Computers and Mathematics with Applications 176 (2024) 324–348

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An implementation of *hp*-FEM for the fractional Laplacian

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ABSTRACT

We consider the discretization of the $1d$-integral Dirichlet fractional Laplacian by *hp*-finite elements. We present quadrature schemes to set up the stiffness matrix and load vector that preserve the exponential convergence of *hp*-FEM on geometric meshes. The schemes are based on Gauss-Jacobi and Gauss-Legendre rules. We show that taking a number of quadrature points slightly exceeding the polynomial degree is enough to preserve root exponential convergence. The total number of algebraic operations to set up the system is $\mathcal{O}(N^{5/2})$, where N is the problem size. Numerical examples illustrate the analysis. We also extend our analysis to the fractional Laplacian in higher dimensions for *hp*-finite element spaces based on shape regular meshes.

1. Introduction

Fractional differential equations have become an important modelling tool, which sparked significant research in analysis and design and analysis of numerical methods, see, e.g., [8] and, for numerical methods, [4,6,25,14,20,21,36] and references therein.

We consider the fractional differential equation

$$(-\Delta)^s u = f \qquad \text{in } \Omega := (-1,1) \subset \mathbb{R}, \tag{1.1a}$$

$$u = 0 \qquad \text{in } \Omega^c := \mathbb{R} \setminus \overline{\Omega}, \tag{1.1b}$$

where $s \in (0,1)$, and f is analytic in $\overline{\Omega}$. Here, the operator $(-\Delta)^s$ is the Dirichlet integral fractional Laplacian, defined in (2.1) below. Among the discretization techniques, methods like the *hp*-finite element method (FEM) stand out as they achieve exponential convergence, [5,17], so that significantly fewer degrees of freedom are required to achieve the same accuracy compared to fixed order methods such as the classical *h*-FEM. This is particularly interesting for non-local problems such as fractional PDEs since there the stiffness matrices are fully populated with corresponding high memory requirements and high complexity to set up the matrices. In fact, [5] considers *hp*-FEM approximations on suitably designed geometric meshes in one space dimension and shows, for the *hp*-FEM approximation u_N to the solution u of (1.1), the energy-norm error estimate

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \le C \exp(-b\sqrt{N}), \tag{1.2}$$

where $b, C > 0$ are constants independent of the problem size N. Such exponential convergence results generalize to higher dimensions, e.g., in two space dimensions [17] asserts a similar convergence estimate where the square root in the exponent is replaced by $N^{1/4}$.

The exponential convergence in [5,17] is asserted ignoring variational crimes, in particular, it is shown under the assumption that u_N is the exact *hp*-finite element Galerkin approximation to u. However, a practical realization of the Galerkin method (2.2) requires

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https://doi.org/10.1016/j.camwa.2024.10.005

the evaluation of singular integrals by numerical quadrature. In the present work we develop and analyze quadrature schemes that preserve the exponential convergence (1.2). The quadratures are based on Gauss-Legendre and Gauss-Jacobi rules, and the analysis is performed in the framework of the First Strang Lemma. The key observation is that the hyper-singular integrand can be transformed such that singularities are aligned with coordinate axes, which allows for efficient treatment with Gauss-Jacobi rules.

The issue of evaluating singular integrals has already appeared in the context of boundary element methods (BEMs), [32]. For the kernels of BEM-operators arising from second order elliptic boundary value problems, regularizing transformations for the singular integrals have been devised that fully remove the singularity so that standard quadrature techniques can be brought to bear and a full quadrature error analysis is available, [32, Chap. 5]. For meshes with certain structure it is even possible to evaluate the stiffness matrices of hp-BEM explicitly, [27,19].

Generalizing the quadrature techniques described in [32, Chap. 5] the works [12,10,9,13] present and analyze regularizing transformations for a class of integrands that includes products of analytic/Gevrey-regular functions and singular functions; computationally, an essential point of these transformations is that they lead to the use of products of Gauss-Legendre and hp-quadrature or Gauss-Jacobi quadrature. Using similar transformations (in $1d$) and building on these works (for $d > 1$), our analysis considers the specific case of hp-FEM for the fractional Laplacian, explicitly works out the dependence on the polynomial degree p of the ansatz space, and asserts exponential convergence of the fully discrete method. The work to set up the stiffness matrix is algebraic in the problem size.

Implementations of the spectral fractional Laplacian have been proposed in the literature. Low order (for $d \geq 1$) Galerkin methods include [2,3,16] and typically exploit that a specific choice of basis is made in contrast to the present quadrature-based approach. Especially for $1d$ fractional differential equations, spectral and spectral element methods are available in the literature, see, e.g., [21,26,33,35,36,25,30,11] and references therein. The $1d$ quadrature techniques employed in the present work on shape regular meshes are closely related to those presented independently in [30]. Compared to these works, an important novel aspect of the present work is the full quadrature error analysis that rigorously establishes that taking $n \geq p+1$ quadrature points ($p > 0$ denoting the employed polynomial degree) is sufficient to retain the exponential convergence of hp-FEM.

In the present article, we consider the $1d$ case in great detail to make key concepts appear clearly. Extensions to $d > 1$ are possible, but come with additional (technical) difficulties. We present an analysis for $d > 1$ for shape regular meshes based on the regularizing transformations of [10] in Section 6. We hasten to add that exponential convergence (both in terms of error versus number of degrees of freedom and error versus computational work) of hp-FEM in $d \geq 2$ requires anisotropic elements with large aspect ratio, [17]. A quadrature error analysis for meshes including anisotropic elements is the topic of a forthcoming work.

The present article is structured as follows: In Section 2, we introduce our model problem and formulate the main result, exponential convergence of hp-FEM in the presence of quadrature, in Theorem 2.4. Section 3 specifies the Gaussian quadrature rules and the resulting approximation of the bilinear and linear forms in the weak formulation of the model problem. Section 3.1 shows stability of the method under quadrature. Section 4 provides the proofs of our main results using the First Strang Lemma, while the consistency analysis is postponed to Section 5. Section 6 extends the $1d$-analysis to higher dimensions for shape regular meshes based on the quadrature techniques developed in [12,10].

Finally, Section 7 provides numerical examples illustrating the performance of the quadrature scheme.

2. Main results

For $s \in (0,1)$, we consider the integral fractional Laplacian defined for univariate functions u pointwise as the principal value singular integral

$$(-\Delta)^s u(x) := C(s)\,\text{P.V.} \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\,dy \quad \text{with} \quad C(s) := -2^{2s}\frac{\Gamma(s+1/2)}{\pi^{1/2}\Gamma(-s)}, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Appropriate function spaces for fractional differential equations are fractional Sobolev spaces, defined for $t \in (0,1)$ and any open set $\omega \subset \mathbb{R}^d$ by means of the Aronstein-Slobodeckij seminorm

$$|v|^2_{H^t(\omega)} = \int_\omega \int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{d+2t}}\,dy\,dx, \qquad \|v\|^2_{H^t(\omega)} = \|v\|^2_{L^2(\omega)} + |v|^2_{H^t(\omega)}.$$

In order to incorporate the exterior Dirichlet condition, we define $r(x) := \text{dist}(x, \partial\Omega)$ and introduce the space $\widetilde{H}^t(\Omega) := \left\{ u \in H^t(\mathbb{R}^d) : u \equiv 0 \text{ on } \mathbb{R}^d \backslash \overline{\Omega} \right\}$ with norm

$$\|v\|^2_{\widetilde{H}^t(\Omega)} := \|v\|^2_{H^t(\Omega)} + \|v/r^t\|^2_{L^2(\Omega)}.$$

With the exception of Section 6 the domain $\Omega = (-1,1)$ always denotes the bounded open interval from our model problem (1.1); in Section 6, we will consider polyhedral $\Omega \subset \mathbb{R}^d$. We will use the fact that the norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$ and the seminorm $|\cdot|_{H^s(\mathbb{R})}$ are equivalent on $\widetilde{H}^s(\Omega)$, [28]. The weak form of the fractional PDE (1.1) reads: find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := \frac{C(s)}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\,dy\,dx = \langle f, v\rangle_{L^2(\Omega)} =: l(v) \qquad \forall v \in \widetilde{H}^s(\Omega). \tag{2.2}$$

Since $a(\cdot,\cdot) : \widetilde{H}^s(\Omega)\times\widetilde{H}^s(\Omega)\to\mathbb{R}$ is continuous and coercive on $\widetilde{H}^s(\Omega)$, (2.2) is uniquely solvable by the Lax-Milgram Lemma, see [1, Sec. 2.1].

For the discretization of the weak formulation, we employ piecewise polynomials on shape regular meshes.

Definition 2.1 *(Shape regular meshes and spline spaces).* For an interval $\mathcal{I}\subset\mathbb{R}$, we denote its length by $h_{\mathcal{I}} := \operatorname{diam}(\mathcal{I})$. For a bounded interval $\Omega=(x_0,x_M)$, let the points $x_0<x_1<\cdots<x_M$ determine the mesh $\mathcal{T}_\gamma=\{T_i := (x_{i-1},x_i) : i=1,\dots,M\}$. The mesh $\mathcal{T}_\gamma$ is said to be γ-shape regular, if

$$\gamma\, h_{T_i}\le h_{T_j}\quad\text{for all}\quad T_i,T_j\in\mathcal{T}_\gamma\quad\text{with}\quad \overline{T_i}\cap\overline{T_j}\neq\emptyset. \tag{2.3}$$

Based on $\mathcal{T}_\gamma$, we define finite dimensional spline spaces by

$$S^{p,1}(\mathcal{T}_\gamma) := \{u\in H^1(\Omega) : u|_T\in\mathcal{P}_p(T)\text{ for all }T\in\mathcal{T}_\gamma\},$$
$$S_0^{p,1}(\mathcal{T}_\gamma) := S^{p,1}(\mathcal{T}_\gamma)\cap H_0^1(\Omega).$$

Here, $\mathcal{P}_p(T)$ denotes the space of all polynomials of degree at most $p\in\mathbb{N}$ on T. The standard basis for $S_0^{p,1}(\mathcal{T}_\gamma)$ is given by

$$\mathcal{B}=\mathcal{B}^{lin}\cup\mathcal{B}^{Leg}, \tag{2.4}$$

where $\mathcal{B}^{lin} := \{\varphi_i : i=1,\dots,M-1\}$ are the hat functions associated with the interior nodes x_i, $i=1,\dots,M-1$, and the higher order modes are collected in $\mathcal{B}^{Leg} := \cup_{T\in\mathcal{T}_\gamma}\mathcal{B}_T$ with *element bubble functions* from $\mathcal{B}_T := \{\varphi_{T,i} : i=2,\dots,p\}$. For an element $T=(x_\ell,x_r)$ of length $h_T=x_r-x_\ell$, the element bubble functions are given by

$$\varphi_{T,i}(x)=\begin{cases}\int_{-1}^{-1+2(x-x_\ell)/h_T} P_{i-1}(t)\,dt & x\in T,\\ 0 & x\in\Omega\setminus\overline{T},\end{cases} \tag{2.5}$$

where P_i is the i-th Legendre polynomial.

The hp-FEM approximation u_N is given by Galerkin discretization of (2.2): Find $u_N\in S_0^{p,1}(\mathcal{T}_\gamma)$ such that

$$a(u_N,v_N)=l(v_N)\quad\text{for all } v_N\in S_0^{p,1}(\mathcal{T}_\gamma). \tag{2.6}$$

For a given basis $\mathcal{B} := \{\varphi_1,\dots,\varphi_N\}$ of $S_0^{p,1}(\mathcal{T}_\gamma)$, finding the solution $u_N := \sum_{i=1}^N x_i\varphi_i$ is equivalent to solving the linear system

$$Ax=b, \tag{2.7}$$

where $A\in\mathbb{R}^{N\times N}$ with $A_{ij}=a(\varphi_j,\varphi_i)$ and $b\in\mathbb{R}^N$ with $b_i := \langle f,\varphi_i\rangle_{L^2(\Omega)}$. Setting up the linear system requires evaluating the bilinear form $a(\cdot,\cdot)$ for all pairs of basis functions, which means calculating (singular) double integrals. Computing the linear form $l(\cdot)$ for all basis functions leads to a routine problem of calculating integrals involving f.

Our main convergence results are formulated for a specific kind of shape regular meshes, so-called geometric meshes, defined in the following Definition 2.2. However, we emphasize that the analysis of the consistency errors of the bilinear and linear forms in Chapter 5 hold for arbitrary shape regular meshes.

Definition 2.2 *(Geometric mesh $\mathcal{T}^L_{geo,\sigma}$ and basis $\mathcal{B}^{geo}$ of the spline space $S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$).* Given a grading factor $\sigma\in(0,1)$ and a number $L\in\mathbb{N}$ of layers, the *geometric mesh* $\mathcal{T}^L_{geo,\sigma}=\{T_i : i=1,...,2L+2\}$ with $2L+2$ elements $T_i=(x^{geo}_{i-1},x^{geo}_i)$ is defined by the nodes

$$x_0^{geo} := -1,\quad x_i^{geo}=-1+\sigma^{L-i+1}\text{ for } i=1,\dots,L,$$
$$x_{i+1}^{geo}=1-\sigma^{i-L}\text{ for } i=L,\dots,2L,\quad x_{2L+2}^{geo} := 1.$$

We note that $N := \dim S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})\sim pL$ and that $\mathcal{T}^L_{geo,\sigma}$ is shape regular with $\gamma=\sigma$. The basis $\mathcal{B}^{geo}$ for $S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$ is taken as the basis of Definition 2.1 for the mesh $\mathcal{T}^L_{geo,\sigma}$.

In [5] the following exponential convergence result for the difference in the energy norm between the solution u in (2.2) and its hp-FEM approximation u_N from (2.6) on geometric meshes $\mathcal{T}_\gamma=\mathcal{T}^L_{geo,\sigma}$ is shown:

Proposition 2.3 *([5]). Let $\mathcal{T}^L_{geo,\sigma}$ be a geometric mesh on the interval $\Omega=(-1,1)$ with grading factor $\sigma\in(0,1)$ and L layers of refinement towards the boundary points. Let the data f be analytic in $\overline{\Omega}$. Let $u_N\in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$ solve (2.6) with $\mathcal{T}_\gamma=\mathcal{T}^L_{geo,\sigma}$ and u solve (2.2). Then, there are b, $C>0$ and for all $\varepsilon>0$ there is $C_\varepsilon>0$ such that for all p and L there holds*

$$\|u-u_N\|_{\widetilde{H}^s(\Omega)}\le Ce^{-bp}+C_\varepsilon\sigma^{(1/2-\varepsilon)L}. \tag{2.8}$$

The choice $L \sim p$ leads to convergence $\|u-u_N\|_{\widetilde{H}^s(\Omega)} \le C\exp(-b'\sqrt{N})$, where N is the dimension of $S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$ and C, b' are constants independent of N.

In practice, it is not possible to set up the linear system of equations corresponding to (2.6) exactly due to the presence of the kernel function $|x-y|^{-1-2s}$. To implement the hp-FEM method, we therefore have to work with computable numerical approximations $\widetilde{a}_n(\cdot,\cdot)$ and $\widetilde{l}_n(\cdot)$ of the bilinear form $a(\cdot,\cdot)$ and the right-hand side $l(\cdot)$, respectively. The fully discrete problem then reads: Find $\widetilde{u}_{N,n} \in S_0^{p,1}(\mathcal{T}_\gamma)$ such that

$$\widetilde{a}_n(\widetilde{u}_{N,n}, v_N) = \widetilde{l}_n(v_N) \quad \text{for all } v_N \in S_0^{p,1}(\mathcal{T}_\gamma). \tag{2.9}$$

In Section 3 below, we specify the approximations $\widetilde{a}_n(\cdot,\cdot)$ and $\widetilde{l}_n(\cdot)$ based on (weighted) Gaussian quadrature rules with n points. Our main result formulated in the following states that the exponential convergence rate of $\widetilde{u}_N$ to the solution u is preserved.

Theorem 2.4 *(Exponential convergence under quadrature). Let $\mathcal{T}^L_{geo,\sigma}$ be a geometric mesh on the interval $\Omega := (-1,1)$ with grading factor $\sigma \in (0,1)$ and L layers of refinement towards the boundary points. Let f be analytic in $\overline{\Omega}$, denote by $u \in \widetilde{H}^s(\Omega)$ the solution to (2.2) and by $\widetilde{u}_{N,n} \in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$ the solution to (2.9) with $\mathcal{T}_\gamma = \mathcal{T}^L_{geo,\sigma}$, where $\widetilde{a}_n(\cdot,\cdot)$ and $\widetilde{l}_n(\cdot)$ are defined in (3.11) and (3.3), respectively. The index n indicates the number of quadrature points that are used per integral and element.*

There are constants C, $b > 0$ and, for each $\varepsilon > 0$, a constant C_ε (depending on f, s, and σ) such that for any $n \ge p+1$, p, $L \in \mathbb{N}$ and $r \in \{1,\dots,p\}$, there holds

$$\|u-\widetilde{u}_{N,n}\|_{\widetilde{H}^s(\Omega)} \le Ce^{-br} + C_\varepsilon \sigma^{(1/2-\varepsilon)L} + CL^2r^3p^3\rho^{1+p+r-2n}. \tag{2.10}$$

For $L \sim p$ and $n \ge p+1$ there holds in terms of the problem size $N := \dim S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})$ for some C, $b' > 0$ independent of N and n

$$\|u-\widetilde{u}_{N,n}\|_{\widetilde{H}^s(\Omega)} \le C\exp(-b'\sqrt{N}). \tag{2.11}$$

For $L \sim p \sim n$ and the basis $\mathcal{B}^{geo}$ from Definition 2.2, the number of algebraic operations to set up the linear system (2.7) is $\mathcal{O}(L^5) = \mathcal{O}(N^{5/2})$.

3. Quadrature approximations

Throughout this section, we consider γ-shape regular meshes $\mathcal{T}_\gamma$. We start with some general definitions and notations. $\widehat{T} := (0,1)$ denotes the reference element and, for each element $T := (x_\ell, x_r) \in \mathcal{T}_\gamma$, we define the affine element map by

$$F_T : \widehat{T} \to T, \quad x \mapsto x_\ell + x\,h_T. \tag{3.1}$$

With a slight abuse of notation, we will naturally extend F_T to an affine function $\mathbb{C} \to \mathbb{C}$ when needed. For a function v defined on T, we write $\hat{v}_T$ for its pullback to the reference element

$$\hat{v}_T := v \circ F_T. \tag{3.2}$$

Our approximations to $a(\cdot,\cdot)$ and $l(\cdot)$ are based on the following (weighted) Gaussian quadrature rules. Let $\omega : (0,1) \to \mathbb{R}$ be a positive, integrable weight function. Then, we approximate

$$I(\Phi) := \int_0^1 \Phi(x)\omega(x)\,dx \approx \sum_{i=1}^n \omega_i \Phi(\xi_i) =: G_n(\Phi),$$

where ξ_i are the Gaussian quadrature nodes (zeros of orthogonal polynomials w.r.t. the ω-weighted L^2-inner product) and $\omega_i = \int_0^1 \omega(x)L_i(x)dx$ with the i-th Lagrange interpolation polynomials $L_i(x) = \prod_{j=1, j\neq i}^n \frac{x-\xi_j}{\xi_i-\xi_j}$ associated with the quadrature nodes $\xi_1,\dots,\xi_n$.

For $\omega \equiv 1$, we write $GL_n(\Phi)$ (Gauss-Legendre quadrature) for the quadrature rule. For integrands with singularities at the boundaries, we take $\omega(x) = (1-x)^\alpha x^\beta$, $\alpha,\beta > -1$ and write $GJ_n^{\alpha,\beta}(\Phi)$ (Gauss-Jacobi quadrature). For multivariate functions $\Phi(x,y)$, we will indicate by the subscript x, y the variable to which the quadrature rule is applied.

