_2$, then also $h \smile \mathbf{Q}, \mathsf{v} : G_1$ and $h \smile \mathbf{Q}, \mathsf{v} : G_2$ are contained,
  - $G = G_1 \wedge G_2$, then also $h \smile \mathbf{Q}, \mathsf{v} : G_1$ and $h \smile \mathbf{Q}, \mathsf{v} : G_2$ are contained,
  - $G = G_1 \to G_2$, then also $h \smile \bar{\mathbf{Q}}, \mathsf{v} : G_1$ and $h \smile \mathbf{Q}, \mathsf{v} : G_2$ are contained[a],
  - $G = \neg G'$, then also $h \smile \bar{\mathbf{Q}}, \mathsf{v} : G'$ is contained,
  - $G = @_i G'$, then also $h \smile \mathbf{Q}, \mathsf{g}(i) : G'$ is contained,
  - $G = \forall i.G'$, then also $h \smile \mathbf{Q}, \mathsf{v} : G[i/j]$ is contained for all $j \in N$,
  - $G = \Box G'$, then also $h \smile \mathbf{Q}, \mathsf{u} : G'$ is contained, for all $\mathsf{u} \in \mathsf{vR}$,
  - $G = \Diamond G'$, then also $h \smile \mathbf{Q}, \mathsf{u} : G'$ is contained, for all $\mathsf{u} \in \mathsf{vR}$.
- Non-terminal histories ending in the states of the following forms are labeled as specified in the following table:

labeled "I"	labeled "Y"
$\mathbf{O} : G$	$\mathbf{P} : G$
$\mathbf{P} : G_1 \vee G_2$	$\mathbf{O} : G_1 \vee G_2$
$\mathbf{O} : G_1 \wedge G_2$	$\mathbf{P} : G_1 \wedge G_2$
$\mathbf{P} : G_1 \rightarrow G_2$	$\mathbf{O} : G_1 \rightarrow G_2$
$\mathbf{P} : \neg G'$	$\mathbf{O} : \neg G'$
$\mathbf{P} : @_i G'$	$\mathbf{O} : @_i G'$
$\mathbf{O} : \forall i.G'$	$\mathbf{P} : \forall i.G'$
$\mathbf{O} : \Box G'$	$\mathbf{P} : \Box G'$
$\mathbf{P} : \Diamond G'$	$\mathbf{O} : \Diamond G'$

- The payoff function $\wp_{\mathcal{M}}$ maps terminal histories to the domain $\{-1, 1\}$, where $-1 \lhd 1$. Terminal histories ending in the following states are mapped to the following values:

mapped to 1		mapped to -1	
$\mathbf{P}, \mathsf{w} : el,$	if $\mathcal{M}, \mathsf{w} \models el$	$\mathbf{P}, \mathsf{w} : el,$	if $\mathcal{M}, \mathsf{w} \not\models el$
$\mathbf{O}, \mathsf{w} : el,$	if $\mathcal{M}, \mathsf{w} \not\models el$	$\mathbf{O}, \mathsf{w} : el,$	if $\mathcal{M}, \mathsf{w} \models el$
$\mathbf{P}, \mathsf{w} : \Box G'$		$\mathbf{O}, \mathsf{w} : \Box G'$	
$\mathbf{O}, \mathsf{w} : \Diamond G'$		$\mathbf{P}, \mathsf{w} : \Diamond G'$	

If the payoff is 1, then *I* win and *You* lose. If it is -1, *I* lose and *You* win.

[a] If $\mathbf{Q} = \mathbf{P}$, then $\bar{\mathbf{Q}} = \mathbf{O}$, and if $\mathbf{Q} = \mathbf{O}$, then $\bar{\mathbf{Q}} = \mathbf{P}$.

Let us convince ourselves that this description matches our intuitive understanding of the game in terms of the rules given above: for example, if the current history h ends in the game state $g = \mathbf{P}, \mathsf{v} : G_1 \vee G_2$ (i.e. *I* am currently the Proponent of $G_1 \vee G_2$ in v), then h is labeled "I". This means *I* can choose from the two minimal histories extending h, namely $h \smile \mathbf{P}, \mathsf{v} : G_1$ and $h \smile \mathbf{P}, \mathsf{v} : G_2$ which will be the new current history. If h ends in $\mathbf{P}, \mathsf{v} : \Box G'$ (i.e. *I* am currently the proponent of $\Box G'$ in v) and there are no successors of v, then h is a terminal history with payoff 1, i.e. *I* win. Otherwise, h is labeled "Y", and the histories extending h are of the form $h \smile \mathbf{P}, \mathsf{u} : G'$, where u ranges over the successors of v. This means, *You* pick such a u and the new history is $h \smile \mathbf{P}, \mathsf{u} : G'$. At $\mathbf{P}, \mathsf{w} : \neg G'$ the labeling is inessential as there are no choices involved: the new history is $h \smile \mathbf{O} : G'$, i.e. the game automatically continues with a role switch. We again point out that the order of moves, their availability, or who wins at a terminal state is determined solely by the last state of the history.

Remember that it is useful to think of a game as a tree, see 3.6. In fact, we will use game trees for illustrating all our examples.

Example 3.3.3 (3.3.1 continued). The game tree in Figure 3.4 represents all possible choices of the two players at any point. The game states are labeled "I" or "Y", depending on

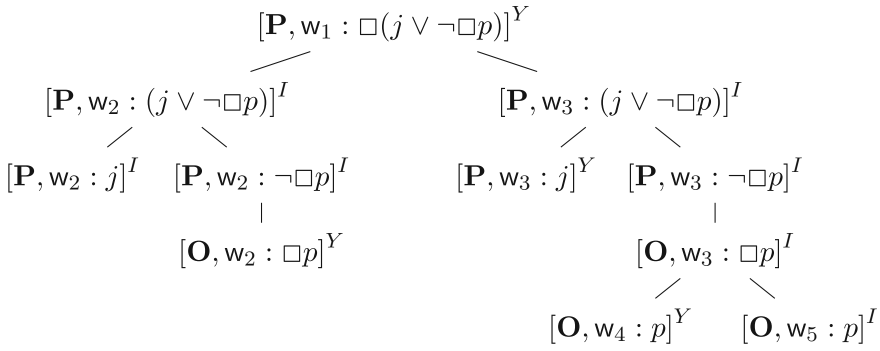


Figure 3.4: The game tree $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$.

which player is to move. We also label every leaf with the player who wins the run there. All leaves of that tree are elementary game states, except for $\mathbf{O}, \mathsf{w}_2 : \Box p$, where I immediately win the game because there are no successors of w_2. The run of the game from Example 3.3.1 can be found as a path through this tree.

Although not all possible runs of a game end in a winning state for *Me*, *I* can make choices that will guarantee that the game will end in *My* victory. Generally, if *I* can always enforce a given game to end in a winning state for *Me*, we say that *I* have a *winning strategy* for that game. We will make this notion more formal in the following subsection.

We now show that the corresponding game tree of the evaluation game is of finite height. Intuitively this means that every run of the game lasts only finitely many rounds:

Proposition 3.3.4

Let g be a game state. All histories of $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ are finite.

Proof. We define the degree $\delta(F)$ of a formula F to be 0 if F is elementary, $\delta(F * G) = \max\{\delta(F), \delta(G)\} + 1$ for $*$ a logical connective and $\delta(\triangle F) = \delta(\triangle F) + 1$ for $\triangle$ a modal operator (that includes $@$), $\neg$, or $\forall i$. We extend δ to game states by setting $\delta(\mathbf{Q}, \mathsf{w} : F) = \delta(F)$ and $\delta(\mathbf{Q} : F) = \delta(F) + 1$ for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$. The proof now follows from the fact that every move in the game strictly reduces the degree of the game state. $\square$

3.3.1 Winning Strategies

To precisely describe the scenario where *I* can enforce a winning outcome, we use the notion of a winning strategy. Remember, that strategies for *Me* were already formally defined in Definition 2.1.5 as functions mapping every I-history h to a history $h \smile g$. Winning strategies were defined in Subsection 2.1.3. We, therefore, content

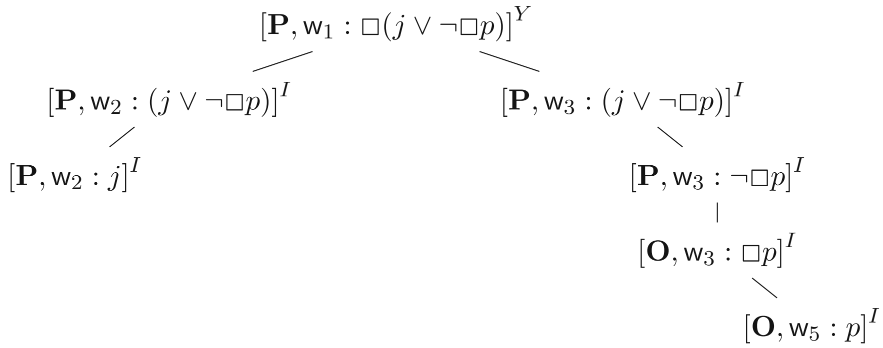


Figure 3.5: A winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$

ourselves with recalling the description of a strategy as a subtree of the game tree and an illuminating example.

A strategy σ for *Me* is a subtree of the game tree of $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ such that g is in σ and for every game state g' appearing in σ,

- if g' is labeled "I", then exactly one successor of g' is in σ,
- if g' is labeled "Y", then all successors of g' are in σ.

The strategy σ is called winning if all leaves in σ are winning for *Me*. (Winning) strategies for *You* are defined symmetrically.

Example 3.3.5. Continuing the game from Example 3.3.3, we can now make precise our observation that *I* can make choices such that the game will always end in winning states for *Me*. A strategy for *Me* for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$ is depicted in Figure 3.5 as a subtree of the game tree in Figure 3.4. Note that all leaves are winning for *Me* (and thus labeled "I"). Therefore, this strategy is a winning strategy for *Me*.

We will see another example of a game, where *You* have a winning strategy in Example 3.3.10 at the end of the section.

Remark 3.3.6. Let h be a history of the evaluation game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$. We write $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ for the subgame $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)@h$, see Defininition 2.1.13. Let $h = h' \smile g'$. Since the winning conditions of the evaluation game depend only on the last game state in the history, but not the entire history itself, we get that the games $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h' \smile g')$ and $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g')$ are strategically equivalent. Where it does not cause confusion, we will therefore often identify these two games.

3.3.2 Adequacy

This subsection is devoted to proving the adequacy of the evaluation game, i.e., that the existence of winning strategies for *Me* and truth in a model coincide. We use the following handy notation: we write $\mathcal{M} \models \mathbf{P}, \mathsf{w} : F$ if $\mathcal{M}, \mathsf{w} \models F$ and $\mathcal{M} \models \mathbf{O}, \mathsf{w} : F$ otherwise. Similarly, we write $\mathcal{M} \models \mathbf{P} : F$ if $\mathcal{M} \models F$ and $\mathcal{M} \models \mathbf{O} : F$ otherwise. The central theorem is the following:

Theorem 3.3.7: Adequacy of the evaluation game

Let $\mathcal{M}$ be a model, and g a game state.

1. *I* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ iff $\mathcal{M} \models g$.
2. *You* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ iff $\mathcal{M} \not\models g$.

As an immediate corollary, we obtain that the evaluation game is determined, which means that exactly one of the players has a winning strategy[7]. To make the connection to the model-theoretic semantics entirely clear, we spell out our abbreviations from above:

Corollary 3.3.8

I have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : F)$ iff $\mathcal{M}, \mathsf{w} \models F$. *I* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P} : F)$ iff $\mathcal{M} \models F$.

Proof Theorem 3.3.7. We show both directions of 1 and 2 simultaneously by induction on the degree of the game state, as in the proof of Proposition 3.3.4. Let us show some of the cases. If g is of the form $\mathsf{w} : el$, then everything follows from the winning conditions.

If $g = \mathbf{P} : G_1 \wedge G_2$, then *I* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : G_1 \wedge G_2)$ iff[8] *I* have winning strategies for both $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : G_1)$ and $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : G_2)$. By the induction hypothesis, this is the case iff $\mathcal{M}, \mathsf{w} \models G_1$ and $\mathcal{M}, \mathsf{w} \models G_2$, which is equivalent to $\mathcal{M}, \mathsf{w} \models G_1 \wedge G_2$ and $\mathcal{M} \models g$.

In $g = \mathbf{P}, \mathsf{w} : \Box G$, *You* choose a successor v and the game continues at $\mathcal{M}, \mathsf{v} : G$. Consequently, if *I* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : \Box G)$, then *I* must have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{v} : G)$, for every R-successor v of w. By the inductive hypothesis, this is the case iff $\mathcal{M}, \mathsf{v} \models G$ for every $v \in \mathsf{wR}$, which in turn is equivalent to $\mathcal{M}, \mathsf{w} \models \Box G$ and $\mathcal{M} \models g$. Note that this covers the case, where w has no successors.

[7] This also follows from the fact that the game is finite-valued and Theorem 2.1.11

[8] This simple equivalence is actually justified by Lemma 2.1.14 and Remark 3.3.6. We will not mention their further applications in this proof.

In $g = \mathbf{P}, \mathsf{w} : \neg G$, the game continues with $\mathbf{O}, \mathsf{w} : G$. Thus, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : \neg G)$ iff *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{O}, \mathsf{w} : G)$. By the inductive hypothesis, this is the case iff $\mathcal{M} \models \mathbf{O}, \mathsf{w} : G$ iff $\mathcal{M}, \mathsf{w} \not\models G$ iff $\mathcal{M}, \mathsf{w} \models \neg G$ iff $\mathcal{M} \models g$.

The other cases are similar. □

3.3.3 Internalizing Nominals

We conclude this section with an important observation about the game for named models. Remember that in a named model, every world w has a name i, i.e. $\mathsf{g}(i) = \mathsf{w}$. Therefore, for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, it is unambiguous if we write $\mathbf{Q}, i : F$ for the game state $\mathbf{Q}, \mathsf{w} : F$. We can thus safely alter the set of game states to the set $\mathrm{Stat}^{\mathrm{Hyb}}$ containing such game states along game states of the form $\mathbf{Q} : F$. Note that $\mathrm{Stat}^{\mathrm{Hyb}}$ is now independent of the underlying model $\mathcal{M}$. This, together with the fact that $\mathcal{M} \models R(i, j)$ iff $\mathsf{g}(i)\mathsf{R}\mathsf{g}(j)$ gives us the following equivalent formulations of the rules for $\Box$, $\Diamond$ and U:

($\mathbf{P}_\Box$) At $\mathbf{P}, i : \Box G$, *You* choose a nominal j and the game continues with $\mathbf{P}, j : R(i, j) \to G$.

($\mathbf{O}_\Box$) At $\mathbf{O}, i : \Box G$, *I* choose a nominal j and the game continues with $\mathbf{P}, j : R(i, j) \to G$.

($\mathbf{P}_\Diamond$) At $\mathbf{P}, i : \Diamond G$, *I* choose a nominal j and the game continues with $\mathbf{P}, j : R(i, j) \land G$.

($\mathbf{O}_\Diamond$) At $\mathbf{O}, i : \Diamond G$, *You* choose a nominal j and the game continues with $\mathbf{P}, j : R(i, j) \land G$.

($\mathbf{P}_U$) At $\mathbf{P} : G$, *You* choose a nominal i and the game continues with $\mathbf{P}, i : G$.

($\mathbf{O}_U$) At $\mathbf{O} : G$, *I* choose a nominal i and the game continues with $\mathbf{P}, i : G$.

For a proof later in this chapter we need an even more general case. Let $N' \subseteq N$ be such that every world in $\mathcal{M}$ has a name in N', i.e., g restricted to N' is surjective. In this case, we say that $\mathcal{M}$ is N'-named. We consider the game $\mathbf{G}^{\mathrm{Hyb},N'}_{\mathcal{M}}(g)$, where branching over the nominals is restricted to N' in the above rules. In the formal description of Definition 3.3.2 we change the definition of the histories to refer to nominals in N', not worlds. Additionally, we have the following changes:

- *If $h = \langle \mathbf{Q} : G \rangle$ is contained, then so is $h \smile \mathbf{Q}, i : G$ for every nominal $i \in N'$. If $h = h' \smile \mathbf{Q}, i : G$ is contained and G is of the form*
  - *$G = \Box G$, then also $h \smile \mathbf{Q}, j : R(i, j) \to G$ is contained, for all nominals $j \in N'$,*
  - *$G = \Diamond G$, then also $h \smile \mathbf{Q}, j : R(i, j) \land G$ is contained, for all nominals $j \in N'$.*

The rest of the definition is the same, except that we remove the payoff values at game states involving boxed formulas. Now all terminal histories end in elementary game states.

Proposition 3.3.9

Let $\mathcal{M}$ be N'-named. For every $g = \mathbf{Q}, \mathsf{w} : F$ let g' be $\mathbf{Q}, i : F$, for some $i \in N'$ with $\mathsf{g}(i) = \mathsf{w}$. If $g = \mathbf{Q} : F$, then $g' = g$. Then the games $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ and $\mathbf{G}^{\mathrm{Hyb},N'}_{\mathcal{M}}(g')$ are strategically equivalent.

Proof. By induction on the degree of g. The interesting cases are for $\Box$, $\Diamond$ and U. For example, suppose *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, \mathsf{w} : \Box G)$ and let $\mathsf{g}(i) = \mathsf{w}$. In the first round of $\mathbf{G}^{\mathrm{Hyb},N'}_{\mathcal{M}}(\mathbf{P}, i : \Box G)$, *You* choose a nominal $j \in N'$ and the game continues with $\mathbf{P}, j : R(i, j) \to G$. If $\mathsf{g}(j)$ is not an R-successor of w, then *I* go to $\mathbf{O}, j : R(i, j)$ and win the game. Otherwise, *I* go to $\mathbf{P}, j : G$. By assumption, *I* have a winning strategy in $\mathbf{P}, \mathsf{g}(j) : G$ and thus in $\mathbf{P}, j : G$, by the induction hypothesis. Since j was arbitrary, *I* have a winning strategy for $\mathbf{P}, i : \Box G$.

For the other direction, suppose *I* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb},N'}_{\mathcal{M}}(\mathbf{P}, i : \Box G)$. We must show that for every R-successor v of w, *I* have a winning strategy in $\mathbf{P}, \mathsf{v} : G$. To this end, let $j \in N'$ be a name for v. Then by assumption, *I* have a winning strategy in $\mathbf{P}, j : R(i, j) \to G$. Since $\mathbf{O}, j : R(i, j)$ is losing for *Me*, *I* must have a winning strategy in $\mathbf{P}, j : G$. The inductive hypothesis gives *Me* a winning strategy in $\mathbf{P}, \mathsf{v} : G$.

The other cases are similar. □

In particular, for named models, Proposition 3.3.4[9], Theorem 3.3.7 remains valid. From now on, unless explicitly noted otherwise, we assume that all models are named. Also, we use only the modified game $\mathbf{G}^{\mathrm{Hyb},N}_{\mathcal{M}}$ with branching over the nominal, refer to it as the evaluation game, and write $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}$. This is justified by Lemma 3.2.2 and by the proposition above. As discussed in the introduction, the advantage of the modified game is that game trees are now uniform and semantic information from the model is only needed at elementary states. Note, however, that the tree underlying the evaluation game is now infinitely branching, in general. Let us consider another example where we use the reformulated rules.

Example 3.3.10. Let $\mathcal{M}$ be the model in Figure 3.6. Because $\mathcal{M}$ is not transitive, we expect *You* to have a winning strategy in the game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$. A (compact representation) of *Your* winning strategy is shown in Figure 3.7. If *I* go to $\mathbf{P}, i : \Diamond p$ in the first turn, *I* will then choose a nominal l and the game continues at $\mathbf{P}, l : R(i, l) \wedge p$. If $l \neq k$, then *You* should go to $\mathbf{P}, l : p$, otherwise to $\mathbf{P}, k : R(i, k)$. In both cases, *You* win the game.

A more direct way of saying that the model is a countermodel to transitivity is by making use of the hybrid language: Clearly, $R(i, j) \wedge R(j, k) \to R(i, k)$ is false at the world $\mathsf{g}(i)$.

[9]For the modified rules, we need to change the definition of degree: $\delta(\triangle F) = \max\{1, \delta(F)\} + 1$ for modal operators $\triangle$ to deal with the occurrence of $R(i, j)$ in the successor state.

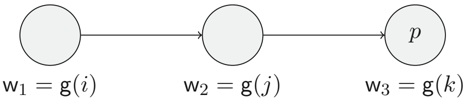


Figure 3.6: A non-transitive model.

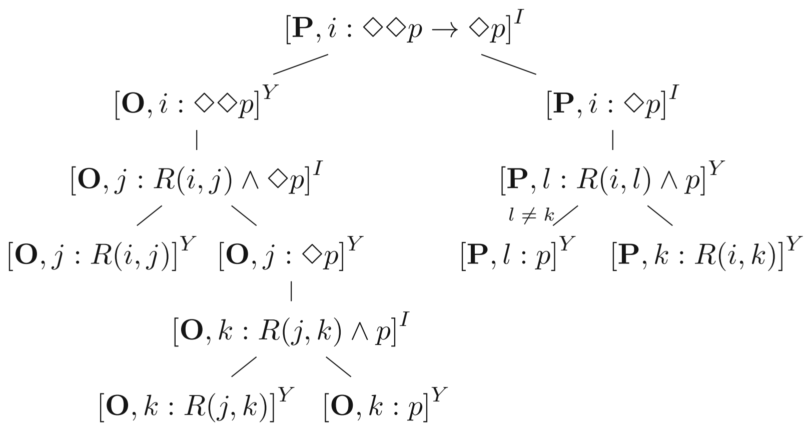


Figure 3.7: A winning strategy for *You* in the game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$.

Therefore, *You* have a winning strategy for the game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(\mathbf{P}, i : R(i, j) \wedge R(j, k) \to R(i, k))$.

3.4 The Disjunctive Game as a Semantic Game

This section describes the disjunctive game, an extension of the evaluation game, and shows the connection between the two games. Similarly to the evaluation game, the disjunctive game is played over a fixed model $\mathcal{M}$. However, in every history h, *I* now have an extra option: instead of moving according to the rules of the evaluation game, *I* can decide to create a "backup-copy" of h and continue playing at the *disjunctive game state* (or *disjunctive state*) $h \bigvee h$. If the game is unfavorable for *Me* in one copy, *I* can always come back to have another shot at the other copy. Formally, disjunctive game states are finite multisets of histories of the evaluation game. We prefer to write $h_1 \bigvee ... \bigvee h_n$ for the disjunctive game state $\{h_1, ..., h_n\}$, but keep the convenient notation $h \in D$ if h belongs to the multiset set D. We write $D_1 \bigvee D_2$ for the multiset sum $D_1 + D_2$ and $D \bigvee h$ for $D + \{h\}$. A disjunctive state is called *elementary* if all its histories end in elementary game states. *My* goal is to win at least one backup copy (hence the "disjunctive"). Therefore, disjunctive states are winning for *Me* if they contain at least one winning history.

Note that due to the design of the game, runs of the game can now be infinite, as *I* can duplicate histories infinitely often. All infinite runs will be considered winning for *You*.

Additionally, *I* take the role of a scheduler: in a disjunctive state $D \bigvee h$, *I* can point to the history h, coded by underlining: $D \bigvee \underline{h}$. Afterward, the corresponding player takes their turn in the evaluation game. Say they move to g, then the new disjunctive game state is $D \bigvee h \smile g$. Alternatively, *I* can decide to end the game. The winner is then determined as described above.

We now give a semi-formal description of the disjunctive game. Let D be a disjunctive state. Let D^{ter} consist of the terminal histories of D. We say that D is terminal if $D = D^{ter}$, or if *I* have decided to end the game.

(End) If no histories in D are underlined, *I* can end the game and D becomes terminal.

(Dupl) If no histories in D are underlined and D is not terminal, *I* can *duplicate* an $h \in D$ and the game continues with $D \bigvee h$.

(Sched) If no histories in $D = D' \bigvee h$ are underlined and D is not terminal, *I* can *underline* a non-terminal $h \in D$ and the game continues with $D' \bigvee \underline{h}$.

(Move) If $D = D' \bigvee \underline{h}$ then the player who is to move in the evaluation game $\mathbf{G}_{\mathcal{M}}^{\mathrm{Hyb}}(h)$ at the history h makes a legal move to the game state g and the game continues with $D \bigvee h \smile g$. For example, if h ends in $\mathbf{P}, i : G_1 \wedge G_2$, then *You* chose a $k \in \{1, 2\}$ and the game continues with $D \bigvee h \smile \mathbf{P}, i : G_k$.

(Win) If D is terminal, then *I* win iff *I* win the evaluation game in some $h \in D^{ter}$, and *You* lose. Otherwise, *You* win and *I* lose.

Additionally, we require that if no history of D is underlined, *I* must move according to **(End)**, **(Dupl)**, or **(Sched)**. **(Dupl)** is referred to as the *duplication rule* and **(Sched)** as the *scheduling*, or *underlining rule*.

Example 3.4.1. In Example 3.3.5, we have seen the evaluation game[10] $\mathbf{G}_{\mathcal{M}}^{\mathrm{Hyb}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$, where *I* have a winning strategy. It, therefore, makes no sense for *Me* to use the duplication rule in the corresponding disjunctive game $\mathbf{DG}_{\mathcal{M}}^{\mathrm{Hyb}}(\mathbf{P}, \mathsf{w}_1 : \Box(j \vee \neg\Box p))$. In Example 3.3.10, *You* have a winning strategy for $\mathbf{G}_{\mathcal{M}}^{\mathrm{Hyb}}(\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$. In the disjunctive game $\mathbf{DG}_{\mathcal{M}}^{\mathrm{Hyb}}(\mathbf{P}, i : \Diamond\Diamond p \to p)$, using the duplication rule only produces backup copies of game states, where *You* have a winning strategy and therefore does not help *Me*.

Finally, let us look at a case where the initial state consists of more than one history. With the non-transitive model $\mathcal{M}$ from Example 3.3.10, consider the disjunctive game over $\mathcal{M}$ starting at

$$\mathbf{O} : \forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k)) \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p$$

[10] We often write g instead of the history $\langle g \rangle$

Since I have a winning strategy for the evaluation game starting in $\mathbf{O}, i : R(i,j) \wedge R(j,k) \to R(i,k)$, I will win the disjunctive game, without having to schedule the other game states. This illustrates the fact that for *Me* to win the disjunctive game, it is enough to have a winning strategy for some game state in the disjunctive state. This fact is also formally proved below. Furthermore, *My* winning strategy for this game can be seen as a witness to the semantic fact

$$R(i,j) \wedge R(j,k) \to R(i,k) \models @_i(\Diamond\Diamond p \Rightarrow \Diamond p).$$

A depiction of *My* winning strategy for this game can be found in Figure 3.8. For the sake of readability, we only write the last game state in each history in disjunctive states. Let D be the multiset containing the relational claims. Furthermore, we left applications of the rule **(Sched)** and **(End)** implicit. Where marked, the strategy branches over all nominals. This represents *Your* choice of nominal in the modal rule for the evaluation game. The strategy starting from that choice is uniform in the chosen nominal. In the state $\mathbf{O}, j : R(i,j) \wedge \Diamond p \bigvee \mathbf{P}, i : \Diamond p$, I use the duplication rule, and go to $\mathbf{O}, j : R(i,j)$ in one, and to $\mathbf{O}, j : \Diamond p$ in the other copy. We do not further mention the other applications of this trick. H is an abbreviation for the disjunctive state $\mathbf{O} : T \bigvee \mathbf{O}, j : R(i,j) \bigvee \mathbf{O}, k : R(j,k)$. At the right leaf, $H \bigvee \mathbf{O}, k : p \bigvee \mathbf{P}, k : p$, I end the game and win. At the other disjunctive state, the game tree continues in Figure 3.9. Here, K is $\mathbf{O}, k : p \bigvee \mathbf{P}, k : R(i,k)$. I begin with an application of the rule (U) and three applications of $(\mathbf{O}_\forall)$ I win the game in all leaves, which shows that the strategy is indeed winning. K_1, K_2 and K_3 are K with $\mathbf{O}, j : R(i,j)$, $\mathbf{O}, k : R(j,k)$, and $\mathbf{P}, k : R(i,k)$ removed, respectively.

This winning strategy may seem a bit overblown since it includes many non-essential moves. For example, it was not necessary for *Me* to move into $\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p$ at all, since I am unable to defend this state over the non-transitive model $\mathcal{M}$. However, we will see that this strategy contains the necessary information to win the disjunctive game over *all* models. Indeed, the leaves are not only winning in the model game over the model $\mathcal{M}$ but over every possible model. Therefore, this winning strategy can even be seen as *proof* of the above semantic fact. Strategies like this will be the topic of the next section.

We now give the full game-theoretic definition of the disjunctive game in terms of Definition 2.1.1. If h is a history of the game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$, let us write $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ for the subgame $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)@h$, see Definition 2.1.13.

Definition 3.4.2: Disjunctive game as Semantic Game

Let $\mathcal{M}$ be a model. Disjunctive states[a] are multisets of histories of the evaluation game, where none or exactly one history is underlined, or the dummy state $*$. Let D be a disjunctive state containing histories of the evaluation game. The disjunctive game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ is defined as follows:

- The initial history is D.

- If $\mathfrak{h}$ is a disjunctive history and no state in the disjunctive state H is underlined, and
  - $\mathfrak{h} = \mathfrak{h}' \smile H \bigvee h$, then $\mathfrak{h} \smile *$ is a disjunctive history. If additionally, h is not terminal, then $\mathfrak{h} \smile H \bigvee h \bigvee h$ and $\mathfrak{h} \smile H \bigvee \underline{h}$ are disjunctive histories[b].
  - $\mathfrak{h} = \mathfrak{h}' \smile H \bigvee \underline{h}$, then $\mathfrak{h} \smile H \bigvee (h \smile g)$ is a disjunctive history if $h \smile g$ is a history of the evaluation game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$.
- Non-terminal disjunctive histories ending in a disjunctive state H with no underlined histories are labeled "I". If $H = H' \bigvee \underline{h}$, then H is labeled the same as h in the evaluation game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$.
- The payoff function maps terminal disjunctive histories to the domain $\{-1, 1\}$, where $-1 \prec 1$. Infinite terminal disjunctive histories are mapped to -1. Terminal disjunctive histories ending in $\langle \ldots, D, * \rangle$ are mapped to
  - 1, if some $h \in D$ is winning in the evaluation game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$. We say that D is winning for *Me* and losing for *You*.
  - -1, else. In this case, D is losing for *Me* and winning for *You*.

[a] To make the distinction easier, we always refer to game states of the disjunctive game as *disjunctive states* and histories of the disjunctive game as *disjunctive histories*.
[b] Note that there is implicit quantification over h.

Remark 3.4.3. (1) It follows from the definition that all terminal disjunctive histories end in $*$.

(1) (Winning) strategies for the disjunctive game are well-defined in light of Definition 2.1.5 and can again be thought of as subtrees of the game tree. To distinguish from the evaluation game, we will speak of *disjunctive (winning) strategies*.

(2) Due to infinite disjunctive states and the duplication rule, runs of the game can now be infinite, resulting in a winning outcome for *Me*. However, our games retain a vital property: By the Gale-Stewart Theorem [32], every instance of the disjunctive game is *determined*, i.e. exactly one of the two players has a winning strategy. Note that this also follows from the fact that the game is finite-valued and Theorem 2.1.11. A direct proof follows from the two propositions below.

(3) In contrast to the evaluation game, the disjunctive game is not fully symmetric. This is due to the duplication rule, the winning conditions, and *My* role as a scheduler, i.e. the scheduling rule. At least the last asymmetry can be eliminated. In Chapter 5, we discuss a general framework for the disjunctive game, where the scheduling is done by a *regulation function*, which can be thought of as a third, non-strategic player. Under certain conditions, the disjunctive game retains its nice properties which we discuss in the present chapter.

$$[\mathbf{O}: T \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p]^I$$

$$\Big[\mathbf{O}: T \bigvee \underline{\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p} \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p\Big]^I$$

$$\Big[\mathbf{O}: T \bigvee \mathbf{O}, i : \Diamond\Diamond p \bigvee \underline{\mathbf{P}, i : \Diamond\Diamond p \to \Diamond p}\Big]^I$$

$$\Big[\mathbf{O}: T \bigvee \underline{\mathbf{O}, i : \Diamond\Diamond p} \bigvee \mathbf{P}, i : \Diamond p\Big]^Y$$

j ranges over N

$$\Big[\mathbf{O}: T \bigvee \underline{\mathbf{O}, j : R(i,j) \wedge \Diamond p} \bigvee \mathbf{P}, i : \Diamond p\Big]^I$$

$$\Big[\mathbf{O}: T \bigvee \mathbf{O}, j : R(i,j) \bigvee \underline{\mathbf{O}, j : \Diamond p} \bigvee \mathbf{P}, i : \Diamond p\Big]^Y$$

k ranges over N

$$\Big[\mathbf{O}: T \bigvee \mathbf{O}, j : R(i,j) \bigvee \underline{\mathbf{O}, k : R(j,k) \wedge p} \bigvee \mathbf{P}, i : \Diamond p\Big]^I$$

$$\Big[\mathbf{O}: T \bigvee \mathbf{O}, j : R(i,j) \bigvee \mathbf{O}, k : R(j,k) \bigvee \mathbf{O}, k : p \bigvee \underline{\mathbf{P}, i : \Diamond p}\Big]^I$$

$$\Big[H \bigvee \mathbf{O}, k : p \bigvee \underline{\mathbf{P}, k : R(i,k) \wedge p}\Big]^Y$$

$$H \bigvee \mathbf{O}, k : p \bigvee \mathbf{P}, k : R(i,k) \qquad\qquad H \bigvee \mathbf{O}, k : p \bigvee \mathbf{P}, k : p$$

Figure 3.8: A winning strategy for the game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{O} : T \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$, where T characterizes transitive frames.

We now compare the disjunctive game to the evaluation game from a strategic viewpoint. Essentially, a winning disjunctive strategy for *Me* for the disjunctive game over a model is nothing more but a disjunction of strategies for the evaluation game:

Proposition 3.4.4: *My* disjunctive strategy = disjunction of *My* strategies

I have a disjunctive winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ iff *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ for some $h \in D$.

Proof. "⇒": Let σ be a winning strategy for *Me* for $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$. By backward induction

$$\left[K \bigvee \underline{\mathbf{O} : \forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k))}\right]^I$$

$$\left[K \bigvee \underline{\mathbf{O}, i : \forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k))}\right]^I$$

$$\left[K \bigvee \underline{\mathbf{O}, i : \forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k))}\right]^I$$

$$\left[K \bigvee \underline{\mathbf{O}, i : (R(i,j) \wedge R(j,k) \to R(i,k))}\right]^I$$

$$\left[K \bigvee \underline{\mathbf{O}, i : R(i,j) \wedge R(j,k) \to R(i,k)}\right]^I$$

$$\left[K \bigvee \underline{\mathbf{O}, i : R(i,j) \wedge R(j,k)} \to R(i,k)\right]^I$$

$$\left[K \bigvee \underline{\mathbf{P}, i : R(i,j) \wedge R(j,k)}\right]^Y \qquad K_3 \bigvee \mathbf{P}, k : R(i,k) \bigvee \mathbf{O}, i : R(i,k)$$

$$K_1 \bigvee \mathbf{O}, j : R(i,j) \bigvee \mathbf{P}, i : R(i,j) \qquad K_2 \bigvee \mathbf{O}, k : R(j,k) \bigvee \mathbf{P}, i : R(j,k)$$

Figure 3.9: Continuation of the strategy in Figure 3.8.

on the tree structure of σ we show[11] that for every $H \in \sigma$ there is $h \in H$ such that *I* have a winning strategy σ_H in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$. The proposition then follows for the case where $H = D$.

By assumption, all leaves $*$ have a predecessor H such that there is some winning $h \in H$. If H is not followed by $*$ and is labeled "Y", then H is of the form $H' \bigvee \underline{h}$. The successors of H are $H \bigvee h'$, where h' are the successors of h in the evaluation game. By the inductive hypothesis, there are winning strategies $\sigma_{H' \bigvee h'}$ for all h'. If for some h', $\sigma_{H' \bigvee h'}$ is a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(k)$, where $k \in H'$, then we can simply set $\sigma_H = \sigma_{H' \bigvee h'}$. Otherwise, every $\sigma_{H' \bigvee h'}$ is a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h')$. Lemma 2.1.14 gives us a winning strategy σ_H for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$.

If H is labeled "I" and is of the form $H' \bigvee h$, and according to σ, *I* move to $H' \bigvee h \bigvee h$, then we simply set $\sigma_H = \sigma_{H \bigvee h \bigvee h}$ and use the inductive hypothesis. We proceed similarly if *I* move to $H \bigvee \underline{h}$. Finally if $H = H' \bigvee \underline{h}$ and *I* move to $H' \bigvee h'$, the inductive hypothesis gives us a winning strategy for $\sigma_{H' \bigvee h'}$ for some $k \in H' \bigvee h'$. If $k \in H'$, we set $\sigma_H = \sigma_{H' \bigvee h'}$ and are done. If $k = h'$, we use Lemma 2.1.14 to obtain a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$.

[11] Since infinite runs are winning for *You*, this tree is necessarily of finite height.

"⇐": Suppose, *I* have a winning strategy σ for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ for some $h \in D$. The idea is as follows. Since *I* can win the evaluation game starting at h, *I* can win the disjunctive game by only ever playing on h and not touching the other histories in D. By induction on the tree structure of σ, we define a strategy μ for *Me* with the following property: (*) every disjunctive state appearing in μ is of the form $H \bigvee k$, where k is a history in σ[12]. The base case follows from the assumption.

If the current disjunctive state is $H \bigvee k$ with k as required, and there are no game states underlined in H, then underline k and (*) follows immediately from the inductive hypothesis. If the current disjunctive state is $H \bigvee \underline{k}$ and k is labeled "Y", then *You* proceed to some $H \bigvee k \smile g$. Since k is labeled "Y", σ contains all immediate successors of k, hence $k \smile g$ must be a history in σ. If k is labeled "I", then *I* move to $H \bigvee \sigma(k)$. Clearly (*) holds for $\sigma(k)$. Eventually, the game reaches a state $H \bigvee k$, where k is a leaf of σ, and thus winning for *Me*. *I*, therefore, end the game and win. □

If D consists of a single history h, then the proposition gives the strategic equivalence of the games $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ and $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$. This shows that the clear differentiation between the evaluation and the disjunctive game is somewhat artificial. If anything, the disjunctive game played over a model should be itself considered a semantic game, as defined in Chapter 1.

We introduced the disjunctive game over a model for two reasons. First, the previous proposition shows that a strategy in the disjunctive game can really be thought of as a disjunction of strategies for the evaluation game, thus giving a strong motivation and intuition, which is useful later on. Second, in the next section, we will see a version of the disjunctive game played over all models simultaneously. Having formulated the disjunctive game makes the formulation, and especially the proofs, much easier

We conclude this section with another characterization of the disjunctive game in terms of *Your* winning strategies. It shows that a disjunctive strategy for *You* is really a *conjunction* of *Your* strategies in the evaluation game.

Proposition 3.4.5: *Your* disjunctive strategy = conjunction of *Your* strategies

You have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ iff *You* have winning strategies in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(h)$ for all $h \in D$.

This fact follows immediately from the determinacy of the game and Proposition 3.4.4. However, there is also a direct constructive proof:

Proof of Proposition 3.4.5. "⇒": Let μ be a disjunctive winning strategy for *You* in the game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ and let $D = D' \bigvee h$. The idea is that *You* can use μ to win the run of the game where *I* only ever schedule h and its successors. The behavior of μ contains

[12]Actually: a path through the tree structure of σ

all the necessary information to define a strategy μ' for *You* in the evaluation game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(h)$. For a history $k = \langle k_1, \dots k_n \rangle$ of the evaluation game labeled "Y", let $\text{disj}(k)$ be the disjunctive history

$$D' \bigvee \langle k_1 \rangle, D' \bigvee \underline{\langle k_1 \rangle}, \dots, D' \bigvee \langle k_1, \dots, k_n \rangle, D' \bigvee \underline{\langle k_1, \dots, k_n \rangle}.$$

If k is labeled "Y", then $\text{disj}(k)$ is mapped to some $\text{disj}(g) \smile D' \bigvee (k \smile g)$ under μ. Consequently, we define $\mu'(k) = k \smile g$.

