Modern Automata Theory

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Chapter 0

Preface

In this book we will give a survey on several topics in language and automata theory and will report on generalizations of some classical results on formal languages, formal tree languages, formal languages with finite and infinite words, automata, tree automata, etc. These generalizations are achieved by an algebraic treatment using semirings, formal power series, formal tree series, fixed point theory and matrices. By the use of these mathematical constructs, definitions, constructions, and proofs are obtained that are very satisfactory from a mathematical point of view. The use of these mathematical constructs yields the following advantages:

(i) The constructions needed in the proofs are mainly the usual ones.

(ii) The descriptions of the constructions by formal series and matrices do not need as much indexing as the usual descriptions.

(iii) The proofs are separated from the constructions and do not need the intuitive contents of the constructions. Often they are shorter than the usual proofs.

(iv) The results are more general than the usual ones. Depending on the semiring used, the results are valid for classical grammars and automata, classical grammars and automata with ambiguity considerations, probabilistic grammars or automata, etc.

The prize to pay for these advantages is a knowledge of the basics of semiring theory (see Kuich, Salomaa [88], Kuich [78]) and fixed point theory (see Bloom, Ésik [10]). The reader is assumed to have some basic knowledge of formal languages and automata (see Hopcroft, Ullman [65], Salomaa [106], Gluschkow, Zeitlin, Justschenko [55]).

Many results in the theory of automata and languages depend only on a few equational axioms. For example, Conway [25] has shown that Kleene's fundamental theorem equating the recognizable languages with the regular ones follows from a few simple equations defining Conway semirings. Such semirings are equipped with a star operation subject to the sum-star-equation and product-star-equation.

The use of equations has several advantages. Proofs can be separated into two parts, where the first part establishes the equational axioms, and the second is based on simple equational reasoning. Such proofs have a transparent structure and are usually very easy to understand, since manipulating equations is one of the most common ways of mathematical reasoning. Moreover, since many results depend on the same equations, the first part of such proofs usually provides a basis to several results. Finally, the results obtained by equational reasoning have a much broader scope, since many models share the same equations.

Essentially, this book is a compilation from Ésik, Kuich [40, 41, 43, 42, 45, 38], Sections 3, 4, 5, 7 of Kuich [78], Kuich [79] and Karner, Kuich [69].

The first chapter of this book deals with the basic results in the theory of finite automata. The presentation and the proofs of these results are based on Conway semirings. A Conway semiring is a starsemiring that satisfies the sum-star-equation and the product-star-equation. We introduce semirings, formal power series and matrices, define Conway semirings and state some of their important properties. Then we prove a Kleene Theorem for Conway semirings. Eventually, we discuss the computation of the star of a matrix with entries in a Conway semiring. The presentation of this chapter follows the lines of Ésik, Kuich [40, 41].

The second chapter deals with the basic results in the theory of algebraic systems. These are a generalization of context-free grammars. The presentation and the proofs of these results are based on continuous semirings. We introduce continuous semirings, state some of their important properties, and report on the basics of fixed point theory. Then we consider the components of the least solutions of algebraic systems. These are a generalization of the context-free languages. Eventually, we show equivalence results for the Chomsky normal form, the operator normal form and the Greibach normal form. Parts of the presentation of this chapter follow the lines of Sections 3 and 5 of Kuich [78].

The third chapter introduces pushdown automata and proves that pushdown automata and algebraic systems are mechanisms of the same power both characterizing the algebraic power series. Furthermore, a characterization of algebraic power series by means of algebraic expressions is given. The presentation of this chapter follows the lines of Section 4 of Kuich [78] and Kuich [79].

The fourth chapter deals with a generalized version of rational transducers. This leads to a generalization of the concept of a (full) abstract family of languages to the concept of an abstract family of elements. The presentation of this chapter follows the lines of Section 7 of Kuich [78] and Karner, Kuich [69].

The fifth chapter deals with the basic results in the theory of finite automata and ω-algebraic systems over quemirings generalizing the classical finite automata accepting and the classical context-free grammars generating languages over finite and infinite words. The presentation of these results is based on semiring-semimodule pairs, especially Conway semiring-semimodule pairs. A Conway semiring-semimodule pair is a pair consisting of a Conway semiring and a semimodule that satisfies the sum-omega-equation and the product-omega-equation. We define these Conway semiring-semimodule pairs and state some of their important properties. Then we introduce finite automata over quemirings and prove a Kleene Theorem. Furthermore, we introduce linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations, and connect certain solutions of these linear systems with the behavior of finite automata over quemirings. We then define ω-algebraic systems and characterize their solutions of order k by behaviors of algebraic finite automata. These solutions are then set in correspondence to ω-context-free languages. Eventually, we introduce rational and algebraic transducers, and abstract ω-families of power series over quemirings and prove that rational and algebraic power series of finite and infinite words constitute such abstract ω-families of power series. The presentation of this chapter follows the lines of Ésik, Kuich [43, 42, 45].

The sixth chapter deals with tree (series) automata and systems of equations over tree series. The main topics of this chapter are the following:

1. Tree automata (resp. finite, polynomial tree automata), whose behaviors are tree series over a semiring, and systems of equations (resp. finite, polynomial systems of equations), whose least solutions are tuples of tree series over a semiring, are equivalent.

2. A Kleene result: the class of recognizable tree series is characterized by rational tree series expressions.

3. Pushdown tree automata, whose behaviors are tree series over a semiring, and algebraic tree systems are equivalent; moreover, the class of algebraic tree series is characterized by algebraic tree series expressions (a Kleene result).

4. The class of recognizable tree series is closed under nondeterministic simple recognizable tree series transductions.

5. The families of recognizable tree series and of algebraic tree series are full abstract families of tree series (full AFTs).

6. The macro power series, a generalization of the indexed languages, and the algebraic power series are exactly the yields of algebraic tree series and of recognizable tree series, respectively; there is a Kleene result for macro power series; the yield of a full AFT is a full abstract family of power series.

The presentation of this chapter follows the lines of Ésik, Kuich [38].

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Chapter 1

Finite automata

1.1 Introduction

This chapter consists of this and three more sections. In Section 2 we introduce semirings, formal power series and matrices. Then we define Conway semirings and state their important properties: the matrix-star-equation is satisfied, and power series semirings and matrix semirings over Conway semirings are again Conway semirings. Very important classes of Conway semirings are complete semirings and k-closed semirings.

In Section 3 we define A'-finite automata, A' being a subset of a Conway semiring A, and prove a Kleene Theorem: the collection of all behaviors of A'-finite automata coincides with the substarsemiring of A generated by A'. Moreover, we construct for each finite automaton an equivalent finite automaton without ε-moves.

In Section 4 we discuss the computation of the star of a matrix. We state an algorithm valid for all matrices with entries in Conway semirings, compute its complexity and compare it to known algorithms valid for all matrices with entries in a complete semiring. We extend then a theorem of Mehlhorn [94] from complete semirings to Conway semirings, stating that the computation of the product of two matrices and the computation of the star of a matrix are of equal complexity. Eventually, we discuss the all-pairs shortest-distance problem, the all-pairs k-shortest distance problem and the all-pairs k-distinct-shortest distance problem for directed graphs.

The presentation of this chapter follows the lines of Ésik, Kuich [40, 41].

We now give a typical example which will be helpful for readers with some background in semiring theory and automata theory. Readers without this background should consult it when the used structures are defined in the following sections.

Example 1.1.1. Let $\mathcal{A} = (Q, \Sigma, \delta, 1, \{1\})$ be a finite automaton (a definition is given at the end of Section 3), where $Q = \{1, 2\}$, $\Sigma = \{a, b, c, d\}$, and $\delta(1, a) = \{1\}$, $\delta(1, b) = \{2\}$, $\delta(2, c) = \{1\}$, $\delta(2, d) = \{2\}$ are the only non-empty images of δ.

The graph of $\mathcal{A}$ is

and the adjacency matrix of this graph is

$$M = \begin{pmatrix} \{a\} & \{b\} \\ \{c\} & \{d\} \end{pmatrix} .$$

(Whenever we use matrices in this example, they are 2×2-matrices and their entries are formal languages over Σ.)

Consider the powers of M, e. g.,

$$M^2 = \begin{pmatrix} \{aa, bc\} & \{ab, bd\} \\ \{ca, dc\} & \{cb, dd\} \end{pmatrix}, M^3 = \begin{pmatrix} \{aaa, abc, bca, bdc\} & \{aab, abd, bcb, bdd\} \\ \{caa, cbc, dca, ddc\} & \{cab, cbd, dcb, ddd\} \end{pmatrix} .$$

It is easily proved by induction on k that $(M^k)_{ij}$ is the language of inscriptions of the paths of length k from state i to state j, $k \geq 0$, $1 \leq i, j \leq 2$. Define M^* by $(M^*)_{ij} = \bigcup_{k \geq 0} (M^k)_{ij}$, $1 \leq i, j \leq 2$. Then $(M^*)_{ij}$ is the language of inscriptions of all the paths from state i to state j, $1 \leq i, j \leq 2$.

We now construct regular expressions for the entries of M^*. Consider the inscriptions of paths from 1 to 1 not passing 1: they are a and $bd^n c$. Hence, the language of inscriptions of these paths is $\{a\} \cup \{b\}\{d\}^*\{c\}$. Consider now the language of inscriptions of paths from 1 to 1: it is $(\{a\} \cup \{b\}\{d\}^*\{c\})^*$. Hence, $(M^*)_{11} = (\{a\} \cup \{b\}\{d\}^*\{c\})^*$. We obtain $(M^*)_{12}$, if we concatenate $(M^*)_{11}$ with the language of inscriptions of all paths from 1 to 2 not passing through 1: $\{b\}\{d\}^*$. Hence, $(M^*)_{12} = (M^*)_{11}\{b\}\{d\}^*$. By symmetry, we obtain $(M^*)_{22} = (\{d\} \cup \{c\}\{a\}^*\{b\})^*$ and $(M^*)_{21} = (M^*)_{22}\{c\}\{a\}^*$. Hence,

$$M^* = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^* & (\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^* \\ (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^* & (\{d\} \cup \{c\}\{a\}^*\{b\})^* \end{pmatrix}$$

The language $||\mathcal{A}||$ accepted by the finite automaton $\mathcal{A}$ is the language of inscriptions of all paths from the initial state 1 to the final state 1, i. e., $||\mathcal{A}|| = (M^*)_{11} = (\{a\} \cup \{b\}\{d\}^*\{c\})^*$. $\square$

1.2 Semirings and formal power series

By a *semiring* we mean a set A together with two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that

(i) $\langle A, +, 0 \rangle$ is a commutative monoid,

(ii) $\langle A, \cdot, 1 \rangle$ is a monoid,

(iii) the distribution laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for every a, b, c,

(iv) $0 \cdot a = a \cdot 0 = 0$ for every a.

A semiring is called *commutative* if $a \cdot b = b \cdot a$ for every a and b. It is called *idempotent* if $1 + 1 = 1$, i. e., $a + a = a$ for every a.

If the operations and the constant elements of A are understood then we denote the semiring simply by A. Otherwise, we use the notation $\langle A, +, \cdot, 0, 1\rangle$. In the sequel, A will denote a semiring.

Intuitively, a semiring is a ring (with unity) without subtraction. A typical example is the semiring of nonnegative integers $\mathbb{N}$. A very important semiring in connection with language theory is the *Boolean* semiring $\mathbb{B} = \{0, 1\}$ where $1+1 = 1\cdot 1 = 1$. Clearly, all rings (with unity), as well as all fields, are semirings, e. g., integers $\mathbb{Z}$, rationals $\mathbb{Q}$, reals $\mathbb{R}$, complex numbers $\mathbb{C}$ etc.

Let $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$. Then $\langle \mathbb{N}^\infty, +, \cdot, 0, 1\rangle$ and $\langle \mathbb{N}^\infty, \min, +, \infty, 0\rangle$, where $+$, $\cdot$ and min are defined in the obvious fashion (observe that $0 \cdot \infty = \infty \cdot 0 = 0$), are semirings.

Let $\mathbb{R}_+ = \{a \in \mathbb{R} \mid a \geq 0\}$ and $\mathbb{R}_+^\infty = \mathbb{R}_+ \cup \{\infty\}$. Then $\langle \mathbb{R}_+, +, \cdot, 0, 1\rangle$, $\langle \mathbb{R}_+^\infty, +, \cdot, 0, 1\rangle$ and $\langle \mathbb{R}_+^\infty, \min, +, \infty, 0\rangle$ are semirings. The semirings $\langle \mathbb{N}_+^\infty, \min, +, \infty, 0\rangle$, $\langle \mathbb{R}_+^\infty, \min, +, \infty, 0\rangle$ are called *tropical semirings*.

Let Σ be an alphabet and define, for formal languages $L_1, L_2 \subseteq \Sigma^*$, the *product* of L_1 and L_2 by

$$L_1 \cdot L_2 = \{w_1 w_2 \mid w_1 \in L_1,\, w_2 \in L_2\}.$$

Then $\langle 2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\varepsilon\}\rangle$ is a semiring, called the *semiring of formal languages over* Σ. Here 2^S denotes the power set of a set S and $\emptyset$ denotes the empty set.

If S is a set, $2^{S\times S}$ is the set of binary relations over S. Define, for two relations R_1 and R_2, the product $R_1 \cdot R_2 \subseteq S \times S$ by

$$\begin{aligned} R_1 \cdot R_2 = \{(s_1, s_2) \mid{} & \text{there exists an } s \in S \text{ such that} \\ & (s_1, s) \in R_1 \text{ and } (s, s_2) \in R_2\} \end{aligned}$$

and, furthermore, define

$$\Delta = \{(s, s) \mid s \in S\}.$$

Then $\langle 2^{S\times S}, \cup, \cdot, \emptyset, \Delta\rangle$ is a semiring, called the *semiring of binary relations over* S.

A *starsemiring* is a semiring equipped with an additional unary operation *. The following semirings are starsemirings:

(i) The Boolean semiring $\langle \mathbb{B}, +, \cdot, ^*, 0, 1\rangle$ with $0^* = 1^* = 1$.

(ii) The semiring $\langle \mathbb{N}^\infty, +, \cdot, ^*, 0, 1\rangle$ with $0^* = 1$ and $a^* = \infty$ for $a \neq 0$.

(iii) The tropical semiring $\langle \mathbb{N}^\infty, \min, +, ^*, \infty, 0\rangle$ with $a^* = 0$ for all $a \in \mathbb{N}^\infty$.

(iv) The semiring $\langle \mathbb{R}_+^\infty, +, \cdot, ^*, 0, 1\rangle$ with $a^* = 1/(1 - a)$ for $0 \leq a < 1$ and $a^* = \infty$ for $a \geq 1$.

(v) The tropical semiring $\langle \mathbb{R}_+^\infty, \min, +, ^*, \infty, 0\rangle$ with $a^* = 0$ for all $a \in \mathbb{R}_+^\infty$.

(vi) The semiring $\langle 2^{\Sigma^*}, \cup, \cdot, {}^*, \emptyset, \{\varepsilon\}\rangle$ of formal languages over Σ with $L^* = \bigcup_{n\geq 0} L^n$ for all $L \subseteq \Sigma^*$.

(vii) The semiring $\langle 2^{S\times S}, \cup, \cdot, {}^*, \emptyset, \Delta\rangle$ of binary relations over S with $R^* = \bigcup_{n\geq 0} R^n$ for all $R \subseteq S \times S$. The relation R^* is called the *reflexive and transitive closure* of R, i. e., the smallest reflexive and transitive binary relation over S containing R.

(viii) The idempotent commutative semiring $\langle \{0, 1, a, \infty\}, +, \cdot, {}^*, 0, 1\rangle$, with $1 + a = a$, $1 + \infty = a + \infty = \infty$, $a \cdot a = a$, $a \cdot \infty = \infty \cdot \infty = \infty$, $0^* = 1^* = 1$, $a^* = \infty^* = \infty$.

The semirings (i)–(v) and (viii) are commutative, the semirings (i), (iii), (v), (vi), (vii), (viii) are idempotent.

A morphism of semirings is a mapping that preserves the semiring operations and constants. Let A and A' be semirings. Then a mapping $h : A \to A'$ is a *morphism* from A into A' if $h(0) = 0$, $h(1) = 1$, $h(a + b) = h(a) + h(b)$ and $h(a \cdot b) = h(a) \cdot h(b)$ for all $a, b \in A$. A morphism of starsemirings is a mapping that preserves additionally the star operation, i. e., $h(a^*) = h(a)^*$ for all $a \in A$.

We now define formal power series (see Kuich, Salomaa [88]). Let Σ be a (finite) alphabet. Mappings r from Σ^* into A are called *(formal) power series*. The values of r are denoted by (r, w), where $w \in \Sigma^*$, and r itself is written as a formal sum

$$r = \sum_{w\in\Sigma^*} (r, w)w.$$

The values (r, w) are also referred to as the *coefficients* of the series. The collection of all power series r as defined above is denoted by $A\langle\langle\Sigma^*\rangle\rangle$.

This terminology reflects the intuitive ideas connected with power series. We call the power series “formal” to indicate that we are not interested in summing up the series but rather, for instance, in various operations defined for series.

Given $r \in A\langle\langle\Sigma^*\rangle\rangle$, the subset of Σ^* defined by

$$\{w \mid (r, w) \neq 0\}$$

is termed the *support* of r and denoted by $\mathrm{supp}(r)$. The subset of $A\langle\langle\Sigma^*\rangle\rangle$ consisting of all series with a finite support is denoted by $A\langle\Sigma^*\rangle$. Series of $A\langle\Sigma^*\rangle$ are referred to as *polynomials*.

Examples of polynomials belonging to $A\langle\Sigma^*\rangle$ for every A are 0, w, aw, $a \in A$, $w \in \Sigma^*$, defined by:

$$\begin{array}{l}(0, w) = 0 \text{ for all } w,\\ (w, w) = 1 \text{ and } (w, w') = 0 \text{ for } w \neq w',\\ (aw, w) = a \text{ and } (aw, w') = 0 \text{ for } w \neq w'.\end{array}$$

Note that w equals $1w$.

We now introduce two operations inducing a semiring structure to power series. For $r_1, r_2 \in A\langle\langle\Sigma^*\rangle\rangle$, we define the *sum* $r_1 + r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ by $(r_1 + r_2, w) =$

$(r_1, w) + (r_2, w)$ for all $w \in \Sigma^*$. For $r_1, r_2 \in A\langle\langle\Sigma^*\rangle\rangle$, we define the *(Cauchy) product* $r_1 r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ by $(r_1 r_2, w) = \sum_{w_1 w_2 = w}(r_1, w_1)(r_2, w_2)$ for all $w \in \Sigma^*$. Clearly, $\langle A\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, \varepsilon\rangle$ and $\langle A\langle\Sigma^*\rangle, +, \cdot, 0, \varepsilon\rangle$ are semirings.

For $a \in A$, $r \in A\langle\langle\Sigma^*\rangle\rangle$, we define the *scalar products* $ar, ra \in A\langle\langle\Sigma^*\rangle\rangle$ by $(ar, w) = a(r, w)$ and $(ra, w) = (r, w)a$ for all $w \in \Sigma^*$. Observe that $ar = (a\varepsilon)r$ and $ra = r(a\varepsilon)$. If A is commutative then $ar = ra$.

A series $r \in A\langle\langle\Sigma^*\rangle\rangle$, where every coefficient equals 0 or 1, is termed the *characteristic series* of its support L, in symbols, $r = \mathrm{char}(L)$.

The *Hadamard product* of two power series r_1 and r_2 belonging to $A\langle\langle\Sigma^*\rangle\rangle$ is defined by

$$r_1 \odot r_2 = \sum_{w \in \Sigma^*}(r_1, w)(r_2, w)w.$$

It will be convenient to use the notations $A\langle\Sigma \cup \varepsilon\rangle$, $A\langle\Sigma\rangle$ and $A\langle\varepsilon\rangle$ for the collection of polynomials having their supports in $\Sigma \cup \{\varepsilon\}$, Σ and $\{\varepsilon\}$, respectively. In the sequel Σ will denote a finite alphabet.

Clearly, 2^{Σ^*} is a semiring isomorphic to $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$. Essentially, a transition from 2^{Σ^*} to $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ and vice versa means a transition from L to $\mathrm{char}(L)$ and from r to $\mathrm{supp}(r)$, respectively. The operation corresponding to the Hadamard product is the intersection of languages. If r_1 and r_2 are the characteristic series of the languages L_1 and L_2 then $r_1 \odot r_2$ is the characteristic series of $L_1 \cap L_2$.

Let $r_i \in A\langle\langle\Sigma^*\rangle\rangle$, $i \in I$, where I is an arbitrary index set. Define, for $w \in \Sigma^*$, $I_w = \{i \mid (r_i, w) \neq 0\}$. Assume now that for all $w \in \Sigma^*$, I_w is finite. Then we can define the sum $\sum_{i \in I} r_i$ by

$$(\sum_{i \in I} r_i, w) = \sum_{i \in I_w}(r_i, w)$$

for all $w \in \Sigma^*$.

The *Hurwitz product* (also called *shuffle product*) is defined as follows. For $w_1, w_2 \in \Sigma^*$ and $x_1, x_2 \in \Sigma$, we define $w_1 \sqcup\!\sqcup\, w_2 \in A\langle\langle\Sigma^*\rangle\rangle$ by

$$w_1 \sqcup\!\sqcup\, \varepsilon = w_1\,, \qquad \varepsilon \sqcup\!\sqcup\, w_2 = w_2\,,$$

and

$$w_1 x_1 \sqcup\!\sqcup\, w_2 x_2 = (w_1 x_1 \sqcup\!\sqcup\, w_2)x_2 + (w_1 \sqcup\!\sqcup\, w_2 x_2)x_1\,.$$

For $r_1, r_2 \in A\langle\langle\Sigma^*\rangle\rangle$, the Hurwitz product $r_1 \sqcup\!\sqcup\, r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ of r_1 and r_2 is then defined by

$$r_1 \sqcup\!\sqcup\, r_2 = \sum_{w_1, w_2 \in \Sigma^*}(r_1, w_1)(r_2, w_2)w_1 \sqcup\!\sqcup\, w_2\,.$$

This Hurwitz product is welldefined since

$$(r_1 \sqcup\!\sqcup\, r_2, w) = \sum_{|w_1|+|w_2|=|w|}(r_1, w_1)(r_2, w_2)(w_1 \sqcup\!\sqcup\, w_2, w)$$

is a finite sum for all $w \in \Sigma^*$.