We start by deriving an approximation to the right-hand side $l(v) := \langle f, v\rangle_{L^2(\Omega)}$ in (2.6). Dividing the integration domain Ω into the elements $T \in \mathcal{T}_\gamma$, transforming them to the reference element $\widehat{T}$, and using Gauss-Legendre quadrature for each integral defines the linear form $\widetilde{l}_n(v_N)$ for $v_N \in S_0^{p,1}(\mathcal{T}_\gamma)$ by

$$l(v_N) = \int_\Omega v_N(x)f(x)\,dx = \sum_{T\in\mathcal{T}_\gamma} h_T \int_{\widehat{T}} \hat{v}_{N,T}(x)\hat{f}_T(x)\,dx \approx \sum_{T\in\mathcal{T}_\gamma} h_T\; GL_n(\hat{v}_{N,T}(x)\hat{f}_T(x)) =: \widetilde{l}_n(v_N). \tag{3.3}$$

The approximation of the bilinear form $a(\cdot,\cdot)$ is more involved since we have to deal with hyper-singular double integrals. Using symmetry and dividing the integration domain $\mathbb{R}\times\mathbb{R}$ into the elements $T\in\mathcal{T}_\gamma$ and the complementary set Ω^c leads to

$$a(v_N,w_N)=\frac{C(s)}{2}\Big(\sum_{T\in\mathcal{T}_\gamma}\sum_{T'\in\mathcal{T}_\gamma} I_{T,T'}(v_N,w_N)+2\sum_{T\in\mathcal{T}_\gamma} I_{T,\Omega^c}(v_N,w_N)\Big),$$

where, for arbitrary sets $A,\ B\subset\mathbb{R}$, the symbol $I_{A,B}(v_N,w_N)$ denotes

$$I_{A,B}(v_N,w_N):=\int_A\int_B \frac{(v_N(x)-v_N(y))(w_N(x)-w_N(y))}{|x-y|^{1+2s}}\,dy\,dx. \tag{3.4}$$

The integral over Ω^c can be integrated explicitly. All other integrals have to be transformed to a reference square and then approximated by a suitable quadrature rule, which leads to four cases.

Identical elements ($T=T'$):

We transform the double integral $I_{T,T}(u,v)$ to the reference square $\widehat{T}\times\widehat{T}$ and divide this integration domain into the triangles $A_1:=\{(x,y)\mid 0<x<1,\ 0<y<x\}$ and $A_2:=\{(x,y)\mid 0<y<1,\ 0<x<y\}$. As the integrand is invariant under the transformation $(x,y)\mapsto(y,x)$, we notice that both integrals are the same. Employing the Duffy transformation, i.e., $(x,y)\mapsto(x,xy)$, leads to

$$\begin{aligned}I_{T,T}(v_N,w_N)&=2h_T^{1-2s}\int_{\widehat{T}}\int_{\widehat{T}}\frac{(\hat v_{N,T}(x)-\hat v_{N,T}(xy))(\hat w_{N,T}(x)-\hat w_{N,T}(xy))}{|x-xy|^2}x^{2-2s}(1-y)^{1-2s}\,dy\,dx\\&\approx 2h_T^{1-2s}GJ_{n,x}^{0,2-2s}\circ GJ_{n,y}^{1-2s,0}\left(\frac{(\hat v_{N,T}(x)-\hat v_{N,T}(xy))(\hat w_{N,T}(x)-\hat w_{N,T}(xy))}{|x-xy|^2}\right)\\&=:Q^n_{T,T}(v_N,w_N).\end{aligned}\tag{3.5}$$

We note that after the separation of the weight function, the integrand in (3.5) is a polynomial since only removable singularities are left.

Remark 3.1. Our choice of the Gauss-Jacobi weight function is not the only possible option, as, e.g., one could cancel one additional power of x in the first equality in (3.5). However, our choice is optimal in the sense that it decreases the polynomial degree of the integrand as much as possible. ■

Adjacent elements ($T\neq T'$ with $\overline{T}\cap\overline{T'}\neq\emptyset$):

Without loss of generality, we may assume that T is the left neighbor of T', otherwise T and T' change their roles. Then, the element maps transform the singularity at $\overline{T}\cap\overline{T'}$ to the point $(1,0)$ in the reference square. With an additional transformation $(x,y)\mapsto(1-x,y)$ we are now in a similar setting as in the previous case. The integral can be split into integrals over A_1 and A_2 and employing the Duffy transformation on A_1 (for A_2 we take $(x,y)\mapsto(xy,y)$) leads to

$$\begin{aligned}I_{T,T'}(v_N,w_N)=h_Th_{T'}\Bigg(&\int_{\widehat{T}}\int_{\widehat{T}}\frac{(\hat v_{N,T}(1-x)-\hat v_{N,T'}(xy))(\hat w_{N,T}(1-x)-\hat w_{N,T'}(xy))}{|h_T+yh_{T'}|^{1+2s}\,x^2}x^{2-2s}\,dy\,dx\\&+\int_{\widehat{T}}\int_{\widehat{T}}\frac{(\hat v_{N,T}(1-xy)-\hat v_{N,T'}(y))(\hat w_{N,T}(1-xy)-\hat w_{N,T'}(y))}{|xh_T+h_{T'}|^{1+2s}\,y^2}y^{2-2s}\,dy\,dx\Bigg).\end{aligned}\tag{3.6}$$

The singularities appear only in one variable in each integral, for which we employ Gauss-Jacobi quadrature, while in the other variable Gauss-Legendre quadrature is sufficient. This gives the approximation

$$\begin{aligned}Q^n_{T,T'}(v_N,w_N):=h_Th_{T'}\Bigg(&GJ_{n,x}^{0,2-2s}\circ GL_{n,y}\left(\frac{(\hat v_{N,T}(1-x)-\hat v_{N,T'}(xy))(\hat w_{N,T}(1-x)-\hat w_{N,T'}(xy))}{|h_T+yh_{T'}|^{1+2s}\,x^2}\right)\\&+GL_{n,x}\circ GJ_{n,y}^{0,2-2s}\left(\frac{(\hat v_{N,T}(1-xy)-\hat v_{N,T'}(y))(\hat w_{N,T}(1-xy)-\hat w_{N,T'}(y))}{|xh_T+h_{T'}|^{1+2s}\,y^2}\right)\Bigg).\end{aligned}\tag{3.7}$$

Separated elements ($\overline{T}\cap\overline{T'}=\emptyset$):

This time, the integrand is not singular. Therefore, one can directly transform the double integral to the reference square and employ tensor product Gauss-Legendre quadrature, which produces as the approximation of $I_{T,T'}(v_N,w_N)$ the expression

$$Q^n_{T,T'}(v_N,w_N):=h_Th_{T'}\ GL_{n,x}\circ GL_{n,y}\left(\frac{(\hat v_{N,T}(x)-\hat v_{N,T'}(y))(\hat w_{N,T}(x)-\hat w_{N,T'}(y))}{|(1-x)h_T+\mathrm{dist}_{T,T'}+yh_{T'}|^{1+2s}}\right),$$

where $\mathrm{dist}_{T,T'}$ denotes the Euclidean distance between the elements T and T'.

Complement part (I_{T,Ω^c}):

The inner integral over Ω^c can be calculated explicitly exploiting that the functions $v_N, w_N \in S_0^{p,1}(\mathcal{T}_\gamma)$ vanish outside of $\Omega = (-1,1)$. The outer integral can be transformed to the reference element $\widehat{T}$, which gives

$$I_{T,\Omega^c}(v_N,w_N) := \frac{h_T}{2s} \int_{\widehat{T}} \frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{-1\}} + x h_T|^{2s}} + \frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{1\}} + (1-x) h_T|^{2s}}\, dx. \tag{3.8}$$

If T is an interior element, i.e., $\overline{T} \cap \partial\Omega = \emptyset$, we employ Gauss-Legendre quadrature

$$Q^n_{T,\Omega^c}(v_N,w_N) := \frac{h_T}{2s}\left(GL_n\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{-1\}} + x h_T|^{2s}} \right) + GL_n\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{1\}} + (1-x) h_T|^{2s}} \right)\right).$$

For $\overline{T} \cap \partial\Omega = \{-1\}$, we set

$$Q^n_{T,\Omega^c}(v_N,w_N) := \frac{h_T}{2s}\left(GJ_n^{0,2-2s}\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{x^2 h_T^{2s}} \right) + GL_n\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{1\}} + (1-x) h_T|^{2s}} \right)\right), \tag{3.9}$$

and for $\overline{T} \cap \partial\Omega = \{1\}$

$$Q^n_{T,\Omega^c}(v_N,w_N) := \frac{h_T}{2s}\left(GL_n\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{|\operatorname{dist}_{T,\{-1\}} + x h_T|^{2s}} \right) + GJ_n^{2-2s,0}\left(\frac{\hat{v}_{N,T}(x)\hat{w}_{N,T}(x)}{(1-x)^2 h_T^{2s}} \right)\right). \tag{3.10}$$

Now, having defined $Q^n_{A,B}(v_N,w_N)$ for all cases of integrals $I_{A,B}(v_N,w_N)$, we obtain the approximated bilinear form as

$$\widetilde{a}_n(v_N,w_N) := \frac{C(s)}{2}\left(\sum_{T\in\mathcal{T}_\gamma}\sum_{T'\in\mathcal{T}_\gamma} Q^n_{T,T'}(v_N,w_N) + 2\sum_{T\in\mathcal{T}_\gamma} Q^n_{T,\Omega^c}(v_N,w_N)\right). \tag{3.11}$$

3.1. Stability of the quadrature rule

Positivity of the kernel function $(x,y) \mapsto |x-y|^{-1-2s}$ and the Gauss-Legendre/Gauss-Jacobi weights as well as exactness of the Gauss-Legendre/Gauss-Jacobi quadrature allow us to prove the following stability result:

Lemma 3.2. *Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh, $u \in S^{p,1}(\mathcal{T}_\gamma)$. Then, the following holds:*

(i) For all $n \geq 1$ and all $T, T' \in \mathcal{T}_\gamma \cup \{\Omega^c\}$, we have $Q^n_{T,T'}(u,u) \geq 0$.

(ii) Let $n \geq p$ and $u \in S_0^{p,1}(\mathcal{T}_\gamma)$. Then $\widetilde{a}_n(u,u) = 0$ implies $u = 0$. In particular, the stiffness matrix A in (2.7) is symmetric positive definite.

Furthermore, there is $C_{coer} > 0$ depending only on γ and s such that for all $u \in S_0^{p,1}(\mathcal{T}_\gamma)$ the following assertions hold:

(iii) (Identical elements) For $n \geq p$ and $T \in \mathcal{T}_\gamma$: $Q^n_{T,T}(u,u) = I_{T,T}(u,u)$.

(iv) (Adjacent elements) For $n \geq p+1$ and $(T,T') \in \mathcal{T}_\gamma \times (\mathcal{T}_\gamma \cup \{\Omega^c\})$ with $T \neq T'$ and $\overline{T} \cap \overline{T'} \neq \emptyset$: $Q^n_{T,T'}(u,u) \geq C_{coer} I_{T,T'}(u,u) \geq 0$.

(v) (Separated elements) For $n \geq p+1$ and $(T,T') \in \mathcal{T}_\gamma \times (\mathcal{T}_\gamma \cup \{\Omega^c\})$ with $\overline{T} \cap \overline{T'} = \emptyset$: $Q^n_{T,T'}(u,u) \geq C_{coer} I_{T,T'}(u,u) \geq 0$.

Proof. *Proof of (i):* This follows from the positivity of the kernel and the Gauss-Legendre/Gauss-Jacobi weights.

Proof of (ii): From (i), we get for $u \in S_0^{p,1}(\mathcal{T}_\gamma)$ with $\widetilde{a}_n(u,u) = 0$

$$0 = \frac{2}{C(s)}\widetilde{a}_n(u,u) = \sum_{T\in\mathcal{T}_\gamma\cup\{\Omega^c\}}\sum_{T'\in\mathcal{T}_\gamma\cup\{\Omega^c\}} Q^n_{T,T'}(u,u) \overset{(i)}{\geq} \sum_{T\in\mathcal{T}_\gamma} Q^n_{T,T}(u,u) \overset{(iii)}{=} \sum_{T\in\mathcal{T}_\gamma} I_{T,T}(u,u) \geq 0.$$

Hence, $|u|_{H^s(T)} = 0$ for each $T \in \mathcal{T}_\gamma$ so that u is constant on each element. By continuity of u, it is constant on Ω, and the boundary conditions then imply $u = 0$.

Proof of (iii): For $n \geq p$, the univariate Gauss-Jacobi quadrature in (3.5) is exact for polynomials of degree $2p-1$. Inspection of (3.5) shows that the argument is the square of a polynomial of degree $p-1$ in each variable.

Proof of (iv): For $n \geq p+1$, the univariate Gauss-Jacobi quadratures in (3.7) are exact for polynomials of degree $2p+1$. We study the cases $(T,T') \in \mathcal{T}_\gamma \times \mathcal{T}_\gamma$ and $(T,T') \in \mathcal{T}_\gamma \times \{\Omega^c\}$ separately, starting with $(T,T') \in \mathcal{T}_\gamma \times \mathcal{T}_\gamma$. We only consider the first term in (3.7), the other one being handled analogously. Let $u \in S_0^{p,1}(\mathcal{T}_\gamma)$. For the pull-backs $\hat{u}_T$, $\hat{u}_{T'}$ to the reference element $\widehat{T}$ of the functions $u|_T$, $u|_{T'}$, we get by continuity of u at $\overline{T} \cap \overline{T'}$ that $\hat{u}_T(1) = \hat{u}_{T'}(0)$. Hence,

$$U(x,y) := \frac{\hat{u}_T(1-x) - \hat{u}_{T'}(xy)}{x}$$

is a polynomial of degree $p-1$ in x and of degree p in y. Using the positivity of the quadrature weights and the exactness of the quadrature rules (U^2 is a polynomial of degree $2p \leq 2p+1$ in each variable)

$$h_T h_{T'} GJ^{0,2-2s}_{n,x} \circ GL_{n,y} \left(\frac{(\hat{u}_T(1-x) - \hat{u}_{T'}(xy))^2}{x^2(h_T + yh_{T'})^{1+2s}} \right) = h_T h_{T'} GJ^{0,2-2s}_{n,x} \circ GL_{n,y} \left(U^2(x,y)(h_T + yh_{T'})^{-(1+2s)} \right)$$
$$\geq h_T h_{T'} GJ^{0,2-2s}_{n,x} \circ GL_{n,y} \left(U^2(x,y)(h_T + h_{T'})^{-(1+2s)} \right) = h_T h_{T'} \int_{x\in\hat{T}} \int_{y\in\hat{T}} U^2(x,y)(h_T + h_{T'})^{-(1+2s)} x^{2-2s}\, dy\, dx$$
$$\geq (1 + h_{T'}/h_T)^{-(1+2s)} h_T h_{T'} \int_{x\in\hat{T}} \int_{y\in\hat{T}} U^2(x,y)(h_T + yh_{T'})^{-(1+2s)} x^{2-2s}\, dy\, dx,$$

where we used in the last inequality that $h_T^{-(1+2s)} \geq (h_T + yh_{T'})^{-(1+2s)}$. We conclude in view of (3.6)

$$Q^n_{T,T'}(u,u) \geq C_{coer}\, I_{T,T'}(u,u),$$

where $C_{coer} := \inf\{(1 + h_{T'}/h_T)^{-(1+2s)} \,|\, T \in \mathcal{T}_\gamma, T' \text{ adjacent to } T\}$ depends only the shape regularity constant γ and s. The case $(T,T') \in \mathcal{T}_\gamma \times \{\Omega^c\}$ leads to two terms of the form (3.9) or (3.10). One term can always be analyzed in similar fashion as above and the other one can be treated as in the following case (v).

Proof of (v): This is handled similarly to the case of adjacent elements in (iv). We consider only the case $(T,T') \in \mathcal{T}_\gamma \times \mathcal{T}_\gamma$, the case $(T,T') \in \mathcal{T}_\gamma \times \{\Omega^c\}$ is handled similarly.

With $\hat{u}_T$, $\hat{u}_{T'}$ as above and using that polynomials of degree $2p+1$ are integrated exactly for $n \geq p+1$ we estimate

$$Q^n_{T,T'}(u,u) = h_T h_{T'} GL_{n,x} \circ GL_{n,y} \left((\hat{u}_T(x) - \hat{u}_{T'}(y))^2 ((1-x)h_T + \mathrm{dist}_{T,T'} + yh_{T'})^{-(1+2s)} \right)$$
$$\geq h_T h_{T'} GL_{n,x} \circ GL_{n,y} \left((\hat{u}_T(x) - \hat{u}_{T'}(y))^2 (h_T + \mathrm{dist}_{T,T'} + h_{T'})^{-(1+2s)} \right)$$
$$= h_T h_{T'} \int_{x\in\hat{T}} \int_{y\in\hat{T}} (\hat{u}_T(x) - \hat{u}_{T'}(y))^2 (h_T + \mathrm{dist}_{T,T'} + h_{T'})^{-(1+2s)}\, dy\, dx$$
$$\geq \left(\frac{\mathrm{dist}_{T,T'}}{h_T + \mathrm{dist}_{T,T'} + h_{T'}} \right)^{1+2s} h_T h_{T'} \int_{x\in\hat{T}} \int_{y\in\hat{T}} \frac{(\hat{u}_T(x) - \hat{u}_{T'}(y))^2}{((1-x)h_T + \mathrm{dist}_{T,T'} + yh_{T'})^{1+2s}}\, dy\, dx$$
$$\geq C_{coer} I_{T,T'}(u,u),$$

for a $C_{coer} > 0$ that depends solely on the shape regularity constant γ and s. □

Remark 3.3. The proof shows that the condition $n \geq p+1$ for the case of adjacent elements could be weakened in that p points suffice in one variable whereas $p+1$ point should be used in the other one. ■

Corollary 3.4. *Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh. There is $c_{coer} > 0$ depending only on γ and s such that for $n \geq p+1$*

$$\widetilde{a}_n(u,u) \geq c_{coer} \|u\|^2_{\widetilde{H}^s(\Omega)} \quad \forall u \in S^{p,1}_0(\mathcal{T}_\gamma). \tag{3.12}$$

Proof. We write

$$\widetilde{a}_n(u,u) = \frac{C(s)}{2} \sum_{T\in\mathcal{T}_\gamma\cup\{\Omega^c\}} \sum_{T'\in\mathcal{T}_\gamma\cup\{\Omega^c\}} Q^n_{T,T'}(u,u)$$

and use Lemma 3.2 to bound $Q^n_{T,T'}(u,u) \geq C_{coer} I^n_{T,T'}(u,u)$ for a $C_{coer} > 0$ depending only on γ and s. □

Remark 3.5. (i) Lemma 3.2 shows that it suffices to use $n \geq p$ quadrature points for the quadrature $Q^n_{T,T}$ to ensure solvability of the linear system (2.7). The condition $n \geq p+1$ stipulated in Corollary 3.4 leads to uniform (in p and $\mathcal{T}_\gamma$) coercivity. (ii) Remark 3.3 shows that for adjacent elements a "mixed" quadrature order could be employed to slightly reduce the number of quadrature points. (iii) The stability result Corollary 3.4 exploits positivity of the kernel and weights as well as a certain exactness property of the Gauss-Legendre/Gauss-Jacobi quadratures. One can avoid exploiting these properties and rely on a perturbation argument that uses consistency error estimates for the quadratures and the coercivity of the continuous bilinear form $a(\cdot,\cdot)$. This approach, which results in the stronger requirement $n \geq p + O(\log((p+1)(\#\mathcal{T}_\gamma + 1)))$ is discussed in Lemma 4.4 below. ■

4. Proof of Theorem 2.4

The proof is based on the classical Strang Lemma, see, e.g., [7, Chap. 3]. In the present setting, it takes the following form:

Lemma 4.1 *(First Strang Lemma). Let $\mathcal{T}$ be a mesh on Ω and let $\widetilde{\alpha}_n > 0$ be such that $\widetilde{a}_n$ satisfies*

$$\widetilde{\alpha}_n \|v_N\|^2_{\widetilde{H}^s(\Omega)} \leq \widetilde{a}_n(v_N, v_N) \quad \textit{for all } v_N \in S^{p,1}_0(\mathcal{T}). \tag{4.1}$$

Then, with the continuity constant C_a of the bilinear form a, the difference $u-\widetilde{u}_{N,n}$ between the solutions $u \in \widetilde{H}^s(\Omega)$ of (2.2) and $\widetilde{u}_{N,n} \in S_0^{p,1}(\mathcal{T})$ of (2.9) satisfies

$$\|u-\widetilde{u}_{N,n}\|_{\widetilde{H}^s(\Omega)} \le \left(1+\frac{C_a}{\widetilde{\alpha}_n}\right)\left(\inf_{v\in S_0^{p,1}(\mathcal{T})}\left(\|u-v\|_{\widetilde{H}^s(\Omega)} + \sup_{w\in S_0^{p,1}(\mathcal{T})}\frac{|a(v,w)-\widetilde{a}_n(v,w)|}{\|w\|_{\widetilde{H}^s(\Omega)}}\right) + \sup_{w\in S_0^{p,1}(\mathcal{T})}\frac{|l(w)-\widetilde{l}_n(w)|}{\|w\|_{\widetilde{H}^s(\Omega)}}\right).$$

Lemma 4.1 indicates that we have to show lower bounds for the coercivity of $\widetilde{a}_n(\cdot,\cdot)$ as well as derive bounds for the consistency errors $|l(w)-\widetilde{l}_n(w)|$ and $|a(v,w)-\widetilde{a}_n(v,w)|$. This is the subject of the following two lemmas, whose proofs are postponed to Section 5.

