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FEM-BEM coupling in fractional diffusion

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We derive and analyze a fully computable discrete scheme for fractional partial differential equations posed on the full space $\mathbb{R}^d$. Based on a reformulation using the well-known Caffarelli–Silvestre extension, we study a modified variational formulation to obtain well-posedness. Our scheme is obtained by combining a diagonalization procedure with a reformulation using boundary integral equations and a coupling of finite elements and boundary elements. For our discrete method we present *a-priori* estimates as well as numerical examples.

1. Introduction

In this work, we study stationary fractional partial differential equations posed on the full space $\mathbb{R}^d$ with $d = 2, 3$ of the form

$$\mathcal{L}^\beta u + su = f \qquad \text{in } \mathbb{R}^d, \qquad \mathcal{L}u := -\operatorname{div}\left(\mathfrak{A}\nabla u\right) \tag{1.1}$$

with $s \geq 0$, and $\beta \in (0, 1)$. Fractional PDEs of this type are oftentimes used to model nonlocal effects in physics, finance or image processing, Bucur & Valdinoci (2016); Sun *et al.* (2018).

Regarding the formal definition of noninteger powers $\mathcal{L}^\beta$ of differential operators, there are various different descriptions in literature such as Fourier transformation, semigroup approaches, singular integrals or spectral calculus, see Lischke *et al.* (2020). A distinct advantage of full-space formulations as in (1.1) is that all of these definitions are equivalent, Kwaśnicki (2017), while there are significant differences in the definitions, if one restricts the problem to a bounded domain.

Nonetheless, there are usually no closed form solutions to these problems available and therefore numerical approximations are used. In order to derive a computable approximation, most numerical methods employ formulations on bounded domains, for which there is a fairly well developed literature. We mention the surveys (Bonito *et al.*, 2018; Lischke *et al.*, 2020) as well as finite element methods for the integral definition of the fractional Laplacian (Acosta & Borthagaray, 2017; Acosta *et al.*, 2019; Faustmann *et al.*, 2022a), for the spectral definition (Nochetto *et al.*, 2015, 2016) and semigroup approaches (Bonito & Pasciak, 2015; Bonito *et al.*, 2019). We especially mention the very influential reformulation using the extension approach by Caffarelli & Silvestre (2007) (see also Stinga & Torrea, 2010, for a more general setting), which allows to use PDE techniques in the analysis. This approach paired with an *hp*-FEM approach in the extended direction has proven to be an effective strategy both for elliptic (Meidner *et al.*, 2018; Banjai *et al.*, 2019, 2023; Faustmann *et al.*, 2022b, 2023) as well as parabolic (Nochetto *et al.*, 2016; Melenk & Rieder, 2021) and hyperbolic problems (Banjai & Otárola, 2019).

Many numerical approaches for the full-space formulation, like Achleitner *et al.* (2021) for the fractional Allen–Cahn equation, rely on truncation of the full-space problem to a bounded domain, which induces an additional truncation error that needs to be investigated. A different approach that avoids any truncation errors is the use of a coupling of finite elements on a truncated domain and boundary elements appearing from a reformulation of the unbounded exterior part as a boundary integral equation. We refer to the classical works (Johnson & Nédélec, 1980; Costabel, 1988; Han, 1990) for the one-equation/Johnson–Nédélec coupling and the symmetric coupling for elliptic transmission problems. For the standard Laplacian these methods are well-posed and thoroughly analyzed, see Sayas (2009); Steinbach (2011); Aurada *et al.* (2013).

In this work, we introduce a method for elliptic full-space fractional operators that combines the mentioned Caffarelli–Silvestre extension approach with FEM-BEM coupling techniques. More precisely, inspired by Laliena & Sayas (2009); Sayas (2009), we reformulate the extension problem as a variational problem, where the solution on a bounded domain and an exterior solution in an exotic Hilbert space are sought. Using suitable Poincaré inequalities, we show well-posedness of the continuous formulation. In order to obtain a computable approximation, we then use the diagonalization procedure of Banjai *et al.* (2019), which leads to a sequence of Helmholtz-type transmission problems. For those, we employ a standard coupling of FEM and BEM of symmetric type, as proposed by Costabel (1988); Han (1990). Finally, we present an *a priori* analysis for a discretization with *hp*-FEM in the extended variable. To our knowledge, this work is the first paper that considers a FEM-BEM discretization for fractional PDEs posed on the full-space.

We note that our previous and recent work (Faustmann & Rieder, 2023) considers the same continuous model problem and provides some essential analytical results for the full-space Caffarelli–Silvestre extension problem, such as well-posedness in an appropriate weighted Sobolev space as well as decay estimates and regularity results for the analytical solution. While the understanding of these properties of the full-space solutions are crucial, Faustmann & Rieder (2023) does not provide any discretization schemes or FEM-BEM formulations. Apart from well-posedness, the derivation of the FEM-BEM formulation in the present paper does not need any results from Faustmann & Rieder (2023), regularity and decay estimates, however, are essential to derive error estimates between the analytical and discrete solutions.

1.1 *Layout*

The present paper is structured as follows: In the remainder of Section 1, we introduce our model problem as well as necessary notation and most notably, the Caffarelli–Silvestre extension problem. In Section 2, we formulate our main results: well-posedness of our formulation, the fully-discrete scheme using the diagonalization procedure together with the symmetric FEM-BEM coupling, and, finally, a best-approximation result. Section 3 provides the proofs for the well-posedness and the diagonalization procedure and, most notably, a Poincaré type estimate. Section 4 contains the proofs for the *a-priori* analysis of the fully discrete formulation using *hp*-finite elements in the extended variable, which builds upon the regularity and decay properties of Faustmann & Rieder (2023). Finally, Section 5 presents some numerical examples that validate the proposed method.

1.2 *Notations*

Throughout the text we use the symbol $a \lesssim b$ meaning that $a \leq Cb$ with a generic constant $C > 0$ that is independent of any crucial quantities in the analysis. Moreover, we write $\simeq$ to indicate that both estimates $\lesssim$ and $\gtrsim$ hold.

We employ classical integer order Sobolev spaces $H^k(\Omega)$ on (bounded) Lipschitz domains Ω and the fractional Sobolev spaces $H^t(\mathbb{R}^d)$ for $t \in \mathbb{R}$ defined, e.g., via Fourier transformation. We also need Sobolev spaces on the boundary $\Gamma := \partial\Omega$ of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, denoted by $H^t(\Gamma)$

with $t \in [-1, 1]$. One way to properly define them is by using local charts, see Sauter & Schwab (2011) for details.

1.3 *Assumptions on the model problem*

Let $d = 2, 3$ and $\beta \in (0, 1)$. We consider the fractional PDE

$$\mathcal{L}^\beta u + su = f \qquad \text{in } \mathbb{R}^d, \qquad \mathcal{L}u := -\operatorname{div}\left(\mathfrak{A}\nabla u\right).$$

For the data in the model problem, we assume:

1. $\mathfrak{A} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is pointwise symmetric and positive definite in the sense that there exists $\mathfrak{A}_0 > 0$ such that

$$(\mathfrak{A}(x)y, y)_2 \geq \mathfrak{A}_0 \, \|y\|_2^2 \qquad \forall y \in \mathbb{R}^d.$$

2. $s \geq 0$ and additionally $s \geq \sigma_0 > 0$ for the case $d = 2$ to avoid several technical difficulties due to decay conditions at infinity, most notably, the lack of the Poincaré-type estimate of Lemma 3.2 below.
3. $f \in L^2(\mathbb{R}^d)$.
4. f is supported in a bounded domain and $\mathfrak{A}$ is constant outside of a bounded domain. We formulate these two requirements as follows: there exists a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ such that
   a. $\operatorname{supp} f \subseteq \Omega$,
   b. $\mathfrak{A} \equiv I$ in $\mathbb{R}^d \setminus \overline{\Omega}$.

We note that the last requirement is necessary in order to be able to apply boundary element techniques to the full space problem, as a reformulation of a PDE as a boundary integral equation requires the existence of a fundamental solution, which is only guaranteed in the constant coefficient case.

There are multiple ways to define the fractional power $\mathcal{L}^\beta$, which on the full space $\mathbb{R}^d$ turn out to be equivalent, Kwaśnicki (2017). A convenient definition, Lischke *et al.* (2020), for sufficiently smooth $u \in L^2(\mathbb{R}^d)$, by using spectral calculus reads as

$$\mathcal{L}^\beta u := \int_{\sigma(\mathcal{L})} z^\beta \, \mathrm{d}E \, u,$$

where E is the spectral measure of $\mathcal{L}$ and $\sigma(\mathcal{L})$ is the spectrum of $\mathcal{L}$. Using standard techniques this definition can be extended to tempered distributions. We note that the spectrum of the self-adjoint operator $\mathcal{L}$ may be continuous on the full-space.

In the following, we will never use the explicit definition of $\mathcal{L}^\beta$, but only employ a reformulation of our model problem specified in the following subsection.

We note that, compared to the integral fractional Laplacian defined on bounded domains, we *do not* impose an homogeneous exterior Dirichlet boundary condition as we are considering a true full-space formulation.

1.4 *Degenerate elliptic extension*

We use a reformulation of the fractional PDE as the Dirichlet-to-Neumann mapping for a degenerate elliptic PDE in a half space in $\mathbb{R}^{d+1}$, the so-called Caffarelli–Silvestre extension, Caffarelli & Silvestre (2007); Stinga & Torrea (2010).

We specify the function space used for the extension problem in the following. For any bounded open subset $D \subset \mathbb{R}^d \times \mathbb{R}^+$, we define $L^2(y^\alpha, D)$ as the space of square integrable functions with respect to the weight y^α and the Sobolev space $H^1(y^\alpha, D) \subset L^2(y^\alpha, D)$ of functions with finite norm

$$\|\mathcal{U}\|^2_{H^1(y^\alpha,D)} := \int\int_D y^\alpha \left(\left|\nabla\mathcal{U}(x,y)\right|^2 + \left|\mathcal{U}(x,y)\right|^2\right) \mathrm{d}x\,\mathrm{d}y.$$

We also employ the spaces $L^2(y^\alpha, (0,\mathcal{Y}))$ and $H^1(y^\alpha, (0,\mathcal{Y}))$ for $\mathcal{Y} \in (0,\infty]$ defined in an analogous way by omitting the x-integration.

For *unbounded* sets D, we additionally use the weight

$$\rho(x,y) := (1 + |x|^2 + |y|^2)^{1/2} \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^+$$

to take care of the behaviour at infinity, see also Remark 2.1 below. In this case, we define the space $H^1_\rho(y^\alpha, D)$ as the space of all square integrable functions $\mathcal{U}$ (with respect to the weight function $y^\alpha\rho^{-2}$) such that the norm

$$\|\mathcal{U}\|^2_{H^1_\rho(y^\alpha,D)} := \int\int_D y^\alpha \left(\left|\nabla\mathcal{U}(x,y)\right|^2 + \rho(x,y)^{-2}\left|\mathcal{U}(x,y)\right|^2\right) \mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

is finite. Commonly used cases are $D = \mathbb{R}^d \times \mathbb{R}^+$ (full space), $D = \mathbb{R}^d \times (0,\mathcal{Y})$ for $\mathcal{Y} > 0$ (corresponding to truncation in y-direction), or $D = \omega \times (0,\mathcal{Y})$ for $\omega \subset \mathbb{R}^d$ and $\mathcal{Y} > 0$.

Moreover, we also employ spaces acting only in x. Using the weight

$$\rho_x(x) := \rho(x,0),$$

we introduce $L^2_{\rho_x}(\mathbb{R}^d)$ and $H^1_{\rho_x}(\mathbb{R}^d)$ as in (1.2) by omitting the y-integration.

For functions $\mathcal{U}$ in $H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$, one can give meaning to their trace at $y = 0$, which we denote by $\mathrm{tr}_0\mathcal{U}$. In fact, by general theory for interpolation of trace spaces, see (Bergh & Löfström, 1976, Sec. 6.6), or in particular (Karkulik & Melenk, 2019, Lemma 3.8) and (Faustmann & Rieder, 2023, Lem. 3.1), we have the trace estimates

$$\begin{aligned} \left|\mathrm{tr}_0\mathcal{U}\right|_{H^\beta(\mathbb{R}^d)} &\lesssim \|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathbb{R}^d\times\mathbb{R}^+)} \\ \left\|(1+|x|^2)^{-\beta/2}\mathrm{tr}_0\mathcal{U}\right\|_{L^2(\mathbb{R}^d)} &\lesssim \|\nabla\mathcal{U}\|_{L^2(y^\alpha,\mathbb{R}^d\times\mathbb{R}^+)} \qquad \text{if } d = 3. \end{aligned} \tag{1.3}$$

Here, $|\cdot|_{H^\beta(\mathbb{R}^d)}$ denotes the Aronstein–Slobodeckij seminorm for $\beta \in (0,1)$.

Then, the extension problem reads as: find $\mathcal{U} \in H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$ such that

$$-\operatorname{div}\left(y^\alpha \mathfrak{A}_x \nabla\mathcal{U}\right) = 0 \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \tag{1.4a}$$

$$d_\beta^{-1}\partial_{\nu^\alpha}\mathcal{U} + s\mathrm{tr}_0\mathcal{U} = f \qquad \text{in } \mathbb{R}^d, \tag{1.4b}$$

where $d_\beta := 2^{1-2\beta}\Gamma(1-\beta)/\Gamma(\beta)$, $\alpha := 1-2\beta \in (-1,1)$, $\partial_{\nu^\alpha}\mathcal{U}(x) := -\lim_{y\to 0} y^\alpha\partial_y\mathcal{U}(x,y)$ and $\mathfrak{A}_x = \begin{pmatrix}\mathfrak{A} & 0 \\ 0 & 1\end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}$. By Stinga & Torrea (2010), the solution to (1.1) is then given by $u = \mathrm{tr}_0\mathcal{U}$.

We note that the bounded domain Ω only enters the problem (1.4) via the coefficients $\mathfrak{A}_x$ and f, i.e., we know that $\mathfrak{A}_x \equiv I \in \mathbb{R}^{(d+1)\times(d+1)}$ on $\Omega^c \times \mathbb{R}^+$ and the boundary condition (1.4b) is homogeneous outside of Ω.

For the domain Ω with boundary $\Gamma := \partial\Omega$, we also introduce the usual trace operators γ_Γ^- (denoting the trace coming from the interior of Ω) and γ_Γ^+ (denoting the trace coming from $\mathbb{R}^d\backslash\overline{\Omega}$) and correspondingly the normal derivative operators $\partial_{\nu,\Gamma}^{\pm}$ (see Sauter & Schwab, 2011, for details). The normal vector ν is always assumed to face out of Ω. With theses operators, the jumps across Γ are defined as

$$[\![\gamma u]\!] = \gamma_\Gamma^+ u - \gamma_\Gamma^- u, \qquad [\![\partial_\nu u]\!] = \partial_{\nu,\Gamma}^+ u - \partial_{\nu,\Gamma}^- u. \tag{1.5}$$

We will apply these operators for functions in $H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times \mathbb{R}^+)$, where they are to be understood pointwise with respect to y. This is equivalent to taking the trace and normal derivative along the lateral boundary $\Gamma \times \mathbb{R}^+$.

2. Main results

Our main results are formulated for weak solutions of the degenerate PDE (1.4) under the assumptions of Section 1.3. Taking traces in the extended variable using (1.3) directly gives the corresponding results for solutions to the fractional PDE in the full-space.

2.1 *Variational formulation*

The weak formulation of (1.4) in $H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$ reads as finding $\mathcal{U} \in H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$ such that

$$A(\mathcal{U}, \mathcal{V}) := \int_0^\infty y^\alpha \int_{\mathbb{R}^d} \mathfrak{A}_x(x)\nabla\mathcal{U} \cdot \nabla\mathcal{V}\, \mathrm{d}x\, \mathrm{d}y + s d_\beta \int_{\mathbb{R}^d} \mathrm{tr}_0\mathcal{U}\mathrm{tr}_0\mathcal{V}\, \mathrm{d}x = \mathrm{d}_\beta(f, \mathrm{tr}_0\mathcal{V})_{L^2(\mathbb{R}^d)} \tag{2.1}$$

for all $\mathcal{V} \in H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$.