Let σ be a strategy for *Me* in $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(h)$. We have to show that *You* win against σ by playing μ'. To this end, we consider the run of the disjunctive game starting at D where *You* play μ and *I* play according to the following strategy: let $k = (\sigma; \mu')^n(h)$ for some n. If the current disjunctive state is $D' \bigvee k$, then *I* underline k. If $\text{disj}(k)$ is labeled "I", then *I* go to $D' \bigvee \sigma(k)$. If $\text{disj}(k)$ is labeled "Y", and *You* play according to μ, then the next disjunctive state is $D' \bigvee \mu'(k)$. Eventually, the game reaches the disjunctive state $D' \bigvee (\sigma; \mu')^m(h)$, where $(\sigma; \mu')^m(h) = O(\sigma, \mu')$ is terminal. Since μ is winning for *You*, this disjunctive state cannot be winning for *Me*. Thus, $O(\sigma, \mu')$ is winning for *You*. Since σ was arbitrary, μ' is winning.

"$\Leftarrow$": For every $h \in D$, let μ_h be a winning strategy for *You* in $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(h)$. *Your* strategy μ in $\mathbf{DG}^{\text{Hyb}}_{\mathcal{M}}(D)$ is as follows: in a disjunctive state $H \bigvee \underline{k}$ labeled "Y", k is a history in $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(h)$, for some h. Hence, *I* can use μ_h and go to $H \bigvee \sigma_h(k)$. Playing this way ensures that all game states contained in every resulting disjunctive state consist of histories of the σ_hs, against any opposing strategy from *Me*. By assumption, every such history that is also terminal is winning for *You*. Hence, the game cannot end in a winning disjunctive state for *Me*, which shows that μ is winning for *You*. □

3.5 The Disjunctive Game as a Provability Game

In this section, we lift the disjunctive game to a provability game and prove the adequacy of the resulting game. Intuitively, the new game $\mathbf{DG}^{\text{Hyb}}(D)$ can be interpreted as the scenario where the players of the evaluation game $\mathbf{DG}^{\text{Hyb}}_{\mathcal{M}}(D)$ forgot – or have not been informed – about the structure of the model $\mathcal{M}$. The goal of both players is to come up with strategies that guarantee them a win, independent of what the model $\mathcal{M}$ looks like.

Note that this "playing over a model blindly" is only possible because of the fact that the game trees of the disjunctive game $\mathbf{DG}^{\text{Hyb}}_{\mathcal{M}}(D)$ are the same, independent of $\mathcal{M}$. The only place where $\mathcal{M}$ comes into play is at the winning conditions. It is exactly these winning conditions that we need to alter to capture our intuition of *My* strategy being winning over all models:

(Win) Let D be terminal. *I* win and *You* lose the game if, for every model $\mathcal{M}$, *I* win the game $\mathbf{G}^{\text{Hyb}}_{\mathcal{M}}(D)$. Otherwise *You* win and *I* lose.

Note that this uniformity was achieved essentially by our change of the rules (R_U), $(R_\Box)$, and $(R_\Diamond)$, i.e. the use of the hybrid language.

Example 3.5.1. In Example 3.4.1, we saw a winning strategy for *Me* for the game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{O} : T \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$, where T was $\forall i.\forall j \forall k.\, (R(i,j) \wedge R(j,k) \to R(i,k))$. It is also a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(\mathbf{O} : T \bigvee \mathbf{P}, i : \Diamond\Diamond p \to \Diamond p)$: in the depicted strategy, the leaves are $\mathbf{O}, j : R(i,j) \bigvee \ldots \bigvee \mathbf{P}, i : R(i,j)$, $\mathbf{O}, k : R(j,k) \bigvee \ldots \bigvee \mathbf{P}, i : R(j,k)$ and $\mathbf{O}, k : p \bigvee \ldots \bigvee \mathbf{P}, k : p$ which are indeed winning for *Me* in all models.

In the formal definition of the game it is also enough to change the payoff function:

Definition 3.5.2: Disjunctive Game as Provability Game

The game $\mathbf{DG}^{\mathrm{Hyb}}(D)$ is the same as the game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ in Definition 3.4.2, except for the payoff function:

- Terminal disjunctive histories $\mathfrak{h}$ ending in $\langle \ldots D, * \rangle$ are mapped to
  - 1, if for every model $\mathcal{M}$, $\mathfrak{h}$ is winning in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$.
  - -1, otherwise.

Intuitively, a winning strategy for $\mathbf{DG}^{\mathrm{Hyb}}(D)$ should contain the information how to win $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ for every model $\mathcal{M}$. Indeed, this is true for *My* winning strategies:

Theorem 3.5.3: Adequacy, ltr

If σ is a disjunctive winning strategy for *Me* in $\mathbf{DG}^{\mathrm{Hyb}}(D)$, then σ is also a disjunctive winning strategy for *Me* in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ for every model $\mathcal{M}$.

Proof. The two games are identical, except maybe for the payoffs. Thus, *I* can use σ to play in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$. By assumption, every outcome resulting from playing σ is winning $\mathbf{DG}^{\mathrm{Hyb}}(D)$, and hence in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$. □

On the level of existence, the contraposition of the theorem is "If *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ for one model $\mathcal{M}$, then *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D)$". Using a similar proof, however, one can show the following constructive result:

Theorem 3.5.4: Adequacy, ltr (*You*-version)

If σ is a disjunctive winning strategy for *You* in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$ for one model $\mathcal{M}$, then it is also a disjunctive winning strategy for *You* in $\mathbf{DG}^{\mathrm{Hyb}}(D)$.

To complete the adequacy result, we want to show that *Your* winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D)$ gives rise to a model $\mathcal{M}$ and a winning strategy for *You* in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$.

Theorem 3.5.5: Adequacy, rtl (*You*-version)

If *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D)$, then there is a model $\mathcal{M}$ such that *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D)$.

The proof will be the subject of the following subsection. It uses techniques from [9] and [47]. Note that the proof is slightly easier compared to [26] and [28], since we do not care about regulations here, as the scheduling is done by *Me*. We will derive some general results on the disjunctive game and regulations in Chapter 5.

My best way to play

We will now describe a strategy σ for *Me* for the game $\mathbf{DG}^{\mathrm{Hyb}}(D_0)$. This strategy is – in a way – the optimal way to play the disjunctive game. Intuitively σ exploits all of *My* possible choices without sacrificing *My* winning chances.

Let us fix an enumeration of pairs (h, g) of histories h and game states g of the evaluation game such that every pair appears in this enumeration infinitely often[13]. Let us denote by $\#(h, g)$ the number of the pair (h, g) under this enumeration. Throughout the game, let us keep track of the number of execution steps n of σ. At D_0, $n = 0$. The strategy σ is as follows:

1. If in the current disjunctive state D, D^{ter} is winning, *I* end the game.
2. Otherwise, let $n = \#(h, g)$. If $D = D' \bigvee h$, and h is labeled
   a) "Y" (otherwise, skip), then underline h and *You* make *Your* move.
   b) "I" and $h \smile g$ is a history of the evaluation game (otherwise, skip), then duplicate h, schedule a copy of h, and go to $h \smile g$ in that copy, i.e., the new disjunctive state is $D' \bigvee h \bigvee h \smile g$.

   Increase n by 1, go to 1.

To put it into words: until the game reaches a winning disjunctive state, *My* strategy is to play in a way such that *I* always duplicate a state, then play by exhausting all possible moves in that state. The only fact from this construction that we need is the following: Let $\mathfrak{h}$ be the outcome in $\mathbf{DG}^{\mathrm{Hyb}}(D_0)$ resulting from *You* playing according to *Your* winning strategy and *Me* playing *My* best way σ. We say that a history h *appears along* $\mathfrak{h}$, and write $h \in \mathfrak{h}$ if it occurs in a disjunctive state in $\mathfrak{h}$. We say that h *disappears*, if $h \in \mathfrak{h}_n$ and for some $m > n$, $h \notin \mathfrak{h}_m$. We have the lemma:

[13]Both the set of histories and the set of game states are countably infinite, i.e. we can enumerate both. Interpreting natural numbers as coordinates in a table, we count them diagonally, which gives us an enumeration of pairs of natural numbers. Finally, the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$ is an enumeration of the set of natural numbers where every number appears infinitely often.

Lemma 3.5.6

Let $\mathfrak{h}$ be as above. Then:

1. Let $h \in \mathfrak{h}$ be a non-terminal history labeled "Y" in the evaluation game. Then at least one immediate successor of h appears along $\mathfrak{h}$.
2. Let $h \in \mathfrak{h}$ be a non-terminal history labeled "I" in the evaluation game. Then all immediate successors of h appear along $\mathfrak{h}$.

Proof. First, note that since *You* play according to *Your* winning strategy, $\mathfrak{h}$ is not a terminal winning disjunctive history for *Me*. This means Case 1 in the definition of σ is never reached.

1. Suppose h appeared in $\mathfrak{h}$ at stage $n \geq 0$ in the above construction. Since every pair appears in the enumeration infinitely often, there is some minimal $m \geq n$ such that $m = \#(h, g)$, for some g. At step m in the execution of σ against *Your* winning strategy, the current disjunctive state is of the form $D' \bigvee h$. According to σ, *I* underline h and *You* move to some successor h', according to *Your* winning strategy. This means the new game state is of the form $D' \bigvee h'$, hence h' is the successor of h appearing along $\mathfrak{h}$.
2. As before, suppose h appeared in $\mathfrak{h}$ at stage $n \geq 0$. Now we additionally assume that $h = h' \smile g$ and fix an arbitrary g' such that $h \smile g'$ is a history of the evaluation game. By the properties of $\#$, there is a minimal $m \geq n$ such that $m = \#(h, g')$. Since *I* always first duplicate histories labeled "I", before *I* make a move into them, h does not disappear. Hence, at step m in the execution of σ, the current disjunctive state is of the form $D' \bigvee h$. According to σ, *I* duplicate h and go to $h \smile g'$ in one copy, i.e. the new disjunctive state is $D' \bigvee h \bigvee (h \smile g')$, which shows that $h \smile g'$ appears along $\mathfrak{h}$.

□

We can now show that $\mathfrak{h}$ gives rise to a model $\mathcal{M}_{\mathfrak{h}}$ with the property that *You* have a winning strategy for every h appearing along $\mathfrak{h}$. In particular, this gives *You* a winning strategy for $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(D_0)$ by Proposition 3.4.5. We will need the following notions in a later section, hence we give the definition a bit more general than is needed here.

Definition 3.5.7: Model defined by elementary game states

Let $\mathcal{E}$ be a set of elementary game states. We define the relation $i \sim_{\mathcal{E}} j$ between two nominals i and j iff $\mathbf{O}, i : j$ or $\mathbf{O}, j : i$ appear in $\mathcal{E}$. Let $\approx_{\mathcal{E}}$ be the symmetric, reflexive and transitive closure of $\sim_{\mathcal{E}}$. We write $[i]$ for the equivalence class of i. Let $\mathcal{M}_{\mathcal{E}}$ be the following named model:

- Worlds W: Equivalence classes of nominals,
- Accessibility relation $R_{\mathcal{E}}$: We have $[i]R_{\mathcal{E}}[j]$ iff for some $i' \in [i]$, $j' \in [j]$, $k \in N$, the state $\mathbf{O}, k : R(i', j')$ is in $\mathcal{E}$.
- Valuation function $V_{\mathcal{E}}$: $[i] \in V_{\mathcal{E}}(p)$ iff for some $i' \in [i]$, the state $\mathbf{O}, i' : p$ is in $\mathcal{E}$.
- Assignment $g_{\mathcal{E}}$: $g_{\mathcal{E}}(i) = [i]$, for all $i \in N$.

We are interested in the case where $\mathcal{E}$ is the set of all elementary game states appearing as final game states of terminal histories in $\mathfrak{h}$ and write $\mathcal{M}_{\mathfrak{h}}$ instead of $\mathcal{M}_{\mathcal{E}}$. Let us look at a simple example. We consider the game $\mathbf{DG}^{\mathrm{Hyb}}(\mathbf{O} : \forall k.R(k,k) \bigvee \mathbf{O}, i^* : j^* \bigvee \mathbf{O}, i^* : p)$. *My* best way to play is to keep moving into $\mathbf{O} : \forall k.R(k,k)$ by choosing a nominal. This procedure never leads to an elementary winning disjunctive state for *Me*, but results in the infinite outcome $\mathfrak{h}$, containing all game states of the form $\mathbf{O}, k' : R(k,k)$. The model $\mathcal{M}_{\mathfrak{h}}$ looks as follows: Its worlds are the equivalence classes $[i]$, where $[i] = [j]$ if and only if i and j are equal or if one is i^* and the other j^*. For every world $[k]$, we have $[k]R_{\mathfrak{h}}[k]$, i.e. the model is reflexive. The valuation function satisfies $[i^*] \in V_{\mathfrak{h}}(p)$.

Lemma 3.5.8

Let $\mathcal{M}_{\mathfrak{h}}$ be the model $\mathcal{M}_{\mathcal{E}}$ from the above definition, where $\mathcal{E}$ is the set of all elementary game states appearing along $\mathfrak{h}$. If h appears along $\mathfrak{h}$, then *You* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(h)$.

Proof. We prove this lemma by induction on the degree of the last game state g of h. The elementary cases where g is of the form $\mathbf{O}, i : el$ follow directly from the definition of $\mathcal{M}_{\mathfrak{h}}$. Assume, $g = \mathbf{P}, i : p$ appears along $\mathfrak{h}$, but $\mathcal{M}_{\mathfrak{h}}, [i] \models p$. The latter implies that for some $j \in [i]$, $\mathbf{O}, j : p$ appears along $\mathfrak{h}$. Since the elementary states of $\mathfrak{h}$ are cumulative, there is a disjunctive state D in $\mathfrak{h}$ containing the three states $\mathbf{P}, i : p$, $\mathbf{O}, j : p$ and one of $\mathbf{O}, i : j$ or $\mathbf{O}, j : i$. Clearly, there is no model $\mathcal{M}$ satisfying $\mathcal{M}, g(i) \not\models p$, $\mathcal{M}, g(j) \models p$ and $g(i) = g(j)$ at the same time. Thus, *I* would have won the game at D, a contradiction to the fact that $\mathfrak{h}$ resulted from *You* playing *Your* winning strategy. The cases for $\mathbf{P}, k : R(i,j)$ and $\mathbf{P}, i : j$ are similar.

For the inductive step, let $h \in \mathfrak{h}$ be non-terminal with $h = h' \smile g$ and label "Y". By Lemma 3.5.6, some successor $h \smile g'$ of h appears along $\mathfrak{h}$. By inductive hypothesis, there is a winning *You* for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(h \smile g')$. Thus, *You* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(h)$, by Lemma 2.1.14.

If h is non-terminal with $h = h' \smile g$ and label "I", then, by Lemma 3.5.6, all immediate successors $h \smile g'$ of h appear along $\mathfrak{h}$. For each $h \smile g'$ there is a winning strategy for *You* in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(h \smile g')$. Thus, *You* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(h)$, by Lemma 2.1.14. □

Proof of Theorem 3.5.5. Suppose, *You* have a winning strategy for the game $\mathbf{DG}^{\mathrm{Hyb}}(D)$. Let *You* play according to this strategy and *Me* according to the strategy σ from above. Let $\mathfrak{h}$ be the corresponding outcome of the game and $\mathcal{M}_{\mathfrak{h}}$ the model form Definition 3.5.7. By Lemma 3.5.8, *You* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{M_{\mathfrak{h}}}(h)$ for all $h \in D$. Therefore, by Proposition 3.4.5, *You* have a winning strategy for $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}_{\mathfrak{h}}}(D)$. □

3.6 Games for Hybrid Logic

In this section, we use our results so far to present several games for hybrid logic in terms of the disjunctive game from the previous subsection. For each game, we state the corresponding adequacy result. Remember that we assume that all models are named.

1. **A game for validity**: Let G be a formula.

   $$I \text{ have a winning strategy in } \mathbf{DG}^{\mathrm{Hyb}}(\mathbf{P} : G) \iff G \text{ is valid.}$$

2. **A game for entailment**: Let $\mathcal{T}$ be a finite set of formulas and set $D_i(\mathcal{T}) = \{\mathbf{O}, i : T \mid T \in \mathcal{T}\}$. Let $\mathbf{E}(\mathcal{T}, G)$ be the following game: In the first round, *You* choose a nominal i. The game then continues with $\mathbf{DG}^{\mathrm{Hyb}}(D_i(\mathcal{T}) \bigvee \mathbf{P}, i : G)$.

   $$I \text{ have a winning strategy in } \mathbf{E}(\mathcal{T}, G) \iff \mathcal{T} \models G.$$

3. **A game for global consequence**: We say that $\mathcal{T}$ *globally entails* G and write $\mathcal{T} \models_g G$ iff for every model $\mathcal{M} \models \mathcal{T}$ implies $\mathcal{M} \models G$. Let $D_g(\mathcal{T}) = \{\mathbf{O} : T \mid T \in \mathcal{T}\}$.

   $$I \text{ have a winning strategy in } \mathbf{DG}^{\mathrm{Hyb}}(D_g(\mathcal{T}) \bigvee \mathbf{P} : G) \iff \mathcal{T} \models_g G.$$

4. **A game for validity over frames**: Let $\mathfrak{F}$ be a class of frames characterized by the finite set of formulas $\mathcal{F}$.

   $$I \text{ have a winning strategy in } \mathbf{DG}^{\mathrm{Hyb}}(D_g(\mathcal{F}) \bigvee \mathbf{P} : G) \iff \models_{\mathfrak{F}} G.$$

5. **A game for entailment over frames**: Let $\mathbf{EF}(\mathcal{F}, \mathcal{T}, G)$ be the following game: In the first round, *You* choose a nominal i. The game then continues with the disjunctive game $\mathbf{DG}^{\mathrm{Hyb}}(D_g(\mathcal{F}) \bigvee D_i(\mathcal{T}) \bigvee \mathbf{P}, i : F)$

   $$I \text{ have a winning strategy in } \mathbf{EF}(\mathcal{F}, \mathcal{T}, F) \iff \mathcal{T} \models_{\mathfrak{F}} F.$$

The proofs readily follow from the previous results. For example, let us consider the game for entailment. Suppose, *I* have a winning strategy in $\mathbf{E}(\mathcal{T}, F)$. Let $\mathcal{M}$ be a model and w a world, and let i be a name of w. By assumption, *I* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D_i(T) \bigvee \mathbf{P}, i : F)$, since *You* may choose i in the first round. By Theorem 3.5.3, *I* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}(D_i(T) \bigvee \mathbf{P}, i : F)$. By Proposition 3.4.4, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, i : F)$ or in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{O}, i : T)$ for some $T \in \mathcal{T}$. In the first case, $\mathcal{M}, \mathrm{g}(i) \models F$ and in the second case $\mathcal{M}, \mathrm{g}(i) \not\models \mathcal{T}$, both by Theorem 3.3.7. If, on the other hand, *You* have a winning strategy in $\mathbf{E}(\mathcal{T}, F)$, then for some i, *You* must have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D_i(T) \bigvee \mathbf{P}, i : F)$. Hence, by Theorem 3.5.4 and Proposition 3.4.4, there is a model $\mathcal{M}$ such that*You* have winning strategies in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, i : F)$ and in all $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{O}, i : T)$ where $T \in \mathcal{T}$. By Theorem 3.3.7, $\mathcal{M}, \mathrm{g}(i) \models \mathcal{T}$ and $\mathcal{M}, \mathrm{g}(i) \not\models F$.

Example 3.6.1. The singleton set $\mathcal{F} = \{\forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k))$ characterizes the class of transitive frames. *My* winning strategy in the game $\mathbf{DG}^{\mathrm{Hyb}}(D_g(\mathcal{F}) \bigvee \mathbf{P} : \Diamond\Diamond p \to \Diamond p)$ is to first schedule $\mathbf{P} : \Diamond\Diamond p \to \Diamond p$. *You* answer with some nominal i and the game continues as depicted in Figures 3.8 and 3.9. This winning strategy can be seen as a proof that $\Diamond\Diamond p \to \Diamond p$ is valid over transitive frames. In fact, we will develop a sequent-style proof system representing *My* winning strategies in the next section.

3.7 From Strategies to Proofs

As we see in Example 3.6.1, winning strategies for *Me* in the probability version of the disjunctive game can be seen as proofs of statements of validity, or entailment. In this section, we will make these observations formal by introducing a sequent calculus $\mathbf{DS}^{\mathrm{Hyb}}$ (disjunctive strategies), where proofs exactly correspond to *My* winning strategies in the disjunctive game (hence the name).

First, we will demonstrate that winning strategies, although by definition infinite, can be finitized. To show that $\mathbf{DS}^{\mathrm{Hyb}}$ is a useful proof system it is also necessary to prove that checking whether elementary disjunctive states are winning for *Me* is tractable. This is done in Subsection 3.7.3. $\mathbf{DS}^{\mathrm{Hyb}}$ itself is introduced in Subsection 3.7.2 and some connections to existing systems are drawn in Subsection 3.7.4.

3.7.1 *Your* optimal choices

In this section, we want to modify the disjunctive game so that it becomes finitely branching in "Y"-nodes. This alteration will help us conveniently formulate the disjunctive game as a calculus. Infinite branching occurs only in the case of the rules (R_U), $(R_\Box)$, $(R_\forall)$, and $(R_\Diamond)$, where branching is parametrized by the nominals. We will show that in these situations, there is an optimal choice for *You*, so *I* can expect *You* to play according to this choice.

First, we need to define substitutions formally. For a sequence x, let x_n denote its n-th element. Let F be a formula and a and b two sequences of nominals of the same length, where every nominal occurs only once in each sequence. We define $F[a/b]$ as the formula obtained by simultaneously substituting for every number n all occurrences of a_n in F with b_n. For example, let $a = \langle i, j\rangle$, $b = \langle k, l\rangle$ and $F = @_i(k \vee j)$. Then $F[a/b] = @_k(k \vee l)$. Note that a substitution also happened in the index of the @-operator. As another example let $a = \langle i_1, i_2, ...\rangle$ and $b = \langle i_2, i_3, ...\rangle$. Then $i_1[a/b] = i_2$, because substitution happens simultaneously. We need to extend the notion of substitution to game states:

for a game state $g = \mathbf{Q}, i : F$ of the evaluation game and two sequences of nominals a, b, we define the substitution $g[a/b]$ as $\mathbf{Q}, i[a/b] : F[a/b]$. Similarly, we extend this definition to histories, strategies, and disjunctive states.

Proposition 3.7.1: *Your* optimal choice

Let j be nominal not occurring in D or F and different from i. Then:

1. *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P} : F)$ iff *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, j : F)$.

2. *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, i : \Box F)$ iff *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, j : R(i,j) \rightarrow F)$.

3. *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{O}, i : \Diamond F)$ iff *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, j : R(i,j) \wedge F)$.

4. *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, i : \forall k.F)$ iff *You* have a winning strategy in $\mathbf{DG}^{\text{Hyb}}(D \bigvee \mathbf{P}, i : F[k/j])$.

This result implies that *My* winning strategies in the disjunctive game can be finitely represented: in every disjunctive state whose children branch over the nominals, it is enough to consider a single child only, given by a nominal j not appearing in that disjunctive state. The following result shows that the same can be achieved for the games for entailment and entailment over frames, where *You* choose a nominal in the first round:

Proposition 3.7.2

Let i be a nominal not occurring in $\mathcal{T}$ or F. Then:

1. *You* have a winning strategy in $\mathbf{E}(\mathcal{T}, F)$ iff *You* have a winning strategy for $\mathbf{DG}^{\text{Hyb}}(D_i(\mathcal{T}) \bigvee \mathbf{P}, i : F)$.

2. *You* have a winning strategy in $\mathbf{EF}(\mathcal{F}, \mathcal{T}, F)$ iff *You* have a winning strategy for $\mathbf{DG}^{\text{Hyb}}(\mathcal{D}_g(\mathcal{F}) \bigvee D_i(\mathcal{T}) \bigvee \mathbf{P}, i : F)$.

Example 3.7.3. In Example 3.6.1, we discussed the disjunctive game $\mathbf{DG}^{\text{Hyb}}(\mathbf{O} : T \bigvee \mathbf{P} : \Diamond\Diamond p \rightarrow \Diamond p)$, where T was $\forall i.\forall j \forall k.\,(R(i,j) \wedge R(j,k) \rightarrow R(i,k))$. The choices of i (in the game's first move starting in $\mathbf{P} : \Diamond\Diamond p \rightarrow \Diamond p$), j and k are optimal for *You* in Figure 3.8. Therefore, this finite tree should be accepted as a winning strategy for the infinitely branching game $\mathbf{DG}^{\text{Hyb}}(\mathbf{O} : T \bigvee \mathbf{P} : \Diamond\Diamond p \rightarrow \Diamond p)$, even without the labels "i/j branches over N".

The rest of the subsection is devoted to proving Proposition 3.7.1. The proof is rather technical and split up into a series of smaller lemmas.

Lemma 3.7.4

Let $\mathcal{M}_1 = (\mathsf{W}, \mathsf{R}, \mathsf{V}, \mathsf{g}_1)$ and $\mathcal{M}_2 = (\mathsf{W}, \mathsf{R}, \mathsf{V}, \mathsf{g}_2)$ be named and $\mathsf{g}_2(i[b/a]) = \mathsf{g}_1(i)$ for all nominals i. Then for all game states g, $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_1}(g) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_2}(g[b/a])$.

Proof. By the assumption, g_2 is surjective, even if restricted to $N[b/a] = \{i[b/a] : i \in N\}$. By Proposition 3.3.9 it is therefore enough to prove $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_1}(g) \cong \mathbf{G}^{\mathrm{Hyb},N[b/a]}_{\mathcal{M}_2}(g[b/a])$. As usual, we show the claim by induction on the degree of g.

If g is elementary and of the form $\mathbf{P}, i : j$ then it is winning for *Me* over $\mathcal{M}_1$ if and only if $\mathsf{g}_1(i) = \mathsf{g}_1(j)$. By assumption, this is equivalent to $\mathsf{g}_2(i[b/a]) = g_2(i[b/a])$, which means that $\mathbf{O}, i[b/a] : j[b/a]$ is winning for *Me* over $\mathcal{M}_2$. The other elementary cases are similar.

As an example of a simple induction step, we consider $\mathbf{P}, i : G_1 \vee G_2$. By Lemma 2.1.14, if *I* have a winning strategy in that game state over $\mathcal{M}_1$ then there is some $k \in \{1, 2\}$ such that *I* have a winning strategy in $\mathbf{P}, i : G_k$. By the inductive hypothesis, *I* have a winning strategy in $\mathbf{P}, i[b/a] : G_k[b/a]$ over $\mathcal{M}_2$, and hence, by Lemma 2.1.14, in $(\mathbf{P}, i : G_1 \vee G_2)[b/a]$. The other direction is similar. For the rest of the proof, we do not mention the application of Lemma 2.1.14 again.

The most interesting induction step is for the modal rules, so let us consider $\mathbf{P}, i : \Box G$. Suppose, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_1}(\mathbf{P}, i : \Box G)$. Then, for every nominal j, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}_1}(\mathbf{P}, j : R(i, j) \to G)$. By the inductive hypothesis, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb},N[b/a]}_{\mathcal{M}_2}(\mathbf{P}, j[b/a] : R(i[b/a], j[b/a]) \to G[b/a])$. In other words, *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb},N[b/a]}_{\mathcal{M}_2}(\mathbf{P}, k : R(i[b/a], k) \to G[b/a])$, for every $k \in N[b/a]$. Since branching in this game is restricted over $N[b/a]$, we conclude that *I* have a winning strategy in $\mathbf{G}^{\mathrm{Hyb},N[b/a]}_{\mathcal{M}_2}((\mathbf{P}, i : \Box G)[b/a])$. The other direction, as well as the other cases of induction steps, are similar. □

Lemma 3.7.5

If $\mathsf{g}(k) = \mathsf{g}(l)$, then $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g[k/l])$.

Proof. We show that $\mathsf{g}(i[k/l]) = \mathsf{g}(i)$ for all nominals i. If $i \neq k$, then $\mathsf{g}(i[k/l]) = \mathsf{g}(i)$. If $i = k$, then by the assumption, $\mathsf{g}(i[k/l]) = \mathsf{g}(l) = \mathsf{g}(k) = \mathsf{g}(i)$. The statement of the lemma follows from this fact and Lemma 3.7.4. □

For a model $\mathcal{M}$, and two sequences of nominals a, b, let $\mathcal{M}[a/b]$ be the same as $\mathcal{M}$, except for the denotation function: $\mathsf{g}_{[a/b]}(i) = \mathsf{g}(i[a/b])$.

Lemma 3.7.6

Let $\mathcal{M}$ be named and a, b two sequences of nominals with $\mathrm{range}(a) \subseteq \mathrm{range}(b)$. Then $\mathcal{M}_{[a/b]}$ is $N[b/a]$-named. Furthermore, $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(g[b/a])$.

Proof. We prove that $\mathsf{g}_{[a/b]}$ is surjective when restricted to $N[b/a] = \{i[b/a] : i \in N\}$. Let w be a world and i its name under g. If $i \notin \mathrm{range}(b)$, then $i \notin \mathrm{range}(b)$ and we have

$$i[b/a][a/b] = i[a/b] = i.$$

If $i \in \mathrm{b}$, then $i = b_m$ for some m. Then

$$i[b/a][a/b] = b_m[b/a][a/b] = a_m[a/b] = b_m = i.$$

This shows that $\mathsf{g}_{[a/b]}(i[b/a]) = \mathsf{g}(i[b/a][a/b]) = \mathsf{g}(i)$, i.e. w has a name in $N[b/a]$ under $g_{[a/b]}$. This identity together with Lemma 3.7.4 also implies the strategic equivalence of the game $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ and $\mathbf{G}(\mathcal{M}[a/b], g[b/a])$. □

We are now ready to prove Proposition 3.7.1.

Proof of Proposition 3.7.1. We will show 2. The direction from right to left is clear, so let us assume *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(D \bigvee \mathbf{P}, i : \Box F)$ with j as in the assumption. By Theorem 3.5.5 and Proposition 3.4.5, there is a named model $\mathcal{M}$ such that *You* have winning strategies for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g)$ for all $g \in D$ and for $\mathbf{P}, i : \Box F$. The latter implies that *You* have a winning strategy for $\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, k : R(i,k) \to F)$ for some nominal k.

Let $j_1, j_2, ...$ be a sequence of nominals not occurring in D or F and different from k, j, and i. Let $a = \langle j, j_1, j_2, ... \rangle$ and $b = \langle k, j, j_1, j_2, ... \rangle$. We have that $\mathrm{range}(a) \subseteq \mathrm{range}(b)$, therefore Lemma 3.7.6 applies. We have the following chain of equivalences:

$$\begin{aligned}
&\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(\mathbf{P}, k : R(i,k) \to F) && \\
\cong\ &\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(\mathbf{P}, k[b/a] : R(i[b/a], k[b/a]) \to F[b/a]) && \text{by Lemma 3.7.6} \\
=\ &\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(\mathbf{P}, j : R(i,j) \to F[k/j]) && \text{by conditions on } i, j, k \\
=\ &\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(\mathbf{P}, j[k/j] : R(i[k/j], j[k/j]) \to F[k/j]) && \\
\cong\ &\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(\mathbf{P}, j : R(i,k) \to F) && \text{by Lemma 3.7.5} \\
& && \text{and } \mathsf{g}_{[a/b]}(j) = \mathsf{g}_{[a/b]}(k)
\end{aligned}$$

By this equivalence and the assumption, *You* have a winning strategy for $\mathbf{P}, j : R(i,j) \to F$ over $\mathcal{M}[a/b]$.

Similarly, we obtain a winning strategy for *You* for $g \in D$ by using the equivalence

$$\mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}}(g) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(g[b/a]) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(g[k/j]) \cong \mathbf{G}^{\mathrm{Hyb}}_{\mathcal{M}[a/b]}(g),$$

the same lemmas as before and the fact that no nominals from a appear in g.

Putting all things together, *You* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}[a,b]}(D \bigvee \mathbf{P}, i : R(i,j) \to F)$, by Proposition 3.4.5. By Theorem 3.5.5, *You* have a winning strategy in the disjunctive game $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}[a,b]}(D \bigvee \mathbf{P}, i : R(i,j) \to F)$. □

3.7.2 The proof system DS$^{\mathrm{Hyb}}$

We are now ready to formulate the sequent calculus $\mathbf{DS}^{\mathrm{Hyb}}$ (Figure 3.1). We say that a string $i : F$ consisting of a hybrid logic formula F and a nominal i is a *labeled formula*. We call an object $\Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulas and labeled formulas, a *sequent*. A disjunctive state D can be rewritten as a sequent $\Gamma \Rightarrow \Delta$ where

$$\Gamma = \{F : \langle \ldots, \mathbf{O} : F\rangle \in D\} + \{i, F : \langle \ldots, \mathbf{O}, i : F\rangle \in D\},$$
$$\Delta = \{F : \langle \ldots, \mathbf{P} : F\rangle \in D\} + \{i : F : \langle \ldots, \mathbf{P}, i : F\rangle \in D\}.$$

i,e., Γ comprises all (labeled) formulas that are final states of a history in D and have the prefix $\mathbf{O}$ and Δ comprises all such formulas with the prefix $\mathbf{P}$ in D. For example, the disjunctive game state $\langle \ldots, \mathbf{O}, i : \Box p\rangle \bigvee \langle \ldots, \mathbf{P} : p\rangle$ becomes the sequent $i : \Box p \Rightarrow p$. Hence, we omit writing down entire histories in disjunctive game states, as in the examples in the previous sections. This is justified by the fact that the winnability of a history of the evaluation game depends on its last game state only. Given this correspondence, we will use the notation for disjunctive game states and sequents interchangeably.

Table 3.1: The proof system $\mathbf{DS}^{\mathrm{Hyb}}$. In the rules $(R_\Box)$, $(L_\Diamond)$, $(R_\forall)$, and (R_U), the nominal j is a nominal not occurring in the lower sequent and different from i. This condition corresponds to *Your* optimal choice.

Axioms

$\Gamma \Rightarrow \Delta$, if it is winning

this means for every model $\mathcal{M}$ there is some elementary $g \in \Gamma \Rightarrow \Delta$ such that $\mathcal{M} \models g$

Structural Rules

$$\frac{\Gamma, i : F, i : F \Rightarrow \Delta}{\Gamma, i : F \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow i : F, i : F, \Delta}{\Gamma \Rightarrow i : F, \Delta}\ (R_c)$$

Propositional rules

$$\frac{\Gamma, i : F \Rightarrow \Delta \qquad \Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \vee G \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow i : F, \Delta}{\Gamma \Rightarrow i : F \vee G, \Delta}\ (R^1_\vee)$$

$$\frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma, i : F \wedge G \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \vee G, \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \wedge G \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow i : F, \Delta \quad \Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \wedge G, \Delta}\ (R_\wedge)$$

$$\frac{\Gamma \Rightarrow i : F, \Delta \quad \Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \to G \Rightarrow \Delta}\ (L_\to) \qquad \frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma \Rightarrow i : F \to G, \Delta}\ (R^1_\to)$$

$$\frac{\Gamma \Rightarrow, i : F, \Delta}{\Gamma, i : \neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \to G, \Delta}\ (R^2_\to)$$

$$\frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma \Rightarrow i : \neg F, \Delta}\ (R_\neg)$$

Modal rules

$$\frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma, F \Rightarrow \Delta}\ (L_U) \qquad \frac{\Gamma \Rightarrow i : F, \Delta}{\Gamma \Rightarrow F, \Delta}\ (R_U)$$

$$\frac{\Gamma, j : R(i,j) \to F \Rightarrow \Delta}{\Gamma, i : \Box F \Rightarrow \Delta}\ (L_\Box) \qquad \frac{\Gamma \Rightarrow j : R(i,j) \to F, \Delta}{\Gamma \Rightarrow i : \Box F, \Delta}\ (R_\Box)$$

$$\frac{\Gamma, j : R(i,j) \wedge F \Rightarrow \Delta}{\Gamma, i : \Diamond F \Rightarrow \Delta}\ (L_\Diamond) \qquad \frac{\Gamma \Rightarrow j : R(i,j) \wedge F, \Delta}{\Gamma \Rightarrow i : \Diamond F, \Delta}\ (R_\Diamond)$$

$$\frac{\Gamma, j : F \Rightarrow \Delta}{\Gamma, i : @_j F \Rightarrow \Delta}\ (L_@) \qquad \frac{\Gamma \Rightarrow j : F, \Delta}{\Gamma \Rightarrow i : @_j F, \Delta}\ (R_@)$$

Quantifier rules

$$\frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma, F \Rightarrow \Delta}\ (L_U) \qquad \frac{\Gamma \Rightarrow i : F, \Delta}{\Gamma \Rightarrow F, \Delta}\ (R_U)$$

$$\frac{\Gamma, i : F[k/j] \Rightarrow \Delta}{\Gamma, i : \forall k.F \Rightarrow \Delta}\ (L_\forall) \qquad \frac{\Gamma \Rightarrow i : F[k/j], \Delta}{\Gamma \Rightarrow i : \forall k.F, \Delta}\ (R_\forall)$$

Apart from this encoding of disjunctive states as sequents and the traditional bottom-up notation of proof trees, proofs in $\mathbf{DS}^{\mathrm{Hyb}}$ exactly correspond to *My* winning strategies in the disjunctive game: the user of the proof system takes the role of *Me*, scheduling game states and choosing moves in "I"-states. Branching in the proof tree corresponds to branching in the winning strategy, i.e., *Your* possible moves. Infinite branching is modified according to the discussion in the previous subsection. Duplication in the game takes the form of left and right contraction rules. The axioms are exactly the (encoding of) winning[14] disjunctive states. Using this correspondence we immediately get the

[14] We say that $\Gamma \Rightarrow \Delta$ is *winning* if for its corresponding disjunctive game state D the disjunctive history $\langle \ldots, D, * \rangle$ is winning for *Me*. In other words: at D, *I* can end the game and win.

fact that the disjunctive game is adequate with respect to the calculus, as well as the following reformulations of the results in Section 3.6:

Theorem 3.7.7: proofs = winning strategies

$\Gamma \Rightarrow \Delta$ is provable in $\mathbf{DS}^{\mathrm{Hyb}}$ iff *I* have a winning strategy in $\mathbf{DG}^{\mathrm{Hyb}}(\Gamma \Rightarrow \Delta)$.

Corollary 3.7.8

Let F be a formula, and $\mathfrak{F}$ a class of frames characterized by $\mathcal{F}$. Then

1. $\vdash_{\mathbf{DS}^{\mathrm{Hyb}}} F$ iff F is valid,
2. $\{i : T \mid T \in \mathcal{T}\} \vdash_{\mathbf{DS}^{\mathrm{Hyb}}} i : F$ iff $\mathcal{T} \models F$,
3. $\mathcal{T} \vdash_{\mathbf{DS}^{\mathrm{Hyb}}} F$ iff $\mathcal{T} \models_g F$,
4. $\mathcal{F} \vdash_{\mathbf{DS}^{\mathrm{Hyb}}} F$ iff $\models_{\mathfrak{F}} F$,
5. $\mathcal{F} + \{i : T \mid T \in \mathcal{T}\} \vdash_{\mathbf{DS}^{\mathrm{Hyb}}} i : F$ iff $\mathcal{T} \models_{\mathfrak{F}} F$.

In 2 and 5, i is a nominal not occurring in F, $\mathcal{T}$, or $\mathcal{F}$.

Here, we write $\Gamma \vdash_{\mathbf{DS}^{\mathrm{Hyb}}} \Delta$ iff $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{DS}^{\mathrm{Hyb}}$.