Lemma 4.2 *(Consistency error for l). Let f be analytic in $\overline{\Omega}$, and let $\mathcal{T}_\gamma$ be a γ-shape regular mesh. Let $l(v)=\langle f,v\rangle_{L^2(\Omega)}$ and let its approximation $\widetilde{l}_n(\cdot)$ be defined by (3.3). Then, there exists a constant $\rho>1$ depending only on f such that*

$$|l(v)-\widetilde{l}_n(v)| \le C_{s,f}\,\rho^{p-2n+1} p\|v\|_{\widetilde{H}^s(\Omega)} \qquad \text{for all } \ v\in S_0^{p,1}(\mathcal{T}_\gamma), \tag{4.2}$$

where $C_{s,f}>0$ is a constant that depends only on s and f.

Lemma 4.3 *(Consistency error for a). Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh, $a(\cdot,\cdot)$ be the bilinear form of (2.2) and $\widetilde{a}_n(\cdot,\cdot)$ be its approximation (3.11). Then, there exists a constant $\rho>1$ that depends only on the shape regularity constant γ such that for all $u\in S_0^{r,1}(\mathcal{T}_\gamma)$ and $v\in S_0^{p,1}(\mathcal{T}_\gamma)$ there holds*

$$|a(u,v)-\widetilde{a}_n(u,v)| \le C_{s,\gamma}(\#\mathcal{T}_\gamma)^2 \rho^{r+p-2n+1} r^3 p^3 \|u\|_{\widetilde{H}^s(\Omega)}\|v\|_{\widetilde{H}^s(\Omega)}, \tag{4.3}$$

where $C_{s,\gamma}$ is a constant that depends only on γ and s.

As pointed out in Remark 3.5, the consistency error $a-\widetilde{a}_n$ allows one to infer uniform coercivity by a perturbation argument:

Lemma 4.4 *(Uniform coercivity). Let the assumptions of Lemma 4.3 hold. Then, there are constants $\widetilde{\alpha},\lambda_1,\lambda_2>0$ depending only on the shape regularity constant γ and s such that for $n\ge p+\lambda_1\ln(p+1)+\lambda_2\ln(\#\mathcal{T}_\gamma+1)$ there holds*

$$\widetilde{\alpha}\|v_N\|^2_{\widetilde{H}^s(\Omega)} \le \widetilde{a}_n(v_N,v_N) \quad \text{for all } v_N\in S_0^{p,1}(\mathcal{T}_\gamma). \tag{4.4}$$

Proof. The coercivity of $a(\cdot,\cdot)$, the triangle inequality and Lemma 4.3 applied with $r=p$ give

$$\alpha\|v_N\|^2_{\widetilde{H}^s(\Omega)} \le a(v_N,v_N) \le \widetilde{a}_n(v_N,v_N)+|a(v_N,v_N)-\widetilde{a}_n(v_N,v_N)| \le \widetilde{a}_n(v_N,v_N)+C_{s,\gamma}\,(\#\mathcal{T}_\gamma)^2\rho^{2p-2n+1}p^6\|v_N\|^2_{\widetilde{H}^s(\Omega)}.$$

As the second term on the right-hand side tends to zero for $n\to\infty$, we may ensure for $n\ge p+\lambda_1\ln(p+1)+\lambda_2\ln(\#\mathcal{T}_\gamma+1)$ with large enough constants λ_1,λ_2 that

$$C_{s,\gamma}\,(\#\mathcal{T}_\gamma)^2 p^6 \rho^{1-2\lambda_1\ln(p+1)-2\lambda_2\ln(\#\mathcal{T}_\gamma+1)} \le \frac{\alpha}{2} \tag{4.5}$$

so that coercivity of $\widetilde{a}_n$ follows with coercivity constant $\widetilde{\alpha}:=\alpha/2$. To give more details: we note that λ_1, λ_2 can be chosen to be independent of p and $\#\mathcal{T}_\gamma$ as

- $\lambda_1 \ge \dfrac{3}{\ln(\rho)} \quad \Longrightarrow \quad (p+1)^{6-2\lambda_1\ln(\rho)}\le 1,$
- $\lambda_2 \ge \dfrac{2}{\ln(\rho)} \quad \Longrightarrow \quad (\#\mathcal{T}_\gamma+1)^{2-\lambda_2\ln(\rho)}\le 1,$
- $\lambda_2 \ge \max\left(\dfrac{\ln(2\rho\, C_{s,\gamma})-\ln(\alpha)}{\ln(\rho)\ln(2)},0\right) \quad \Longrightarrow \quad C_{s,\gamma}\,\rho\,(\#\mathcal{T}_\gamma+1)^{-\lambda_2\ln(\rho)}\le\dfrac{\alpha}{2},$

which directly gives (4.5). □

Proof of Theorem 2.4. *Proof of (2.10):* Under the assumptions made, we can apply the stability result Corollary 3.4 with $\mathcal{T}_\gamma=\mathcal{T}^L_{geo,\sigma}$ noting that $\#\mathcal{T}^L_{geo,\sigma}=2L+2$. Hence, for $r\in\{1,\dots,p\}$, we can use the First Strang Lemma to estimate

$$\|u-\widetilde{u}_{N,n}\|_{\widetilde{H}^s(\Omega)} \le C\left(\inf_{u_r\in S_0^{r,1}(\mathcal{T}^L_{geo,\sigma})}\left(\|u-u_r\|_{\widetilde{H}^s(\Omega)}+\sup_{w\in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})}\frac{|a(u_r,w)-\widetilde{a}_n(u_r,w)|}{\|w\|_{\widetilde{H}^s(\Omega)}}\right)+\sup_{w\in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})}\frac{|l(w)-\widetilde{l}_n(w)|}{\|w\|_{\widetilde{H}^s(\Omega)}}\right).$$

Taking $u_r\in S_0^{r,1}(\mathcal{T}^L_{geo,\sigma})$ as the hp-FEM approximation of (2.6) for the space $S_0^{r,1}(\mathcal{T}^L_{geo,\sigma})$, we get from Proposition 2.3 for the first term

$$\|u-u_r\|_{\widetilde{H}^s(\Omega)} \le Ce^{-br}+C_\varepsilon\sigma^{(1/2-\varepsilon)L}. \tag{4.6}$$

Lemma 4.3 and the *a priori* estimate $\|u_r\|_{\widetilde{H}^s(\Omega)} \le C\|f\|_{L^2(\Omega)}$ lead to

$$\sup_{w\in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})} \frac{|a(u_r,w)-\widetilde{a}_n(u_r,w)|}{\|w\|_{\widetilde{H}^s(\Omega)}} \le C_{s,\sigma,f}\; L^2\rho^{r+p-2n+1}r^3p^3. \tag{4.7}$$

Finally, Lemma 4.2 provides

$$\sup_{w\in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})} \frac{|l(w)-\widetilde{l}_n(w)|}{\|w\|_{\widetilde{H}^s(\Omega)}} \le C_f\; \rho^{p-2n+1}p. \tag{4.8}$$

This proves the convergence result (2.10).

Proof of (2.11): Follows from (2.10) by taking $r=p/2$.

Proof of the complexity estimate: We are left to show that, for $L\sim p\sim n$ and the basis $\mathcal{B}^{geo}=\mathcal{B}^{lin}\cup\mathcal{B}^{Leg}$ from Definition 2.2, the number of algebraic operations to set up the linear system $Ax=b$ is $\mathcal{O}(L^5)$, where $A\in\mathbb{R}^{N\times N}$ with $A_{ij}=a(\varphi_j,\varphi_i)$ and $b\in\mathbb{R}^N$ with $b_i:=\langle f,\varphi_i\rangle_{L^2(\Omega)}$. The key to this improved complexity estimate is that the evaluation of the $p+1$ shape functions at the n quadrature points always happens on the reference element $\widehat{T}$ and can therefore be precomputed. This precomputation can be realized in $\mathcal{O}(np)$ operations using three-term recurrence relations by noting that the integrated Legendre polynomials are orthogonal polynomials (see, e.g., [24, (A.3), (A.9)]).

We recall that the support of the basis functions consists of two mesh elements for $\mathcal{B}^{lin}$ and one for $\mathcal{B}^{Leg}$. If the supports of φ_i and φ_j are separated then in the definition of the approximated bilinear form

$$\widetilde{a}_n(\varphi_i,\varphi_j)=\frac{C(s)}{2}\left(\sum_{T\in\mathcal{T}^L_{geo,\sigma}}\sum_{T'\in\mathcal{T}^L_{geo,\sigma}} Q^n_{T,T'}(\varphi_i,\varphi_j)+2\sum_{T\in\mathcal{T}^L_{geo,\sigma}} Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)\right),$$

most of the summands are zero; in fact, only $\mathcal{O}(1)$ terms have to be calculated. If the supports of φ_i and φ_j have non-trivial intersection, we have to calculate $\mathcal{O}(L)$ summands. Before we derive the stated complexity bound, we show that a direct implementation is insufficient to achieve complexity $\mathcal{O}(L^5)$.

Direct implementation: In terms of computational effort, the evaluation of the stiffness matrix A_{ij} dominates the computation of the load vector b_i (which is of order $\mathcal{O}(Lpn)$ by the same reasoning as below). For the stiffness matrix, a straight-forward implementation contains several nested loops:

- 2 loops over the $\mathcal{O}(p\,L)$ basis functions $\mathcal{B}^{geo}$ to calculate each of the $\mathcal{O}(p^2L^2)$ entries $\widetilde{a}_n(\varphi_i,\varphi_j)$:
  - $\mathcal{O}(p^2L)$ of these pairs (φ_i,φ_j) of basis functions are pairs whose supports intersect non-trivially, which leads to $\mathcal{O}(L)$ evaluations of the type $Q^n_{T,T'}(\varphi_i,\varphi_j)$ or $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$;
  - the remainder of the $\mathcal{O}(p^2L^2)$ entries of the stiffness matrix result from pairs of basis functions with separated supports, which leads to $\mathcal{O}(1)$ evaluations of the type $Q^n_{T,T'}(\varphi_i,\varphi_j)$;
- evaluation of each $Q^n_{T,T'}(\varphi_i,\varphi_j)$ and $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$: 2 loops over the quadrature points with complexity $\mathcal{O}(n^2)$.

In total this leads to a complexity of $\mathcal{O}(p^2L^2n^2)=\mathcal{O}(L^6)$, since $L\sim p\sim n$. We now show that the complexity of setting up the stiffness matrix and therefore the overall complexity, can actually be reduced from $\mathcal{O}(L^6)$ to $\mathcal{O}(L^5)$.

The reduction in complexity comes from precomputing the terms $Q^n_{T,T'}(\varphi_i,\varphi_j)$ and $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$ before the stiffness matrix is assembled by looping over all basis functions.

We start by noting that $Q^n_{T,T'}(\varphi_i,\varphi_j)=0$, if neither T nor T' is in the support of one of the basis functions φ_i or φ_j.

Step 1 (precomputation of $Q^n_{T,T'}(\varphi_i,\varphi_j)$): Let $T,T'\in\mathcal{T}^L_{geo,\sigma}$ be a pair of elements. We distinguish two cases: the $\mathcal{O}(L)$ coinciding or adjacent pairs T',T, and the $\mathcal{O}(L^2)$ well-separated pairs.

For the first case of adjacent pairs or identical pairs, one has to consider $\mathcal{O}(p^2)$ pairs of basis functions from $\mathcal{B}^{geo}$ with the property that $Q^n_{T,T'}(\varphi_i,\varphi_j)$ is non-zero. This leads to a total complexity of $\mathcal{O}(L\,p^2n^2)=\mathcal{O}(L^5)$ to precompute all $Q^n_{T,T'}(\varphi_i,\varphi_j)$ of this type.

For well-separated pairs $\overline{T}\cap\overline{T'}=\emptyset$, we differentiate between two possibilities for non-zero contributions $Q^n_{T,T'}(\varphi_i,\varphi_j)$:

- $T\subseteq\operatorname{supp}(\varphi_i)\cap\operatorname{supp}(\varphi_j)$ (or $T'\subseteq\operatorname{supp}(\varphi_i)\cap\operatorname{supp}(\varphi_j)$, which is handled analogously):

$$\frac{Q^n_{T,T'}(\varphi_i,\varphi_j)}{h_Th_{T'}}=GL_{n,x}\circ GL_{n,y}\left(\frac{\widehat{\varphi}_i(x)\widehat{\varphi}_j(x)}{|(1-x)h_T+\operatorname{dist}_{T,T'}+yh_{T'}|^{1+2s}}\right); \tag{4.9}$$

- $T\subseteq\operatorname{supp}(\varphi_i)$ and $T'\subseteq\operatorname{supp}(\varphi_j)$ (interchanging the roles of φ_i,φ_j leads to the same case):

$$\frac{Q^n_{T,T'}(\varphi_i,\varphi_j)}{h_Th_{T'}}=GL_{n,x}\circ GL_{n,y}\left(\frac{\widehat{\varphi}_i(x)\widehat{\varphi}_j(y)}{|(1-x)h_T+\operatorname{dist}_{T,T'}+yh_{T'}|^{1+2s}}\right). \tag{4.10}$$

For (4.9) the term $Q^n_{T,T'}(\varphi_i,\varphi_j)$ is calculated by

$$\frac{Q^n_{T,T'}(\varphi_i,\varphi_j)}{h_T h_{T'}} = (\omega_k\,\hat{\varphi}_i(x_k)\,\hat{\varphi}_j(x_k))^\top_{k=1,\dots,n}\cdot(k_{T,T'}(x_k,y_l))_{k,l=1,\dots,n}\cdot(\omega_l)_{l=1,\dots,n}, \tag{4.11}$$

where $k_{T,T'}(x,y) := (|(1-x)h_T + \mathrm{dist}_{T,T'} + yh_{T'}|^{1+2s})^{-1}$. As mentioned in the beginning, the vectors $(\hat{\varphi}_i(x_k))_{k=1,\dots,n}$ can be precomputed in $\mathcal{O}(np)$. Remember that $\hat{\varphi}_i,\hat{\varphi}_j$ are independent of T since the evaluation happens on the reference element $\hat{T}$. Thus, with additional $\mathcal{O}(p^2n)$ operations we can precompute the vectors $(\omega_k\,\hat{\varphi}_i(x_k)\,\hat{\varphi}_j(x_k))^\top_{k=1,\dots,n}$. The precomputation of $V := (k_{T,T'}(x_k,y_l))_{k,l=1,\dots,n}\cdot(\omega_l)_{l=1,\dots,n}$ for all T,T' takes $\mathcal{O}(L^2n^2)$ operations. Therefore, we can compute the products in (4.11) as: Multiply each of the $\mathcal{O}(p^2)$ possible vectors $(\omega_k\,\hat{\varphi}_i(x_k)\,\hat{\varphi}_j(x_k))^\top_{k=1,\dots,n}$ with each of the $\mathcal{O}(L^2)$ suitable vectors V in $\mathcal{O}(n)$ operations. For all $Q^n_{T,T'}(\varphi_i,\varphi_j)$ of the form (4.11), this leads to a total complexity of $\mathcal{O}(p^2L^2\,n) = \mathcal{O}(L^5)$.

For (4.10) the term $Q^n_{T,T'}(\varphi_i,\varphi_j)$ is calculated by

$$\frac{Q^n_{T,T'}(\varphi_i,\varphi_j)}{h_T h_{T'}} = (\omega_k\,\hat{\varphi}_i(x_k))^\top_{k=1,\dots,n}\cdot(k_{T,T'}(x_k,y_l))_{k,l=1,\dots,n}\cdot(\omega_l\,\hat{\varphi}_j(y_l))_{l=1,\dots,n}, \tag{4.12}$$

where $k_{T,T'}(x,y) := (|(1-x)h_T + \mathrm{dist}_{T,T'} + yh_{T'}|^{1+2s})^{-1}$. All three terms in the product (4.12) can be precomputed similarly to the case (4.11). Thus, we can compute the products in (4.12) as: For all pairs of separated elements T, T' and all $\varphi_i\in\mathcal{B}^{geo}$ with $T\subseteq\mathrm{supp}(\varphi_i)$, compute the vectors

- $M := (\omega_k\,\hat{\varphi}_i(x_k))^\top_{k=1,\dots,n}\cdot(k_{T,T'}(x_k,y_l))_{k,l=1,\dots,n}$ in $\mathcal{O}(n^2)$;
- then, loop over all basis functions $\varphi_j\in\mathcal{B}^{geo}$ with $T'\subseteq\mathrm{supp}(\varphi_j)$ and compute the scalar product $M\cdot(\omega_l\,\hat{\varphi}_j(y_l))_{l=1,\dots,n}$ in $\mathcal{O}(n)$.

This leads to a total complexity of $\mathcal{O}(L^2\,p\,(n^2+p\,n)) = \mathcal{O}(L^5)$.

Step 2 (precomputation of $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$): There are $\mathcal{O}(L\,p^2)$ constellations such that $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$ is non-zero with $\mathcal{O}(n)$ operations to calculate so that the total complexity for this step is $\mathcal{O}(L\,p^2n) = \mathcal{O}(L^4)$.

Step 3 (assembling the stiffness matrix): To calculate the $\mathcal{O}(L^2p^2)$ matrix entries $\tilde{a}_n(\varphi_i,\varphi_j)$, we have to sum for each entry over $\mathcal{O}(L)$ non-zero, precomputed terms $Q^n_{T,T'}(\varphi_i,\varphi_j)$ and $Q^n_{T,\Omega^c}(\varphi_i,\varphi_j)$. This proves the total complexity of $\mathcal{O}(L^2p^2L) = \mathcal{O}(L^5)$. □

5. Consistency errors

In this chapter, we present the proofs for the consistency error estimates in Lemmas 4.2 and 4.3.