In language theory, the shuffle product is customarily defined for languages L and L' by

$$L \sqcup\!\sqcup L' = \{w_1 w_1' \dots w_n w_n' \mid w_1 \dots w_n \in L,\ w_1' \dots w_n' \in L',\ n \geq 1\}\,.$$

If $r_1, r_2 \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ then this definition is "isomorphic" to that given above for formal power series.

Generalizations of formal power series over the free monoid to formal power series over graded monoids and skew formal power series can be found in Sakarovitch [105], Droste, Kuske [28] and Kuich [87], respectively.

In a starsemiring, we define, for $r \in A\langle\langle\Sigma^*\rangle\rangle$, the star $r^* \in A\langle\langle\Sigma^*\rangle\rangle$ of r inductively as follows:

$$(r^*, \varepsilon) = (r, \varepsilon)^*, \quad (r^*, w) = (r, \varepsilon)^* \sum_{uv=w,\ u\neq\varepsilon} (r, u)(r^*, v),\ w \in \Sigma^*,\ w \neq \varepsilon\,.$$

(See Theorem 3.5 of Kuich, Salomaa [88] and the forthcoming Theorem 2.27.) If $\langle A, +, \cdot, ^*, 0, 1\rangle$ is a starsemiring then the *star operation* in the starsemiring $\langle A\langle\langle\Sigma^*\rangle\rangle, +, \cdot, ^*, 0, \varepsilon\rangle$ will be always defined as above.

We now introduce matrices. Let $m, n \geq 1$. Mappings M from $\{1, \dots, m\} \times \{1, \dots, n\}$ into a semiring A are called *matrices*. The values of M are denoted by M_{ij}, where $1 \leq i \leq m$, $1 \leq j \leq n$. The values M_{ij} are also referred to as the *entries* of the matrix M. In particular, M_{ij} is called the (i,j)-*entry* of M. The collection of all matrices as defined above is denoted by $A^{m\times n}$. If $m = 1$ or $n = 1$ then M is called *row* or *column vector*, respecively.

We introduce some operations and special matrices inducing a monoid or semiring structure to matrices. For $M_1, M_2 \in A^{m\times n}$ we define the *sum* $M_1 + M_2 \in A^{m\times n}$ by $(M_1 + M_2)_{ij} = (M_1)_{ij} + (M_2)_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. Furthermore, we introduce the *zero matrix* $0 \in A^{m\times n}$. All entries of the zero matrix 0 are 0. By these definitions, $\langle A^{m\times n}, +, 0\rangle$ is a commutative monoid.

For $M_1 \in A^{m\times n}$ and $M_2 \in A^{n\times p}$ we define the *product* $M_1 M_2 \in A^{m\times p}$ by

$$(M_1 M_2)_{i_1 i_3} = \sum_{1\leq i_2\leq n} (M_1)_{i_1 i_2} (M_2)_{i_2 i_3} \qquad \text{for all } 1 \leq i_1 \leq m,\ 1 \leq i_3 \leq p.$$

Furthermore, we introduce the *matrix of unity* $E \in A^{n\times n}$. The diagonal entries E_{ii} of E are equal to 1, the off-diagonal entries $E_{i_1 i_2}$, $i_1 \neq i_2$, of E are equal to 0, $1 \leq i, i_1, i_2 \leq n$.

It is easily shown that matrix multiplication is associative, the distribution laws are valid for matrix addition and multiplication, E is a multiplicative unit and 0 is a multiplicative zero. So we infer that $\langle A^{n\times n}, +, \cdot, 0, E\rangle$ is a semiring for each $n \geq 1$.

Let A be a starsemiring. Then for $M \in A^{n\times n}$ we define $M^* \in A^{n\times n}$ inductively as follows:

(i) For $n = 1$ and $M = (a)$, $a \in A$, we define $M^* = (a^*)$.

(ii) For $n > 1$ we partition M into blocks $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $M^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $a, \alpha \in A^{1\times 1}$, $b, \beta \in A^{1\times(n-1)}$, $c, \gamma \in A^{(n-1)\times 1}$, $d, \delta \in A^{(n-1)\times(n-1)}$, by

$$\alpha = (a + bd^*c)^*, \quad \beta = \alpha bd^*, \quad \gamma = \delta ca^*, \quad \delta = (d + ca^*b)^* .$$

(See Theorem 3.3 of Conway [25], Theorem 4.21 of Kuich, Salomaa [88], Theorem 2.5 of Kuich [78], the forthcoming Theorem 2.26 and Example 1.1.) If $\langle A, +, \cdot, ^*, 0, 1\rangle$ is a starsemiring then the *star operation* in the starsemiring $\langle A^{n\times n}, +, \cdot, ^*, 0, E\rangle$ will always be defined as above.

Three equations are important for applications in automata theory:

(i) The *sum-star-equation* is valid in the starsemiring A if

$$(a + b)^* = (a^*b)^*a^*$$

for all $a, b \in A$. (See the forthcoming Theorem 2.24 and Example 2.2.1.)

(ii) The *product-star-equation* is valid in the starsemiring A if

$$(ab)^* = 1 + a(ba)^*b$$

for all $a, b \in A$. (See the forthcoming Theorem 2.24 and Example 2.2.2.)

(iii) Let M and M^* be given as in (ii) of the definition of M^* above, but with $a, \alpha \in A^{n_1\times n_1}$, $b, \beta \in A^{n_1\times n_2}$, $c, \gamma \in A^{n_2\times n_1}$, $d, \delta \in A^{n_2\times n_2}$, $n_1 + n_2 = n$. Then the *matrix-star-equation* is valid in the starsemiring A if M^* is independent of the partition of n in summands. (See the forthcoming Theorems 2.26 and 3.2.1 and Example 2.2.3.)

A *Conway semiring* is now a starsemiring that satisfies the sum-star-equation and the product-star-equation. All the starsemirings in (i)–(viii) are Conway semirings.

Conway [25] und Bloom, Ésik [9, 10] have shown that $A^{n\times n}$ and $A\langle\langle\Sigma^*\rangle\rangle$ are Conway semirings if A is a Conway semiring. Moreover, they have shown that the matrix-star-equation is valid for Conway semirings.

Since complete elementary proofs of these results are not well documented, we give complete elementary proofs that $A^{n\times n}$ and $A\langle\langle\Sigma^*\rangle\rangle$ are Conway semirings and that the matrix-star-equation is valid for Conway semirings.

In the proofs, we use the notation $a^+ = aa^* = a^*a$, $a \in A$.

We will prove in the sequel that the sum-star-equation and the product-star-equation hold in $A^{(n+1)\times(n+1)}$, $n \geq 0$, if A is a Conway semiring. This means that $A^{(n+1)\times(n+1)}$ is again a Conway semiring. The proofs are by induction on n. There are no problems with $n = 0$. So we assume, up to Theorem 2.13 that $n \geq 1$. Furthermore, we will prove that the matrix-star-equation is valid for Conway semirings.

Firstly, we prove that some particular cases of the sum-star-equation are satisfied in the matrix semiring $A^{(n+1)\times(n+1)}$.

Lemma 1.2.1 *Let A be a Conway semiring. Then, for $a, f \in A^{1\times 1}$, $g \in A^{1\times n}$, $h \in A^{n\times 1}$, $d, i \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* = \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* .$$

Proof. The left side and the right side of the equality are equal to

$$\begin{pmatrix} \alpha & (a+f)^* g\delta \\ (d+i)^* h\alpha & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha' & (a^* f)^* a^* g\delta' \\ (d^* i)^* d^* h\alpha' & \delta' \end{pmatrix},$$

respectively, where $\alpha = (a + f + g(d + i)^* h)^*$, $\delta = (d + i + h(a + f)^* g)^*$, $\alpha' = (a^* f + a^* g(d^* i)^* d^* h)^* a^*$, $\delta' = (d^* i + d^* h(a^* f)^* a^* g)^* d^*$.

We now obtain $\alpha = (a^* f + a^* g(d+i)^* h)^* a^* = (a^* f + a^* g(d^* i)^* d^* h)^* a^* = \alpha'$ and $(d+i)^* h\alpha = (d^* i)^* d^* h\alpha'$. The substitution $d \leftrightarrow a$, $i \leftrightarrow f$, $h \leftrightarrow g$ shows the symmetry of the proof for the remaining two entries of the matrices. $\square$

Lemma 1.2.2 *Let A be a Conway semiring. Then, for $f \in A^{1\times 1}$, $b, g \in A^{1\times n}$, $c, h \in A^{n\times 1}$, $i \in A^{n\times n}$, the following equalities are satisfied:*

$$\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* = \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* ,$$

$$\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* = \left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^* \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^* .$$

Proof. The left side and the right side of the first equality are equal to

$$\begin{pmatrix} \alpha & \alpha(g+b)i^* \\ \delta h f^* & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha' & \alpha' b + \alpha'(g+bi)i^* \\ \delta' h(f+bh)^* & \delta' h(f+bh)^* b + \delta' \end{pmatrix},$$

respectively, where $\alpha = (f + (g+b)i^* h)^*$, $\delta = (i + hf^*(g+b))^*$, $\alpha' = (f + bh + (g+bi)i^* h)^*$, $\delta' = (i + h(f+bh)^*(g+bi))^*$.

We now obtain $\alpha' = (f + bh + gi^* h + bii^* h)^* = (f + gi^* h + bi^* h)^* = \alpha$, $\alpha' b + \alpha'(g+bi)i^* = \alpha' b + \alpha' gi^* + \alpha' bii^* = \alpha' bi^* + \alpha' gi^* = \alpha(g+b)i^*$, $\delta' h(f+bh)^* = (i + h(f+bh)^*(g+bi))^* h(f+bh)^* = (i + h(f^* bh)^* f^*(g + bi))^* h(f^* bh)^* f^* = (i + (hf^* b)^* hf^*(g+bi))^*(hf^* b)^* hf^* = (i + (hf^* b)^* hf^* g + (hf^* b)^* hf^* bi)^*(hf^* b)^* hf^* = ((hf^* b)^* hf^* g + (hf^* b)^* i)^*(hf^* b)^* hf^* = (hf^* b + hf^* g + i)^* hf^* = \delta hf^*$, $\delta' h(f+bh)^* b + \delta' = ((hf^* b)^* hf^* g + (hf^* b)^* i)^*(hf^* b)^* hf^* b + (i + h(f^* bh)^* f^*(g+bi))^* = ((hf^* b)^* hf^* g + (hf^* b)^* i)^*(hf^* b)^* hf^* b + ((hf^* b)^* hf^* g + (hf^* b)^* i)^* + ((hf^* b)^* hf^* g + (hf^* b)^* i)^*(hf^* b)^* = (hf^* b + hf^* g + i)^* = \delta$.

The left side and the right side of the second equality are equal to

$$\begin{pmatrix} \alpha & \alpha g i^* \\ \delta(c+h)f^* & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha' + \alpha' g(cg+i)^* c & \alpha' g(cg+i)^* \\ \delta'(cf+h)f^* + \delta' & \delta' \end{pmatrix},$$

respectively, where $\alpha = (f + gi^*(c+h))^*$, $\delta = (i + (c+h)f^* g)^*$, $\alpha' = (f + g(cg + i)^*(cf+h))^*$, $\delta' = (cg + i + (cf+h)f^* g)^*$. The substitution $f \leftrightarrow i$, $h \leftrightarrow g$, $b \leftrightarrow c$ shows the symmetry to the first equality of the lemma. $\square$

Lemma 1.2.3 *Let A be a Conway semiring. Then, for $b \in A^{1\times n}$, $c \in A^{n\times 1}$ and $M \in A^{(n+1)\times(n+1)}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^* = \left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* M\right)^* \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* .$$

Proof.

$$\begin{aligned}
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^* = \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + M\right)^* = \\
&\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* M\right)^* \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* = \\
&\left(\begin{pmatrix} bc & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} = \\
&\left(\begin{pmatrix} bc & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} = \\
&\left(\begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^* \cdot \\
&\qquad \begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} = \\
&\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} = \\
&\left(\begin{pmatrix} 1 & 0 \\ c & E \end{pmatrix} \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} 1 & 0 \\ c & E \end{pmatrix} \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} = \\
&\left(\begin{pmatrix} (bc)^* & (bc)^*b \\ c(bc)^* & (bc)^* \end{pmatrix} M\right)^* \begin{pmatrix} (bc)^* & (bc)^*b \\ c(bc)^* & (bc)^* \end{pmatrix} = \\
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* M\right)^* \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* .
\end{aligned}$$

□

Theorem 1.2.4 *Let A be a Conway semiring. Then the sum-star-equation holds in the semiring $A^{(n+1)\times(n+1)}$.*

Proof. Let $a \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d \in A^{n\times n}$, $M \in A^{(n+1)\times(n+1)}$. Then we obtain

$$
\begin{aligned}
&\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + M\right)^* = \\
&\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* M\right)^* \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* = \\
&\left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix} + \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^* \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} = \\
&\left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^* \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^* \begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^* \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} = \\
&\left(\begin{pmatrix} (a^*bd^*c)^*a^* & a^*b(d^*ca^*b)^*d^* \\ d^*c(a^*bd^*c)^*a^* & (d^*ca^*b)^*d^* \end{pmatrix} M\right)^* \cdot \\
&\quad \begin{pmatrix} (a^*bd^*c)^*a^* & a^*b(d^*ca^*b)^*d^* \\ d^*c(a^*bd^*c)^*a^* & (d^*ca^*b)^*d^* \end{pmatrix} = \\
&\left(\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} M\right)^* \cdot \\
&\quad \begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} = \\
&\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* M\right)^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* .
\end{aligned}
$$

□

Secondly, we prove that some particular cases of a variant of the sum-star-equality are satisfied in the matrix semiring $A^{(n+1)\times(n+1)}$. The variant is $(a+b)^* = a^*(ba^*)^*$, $a, b \in A$.

Lemma 1.2.5 *Let A be a Conway semiring. Then, for $a, f \in A^{1\times 1}$, $g \in A^{1\times n}$, $h \in A^{n\times 1}$, $d, i \in A^{n\times n}$, the following equality is satisfied:*

$$
\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^*\right)^* .
$$

Proof. The left side and the right side of the equality are equal to

$$
\begin{pmatrix} \alpha & (a+f)^*g\delta \\ (d+i)^*h\alpha & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha' & (fa^*)^*gd^*\delta' \\ (id^*)^*ha^*\alpha' & \delta' \end{pmatrix},
$$

respectively, where $\alpha = (a + f + g(d + i)^*h)^*$, $\delta = (d + i + h(a + f)^*g)^*$, $\alpha' = a^*(fa^* + gd^*(id^*)^*ha^*)^*$, $\delta' = d^*(id^* + ha^*(fa^*)^*gd^*)^*$.

Since $\alpha' = (a^*f + a^*g(d^*i)^*d^*h)^*a^*$ and $\delta' = (d^*i + d^*h(a^*f)^*a^*g)^*d^*$, the rest of the proof is identical with the proof of Lemma 2.1. □

Lemma 1.2.6 *Let A be a Conway semiring. Then, for $f \in A^{1\times 1}$, $b, g \in A^{1\times n}$, $c, h \in A^{n\times 1}$, $i \in A^{n\times n}$, the following equalities are satisfied:*

$$
\begin{aligned}
\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* \left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^*\right)^* , \\
\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* &= \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^* \left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^*\right)^* .
\end{aligned}
$$

Proof. The left side and the right side of the first equality are equal to

$$\begin{pmatrix} \alpha & f^*(g+b)\delta \\ i^*h\alpha & \delta \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha' + b(hb+i)^*h\alpha' & f^*(fb+g)\delta' + b\delta' \\ (hb+i)^*h\alpha' & \delta' \end{pmatrix},$$

respectively, where $\alpha = (f + (g+b)i^*h)^*$, $\delta = (i + hf^*(g+b))^*$, $\alpha' = (f + (fb+g)(hb+i)^*h)^*$, $\delta' = (hb + i + hf^*(fb+g))^*$.

We now obtain $\alpha' + b(hb+i)^*h\alpha' = (1 + b(i^*hb)^*i^*h)\alpha' = (bi^*h)^*\alpha' = (bi^*h)^*(f^*(fb+g)i^*(hbi^*)^*h)^*f^* = (bi^*h)^*(f^*(fb+g)i^*h(bi^*h)^*)^*f^* = (bi^*h + f^*fbi^*h + f^*gi^*h)^*f^* = (f^*bi^*h + f^*gi^*h)^*f^* = (f + bi^*h + gi^*h)^* = \alpha$, $(hb+i)^*h\alpha' = i^*(hbi^*)^*h\alpha' = i^*h(bi^*h)^*\alpha' = i^*h\alpha$, $\delta' = (hb + i + hf^*(fb+g))^* = (i + hf^*b + hf^*g)^* = \delta$, $f^*(fb+g)\delta' + b\delta' = f^*b\delta' + f^*g\delta' = f^*(g+b)\delta$.

The left side and the right side of the second equality are equal to

$$\begin{pmatrix} \alpha & f^*g\delta \\ i^*(c+h)\alpha & \delta \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha' & (f+gc)^*g\delta' \\ c\alpha' + i^*(h+ic)\alpha' & c(f+gc)^*g\delta' + \delta' \end{pmatrix},$$

respectively, where $\alpha = (f + gi^*(c+h))^*$, $\delta = (i + (c+h)f^*g)^*$, $\alpha' = (f + gc + gi^*(h+ic))^*$, $\delta' = (i + (h+ic)(f+gc)^*g)^*$. The substitution $f \leftrightarrow i$, $h \leftrightarrow g$, $b \leftrightarrow c$ shows the symmetry to the first equality of the lemma. □

Lemma 1.2.7 *Let A be a Conway semiring. Then, for $b \in A^{1\times n}$, $c \in A^{n\times 1}$ and $M \in A^{(n+1)\times(n+1)}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^* = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* \left(M \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^*\right)^*.$$

Proof.

$$
\begin{aligned}
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^* = \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + M\right)^* = \\
&\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* \left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* + M\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^*\right)^* = \\
&\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ c & cb \end{pmatrix} + M\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\right)^* = \\
&\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & cb \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + M\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\right)^* = \\
&\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (cb)^* \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (cb)^* \end{pmatrix} + \right. \\
&\quad \left. M\begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (cb)^* \end{pmatrix}\right)^* = \\
&\begin{pmatrix} 1 & b(cb)^* \\ 0 & (cb)^* \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + M\begin{pmatrix} 1 & b(cb)^* \\ 0 & (cb)^* \end{pmatrix}\right)^* = \\
&\begin{pmatrix} 1 & b(cb)^* \\ 0 & (cb)^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\left(M\begin{pmatrix} 1 & b(cb)^* \\ 0 & (cb)^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right)^* = \\
&\begin{pmatrix} (bc)^* & b(cb)^* \\ (cb)^*c & (cb)^* \end{pmatrix}\left(M\begin{pmatrix} (bc)^* & b(cb)^* \\ (cb)^*c & (cb)^* \end{pmatrix}\right)^* = \\
&\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^*\left(M\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^*\right)^*.
\end{aligned}
$$

$\square$

Theorem 1.2.8 *Let A be a Conway semiring. Then, for $a \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d \in A^{n\times n}$, and $M \in A^{(n+1)\times(n+1)}$ the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + M\right)^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\left(M\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\right)^*.$$

Proof.

$$
\begin{array}{l}
\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + M\right)^* = \\
\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* + M\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^*\right)^* = \\
\begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix}\left(\begin{pmatrix} 0 & bd^* \\ ca^* & 0 \end{pmatrix} + M\begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix}\right)^* = \\
\begin{pmatrix} a^*(bd^*ca^*)^* & a^*bd^*(ca^*bd^*)^* \\ d^*ca^*(bd^*ca^*)^* & d^*(ca^*bd^*)^* \end{pmatrix} \cdot \\
\quad \left(M\begin{pmatrix} a^*(bd^*ca^*)^* & a^*bd^*(ca^*bd^*)^* \\ d^*ca^*(bd^*ca^*)^* & d^*(ca^*bd^*)^* \end{pmatrix}\right)^* = \\
\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} \cdot \\
\quad \left(M\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix}\right)^* = \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \left(M\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\right)^* .
\end{array}
$$

□

The next theorem splits the product-star-equation into two more simple equations. The proof is easy and is omitted.

Theorem 1.2.9 *Let A be a starsemiring. Then the equalities*

$$a^* = aa^* + 1 \quad \textit{and} \quad (ab)^*a = a(ba)^* ,$$

$a, b \in A$, are equivalent to the product-star-equation.

Theorem 1.2.10 *Let A be a Conway semiring. Then, for $M \in A^{(n+1)\times(n+1)}$,*

$$M^* = MM^* + E .$$

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d \in A^{n\times n}$. Then

$$
\begin{array}{l}
MM^* + E = \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} = \\
\begin{pmatrix} (a+bd^*c)(a+bd^*c)^* + 1 & (aa^*b+b)(d+ca^*b)^* \\ (c+dd^*c)(a+bd^*c) & (ca^*b+d)(d+ca^*b)^* + E \end{pmatrix} = \\
\begin{pmatrix} a+bd^*c & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} = M^* .
\end{array}
$$

□

Thirdly, we prove that some particular cases of the product-star-equation are satisfied in the matrix semiring $A^{(n+1)\times(n+1)}$.