The calculus $\mathbf{DS}^{\mathrm{Hyb}}$ takes a familiar form: The rules for the logical connectives are the (labeled) versions of the usual propositional rules of a sequent calculus for classical logic. The modal operator rules come in the form of their first-order translations. Apart from the axioms, $\mathbf{DS}^{\mathrm{Hyb}}$ can therefore be seen as a fragment of the usual sequent system for first-order logic. In turn, $\mathbf{DS}^{\mathrm{Hyb}}$ is an extension of the sequent calculus **G3K** [47] to hybrid logic. Similarly to **G3K**, a failed proof search in $\mathbf{DS}^{\mathrm{Hyb}}$ directly gives rise to a countermodel. This follows from our proof of Theorem 3.5.5, where the explicit countermodel $\mathcal{M}_{\mathfrak{h}}$ was constructed. We will investigate the connections to other calculi in Subsection 3.7.4.

Example 3.7.9. Let $\mathcal{F}$ characterize the set of transitive frames, i.e., $\mathcal{F}$ consists of the single formula $\forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k))$. Let us write $\Gamma \Rightarrow_{\mathfrak{F}} \Delta$ for $\Gamma \cup \mathcal{F} \Rightarrow \Delta$. We show that the following rule scheme for transitivity is admissible in $\mathbf{DS}^{\mathrm{Hyb}}$:

$$\frac{\Gamma \Rightarrow_{\mathfrak{F}} R(i,j), \Delta \qquad \Gamma \Rightarrow_{\mathfrak{F}} R(j,k), \Delta \qquad \Gamma, , R(i,k) \Rightarrow_{\mathfrak{F}} \Delta}{\Gamma \Rightarrow_{\mathfrak{F}} \Delta}\ (trans)$$

The following derivation resembles *My* winning strategy in Figure 3.9.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow_{\mathfrak{F}} R(i,j), \Delta \quad \Gamma \Rightarrow_{\mathfrak{F}} R(j,k), \Delta}{\Gamma \Rightarrow_{\mathfrak{F}} R(i,j) \wedge R(j,k), \Delta}(R_\wedge) \quad \Gamma, R(i,k) \Rightarrow_{\mathfrak{F}} \Delta}{\Gamma, R(i,j) \wedge R(j,k) \to R(i,k) \Rightarrow_{\mathfrak{F}} \Delta}(L_\to)}{\Gamma, \forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k)) \Rightarrow_{\mathfrak{F}} \Delta}(L_\forall) \times 3}{\Gamma \Rightarrow_{\mathfrak{F}} \Delta}(L_c)$$

In the step at the bottom, we apply (L_c) to $\forall i.\forall j.\forall k.\,(R(i,j) \wedge R(j,k) \to R(i,k)) \in \mathcal{F}$. Also, note that we avoid writing the label of relational atoms and propositional formulas built from relational atoms since the truth of these formulas does not depend on the particular world they are evaluated in.

This example shows that we can use the game-theoretic approach to formulate structural rules for classes of frames that are characterizable in the hybrid language. We could therefore reformulate our proof system with these structural rules in place of a set of formulas characterizing a frame condition (to show that $\mathbf{DS}^{\mathrm{Hyb}} + (trans)$ derives the transitivity-axioms, we can use invertibility of the logical rules), similar to **G3K** [47]. The rule for transitivity there is easily seen to be derivable using $(trans)$, and $(trans)$ is admissible in **G3K** plus the transitivity rule.

Example 3.7.10. We show how to derive $\Rightarrow_{\mathfrak{F}} \Diamond\Diamond p \to \Diamond p$ in $\mathbf{DS}^{\mathrm{Hyb}}$, with $\Rightarrow_{\mathfrak{F}}$ as in the previous example. The following proof is just a rewriting of *My* winning strategy in the game $\mathbf{DG}^{\mathrm{Hyb}}(\mathcal{D}_g(\mathcal{F}) \bigvee \mathbf{P} : \Diamond\Diamond p \to \Diamond p)$ from Example 3.6.1. In all derivations, we omit writing down the label of relational formulas:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R(i,j), R(j,k), k:p \Rightarrow_{\mathfrak{F}} R(i,k) \quad R(i,j), R(j,k), k:p \Rightarrow_{\mathfrak{F}} k:p}{R(i,j), R(j,k), k:p \Rightarrow_{\mathfrak{F}} k : R(i,k) \wedge p}(L_\wedge)}{R(i,j), R(j,k), k:p \Rightarrow_{\mathfrak{F}} i : \Diamond p}(R_\Diamond)}{R(i,j), k : R(j,k) \wedge p \Rightarrow_{\mathfrak{F}} i : \Diamond p}(L_\wedge) \times 2, (R_c) \times 2}{R(i,j), j : \Diamond p \Rightarrow_{\mathfrak{F}} i : \Diamond p}(L_\Diamond)}{j : R(i,j) \wedge \Diamond p \Rightarrow_{\mathfrak{F}} i : \Diamond p}(L_\wedge) \times 2, (R_c) \times 2}{i : \Diamond\Diamond p \Rightarrow_{\mathfrak{F}} i : \Diamond p}(L_\Diamond)}{\Rightarrow_{\mathfrak{F}} i : \Diamond\Diamond p \Rightarrow_{\mathfrak{F}} \Diamond p}(R_\to) \times 2, (R_c) \times 2}{\Rightarrow_{\mathfrak{F}} \Diamond\Diamond p \Rightarrow_{\mathfrak{F}} \Diamond p}(R_U)$$

The proof at the left topmost sequent continues with an application of the $(trans)$-rule from Example 3.7.9 with $\Gamma = \{R(i,j), R(j,k), k:p\}$ and $\Delta = \{R(i,k)\}$.

3.7.3 Checkability of Initial Sequents is tractable

To establish $\mathbf{DS}^{\mathrm{Hyb}}$ as a suitable proof system, we must demonstrate that we can effectively check whether given elementary sequents are indeed winning, i.e. initial sequents[15]. As an upper limit, checking elementary sequents must be easier than checking validity in hybrid logic, which is known to be PSPACE-complete. In fact, checking elementary sequents turns out to be in P:

Proposition 3.7.11: Checking axioms is in P

Checking whether a disjunctive state D is winning for *Me* is in P.

The proof of this proposition is the topic of the remaining subsection.

Definition 3.7.12: Homomorphism of models

Let $\mathcal{M} = (\mathsf{W}^{\mathcal{M}}, \mathsf{R}^{\mathcal{M}}, \mathsf{V}^{\mathcal{M}}, \mathsf{g}^{\mathcal{M}})$ and $\mathcal{N} = (\mathsf{W}^{\mathcal{N}}, \mathsf{R}^{\mathcal{N}}, \mathsf{V}^{\mathcal{N}}, \mathsf{g}^{\mathcal{N}})$ be two models. We call a function $\mathsf{f} : \mathsf{W}^{\mathcal{M}} \to \mathsf{W}^{\mathcal{N}}$ a homomorphism and write $\mathsf{f} : \mathcal{M} \to \mathcal{N}$ iff

1. $(\mathsf{g}^{\mathcal{M}}(i)) = \mathsf{g}^{\mathcal{N}}(i)$, for all nominals i,
2. $\mathsf{w}\mathsf{R}^{\mathcal{M}}\mathsf{v}$ implies $(\mathsf{w})\mathsf{R}^{\mathcal{N}}(\mathsf{v})$, for all worlds w, v,
3. $\mathsf{w} \in \mathsf{V}^{\mathcal{M}}(p)$ implies $(\mathsf{w}) \in \mathsf{V}^{\mathcal{N}}(p)$ for all worlds w and variables p.

Let $\mathcal{E}$ be a set of elementary game states. We call a model $\mathcal{M}$ an *$\mathcal{E}$-countermodel* if all states of $\mathcal{E}$ are winning for *You* in the evaluation game over $\mathcal{M}$. The following proposition shows that the model $\mathcal{M}_{\mathcal{E}}$ (see Definition 3.5.7) is, in a way, the most general $\mathcal{E}$-countermodel.

Proposition 3.7.13: $\mathcal{M}_{\mathcal{E}}$ is the least countermodel

Let $\mathcal{N}$ be an $\mathcal{E}$-countermodel. Then there is a (unique) homomorphism : $\mathcal{M}_{\mathcal{E}} \to \mathcal{N}$. In this case, $\mathcal{M}_{\mathcal{E}}$ is an $\mathcal{E}$-countermodel, too.

Proof. The only way to define such a homomorphism is by setting $([i]) = \mathsf{g}^{\mathcal{N}}(i)$. To check that this function is well-defined, we observe that $i \approx_{\mathcal{E}} j$ implies $\mathsf{g}^{\mathcal{N}}(i) = \mathsf{g}^{\mathcal{N}}(j)$: Let $i \sim_{\mathcal{E}} j$, then either $\mathbf{O}, i : j$ or $\mathbf{O}, j : i$ appears in $\mathcal{E}$. Since $\mathcal{N}$ is an $\mathcal{E}$-countermodel, $\mathsf{g}^{\mathcal{N}}(i) = \mathsf{g}^{\mathcal{N}}(j)$. The claim for $\approx_{\mathcal{E}}$ follows by transitivity, reflexivity and symmetry of equality.

To see that is a homomorphism, let $[i]\mathsf{R}^{\mathcal{E}}[j]$. This means, for some $i' \approx i$ and $j' \approx j$ and some nominal k, $\mathbf{O}, k : R(i, j)$ is in $\mathcal{E}$. Since $\mathcal{N}$ is an $\mathcal{E}$-countermodel, $\mathsf{g}^{\mathcal{N}}(i')\mathsf{R}^{\mathcal{N}}\mathsf{g}^{\mathcal{N}}(j')$. By the above observation, $\mathsf{g}^{\mathcal{N}}(i)\mathsf{R}^{\mathcal{N}}\mathsf{g}^{\mathcal{N}}(j)$. Showing that $[i] \in \mathsf{V}^{\mathcal{E}}(p)$ implies $\mathsf{g}^{\mathcal{N}}(i) \in \mathsf{V}^{\mathcal{N}}(p)$ is similar.

It follows that $\mathcal{M}_{\mathcal{E}}$ is an $\mathcal{E}$-countermodel: by construction, all states in $\mathcal{E}$ of the form $\mathbf{O}, i : F$ are labeled "Y" in the evaluation game over $\mathcal{M}_{\mathcal{E}}$. If $\mathbf{P}, i : j$ is in $\mathcal{E}$, then

[15] Alternatively, one could restrict initial sequents to trivial instances and extend the proof system by structural rules, cf. Section 3.7.4

$g^{\mathcal{N}}(i) \neq g^{\mathcal{N}}(j)$, since $\mathcal{N}$ is an $\mathcal{E}$-countermodel. This implies $[i] \neq [j]$ and hence $\mathbf{P}, i : j$ is labeled "Y" in $\mathcal{M}_{\mathcal{E}}$. If $\mathbf{P}, k : R(i,j)$ is in $\mathcal{E}$, then $g^{\mathcal{N}}(i) \not{R}^{\mathcal{N}} g^{\mathcal{N}}(j)$ and therefore $[i] \not{R}^{\mathcal{E}} [j]$. The case for $\mathbf{P}, i : p$ is similar. □

Proof of Proposition 3.7.11. Equivalently, we can check whether there is a countermodel of H, the elementary part of D. If there is a countermodel of H, then by Proposition 3.7.13, $\mathcal{M}_H$ is already a countermodel. Thus, it suffices to construct $\mathcal{M}_H$ and check whether the $\mathbf{P}$-part of H is false there since the $\mathbf{O}$-part is false by construction. Although the model $\mathcal{M}_H$ is infinite, it can be finitely represented using the $\mathbf{O}$-part of H:

- For every nominal i, create an equivalence class (list) containing only i.
- For every $\mathbf{O}, i : j \in H$, merge the equivalence classes of i and j.
- For every $\mathbf{O}, k : R(i,j) \in H$, draw an R-arrow from $[i]$ to $[j]$.
- For every $\mathbf{O}, i : p \in H$, mark $[i]$ with the label p.

Constructing this representation takes polynomial time. Checking whether the $\mathbf{P}$-part of H is winning for *You* in the evaluation game over $\mathcal{M}_H$ can be done in polynomial time, too:

- For every $\mathbf{P}, i : j \in H$, check that $j \notin [i]$.
- For every $\mathbf{P}, k : R(i,j) \in H$, check that there is no R-arrow from $[i]$ to $[j]$.
- For every $\mathbf{P}, i : p \in H$, check that $[i]$ does not have the label p.

The disjunctive state D is winning iff this check is negative. □

3.7.4 Connections to existing calculi

Our calculus $\mathbf{DS}^{\text{Hyb}}$[16] is closely related to several existing sequent systems in the literature, like Negri's labeled system for orthodox modal logic [47]. In [9], Blackburn reformulates his tableau system for hybrid logic as a sequent calculus, and in [13], Braüner does the same with his natural deduction system. In this section, we investigate the connections to the latter system $\mathbf{G}_{\mathcal{H}}$ (see Figure 3.2), as it is the most similar to $\mathbf{DS}^{\text{Hyb}}$. Most of the discussion, however, also applies to the other systems. We will argue that $\mathbf{DS}^{\text{Hyb}}$ and $\mathbf{G}_{\mathcal{H}}$ are essentially equivalent. The differences are:

[16] In this subsection, we consider $\mathbf{DS}^{\text{Hyb}}$ minus the rules (R_U) and (L_U).

Table 3.2: The proof system $\mathbf{G}_{\mathcal{H}}$. In the rules $(R_\Box)$ and $(R_\forall)$, the nominal j must not occur in the lower sequent, in the rule $(Nom1)$, F must be a propositional variable or a nominal.

Axioms

$$\Gamma, i : F \Rightarrow i : F, \Delta \qquad\qquad \Gamma, i : \bot \Rightarrow \Delta$$

Propositional rules

$$(L_\wedge)\ \frac{\Gamma, i : F, i : G \Rightarrow \Delta}{\Gamma, i : F \wedge G \Rightarrow \Delta} \qquad (R_\wedge)\ \frac{\Gamma \Rightarrow i : F, \Delta \qquad \Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \wedge G, \Delta}$$

$$(L_\rightarrow)\ \frac{\Gamma \Rightarrow i : F, \Delta \qquad \Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \rightarrow G \Rightarrow \Delta} \qquad (R_\rightarrow)\ \frac{\Gamma, i : F \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \rightarrow G, \Delta}$$

Modal rules

$$(L_\Box)\ \frac{\Gamma \Rightarrow i : \Diamond j, \Delta \qquad \Gamma, j : F \Rightarrow \Delta}{\Gamma, i : \Box F \Rightarrow \Delta} \qquad (R_\Box)\ \frac{\Gamma, i : \Diamond j \Rightarrow j : F, \Delta}{\Gamma \Rightarrow i : \Box F, \Delta}$$

$$(L_@)\ \frac{\Gamma, j : F \Rightarrow \Delta}{\Gamma, i : @_j F \Rightarrow \Delta} \qquad (R_@)\ \frac{\Gamma \Rightarrow j : F, \Delta}{\Gamma \Rightarrow i : @_j F, \Delta}$$

Quantifier rules

$$(L_\forall)\ \frac{\Gamma, i : F[j/k] \Rightarrow \Delta}{\Gamma, i : \forall k.F \Rightarrow \Delta} \qquad (R_\forall)\ \frac{\Gamma \Rightarrow i : F[j/k], \Delta}{\Gamma \Rightarrow i : \forall k.F, \Delta}$$

Rules for nominals

$$(Ref)\ \frac{\Gamma, i : i \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (Nom1)\ \frac{\Gamma \Rightarrow i : j, \Delta \qquad \Gamma \Rightarrow i : F, \Delta}{\Gamma \Rightarrow j : F, \Delta}$$

$$(Nom2)\ \frac{\Gamma \Rightarrow i : k, \Delta \qquad \Gamma \Rightarrow i : \Diamond j, \Delta \qquad \Gamma, k : \Diamond j \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

- In the original system $\mathbf{G}_{\mathcal{H}}$ in [13], formulas are not labeled but instead prefixed by the @-operator. As this is clearly equivalent, we rephrased the system for the convenience of comparison.
- In $\mathbf{G}_{\mathcal{H}}$, sequents $\Gamma \Rightarrow \Delta$ consist of finite *sets* Γ and Δ, rather than multisets in $\mathbf{DS}^{\mathrm{Hyb}}$. Consequently, while we make use of contraction rules in $\mathbf{DS}^{\mathrm{Hyb}}$, they are "built-in" to $\mathbf{G}_{\mathcal{H}}$.

- In $\mathbf{G}_{\mathcal{H}}$, the connectives $\vee, \neg, \Diamond$ are treated as defined. In $\mathbf{DS}^{\mathrm{Hyb}}$, relational claims $R(i,j)$ are treated as elementary, in $\mathbf{G}_{\mathcal{H}}$ relational claims are written as complex formulas $@_i\Diamond j$.

- The propositional rules of $\mathbf{G}_{\mathcal{H}}$ are derivable in $\mathbf{DS}^{\mathrm{Hyb}}$. The propositional rules of $\mathbf{DS}^{\mathrm{Hyb}}$ can be shown to be admissible in $\mathbf{G}_{\mathcal{H}}$ using the admissibility of weakening.

- Replacing "$R(i,j)$" with "$@_i\Diamond j$", the modal rules of $\mathbf{G}_{\mathcal{H}}$ are derivable in $\mathbf{DS}^{\mathrm{Hyb}}$. Writing "$@_i\Diamond j$" instead of "$R(i,j)$", the modal rules of $\mathbf{DS}^{\mathrm{Hyb}}$ can be shown to be admissible in $\mathbf{G}_{\mathcal{H}}$ using the invertibility of the propositional rules.

- Axioms are more complex in $\mathbf{DS}^{\mathrm{Hyb}}$: There, we allow all winning sequents as axioms. In $\mathbf{G}_{\mathcal{H}}$, there are only "trivial" axioms, but also extra rules (Ref), $(Nom1)$, and $(Nom2)$.

The last point is perhaps the most interesting one: By completeness of $\mathbf{G}_{\mathcal{H}}$, all the initial sequents in $\mathbf{DS}^{\mathrm{Hyb}}$ can be derived in $\mathbf{G}_{\mathcal{H}}$ from trivial axioms by using the rules (Ref), $(Nom1)$, $(Nom2)$ and the diamond rules restricted to instances where F is a nominal or a propositional variable. Putting things together, we have the following correspondence:

Proposition 3.7.14: $\mathbf{DS}^{\mathrm{Hyb}}$ and $\mathbf{G}_{\mathcal{H}}$ are equivalent.

1. Let Γ and Δ be finite multisets of labeled formulas. Then $\vdash_{\mathbf{DS}^{\mathrm{Hyb}}} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{G}_{\mathcal{H}}} \Gamma' \Rightarrow \Delta'$, where Γ' and Δ' is the support of Γ and Δ, respectively, and every subformula $R(i,j)$ is replaced by $@_i\Diamond j$.

2. Let Γ and Δ be finite sets of labeled formulas. Then $\vdash_{\mathbf{G}_{\mathcal{H}}} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{DS}^{\mathrm{Hyb}}} \Gamma' \Rightarrow \Delta'$, where Γ' and Δ' are obtained from Γ and Δ by replacing every subformula $@_i\Diamond j$ by $R(i,j)$ and every labeled formula $i : \Diamond j$ by $j : R(i,j)$.

In particular, this correspondence implies that every proof in $\mathbf{G}_{\mathcal{H}}$ can be structured as follows: Starting from the root, apply only propositional rules and modal rules to complex formulas until all leaves are elementary. This is essentially a proof in $\mathbf{DS}^{\mathrm{Hyb}}$. Then, apply nominal and restricted diamond rules to arrive at trivial axioms.

It must be noted that, in principle, nominal rules can be included in the disjunctive game. For example, the rule corresponding to $(Nom1)$ would look as follows: if in the current disjunctive state $D \bigvee \underline{\mathbf{P}, j : F}$, it is *My* turn, we can allow *Me* to choose a nominal i and then *You* choose whether to continue the game with $D \bigvee \mathbf{P}, i : j$ or with $D \bigvee \mathbf{P}, i : F$. However, we chose not to include these rules, as they seem to lack a clear motivation in terms of the evaluation game. Another way to reformulate the disjunctive game with trivial initial sequents would be to include nominal axioms in Γ, for example, $\forall i.\forall j.\,(@_i j \to @_j i)$, $\forall i.\forall j.\forall k.\,(R(i,j) \wedge @_i k \to R(k,j))$, etc.

3.8 Conclusion and Future Work

In this chapter, we investigated the possibility of lifting a version of Hintikka's evaluation game for modal logic. The challenge in this endeavor is that the syntax of modal logic does not represent an essential part of its semantics. This fact is also reflected in the evaluation game. Consequently, games over the same formula, but different models, may have game trees of different shapes. This makes the lifting technique inapplicable.

We overcame this problem by turning to hybrid logic – an extension of modal logic that makes it possible to represent worlds and the accessibility relation of the underlying model within the syntax. A version of the evaluation game for hybrid logic has uniform game trees, but this comes at the price of infinite branching, even if one restricts attention to evaluation over finite models.

On the game-theoretic side, this makes the adequacy proofs for the resulting provability game more complex. Additionally, scheduling is more important in the infinite case – a fact that we explore in Chapter 5, where we give a class of scheduling functions (called *regulations* there) that enable the adequacy of the disjunctive provability game.

Concerning the proof theory, we showed that infinite branching in the rules for modalities can be eliminated by considering *Your* optimal choices only. These moves correspond to an eigenvariable condition, as known from similar first-order systems. In the resulting sequent calculus, proofs are nothing but notational variants of *My* winning strategies in the disjunctive game. By construction, this system is cut-free.

A few words must be said about the design choices in this chapter. For many possible applications of our technique in the realm of modal logic, the full expressiveness of the hybrid language used in this chapter is perhaps a little overblown. After all, the satisfiability problem of hybrid logic using $@$ and $\forall$ is undecidable. In these cases, it can be desirable to restrict the use of $\forall$ to instances of formulas characterizing certain frame properties. The reason why we chose this highly general approach is that it gives, in one sweep, robust proof systems for many scenarios.

Another possible design would be to include frame properties as explicit rules of the disjunctive game and hence the proof system. These rules can be directly derived from the standard game rules applied to the characterizing formulas. For instance, the characterizing formula for reflexivity $\forall i.R(i,i)$ results in the rule

$$\frac{\Gamma, R(i,i) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Another possibility to obtain an adequate provability game over a class of models $\mathfrak{M}$ is to evaluate elementary disjunctive states over models in $\mathfrak{M}$ only. That is, the disjunctive state D is winning for *Me* iff for all models $\mathcal{M} \in \mathfrak{M}$ there is some $h \in D$ such that *I* win in $\mathbf{G}^{\mathrm{Hyb}}(h)$. While this would result in a sound and complete proof system, the complexity of checking validity of initial sequents is unclear. It would be an exciting research question to characterize the classes of models for which checking initial sequents remains tractable.

Another interesting exercise would be to study the simplest modal logic to which our ideas can be applied. We conjecture that one can a) restrict nominals to occur only in relational formulas $R(i, j)$, and b) restrict relational formulas to never occur in the scope of any logical connective, modal operator, or quantifier, without running into technical or conceptual problems. Indeed, the semantic game can be adjusted to cover these restrictions in a straightforward way. Thus, the simplest "liftable" modal logic is indeed the simplest normal modal logic K.

Going in the other direction, the obvious first candidate for extending the language, the first candidate is first-order hybrid logic. Its well-behaved proof theory [14] indicates that the provability game could be extended to cover this case as well.

Various semantic games for fuzzy logics are known, like Giles's game for Łukasiewicz logic [33, 24], and truth-degree comparison for Gödel-logics [22, 23]. These evaluation games could be combined with the game for hybrid logic to capture the corresponding fuzzy modal logics. Furthermore, the techniques for managing infinite branching could possibly be applied to develop a game-theoretic approach to the proof theory for first-order fuzzy logics.

CHAPTER 4

Choice Logics

4.1 Introduction

Preferences are a key research area in artificial intelligence, and thus, many preference formalisms have been described in the literature [49]. An interesting example is Qualitative Choice Logic (QCL) [15], which extends classical propositional logic by the connective $\vec{\times}$ called ordered disjunction. The *choice formula* $F \vec{\times} G$ states that F or G should be satisfied, but satisfying F is preferable to satisfying only G. This allows us to express soft constraints (preferences) and hard constraints (truth) in a single language.

For example, say we want to formalize our choice of ice cream flavors: we definitely want apricot (a). Moreover, we want either banana (b) or caramel (c), but preferably banana. This can easily be expressed in QCL via the choice formula $a \wedge (b \vec{\times} c)$. This formula has three models in QCL, namely $M_1 = \{a, b, c\}$, $M_2 = \{a, b\}$, and $M_3 = \{a, c\}$. QCL-semantics then ranks these models via so-called satisfaction degrees. The lower this degree, the more preferable the model. In this case, M_1 and M_2 are assigned a degree of 1, and M_3 is assigned a degree of 2, i.e., M_1 and M_2 are the preferred models of this formula.

In the literature, QCL has been studied with regard to possible applications [51], computational properties [7], and proof systems [6].

Not all aspects of QCL are uncontroversial. Although the degree semantics coincide with classical Tarskian semantics when restricted to formulas without $\vec{\times}$, it cannot be seen as a "classical" extension of classical logic, i.e., an extension where negation behaves as in classical logic. For example, the degree of a choice formula F is not equal to that of its double negation $\neg\neg F$, as all information about preferences is erased by $\neg$. This issue has been addressed by Prioritized QCL (PQCL) [5], which defines ordered disjunction the same way as QCL but changes the meaning of the classical connectives, including negation. While PQCL solves QCL's problem with double negation, it, in turn, introduces other controversial behavior, e.g., a formula F and its negation $\neg F$ can be satisfied by the same interpretation. No alternative semantics for QCL is known to us that addresses both of these issues simultaneously.

To tackle these issues, we develop a novel game semantics for the language of QCL. In this extension of Hintikka's game, we capture not only truth but also preferences by introducing more fine-grained payoffs and giving a game-theoretic interpretation of ordered disjunction. Intuitively, the rule for dealing with the choice connective $\vec{\times}$ is as follows: in $F \vec{\times} G$, the current proponent chooses whether to continue the game with F or with G, but that player prefers F. What is essential for the classicality of negation is that we treat negation as *dual negation*, [54], i.e., at $\neg F$, the game continues with a role switch.

We show that the value of the resulting game can be characterized via functional degree-based semantics, similar to QCL. In turn, the degree function of QCL can be adequately recaptured in our game by interpreting negation not only as a role switch. To model the aforementioned asymmetries in our evaluation game, we additionally erase all preferences at $\neg F$.

Game-theoretically speaking, the validity of a formula F can be characterized as the existence of winning strategies for the evaluation games starting at F over all possible interpretations. Equivalently, the degree of validity of F is the least value of these evaluation games. Analogously, we define the *degree validity of* F as the least value of the evaluation game starting in F over all possible interpretations. Hence, in the resulting logic GCL (game-induced choice logic), the notion of validity is degree-based rather than binary.

To capture this degree of validity, we lift the new semantic game to a provability game, using the technique of the disjunctive game. The novelty of this approach is that both the evaluation game as well as the disjunctive game allow for payoff values different from the usual win/lose. So far, even the semantic games for multi-valued logics [24, 22, 50] were designed with binary outcomes.

In accordance with the technique, proofs in the resulting (labeled) sequent calculus represent (encodings of) winning strategies for *Me* in the disjunctive game. However, in the case of GCL, proofs come in degrees that represent the degree of the encoded winning strategy. In a version of this system, $\mathbf{ODS}^{\mathrm{GCL}}$, with invertible rules, the represented strategies are automatically optimal. Hence, positive degrees of proofs represent proofs of validity, while negative degrees represent refutations of a formula. We show that preferred models can be extracted from proofs in $\mathbf{ODS}^{\mathrm{GCL}}$.

This chapter is structured as follows: in Section 3.2, we recall some basic definitions and facts about QCL, and discuss some of the behavior concerning negations mentioned above. In Section 4.3, we design our evaluation game for the language of QCL and demonstrate that it is a more suitable classical extension of classical logic by the choice connective $\vec{\times}$. We provide an adequate alternative degree semantics and show that QCL can be captured in this game using a suitable translation of formulas. The lifting is done in two steps in Sections 4.4 and 4.5 and Section 4.6 gives the interpretation of *My* strategies as proofs with degrees in a sequent system.

4.2 Preliminaries

In this section, we recall the language and semantics of QCL and discuss some of its behavior concerning negations. From the issues raised in this discussion, we formulate desiderata for a more suitable choice logic.

QCL [15] is the most prominent choice logic which adds ordered disjunction ($\vec{\times}$) to classical propositional logic. The language of choice logic is as follows. We have a countably infinite set of propositional variables usually denoted "a, b, ...". Choice formulas F are built according to the following grammar:

$$F ::= \bot \mid a \mid \neg F \mid F \wedge F \mid F \vee F \mid F \vec{\times} F$$

In this chapter, we say that F is a *propositional formula* if it has no occurrences of $\vec{\times}$.

The semantics of QCL is based on two functions, namely optionality and satisfaction degree. The satisfaction degree of a formula can be either a positive integer or -1 and is used to rank interpretations[1] using the order $\trianglelefteq$. This order is the inverse of the natural ordering on $\mathbb{Z}^+$ and $-1 \triangleleft k$, for all $k \in \mathbb{Z}^+$, i.e. we have $-1 \triangleleft \cdots \triangleleft 3 \triangleleft 2 \triangleleft 1$. The optionality of a formula represents the maximum finite satisfaction degree this formula can obtain (as we will see in Lemma 4.2.4) and is used to penalize interpretations that do not satisfy the preferred option F in an ordered disjunct $F \vec{\times} G$.

Definition 4.2.1: Optionality

The optionality of choice formulas is defined recursively as follows:

$$\begin{aligned}
\text{opt}(x) &= 1 \text{ for every propositional variable } x,\\
\text{opt}(\neg F) &= 1,\\
\text{opt}(F \wedge G) &= \max(\text{opt}(F), \text{opt}(G)),\\
\text{opt}(F \vee G) &= \max(\text{opt}(F), \text{opt}(G)),\\
\text{opt}(F \vec{\times} G) &= \text{opt}(F) + \text{opt}(G).
\end{aligned}$$

Here, $\max$ is relative to the natural ordering on the integers.

[1] We use -1 instead of ∞ which is common in the literature. We do this in order to make the presentation easier when comparing our semantic approach to QCL.

Definition 4.2.2: Degree under an interpretation

An *interpretation* $\mathcal{I}$ is a set of propositional variables. The satisfaction degree of choice formulas is defined recursively as follows:

$$\deg_{\mathcal{I}}(a) = \begin{cases} 1 & \text{if } x \in \mathcal{I}, \\ -1 & \text{if } x \notin \mathcal{I}, \end{cases}$$
$$\deg_{\mathcal{I}}(\neg F) = \begin{cases} 1 & \text{if } \deg_{\mathcal{I}}(F) = -1, \\ -1 & \text{if } \deg_{\mathcal{I}}(F) \in \mathbb{Z}^+, \end{cases}$$
$$\deg_{\mathcal{I}}(F \wedge G) = \min_{\trianglelefteq}(\deg_{\mathcal{I}}(F), \deg_{\mathcal{I}}(G)),$$
$$\deg_{\mathcal{I}}(F \vee G) = \max_{\trianglelefteq}(\deg_{\mathcal{I}}(F), \deg_{\mathcal{I}}(G)),$$
$$\deg_{\mathcal{I}}(F \vec{\times} G) = \begin{cases} \deg_{\mathcal{I}}(F) & \text{if } \deg_{\mathcal{I}}(F) \in \mathbb{Z}^+. \\ \text{opt}(F) + \deg_{\mathcal{I}}(G) & \text{if } \deg_{\mathcal{I}}(F) = -1 \text{ and } \deg_{\mathcal{I}}(G) \in \mathbb{Z}^+, \\ -1 & \text{otherwise.} \end{cases}$$

If $\deg_{\mathcal{I}}(F) = k$ we say that $\mathcal{I}$ satisfies F to a degree of k. If $\deg_{\mathcal{I}}(F) \in \mathbb{Z}^+$ we say that $\mathcal{I}$ classically satisfies F, or that $\mathcal{I}$ is a model of F and write $\mathcal{I} \models F$. Note that the question of satisfiability in QCL reduces to classical propositional logic: let F be a choice formula and F^* be F with all $\vec{\times}$s replaced by $\vee$s. Then $\mathcal{I} \models F$ in QCL iff $\mathcal{I} \models F^*$ in classical logic.

Example 4.2.3. An illuminating example is the formula $F = G_1 \vec{\times} G_2 \vec{\times} \cdots \vec{\times} G_n$, where the G_i are propositional formulas, i.e., they do not contain the choice connective $\vec{\times}$. Let $\mathcal{I}$ be an interpretation. If $\mathcal{I} \models G_1$, then $\deg_{\mathcal{I}}(F) = 1$. If $\mathcal{I} \not\models G_1$, but $\mathcal{I} \models G_2$, then $\deg_{\mathcal{I}}(F) = 2$. In general, $\deg_{\mathcal{I}}(F)$ is the least i such that $\mathcal{I} \models G_i$ and -1 if there is no such i.

To fully understand QCL semantics, we must take note that satisfaction degrees are bounded by optionality, as intended:

Lemma 4.2.4: (from [15]) Degree bounded by optionality

For all choice formulas F and all interpretations $\mathcal{I}$, $\deg_{\mathcal{I}}(F) \trianglerighteq \text{opt}(F)$ or $\deg_{\mathcal{I}}(F) = -1$.

Indeed, inspecting Definition 4.2.1 in view of Lemma 4.2.4 shows how optionality is used to penalize non-satisfaction: given $F \vec{\times} G$, if some interpretation $\mathcal{I}$ classically satisfies F, i.e., $\deg_{\mathcal{I}}(F) \in \mathbb{Z}^+$, we get $\deg_{\mathcal{I}}(F \vec{\times} G) = \deg_{\mathcal{I}}(F) \trianglerighteq \text{opt}(F)$; if $\mathcal{I}$ does not classically satisfy F, i.e., $\deg_{\mathcal{I}}(F) = -1$, we get $\deg_{\mathcal{I}}(F \vec{\times} G) = \text{opt}(F) + \deg_{\mathcal{I}}(G) \triangleleft \text{opt}(F)$.

We now define the central notion of preferred models and then give a small example of QCL semantics in action.

Definition 4.2.5: Preferred model

Let F be a choice formula. $\mathcal{I}$ is a preferred model of F iff $\deg_{\mathcal{I}}(F) \in \mathbb{Z}^{+}$ and $\deg_{\mathcal{I}}(F) \trianglerighteq \deg_{\mathcal{J}}(F)$ for all other interpretations $\mathcal{J}$.

Example 4.2.6. The choice formula $F = (a \wedge b)\vec{\times}a\vec{\times}b$ expresses that satisfying both a and b is preferable to satisfying only a, which in turn is preferable to satisfying only b. First, observe that $\text{opt}(F) = 3$. Moreover, $\deg_{\emptyset}(F) = -1$, $\deg_{\{b\}}(F) = 3$, $\deg_{\{a\}}(F) = 2$, and $\deg_{\{a,b\}}(F) = 1$. Thus, $\{a, b\}$ is a preferred model of F.

Now consider $F' = ((a \wedge b)\vec{\times}a\vec{\times}b) \wedge \neg(a \wedge b)$, which is similar to F, but with the additional information that a and b can not be jointly satisfied. Again, $\deg_{\emptyset}(F') = -1$, $\deg_{\{b\}}(F') = 3$, and $\deg_{\{a\}}(F') = 2$. However, $\deg_{\{a,b\}}(F') = -1$, i.e., $\{a, b\}$ does not satisfy F'. Since it is not possible to satisfy F' to a degree of 1, $\{a\}$ is a preferred model of F'.

Note that ordered disjunction is associative under QCL-semantics, which means that we can simply write $A_1\vec{\times}A_2\vec{\times}\ldots\vec{\times}A_n$ to express that we must satisfy at least one of $A_1, \ldots, A_n$, and that we prefer A_i to A_j for $i < j$. Formally, this is expressed by the following lemma:

Lemma 4.2.7: (from [15]): Ordered disjunction is associative

Let F, G, and H be choice formulas. Then $(F\vec{\times}(G\vec{\times}H))$ and $((F\vec{\times}G)\vec{\times}H)$ have the same optionality and the same satisfaction degree under all interpretations.

As mentioned in the introduction, an alternative semantics for QCL has been proposed in the form of PQCL [5]. Specifically, PQCL changes the semantics for the classical connectives $(\neg, \vee, \wedge)$, but defines ordered disjunction $(\vec{\times})$ in the same way as QCL. For our purposes, it is not necessary to formally define PQCL. Rather, it suffices to note that, in PQCL, negation propagates to the atom level, meaning that $\neg(F \wedge G)$ is simply assigned the satisfaction degree of $\neg F \vee \neg G$, $\neg(F \vee G)$ is assigned the degree of $\neg F \wedge \neg G$, and $\neg(F\vec{\times}G)$ is assigned the degree of $\neg F\vec{\times}\neg G$.

4.2.1 Comments on Negation in QCL and PQCL

While choice logics are a useful formalism to express both soft constraints (preferences) and hard constraints (truth) in a single language, existing semantics (such as QCL and PQCL) are not entirely uncontroversial. Table 4.1 shows how negation acts on ordered disjunction in both systems: negation in QCL erases preferences, while in PQCL it is possible to satisfy a formula and its negation at the same time ($\{a\}$ and $\{b\}$ classically satisfy both $a\vec{\times}b$ and $\neg a\vec{\times}\neg b$). Moreover, in PQCL, the satisfaction degree of $\neg F$ does not only depend on the degree and optionality of F ($\{a\}$ and $\{a, b\}$ satisfy $a\vec{\times}b$ to degree 1, but $\{a\}$ satisfies $\neg a\vec{\times}\neg b$ to degree 2 while $\{a, b\}$ does not satisfy $\neg a\vec{\times}\neg b$ at all).

Table 4.1: Truth table showing the satisfaction degrees of $\neg(a \vec{\times} b)$ in QCL (equivalent to $\neg a \wedge \neg b$) and PQCL (equivalent to $\neg a \vec{\times} \neg b$).

$\mathcal{I}$	$a \vec{\times} b$	$\neg a \wedge \neg b$	$\neg a \vec{\times} \neg b$
$\emptyset$	-1	1	1
$\{b\}$	2	-1	1
$\{a\}$	1	-1	2
$\{a, b\}$	1	-1	-1

In Section 4.3, we will make use of game-theoretic negation to define alternative semantics for the language of QCL. Our goal there is to define a negation that acts both on hard constraints as in QCL and soft constraints as in PQCL. Specifically, we will ensure that

1. the satisfaction degree of $\neg F$ depends only on the degree of F,
2. formulas and their negation can not be classically satisfied by the same interpretation,
3. formulas are equivalent to their double negation,
4. De Morgan's laws hold.

It must be noted that QCL and PQCL are not the only semantics for propositional logic extended with ordered disjunction. Another example is the work by Maly and Woltran [44], where the semantics "directly" induces a partial order among interpretations (instead of using satisfaction degrees). However, negation is handled in the same way as in QCL, i.e., all information about preferences is lost and formulas can have different degrees than their double negations.

4.3 A Semantic Game for Choice Logic

In this section, we define our evaluation game for the language of QCL. In this game, the payoffs represent how well a formula is satisfied with respect to both hard constraints (truth) as well as soft constraints (preferences). We recall the central notion of a strategy in Section 4.3.1. Validity in the resulting logic, GCL, is degree-based as well. We introduce this logic in Section 4.3.2 and demonstrate that it satisfies the requirements specified in Section 4.2.1. We give a QCL-style degree semantics for GCL in Section 4.3.3 and prove its adequacy. Finally, in Section 4.3.4, we formally prove that our evaluation game is a refinement of Hintikka's game. Moreover, an evaluation game for QCL can be obtained under a suitable translation of formulas.

We start by giving an informal account of the evaluation game[2] $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$.