We start with a well-known basic error estimate for Gaussian quadrature. Recall that

$$I(\Phi) := \int_{\hat{T}} \Phi(x)\omega(x)\,dx \approx \sum_{i=1}^n \omega_i\Phi(x_i) =: G_n(\Phi),$$

with $\sum_i \omega_i = C_\omega := \int_{\hat{T}}\omega dx$ and that the numerical integration is exact for $\Pi\in\mathcal{P}_{2n-1}(\hat{T})$. Thus, for an arbitrary polynomial $\Pi\in\mathcal{P}_{2n-1}(\hat{T})$ we get (using also the positivity of the weights ω_i)

$$E_n := |I(\Phi)-G_n(\Phi)| = |I(\Phi-\Pi)-G_n(\Phi-\Pi)| \le C_\omega\|\Phi-\Pi\|_{L^\infty(\hat{T})} + \sum_{i=1}^n \omega_i\,\|\Phi-\Pi\|_{L^\infty(\hat{T})} \le 2C_\omega\|\Phi-\Pi\|_{L^\infty(\hat{T})},$$

which gives the best approximation estimate

$$E_n \le 2C_\omega \inf_{\Pi\in\mathcal{P}_{2n-1}(\hat{T})}\|\Phi-\Pi\|_{L^\infty(\hat{T})}. \tag{5.1}$$

By tensorization, this result for univariate Gaussian quadrature can be extended to the $2d$-case. For the special case $\omega\equiv 1$ and

$$I^{2D}(\Phi) := \int_{\hat{T}}\int_{\hat{T}} \Phi(x,y)\,dy\,dx \approx G_n^{2D}(\Phi) := G_{n,x}\circ G_{n,y}(\Phi) = G_{n,y}\circ G_{n,x}(\Phi) = \sum_{i,j=1}^n \omega_i\omega_j\Phi(x_i,y_j)$$

we estimate the quadrature error using $C_\omega = 1$ for $\omega\equiv 1$:

$$\begin{aligned} E_n^{2D} := |I^{2D}(\Phi)-G_n^{2D}(\Phi)| &= \left|\int_{\hat{T}}\int_{\hat{T}}\Phi(x,y)\,dy - G_{n,y}(\Phi(x,\cdot))\,dx\right| + \left|\int_{\hat{T}} G_{n,y}(\Phi(x,\cdot))\,dx - G_{n,y}\circ G_{n,x}(\Phi)\right| \\ &\le \sup_{x\in\hat{T}}|I(\Phi(x,\cdot))-G_{n,y}(\Phi(x,\cdot))| + \left|G_{n,y}\Big(\int_{\hat{T}}\Phi(x,\cdot)\,dx - G_{n,x}(\Phi)\Big)\right| \end{aligned} \tag{5.2}$$

$$\le \sup_{x\in\widehat{T}} |I(\Phi(x,\cdot)) - G_{n,y}(\Phi(x,\cdot))| + \sum_{i=1}^{n} \omega_i \left| \int_{\widehat{T}} \Phi(x,y_i)\,dx - G_{n,x}(\Phi(\cdot,y_i)) \right|$$

$$\overset{\sum_i \omega_i = 1}{\le} \sup_{x\in\widehat{T}} |I(\Phi(x,\cdot)) - G_{n,y}(\Phi(x,\cdot))| + \sup_{y\in\widehat{T}} \left| I(\Phi(\cdot,y)) - G_{n,x}(\Phi(\cdot,y)) \right|.$$

In view of (5.1), these two univariate integration errors are estimated by best approximation errors. For analytic integrands, the best approximation errors will be quantified in Proposition 5.2.

Definition 5.1 *(Bernstein ellipse).* For $\rho > 1$, we define the Bernstein ellipse $\mathcal{E}_\rho$ and its scaled version $\widehat{\mathcal{E}}_\rho$ by

$$\mathcal{E}_\rho := \{ z \in \mathbb{C} : |z-1| + |z+1| < \rho + \rho^{-1} \}, \tag{5.3}$$

$$\widehat{\mathcal{E}}_\rho := F^{-1}_{(-1,1)}(\mathcal{E}_\rho), \tag{5.4}$$

where $F_{(-1,1)} : \mathbb{C} \to \mathbb{C}$, $x \mapsto 2x - 1$ is the affine map transforming $(-1,1)$ to $(0,1)$. We note that the focal points of $\widehat{\mathcal{E}}_\rho$ are 0 and 1.

Proposition 5.2. *Let Φ be holomorphic on $\widehat{\mathcal{E}}_{\widetilde{\rho}}$, $\widetilde{\rho} > 1$. Then, for every $1 < \rho < \widetilde{\rho}$, we have*

$$\inf_{v\in\mathcal{P}_n} \|\Phi - v\|_{L^\infty(0,1)} \le \frac{2}{\rho - 1} \rho^{-n} \|\Phi\|_{L^\infty(\widehat{\mathcal{E}}_\rho)}. \tag{5.5}$$

Proof. This proposition is just a transformed version of [15, Chap. 7, Thm. 8.1]. □

With this estimate for the best approximation error, we obtain exponential convergence for the quadrature errors.

Lemma 5.3. *Let $\widetilde{\rho} > 1$.*

(i) Let $\Phi : \widehat{\mathcal{E}}_{\widetilde{\rho}} \to \mathbb{C}$ be holomorphic. Then, for every $1 < \rho < \widetilde{\rho}$, the quadrature error can be estimated by

$$|I(\Phi) - G_n(\Phi)| \le C \rho^{-2n+1} \|\Phi\|_{L^\infty(\widehat{\mathcal{E}}_\rho)}, \tag{5.6}$$

where the constant C is independent of n and Φ.

(ii) Let $\Phi : \widehat{\mathcal{E}}_{\widetilde{\rho}} \times \widehat{\mathcal{E}}_{\widetilde{\rho}} \to \mathbb{C}$ be such that for each $y \in (0,1)$ the function $\Phi(\cdot, y)$ is holomorphic on $\widehat{\mathcal{E}}_{\widetilde{\rho}}$ and such that for each $x \in (0,1)$, the function $\Phi(x,\cdot)$ is holomorphic on $\widehat{\mathcal{E}}_{\widetilde{\rho}}$. Then, for every $1 < \rho < \widetilde{\rho}$, the quadrature error can be estimated by

$$|I^{2D}(\Phi) - G_n^{2D}(\Phi)| \le C \rho^{-2n+1} \left(\sup_{y\in(0,1)} \|\Phi(\cdot,y)\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} + \sup_{x\in(0,1)} \|\Phi(x,\cdot)\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \right), \tag{5.7}$$

where the constant C is independent of n and Φ.

The norms in the previous estimates do not involve the $\|\cdot\|_{\widetilde{H}^s(\Omega)}$ norm required in the Strang Lemma. This is achieved with an inverse estimate or a Poincaré type estimate.

Lemma 5.4. *Let $\mathcal{I} = (x_\ell, x_\ell + h_\mathcal{I}) \subset \mathbb{R}$ be an interval with diameter $h_\mathcal{I} := \operatorname{diam}(\mathcal{I}) < \infty$.*

(i) There is a constant independent of $\mathcal{I}$ such that for every $\rho > 1$ and $p \in \mathbb{N}$ there holds for all polynomials $v \in \mathcal{P}_p(\mathcal{I})$ and their pullbacks $\hat{v} := v \circ F_\mathcal{I}$

$$\left\| \frac{d}{dx} \hat{v} \right\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \le C \rho^p p^3 h_\mathcal{I}^{s-1/2} |v|_{H^s(\mathcal{I})}, \tag{5.8}$$

$$\|\hat{v}\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \le C \rho^p p\, h_\mathcal{I}^{-1/2} \|v\|_{H^s(\mathcal{I})}. \tag{5.9}$$

(ii) Denote $\mathcal{I}_{sym} = (x_\ell - h_\mathcal{I}, x_\ell + h_\mathcal{I})$ and let $v \in H^s(\mathcal{I}_{sym})$ with $v|_{(x_\ell - h_\mathcal{I}, x_\ell)} = 0$. Then, there is $C > 0$ depending only on s such that

$$\|v\|_{L^2(\mathcal{I})} \le C\, h_\mathcal{I}^s |v|_{H^s(\mathcal{I}_{sym})}. \tag{5.10}$$

The same estimate holds for $\mathcal{I}_{sym} = (x_\ell, x_\ell + 2h_\mathcal{I})$ and $v \in H^s(\mathcal{I}_{sym})$ with $v|_{(x_\ell + h_\mathcal{I}, x_\ell + 2h_\mathcal{I})} = 0$.

Proof. With the Bernstein inequality [15, Chap. 4, Thm. 2.2]

$$\|q\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \le \rho^p \|q\|_{L^\infty(0,1)} \quad \text{for all } q \in \mathcal{P}_p(0,1)$$

and inserting the mean $\overline{\hat{v}} := \int_0^1 \hat{v}(x)\,dx$, we obtain

$$\left\| \frac{d}{dx}\hat{v} \right\|_{L^\infty(\hat{\mathcal{E}}_\rho)} \le C\rho^p \left\| \frac{d}{dx}\hat{v} \right\|_{L^\infty(0,1)} = C\rho^p \left\| \frac{d}{dx}(\hat{v}-\overline{\hat{v}}) \right\|_{L^\infty(0,1)}.$$

Employing inverse inequalities of Markov type, see [31, Thm 3.91, Thm. 3.92] together with a fractional Poincaré inequality, see [18], and a scaling argument, we arrive at

$$\left\| \frac{d}{dx}(\hat{v}-\overline{\hat{v}}) \right\|_{L^\infty(0,1)} \le Cp^2 \left\| \hat{v}-\overline{\hat{v}} \right\|_{L^\infty(0,1)} \le Cp^3 \left\| \hat{v}-\overline{\hat{v}} \right\|_{L^2(0,1)} \le Cp^3 |\hat{v}|_{H^s(0,1)} \le Cp^3 h_{\mathcal{I}}^{s-1/2} |v|_{H^s(\mathcal{I})}. \tag{5.11}$$

This shows (5.8). Inequality (5.9) follows with the same arguments.

The fractional Poincaré inequality (5.10) can be shown by a scaling argument and the compact embedding $H^s \subset L^2$; the fact that the seminorm appears on the right-hand side of (5.10) is a consequence of the fact that v is assumed to vanish on parts of $\mathcal{I}_{sym}$. See also [1] for the proof of a closely related result. □

The following lemma provides the key technical estimates for the quadrature errors appearing in the approximated bilinear and linear forms.

Lemma 5.5. *Let* $\mathrm{co}(T,T')$ *denote the convex hull of two sets* T *and* T'. *Let* $\mathcal{T}_\gamma$ *be a* γ*-shape regular mesh on* Ω. *There exists a constant* $\rho > 1$ *that depends only on* γ *and* s *such that for all* $v \in S_0^{r,1}(\mathcal{T}_\gamma)$, $w \in S_0^{p,1}(\mathcal{T}_\gamma)$ *and* $T, T' \in \mathcal{T}_\gamma$ *there holds*

$$\left| I_{T,T'}(v,w) - Q^n_{T,T'}(v,w) \right| \le C_{s,\rho,\gamma} r^3 p^3 \rho^{r+p-2n+1} |v|_{H^s(\mathrm{co}(T,T'))} |w|_{H^s(\mathrm{co}(T,T'))}, \tag{5.12}$$

$$\left| I_{T,\Omega^c}(v,w) - Q^n_{T,\Omega^c}(v,w) \right| \le C_{s,\rho,\gamma} r^3 p^3 \rho^{r+p-2n+1} |v|_{\widetilde{H}^s(\Omega)} |w|_{\widetilde{H}^s(\Omega)}. \tag{5.13}$$

Proof. We distinguish the cases of pairs of adjacent elements, identical pairs, well-separated pairs, and combinations of elements T with Ω^c.

Case of adjacent elements: We start with the case for adjacent elements $T \ne T'$ with $\overline{T} \cap \overline{T'} \ne \emptyset$. Due to Lemma 5.3 it is sufficient to estimate the L^∞-norms of the integrands in (3.6). As both integrands can be treated in the same way, we only consider the first one

$$\hat{g}_1(x,y) := h_T^{-1-2s} \frac{(\hat{v}_T(1-x) - \hat{v}_{T'}(xy))}{x} \cdot \frac{(\hat{w}_T(1-x) - \hat{w}_{T'}(xy))}{x} \cdot \frac{1}{|1+yh_{T'}/h_T|^{1+2s}}.$$

Note that the first two fractions of the product on the right-hand side have removable singularities and are therefore holomorphic on $\mathbb{C}$ in each variable. The function $y \mapsto |1+yh_{T'}/h_T| = \sqrt{(1+yh_{T'}/h_T)^2} > 0$ on the closed interval $[0,1]$ and therefore has a holomorphic extension to an ellipse $\hat{\mathcal{E}}_\rho$ for some $\rho > 1$ that solely depends on γ since $h_{T'}/h_T \le 1/\gamma$ by shape regularity. We conclude that $\hat{g}_1(\cdot,y)$ is holomorphic on $\mathbb{C}$ for fixed $y \in [0,1]$ and $\hat{g}_1(x,\cdot)$ is holomorphic on $\hat{\mathcal{E}}_\rho$ for fixed $x \in [0,1]$. Using that $\hat{v}_T(1) = \hat{v}_{T'}(0)$, the fundamental theorem of calculus implies for $(x,y) \in [0,1] \times \hat{\mathcal{E}}_\rho$ and for $(x,y) \in \hat{\mathcal{E}}_\rho \times [0,1]$

$$\left| \frac{1}{x}(\hat{v}_T(1-x) - \hat{v}_{T'}(xy)) \right| = \left| \frac{1}{x}\left(\int_0^x \frac{d}{dz}\hat{v}_T(1-z)\,dz - \int_0^{xy} \frac{d}{dz}\hat{v}_{T'}(z)\,dz \right) \right| \le 2\,\mathrm{diam}(\hat{\mathcal{E}}_\rho) \max\left(\left\| \frac{d}{dz}\hat{v}_T \right\|_{L^\infty(\hat{\mathcal{E}}_\rho)}, \left\| \frac{d}{dz}\hat{v}_{T'} \right\|_{L^\infty(\hat{\mathcal{E}}_\rho)} \right).$$

Analogously, the same can be shown for the function $\hat{w}$. With Lemma 5.4, this implies

$$\sup_{y\in(0,1)} \|\hat{g}_1(\cdot,y)\|_{L^\infty(\hat{\mathcal{E}}_\rho)} + \sup_{x\in(0,1)} \|\hat{g}_1(x,\cdot)\|_{L^\infty(\hat{\mathcal{E}}_\rho)} \le C_{s,\rho,\gamma} r^3 p^3 \rho^{r+p} \max(|v|_{H^s(T)}, |v|_{H^s(T')}) \max(|w|_{H^s(T)}, |w|_{H^s(T')}).$$

Together with (5.7) and $\max(|v|_{H^s(T)}, |v|_{H^s(T')}) \le |v|_{H^s(\mathrm{co}(T,T'))}$, this finishes the proof for the case of adjacent elements T, T'.

Case of identical elements: The case $T = T'$ follows with similar arguments. We note that in this case the integrand

$$\hat{g}_2(x,y) := \frac{(\hat{u}_T(x) - \hat{u}_T(xy))(\hat{v}_T(x) - \hat{v}_T(xy))}{|x-xy|^2} h_T^{-1-2s} \tag{5.14}$$

is a polynomial of degree $\le r+p-1$ and thus is integrated exactly for $n \ge \max(r,p)$.

Case of well-separated elements: For separated elements $\overline{T} \cap \overline{T'} = \emptyset$ the integrand is continuous. Thus, by [29, Lem. 4.6], the Gaussian quadrature error can be estimated by the best approximation error for the function $\hat{g}_3(x,y) := |\,\mathrm{dist}_{T,T'} + (1-x)h_T + yh_{T'}|^{-1-2s}$ in L^∞ using polynomials of maximal degree $r_c := 2n-p-r-1$ and L^2-norms of the polynomials $\hat{v}_T(x) - \hat{v}_{T'}(y)$ and $\hat{w}_T(x) - \hat{w}_{T'}(y)$:

$$\left| I_{T,T'}(v,w) - Q^n_{T,T'}(v,w) \right| \le Cp^2 h_T h_{T'} \inf_{\hat{q}\in Q_{r_c}((0,1)^2)} \|\hat{g}_3 - \hat{q}\|_{L^\infty(\hat{T}\times\hat{T})} \cdot \left(\int_{\hat{T}} \int_{\hat{T}} (\hat{v}_T(x) - \hat{v}_{T'}(y))^2\,dy\,dx \right)^{1/2} \left(\int_{\hat{T}} \int_{\hat{T}} (\hat{w}_T(x) - \hat{w}_{T'}(y))^2\,dy\,dx \right)^{1/2}, \tag{5.15}$$

where $\mathcal{Q}_{r_c}((0,1)^2)$ denotes the tensor product space $\mathcal{P}_{r_c}(0,1)\otimes\mathcal{P}_{r_c}(0,1)=\operatorname{span}\{(x,y)\mapsto x^i y^j : 0\le i,j\le r_c\}$. Similarly to the case of adjacent elements, the function $\hat{g}$ admits a holomorphic extension to $\hat{\mathcal{E}}_\rho\times\hat{\mathcal{E}}_\rho$ for some $\rho>1$ since $\hat{g}_3(x,y)=((\operatorname{dist}_{T,T'}+(1-x)h_T+yh_{T'})^2)^{-1/2-s}$ and the argument of $(\cdot)^{-1-2s}$ is bounded away from 0 for $(x,y)\in[0,1]^2$. In fact, we only require that for each fixed $x\in[0,1]$ the function $\hat{g}_3(x,\cdot)$ can be extended holomorphically to $\hat{\mathcal{E}}_\rho$ and for each fixed $y\in[0,1]$ the function $\hat{g}_3(\cdot,y)$ can be extended holomorphically to $\hat{\mathcal{E}}_\rho$. As in the case of adjacent elements, we have by shape regularity $h_T/\operatorname{dist}_{T,T'}\le 1/\gamma$ and $h_{T'}/\operatorname{dist}_{T,T'}\le 1/\gamma$.