Remark 2.1 Using the weight ρ in the space $H^1_\rho(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$ in (2.1) is essential for well-posedness. Solutions $\mathcal{U}$ to (2.1) are in $L^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^+)$; however, they are not necessarily in $L^2(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$, thus solvability in $H^1(y^\alpha, \mathbb{R}^d \times \mathbb{R}^+)$ is not expected to hold (compare Amrouche *et al.*, 1994, for the full-space Laplacian).

Unique solvability of the continuous formulation in spaces involving the weight ρ follows from Faustmann & Rieder (2023, Prop 2.3). We also mention that, on bounded domains $D \subset \mathbb{R}^d \times \mathbb{R}$, the spaces $H^1_\rho(y^\alpha, D)$ and $H^1(y^\alpha, D)$ coincide.

In order to obtain a computable formulation, we will be cutting the problem from the infinite cylinder $\mathbb{R}^d \times \mathbb{R}^+$ to a finite cylinder in the y-direction $\mathbb{R}^d \times (0, \mathcal{Y})$ with a fixed parameter $\mathcal{Y} > 0$ to be chosen later.

Since we want to include our discretization scheme for both (2.1) and the truncated problem, we work in a slightly expanded variational form, inspired by Laliena & Sayas (2009); Sayas (2009). In short, one can formulate an equivalent problem for the solution inside Ω and a function $\mathcal{U}_\star$ on $\mathbb{R}^d$ defined in a modified Hilbert space. Both functions are matched on the interface Γ by appropriate jump conditions.

Definition 2.2 Fix $\mathcal{Y} \in (0, \infty]$. We consider the space

$$\mathbb{H}_\mathcal{Y} := \Big\{(\mathcal{U}_\Omega, \mathcal{U}_\star) \in H^1(y^\alpha, \Omega \times (0, \mathcal{Y})) \times H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0, \mathcal{Y})) : \\ [\![\gamma\mathcal{U}_\star]\!] = \gamma_\Gamma^- \mathcal{U}_\Omega,\ \gamma_\Gamma^- \mathcal{U}_\star = 0,\ s\,\mathrm{tr}_0\mathcal{U}_\star \in L^2(\mathbb{R}^d)\Big\}$$

equipped with the norm

$$\|\mathcal{U}\|^2_{\mathbb{H}_{\mathcal{Y}}} := \|(\mathcal{U}_\Omega, \mathcal{U}_\star)\|^2_{\mathbb{H}_{\mathcal{Y}}}$$
$$:= \|\mathcal{U}_\Omega\|^2_{H^1(y^\alpha, \Omega\times(0,\mathcal{Y}))} + \|\mathcal{U}_\star\|^2_{H^1_\rho(y^\alpha, \mathbb{R}^d\setminus\Gamma\times(0,\mathcal{Y}))} + s\|\mathrm{tr}_0\mathcal{U}_\Omega\|^2_{L^2(\Omega)} + s\|\mathrm{tr}_0\mathcal{U}_\star\|^2_{L^2(\mathbb{R}^d)}.$$

We note that, by definition, the additional condition of $\mathrm{tr}_0\mathcal{U}_\star$ being in $L^2(\mathbb{R}^d)$ is only needed for $s \neq 0$ as in this case the norm in $\mathbb{H}_{\mathcal{Y}}$ contains said L^2-term, which has to be finite. The conditions on the traces of $\mathcal{U}_\star$ and $\mathcal{U}_\Omega$ also imply $\gamma_\Gamma^+\mathcal{U}_\star = \gamma_\Gamma^-\mathcal{U}_\Omega$.

With $\mathcal{U} = (\mathcal{U}_\Omega, \mathcal{U}_\star) \in \mathbb{H}_\infty$ and $\mathcal{V} = (\mathcal{V}_\Omega, \mathcal{V}_\star) \in \mathbb{H}_\infty$, we define the bilinear form $B : \mathbb{H}_\infty \times \mathbb{H}_\infty \to \mathbb{R}$ as

$$B(\mathcal{U}, \mathcal{V}) := \int_0^\infty \int_\Omega y^\alpha \mathfrak{A}_x(x)\nabla\mathcal{U}_\Omega \cdot \nabla\mathcal{V}_\Omega \,\mathrm{d}x\,\mathrm{d}y + \int_0^\infty \int_{\mathbb{R}^d} y^\alpha \nabla\mathcal{U}_\star \cdot \nabla\mathcal{V}_\star \,\mathrm{d}x\,\mathrm{d}y$$
$$+ sd_\beta \int_\Omega \mathrm{tr}_0\mathcal{U}_\Omega \mathrm{tr}_0\mathcal{V}_\Omega \,\mathrm{d}x + s\,\mathrm{d}_\beta \int_{\mathbb{R}^d} \mathrm{tr}_0\mathcal{U}_\star \mathrm{tr}_0\mathcal{V}_\star \,\mathrm{d}x. \tag{2.2}$$

For $f \in L^2(\Omega)$, the weak formulation is given as the problem of finding $\mathcal{U} \in \mathbb{H}_\infty$ such that

$$B(\mathcal{U}, \mathcal{V}) = \mathrm{d}_\beta \int_\Omega f\,\mathrm{tr}_0\mathcal{V}_\Omega \,\mathrm{d}x \qquad \forall\, \mathcal{V} = (\mathcal{V}_\Omega, \mathcal{V}_\star) \in \mathbb{H}_\infty. \tag{2.3}$$

Problems (1.4) and (2.3) are connected as follows: if $\mathcal{U}_\infty = (\mathcal{U}_\Omega, \mathcal{U}_\star) \in \mathbb{H}_\infty$ solves (2.3), then the function $\mathcal{U} := \begin{cases} \mathcal{U}_\Omega, & \text{in } \Omega \\ \mathcal{U}_\star, & \text{in } \mathbb{R}^d\setminus\overline{\Omega} \end{cases}$ solves (2.1). We note that by unique solvability of the Caffarelli–Silvestre extension problem on bounded domains, we obtain that $\mathcal{U}_\star = 0$ in $\Omega \times \mathbb{R}^+$, as it solves the homogeneous extension problem with homogeneous (Dirichlet on $\Gamma \times \mathbb{R}^+$ and Neumann on $\mathbb{R}^d \times \{0\}$) boundary conditions.

Now, cutting the integration in y at $\mathcal{Y} > 0$, we introduce the truncated bilinear forms

$$A^{\mathcal{Y}}_\Omega(\mathcal{U}, \mathcal{V}) := \int_0^{\mathcal{Y}} y^\alpha \int_\Omega \mathfrak{A}_x(x)\nabla\mathcal{U} \cdot \nabla\mathcal{V} \,\mathrm{d}x\,\mathrm{d}y + sd_\beta \int_\Omega \mathrm{tr}_0\mathcal{U}\mathrm{tr}_0\mathcal{V}\,\mathrm{d}x,$$
$$A^{\mathcal{Y}}_{\mathbb{R}^d\setminus\Gamma}(\mathcal{U}, \mathcal{V}) := \int_0^{\mathcal{Y}} y^\alpha \int_{\mathbb{R}^d\setminus\Gamma} \nabla\mathcal{U} \cdot \nabla\mathcal{V} \,\mathrm{d}x\,\mathrm{d}y + sd_\beta \int_{\mathbb{R}^d\setminus\Gamma} \mathrm{tr}_0\mathcal{U}\mathrm{tr}_0\mathcal{V}\,\mathrm{d}x.$$

The ‘big’ bilinear form is then given by

$$B^{\mathcal{Y}}\big((\mathcal{U}_\Omega, \mathcal{U}_\star), (\mathcal{V}_\Omega, \mathcal{V}_\star)\big) := A^{\mathcal{Y}}_\Omega(\mathcal{U}_\Omega, \mathcal{V}_\Omega) + A^{\mathcal{Y}}_{\mathbb{R}^d\setminus\Gamma}(\mathcal{U}_\star, \mathcal{V}_\star),$$

and the cutoff problem reads as: find $\mathcal{U}^{\mathcal{Y}} = (\mathcal{U}^{\mathcal{Y}}_\Omega, \mathcal{U}^{\mathcal{Y}}_\star) \in \mathbb{H}_{\mathcal{Y}}$ such that

$$B^{\mathcal{Y}}\big(\mathcal{U}^{\mathcal{Y}}, \mathcal{V}^{\mathcal{Y}}\big) = d_\beta\big(f, \mathrm{tr}_0\mathcal{V}^{\mathcal{Y}}_\Omega\big)_{L^2(\mathbb{R}^d)} \quad \text{for all } \mathcal{V}^{\mathcal{Y}} = (\mathcal{V}^{\mathcal{Y}}_\Omega, \mathcal{V}^{\mathcal{Y}}_\star) \in \mathbb{H}_{\mathcal{Y}}. \tag{2.4}$$

By the following theorem, we obtain well-posedness of the weak formulation of both variational formulations.

THEOREM 2.3 Assume either $d = 3$ or $s > 0$. Then, problem (2.3) has a unique solution $\mathcal{U} \in \mathbb{H}_\infty$, satisfying

$$\|\mathcal{U}\|_{\mathbb{H}_\infty} \le C \min(1, s^{-1})\, \|f\|_{L^2(\Omega)}.$$

Fix $\mathcal{Y} \in (0,\infty)$. Then, the truncated problem (2.4) has a unique solution $\mathcal{U}^{\mathcal{Y}} \in \mathbb{H}_{\mathcal{Y}}$, for which the estimate

$$\left\|\mathcal{U}^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}} \leq C\left(1+\frac{1}{\mathcal{Y}}\right)\min(1,s^{-1})\,\|f\|_{L^2(\Omega)}$$

holds. Additionally, the bilinear forms in (2.3) and (2.4) are coercive.

The proof of the theorem is given in Section 3 and essentially reduces to the application of suitable Poincaré inequalities.

Note that the assumption $s>0$ for $d=2$ is necessary to avoid a critical exponent in a Hardy inequality or in other words, the employed weights are not sufficient to capture the correct behavior at infinity, compare Sayas (2009), where an additional logarithmic weight for the case of the Laplacian was needed.

2.2 *The discrete scheme*

In this section, we describe our discrete scheme to approximate solutions to the truncated variational formulation (2.4). The main idea is to employ a tensor product structure for the approximation by using the diagonalization procedure described in Banjai *et al.* (2019), which leads to a sequence of modified Helmholtz problems. As these are still posed on the full-space, boundary integral formulations are used.

Let $\mathbb{V}_h^y$ be an arbitrary finite dimensional subspace of $L^2(y^\alpha,(0,\mathcal{Y}))$ of dimension N_y+1. Following the ideas of Banjai *et al.* (2019), we chose an orthonormal basis $(\varphi_j)_{j=0}^{N_y}$ of $\mathbb{V}_h^y$ in $L^2(y^\alpha,(0,\mathcal{Y}))$ and generalized eigenvalues $\mu_j \geq 0$, satisfying

$$\int_0^{\mathcal{Y}} y^\alpha \varphi_i'\varphi_j'\,\mathrm{d}y + s\varphi_i(0)\varphi_j(0) = \mu_j\int_0^{\mathcal{Y}} y^\alpha\varphi_i\varphi_j\,\mathrm{d}y = \mu_j\delta_{ij} \qquad \forall\, 0\leq i,j\leq N_y. \tag{2.5}$$

It is easy to see that, for $s=0$, the constant function is an eigenfunction corresponding to the eigenvalue $\mu=0$. Moreover, the assumption $s>0$ for $d=2$ guarantees that there is not a zero eigenvalue. If zero is an eigenvalue (for $d=3$), we assume that the eigenvalues are ordered such that $\mu_0=0$.

We now give a formal definition of the (semi-)discrete space used for the discrete formulation, which has tensor product structure with respect to the variables x, y.

Definition 2.4 Let $\mathbb{V}_h^x \subset H^1(\Omega)$ and $\mathbb{V}_h^\lambda \subset H^{-1/2}(\Gamma)$ be finite dimensional spaces and $\mathcal{Y}\in(0,\infty)$. Additionally, assume that $1 \in \mathbb{V}_h^\lambda$. We introduce the closed subspace $\mathbb{H}_{h,\mathcal{Y}} \subset H^1(y^\alpha,\Omega\times(0,\mathcal{Y})) \times H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\mathcal{Y}))$ as

$$\mathbb{H}_{h,\mathcal{Y}} := \operatorname{cls}\Bigg\{\mathcal{U}_h=(\mathcal{U}_\Omega,\mathcal{U}_\star):\quad \mathcal{U}_\Omega(x,y)=\sum_{j=0}^{N_y}u_{j,\Omega}(x)\varphi_j(y) \text{ with } u_{j,\Omega}\in\mathbb{V}_h^x,$$

$$\mathcal{U}_\star(x,y)=\sum_{j=0}^{N_y}u_{j,\star}(x)\varphi_j(y) \text{ with } u_{j,\star}\in H^1_{\rho_x}(\mathbb{R}^d\setminus\Gamma), \tag{2.6}$$

$$[\![\gamma u_{j,\star}]\!]=\gamma_\Gamma^- u_j,\quad \gamma_\Gamma^- u_{j,\star}\in\left(\mathbb{V}_h^\lambda\right)^\circ,\quad s\,u_{j,\star}\in L^2(\mathbb{R}^d)\Bigg\}.$$

Remark 2.5 We note that the spaces $\mathbb{H}_{\mathcal{Y}}$ and $\mathbb{H}_{h,\mathcal{Y}}$ for the continuous and semidiscrete problem are not nested, as, in contrast to the definition of the space $\mathbb{H}_{\mathcal{Y}}$, the boundary condition $\gamma_\Gamma^- u_{j,\star}\in(\mathbb{V}_h^\lambda)^\circ$ appears only in a weak sense in $\mathbb{H}_{h,\mathcal{Y}}$. This is identical to the case of the integer order Laplacian, Laliena & Sayas (2009), and induces consistency error terms in the *a-priori* analysis, see Lemma 4.7 below.

The semidiscrete problem now reads as: find $\mathcal{U}_h^{\mathcal{Y}} = (\mathcal{U}_\Omega^{\mathcal{Y}}, \mathcal{U}_\star^{\mathcal{Y}}) \in \mathbb{H}_{h,\mathcal{Y}}$ such that

$$B^{\mathcal{Y}}\left(\mathcal{U}_h^{\mathcal{Y}}, \mathcal{V}_h^{\mathcal{Y}}\right) = d_\beta\left(f, \operatorname{tr}_0 \mathcal{V}_\Omega^{\mathcal{Y}}\right)_{L^2(\mathbb{R}^d)} \quad \text{for all } \mathcal{V}_h^{\mathcal{Y}} = \left(\mathcal{V}_\Omega^{\mathcal{Y}}, \mathcal{V}_\star^{\mathcal{Y}}\right) \in \mathbb{H}_{h,\mathcal{Y}}. \tag{2.7}$$

Using the orthogonal basis for the y-direction, we can actually diagonalize some of the bilinear forms and obtain an equivalent sequence of scalar problems. In fact, functions $(\mathcal{U}_\Omega, \mathcal{U}_\star) \in \mathbb{H}_{h,\mathcal{Y}}$ solve (2.7), if and only if they can be written as

$$\mathcal{U}_\Omega(x,y) = \sum_{j=0}^{N_y} u_{j,\Omega}(x)\varphi_j(y), \qquad \mathcal{U}_\star(x,y) = \sum_{j=0}^{N_y} u_{j,\star}(x)\varphi_j(y), \tag{2.8}$$

with $u_{j,\Omega} \in \mathbb{V}_h^x$, $u_{j,\star} \in H^1_{\rho_x}(\mathbb{R}^d \setminus \Gamma)$, where the functions $u_{j,\Omega}$, $u_{j,\star}$ solve

$$\left(\mathfrak{A}\nabla u_{j,\Omega}, \nabla v\right)_{L^2(\Omega)} + \mu_j\left(u_{j,\Omega}, v\right)_{L^2(\Omega)} - \left\langle \partial_{\nu,\Gamma}^- u_{j,\star}, \gamma_\Gamma^- v\right\rangle_{L^2(\Gamma)} = \mathrm{d}_\beta \varphi_j(0)(f,v)_{L^2(\Omega)} \quad \forall v \in \mathbb{V}_h^x, \tag{2.9a}$$

and

$$-\Delta u_{j,\star} + \mu_j u_{j,\star} = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma \tag{2.9b}$$

$$[\![\gamma u_{j,\star}]\!] = \gamma_\Gamma^- u_j, \quad \gamma_\Gamma^- u_{j,\star} \in \left(\mathbb{V}_h^\lambda\right)^\circ. \tag{2.9c}$$

We refer to Lemma 3.3 for a proof of this statement.