[2]The resulting logic is called GCL (game-induced choice logic), see Subsection 4.3.2, which explains the superscript.

The game is played by two players, *Me* and *You*, play the game over a fixed interpretation $\mathcal{I}$. Both players can be in the role of the proponent (**P**) or the opponent (**O**) of a given choice formula F. We encode the situation where *I* am the proponent of F, and *You* are the opponent as by the *game state* $\mathbf{P} : F$, and similarly with $\mathbf{O} : F$ for the opposite situation. Let g be a game state. The game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$ starts at g. It proceeds according to the rules below. The rules for the classical connectives are just as in Hintikka's game for classical logic.

($\mathbf{P}_\vee$) At game states of the form $\mathbf{P} : G_1 \vee G_2$, *I* choose between the game states $\mathbf{P} : G_1$ and $\mathbf{P} : G_2$ to continue the game.

($\mathbf{O}_\vee$) At $\mathbf{O} : G_1 \vee G_2$, *You* choose between $\mathbf{O} : G_1$ and $\mathbf{O} : G_2$.

($\mathbf{P}_\wedge$) At $\mathbf{P} : G_1 \wedge G_2$, *You* choose between $\mathbf{P} : G_1$ and $\mathbf{P} : G_2$.

($\mathbf{O}_\wedge$) At $\mathbf{O} : G_1 \wedge G_2$, *I* choose between $\mathbf{O} : G_1$ and $\mathbf{O} : G_2$.

($\mathbf{P}_\neg$) At $\mathbf{P} : \neg G$, the game continues with $\mathbf{O} : G$.

($\mathbf{O}_\neg$) At $\mathbf{O} : \neg G$, the game continues with $\mathbf{P} : G$.

The new ingredient is the interpretation of the choice connective $\vec{\times}$. Remember that $G_1 \vec{\times} G_2$ encodes not only a hard constraint (at least one of G_1, G_2 has to be true), but also a soft constraint, or preference: it is preferable that G_1 is true. We encode this by an explicit preference relation $\ll$. This relation represents *My* preferences over outcomes:

($\mathbf{P}_{\vec{\times}}$) At game states of the form $\mathbf{P} : G_1 \vec{\times} G_2$, *I* choose between $\mathbf{P} : G_1$ and $\mathbf{P} : G_2$ to continue the game. All outcomes of the $\mathbf{P} : G_1$-subgame are in $\gg$-relation to all outcomes of the $\mathbf{P} : G_2$-subgame.

($\mathbf{O}_{\vec{\times}}$) At $\mathbf{O} : G_1 \vec{\times} G_2$, *You* choose between $\mathbf{O} : G_1$ and $\mathbf{O} : G_2$. All outcomes of the $\mathbf{P} : G_1$-subgame are in $\ll$-relation to all outcomes of the $\mathbf{P} : G_2$-subgame.

Note that the rule ($\mathbf{O}_{\vec{\times}}$) is in complete symmetry to the rule ($\mathbf{P}_{\vec{\times}}$). By its design, the current proponent always prefers outcomes of the G_1-game over outcomes in the G_2-game.

Finally, a sensible payoff must respect both truth (winning conditions) and preferences (the relation $\ll$). Our payoff values are in the domain $Z := (\mathbb{Z} \setminus \{0\}, \trianglelefteq)$. The ordering $\trianglelefteq$ is the inverse ordering on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$; for $a \in \mathbb{Z}^+, b \in \mathbb{Z}^-$ we set $b \triangleleft a$, i.e. $-1 \triangleleft -2 \triangleleft \ldots 2 \triangleleft 1$. The domain Z is depicted in Figure 4.1. For each outcome o, let $\pi_{\ll}(o)$ be the longest $\ll$-chain starting in o, i.e., pairwise different outcomes $o_1, \ldots, o_n$ such that $o = o_1 \ll \cdots \ll o_n$. Let $|\pi_{\ll}(o)| = n$ denote its length. We call game states of the form $\mathbf{Q} : a$ where a is a propositional variable *elementary*. For elementary game states, let us write $\mathcal{I} \models \mathbf{P} : a$ if $\mathcal{I} \models a$ and $\mathcal{I} \models \mathbf{O} : a$ if $\mathcal{I} \not\models a$.

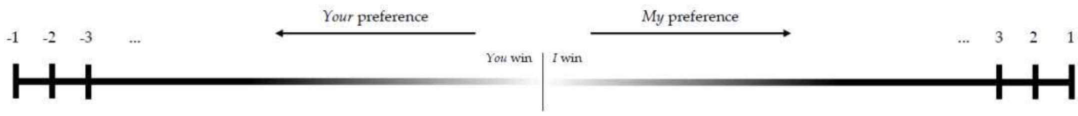


Figure 4.1: The domain of payoffs for $\mathbf{G}^{\mathrm{GCL}}$.

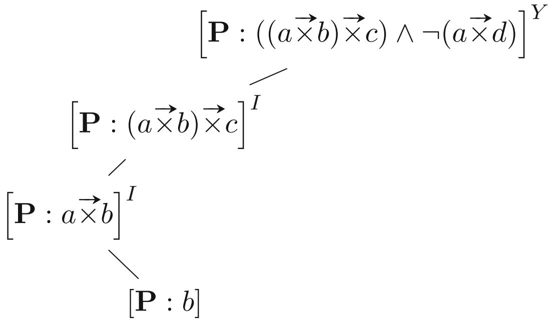


Figure 4.2: A run of the game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}$.

($\mathbf{P}_a$) Let a be a propositional variable. *I* win and *You* lose at $\mathbf{P} : a$ iff $\mathcal{I} \models a$. Otherwise, *You* win and *I* lose. If *I* win, then the payoff is $|\pi_{\ll}(\mathbf{P} : a)|$. If *I* lose, the payoff is[3] $-|\pi_{\gg}(\mathbf{P} : a)|$.

($\mathbf{O}_a$) At $\mathbf{O} : a$, *I* win and *You* lose iff $\mathcal{I} \not\models a$. Otherwise, *You* win and *I* lose. The payoff is as above.

The game can be thus seen as a refinement of Hintikka's game. Indeed, let F^* be F with all $\vec{\times}$s replaced by $\vee$s. Then *I* have a winning strategy for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{P} : F)$ iff *I* have a winning strategy for F^* in Hintikka's game over $\mathcal{I}$. We prove this result formally in Section 4.3.4. Furthermore, the payoff respects the relation $\ll$: if $o_1 \ll o_2$ and both are winning (or both are losing) for *Me*, then the outcome at o_1 is strictly less than the outcome at o_2.

Example 4.3.1. Let $F = ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)$ and $\mathcal{I} = \{b\}$. Let us consider the game starting at $\mathbf{P} : F$. If *You* go to $\mathbf{P} : ((a \vec{\times} b) \vec{\times} c)$, then *I* can go to $\mathbf{P} : a \vec{\times} b$, and then to the winning outcome $\mathbf{P} : b$. Since $\mathbf{P} : a \ll \mathbf{P} : b$, *I* receive a payoff of 2. This run of the game is depicted in Figure 4.2.

We now formally define the evaluation game in terms of Definition 2.1.1. Game states are of the form $\mathbf{Q} : F$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ and F is a choice formula. The set of all game states is denoted $\mathrm{Stat}^{\mathrm{GCL}}$.

[3]Notice the flipped $\ll$-sign.

Definition 4.3.2: Evaluation game for QCL

Let $\mathcal{I}$ be an interpretation and g a game state in $\mathrm{Stat}^{\mathrm{GCL}}$. The evaluation game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$ is defined as follows:

- The initial history is $\langle g \rangle$.
- The set of histories $\mathrm{Hist}_g^{\mathrm{GCL}}$ is the minimal set containing $\langle g \rangle$ and satisfying the following conditions: if $h = h' \smile \mathbf{Q} : G$ is contained, and G is of the form
  - $G = G_1 \vee G_2$, then also $h \smile \mathbf{Q} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = G_1 \wedge G_2$, then also $h \smile \mathbf{Q} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = G_1 \vec{\times} G_2$, then also $h \smile \mathbf{Q} : G_1$ and $h \smile \mathbf{Q} : G_2$ are contained,
  - $G = \neg G'$, then also $h \smile \bar{\mathbf{Q}} : G'$ is contained[a].
- Non-terminal histories ending in states of the following forms are labeled:

labeled "I"	labeled "Y"
$\mathbf{P} : G_1 \vee G_2$	$\mathbf{O} : G_1 \vee G_2$
$\mathbf{O} : G_1 \wedge G_2$	$\mathbf{P} : G_1 \wedge G_2$
$\mathbf{P} : G_1 \vec{\times} G_2$	$\mathbf{O} : G_1 \vec{\times} G_2$
$\mathbf{P} : \neg G'$	$\mathbf{O} : \neg G'$

- The payoff function $\delta_{\mathcal{I}}$ takes values in the domain Z described above. The preference relation $\ll$ is formally defined as follows:

$$h_1 \smile \mathbf{P} : G_1 \vec{\times} G_2 \smile \mathbf{P} : G_1 \smile h_2 \gg h_1 \smile \mathbf{P} : G_1 \vec{\times} G_2 \smile \mathbf{P} : G_2 \smile h_2',$$

  and

$$h_1 \smile \mathbf{O} : G_1 \vec{\times} G_2 \smile \mathbf{O} : G_1 \smile h_2 \ll h_1 \smile \mathbf{O} : G_1 \vec{\times} G_2 \smile \mathbf{O} : G_2 \smile h_2'.$$

  For a history h, let $\pi_{\ll}(h)$ be the[b] longest $\ll$-path starting in h and similarly for $\pi_{\gg}(h)$, and let $|\pi_{\ll}(h)|$ and $|\pi_{\gg}(h)|$ denote their lengths. Terminal histories h ending in $\mathbf{Q} : a$ are mapped to the following values:

$$\delta_{\mathcal{I}}(h) = \begin{cases} |\pi_{\ll}(h)| & \text{if } \mathcal{I} \models \mathbf{Q} : a, \\ -|\pi_{\gg}(h)| & \text{if } \mathcal{I} \not\models \mathbf{Q} : a. \end{cases}$$

  If the payoff is in $\mathbb{Z}^+$, then *I* win and *You* lose. If it is in $\mathbb{Z}^-$, *I* lose, and *You* win.

[a] If $\mathbf{Q} = \mathbf{P}$, then $\bar{\mathbf{Q}} = \mathbf{O}$, and if $\mathbf{Q} = \mathbf{O}$, then $\bar{\mathbf{Q}} = \mathbf{P}$.
[b] To be pedantic, it should be "a" longest path, i.e., some path with maximal length.

Here is how to read this formal description. The final state g of a history h is the current game state. The other states are the game states leading up to g. If, for example, g is of the form $\mathbf{P} : G_1 \wedge G_2$, it is labeled "Y". That means, *You* choose from the minimal histories extending h, i.e., $h \smile \mathbf{P} : G_1$ and $h \smile \mathbf{P} : G_2$. If $g = \mathbf{P} : G_1 \vec{\times} G_2$, then *I* choose between $h \smile \mathbf{P} : G_1$ and $h \smile \mathbf{P} : G_2$.

Eventually, the game reaches a final history h, where h is a game state $\mathbf{Q} : a$. Now, the preference relation $\ll$ comes into play. If h is of the form $h_1 \smile \mathbf{P} : G_1 \vec{\times} G_2 \smile \mathbf{P} : G_2 \smile h_2'$, then at some point during the game *I* decided to go to $\mathbf{P} : G_2$, which is in conflict with *My* explicit preference expressed by $\vec{\times}$. Consequently, all histories of the form $h_1 \smile \mathbf{P} : G_1 \vec{\times} G_2 \smile \mathbf{P} : G_1 \smile h_2$ are in $\gg$-relation to h. Note that *I* will only make this decision if *I* cannot win the game if *I* go to $\mathbf{P} : G_1$. If *I* win at h, then *My* payoff is $|\pi_{\ll}(h)|$. Intuitively, that means that *I* win, but there is a chain of $|\pi_{\ll}(h)|$-many outcomes that *I* would have preferred over the given one. If *I* lose then *My* payoff is $-|\pi_{\gg}(h)|$. Even though *You* win, there is a chain of $|\pi_{\gg}(h)|$-many outcomes that *You* would have preferred over h.

Example 4.3.3. As in Example 4.3.1, consider the formula $F = ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)$. The game tree, where *I* am initially the Proponent can be found in Figure 4.3. The order on outcomes is $\mathbf{P} : c \ll \mathbf{P} : b \ll \mathbf{P} : a$ and $\mathbf{O} : a \ll \mathbf{O} : d$. Or, to be precise, for the terminal histories

$$
\begin{aligned}
h_1 &= \langle \mathbf{P} : ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d), \mathbf{P} : (a \vec{\times} b) \vec{\times} c, \mathbf{P} : a \vec{\times} b, \mathbf{P} : a \rangle,\\
h_2 &= \langle \mathbf{P} : ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d), \mathbf{P} : (a \vec{\times} b) \vec{\times} c, \mathbf{P} : a \vec{\times} b, \mathbf{P} : b \rangle,\\
h_3 &= \langle \mathbf{P} : ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d), \mathbf{P} : (a \vec{\times} b) \vec{\times} c, \mathbf{P} : c \rangle,\\
h_4 &= \langle \mathbf{P} : ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d), \mathbf{P} : \neg(a \vec{\times} d), \mathbf{O} : a \vec{\times} d, \mathbf{O} : a \rangle,\\
h_5 &= \langle \mathbf{P} : ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d), \mathbf{P} : \neg(a \vec{\times} d), \mathbf{O} : a \vec{\times} d, \mathbf{O} : d \rangle,
\end{aligned}
$$

we have $h_3 \ll h_2 \ll h_1$ and $h_4 \ll h_5$. For the valuation $\mathcal{I} = \{b\}$, the winning outcomes are $\mathbf{P} : b$, $\mathbf{O} : a$, and $\mathbf{O} : d$ (or: h_2, h_4, and h_5), and the payoffs are $-1, 2, -3$ at $\mathbf{P} : c$, $\mathbf{P} : b$, $\mathbf{P} : a$ (or: h_3, h_2, h_1), respectively, and $2, 1$ at $\mathbf{O} : a$, $\mathbf{O} : d$ (or: h_4, h_5), respectively.

Although the domain of payoffs is infinite, we can show that the payoff function assumes only finitely many values for every instance of the game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$. By Theorem 2.1.11, this proves that the game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$ is determined.

Lemma 4.3.4: $\mathbf{G}^{\mathrm{GCL}}$ is finite valued

Let $\mathcal{I}$ be an interpretation and g a game state. The game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$ is finite-valued.

Proof. For every choice formula F, we define the number $\mathrm{rank}(F)$, the rank of F, recursively: $\mathrm{rank}(a) = 1$, for propositional variables, $\mathrm{rank}(F_1 \circ F_2) = \max\{\mathrm{rank}(F_1), \mathrm{rank}(F_2)\}$ for $\circ \in \{\vee, \wedge, \vec{\times}\}$, and $\mathrm{rank}(\neg G) = \mathrm{rank}(G) + 1$. We extend the function to game states by setting $\mathrm{rank}(\mathbf{Q} : F) = \mathrm{rank}(F)$ for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$.

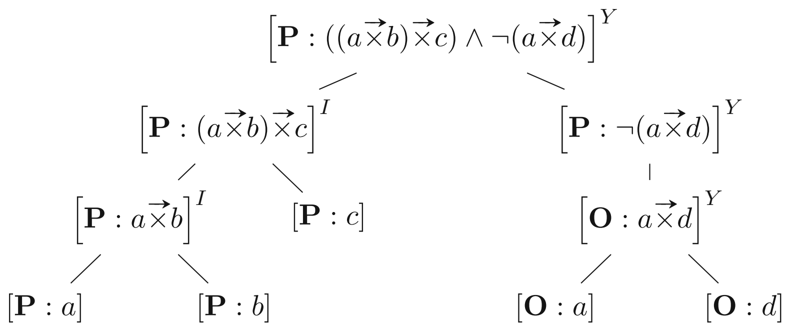


Figure 4.3: A game tree for $\mathbf{G}^{\mathrm{GCL}}$.

Since every history has at most 2 immediate successors, and the length of each history is bounded by $\mathrm{rank}(g)$, we conclude that there are at most $2^{\mathrm{rank}(g)}$-many terminal histories in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$. Hence, the game is finite-valued. □

4.3.1 Strategies

Strategies, k-strategies, winning strategies, and the value of the game were formally defined in Chapter 2. Here, we remind the reader of how to think of strategies as a subtree of the game tree: Every strategy σ for *Me* is represented by a subtree of the game tree that results from pruning all but one successor from I-nodes and leaving all successors of Y-nodes intact. The strategy σ is a k-strategy if all leaves of this subtree have payoff $\trianglerighteq k$, and winning if all leaves are winning, i.e., have a payoff in $\mathbb{Z}^+$.

Example 4.3.5. In Example 4.3.3, we saw the game tree of $F = ((a\overrightarrow{\times}b)\overrightarrow{\times}c) \wedge \neg(a\overrightarrow{\times}d)$ over the interpretation $\mathcal{I} = \{b\}$. If *You* go to the left subgame in the root, then *I* can enforce the game to end in a $\mathbf{P} : b$, resulting in a payoff of 2. This run of the game was discussed in Example 4.3.1. In fact, *I* have a 2-strategy for this game: if *You* go to the right, then *You* have the choice between the outcomes $\mathbf{O} : a$ with payoff 1 and $\mathbf{O} : d$ with payoff 2. Note that in this case, *You* will always choose $\mathbf{O} : d$. This shows that the value of the game is 2. The corresponding strategy for *Me* is depicted in Figure 4.4. In terms of strategies as mappings of histories, let $h = \langle \mathbf{P} : F, \mathbf{P} : (a\overrightarrow{\times}b)\overrightarrow{\times}c \rangle$. Any strategy σ for *Me* with $\sigma(h) = h \smile \mathbf{P} : a\overrightarrow{\times}b$ and $\sigma(h \smile \mathbf{P} : a\overrightarrow{\times}b) = h \smile \mathbf{P} : a\overrightarrow{\times}b \smile \mathbf{P} : b$ is a -2 strategy.

Example 4.3.6. Let us consider the game over F and $\{b\}$ as in the previous example, but now the game starts with *Me* in the role of the opponent. The game tree is the same as in Figure 4.3, but all $\mathbf{P}$s and $\mathbf{O}$s are swapped, which means swapped labels and mirrored $\ll$-relations, too. To be explicit, the order on outcomes is $\mathbf{O} : a \ll \mathbf{O} : b \ll \mathbf{O} : c$ and $\mathbf{P} : a \gg \mathbf{P} : d$. This game has value -2: if *I* go to the left at the root, then *You* go to $\mathbf{O} : b$, giving *Me* a payoff of -2. If *I* to go to the right *I* can reach the outcome of $\mathbf{P} : a$ with the

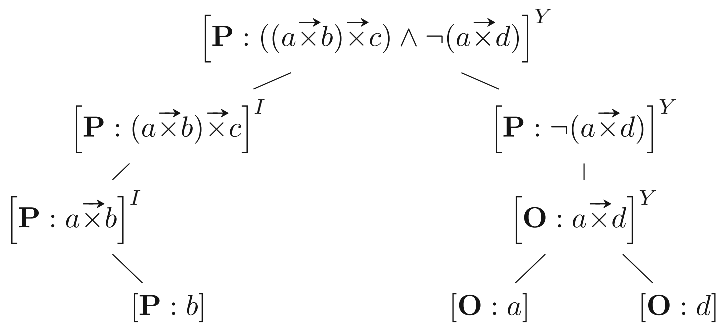


Figure 4.4: A 2-strategy for *Me* over $\mathcal{I} = \{b\}$.

same payoff of -2. Hence, if we swap **P**s and **O**s and labels in Figure 4.4, then this tree represents the corresponding -2-strategy for *You*.

4.3.2 The logic GCL and its negation

Usually, in a semantic view of a logic, validity of a formula F is defined as truth of F in all interpretations. In our context of graded truth, however, we can refine this notion to *graded validity*. Thus, we define the *degree (of validity)* of F to be the least possible value of F in all evaluation games:

$$v(F) = \min_{\mathcal{I}} v_{\mathcal{I}}(F),$$

where $v_{\mathcal{I}}(F)$ is an abbreviation for $v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F)$.

In Sections 4.4 and Section 4.5, we give a fully game-theoretic characterization of this degree. For now, let us check that our new logic satisfies the requirements formulated in Subsection 4.2.1. To this end, define $F \equiv G$ iff $v_{\mathcal{I}}(F) = v_{\mathcal{I}}(G)$. In the below proposition, Point 1 shows that the value of $\neg F$ depends only on the value of F, 2 shows that either F, or its negation (but not both) is winning, and 3 shows that every formula is equivalent to its double negation. Finally, 4 and 5 show that De Morgan's laws hold for GCL, and 6 shows associativity.

Proposition 4.3.7: ¬-requirements, De Morgan, associativity

The following holds:

1. $v_{\mathcal{I}}(F) = v_{\mathcal{J}}(F) \iff v_{\mathcal{I}}(\neg F) = v_{\mathcal{J}}(\neg F)$
2. $v_{\mathcal{I}}(F) \in \mathbb{Z}^+ \iff v_{\mathcal{I}}(\neg F) \in \mathbb{Z}^-$
3. $F \equiv \neg\neg F$

4. $\neg(F \wedge G) \equiv \neg F \vee \neg G$
5. $\neg(F \vee G) \equiv \neg F \wedge \neg G$
6. $((F \circ G) \circ H) \equiv (F \circ (G \circ H))$ for $\circ \in \{\wedge, \vee, \vec{\times}\}$

For the proof of the proposition we need the following lemma:

Lemma 4.3.8: role switch = negated value

$v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{P} : F)) = -v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F))$.

Proof. We fix $\mathcal{I}$ and F. For every history h in $v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F))$, let h' be the history with all $\mathbf{P}$s and $\mathbf{O}$s swapped. We write $v^{\mathbf{P}}(h)$ and $v^{\mathbf{O}}(h')$ for the values of the subgames $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{P} : F))@h$ and $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F))@h'$, respectively, and prove the claim by induction on h. The case for the initial history $\langle \mathbf{P} : F \rangle$ shows the claim.

Let h be terminal and end in $\mathbf{Q} : a$. Note that in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F))$, all $\mathbf{P}$s and $\mathbf{O}$ are swapped. Hence, if h is winning, i.e., $\mathcal{I} \models \mathbf{Q} : a$, then, since $\mathcal{I} \not\models \bar{\mathbf{Q}} : a$, h' is losing. We have

$$-v^{\mathbf{O}}(h) = |\pi_{\gg}^{\mathbf{O}}(h')| = |\pi_{\ll}^{\mathbf{P}}(h)| = v^{\mathbf{P}}(h),$$

where we write $\pi_{\gg}^{\mathbf{O}}$ for the longest $\gg$-path in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F)$ and $\pi_{\ll}^{\mathbf{P}}$ for the longest $\ll$-path in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{P} : F)$. The case where h is losing is symmetric.

Let h be labeled "I". We first show $v^{\mathbf{O}}(h) \trianglelefteq -v^{\mathbf{P}}(h')$. By Lemma 2.1.14, there is some $l = h \smile g$ such that *I* have a $v^{\mathbf{P}}(h)$-strategy in the subgame starting at l, which means $v^{\mathbf{P}}(l) \trianglerighteq v^{\mathbf{P}}(h)$. By the inductive hypothesis, $v^{\mathbf{O}}(l') = -v^{\mathbf{P}}(l)$, hence *You* have a $-v^{\mathbf{P}}(l)$-strategy in the l'-subgame. By Lemma 2.1.14, *You* have a $-v^{\mathbf{P}}(l)$-strategy $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F)@h'$, hence $v^{\mathbf{O}}(h') \trianglelefteq -v^{\mathbf{P}}(l)$. Since $-v^{\mathbf{P}}(l) \trianglelefteq -v^{\mathbf{P}}(h)$, we have $v^{\mathbf{O}}(h') \trianglelefteq -v^{\mathbf{P}}(h)$.

For the other inequality, we use Lemma 2.1.14 showing that for all $l = h \smile g$, *You* have a $v^{\mathbf{P}}(h)$-strategy in the subgame starting at l, hence $v^{\mathbf{P}}(l) \trianglelefteq v^{\mathbf{P}}(h)$. By the inductive hypothesis, $v^{\mathbf{O}}(l') = -v^{\mathbf{P}}(l)$, hence *I* have $-v^{\mathbf{O}}(l')$-strategy in every such l'-subgame. Since $v^{\mathbf{O}}(l') = -v^{\mathbf{P}}(l) \trianglerighteq -v^{\mathbf{P}}(h)$, *I* have a $-v^{\mathbf{P}}(h)$-strategy in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : P)@h'$ by Lemma 2.1.14. Therefore, $v^{\mathbf{O}}(h') \trianglerighteq -v^{\mathbf{P}}(h)$.

The inductive step where h is labeled "Y" is symmetric. □

Proof of Proposition 4.3.7. 1 and 2 follow directly from Lemma 4.3.8, as $v_{\mathcal{I}}(\neg F) = -v_{\mathcal{I}}(F)$. The other points are simple game equivalences: For example, the game starting at $\mathbf{P} : \neg(F \wedge G)$ automatically proceeds to $\mathbf{O} : F \wedge G$, where *I* choose between $\mathbf{O} : F$ and $\mathbf{O} : G$. In the game starting at $\neg F \vee \neg G$, *I* chose between continuing the game with $\mathbf{P} : \neg F$ and hence $\mathbf{O} : F$, or with $\mathbf{P} : \neg G$ and hence $\mathbf{O} : G$. Histories in both games are thus in a 1-1 correspondence, which shows their strategical equivalence. □

4.3.3 A degree semantics for GCL

We now extract a novel degree-based semantics for the language of GCL from our game $\mathbf{G}^{\mathrm{GCL}}$, as usual in choice logics. The definition of degree depends on the following syntactic notion of optionality:

Definition 4.3.9: Optionality

The optionality of choice formulas is defined recursively as follows:

$$\begin{aligned}
\mathrm{opt}^{\mathcal{G}}(x) &= 1 \text{ for every propositional variable } x,\\
\mathrm{opt}^{\mathcal{G}}(\neg F) &= \mathrm{opt}^{\mathcal{G}}(F),\\
\mathrm{opt}^{\mathcal{G}}(F \wedge G) &= \max(\mathrm{opt}^{\mathcal{G}}(F), \mathrm{opt}^{\mathcal{G}}(G)),\\
\mathrm{opt}^{\mathcal{G}}(F \vee G) &= \max(\mathrm{opt}^{\mathcal{G}}(F), \mathrm{opt}^{\mathcal{G}}(G)),\\
\mathrm{opt}^{\mathcal{G}}(F \vec{\times} G) &= \mathrm{opt}^{\mathcal{G}}(F) + \mathrm{opt}^{\mathcal{G}}(G).
\end{aligned}$$

The difference to QCL is that negation does not trivialize optionality. Optionality is used in the recursive definition of degrees in the case of the choice connective $\vec{\times}$:

Definition 4.3.10: Degree under an interpretation

The satisfaction degree of choice formulas is defined recursively as follows:

$$\begin{aligned}
\deg^{\mathcal{G}}_{\mathcal{I}}(a) &= \begin{cases} 1 & \text{if } x \in \mathcal{I},\\ -1 & \text{if } x \notin \mathcal{I},\end{cases}\\
\deg^{\mathcal{G}}_{\mathcal{I}}(\neg F) &= -\deg^{\mathcal{G}}_{\mathcal{I}}(F)\\
\deg^{\mathcal{G}}_{\mathcal{I}}(F \wedge G) &= \min\nolimits_{\trianglelefteq}(\deg^{\mathcal{G}}_{\mathcal{I}}(F), \deg^{\mathcal{G}}_{\mathcal{I}}(G)),\\
\deg^{\mathcal{G}}_{\mathcal{I}}(F \vee G) &= \max\nolimits_{\trianglelefteq}(\deg_{\mathcal{I}}(F), \deg^{\mathcal{G}}_{\mathcal{I}}(G)),\\
\deg^{\mathcal{G}}_{\mathcal{I}}(F \vec{\times} G) &= \begin{cases} \deg^{\mathcal{G}}_{\mathcal{I}}(F) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^{+}\\ \mathrm{opt}^{\mathcal{G}}(F) + \deg^{\mathcal{G}}_{\mathcal{I}}(G) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^{-} \text{ and } \deg^{\mathcal{G}}_{\mathcal{I}}(G) \in \mathbb{Z}^{+},\\ \deg^{\mathcal{G}}_{\mathcal{I}}(F) - \mathrm{opt}^{\mathcal{G}}(G) & \text{otherwise}\end{cases}
\end{aligned}$$

Analogously to QCL, $\mathcal{I}$ is a preferred model of F iff $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^{+}$ and $\deg^{\mathcal{G}}_{\mathcal{J}}(F) \trianglelefteq \deg^{\mathcal{G}}_{\mathcal{I}}(F)$ for all $\mathcal{J}$.

An interesting fact about our game semantics is that the notion of optionality, which must be defined a-priori in degree-based semantics, arises naturally in our game.

Lemma 4.3.11: opt **captures** $\pi_{\ll}$

For every game state g, let

$$|\pi_{\ll}(g)| = \max\{|\pi_{\ll}(h)| : h \text{ is a terminal history in } \mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)\}.$$

Then $|\pi_{\ll}(\mathbf{Q} : F)| = \mathrm{opt}^{\mathcal{G}}(F)$ for both $\mathbf{Q} = \mathbf{P}, \mathbf{O}$. The same equality holds for $\pi_{\gg}$.

Proof. First note that the definition of $|\pi_{\ll}(g)|$ does not depend on $\mathcal{I}$. The lemma is proved by induction on F. If $g = \mathbf{Q} : a$, then the relation $\ll$ is trivial and $\mathrm{opt}^{\mathcal{G}}(a) = 1$.

If $g = \mathbf{Q} : F_1 \vee F_2$, then all histories in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)$ can be written as $g \smile h_i$, where h_i is a history in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{Q} : F_i)$ for some $i = 1, 2$. As for the $\ll$-relation in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)$, we have that $h \ll k$ iff $h_i \ll k_i$, i.e., both histories go to the same subgame in the first move and the resulting histories are in $\ll$-relation. Thus, a longest $\ll$-path in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)$ is either $g \smile h_1^1 \ll \ldots \ll g \smile h_1^n$, where $h_1^1 \ll \cdots \ll h_1^n$ is $\pi_{\ll}(\mathbf{Q} : F_1)$, or $g \smile h_2^1 \ll \ldots \ll g \smile h_2^m$, where $h_2^1 \ll \cdots \ll h_2^m$ is $\pi_{\ll}(\mathbf{Q} : F_2)$, whichever is the longest. Thus, $|\pi_{\ll}(g)| = \max\{n, m\} = \max\{|\pi_{\ll}(\mathbf{Q} : F_1)|, |\pi_{\ll}(\mathbf{Q} : F_2)|\}$. By the inductive hypothesis, this is equal to $\max\{\mathrm{opt}^{\mathcal{G}}(F_1)|, \mathrm{opt}^{\mathcal{G}}(F_2)|\} = \mathrm{opt}^{\mathcal{G}}(F_1 \vee F_2)$. The case $\mathbf{Q} : F_1 \wedge F_2$ is similar.

Let $g = \mathbf{P} : \neg G$. By the inductive hypothesis, the longest path in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{O} : G)$ has length $|\pi_{\ll}(\mathbf{O} : G)| = \mathrm{opt}^{\mathcal{G}}(G)$. For every $\ll$-path $h_1 \ll \ldots \ll h_n$, the path $g \smile h_1 \ll \ldots \ll g \smile h_n$ is a $\ll$-path in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : \neg G)$, and vice versa. Hence, $|\pi_{\ll}(\mathbf{P} : \neg G)| = \mathrm{opt}^{\mathcal{G}}(G) = \mathrm{opt}^{\mathcal{G}}(\neg G)$.

The most interesting case is for $g = \mathbf{Q} : F_1 \vec{\times} F_2$. As before, we write the terminal histories h in this game as $g \smile h_i$, where h_i is a terminal history in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{Q} : F_i)$ for some $i = 1, 2$. The difference is that now we have $g \smile h_1 \gg g \smile h_2$ if $\mathbf{Q} = \mathbf{P}$. In this case, the longest $\ll$-path in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)$ is

$$\pi_{\ll}(g) = g \smile h_2^1 \ll \ldots \ll g \smile h_2^m \ll g \smile h_1^1 \ll \ldots \ll g \smile h_1^n,$$

where $h_2^1 \ll \cdots \ll h_2^m$ is $\pi_{\ll}(\mathbf{P} : F_2)$ and $h_1^1 \ll \cdots \ll h_1^n$ is $\pi_{\ll}(\mathbf{P} : F_1)$. Hence, $|\pi_{\ll}(g)| = |\pi_{\ll}(\mathbf{P} : F_1)| + |\pi_{\ll}(\mathbf{P} : F_2)| = \mathrm{opt}^{\mathcal{G}}(F_1) + \mathrm{opt}^{\mathcal{G}}(F_2) = \mathrm{opt}^{\mathcal{G}}(F_1 \vec{\times} F_2)$. In the case $\mathbf{Q} = \mathbf{O}$, we have $g \smile h_1 \ll g \smile h_2$ and we can again connect the two paths. □

Lemma 4.3.12

Let h be a history of $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g)$ ending in the game state g'. Then

$$v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(h)) = \begin{cases} v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g')) + |\pi_{\ll}(h)| - 1, & \text{if } v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g')) \in \mathbb{Z}^+, \\ v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g')) - |\pi_{\gg}(h)| + 1, & \text{if } v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(g')) \in \mathbb{Z}^-. \end{cases}$$

Proof. We show the first case where I can win $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g')$. Let $\ll_{g'}$ be the relation $\ll$ in that game. We notice that (1) for every pair of histories $h' \ll_{g'} h''$ we have that $h \smile h' \ll h \smile h''$. Furthermore, (2) for every k with $h \ll k$ and every history h' in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g')$, we have $h \smile h' \ll k$. This instructs us how to construct a maximal $\ll$-path starting in h': take $\pi_{\ll_{g'}}(h')$, i.e., a maximal $\ll_{g'}$-path starting in h': $h' = h'_1 \ll_{g'} \cdots \ll_{g'} h'_n$, and with $\pi_{\ll}(h)$, i.e. a maximal $\ll$-path starting in h: $h = h_1 \ll h_2 \ll \cdots \ll h_m$. Then the path

$$h \smile h'_1 \ll \cdots \ll h \smile h'_n \ll h_2 \ll \cdots \ll h_m,$$

is a maximal $\ll$-path starting at $h \smile h'$ and has length $|\pi_{\ll_{g'}}(h')| + |\pi_{\ll}(h)| - 1$. If h' is a terminal history, this translates to payoffs: $\delta_{\mathcal{I}}(h \smile h') = \delta_{\mathcal{I}}^{g'}(h') + |\pi_{\ll}(h)| + 1$, where $\delta_{\mathcal{I}}^{g'}$ denotes the payoff function in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g')$. If h' is the result of *Me* playing *My* winning minimax-strategy and *You* playing *Your* maximin-strategy in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g')$, we obtain $v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)) = v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g')) + |\pi_{\ll}(h)| + 1$, as desired. ☐

Theorem 4.3.13: Value = degree

For every choice formula F, the value of $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{P} : F)$ is $\deg_{\mathcal{I}}^{\mathcal{G}}(F)$. The value of $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(\mathbf{O} : F)$ is $-\deg_{\mathcal{I}}^{\mathcal{G}}(F)$.

Proof. Let us fix $\mathcal{I}$ and write $v(g)$ for the value of $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)$ and $v(h)$ for the value of $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(g)@h$, where h is a history starting in g. We only need to show that $v(\mathbf{P} : F) = \deg_{\mathcal{I}}^{\mathcal{G}}(F)$, as the other claim $v(\mathbf{O} : F) = -\deg_{\mathcal{I}}^{\mathcal{G}}(F)$ follows from Lemma 4.3.8. The proof proceeds by induction on F.

For the base case $F = a$, the game consists of the initial history $\langle \mathbf{Q} : a \rangle$ only. The longest $\ll$-path, therefore, has length 1. Consequently, $v(\mathbf{Q} : a) = 1$ if $\mathcal{I} \models \mathbf{Q} : a$ and $v_{\mathbf{Q}:a} = -1$, otherwise.

In the first round of the game starting at $\mathbf{P} : F_1 \vee F_2$, I choose between $\mathbf{P} : F_1$ and $\mathbf{P} : F_2$. To compute the values of these subgames with respect to the games starting at the $\mathbf{P} : F_i$, let $h_i = \langle \mathbf{P} : F_1 \vee \mathbf{P} : F_2, \mathbf{P} : F_i \rangle$, for $i = 1, 2$. Using Lemma 4.3.12, we have

$$v(h_i) = \begin{cases} v(\mathbf{P} : F_i) + |\pi_{\ll}(h_i)| - 1, & \text{if } v(\mathbf{P} : F_i) \in \mathbb{Z}^+, \\ v(\mathbf{P} : F_i) - |\pi_{\gg}(h_i)| + 1, & \text{if } v(\mathbf{P} : F_i) \in \mathbb{Z}^-. \end{cases}$$

Since $|\pi_{\ll}(h_i)| = |\pi_{\gg}(h_i)| = 1$, we get $v(h_i) = v(\mathbf{P} : F_i)$. *My* best strategy is to move to the subgame with maximal value (Lemma 2.1.14), hence:

$$\begin{aligned} v(\mathbf{P} : F_1 \vee F_2) &= \max_{\trianglelefteq}\{v(h_1), v(h_2)\} \\ &= \max_{\trianglelefteq}\{v(\mathbf{P} : F_1), v(\mathbf{P} : F_2)\} \\ &= \max_{\trianglelefteq}\{\deg_{\mathcal{I}}^{\mathcal{G}}(F_1), \deg_{\mathcal{I}}^{\mathcal{G}}(F_2)\} \\ &= \deg_{\mathcal{I}}^{\mathcal{G}}(F_1 \vee F_2), \end{aligned}$$

where we used the inductive hypothesis in Step 3.

At $\mathbf{P} : F_1 \wedge \mathbf{P} : F_2$, *You* choose between $\mathbf{P} : F_1$ and $\mathbf{P} : F_2$. As before, we have $v(h_i) = v(\mathbf{P} : F_i)$ for $i = 1, 2$. Since *You* seek to minimize *My* payoff, *Your* best strategy is to move to the subgame with minimal value, hence:

$$\begin{aligned} v(\mathbf{P} : F_1 \wedge F_2) &= \min_{\trianglelefteq}\{v(h_1), v(h_2)\} \\ &= \min_{\trianglelefteq}\{v(\mathbf{P} : F_1), v(\mathbf{P} : F_2)\} \\ &= \min_{\trianglelefteq}\{\deg^{\mathcal{G}}_{\mathcal{I}}(F_1), \deg^{\mathcal{G}}_{\mathcal{I}}(F_2)\} \\ &= \deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \wedge F_2). \end{aligned}$$

At $\mathbf{P} : \neg G$, the game continues at $\mathbf{O} : G$, hence $v(\mathbf{P} : \neg G) = v(\mathbf{O} : G)$. By the second claim and the inductive hypothesis, this equals $-v(\mathbf{O} : G) = -\deg^{\mathcal{G}}_{\mathcal{I}}(F) = \deg^{\mathcal{G}}_{\mathcal{I}}(\neg G)$.

