Projective metric geometry and Clifford algebras

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Abstract

Let $\boldsymbol{V}$ be a finite dimensional vector space over any field F. Also, let $\boldsymbol{V}$ be endowed with a quadratic form Q. There is a widespread literature dealing with the representation of orthogonal transformations of $(\boldsymbol{V}, Q)$ in terms of the associated Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, Q)$. Thereby, several substructures of $\mathrm{Cl}(\boldsymbol{V}, Q)$ play a crucial role. These include the Lipschitz monoid $\mathrm{Lip}(\boldsymbol{V}, Q)$ and the Lipschitz group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$. See [1] for an extensive bibliography.

Our aim is to interpret these results in projective terms. For example, we consider the quotient of the Lipschitz group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ by the multiplicative group of F. This quotient group, say $\mathcal{G}(\boldsymbol{V}, Q)$, can be viewed as a set of points in the projective space on $\mathrm{Cl}(\boldsymbol{V}, Q)$. Furthermore, the group $\mathcal{G}(\boldsymbol{V}, Q)$ acts on the projective metric space $\mathbb{P}(\boldsymbol{V}, Q)$ as a group of motions.

However, the following issue crops up: If Q is replaced by a non-zero multiple, say cQ with $c \in F \setminus \{0\}$, then this does not affect the geometry of $\mathbb{P}(\boldsymbol{V}, Q)$. On the other hand, the Clifford algebras $\mathrm{Cl}(\boldsymbol{V}, Q)$ and $\mathrm{Cl}(\boldsymbol{V}, cQ)$ (and likewise the respective Lipschitz groups) need not be isomorphic. Still, by virtue of a particular linear bijection $\mathrm{Cl}(\boldsymbol{V}, Q) \to \mathrm{Cl}(\boldsymbol{V}, cQ)$, the problem can be resolved. The point sets $\mathcal{G}(\boldsymbol{V}, Q)$ and $\mathcal{G}(\boldsymbol{V}, cQ)$ turn out to be projectively equivalent. Furthermore, the groups $\mathcal{G}(\boldsymbol{V}, Q)$ and $\mathcal{G}(\boldsymbol{V}, cQ)$ are isomorphic and they act in an isomorphic way on $\mathbb{P}(\boldsymbol{V}, Q)$.

Keywords: projective metric space, Clifford algebra, Lipschitz monoid, Lipschitz group

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References

[1] H. Havlicek. Projective metric geometry and Clifford algebras, *Result. Math.*, **76**(4):Art. No. 219, 2021.