The equation for $u_{j,\star}$ is still posed on an unbounded domain. We will replace this with boundary integral equations. Therefore, given $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) \geq 0$, we introduce the fundamental solutions

$$G(z;\mu) := \begin{cases} \frac{i}{4} H_0^{(1)}(i\mu\,|z|), & \text{for } d=2, \\ \frac{e^{-\mu|z|}}{4\pi\,|z|}, & \text{for } d=3, \end{cases} \quad \text{for } \mu \neq 0 \text{ and} \quad G(z;0) := \begin{cases} \frac{-1}{2\pi}\ln(|z|), & \text{for } d=2, \\ \frac{1}{4\pi\,|z|}, & \text{for } d=3, \end{cases}$$

where $H_0^{(1)}$ denotes the first kind Hankel function of order 0. The corresponding single-layer potential $\widetilde{V}(\mu) : H^{-1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and double-layer potential $\widetilde{K}(\mu) : H^{1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^d \backslash \Gamma)$ are then defined as

$$\left(\widetilde{V}(\mu)\varphi\right)(x) := \int_\Gamma G(x-z;\mu)\varphi(z)\,\mathrm{d}z, \qquad \left(\widetilde{K}(\mu)\psi\right)(x) := \int_\Gamma \partial_{\nu,\Gamma}^- G(x-z;\mu)\psi(z)\,\mathrm{d}z.$$

Taking traces at Γ produces the boundary integral operators

$$V(\mu) := \gamma_\Gamma^\pm \widetilde{V}(\mu), \quad K(\mu) := \frac{1}{2}\left(\gamma_\Gamma^+ \widetilde{K}(\mu) + \gamma_\Gamma^- \widetilde{K}(\mu)\right), \tag{2.10}$$

$$K'(\mu) := \frac{1}{2}\left(\partial_{\nu,\Gamma}^+ \widetilde{V}(\mu) + \partial_{\nu,\Gamma}^- \widetilde{V}(\mu)\right), \quad W(\mu) := -\partial_{\nu,\Gamma}^\pm \widetilde{K}(\mu). \tag{2.11}$$

Making the ansatz $u_{j,\star}(x) := \widetilde{K}(\mu_j)\gamma_\Gamma^- u_{j,\Omega}(x) - \widetilde{V}(\mu_j)\lambda_j(x)$ for the exterior solution leads to a symmetric FEM-BEM formulation, similarly to Costabel (1988); Han (1990). Thus, we have derived a computable approximation of (1.1) that only relies on well-known operators.

THEOREM 2.6 Let φ_j, μ_j be the generalized eigenfunctions and eigenvalues from (2.5). For all $j = 0, \dots, N_y$, let $(u_j, \lambda_j) \in \mathbb{V}_h^x \times \mathbb{V}_h^\lambda$ solve

$$\begin{aligned}\big(\mathfrak{A}\nabla u_j, \nabla v_h\big)_{L^2(\Omega)} + \mu_j\big(u_j, v_h\big)_{L^2(\Omega)} + \big\langle W(\mu_j)\gamma_\Gamma^- u_j + (-1/2 + K'(\mu_j))\lambda_j, \gamma_\Gamma^- v_h\big\rangle_{L^2(\Gamma)}& \\ = \mathrm{d}_\beta\varphi_j(0)\big(f, v_h\big)_{L^2(\Omega)},& \qquad (2.12a)\end{aligned}$$

$$\big\langle(1/2 - K(\mu_j))\gamma_\Gamma^- u_j, \xi_h\big\rangle_{L^2(\Gamma)} + \big\langle V(\mu_j)\lambda_j, \xi_h\big\rangle_{L^2(\Gamma)} = 0 \qquad (2.12b)$$

for all $v_h \in \mathbb{V}_h^x$ and $\xi_h \in \mathbb{V}_h^\lambda$. Then,

$$\mathcal{U}_\Omega(x,y) := \sum_{j=0}^{N_y} u_j(x)\varphi_j(y), \qquad \mathcal{U}_\star(x,y) := \sum_{j=0}^{N_y}\Big(\widetilde{V}(\mu_j)\lambda_j(x) - \widetilde{K}(\mu_j)\gamma_\Gamma^- u_j(x)\Big)\varphi_j(y)$$

solves (2.7). We thus have a computable representation of our discrete approximation.

The problems (2.12) are standard FEM-BEM coupling problems for what is often called the modified Helmholtz or Yukawa equation. As such, existence and uniqueness of solutions $(u_j, \lambda_j) \in \mathbb{V}_h^x \times \mathbb{V}_h^\lambda$ is well-known, see Laliena & Sayas (2009, Sect. 7). Consequently, we also obtain well-posedness of the semidiscrete formulation (2.7) as we have constructed a solution in $\mathbb{H}_{h,\mathcal{Y}}$. Uniqueness follows from coercivity of the bilinear form.

COROLLARY 2.7 Fix $\mathcal{Y} \in (0,\infty)$. Let $\mathbb{V}_h^x \subseteq H^1(\Omega)$, $\mathbb{V}_h^\lambda \subseteq H^{-1/2}(\Gamma)$, $\mathbb{V}_h^y \subseteq H^1(y^\alpha, (0,\mathcal{Y}))$ be finite dimensional subspaces. Assume that $1 \in \mathbb{V}_h^\lambda$, i.e., the space $\mathbb{V}_h^\lambda$ contains the constant functions, and either $d = 3$ or $s > 0$. Then, the truncated problem (2.7) has a unique solution $\mathcal{U}_h^{\mathcal{Y}} \in \mathbb{H}_{h,\mathcal{Y}}$.

REMARK 2.8 Due to the construction in Theorem 2.6, we mention that our discrete approximation can very easily be computed with the use of existing FEM/BEM libraries. We refer to Section 5 for a description of the implementation used in the numerical examples therein.

2.3 A-priori *convergence estimates*

In the extended variable y, we employ a *hp*-FEM discretization and choose the truncation parameter $\mathcal{Y}$ accordingly to obtain an algebraically convergent method with reasonable computational effort.

Let $\mathcal{Y} > 0$ and $\mathcal{T}_y$ be a geometric grid on $(0,\mathcal{Y})$ with mesh grading factor σ, L-refinement layers towards 0 and $M = \lfloor \ln(\mathcal{Y})/\ln(\sigma) \rfloor$ levels of growth towards $\mathcal{Y}$. More precisely, we define the grid points as

$$x_0 := 0, \qquad x_\ell := \sigma^{L-\ell} \text{ for } \ell = 0, \dots, L+M, \qquad x_{L+M+1} := \mathcal{Y}. \qquad (2.13)$$

By

$$\mathcal{S}^{p,1}(\mathcal{T}_y) := \Big\{u \in C(0,\mathcal{Y}) \;:\; u|_{(x_\ell, x_{\ell+1})} \in P_p \;\forall \ell = 0, \dots, L+M\Big\}$$

we denote the space of continuous, piecewise polynomials of degree up to p.

The following proposition provides a best-approximation estimate for the *hp*-semidiscretization in y including the cut-off error at $\mathcal{Y}$.

THEOREM 2.9 (Best-Approximation). Let $\mathcal{Y} \in (0,\infty)$ and $p \in \mathbb{N}$. Let $\mathcal{U}$ solve (2.3) and $\mathcal{U}^{\mathcal{Y}} = (\mathcal{U}^{\mathcal{Y}}_{\Omega}, \mathcal{U}^{\mathcal{Y}}_{\star})$ solve the cutoff problem (2.4). Set $\lambda := \partial_{\nu}^{+}\mathcal{U}^{\mathcal{Y}}_{\star}$. Let $\mathcal{T}_y$ be a geometric grid on $(0,\mathcal{Y})$ with $L = p$. Let $\mathcal{U}^{\mathcal{Y}}_h$ solve (2.7) with arbitrary finite dimensional subspaces $\mathbb{V}^x_h \subseteq H^1(\Omega)$, $\mathbb{V}^{\lambda}_h \subseteq H^{-1/2}(\Gamma)$ and the choice $\mathbb{V}^y_h := \mathcal{S}^{p,1}(\mathcal{T}_y)$. Let $\pi_{\Omega} : L^2(\Omega) \to \mathbb{V}^x_h$ be an arbitrary linear operator that is stable in $L^2(\Omega)$ and $H^1(\Omega)$. Then, for any $\lambda_h : \mathbb{R}_+ \to \mathbb{V}^{\lambda}_h$, there exist $\varepsilon > 0$, $\kappa > 0$ such that there holds

$$\left\|\mathcal{U} - \mathcal{U}^{\mathcal{Y}}_h\right\|^2_{\mathbb{H}_{\mathcal{Y}}} \lesssim \int_0^{\mathcal{Y}} y^{\alpha}\left(\left\|(I-\pi_{\Omega})\mathcal{U}^{\mathcal{Y}}_{\Omega}(y)\right\|^2_{H^1(\Omega)} + \|\lambda(y) - \lambda_h(y)\|^2_{H^{-1/2}(\Gamma)}\right)\mathrm{d}y + \mathcal{Y}^{2\varepsilon}e^{-2\kappa p} + \mathcal{Y}^{-\mu}\,\|f\|^2_{L^2(\mathbb{R}^d)}$$

with $\mu := \begin{cases} 1+|\alpha| & \text{for } s>0 \\ 1+\alpha & \text{for } s=0 \end{cases}$ and all constants independent of $\mathcal{Y}, p$.

REMARK 2.10 A possible choice for the spatial discretization is $\mathbb{V}^x_h := \mathcal{S}^{1,1}(\mathcal{T}_x)$, i.e., continuous, piecewise linear polynomials on some (quasi-uniform) mesh $\mathcal{T}_x$ of Ω. For the operator π_{Ω} one could take the Scott–Zhang projection mapping onto $\mathcal{S}^{1,1}(\mathcal{T}_x)$, see Scott & Zhang (1990). In addition to the required $L^2(\Omega)$- and $H^1(\Omega)$-stabilities, the operator has first order approximation properties in $H^1(\Omega)$, provided the input function is sufficiently regular.

Using first-order approximation properties of the Scott–Zhang projection together with best-approximation of the BEM part (which converges of order $h^{3/2}$ assuming sufficient regularity, see Sauter & Schwab, 2011) and correct choice of the cut-off parameter $\mathcal{Y}$ and polynomial degree p, the best-approximation estimate for the semidiscretization in Theorem 2.9 directly gives first order convergence in h.

COROLLARY 2.11 Let the assumptions of Theorem 2.9 hold. Assume $\mathfrak{A} \in C^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $f \in H^1(\Omega)$ and assume Ω has piecewise smooth boundary. Choose $\mathbb{V}^x_h := \mathcal{S}^{1,1}(\mathcal{T}_x)$ with a quasi-uniform mesh $\mathcal{T}_x$ of Ω of maximal mesh-width h and take π_{Ω} to be the Scott–Zhang projection. Let $\mathbb{V}^{\lambda}_h := S^{0,0}(\mathcal{T}_{\Gamma})$ be the space of piecewise constants on the trace mesh $\mathcal{T}_{\Gamma}$ of $\mathcal{T}_x$. Moreover, choose $p = -c_{\kappa,\mu,\varepsilon}\ln h$ with a sufficiently large constant $c_{\kappa,\mu,\varepsilon}$ depending only on κ, μ and ε, and $\mathcal{Y} \sim h^{-2/\mu}$. Then,

$$\|\mathcal{U} - \mathcal{U}^{\mathcal{Y}}_h\|_{\mathbb{H}_{\mathcal{Y}}} \leq Ch.$$

We note that the algebraic convergence of the cut-off error induces the condition $\mathcal{Y} \sim h^{-2/\mu}$. This is in fact the reason, why a *hp*-semidiscretization in y is employed. As *hp*-FEM converges exponentially with algebraic computational cost, we can recuperate any algebraic convergence rates of the discretization in x without destroying the overall complexity of the discrete scheme.

REMARK 2.12 The FEM-BEM coupling formulation (2.12) leads to a block system matrix $\begin{pmatrix} \mathbf{A}+\mathbf{W} & \mathbf{K}^T - \frac{1}{2}\mathbf{M}^T \\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{V} \end{pmatrix}$, where $\mathbf{A}$ denotes the FEM-block, $\mathbf{W}$ the discretization of the hyper-singular operator, $\mathbf{V}$ the discretization of the single-layer operator and $\mathbf{M}$ a mass matrix. In the setting of

Corollary 2.11, the computation of the sparse matrices **A** and **M** take $\mathcal{O}(h^{-d})$ operations. The dense matrices **V** and **W** would require computational effort of $\mathcal{O}(h^{-2(d-1)})$, which can be significantly reduced by employing compression techniques such as the fast multipole method, Greengard & Rokhlin (1997), or hierarchical matrices, Hackbusch (2015), to $\mathcal{O}(h^{1-d}(\log h)^2)$.

For the discretization in y-direction, the space $\mathcal{S}^{p,1}(\mathcal{T}_y)$ has $N_y+1 = \mathcal{O}((L+M)p)$ degrees of freedom with $L \sim p$ and $M \sim \log \mathcal{Y}$. With the assumptions of Corollary 2.11 this gives $N_y = \mathcal{O}((\log h)^2)$. Thus, the computation of $\mathcal{U}_\Omega$ and $\mathcal{U}_\star$ in Theorem 2.6 takes a total effort of $\mathcal{O}(h^{-d}(\log h)^2)$, provided matrix compression techniques are employed.

Remark 2.13 We note that our main results are valid for more general fractional PDEs as well. Using the same techniques, one obtains the statements also for:

1. $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$;
2. operators containing lower order terms, i.e.,

$$\mathcal{L}u := -\operatorname{div}\left(\mathfrak{A}\nabla u\right) + \mathfrak{c}u,$$

where $\mathfrak{c} : \mathbb{R}^d \to \mathbb{R}$ with $\mathfrak{c} \geq 0$ is smooth and satisfies $\mathfrak{c} \equiv \mathfrak{c}_0 \in \mathbb{R}$ in $\mathbb{R}^d \setminus \overline{\Omega}$.

3. Well-posedness and FEM-BEM formulation

In this section, we provide the proofs of Theorem 2.3 and Theorem 2.6.

3.1 *Poincaré inequalities*

We now show the well-posedness of our variational formulations. The main ingredient is a Poincaré type estimate, which uses the following compactness result.

Lemma 3.1 Let $D \subseteq \mathbb{R}^d \times \mathbb{R}^+$ be a bounded Lipschitz domain. Assume $u_n \rightharpoonup 0$ weakly in $H^1(y^\alpha, D)$ and $\|\nabla u_n\|_{L^2(y^\alpha,D)} \to 0$. Then, $u_n \to 0$ in $L^2(y^\alpha, D)$.

Proof. We can cover the Lipschitz domain D by a finite number of Lipschitz domains $D_1, \ldots, D_m$ that are starshaped with respect to a ball, see for example Maz'ya (2011, Sect. 1.1.9, Lemma 1). Thus, without loss of generality we may assume that D is starshaped with respect to a ball. With $c_n := \int_D u_n$ we compute

$$\begin{aligned}
\|u_n\|^2_{L^2(y^\alpha,D)} &= \|u_n - c_n\|^2_{L^2(y^\alpha,D)} + 2(u_n, c_n)_{L^2(y^\alpha,D)} - \|c_n\|^2_{L^2(y^\alpha,D)} \\
&\lesssim \|\nabla u_n\|^2_{L^2(y^\alpha,D)} + 2\left|(u_n, c_n)_{L^2(y^\alpha,D)}\right| \\
&\leq \|\nabla u_n\|^2_{L^2(y^\alpha,D)} + 2\left|c_n\right|\left|(u_n, 1)_{L^2(y^\alpha,D)}\right| \to 0,
\end{aligned}$$

where we used the Poincaré estimate of Nochetto *et al.* (2015, Corollary 4.4) and the assumed weak convergence. □

Using a jump condition in a weak sense to fix constants, the following weighted Poincaré type estimate in the full space holds. Note that including (powers of) the weight ρ is essential here.