Finally, we consider $\mathbf{P} : F_1 \vec{\times} F_2$. Again, *I* choose between $\mathbf{P} : F_1$ and $\mathbf{P} : F_2$, but this time, the values of these subgames do not coincide with those of the games starting at $\mathbf{P} : F_i$. As before, we apply Lemma 4.3.12 with $h_i = \langle \mathbf{P} : F_1 \vec{\times} F_2, \mathbf{P} : F_i \rangle$:

$$v(h_i) = \begin{cases} v(\mathbf{P} : F_i) + |\pi_{\ll}(h_i)| - 1, & \text{if } v(\mathbf{P} : F_i) \in \mathbb{Z}^+, \\ v(\mathbf{P} : F_i) - |\pi_{\gg}(h_i)| + 1, & \text{if } v(\mathbf{P} : F_i) \in \mathbb{Z}^-. \end{cases}$$

For h_1, we have $|\pi_{\ll}(h_1)| = 1$, and $|\pi_{\gg}(h_1)| = \mathrm{opt}(F_2) + 1$, by Lemma 4.3.11, since all outcomes of the subgame starting at h_1 are in $\gg$-relation with all outcomes of the h_2-subgame. Therefore, $v(h_1) = v(\mathbf{P} : F_1)$ if $v(\mathbf{P} : F_1) \in \mathbb{Z}^+$, and $v(\mathbf{P} : F_1) - \mathrm{opt}(F_2)$, otherwise. A similar computation shows that $v(h_2) = v(\mathbf{P} : F_2) + \mathrm{opt}(F_2)$ if $v(\mathbf{P} : F_2) \in \mathbb{Z}^+$, and $v(\mathbf{P} : F_2)$, otherwise. Since *I* seek to maximize *My* payoff (to be explicit: *I* prefer winning over losing and satisfying as many $\ll$-preferences as possible) *I* move to $\mathbf{P} : F_1$ if $v(\mathbf{P} : F_1) \in \mathbb{Z}^+$, to $\mathbf{P} : F_2$ if $v(\mathbf{P} : F_1) \in \mathbb{Z}^-$ and $v(\mathbf{P} : F_2) \in \mathbb{Z}^+$, and to $\mathbf{P} : F_1$ if $v(\mathbf{P} : F_1) \in \mathbb{Z}^-$ and $v(\mathbf{P} : F_2) \in \mathbb{Z}^-$.

$$v(\mathbf{P} : F_1 \vec{\times} F_2) = \begin{cases} v(\mathbf{P} : F_1) & \text{if } v(\mathbf{P} : F_1) \in \mathbb{Z}^+, \\ v(\mathbf{P} : F_2) + \mathrm{opt}(F_1) & \text{if } v(\mathbf{P} : F_1) \in \mathbb{Z}^- \text{ and } v(\mathbf{P} : F_2) \in \mathbb{Z}^+, \\ v(\mathbf{P} : F_1) - \mathrm{opt}(F_2) & \text{if } v(\mathbf{P} : F_1) \in \mathbb{Z}^- \text{ and } v(\mathbf{P} : F_2) \in \mathbb{Z}^-, \end{cases}$$

and using the inductive hypothesis,

$$v(\mathbf{P} : F_1 \vec{\times} F_2) = \begin{cases} \deg^{\mathcal{G}}_{\mathcal{I}}(F_1) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F_1) \in \mathbb{Z}^+, \\ \deg^{\mathcal{G}}_{\mathcal{I}}(F_2) + \mathrm{opt}(F_1) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F_1) \in \mathbb{Z}^- \text{ and } \deg^{\mathcal{G}}_{\mathcal{I}}(F_2) \in \mathbb{Z}^+, \\ \deg^{\mathcal{G}}_{\mathcal{I}}(F_1) - \mathrm{opt}(F_2) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F_1) \in \mathbb{Z}^- \text{ and } \deg^{\mathcal{G}}_{\mathcal{I}}(F_2) \in \mathbb{Z}^-, \end{cases}$$

which is equal to $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \vec{\times} F_2)$. □

As a corollary, we get a degree-based characterization of the degree of validity of a formula introduced in Section 4.3.2.

Corollary 4.3.14: Degree of validity, degree version

Let F be a choice formula. Then

$$v(F) = \min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F).$$

4.3.4 Capturing CL and QCL

This subsection aims at clarifying the connections of our game semantics with classical logic and QCL. To this end, we formally prove that $\mathbf{G}^{\mathrm{GCL}}$ is a refinement of Hintikka's game. We adjust our game semantics to provide an adequate game model for QCL, too. The resulting semantic game, $\mathbf{G}^{\mathrm{QCL}}$, illustrates the the asymmetric behavior of negation, as discussed in Section 4.2.1, in a game-theoretic way.

$\mathbf{G}^{\mathrm{GCL}}$ as a refinement of Hintikka's game

We start with the following recursive translation of choice formulas F:

$$\begin{aligned} a^* &= a \\ (F \wedge G)^* &= F^* \wedge G^* \\ (F \vee G)^* &= F^* \vee G^* \\ (F \vec{\times} G)^* &= F^* \vee G^* \\ (\neg F)^* &= \neg F^* \end{aligned}$$

The $*$-translation of a choice formula F gives the "classical content" of F by removing all soft constraints, as every $\vec{\times}$ is replaced by $\vee$. Hence, preferences do not matter in the evaluation game over F^*, and payoffs are restricted to -1 (full dissatisfaction) and 1 (full satisfaction). We start by showing that the winning strategies in $\mathbf{G}^{\mathrm{GCL}}$ correspond to winning strategies in Hintikka's game. To this end, we extend the translation to game states by setting $(\mathbf{Q} : F)^* = \mathbf{Q} : F^*$ for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$.

Theorem 4.3.15

Let F be a choice formula. Then I have a winning strategy in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{Q} : F)$ iff I have a winning strategy in $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}(\mathbf{Q} : F^*)$.

Proof. By induction on F, we show that $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+$ iff $\mathcal{I} \models F^*$. By Theorem 4.3.13 the left side of the equivalence is itself equivalent to *Me* having a winning strategy in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{Q} : F)$, and, by the adequacy for $\mathbf{G}^{\mathrm{CL}}$ (Example 2.1.9), the right side is equivalent to *Me* having a winning strategy in $\mathbf{G}^{\mathrm{CL}}_{\mathcal{I}}((\mathbf{Q} : F)^*)$, giving us the claim.

If F is a variable, the value of $\deg^{\mathcal{G}}_{\mathcal{I}}(a) = 1$ iff $a = a^*$ is true under $\mathcal{I}$.

If $F = F_1 \circ F_2$, for $\circ \in \{\lor, \overrightarrow{\times}\}$, then $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \circ F_2) \in \mathbb{Z}^+$ iff for some i, $\deg^{\mathcal{G}}_{\mathcal{I}}(F_i) \in \mathbb{Z}^+$. By the inductive hypothesis, this is the case iff $\mathcal{I} \models F_i^*$ for some i, with is equivalent to $\mathcal{I}$ satisfying $F_1^* \circ F_2^* = F^*$.

If $F = F_1 \land F_2$, then $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \land F_2) \in \mathbb{Z}^+$ iff both $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1), \deg^{\mathcal{G}}_{\mathcal{I}}(F_2) \in \mathbb{Z}^+$. By the inductive hypothesis, this is equivalent to $\mathcal{I} \models F_1^*$ and $\mathcal{I} \models F_2^*$, which in turn is equivalent to $\mathcal{I}$ satisfying $F_1^* \land F_2^* = F^*$. □

As an immediate consequence, we get that the evaluation game for GCL, when played over a propositional formula, agrees with Hintikka's game for classical propositional logic:

Corollary 4.3.16

Let $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, then for every propositional formula F,

$$v(\mathbf{G}^{\mathrm{GCL}}(\mathbf{Q} : F)) = v(\mathbf{G}^{\mathrm{CL}}(\mathbf{Q} : F))$$

Proof. The first equality follows by Theorem 4.3.16 by the facts that $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{Q} : F^*)$ has value 1, or -1, and $F^* = F$, if F is propositional. □

A Game for QCL

We now describe an adequate game for QCL. Intuitively, the asymmetries of negation discussed in Section 4.2.1 translate into the following asymmetric rule for negation:

($\mathbf{P}_\neg$) At $\mathbf{P} : \neg G$, the game automatically continues with $\mathbf{O} : G$, and all preferences are erased.

This means, that if the game starts at a game state $\mathbf{P} : F$, and comes to a a negation, $\mathbf{O} : \neg G$, roles are switched and from this point on, the game is a win/lose game only. Additionally, if the game starts at $\mathbf{O} : F$, then it is a win/lose game from the very start.

Note that by interpreting negation via this rule, some of the behavior of negation mentioned in Section 4.2.1 becomes immediately clear. For example, the games $\mathbf{P} : F$ and $\mathbf{P} : \neg\neg F$ are equivalent, when it comes to who wins and who loses, since the second game continues at $\mathbf{P} : F$, after two role switches. However, now all preferences are lost which could influence the induced payoff.

Formally, the most elegant way to prove that the new semantics is adequate for QCL is to use our game for GCL to model preferences and truth and model the asymmetries present in QCL by a suitable translation of formulas. This translation is recursively defined as follows[4]:

$$
\begin{aligned}
a^{\triangledown} &= a \\
(F \wedge G)^{\triangledown} &= F^{\triangledown} \wedge G^{\triangledown} \\
(F \vee G)^{\triangledown} &= F^{\triangledown} \vee G^{\triangledown} \\
(F \vec{\times} G)^{\triangledown} &= F^{\triangledown} \vec{\times} G^{\triangledown} \\
(\neg F)^{\triangledown} &= \neg F^{*}
\end{aligned}
$$

The $\triangledown$-translation aims at capturing QCL by removing all preferences from negated formulas but keeping unnegated preferences intact. We extend this translation to game states by setting $(\mathbf{P} : F)^{\triangledown} = \mathbf{P} : F^{\triangledown}$, $(\mathbf{O} : F)^{\triangledown} = \mathbf{O} : F^{*}$.

Theorem 4.3.17: $\mathbf{G}^{\mathrm{GCL}}$+$\triangledown$ characterizes deg

For every choice formula F, *I* have a winning strategy in $\mathbf{G}^{\mathrm{GCL}}(\mathbf{P} : F^{\triangledown})$ iff $\deg_{\mathcal{I}}(F) \in \mathbb{Z}^{+}$. In this case,

$$v(\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F^{\triangledown})) = \deg_{\mathcal{I}}(F).$$

You have a winning strategy in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F^{\triangledown})$ iff $\deg_{\mathcal{I}}(F) = -1$.

The theorem implies that the following game $\mathbf{G}^{\mathrm{QCL}}_{\mathcal{I}}(\mathbf{P} : F)$ over an interpretation $\mathcal{I}$ and a choice formula F is adequate[5]. The game proceeds as $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F^{\triangledown})$, except for the payoffs: if the payoff in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F^{\triangledown})$ is $k \in \mathbb{Z}^{+}$, then $\mathbf{G}^{\mathrm{QCL}}_{\mathcal{I}}(\mathbf{P} : F)$ has the same payoff. If the payoff in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P} : F^{\triangledown})$ is in $\mathbb{Z}^{-}$, then the payoff in $\mathbf{G}^{\mathrm{QCL}}_{\mathcal{I}}(\mathbf{P} : F)$ is -1. From the theorem, we immediately get the adequacy of this game:

Corollary 4.3.18: Adequacy of $\mathbf{G}^{\mathrm{QCL}}_{\mathcal{I}}$

Let F be a choice formula and $\mathcal{I}$ an interpretation. Then

$$v(\mathbf{G}^{\mathrm{QCL}}_{\mathcal{I}}(\mathbf{P} : F)) = \deg_{\mathcal{I}}(F).$$

We prove Theorem 4.3.17 by using the degree-based semantics from the previous subsection. We need the following lemma:

Lemma 4.3.19

For all choice formulas F, $\mathrm{opt}(F) = \mathrm{opt}^{\mathcal{G}}(F^{\triangledown})$.

[4]The use of the symbol "$\triangledown$" is inspired by Baaz's projection modality for Gödel logic[2]. The degree of $\triangledown$-translated formulas behaves similarly to that operator, as demonstrated in Theorem 4.3.17.

[5]We do not deal with this game again, hence we skip a fully formal definition.

Proof. By a straightforward induction on F. If F is a variable, or of the form $F_1 \vee F_2$, $F_1 \vee F_2$, or $F_1 \vec{\times} F_2$, then opt and $\text{opt}^{\mathcal{G}}$ are defined the same. If $F = \neg G$, then $\text{opt}^{\mathcal{G}}(F^{\triangledown}) = \text{opt}^{\mathcal{G}}(G^*) = 1$, using the fact that $\text{opt}^{\mathcal{G}}$ gives 1 for all propositional formulas. □

Proof of Theorem 4.3.17. By induction on F, we show $\deg^{\mathcal{G}}_{\mathcal{I}}(F^{\triangledown}) = \deg_{\mathcal{I}}(F)$, if $\deg_{\mathcal{I}}(F) \in \mathbb{Z}^+$, and $\deg^{\mathcal{G}}_{\mathcal{I}}(F^{\triangledown}) \in \mathbb{Z}^-$ iff $\deg_{\mathcal{I}}(F) = -1$. Theorem 4.3.13 then gives the game-theoretic characterization.

If $F = a$, then $\deg^{\mathcal{G}}_{\mathcal{I}}(a^{\triangledown}) = \deg^{\mathcal{G}}_{\mathcal{I}}(a) = \deg_{\mathcal{I}}(a)$.

If $F = F_1 \vee F_2$, then $\deg_{\mathcal{I}}(F_1 \vee F_2) \in \mathbb{Z}^+$ iff for some i, $\deg_{\mathcal{I}}(F_i) \in \mathbb{Z}^+$. By the inductive hypothesis, this is the case iff $\deg^{\mathcal{G}}_{\mathcal{I}}(F_i) \in \mathbb{Z}^+$, which is equivalent to $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \vee F_2) \in \mathbb{Z}^+$. In this case,

$$\begin{aligned}\deg_{\mathcal{I}}(F_1 \vee F_2) &= \max_{\trianglelefteq}\{\deg_{\mathcal{I}}(F_1), \deg_{\mathcal{I}}(F_2)\} \\ &= \max_{\trianglelefteq}\{\deg^{\mathcal{G}}_{\mathcal{I}}(F_1), \deg^{\mathcal{G}}_{\mathcal{I}}(F_2)\} = \deg^{\mathcal{G}}_{\mathcal{I}}(F_1 \vee F_2),\end{aligned}$$

where we applied the inductive hypothesis to those F_i with $\deg_{\mathcal{I}}(F_i) \in \mathbb{Z}^+$.

If $F = \neg G$, then $\deg_{\mathcal{I}}(\neg G) = 1$ iff (1) $\deg_{\mathcal{I}}(G) = -1$, and $\deg_{\mathcal{I}}(\neg G) = -1$ iff (2) $\deg_{\mathcal{I}}(G) \in \mathbb{Z}^+$. In Case 1, $\deg^{\mathcal{G}}_{\mathcal{I}}(G^{\triangledown}) \in \mathbb{Z}^-$, by the inductive hypothesis. Hence,

$$\deg^{\mathcal{G}}_{\mathcal{I}}((\neg G)^{\triangledown}) = \deg^{\mathcal{G}}_{\mathcal{I}}(\neg G^*) = -\deg^{\mathcal{G}}_{\mathcal{I}}(G^*) = 1,$$

where the last equality follows from Theorems 4.3.13 and 4.3.15, and Corollary 4.3.16: $\deg^{\mathcal{G}}_{\mathcal{I}}(G^{\triangledown}) \in \mathbb{Z}^-$ iff *You* have a winning strategy in $\mathbf{G}^{\text{CL}}_{\mathcal{I}}((\mathbf{P} : G^{\triangledown})^*) = \mathbf{G}^{\text{CL}}_{\mathcal{I}}(\mathbf{P} : G^*)$. In this case, the values of $\mathbf{G}^{\text{GCL}}(\mathbf{P} : G^*)$ and $\mathbf{G}^{\text{CL}}(\mathbf{P} : G^*)$ are both -1, which implies $\deg^{\mathcal{G}}_{\mathcal{I}}(G^*) = -1$. Case 2 is similar.

If $F = F_1 \vec{\times} F_2$ and $\deg_{\mathcal{I}}(F_1 \vec{\times} F_2) = -1$, then both F_1 and F_2 have degree -1. By the inductive hypothesis, $\deg^{\mathcal{G}}_{\mathcal{I}}(F_i^{\triangledown}) \in \mathbb{Z}^-$ for $i = 1, 2$. But then $\deg^{\mathcal{G}}_{\mathcal{I}}((F_1 \vec{\times} F_2)^{\triangledown}) = \deg^{\mathcal{G}}_{\mathcal{I}}(F_1^{\triangledown} \vec{\times} F_2^{\triangledown}) \in \mathbb{Z}^-$. If $\deg(F_1 \vec{\times} F_2) \in \mathbb{Z}^+$, then at least one of F_1 and F_2 have a degree in $\mathbb{Z}^+$. If $\deg_{\mathcal{I}}(F_1) \in \mathbb{Z}^+$, then, by the inductive hypothesis, $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1^{\triangledown}) = \deg_{\mathcal{I}}(F_1)$. Hence, $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1^{\triangledown} \vec{\times} F_2^{\triangledown}) = \deg^{\mathcal{G}}_{\mathcal{I}}(F_1^{\triangledown}) = \deg_{\mathcal{I}}(F_1) = \deg_{\mathcal{I}}(F_1 \vec{\times} F_2$. Otherwise, $\deg_{\mathcal{I}}(F_2) \in \mathbb{Z}^+$, and equals $\deg^{\mathcal{G}}_{\mathcal{I}}(F_2^{\triangledown})$. Hence, using Lemma 4.3.19, $\deg^{\mathcal{G}}_{\mathcal{I}}(F_1^{\triangledown} \vec{\times} F_2^{\triangledown}) = \deg^{\mathcal{G}}_{\mathcal{I}}(F_2^{\triangledown}) + \text{opt}^{\mathcal{G}}(F_1^{\triangledown}) = \deg_{\mathcal{I}}(F_2) + \text{opt}(F_1) = \deg_{\mathcal{I}}(F_1 \vec{\times} F_2)$. □

4.4 The Disjunctive Game as Semantic Game

As in Chapter 3, we start the lifting of the evaluation game to a provability game by extending it to the disjunctive game over an evaluation $\mathcal{I}$.

In the new game, *I* have an extra option: instead of moving according to the rules of the evaluation game, *I* can decide to create a "backup-copy" of h and continue playing at the *disjunctive game state* (or *disjunctive state*) $h \bigvee h$. If the game is unfavorable for *Me* in one copy, *I* can always come back to have another shot at the other copy. Formally, disjunctive game states are finite multisets of histories of the evaluation game. We prefer to write $h_1 \bigvee ... \bigvee h_n$ for the disjunctive game state $\{h_1, ..., h_n\}$, but keep the convenient notation $h \in D$ if h belongs to the multiset set D. We write $D_1 \bigvee D_2$ for the multiset sum $D_1 + D_2$ and $D \bigvee h$ for $D + \{h\}$. A disjunctive state is called *elementary* if all its histories end in elementary game states.

My goal is to win at least one backup copy. The game states of this game can be thus read as disjunctions, and are therefore called *disjunctive game states*[6], (hence the name of the game). The sign "$\bigvee$", can also be read as maximum: the payoff at D is the maximum of the payoff of its terminal histories.

Note that due to the design of the game, runs of the game can now be infinite. (*I* can duplicate histories infinitely often). All infinite runs will be considered winning for *You*.

Additionally, *I* take the rule of a scheduler who decides which copy is played upon next[7]. At the disjunctive state $D \bigvee h$, *I* can point to the history h, coded by underlining: $D \bigvee \underline{h}$. Afterward, the corresponding player takes their turn in the evaluation game. Say they move to g, then the new disjunctive game state is $D \bigvee h \smile g$. Alternatively, *I* can decide to end the game. The winner is then determined as described above.

We now give a semi-formal description of the disjunctive game. Let D be a disjunctive state. Let D^{ter} consist of the terminal histories of D. We say that D is terminal if $D = D^{ter}$, or if *I* have decided to end the game.

(End) If no histories in D are underlined, *I* can end the game, and D becomes terminal.

(Dupl) If no histories in D are underlined and D is not terminal, *I* can *duplicate* an $h \in D$ and the game continues with $D \bigvee h$.

(Sched) If no histories in $D = D' \bigvee h$ are underlined and D is not terminal, *I* can *underline* a non-terminal $h \in D$ and the game continues with $D' \bigvee \underline{h}$.

(Move) If $D = D' \bigvee \underline{h}$, then the player who is to move in the evaluation game $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(h)$ at the history h makes a legal move to the game state g and the game continues with $D \bigvee h \smile g$. For example, if h ends in $\mathbf{P} : G_1 \wedge G_2$, then *You* chose a $k \in \{1, 2\}$, and the game continues with $D \bigvee h \smile \mathbf{P} : G_k$.

(Pay) If D is terminal, then the payoff is the maximum of all the payoffs of the histories in D. In particular, *I* win iff *I* win the evaluation game in some $h \in D^{ter}$, and *You* lose. Otherwise, *You* win, and *I* lose.

[6]To avoid confusion, we always refer to game states of the disjunctive game $\mathbf{DG}^{\mathrm{GCL}}$ as "*disjunctive (game) states*". "*(Game) states*" is reserved for the semantic game $\mathbf{G}^{\mathrm{GCL}}$.

[7]We could be more general here and give the task of scheduling to *You*, or even a non-strategic player, or so-called *regulation function*. In terms of game values, all of these variants are equivalent – a result that holds for all finite games. In order to keep the presentation here more streamlined we will discuss the topic of regulations in full generality in Chapter 5

Additionally, we require that if no history of D is underlined, *I* must move according to **(End)**, **(Dupl)** or **(Sched)**. **(Dupl)** is referred to as the *duplication rule* and **(Sched)** as the *scheduling*, or *underlining rule*.

We point out the similarities of this game to the disjunctive game for hybrid logic. The rules **(End)**, **(Dupl)** or **(Sched)** are exactly as in $\mathbf{DG}^{\mathrm{Hyb}}$. The difference lies in the payoffs: while payoffs in $\mathbf{DG}^{\mathrm{Hyb}}$ are restricted to $-1, 1$, we now have a richer domain. Furthermore, the game for GCL does not have infinite branching. Note, however, that these differences are imports of the respective evaluation games, and are not introduced at the level of the disjunctive game. We will formally investigate the result of lifting a general semantic game to a disjunctive game in Chapter 5. Another difference is that we do not consider infinite disjunctive states here. For hybrid logic, we needed infinite disjunctive states to model frame properties that should hold at every nominal.

Example 4.4.1. Let F be $((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} b)$, and $\mathcal{I} = \{b\}$, as in Example 4.3.6 . Remember, in the evaluation game starting in $\mathbf{O} : F$ the prefrence on outcomes was $\mathbf{O} : a \ll \mathbf{O} : b \ll \mathbf{O} : c$ and $\mathbf{P} : a \gg \mathbf{P} : d$. Figure 4.5 shows a compact representation of a strategy for *Me* for the game $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{O} : F)$, where we write the last game state of every history, instead of entire histories. For example, the game state $\mathbf{O} : a$ in the leftmost leaf stands for the history

$$\langle \mathbf{O} : F, \mathbf{O} : (a \vec{\times} b) \vec{\times} c, \mathbf{O} : a \vec{\times} b, \mathbf{O} : a \rangle.$$

Underlining moves are clear from the context and are therefore, hidden. First, *I* duplicate $\mathbf{O} : F$ and move to $\mathbf{P} : ((a \vec{\times} b) \vec{\times} c)$ in one copy and to $\mathbf{O} : \neg(a \vec{\times} d)$ in the other. The latter is immediately converted to $\mathbf{P} : a \vec{\times} d$, for which *I* repeat the strategy of duplicating and moving into both options. Finally, *I* point to $\mathbf{O} : (a \vec{\times} b) \vec{\times} c$, where it is *Your* turn. All *Your* possible choices are shown in the strategy. The payoffs are

$$\begin{aligned}
\delta_{\mathcal{I}}(\mathbf{O} : a \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) &= \max\{\delta_{\mathcal{I}}(\mathbf{O} : a), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\} \\
&= \max\{3, -2, -1\} = 3, \\
\delta_{\mathcal{I}}(\mathbf{O} : b \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) &= \max\{\delta_{\mathcal{I}}(\mathbf{O} : b), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\} \\
&= \max\{-2, -2, -1\} = -2, \\
\delta_{\mathcal{I}}(\mathbf{O} : c \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) &= \max\{\delta_{\mathcal{I}}(\mathbf{O} : c), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\} \\
&= \max\{1, -2, -2\} = 1.
\end{aligned}$$

Given these payoffs, *You* prefer the second outcome, giving *Me* a payoff of -2. Hence, the depicted strategy is a -2-strategy. We note two things. First, *I* cannot do better by playing another strategy. If the outcomes do not contain game states resulting from $\mathbf{O} : (a \vec{\times} b) \vec{\times} c$, then their pay-offs are the same or even less. Hence, we can conclude that the value of the game is -2. Although in this example, the strategy of first duplicating, then exploiting all possible moves produces many unnecessary moves, it is – in a way – optimal for *Me*. We will see in the following sections that this strategy is, in fact a -2-strategy over *all* interpretations $\mathcal{I}$.

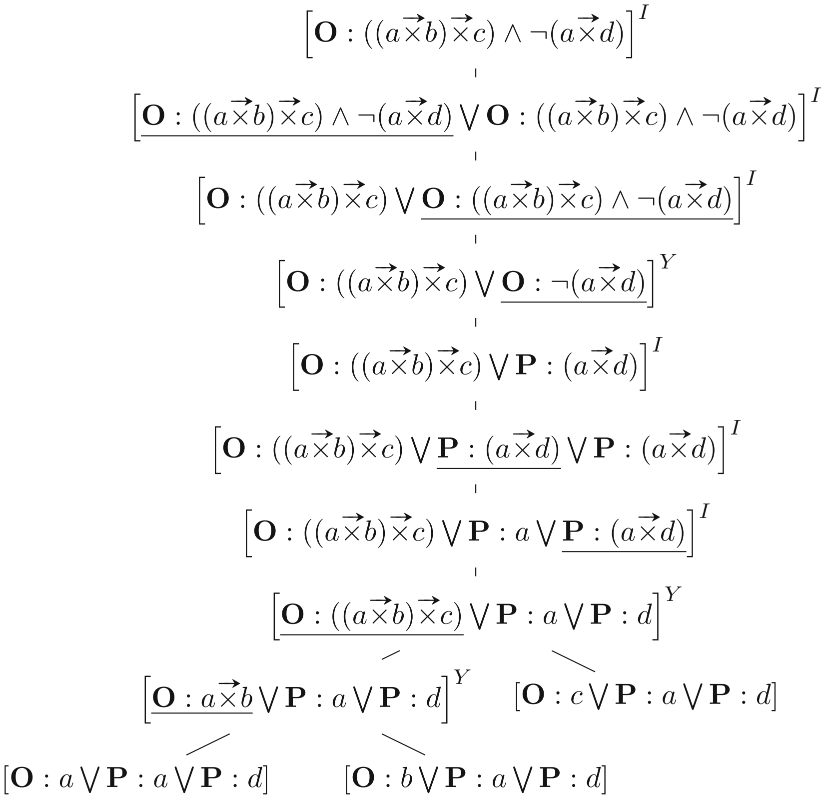


Figure 4.5: A compact representation of the strategy for *Me* for an instance of $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}$

We now come to the formal definition of the disjunctive game for GCL.

Definition 4.4.2: Disjunctive game as Semantic Game

Let $\mathcal{I}$ be an interpretation. Disjunctive states[a] are multisets of histories of the evaluation game, where none or exactly one history is underlined or the dummy state $*$. Let D be a disjunctive state consisting of histories of the evaluation game. The disjunctive game $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ is defined as follows:

- The initial history is $\langle D \rangle$.
- If $\mathfrak{h}$ is a disjunctive history and no state in the disjunctive state H is underlined, and
  - $\mathfrak{h} = \mathfrak{h}' \smile H \bigvee h$, then $\mathfrak{h} \smile *$ is a disjunctive history. If, additionally, h is not terminal, then $\mathfrak{h} \smile H \bigvee h \bigvee h$ and $\mathfrak{h} \smile H \bigvee \underline{h}$ are disjunctive

histories[b].

– $\mathfrak{h} = \mathfrak{h}' \smile H \bigvee \underline{h}$, then $\mathfrak{h} \smile H \bigvee (h \smile g)$ is a disjunctive history if $h \smile g$ is a history of the evaluation game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$.

- Non-terminal disjunctive histories ending in a disjunctive state H with no underlined histories are labeled "I". If $H = H' \bigvee \underline{h}$, then H is labeled the same as h in the evaluation game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$.
- As for the evaluation game, the payoff function $\delta_{\mathcal{I}}$ maps terminal disjunctive histories to the domain Z. Terminal disjunctive histories ending in $\langle \ldots, H, * \rangle$ are mapped to

$$\delta_{\mathcal{I}}(\langle \ldots, H, * \rangle) = \max_{h \in H} \delta_{\mathcal{I}}(h).$$

[a] To make the distinction easier, we always refer to game states of the disjunctive game as *disjunctive states* and histories of the disjunctive game as *disjunctive histories*.

[b] Note that there is implicit quantification over h.

Remark 4.4.3. (1) It follows from the definition that all terminal disjunctive histories end in $*$.

(2) Winning strategies and k-strategies for the disjunctive game are well-defined in light of Definition 2.1.5 and can again be considered subtrees of the game tree. We will speak of *disjunctive (winning) strategies* to distinguish from the evaluation game.

(3) Due to the duplication rule, runs of the game can now be infinite, resulting in a winning outcome for *Me*. Even though the game is infinite, the payoff values for every instance $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ is restricted to finitely many values. This follows from Lemma 4.3.4. By Theorem 2.1.11, this implies that for some k, both *I* and *You* have k-strategies. A direct proof follows from the two propositions below.

(4) In contrast to the evaluation game, the disjunctive game is not fully symmetric. This is due to the duplication rule, the winning conditions, and *My* role as a scheduler, i.e. the scheduling rule. At least the last asymmetry can be eliminated. In Chapter 5, we discuss a general framework for the disjunctive game, where the scheduling is done by a *regulation function*, which can be thought of as a third, non-strategic player. Under certain conditions, the disjunctive game retains its nice properties that we discuss in the present chapter.

We now compare the disjunctive game to the evaluation game from a strategic viewpoint. Essentially, a disjunctive k-strategy for *Me* in the game over a model is nothing more but a disjunction of strategies for the evaluation game:

Proposition 4.4.4: *My* disjunctive strategy = disjunction of *My* strategies

I have a disjunctive k-strategy in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ iff *I* have a k-strategy in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$ for some $h \in D$.

Proof. "$\Rightarrow$": Let σ be a disjunctive k-strategy for *Me* in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$. The case for $k = -1$ is trivial. If $k \triangleright -1$, then, by definition, all terminal histories in σ are finite. In this case, we can use backward induction on the tree structure of σ to show that for every $H \in \sigma$ there is $h \in H$ such that *I* have a winning strategy σ_H in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. The proposition then follows for the case where $H = D$.

By assumption, all leaves $*$ have a predecessor H such that there is some $h \in H$ with payoff $\trianglerighteq k$. If H is not followed by $*$ and is labeled "Y", then H is of the form $H' \bigvee \underline{h}$. The successors of H are $H \bigvee h'$, where h' are the successors of h in the evaluation game. By the inductive hypothesis, there are k-strategies $\sigma_{H' \bigvee h'}$ for all h'. If for some h', $\sigma_{H' \bigvee h'}$ is a k-strategy for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(l)$, where $l \in H'$, then we can simply set $\sigma_H = \sigma_{H' \bigvee h'}$. Otherwise, every $\sigma_{H' \bigvee h'}$ is a k-strategy for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h')$. Lemma 2.1.14 gives us a k-strategy σ_H for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$.

If H is labeled "I" and is of the form $H' \bigvee h$, and according to σ, *I* move to $H' \bigvee h \bigvee h$, then we simply set $\sigma_H = \sigma_{H \bigvee h \bigvee h}$ and use the inductive hypothesis. We proceed similarly if *I* move to $H \bigvee \underline{h}$. Finally if $H = H' \bigvee \underline{h}$ and *I* move to $H' \bigvee h'$, the inductive hypothesis gives us a k-strategy for $\sigma_{H' \bigvee h'}$ for some $l \in H' \bigvee h'$. If $l \in H'$, we set $\sigma_H = \sigma_{H' \bigvee h'}$ and are done. If $l = h'$, we use Lemma 2.1.14 to obtain a k-strategy for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$.

"$\Leftarrow$": Suppose, *I* have a k-strategy σ for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$ for some $h \in D$. The idea is as follows. *I* can enforce a payoff of $\trianglerighteq k$ in the disjunctive game by only ever playing on h and not touching the other histories in D. By induction on the tree structure of σ, we define a disjunctive strategy μ for *Me* with the following property: (*) every disjunctive state appearing in μ is of the form $H \bigvee l$, where l is a history in σ[8]. The base case follows from the assumption.

If the current disjunctive state is $H \bigvee l$ with l as required, and there are no game states underlined in H, then underline l and (*) follows immediately from the inductive hypothesis. If the current disjunctive state is $H \bigvee \underline{l}$ and l is labeled "Y", then *You* proceed to some $H \bigvee l \smile g$. Since l is labeled "Y", σ contains all immediate successors of l, hence $l \smile g$ must be a history in σ. If l is labeled "I", then *I* move to $H \bigvee \sigma(l)$. Clearly (*) holds for $\sigma(l)$. Eventually, the game reaches a state $H \bigvee l$, where l is a leaf of σ, and thus has payoff $\trianglerighteq k$. *I*, therefore, end the game and receive a payoff of at least k. $\square$

As an immediate consequence, we have that the degree of the disjunctive game is the

[8]Actually: a path through the tree structure of σ

maximum of the degrees of the histories in D. In particular, if D consists of a single history h, then the proposition gives the strategic equivalence of the games $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$ and $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. Hence, the disjunctive game played over a model should be itself considered a semantic game, as defined in Chapter 1.

Corollary 4.4.5: *My* disjunctive strategy = Maximum of *My* strategies

Let D be a disjunctive state, then

$$v(\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)) = \max_{h \in D} v(\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)).$$

Proof. Let v_l and v_r denote the two sides of the equality. By the proposition, *I* have v_l-strategy for some of the $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. Hence, $v_l \trianglelefteq v_r$. On the other hand, v_r is the maximal value of the $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. By the proposition, $v_l \trianglerighteq v_r$. □

We introduced the disjunctive game over a model for two reasons. First, the previous proposition shows that a strategy in the disjunctive game can be thought of as a disjunction, or maximum, of strategies for the evaluation game, thus giving a solid motivation and intuition, which is useful later on. Second, in the next section, we will see a version of the disjunctive game played over all models simultaneously. Formulating the disjunctive game makes the formulation, especially the proofs, much more accessible.

As in the previous chapter, we give a constructive proof of the following classically equivalent formulation of Proposition 4.4.4 in terms of *Your* strategies:

Proposition 4.4.6: *Your* disjunctive strategy = conjunction of *Your* strategies

You have a disjunctive k-strategy in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ iff *You* have k-strategies in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$ for all $h \in D$.

Proof of Proposition 3.4.5. "⇒": Let μ be a disjunctive k-strategy for *You* in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ and let $D = D' \bigvee h$. The idea is that *You* can use μ to enforce a payoff of $\trianglerighteq k$ in the run of the game where *I* only ever schedule h and its successors. The behavior of μ contains all the necessary information to define a strategy μ' for *You* in the evaluation game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. For a history $l = \langle l_1, \ldots l_n \rangle$ of the evaluation game labeled "Y", let $\mathrm{disj}(l)$ be the disjunctive history

$$D' \bigvee \langle l_1 \rangle, D' \bigvee \underline{\langle l_1 \rangle}, \ldots, D' \bigvee \langle l_1, \ldots, l_n \rangle, D' \bigvee \underline{\langle l_1, \ldots, l_n \rangle}.$$

If k is labeled "Y", then $\mathrm{disj}(l)$ is mapped to some $\mathrm{disj}(g) \smile D' \bigvee (l \smile g)$ under μ. Consequently, we define $\mu'(l) = l \smile g$.

Let σ' be a strategy for *Me* in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. We have to show that the payoff of playing σ' is $\trianglelefteq k$. To this end, we consider the run of the disjunctive game starting at D where *You* play μ and I play according to the following strategy σ: let $l = (\sigma'; \mu')^n(h)$ for some n. If the current disjunctive state is $D' \bigvee l$, then I underline l. If $\mathrm{disj}(l)$ is labeled "I", then I go to $D' \bigvee \sigma(l)$. For all other disjunctive states, the strategy is arbitrary. We compute the outcome of $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ where I play this strategy against *Your* μ. By induction on n, we show that $(\sigma, \mu)^n(D) = \mathrm{disj}((\sigma', \mu')(h))$. The case for $n = 0$ follows by definition. If $\mathrm{disj}(l)$ is labeled "Y", and *You* play according to μ, then the next disjunctive state is $D' \bigvee \mu'(l)$. Eventually, the game reaches the disjunctive state $D' \bigvee (\sigma; \mu')^m(h)$, where $(\sigma; \mu')^m(h) = O(\sigma, \mu')$ is terminal. Since μ is a k-strategy for *You*, this disjunctive state cannot have a payoff of more than k. Thus, the payoff at $O(\sigma, \mu')$ is $\trianglelefteq k$. Since σ was arbitrary, μ' is a k-strategy.

"$\Leftarrow$": For every $h \in D$, let μ_h be a k-strategy for *You* in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$. *Your* strategy μ in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ is as follows: in a disjunctive state $H \bigvee \underline{l}$ labeled "Y", l is a history in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$, for some h. Hence, *You* can use μ_h and go to $H \bigvee \sigma_h(l)$. Playing this way ensures that all histories contained in every resulting disjunctive state consist of histories of the σ_hs against any opposing strategy from *Me*. By assumption, every such history that is also terminal has payoff $\trianglelefteq k$. Hence, the game can only end in disjunctive states with a payoff of at most k, which shows that μ is a k-strategy for *Me*. □

4.5 The Disjunctive Game as Provability Game

In this section, we lift the disjunctive game to a provability game and prove the adequacy of the resulting game.

Intuitively, the new game $\mathbf{DG}^{\mathrm{GCL}}(D)$ can be interpreted as the scenario where the players of the evaluation game $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ forgot – or have not been informed – about the structure of the interpretation $\mathcal{I}$. Both players' goal is to develop strategies that guarantee them a win, independent of what the interpretation $\mathcal{I}$ looks like.

Note that this "playing over a model blindly" is only possible because the game trees of the disjunctive game $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ are the uniform[9] in $\mathcal{I}$. The only place where $\mathcal{I}$ comes into play is at the winning conditions. It is precisely these winning conditions that we need to alter to capture our intuition of *My* strategy being winning over all models:

(Pay) Let D be terminal. The payoff at D is the minimal payoff $\delta_{\mathcal{I}}(D)$ of $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}$, where $\mathcal{I}$ ranges over all interpretations. In particular, I win and *You* lose the game if, for every interpretation $\mathcal{I}$, I win the game $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(D)$. Otherwise *You* win, and I lose.

Example 4.5.1. We continue Example 4.4.1, but now we consider the same game played over all models, i.e., $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$, with $F = ((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} b)$. Figure 4.5 shows a compact representation of *My* disjunctive strategy. The new payoffs are

$$\delta(\mathbf{O} : a \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) = \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O} : a), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\}$$

[9] By this we mean that the game trees (or: set of histories) for the games $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$ and $\mathbf{DG}_{\mathcal{J}}^{\mathrm{GCL}}(D)$ are the same, for any two interpretations $\mathcal{I}$ and $\mathcal{J}$

$$= \max\{\delta_\emptyset(\mathbf{O}:a), \delta_\emptyset(\mathbf{P}:a), \delta_\emptyset(\mathbf{P}:d)\} = \max\{3,-2,-1\} = 3,$$
$$\delta(\mathbf{O}:b \bigvee \mathbf{P}:a \bigvee \mathbf{P}:d) = \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O}:b), \delta_{\mathcal{I}}(\mathbf{P}:a), \delta_{\mathcal{I}}(\mathbf{P}:d)\}$$
$$= \max\{\delta_{\{b\}}(\mathbf{O}:b), \delta_{\{b\}}(\mathbf{P}:a), \delta_{\{b\}}(\mathbf{P}:d)\} = \max\{-2,-2,-1\} = -2,$$
$$\delta(\mathbf{O}:c \bigvee \mathbf{P}:a \bigvee \mathbf{P}:d) = \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O}:c), \delta_{\mathcal{I}}(\mathbf{P}:a), \delta_{\mathcal{I}}(\mathbf{P}:d)\}$$
$$= \max\{\delta_{\{c\}}(\mathbf{O}:c), \delta_{\{c\}}(\mathbf{P}:a), \delta_{\{c\}}(\mathbf{P}:d)\} = \max\{-3,-2,-1\} = -3.$$

Given these payoffs, *You* prefer the second outcome, giving *Me* a payoff of -2. As in the previous example, we conclude that the value of the game is indeed -2.