Lemma 1.2.11 *Let A be a Conway semiring. Then, for $a \in A^{1\times 1}$, $b, g \in A^{1\times n}$, $c, h \in A^{n\times 1}$, $d \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} \right)^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^* .$$

Proof. The left side and the right side of the equality are equal to

$$\begin{pmatrix} \alpha a(gc)^* & \alpha b + \alpha a g(cg)^* d \\ \delta d h(bh)^* a + \delta c & \delta d(hb)^* \end{pmatrix}$$

and

$$\begin{pmatrix} (bh)^* a\alpha' & a(gc)^* g d\delta' + b\delta' \\ c\alpha' + d(hb)^* h a\alpha' & (cg)^* d\delta' \end{pmatrix},$$

respectively, where $\alpha = (bh + ag(cg)^*dh)^*$, $\delta = (cg + dh(bh)^*ag)^*$, $\alpha' = (gc + gd(hb)^*ha)^*$, $\delta' = (hb + ha(gc)^*gd)^*$.

We now obtain $\alpha a(gc)^* = ((bh)^*ag(cg)^*dh)^*(bh)^*a(gc)^* = (bh)^*a((gc)^*gdh(bh)^*a)^*(gc)^* = (bh)^*a\alpha'$; moreover, since $h\alpha = h((b+ag(cg)^*d)h)^* = (h(b+a(gc)^*gd))^*h = \delta' h$, we obtain $\alpha b + \alpha a g(cg)^*d = \alpha(b + ag(cg)^*d) = (1 + (bh + a(gc)^*gdh)\alpha)(b + a(gc)^*gd) = b + a(gc)^*gd + bh\alpha(b + a(gc)^*gd) + a(gc)^*gdh\alpha(b + a(gc)^*gd) = b + a(gc)^*gd + b\delta' h(b + a(gc)^*gd) + a(gc)^*gd\delta' h(b + a(gc)^*gd) = b(1 + \delta' h(b + a(gc)^*gd)) + a(gc)^*gd(1+\delta' h(b+a(gc)^*gd)) = b\delta' + a(gc)^*gd\delta'$. The substitution $b \leftrightarrow c$, $h \leftrightarrow g$, $a \leftrightarrow d$ yields $\alpha \leftrightarrow \delta$ and $\alpha' \leftrightarrow \delta'$. Hence, we obtain by analogous computations $\delta d(hb)^* = (cg)^*d\delta'$ and $\delta dh(bh)^*a + \delta c = c\alpha' + d(hb)^*ha\alpha'$. □

Lemma 1.2.12 *Let A be a Conway semiring. Then, for $a, f \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d, i \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} \right)^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^*$$

Proof. The left side and the right side of the equality are equal to

$$\begin{pmatrix} \alpha a + \alpha b i(di)^* c & \alpha b(id)^* \\ \delta c(fa)^* & \delta c(af)^* b + \delta d \end{pmatrix}$$

and

$$\begin{pmatrix} a\alpha' + b(id)^* i c\alpha' & (af)^* b\delta' \\ (di)^* c\alpha' & c(fa)^* f b\delta' + d\delta' \end{pmatrix},$$

respectively, where $\alpha = (af + bi(di)^*cf)^*$, $\delta = (di + cf(af)^*bi)^*$, $\alpha' = (fa + fb(id)^*ic)^*$, $\delta' = (id + ic(fa)^*fb)^*$.

Since $f\alpha = f((a + bi(di)^*c)f)^* = (f(a + b(id)^*ic))^*f = \alpha' f$, we obtain $\alpha a + \alpha bi(di)^*c = (1 + (af + bi(di)^*cf)\alpha)(a + bi(di)^*c) = a + b(id)^*ic + (a +$

$b(id)^*ic)\alpha' f(a + b(id)^*ic) = a(1 + \alpha' f(a + b(id)^*ic)) + b(id)^*ic(1 + \alpha' f(a + b(id)^*ic) = a\alpha' + b(id)^*ic\alpha'$; moreover, we obtain $\alpha b(id)^* = (af + bi(di)^*cf)^*b(id)^* = (af)^*(b(id)^*icf(af)^*)^*b(id)^* = (af)^*b((id)^*ic(fa)^*fb)^*(id)^* = (af)^*b(id + ic(fa)^*fb)^* = (af)^*b\delta'$. The substitution $a \leftrightarrow d$, $f \leftrightarrow i$, $b \leftrightarrow c$ yields $\alpha \leftrightarrow \delta$ and $\alpha' \leftrightarrow \delta'$. Hence, we obtain by analogous computations $\delta c(af)^*b + \delta d = c(fa)^*fb\delta' + d\delta'$ and $\delta c(fa)^* = (di)^*c\alpha'$. □

Theorem 1.2.13 *Let A be a Conway semiring, and let $M \in A^{(k+1)\times(m+1)}$ and $M' \in A^{(m+1)\times(k+1)}$, $k, m \geq 0$. Then the following equality is satisfied:*

$$(MM')^*M = M(M'M)^* .$$

Proof. We consider three cases: (i) $k = m = n$, (ii) $k = n > m$, (iii) $m = n > k$. In all three cases the proof is by induction on n.

(i) Denote $M' = \begin{pmatrix} f & g \\ h & i \end{pmatrix}$, where $f \in A^{1\times1}, g \in A^{1\times n}, h \in A^{n\times1}, i \in A^{n\times n}$. Then we obtain

$$\begin{aligned}
&\left(M\begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* M = \left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} + M\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^* M = \\
&\left(\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^* M\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^* \left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^* M = \\
&\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^* M\left(\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^* M\right)^* = \\
&M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^* \left(\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^*\right)^* = \\
&M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M + \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\right)^* = M\left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} M\right)^* .
\end{aligned}$$

(ii) Partition M and M' into

$$M = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \qquad \text{and} \qquad M' = \begin{pmatrix} f & g \\ 0 & 0 \end{pmatrix},$$

where $a, f \in A^{(m+1)\times(m+1)}$, $b \in A^{(n-m)\times(m+1)}$ and $g \in A^{(m+1)\times(n-m)}$. We obtain

$$\begin{aligned}
&(MM')^*M = \begin{pmatrix} af & ag \\ bf & bg \end{pmatrix}^* \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \\
&\begin{pmatrix} (af + ag(bg)^*bf)^*(a + ag(bg)^*b) & 0 \\ (bg + bf(af)^*ag)^*(bf(af)^*a + b) & 0 \end{pmatrix} = \\
&\begin{pmatrix} (a(gb)^*f)^*a(gb)^* & 0 \\ (b(fa)^*g)^*b(fa)^* & 0 \end{pmatrix} = \begin{pmatrix} a(gb + fa)^* & 0 \\ b(fa + gb)^* & 0 \end{pmatrix},
\end{aligned}$$

$$M(M'M)^* = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} fa + gb & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} a(fa + gb)^* & 0 \\ b(fa + gb)^* & 0 \end{pmatrix}.$$

(iii) Partition M and M' into

$$M = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M' = \begin{pmatrix} f & 0 \\ h & 0 \end{pmatrix},$$

where $a, f \in A^{(k+1)\times(k+1)}$, $c \in A^{(k+1)\times(n-k)}$ and $h \in A^{(n-k)\times(k+1)}$. We obtain

$$(MM')^* M = \begin{pmatrix} af + ch & 0 \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (af+ch)^* a & (af+ch)^* c \\ 0 & 0 \end{pmatrix},$$

$$M(M'M)^* = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} fa & fc \\ ha & hc \end{pmatrix}^* = \begin{pmatrix} (a + c(hc)^*ha)(fa + fc(hc)^*ha)^* & (a(fa)^*fc + c)(hc + ha(fa)^*fc)^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (ch)^*a(f(ch)^*a)^* & (af)^*c(h(af)^*c)^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (ch+af)^*a & (af+ch)^*c \\ 0 & 0 \end{pmatrix}.$$

$\square$

Corollary 1.2.14 *Let A be a Conway semiring. Then the product-star-equation holds in the semiring $A^{(n+1)\times(n+1)}$, $n \geq 0$.*

In fact, Theorems 2.10 and 2.13 prove that

$$(MM')^* = E + M(M'M)^* M'$$

holds for all rectangular matrices $M \in A^{(k+1)\times(m+1)}$ and $M' \in A^{(m+1)\times(k+1)}$.

Corollary 1.2.15 (Conway [25]) *If A is a Conway semiring then, for $n \geq 0$, $A^{(n+1)\times(n+1)}$ again is a Conway semiring.*

We prove now the matrix-star-equation.

Lemma 1.2.16 *Let A be a Conway semiring. Let, for $n_1, n_2, n_3 \geq 0$, $f \in A^{n_1\times n_1}$, $g \in A^{n_1\times n_2}$, $h \in A^{n_1\times n_3}$, $i \in A^{n_2\times n_1}$, $a \in A^{n_2\times n_2}$, $b \in A^{n_2\times n_3}$, $j \in A^{n_3\times n_1}$, $c \in A^{n_3\times n_2}$, $d \in A^{n_3\times n_3}$. Then $f + g(a+bd^*c)^*i + ga^*b(d+ca^*b)^*j + hd^*c(a+bd^*c)^*i + h(d+ca^*b)^*j = f + hd^*j + g(a+bd^*c)^*i + g(a+bd^*c)^*bd^*j + hd^*c(a+bd^*c)^*i + hd^*c(a+bd^*c)^*bd^*j$.*

Proof. We denote the i-th term on the left side by L_i, $1 \leq i \leq 5$, and the i-th term on the right side by R_i, $1 \leq i \leq 6$. Then we obtain $L_1 = R_1$, $L_2 = R_3$, $L_4 = R_5$, $L_3 = R_4$ by $ga^*b(d+ca^*b)^*j = ga^*b(d^*ca^*b)^*d^*j = g(a^*bd^*c)^*a^*bd^*j = g(a+bd^*c)^*bd^*j$ and $L_5 = R_2 + R_6$ by $h(d+ca^*b)^*j = h(d^*ca^*b)^*d^*j = hd^*j + h(d^*ca^*b)^*d^*ca^*bd^*j = hd^*j + hd^*c(a^*bd^*c)^*a^*bd^*j = hd^*j + hd^*c(a+bd^*c)^*bd^*j$. $\square$

Lemma 1.2.17 *Let A be a Conway semiring. Let $f, g, h, i, a, b, j, c, d$ be as in Lemma 2.16. Then $a + if^*g + b(d + jf^*h)^*c + if^*h(d + jf^*h)^*c + b(d + jf^*h)^*jf^*g + if^*h(d + jf^*h)^*jf^*g = a + bd^*c + i(f + hd^*j)^*g + bd^*j(f + hd^*j)^*g + i(f + hd^*j)^*hd^*c + bd^*j(f + hd^*j)^*hd^*c$.*

Proof. We denote the i-th term on the left (resp. right) side by L_i (resp. R_i), $1 \le i \le 6$. Then, similarly to the proof of Lemma 2.16, we obtain the following equalities: $L_1 = R_1$, $L_2 + L_6 = R_3$, $L_3 = R_2 + R_6$, $L_4 = R_5$, $L_5 = R_4$. □

Theorem 1.2.18 (Conway [25]) *Let A be a Conway semiring. Then the matrix-star-equation holds.*

Proof. The proof uses the proof idea of the proof of Theorem 2 of Conway [25] on page 110. It is by induction on the dimension of the matrix. For 2×2-matrices there is no problem. Let $M \in A^{n \times n}$, $n \ge 3$, and partition M into nine blocks

$$M = \begin{pmatrix} f & g & h \\ i & a & b \\ j & c & d \end{pmatrix}$$

with dimensions as in Lemma 2.16. The proof reduces then to showing that when we compute * of

$$M = \left(\begin{array}{c|cc} f & g & h \\ \hline i & a & b \\ j & c & d \end{array}\right) \quad \text{and} \quad M' = \left(\begin{array}{cc|c} f & g & h \\ i & a & b \\ \hline j & c & d \end{array}\right)$$

in the indicated ways we get the same result. Hence, we have to verify nine equalities in nine variables.

(i) First we compute M^*. We denote the blocks of M^* by $(M^*)_{ij}$, $1 \le i, j \le 3$. We obtain

$$(M^*)_{11} = \left(f + (g\ h)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} i \\ j \end{pmatrix}\right)^* =$$
$$\left(f + (g\ h)\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\right)^* =$$
$$(f + g(a+bd^*c)^*i + ga^*b(d+ca^*b)^*j + hd^*c(a+bd^*c)^*i + h(d+ca^*b)^*j)^*\,,$$

$$\begin{pmatrix} (M^*)_{21} \\ (M^*)_{31} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} i \\ j \end{pmatrix} (M^*)_{11} = \begin{pmatrix} (a+bd^*c)^*i + a^*b(d+ca^*b)^*j \\ d^*c(a+bd^*c)^*i + (d+ca^*b)^*j \end{pmatrix} (M^*)_{11}\,,$$

$$\begin{pmatrix} (M^*)_{22} & (M^*)_{23} \\ (M^*)_{32} & (M^*)_{33} \end{pmatrix} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix} f^*(g\ h)\right)^* =$$
$$\begin{pmatrix} a + if^*g & b + if^*h \\ c + jf^*g & d + jf^*h \end{pmatrix}^*.$$

Hence,

$$\begin{aligned}
(M^*)_{22} &= (a + if^*g + (b + if^*h)(d + jf^*h)^*(c + jf^*g))^*\,,\\
(M^*)_{23} &= (a + if^*g)^*(b + if^*h)(M^*)_{33}\,,\\
(M^*)_{32} &= (d + jf^*h)^*(c + jf^*g)(M^*)_{22}\,,\\
(M^*)_{33} &= (d + jf^*h + (c + jf^*g)(a + if^*g)^*(b + if^*h))^*\,.
\end{aligned}$$

Eventually, we obtain

$$\begin{array}{l}
((M^*)_{12}, (M^*)_{13}) = f^*(g\ h)\begin{pmatrix} (M^*)_{22} & (M^*)_{23} \\ (M^*)_{32} & (M^*)_{33} \end{pmatrix} = \\
(f^*g(M^*)_{22} + f^*h(M^*)_{32}, f^*g(M^*)_{23} + f^*h(M^*)_{33} = \\
((f^*g + f^*h(d + jf^*h)^*(c + jf^*g))(M^*)_{22}, \\
\qquad (f^*g(a + if^*g)^*(b + if^*h) + f^*h)(M^*)_{33}).
\end{array}$$

(ii) We now compute M'^*. We denote the blocks of M'^* by $(M'^*)_{ij}$, $1 \leq i, j \leq 3$. We obtain

$$\begin{array}{l}
\begin{pmatrix} (M'^*)_{11} & (M'^*)_{12} \\ (M'^*)_{21} & (M'^*)_{22} \end{pmatrix} = \left(\begin{pmatrix} f & g \\ i & a \end{pmatrix} + \begin{pmatrix} h \\ b \end{pmatrix} d^*(j\ c)\right)^* = \\
\begin{pmatrix} f + hd^*j & g + hd^*c \\ i + bd^*j & a + bd^*c \end{pmatrix}^*.
\end{array}$$

Hence,

$$\begin{aligned}
(M'^*)_{11} &= (f + hd^*j + (g + hd^*c)(a + bd^*c)^*(i + bd^*j))^*\,,\\
(M'^*)_{21} &= (a + bd^*c)^*(i + bd^*j)(M'^*)_{11}\,,\\
(M'^*)_{12} &= (f + hd^*j)^*(g + hd^*c)(M'^*)_{22}\,,\\
(M'^*)_{22} &= (a + bd^*c + (i + bd^*j)(f + hd^*j)^*(g + hd^*c))^*\,.
\end{aligned}$$

Furthermore, we obtain

$$\begin{array}{l}
((M'^*)_{31}, (M'^*)_{32}) = d^*(j\ c)\begin{pmatrix} (M'^*)_{11} & (M'^*)_{12} \\ (M'^*)_{21} & (M'^*)_{22} \end{pmatrix} = \\
((d^*j + d^*c(a + bd^*c)^*(i + bd^*j))(M'^*)_{11}, \\
\qquad (d^*j(f + hd^*j)^*(g + hd^*c) + d^*c)(M'^*)_{22}\,,
\end{array}$$

$$\begin{array}{l}
\begin{pmatrix} (M'^*)_{13} \\ (M'^*)_{23} \end{pmatrix} = \begin{pmatrix} (f + ga^*i)^* & f^*g(a + if^*g)^* \\ a^*i(f + ga^*i)^* & (a + if^*g)^* \end{pmatrix}\begin{pmatrix} h \\ b \end{pmatrix}(M'^*)_{33} = \\
\begin{pmatrix} ((f + ga^*i)^*h + f^*g(a + if^*g)^*b)(M'^*)_{33} \\ (a^*i(f + ga^*i)^*h + (a + if^*g)b)(M'^*)_{33} \end{pmatrix},
\end{array}$$

and eventually

$$\begin{array}{l}
(M'^*)_{33} = \left(d + (j\ c)\begin{pmatrix} f & g \\ i & a \end{pmatrix}^*\begin{pmatrix} h \\ b \end{pmatrix}\right)^* = \\
\left(d + (j\ c)\begin{pmatrix} (f + ga^*i)^* & f^*g(a + if^*g)^* \\ a^*i(f + ga^*i)^* & (a + if^*g)^* \end{pmatrix}\begin{pmatrix} h \\ b \end{pmatrix}\right)^* = \\
(d + j(f + ga^*i)^*h + jf^*g(a + if^*g)^*b + ca^*i(f + ga^*i)^*h + \\
\quad c(a + if^*g)^*b)^*\,.
\end{array}$$

(iii) We now show the equality of $(M^*)_{ij} = (M'^*)_{ij}$, $1 \le i, j \le 3$. We first compare the diagonal entries and obtain $(M^*)_{11} = (M'^*)_{11}$ by Lemma 2.16, $(M^*)_{22} = (M'^*)_{22}$ by Lemma 2.17, and $(M^*)_{33} = (M'^*)_{33}$ by Lemma 2.16 with the substitution $f \leftrightarrow d$, $h \leftrightarrow j$, $g \leftrightarrow c$, $i \leftrightarrow b$, $a \leftrightarrow a$.

Next, we obtain $(M^*)_{12} = (M'^*)_{12}$ by $f^*g + f^*h(d + jf^*h)^*(c + jf^*g) = f^*g + f^*h(d^*jf^*h)^*d^*c + f^*h(d^*jf^*h)^*d^*jf^*g = f^*g + (f^*hd^*j)^*f^*hd^*c + f^*hd^*j(f^*hd^*j)^*f^*g = (f + hd^*j)^*(g + hd^*c)$,

$(M^*)_{13} = (M'^*)_{13}$ by $f^*g(a + if^*g)^*(b + if^*h) + f^*h = f^*g(a + if^*g)^*b + f^*g(a^*if^*g)^*a^*if^*h + f^*h = f^*g(a + if^*g)^*b + f^*ga^*i(f^*ga^*i)^*f^*h + f^*h = f^*g(a + if^*g)^*b + (f + ga^*i)^*h$,

$(M^*)_{21} = (M'^*)_{21}$ by $(a+bd^*c)^*i+a^*b(d+ca^*b)^*j = (a+bd^*c)^*i+a^*b(d^*ca^*b)^*d^*j = (a + bd^*c)^*i + (a^*bd^*c)^*a^*bd^*j = (a + bd^*c)^*i + (a + bd^*c)^*bd^*j$,

$(M^*)_{31} = (M'^*)_{31}$ by $d^*c(a + bd^*c)^*i + (d + ca^*b)^*j = d^*c(a + bd^*c)^*i + (d^*ca^*b)^*d^*j = d^*c(a + bd^*c)^*i + d^*j + (d^*ca^*b)^*d^*ca^*bd^*j = d^*c(a + bd^*c)^*i + d^*j + d^*c(a^*bd^*c)^*a^*bd^*j = d^*c(a + bd^*c)^*i + d^*j + d^*c(a + bd^*c)^*bd^*j$,

$(M^*)_{23} = (M'^*)_{23}$ by $(a + if^*g)^*(b + if^*h) = (a + if^*g)^*b + (a^*if^*g)^*a^*if^*h = (a + if^*g)^*b + a^*i(f^*ga^*i)^*f^*h = (a + if^*g)^*b + a^*i(f + ga^*i)^*h$,

and eventually $(M^*)_{32} = (M'^*)_{32}$ by $(d + jf^*h)^*(c + jf^*g) = (d^*jf^*h)^*d^*c + (d^*jf^*h)^*d^*jf^*g = d^*c + (d^*jf^*h)^*d^*jf^*hd^*c + d^*j(f^*hd^*j)^*f^*g = d^*c + d^*j(f + hd^*j)^*hd^*c + d^*j(f + hd^*j)^*g$. □

We now prove that $A\langle\langle\Sigma^*\rangle\rangle$ is a Conway semiring if A is a Conway semiring and Σ is an alphabet. The proofs are by induction on the length of words.

Theorem 1.2.19 *Let A be a Conway semiring an Σ be an alphabet. Then the sum-star-equation holds in $A\langle\langle\Sigma^*\rangle\rangle$.*

Proof. Let $r, s \in A\langle\langle\Sigma^*\rangle\rangle$. Then we proof by induction on the length of $w \in \Sigma^*$ that $((r + s)^*, w) = ((r^*s)^*r^*, w)$. The case $w = \varepsilon$ is clear. Assume now $w \ne \varepsilon$. Then we obtain $((r + s)^*, w) = ((r + s)^*, \varepsilon) \sum_{uv=w,\ u\ne\varepsilon}(r + s, u)((r + s)^*, v) = ((r + s)^*, \varepsilon) \sum_{uv=w,\ u\ne\varepsilon}(r, u)((r + s)^*, v) + ((r + s)^*, \varepsilon) \sum_{uv=w,\ u\ne\varepsilon}(s, u)((r + s)^*, v)$. We call the first and second of these terms L_1 and L_2, respectively. Moreover, we obtain

$$
\begin{aligned}
&((r^*s)^*r^*, w) = \textstyle\sum_{w_1w_2=w}((r^*s)^*, w_1)(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + \textstyle\sum_{w_1w_2=w,\ w_1\ne\varepsilon}((r^*s)^*, w_1)(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + \\
&\quad ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\ u_1\ne\varepsilon}(r^*s, u_1)((r^*s)^*, v_1)(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + ((r^*s)^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\, u_1\ne\varepsilon} \sum_{w_3w_4=u_1}(r^*, w_3)(s, w_4)((r^*s)^*, v_1)(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + \\
&\quad ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\ u_1\ne\varepsilon}(r^*, \varepsilon)(s, u_1)((r^*s)^*, v_1)(r^*, w_2) + \\
&\quad ((r^*s)^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1} \sum_{w_3w_4=u_1,\, w_3\ne\varepsilon}(r^*, w_3)(s, w_4)((r^*s)^*, v_1)(r^*, w_2).
\end{aligned}
$$

We call the first, second and third of these terms R_1, R_2 and R_3, respectively.