LEMMA 3.2 Fix $\mathcal{Y} \in (0, \infty]$. Let $\mathcal{U} \in H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0, \mathcal{Y}))$ with $\int_\Gamma [\![\gamma \mathcal{U}]\!] \, \mathrm{d}s_x = 0$ for almost every $y \in (0, \mathcal{Y})$.

1. Let $0 \le \mu \le 2$ and $\mathcal{Y} = \infty$. There holds

$$\int_0^\infty \int_{\mathbb{R}^d} y^\alpha \rho^{\mu-2} |\mathcal{U}|^2 \, \mathrm{d}x \, \mathrm{d}y \le C \int_0^\infty \int_{\mathbb{R}^d \setminus \Gamma} y^\alpha \rho^\mu |\nabla \mathcal{U}|^2 \, \mathrm{d}x \, \mathrm{d}y \tag{3.1}$$

provided the right-hand side is finite.

2. Let $\mathcal{Y} \in (0, \infty)$. There exists $\mu_0 > 0$ such that for all $\mu \in [0, \mu_0)$ there holds

$$\int_0^{\mathcal{Y}} \int_{\mathbb{R}^d} y^\alpha \rho^{\mu-2} |\mathcal{U}|^2 \, \mathrm{d}x \, \mathrm{d}y \le C \left(\int_0^{\mathcal{Y}} \int_{\mathbb{R}^d \setminus \Gamma} y^\alpha \rho^\mu |\nabla \mathcal{U}|^2 \, \mathrm{d}x \, \mathrm{d}y + |3 - d| \|\mathrm{tr}_0 \mathcal{U}\|^2_{L^2(\mathbb{R}^d)} \right) \tag{3.2}$$

provided the right-hand side is finite.

Proof. The estimates follow from techniques employed in Amrouche *et al.* (1994, Theorem 3.3) using a proof by contradiction. In the first step, we show (3.1) (which essentially is covered by Amrouche *et al.*, 1994, Theorem 3.3, we only account for the additional weight y^α) and (3.2) for functions vanishing inside a ball containing the origin. Finally, using a compactness argument, this assumption is removed in the second step.

Step 1: First, assume that $\mathcal{U} \equiv 0$ on a sufficiently large (half) ball $B_R(0) \subset \mathbb{R}^{d+1}$ and has compact support.

We employ spherical coordinates in $\mathbb{R}^d \times \mathbb{R}^+$, chosen such that $y = r\cos(\varphi)$ and collect the remaining $d-1$ angles into $\hat{\varphi}$. Using $\rho^{\mu-2} = (1 + |x|^2 + y^2)^{-(\mu-2)/2} < r^{\mu-2}$ for $\mu \le 2$, we calculate

$$\int_0^\infty y^\alpha \int_{\mathbb{R}^d} \rho^{\mu-2} |\mathcal{U}(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \lesssim \int_{\hat{\varphi}} \int_{\varphi=-\pi/2}^{\pi/2} \int_R^\infty r^{d+\alpha+\mu-2} \cos(\varphi)^\alpha |\mathcal{U}(x,y)|^2 |J(\varphi,\hat{\varphi})| \, \mathrm{d}r \, \mathrm{d}\varphi \mathrm{d}\hat{\varphi},$$

where we denoted by $J(\varphi, \hat{\varphi})$ the angular components of the Jacobian in the transformation theorem. Integration by parts in r and using the assumed support properties of $\mathcal{U}$ gives

$$\begin{aligned}
&\int_{\hat{\varphi}} \int_{-\pi/2}^{\pi/2} \int_R^\infty r^{d+\alpha+\mu-2} \cos(\varphi)^\alpha |\mathcal{U}(x,y)|^2 |J(\varphi,\hat{\varphi})| \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\hat{\varphi} \\
&\quad \lesssim \int_{\hat{\varphi}} \int_{-\pi/2}^{\pi/2} \int_R^\infty r^{d+\alpha+\mu-1} \cos(\varphi)^\alpha |\mathcal{U}(x,y)| |\nabla \mathcal{U}(x,y)| |J(\varphi,\hat{\varphi})| \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\hat{\varphi} \\
&\quad \lesssim \left(\int_{\hat{\varphi}} \int_{-\pi/2}^{\pi/2} \int_R^\infty r^{d+\alpha+\mu-2} \cos(\varphi)^\alpha \mathcal{U}(x,y)^2 |J(\varphi,\hat{\varphi})| \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\hat{\varphi} \right)^{1/2} \\
&\qquad \times \left(\int_{\hat{\varphi}} \int_{-\pi/2}^{\pi/2} \int_R^\infty r^{d+\alpha+\mu} \cos(\varphi)^\alpha \, |\nabla \mathcal{U}(x,y)|^2 \, |J(\varphi,\hat{\varphi})| \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\hat{\varphi} \right)^{1/2}.
\end{aligned}$$

Transforming back to (x,y)-variables and using $r^\mu \le \rho^\mu$ for $\mu \ge 0$ this gives the desired bound. By density, we can remove the requirement of compact support of $\mathcal{U}$.

Step 2: In order to get rid of the requirement that $\mathcal{U}$ vanishes on the ball $B_R(0)$, we use a compactness argument. To keep the notation succinct we set $\mu = 0$ in the following, the general case $\mu > 0$ can be done with the exact same arguments. Assume that (3.1) does not hold, i.e., there exists a sequence $\mathcal{U}_n \in H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0,\infty))$ such that

$$\|\mathcal{U}_n\|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} = 1, \qquad \int_0^\infty y^\alpha \|\nabla \mathcal{U}_n(y)\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)}\,\mathrm{d}y \le \frac{1}{n}.$$

Since $\mathcal{U}_n$ is a bounded sequence in the Hilbert space $H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0,\infty))$, there exists a weakly convergent subsequence (also denoted by $\mathcal{U}_n$) and we denote the weak limit by $\mathcal{U}$.

Because the seminorm is weakly lower semicontinuous, we get $|\mathcal{U}|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} = 0$. A simple calculation (using polar coordinates, similar to the estimate above) shows that—as we are in a half-space in $\mathbb{R}^{d+1}$ with $d+1 > 2$ and $\int_\Gamma [\![\gamma\mathcal{U}]\!]\,\mathrm{d}s_x = 0$—the space $H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0,\mathcal{Y}))$ does not contain piecewise constant functions except for 0, which means that $\mathcal{U} = 0$.

We now show strong convergence of the sequence to $\mathcal{U} = 0$. To that end, fix a ball $B_R := B_R(0) \subset \mathbb{R}^{d+1}$ with sufficiently large R such that $\Omega \times \{0\} \subset B_R$ and consider a smooth cutoff function $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\psi \equiv 1$ on B_R and $\psi \equiv 0$ on B_{2R}. We thus decompose $\mathcal{U}_n$ as

$$\mathcal{U}_n = \psi\mathcal{U}_n + (1-\psi)\mathcal{U}_n =: \mathcal{U}_n^1 + \mathcal{U}_n^2.$$

From the compactness result of Lemma 3.1 applied to $(\Omega \times \mathbb{R}^+) \cap B_{\widetilde{R}}$ and $B_{\widetilde{R}} \setminus (\overline{\Omega} \times \mathbb{R}^+)$ separately, we deduce that $\mathcal{U}_n \to \mathcal{U}$ in $L^2(y^\alpha, B_{\widetilde{R}})$ and thus $\mathcal{U}_n^1 \to \psi\mathcal{U} = 0$ in $L^2(y^\alpha, B_{\widetilde{R}})$ on all bounded half balls $B_{\widetilde{R}}$ with sufficiently large $\widetilde{R}$.

Since $\mathcal{U}_n^2$ vanishes on B_R, we can apply step 1 of the proof to determine:

$$\begin{aligned}
\left\|\mathcal{U}_n^2\right\|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} &\lesssim \left|\mathcal{U}_n^2\right|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} \\
&\lesssim \left|(1-\psi)\mathcal{U}_n\right|_{H^1_\rho(y^\alpha,B_{2R}(0)\setminus\Gamma\times(0,\infty))} + \left|\mathcal{U}_n\right|_{H^1_\rho(y^\alpha,B_{2R}(0)^c\setminus\Gamma\times(0,\infty))} \\
&\lesssim \|\mathcal{U}_n\|_{L^2(y^\alpha,B_{2R}(0))} + \left|\mathcal{U}_n\right|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} \to 0.
\end{aligned}$$

Overall, we get that $\mathcal{U}_n \to 0$ in $H^1_\rho(y^\alpha, \mathbb{R}^d \setminus \Gamma \times (0,\infty))$, which is a contradiction to the assumption $\|\mathcal{U}_n\|_{H^1_\rho(y^\alpha,\mathbb{R}^d\setminus\Gamma\times(0,\infty))} = 1$ for all $n \in \mathbb{N}$.

Step 3: Estimate (3.2) for the case $d = 3$ follows directly from multiplying a full-space Poincaré-inequality (see for example Amrouche *et al.*, 1994, Theorem 3.3 for $\mu = 0$ and a similar calculation to step 1 for $0 < \mu \le 2$ with polar coordinates only in x) applied only in x with y^α and integrating over $(0, \mathcal{Y})$.

Step 4: It remains to show (3.2) for $d = 2$, which is (Faustmann & Rieder, 2023, Lem. 3.2), but for sake of completeness we repeat the proof here. We write $\mathcal{U}(x,y) = \mathcal{U}(x,0) + \int_0^y \partial_\tau\mathcal{U}(x,\tau)\,\mathrm{d}\tau$, which gives

$$\int_0^{\mathcal{Y}}\int_{\mathbb{R}^d} y^\alpha\rho^{\mu-2}|\mathcal{U}|^2\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_0^{\mathcal{Y}}\int_{\mathbb{R}^d} y^\alpha\rho^{\mu-2}|\mathcal{U}(x,0)|^2 + y^\alpha\rho^{\mu-2}\Big(\int_0^y \partial_\tau\mathcal{U}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\int_0^{\mathcal{Y}} y^\alpha \rho^{\mu-2} \mathrm{d}y \lesssim 1$ for sufficiently small $\mu < \mu_0$ with μ_0 depending only on α, the first term on the left-hand side can be bounded by $C \left\| \mathrm{tr}_0 \mathcal{U} \right\|^2_{L^2(\mathbb{R}^d)}$. For the second term, we employ a weighted Hardy-inequality, see e.g., Muckenhoupt (1972), to obtain

$$\int_0^{\mathcal{Y}} \int_{\mathbb{R}^d} y^\alpha \rho^{\mu-2} \Big(\int_0^y \partial_\tau \mathcal{U}(x,\tau)\, \mathrm{d}\tau \Big)^2 \mathrm{d}x\, \mathrm{d}y \lesssim \int_{\mathbb{R}^d} \int_0^{\mathcal{Y}} y^\alpha \rho^\mu |\partial_y \mathcal{U}|^2 \,\mathrm{d}y\, \mathrm{d}x,$$

which shows the claimed inequality. □

We can now look at the well-posedness of the reformulation of the Caffarelli–Silvestre extension in the modified Hilbert space $\mathbb{H}_{\mathcal{Y}}$.

Proof of Theorem 2.3. Let $\mathcal{Y} \in (0,\infty]$ and $(\mathcal{U}^{\mathcal{Y}}_\Omega, \mathcal{U}^{\mathcal{Y}}_\star) \in \mathbb{H}_{\mathcal{Y}}$. On the interior domain Ω, we integrate a standard Poincaré-like estimate using the boundary condition $\gamma^-_\Gamma \mathcal{U}^{\mathcal{Y}}_\star = 0$ in the definition of $\mathbb{H}_{\mathcal{Y}}$, to obtain

$$\int_0^{\mathcal{Y}} y^\alpha \int_\Omega \rho^{-2} \left| \mathcal{U}^{\mathcal{Y}}_\star \right|^2 \mathrm{d}x\, \mathrm{d}y \le \int_0^{\mathcal{Y}} y^\alpha \int_\Omega \left| \mathcal{U}^{\mathcal{Y}}_\star \right|^2 \mathrm{d}x\, \mathrm{d}y \lesssim \int_0^{\mathcal{Y}} y^\alpha \int_\Omega \left| \nabla_x \mathcal{U}^{\mathcal{Y}}_\star \right|^2 \mathrm{d}x\, \mathrm{d}y. \tag{3.3}$$

By the condition $\gamma^+_\Gamma \mathcal{U}^{\mathcal{Y}}_\star = \gamma^-_\Gamma \mathcal{U}^{\mathcal{Y}}_\Omega$, we observe that the function $\mathcal{U}^{\mathcal{Y}} := \begin{cases} \mathcal{U}^{\mathcal{Y}}_\Omega, & \text{in } \Omega \\ \mathcal{U}^{\mathcal{Y}}_\star, & \text{in } \mathbb{R}^d \backslash \overline{\Omega} \end{cases}$ has a vanishing jump across $\partial\Omega$. Applying Lemma 3.2 to $\mathcal{U}^{\mathcal{Y}}$, we get with (3.3) that

$$\begin{aligned}
B^{\mathcal{Y}}\left(\mathcal{U}^{\mathcal{Y}}_h, \mathcal{U}^{\mathcal{Y}}_h\right) &\gtrsim \int_0^{\mathcal{Y}} \int_\Omega y^\alpha \left| \nabla \mathcal{U}^{\mathcal{Y}}_\Omega \right|^2 \mathrm{d}x\, \mathrm{d}y + \int_0^{\mathcal{Y}} \int_{\mathbb{R}^d \backslash \Gamma} y^\alpha \left| \nabla \mathcal{U}^{\mathcal{Y}}_\star \right|^2 \mathrm{d}x\, \mathrm{d}y \\
&\quad + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\Omega \right\|^2_{L^2(\Omega)} + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\star \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \int_0^{\mathcal{Y}} \int_{\mathbb{R}^d \backslash \Gamma} y^\alpha \left| \nabla \mathcal{U}^{\mathcal{Y}} \right|^2 \mathrm{d}x\, \mathrm{d}y + \int_0^{\mathcal{Y}} \int_\Omega y^\alpha \left| \nabla \mathcal{U}^{\mathcal{Y}}_\star \right|^2 \mathrm{d}x\, \mathrm{d}y \\
&\quad + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\Omega \right\|^2_{L^2(\Omega)} + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\star \right\|^2_{L^2(\mathbb{R}^d)} \\
&\gtrsim \left\| \mathcal{U}^{\mathcal{Y}} \right\|^2_{H^1_\rho(y^\alpha, \mathbb{R}^d \backslash \Gamma \times (0,\mathcal{Y}))} + \left\| \mathcal{U}^{\mathcal{Y}}_\star \right\|^2_{H^1_\rho(y^\alpha, \Omega \times (0,\mathcal{Y}))} + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\Omega \right\|^2_{L^2(\Omega)} + s \left\| \mathrm{tr}_0 \mathcal{U}^{\mathcal{Y}}_\star \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \left\| \left(\mathcal{U}^{\mathcal{Y}}_\Omega, \mathcal{U}^{\mathcal{Y}}_\star \right) \right\|^2_{\mathbb{H}_{\mathcal{Y}}},
\end{aligned}$$

which shows coercivity.

In order to bound the right-hand side in (2.4), we distinguish three cases.