In the formal definition of the game, it is also enough to change the payoff function:

Definition 4.5.2: Disjunctive Game as Provability Game

The game $\mathbf{DG}^{\mathrm{GCL}}(D)$ is the same as the game $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}(D)$ in Definition 4.4.2, except for the payoff function:

- Terminal disjunctive histories $\mathfrak{h}$ ending in $\langle \dots D, * \rangle$ are mapped to

$$\delta(\langle \dots D, * \rangle) = \min\nolimits_{\trianglelefteq}\{\delta_{\mathcal{I}}(D) : \mathcal{I} \text{ is an interpretation}\}.$$

The remainder of this section is devoted to proving the following central theorem:

Theorem 4.5.3

I have a disjunctive k-strategy in $\mathbf{DG}^{\mathrm{GCL}}(D)$ iff *I* have disjunctive k-strategies for $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}(D)$, for all interpretations $\mathcal{I}$. *You* have a disjunctive k-strategy in $\mathbf{DG}^{\mathrm{GCL}}(D)$ iff *You* have a disjunctive k-strategy for $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}(D)$, for some $\mathcal{I}$.

As a corollary, we get the adequacy of the disjunctive game with respect to the degree-based semantics introduced in Section 4.3.2. Remember, that the degree of validity of F was defined as the least value of the games $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(\mathbf{P}:F)$ where $\mathcal{I}$ ranges over all interpretations, or equivalently, the minimal possible value of $\deg^{\mathcal{G}}_{\mathcal{I}}(F)$.

Corollary 4.5.4: $\mathbf{DG}^{\mathrm{GCL}}$ characterizes degree of validity

Let F be a choice formula. Then

$$v(F) = v(\mathbf{DG}^{\mathrm{GCL}}(\mathbf{P}:F)).$$

Adequacy

For a disjunctive history $\mathfrak{h}$ we say that a history h appears in $\mathfrak{h}$, and write $h \in \mathfrak{h}$, if for some i, $\mathfrak{h}_i = D$ and $h \in D$.

Lemma 4.5.5

Let $\mathfrak{h}$ be a finite terminal history of the game $\mathbf{DG}^{\mathrm{GCL}}(D)$ such that for every history g in $\mathfrak{h}$ labeled "I", all of its immediate successors appear in $\mathfrak{h}$, too. Let k be the payoff of $\mathfrak{h}$. Then there is an interpretation $\mathcal{I}$ such that *You* have a k-strategy for $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h)$, for each $h \in D$.

Proof. Let k be the payoff of $\mathfrak{h}$. By definition, there is some interpretation $\mathcal{I}$ such that $\delta_{\mathcal{I}}(\mathfrak{h}) \trianglelefteq k$. Let us fix $h_0 \in D$. We define a strategy μ for *You* in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h_0)$ as follows. Let h be a successor of h_0 labeled "Y" and appearing in $\mathfrak{h}$. Since $\mathfrak{h}$ is terminal, there is some $D \bigvee \underline{h}$ in $\mathfrak{h}$. By the rules of the disjunctive game, its immediate successor is $D \bigvee h'$, where h' is an immediate successor of h in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h_0)$. We can therefore set $\mu(h) = h'$. For all other states, we let μ be arbitrary. We sow that μ is a k-strategy for *You*.

To this end, let σ be a strategy for *Me* in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h_0)$. We show by induction on n that $(\sigma;\mu)^n(h_0)$ appears in $\mathfrak{h}$. The base case $n = 0$ is clear. For the inductive step, suppose, we have shown the claim for n, i.e., $h = (\sigma;\mu)^n(h_0)$ appears in $\mathfrak{h}$. If h is labeled "I", then by the assumption, all immediate successors of h appear in $\mathfrak{h}$, in particular, $\sigma(h)$, hence $(\sigma;\mu)^{n+1}(h_0) = \sigma(h)$ appears in $\mathfrak{h}$. If h is labeled "Y", then everything follows from the definition of μ.

Let n be such that $h^* = (\sigma;\mu)^n(h_0)$ is terminal. Since $\delta_{\mathcal{I}}(h^*) \trianglelefteq \delta_{\mathcal{I}}(\mathfrak{h}) \trianglelefteq k$, and σ was arbitrary, we conclude that μ is a k-strategy for *You* in $\mathbf{G}_{\mathcal{I}}^{\mathrm{GCL}}(h_0)$. □

Proof of Theorem 4.5.3. We prove the left-to-right directions (ltr) of both statements. The right-to-left directions (rtl) then follow easily: for example, suppose, for every $\mathcal{I}$, *I* have a k-strategy in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$. Let $l \triangleleft k$ be maximal. Then *You* do not have an l-strategy in $\mathbf{DG}_{\mathcal{I}}^{\mathrm{GCL}}(D)$. By ltr of Statement 2, *You* do not have an l-strategy for $\mathbf{DG}^{\mathrm{GCL}}(D)$. Since *You* cannot enforce the payoff to be below k, *I* have a k-strategy. The rtl of the other statement is similar.

Ltr of Statement 1: Let σ be a disjunctive k-strategy for *Me* in $\mathbf{DG}^{\mathrm{GCL}}(D)$ and fix $\mathcal{I}_0$. Since the games $\mathbf{DG}^{\mathrm{GCL}}(D)$ and $\mathbf{DG}_{\mathcal{I}_0}^{\mathrm{GCL}}(D)$ are identical, except maybe for the payoffs, *I* can use σ to play in the latter game. Let μ be a disjunctive strategy for *You* in $\mathbf{DG}_{\mathcal{I}_0}^{\mathrm{GCL}}(D)$. Then

$$\delta_{\mathcal{I}}(\sigma,\mu) \trianglerighteq \min_{\mathcal{I}} \delta_{\mathcal{I}}(\sigma,\mu) = \delta(\sigma,\mu) \trianglerighteq k.$$

Since μ was arbitrary, σ is a k-strategy.

Ltr of Statement 2: Let μ be a disjunctive k-strategy for *You* in $\mathbf{DG}^{\mathrm{GCL}}(D)$. Let $\mathfrak{h}$ be the terminal run resulting from *You* playing μ and *Me* playing according to the following strategy: if the current disjunctive state is D', *I* underline an arbitrary $h \in D'$. If h is an I-history and has only one immediate successor h', *I* go to that successor in the corresponding copy. If h has two successors h_1 and h_2, *I* first duplicate h, then go to h_1 in the first and to h_2 in the second copy. By the assumption, $\delta(\mathfrak{h}) \trianglelefteq k$. By construction,

$\mathfrak{h}$ contains all the immediate successors of every I-history appearing in $\mathfrak{h}$. We apply Lemma 4.5.5 to obtain an interpretation $\mathcal{I}$ and k-strategies for *You* in $\mathbf{G}^{\mathrm{GCL}}_{\mathcal{I}}(h)$, for each $h \in D$. □

Corollary 4.5.6

The values of the games $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{P} : F)$ and $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$ are given by $\deg^{\mathcal{G}}_{\mathcal{I}}(F) = \min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$ and $-\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$, respectively.

Proof. For each interpretation $\mathcal{I}$, let $v_{\mathcal{I}}$ be the value of $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}(D)$. It follows from the theorem that the value of $\mathbf{DG}^{\mathrm{GCL}}(D)$ is $\min_{\mathcal{I}} v_{\mathcal{I}}$. Thus, by Corollary 4.4.5 and Theorem 4.3.13, the values of $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{P} : F)$ and $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$ are $\min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$ and $\min_{\mathcal{I}} -\deg^{\mathcal{G}}_{\mathcal{I}}(F)) = -\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$, respectively. □

Corollary 4.5.7

Let $\mathcal{I}$ be a preferred model of F and let k be the value of $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$. Then $k = -\deg^{\mathcal{G}}_{\mathcal{I}}(F)$ and a preferred model of F can be extracted from *Your* k-strategy for $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$.

Proof. The first statement immediately follows from Corollary 4.5.6. Let μ be *Your* k-strategy in $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F)$. Since there is an interpretation making F true, k must be negative and thus winning for *You*. By the proof of Theorem 4.5.3, all the information for a preferred model is contained in the outcome of the run of the game, where *I* play according to the strategy sketched in that proof and *You* play according to μ. Let $\mathfrak{h}$ be the resulting terminal history which is necessarily winning for *You*. We, therefore, set $\mathcal{I}^{\mathfrak{h}} = \{a \mid \mathbf{O} : a \in \mathfrak{h}\}$ and obtain a k-strategy for *You* for $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}^{\mathfrak{h}}}(\mathbf{O} : F)$. Let v be the value of that game. We have that $v \trianglelefteq k$, by the existence of *Your* k-strategy and $v \trianglerighteq k$, since by Theorem 4.3.13 and Corollaries 4.4.5 and 4.5.6, $v = -\deg^{\mathcal{G}}_{\mathcal{I}^{\mathfrak{h}}}(F) \trianglerighteq -\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F) = k$. This shows $\deg^{\mathcal{G}}_{\mathcal{I}^{\mathfrak{h}}}(F) = \max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$, i.e., $\mathcal{I}^{\mathfrak{h}}$ is a preferred model of F. □

4.6 From Strategies to Proofs

In this section, we study the proof-theoretic content of the provability game by reinterpreting strategies as proofs in three suitable labeled sequent calculi.

As in Chapter 3, the idea is to encode disjunctive states D by the *sequent*

$$\{F \mid h = \langle \ldots, \mathbf{O} : F \rangle \in D\} \Rightarrow \{F \mid h = \langle \ldots, \mathbf{P} : F \rangle \in D\}.$$

In contrast to Chapter 3, the value of a history of the evaluation game now depends not solely on its final game state, but additionally on the preference relation $\ll$. In light of

Lemma 4.3.12, however, it suffices to keep track of the last state of a history h and the numbers $|\pi_{\ll}(h)|$ and $|\pi_{\gg}(h)|$. Hence, we code a disjunctive state D as a *labeled sequent*

$$\{{}^k_l F \mid h = \langle \dots, \mathbf{O} : F\rangle \in D\} \Rightarrow \{{}^k_l F \mid h = \langle \dots, \mathbf{P} : F\rangle \in D\},$$

where $l = |\pi_{\ll}(h)|$ and $k = |\pi_{\gg}(h)|$.

By Lemma 4.3.12 and Theorem 4.3.13, the degree of a history $h = \langle \dots, \mathbf{O} : F\rangle$ is $-\deg^{\mathcal{G}}_{\mathcal{I}}(F) + l - 1$ if $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-$, and $-\deg^{\mathcal{G}}_{\mathcal{I}}(F) - k + 1$ if $\mathbb{Z}^+$. Similarly, the degree of a history $h = \langle \dots, \mathbf{P} : F\rangle$ is $\deg^{\mathcal{G}}_{\mathcal{I}}(F) + l - 1$ if $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+$, and $\deg^{\mathcal{G}}_{\mathcal{I}}(F) - k + 1$ if $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-$. Thus, we assign degrees to labeled sequents $\Gamma \Rightarrow \Delta$ as follows: for each interpretation $\mathcal{I}$,

$$\text{if } {}^k_l F \in \Delta, \text{ we set } \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F) = \begin{cases} l + \deg^{\mathcal{G}}_{\mathcal{I}}(F) - 1, & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+, \\ -k + \deg^{\mathcal{G}}_{\mathcal{I}}(F) + 1, & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-, \end{cases}$$

$$\text{if } {}^k_l F \in \Gamma, \text{ we set } \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F) = \begin{cases} l - \deg^{\mathcal{G}}_{\mathcal{I}}(F) - 1 & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-, \\ -k - \deg^{\mathcal{G}}_{\mathcal{I}}(F) + 1 & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+. \end{cases}$$

We then set

$$\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) = \min_{\mathcal{I}} \max_{{}^k_l F \in \Gamma \cup \Delta} \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F).$$

In the simplest case, $\deg^{\mathcal{G}}(\Rightarrow {}^1_1 F)$ coincides with $\deg^{\mathcal{G}}(F)$. We now have all ingredients to present our proof systems.

The first proof system, $\mathbf{DS}^{\mathrm{GCL}}$ (disjunctive strategies) in Figure 4.3, is closer to the game-theoretic view. Proofs are (bottom-up) representations of *My* strategies for the disjunctive game. Branching corresponds to *Your* choices in the disjunctive game which must be kept intact in this encoding of *My* strategy. The contraction rules represent applications of the duplication rule in the disjunctive game, whereas the propositional and choice rules represent the rules of the underlying evaluation game. Note that only the choice rules increase the parameters k and l in accordance with Lemma 4.3.12 and the above definitions.

What is unusual is that *all* sequents are allowed as initial sequents. A proof where the elementary part $\Gamma^{el} \Rightarrow \Delta^{el}$ of all initial sequents $\Gamma \Rightarrow \Delta$ has degree $\trianglerighteq k$, therefore, represents a k-strategy for *Me*. Hence, in this case, we speak of a *k-proof*. Note that in accordance with a k-strategy, k-proofs are not per se optimal: they merely witness that the degree of the proved sequent is at least k. In particular, every k-proof is also an l-proof, if $k \trianglerighteq l$.

Table 4.3: Proof systems $\mathbf{DS}^{\mathrm{GCL}}$ and $\mathbf{DS}^{\mathrm{GCL}}_k$.

Initial Sequents for GS

$\Gamma \Rightarrow \Delta$ has degree $\deg^{\mathcal{G}}(\Gamma^{el} \Rightarrow \Delta^{el})$

Axioms for $\mathbf{S}^k$

$\Gamma \Rightarrow \Delta$, if $\deg^G(\Gamma^{el} \Rightarrow \Delta^{el}) \trianglerighteq k$,

Structural Rules

$$\frac{\Gamma, {}^k_l F, {}^k_l F \Rightarrow \Delta}{\Gamma, {}^k_l F \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow {}^k_l F, {}^k_l F, \Delta}{\Gamma \Rightarrow {}^k_l F, \Delta}\ (R_c)$$

Propositional rules

$$\frac{\Gamma, {}^k_l F \Rightarrow \Delta \qquad \Gamma, {}^k_l G \Rightarrow \Delta}{\Gamma, {}^k_l (F \vee G) \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow {}^k_l F, \Delta}{\Gamma \Rightarrow {}^k_l (F \vee G), \Delta}\ (R^1_\vee)$$

$$\frac{\Gamma, {}^k_l F \Rightarrow \Delta}{\Gamma, {}^k_l (F \wedge G) \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma \Rightarrow {}^k_l G, \Delta}{\Gamma \Rightarrow {}^k_l (F \vee G), \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, {}^k_l G \Rightarrow \Delta}{\Gamma, {}^k_l (F \wedge G) \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow {}^k_l F, \Delta \qquad \Gamma \Rightarrow {}^k_l G, \Delta}{\Gamma \Rightarrow {}^k_l (F \wedge G), \Delta}\ (R_\wedge)$$

$$\frac{\Gamma \Rightarrow {}^k_l F, \Delta}{\Gamma, {}^k_l \neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, {}^k_l F \Rightarrow \Delta}{\Gamma \Rightarrow {}^k_l \neg F, \Delta}\ (R_\neg)$$

Choice rules

$$\frac{\Gamma, {}^{k}_{l+\mathrm{opt}(G)} F \Rightarrow \Delta \qquad \Gamma, {}^{k+\mathrm{opt}(F)}_{l} G \Rightarrow \Delta}{\Gamma, {}^k_l (F \overrightarrow{\times} G) \Rightarrow \Delta}\ (L_{\overrightarrow{\times}}) \qquad \frac{\Gamma \Rightarrow {}^{k+\mathrm{opt}(G)}_{l} F, \Delta}{\Gamma \Rightarrow {}^k_l (F \overrightarrow{\times} G), \Delta}\ (R^1_{\overrightarrow{\times}})$$

$$\frac{\Gamma \Rightarrow {}^{k}_{l+\mathrm{opt}(G)} G, \Delta}{\Gamma \Rightarrow {}^k_l (F \overrightarrow{\times} G), \Delta}\ (R^2_{\overrightarrow{\times}})$$

The second proof system is a proof-theoretically more orthodox system. In fact, it is actually a family of proof systems: for each $k \in Z$, the system $\mathbf{DS}^{\mathrm{GCL}}_k$ is defined in Table 4.3. These proof systems share all the rules with $\mathbf{DS}^{\mathrm{GCL}}$, but initial sequents are valid iff their elementary part has degree at least k. Such initial sequents are axioms in the usual sense.

The conceptual difference between the two approaches is as follows: in $\mathbf{DS}^{\mathrm{GCL}}$, the value k can be *computed* from the initial sequents. In the second approach, k is *guessed* (implicitly, by picking the proof system $\mathbf{DS}^{\mathrm{GCL}}_k$, for a concrete k).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{1}_{3}a \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d \qquad {}^{2}_{2}b \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{2}_{1}(a \vec{\times} b) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\vec{\times}}) \qquad {}^{3}_{1}c \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\vec{\times}})}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{1}(a \vec{\times} d)}\,(R^{2}_{\vec{\times}})}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c) \Rightarrow {}^{1}_{1}(a \vec{\times} d), {}^{1}_{1}(a \vec{\times} d)}\,(R^{1}_{\vec{\times}})}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c) \Rightarrow {}^{1}_{1}(a \vec{\times} d)}\,(R_{C})}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c), {}^{1}_{1}(\neg(a \vec{\times} d)) \Rightarrow}\,(L_{\neg})}{{}^{1}_{1}((a \vec{\times} b) \vec{\times} c), {}^{1}_{1}(((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)) \Rightarrow}\,(L_{\wedge})}{{}^{1}_{1}(((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)), {}^{1}_{1}(((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)) \Rightarrow}\,(L_{\wedge})}{{}^{1}_{1}(((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d)) \Rightarrow}\,(L_{C})}
$$

Figure 4.6: A -2-proof in $\mathbf{DS}^{\mathrm{GCL}}$.

Example 4.6.1. Figure 4.6 shows a derivation of $((a \vec{\times} b) \vec{\times} c) \wedge \neg(a \vec{\times} d) \Rightarrow$ in $\mathbf{DS}^{\mathrm{GCL}}$. Essentially, it is *My* strategy from Example 4.4.1 bottom-up. The degrees of the initial sequents are:

$$
\begin{aligned}
\deg^{\mathcal{G}}({}^{3}_{1}a \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) &= \deg^{\mathcal{G}}_{\{a\}}({}^{3}_{1}a \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) = 2, \\
\deg^{\mathcal{G}}({}^{2}_{2}b \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) &= \deg^{\mathcal{G}}_{\{b\}}({}^{2}_{2}b \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) = -2, \\
\deg^{\mathcal{G}}({}^{1}_{3}c \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) &= \deg^{\mathcal{G}}_{\{c\}}({}^{1}_{3}c \Rightarrow {}^{1}_{2}a, {}^{2}_{1}d) = -3.
\end{aligned}
$$

Therefore, the derivation is a -2-proof and thus a proof in $\mathbf{DS}^{\mathrm{GCL}}_{-2}$.

It follows directly from the translation of *My* strategies into proofs:

Theorem 4.6.2

The following are equivalent:

1. *I* have a k-strategy for $\mathbf{DG}^{\mathrm{GCL}}(\mathbf{O} : F_1 \bigvee \ldots \bigvee \mathbf{O} : F_n \bigvee \mathbf{P} : G_1 \bigvee \ldots \bigvee \mathbf{P} : G_m)$.
2. $\deg^{\mathcal{G}}({}^{1}_{1}F_1, \ldots, {}^{1}_{1}F_n \Rightarrow {}^{1}_{1}G_1, \ldots, {}^{1}_{1}G_m) \trianglerighteq k$.
3. There is a k-proof of ${}^{1}_{1}F_1, \ldots, {}^{1}_{1}F_n \Rightarrow {}^{1}_{1}G_1, \ldots, {}^{1}_{1}G_m$ in $\mathbf{DS}^{\mathrm{GCL}}$.
4. There is a proof of ${}^{1}_{1}F_1, \ldots, {}^{1}_{1}F_n \Rightarrow {}^{1}_{1}G_1, \ldots, {}^{1}_{1}G_m$ in $\mathbf{DS}^{\mathrm{GCL}}_{k}$.

Corollary 4.6.3

Let $k \in \mathbb{Z}^-$. Then there is a k-proof of ${}^1_1F \Rightarrow$ in $\mathbf{DS}^{\mathrm{GCL}}$ iff there is a proof of ${}^1_1F \Rightarrow$ in $\mathbf{DS}_k^{\mathrm{GCL}}$ iff the degree of F in a preferred model is at most $-k$.

My strategy in Example 4.4.1 is not only a -2-strategy but also a minmax-strategy for *Me*. This implies that *I* cannot do better than -2, i.e. the value of the game is -2. How does this translate into the proof-theoretic interpretation of Example 4.6.1? There, the minmax-strategy takes the form of invertibility of rule applications: rule applications S'/S and $(S_1, S_2)/S$ are called *invertible* iff $\deg^{\mathcal{G}}(S') = \deg^{\mathcal{G}}(S)$ and $\min\{\deg^{\mathcal{G}}(S_1), \deg^{\mathcal{G}}(S_2)\} = \deg^{\mathcal{G}}(S)$. In Example 4.6.1 only invertible rule applications are used.

In Table 4.4 we give a calculus $\mathbf{ODS}^{\mathrm{GCL}}$ (optimal disjunctive strategies) which is equivalent to $\mathbf{DS}^{\mathrm{GCL}}$ but has only invertible rules, i.e. all rule applications are invertible. The contraction rules are admissible in this system. The motivation behind this calculus is the same as in *My* optimal strategy from the adequacy proof of $\mathbf{DG}^{\mathrm{GCL}}$: in every *I*-state, *I* first duplicate and then exhaustively take all the available options. Every proof produced in this system corresponds to an optimal strategy and has, therefore, an optimal degree. The below results follow directly from the invertibility of the rules:

Table 4.4: The proof system $\mathbf{ODS}^{\mathrm{GCL}}$ for GCL with invertible rules.

Initial Sequents

$\Gamma \Rightarrow \Delta$ has degree $\deg^{\mathcal{G}}(\Gamma^{el} \Rightarrow \Delta^{el})$

Propositional rules

$$\frac{\Gamma, {}^k_lF \Rightarrow \Delta \qquad \Gamma, {}^k_lG \Rightarrow \Delta}{\Gamma, {}^k_l(F \vee G) \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow {}^k_lF, {}^k_lG, \Delta}{\Gamma \Rightarrow {}^k_l(F \vee G), \Delta}\ (R_\vee)$$

$$\frac{\Gamma, {}^k_lF, {}^k_lG, \Rightarrow \Delta}{\Gamma, {}^k_l(F \wedge G) \Rightarrow \Delta}\ (L_\wedge) \qquad \frac{\Gamma \Rightarrow {}^k_lF, \Delta \qquad \Gamma \Rightarrow {}^k_lG, \Delta}{\Gamma \Rightarrow {}^k_l(F \wedge G), \Delta}\ (R_\wedge)$$

$$\frac{\Gamma \Rightarrow {}^k_lF, \Delta}{\Gamma, {}^k_l\neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, {}^k_lF \Rightarrow \Delta}{\Gamma \Rightarrow {}^k_l\neg F, \Delta}\ (R_\neg)$$

Choice rules

$$\frac{\Gamma, {}^{k}_{l+\mathrm{opt}(G)}F \Rightarrow \Delta \qquad \Gamma, {}^{k+\mathrm{opt}(F)}_{l}G \Rightarrow \Delta}{\Gamma, {}^k_l(F \overrightarrow{\times} G) \Rightarrow \Delta}\ (L_{\overrightarrow{\times}})$$

$$\frac{\Gamma \Rightarrow {}^{k+\mathrm{opt}(G)}_{l}F, {}^{k}_{l+\mathrm{opt}(F)}G, \Delta}{\Gamma \Rightarrow {}^k_l(F \overrightarrow{\times} G), \Delta}\ (R_{\overrightarrow{\times}})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{1}_{3}a \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d \qquad {}^{2}_{2}b \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{2}_{1}(a \overrightarrow{\times} b) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\overrightarrow{\times}}) \qquad {}^{3}_{1}c \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\overrightarrow{\times}})}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d)}\,(R_{\overrightarrow{\times}})}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}\neg(a \overrightarrow{\times} d) \Rightarrow}\,(L_{\neg})}{{}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}\,(L_{\wedge})$$

Figure 4.7: A proof in $\mathbf{ODS}^{\mathrm{GCL}}$.

Proposition 4.6.4

Every $\mathbf{ODS}^{\mathrm{GCL}}$-proof of a sequent S has degree $\deg^{\mathcal{G}}(S)$.

Corollary 4.6.5

Let $k = \deg^{\mathcal{G}}({}^{1}_{1}F \Rightarrow) \in \mathbb{Z}^{-}$. Then the degree of F in a preferred model is equal to $-k$. Furthermore, a preferred model of F can be extracted from every $\mathbf{ODS}^{\mathrm{GCL}}$-proof of ${}^{1}_{1}F \Rightarrow$.

Example 4.6.6. Figure 4.7 shows a $\mathbf{ODS}^{\mathrm{GCL}}$-proof of ${}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d) \Rightarrow$. The proof is essentially a compact representation of the proof in Figure 4.6, and has therefore degree -2. We conclude that in a preferred model, $((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)$ has degree 2. Furthermore, we can extract the preferred model $\{b\}$ from the position where the $\deg^{\mathcal{G}}$-function is minimal on the initial sequents, as computed in Example 4.6.1.

We note that the following degree version of the cut rule does not hold. The existence of k-strategies for $D \bigvee \mathbf{P} : F$ and $D \bigvee \mathbf{O} : F$ does not imply that a k-strategy for D exists. For example, the values of $\mathbf{O} : \top \bigvee \mathbf{O} : \bot \overrightarrow{\times} \top$ and $\mathbf{O} : \top \bigvee \mathbf{P} : \bot \overrightarrow{\times} \top$ are -2 and 2, respectively. But the value of the "conclusion" of the cut, $\mathbf{O} : \top$, has value -1.

What is more, there is no function computing the value of the conclusion of cut from the values of the premises. To see this, note that the values of $\mathbf{O} : \bot \overrightarrow{\times} \top \bigvee \mathbf{O} : \bot \overrightarrow{\times} \top$ and $\mathbf{O} : \bot \overrightarrow{\times} \top \bigvee \mathbf{P} : \bot \overrightarrow{\times} \top$ are -2 and 2 respectively, as in the above example. However, in contrast to the above example, the conclusion of this cut, $\mathbf{O} : \bot \overrightarrow{\times} \top$, has value -2.

Lastly, we demonstrate that $\mathbf{DS}^{\mathrm{GCL}}$ and $\mathbf{DS}^{\mathrm{GCL}}_{k}$ are useful systems, i.e. that computing the degree of elementary sequents is easier than the degree of general sequents.

Proposition 4.6.7

Deciding whether $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) \trianglerighteq k$ is coNP-hard in general. If $\Gamma \Rightarrow \Delta$ is elementary, then $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta)$ can be computed in polynomial time.

Proof. coNP-hardness of deciding $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) \trianglerighteq k$ follows by coNP-hardness of the validity problem in classical logic: if F is a classical formula, then it holds that $\deg^{\mathcal{G}}(\Rightarrow {}^{1}_{1}F) \in \mathbb{Z}^{+}$ if and only if F is valid (true under all interpretations).

We now show that $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta)$ can be computed in polynomial time if $\Gamma \Rightarrow \Delta$ is elementary. We apply the following helpful fact: setting a variable x to true or false does not influence the degree ascribed to formulas not containing x. We start with the empty interpretation $\mathcal{I} = \emptyset$. Now, go through every variable x occurring in $\Gamma \Rightarrow \Delta$. Consider $\Gamma_x \Rightarrow \Delta_x$ where ${}^{l}_{k}x \in \Gamma_x$ iff ${}^{l}_{k}x \in \Gamma$ and ${}^{l}_{k}x \in \Delta_x$ iff ${}^{l}_{k}x \in \Delta$. If we have $\deg^{\mathcal{G}}_{\{x\}}(\Gamma_x \Rightarrow \Delta_x) \triangleleft \deg^{\mathcal{G}}_{\emptyset}(\Gamma_x \Rightarrow \Delta_x)$, then let $\mathcal{I} = \mathcal{I} \cup \{x\}$, otherwise leave $\mathcal{I}$ unchanged. In other words, since $\Gamma \Rightarrow \Delta$ is elementary, we can choose the "better" option for any variable x without side effects. Thus, this procedure gives us the minimal $\mathcal{I}$ for $\Gamma \Rightarrow \Delta$. □

4.7 Conclusion and Future Work

QCL is a multi-valued logic designed for representing both hard constraints (truth) and soft constraints (preferences). It extends classical logic with the ordered disjunction $\vec{\times}$. Intuitively, the proposition $F \vec{\times} G$ stands for "F or G should be satisfied, but preferably F". In this chapter, we gave a new evaluation game, $\mathbf{G}^{\mathrm{GCL}}$, for the language of QCL. In this refinement of Hintikka's game for classical logic, the proponent of $F \vec{\times} G$ chooses between continuing the game with F or with G, but that player prefers F. All other rules remain the same as in Hintikka's game. In particular, negation is interpreted as a role switch: at $\neg F$, the game continues with F and reversed roles. We showed that this natural reinterpretation of the choice connective gives a new semantics that solves some conceptual issues regarding the "classical" behavior of negation present in QCL. For example, the formula $\neg\neg F$ is equivalent to F in our evaluation game but not necessarily in QCL.

Notably, our preference modeling is accomplished by refining the two values of Hintikka's game (win/lose) with a richer structure of winning and losing payoffs. To our knowledge, $\mathbf{G}^{\mathrm{GCL}}$ is thus the first evaluation game for multi-valued logic where the payoff values directly correspond to the truth degrees. So far, these games internalized multiple truth values by comparing (combined) truth values [24, 23, 23], or introducing multiple players [4]. It would be interesting to see which logics or formalisms, especially from AI applications, can be modeled using multi-valued evaluation games. Furthermore, we conjecture that our evaluation game can be easily extended to the first-order level.

The induced notion of validity in the resulting logic GCL is degree-based, too: the degree of validity of F, $v(F)$ is the least game value of all evaluation games starting at F over every interpretation. We show that the disjunctive game adequately models this degree. Strategies in the disjunctive game can be reinterpreted as proofs in three labeled sequent calculi: $\mathbf{DS}^{\mathrm{GCL}}$, which is closest to the game-theoretic view, and proofs directly correspond to strategies and are thus graded, $\mathbf{DS}_k^{\mathrm{GCL}}$ which is a more orthodox proof system, but a "degree-profile" must be guessed similar to [6], and $\mathbf{ODS}^{\mathrm{GCL}}$, where proofs correspond to optimal strategies and preferred models (interpretations with maximal degrees) can be directly extracted from a corresponding proof.

As for future work, it would be interesting to design a natural game model of the central notion of *preferred model entailment*: $T \vdash_{pref} F$ if F is true in every preferred model of T. Depending on how a preferred model of a *set* T is interpreted (in the literature, often a lexicographic order is used [16]), this might require us to compute an overall payoff from subgames where payoffs have different formats (for instance the usual ordering on Z vs lexicographic). This would provide a game-theoretic approach to preferred model entailment, complementing the proof-theoretic analysis in [6].

A few future research directions are inspired by common topics in game theory. For instance, it would be interesting to investigate settings where the player's preferences are not strictly opposed. This also opens up the direction to multiple-player frameworks. In this light, different solution methods, like elimination of dominated strategies and Nash equilibria might be relevant.

Finally, one should be open to a new approach that deals jointly with truth and preferences. After all, the intuitive reading of $\vec{\times}$ does not presuppose that all models are necessarily totally ordered by preferences. Nevertheless, all approaches discussed in this chapter – QCL, PQCL, and GCL – force values in a linear domain. However, we point out that allowing incomparable models implies the necessity of a non-linear degree domain and, thus, possibly, a departure from game theory.

CHAPTER 5

A General Framework

5.1 Introduction

In the previous chapters, we applied the lifting technique to two semantic games, $\mathbf{G}^{\mathrm{Hyb}}$ and $\mathbf{G}^{\mathrm{GCL}}$. Although the games are fairly distinct and arise from different semantic motivations, we can observe several similarities regarding the corresponding lifting steps. For example, the definition of the disjunctive game was almost identical (except for the payoff, which is non-binary in the case of GCL). Still, in both cases, the value associated with a disjunctive state is the maximum of the values of its contained histories. The provability games, too, are defined similarly. The adequacy proofs follow the same pattern of considering an optimal strategy for *Me*. Proofs in the resulting analytic calculi are interpreted as notational variants of *My* strategies.

This begs the question if there is a general mechanism for lifting semantic games to analytic calculi. In this chapter, we aim to answer this question. To this end, we give a suitable definition of a *semantic game* – a term that has been used informally, so far. This definition is general enough to capture the different features of the semantic games presented in this thesis, like non-binary payoff values and infinite branching. At the same time, it is specific enough to capture the similarities crucial for applying a general form of the lifting technique and executing the adequacy results, like the uniformity of game trees. We obtain adequacy proofs following the same pattern as in the previous chapters. Along the way, we identify the conditions for a general semantic game for the lifting technique to work.

Another abstraction from the previous chapters is the new role of turn distribution. Remember, in the disjunctive game, game states are of the form

$$D = h_1 \bigvee \ldots \bigvee h_n,$$

where the h_i are histories of the underlying semantic game. If the current disjunctive state is D, *I* choose an i, and the player who moves in the semantic game at h_i, moves to some $h_i \smile g$. This means that the scheduling role falls to *Me*, creating some asymmetry in the disjunctive game. In the present chapter, we give the role of scheduling to a regulation function ρ, a concept first employed in [**?**]. This regulation can be seen as a non-strategic actor, who, at each disjunctive state D, schedules an $h \in D$, where the corresponding player has to move. Interestingly, adequacy fails if ρ is ill-behaved.

We give special attention to the important class of finite games. For these games, optimal strategies for *Me* yield a calculus with finitely branching invertible rules. To demonstrate that our framework and general lifting technique work not only for the two semantic games presented in this thesis, we conduct a case study and investigate the *truth-degree comparison game* for Gödel logic. We apply the general framework to this semantic game and obtain a sequent-of-relations-style calculus.

The chapter is structured as follows: We define semantic games in Section 5.2. In Section 5.3, we introduce the general disjunctive game, and in Section 5.4 the general provability game and prove the adequacy theorems. The last part of the lifting is done in Section 5.5, where we obtain the resulting general calculus. We focus on finite games in Section 5.6 and draw some conclusions and sketch some future work in Section 5.7.

5.2 The Semantic Game

In this section, we give a general definition of a semantic game and demonstrate that this definition captures the games in the previous chapters. We introduce the notion of a complete semantic game which will be necessary for the adequacy proofs.

Definition 5.2.1: Semantic Game

A *semantic game* $\mathbf{G}$ is a tuple $(\mathrm{Stat}, \mathrm{Hist}, \ell, \mathcal{P})$ such that

- Stat is a set whose elements are called *game states*,
- Hist consists of (finite or infinite) sequences of game states. Hist is partially ordered by $\sqsubseteq$. We require that Hist is downwards closed with respect to $\sqsubseteq$ and contains $\langle g \rangle$, for every $g \in \mathrm{Stat}$. We write Hist_g for the set of histories extending $\langle g \rangle$. Furthermore, Hist is closed under ascending $\sqsubseteq$-chains. Maximal elements in Hist are called *terminal*. The set of terminal histories is denoted Ter.
- $\ell : (\mathrm{Hist} \setminus \mathrm{Ter}) \to \{I, Y\}$ is a *labeling function*.
- $\wp : \mathrm{Ter} \to Z$ for every $\wp \in \mathcal{P}$. Here, $(Z, \trianglelefteq)$ is a linear order with a least element, -1. The function $\wp$ is called a *payoff function*.

If $\wp \in \mathcal{P}$ and $g \in \mathrm{Stat}$, we write $\mathbf{G}_\wp(g)$ for the game $(\mathrm{Hist}_g, \langle g \rangle, \ell, \wp)$.

Note that $\mathbf{G}_\wp(g)$ forms a game in the sense of Definition 2.1.1. A semantic game is of *finite height* if all histories are finite sequences. It is *finite* if Hist_g is finite for every game state g.

Example 5.2.2. The semantic game $\mathbf{G}^{\mathrm{Hyb}}$ of Chapter 3 fits the above framework. Following Definition 3.3.2 and the discussion in Section 3.3.3, let $\mathcal{F}^{\mathrm{Hyb}}$ denote the set of formulas of hybrid logic. Then the set of game states $\mathrm{Stat}^{\mathrm{Hyb}}$ contains all elements of the form $\mathbf{Q} : F$ and $\mathbf{Q}, i : F$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, i is a nominal, and $F \in \mathcal{F}^{\mathrm{Hyb}}$. The set of histories $\mathrm{Hist}^{\mathrm{Hyb}}$ is $\bigcup_{g \in \mathrm{Stat}^{\mathrm{Hyb}}} \mathrm{Hist}_g^{\mathrm{Hyb}}$, where $\mathrm{Hist}_g^{\mathrm{Hyb}}$ was defined in Definition 3.3.2. $\mathcal{P}^{\mathrm{Hyb}}$ is the set of the $\wp_{\mathcal{M}}$s, where $\mathcal{M}$ ranges over all models. The payoff function $\wp_{\mathcal{M}}$ as well as the labeling function ℓ were defined in 3.3.2. Note that it is crucial that we consider the reformulated version of the game from Section 3.3.3: in our definition of a semantic game, we require the games $\mathbf{G}^{\mathrm{Hyb}}_{\wp_{\mathcal{M}}}(g)$ and $\mathbf{G}^{\mathrm{Hyb}}_{\wp_{\mathcal{N}}}(g)$ to have the same set of histories, even if $\mathcal{M} \neq \mathcal{N}$. However, this is not the case in the presence of the original rules for the modal operators. An example of this dependence on the underlying model can be seen in Figure 3.1.

Example 5.2.3. As another example, take the semantic game $\mathbf{G}^{\mathrm{GCL}}$ from Chapter 4. Let $\mathcal{F}^{\mathrm{GCL}}$ be the set of choice formulas, then $\mathrm{Stat}^{\mathrm{GCL}}$ contains all elements of the form $\mathbf{Q} : F$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ and $F \in \mathcal{F}^{\mathrm{GCL}}$. The labeling is as in Definition 4.3.2, and the set of histories is $\mathrm{Hist}^{\mathrm{GCL}}$ is $\bigcup_{g \in \mathrm{Stat}^{\mathrm{GCL}}} \mathrm{Hist}_g^{\mathrm{GCL}}$. The set of payoff functions, $\mathcal{P}^{\mathrm{GCL}}$, consists of all $\delta_{\mathcal{I}}$, where $\mathcal{I}$ ranges over all interpretations. Each $\delta_{\mathcal{I}}$ takes values in the domain $-1 \lhd -2 \lhd \cdots \lhd 2 \lhd 1$.

As usual, if $h \in \mathrm{Hist}_g$, we write $\mathbf{G}_\wp(h)$ instead of $\mathbf{G}_\wp(g)@h$; see Definition 2.1.13. $\mathbf{G}$ is *determined* if the game $\mathbf{G}_\wp(h)$ is determined for every $\wp \in \mathcal{P}$ and history h, i.e., it has a value (Definition 2.1.10). A set D of histories is *compact* if it has a finite base, i.e., there are histories $h_1, \ldots, h_n$ such that for every $h \in D$, $h \sqsupseteq h_i$ for some i.