We may employ Proposition 5.2 and a tensor product argument akin to that employed in (5.2) to get with inequality (5.2) the existence of $\rho>1$ such that

$$\inf_{\hat{q}\in\mathcal{Q}_{r_c}((0,1)^2)}\|\hat{g}_3-\hat{q}\|_{L^\infty(\hat{T}\times\hat{T})}\le C_{s,\rho,\gamma}\rho^{-r_c}\operatorname{dist}_{T,T'}^{-1-2s}. \tag{5.16}$$

For the remaining terms in (5.15), we transform back to the physical elements, insert the mean $\overline{v_{\mathrm{co}(T,T')}}:=\int_{\mathrm{co}(T,T')}v(x)\,dx/h_{\mathrm{co}(T,T')}$ over the convex hull $co(T,T')$ of T and T' and integrate in one variable to obtain

$$\int_{\hat{T}}\int_{\hat{T}}(\hat{v}_T(x)-\hat{v}_{T'}(y))^2\,dy\,dx=h_T^{-1}h_{T'}^{-1}\int_T\int_{T'}(v_T(x)-v_{T'}(y))^2\,dy\,dx\le 2h_T^{-1}h_{T'}^{-1}\int_T\int_{T'}(v(x)-\overline{v_{\mathrm{co}(T,T')}})^2+(\overline{v_{\mathrm{co}(T,T')}}-v(y))^2dy\,dx$$
$$=2h_T^{-1}\|v-\overline{v_{\mathrm{co}(T,T')}}\|^2_{L^2(T)}+2h_{T'}^{-1}\|v-\overline{v_{\mathrm{co}(T,T')}}\|^2_{L^2(T')}. \tag{5.17}$$

Both terms can be treated in the same way, we thus only focus on the first one. Increasing the domain of integration to the convex hull $\mathrm{co}(T,T')$ and employing a Poincaré inequality, see [18, Prop. 2.2], gives

$$\|v-\overline{v_{\mathrm{co}(T,T')}}\|^2_{L^2(T)}\le\|v-\overline{v_{\mathrm{co}(T,T')}}\|^2_{L^2(\mathrm{co}(T,T'))}\le C_s h^{2s}_{\mathrm{co}(T,T')}|v|^2_{H^s(\mathrm{co}(T,T'))}. \tag{5.18}$$

Inserting everything into (5.15) gives

$$\left|I_{T,T'}(v,w)-Q^n_{T,T'}(v,w)\right|\le Cp^2\rho^{r+p-2n+1}\operatorname{dist}_{T,T'}^{-1-2s}(h_T+h_{T'})h^{2s}_{\mathrm{co}(T,T')}|v|_{H^s(\mathrm{co}(T,T'))}|w|_{H^s(\mathrm{co}(T,T'))}. \tag{5.19}$$

We note that, for shape regular meshes, we can estimate

$$h_T\le\gamma^{-1}\operatorname{dist}_{T,T'},\qquad h_{T'}\le\gamma^{-1}\operatorname{dist}_{T,T'},\qquad h_{\mathrm{co}(T,T')}\le\operatorname{dist}_{T,T'}+h_T+h_{T'}\le\operatorname{dist}_{T,T'}\left(1+\frac{2}{\gamma}\right).$$

Thus, there holds $\operatorname{dist}_{T,T'}^{-1-2s}(h_{T'}+h_T)h^{2s}_{\mathrm{co}(T,T')}\le(2/\gamma)(1+2/\gamma)^{2s}$, which concludes the argument for the case of separated elements.

Case of combination of T with Ω^c: For the complementary part, see (3.8), we consider integrals of the form

$$h_T\int_{\hat{T}}\frac{\hat{v}_T(x)\hat{w}_T(x)}{|\operatorname{dist}_{T,\{-1\}}+xh_T|^{2s}}\,dx. \tag{5.20}$$

We have to distinguish two cases. If T is at the left boundary, $-1\in\overline{T}$ and therefore $\operatorname{dist}_{T,\{-1\}}=0$, we can treat the singular integral (5.20) as a one dimensional version of the adjacent case. If $\operatorname{dist}_{T,\{-1\}}>0$, the proof uses similar techniques as the separated case. The only difference is that, instead of the convex hull of two elements, the convex hull of the element and the boundary point -1 is used and [18, Prop. 2.2] is replaced with (5.10) to bound the L^2-norms

$$\|v\|_{L^2(T)}\le\|v\|_{L^2(\mathrm{co}(T,\{-1\}))}\le C\,h^s_{\mathrm{co}(T,\{-1\})}\|v\|_{H^s(\Omega)}.\quad\square \tag{5.21}$$

The consistency errors follow from summation of the elementwise contributions.

Proof of Lemma 4.3. With the triangle inequality, basic integration and Lemma 5.5 we obtain

$$|a(v,w)-\tilde{a}_n(v,w)|\le\frac{C(s)}{2}\left(\sum_{T\in\mathcal{T}_\gamma}\sum_{T'\in\mathcal{T}_\gamma}\left|I_{T,T'}(v,w)-Q^n_{T,T'}(v,w)\right|+2\sum_{T\in\mathcal{T}_\gamma}\left|I_{T,\Omega^c}(v,w)-Q^n_{T,\Omega^c}(v,w)\right|\right)$$
$$\le C_{s,\rho,\gamma}r^3p^3\rho^{r+p-2n+1}\left(\sum_{T\in\mathcal{T}_\gamma}\sum_{T'\in\mathcal{T}_\gamma}|v|_{H^s(\mathrm{co}(T,T'))}|w|_{H^s(\mathrm{co}(T,T'))}+2\sum_{T\in\mathcal{T}_\gamma}\|v\|_{\tilde{H}^s(\Omega)}\|w\|_{\tilde{H}^s(\Omega)}\right)$$
$$\le C_{s,\rho,\gamma}(\#\mathcal{T}_\gamma)^2r^3p^3\rho^{r+p-2n+1}\|v\|_{\tilde{H}^s(\Omega)}\|w\|_{\tilde{H}^s(\Omega)}, \tag{5.22}$$

which finishes the proof. $\square$

Proof of Lemma 4.2. As f is analytic on $[0,1]$ there exists an analytic extension to a Bernstein ellipse $\hat{\mathcal{E}}_\rho$ for some $\rho>1$. Using (5.6) of Lemma 5.3 gives for each element

$$\left|\int_{\widehat{T}} \widehat{f}_T(x)\widehat{v}_T(x)dx - GL_n(\widehat{f}_T\widehat{v}_T)\right| \leq C\rho^{-2n+1}\|\widehat{f}_T\widehat{v}_T\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \leq C_{\rho,f}\rho^{-2n+1}\|\widehat{v}_T\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \overset{(5.9)}{\leq} C_{s,\rho,f}\,p\rho^{p-2n+1}h_T^{-1/2}\|v\|_{H^s(T)}.$$

Summation over all elements $T \in \mathcal{T}_\gamma$ together with the Cauchy-Schwarz inequality gives

$$\begin{aligned}|l(v) - \widetilde{l}_n(v)| &\leq \sum_{T\in\mathcal{T}_\gamma} h_T \left|\int_{\widehat{T}} \widehat{f}_T(x)\widehat{v}_T(x)\,dx - GL_n(\widehat{f}_T\widehat{v}_T)\right| \leq C_{s,\rho,f}\,p\rho^{p-2n+1}\sum_{T\in\mathcal{T}_\gamma} h_T^{1/2}\|v\|_{H^s(T)} \\ &\leq C_{s,\rho,f}\,p\rho^{p-2n+1}\sqrt{|\Omega|}\Big(\sum_{T\in\mathcal{T}_\gamma}\|v\|^2_{H^s(T)}\Big)^{1/2} \leq C_{s,\rho,f}\;p\rho^{p-2n+1}\|v\|_{H^s(\Omega)}. \quad \square\end{aligned}$$

6. Outlook: the multidimensional case on shape regular meshes

In this section, we discuss how the preceding $1d$-analysis can be generalized to the multidimensional case $d > 1$ for bounded polyhedral Lipschitz domains $\Omega \subset \mathbb{R}^d$. In this case, the weak formulation is given by: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := \frac{C(s,d)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(\vec{x}) - u(\vec{y}))(v(\vec{x}) - v(\vec{y}))}{|\vec{x}-\vec{y}|^{d+2s}}\,d\vec{y}\,d\vec{x} = \langle f, v\rangle_{L^2(\Omega)} =: l(v) \qquad \forall v \in \widetilde{H}^s(\Omega), \tag{6.1}$$

where $C(s,d) := 2^{2s}s\Gamma(s+d/2)/(\pi^{d/2}\Gamma(1-s))$ (see, e.g., [1]). Thus, we have to numerically compute integrals of the form

$$I_{S_1,S_2}(v,w) := \int_{S_1}\int_{S_2} \frac{(v(\vec{x}) - v(\vec{y}))(w(\vec{x}) - w(\vec{y}))}{|\vec{x}-\vec{y}|^{d+2s}}\,d\vec{y}\,d\vec{x}, \tag{6.2}$$

$$I_{S_1,\Omega^c}(v,w) := \int_{S_1} v(\vec{x})w(\vec{x})\int_{\Omega^c} \frac{1}{|\vec{x}-\vec{y}|^{d+2s}}\,d\vec{y}\,d\vec{x}, \tag{6.3}$$

where S_1 and S_2 denote d-dimensional simplices.

In the following, we will consider regular, γ-shape regular triangulations $\mathcal{T}_\gamma$ of Ω, i.e., decompositions of Ω into simplices. As usual, γ-shape regularity means that $\gamma > 0$ is independent of the number of elements in the mesh and there holds $\max_{S\in\mathcal{T}_\ell}(h_S/|S|^{1/d}) \leq \gamma < \infty$, where $h_S := \mathrm{diam}(S)$ and $|S|$ denotes the d-dimensional Lebesgue volume. We set $S_0^{p,1}(\mathcal{T}_\gamma) := \{u \in H_0^1(\Omega)\,|\,u|_S \in \mathcal{P}_p(\mathbb{R}^d) \quad \forall S \in \mathcal{T}_\gamma\}$, where $\mathcal{P}_p(\mathbb{R}^d)$ denotes the space of d-variate polynomials of (total) degree p. We will also require the tensor-product space $\mathcal{Q}_p(\mathbb{R}^d) := \mathrm{span}\{x_1^{\alpha_1}\cdots x_d^{\alpha_d}\,|\,0 \leq \alpha_1,\ldots,\alpha_d \leq p\}$.

6.1. Quadrature on pairs of simplices

In the present case of shape regular triangulations, techniques developed in [10] can be adapted to numerically integrate (6.2). Similarly to the case $d = 1$ in the previous sections, singularities in the integrand can be transformed such that suitable combinations of Gauss-Legendre and Gauss-Jacobi quadrature can be employed. In the following we state the main result of [10] regarding numerical integration of certain singular integrals.

Proposition 6.1 *([10]). Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh and S_1, $S_2 \in \mathcal{T}_\gamma$ be closed simplices in $\mathbb{R}^d$ with $k := dim(S_1 \cap S_2)$ (setting $k := -1$ if $S_1 \cap S_2 = \emptyset$) and consider integrals of the form*

$$I = \int_{S_1}\int_{S_2} |\vec{x}-\vec{y}|^\alpha F(\vec{x},\vec{y},\vec{x}-\vec{y})\,d\vec{y}\,d\vec{x}, \tag{6.4}$$

where $\alpha \in \mathbb{R}$ and F is a real analytic function, i.e., $F \in C^\omega(S_1 \times S_2 \times (S_2 - S_1))$.

Then, there exist $K_k \in \mathbb{N}$ depending only on k and polynomial transformations Φ_j, $j = 0,\ldots,K_k$ of degree $q_\Phi := \max_j \deg(\Phi_j)$, depending only on d, such that the integral I takes the form

$$I = \sum_{j=0}^{K_k}\int_{[0,1]^{2d}} F\circ\Phi_j(\vec{t})\,\mathcal{R}_j(\vec{t})\,J_{\Phi_j}^{rem}(\vec{t})\,t_1^{\alpha+2d-k-1}\,d\vec{t}, \tag{6.5}$$

where $\mathcal{R}_j \in C^\omega([0,1]^{2d})$ are real analytic functions given by

$$\mathcal{R}_j(\vec{t}) := \frac{|(\vec{x}-\vec{y})\circ\Phi_j(\vec{t})|^\alpha}{t_1^\alpha}, \tag{6.6}$$

$J^{rem}_{\Phi_j} := J_{\Phi_j}/t_1^{2d-k-1}$ are polynomials of degree at most $d(q_\Phi - 1)$, and J_{Φ_j} are the Jacobians of the transformations Φ_j. In particular, the condition $\alpha > k - 2d$ ensures that (6.5) is integrable, and $t_1^{\alpha+2d-k-1}$ can be used as a Gauss-Jacobi weight function.

Proof. See [10, Sec. 3] for the explicit construction of the transformations Φ_j and the resulting polynomial degree q_Φ. We also refer to [10, Thm. 4.1, Rem. 2], where a slightly different formulation is given, which even includes the more general case that F is in a Gevrey class. In [10] the condition $\alpha > k - 2d$ is required, but it follows from inspection of the proof that it is only needed to ensure integrability of the integrand. □

Remark 6.2.

(i) The transformations Φ_j are, similarly to the case $d = 1$, combinations of affine transformations and Duffy-like transformations that transform simplices to hypercubes and thus are polynomials. The parameter $K_k \in \mathbb{N}$ accounts for different cases that have to be treated with different transformations (as can be seen in the case $d = 1$ as well, compare (3.5) and (3.6)). For $d > 1$, this requires even more cases; however, structurally they are all similar, which allows for the compact notation.

(ii) An important observation of (6.5) is that the transformations (by employing relative coordinates) can be constructed such that the singularity of the function $|\vec{x} - \vec{y}|^{-d-2s}$ appears after transformation and permutations of the variables only in a single variable labelled t_1.

(iii) Since the term $t_1^{\alpha+2d-k-1}$ with $\alpha + 2d - k - 1 > -1$ can be handled as a weight function with Gauss-Jacobi quadrature, an approximation to (6.5) can be achieved by a tensor quadrature rule. ■

Unfortunately, the integrals in (6.2) do not fulfill the requirement of the final statement of Proposition 6.1 to be integrable since $\alpha = -d - 2s > k - 2d$ does not hold for all $0 \le k \le d$ and $s \in (0, 1)$. Therefore, we have to modify the analysis of [10] to suit our integrand by showing that, after application of the transformations Φ_j, the term $(v(\vec{x}) - v(\vec{y}))(w(\vec{x}) - w(\vec{y}))$ takes the form $t_1^2 q(t_1, \dots, t_{2d})$ where q is a polynomial in $2d$ variables, i.e., $q \in \mathcal{P}_k(\mathbb{R}^{2d})$ for some k. Consequently, the singular term in the integral takes the form $t_1^{\tilde{\alpha}}$ with $\tilde{\alpha} := \alpha + 2d - k + 1 > -1$. More precisely, we have the following Corollary 6.3, which can be seen as an extension of [10, Thm. 4.1] to the present specific case (6.2).

Corollary 6.3. *Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh and $S_1, S_2 \in \mathcal{T}_\gamma$ be closed simplices in $\mathbb{R}^d$ with $k := dim(S_1 \cap S_2)$ (setting $k := -1$ if $S_1 \cap S_2 = \emptyset$) and, for $v, w \in S_0^{p,1}(\mathcal{T}_\gamma)$, consider the integral*

$$I_{S_1,S_2}(v,w) := \int_{S_1}\int_{S_2} \frac{(v|_{S_1}(\vec{x}) - v|_{S_2}(\vec{y}))(w|_{S_1}(\vec{x}) - w|_{S_2}(\vec{y}))}{|\vec{x} - \vec{y}|^{d+2s}}\, d\vec{y}\, d\vec{x}. \tag{6.7}$$

Then, employing, for $k \ge 0$, the polynomial transformations Φ_j of Proposition 6.1 of degree (at most) q_Φ the integral $I_{S_1,S_2}(v,w)$ takes the form

$$I_{S_1,S_2}(v,w) = \int_{[0,1]^{2d}} \sum_{j=0}^{K_k} \underbrace{P_{v,j}(\vec{t})\, P_{w,j}(\vec{t})\, \mathcal{R}_j(\vec{t})\, J^{rem}_{\Phi_j}(\vec{t})}_{=:F_j}\, t_1^{2-2s+d-k-1}\, d\vec{t}. \tag{6.8}$$

Here, $J^{rem}_{\Phi_j} := J_{\Phi_j}/t_1^{2d-k-1}$ are polynomials of degree at most $d(q_\Phi - 1)$, the transformations Φ_j have Jacobians J_{Φ_j}, and the $\mathcal{R}_j \in C^\omega([0,1]^{2d})$ are analytic functions given by

$$\mathcal{R}_j(\vec{t}) := \frac{t_1^{d+2s}}{|(\vec{x} - \vec{y})\circ\Phi_j(\vec{t})|^{d+2s}}, \tag{6.9}$$

and $P_{v,j}$, $P_{w,j} \in \mathcal{P}(\mathbb{R}^{2d})$ are polynomials of degree (at most) $\le pq_\Phi - 1$, defined by

$$P_{v,j}(\vec{t}) := \frac{(v|_{S_1} - v|_{S_2})\circ\Phi_j(\vec{t})}{t_1} \quad \text{and} \quad P_{w,j}(\vec{t}) := \frac{(w|_{S_1} - w|_{S_2})\circ\Phi_j(\vec{t})}{t_1}. \tag{6.10}$$

For $k = -1$, we get the form

$$I_{S_1,S_2}(v,w) = \int_{[0,1]^{2d}} \underbrace{P_{v,-1}(\vec{t})\, P_{w,-1}(\vec{t})\, \mathcal{R}_{-1}(\vec{t})\, J_{\Phi_{-1}}(\vec{t})}_{=:F_{-1}}\, d\vec{t}, \tag{6.11}$$

with a polynomial transformation Φ_{-1} and its polynomial Jacobian $J_{\Phi_{-1}}$, $\mathcal{R}_{-1}(\vec{t}) := |(\vec{x} - \vec{y})\circ\Phi_{-1}(\vec{t})|^{-d-2s}$ analytic and polynomials $P_{v,-1}$, $P_{w,-1} \in \mathcal{P}_{pq_\Phi-1}(\mathbb{R}^{2d})$ defined by

$$P_{v,-1}(\vec{t}) := (v|_{S_1} - v|_{S_2})\circ\Phi_{-1}(\vec{t}) \quad \text{and} \quad P_{w,-1}(\vec{t}) := (w|_{S_1} - w|_{S_2})\circ\Phi_{-1}(\vec{t}). \tag{6.12}$$

Proof. For $k \geq 0$, with Proposition 6.1 it is only left to show that $P_{v,j}$ and $P_{w,j}$ from (6.10) are polynomials. We only prove the statement for $P_{v,j}$.

Since $v \in S_0^{p,1}(\mathcal{T}_\gamma)$ is a piecewise continuous polynomial, the singularity points $\vec{x} = \vec{y}$ of $|\vec{x} - \vec{y}|^{-d-2s}$ are a subset of the roots of the polynomial $(v|_{S_1}(\vec{x}) - v|_{S_2}(\vec{y}))$. Since Φ_j is a polynomial, it follows that $(v|_{S_1} - v|_{S_2})\circ\Phi_j$ is also a polynomial (of degree bounded by pq_Φ) that vanishes at the singularities of $|\vec{x} - \vec{y}|^{-d-2s}\circ\Phi_j$. So the separated singularity t_1 has to be a root of $(v|_{S_1} - v|_{S_2})\circ\Phi_j$. The fundamental theorem of algebra finishes the proof.

For $k = -1$ the proof follows immediately from Step 1 and 2 of the transformations of [10, Sec. 3]. □

[10, Thm. 5.4] also asserts exponential convergence of a suitable combination of Gauss-Jacobi and Gauss-Legendre quadrature employed to integrands covered by Proposition 6.1.

Proposition 6.4. *Let $\mathcal{R} \in C^\omega([0,1]^{d'})$ and $\beta_1 > -1$. Then, there exist $C,\ b > 0$ independent of d' such that for all $n \in \mathbb{N}$ there holds*

$$\left| \int_{[0,1]^{d'}} t_1^{\beta_1} \mathcal{R}(\vec{t})\, d\vec{t} - GJ_{n,t_1}^{0,\beta_1} \circ GL_{n,t_2} \circ \cdots \circ GL_{n,t_{d'}}(\mathcal{R}) \right| \leq C \exp(-b\, N^{1/d'}), \tag{6.13}$$

where $N = \mathcal{O}(n^{d'})$ is the total number of quadrature points.

Propositions 6.1 and 6.4 are formulated for fairly general integrands. However, in order to obtain exponential convergence results for hp-FEM discretizations, as in the case $d = 1$, an explicit dependence of the convergence rate on the employed polynomial degree has to be derived, which is not directly deducible from Proposition 6.4.