Case $s > 0$**:** We directly use the definition of the $\mathbb{H}_{\mathcal{Y}}$-norm together with $\mathrm{supp} f \subset \Omega$, to obtain

$$\int_{\mathbb{R}^d} f \mathrm{tr}_0 \mathcal{V}^{\mathcal{Y}}_\Omega \,\mathrm{d}x \le s^{-1} \|f\|_{L^2(\Omega)} s \left\| \mathrm{tr}_0 \mathcal{V}^{\mathcal{Y}}_\Omega \right\|_{L^2(\Omega)} \le s^{-1} \|f\|_{L^2(\Omega)} \left\| \mathcal{V}^{\mathcal{Y}} \right\|_{\mathbb{H}_{\mathcal{Y}}}.$$

Case $s = 0$, $\mathcal{Y} = \infty$: By assumption, this means $d = 3$, and the trace estimate (1.3) gives

$$\int_{\mathbb{R}^d} f \operatorname{tr}_0 \mathcal{V}_\Omega \,\mathrm{d}x \le \|\rho(x,0)^\beta f\|_{L^2(\Omega)} \left\|\rho(x,0)^{-\beta} \operatorname{tr}_0 \mathcal{V}_\Omega\right\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\Omega)} \left\|\nabla \mathcal{V}_\Omega\right\|_{L^2(y^\alpha,\mathbb{R}^d\times\mathbb{R}^+)}$$
$$\le \|f\|_{L^2(\Omega)} \|\mathcal{V}\|_{\mathbb{H}_\infty}.$$

Case $s = 0$, $\mathcal{Y} < \infty$: We use a cut-off function χ satisfying $\chi \equiv 1$ on $(0, \mathcal{Y}/2)$, $\operatorname{supp}\chi \subset (0, \mathcal{Y})$ and $\|\nabla\chi\|_{L^\infty(\mathbb{R}^+)} \lesssim \mathcal{Y}^{-1}$. As Ω is bounded, this gives with the trace estimate (Karkulik & Melenk, 2019, Lem. 3.7)

$$\begin{aligned}
\int_{\mathbb{R}^d} f \operatorname{tr}_0 \mathcal{V}_\Omega^{\mathcal{Y}} \,\mathrm{d}x &\le \|f\|_{L^2(\Omega)} \left\|\operatorname{tr}_0\left(\chi \mathcal{V}_\Omega^{\mathcal{Y}}\right)\right\|_{L^2(\Omega)} \\
&\lesssim \|f\|_{L^2(\Omega)} \left(\left\|\chi \mathcal{V}_\Omega^{\mathcal{Y}}\right\|_{L^2(y^\alpha,\Omega\times(0,\mathcal{Y}))} + \left\|\nabla\left(\chi \mathcal{V}_\Omega^{\mathcal{Y}}\right)\right\|_{L^2(y^\alpha,\Omega\times(0,\mathcal{Y}))}\right) \\
&\lesssim \|f\|_{L^2(\Omega)} \left(\left(1 + \frac{1}{\mathcal{Y}}\right) \left\|\mathcal{V}_\Omega^{\mathcal{Y}}\right\|_{L^2(y^\alpha,\Omega\times(0,\mathcal{Y}))} + \left\|\nabla \mathcal{V}_\Omega^{\mathcal{Y}}\right\|_{L^2(y^\alpha,\Omega\times(0,\mathcal{Y}))}\right) \\
&\le C\left(1 + \frac{1}{\mathcal{Y}}\right) \|f\|_{L^2(\Omega)} \left\|\mathcal{V}^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}},
\end{aligned}$$

which finishes the proof. □

3.2 *Diagonalization*

We now apply the diagonalization procedure of Banjai *et al.* (2019) to show that solutions of (2.7) can be written as in (2.8), where the coefficient functions $u_{j,\Omega}$ and $u_{j,\star}$ satisfy certain equations. We recall that $(\varphi_j)_{j=0}^{N_y}$ is the orthonormal basis of eigenfunctions from (2.5) with corresponding eigenvalues μ_j.

Lemma 3.3 Functions $(\mathcal{U}_\Omega^{\mathcal{Y}}, \mathcal{U}_\star^{\mathcal{Y}}) \in \mathbb{H}_{h,\mathcal{Y}}$ solve (2.7), if and only if they can be written as

$$\mathcal{U}_\bullet^{\mathcal{Y}}(x,y) = \sum_{j=0}^{N_y} u_{j,\bullet}(x)\varphi_j(y),$$

where $\bullet \in \{\Omega, \star\}$ and

$$u_{j,\Omega} \in \mathbb{V}_h^x, \qquad u_{j,\star} \in H^1_{\rho_x}(\mathbb{R}^d \setminus \Gamma) \quad \forall j \ge 0$$

solve for all $v \in \mathbb{V}_h^x$

$$\left(\mathfrak{A}\nabla u_{j,\Omega}, \nabla v\right)_{L^2(\Omega)} + \mu_j\left(u_{j,\Omega}, v\right)_{L^2(\Omega)} - \left\langle \partial^-_{\nu,\Gamma} u_{j,\star}, \gamma_\Gamma^- v\right\rangle_{L^2(\Gamma)} = \mathrm{d}_\beta \varphi_j(0)(f, v)_{L^2(\Omega)} \tag{3.4a}$$

$$-\Delta u_{j,\star} + \mu_j u_{j,\star} = 0 \qquad \text{in } \mathbb{R}^d \setminus \Gamma, \tag{3.4b}$$

$$[\![\gamma u_{j,\star}]\!] = \gamma_\Gamma^- u_j, \quad \gamma_\Gamma^- u_{j,\star} \in \left(\mathbb{V}_h^\lambda\right)^\circ. \tag{3.4c}$$

Proof. At first, we show unique solvability of (3.4). For that, we consider the weak formulation of (3.4) given by

$$\left(\mathfrak{A}\nabla u_{j,\Omega},\nabla v_{j,\Omega}\right)_{L^2(\Omega)}+\mu_j(u_{j,\Omega},v_{j,\Omega})_{L^2(\Omega)}+\left(\nabla u_{j,\star},\nabla v_{j,\star}\right)_{L^2(\mathbb{R}^d\backslash\Gamma)}+\mu_j\left(u_{j,\star},v_{j,\star}\right)_{L^2(\mathbb{R}^d)}$$
$$=\mathrm{d}_\beta\varphi_j(0)(f,v_{j,\Omega})_{L^2(\mathbb{R}^d)}. \quad (3.5)$$

The equivalence between the weak form and the strong form follows from standard arguments, and we refer to Laliena & Sayas (2009, Sect.7). Coercivity of the weak formulation in $H^1(\Omega)\times H^1(\mathbb{R}^d\backslash\Gamma)$ is clear for $\mu_j>0$ as $\mathfrak{A}$ is positive definite. For $\mu_j=0$, one can employ Poincaré estimates on Ω and $\mathbb{R}^d$ (with weights) to obtain coercivity in $H^1(\Omega)\times H^1_{\rho_x}(\mathbb{R}^d\backslash\Gamma)$. Therefore, for each j, a unique solution $(u_{j,\Omega},u_{j,\star})\in\mathbb{V}_h^x\times H^1_{\rho_x}(\mathbb{R}^d\backslash\Gamma)\subset H^1(\Omega)\times H^1_{\rho_x}(\mathbb{R}^d\backslash\Gamma)$ exists.

We now show that, if the $u_{j,\bullet}$ solve (3.4), then $\mathcal{U}_h^{\mathcal{Y}}:=(\mathcal{U}_\Omega^{\mathcal{Y}},\mathcal{U}_\star^{\mathcal{Y}})$ with $\mathcal{U}_\bullet^{\mathcal{Y}}:=\sum_{j=0}^{N_y}u_{j,\bullet}\varphi_j$ solves (2.7). By construction, we have $\mathcal{U}_h^{\mathcal{Y}}\in\mathbb{H}_{h,\mathcal{Y}}$. We next look at the weak formulation of (2.7). First, we focus on the $\star$-contribution. Taking $\mathcal{V}_\star^{\mathcal{Y}}=v_{j,\star}(x)\varphi_j(y)$ with arbitrary $v_{j,\star}\in\mathbb{V}_h^x$ as test function, we compute

$$\begin{aligned}A^{\mathcal{Y}}_{\mathbb{R}^d\backslash\Gamma}\left(\mathcal{U}_\star^{\mathcal{Y}},\mathcal{V}_\star^{\mathcal{Y}}\right)&=\sum_{\ell=0}^{N_y}\int_{\mathbb{R}^d}u_{\ell,\star}(x)v_{j,\star}(x)\,\mathrm{d}x\int_0^{\mathcal{Y}}y^\alpha\varphi_\ell'(y)\varphi_j'(y)\,\mathrm{d}y\\&\quad+\int_{\mathbb{R}^d\backslash\Gamma}\nabla u_{\ell,\star}(x)\nabla v_{j,\star}(x)\,\mathrm{d}x\int_0^{\mathcal{Y}}y^\alpha\varphi_\ell(y)\varphi_j(y)\,\mathrm{d}y\\&\quad+\int_{\mathbb{R}^d}u_{\ell,\star}(x)v_{j,\star}(x)\,\mathrm{d}x\cdot s\varphi_\ell(0)\varphi_j(0)\\&=\mu_j\int_{\mathbb{R}^d}u_{j,\star}(x)v_{j,\star}(x)\,\mathrm{d}x+\int_{\mathbb{R}^d\backslash\Gamma}\nabla u_{j,\star}(x)\nabla v_{j,\star}(x)\,\mathrm{d}x.\end{aligned}$$

For the interior contribution, the same diagonalization procedure gives for $\mathcal{V}_\Omega^{\mathcal{Y}}:=v_{j,\Omega}(x)\varphi_j(y)$

$$A^{\mathcal{Y}}_\Omega\left(\mathcal{U}_\Omega^{\mathcal{Y}},\mathcal{V}_\Omega^{\mathcal{Y}}\right)=\left(\mathfrak{A}\nabla u_{j,\Omega},\nabla v_{j,\Omega}\right)_{L^2(\Omega)}+\mu_j(u_{j,\Omega},v_{j,\Omega})_{L^2(\Omega)}.$$

Summing up, and using the weak form (5), we get that

$$B^{\mathcal{Y}}\left(\mathcal{U}_h^{\mathcal{Y}},\mathcal{V}_h^{\mathcal{Y}}\right)=d_\beta\left(f,\mathrm{tr}_0\mathcal{V}_\Omega^{\mathcal{Y}}\right)_{L^2(\mathbb{R}^d)}\qquad\text{for all }\mathcal{V}_h^{\mathcal{Y}}=\left(\mathcal{V}_\Omega^{\mathcal{Y}},\mathcal{V}_\star^{\mathcal{Y}}\right)=\left(\sum_{j=0}^{N_y}v_{j,\Omega}\varphi_j,\sum_{j=0}^{N_y}v_{j,\star}\varphi_j\right).$$

By density, we can extend this equality to all test functions $\mathcal{V}_h^{\mathcal{Y}}$ in the space $\mathbb{H}_{h,\mathcal{Y}}$ and get (2.7). Since the bilinear form $B^{\mathcal{Y}}(\cdot,\cdot)$ is coercive, we get that the constructed function $\mathcal{U}^{\mathcal{Y}}$ is the only solution to (2.7), which establishes the stated equivalence. □

Proof of Theorem 2.6. The statement follows from Lemma 3.3 and (Laliena & Sayas, 2009, Section 7), as defining $u_{j,\star}(x):=\widetilde{K}(\mu_j)\gamma_\Gamma^- u_{j,\Omega}(x)-\widetilde{V}(\mu_j)\lambda_j(x)$ and plugging that into (3.4) gives the stated equations using classical properties of the layer potentials, such as definition of the boundary integral operators (2.10) and jump conditions of the potentials.

By definition and decay of the layer potentials, we have that $u_{j,\star}\in H^1(\mathbb{R}^d\backslash\Gamma)$ for all $j\in\mathbb{N}_0$ such that $\mu_j\neq0$, which gives $u_{j,\star}\in H^1_{\rho_x}(\mathbb{R}^d\backslash\Gamma)$ as well. If $s>0$, no zero eigenvalue is possible. This matches with the the requirement $s\,\mathrm{tr}_0\mathcal{U}_\star \in L^2(\mathbb{R}^d)$ in the definition of the space $\mathbb{H}_{\mathcal{Y}}$. The case $s = 0$ is only allowed for $d = 3$. Here, $\mathrm{tr}_0\mathcal{U}_\star$ is not required to be in $L^2(\mathbb{R}^d)$, which would not hold. However, in this case the decay property of the layer potentials for the Poisson equation, see e.g., Sauter & Schwab (2011), give $u_{j,\star} \in H^1_{\rho_x}(\mathbb{R}^d\backslash\Gamma)$. In short, we have that our constructed solution is in the semidiscrete space $\mathbb{H}_{h,\mathcal{Y}}$.

Notably, (3.4) is just the 'non-standard transmission problem' corresponding to the standard symmetric FEM-BEM coupling given by Theorem 2.6. □

4. Error analysis

The key to the error analysis are the decay and regularity properties shown in Faustmann & Rieder (2023). In order to make the present paper more accessible, we summarize the key results of Faustmann & Rieder (2023) in the following.

4.1 *Decay and regularity*

The solution to the truncated problem is in fact a weak solution to a Neumann problem. Thus, in this section, we consider solutions $\mathcal{U}^{\mathcal{Y}}$ to the following truncated problem:

$$-\operatorname{div}\left(y^\alpha \mathfrak{A}_x \nabla \mathcal{U}^{\mathcal{Y}}\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0,\mathcal{Y}), \tag{4.1a}$$

$$d_\beta^{-1}\partial_{\nu^\alpha}\mathcal{U}^{\mathcal{Y}} + s\mathrm{tr}_0\mathcal{U}^{\mathcal{Y}} = f \qquad \text{on } \mathbb{R}^d \times \{0\}, \tag{4.1b}$$

$$\partial_y \mathcal{U}^{\mathcal{Y}} = 0 \qquad \text{on } \mathbb{R}^d \times \{\mathcal{Y}\}. \tag{4.1c}$$

Then, the truncation error can be controlled via the following proposition.

PROPOSITION 4.1 (Decay in y, Faustmann & Rieder, 2023, Prop. 2.5). Fix $\mathcal{Y} > 0$. Let $\mathcal{U}$ solve (1.4) and $\mathcal{U}^{\mathcal{Y}}$ solve (4.1). Let μ be given by $\mu := \begin{cases} 1+|\alpha| & s>0 \\ 1+\alpha & s=0 \end{cases}$. Then, the following estimate holds:

$$\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}\|^2_{H^1_\rho(y^\alpha,\mathbb{R}^d\times(0,\mathcal{Y}))} + s\left\|\mathrm{tr}_0(\mathcal{U}^{\mathcal{Y}} - \mathcal{U})\right\|^2_{L^2(\mathbb{R}^d)} \lesssim \mathcal{Y}^{-\mu}\,\|f\|^2_{L^2(\Omega)}.$$

We note that in contrast to Banjai *et al.* (2019), which exploits a known closed form representation of the solution for the problem on a bounded domain, we only obtain algebraic convergence of the truncated solution rather than exponential convergence. For the full-space problem to our knowledge, no closed form representation is available and as there is no spectral gap, exponential convergence does likely not hold in this case. We also stress that the techniques employed in Faustmann & Rieder (2023) are purely variational and thus can be applied more generally to other model problems without known closed form representations.

The goal in the following is to employ *hp*-FEM in the extended variable y. Therefore, weighted analytic regularity estimates are the key to the *a-priori* analysis.

PROPOSITION 4.2 (Regularity in y, Faustmann & Rieder, 2023, Prop. 2.6). Fix $\mathcal{Y} \in (0,\infty]$ and let $\ell \in \mathbb{N}$. Let $\mathcal{U}$ solve (4.1). Then, there exist constants $C, K > 0$ and $\varepsilon \in (0,1)$ such that the following estimate holds:

$$\left\|y^{\ell-\varepsilon}\nabla\partial_y^\ell\mathcal{U}\right\|_{L^2(y^\alpha,\mathbb{R}^d\times(0,\mathcal{Y}))} \le CK^\ell \ell!\,\|f\|_{L^2(\Omega)}.$$

All constants are independent of ℓ, $\mathcal{Y}$ and $\mathcal{U}$.