We will require our semantic games to have the following useful property:

Definition 5.2.4: Complete Semantic Game

The semantic game $\mathbf{G}$ is *complete* if, for every compact D consisting of terminal histories, every $P \subseteq \bigcup_{\wp \in \mathcal{P}} \{\wp(h) : h \in D\}$ has a minimum and a maximum.

For example, complete semantic games are always determined:

Proposition 5.2.5: Complete semantic games are determined

Every complete semantic game is determined.

Proof. Fix a payoff function $\wp$ and a history h. Let Σ_I^h denote the set of *My* strategies restricted to histories extending h. Similarly, we define Σ_Y^h. Fix $\sigma_I \in \Sigma_I^h$. Then $\{O(\sigma_I, \sigma_Y) : \sigma_Y \in \Sigma_Y^h\}$ has base $\{h\}$ and is thus compact. By completeness, the set

$$S(\sigma_Y^I) = \{\wp(\sigma_I, \sigma_Y) : \sigma_Y \in \Sigma_Y^h\}$$

has a minimum, let us denote it by $v(\sigma_I)$. Similarly, the set

$$\{v(\sigma_I) : \sigma_Y \in \Sigma_I^h\}$$

has a maximum, v. Putting things together, we proved that

$$\begin{aligned} v &= \max_{\sigma_I \in \Sigma_I^h} v(\sigma_I) \\ &= \max_{\sigma_I \in \Sigma_I^h} \min_{\sigma_Y \in \Sigma_Y^h} \wp(\sigma_I, \sigma_Y) \end{aligned}$$

Hence, the sup/inf in the definition of the maximin value (2.1.10) are max/min. We are thus in the situation of the proof of Theorem 2.1.11, where the fact that the sups and infs were witnessed was enough to show that v is the value of the game. □

Example 5.2.6. The game $\mathbf{G}^{\mathrm{Hyb}}$ is finite-valued and thus complete.

Example 5.2.7. In finite games, like GCL, there are only finitely many histories extending a given history h. Hence, the existence of a finite base for a compact set of histories D actually implies that D is finite. Remember that in GCL the payoff function $\delta_{\mathcal{I}}$ for a terminal history h ending in the game state $\mathbf{Q} : a$ takes the value $|\pi_{\ll}(h)|$ if $\mathcal{I} \models \mathbf{Q} : a$, and $-|\pi_{\gg}(h)|$, otherwise. Hence, the set $\bigcup_{\mathcal{I}} \delta_{\mathcal{I}}(h)$ has a cardinality of at most 2. This shows that $|\bigcup_{\mathcal{I}} \{\delta_{\mathcal{I}}(h) : h \in D\}| \leq 2|D|$ is finite, for every compact (and thus finite) D. Every subset of this finite set must have a minimum and a maximum, therefore, $\mathbf{G}^{\mathrm{GCL}}$ is complete.

5.3 The Disjunctive Game as Semantic Game

As in Section 3.4 and 4.4 we now lift the semantic game to the disjunctive game, which in turn can be seen as a semantic game. The difference to the previous chapter *I* am now relieved from my duty as the scheduler. Instead, we wish to investigate a more general case where the scheduling is done by a predefined *regulation function*.

For the rest of the chapter, let us fix a semantic game $\mathbf{G} = (\mathrm{Stat}, \mathrm{Hist}, \ell, \mathcal{P})$ of finite height. As in the previous chapters, the main ingredient in lifting the semantic game to the disjunctive game is *My* option to create backup copies: in a history h, instead of moving according to the rules of the semantic game, *I* can decide to create a "backup-copy" of h and continue playing at the *disjunctive game state* (or *disjunctive state*) $h \bigvee h$. Game states of the disjunctive game are thus multisets of histories of the semantic game $D = \{h_1, \ldots, h_n\}$. Due to our intuition behind a disjunctive state as a disjunction, or maximum, we prefer to write $h_1 \bigvee \ldots \bigvee h_n$ for D. As before, we keep the multiset theoretic notation $h \in D$ if $h = h_i$ for some i and write $D \bigvee h$ for the multiset sum $D + \{h\}$. D is called *elementary* if it consists of terminal histories.

Following the intuition of a disjunctive state as the maximum of its histories, the payoff at an elementary disjunctive state is $\delta_{\wp}(D) = \max_{h \in D} \wp(h)$.

The new ingredient is the *regulation function* ρ. Intuitively, ρ picks for every current disjunctive state D a non-terminal history $h \in D$ which is to be played on next. If the current disjunctive state is $D = H \bigvee h$, we like to write $H \bigvee \underline{h}$, if ρ selects h. We denote the disjunctive game over the payoff function $\wp$ and under the regulation ρ by $\mathbf{DG}^{\rho}_{\wp}(D)$. It must be noted that the behavior of ρ is known to both players in advance.

In the style of the previous chapters, let us give a semi-formal description of the rules of the disjunctive game $\mathbf{DG}^{\rho}_{\wp}(D)$. As mentioned above, we highlight the history selected by ρ by underlining.

(Dupl) If the current disjunctive state is $D = D' \bigvee \underline{h}$, and h is labeled "I", then I can *duplicate* h, and the game continues with $D' \bigvee h \bigvee h$.

(Move) If the current disjunctive state is $D = D' \bigvee \underline{h}$ then the player who is to move in the semantic game $\mathbf{G}$ at the history h can make a legal move to the game state g and the game continues with $D \bigvee h \smile g$.

(Pay) If D is elementary, then the payoff is the maximum of all the payoffs of the histories in D, i.e., $\delta_{\wp}(D) = \max_{h \in D} \wp(D)$. Infinite disjunctive histories are assigned a payoff of -1.

Additionally, we require that in $D = D' \bigvee h$ if h is labeled "I", then I have to move according to **(Dupl)** or **(Move)**. If h is labeled "Y", then *You* have to move according to **(Move)**.

Note that this description is shorter than the corresponding descriptions for the games for Hyb and GCL. The reason is that here we do not require the rule **(Sched)**, since scheduling is now done by ρ. Also, the rule **(End)** is redundant. Instead, we require that all histories in the current disjunctive state are played until the end.

Example 5.3.1. In the games $\mathbf{DG}^{\mathrm{Hyb}}_{\mathcal{M}}$ and $\mathbf{DG}^{\mathrm{GCL}}_{\mathcal{I}}$, *I* am the scheduler. However, these games fit into our framework by interpreting them as $\mathbf{DG}^{\mathrm{Hyb},\rho}_{\wp\mathcal{M}}$, and $\mathbf{DG}^{\mathrm{GCL},\rho}_{\wp\mathcal{I}}$, respectively. Here, ρ can be taken as any regulation acting according to *My* interests. We will see later that – under certain conditions – it can be assumed that the regulation always behaves this way. Regulation independence is the topic of Section 5.6.

We now come to the formal description of $\mathbf{DG}^{\rho}_{\wp}$. The set of game states of the disjunctive game is

$$\mathrm{DStat} = \left\{ g_1 \bigvee \ldots \bigvee g_n \mid g_1, \ldots, g_n \in \mathrm{Hist} \right\}.$$

For a disjunctive state D, let D^{ter} contain the terminal histories of D, i.e., $D = \{g \in D : g \in \mathrm{Hist} \setminus \mathrm{Ter}\}$. D is called *terminal* if it consists of terminal histories, $D = D^{ter}$. A *regulation* ρ is a function mapping finite sequences of disjunctive states ending in a non-terminal disjunctive state to non-terminal histories of the semantic game $\mathbf{G}$ such that $\rho(\langle \ldots, D \rangle) \in D \setminus D^{ter}$.

Definition 5.3.2: Disjunctive game as Semantic Game

Let ρ be a regulation and $D \in \mathrm{DStat}$. The disjunctive game $\mathbf{DG}^{\rho}_{\wp}(D)$ is defined as follows:

- The initial history is $\langle D \rangle$.
- The set of *disjunctive histories*, DHist^{ρ} is the smallest set containing $\langle D \rangle$ and closed under the following conditions: If $\mathfrak{h} = \mathfrak{h}' \smile H \bigvee h$ is a disjunctive history with $\rho(\mathfrak{h}) = h$ the sequence $\mathfrak{h} \smile H \bigvee (h \smile g)$ is a disjunctive history if $h \smile g \in \mathrm{Hist}$. If h is labeled "I", then also $\mathfrak{h} \smile H \bigvee h \bigvee h$ is a disjunctive history.
- Non-terminal disjunctive histories $\mathfrak{h}$ are labeled $\ell(\rho(\mathfrak{h}))$, the same as $\rho(\mathfrak{h})$ in the semantic game.
- As for the semantic game, the payoff function $\delta_{\wp}$ maps terminal disjunctive histories to the domain Z. Terminal disjunctive histories ending in $\langle \dots, H \rangle$ are mapped to
$$\delta_{\wp}(\langle \dots, H \rangle) = \delta_{\wp}(H) = \max_{h \in H} \delta_{\wp}(h).$$
Infinite terminal histories are mapped to -1.

Note that for a fixed ρ, $\mathbf{DG}^{\rho} = (\mathrm{DStat}, \mathrm{DHist}^{\rho}, \ell \circ \rho, \{\delta_{\wp} : \wp \in \mathcal{P}\})$ is a semantic game in terms of Definition 5.2.1, albeit not one of finite height. Strategies and related notions were already defined in Chapter 2. To avoid confusion, we will talk of *disjunctive strategies* when referring to the disjunctive game, and strategies when referring to the underlying semantic game. We now show that the disjunctive game inherits completeness from the semantic game.

Proposition 5.3.3: disjunctive game is complete

If $\mathbf{G}$ is complete, then the semantic game $\mathbf{DG}$ is complete, too.

Proof. Let Δ be a compact set of terminal histories, $Q = \bigcup_{\wp \in \mathcal{P}} \{\delta_{\wp}(\mathfrak{h}) : \mathfrak{h} \in \Delta\}$, and $P \subseteq Q$. Let us show that P has a minimum. If some $\mathfrak{h} \in \Delta$ is infinite, then $\min P = -1$. Otherwise, let all histories in Δ be finite. For every disjunctive history $\mathfrak{h} \in \Delta$ ending in the disjunctive state D, let $h_D \in D$ be such that $\wp(h_D) = \delta_{\wp}(D)$. Let $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ be a base of Δ, then $\bigcup_{i=1,\dots,n} \mathrm{last}(\mathfrak{h}_i)$ is a base of $\{h_{\mathrm{last}(\mathfrak{h})} : \mathfrak{h} \in \Delta\}$. Thus, every subset of

$$\bigcup_{\wp \in \mathcal{P}} \{\wp(h_{\mathrm{last}(\mathfrak{h})}) : \mathfrak{h} \in \Delta\} = Q$$

has a minimum, in particular, P. □

This implies that the disjunctive game is determined. The below results are the generalizations of Propositions 3.4.4 and 4.4.4 showing that *My* disjunctive strategy can be seen as a disjunction (or maximum) of *My* strategies for the semantic game. The proof closely follows that of the corresponding propositions in the earlier chapters. In particular, this proposition provides a characterization of the value of the disjunctive game in terms of the value of the underlying semantic game, see Corollary 5.3.6 below.

Proposition 5.3.4: *My* disjunctive strategy = disjunction of *My* strategies

I have a disjunctive k-strategy in $\mathbf{DG}^{\rho}_{\wp}(D)$ iff *I* have a k-strategy in $\mathbf{G}_{\wp}(h)$ for some $h \in D$.

Proof. "$\Rightarrow$": Let σ be a disjunctive k-strategy for *Me* in $\mathbf{DG}^{\rho}_{\wp}(D)$. The case for $k = -1$ is trivial. If $k \rhd -1$, then, by definition, all terminal histories in σ are finite. In this case, we can use backward induction on the tree structure of σ to show that for every $H \in \sigma$ there is $h \in H$ such that *I* have a winning strategy σ_H in $\mathbf{G}_{\wp}(h)$. The proposition then follows for the case where $H = D$.

By assumption, all leaves for all leaves H there is some $h \in H$ with payoff $\trianglerighteq k$. If H is non-elementary labeled "Y", then H is of the form $H' \bigvee \underline{h}$. The successors of H are $H \bigvee h'$, where h' are the successors of h in the evaluation game. By the inductive hypothesis, there are k-strategies $\sigma_{H' \bigvee h'}$ for all h'. If for some h', $\sigma_{H' \bigvee h'}$ is a k-strategy for $\mathbf{G}_{\wp}(l)$, where $l \in H'$, then we can simply set $\sigma_H = \sigma_{H' \bigvee h'}$. Otherwise, every $\sigma_{H' \bigvee h'}$ is a k-strategy for $\mathbf{G}_{\wp}(h')$. Lemma 2.1.14 gives us a k-strategy σ_H for $\mathbf{G}_{\wp}(h)$.

If H is labeled "I" and is of the form $H' \bigvee h$, and according to σ, *I* move to $H' \bigvee h \bigvee h$, then we simply set $\sigma_H = \sigma_{H \bigvee h \bigvee h}$ and use the inductive hypothesis. Finally if $H = H' \bigvee \underline{h}$ and *I* move to $H' \bigvee h'$, the inductive hypothesis gives us a k-strategy for $\sigma_{H' \bigvee h'}$ for some $l \in H' \bigvee h'$. If $l \in H'$, we set $\sigma_H = \sigma_{H' \bigvee h'}$ and are done. If $l = h'$, we use Lemma 2.1.14 to obtain a k-strategy for $\mathbf{G}_{\wp}(h)$.

"$\Leftarrow$": Suppose, *I* have a k-strategy σ for $\mathbf{G}_{\wp}(h)$ for some $h \in D$. The idea is as follows: by playing σ on h and making arbitrary moves on all other histories of D, the game will terminate with a payoff at least as good as σ's. By induction on the tree structure of σ, we define a disjunctive strategy μ for *Me* with the following property: (*) every disjunctive state appearing in μ is of the form $H \bigvee l$, where l is a history in σ[1]. The base case follows from the assumption.

If the current disjunctive state is $H \bigvee \underline{l}$ where l appears in σ and is labeled "Y", then *You* proceed to some $H \bigvee l \smile g$. Since l is labeled "Y", σ contains all immediate successors of l; hence $l \smile g$ must be a history in σ. If l is labeled "I", then *I* move to $H \bigvee \sigma(l)$. Clearly, (*) holds for $\sigma(l)$. If (*) holds for $H \bigvee \underline{l}$ and l does not appear in σ, then it holds for $H \bigvee l \smile g$ for every g. Thus, if l is labeled "I" it suffices for *Me* to make an arbitrary

[1]Actually: a path through the tree structure of σ

move. Eventually, the game reaches a state $H \bigvee l$, where l is a leaf of σ, thus has payoff $\trianglerighteq k$. Therefore, *My* payoff in the disjunctive game is at least k. □

The equivalent *You*-formulation of Proposition 5.3.4 also admits a constructive proof.

Proposition 5.3.5: *Your* disjunctive strategy = conjunction of *Your* strategies

You have a disjunctive k-strategy in $\mathbf{DG}^{\rho}_{\wp}(D)$ iff *You* have k-strategies in $\mathbf{G}_{\wp}(h)$ for all $h \in D$.

Proof. "⇒": Let μ be a disjunctive k-strategy for *You* in $\mathbf{DG}^{\rho}_{\wp}(D \bigvee h)$. We recursively define a strategy μ' for *You* in $\mathbf{G}_{\wp}(h)$. Assume μ' has been defined for histories labeled "Y" up to a length of n. Let *Me* play in $\mathbf{DG}^{\rho}_{\wp}(D \bigvee h)$ by never using the duplication rule. This ensures that at some point, the current disjunctive state is $D' \bigvee \underline{h'}$, where $h' \sqsupseteq h$ is either terminal or has a length of n and is labeled "Y". In the latter case, according to μ, *You* move to some $D' \bigvee (h' \smile g)$. Correspondingly, we define $\mu'(h') = h' \smile g$.

We now show that μ' is indeed a k-strategy. Let σ' be a strategy for *Me* in $\mathbf{G}_{\wp}(h)$ and let σ be the following strategy for *Me* in $\mathbf{DG}^{\rho}_{\wp}(D \bigvee h)$: if the current state is $H \bigvee h'$ with $h' \sqsupseteq h$ and $\rho(H \bigvee h') \in H$, *I* make an arbitrary move but do not use the duplication rule. If $\rho(H \bigvee h') = h'$, *I* move to $H \bigvee \sigma'(h)$. If *I* play according to this strategy and *You* play μ, a simple induction shows that the current disjunctive state is always of the form $H \bigvee (\sigma'; \mu')^n(h)$, for some n. Since *I* do not use the duplication rule, the game ends after finitely many rounds in some disjunctive state $D' \bigvee O(\sigma', \mu')$. By assumption, $\wp(O(\sigma', \mu')) \trianglelefteq \delta_{\wp}(H \bigvee O(\sigma', \mu')) \trianglelefteq k$.

"⇐": For every $h \in D$, let μ_h be a k-strategy for *You* in $\mathbf{G}_{\wp}(h)$. *Your* strategy μ in $\mathbf{DG}^{\rho}_{\wp}(D)$ is as follows: in a disjunctive state $H \bigvee \underline{l}$ labeled "Y", l is a history in $\mathbf{G}_{\wp}(h)$, for some h. Hence, *I* can use μ_h and go to $H \bigvee \sigma_h(l)$. Playing this way ensures that all game states contained in every resulting disjunctive state consist of histories of the σ_hs, against any opposing strategy from *Me*. By assumption, every such history that is also terminal is winning for *You*. Hence, the game cannot end in a terminal disjunctive state with payoff $\triangleright k$, which shows that μ is a k-strategy for *You*. □

If $\mathbf{G}$ is clear from context, let us write $v_{\wp}(h)$ for $v(\mathbf{G}_{\wp}(h))$, the value of the game $\mathbf{G}_{\wp}(h)$, and similarly $v^{\rho}_{\wp}(D)$ for $v(\mathbf{DG}^{\rho}_{\wp}(D))$. We sum up the results of this section:

Corollary 5.3.6: Value of the disjunctive game

Let $\mathbf{G}$ be a complete semantic game. Then $v^{\rho}_{\wp}(D) = \max_{h \in D} v_{\wp}(h)$.

In particular, this shows that disjunctive game $\mathbf{DG}^{\rho}_{\wp}$ is fully *regulation-independent*: the value of the game does not depend on the underlying regulation ρ. This means we can always consider regulations that act in *My* interest or even give the task of scheduling to *Me*, just as in the previous chapters. We will see in the next section that regulation-independence fails at the level of $\mathbf{DG}^\rho$ as a provability game.

5.4 The Disjunctive Game as Provability Game

In analogy to the previous chapters, as well as model-theoretic semantics, the following definition gives the sensible notion of a *game-induced validity degree*:

$$\deg^{\mathbf{G}}(D) = \min_{\wp \in \mathcal{P}} \max_{h \in D} v_\wp(h).$$

In this section, we are interested in lifting **DG** (and by extension, **G**) to a game characterizing this degree. First, we must show that $\deg^{\mathbf{G}}$ is well-defined, i.e., that the $\min$ is taken at some $\wp$.

Lemma 5.4.1: $\deg^{\mathbf{G}}$ is well-defined

We have $\inf_{\wp \in \mathcal{P}} \max_{h \in D} v_\wp(h) = \min_{\wp \in \mathcal{P}} \max_{h \in D} v_\wp(h)$, for every complete $\mathbf{G}$, regulation ρ, and disjunctive state D.

Proof. By Corollary 5.3.6, we have $v_\wp^\rho(D) = \max_{h \in D} v_\wp(h)$. Hence, it suffices to show that the set $\{v_\wp^\rho(D) : \wp \in \mathcal{P}\}$ has a minimum. By Propositions 5.3.3 and 5.2.5, the game $\mathbf{DG}_\wp^\rho(D)$ is determined. For every $\wp$, let $\mathfrak{h}_\wp$ be the maximin-outcome of $\mathbf{DG}^\rho(D)$, that means $v_\wp^\rho(D) = \delta_\wp(\mathfrak{h}_\wp)$. Since $\mathbf{DG}^\rho$ is complete, the set $P = \{\delta_\wp(\mathfrak{h}_\wp) : \wp \in \mathcal{P}\}$ has a minimum. The claim now follows since $\min P = \min\{v_\wp^\rho(D) : \wp \in \mathcal{P}\}$. □

We now define the provability game $\mathbf{DG}^\rho$. Unlike the disjunctive game in the previous section, this game is not played over a fixed payoff function $\wp$, but can rather be seen as a simultaneous play over all payoff functions. Accordingly, the rules of the game $\mathbf{DG}^\rho$ are exactly like for $\mathbf{DG}_\wp^\rho$, except for the payoff:

(Pay) If D is elementary, then the payoff $\delta(D)$ is the minimal payoff $\delta_\wp(D)$ of $\mathbf{DG}_\wp^\rho$, where $\wp$ ranges over all payoff functions, i.e., $\delta(D) = \min_{\wp \in \mathcal{P}} \delta_\wp(D)$. Infinite disjunctive histories are assigned a payoff of -1.

We make the same changes in the formal definition of $\mathbf{DG}^\rho$:

Definition 5.4.2: Disjunctive Game as Provability Game

The game $\mathbf{DG}^\rho(D)$ is the same as the game $\mathbf{DG}_\wp^\rho(D)$ in Definition 5.3.2, except for the payoff function:

- Terminal disjunctive histories $\mathfrak{h}$ ending in $\langle \dots H \rangle$ are mapped to

$$\delta(\langle \dots, H \rangle) = \min_{\trianglelefteq}\{\delta_\wp(\langle \dots H \rangle) : \wp \in \mathcal{P}\}$$

Infinite terminal histories are mapped to -1.

It follows directly from the completeness of $\mathbf{DG}^{\rho}_{\wp}$ that δ is well-defined. Note that strictly speaking, for a fixed regulation ρ, the provability game $\mathbf{DG}^{\rho} = (\mathrm{DStat}, \mathrm{DHist}, \ell \circ \rho, \{\delta\})$ is a semantic game in terms of Definition 5.2.1. However, given that there is only one payoff function in this semantic game, we prefer to speak of it as a *provability game* instead. This also makes the distinction to the game of the previous subsection easier: whenever we use the term *disjunctive game* we mean the game of Section 5.3, *provability game* refers to the game introduced in the current section.

As in the previous sections, the determinacy of the provability game follows from its completeness and Proposition 5.2.5.

Proposition 5.4.3

If the disjunctive game is complete, then so is the provability game.

Proof. Let Δ be a compact set of disjunctive terminal histories and $P \subseteq \{\delta(\mathfrak{h}) : \mathfrak{h} \in \Delta\}$. Let us show that P has a maximum. If Δ contains an infinite history, then $\min P = -1$. Otherwise, let all histories in Δ be finite. Since the semantic game $\mathbf{DG}$ is complete, for each terminal $\mathfrak{h}$, $\delta(\mathfrak{h}) = \min_{\wp \in \mathcal{P}} \delta_{\wp}(\mathfrak{h})$, so let $\wp_{\mathfrak{h}}$ be such that $\delta_{\wp_{\mathfrak{h}}}(\mathfrak{h}) = \delta(\mathfrak{h})$. But then

$$P \subseteq \{\delta_{\wp_{\mathfrak{h}}}(\mathfrak{h}) : \mathfrak{h} \in \Delta\}\} \subseteq \bigcup_{\wp \in \mathcal{P}} \{\delta_{\wp}(\mathfrak{h}) : \mathfrak{h} \in \Delta\},$$

and hence must have a maximum, by completeness of $\mathbf{DG}$. □

We are now ready to prove the adequacy of the provability game. The left-to-right direction is easy and is stated and proved below. The right-to-left direction is harder and will require another assumption on the semantic game and the regulation. This direction is the subject of the following subsection.

Theorem 5.4.4: Adequacy, ltr, *I*-formulation

Let $\mathbf{G}$ be of finite height. If I have a k-strategy in $\mathbf{DG}^{\rho}(D)$, then I have k-strategies in $\mathbf{DG}^{\rho}_{\wp}(D)$ for every $\wp \in \mathcal{P}$.

Proof. The two games are identical, except maybe for the payoffs. Thus, I can use σ to play in $\mathbf{DG}^{\rho}_{\wp}(D)$. By assumption, every outcome resulting from playing σ has payoff $\trianglerighteq k$ in $\mathbf{DG}^{\rho}(D)$, and hence in $\mathbf{DG}^{\rho}_{\wp}(D)$. □

A similar argument shows:

Theorem 5.4.5: Adequacy, ltr, *You*-formulation

Let $\mathbf{G}$ be of finite height. If *You* have a k-strategy in $\mathbf{DG}^{\rho}_{\wp}(D)$, for some $\wp \in \mathcal{P}$, then *You* have a k-strategy in $\mathbf{DG}^{\rho}(D)$.

My best way to play

We start this subsection by identifying a property of the regulation ρ that is necessary in order for $\mathbf{DG}^{\rho}$ to adequately model the degree of validity defined at the beginning of this section. To this end, we give an example of an ill-behaved regulation causing adequacy to fail.

Example 5.4.6. Let us consider the game $\mathbf{DG}^{\mathrm{Hyb},\rho}(D)$, where[2] $D = \mathbf{P} : p \bigvee \mathbf{O} : p$. Remember that according to the rules of $\mathbf{G}^{\mathrm{Hyb}}$, in the game state $\mathbf{P} : p$ is: *You* choose one of countably infinite many nominals, say i, and the game ends in $\mathbf{P}, i : p$. Over the model $\mathcal{M} = (\mathsf{W}, \mathsf{R}, \mathsf{V}, \mathsf{g})$, *I* win and *You* lose in this state if $\mathsf{g}(i) \in \mathsf{V}(p)$. In $\mathbf{O} : p$, *I* choose a nominal i and the game ends in $\mathbf{O}, i : p$. *I* win and *You* lose if $\mathsf{g}(i) \notin \mathsf{V}(p)$. Note that over every model $\mathcal{M}$, *I* can win the game $\mathbf{DG}^{\mathrm{Hyb},\rho}_{\mathcal{M}}(D)$, independent of the regulation ρ: if there is a nominal i with $\mathsf{g}(i) \notin \mathsf{V}(p)$, *I* go to that nominal in $\mathbf{O} : p$. Otherwise, any move *You* make in $\mathbf{P} : p$ results in a win for *Me*. Note that the order of moves, i.e., the regulation ρ is irrelevant here. Hence, $\deg^{\mathbf{G}^{\mathrm{Hyb}}}(D) = 1$.

Let us turn to the provability game: if the regulation ρ is fair, it picks the left state in D: $\underline{\mathbf{P} : p} \bigvee \mathbf{O} : p$, *You* pick some i, and the game continues at $\mathbf{P}, i : p \bigvee \underline{\mathbf{O} : p}$. *I* go to $\mathbf{P}, i : p \bigvee \mathbf{O}, i : p$, which is winning over every model. Hence, the value of the provability game with this regulation is 1 and adequately captures $\deg^{\mathbf{G}^{\mathrm{Hyb}}}(D)$.

Now, let us consider a "nasty" regulation picking $\mathbf{O} : p$, whenever possible. If, at $\mathbf{P} : p \bigvee \underline{\mathbf{O} : p}$, *I* pick some i, then the game continues at $\underline{\mathbf{P} : p} \bigvee \mathbf{O}, i : p$. *You* pick some $j \neq i$ and the game ends at $\mathbf{P}, j : p \bigvee \mathbf{O}, i : p$, which is winning for *You* since there are models falsifying both $\mathsf{g}(j) \in \mathsf{V}(p)$ and $\mathsf{g}(i) \notin \mathsf{V}(p)$. The situation does not get better if *I* apply the duplication rule finitely many times. In this case, the game arrives at a disjunctive state of the form

$$\underline{\mathbf{P} : p} \bigvee \mathbf{O}, i_1 : p \bigvee \ldots \bigvee \mathbf{O}, i_n : p,$$

which *You* easily win by choosing a nominal different from $i_1, \ldots, i_n$. Of course, if *I* apply the duplication rule infinitely often, *I* lose this infinite run by definition. With this unfair regulation, the value of the provability game is -1 and adequacy fails.

This example demonstrates that the "nasty" behavior of the regulation ρ must be prevented in order to guarantee adequacy. A *fair* regulation ρ picks eventually every history in a disjunctive state, independent of the behavior of the players. The regulation in

[2] In $\mathbf{G}^{\mathrm{Hyb}}$, the value of a history depends only on its last game state. Hence, we omit writing down the whole history in a disjunctive state.

the above example is not fair, since it picks $\mathbf{P} : p$ only if $\mathbf{O} : p$ does not occur in the disjunctive state.

In fact, we need a more refined notion. Let σ be a strategy for *Me* in $\mathbf{DG}^{\rho}(D)$. The regulation ρ is called *fair for* σ, if for all non-terminal histories h, and disjunctive histories $\mathfrak{h}$, whenever $h \in \text{last}(\mathfrak{h})$ then for all strategies μ for *You* with $\mathfrak{h} \sqsubseteq O(\sigma, \mu)$, there is some $\mathfrak{h}'$ with $\mathfrak{h} \sqsubseteq \mathfrak{h}' \sqsubseteq O(\sigma, \mu)$ and $\rho(\mathfrak{h}') = h$. In other words, ρ is fair for σ, if at every current disjunctive state $D \bigvee h$, if h is non-terminal, then, no matter how *You* play, there is a point where h is selected by ρ if *I* play according to σ. If this holds even if *I* do not play according to σ we simply say that ρ is fair: the regulation ρ is called *fair* if for every non-terminal history h and disjunctive history $\mathfrak{h}$, whenever $h \in \text{last}(\mathfrak{h})$ then for all strategies σ for *Me* and μ for *You* with $\mathfrak{h} \sqsubset O(\sigma, \mu)$ there is some $\mathfrak{h}'$ with $\mathfrak{h} \sqsubseteq \mathfrak{h}' \sqsubseteq O(\sigma, \mu)$. This corresponds to the intuitive notion of "fair" discussed above.

We will now describe a strategy σ for *Me* for the game $\mathbf{DG}^{\text{Hyb}}(D_0)$. This strategy is – in a way – the optimal way to play the disjunctive game. Intuitively σ exploits all of *My* possible choices without sacrificing *My* winning chances.

We say that a history h *appears along* a disjunctive history $\mathfrak{h}$ and write $h \in \mathfrak{h}$ if it occurs in a disjunctive state in $\mathfrak{h}$. Let us fix an enumeration of game states of the semantic game such that every game state g appears in this enumeration infinitely often. Let us denote by $\#g$ the number of g under this enumeration. For every disjunctive history $\mathfrak{h}$ and non-terminal history h, let $N_{\mathfrak{h}}(h) = \{\#g : h \smile g \notin \mathfrak{h}\}$ be the set of numbers of histories extending h and not appearing along $\mathfrak{h}$. The strategy σ_k is as follows:

If the current non-terminal history $\mathfrak{h}$ ends in $D = D' \bigvee \underline{h}$, the history h is labeled "I", and

1. $|N_{\mathfrak{h}}(h)| \geq 1$, and $\delta(D^{ter}) \trianglelefteq k$,
   a) if $|N_{\mathfrak{h}}(h)| \geq 2$ and $h \notin D'$, then *I* duplicate h, i.e. the game continues at $D' \bigvee h \bigvee h$.
   b) otherwise, *I* move to $D' \bigvee h \smile g$, where $\#g = \min N_{\mathfrak{h}}(h)$.
2. Otherwise, move to an arbitrary $D' \bigvee h \smile g$.

Lemma 5.4.7

Let ρ be fair for σ_k and let $\mathfrak{h}$ be the outcome of $\mathbf{DG}^{\rho}(D)$ given by *Me* playing σ_k and *You* playing some strategy μ such that $\delta(\mathfrak{h}) \trianglelefteq k$, Then $\delta(\mathfrak{h}_i^{ter}) \trianglelefteq k$ for every $i \leq \text{length}(\mathfrak{h})$, and for every h appearing along $\mathfrak{h}$, if

1. $\ell(h) = Y$, there is at least one immediate successor of h appearing along $\mathfrak{h}$,
2. $\ell(h) = I$, all immediate successors of h appear along $\mathfrak{h}$.

Proof. Suppose that there is some i with $k \triangleleft \delta(\mathfrak{h}_i^{ter})$. By definition of σ_k, *I* do not use the duplication rule after $\mathfrak{h}_i$. Since the semantic game is of finite height, the run terminates at some $\mathfrak{h}_j$ with $j \geq i$. We have that $\mathfrak{h}_i^{ter} \subseteq \mathfrak{h}_j$ and thus $\delta_\wp(\mathfrak{h}_i^{ter}) \trianglelefteq \delta_\wp(\mathfrak{h}_j)$, for every $\wp \in \mathcal{P}$. Minimizing over $\wp$, $k \triangleleft \delta(\mathfrak{h}_i^{ter}) \trianglelefteq \delta(\mathfrak{h}_j) = \delta(\mathfrak{h})$, which is a contradiction.

For 1, let h appear along $\mathfrak{h}$. It suffices to note that, since ρ is fair, there is some $\mathfrak{h}_n = H \bigvee \underline{h}$. According to μ, *You* move to some $H \bigvee h \smile g$, i.e., the history $h \smile g$ appears along $\mathfrak{h}$.

For every n, let $\mathfrak{h}_{-n}$ denote the initial segment of $\mathfrak{h}$ up to $\mathfrak{h}_n$. To prove 2, assume towards a contradiction, that $h \in \mathfrak{h}$ but there are immediate h-successors not appearing along $\mathfrak{h}$, i.e., $|N_\mathfrak{h}(h)| \neq 0$. Let $\#g = \min N_\mathfrak{h}(h)$. By assumption, there is some number n such that $h \smile g'$ appears along $\mathfrak{h}_{-n}$ for all game states g' with $\#g' < \#g$. Since ρ is fair, we may even assume that $\mathfrak{h}_n = H \bigvee \underline{h}$. If $|N_{\mathfrak{h}_{-n}}(h)| = 1$, or $h \in D$, then, according to σ_k, *I* move to $\mathfrak{h}_{n+1} = H \bigvee h \smile g$, since $\#g = \min N_{\mathfrak{h}_{-n}}(h)$. This contradicts the fact that $h \smile g$ does not appear along $\mathfrak{h}$. Hence, $|N_{\mathfrak{h}_{-n}}(h)| \geq 2$, and $h \notin D$. According to σ_k, *I* duplicate h, and the game continues with $H \bigvee h \bigvee h$. Since ρ is fair, there is some minimal $m > n$ such that $\mathfrak{h}_m = H' \bigvee \underline{h}$, i.e. h is scheduled again. Note that $|N_{\mathfrak{h}_{-m}}(h)| \geq 2$. Since g has the minimal number of all game states g' with $h \smile g' \notin \mathfrak{h}_{-m}$, by the construction of σ_k, *I* move to $H' \bigvee h \smile g$. But then we have the same contradiction as before, which concludes the proof. □

Following the idea of proofs of the corresponding theorems in Chapter 3 and 4, we now want to extract from $\mathfrak{h}$ a payoff function $\wp$ and k-strategies for *You* for every $\mathbf{G}_\wp(h)$, where h appears along $\mathfrak{h}$. However, at this point, our framework is too general to accomplish this.

Example 5.4.8. Consider the game $\mathbf{DG}^{\mathrm{Hyb},\rho}(\mathbf{O} : p)$, but over the class of models excluding $\mathcal{M}^*$ with $\mathsf{g}(i) \in \mathsf{V}(p)$, for all nominals i. Then *My* strategy σ_{-1} is to repeatedly use the duplication rule and move to a new $\mathbf{O}, i : p$. For example, after n turns the current disjunctive state will look like.

$$\mathbf{O} : p \bigvee \mathbf{O}, i_1 : p \bigvee \ldots \bigvee \mathbf{O}, i_n : p$$

We always have $\delta(D^{ter}) = -1$, since there are, for every n, models $\mathcal{M}_n$ with $\mathsf{g}(i_j) \in \mathsf{V}(p)$ for every $j = 1, ..., n$. However, there is no model $\mathcal{M}$ making all $\mathbf{O}, i : p$ winning for *You* for every i.

The example demonstrates that we need to add another requirement for the set of payoff function $\mathcal{P}$. This property is easiest stated in terms of a sort of continuity of the function δ.

Definition 5.4.9: δ is continuous

We call δ *continuous* if the following holds: for every k, every compact D, if $\delta(D') \trianglelefteq k$ for every finite $D' \subseteq D$, then $\delta(D) \trianglelefteq k$.

The existence of the required $\wp$ (or $\mathcal{M}^*$ in the example) then follows by completeness, as shown in the proof of the following lemma.

game state	assoc. with	game state	assoc. with
$\mathbf{P}, i : p$	$\neg V_p(i)$	$\mathbf{O}, i : p$	$V_p(i)$
$\mathbf{P}, i : j$	$\neg(i = j)$	$\mathbf{O}, i : j$	$i = j$
$\mathbf{P}, i : R(j, k)$	$\neg R(j, k)$	$\mathbf{O}, i : R(j, k)$	$R(j, k)$

Table 5.1: How to show δ is continuous in $\mathbf{G}^{\text{Hyb}}$.

Example 5.4.10. Let D be a set (does not need to be compact) consisting of elementary game states of the semantic game for hybrid logic, i.e., of states of the form $\mathbf{Q}, i : p$, $\mathbf{Q}, i : j$, or $\mathbf{Q}, i : R(j, k)$, where $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, with $\delta(D') \trianglelefteq -1$, for every finite $D' \subseteq D$. This means that for every such D', there is a model $\mathcal{M}_{D'}$ such that every $g \in D'$ is winning for *You* over this model. We must find a $\mathcal{M}^*$ making all of D winning for *You*. This is a simple application of the compactness theorem for classical first-order logic with equality.

We consider the following first-order theory of classical logic with equality. In our language, we include for every nominal i a corresponding constant (which, for simplicity we also name) i, for every propositional variable p of hybrid logic a unary predicate V_p and a binary relational symbol R. Note that every model of this first-order language directly corresponds to a model of hybrid logic.

To every game state in D associate a formula of first-order logic as shown in Table 5.1. Let T be a set containing all these formulas. Let T_D be the set of all associated formulas. Then for every finite $T' \subseteq T_D$ has a model satisfying all of T'. By the compactness theorem, there is a model of all T_D, which can be easily translated to a model of hybrid logic making all of D wining for *You*.

Example 5.4.11. In finite games, "compact" means "finite", hence for finite semantic games like $\mathbf{G}^{\text{GCL}}$, δ is always continuous.

Lemma 5.4.12

Let δ be continuous, $\mathbf{G}$ complete, and let $\mathfrak{h}$ be as above. Then there is some $\wp \in \mathcal{P}$ such that *You* have a k-strategy in $\mathbf{G}_\wp(h)$, for every h appearing along $\mathfrak{h}$.

Proof. Where defined, let D_i consist of the terminal histories in the disjunctive state $\mathfrak{h}_i$ (otherwise, let $D_i = \emptyset$). Then $D = \bigcup_{i \in \omega} D_i$ is compact. By Lemma 5.4.7, $\delta(D') \trianglelefteq k$, for all finite $D' \subseteq D$. Hence $\delta(D) \trianglelefteq k$, By continuity of δ. Using the completeness of $\mathbf{G}$, we define

1. For each $\wp$, let $\wp(h_\wp) = \max\{\wp(h) : h \in D\}$,

2. $\wp^*(h_{\wp*}) = \min\{\wp(h_\wp) : \wp \in \mathcal{P}\}$.

The latter is well-defined because of the set inclusion

$$\{\wp(h_\wp) : \wp \in \mathcal{P}\} \subseteq \bigcup_{\wp \in \mathcal{P}} \{\wp(h) : h \in D\}.$$

With this $\wp^*$, we now show the claim by bottom-up induction on h. If h is terminal, then it appears in D. We have

$$k \trianglerighteq \delta(D) = \inf_{\wp \in \mathcal{P}} \sup_{h \in D} \wp(h) = \inf_{\wp \in \mathcal{P}} \wp(h_\wp) = \wp^*(h_{\wp^*}) \trianglerighteq \wp^*(h),$$

which shows that the payoff at h is at most k.