Eventually, we obtain

$$\begin{aligned}
&R_2 = ((r+s)^*,\varepsilon) \textstyle\sum_{u_1 z = w,\ u_1 \neq \varepsilon} (s,u_1)((r^*s)^*r^*, z) = \\
&((r+s)^*,\varepsilon) \textstyle\sum_{u_1 z = w,\ u_1 \neq \varepsilon} (s,u_1)((r+s)^*, z) = L_2
\end{aligned}$$

and

$$\begin{aligned}
&R_1 + R_3 = ((r^*s)^*,\varepsilon)(r^*,\varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r,u)(r^*,v) + \\
&\quad ((r^*s)^*,\varepsilon) \textstyle\sum_{w_1 w_2 = w} \sum_{u_1 v_1 = w_1} \sum_{w_3 w_4 = u_1} (r^*,\varepsilon) \cdot \\
&\quad \textstyle\sum_{u_2 v_2 = w_3,\ u_2 \neq \varepsilon} (r,u_2)(r^*,v_2)(s,w_4)((r^*s)^*,v_1)(r^*,w_2) = \\
&((r+s)^*,\varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r,u)(r^*,v) + \\
&\quad ((r+s)^*,\varepsilon) \textstyle\sum_{u_2 z = w,\ u_2\neq\varepsilon} (r,u_2)((r^*s)^+r^*, z) = \\
&((r+s)^*,\varepsilon) \textstyle\sum_{u_2 z = w,\ u_2\neq\varepsilon} (r,u_2)((r^*s)^*r^*, z) = \\
&((r+s)^*,\varepsilon) \textstyle\sum_{u_2 z = w,\ u_2\neq\varepsilon} (r,u_2)((r+s)^*, z) = L_1 \,.
\end{aligned}$$

Hence, $L_1 + L_2 = R_1 + R_2 + R_3$ and the sum-star-equality holds in $A\langle\langle\Sigma^*\rangle\rangle$. $\square$

Theorem 1.2.20 *Let A be a Conway semiring and Σ be an alphabet. Then, for $r \in A\langle\langle\Sigma^*\rangle\rangle$, the following equation is satisfied:*

$$r^* = \varepsilon + rr^* \,.$$

Proof. We prove by induction on the length of $w \in \Sigma^*$ that $(r^*, w) = (\varepsilon + rr^*, w)$. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain

$$\begin{aligned}
&(\varepsilon + rr^*, w) = \textstyle\sum_{w_1 w_2 = w} (r,w_1)(r^*,w_2) = \\
&(r,\varepsilon)(r^*,w) + \textstyle\sum_{w_1 w_2 = w,\ w_1\neq\varepsilon} (r,w_1)(r^*,w_2) = \\
&(r,\varepsilon)(r^*,\varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r,u)(r^*v) + \sum_{w_1 w_2 = w,\ w_1\neq\varepsilon} (r,w_1)(r^*,w_2) = \\
&(r^+,\varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r,u)(r^*v) + \sum_{w_1 w_2 = w,\ w_1\neq\varepsilon} (r,w_1)(r^*,w_2) = \\
&(r^*,\varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r,u)(r^*v) = (r^*,w) \,.
\end{aligned}$$

$\square$

Theorem 1.2.21 *Let A be a Conway semiring and Σ be an alphabet. Then, for $r, s \in A\langle\langle\Sigma^*\rangle\rangle$, the following equation is satisfied:*

$$r(sr)^* = (rs)^*r \,.$$

Proof. We prove by induction on the length of $w \in \Sigma^*$ that $(r(sr)^*, w) = ((rs)^*r, w)$. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain

$$
\begin{aligned}
&(r(sr)^*, w) = \textstyle\sum_{w_1w_2=w}(r, w_1)((sr)^*, w_2) = \\
&(r, \varepsilon)((sr)^*, w) + \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}(r, w_1)((sr)^*, w_2) = \\
&(r, \varepsilon)((sr)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(sr, u)((sr)^*, v) + \\
&\quad \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}(r, w_1)((sr)^*, w_2) = \\
&(r(sr)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}\sum_{w_3w_4=u}(s, w_3)(r, w_4)((sr)^*, v) + \\
&\quad \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}(r, w_1)((sr)^*, w_2) = \\
&(r(sr)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(s, \varepsilon)(r, u)((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{w_3w_4=u,\ w_3\neq\varepsilon}(s, w_3)(r, w_4)((sr)^*, v) + \\
&\quad \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}(r, w_1)((sr)^*, w_2) = \\
&((rs)^+, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(r, u)((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{w_3w_4=u,\ w_3\neq\varepsilon}(s, w_3)(r, w_4)((sr)^*, v) + \\
&\quad \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}(r, w_1)((sr)^*, w_2) = \\
&((rs)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(r, u)((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon)\textstyle\sum_{w_3z=w,\ w_3\neq\varepsilon}(r(sr)^*, z)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&((rs)^*r, w) = \textstyle\sum_{w_1w_2=w}((rs)^*, w_1)(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon}((rs)^*, w_1)(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \\
&\quad \textstyle\sum_{w_1w_2=w}((rs)^*, \varepsilon)\sum_{uv=w_1,\ u\neq\varepsilon}(rs, u)((rs)^*, v)(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \textstyle\sum_{w_1w_2=w}((rs)^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{uv=w_1,\ u\neq\varepsilon}\sum_{w_3w_4=u}(r, w_3)(s, w_4)((rs)^*, v)(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \textstyle\sum_{w_1w_2=w}((rs)^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{uv=w_1,\ u\neq\varepsilon}(r, \varepsilon)(s, u)((rs)^*, v)(r, w_2) + \sum_{w_1w_2=w}((rs)^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{uv=w_1}\sum_{w_3w_4=u,\ w_3\neq\varepsilon}(r, w_3)(s, w_4)((rs)^*, v)(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + ((rs)^*r, \varepsilon)\textstyle\sum_{uz=w,\ u\neq\varepsilon}(s, u)((rs)^*r, z) + \\
&\quad ((rs)^*, \varepsilon)\textstyle\sum_{w_3z=w,\ w_3\neq\varepsilon}(r, w_3)((sr)^+, z) = \\
&((rs)^*, \varepsilon)(r, w) + ((rs)^*r, \varepsilon)\textstyle\sum_{uz=w,\ u\neq\varepsilon}(s, u)((rs)^*r, z) + \\
&\quad ((rs)^*, \varepsilon)\textstyle\sum_{w_3z=w,\ w_3\neq\varepsilon,\ z\neq\varepsilon}(r, w_3)((sr)^*, z) + \\
&\quad ((rs)^*, \varepsilon)(r, w)((sr)^+, \varepsilon) = \\
&((rs)^*r, \varepsilon)\textstyle\sum_{uz=w,\ u\neq\varepsilon}(s, u)((rs)^*r, z) + \\
&\quad ((rs)^*, \varepsilon)\textstyle\sum_{w_3z=w,\ w_3\neq\varepsilon}(r, w_3)((sr)^*, z)\,.
\end{aligned}
$$

Hence, $(r(sr)^*, w) = ((rs)^*r, w)$. □

Corollary 1.2.22 (Bloom, Ésik [9]) *If A is a Conway semiring and Σ is an alphabet then $A\langle\langle\Sigma^*\rangle\rangle$ is again a Conway semiring.*

Corollary 1.2.23 *Let A be a Conway semiring, Σ be an alphabet and $n \geq 1$. Then $(A\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ is again a Conway semiring.*

A semiring A is called *complete* if it is possible to define sums for all families $(a_i \mid i \in I)$ of elements of A, where I is an arbitrary index set, such that the following conditions are satisfied (see Conway [25], Eilenberg [29], Kuich [78]):

(i) $\sum_{i\in\emptyset} a_i = 0, \qquad \sum_{i\in\{j\}} a_i = a_j, \qquad \sum_{i\in\{j,k\}} a_i = a_j + a_k$ for $j \neq k$,

(ii) $\sum_{j\in J}\Big(\sum_{i\in I_j} a_i\Big) = \sum_{i\in I} a_i$, if $\bigcup_{j\in J} I_j = I$ and $I_j \cap I_{j'} = \emptyset$ for $j \neq j'$,

(iii) $\sum_{i\in I}(c\cdot a_i) = c\cdot\Big(\sum_{i\in I} a_i\Big), \qquad \sum_{i\in I}(a_i\cdot c) = \Big(\sum_{i\in I} a_i\Big)\cdot c.$

This means that a semiring A is complete if it is possible to define "infinite sums" (i) that are an extension of the finite sums, (ii) that are associative and commutative and (iii) that satisfy the distribution laws.

In complete semirings for each element a, the *star* a^* of a is defined by

$$a^* = \sum_{j\geq 0} a^j .$$

Hence, each complete semiring is a starsemiring called a *complete starsemiring.* The semirings (i)–(vii) are complete starsemirings. The semiring (viii) is a complete semiring, but not a complete starsemiring.

If $\langle A, +, \cdot, 0, 1\rangle$ is a complete semiring, then so are $\langle A\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, \varepsilon\rangle$ and $\langle A^{n\times n}, +, \cdot, 0, E\rangle$ by the following definitions:
If $r_i \in A\langle\langle\Sigma^*\rangle\rangle$ for $i \in I$, then $\sum_{i\in I} r_i = \sum_{w\in\Sigma^*}\Big(\sum_{i\in I}(r_i, w)\Big)w$;
if $M_i \in A^{n\times n}$ for $i \in I$, then $\Big(\sum_{i\in I} M_i\Big)_{kj} = \sum_{i\in I}(M_i)_{kj}$ for $1 \leq k, j \leq n$.
Here I is an arbitrary index set. Moreover, each complete starsemiring is a Conway semiring (see Conway [25], Bloom, Ésik [10], Kuich [76], Hebisch [64]) and the star operation in the complete semirings $A\langle\langle\Sigma^*\rangle\rangle$ and $A^{n\times n}$ is the same as the star operation in the Conway semirings $A\langle\langle\Sigma^*\rangle\rangle$ and $A^{n\times n}$, respectively. We prove the first statement.

Theorem 1.2.24 *Each complete starsemiring is a Conway semiring.*

Proof. Let A be a complete starsemiring and let $a, b \in A$. Let $\bar{a}, \bar{b}$ be letters. Note that to each word $\bar{w} = \bar{c_1}\bar{c_2}\ldots\bar{c_n}$, $\bar{c_i} \in \{\bar{a}, \bar{b}\}$, $1 \leq i \leq n$, there corresponds the element $w = c_1c_2\ldots c_n \in A$. Let A' be the complete starsemiring generated by 1 and consider the complete starsemiring $A'\langle\langle\{\bar{a}, \bar{b}\}^*\rangle\rangle$. Since A' is commutative, $\bar{a} \mapsto a$, $\bar{b} \mapsto b$ induces a starsemiring morphism from the complete starsemiring $A'\langle\langle\{\bar{a}, \bar{b}\}^*\rangle\rangle \to A$.

By straightforward proofs by induction the following equalities can be proved for $a, b \in A$, $n, m \geq 0$:

$$(\bar{a} + \bar{b})^n = \sum_{0\leq j\leq n} \bar{a}^j \sqcup\!\sqcup\, \bar{b}^{n-j} ,$$

$$\bar{a}^n \sqcup\!\sqcup\, \bar{b}^m = \sum_{0\leq j\leq n} (\bar{a}^j \sqcup\!\sqcup\, \bar{b}^{m-1})\bar{b}\bar{a}^{n-j}$$

and

$$\bar{a}^* \sqcup\!\sqcup\, \bar{b}^n = \sum_{j\geq 0} \bar{a}^j \sqcup\!\sqcup\, \bar{b}^n = (\bar{a}^*\bar{b})^n \bar{a}^* .$$

Hence, we infer the equality

$$(\bar{a}+\bar{b})^* = \sum_{n\geq 0}\sum_{j\geq 0} \bar{a}^j \sqcup\!\sqcup\, \bar{b}^n = \bar{a}^* \sqcup\!\sqcup\, \bar{b}^* ,$$

which implies immediately

$$(\bar{a}+\bar{b})^* = (\bar{a}^*\bar{b})^*\bar{a}^* .$$

Moreover, we obtain

$$\begin{aligned}(\bar{a}\bar{b})^* &= 1+\textstyle\sum_{n\geq 0}(\bar{a}\bar{b})^n = 1+\bar{a}\left(\textstyle\sum_{n\geq 0}(\bar{b}\bar{a})^n\right)\bar{b} = \\ & \quad 1+\bar{a}(\bar{b}\bar{a})^*\bar{b} .\end{aligned}$$

Applying the starsemiring morphism defined above we obtain the sum-star-equation and the product-star-equation in A:

$$(a+b)^* = (a^*b)^*a^* \qquad \text{and} \qquad (ab)^* = 1+a(ba)^*b .$$

□

We now prove the first part of the second statement before Theorem 2.24 after the next lemma.

Lemma 1.2.25 *Let A be a complete starsemiring. Then, for all $f,g \in A$,*

$$(f+g)^* = (f+gf^*g)^*(1+gf^*) .$$

Proof. $(f+gf^*g)^*(1+gf^*) = (f^*gf^*g)^*f^*(1+gf^*) = \sum_{n\geq 0}(f^*g)^{2n}f^* + \sum_{n\geq 0}(f^*g)^{2n+1}f^* = (f^*g)^*f^* = (f+g)^*$. □

Theorem 1.2.26 *If A is a complete starsemiring so is $A^{n\times n}$, $n \geq 1$.*

Proof. The case $n=1$ is clear. Let now $n \geq 2$ and partition $M \in A^{n\times n}$ into

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where, for some $n_1, n_2 \geq 1$ with $n_1+n_2 = n$, a is an $n_1 \times n_1$-matrix and d is an $n_2 \times n_2$-matrix. Consider now the matrices

$$f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} .$$

The computation of $(f+gf^*g)^*(E+gf^*)$ and application of Lemma 2.25 prove our theorem. □

Observe that the proof of Theorem 2.26 shows again that the matrix-star-equation is valid in the complete starsemiring $A^{n\times n}$.

We now prove the second part of the second statement before Theorem 2.24 after some definitions needed in the proof.

For $w \in \Sigma^*$, $w \neq \varepsilon$, we define

$$\mathfrak{p}(w) = \{(u, v, \ldots, z) \mid uv \ldots z = w,\ u, w, \ldots, z \neq \varepsilon\}\,.$$

Observe that, for $w \in \Sigma^*$, $w \neq \varepsilon$, and $x \in \Sigma$, we obtain

$$\begin{aligned}\mathfrak{p}(xw) = \{&(x, u, v, \ldots, z) \mid (u, v, \ldots, z) \in \mathfrak{p}(w)\} \cup \\ \{&(xu, v, \ldots, z) \mid (u, v, \ldots, z) \in \mathfrak{p}(w)\}\,.\end{aligned}$$

Theorem 1.2.27 *If A is a complete starsemiring and Σ is an alphabet then $A\langle\langle\Sigma^*\rangle\rangle$ is again a complete starsemiring.*

Proof. We first prove by induction that, for $r \in A\langle\langle\Sigma^*\rangle\rangle$ and $w \in \Sigma^*$, $w \neq \varepsilon$,

$$(r^*, w) = \sum_{(u,v,\ldots,z)\in\mathfrak{p}(w)} (r, \varepsilon)^*(r, u)(r, \varepsilon)^*(r, v) \ldots (r, \varepsilon)^*(r, z)(r, \varepsilon)^*\,.$$

The case $w \in \Sigma$ is clear by the definition of r^*. Let now $x \in \Sigma$ and $w \in \Sigma^*$, $w \neq \varepsilon$. Then

$$\begin{aligned}&(r^*, xw) = (r, \varepsilon)^*(r, x)(r, w)^* + (r, \varepsilon)^* \textstyle\sum_{pq=w,\ p\neq\varepsilon}(r, xp)(r^*, q) = \\ &(r, \varepsilon)^*(r, x) \textstyle\sum_{(u,v,\ldots,z)\in\mathfrak{p}(w)}(r, \varepsilon)^*(r, u)(r, \varepsilon)^*(r, v) \ldots (r, \varepsilon)^*(r, z)(r, \varepsilon)^* + \\ &(r, \varepsilon)^* \textstyle\sum_{(p,v,\ldots,z)\in\mathfrak{p}(w)}(r, xp)(r, \varepsilon)^*(r, v)(r, \varepsilon)^* \ldots (r, \varepsilon)^*(r, z)(r, \varepsilon)^* = \\ &\textstyle\sum_{(u,v,\ldots,z)\in\mathfrak{p}(xw)}(r, \varepsilon)^*(r, u)(r, \varepsilon)^*(r, v) \ldots (r, \varepsilon)^*(r, z)(r, \varepsilon)^*\,.\end{aligned}$$

We now show that, for $r \in A\langle\langle\Sigma^*\rangle\rangle$ and $w \in \Sigma^*$,

$$(r^*, w) = \sum_{n\geq 0}(r^n, w)\,.$$

By definition,

$$(r^*, \varepsilon) = (r, \varepsilon)^* = \sum_{n\geq 0}(r, \varepsilon)^n = (\sum_{n\geq 0} r^n, \varepsilon)\,.$$

Let now $w \neq \varepsilon$. Then we obtain

$$\begin{aligned}&(\textstyle\sum_{n\geq 0} r^n, w) = \textstyle\sum_{n\geq 0}(r^n, w) = \textstyle\sum_{n\geq |w|} \textstyle\sum_{u_1\ldots u_n=w}(r, u_1) \ldots (r, u_n) = \\ &\textstyle\sum_{n\geq |w|} \textstyle\sum_{(u,\ldots,z)\in\mathfrak{p}(w)} \textstyle\sum_{i_u+\ldots+i_z+i=n-|w|}(r, \varepsilon)^{i_u}(r, u) \ldots (r, \varepsilon)^{i_z}(r, z)(r, \varepsilon)^i = \\ &\textstyle\sum_{(u,\ldots,z)\in\mathfrak{p}(w)}(\textstyle\sum_{i_u\geq 0}(r, \varepsilon)^{i_u})(r, u) \ldots (\textstyle\sum_{i_z\geq 0}(r, \varepsilon)^{i_z})(r, z) \textstyle\sum_{i\geq 0}(r, \varepsilon)^i = \\ &\textstyle\sum_{(u,\ldots,z)\in\mathfrak{p}(w)}(r, \varepsilon)^*(r, u) \ldots (r, \varepsilon)^*(r, z)(r, \varepsilon)^* = (r^*, w)\,.\end{aligned}$$

$\square$

A semiring A is *k-closed*, $k \geq 0$, if for each $a \in A$,

$$1 + a + \ldots + a^k = 1 + a + \ldots + a^k + a^{k+1}\,.$$

(See Carré [19], Mohri [95], Ésik, Kuich [37].) If $\langle A, +, \cdot, 0, 1\rangle$ is a k-closed semiring, then define the star of $a \in A$ by

$$a^* = 1 + a + \ldots + a^k .$$

By Ésik, Kuich [37] the starsemiring $\langle A, +, \cdot, 0, 1\rangle$ is then a Conway semiring if $k = 0, 1$ or if A is commutative.

1.3 Kleene's Theorem for Conway semirings

In the sequel, A denotes a Conway semiring and A' denotes a subset of A.

A *finite A'-automaton* $\mathfrak{A} = (n, M, S, P)$, $n \geq 1$ is given by

(i) a transition matrix $M \in (A' \cup \{0, 1\})^{n\times n}$,

(ii) an initial state vector $S \in (A' \cup \{0, 1\})^{1\times n}$,

(iii) a final state vector $P \in (A' \cup \{0, 1\})^{n\times 1}$.

The *behavior* $||\mathfrak{A}||$ of $\mathfrak{A}$ is defined by

$$||\mathfrak{A}|| = \sum_{1\leq i_1,i_2\leq n} S_{i_1}(M^*)_{i_1,i_2}P_{i_2} = SM^*P .$$

The *(directed) graph of* $\mathfrak{A}$ is constructed in the usual manner. It has nodes $1, \ldots, n$ and an edge from node i to node j if $M_{ij} \neq 0$. The *weight* of this edge is $M_{ij} \in A'$. The *initial* (resp. *final*) *weight* of a node i is given by S_i (resp. P_i). A node is called *initial* (resp. *final*) if its initial (resp. final) weight is unequal to 0. The *weight* of a path is the product of the weigths of its edges. It is easily shown that $(M^k)_{ij}$ is the sum of the weights of paths of length k from node i to node j. If A is a complete semiring and, hence, $(M^*)_{ij} = \sum_{k\geq 0}(M^k)_{ij}$, then $(M^*)_{ij}$ is the sum of the weights of the paths from node i to node j. Hence, $S_{i_1}(M^*)_{i_1,i_2}P_{i_2}$ is this sum for nodes i_1 and i_2, properly multiplied on the left and right by the initial weight of node i_1 and the final weight of node i_2, respectively. Eventually, the behavior of $\mathfrak{A}$ is the sum of all these terms with summation over all initial states i_1 and all final states i_2.