In the following we extend our $1d$-quadrature analysis, which was explicit in p, to higher dimension $d > 1$ specifically for the easier case of γ-shape regular meshes $\mathcal{T}_\gamma$ with a finite number of patch configurations. We will make the following assumption on the structure of the underlying triangulation of Ω:

Assumption 6.5. The triangulation $\mathcal{T}_\gamma$ is γ-shape regular and there exists, up to dilations, rotations, and translations, a finite number (independent on the number of elements in the mesh) of different patches (i.e., unions of elements sharing a vertex). This is, for example, ensured for $d \in \{2, 3\}$, if the mesh is generated from a coarse mesh by "newest vertex bisection", [23,34].

Remark 6.6. For exponential convergence results in terms of "error vs. number of degrees of freedom" as in Proposition 2.3 or Theorem 2.4, special geometric meshes $\mathcal{T}_{geo}$ are required that include anisotropic elements, [17]. A quadrature analysis on such meshes requires a more careful analysis of elements with large aspect ratio and is postponed to a forthcoming work. ■

6.2. Consistency error analysis

We start with a standard quadrature rule on a simplex S. To that end, we can also use the affine transformation [10, Sec. 3 (Step 1)] to map a given simplex S to the reference simplex $\hat{S}_d := \{(x_1, \ldots, x_d) \mid x_i \geq 0\, \forall i = 1, \ldots, d,\ x_1 + \cdots + x_d \leq 1\}$ and afterwards with the Duffy type transformation [10, (2.12)] to $[0,1]^d$. This then allows one to use tensor product Gauss-Legendre rules to obtain

$$\int_S f(\vec{x})\, d\vec{x} = \int_{[0,1]^d} f\circ\Phi_S(\vec{t})\, J_{\Phi_S}(\vec{t})\, d\vec{t} \approx GL_{n,t_1} \circ \cdots \circ GL_{n,t_d}(f\circ\Phi_S\, J_{\Phi_S}) =: GL_S^n(f), \tag{6.14}$$

where Φ_S denotes the composed polynomial transformations [10, Sec. 3 (Step 1) with (2.12)] depending only on the simplex S with its polynomial Jacobian J_{Φ_S}. Since Φ_S is an affine transformation composed with a Duffy type transformation, it holds for polynomials $u \in \mathcal{P}_p(S)$ that $u\circ\Phi_S \in \mathcal{Q}_p(\mathbb{R}^d)$.

The approximation of the right-hand side $l(v) := \langle f, v\rangle_{L^2(\Omega)}$ follows immediately.

Definition 6.7 *(Approximate linear form for $d > 1$)*. For a piecewise polynomial $v \in S_0^{p,1}(\mathcal{T}_\gamma)$, define the approximate linear form by

$$l(v) := \langle f, v\rangle_{L^2(\Omega)} = \sum_{S\in\mathcal{T}_\gamma} \int_S f(\vec{x}) v(\vec{x})\, d\vec{x} \approx \sum_{S\in\mathcal{T}_\gamma} GL_S^n\big(f\, v\big) =: \widetilde{l}_n(v), \tag{6.15}$$

where GL_S^n denotes the tensor product Gauss-Legendre rule (6.14).

Consistency error estimates for the linear form l follow with the same arguments as for the one dimensional case in Lemma 4.2.

Lemma 6.8 *(Consistency error for l). Let f be analytic in $\overline{\Omega}$, and let $\mathcal{T}_\gamma$ be a γ-shape regular mesh on $\Omega \subseteq \mathbb{R}^d$. Let $l(v) := \langle f, v\rangle_{L^2(\Omega)}$ and let its approximation $\widetilde{l}_n(\cdot)$ be defined by (6.15). Then, there exist constants $\rho > 1$ and $C_{f,\gamma,s} > 0$ depending only on f, γ, s, and Ω such that*

$$|l(v) - \widetilde{l}_n(v)| \leq C_{f,\gamma,s}\, p\, \rho^{p-2n+1} \|v\|_{\widetilde{H}^s(\Omega)} \qquad \text{for all } v \in S_0^{p,1}(\mathcal{T}_\gamma). \tag{6.16}$$

Next, we define the approximation to the bilinear form $a(\cdot,\cdot)$.

Definition 6.9 *(Approximate bilinear form for $d>1$).* Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh and $S_1, S_2 \in \mathcal{T}_\gamma$ be closed simplices in $\mathbb{R}^d$ with $k := \dim(S_1 \cap S_2)$ (setting $k := -1$ if $S_1 \cap S_2 = \emptyset$). For piecewise polynomials $v \in S_0^{p,1}(\mathcal{T}_\gamma)$, $w \in S_0^{r,1}(\mathcal{T}_\gamma)$, using the notations F_j, $j=-1,\dots,K_k$ from Corollary 6.3, we define the following tensor product quadrature rules

$$Q^n_{S_1,S_2}(v,w) := GJ^{0,\beta_1}_{n,t_1} \circ GL_{n,t_2} \circ \cdots \circ GL_{n,t_d}\left(\textstyle\sum_{j=0}^{K_k} F_j\right) \qquad \text{for } k \ge 0, \tag{6.17}$$

$$Q^n_{S_1,S_2}(v,w) := GL_{n,t_1} \circ \cdots \circ GL_{n,t_d}\left(F_{-1}\right) \qquad \text{for } k = -1, \tag{6.18}$$

where $\beta_1 := 1-2s+d-k$.

The final approximation to the bilinear form $a(\cdot,\cdot)$ reads

$$\widetilde{a}_n(v,w) := \frac{C(s,d)}{2} \sum_{S_1 \in \mathcal{T}_\gamma} \sum_{S_2 \in \mathcal{T}_\gamma} Q^n_{S_1,S_2}(v,w) + C(s,d) \sum_{S_1 \in \mathcal{T}_\gamma} Q^n_{S_1,\Omega^c}(v,w). \tag{6.19}$$

Here $Q^n_{S_1,\Omega^c}(v,w)$ denotes an approximation to $I_{S_1,\Omega^c}(v,w)$ given by (6.39).

Next, we employ scaling arguments to work out the dependence on the element sizes and the polynomial degree when estimating $|a(\cdot,\cdot) - \widetilde{a}_n(\cdot,\cdot)|$.

Adjacent or identical simplices

We start with the case of two simplices S_1, S_2 with $k := \dim(S_1 \cap S_2) \ge 0$. We define the reference simplex as $\widehat{S}_d := \{(x_1,\dots,x_d) \mid x_i \ge 0\ \forall i=1,\dots,d,\ x_1+\cdots+x_d \le 1\}$. As the simplices S_i, $i=1,2$, share, by assumption, $k+1$ vertices, we may label the vertices $\vec{v}^{(i,\ell)}$ of S_i such that $\vec{v}^{(1,\ell)} = \vec{v}^{(2,\ell)}$ for all $0 \le \ell \le k$ and $\vec{v}^{(1,\ell)} \ne \vec{v}^{(2,\ell)}$ for all $k+1 \le \ell \le d$. With the $d \times d$-matrices

$$A^{(i)} := \left(\vec{v}^{(i,1)} - \vec{v}^{(i,0)} \ \cdots\ \vec{v}^{(i,d)} - \vec{v}^{(i,0)}\right), \quad i=1,2, \tag{6.20}$$

the pullback transformation $F_{S_1\times S_2}$ is given by

$$F_{S_1\times S_2} : \widehat{S}_d \times \widehat{S}_d \to S_1 \times S_2, \quad (\vec{x},\vec{y}) \mapsto \left(F_{S_1}(\vec{x}), F_{S_2}(\vec{y})\right) := \left(\vec{v}^{(1,0)} + A^{(1)}\vec{x},\ \vec{v}^{(2,0)} + A^{(2)}\vec{y}\right) \tag{6.21}$$

with its Jacobian $J_{F_{S_1\times S_2}} = |\det A^{(1)} \det A^{(2)}|$. Denoting by $\hat{v}_{S_i} := v|_{S_i} \circ F_{S_i}$ and $\hat{w}_{S_i} := w|_{S_i} \circ F_{S_i}$ for $i=1,2$, the pullbacks to the reference simplex $\widehat{S}_d$, the map $F_{S_1\times S_2}$ transforms the integral (6.7) to

$$I_{S_1,S_2}(v,w) = \int_{\widehat{S}_d} \int_{\widehat{S}_d} \frac{(\hat{v}_{S_1}(\vec{x}) - \hat{v}_{S_2}(\vec{y}))\,(\hat{w}_{S_1}(\vec{x}) - \hat{w}_{S_2}(\vec{y}))}{|F_{S_1}(\vec{x}) - F_{S_2}(\vec{y})|^{d+2s}} J_{F_{S_1\times S_2}}\, d\vec{y}\, d\vec{x}. \tag{6.22}$$

As, for all elements in a γ-shape regular mesh $\mathcal{T}_\gamma$, the lengths of all edges $|\vec{v}^{(j,i)} - \vec{v}^{(j,0)}|$ are controlled by the element diameter h_{S_j}, we obtain $J_{F_{S_1\times S_2}} \le C_{\gamma,d}\, h^d_{S_1} h^d_{S_2}$ with a constant $C_{\gamma,d}$ that depends only on γ and the dimension d.

To simplify the notation we introduce $\mathcal{N}_{\hat{v}}(\vec{x},\vec{y}) := \hat{v}_{S_1}(\vec{x}) - \hat{v}_{S_2}(\vec{y})$ and $\mathcal{N}_{\hat{w}}(\vec{x},\vec{y}) := \hat{w}_{S_1}(\vec{x}) - \hat{w}_{S_2}(\vec{y})$. Corollary 6.3 yields for (6.22)

$$I_{S_1,S_2}(v,w) = J_{F_{S_1\times S_2}} \int_{[0,1]^{2d}} \underbrace{\sum_{j=0}^{K_k} \frac{\mathcal{N}_{\hat{v}} \circ \Phi_j(\vec{t})}{t_1} \frac{\mathcal{N}_{\hat{w}} \circ \Phi_j(\vec{t})}{t_1} \frac{t_1^{d+2s}}{|(A^{(1)}\vec{x} - A^{(2)}\vec{y}) \circ \Phi_j(\vec{t})|^{d+2s}} J^{rem}_{\Phi_j}(\vec{t})\, t_1^{1-2s+d-k}}_{=:\mathcal{I}(\vec{t})} d\vec{t}. \tag{6.23}$$

The estimate of the consistency error is again based on Lemma 5.3, which directly generalizes to higher dimensions. Corollary 6.3 shows that $\mathcal{I}$ allows for a holomorphic extension to a Bernstein ellipse $\widehat{\mathcal{E}_\rho}$ in each variable with fixed $\rho_{A^{(1)},A^{(2)}} > 1$, ostensibly dependent on the transformation matrices $A^{(1)}, A^{(2)}$ but independent of $v \in S_0^{p,1}(\mathcal{T}_\gamma)$, $w \in S_0^{r,1}(\mathcal{T}_\gamma)$. By Assumption 6.5, there is only a finite number of patch configurations in $\mathcal{T}_\gamma$, which leads, up to scaling, to a finite number of different matrices $A^{(1)}, A^{(2)}$. To remove the scaling dependence, we note that

$$A^{(1)}\vec{x} - A^{(2)}\vec{y} = h_{S_1}\left(h^{-1}_{S_1} A^{(1)}\vec{x} - h^{-1}_{S_1} A^{(2)}\vec{y}\right). \tag{6.24}$$

For γ-shape regular meshes we have $h_{S_1} \sim h_{S_2}$ and the diameter of each simplex is proportional to all edge lengths, which leads for $\widehat{A}^{(i)} := h^{-1}_{S_1} A^{(i)}$ to $\|\widehat{A}^{(i)}\|_1 = \mathcal{O}(1)$ for $i=1,2$ and subsequently to a finite number of different values $\rho_{\widehat{A}^{(1)},\widehat{A}^{(2)}} > 1$. Thus, we have a holomorphic extension of $\mathcal{I}$ to a Bernstein ellipse $\widehat{\mathcal{E}}_\rho$ with a fixed $\rho := \min_{\widehat{A}^{(1)},\widehat{A}^{(2)}}(\rho_{\widehat{A}^{(1)},\widehat{A}^{(2)}}) > 1$. To finish the estimate of the consistency error, it suffices to bound each of the three quotients in (6.23) in the norms $\|\cdot\|_{L^\infty(\widehat{\mathcal{I}}^{2d\setminus\ell} \times \widehat{\mathcal{E}}^\ell_\rho)}$, where $\widehat{\mathcal{I}} := (0,1)$ and $\widehat{\mathcal{I}}^{2d\setminus\ell} \times \widehat{\mathcal{E}}^\ell_\rho := \widehat{\mathcal{I}} \times \cdots \times \widehat{\mathcal{I}} \times \widehat{\mathcal{E}}_\rho \times \widehat{\mathcal{I}} \times \cdots \times \widehat{\mathcal{I}}$ denotes the set where the ℓ-th component of $\widehat{\mathcal{I}}^{2d}$ is extended to the Bernstein ellipse $\widehat{\mathcal{E}}_\rho$.

Using $\hat{v}_{S_1}(\vec{0}) = \hat{v}_{S_2}(\vec{0})$ for $k \geq 0$, the first term can be bounded as in Lemma 5.4 using the Bernstein and Markov inequalities by

$$\begin{aligned}\left\|\mathcal{N}_{\hat{v}}\circ\Phi_j\, t_1^{-1}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} &= \left\|\int_0^{t_1} \partial_\tau\left(\mathcal{N}_{\hat{v}}\circ\Phi_j(\tau,t_2,\dots,t_d)\right)d\tau\, t_1^{-1}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} \leq \left\|\partial_{t_1}\left(\mathcal{N}_{\hat{v}}\circ\Phi_j(\vec{t})\right)\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)}\\
&\leq \rho^{q_\Phi p}\left\|\partial_{t_1}\left(\mathcal{N}_{\hat{v}}\circ\Phi_j(\vec{t})\right)\right\|_{L^\infty(\hat{I}^{2d})} \lesssim (q_\Phi p)^2\rho^{q_\Phi p}\left\|\mathcal{N}_{\hat{v}}\circ\Phi_j(\vec{t})\right\|_{L^\infty(\hat{I}^{2d})} = (q_\Phi p)^2\rho^{q_\Phi p}\left\|\mathcal{N}_{\hat{v}}\right\|_{L^\infty(\Phi_j(\hat{I}^{2d}))} \leq (q_\Phi p)^2\rho^{q_\Phi p}\left\|\mathcal{N}_{\hat{v}}\right\|_{L^\infty(\hat{S}_d\times\hat{S}_d)}\\
&= (q_\Phi p)^2\rho^{q_\Phi p}\left\|\hat{v}_{S_1}-\hat{v}_{S_1}(\vec{0})+\hat{v}_{S_2}(\vec{0})-\hat{v}_{S_2}\right\|_{L^\infty(\hat{S}_d\times\hat{S}_d)} \leq (q_\Phi p)^2\rho^{q_\Phi p}\left(\left\|\hat{v}_{S_1}-\hat{v}_{S_1}(\vec{0})\right\|_{L^\infty(\hat{S}_d)}+\left\|\hat{v}_{S_2}-\hat{v}_{S_2}(\vec{0})\right\|_{L^\infty(\hat{S}_d)}\right),\end{aligned} \tag{6.25}$$

where, again, q_Φ is the maximal degree of the polynomial transformations Φ_j. On the reference simplex, there holds by Markov's inequality and inductive application of the inverse inequality from [31, Thm. 3.92] that

$$\begin{aligned}\left\|\hat{v}_{S_1}-\hat{v}_{S_1}(\vec{0})\right\|_{L^\infty(\hat{S}_d)} &\lesssim \left\|\nabla\hat{v}_{S_1}\right\|_{L^\infty(\hat{S}_d)} = \left\|\nabla(\hat{v}_{S_1}-\overline{\hat{v}_{S_1}})\right\|_{L^\infty(\hat{S}_d)} \lesssim (q_\Phi p)^2\left\|\hat{v}_{S_1}-\overline{\hat{v}_{S_1}}\right\|_{L^\infty(\hat{S}_d)}\\
&\lesssim (q_\Phi p)^{2+d}\left\|\hat{v}_{S_1}-\overline{\hat{v}_{S_1}}\right\|_{L^2(\hat{S}_d)} \lesssim (q_\Phi p)^{2+d}\left|\hat{v}_{S_1}\right|_{H^s(\hat{S}_d)} \leq C_{\gamma,d,s}(q_\Phi p)^{2+d}h_{S_1}^{s-d/2}\left|v|_{S_1}\right|_{H^s(S_1)},\end{aligned} \tag{6.26}$$

where $C_{\gamma,d,s}$ is a constant that depends only on γ, d, s. This finishes the upper bound for the first quotient in (6.23)

$$\begin{aligned}\left\|\mathcal{N}_{\hat{v}}\circ\Phi_j\, t_1^{-1}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} &\leq C_{\gamma,d,s}\,\rho^{q_\Phi p}\,(q_\Phi p)^{4+d}\max\left(h_{S_1}^{s-d/2}\left|v|_{S_1}\right|_{H^s(S_1)},\, h_{S_2}^{s-d/2}\left|v|_{S_2}\right|_{H^s(S_2)}\right)\\
&\lesssim C_{\gamma,d,s}\,\rho^{q_\Phi p}\,(q_\Phi p)^{4+d}h_{S_1}^{s-d/2}\,|v|_{H^s(co(S_1,S_2))},\end{aligned} \tag{6.27}$$

where $co(S_1,S_2)$ denotes the convex hull of S_1 and S_2.

The second factor in the integrand in (6.23) can be treated in the same way. The estimate for the third factor in the integrand follows again, as discussed above, by Assumption 6.5 and (6.24)

$$\left\|\frac{t_1^{d+2s}}{|(A^{(1)}\vec{x}-A^{(2)}\vec{y})\circ\Phi_j(\vec{t})|^{d+2s}}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} = h_{S_1}^{-d-2s}\left\|\frac{t_1^{d+2s}}{\left|\left(h_{S_1}^{-1}A^{(1)}\vec{x}-h_{S_1}^{-1}A^{(2)}\vec{y}\right)\circ\Phi_j(\vec{t})\right|^{d+2s}}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} \leq C_{\gamma,s,\rho,d}\,h_{S_1}^{-d-2s},$$

where the last estimate follows from the observation that we only have a finite number of cases for the function inside the norm. Now, we have deduced the appropriate scaling in terms of the element sizes for each factor in (6.23) in the L^∞-norm and inserting everything into the higher-dimensional analog of Lemma 5.3 yields

$$|I_{S_1,S_2}(v,w)-Q^n_{S_1,S_2}(v,w)| \leq C_{s,\gamma,\rho,d}\,(q_\Phi p)^{d+4}(q_\Phi r)^{d+4}\rho^{q_\Phi(p+r)-2n+1}|v|_{H^s(co(S_1,S_2))}|w|_{H^s(co(S_1,S_2))} \tag{6.28}$$

for adjacent or identical simplices S_1, S_2.