Denoting by $L^2(y^\alpha,(0,\mathcal{Y});X)$ the Bochner spaces of square integrable functions (with respect to the weight y^α) and values in the Banach space X, the regularity results of the previous Proposition can be captured by the solution being in some countably normed space. For constants $C,K>0$, we introduce

$$\mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};X) := \Big\{\mathcal{V}\in C^\infty((0,\mathcal{Y});X) : \|\mathcal{V}\|_{L^2(y^\alpha,(0,\mathcal{Y});X)} < C, \\ \left\|y^{\ell+1-\varepsilon}\mathcal{V}^{(\ell+1)}\right\|_{L^2(y^\alpha,(0,\mathcal{Y});X)} < CK^{\ell+1}(\ell+1)!\ \forall \ell\in\mathbb{N}_0\Big\}.$$

Corollary 4.3 Fix $\mathcal{Y}\in(0,\infty]$ and let $\mathcal{U}^{\mathcal{Y}}$ solve (4.1). Then, there are constants $C,K>0$ such that there holds

$$\partial_y\mathcal{U}^{\mathcal{Y}}\in\mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d)) \qquad \text{and} \qquad \nabla_x\mathcal{U}^{\mathcal{Y}}\in\mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d)). \tag{4.2}$$

4.2 *Fully discrete analysis*

In order to derive error bounds, we employ the reformulation in (2.2) together with the already established decay bounds for the truncation in $\mathcal{Y}$.

We will need two quasi-interpolation operators—one for the x-variables and one for the y-direction. Their construction and properties are the subject of the next two lemmas.

Lemma 4.4 (Interpolation in x). Let $\mathbb{V}_h^x\subset H^1(\Omega)$ and $\mathbb{V}_h^\lambda\subset H^{-1/2}(\Gamma)$ be finite dimensional and $\pi_\Omega: L^2(\Omega)\to\mathbb{V}_h^x$ be a linear operator. Then, there exists a linear operator $\Pi_x: L^2(\Omega)\times L^2_{\rho_x}(\mathbb{R}^d)\to\mathbb{V}_h^x\times L^2_{\rho_x}(\mathbb{R}^d)$ such that the following properties hold for $(u^h,u^h_\star):=\Pi_x(u,u_\star)$ with $u\in H^1(\Omega)$, $u_\star\in H^1_{\rho_x}(\mathbb{R}^d\setminus\Gamma)$ satisfying $u_\star|_\Omega=0$ and $[\![\gamma u_\star]\!]=\gamma^- u$:

i. $\gamma^- u^h_\star\in(\mathbb{V}_h^\lambda)^\circ$.

ii. $[\![\gamma u^h_\star]\!]=\gamma^- u^h$.

iii. If π_Ω is stable in the $H^1(\Omega)$-norm, then

$$\|u^h\|^2_{H^1(\Omega)}+\left\|u^h_\star\right\|^2_{H^1_{\rho_x}(\mathbb{R}^d\setminus\Gamma)}\lesssim\|u\|^2_{H^1(\Omega)}+\left\|u_\star\right\|^2_{H^1_{\rho_x}(\mathbb{R}^d\setminus\Gamma)}.$$

If π_Ω is stable in the $L^2(\Omega)$-norm, and $u_\star\in L^2(\mathbb{R}^d)$ then

$$\|u^h\|^2_{L^2(\Omega)}+\left\|u^h_\star\right\|^2_{L^2(\mathbb{R}^d)}\lesssim\|u\|^2_{L^2(\Omega)}+\left\|u_\star\right\|^2_{L^2(\mathbb{R}^d)}.$$

iv. There hold the approximation properties:

$$\|u^h-u\|^2_{L^2(\Omega)}+\left\|u^h_\star-u_\star\right\|^2_{L^2_{\rho_x}(\mathbb{R}^d\setminus\Gamma)}\lesssim\|u-\pi_\Omega u\|^2_{L^2(\Omega)},$$

$$\|u^h-u\|^2_{H^1(\Omega)}+\left\|u^h_\star-u_\star\right\|^2_{H^1_{\rho_x}(\mathbb{R}^d\setminus\Gamma)}\lesssim\|u-\pi_\Omega u\|^2_{H^1(\Omega)}.$$

Proof. We note that a very similar operator is introduced in Melenk & Rieder (2017, Lemma 4.3). We define:

$$u^h:=\pi_\Omega u, \qquad u^h_\star:=u_\star+\delta,$$

where $\delta = -u_\star$ in Ω and $\delta := -\mathcal{E}(u - \pi_\Omega u)$ in $\mathbb{R}^d \backslash \Omega$, where $\mathcal{E} : L^2(\Omega) \to L^2(\mathbb{R}^d)$ denotes the Stein extension operator (Stein, 1970, Chapter VI.3) that is stable both in $L^2(\Omega)$ and $H^1(\Omega)$.

By construction, we have i., since $u^h_\star = 0$ in the interior. Since $\gamma^+\mathcal{E}v = \gamma^- v$ due to the extension property, we get (ii) by

$$[\![\gamma u^h_\star]\!] = \gamma^+ u_\star - \gamma^+\mathcal{E}(u - \pi_\Omega u) = \gamma^- u - \gamma^- u + \gamma^- \pi_\Omega u = \gamma^- u^h.$$

The stability estimates follow from the stability of the extension operator and the assumed stabilities of π_Ω as

$$\begin{aligned} \|u^h\|^2_{L^2(\Omega)} + \|u^h_\star\|^2_{L^2(\mathbb{R}^d)} &\lesssim \|u\|^2_{L^2(\Omega)} + \left\|u_\star\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\mathcal{E}(u - \pi_\Omega u)\right\|^2_{L^2(\mathbb{R}^d)} \\ &\lesssim \|u\|^2_{L^2(\Omega)} + \left\|u_\star\right\|^2_{L^2(\mathbb{R}^d)} + \left\|u - \pi_\Omega u\right\|^2_{L^2(\Omega)} \lesssim \|u\|^2_{L^2(\Omega)} + \left\|u_\star\right\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

The approximation property can be seen in a similar fashion using $\rho_x^{-2} < 1$

$$\begin{aligned} \|u^h - u\|^2_{L^2(\Omega)} + \|u^h_\star - u_\star\|^2_{L^2_{\rho_x}(\mathbb{R}^d)} &\le \|\pi_\Omega u - u\|^2_{L^2(\Omega)} + \|\mathcal{E}(u - \pi_\Omega u)\|^2_{L^2_{\rho_x}(\mathbb{R}^d)} \\ &\le \|\pi_\Omega u - u\|^2_{L^2(\Omega)} + \|\mathcal{E}(u - \pi_\Omega u)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|\pi_\Omega u - u\|^2_{L^2(\Omega)}. \end{aligned}$$

The H^1-estimates follows analogously. □

Lemma 4.5 (Interpolation in y). Let $\mathcal{Y} \in (0, \infty)$ and $\mathcal{U}^{\mathcal{Y}}$ solve (4.1). Let $\mathcal{T}_y$ be a geometric grid on $(0, \mathcal{Y})$ with mesh grading factor σ, and L-refinement layers towards 0 as given by (2.13). Let $\varepsilon > 0$ be given by Proposition 4.2. Then, choosing $L = p$, there exists an operator $\varPi_y : H^1_\rho(y^\alpha, \mathbb{R}^d \times (0, \mathcal{Y})) \to H^1_\rho(y^\alpha, \mathbb{R}^d \times (0, \mathcal{Y}))$ such that $\varPi_y \mathcal{U}(x, \cdot) \in \mathcal{S}^{p,1}(\mathcal{T}_y)$ for almost all $x \in \mathbb{R}^d$, and such that the following estimate holds:

$$\int_0^{\mathcal{Y}} \int_{\mathbb{R}^d} y^\alpha \left|\nabla\left(\mathcal{U}^{\mathcal{Y}} - \varPi_y \mathcal{U}^{\mathcal{Y}}\right)\right|^2 \mathrm{d}x\, \mathrm{d}y \le Ce^{-2\kappa p}\mathcal{Y}^{2\varepsilon}.$$

The constants $C, \kappa > 0$ are independent of $p, \mathcal{Y}$.

Proof. We use the *hp*-interpolation operator from Banjai *et al.* (2019, Sec. 5.5.1) for $\varPi_y$. This operator is constructed on a geometric mesh in an element-by-element way. On the first element a linear interpolant in $\sigma^L/2$ and σ^L is used, while the remaining elements are mapped to the reference element, on which a polynomial approximation operator that has exponential convergence properties (in the polynomial degree) for analytic functions is used.

For the operator on the reference element, we take the Babuška–Szabó polynomial approximation operator $\widehat{\varPi}_p$ on $(-1, 1)$, Szabó & Babuška (1991), defined as

$$\widehat{\varPi}_p v(y) := v(-1) + \int_{-1}^{y} \varPi^{L^2}_{p-1} v'(t)\, \mathrm{d}t,$$

where $\Pi^{L^2}_{p-1} : L^2(-1,1) \to P_{p-1}$ denotes the L^2-orthogonal projection, see e.g., Apel & Melenk (2015, Exa. 3.17). By construction, this operator has the commutator property

$$(\widehat{\Pi}_p v)' = \Pi^{L^2}_{p-1} v'.$$

Regularity in countably normed spaces gives exponential error bounds for Π_y. In fact, for functions in $\mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d))$, one obtains a bound in $L^2(y^\alpha,\mathbb{R}^d \times (0,\mathcal{Y}))$. Consequently, we can employ Proposition 4.2 to obtain $\nabla_x \mathcal{U}^{\mathcal{Y}} \in \mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d))$ and together with Banjai *et al.* (2019, Lem. 11(i)) this gives the error estimate

$$\int_0^{\mathcal{Y}} y^\alpha \left\| \nabla_x \mathcal{U}^{\mathcal{Y}}(\cdot,y) - \Pi_y \nabla_x \mathcal{U}^{\mathcal{Y}}(\cdot,y) \right\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}y \le C e^{-2\kappa p} \mathcal{Y}^{2\varepsilon}$$

for a constant $\kappa > 0$. Interchanging Π_y and ∇_x gives the estimate for the x-derivatives.

For the y-derivatives the situation is a bit more involved, as the same argument cannot be made as Π_y and ∂_y do not commute. Banjai *et al.* (2019, Lem. 11(ii)) gives an exponentially convergent error bound for the y-derivative provided $\mathcal{U}^{\mathcal{Y}} \in \mathcal{B}^2_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d))$ (essentially meaning $\partial_y \mathcal{U}^{\mathcal{Y}} \in \mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d))$ and $\mathcal{U}^{\mathcal{Y}} \in L^2(y^\alpha,\mathbb{R}^d \times (0,\mathcal{Y}))$). However, in our setting, the requirement $\mathcal{U}^{\mathcal{Y}} \in L^2(y^\alpha,\mathbb{R}^d \times (0,\mathcal{Y}))$ does not hold (see Remark 2.1). Nonetheless, we have Corollary 4.3 giving $\partial_y \mathcal{U}^{\mathcal{Y}} \in \mathcal{B}^1_{\varepsilon,0}(C,K,\mathcal{Y};L^2(\mathbb{R}^d))$, which is enough to regain the exponential estimate as seen in the following.

On the first element $(0,\sigma^L) \in \mathcal{T}_y$, the definition of the piecewise linear interpolation gives

$$\partial_y \Pi_y v(y) = \frac{v(\sigma^L) - v(\sigma^L/2)}{\sigma^L/2} = \frac{2}{\sigma^L} \int_{\sigma^L/2}^{\sigma^L} \partial_\tau v(\tau)\, \mathrm{d}\tau,$$

which is nothing else than the L^2-orthogonal projection of $\partial_y v$ on $(\sigma^L/2,\sigma^L)$. By choice of the Babŭska–Szabó operator and denoting by $\widetilde{\Pi}^{L^2}_{p-1}$ the mapped L^2-projection onto an element in $\mathcal{T}_y$, we have due to the commutator property and the preceding discussion

$$\partial_y \left(\Pi_y \mathcal{U}^{\mathcal{Y}}\right)\Big|_{\mathbb{R}^d \times K_i} = \widetilde{\Pi}^{L^2}_{p-1} \partial_y \mathcal{U}^{\mathcal{Y}}\Big|_{\mathbb{R}^d \times K_i} \in L^2\left(y^\alpha,\mathbb{R}^d \times K_i\right) \qquad \forall K_i \in \mathcal{T}_y$$

since $\partial_y \mathcal{U} \in L^2(y^\alpha,\mathbb{R}^d \times K_i)$, which implies that $\partial_y \Pi_y \mathcal{U}^{\mathcal{Y}} \in L^2(y^\alpha,\mathbb{R}^d \times (0,\mathcal{Y}))$. The error estimate for the y-derivative follows from scaling arguments. More precisely, we decompose

$$\left\| \partial_y \left(\mathcal{U}^{\mathcal{Y}} - \Pi_y \mathcal{U}^{\mathcal{Y}}\right)\right\|^2_{L^2(y^\alpha,\mathbb{R}^d \times (0,\mathcal{Y}))} = \sum_{K_i \in \mathcal{T}_y} \left\| \partial_y \left(\mathcal{U}^{\mathcal{Y}} - \Pi_y \mathcal{U}^{\mathcal{Y}}\right)\right\|^2_{L^2(y^\alpha,\mathbb{R}^d \times K_i)},$$

where $K_i = (x_i, x_{i+1})$. Using a Hardy inequality, one obtains a bound for the approximation error on the first element using second derivatives only; see Banjai *et al.* (2019, Lem. 15). Together with a scaling argument this leads to

$$\left\| \partial_y \left(\mathcal{U}^{\mathcal{Y}} - \Pi_y \mathcal{U}^{\mathcal{Y}}\right)\right\|^2_{L^2(y^\alpha,\mathbb{R}^d \times (0,\sigma^L))} \lesssim \sigma^{\varepsilon L} \left\| \partial_{yy} \mathcal{U}^{\mathcal{Y}} \right\|^2_{L^2(y^{\alpha+2-2\varepsilon},\mathbb{R}^d \times (0,\sigma^L))}.$$

By Corollary 4.3 we can bound the right-hand side. For the remaining elements, we employ a scaling argument from Apel & Melenk (2015, Thm. 3.13). Denoting by h_{K_i} the diameter of K_i, we infer $y \sim h_{K_i}$

on K_i for $i > 0$. For any univariate function v satisfying $\left\| y^{\ell-\varepsilon} v^{(\ell+1)} \right\|_{L^2(y^\alpha,(0,\mathcal{Y}))} < CK^\ell \ell!$ for all $\ell \in \mathbb{N}_0$ there holds

$$\left\| \widehat{v}^{(\ell+1)} \right\|^2_{L^2(-1,1)} = \frac{2}{h_{K_i}} h_{K_i}^{2(\ell+1)} \left\| v^{(\ell+1)} \right\|^2_{L^2(K_i)} \lesssim h_{K_i}^{2\varepsilon-\alpha+1} \left\| y^{\ell-\varepsilon} v^{(\ell+1)} \right\|^2_{L^2(y^\alpha,K_i)}$$
$$\lesssim h_{K_i}^{2\varepsilon-\alpha+1} K^\ell \ell! \,, \tag{4.3}$$

where $\widehat{v}$ is the pull-back of v to the reference element. The exponential approximation properties of the Babŭska–Szabó polynomial approximation operator then provides

$$\left\| \widehat{v} - \widehat{\Pi}_p \widehat{v} \right\|^2_{H^1(-1,1)} \lesssim h_{K_i}^{2\varepsilon-\alpha+1} e^{-\kappa p}. \tag{4.4}$$

Together with

$$\left\| (v - \Pi_y v)' \right\|^2_{L^2(y^\alpha,K_i)} \lesssim h_{K_i}^{\alpha-1} \left\| (\widehat{v} - \widehat{\Pi}_p \widehat{v})' \right\|^2_{L^2(-1,1)},$$

we can employ (4.4) for $v(y) = \mathcal{U}(\cdot, y)$ and square integrate over $\mathbb{R}^d$, noting that (4.3) holds due to Corollary 4.3. Summing over i and using $\sum_i h_{K_i}^{2\varepsilon} \lesssim \mathcal{Y}^{2\varepsilon}$ shows the claimed estimate.