Two finite A'-automata $\mathfrak{A}$ and $\mathfrak{A}'$ are *equivalent* if $||\mathfrak{A}|| = ||\mathfrak{A}'||$. A finite A'-automaton $\mathfrak{A} = (n, M, S, P)$ is called *normalized* if $n \geq 2$ and

(i) $S_1 = 1$, $S_i = 0$, $2 \leq i \leq n$;

(ii) $P_n = 1$, $P_i = 0$, $1 \leq i \leq n - 1$;

(iii) $M_{i,1} = M_{n,i} = 0$, $1 \leq i \leq n$.

Hence, the directed graph of a normalized finite A'-automaton has the unique initial node 1 and the unique final node n, both with weight 1; moreover, no edges are leading to the initial node and no edges are leaving the final node.

Theorem 1.3.1 *Each finite A'-automaton is equivalent to a normalized finite A'-automaton.*

Proof. Let $\mathfrak{A} = (n, M, S, P)$ be a finite A'-automaton. Define the finite A'-automaton $\mathfrak{A}'$ by

$$\mathfrak{A}' = (1+n+1, \begin{pmatrix} 0 & S & 0 \\ 0 & M & P \\ 0 & 0 & 0 \end{pmatrix}, (1\ 0\ 0), \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}).$$

Then $\mathfrak{A}'$ is normalized. Applying the matrix-star-equation yields the proof that $||\mathfrak{A}'|| = ||\mathfrak{A}||$. $\square$

The substarsemiring of A that is generated by A' is denoted by $\mathfrak{Rat}(A')$. The collection of all behaviors of finite A'-automata is denoted by $\mathfrak{Rec}(A')$. The classical Theorem of Kleene essentially states that, in the semiring 2^{Σ^*} of formal languages over Σ, $\mathfrak{Rat}(\Sigma)$ and $\mathfrak{Rec}(2^{\Sigma})$ coincide.

As a generalization of this Theorem of Kleene we show that $\mathfrak{Rat}(A') = \mathfrak{Rec}(A')$. (See Schützenberger [108], Conway [25], Kuich [78], Berstel, Reutenauer [7], Ésik, Kuich [39]).

Theorem 1.3.2 *Let A be a Conway semiring and $A' \subseteq A$. Then $\mathfrak{Rat}(A') = \mathfrak{Rec}(A')$.*

Proof. (i) An easy proof by induction on n using the matrix-star-equation shows that $M^* \in \mathfrak{Rat}(A')^{n\times n}$ if $M \in A'^{n\times n}$. This implies immediately $\mathfrak{Rec}(A') \subseteq \mathfrak{Rat}(A')$.

(ii) Easy constructions yield $A' \cup \{0,1\} \subseteq \mathfrak{Rec}(A')$. Let now $\mathfrak{A} = (n, M, S, P)$ and $\mathfrak{A}' = (n', M', S', P')$ be normalized finite A'-automata. Then we define finite A'-automata $\mathfrak{A} + \mathfrak{A}'$, $\mathfrak{A} \cdot \mathfrak{A}'$ and $\mathfrak{A}^*$ with behaviors $||\mathfrak{A}|| + ||\mathfrak{A}'||$, $||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$ and $||\mathfrak{A}||^*$, respectively:

$$\mathfrak{A} + \mathfrak{A}' = (n+n', \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}, (S\ S'), \begin{pmatrix} P \\ P' \end{pmatrix}),$$

$$\mathfrak{A} \cdot \mathfrak{A}' = (n+n', \begin{pmatrix} M & PS' \\ 0 & M' \end{pmatrix}, (S\ 0), \begin{pmatrix} 0 \\ P' \end{pmatrix}),$$

$$\mathfrak{A}^* = (1+n, \begin{pmatrix} 0 & S \\ P & M \end{pmatrix}, (1\ 0), \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Application of the matrix-star-equality shows that the equations $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$, $||\mathfrak{A} \cdot \mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$ und $||\mathfrak{A}^*|| = ||\mathfrak{A}||^*$ are valid. $\square$

A finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton $\mathfrak{A} = (n, M, S, P)$ is called *finite automaton (over A and Σ) without ε-moves* if $M \in (A\langle\Sigma\rangle)^{n\times n}$, $S \in (A\langle\varepsilon\rangle)^{1\times n}$ with $S_1 = \varepsilon$, $S_j = 0$ for $2 \le j \le n$, $P \in (A\langle\varepsilon\rangle)^{n\times 1}$. For $A = \mathbb{B}$ this is the usual definition, i. e., such a finite $\mathbb{B}\langle\Sigma\rangle$-automaton is a finite automaton without ε-moves in the classical sense.

We now show that each finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton is equivalent to a finite automaton without ε-moves.

Theorem 1.3.3 *Each finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton is equivalent to a finite automaton over A and Σ without ε-moves.*

Proof. For each finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton there exits, by Theorem 3.1, an equivalent normalized finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton. Let $\mathfrak{A} = (n, M, S, P)$ be such a normalized finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton. Let $M_0 = (M, \varepsilon)\varepsilon$ and $M_1 = \sum_{x\in\Sigma}(M, x)x$ and define the finite automaton without ε-moves $\mathfrak{A}' = (n, M_0^* M_1, S, M_0^* P)$. Then

$$||\mathfrak{A}'|| = S(M_0^* M_1)^* M_0^* P = S(M_0 + M_1)^* P = SM^* P = ||\mathfrak{A}|| \,.$$

Here we have applied in the second equality the sum-star-equation. □

Corollary 1.3.4 $\mathfrak{Rat}(A\langle\Sigma \cup \varepsilon\rangle) = \mathfrak{Rec}(A\langle\Sigma \cup \varepsilon\rangle) = \{||\mathfrak{A}|| \mid \mathfrak{A}$ *is a finite automaton over A and Σ without ε-moves*$\}$.

Corollary 1.3.5 $\mathfrak{Rat}(\mathbb{B}\langle\Sigma\rangle) = \mathfrak{Rec}(\mathbb{B}\langle\Sigma\rangle) = \{||\mathfrak{A}|| \mid \mathfrak{A}$ *is a finite automaton over $\mathbb{B}$ and Σ without ε-moves*$\}$.

For the definition of nondeterministic finite automata, we need matrices which are indexed by $Q \times Q'$, Q, Q' finite index sets, and whose entries are in the semiring 2^{Σ^*}. The collection of all these matrices is denoted by $(2^{\Sigma^*})^{Q\times Q'}$. If $|Q| = n$, then $\langle(2^{\Sigma^*})^{Q\times Q}, +, \cdot, 0, E\rangle$ and $\langle(2^{\Sigma^*})^{n\times n}, +, \cdot, 0, E\rangle$ are isomorphic. Furthermore, for $M \in (2^{\Sigma^*})^{n\times n}$ and its isomorphic copy $M' \in (2^{\Sigma^*})^{Q\times Q}$ we obtain that M'^* is the isomorphic copy of M^*.

Usually, a nondeterministic finite automaton without ε-moves is defined as follows (see Hopcroft, Ulman [65]). A nondeterministic finite automaton (in the classical sense)

$$\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$$

is given by

(i) a finite non-empty *set of states* Q,

(ii) an *input alphabet* Σ,

(iii) a *transition function* $\delta : Q \times \Sigma \to 2^Q$,

(iv) an *initial state* $q_1 \in Q$,

(v) a *set of final states* $F \subseteq Q$.

The transition function δ is extended to a mapping $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ by

$$\hat{\delta}(q, \varepsilon) = \{q\}, \quad \hat{\delta}(q, wx) = \{p \mid p \in \delta(r, x) \text{ for some } r \in \hat{\delta}(q, w)\} \,,$$

for $q \in Q$, $w \in \Sigma^*$ and $x \in \Sigma$.

A word $w \in \Sigma^*$ is *accepted* by $\mathcal{A}$ if $\hat{\delta}(q_1, w) \cap F \neq \emptyset$. The *language* $||\mathcal{A}||$ *accepted by* $\mathcal{A}$, is defined by

$$||\mathcal{A}|| = \{w \in \Sigma^* \mid \hat{\delta}(q_1, w) \cap F \neq \emptyset\} \,.$$

We now connect the notion of a finite automaton $\mathfrak{A}$ over S and Σ without ε-moves, $||\mathfrak{A}|| \subseteq 2^{\Sigma^*}$, with the notion of a nondeterministic finite automaton $\mathcal{A}$ as defined above.

Assume that $\mathfrak{A} = (n, M, S, P)$ and $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$. Then $\mathfrak{A}$ and $\mathcal{A}$ *correspond to each other* if the following conditions are satisfied:

(i) $|Q| = n$; so we may assume $Q = \{q_1, \ldots, q_n\}$, where i corresponds to q_i, $1 \le i \le n$.

(ii) $x \in M_{ij} \Leftrightarrow q_j \in \delta(q_i, x)$, $1 \le i, j \le n$, $x \in \Sigma$.

(iii) $S_{q_1} = \{\varepsilon\}$, $S_{q_i} = \emptyset$, $2 \le i \le n$.

(iv) $P_q = \{\varepsilon\} \Leftrightarrow q \in F$, $P_q = \emptyset \Leftrightarrow q \notin F$.

It is easily seen that $||\mathfrak{A}|| = ||\mathcal{A}||$ if $\mathfrak{A}$ and $\mathcal{A}$ correspond to each other. This is due to the fact that

$$w \in (M^k)_{ij} \Leftrightarrow q_j \in \hat{\delta}(q_i, w), \quad 1 \le i, j \le n,\ k \ge 0,\ w \in \Sigma^*,\ |w| = k\,,$$

and

$$w \in (M^*)_{ij} \Leftrightarrow q_j \in \hat{\delta}(q_i, w), \quad 1 \le i, j \le n,\ w \in \Sigma^*\,.$$

(In the complete starsemiring $(2^{\Sigma^*})^{n \times n}$ we have $M^* = \bigcup_{n \ge 0} M^n$.) Hence,

$$\begin{aligned} ||\mathfrak{A}|| \quad &= \quad SM^*P = \textstyle\bigcup_{1 \le i,j \le n} S_i (M^*)_{ij} P_j = \bigcup_{q_j \in F} (M^*)_{1j} = \\ & \quad\quad \textstyle\bigcup_{q_j \in F} \{w \mid q_j \in \hat{\delta}(q_1, w)\} = \{w \mid \hat{\delta}(q_1, w) \cap F \ne \emptyset\} = ||\mathcal{A}||\,. \end{aligned}$$

Corollary 1.3.6 (Kleene's Theorem) *In the semiring* 2^{Σ^*}, $\mathfrak{Rat}(\Sigma) = \mathfrak{Rec}(2^{\Sigma}) = \{||\mathcal{A}|| \mid \mathcal{A}$ *is a nondeterministic finite automaton (in the classical sense)* $\}$.

1.4 The computation of the star of a matrix

In Section 3 we have seen that the computation of M^*, where M is the transition matrix of a finite A'-automaton $\mathfrak{A}$, is essential for the computation of $||\mathfrak{A}||$. We now give an algorithm for computing $M^* \in A^{n \times n}$ for $M \in A^{n \times n}$. The next theorem can be found in Mehlhorn [94], page 145, and Bloom, Ésik [10], page 291.

Theorem 1.4.1 *Let* $M, \bar{M} \in A^{n \times n}$, $n \ge 2$, *be partitioned into blocks* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\bar{M} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$, *where* $a, \bar{\alpha} \in A^{n_1 \times n_1}$, $b, \bar{\beta} \in A^{n_1 \times n_2}$, $c, \bar{\gamma} \in A^{n_2 \times n_1}$, $d, \bar{\delta} \in A^{n_2 \times n_2}$, $n_1 + n_2 = n$. *Assume that* $\bar{\delta} = (d + ca^*b)^*$, $\bar{\gamma} = \bar{\delta} ca^*$, $\bar{\beta} = a^* b \bar{\delta}$, $\bar{\alpha} = a^* + a^* b \bar{\delta} ca^*$. *Then* $\bar{M} = M^*$.

Proof. We start with the matrix-star-equations for M^* and transform the equations stepwise by the sum-star-equation and the matrix-star-equation for matrices.

Let $\alpha, \beta, \gamma, \delta$ be the right sides of the matrix-star-equations for M^*. Then $\delta = \bar{\delta}$, $\gamma = \bar{\gamma}$, $\beta = (a + bd^*c)^*bd^* = (a^*bd^*c)^*a^*bd^* = a^*b(d^*ca^*b)^*d^* = a^*b(d+ca^*b)^* = a^*b\bar{\delta}$, $\alpha = (a+bd^*c)^* = (a^*bd^*c)^*a^* = a^*+(a^*bd^*c)^*a^*bd^*ca^* = a^* + a^*b(d^*ca^*b)^*d^*ca^* = a^* + a^*b(d + ca^*b)ca^* = a^* + a^*b\bar{\delta}ca^*$. □

Theorem 4.1 with $n_1 = 1$, $n_2 = n - 1$, gives rise to the following algorithm computing M^* for $M \in A^{n\times n}$.

Algorithm 1. Compute $\eta_1, \eta_2, \eta_2', \eta_3$ by

(1) $\eta_1 = a^*$,

(2) $\eta_2 = c\eta_1$, $\eta_2' = \eta_1 b$,

(3) $\eta_3 = \eta_2 b = c\eta_2'$ (only one of the computations is needed; take that with the lower complexity),

Compute now $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ by

(4) $\bar{\delta} = (d + \eta_3)^*$,

(5) $\bar{\gamma} = \bar{\delta}\eta_2$, $\bar{\beta} = \eta_2'\bar{\delta}$,

(6) $\bar{\alpha} = \eta_1 + \eta_2'\bar{\gamma} = \eta_1 + \bar{\beta}\eta_2$ (only one of the computations is needed; take that with the lower complexity).

Let $T_*, T_\times, T_+$ be the worst costs performing the operations $^*, \times, +$, respectively, on elements of A. Let T_1^n, $n \geq 1$, be the worst costs of computing the star of $M \in A^{n\times n}$ by Algorithm 1. Then the time complexity of the steps (1)–(6) is as follows:

(1) T_* ,

(2) $2(n-1)T_\times$,

(3) $(n-1)^2 T_\times$,

(4) $(n-1)^2 T_+ + T_1^{n-1}$,

(5) $2(n-1)^2 T_\times + 2(n-1)(n-2)T_+$,

(6) $T_+ + (n-1)T_\times + (n-2)T_+$.

Hence, we obtain, for $n \geq 2$, the recursion

$$T_1^n = T_1^{n-1} + T_* + 3n(n-1)T_\times + (3n-4)(n-1)T_+ \,, \quad \text{with } T_1^1 = T_* \,.$$

It has, for $n \geq 1$, the solution

$$T_1^n = nT_* + (n+1)n(n-1)T_\times + n(n-1)^2 T_+ \,.$$

We now compare our Algorithm 1 with the standard algorithm for computing M^* for $M \in A^{n\times n}$, where A is a complete starsemiring.

Algorithm 2. Compute $d_{ij}^{(k)}$, $(M^*)_{ij}$, for $1 \leq i, j \leq n$, $0 \leq k \leq n$, by

(1) $d_{ij}^{(0)} = M_{ij}$,

(2) For $1 \le k \le n$, $d_{ij}^{(k)} = \begin{cases} (d_{kk}^{(k-1)})^* \text{ for } i = j = k\,, \\ d_{ij}^{(k-1)} + d_{ik}^{(k-1)}(d_{kk}^{(k-1)})^* d_{kj}^{(k-1)} \text{ otherwise}\,. \end{cases}$

(3) $(M^*)_{ij} = d_{ij}^{(n)}$.

This algorihm is originally due to Kleene [70]. We have given the variant of Mehlhorn [94], pages 138 and 139. The worst costs T_2^n, $n \ge 1$, of computing the star of $M \in A^{n\times n}$ by Algorithm 2 are

$$T_2^n = nT_* + 2(n+1)n(n-1)T_\times + (n+1)n(n-1)T_+\,.$$

Hence, $T_1^n < T_2^n$ for all $n \ge 2$. But more important is that Algorithm 1 is valid in all Conway semirings.

We now consider Algorithm 1 for 0-closed semirings A, i. e., for each $a \in A$ we have $a^* = 1 + a = 1$, and call it Algorithm 3. The worst costs T_3^n, $n \ge 1$, of computing the star of $M \in A^{n\times n}$ by Algorithm 3 are then

$$T_3^n = n(n-1)(n-2)(T_\times + T_+)\,,$$

since all the diagonal elements of $\bar{\delta}$ are 1.

The following variant of Algorithm 2 is usually used for computations in the complete semiring $\langle \mathbb{R}_+^\infty, \min, +, \infty, 0\rangle$.

Algorithm 4. Compute $d_{ij}^{(k)}, (M^*)_{ij}$, for $1 \le i, j \le n$, $0 \le k \le n$, by

(1) $d_{ij}^{(0)} = \begin{cases} 0 & \text{for } i = j\,, \\ M_{ij} & \text{for } i \ne j\,, \end{cases}$

(2) For $1 \le k \le n$, $d_{ij}^{(k)} = \begin{cases} d_{ij}^{(k-1)} \text{ for } i = k \text{ or } j = k \text{ or } i = j\,, \\ \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) \text{ otherwise}\,. \end{cases}$

(3) $(M^*)_{ij} = d_{ij}^{(n)}$.

Let $T_{\min}$ and T_+ be the worst costs performing the operations min and $+$ on elements of $\mathbb{R}_+^\infty$, respectively, and let T_4^n, $n \ge 1$, be the worst costs of computing the star of $M \in (\mathbb{R}_+^\infty)^{n\times n}$. Then

$$T_4^n = n(n-1)(n-2)(T_+ + T_{\min}), \quad n \ge 1\,.$$

Hence, if Algorithm 3 is applied in the tropical semiring $\mathbb{R}_+^\infty$ then $T_4^n = T_3^n$ for all $n \ge 1$. But Algorithm 3 has the advantage that it works in all 0-closed semirings.

Let $M \in (\mathbb{R}_+^\infty)^{n\times n}$ be the transition matrix of a finite $\mathbb{R}_+^\infty$-automaton. Then $(M^*)_{ij}$ can be interpreted as the length of a shortest path from node i to node j in its directed graph. Hence, the computation of the star of matrices solves the all-pairs shortest-distance problem for directed graphs with weights in the tropical semiring $\mathbb{R}_+^\infty$, i.e., computes the shortest distances between all pairs of nodes of a directed graph.

If negative distances are to be considered then the complete semiring $\langle \mathbb{R} \cup \{-\infty, \infty\}, \min, +, \infty, 0\rangle$ with $(-\infty) + \infty = \infty + (-\infty) = \infty$, $a^* = -\infty$ for $a < 0$, $a^* = 0$ for $a \geq 0$, has to be taken. Then the computation of the star of matrices solves the all-pairs shortest-distance problem for directed graphs with weights in $\mathbb{R} \cup \{-\infty, \infty\}$. Algorithm 4 yields wrong results in this semiring if a negative cycle appears in the directed graph. Hence, in this case one has to take Algorithm 1 or Algorithm 2.

Theorem 4.1 is also the basis for comparing the complexities of computing the product of two matrices versus computing the star of a matrix. The next two theorems are stated in Mehlhorn [94], on pages 143 and 144 as Theorems 3 and 4 for complete semirings. Inspection of the proofs of these theorems as given by Mehlhorn [94] shows that these proofs are valid also for Conway semirings.

Theorem 1.4.2 *Let A be a Conway semiring and let $T : \mathbb{N} \to \mathbb{R}_+$ be a function with $T(3n) \leq cT(n)$ for some $c \in \mathbb{R}_+$ and all $n \in \mathbb{N}$. If there is an algorithm which computes the star of an $n \times n$-matrix with entries in A with $T(n)$ additions, multiplications and star operations of semiring elements of A then there is an algorithm to multiply two $n \times n$-matrices with entries in A with $O(T(n))$ additions and multiplications of semiring elements of A.*

Theorem 1.4.3 *Let A be a Conway semiring and let $T : \mathbb{N} \to \mathbb{R}_+$ be a function with $T(1) = 1$, $4T(2^{k-1}) \leq T(2^k)$ for all $k \geq 1$, and $T(2n) \leq cT(n)$ for some $c \in \mathbb{R}_+$ and all $n \geq 1$. If the product of two $n \times n$-matrices with entries in A can be computed with $T(n)$ additions and multiplications of semiring elements of A then the star of an $n \times n$-matrix with entries in A can be computed with $O(T(n))$ additions, multiplications and star operations of semiring elements of A.*

We now will consider, for $k \geq 1$, the semirings $\mathbb{T}_k$ and $\mathbb{T}'_k$. They will be used to solve the all-pairs k-shortest distance problem and the all-pairs k-distinct-shortest distance problem for directed graphs by computing the star of a matrix with entries in $\mathbb{T}_k$ and $\mathbb{T}'_k$, respectively (see Mohri [95]).

We first define the semiring $\mathbb{T}_k$ for a fixed $k \geq 1$. Let $(a_1, \ldots, a_m) \in (\mathbb{R}_+^\infty)^m$, $m \geq k$, and define $\min_k(a_1, \ldots, a_m) = (b_1, \ldots, b_k)$, where $(b_1, \ldots, b_k)$ is the ordered list of the k least elements of $(a_1, \ldots, a_m)$ with repetitions using the usual order of $\mathbb{R}_+^\infty$ (e.g., $\min_4(2, 1, 3, 1) = (1, 1, 2, 3)$).