If h appears along $\mathfrak{h}$ and is labeled "Y", then, by Lemma 5.4.7, there is some immediate successor h' of h appears along $\mathfrak{h}$. By the inductive hypothesis, *You* have a k-strategy in $\mathbf{G}_{\wp^*}(h')$. By Lemma 2.1.14, *You* have a k-strategy in $\mathbf{G}_{\wp^*}(h)$.

If h appears along $\mathfrak{h}$ and is labeled "I", then by Lemma 5.4.7, all immediate successors h' of h appear along $\mathfrak{h}$, too. By the inductive hypothesis, *You* have a k-strategy for every $\mathbf{G}_{\wp^*}(h')$. By Lemma 2.1.14, *You* can combine these strategies to a k-strategy in $\mathbf{G}_{\wp^*}(h)$. □

We now have all the ingredients to prove the missing directions of the adequacy theorems:

Theorem 5.4.13: Adequacy, rtl, *You*-formulation

Let $\mathbf{G}$ be a complete semantic game of finite height with δ continuous, and let ρ be fair. If *You* have a k-strategy in $\mathbf{DG}^\rho(D)$, then there is some $\wp \in \mathcal{P}$ such that *You* have a k-strategy in $\mathbf{DG}^\rho_\wp(D)$.

Proof. Let $\mathfrak{h}$ be the outcome of the game given by *Me* playing *My* best way, σ_k, as described above, and *You* playing *Your* k-strategy. By assumption, the payoff, $\delta(\mathfrak{h})$, is at most k. Hence, $\mathfrak{h}$ satisfies the requirements of Lemma 5.4.12, which shows that there must be some $\wp \in \mathcal{P}$ such that *You* have k-strategies in $\mathbf{G}_\wp(h)$, for every h appearing along $\mathfrak{h}$. In particular, *You* have k-strategies in $\mathbf{G}_\wp(h)$, for all $h \in D$. By Proposition 5.3.5, *You* have a winning strategy in $\mathbf{DG}^\rho_\wp(D)$. □

Theorem 5.4.14: Adequacy, rtl, *I*-formulation

Let $\mathbf{G}$ be a complete semantic game of finite height with δ continuous, and let ρ be fair. Then *I* have a k-strategy in $\mathbf{DG}^\rho(D)$ if *I* have k-strategies in $\mathbf{DG}^\rho_\wp(D)$ for every $\wp \in \mathcal{P}$.

Proof. By Propositions 5.3.3 and 5.4.3, the provability game $\mathbf{DG}$ is complete and, by Proposition 5.2.5, determined. Suppose *I* have no k-strategy in $\mathbf{DG}^\rho(D)$. Then, by the determinacy of $\mathbf{DG}$, *You* must have an l-strategy for some $l \lhd k$. By Theorem 5.4.13, *You* have an l-strategy in some $\mathbf{DG}^{\rho}_{\wp}(D)$. We conclude that *I* cannot have a k-strategy in this game. □

Let us write $v^{\rho}(D)$ for $v(\mathbf{DG}^{\rho}(D))$. We sum up our results in this section:

Corollary 5.4.15

Let $\mathbf{G}$ be a complete semantic game of finite height and ρ fair. Then $v^{\rho}(D) = \deg^{\mathbf{G}}(D)$.

Proof. We have $v^{\rho}(D) = \min_{\wp \in \mathcal{P}} v^{\rho}_{\wp}(D)$. By Theorem 5.4.13, $v^{\rho}(D) \unlhd \min_{\wp \in \mathcal{P}} v^{\rho}_{\wp}(D)$, and by Theorem 5.4.14, $v^{\rho}(D) \unrhd \min_{\wp \in \mathcal{P}} v^{\rho}_{\wp}(D)$. □

Remark 5.4.16. Our results imply that fair regulations lead to optimal game values for *Me*. To see this, let ρ_1 and ρ_2 be regulations and let ρ_2 be fair. Let $k = v^{\rho_2}(D)$. By the corollary, there is some $\wp$ such that *You* have a k-strategy in $\mathbf{G}_{\wp}(h)$, for all $h \in D$. By Proposition 5.3.5, *You* have a k-strategy in $\mathbf{DG}^{\rho_1}_{\wp}(D)$, and, by Theorem 5.4.5, *You* have a k-strategy in $\mathbf{DG}^{\rho_1}(D)$. This shows $v^{\rho_1}(D) \unlhd k$.

Example 5.4.17. As demonstrated in the previous examples of this chapter, the semantic games $\mathbf{G}^{\mathrm{Hyb}}$ and $\mathbf{G}^{\mathrm{GCL}}$ are complete and their respective δs are continuous. Hence, we immediately obtain generalizations of the adequacy theorems (3.5.3,3.5.5,4.5.3) of the provability games in Chapters 3 and 4, where scheduling is done by a fair regulation function instead of *Me*.

5.5 From Semantic Games to Proofs

In this section, we extract a sequent-style proof system from the provability game. As in Chapter 4, proofs in this system come in degrees, representing the degree of validity, $\deg^{\mathbf{G}}$, defined in the previous section.

Proofs in the calculus in Table 5.2 represent strategies for *Me* in the provability game $\mathbf{DG}^{\rho}$. The semantic rule $(Ih)_g$ represents the possible choices for *Me* at the I-history h. Branching in the proof arises due to (Yh) and corresponds with branching in the strategies for *Me* at Y-histories h, according to *Your* available moves at h. In general, there are infinitely many options for *You*, hence proofs can be infinitely branching. In some cases, this infinite branching can be reduced, if there is a finite set of optimal choices for *You*, see Section 3.7.1.

The contraction rule (C) represents duplication, and the degree at elementary disjunctive states coincides with the payoff function δ. Proofs of D where all initial sequents have a value $\unrhd k$ are called *k-proofs*. If a k-proof of D exists, we write $\vdash^{k}_{\mathbf{DS}} D$. In contrast to the previous chapters, a k-proof in $\mathbf{DS}$ represents not only a k-strategy for *Me*, but also (implicitly, through the choice of rule application) a regulation ρ. Hence, the existence of a k-proof implies that there is a regulation ρ and a k-strategy for *Me* in the game $\mathbf{DG}^{\rho}(D)$. By Theorem 5.4.4, this implies $\deg^{\mathbf{G}}(D) \trianglerighteq k$. On the other hand, Theorem 5.4.14 tells us that $\deg^{\mathbf{G}}(D) \trianglerighteq k$ implies that there is[3] a regulation ρ and a k-strategy for *Me* in $\mathbf{DG}^{\rho}(D)$, and hence a k-proof of D. We have just proved:

Table 5.2: The proof system **DS**. In the rule $(Ih)_g$, h is labeled "I" and $h \smile g$ is a successor of h. In the rule (Yh), h is labeled "Y" and $h \smile g_1$, $h \smile g_2$, ... are all the immediate successors of h.

Initial sequents

D, where D is terminal and has degree $\delta(D)$

Structural Rule

$$\frac{D \bigvee h \bigvee h}{D \bigvee h}\ (C)$$

Semantic rules

$$\frac{D \bigvee (h \smile g)}{D \bigvee h}\ (Ih)_g \qquad\qquad \frac{D \bigvee (h \smile g_1) \quad D \bigvee (h \smile g_2) \quad \dots}{D \bigvee h}\ (Yh)$$

Theorem 5.5.1: Completeness of DS

Let $\mathbf{G}$ be a complete semantic of finite height game with δ continuous. Then

$$\vdash^{k}_{\mathbf{DS}} D \iff \deg^{\mathbf{G}}(D) \trianglerighteq k.$$

We conclude this section with a remark on scheduling by a regulation instead of scheduling done by *Me*. Remark 5.4.16 implies that fair regulations favor *Me* and enable *Me* to enforce a value of $\deg^{\mathbf{G}}(D)$ in $\mathbf{DG}^{\rho}(D)$. Hence, if we want our provability game to be adequate (that is, if we want the value of $\mathbf{DG}^{\rho}(D)$ to be $\deg^{\mathbf{G}}(D)$), we might as well give the task of scheduling to *Me*, since it is in *My* best interest to schedule fairly.

Returning the task of scheduling to *Me* also restores the interpretation of the user of the proof system as playing the provability game on behalf of *Me* (or rather: planning a strategy for *Me*) in the provability game.

5.6 Finite Games and Regulation-independence

In this section, we discuss the important class of finite semantic games and show that these games are *regulation independent*, i.e., the value of the game $\mathbf{DG}^{\rho}(D)$ is $\deg^{\mathbf{G}}(D)$, even if ρ is not fair. We use that $\mathbf{G}$ is finitely branching to introduce a version of

[3] The theorem says "for every". Since there are always fair regulations, this implies "there are".

the calculus from the previous section where proofs of D automatically have degree $\deg^{\mathbf{G}}(D)$. We conclude the section with a case study for truth-degree comparison games for Gödel-logic.

Theorem 5.6.1: Finite games are adequate and regulation-independent

Let $\mathbf{G}$ be a finite and complete semantic game. Then for every disjunctive state D and every regulation ρ,

$$v^{\rho}(D) = \deg^{\mathbf{G}}(D).$$

For the proof of this theorem, we inspect the proof of Theorem 5.4.13 and find that fairness of ρ is only ever needed in the proof of Lemma 5.4.7, where fairness of ρ for σ_k is sufficient. Additionally, the requirement for δ to be continuous is always fulfilled for finite semantic games. Hence, to prove the theorem, it is enough to prove the following lemma:

Lemma 5.6.2

If $\mathbf{G}$ is a finite game, then every regulation ρ is fair for *My* best k-strategy σ_k.

Proof. Let $\mathfrak{h}$ be a history of $\mathbf{DG}^{\rho}(D)$, $h \in \mathrm{last}(\mathfrak{h})$ non-terminal, and μ a strategy for *You* such that $\mathfrak{h} \sqsubseteq O(\sigma_k, \mu)$. We show that $\mathfrak{t} = O(\sigma_k, \mu)$ is necessarily finite. The claim follows since finite terminal disjunctive histories can only end in disjunctive states containing exclusively terminal histories. Hence, at some point, the game must have continued at the non-terminal h.

Let h be a non-terminal I-history $h \in \mathfrak{t}$, and let $g_1, \ldots, g_n$ be game states such that $\#g_1 < \cdots < \#g_n$ and $\{h \smile g_j : j = 1, \ldots n\}$ are all immediate successors of h. We prove the following claim $(*)$: for every j, if the duplication rule has been applied to h at least $j+1$-times in an initial segment of $\mathfrak{t}$, then the successors $h \smile g_1, \ldots, h \smile g_j$ appear in this segment, too.

Suppose we have shown the claim for all $i < j$, and the duplication rule was applied to h at least $j+1$-many times. By the inductive hypothesis, the immediate successors $h \smile g_1, \ldots, h \smile g_{j-1}$ appear along an initial segment of $\mathfrak{t}$. If $h \smile g_j$ appears there, too, we are done. Otherwise, let k_1 and k_2 be such that the duplication rule is applied to h at $\mathfrak{t}_{k_1}$ for the jth and at $\mathfrak{t}_{k_2}$ for the $j+1$st time. Thus, $\mathfrak{t}_{k_1} = H_1 \bigvee \underline{h}$, $\mathfrak{t}_{k_1+1} = H_1 \bigvee h \bigvee h$, $\mathfrak{t}_{k_2} = H_2 \bigvee \underline{h}$ and $h \notin H_1 \cup H_2$. According to σ_k, the duplication rule is never applied to h if there is more than one copy of h in the current disjunctive state. Hence, there must be some k with $k_1 < k < k_2$ such that $\mathfrak{t}_k = H \bigvee \underline{h}$. According to σ_k, at $\mathfrak{t}_k$, *I* move to $h \smile g_j$.

We can now show that $\mathfrak{t}$ is finite. Towards a contradiction, suppose $\mathfrak{t}$ is infinite. If the duplication rule is used only finitely many times, then from some point on, the duplication rule is never used. Hence, the disjunctive game terminates after finitely many rounds. But since $\mathfrak{t}$ is infinite and there are only finitely many histories extending a history from g, there must be some history h to which the duplication rule is applied infinitely often. Let n be the number of immediate h-successors, and let $D_i = H \bigvee \underline{h}$ be the disjunctive state where the duplication rule is applied to h for the $n+2$nd time. By the claim, all immediate successors appear in $\mathfrak{k} = \langle D_0, \ldots, D_i \rangle$. But the precondition to this duplication in the definition of σ_k is that $|N_{\mathfrak{k}}(h)| \geq 2$, which is a contradiction. ☐

5.6.1 Invertible Calculi

The proof system **ODS** in Table 5.3 produces proofs that resemble *My* optimal strategies, as described for the proof of Theorem 5.4.13. According to this strategy, at I-states $h\ I$ use the duplication rule until all immediate successors $h \smile g_1$,...,$h \smile g_n$ of h appear in the current disjunctive state. An example of (a version) of this proof system is $\mathbf{ODS}^{\mathrm{GCL}}$ in Table 4.4.

Table 5.3: The proof system **ODS**. In the rule (Ih), h is labeled "I" and $h \smile g_1$,...,$h \smile g_n$ are all successors of h. In the rule (Yh), h is labeled "Y" and $h \smile g_1$, ..., $h \smile g_m$ are all the immediate successors of h.

Initial sequents

D, where D is terminal and has degree $\delta(D)$

Semantic rules

$$\frac{D \bigvee (h \smile g_1) \bigvee \ldots \bigvee (h \smile g_n)}{D \bigvee h}\ (Ih) \qquad \frac{D \bigvee (h \smile g_1) \quad \ldots \quad D \bigvee (h \smile g_m)}{D \bigvee h}\ (Yh)$$

In proof-theoretic terms, all rules of the calculus are *invertible*. This means that for a rule

$$\frac{D_1 \quad \ldots \quad D_n}{D}$$

we have $\deg^{\mathbf{G}}(D) = \min_{i=1,\ldots,n} \deg^{\mathbf{G}}(D_i)$.

Theorem 5.6.3: ODS produces optimal proofs

Every proof of D in **ODS** has the degree $\deg^{\mathbf{G}}(D)$.

Proof. We show that the rules of **ODS** are invertible. We start with (I). Let $D = \{k_1, \ldots, k_m\}$ and let h be an I-history with immediate successors $h \smile g_1$,...,$h \smile g_n$. Fix a payoff function $\wp$. Then

$$v_\wp\left(D \bigvee h\right) = \max\{v_\wp(k_1), \ldots, v_\wp(k_m), v_\wp(h)\} \qquad \text{(Corollary 5.3.6)}$$

$$
\begin{aligned}
&= \max\{v_\wp(k_1), \ldots, v_\wp(k_m), \max_{i=1,\ldots n} v_\wp(h \smile g_i)\} && \text{(Lemma 2.1.14)}\\
&= \max\{v_\wp(k_1), \ldots, v_\wp(k_m), v_\wp(h \smile g_1), \ldots, v_\wp(h \smile g_n)\}\\
&= v_\wp\left(D \bigvee (h \smile g_1) \bigvee \ldots \bigvee (h \smile g_n)\right) && \text{(Corollary 5.3.6)}
\end{aligned}
$$

Taking the minimum over $\wp$ on both sides, we get

$$
\deg^{\mathbf{G}}\left(D \bigvee h\right) = \deg^{\mathbf{G}}\left(D \bigvee (h \smile g_1) \bigvee \ldots \bigvee (h \smile g_n)\right).
$$

For (Y), let D and h be as before, only that now h is a Y-history.

$$
\begin{aligned}
v_\wp\left(D \bigvee h\right) &= \max\{v_\wp(k_1), \ldots, v_\wp(k_m), v_\wp(h)\} && \text{(Corollary 5.3.6)}\\
&= \max\{v_\wp(k_1), \ldots, v_\wp(k_m), \min_{i=1,\ldots n} v_\wp(h \smile g_i)\} && \text{(Lemma 2.1.14)}\\
&= \min_{i=1,\ldots,n} \max\{v_\wp(k_1), \ldots, v_\wp(k_m), v_\wp(h \smile g_i)\} && \text{(De Morgan)}\\
&= \min_{i=1,\ldots,n} v_\wp\left(D \bigvee (h \smile g_i)\right) && \text{(Corollary 5.3.6)}
\end{aligned}
$$

Taking the minimum over $\wp$ on both sides (the two minima on the right commute),

$$
\deg^{\mathbf{G}}\left(D \bigvee h\right) = \min_{i=1,\ldots,n} \deg^{\mathbf{G}}\left(D \bigvee (h \smile g_i)\right),
$$

which concludes the proof. □

5.6.2 Case Study: Gödel logic

In this section, we apply the general lifting technique for a semantic game for Gödel logic, developed in [25, 22, ?], called the truth-degree comparison game $\mathbf{G}^{\mathrm{G}}$. The presentation in this section slightly differs from these papers, where the disjunctive game is used only metaphorically, and the central notion used to lift the game to an analytic calculus is that of a disjunctive strategy. In particular, regulations on the level of the disjunctive game, as well as regulation independence play no role there. The motivation for discussing this game is to illustrate the general lifting framework developed in this chapter. Hence, the presentation in this section is often kept semi-formal and makes no claims to be complete. A much more thorough analysis of the game and the resulting calculus, including extensions with the projection operator Δ, involutive negations, and truth constants, as well as connections to other calculi can be found in the mentioned papers.

Next to Łukasiewicz and Product logic, Gödel logic G [3, 35, 19, 52] is one of the three prominent fuzzy logics aimed at formalizing reasoning under vagueness. Following this paradigm, under an interpretation $\mathcal{I}$, formulas in G can take values in the real unit interval $[0, 1]$, where 0 stands for falsity, 1 for truth, and the other values for intermediate degrees for truth. Interestingly, G can also be seen as an extension of intuitionistic logic with the linearity axiom $(A \to B) \vee (B \to A)$, giving it an additional semantics in terms of linear Kripke models.

Its language is built according to the following grammar:

$$F ::= \bot \mid \top \mid a \mid F \wedge F \mid F \vee F \mid F \to F$$

Negation is expressible via implication via $\neg F = F \to \bot$. An *interpretation* $\mathcal{I}$ is a function mapping propositional variables to the real unit interval $[0, 1]$. The function $\mathcal{I}$ extends to compound formulas as follows:

$$\begin{aligned}
\mathcal{I}(\bot) &= 0, \\
\mathcal{I}(\top) &= 1, \\
\mathcal{I}(F \wedge G) &= \min\{\mathcal{I}(F), \mathcal{I}(G)\}, \\
\mathcal{I}(F \vee G) &= \max\{\mathcal{I}(F), \mathcal{I}(G)\}, \\
\mathcal{I}(F \to G) &= \begin{cases} 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B) \\ \mathcal{I}(B) & \text{otherwise.} \end{cases}
\end{aligned}$$

Here, $\min$ and $\max$ are with respect to the usual ordering on the real numbers. A formula F is valid in G if $\mathcal{I}(F) = 1$ for every interpretation $\mathcal{I}$.

Game states of the semantic game $\mathbf{G}^{\mathrm{G}}$ are of the form $F \prec G$, $\underline{F} \prec G$, $F \prec \underline{G}$, for $\prec \in \{<, \leq\}$, or[4] $(F \leq G, H < I)$. The game is played between *Me* and *You* over a fixed interpretation $\mathcal{I}$. Intuitively, at $F \leq G$, *I* am the proponent, and *You* are the opponent[5] of the claim $\mathcal{I}(F) \leq \mathcal{I}(G)$.

(**Sched**) At game states of the form[6] $F \prec G$, *I* choose whether to continue the game with $\underline{F} \prec G$ or with $F \prec \underline{G}$.

($\wedge \prec$) At $\underline{F_1 \wedge F_2} \prec G$, *I* choose whether to continue the game with $F_1 \prec G$ or with $F_2 \prec G$.

($\prec \wedge$) At $F \prec \underline{G_1 \wedge G_2}$, *You* choose between $F \prec G_1$ and $F \prec G_2$.

($\vee \prec$) At $\underline{F_1 \vee F_2} \prec G$, *You* choose whether to continue the game with $F_1 \prec G$ or with $F_2 \prec G$.

($\prec \vee$) At $F \prec \underline{G_1 \vee G_2}$, *I* choose between $F \prec G_1$ and $F \prec G_2$.

($\to\leq$) At $\underline{F_1 \to F_2} \leq G$, *I* choose between

(1) continuing with $\top \leq G$, or

(2) with $(F_2 < F_1, F_2 \leq G)$, where *You* choose whether the game continues with $F_2 < F_1$ or with $F_2 \leq G$.

[4]Game states of this form do not occur in the original formulation of the game, but we need them in order for the game to meet our game-format.

[5]In the original game, the players are called Proponent and Opponent.

[6]In the original game, scheduling is done by a regulation function picking one of the two sides of the inequality. We choose to give the task of scheduling to *Me*, to prevent confusion when it comes to scheduling on the level of the disjunctive game $\mathbf{DG}^{\mathrm{G}}$. At this level, scheduling will be done by a regulation function, as discussed in the previous sections.

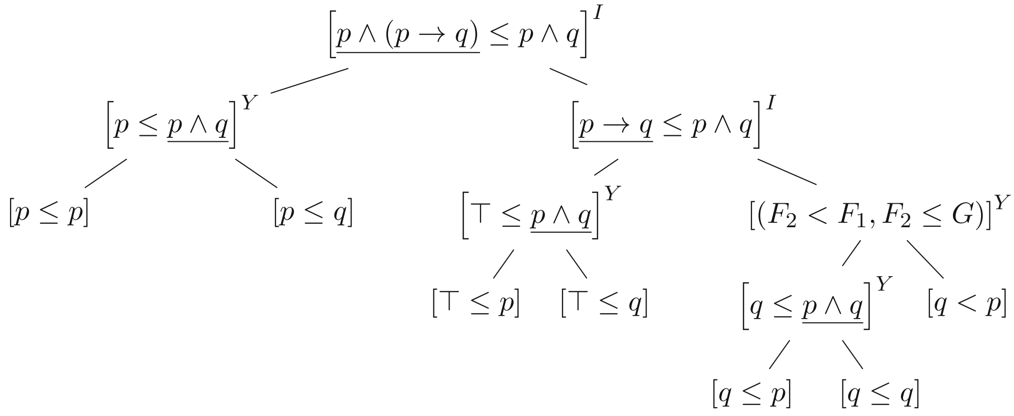


Figure 5.1: The game tree in $\mathbf{G}^{\mathrm{G}}_{\mathcal{I}}\,(p \wedge (p \to q) \leq p \wedge q)$

($\leq\!\to$) At $F \leq \underline{G_1 \to G_2}$, *I* choose between $G_1 \leq G_2$ and $F \leq G_2$.

($\to\!<$) At $\underline{F_1 \to F_2} < G$, *You* choose whether to continue the game with $F_2 < F_1$ or with $F_2 < G$.

($<\!\to$) At $F < \underline{G_1 \to G_2}$, *I* choose between
(1) continuing with $F < G_2$,
(2) or with $(F < \top, G_1 \leq G_2)$, where *You* choose whether to continue the game with $G_1 \leq G_2$, or with $F < \top$.

(**Win**) At $a \prec b$, *I* win and *You* lose iff $\mathcal{I}(a) \prec \mathcal{I}(b)$. Otherwise, *You* win and *I* lose.

Example 5.6.4. Let us consider the game state $p \wedge (p \to q) \leq p \wedge q$ the game tree[7] for $\mathbf{G}^{\mathrm{G}}$ starting at this state is depicted in Figure 5.1. For an interpretation $\mathcal{I}$ with $\mathcal{I}(q) < \mathcal{I}(p) < 1$, say $\mathcal{I}(q) = 0, \mathcal{I}(p) = 0.5$, the winning states are $p \leq p$, $q < p$, $q \leq p$, and $q \leq q$. A winning strategy for *Me* is to go to $\underline{p \to q} \leq p \wedge q$ in the first and to $(F_2 < F_1, F_2 \leq G)$ in the second move. This strategy is depicted in Figure 5.2.

Theorem 5.6.5: [22], Adequacy of $\mathbf{G}^{\mathrm{G}}$

I have a winning strategy in $\mathbf{G}^{\mathrm{G}}_{\mathcal{I}}(F \prec G)$ iff $\mathcal{I}(F) \prec \mathcal{I}(G)$.

Using the general adequacy results of this chapter, we immediately obtain the adequacy theorem and regulation independence of the induced disjunctive game.

[7] Actually, this is not the full game tree, but only a fragment with particular scheduling moves for *Me*. We will see that this scheduling can be extended to a winning strategy for *Me*.

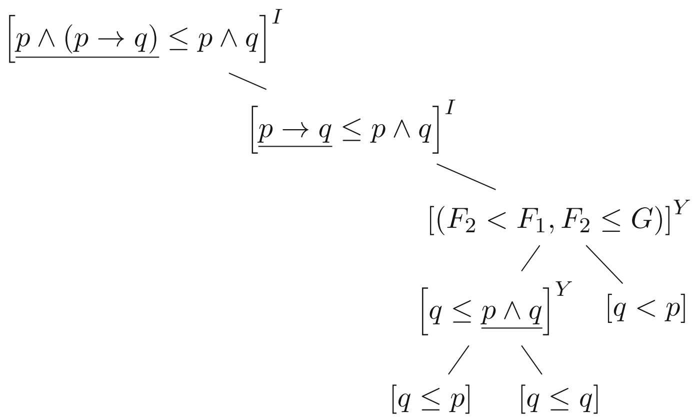


Figure 5.2: A winning strategy for *Me* in $\mathbf{G}^{\mathrm{G}}_{\mathcal{I}}\,(p \wedge (p \to q) \leq p \wedge q)$ with $\mathcal{I}(q) < \mathcal{I}(p)$

Theorem 5.6.6: Adequacy of $\mathbf{DG}^{\mathrm{G}}$

For any ρ, *I* have a winning strategy in $\mathbf{DG}^{\mathrm{G},\rho}(\top \leq F)$ iff F is valid in G.

Proof. The semantic game $\mathbf{G}^{\mathrm{G}}$ is finite, and so is the domain of payoff values. Hence, $\mathbf{G}^{\mathrm{G}}$ is complete, and we can apply Theorem 5.6.1 and Theorem 5.6.5 to obtain the desired equivalence. □

Finally, let us come to the resulting proof systems. The calculus $\mathbf{DS}^{\mathrm{G}}$ in Figure 5.4 is exactly the calculus $\mathbf{DS}$ from Figure 5.2 instantiated to the game $\mathbf{G}^{\mathrm{G}}$, except in the rule $(\to\leq)_2$, we omit writing the game state $(F_2 < F_1, F_2 \leq G)$ and immediately involve the following choice of *You*. Similarly, in $(<\to)$, we omit writing $(F < \top, G_1 \leq G_2)$. The resulting calculus has no rules for these game states, which is unproblematic, as we are primarily interested in proofs of the validity of formulas F, represented by the game state $\top \leq F$. An example of a proof can be found in Figure 5.3.

Table 5.4: The proof system $\mathbf{DS}^{\mathrm{G}}$.

Axioms

D terminal and winning

Structural Rule

$$\frac{D \bigvee F \prec G \bigvee F \prec G}{D \bigvee F \prec G}\ (C)$$

Propositional rules

$$\frac{D \bigvee F_1 \prec G}{D \bigvee F_1 \wedge F_2 \prec G}\ (\wedge \prec)_1 \qquad \frac{D \bigvee F_2 \prec G}{D \bigvee F_1 \wedge F_2 \prec G}\ (\wedge \prec)_2$$

$$\frac{D \bigvee F \prec G_1 \quad D \bigvee F \prec G_2}{D \bigvee F \prec G_1 \wedge G_2}\ (\prec \wedge) \qquad \frac{D \bigvee F_2 \prec G \quad D \bigvee F_1 \prec G}{D \bigvee F_1 \vee F_2 \prec G}\ (\vee \prec)$$

$$\frac{D \bigvee F \prec G_1}{D \bigvee F \prec G_1 \vee G_2}\ (\prec \vee)_1 \qquad \frac{D \bigvee F \prec G_2}{D \bigvee F \prec G_1 \vee G_2}\ (\prec \vee)_2$$

$$\frac{D \bigvee \top \leq G}{D \bigvee F_1 \to F_2 \leq G}\ (\to\leq)_1 \qquad \frac{D \bigvee F_2 < F_1 \quad D \bigvee F_2 \leq G}{D \bigvee F_1 \to F_2 \leq G}\ (\to\leq)_2$$

$$\frac{D \bigvee G_1 \leq G_2}{D \bigvee F \leq G_1 \to G_2}\ (\leq\to)_1 \qquad \frac{D \bigvee F < G_2}{D \bigvee F \leq G_1 \to G_2}\ (\leq\to)_2$$

$$\frac{D \bigvee F_2 < F_1 \quad D \bigvee F_2 < G}{D \bigvee F_1 \to F_2 < G}\ (\to<)$$

$$\frac{D \bigvee F < G_2}{D \bigvee F < G_1 \to G_2}\ (<\to)_1 \qquad \frac{D \bigvee G_1 \leq G_2 \quad D \bigvee F < \top}{D \bigvee F < G_1 \to G_2}\ (<\to)_2$$

The calculus $\mathbf{ODS}^{\mathrm{G}}$ in Figure 5.5 is the calculus presented in [22]. In our case, it arises as an instantiation of **ODS** in Figure 5.3 and corresponds to *My* optimal strategy in the provability game. According to this strategy, for every *I*-history[8] h, *I* first duplicate h and then exhaustively go to all possible immediate successors of h. The rule $(\to\leq)$ arises from this idea and the rules $(\to\leq)_1$ and $(\to\leq)_2$ of the semantic game, as shown in Figure 5.5. Figure 5.6 shows an example of a failed proof in $\mathbf{ODS}^{\mathrm{G}}$. The initial sequent

$$\top \leq p \bigvee p \leq \bot \bigvee \top \leq \bot$$

is falsified by any interpretation $\mathcal{I}$ with $\mathcal{I}(A) < 1$, and $0 < \mathcal{I}(p)$, for example, $\mathcal{I}(p) = 0.5$, which, by invertibility of the rules, gives us a countermodel of $p \vee \neg p$. [22] provides an effective algorithm computing interpretations falsifying any given losing state, thus, countermodels of the original formula.

Table 5.5: The proof system $\mathbf{ODS}^{\mathrm{G}}$.

Axioms

D terminal and winning

[8] As in Chapter 3, it suffices to keep track of the last state of the current history, since the winning condition solely depends on this state.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \leq p \bigvee q \leq p \qquad p \leq q \bigvee q < p}{p \leq p \wedge q \bigvee q < p}\,(\leq \wedge) \qquad p \leq p \wedge q \bigvee q \leq p \wedge q}{p \leq p \wedge q \bigvee p \to q \leq p \wedge q}\,(\to\leq)_2}{p \leq p \wedge q \bigvee p \wedge (p \to q) \leq p \wedge q}\,(\wedge \leq)_2}{p \wedge (p \to q) \leq p \wedge q \bigvee p \wedge (p \to q) \leq p \wedge q}\,(\wedge \leq)_1}{p \wedge (p \to q) \leq p \wedge q}\,(C)}{\top \leq (p \wedge (p \to q)) \to (p \wedge q)}\,(\leq\to)_1$$

Figure 5.3: Proof of $p \wedge (p \to q) \to (p \wedge q)$ in $\mathbf{DS}^{\mathrm{G}}$. The proof continues in Figure 5.4

Propositional rules

$$\dfrac{D \bigvee F_1 \prec G \bigvee F_2 \prec G}{D \bigvee F_1 \wedge F_2 \prec G}\,(\wedge \prec) \qquad \dfrac{D \bigvee F \prec G_1 \qquad D \bigvee F \prec G_2}{D \bigvee F \prec G_1 \wedge G_2}\,(\prec \wedge)$$

$$\dfrac{D \bigvee F_2 \prec G \qquad D \bigvee F_1 \prec G}{D \bigvee F_1 \vee F_2 \prec G}\,(\vee \prec) \qquad \dfrac{D \bigvee F \prec G_1 \bigvee F \prec G_2}{D \bigvee F \prec G_1 \vee G_2}\,(\prec \vee)$$

$$\dfrac{D \bigvee G_1 \leq G_2 \bigvee F \leq G_2}{D \bigvee F \leq G_1 \to G_2}\,(\leq\to) \qquad \dfrac{D \bigvee F_2 < F_1 \qquad D \bigvee F_2 < G}{D \bigvee F_1 \to F_2 < G}\,(\to<)$$

$$\dfrac{D \bigvee \top \leq G \bigvee F_2 < F_1 \qquad D \bigvee \top \leq G \bigvee F_2 \leq G}{D \bigvee F_1 \to F_2 \leq G}\,(\to\leq)$$

$$\dfrac{D \bigvee F < G_2 \bigvee G_1 \leq G_2 \qquad D \bigvee F < G_2 \bigvee F < \top}{D \bigvee F < G_1 \to G_2}\,(<\to)$$

The following theorem immediately follows from the corresponding results in this chapter:

$$\dfrac{\dfrac{p \leq p \bigvee q \leq p \qquad p \leq p \bigvee q \leq q}{p \leq p \bigvee q \leq p \wedge q}\,(\leq \wedge) \qquad \dfrac{p \leq q \bigvee p \leq p \qquad p \leq q \bigvee q \leq q}{p \leq q \bigvee q \leq p \wedge q}\,(\leq \wedge)}{p \leq p \wedge q \bigvee q \leq p \wedge q}\,(\leq \wedge)$$

Figure 5.4: Proof of $p \leq p \wedge q \bigvee q \leq p \wedge q$ in $\mathbf{DS}^{\mathrm{G}}$

$$\dfrac{\dfrac{\dfrac{D \bigvee \top \leq G \bigvee F_2 < F_1 \qquad D \bigvee \top \leq G \bigvee F_2 \leq G}{D \bigvee \top \leq G \bigvee (F_2 < F_1, F_2 \leq G)}\ (\to\leq)_2}{D \bigvee \top \leq G \bigvee F_1 \to F_2 \leq G}\ (\to\leq)_1}{D \bigvee F_1 \to F_2 \leq G}\ (C)$$

Figure 5.5: How to derive the rule $(\to\leq)$

$$\dfrac{\dfrac{\top \leq p \bigvee p \leq \bot \bigvee \top \leq \bot}{\top \leq p \bigvee p \to \bot}\ (\leq\to)}{\top \leq p \vee (p \to \bot)}\ (\leq \vee)$$

Figure 5.6: A failed proof of $A \vee \neg A$ in $\mathbf{ODS}^{\mathrm{G}}$.

Theorem 5.6.7: Completeness of $\mathbf{DS}^{\mathrm{G}}$ and $\mathbf{ODS}^{\mathrm{G}}$

Let F be a formula and ρ a regulation. The following are equivalent:

1. F is valid in Gödel logic
2. *I* have a winning strategy in $\mathbf{DG}^{\mathrm{G},\rho}(\top \leq F)$.
3. There is a proof of $\top \leq F$ in $\mathbf{DS}^{\mathrm{G}}$.
4. There is a proof of $\top \leq F$ in $\mathbf{ODS}^{\mathrm{G}}$

5.7 Conclusion and Future Work

In this chapter, we gave a general framework for lifting semantic games to provability games and further to analytic calculi. We demonstrated that this framework indeed covers the cases of hybrid logic and choice logic. Furthermore, we applied this framework to another semantic game, the truth-degree-comparison game for Gödel logic.

For infinitely branching games, the resulting calculus is infinitely branching, too. However, we have seen that in practice, infinite branching can be reduced: in Chapter 3, we showed that every infinite choice of *You* can be reduced to a single *optimal* choice. Hence, in *My* strategies *I* do not need to care about all of *Your* choices, and can consider *Your* optimal choice instead. Applying the idea to the resulting calculus leads to a version with finitely branching rules. It would be interesting to identify conditions for this finitization, or optimization, of *Your* choices.

We admit that our general framework may not be suitable for all present and future semantic games. In particular, some aspects of our definitions may be too general, while others may be too restrictive. As a candidate for the first, we mention a property that seems central in semantic games. Namely, that game states are essentially formulas of a logic, and the rules of the game are dictated by the shape of the formula. While it is remarkable that the lifting to provability games and analytic calculi does not make use of this strong structural property, it seems promising to investigate which results are obtainable if we manage to grasp this property in a general framework for semantic games.

On the other front of conditions that are a good candidate to be relaxed, we mention our requirement for all histories of the semantic game to be of finite height. In the semantic game in [20] for hybrid logic with fixed points, infinite runs are allowed and are not, per se, losing for *Me*. We believe that the disjunctive game can be straightforwardly adjusted to cover these infinite runs. The payoff of an outcome is then simply the maximal payoff of all of the terminal runs it contains (finite or infinite). We expect the resulting proofs to be easier than for the current framework where infinite runs of the disjunctive game are trivially won by *You* since we do not need to care about termination in *My* strategies. The presentation in the mentioned paper is mostly game-theoretic, too, and results in a cyclic proof system. However, we conjecture that it proceeds like the (generalized) lifting technique.

Another possible generalization is the number of players. Recently, semantic games with more than two players were recently introduced [4] for several non-classical logics. The different players represent payoff values in a non-linear domain. It would be interesting to see whether the lifting technique can be extended to cover these cases. In particular, this may suggest how to deal with the restriction of the linear domain of payoff values, as imposed by game theory.

APPENDIX A

Basic Notions from Mathematics

A.1 Orders

A pair $\mathcal{S} = (S, \preceq)$ where S is a set and $\preceq \subseteq S \times S$ is a binary relation is called a *preorder* if it satisfies for every $a, b, c \in S$:

1. Reflexivity: $a \preceq a$
2. Antisymmetry: if $a \preceq b$ and $b \preceq a$ then $a = b$
3. Transitivity: if $a \preceq b$ and $b \preceq c$, then $a \preceq c$

$\mathcal{S}$ is a *total order* if additionally, it satisfies for all $a, b \in S$;

4. Totality: $a \preceq b$ or $b \preceq a$.

A subset A of S is *bounded from above (below)* if there is an upper (lower) bound, i.e., some $b \in S$ such that $a \preceq b$, for all $a \in A$ ($b \preceq a$, for all $a \in A$). $\mathcal{S}$ is called complete if every subset A that is bounded from above has a least upper bound, $\sup A$. If $\mathcal{S}$ is complete, and A is bounded from below, then there is a greatest lower bound, $\inf A$, of A.

A.2 Sequences

Let S be a set. A *sequence over* S is a partial function h from $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of natural numbers, to S whose domain is downwards-closed with respect to the usual ordering on $\mathbb{N}$. The image of h is denoted $\mathrm{range}(h)$. Instead of $h(i)$, we prefer to write h_i, if i is in the domain of h. We write $\mathrm{length}(h)$ for the maximal number in the domain of h, and $\mathrm{last}(h)$ for $h_{\mathrm{length}(h)}$.

For two sequences h and k, we say that h is an *initial segment* of k and write $h \sqsubseteq k$ if $h_i = k_i$ for all i in the domain of h. If H is a set of sequences over S, then it is partially ordered by $\sqsubseteq$.

For two sequences h, k, where h is a finite and non-empty sequence, then their *concatenation* $h \smile k$, is the following sequence:

$$(h \smile k)_i = \begin{cases} h_i, & \text{if } i \leq \text{length}(h), \\ k_{i-\text{length}(h)-1}, & \text{if } i > \text{length}(h) \end{cases}$$

If $k = \langle s \rangle$ we write $h \smile s$ for $h \smile k$. We call $h \smile s$ an *immediate successor* of h.

A.3 Multisets

A multiset is a pair $M = (A, m)$, where A is a set, called the *support* of M, and m is a function from M to $\mathbb{N}^+$, the set of positive natural numbers. Intuitively, A states which elements occur in M and the function m determines how often each element occurs there.

We need the following operation on multisets: for $M_1 = (A_1, m_1)$ and $M_2 = (A_2, m_2)$ let the *sum* $M_1 + M_2$ be the multiset $(A_1 \cup A_2, m)$ with $m(x) = m_1(x) + m_2(x)$, for all $x \in A_1 \cup A_2$. By a slight abuse of notation, we often write $M_1 \cup M_2$ for $A_1 \cup A_2$.

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