Finally, to show that the operator does indeed map to $H^1_\rho(y^\alpha, \mathbb{R}^d \times (0,\mathcal{Y}))$, we note that by the previous considerations we have $\partial_y \Pi_y \mathcal{U} \in L^2(y^\alpha, \mathbb{R}^d \times (0,\mathcal{Y}))$ as well as $\Pi_y \mathcal{U}(\cdot, y) = \mathcal{U}(\cdot, y) \in L^2_{\rho_x}(\mathbb{R}^d)$ for certain values $y \in (0,\mathcal{Y})$ where it is interpolatory. By the fundamental theorem of calculus, this is sufficient to show that $\Pi_y \mathcal{U} \in L^2_\rho(y^\alpha, \mathbb{R}^d \times (0,\mathcal{Y}))$. □

We can now define an interpolation operator acting on both x and y in a tensor product fashion. In order to keep notation compact, we write $\|\cdot\|_{L^2}$ for the $L^2(\Omega) \times L^2(\mathbb{R}^d \backslash \Gamma)$-norm and $\|\cdot\|_{H^1_{\rho_x}}$ for the $H^1(\Omega) \times H^1_{\rho_x}(\mathbb{R}^d \backslash \Gamma)$-norm.

LEMMA 4.6 (Tensor approximation). Fix $\mathcal{Y} \in (0,\infty)$ and let $\mathcal{U} = (\mathcal{U}_\Omega, \mathcal{U}_\star) \in \mathbb{H}_\mathcal{Y}$. Define $\Pi(\mathcal{U}_\Omega, \mathcal{U}_\star) := \Pi_x \otimes \Pi_y(\mathcal{U}_\Omega, \mathcal{U}_\star) \in \mathbb{H}_{h,\mathcal{Y}}$ with the operators Π_x from Lemma 4.4 and Π_y from Lemma 4.5. Assume that the operator π_Ω in the definition of Π_x is both L^2- and H^1-stable. Then, the following approximation estimate holds

$$\|\mathcal{U} - \Pi\mathcal{U}\|^2_{\mathbb{H}_\mathcal{Y}} \lesssim \int_0^\mathcal{Y} y^\alpha \left(\left\| \nabla(1 - \Pi_y)\mathcal{U}(y) \right\|^2_{L^2} + \left\| \nabla(1 - \pi_\Omega)\mathcal{U}_\Omega(y) \right\|^2_{L^2(\Omega)} \right) \mathrm{d}y.$$

Proof. By the Poincaré inequality (3.2) and the trace inequality (1.3), we only have to estimate the gradient norms. We start with the x-derivatives. Employing the H^1-stability and approximation properties of Π_x from Lemma 4.4 (3) and (4) gives

$$\int_0^\mathcal{Y} y^\alpha \left\| \nabla_x(\mathcal{U} - \Pi\mathcal{U}) \right\|^2_{L^2} \mathrm{d}y \lesssim \int_0^\mathcal{Y} y^\alpha \left\| \nabla_x \mathcal{U} - \nabla_x(\Pi_x \otimes I)\mathcal{U} \right\|^2_{L^2} \mathrm{d}y$$
$$+ \int_0^\mathcal{Y} y^\alpha \left\| \nabla_x(\Pi_x \otimes I)\mathcal{U} - \nabla_x(\Pi_x \otimes \Pi_y)\mathcal{U} \right\|^2_{L^2} \mathrm{d}y$$
$$\lesssim \int_0^\mathcal{Y} y^\alpha \left\| (I - \pi_\Omega)\mathcal{U}_\Omega(y) \right\|^2_{H^1(\Omega)} \mathrm{d}y + \int_0^\mathcal{Y} y^\alpha \left\| (I - \Pi_y)\mathcal{U}(y) \right\|^2_{H^1_{\rho_x}} \mathrm{d}y.$$

Employing again Poincaré inequalities, we can reduce the right-hand side to norms of derivatives only. For the y-derivative, we proceed similarly using the L^2- stability and approximation properties of Π_x

$$\begin{aligned}\int_0^{\mathcal{Y}} y^\alpha \left\|\partial_y(\mathcal{U}-\Pi\mathcal{U})\right\|_{L^2}^2 \mathrm{d}y &\lesssim \int_0^{\mathcal{Y}} y^\alpha \left\|\partial_y\mathcal{U}-\partial_y(\Pi_x\otimes I)\mathcal{U}\right\|_{L^2}^2 \mathrm{d}y \\ &\quad + \int_0^{\mathcal{Y}} y^\alpha \left\|\partial_y(\Pi_x\otimes I)\mathcal{U}-\partial_y(\Pi_x\otimes \Pi_y)\mathcal{U}\right\|_{L^2}^2 \mathrm{d}y \\ &\lesssim \int_0^{\mathcal{Y}} y^\alpha \left\|(1-\pi_\Omega)\partial_y\mathcal{U}_\Omega(y)\right\|_{L^2(\Omega)}^2 + y^\alpha\left\|\partial_y(I-\Pi_y)\mathcal{U}(y)\right\|_{L^2}^2 \mathrm{d}y,\end{aligned}$$

which finishes the proof. □

In order to obtain a best-approximation estimate for the semidiscretization, we observe that the difference $\mathcal{U}-\mathcal{U}_h$ satisfies some form of Galerkin orthogonality.

Lemma 4.7 (Galerkin orthogonality). Let $\mathcal{Y}>0, \mathcal{U}^{\mathcal{Y}}=(\mathcal{U}_\Omega^{\mathcal{Y}},\mathcal{U}_\star^{\mathcal{Y}})\in\mathbb{H}_{\mathcal{Y}}$ be the solution of (2.4) and $\mathcal{U}_h^{\mathcal{Y}}\in\mathbb{H}_{h,\mathcal{Y}}$ solve (2.7). Then, for all $\mathcal{V}_h^{\mathcal{Y}}=(\mathcal{V}_\Omega^{\mathcal{Y}},\mathcal{V}_\star^{\mathcal{Y}})\in\mathbb{H}_{h,\mathcal{Y}}$ and $\lambda_h:\mathbb{R}^+\to\mathbb{V}_h^\lambda$, there holds

$$B^{\mathcal{Y}}\left(\mathcal{U}^{\mathcal{Y}}-\mathcal{U}_h^{\mathcal{Y}},\mathcal{V}_h^{\mathcal{Y}}\right)=\int_0^{\mathcal{Y}} y^\alpha\left\langle[\![\partial_\nu\mathcal{U}_\star^{\mathcal{Y}}]\!]-\lambda_h,\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\right\rangle_{L^2(\Gamma)}\mathrm{d}y.$$

Proof. Compared to 'standard' Galerkin orthogonality, we observe that $\mathcal{V}_h^{\mathcal{Y}}$ is not an admissible test function in (2.4) due to the weak condition of $\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\in(\mathbb{V}_h^\lambda)^\circ$ compared to $\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\in(H^{-1/2}(\Gamma))^\circ=\{0\}$. Also, if we work in the $H_\rho^1(y^\alpha,\mathbb{R}^d\times\mathbb{R}^+)$-setting (i.e., working with global functions instead of pairs), the test function $\mathcal{V}_\Omega^{\mathcal{Y}}\chi_\Omega+\mathcal{V}_\star^{\mathcal{Y}}\chi_{\Omega^c}$ is not continuous along Γ due to a possible jump of size $\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}$. However, if we use the pointwise equation (4.1) and integrate back by parts, we get that

$$B^{\mathcal{Y}}\left(\mathcal{U}^{\mathcal{Y}}-\mathcal{U}_h^{\mathcal{Y}},\mathcal{V}_h^{\mathcal{Y}}\right)=\int_0^{\mathcal{Y}} y^\alpha\left\langle[\![\partial_\nu\mathcal{U}_\star^{\mathcal{Y}}]\!],\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\right\rangle_{L^2(\Gamma)}\mathrm{d}y.$$

Since $\langle\lambda_h,\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\rangle_{L^2(\Gamma)}$ vanishes due to the requirement in $\gamma_\Gamma^-\mathcal{V}_\star^{\mathcal{Y}}\in(\mathbb{V}_h^\lambda)^\circ$, we can subtract such a term from the right-hand side without changing the equality, which shows the stated Galerkin orthogonality. □

Finally, we are in position to show our main result, Theorem 2.9, by combining the decay estimate with the previous two lemmas.

Proof of Theorem 2.9. We start with the triangle inequality

$$\left\|\mathcal{U}-\mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}\leq\|\mathcal{U}-\mathcal{U}^{\mathcal{Y}}\|_{\mathbb{H}_{\mathcal{Y}}}+\left\|\mathcal{U}^{\mathcal{Y}}-\mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}.$$

For the first term, we use the decay properties of Proposition 4.1, to obtain

$$\|\mathcal{U}-\mathcal{U}^{\mathcal{Y}}\|_{\mathbb{H}_{\mathcal{Y}}}\lesssim\left\|\mathcal{U}_\Omega-\mathcal{U}_\Omega^{\mathcal{Y}}\right\|_{H^1(y^\alpha,\Omega\times(0,\mathcal{Y}))}+\left\|\mathcal{U}_\star-\mathcal{U}_\star^{\mathcal{Y}}\right\|_{H_\rho^1(y^\alpha,\mathbb{R}^d\backslash\Gamma\times(0,\mathcal{Y}))}\lesssim\mathcal{Y}^{-\mu/2}\|f\|_{L^2(\Omega)}.$$

For the second term, we employ the coercivity of Theorem 2.3, the Galerkin orthogonality of Lemma 4.7, $\mathcal{U}_\star^{\mathcal{Y}}|_\Omega = 0$, and a trace inequality for Ω, which gives for arbitrary $\lambda_h \in \mathbb{V}_h^\lambda$ and $\mathcal{V}_h^{\mathcal{Y}} = (\mathcal{V}_\Omega^{\mathcal{Y}}, \mathcal{V}_\star^{\mathcal{Y}}) \in \mathbb{H}_{h,\mathcal{Y}}$ that

$$\begin{aligned}
\left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 &\lesssim B^{\mathcal{Y}}\left(\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}, \mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right) \\
&= B^{\mathcal{Y}}\left(\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}, \mathcal{U}^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right) + \int_0^{\mathcal{Y}} y^\alpha \left\langle [\![\partial_\nu \mathcal{U}_\star^{\mathcal{Y}}]\!] - \lambda_h, \gamma_\Gamma^- \left(\mathcal{U}_{h,\star}^{\mathcal{Y}} - \mathcal{V}_\star^{\mathcal{Y}}\right)\right\rangle_{L^2(\Gamma)} \mathrm{d}y \\
&\lesssim \varepsilon \left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 + \varepsilon^{-1}\left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 \\
&\quad + \int_0^{\mathcal{Y}} y^\alpha \left\|[\![\partial_\nu \mathcal{U}_\star^{\mathcal{Y}}]\!] - \lambda_h\right\|_{H^{-1/2}(\Gamma)} \left\|\gamma_\Gamma^-\left(\mathcal{U}_{h,\star}^{\mathcal{Y}} - \mathcal{V}_\star^{\mathcal{Y}}\right)\right\|_{H^{1/2}(\Gamma)} \mathrm{d}y \\
&\lesssim \varepsilon \left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 + \varepsilon^{-1}\left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 \\
&\quad + \varepsilon^{-1}\int_0^{\mathcal{Y}} y^\alpha \left\|[\![\partial_\nu \mathcal{U}_\star^{\mathcal{Y}}]\!] - \lambda_h\right\|_{H^{-1/2}(\Gamma)}^2 \mathrm{d}y + \varepsilon\left\|\mathcal{U}_h^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 \\
&\lesssim 2\varepsilon \left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 + (\varepsilon + \varepsilon^{-1})\left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 + \varepsilon^{-1}\int_0^{\mathcal{Y}} y^\alpha \left\|[\![\partial_\nu \mathcal{U}_\star^{\mathcal{Y}}]\!] - \lambda_h\right\|_{H^{-1/2}(\Gamma)}^2 \mathrm{d}y.
\end{aligned}$$

Taking ε sufficiently small and absorbing the first term in the left-hand side gives

$$\left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{U}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 \lesssim \left\|\mathcal{U}^{\mathcal{Y}} - \mathcal{V}_h^{\mathcal{Y}}\right\|_{\mathbb{H}_{\mathcal{Y}}}^2 + \int_0^{\mathcal{Y}} y^\alpha \left\|[\![\partial_\nu \mathcal{U}_\star^{\mathcal{Y}}]\!] - \lambda_h\right\|_{H^{-1/2}(\Gamma)}^2 \mathrm{d}y.$$

As $\mathcal{V}_h^{\mathcal{Y}} \in \mathbb{H}_{h,\mathcal{Y}}$ was arbitrary, we can take $\mathcal{V}_h^{\mathcal{Y}} = \Pi(\mathcal{U}_\Omega^{\mathcal{Y}}, \mathcal{U}_\star^{\mathcal{Y}}) \in \mathbb{H}_{h,\mathcal{Y}}$ with the operator Π of Lemma 4.6. Then, Lemma 4.6 together with the approximation properties of the *hp*-interpolation in $\mathcal{Y}$ gives

$$\begin{aligned}
\|\mathcal{U}^{\mathcal{Y}} - \Pi\mathcal{U}^{\mathcal{Y}}\|_{\mathbb{H}_{\mathcal{Y}}}^2 &\lesssim \int_0^{\mathcal{Y}} y^\alpha \left(\left\|\nabla(1 - \Pi_y)\mathcal{U}^{\mathcal{Y}}(y)\right\|_{L^2}^2 + \left\|\nabla(1 - \pi_\Omega)\mathcal{U}_\Omega^{\mathcal{Y}}(y)\right\|_{L^2(\Omega)}^2\right) \mathrm{d}y \\
&\lesssim \mathcal{Y}^{2\varepsilon} e^{-2\kappa p} + \int_0^{\mathcal{Y}} y^\alpha \left\|\nabla(1 - \pi_\Omega)\mathcal{U}_\Omega^{\mathcal{Y}}(y)\right\|_{L^2(\Omega)}^2 \mathrm{d}y.
\end{aligned}$$

Combining all estimates gives the stated result. □

Finally, we present the proof of Corollary 2.11 that gives first order convergence for a specific choice of discrete spaces.

Proof of Corollary 2.11. Employing (Faustmann & Rieder, 2023, Pro. 2.8)—which with the same techniques also holds for $\mathcal{Y} < \infty$ and a constant independent of $\mathcal{Y}$—together with the assumptions on $\Omega, \mathfrak{A}$ and f, we obtain control of second order x-derivatives of $\mathcal{U}^{\mathcal{Y}}$.

As $\lambda := \partial_\nu^+ \mathcal{U}_\star^{\mathcal{Y}}$, it is piecewise smooth, depending on the regularity of $\mathcal{U}_\star^{\mathcal{Y}}$. For $m = 0, 1$, denoting by π_{L^2} the L^2-projection onto $\mathcal{S}^{0,0}(\mathcal{T}_\Gamma)$ and using a trace estimate, it holds that

$$\begin{aligned} \|\lambda(y) - \lambda_h(y)\|^2_{H^{-1/2}(\Gamma)} &\overset{\text{(Sauter \& Schwab, 2011, Thm 4.1.33)}}{\lesssim} h^{1/2}\|\lambda(y) - \pi_{L^2}\lambda(y)\|^2_{L^2(\Gamma)} \\ &\overset{\text{(Sauter \& Schwab, 2011, Prop 4.1.31)}}{\lesssim} h^{1/2}h^m \sum_{K\in\mathcal{T}_\Gamma} \|\lambda(y)\|^2_{H^m(K)} \\ &\lesssim h^{1/2}h^m \left\|\mathcal{U}_\star^{\mathcal{Y}}(y)\right\|^2_{H^{m+3/2}(B_R(0)\backslash\Gamma)}. \end{aligned}$$

Interpolating between $m = 0$ and $m = 1$ gives

$$\|\lambda(y) - \lambda_h(y)\|^2_{H^{-1/2}(\Gamma)} \lesssim h\|\mathcal{U}_\star^{\mathcal{Y}}(y)\|^2_{H^2(B_R(0)\backslash\Gamma)}.$$

Multiplying with y^α and integrating with respect to y then controls the second term on the right-hand side of Theorem 2.9. For the first term, the approximation properties of the Scott–Zhang projection together with control of the second order x-derivatives gives first order convergence in h. Finally, the last two terms in Theorem 2.9 can also be bounded by Ch by choice of $\mathcal{Y}$ and p. □

5. Numerical examples

In this section, we present two numerical examples to underline the *a priori* estimates of Theorem 2.9 and Corollary 2.11 as well as one numerical example that illustrates the algebraic decay w.r.t. the truncation parameter $\mathcal{Y}$. As previously already mentioned, a nice feature of our numerical scheme is that software packages developed for integer order differential operators can be employed directly. As such, we implement our method based on a coupling of the libraries NGSolve (Schöberl, 2021, for the FEM-part) and Bempp-cl (Betcke & Scroggs, 2021, for the BEM-part) libraries.