Consider $\mathbb{T}_k = \{(a_1, \ldots, a_k) \mid a_1 \leq \ldots \leq a_k,\ a_i \in \mathbb{R}_+^\infty,\ 1 \leq i \leq k\}$ and define the two operations $\oplus_k$ and $\otimes_k$ over $\mathbb{T}_k$ by

$$
\begin{aligned}
&(a_1, \ldots, a_k) \oplus_k (b_1, \ldots, b_k) = \min_k(a_1, \ldots, a_k, b_1, \ldots, b_k)\,, \\
&(a_1, \ldots, a_k) \otimes_k (b_1, \ldots, b_k) = \\
&\qquad = \min_k(a_1 + b_1, \ldots, a_1 + b_k, \ldots, a_k + b_1, \ldots, a_k + b_k)\,.
\end{aligned}
$$

Define $0_k = (\infty, \ldots, \infty)$ and $1_k = (0, \infty, \ldots, \infty)$. Then by Mohri [95], Proposition 2, $\langle \mathbb{T}_k, \oplus_k, \otimes_k, 0_k, 1_k\rangle$ is a $(k-1)$-closed commutative semiring. Observe that, for $k = 1$, $\mathbb{T}_k = \mathbb{R}_+^\infty$ is the tropical semiring.

Consider now a matrix $M \in \mathbb{T}_k^{n\times n}$ and its directed graph: It has n nodes $\{1,\ldots,n\}$; if $M_{ij} = (a_1,\ldots,a_m,\infty,\ldots,\infty)$, $1 \le i,j \le n$, $a_t \in \mathbb{R}_+$, $1 \le t \le m$, $0 \le m \le k$, then there are m different edges from node i to node j with weights (i. e., lengths) $a_1,\ldots,a_m$. Observe that some or all of the weights can be equal.

The entries $(M^*)_{ij}$, $1 \le i,j \le n$ of the star of $M \in \mathbb{T}_k^{n\times n}$ can be interpreted as follows: if $(M^*)_{ij} = (a_1,\ldots,a_m,\infty,\ldots,\infty)$, $a_t \in \mathbb{R}_+$, $1 \le t \le m$, $0 \le m < k$, then there are exactly m different paths from node i to node j with weights (i. e., lengths) $a_1,\ldots,a_m$; if $(M^*)_{ij} = (a_1,\ldots,a_k)$, $a_t \in \mathbb{R}_+$, $1 \le t \le k$, then the k paths with shortest lengths from node i to node j have lengths $a_1,\ldots,a_k$. Again, some or all of the weights can be equal. Hence, computing the star of a matrix over $\mathbb{T}_k$ solves the all-pairs k-shortest distance problem.

Since $\mathbb{T}_k$ is $(k-1)$-closed, the computation of the star of a matrix over $\mathbb{T}_k$ can be performed by Algorithm 1.

We now define the semiring $\mathbb{T}'_k$ for a fixed $k \ge 1$. Consider $(a_1,\ldots,a_m) \in (\mathbb{R}_+^\infty)^m$, $m \ge 1$. Let $(a_{i_1},\ldots,a_{i_t})$ be the ordered list without repetitions of the non-∞ elements of $(a_1,\ldots,a_m)$ (i. e., $a_{i_1} < \ldots < a_{i_t}$). Define $\min'_k(a_1,\ldots,a_m) = (a_{i_1},\ldots,a_{i_k})$ if $t \ge k$ and $\min'_k(a_1,\ldots,a_m) = (a_{i_1},\ldots,a_{i_t},\infty,\ldots,\infty) \in (\mathbb{R}_+^\infty)^k$ if $t < k$ (e. g., $\min'_4(2,1,3,1) = (1,2,3,\infty)$).

Consider $\mathbb{T}'_k = \{(a_1,\ldots,a_t,\infty,\ldots,\infty) \mid a_1 < \ldots < a_t,\ a_i \in \mathbb{R}_+,\ 1 \le i \le t,\ 0 \le t \le k\}$ and define the two operations $\oplus'_k$ and $\otimes'_k$ over $\mathbb{T}'_k$ by

$$
\begin{aligned}
&(a_1,\ldots,a_k) \oplus'_k (b_1,\ldots,b_k) = \min'_k(a_1,\ldots,a_k,b_1,\ldots,b_k)\,,\\
&(a_1,\ldots,a_k) \otimes'_k (b_1,\ldots,b_k) =\\
&\qquad = \min'_k(a_1+b_1,\ldots,a_1+b_k,\ldots,a_k+b_1,\ldots,a_k+b_k)\,.
\end{aligned}
$$

Then by Mohri [95], Proposition 3, $\langle \mathbb{T}'_k, \oplus'_k, \otimes'_k, 0_k, 1_k\rangle$ is a $(k-1)$-closed commutative semiring. Moreover, $\mathbb{T}'_k$ is idempotent. Similarly as above, computing the star of a matrix over $\mathbb{T}'_k$ solves the all-pairs k-distinct-shortest-distance problem.

Since $\mathbb{T}'_k$ is $(k-1)$-closed, the computation of the star of a matrix over $\mathbb{T}'_k$ can be performed by Algorithm 1.

Chapter 2

Context-Free grammars and algebraic systems

2.1 Introduction

In this chapter, we deal with continuous semirings and algebraic systems. These algebraic systems are a generalization of the context-free grammars. This chapter consists of this and three more sections.

In Section 2, we introduce continuous monoids and semirings and give the basic facts of fixed point theory. These are needed in Section 3 to apply the fixed point theory to algebraic systems in order to get least solutions. The components of the least solutions of algebraic systems are a generalization of the context-free languages. In Section 4, we introduce normal forms and show equivalence results for the Chomsky normal form, the operator normal form and the Greibach normal form.

We now give a typical example which will be helpful for readers with some background in algebraic systems. Readers without this background should consult it when algebraic systems are defined in the following sections.

Example 2.1.1. Let $G = (\{y\}, \Sigma, P, y)$ with $\Sigma = \{x, \bar{x}\}$ and $P = \{y \to xy\bar{x}y,\ y \to \varepsilon\}$ be a context-free grammar. The context-free language $L(G)$ generated by G is called restricted Dyck language (see Berstel [4]) and is the set of wellformed expressions with parantheses $x, \bar{x}$. We consider now the derivation of a word w in $L(G)$ according to the levels of the (unique) derivation tree of w:

$$\begin{aligned}
&y \Rightarrow \varepsilon\,,\\
&y \Rightarrow xy\bar{x}y \Rightarrow^2 x\bar{x}\,,\\
&y \Rightarrow xy\bar{x}y \Rightarrow^2 xxy\bar{x}y\bar{x}xy\bar{x}y \Rightarrow^4 xx\bar{x}\bar{x}x\bar{x}\,,\\
&y \Rightarrow xy\bar{x}y \Rightarrow^2 xxy\bar{x}y\bar{x} \Rightarrow^2 xx\bar{x}\bar{x}\,,\\
&y \Rightarrow xy\bar{x}y \Rightarrow^2 x\bar{x}xy\bar{x}y \Rightarrow^2 x\bar{x}x\bar{x}\,,\ \ldots
\end{aligned}$$

We now associate to G the equation $y = \{x\}y\{\bar{x}\}y \cup \{\varepsilon\}$ and construct its (unique) solution in the semiring 2^{Σ^*} by the following procedure (due to the forthcoming Theorem 2.9): We start with $\sigma^0 = \emptyset$ and substitute σ^n into the right side of the equation to obtain σ^{n+1}. Hence,

$$\begin{aligned}\sigma^0 = \emptyset, \quad \sigma^1 = \{\varepsilon\}, \quad \sigma^2 = \{x\bar{x}, \varepsilon\},\\ \sigma^3 = \{x\}\{x\bar{x}, \varepsilon\}\{\bar{x}\}\{x\bar{x}, \varepsilon\} \cup \{\varepsilon\} =\\ \{xx\bar{x}\bar{x}x\bar{x}, xx\bar{x}\bar{x}, x\bar{x}x\bar{x}, \varepsilon\}, \ldots\end{aligned}$$

Observe that σ^n contains exactly those words of $L(G)$ that are the results of derivation trees of height at most n, $n \geq 1$. The sequence $\sigma^0, \sigma^1, \ldots, \sigma^n, \ldots$ is called approximation sequence. Since it is monotonic, its least upper bound equals $\bigcup_{n \geq 0} \sigma^n$ and coincides by the forthcoming Theorem 2.9 with $L(G)$. $\square$

2.2 Preliminaries

In this section we first consider commutative monoids. The definitions and results on commutative monoids are mainly due—sometimes in the framework of semiring theory—to Eilenberg [29], Goldstern [57], Karner [67, 68], Krob [72, 73], Kuich [76, 78], Kuich, Salomaa [88], Manes, Arbib [92], Sakarovitch [104]. Our notion of continuous monoid is a specialization of the continuous algebras as defined, e. g., in Guessarian [61], Goguen, Thatcher, Wagner, Wright [56], Adamek, Nelson, Reiterman [1].

In the second part of this section we consider algebraic properties of formal power series.

In our book we often will need certain results of fixed point theory. Hence, in the third part of this section, we give a short introduction into the fixed point theory of continuous functions and refer to a few results of this theory.

A commutative monoid $\langle A, +, 0\rangle$ is called *ordered* iff it is equipped with a partial order $\leq$ preserved by the $+$ operation such that $0 \leq a$ holds for all $a \in A$. It then follows that $a \leq a + b$, for all $a, b \in A$. In particular, a commutative monoid $\langle A, +, 0\rangle$ is called *naturally ordered* iff the relation $\sqsubseteq$ defined by: $a \sqsubseteq b$ iff there exists a c such that $a + c = b$, is a partial order. Morphisms of ordered monoids preserve the order.

A monoid $\langle A, +, 0\rangle$ is called *complete* iff it has sums for all families $(a_i \mid i \in I)$ of elements of A, where I is an arbitrary index set, such that the following conditions are satisfied:

(i) $\sum_{i \in \emptyset} a_i = 0$, $\sum_{i \in \{j\}} a_i = a_j$, $\sum_{i \in \{j,k\}} a_i = a_j + a_k$, for $j \neq k$,

(ii) $\sum_{j \in J}(\sum_{i \in I_j} a_i) = \sum_{i \in I} a_i$, if $\bigcup_{j \in J} I_j = I$ and $I_j \cap I_{j'} = \emptyset$ for $j \neq j'$.

A morphism of complete monoids preserves all sums. Note that any complete monoid is commutative.

Recall that a non-empty subset D of a partially ordered set P is called *directed* iff each pair of elements of D has an upper bound in D. Moreover, a function $f : P \to Q$ between partially orderet sets is *continuous* iff it preserves the least upper bound of any directed set, i.e., when $f(\sup D) = \sup f(D)$, for all directed sets $D \subseteq P$ such that $\sup D$ exists. It follows that any continuous function preserves the order.

An ordered commutative monoid $\langle A, +, 0\rangle$ is called a *continuous monoid* iff each directed subset of A has a least upper bound and the $+$ operation preserves the least upper bound of directed sets, i.e., when

$$a + \sup D = \sup(a + D)\,,$$

for all directed sets $D \subseteq A$ and for all $a \in A$. Here, $a + D$ is the set $\{a + x \mid x \in D\}$. A morphism of continuous monoids is a continuous monoid homomorphism.

It is known that an ordered commutative monoid A is continuous iff each chain in A has a least upper bound and the $+$ operation preserves least upper bounds of chains, i. e., when $a + \sup C = \sup(a + C)$ holds for all non-empty chains C in A. (See Markowsky [93].)

Proposition 2.2.1 *Any continuous monoid $\langle A, +, 0\rangle$ is a complete monoid equipped with the following sum operation:*

$$\sum_{i\in I} a_i = \sup\{\sum_{i\in E} a_i \mid E \subseteq I,\ E \text{ finite}\}\,,$$

for all index sets I and all families $(a_i \mid i \in I)$ in A. Any morphism between continuous monoids is a complete monoid morphism.

A semiring $\langle A, +, \cdot, 0, 1\rangle$ is called *ordered* if $\langle A, +, 0\rangle$ is an ordered monoid and multiplication preserves the order. When the order on A is the natural order, $\langle A, +, \cdot, 0, 1\rangle$ is automatically an ordered semiring. A morphism of ordered semirings is an order preserving semiring morphism.

A semiring $\langle A, +, \cdot, 0, 1\rangle$ is called *continuous* if $\langle A, +, 0\rangle$ is a continuous monoid and if multiplication is continuous, i.e.,

$$a \cdot (\sup_{i\in I} a_i) = \sup_{i\in I}(a \cdot a_i) \quad \text{and} \quad (\sup_{i\in I} a_i) \cdot a = \sup_{i\in I}(a_i \cdot a)$$

for all directed sets $\{a_i \mid i \in I\}$. It follows that the distribution laws hold for infinite sums:

$$a \cdot (\sum_{i\in I} a_i) = \sum_{i\in I}(a \cdot a_i) \quad \text{and} \quad (\sum_{i\in I} a_i) \cdot a = \sum_{i\in I}(a_i \cdot a)$$

for all families $(a_i \mid i \in I)$.

A morphism of continuous semirings is a semiring morphism which is a continuous function. Note that every continuous semiring is an ordered semiring and every continuous semiring morphism is an ordered semiring morphism.

Corollary 2.2.2 *Any continuous semiring is complete.*

Corollary 2.2.3 *Any continuous semiring is a Conway semiring.*

Proof. By Theorem 1.2.24. □

Examples of continuous semirings include $\mathbb{B}, \mathbb{N}^\infty, \mathbb{R}_+^\infty$, the tropical semirings, the semiring 2^{Σ^*} of formal languages over Σ, and the semiring $2^{S \times S}$ of binary relations over S.

Suppose that A is a semiring and Σ is an alphabet. Recall from Chapter 1 the definitions of the polynomial semiring $A\langle\Sigma^*\rangle$ and the power series semiring $A\langle\langle\Sigma^*\rangle\rangle$. We now exhibit a universal property of these constructions. Note that $A\langle\langle\Sigma^*\rangle\rangle$ may be equipped with a *scalar multiplication* $A \times A\langle\langle\Sigma^*\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle$, $(a, s) \mapsto as$, defined by $(as, u) = a(s, u)$, for all $u \in \Sigma^*$. Here and in the sequel, Σ denotes a finite alphabet. When $s \in A\langle\Sigma^*\rangle$, then also $as \in A\langle\Sigma^*\rangle$. This operation satisfies the following equations:

$$\begin{aligned} a(bs) &= (ab)s && (2.1)\\ 1s &= s && (2.2)\\ (a+b)s &= as + bs && (2.3)\\ a(s+s') &= as + as' && (2.4)\\ a0 &= 0\,, && (2.5) \end{aligned}$$

for all $a, b \in A$ and $s, s' \in A\langle\langle\Sigma^*\rangle\rangle$. It follows that

$$0s = 0\,,$$

for all s. Moreover, when A is commutative, we also have that

$$(as)(bs') = (ab)(ss') \qquad (2.6)$$

for all $a, b \in A$ and $s, s' \in A\langle\langle\Sigma^*\rangle\rangle$.

Theorem 2.2.4 *Suppose that A is a commutative semiring and S is a semiring equipped with a scalar multiplication $A \times S \to S$, $(a, s) \mapsto as$, which satisfies the equations (6.1)–(6.6). Then any function $\varphi : \Sigma \to S$ extends to a unique semiring morphism $\varphi^\sharp : A\langle\Sigma^*\rangle \to S$ preserving scalar multiplication.*

Proof. It is well-known that φ extends to a unique monoid morphism morphism $\bar{\varphi} : \Sigma^* \to S$. We further extend $\bar{\varphi}$ to $\varphi^\sharp$ by defining

$$\varphi^\sharp(s) = \sum_{u \in \Sigma^*} (s, u)\bar{\varphi}(u)\,,$$

for all $s \in A\langle\Sigma^*\rangle$. It is a routine matter to show that $\varphi^\sharp$ extends φ and is a semiring morphism that preserves scalar multiplication. Since the definition of $\varphi^\sharp$ was forced, the extension is unique. □

A similar result holds when A is a complete commutative semiring, so that $A\langle\langle\Sigma^*\rangle\rangle$ is a complete semiring equipped with a scalar multiplication defined above.

Theorem 2.2.5 *Suppose that A is a complete commutative semiring and S is a complete semiring equipped with a scalar multiplication $A \times S \to S$, $(a, s) \mapsto as$, which satisfies the equations (6.1)–(6.6). Moreover, assume that*

$$(\sum_{i \in I} a_i)s = \sum_{i \in I} a_i s \tag{2.7}$$

$$a \sum_{i \in I} s_i = \sum_{i \in I} a s_i\,, \tag{2.8}$$

for all $a, a_i \in A$ and $s, s_i \in S$, $i \in I$, where I is any index set. Then any function $\varphi : \Sigma \to S$ extends to a unique complete semiring morphism $\varphi^\sharp : A\langle\langle\Sigma^*\rangle\rangle \to S$ preserving scalar multiplication.

Proof. The proof of this result parallels that of Theorem 2.4. First we extend φ to $\overline{\varphi} : \Sigma^* \to S$, and then define

$$\varphi^\sharp(s) = \sum_{u \in \Sigma^*} (s, u)\overline{\varphi}(u),$$

for all $s \in A\langle\langle\Sigma^*\rangle\rangle$. This sum makes sense since S is complete. The details of the proof that $\varphi^\sharp$ is a complete semiring morphism preserving scalar multiplication are routine. The definition of $\varphi^\sharp$ was again forced. □

When A is ordered by $\leq$, we may order $A\langle\langle\Sigma^*\rangle\rangle$, and thus $A\langle\Sigma^*\rangle$, by the pointwise order: We define $s \leq s'$ for $s, s' \in A\langle\langle\Sigma^*\rangle\rangle$ iff $(s, u) \leq (s', u)$ for all $u \in \Sigma^*$. Equipped with this order, both $A\langle\langle\Sigma^*\rangle\rangle$ and $A\langle\Sigma^*\rangle$ are ordered semirings if A is an ordered semiring. Moreover, scalar multiplication preserves the order in both arguments. Finally, if A is a continuous semiring, then $A\langle\langle\Sigma^*\rangle\rangle$ is also continuous, and scalar multiplication preserves least upper bounds of directed sets in both arguments.

Corollary 2.2.6 *Suppose that A is an ordered commutative semiring and S is an ordered semiring equipped with an order preserving scalar multiplication $A \times S \to S$, $(a, s) \mapsto as$, which satisfies the equations (6.1)–(6.6). Then any function $\varphi : \Sigma \to S$ extends to a unique ordered semiring morphism $\varphi^\sharp : A\langle\Sigma^*\rangle \to S$ preserving scalar multiplication. Moreover, when A is a continuous commutative semiring and S is a continuous semiring equipped with a continuous scalar multiplication $A \times S \to S$, $(a, s) \mapsto as$, which satisfies the above equations, then any function $\varphi : \Sigma \to S$ extends to a unique continuous semiring morphism $\varphi^\sharp : A\langle\langle\Sigma^*\rangle\rangle \to S$ preserving scalar multiplication.*

Let A denote a continuous (and thus complete) commutative semiring where sums are defined by Proposition 2.1. Let s be a formal series in $A\langle\langle\Sigma^*\rangle\rangle$, and let S denote a continuous semiring equipped with a scalar multiplication $A \times S \to S$ satisfying (6.1)–(6.6) which is also continuous. The set S^Σ of all functions $\Sigma \to S$ is also a continuous semiring equipped with the pointwise operations and ordering as is the set of all continuous functions $S^\Sigma \to S$. Moreover, it is equipped with the pointwise scalar multiplication which again satisfies (6.1)–(6.6) and is continuous. Now s induces a mapping $s^S : S^\Sigma \to S$, $h \mapsto h^\sharp(s)$ for $h \in S^\Sigma$.

Proposition 2.2.7 *The function s^S is continuous. Moreover, the assignment $s \to s^S$ defines a continuous function of s.*

Proof. It is known that when $u \in \Sigma^*$, then the function $u^S : S^\Sigma \to S$ induced by u is continuous, since it can be constructed from continuous functions (namely, the projections and product operation of S) by function composition, see, e.g., Guessarian [61]. Since scalar multiplication and $+$ are continuous, so is any function induced by a series in $A\langle\Sigma^*\rangle$. But s^S is the pointwise supremum of the functions induced by the polynomials $\sum_{u \in F}(s, u)u$, where F is a finite subset of Σ^*. Since the pointwise supremum of continuous functions is continuous, see Guessarian [61], the result follows.

To show that the assignment $s \mapsto s^S$ defines a continuous function, let D denote a directed set in $A\langle\langle\Sigma^*\rangle\rangle$. We need to prove that

$$(\sup_{s \in D} s)^S \quad = \quad \sup_{s \in D} s^S.$$

But for any $h : \Sigma \to S$,

$$\begin{aligned}
(\sup_{s \in D} s)^S(h) &= h^\sharp(\sup_{s \in D} s) \\
&= \sup_{s \in D} h^\sharp(s) \\
&= \sup_{s \in D} s^S(h) \\
&= (\sup_{s \in D} s^S)(h).
\end{aligned}$$

□

From now on we will write just h for $h^\sharp$ and denote s^S by just s.