Separated simplices

For the case $S_1\cap S_2=\emptyset$, i.e. $k=-1$, we start with the same transformation as in (6.22), where we labelled the vertices such that there holds $\|\vec{v}^{(1,0)}-\vec{v}^{(2,0)}\|_2 = \min_{i,j}\|\vec{v}^{(1,i)}-\vec{v}^{(2,j)}\|_2$ is the shortest Euclidean distance between vertices of S_1 and S_2. Corollary 6.3 yields for (6.22)

$$I_{S_1,S_2}(v,w) = J_{F_{S_1\times S_2}}\int_{[0,1]^{2d}}\mathcal{N}_{\hat{v}}\circ\Phi_{-1}(\vec{t})\,\mathcal{N}_{\hat{w}}\circ\Phi_{-1}(\vec{t})\,|(F_{S_1}(\vec{x})-F_{S_2}(\vec{y}))\circ\Phi_{-1}(\vec{t})|^{-d-2s}\,J_{\Phi_{-1}}(\vec{t})\,d\vec{t}. \tag{6.29}$$

For simplices S_1, S_2 define $d_{S_1,S_2} := \operatorname{dist}(S_1,S_2)$ and pick a closed ball B_{S_1,S_2} with S_1, $S_2\subseteq B_{S_1,S_2}$ and $\operatorname{diam}B_{S_1,S_2}\leq h_{S_1}+h_{S_2}+d_{S_1,S_2}$. The integrand can be estimated with a combination of arguments applied to the case $k\geq 0$ and the case $d=1$ in Lemma 5.5. Inserting the mean $\overline{v_{B_{S_1,S_2}}} := \int_{B_{S_1,S_2}}v(x)dx/|B_{S_1,S_2}|$ gives

$$\begin{aligned}\left\|\mathcal{N}_{\hat{v}}\circ\Phi_{-1}\right\|_{L^\infty(\hat{I}^{2d}\setminus^\ell\times\hat{\mathcal{E}}^\ell_\rho)} &\leq \rho^{q_\Phi p}\left\|\mathcal{N}_{\hat{v}}\circ\Phi_{-1}\right\|_{L^\infty(\hat{I}^{2d})} \leq C\rho^{q_\Phi p}\left\|\hat{v}_{S_1}-\hat{v}_{S_2}\right\|_{L^\infty(\hat{S}_d\times\hat{S}_d)}\\
&\leq C\rho^{q_\Phi p}\left(\left\|\hat{v}_{S_1}-\overline{v_{B_{S_1,S_2}}}\right\|_{L^\infty(\hat{S}_d)}+\left\|\hat{v}_{S_2}-\overline{v_{B_{S_1,S_2}}}\right\|_{L^\infty(\hat{S}_d)}\right).\end{aligned} \tag{6.30}$$

With an L^∞–L^2 inverse estimate on the reference simplex and a Poincaré type estimate for the ball B_{S_1,S_2} there holds

$$\begin{aligned}\left\|\hat{v}_{S_1}-\overline{v_{B_{S_1,S_2}}}\right\|_{L^\infty(\hat{S}_d)} &\leq C_d(q_\Phi p)^d\left\|\hat{v}_{S_1}-\overline{v_{B_{S_1,S_2}}}\right\|_{L^2(\hat{S}_d)} \leq C_{\gamma,d,s}\,h_{S_1}^{-d/2}(q_\Phi p)^d\left\|v-\overline{v_{B_{S_1,S_2}}}\right\|_{L^2(S_1)}\\
&\leq C_{\gamma,d,s}\,h_{S_1}^{-d/2}(h_{S_1}+h_{S_2}+d_{S_1,S_2})^s(q_\Phi p)^d\,|v|_{H^s(B_{S_1,S_2})},\end{aligned} \tag{6.31}$$

where $C_{\gamma,d,s}$ is a constant that depends only on γ, d, s. For the third factor in the integrand in (6.29), we note

$$|(F_{S_1}(\vec{x})-F_{S_2}(\vec{y}))\circ\Phi_{-1}(\vec{t})|^{-d-2s} = |(\vec{v}^{(1,0)}-\vec{v}^{(2,0)}+A^{(1)}\vec{x}-A^{(2)}\vec{y})\circ\Phi_{-1}(\vec{t})|^{-d-2s}. \tag{6.32}$$

It follows that

$$\begin{aligned}&\left\| |(F_{S_1}(\vec{x}) - F_{S_2}(\vec{y}))\circ\Phi_{-1}(\vec{t})|^{-d-2s} \right\|_{L^\infty(\hat{\mathcal{I}}^{2d}\setminus\ell\times\hat{\mathcal{E}}_\rho^\ell)} \\ &\leq d_{S_1,S_2}^{-d-2s} \left\| |(d_{S_1,S_2}^{-1}(\vec{v}^{(1,0)} - \vec{v}^{(2,0)}) + d_{S_1,S_2}^{-1}A^{(1)}\vec{x} - d_{S_1,S_2}^{-1}A^{(2)}\vec{y})\circ\Phi_{-1}(\vec{t})|^{-d-2s} \right\|_{L^\infty(\hat{\mathcal{I}}^{2d}\setminus\ell\times\hat{\mathcal{E}}_\rho^\ell)}. \end{aligned} \tag{6.33}$$

By Assumption 6.5, there is only a finite number of patch configurations in $\mathcal{T}_\gamma$, which leads, up to scaling, to a finite number of different matrices $A^{(1)}, A^{(2)}$. The γ-shape regularity and choice of numbering of the vertices yield $|(\vec{v}^{(1,0)} - \vec{v}^{(2,0)})| \sim d_{S_1,S_2}$ and $\|d_{S_1,S_2}^{-1}A^{(1)}\|_1$, $\|d_{S_1,S_2}^{-1}A^{(2)}\|_1 = \mathcal{O}(1)$. This leads to a finite number of holomorphic extensions. Hence, there is a $\rho > 1$ depending only on γ and Ω for which, in each variable, a holomorphic extension to the Bernstein ellipse $\hat{\mathcal{E}}_\rho$ is possible, and this extension can be bounded by

$$\left\| |(F_{S_1}(\vec{x}) - F_{S_2}(\vec{y}))\circ\Phi_{-1}(\vec{t})|^{-d-2s} \right\|_{L^\infty(\hat{\mathcal{I}}^{2d}\setminus\ell\times\hat{\mathcal{E}}_\rho^\ell)} \leq C_{\gamma,s,d}\, d_{S_1,S_2}^{-d-2s}. \tag{6.34}$$

Inserting everything into the higher-dimensional analog of Lemma 5.3 yields for separated simplices S_1, S_2

$$|I_{S_1,S_2}(v,w) - Q^n_{S_1,S_2}(v,w)| \leq C_{s,\gamma,d}\,(q_\Phi p)^d (q_\Phi r)^d \rho^{q_\Phi(p+r)-2n+1} |v|_{H^s(B_{S_1,S_2})} |w|_{H^s(B_{S_1,S_2})}, \tag{6.35}$$

where we used that for γ-shape regular meshes there holds $d_{S_1,S_2} \geq C\max\{h_{S_1}, h_{S_2}\}$ for some $C > 0$ depending on γ so that the combined effect of the scaling parameters of all contributions in (6.29) can be uniformly bounded by

$$d_{S_1,S_2}^{-d-2s}\, h_{S_1}^d\, h_{S_2}^d (h_{S_1} + h_{S_2} + d_{S_1,S_2})^{2s} \left(h_{S_1}^{-d/2} + h_{S_2}^{-d/2}\right)^2 \leq C_{\gamma,s}.$$

Combining the estimates for all cases with the simple observation $co(S_1, S_2) \subseteq B_{S_1,S_2}$ yields the following lemma for the quadrature error.

Lemma 6.10. *Let $\mathcal{T}_\gamma$ be a γ-shape regular mesh satisfying Assumption 6.5. Let S_1, $S_2 \in \mathcal{T}_\gamma$ be closed simplices in $\mathbb{R}^d$ and denote by B_{S_1,S_2} a closed ball with $\operatorname{diam} B_{S_1,S_2} \leq h_{S_1} + h_{S_2} + \operatorname{dist}(S_1, S_2)$ that contains the simplices S_1, $S_2 \subseteq B_{S_1,S_2}$. Then, for the integral $I_{S_1,S_2}(v,w)$ from (6.7) and its approximation $Q^n_{S_1,S_2}(v,w)$ by quadrature, there exists a constant $\rho > 1$ that depends only on γ and Ω such that for all $v \in S_0^{p,1}(\mathcal{T}_\gamma)$, $w \in S_0^{r,1}(\mathcal{T}_\gamma)$ there holds*

$$|I_{S_1,S_2}(v,w) - Q^n_{S_1,S_2}(v,w)| \leq C_{s,\gamma,d}\,(q_\Phi p)^{d+4} (q_\Phi r)^{d+4} \rho^{q_\Phi(p+r)-2n+1} |v|_{H^s(B_{S_1,S_2})} |w|_{H^s(B_{S_1,S_2})} \tag{6.36}$$

with the constant $C_{s,\gamma,d}$ depending only on s, γ, d, and Ω; q_Φ is given by Proposition 6.1.

6.3. Treatment of Ω^c

In this section, we discuss the issue that the evaluation of the bilinear form $a(\cdot,\cdot)$ requires the evaluation of I_{S_1,Ω^c} given by (6.3). This is addressed using two ingredients:

(i) we select an open set B_R with $\overline{\Omega} \subset B_R$ (for convenience, this set will be taken to be a hypercube $(-R,R)^d$ below) and extend the mesh $\mathcal{T}_\gamma$ to a triangulation $\mathcal{T}_\gamma^R$ of B_R satisfying Assumption 6.5. For this triangulation, we may employ the quadrature technique used above.

(ii) We develop a quadrature rule for integration over $B_R^c := \mathbb{R}^d \setminus B_R$ and exploit that $\operatorname{dist}(S_1, B_R^c) \geq \operatorname{dist}(\Omega, B_R^c) > 0$ together with analyticity of the integrand.

We focus on (ii). Let $B_R := (-R,R)^d$ for a fixed $R > 0$. Introduce the cones $\mathcal{C}_1 := \{(y_1, y_1 y') \,|\, y_1 > R, y' \in [-1,1]^{d-1}\}$ as well as $\mathcal{C}_i$, $i = 2,\dots,2d$ obtained by rotating $\mathcal{C}_1$ so that the centerline of $\mathcal{C}_i$ is aligned with one of the unit vectors $(\pm 1, 0, \dots, 0)$, $(0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1) \in \mathbb{R}^d$. An integral of the kernel function over $\mathcal{C}_1$ can be evaluated using the transformation $\eta = 1/y_1$ as follows:

$$\begin{aligned} G_1(\vec{x}) &:= \int_{\vec{y}\in\mathcal{C}_1} |\vec{x} - \vec{y}|^{-(d+2s)}\, d\vec{y} = \int_{y'\in[-1,1]^{d-1}} \int_{y_1=R}^{\infty} |\vec{x} - y_1(1,y')^\top|^{-(d+2s)} y_1^{d-1}\, dy'\, dy_1 \\ &= \int_{y'\in[-1,1]^{d-1}} \int_{\eta=0}^{1/R} \underbrace{|\eta\vec{x} - (1,y')^\top|^{-(d+2s)}}_{=:\mathcal{G}_1(\vec{x},\eta,y')} \eta^{2s-1}\, d\eta\, dy'. \end{aligned}$$

This suggests to use a tensor product quadrature with (product) Gauss-Legendre quadrature in the y'-variables and a Gauss-Jacobi quadrature with weight η^{2s-1} in the η-variable. Key to the performance of the quadrature rule is the analyticity of the function $\mathcal{G}_1$:

Lemma 6.11. *Let $\overline{\Omega} \subset B_R = (-R,R)^d$. Then:*

(i) The function G_1 is analytic on $\overline{\Omega}$.

(ii) The function $\mathcal{G}_1$ is analytic on $\overline{\Omega} \times [0, 1/R] \times [-1, 1]^{d-1}$.

(iii) The functions G_1 and $\mathcal{G}_1$ are positive on $\overline{\Omega}$ and $\overline{\Omega} \times [0, 1/R] \times [-1, 1]^{d-1}$, respectively.

Analyticity of a function G on a closed set $A \subset \mathbb{R}^n$ means that there is a complex neighborhood $A_\varepsilon \subset \mathbb{C}^n$ of A and a function G_ε holomorphic on A_ε with $G_\varepsilon|_A = G$.

Proof. *Proof of (iii):* Consider the function

$$\hat{G}(x_1, \dots, x_d, \eta, y_2', \dots, y_d') := (\eta x_1 - 1)^2 + \sum_{i=2}^{d} (\eta x_i - y_i')^2 = \eta^2 |\vec{x} - \eta^{-1}(1, y')^\top|^2, \tag{6.37}$$

which is an entire function on $\mathbb{C}^{2d}$. We claim that $\hat{G}(\vec{x}, \eta, y') > 0$ on $K := \overline{\Omega} \times [0, 1/R] \times [-1, 1]^{d-1}$. Since $\vec{x} \in \overline{\Omega}$ and $\eta^{-1}(1, y')^\top \in C_1 \subset B_R^c$, we have $\hat{G} \geq (\eta^2 \operatorname{dist}(\Omega, B_R^c))^2 > 0$ for $\eta > 0$. For $\eta = 0$, we have $\hat{G}(\vec{x}, 0, y') = |(1, y')|^2 \geq 1$. Since $\mathcal{G}_1 = \hat{G}^{-(d+2s)/2} > 0$, the integrand of G_1 is positive. This finishes the proof of (iii).

Proof of (ii): By positivity of $\hat{G}$ on the compact set K and the smoothness of $\hat{G}$, there is a complex neighborhood $K_\varepsilon := \cup_{z \in K} B_\varepsilon(z) \subset \mathbb{C}^{2d}$ such that $\operatorname{Re} \hat{G} > 0$ on D_ε. Hence, with the principal branch of the logarithm, the function $\exp(-\frac{d+2s}{2} \log \hat{G}(z))$ is holomorphic on K_ε and coincides with $\mathcal{G}_1$ on K.

Proof of (i): This follows from (ii). $\square$

In total, we have arrived at

$$I_{S_1, \Omega^c}(v, w) = \sum_{S_2 \in \mathcal{T}_\gamma^R \setminus \mathcal{T}_\gamma} I_{S_1, S_2}(v, w) + \sum_{i=1}^{2d} \int_{\vec{x} \in S_1} G_i(\vec{x}) v(\vec{x}) w(\vec{x}) \, d\vec{x},$$

where the functions G_i, $i \geq 2$, are defined as G_1 with C_1 replaced with C_i. Analogous to Lemma 6.11, the functions G_i and the corresponding integrands $\mathcal{G}_i$ are analytic. For a fully discrete approximation of $I_{S_1, \Omega^c}(v, w)$, we denote by $Q^n_{S_1, C_1}(v, w)$ the quadrature rule to evaluate

$$I_{S_1, C_1}(v, w) := \int_{\vec{x} \in S_1} v(\vec{x}) w(\vec{x}) \int_{y' \in [-1,1]^{d-1}} \int_{\eta=0}^{1/R} \mathcal{G}_1(\vec{x}, \eta, y') \eta^{2s-1} \, d\eta \, dy' \, d\vec{x}$$

with a tensor product Gauss-Legendre rule (with n points for each variable) for the integration in y', a Gauss-Jacobi rule (with n points) for the integration in η, and the tensor product Gauss-Legendre rule (6.14) for the integration in $\vec{x}$ over the simplex S_1, i.e.,

$$Q^n_{S_1, C_1}(v, w) := GL^n_{S, \vec{x}}(v \, w \, GJ^{0, 2s-1}_{n, \eta} \circ GL_{n, y_1'} \circ \cdots \circ GL_{n, y_{d-1}'}(\mathcal{G}_1 \circ F_{\eta, y'} \, J_{F_{\eta, y'}})), \tag{6.38}$$

where the pullback transformation $F_{\eta, y'} : [0, 1/R] \times [-1, 1]^{d-1} \to [0, 1]^d$ is defined in a canonical way and $J_{F_{\eta, y'}}$ denotes its Jacobian. Analogously, we define rules $Q^n_{S_1, C_i}$, $i \geq 2$. The fully discrete approximation is then given by

$$I_{S_1, \Omega^c}(v, w) \approx Q^n_{S_1, \Omega^c}(v, w) := \sum_{S_2 \in \mathcal{T}_\gamma^R \setminus \mathcal{T}_\gamma} Q^n_{S_1, S_2}(v, w) + \sum_{i=1}^{2d} Q^n_{S_1, C_i}(v, w). \tag{6.39}$$

Remark 6.12. The function $\vec{x} \mapsto \int_{B_R^c} |\vec{x} - \vec{y}|^{-(d+2s)} \, d\vec{y}$ is analytic on $\overline{\Omega}$. Hence, it could be approximated by a (piecewise) polynomial on a coarse mesh. A computational speed-up is then possible since the evaluation of the $Q^n_{S_1, C_i}(v, w)$ can be replaced with the evaluation of $\int_{S_1} v(\vec{x}) w(\vec{x}) \pi(\vec{x}) \, d\vec{x}$ for some polynomials π. Precomputing on the reference element is an option. ■

6.4. Exponential convergence under quadrature

Combining the approximation results for the integrals $I_{S_1, S_2}(v, w)$ and $I_{S_1, \Omega^c}(v, w)$ from the previous subsections, we directly arrive at an error estimate for the consistency error for the bilinear form $a(\cdot, \cdot)$.

Lemma 6.13 *(Consistency error for bilinear form a for $d > 1$). Let $\mathcal{T}_\gamma$ be a γ-shape regular triangulation of $\Omega \subset \mathbb{R}^d$, $R > 0$ be such that $\overline{\Omega} \subset (-R, R)^d$ and $\mathcal{T}_\gamma^R$ be a γ-shape regular mesh that extends the mesh $\mathcal{T}_\gamma$ to $(-R, R)^d$. Assume $\mathcal{T}_\gamma^R$ satisfies Assumption 6.5. Let $a(\cdot, \cdot)$ be the bilinear form of (6.1) and $\tilde{a}_n(\cdot, \cdot)$ be its approximation given by (6.19). Then, there exists a constant $\rho > 1$ that depends only on the shape regularity constant γ and Ω such that for all $v \in S_0^{p,1}(\mathcal{T}_\gamma)$ and $w \in S_0^{r,1}(\mathcal{T}_\gamma)$ there holds*

$$|a(v, w) - \tilde{a}_n(v, w)| \leq C_{s,\gamma,d} (\# \mathcal{T}_\gamma^R)^2 p^{d+4} r^{d+4} \rho^{q_\Phi (r+p) - 2n + 1} \|v\|_{\tilde{H}^s(\Omega)} \|w\|_{\tilde{H}^s(\Omega)}, \tag{6.40}$$

with constants $C_{s,\gamma,d}$ depending only on s, γ, d, and Ω; q_Φ is given by Proposition 6.1.

Proof. With the triangle inequality, (6.19) and (6.39), we obtain

$$|a(v,w)-\widetilde{a}_n(v,w)| \leq \frac{C(s,d)}{2}\Bigg(\sum_{S_1\in\mathcal{T}_\gamma}\sum_{S_2\in\mathcal{T}_\gamma} |I_{S_1,S_2}(v,w) - Q^n_{S_1,S_2}(v,w)| \\ + 2\sum_{S_1\in\mathcal{T}_\gamma}\sum_{S_2\in\mathcal{T}_\gamma^R\setminus\mathcal{T}_\gamma} |I_{S_1,S_2}(v,w) - Q^n_{S_1,S_2}(v,w)| + 2\sum_{i=1}^{2d} |I_{S_1,C_i}(v,w) - Q^n_{S_1,C_i}(v,w)|\Bigg).$$

The terms $|I_{S_1,S_2}(v,w) - Q^n_{S_1,S_2}(v,w)|$ can be estimated with Lemma 6.10. The other contributions of the form $|I_{S_1,C_i}(v,w) - Q^n_{S_1,C_i}(v,w)|$ correspond to approximation of $\int_{S_1} v(\vec{x})w(\vec{x})G_i(\vec{x})\,d\vec{x}$ with analytic functions G_i and thus take the same form as the integrals involved in the linear form $l(\cdot)$. Thus, a combination of Lemma 6.10 and Lemma 6.8 together with summation over all simplices gives the result. □

With the estimate for the consistency error, we directly obtain uniform coercivity as in the case $d = 1$ by a perturbation argument as described in Lemma 4.4. Note that the integral transformations Φ_j for $d > 1$ induce an additional constant q_Φ in the exponential term in the consistency error. In order to compensate for that the number of quadrature points now has to grow like λp for some $\lambda > 1$.