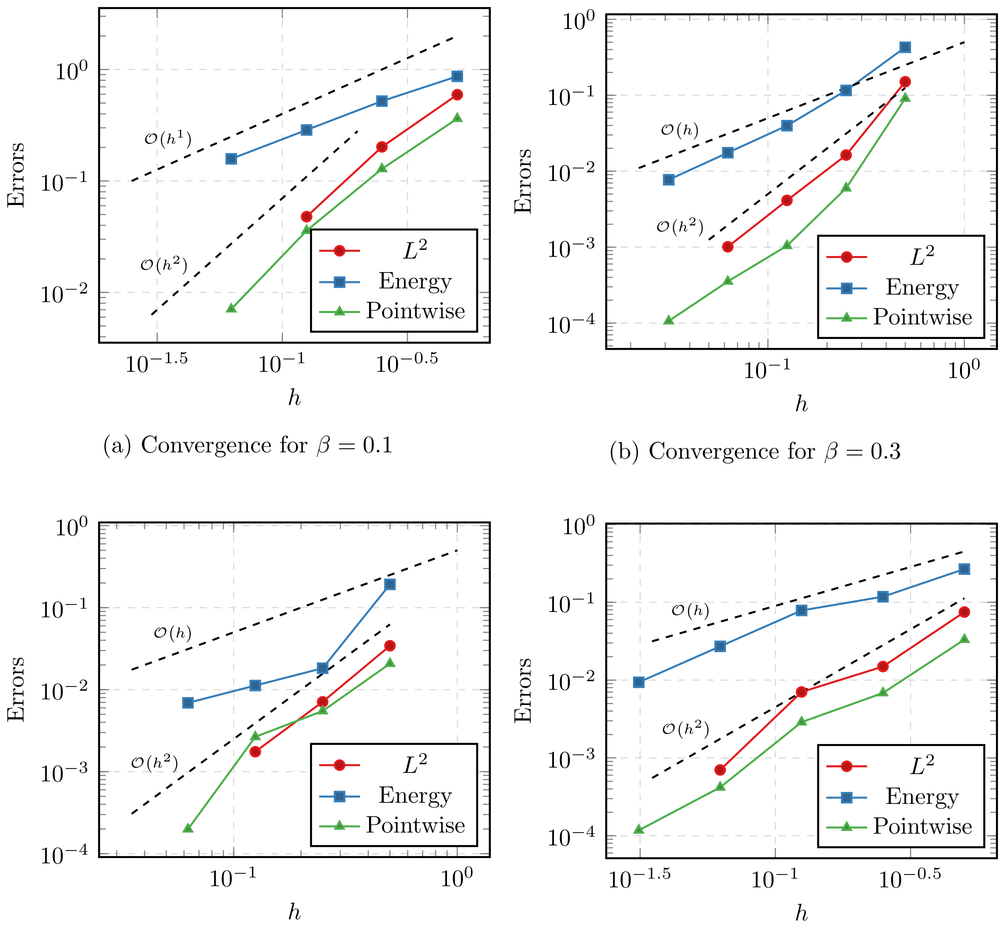
5.1 *Convergence for the fractional Laplacian*

In order to validate our numerical method, we consider the case $s = 0$ and the standard fractional Laplacian, i.e., $\mathfrak{A} = I$. In this case a representation formula is available from Caffarelli & Silvestre (2007). In fact, the fundamental solution for the fractional Laplacian is given by

$$\Psi(x) := \frac{C_{d,\beta}}{|x|^{d-2\beta}} \qquad x \in \mathbb{R}^d\backslash\{0\},\ d \neq 2\beta$$

with $C_{d,\beta} := \frac{\Gamma(d/2-\beta)}{2^{2\beta}\pi^{d/2}\Gamma(\beta)}$. Thus, for $f \in C_0^\infty(\Omega)$ we can write

$$u(x) = C_{d,\beta}\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2\beta}}\,\mathrm{d}y.$$



(a) Convergence for $\beta = 0.1$

(b) Convergence for $\beta = 0.3$

(c) Convergence for $\beta = 0.5$

(d) Convergence for $\beta = 0.7$

FIG. 1. Convergence of our discrete approximation to the exact solution for different fractional powers β in different norms.

For our numerical example, we choose $f(x) := \begin{cases} \exp(-0.1/(1-|x|)) & \text{for}|x| < 1 \\ 0 & \text{for}|x| \geq 0 \end{cases}$. We then calculate $u(x)$ at random sampling points x_j using spherical coordinates and Gauss–Jacobi numerical integration to deal with the singularity at $r = |x - x_j| = 0$, as well as standard Gauss-quadrature for the other coordinate directions.

In order to compute the energy error, we compute the energy differences. For standard Galerkin methods with bilinear form $a(\cdot,\cdot)$ and right-hand side $f(\cdot)$, it is well known that one can compute the energy error by the identity $\|u-u_h\|_E^2 = a(u,u) - a(u_h,u_h) = f(u) - f(u_h)$. Due to the more complicated form of our method, most notably the presence of the cutoff error, such an identity does not hold exactly.

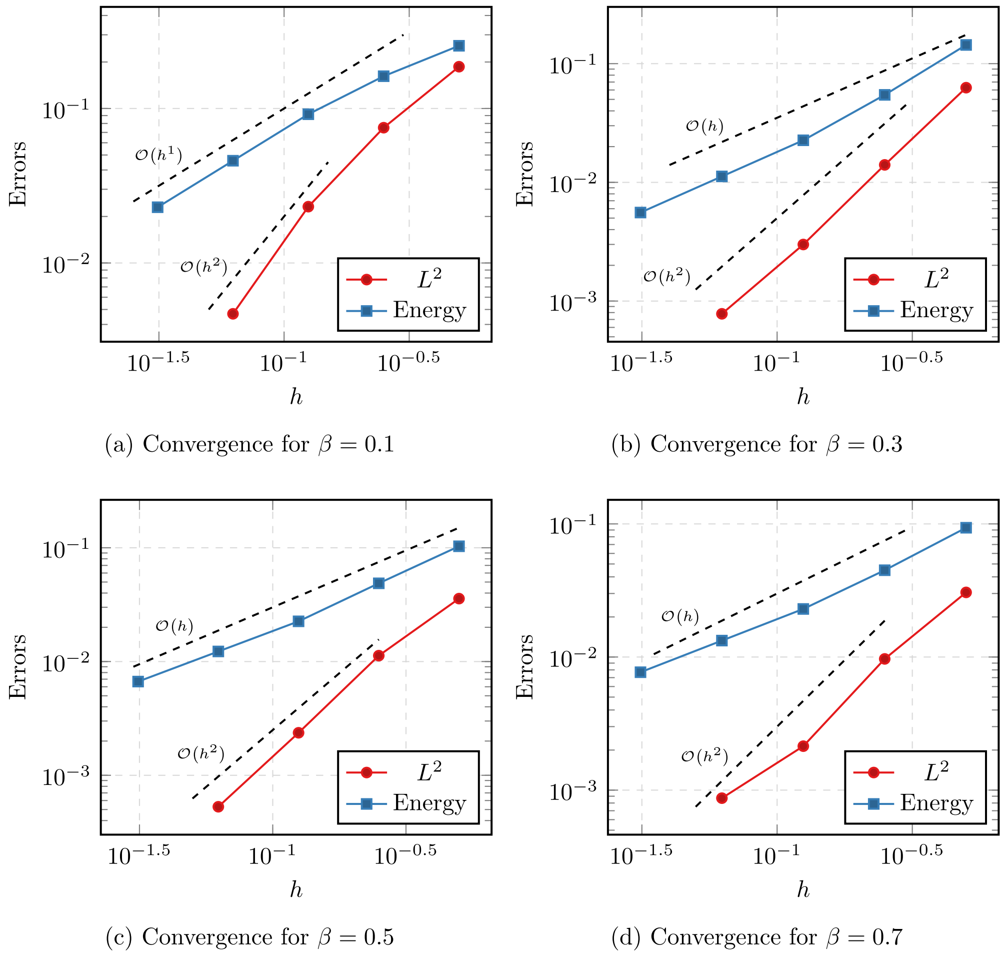


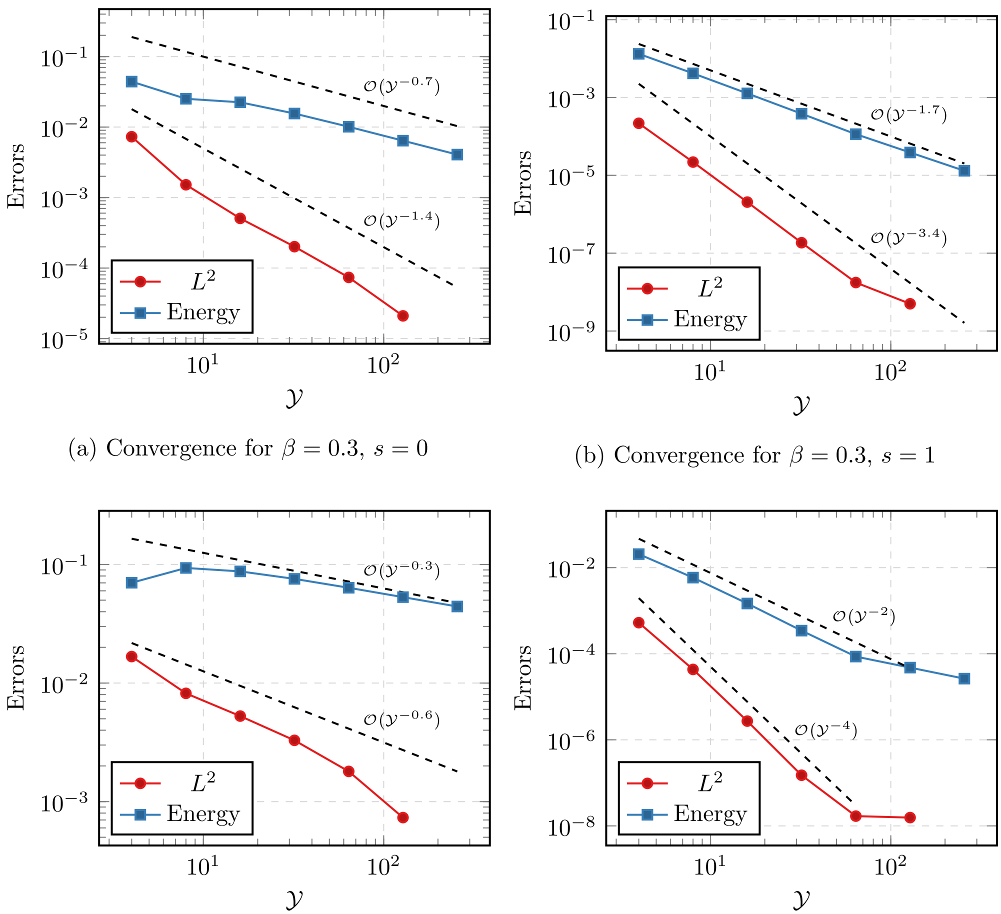
FIG. 2. Convergence of our discrete approximation to the exact solution for different fractional powers β in different norms, the nonconstant coefficient case.

Nevertheless, we expect the following identity to hold approximately

$$\left\|\mathcal{U} - \mathcal{U}_h^{\mathcal{Y}}\right\|^2_{H^1(y^\alpha,\mathbb{R}^d\times(0,\mathcal{Y}))} \approx (f, \mathrm{tr}_0\mathcal{U})_{L^2(\Omega)} - \left(f, \mathrm{tr}_0\mathcal{U}_h^{\mathcal{Y}}\right)_{L^2(\Omega)}.$$

We now further replace the unknown value $(f, \mathrm{tr}_0\mathcal{U})_{L^2(\Omega)}$ by the extrapolation from $(f, \mathrm{tr}_0\mathcal{U}_h^{\mathcal{Y}})_{L^2(\Omega)}$ for different refinements using Aitken's Δ^2-method. This will be our approximation of the true energy error. For the L^2-error, we use the approximation $\mathcal{U}_h^{\mathcal{Y}}$ on the finest grid as our stand-in for the exact solution and compare it to the other approximations by computing the L^2-difference of the traces at $y = 0$ using Gauss quadrature.

For the geometry, we used the unit cube $\Omega := [-1, 1]^3$. In the bounded domain Ω, we use piecewise linear Lagrangian finite elements on a quasi-uniform mesh of maximal mesh width h.



(a) Convergence for $\beta = 0.3$, $s = 0$

(b) Convergence for $\beta = 0.3$, $s = 1$

(c) Convergence for $\beta = 0.7$, $s = 0$

(d) Convergence for $\beta = 0.7$, $s = 1$

FIG. 3. Convergence of our discrete approximation to the exact solution for different fractional powers β in different norms.

In Fig. 1, we study the convergence of the proposed fully discrete method as we reduce the mesh size. In order to reduce all the error contributions, we choose the cutoff point $\mathcal{Y} = h^{-\frac{2}{1+\alpha}}$, which gives $\mathcal{O}(h)$ for the cutoff error in Proposition 4.1. Since the convergence with respect to the polynomial degree is exponential (but with unknown explicit rate), we use $p := \mathrm{round}(2m\log(m+1))$ where m is the number of uniform h-refinements. This gives a decrease of the y-discretization error, which is faster than $\mathcal{O}(h)$. Overall, we expect the energy error to behave like $\mathcal{O}(h)$ by Corollary 2.11. For the pointwise and L^2-errors we did not establish a rigorous theory in this work. Nonetheless, Fig. 1 shows convergence rates for these error measures of roughly order $\mathcal{O}(h^2)$.

5.2 *Convergence for the nonconstant coefficient case*

As a second numerical example, we consider as the domain Ω the unit sphere in $\mathbb{R}^3$. Instead of using the standard Laplacian with constant coefficients, we consider the following diffusion parameter and right-hand side:

$$\mathfrak{A}(x) := \begin{cases} 1 + |x|(1-|x|) & \text{for } |x| < 1 \\ 1 & \text{for } |x| \geq 0 \end{cases}, \qquad \text{and} \qquad f(x) := \begin{cases} |x|(1-|x|) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 0 \end{cases}$$

(with the slight abuse of notation of making $\mathfrak{A}(x)$ scalar valued). Since the coefficients are globally continuous, and we are working with lowest order elements, by Corollary 2.11 we expect to obtain first order convergence. Figure 2 supports the theoretical results. Since in this case the fundamental solution is not available, we cannot compute the pointwise error, but looking at the extrapolated energy and L^2-errors, we get the optimal rates.

5.3 *A study on the y-dependence*

We return to the setting of Section 5.1, i.e., constant coefficients, $\Omega = [-1, 1]^3$ and the same right-hand side f. However, here, we keep the discretization parameter h (and p as well as it depends by our choice on h) fixed as the finest mesh size used in Section 5.1 and vary only the cutoff parameter $\mathcal{Y}$. This should give an indication of the decay as $\mathcal{Y} \to \infty$. Proposition 4.1 predicts at least decay of $\mathcal{Y}^{-\mu/2}$, where $\mu := \begin{cases} 1 + |\alpha| & s > 0 \\ 1 + \alpha & s = 0 \end{cases}$ and $\alpha = 1 - 2\beta$.

In Fig. 3, we plot the decay of the energy error versus the increasing cut-off parameter $\mathcal{Y}$ for different cases of s and β.

We observe that in the case $s = 0$ the predicted energy decay closely matches the rate $\mathcal{Y}^{-\mu/2}$ as predicted in the theory. For the L^2-error, we see roughly a doubling of the convergence rate. In the case $s = 1$, our estimates are not sharp. Instead of the predicted convergence rate $\mathcal{Y}^{-0.7}$ (for both $\beta = 0.3$ and $\beta = 0.7$) we see much better convergence rates. A theoretical justification for these observations remains to be done.

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