In particular, formal series induce continuous mappings called *substitutions* as follows. Let Y denote a non-empty set of variables, where $Y \cap \Sigma = \emptyset$, and consider a mapping $h : Y \to A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$. This mapping can be extended to a mapping $h : (\Sigma \cup Y)^* \to A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$ by setting first $h(x) = x$, $x \in \Sigma$. Now, by the above result, for any series $s \in A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$, the mapping $h \mapsto h(s)$ is a continuous function of h. By the arguments outlined above, $h(s)$ can be constructed as follows. First, extend h to words $u = u_1 \ldots u_k$ with $u_i \in \Sigma \cup Y$ by defining

$$\begin{aligned}
&h(u) = h(u_1) \cdot \ldots \cdot h(u_k) = \\
&\textstyle\sum_{v_1,\ldots,v_k \in (\Sigma \cup Y)^*} (h(u_1), v_1) \ldots (h(u_k), v_k) v_1 \ldots v_k \,.
\end{aligned}$$

One more extension of h yields a mapping $h : A\langle\langle(\Sigma\cup Y)^*\rangle\rangle \to A\langle\langle(\Sigma\cup Y)^*\rangle\rangle$ with $h(s) = \sum_{u \in (\Sigma \cup Y)^*}(s, u)h(u)$, for all $s \in A\langle\langle(\Sigma\cup Y)^*\rangle\rangle$. Now $s(h)$ is just the value of this extended function on s. If $Y = \{y_1, \ldots, y_n\}$ is finite, we use the following notation: $h : Y \to A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$, where $h(y_i) = s_i$, $1 \le i \le n$, is denoted by $(s_i,\ 1 \le i \le n)$ or $(s_1, \ldots, s_n)$ and the value of s with argument h is denoted by $s(s_i,\ 1 \le i \le n)$ or $s(s_1, \ldots, s_n)$. Intuitively, this is simply the *substitution* of the formal series $s_i \in A\langle\langle(\Sigma\cup Y)^*\rangle\rangle$ into the variables y_i, $1 \le i \le n$, of $s \in A\langle\langle(\Sigma\cup Y)^*\rangle\rangle$. By Proposition 2.7, the mapping $s : (A\langle\langle(\Sigma \cup Y)^*\rangle\rangle)^Y \to A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$, i. e., the substitution of formal series into the variables of Y, is a continuous mapping. Moreover, $s(s_1, \ldots, s_n)$ is also continuous in s. (So it is continuous in s and in each s_i.) Observe that $s(s_1, \ldots, s_n) = \sum_{u\in(\Sigma\cup Y)^*}(s, u)u(s_1, \ldots, s_n)$.

In certain situations, formulae are easier to read if we use the notation $s[s_i/y_i,\ 1 \le i \le n]$ for the substitution of the formal series s_i into the variables y_i, $1 \le i \le n$, of s instead of the notation $s(s_i,\ 1 \le i \le n)$. So we will use sometimes this notation $s[s_i/y_i,\ 1 \le i \le n]$.

In the same way, $s \in A\langle\langle(\Sigma \cup Y)^*\rangle\rangle$ also induces a mapping $s : (A\langle\langle\Sigma^*\rangle\rangle)^Y \to A\langle\langle\Sigma^*\rangle\rangle$.

The construction of series and the above freeness results can be generalized to a great extent. Suppose that M is any monoid and A is any complete commutative semiring. Then the set of functions $M \to A$, denoted $A\langle\langle M\rangle\rangle$, is a complete semiring. We call the elements of $A\langle\langle M\rangle\rangle$ series and denote them as $\sum_{m\in M}(s, m)m$, or $\sum_{m\in\mathrm{supp}(s)}(s, m)m$. The sum of any family of series is their pointwise sum. The zero series serves as zero. In this semiring $A\langle\langle M\rangle\rangle$, for each $s_1, s_2 \in A\langle\langle M\rangle\rangle$,

$$s_1 s_2 = \sum_{m\in M}\Big(\sum_{m=m_1m_2}(s_1, m_1)(s_2, m_2)\Big)m\,.$$

Note also that $A\langle\langle M\rangle\rangle$ is equipped with a scalar multiplication $A \times A\langle\langle M\rangle\rangle \to A\langle\langle M\rangle\rangle$. Moreover, equations (6.1)–(6.6) and (6.7), (6.8) hold. When A is a continuous semiring then, equipped with the pointwise order, $A\langle\langle M\rangle\rangle$ is a continuous semiring and scalar multiplication is continuous. We are now ready to state the promised generalization of Theorem 2.5 and Corollary 2.6.

Theorem 2.2.8 *Suppose that A is a complete commutative semiring and S is a complete semiring equipped with a scalar multiplication $A\times S \to S$ which satisfies the equations (6.1)–(6.6) as well as (6.7) and (6.8). Moreover, assume that M is a monoid. Then any monoid morphism $\varphi : M \to S$ extends to a unique complete semiring morphism $\varphi^\sharp : A\langle\langle M\rangle\rangle \to S$ preserving scalar multiplication. When A is a continuous commutative semiring and S is continuous, moreover, the scalar multiplication $A \times S \to S$ is continuous, then so is the function $\varphi^\sharp$.*

Proof. Given φ, we are forced to define

$$\varphi^\sharp(s) \quad = \quad \sum_{m\in M}(s, m)\varphi(m), \tag{2.9}$$

for all $s \in A\langle\langle M\rangle\rangle$. On the other hand, it is a routine matter to verify that (6.9) defines a complete semiring morphism $\varphi^\sharp : A\langle\langle M\rangle\rangle \to S$ that extends φ.

Suppose now that A and S are continuous and that the scalar multiplication $A \times S \to S$ is also continuous. In order to prove that $\varphi^\sharp$ is continuous, suppose that D is a directed set in $A\langle\langle M\rangle\rangle$. Then for each $m \in M$, the set $\{(s,m) : m \in D\}$ is also directed, moreover, $(\sup D, m) = \sup_{d\in D}(d,m)$. Using this, and the continuity of scalar multilication and summation, we have that

$$\begin{aligned}
\varphi^\sharp(\sup D) &= \sum_{m\in M}(\sup D, m)\varphi(m)\\
&= \sum_{m\in M}(\sup_{d\in D}(d,m))\varphi(m)\\
&= \sum_{m\in M}\sup_{d\in D}(d,m)\varphi(m)\\
&= \sup_{d\in D}\sum_{m\in M}(d,m)\varphi(m)\\
&= \sup\{\varphi^\sharp(d) : d\in D\}\\
&= \sup\varphi^\sharp(D).
\end{aligned}$$

□

Theorem 2.4 can be generalized in the same way.

In the sequel, we shall make use of some basic facts about least fixed points of continuous functions that we review next.

A *complete partially ordered set*, or *cpo*, for short, is a partially ordered set P which has a bottom element, usually denoted $\perp$, such that $\sup D$ exists for each directed set $D \subseteq P$. Note that continuous monoids and continuous semirings are cpo's. When P and Q are cpo's, a function $f : P \to Q$ is called continuous if f preserves the least upper bound of directed sets (see also above). It is clear that any composition of continuous functions is continuous, and any direct product of cpo's is a cpo equipped with the pointwise order. Moreover, when P, Q are cpo's, the set of continuous functions $P \to Q$ equipped with the pointwise order is also a cpo.

Suppose that P and $Q_i,\ i \in I$, are cpo's and let $\prod_{i\in I} Q_i$ denote the direct product of the Q_i. Then for any $j \in I$, the jth projection function $\prod_{i\in I} Q_i \to Q_j$ is continuous. Moreover, a function $f : P \to \prod_{i\in I} Q_i$ is continuous iff each "component function" $f_i : P \to Q_i$ is continuous. And when I is finite, say $I = \{1,\ldots,n\}$, then a function $f : \prod_{i\in I} Q_i \to P$ is continuous iff it is continuous separately in each argument, i.e., when

$$f(a_1,\ldots,\sup D,\ldots,a_n) = \sup f(a_1,\ldots,D,\ldots,a_n)$$

holds for each $1 \le i \le n$, $a_j \in Q_j$, $j \ne i$, and for each directed set $D \subseteq Q_i$. In the sequel, we will use these facts without any further mention.

Due to a well-known fixed point theorem, that we recall now, cpo's and continuous functions have been used widely to give semantics to recursive definitions, see, e.g., Bloom, Ésik [10], Guessarian [61].

Theorem 2.2.9 *Suppose that P and Q are cpo's and f is a continuous function $P \times Q \to P$. Then for each $q \in Q$ there is a least $p \in P$ with $p = f(p,q)$, called* *the* least fixed point of f with respect to the parameter q. *Moreover, the function* $Q \to P$ *that takes* q *to the least fixed point* p *is continuous.*

Proof. For the sake of completeness, we give a proof here. For each $q \in Q$, define $f_q : P \to P$ by $f_q(x) = f(x, q)$, for each $x \in P$. Since the partial order on $P \times Q$ is the pointwise order, f_q is also continuous. Now define $x_0^q = \bot$, the least element of P, and $x_{n+1}^q = f_q(x_n^q)$, for all $n \geq 0$. Since f is monotone, it follows that the elements x_n^q, $n \geq 0$ form a chain, hence $x^q = \sup_{n \geq 0} x_n^q$ exists. Moreover, since f_q is continuous, we have $f(x^q) = f(\sup_{n\geq 0} x_n^q) = \sup_{n\geq 0} f(x_n^q) = \sup_{n \geq 1} x_n^q = x^q$, so that x^q is a fixed point of f_q. Also, if $f_q(y) \leq y$, then it follows by induction on n that $x_n^q \leq y$, for all n. Thus, x^q is the least fixed point of f_q and thus the least $y \in P$ with $y = f(y, q)$. Let $f^\dagger$ denote the function $q \mapsto \sup_{n\geq 0} x_n^q$. Then, for every directed set $D \subseteq Q$ and each $n \geq 0$, $\sup_{d \in D,\ n \geq 0} x_n^d$ exists and

$$\sup_{n\geq 0} x_n^{\sup D} = \sup_{d\in D,\ n\geq 0} x_n^d = \sup_{d\in D} \sup_{n \geq 0} x_n^d.$$

This follows by noting that for each n, $x_n^{\sup D} = \sup_{d\in D} x_n^d$. Thus,

$$\begin{aligned} f^\dagger(\sup D) &= \sup_{n\geq 0} x_n^{\sup D} \\ &= \sup_{d\in D} \sup_{n\geq 0} x_n^d \\ &= \sup_{d \in D} f^\dagger(d), \end{aligned}$$

proving that $f^\dagger$ is continuous.

Note that by the above proof, $f^\dagger(q)$ is also the least pre-fixed point of the function $p \mapsto f(p, q)$, $p \in P$. □

In Bloom, Ésik [10], the function $Q \to P$ arising from Theorem 2.9 that provides the parameterized least fixed point for a given continuous function $f : P \times Q \to P$ is denoted $f^\dagger$. Here, we will mainly use the notation $\mu x.f(x, y)$ or, for $f : P \to P$, also $\mathrm{fix}(f)$.

We now recall three very important elementary facts about least fixed points of continuous functions. Theorem 2.10 is independently due to Bekić [3] and De Bakker, Scott [27]. For Proposition 2.11, see also Niwiński [98].

Theorem 2.2.10 *Suppose that* $f : P \times Q \times R \to P$ *and* $g : P \times Q \times R \to Q$ *are continuous functions, where* P, Q, R *are cpo's. Let* $h : P \times Q \times R \to P \times Q$ *denote the "target pairing" of* f *and* g*, so that* $h(x, y, z) = (f(x, y, z), g(x, y, z))$*. Then*

$$\mu(x, y).h(x, y, z) = (\mu x.f(x, k(x, z), z), k(\mu x.f(x, k(x, z), z), z))$$

where $\mu(x, y).h(x, y, z) : R \to P \times Q$ *and* $k(x, z) = \mu y.g(x, y, z) : P \times R \to Q$.

Proposition 2.2.11 *Suppose that $f : P \times P \times Q \to P$ is a continuous function, where P and Q are cpo's. Then*

$$\mu x.\mu y.f(x, y, z) \quad = \quad \mu x.f(x, x, z).$$

Proposition 2.2.12 *Suppose that $f : P \times Q \to P$ and $g : R \to Q$ are continuous functions, where P, Q, R are all cpo's. Then*

$$\mu x.f(x, g(z)) = h(g(z)),$$

where $h(y) = \mu x.f(x, y)$.

We refer to the equation in Theorem 2.10 as the *Bekić-De Bakker-Scott rule.* The equation in Proposition 2.11 is usually referred to as the *diagonal equation*, or the *double iteration equation.* In the terminology of Bloom, Ésik [10], Proposition 2.12 asserts that the *parameter identity* holds.

The above results describe three equational properties of the least fixed point operation on continuous functions. For a complete description, we refer the reader to Bloom, Ésik [10], Ésik [36]. Least fixed points of continuous functions on cpo's are also least pre-fixed points. In Ésik [35], it is shown that the equational properties of the least fixed point operation on continuous functions on cpo's are exactly the same as those of the least pre-fixed point operation on order preserving functions on partially ordered sets in general.

2.3 Algebraic systems

In this section we introduce semiring-polynomials and consider algebraic systems over continuous semirings as a generalization of the context-free grammars. We show that least solutions of these algebraic systems exist. The components of the least solutions of algebraic systems are a generalization of the context-free languages. Our development of the theory concerning algebraic systems parallels that of Eilenberg [30] and uses fixed point theory.

In the sequel, A denotes a *continuous* semiring and $Y = \{y_1, \ldots, y_n\}$ denotes a finite set of *variables.* We denote by $A(Y)$ the *polynomial semiring over the semiring A in the set of variables Y* (see Lausch, Nöbauer [90], Chapter 1.4). To distinguish the polynomials in $A(Y)$ from the polynomials in $A\langle\Sigma^*\rangle$, we call them *semiring-polynomials.*

Each semiring-polynomial has a representation as follows. A *product term t* has the form

$$t(y_1, \ldots, y_n) = a_0 y_{i_1} a_1 \ldots a_{k-1} y_{i_k} a_k, \quad k \geq 0,$$

where $a_j \in A$ and $y_{i_j} \in Y$. The elements a_j are referred to as *coefficients* of the product term. Observe that for $k = 0$ we have $t(y_1, \ldots, y_n) = a_0$. If $k \geq 1$, we do not write down coefficients that are equal to 1; e. g., $y_1 y_2$ stands for

$1 \cdot y_1 \cdot 1 \cdot y_2 \cdot 1$. Each semiring-polynomial p has a *representation* as a finite sum of product terms t_j, i. e.,

$$p(y_1, \ldots, y_n) = \sum_{1 \le j \le m} t_j(y_1, \ldots, y_n).$$

The coefficients of all the product terms t_j, $1 \le j \le m$, are referred to as the *coefficients* of the semiring-polynomial p. For a non-empty subset A' of A we denote the collection of all semiring-polynomials with coefficients in A' by $A'(Y)$.

If the basic semiring is given by $A\langle\langle \Sigma^* \rangle\rangle$ then $A\langle (\Sigma \cup Y)^* \rangle$, the set of polynomials over $\Sigma \cup Y$, can be regarded as a subset of the set $A'(Y)$ of semiring-polynomials, where $A' = \{aw \mid a \in A,\ w \in \Sigma^*\}$.

We are not interested in the algebraic properties of $A(Y)$, but only in the mappings induced by semiring-polynomials. These mappings are called *polynomial functions on* A (see Lausch, Nöbauer [90], Chapter 1.6).

Each product term t (resp. semiring-polynomial p) with variables $y_1, \ldots, y_n$ induces a mapping $\bar{t}$ (resp. $\overline{p}$) from A^n into A. For a product term t represented as above, the mapping $\bar{t}$ is defined by

$$\bar{t}(\sigma_1, \ldots, \sigma_n) = a_0 \sigma_{i_1} a_1 \ldots a_{k-1} \sigma_{i_k} a_k\,;$$

for a semiring-polynomial p, represented by a finite sum of product terms t_j as above, the mapping $\overline{p}$ is defined by

$$\overline{p}(\sigma_1, \ldots, \sigma_n) = \sum_{1 \le j \le m} \bar{t}_j(\sigma_1, \ldots, \sigma_n)$$

for all $(\sigma_1, \ldots, \sigma_n) \in A^n$.

In the sequel we denote the mapping $\overline{p}$ induced by p also by p. This should not lead to any confusion.

We now define the basic notions concerning algebraic systems. Let A' be a non-empty subset of A. An *A'-algebraic system* (with variables in $Y = \{y_1, \ldots, y_n\}$) is a system of equations

$$y_i = p_i, \quad 1 \le i \le n,$$

where each p_i is a semiring-polynomial in $A'(Y)$. A *solution* to the A'-algebraic system $y_i = p_i$, $1 \le i \le n$, is given by $(\sigma_1, \ldots, \sigma_n) \in A^n$ such that

$$\sigma_i = p_i(\sigma_1, \ldots, \sigma_n), \quad 1 \le i \le n.$$

A solution $(\sigma_1, \ldots, \sigma_n)$ of the A'-algebraic system $y_i = p_i$, $1 \le i \le n$, is termed a *least solution* iff

$$\sigma_i \le \tau_i, \quad 1 \le i \le n,$$

for all solutions $(\tau_1, \ldots, \tau_n)$ of $y_i = p_i$, $1 \le i \le n$.

Often it is convenient to write the A'-algebraic system $y_i = p_i$, $1 \le i \le n$, in matrix notation. Defining the two column vectors

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix},$$

we can write our A'-algebraic system in the matrix notation

$$y = p(y) \quad \text{or} \quad y = p.$$

A *solution* to $y = p(y)$ is now given by $\sigma \in A^n$ such that $\sigma = p(\sigma)$. A solution σ of $y = p$ is termed a *least solution* iff $\sigma \le \tau$ for all solutions τ of $y = p$.

One of our main results in this section will be that an A'-algebraic system has a unique least solution.

Since addition and multiplication in a continuous semiring are continuous functions, we have:

Proposition 2.3.1 *Let A be a continuous semiring and let p be a semiring-polynomial in $A(Y)$. Then the mapping $p : A^n \to A$ is continuous.*

Let now $p \in A(Y)^{n \times 1}$, i. e., p is a column vector of semiring-polynomials. Then p induces a mapping $p : A^n \to A^n$ by $(p(a_1, \ldots, a_n))_i = p_i(a_1, \ldots, a_n)$, $1 \le i \le n$, i. e., the i-th component of the value of p at $(a_1, \ldots, a_n) \in A^n$ is given by the value of the i-th component p_i of p at $(a_1, \ldots, a_n)$.

The next corollary follows by the observation that any target tupling of continuous functions is continuous.

Corollary 2.3.2 *Let A be a continuous semiring and let $p \in A(Y)^{n \times 1}$. Then the mapping $p : A^n \to A^n$ is continuous.*

Consider now an A'-algebraic system $y = p$. The least fixpoint of the mapping p is nothing else than the least solution of $y = p$.

Theorem 2.3.3 *Let A be a continuous semiring and A' be a non-empty subset of A. Then the least solution of an A'-algebraic system $y = p$ exists in A^n and equals*

$$\mathrm{fix}(p) = \sup(p^i(0) \mid i \in \mathbb{N}).$$

Proof. By Theorem 2.9. $\square$

Theorem 2.9 indicates how we can compute an approximation to the least solution of an A'-algebraic system $y = p$. The *approximation sequence* σ^0, σ^1, σ^2,…,σ^j,…, where each $\sigma^j \in A^{n \times 1}$, *associated to* an A'-algebraic system $y = p(y)$ is defined as follows:

$$\sigma^0 = 0, \quad \sigma^{j+1} = p(\sigma^j),\ j \in \mathbb{N}.$$

Clearly, $(\sigma^j \mid j \in \mathbb{N})$ is a chain and $\mathrm{fix}(p) = \sup(\sigma^j \mid j \in \mathbb{N})$, i. e., we obtain the least solution of $y = p$ by computing the least upper bound of the approximation sequence associated to it.

The collection of the components of the least solutions of all A'-algebraic systems, where A' is a fixed subset of A, is denoted by $\mathfrak{Alg}(A')$. In the sequel, A' denotes always a subset of A containing 0 and 1. But observe that most of the definitions and some of the results involving a subset A' of A are valid without this restriction as well, i. e., are valid for arbitrary subsets A' of A.

An A'-algebraic system $y_i = p_i$, $1 \le i \le n$, is called *finite A'-linear system* if $p_i = \sum_{1 \le j \le n} M_{ij} y_j + R_i$, where $M_{ij}, R_i \in A'$. Let $M \in A'^{n \times n}$ (resp. $R \in A'^{n \times 1}$) be the matrix (resp. column vector) with entries M_{ij} (resp. R_i). Then such a finite A'-linear system can be written in matrix notation as $y = My + R$. The approximation sequence $\sigma^0, \sigma^1, \sigma^2, \ldots, \sigma^j, \ldots$, where each $\sigma^j \in A^{n \times 1}$, associated to the finite A'-linear system $y = My + R$ is given by:

$$\sigma^0 = 0, \qquad\qquad \sigma^{j+1} = \sum_{0 \le i \le j} M^i R, \quad j \ge 0\,.$$

Hence, we have proved the following result.

Theorem 2.3.4 *Let A be a continuous semiring and A' be a non-empty subset of A. Then M^*R is the least solution of the finite A'-linear system $y = My + R$, where $M \in A'^{n \times n}$ and $R \in A'^{n \times 1}$.*

Corollary 2.3.5 *Let $a \in A$. Then a is a component of the least solution of a finite A'-linear system iff $a \in \mathfrak{Rec}(A')$.*

Proof. By Theorem 1.3.1. □

Example 2.3.1. Let A be a continuous semiring and $a, b \in A$. We consider the A-linear system $y = (a + b)y + 1$ and compute its unique least solution by two different methods.

(i) By Theorem 3.4, the least solution of $y = (a+b)y+1$ is given by $(a+b)^*$.

(ii) We compute the least solution of $y = ay+by+1$ by the diagonal equation, Proposition 2.12: We first compute the least solution of $y = ay + bz + 1$. It is $a^*(bz+1)$. Then we compute the least solution of $z = a^*bz + a^*$. It is $(a^*b)^*a^*$. By the diagonal equation, $(a^*b)^*a^*$ is then the least solution of $y = (a+b)y+1$.

(iii) Since the least solution is unique, we obtain the sum-star-equation

$$(a + b)^* = (a^*b)^*a^*\,.$$

Formally, we obtain

$$\begin{aligned} &(a + b)^* = \mu y.((a + b)y + 1) = \\ &\mu z.\mu y.(ay + bz + 1) = \\ &\mu z.a^*(bz + 1) = (a^*b)^*a^*\,. \end{aligned}$$

□

Example 2.3.2. Let A be a continuous semiring and $a, b \in A$. We consider the A-linear system $y_1 = by_2$, $y_2 = ay_1 + 1$ and compute its unique solution by two different methods.