Theorem 6.14 *(Uniform coercivity, $d > 1$). Let the assumptions of Lemma 6.13 hold. Then, there are constants $\widetilde{\alpha}$, λ_1, $\lambda_2 > 0$ depending only on s, the shape regularity constant γ, the dimension d, and Ω such that for $n \geq \lambda_1 p + \lambda_2 \ln(\#\mathcal{T}_\gamma^R + 1)$ there holds*

$$\widetilde{\alpha}\|v\|^2_{\widetilde{H}^s(\Omega)} \leq \widetilde{a}_n(v,v) \quad \text{for all } v \in S_0^{p,1}(\mathcal{T}_\gamma). \tag{6.41}$$

Employing the Strang Lemma, we can derive a result similar to Theorem 2.4 for $d > 1$ by the same arguments. The error of the fully discrete FEM approximation can be bounded by the exact FEM error and a consistency error that decays exponentially in the number of quadrature points.

Theorem 6.15 *(Exponential convergence under quadrature, $d > 1$). Let $\mathcal{T}_\gamma$ be a γ-shape regular triangulation of the bounded polyhedron $\Omega \subset \mathbb{R}^d$. Let $R > 0$ be such that $\overline{\Omega} \subset (-R,R)^d$, and let $\mathcal{T}_\gamma^R$ be a γ-shape regular mesh that extends the mesh $\mathcal{T}_\gamma$ to $(-R,R)^d$. Assume $\mathcal{T}_\gamma^R$ satisfies Assumption 6.5. Let f be analytic in $\overline{\Omega}$. Denote by $u \in \widetilde{H}^s(\Omega)$ the solution to (6.1), by $u_r \in S_0^{r,1}(\mathcal{T}_\gamma)$ the FEM solution for the exact variational formulation in the space $S_0^{r,1}(\mathcal{T}_\gamma) \subseteq S_0^{p,1}(\mathcal{T}_\gamma)$, and by $\widetilde{u}_{N,n} \in S_0^{p,1}(\mathcal{T}_\gamma)$ the solution to*

$$\widetilde{a}_n(\widetilde{u}_{N,n}, v_N) = \widetilde{l}(v_N) \qquad \forall v_N \in S_0^{p,1}(\mathcal{T}_\gamma),$$

where $\widetilde{a}_n(\cdot,\cdot)$ and $\widetilde{l}_n(\cdot)$ are defined in (6.19) and (6.15). The index n indicates the number of quadrature points that is used per coordinate direction per integral and element.

Then, there exist constants $\rho > 1$, λ_1, λ_2, $C_{s,\gamma,d} > 0$ (depending only on s, Ω, d, γ), such that for all p, $\#\mathcal{T}_\gamma^R$ and n with $n \geq \lambda_1 p + \lambda_2 \ln(\#\mathcal{T}_\gamma^R + 1)$) and $r \in \{1,\dots,p\}$ there holds

$$\|u - \widetilde{u}_{N,n}\|_{\widetilde{H}^s(\Omega)} \leq \|u - u_r\|_{\widetilde{H}^s(\Omega)} + C_{s,\gamma,d}(\#\mathcal{T}_\gamma^R)^2 p^{d+4} r^{d+4} \rho^{q_\Phi(p+r)-2n+1}; \tag{6.42}$$

the constant q_Φ is given by Proposition 6.1. The number of operations to compute the stiffness is $\mathcal{O}((np)^{2d}(\#\mathcal{T}_\gamma^R)^2)$.

Remark 6.16. The treatment of the complementary part Ω^c in the bilinear form induces the appearance of the term $\#\mathcal{T}_\gamma^R$ in the error estimate (6.42). In the context of shape-regular hp-FEM "boundary concentrated meshes" [22] both for $\mathcal{T}_\gamma$ and $\mathcal{T}_\gamma^R$ are a natural choice. The total number of elements is then proportional to the number of elements touching the boundary $\partial\Omega$ and thus $\#\mathcal{T}_\gamma^R$ is proportional to $\#\mathcal{T}_\gamma$. ■

7. Numerical experiments

In this section, we present some numerical examples that underline the theoretical estimates in our main results, Theorem 2.4. We consider

$$(-\Delta)^s u = 1 \quad \text{in } \Omega := (-1,1), \qquad u = 0 \;\text{ on } \Omega^c,$$

with exact solution $u(x) = 2^{-2s}\sqrt{\pi}(\Gamma(s+1/2)\Gamma(1+s))^{-1}(1-x^2)^s$.

In the following, we will present three different approaches to estimate the energy norm error between the exact solution u and the fully discrete hp-FEM approximation $\widetilde{u}_{N,n}$

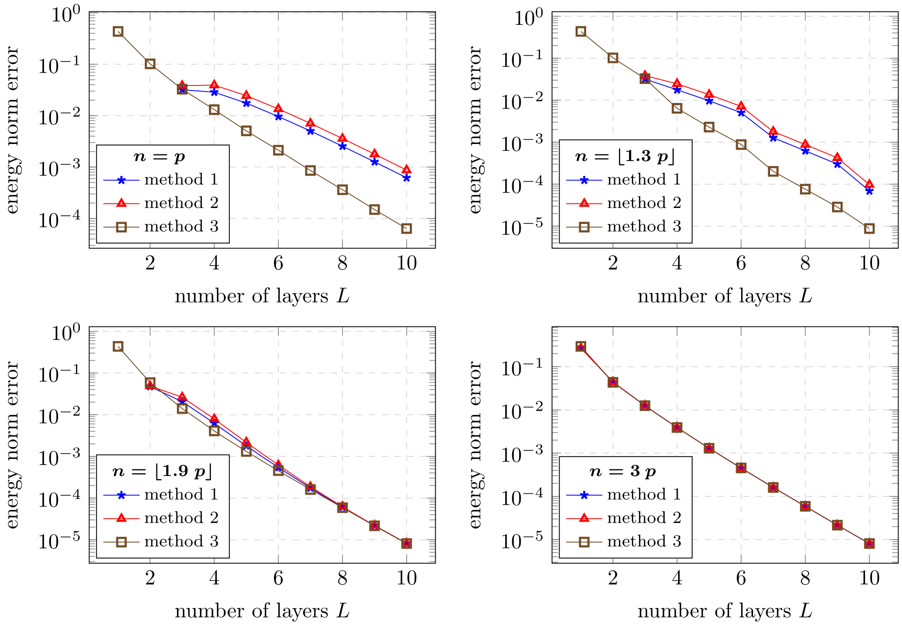

Fig. 1. Three different methods (see Example 7.1) to calculate the energy norm error of *hp*-FEM with $n = \mathcal{O}(p)$ quadrature points on a geometric mesh with grading factor $\sigma = 0.172$, polynomial degree $p = L$, $s = 3/4$.

$$\sqrt{a(u - \widetilde{u}_{N,n}, u - \widetilde{u}_{N,n})} = \sqrt{a(u,u) - a(\widetilde{u}_{N,n}, \widetilde{u}_{N,n}) - 2a(u - \widetilde{u}_{N,n}, \widetilde{u}_{N,n})}.$$

If the quadrature error is ignored, i.e., if it is assumed that $u_N = \widetilde{u}_{N,n}$, then Galerkin orthogonality $a(u - \widetilde{u}_{N,n}, \widetilde{u}_{N,n}) = 0$ holds and, assuming that $a(u,u)$ is known, the error can be computed in the standard way as the square root of the difference between the energies of the exact solution and the Galerkin approximation. The exact energy $a(\widetilde{u}_{N,n}, \widetilde{u}_{N,n})$ of $\widetilde{u}_{N,n}$ can in general only be approximated by quadrature, leading to an error estimate of the form

$$\sqrt{a(u - \widetilde{u}_{N,n}, u - \widetilde{u}_{N,n})} \approx \sqrt{a(u,u) - \widetilde{a}_m(\widetilde{u}_{N,n}, \widetilde{u}_{N,n})}, \tag{7.1}$$

where $m \geq n$ denotes a number of quadrature points used. However, as u and $\widetilde{u}_{N,n}$ solve different variational formulations, Galerkin orthogonality for $\widetilde{u}_{N,n}$ holds only up to the consistency error

$$|a(u - \widetilde{u}_{N,n}, \widetilde{u}_{N,n})| \leq |\widetilde{a}_n(\widetilde{u}_{N,n}, \widetilde{u}_{N,n}) - a(\widetilde{u}_{N,n}, \widetilde{u}_{N,n})| + |l(\widetilde{u}_{N,n}) - \widetilde{l}_n(\widetilde{u}_{N,n})|.$$

For a high number of quadrature points n the consistency error is small compared to the approximation error. However, for n close to the polynomial degree p we need a different approach. The idea is to calculate an additional reference solution $\widetilde{u}_{N,m}$ with an increased number of quadrature points $m \gg n$ and use the triangle inequality to estimate the energy norm error by

$$\sqrt{a(u - \widetilde{u}_{N,n}, u - \widetilde{u}_{N,n})} \leq \sqrt{a(u - \widetilde{u}_{N,m}, u - \widetilde{u}_{N,m})} + \sqrt{a(\widetilde{u}_{N,m} - \widetilde{u}_{N,n}, \widetilde{u}_{N,m} - \widetilde{u}_{N,n})}. \tag{7.2}$$

By choosing m sufficiently large, the Galerkin orthogonality $a(u - \widetilde{u}_{N,m}, \widetilde{u}_{N,m}) = 0$ holds with a negligible consistency error, and we can again use approximation (7.1) for the first term of the right hand-side. The second term can be approximated with the same small consistency error by replacing a with $\widetilde{a}_m$, which leads to an estimate for the energy norm error given by

$$\sqrt{a(u,u) - \widetilde{a}_m(\widetilde{u}_{N,m}, \widetilde{u}_{N,m})} + \sqrt{\widetilde{a}_m(\widetilde{u}_{N,m} - \widetilde{u}_{N,n}, \widetilde{u}_{N,m} - \widetilde{u}_{N,n})}. \tag{7.3}$$

We can interpret the first term in (7.3) as a good approximation to the energy norm error $\sqrt{a(u - u_N, u - u_N)}$ and therefore as the optimum that our implementation can achieve. The second term in (7.3) represents the implementation error due to the quadrature. The following example shows that the difference between the approximation methods (7.1) and (7.3) can be significant.

Example 7.1. We employ a geometric mesh $\mathcal{T}^L_{geo,\sigma}$ with grading factor $\sigma = 0.172$ and take piecewise polynomials of degree $p = L$. In Fig. 1, three different error measures are plotted versus the number of refinement layers L for different numbers of quadrature points $n = \mathcal{O}(p)$ used to calculate the solution $\widetilde{u}_{N,n}$:

- Method 1: Use approximation (7.1) with the same number of quadrature points m for $\widetilde{a}_m(\cdot,\cdot)$ as for the solution $\widetilde{u}_{N,n}$, i.e., $m = n$.
- Method 2: Use approximation (7.1) and increase the number of quadrature points for the bilinear form $\widetilde{a}_m(\cdot,\cdot)$ to $m = 6p$.
- Method 3: Use approximation (7.3) with $m = 6p$ quadrature points for the reference solution $\widetilde{u}_{N,m}$ and the bilinear form $\widetilde{a}_m(\cdot,\cdot)$.

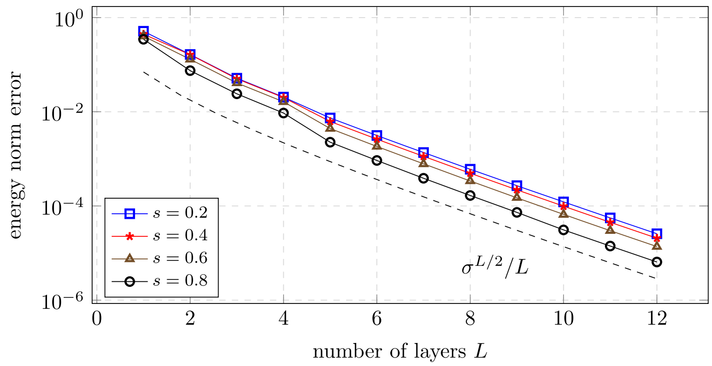

Fig. 2. Exponential convergence in the energy norm (approximation (7.3) with $m = 6p$) of hp-FEM on geometric mesh with grading factor $\sigma = 0.25$, polynomial degree $p = L$, $n = \lfloor 1.2\,p \rfloor$ quadrature points, and different fractional parameters s.

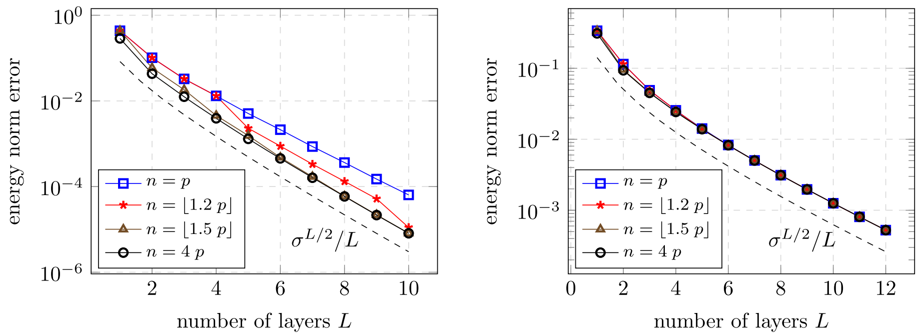

Fig. 3. Exponential convergence in the energy norm (approximation (7.3) with $m = 6p$) of hp-FEM with $n = \mathcal{O}(p)$ quadrature points on geometric mesh, polynomial degree $p = L$, $s = 3/4$. Left: grading factor $\sigma = 0.172$. Right: grading factor $\sigma = 0.5$.

For the cases $n = \lfloor 1.9\,p \rfloor$ and $n = 3\,p$ all three methods produce nearly identical results, whereas for $n = p$ and $n = \lfloor 1.3\,p \rfloor$ the method of calculating the error significantly differs. We observe that Method 1 overestimates the energy norm error significantly and also increasing the number of quadrature points for the norm calculation (Method 2) does not help either. This is consistent with the fact that Method 2 does not decrease the consistency error that is made in the Galerkin orthogonality. We also note that for the cases $n = p$ and $n = \lfloor 1.3p \rfloor$ the computed "energies" were larger than the exact energy so that no errors are reported for these cases in Fig. 1.

The next example is similar to an example in [5] that shows exponential convergence of hp-FEM, where the linear system was assembled using the quadrature approach (2.9) in this article.

Example 7.2. We employ a geometric mesh $\mathcal{T}^L_{geo,\sigma}$ with grading factor $\sigma = 0.25$ and take piecewise polynomials of degree $p = L$. In Fig. 2, the energy norm error (approximation (7.3) with $m = 6p$) is plotted versus the number of refinement layers L for different fractional parameters s. For the number of quadrature points, we used $n := \lfloor 1.2\,p \rfloor$ and, as predicted by Theorem 2.4, we observe exponential convergence with respect to the number of layers L noting that $N \sim L^2$. In fact, the convergence behavior is $\mathcal{O}(\sigma^{L/2}L^{-1})$ and thus slightly faster than asserted by Theorem 2.4. An argument for this observation is given in [5, Sec. 4].

Next, we discuss the number of quadrature points used. Although Theorem 2.4 suggests that a choice of quadrature points $n \geq p+1$ and in particular $n := p + \widetilde{\lambda}\,p$ for any fixed $\widetilde{\lambda} > 0$ suffices to obtain exponential convergence, the rate, or more precisely, the constant in the exponent, is impacted by the choice of $\widetilde{\lambda}$.

Example 7.3. Fig. 3 plots the energy norm error (approximation (7.3) with $m = 6p$) for different numbers of quadrature points $n := p + \widetilde{\lambda}p$ versus the number of layers L for two different choices of grading parameters, $\sigma = 0.172$ and $\sigma = 0.5$. Again, we choose $p = L$ and fix the fractional parameter $s = 3/4$. We notice that the grading factor σ has a direct impact on the number of quadrature points needed to achieve the same accuracy. For the smaller $\sigma = 0.172$, the rate of the exponential convergence depends on the choice of $\widetilde{\lambda}$, while, for $\sigma = 0.5$, the convergence always appears to be $\mathcal{O}(\sigma^{L/2}L^{-1})$. This can also be observed in the theoretical estimates in Theorem 2.4 as the term $L^2p^6\rho^{1+2p-2n}$ may be dominant in the case of small σ.

Finally, we consider the elementwise contributions in Lemma 5.5 and observe exponential convergence for two different configurations.

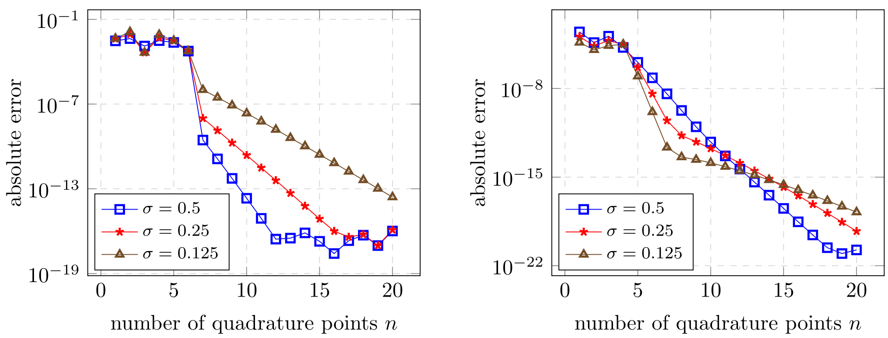

Fig. 4. Exponential convergence of the elementwise contributions $|I_{T,T'}(v,w) - Q^n_{T,T'}(v,w)|$ for the integrated Legendre polynomials (7.4) on geometric meshes $\mathcal{T}^L_{geo,\sigma}$ with $L = 2$ layers and different grading parameters σ. Left: adjacent elements. Right: separated elements.

Example 7.4. Fig. 4 considers the case of adjacent elements $T := (x_0^{geo}, x_1^{geo})$, $T' := (x_1^{geo}, x_2^{geo})$ (left) and separated elements $T := (x_0^{geo}, x_1^{geo}), T' := (x_2^{geo}, x_3^{geo})$ (right) in a geometric mesh $\mathcal{T}^L_{geo,\sigma}$ with $L = 2$ layers and different grading parameters σ (see Definition 2.2). We plot the absolute quadrature errors $|I_{T,T'}(v,w) - Q^n_{T,T'}(v,w)|$ for two integrated Legendre polynomials $v : T \to \mathbb{R}$ and $w : T' \to \mathbb{R}$ versus the number of quadrature points n. On the reference domain $(-1,1)$ they are defined as

$$v(x) = \int_{-1}^{x} P_5(t)dt \quad \text{and} \quad w(y) = \int_{-1}^{y} P_7(t)dt, \tag{7.4}$$

where $P_i(t) \in \mathcal{P}_i$ denotes the i-th Legendre polynomial. We used $Q^{50}_{T,T'}(v,w)$ with 50 quadrature points, as the reference solution $I_{T,T'}(v,w)$ and observe the predicted exponential convergence rate as well as that the rate decreases with σ. This is in line with Lemma 5.5 since $\rho \to 1$ as $\sigma \to 0$. We stress that Fig. 4 shows the absolute error; the final relative error is close to machine precision.

Acknowledgements

BB and JMM gladly acknowledge financial support by Austrian Science Fund (FWF) throughout the special research program *Taming complexity in PDE systems* (grant SFB F65, DOI:10.55776/F65).

Data availability

Data will be made available on request.

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