(i) We solve $y_1 = by_2$. Its least solution is $\tau_1(y_2) = by_2$. Then we substitute $\tau_1(y_2)$ into the second equation and obtain the equation $y_2 = aby_2 + 1$. Its least solution is $\tau_2 = (ab)^*$. Hence, by the Bekić-De Bakker-Scott rule, Theorem 2.10, the least solution of our original system is given by $(b(ab)^*, (ab)^*)$.

(ii) We solve $y_2 = ay_1 + 1$. Its least solution is $\tau_2(y_1) = ay_1 + 1$. Then we substitute $\tau_2(y_1)$ into the first equation and obtain the equation $y_1 = bay_1 + b$. Its least solution is $\tau_1 = (ba)^*b$. Hence, by the Bekić-De Bakker-Scott rule, the least solution of our original system is given by $((ba)^*b, a(ba)^*b + 1)$.

(iii) Since the least solution is unique, we obtain the equality

$$b(ab)^* = (ba)^*b$$

and the product-star-equation

$$(ab)^* = a(ba)^*b + 1\,.$$

Observe that, for $a = 1$, we obtain the equalities

$$bb^* = b^*b$$

and

$$b^* = b^*b + 1\,.$$

We now formalize the computations. The Bekić-De Bakker-Scott rule, Theorem 2.10, where R is a singleton, reads

$$\mu(x, y).(f(x, y), g(x, y)) = (\mu x.f(x, \mu y.g(x, y)), \mu y.g(\mu x.f(x, \mu y.g(x, y)), y)\,.$$

We now substitute y_1 for x, y_2 for y, by_2 for $f(y_1, y_2)$ and $ay_1 + 1$ for $g(y_1, y_2)$ and obtain

$$\begin{aligned}
&\mu(y_1, y_2).(f(y_1, y_2), g(y_1, y_2)) = \\
&(\mu y_1.b\mu y_2.(ay_1 + 1), \mu y_2.(a\mu y_1.b\mu y_2.(ay_1 + 1) + 1) = \\
&(\mu y_1.(bay_1 + b), \mu y_2.(a\mu y_1.(bay_1 + b) + 1)) = \\
&((ba)^*b, \mu y_2.(a(ba)^*b + 1)) = \\
&((ba)^*b, a(ba)^*b + 1)\,.
\end{aligned}$$

Another formulation of the Bekić-De Bakker-Scott rule is obtained by symmetry:

$$\mu(x, y).(f(x, y), g(x, y)) = (\mu x.f(x, \mu y.g(\mu x.f(x, y), y)), \mu y.g(\mu x.f(x, y), y))\,.$$

We obtain, by the same substitution as above,

$$\begin{aligned}
&\mu(y_1, y_2).(f(y_1, y_2), g(y_1, y_2)) = \\
&(\mu y_1.b\mu y_2.(a\mu y_1.by_2 + 1), \mu y_2.(a\mu y_1.by_2 + 1)) = \\
&(\mu y_1.b\mu y_2.(aby_2 + 1), \mu y_2.(aby_2 + 1)) = \\
&(\mu y_1.b(ab)^*, (ab)^*) = \\
&(b(ab)^*, (ab)^*)\,.
\end{aligned}$$

Hence,

$$((ba)^*b, a(ba)^*b+1) = (b(ab)^*, (ab)^*)\,.$$

□

Example 2.3.3. Let A be a continuous semiring. Then, for $n \geq 1$, $A^{n\times n}$ is again a continuous semiring. We consider the $A^{n\times n}$-linear system $y = My + E$ and compute its unique solution by two different methods.

(i) The least solution of $y = My + E$ is given by M^*.

(ii) We partition M into blocks $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \in A^{n_1\times n_1}$, $b \in A^{n_1\times n_2}$, $c \in A^{n_2\times n_1}$, $d \in A^{n_2\times n_2}$, $n_1 + n_2 = n$, and write the equation $y = My + E$ in the form

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Computation of the right side of this matrix equation yields the two systems

$$\begin{aligned} y_{11} &= ay_{11} + by_{21} + 1 \\ y_{21} &= cy_{11} + dy_{21} \end{aligned} \qquad \text{and} \qquad \begin{aligned} y_{12} &= ay_{12} + by_{22} \\ y_{22} &= cy_{12} + dy_{22} + 1 \end{aligned}$$

We now solve the first system and apply the Bekić-De Bakker-Scott rule: We solve $y_{21} = cy_{11} + dy_{21}$. Its least solution is $\tau_{21}(y_{11}) = d^*cy_{11}$. Then we substitute $\tau_{21}(y_{11})$ into the first equation and obtain the equation $y_{11} = ay_{11} + bd^*cy_{11} + 1$. Its least solution is $\tau_{11} = (a + bd^*c)^*$. We now substitute τ_{11} into the second equation $y_{21} = c(a+bd^*c)^* + dy_{21}$. Its least solution is $\tau_{21} = d^*c(a+bd^*c)^*$. Applying the sum-star-equation and the equality $f(gf)^* = (fg)^*f$ of Example 3.2 (iii) yields

$$\tau_{21} = d^*c(a^*bd^*c)^*a^* = (d^*ca^*b)^*d^*ca^* = (d + ca^*b)^*ca^*\,.$$

Hence, $\alpha = (a + bd^*c)^*$, $\gamma = (d + ca^*b)^*ca^*$ is the least solution of the first system. By symmetry, the least solution of the second system is given by $\beta = (a + bd^*c)^*bd^*$, $\delta = (d + ca^*b)^*$.

(iii) Since the least solution is unique, we obtain the matrix-star-equation

$$M^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

(iv) The computations in (ii) can be formalized by substituting into the Bekić-De Bakker-Scott rule, Theorem 2.10, as given in Example 3.2 (iii), y_{11} for x, y_{21} for y, $ay_{11} + by_{21} + 1$ for $f(y_{11}, y_{21})$ and $cy_{11} + dy_{21}$ for $g(y_{11}, y_{21})$:

$$\begin{aligned}
&\mu(y_{11}, y_{21}).(f(y_{11}, y_{21}), g(y_{11}, y_{21})) = \\
&(\mu y_{11}.(ay_{11} + b\mu y_{21}.(cy_{11} + dy_{21}) + 1), \\
&\mu y_{21}.(c\mu y_{11}.(ay_{11} + by_{21}.(cy_{11} + dy_{21}) + 1) + dy_{21})) = \\
&(\mu y_{11}.(ay_{11} + bd^*cy_{11} + 1), \mu y_{21}.(c\mu y_{11}.(ay_{11} + bd^*cy_{11} + 1) + dy_{21})) = \\
&((a + bd^*c)^*, \mu y_{21}.(c(a + bd^*c)^* + dy_{21})) = \\
&((a + bd^*c)^*, d^*c(a + bd^*c)^*) = \\
&((a + bd^*c)^*, (d + ca^*b)^*ca^*)\,.
\end{aligned}$$

□

We are now ready to discuss the connection between algebraic systems and context-free grammars.

Consider a context-free grammar $G = (Y, \Sigma, P, y_1)$. Here $Y = \{y_1, \ldots, y_n\}$ is the set of variables or nonterminal symbols, Σ is the set of terminal symbols, P is the set of productions and y_1 is the initial variable. The language generated by G is denoted by $L(G)$. Changing the initial variable yields the context-free grammars $G_i = (Y, \Sigma, P, y_i)$, $1 \le i \le n$, and the context-free languages $L(G_i)$ generated by them. Clearly, $L(G) = L(G_1)$. We now assume that the basic semiring is given by 2^{Σ^*}. We define a $\{\{w\} \mid w \in \Sigma^*\}$-algebraic system $y_i = p_i$, $1 \le i \le n$, whose least solution is $(L(G_1), \ldots, L(G_n))$:

$$p_i = \bigcup_{y_i \to \gamma \in P} \{\gamma\}.$$

Whenever we speak of a context-free grammar corresponding to a $\{\{w\} \mid w \in \Sigma^*\}$-algebraic system, or vice versa, then we mean the correspondence in the sense of the above definition. The next theorem is due to Ginsburg, Rice [54]. (See also Salomaa, Soittola [107], Theorem IV.1.2 and Moll, Arbib, Kfoury [96], Chapter 6.)

Theorem 2.3.6 (Ginsburg, Rice [54], Theorem 2) *Assume that $G = (Y, \Sigma, P, y_1)$ is a context-free grammar and $y_i = p_i$, $1 \le i \le n$, is the corresponding $\{\{w\} \mid w \in \Sigma^*\}$-algebraic system with least solution $(\sigma_1, \ldots, \sigma_n)$. Let $G_i = (Y, \Sigma, P, y_i)$, $1 \le i \le n$. Then*

$$\sigma_i = L(G_i), \quad 1 \le i \le n.$$

Corollary 2.3.7 *A formal language over Σ is context-free iff it is in $\mathfrak{Alg}(\{\{w\} \mid w \in \Sigma^*\})$.*

We now consider the case where the basic semiring is given by $A\langle\langle\Sigma^*\rangle\rangle$, and A is a commutative continuous semiring. Let $A' = \{aw \mid a \in A,\ w \in \Sigma^*\}$. Then $\mathfrak{Alg}(A')$ is equal to the collection of the components of the least solutions of A'-algebraic systems $y_i = p_i$, $1 \le i \le n$, where p_i is a polynomial in $A\langle(\Sigma \cup Y)^*\rangle$. This is due to the commutativity of A: any polynomial function on $A\langle\langle\Sigma^*\rangle\rangle$ that is induced by a semiring-polynomial of $A'(Y)$ is also induced by a polynomial of $A\langle(\Sigma \cup Y)^*\rangle$. In this case, $\mathfrak{Alg}(A')$ is usually denoted by $A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$. The power series in $A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$ are called *algebraic power series*. Whenever we speak of an *algebraic system* $y_i = p_i$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, in connection with the basic semiring $A\langle\langle\Sigma^*\rangle\rangle$, then we assume that A is *commutative* and mean an A'-algebraic system as described above.

We generalize the connection between algebraic systems and context-free grammars as discussed above. Define, for a given context-free grammar $G = (Y, \Sigma, P, y_1)$, the algebraic system $y_i = p_i$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, by

$$(p_i, \gamma) = 1 \text{ if } y_i \to \gamma \in P \quad \text{and} \quad (p_i, \gamma) = 0, \text{ otherwise.}$$

Conversely, given an algebraic system $y_i = p_i$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, define the context-free grammar $G = (Y, \Sigma, P, y_1)$ by

$$y_i \to \gamma \in P \quad \text{iff} \quad (p_i, \gamma) \neq 0.$$

Whenever we speak of a context-free grammar corresponding to an algebraic system $y_i = p_i$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, or vice versa, then we mean the correspondence in the sense of the above definition. If attention is restricted to algebraic systems with coefficients 0 and 1 then this correspondence is one-to-one. The correspondence between context-free grammars and algebraic systems $y_i = p_i$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, is a generalization of the correspondence between context-free grammars and $\{\{w\} \mid w \in \Sigma^*\}$-algebraic systems defined earlier. This is seen by taking in account the isomorphism between the semirings 2^{Σ^*} and $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$. The next theorem is due to Chomsky, Schützenberger [22].

Theorem 2.3.8 (Salomaa, Soittola [107], Theorem IV.1.5) *Assume that $G = (Y, \Sigma, P, y_1)$ is a context-free grammar and $y_i = p_i$, $p_i \in \mathbb{N}^\infty\langle(\Sigma \cup Y)^*\rangle$, $1 \le i \le n$, is the corresponding algebraic system with least solution $(\sigma_1, \ldots, \sigma_n)$. Denote by $d_i(w)$ the number (possibly ∞) of distinct leftmost derivations of w from the variable y_i, $1 \le i \le n$, $w \in \Sigma^*$. Then*

$$\sigma_i = \sum_{w \in \Sigma^*} d_i(w) w, \quad 1 \le i \le n.$$

Corollary 2.3.9 *Under the assumptions of Theorem 3.8, G is unambiguous iff, for all $w \in \Sigma^*$,*

$$(\sigma_1, w) \le 1$$

Example 2.3.4. (See Chomsky, Schützenberger [22], Kuich [74].) Consider the context-free grammar $G = (\{y\}, \{x\}, \{y \to y^2, y \to x\}, y)$. If the basic semiring is $2^{\{x\}^*}$, the corresponding algebraic system is given by $y = y^2 \cup \{x\}$. The j-th element of the approximation sequence is $\{x^{2^{j-1}}, x^{2^{j-1}-1}, \ldots, x\}$, $j \ge 1$. Hence, $\{x\}^+$ is the least solution of $y = y^2 \cup \{x\}$. Observe that $\{x\}^*$ is also a solution.

If the basic semiring is $\mathbb{N}^\infty\langle\langle\{x\}^*\rangle\rangle$, the corresponding algebraic system is given by $y = y^2 + x$. The first elements of the approximation sequence are $\sigma^0 = 0$, $\sigma^1 = x$, $\sigma^2 = x^2 + x$, $\sigma^3 = x^4 + 2x^3 + x^2 + x$. It can be shown that

$$\sum_{n \ge 0} C_n x^{n+1}, \quad \text{where } C_n = \frac{(2n)!}{n!(n+1)!},\ n \ge 0,$$

is the least solution of $y = y^2 + x$. This means that x^{n+1} has C_n distinct leftmost derivations with respect to G. $\square$

In a continuous semiring A, the three operations $+$, $\cdot$, * are called the *rational operations*. A subsemiring of A is called *rationally closed* iff it is closed under the rational operations.

Theorem 2.3.10 $\langle \mathfrak{Alg}(A'), +, \cdot, 0, 1\rangle$ *is a rationally closed semiring.*

Proof. Let a and a' be in $\mathfrak{Alg}(A')$. Then there exist A'-algebraic systems $y_i = p_i$, $1 \le i \le n$, and $y'_j = p'_j$, $1 \le j \le m$, such that a and a', respectively, are the first components of their least solutions. Assume that the sets of variables are disjoint and consider the A'-algebraic systems

$$\text{(i)}\ \begin{aligned} y_0 &= y_1 + y'_1 \\ y_i &= p_i \\ y'_j &= p'_j \end{aligned} \qquad \text{(ii)}\ \begin{aligned} y_0 &= y_1 y'_1 \\ y_i &= p_i \\ y'_j &= p'_j \end{aligned} \qquad \text{(iii)}\ \begin{aligned} y_0 &= y_1 y_0 + 1 \\ y_i &= p_i \end{aligned}$$

where $1 \le i \le n$, $1 \le j \le m$. Let σ and σ' be the least solutions of $y_i = p_i$, $1 \le i \le n$, and $y'_j = p'_j$, $1 \le j \le m$, respectively. It follows from the Bekić-De Bakker-Scott rule that $(a + a', \sigma, \sigma')$, (aa', σ, σ') and (a^*, σ) are the least solutions of the A'-algebraic systems (i), (ii) and (iii), respectively. □

Corollary 2.3.11 *The family of context-free languages over Σ is closed under the rational operations.*

According to our convention, the continuous semiring A is commutative in the next corollary.

Corollary 2.3.12 $\langle A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle, +, \cdot, 0, \varepsilon \rangle$ *is a rationally closed semiring.*

Our next result shows that $\mathfrak{Alg}$ is an idempotent operator (Berstel [4], Wechler [115]).

Theorem 2.3.13 $\mathfrak{Alg}(\mathfrak{Alg}(A')) = \mathfrak{Alg}(A')$.

Proof. Let $y_i = p_i$, $1 \le i \le n$, be an $\mathfrak{Alg}(A')$-algebraic system with least solution σ. Consider the coefficients a of the semiring-polynomials p_i, $1 \le i \le n$, where $a \in \mathfrak{Alg}(A')$ and $a \notin A'$. For each of these coefficients a there exists an A'-algebraic system $z^a_j = q^a_j$ with least solution τ^a whose first component is equal to a. Perform now the following procedure on the $\mathfrak{Alg}(A')$-algebraic system $y_i = p_i$, $1 \le i \le n$: each coefficient a, $a \in \mathfrak{Alg}(A')$, $a \notin A'$, in p_i is replaced by the variable z^a_1 and the equations $z^a_j = q^a_j$ are added to the system for all these a.

Using the Bekić-De Bakker-Scott rule, it follows that the newly constructed system is an A'-algebraic system whose least solution is given by σ and all the τ^a. Hence, the components of σ are in $\mathfrak{Alg}(A')$. □

In the last part of this section we will obtain similar results for $\mathfrak{Rat}(A')$.

In language theory, a formula telling how a given regular language is obtained from the languages $\{x\}$, $x \in \Sigma$, and $\emptyset$, by the rational operations $+, \cdot, ^*$, is referred to as a regular expression (see Salomaa [106]).

Analogously to these regular expressions we define now rational expressions. Assume that A, Σ and $U = \{+, \cdot, ^*, [,]\}$ are pairwise disjoint. A word E over $A \cup \Sigma \cup U$ is a *rational expression* over (A, Σ) if

(i) E is a symbol of $A \cup \Sigma$, or

(ii) E is of one of the forms $[E_1 + E_2]$, $[E_1 \cdot E_2]$ or $[E_1^*]$, where E_1 and E_2 are rational expressions over (A, Σ).

Each rational expression E over (A, Σ) *denotes* a power series $|E| \in A\langle\langle\Sigma^*\rangle\rangle$ according to the following conventions.

(i) If $E = a \in A$ then $|E| = a\varepsilon$; if $E = x \in \Sigma$ then $|E| = x$.

(ii) For rational expressions E_1 and E_2 we define $|[E_1 + E_2]| = |E_1| + |E_2|$, $|[E_1 \cdot E_2]| = |E_1| \cdot |E_2|$, $|[E_1^*]| = |E_1|^*$.

Corollary 2.3.14 *Let $r \in A\langle\langle\Sigma^*\rangle\rangle$. Then the following statements are equivalent:*

(i) r is a component of the least solution of a finite $A\langle\Sigma \cup \varepsilon\rangle$-linear system,

(ii) $r \in \{||\mathfrak{A}|| \mid \mathfrak{A}$ is a finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton$\}$,

(iii) $r \in \{||\mathfrak{A}|| \mid \mathfrak{A}$ is a finite automaton over A and Σ without ε-moves$\}$,

(iv) r is denoted by a rational expression over (A, Σ).

Proof. (iv) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is implied by Corollary 1.3.4, (i) $\Leftrightarrow$ (ii) follows by Corollary 3.5. $\square$

If $r \in A\langle\langle\Sigma^*\rangle\rangle$ satisfies one condition (and, hence, all conditions) of Corollary 3.14, it is called *rational power series* (over Σ). The collection of all rational power series over Σ is denoted by $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. Clearly, $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = \mathfrak{Rec}(A\langle\Sigma \cup \varepsilon\rangle)$.

The definitions of $\mathfrak{Rat}(A')$, $A' \subseteq A$, and $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ imply the following two corollaries.

Corollary 2.3.15 *$\langle\mathfrak{Rat}(A'), +, \cdot, 0, 1\rangle$ and $\langle A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, \varepsilon\rangle$ are rationally closed semirings.*

Corollary 2.3.16 $\mathfrak{Rat}(\mathfrak{Rat}(A')) = \mathfrak{Rat}(A')$.

This trivial corollary to the definition of $\mathfrak{Rat}(A')$ has a nontrivial corollary.

Corollary 2.3.17 *If $\mathfrak{A}$ is a $\mathfrak{Rat}(A')$-finite automaton then $||\mathfrak{A}||$ is in $\mathfrak{Rat}(A')$. If $\mathfrak{A}$ is a $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$-finite automaton then $||\mathfrak{A}||$ is in $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.*

Proof. We obtain $\mathfrak{Rat}(A') = \mathfrak{Rec}(\mathfrak{Rat}(A'))$ by Theorem 1.3.2. This proves the first statement of our corollary. The second statement follows from the first one by $\mathfrak{Rat}(A\langle\Sigma \cup \varepsilon\rangle = A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. $\square$

Let 2^{Σ^*} be the basic semiring and consider a finite $2^{\Sigma \cup \{\varepsilon\}}$-linear system $y = My + R$, where $M \in (2^{\Sigma \cup \{\varepsilon\}})^{n \times n}$ and $R \in (2^{\Sigma \cup \{\varepsilon\}})^{n \times 1}$. Then the corresponding context-free grammar is in fact an extended regular grammar. Vice versa, the algebraic system corresponding to a regular grammar is a finite $2^{\Sigma \cup \{\varepsilon\}}$-linear system.

A rational expression over $(\mathbb{B}, \Sigma)$ is called *regular expression* over Σ. A regular expression denotes, via the isomorphism between $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ and 2^{Σ^*}, a regular language over Σ.

Theorem 2.3.18 *Let L be a formal language over an alphabet Σ. Then the following statements are equivalent:*

(i) L is a component of the least solution of a finite $2^{\Sigma\cup\{\varepsilon\}}$-linear system,

(ii) L is a regular language,

(iii) L is the behavior of a finite automaton (in the classical sense),

(iv) L is denoted by a regular expression.

Proof. By the above considerations and Corollary 3.14. □

2.4 Normal forms for algebraic systems

In this section, we will show that elements of $\mathfrak{Alg}(A')$ can be defined by A'-algebraic systems that are "simple" in a well-defined sense. In other words, we will exhibit a number of *normal forms* that correspond to well-known normal forms in language theory, e. g. the Chomsky normal form, the operator normal form and the Greibach normal form. Apart from the beginning of this section, we will only consider power series in $A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$.

An A'-algebraic system is in the *canonical two form* (see Harrison [63]) iff its equations have the form

$$y_i = \sum_{1\le k,m\le n} a^i_{km} y_k y_m + \sum_{1\le k\le n} a^i_k y_k + a_i, \quad 1 \le i \le n,$$

where $a^i_{km}, a^i_k \in \{0,1\}$ and $a_i \in A'$.

Consider an A'-algebraic system $y_i = p_i$, $1 \le i \le n$, whose least solution is given by σ. Perform the following procedure on the product terms $a_0 y_{i_1} a_1 \dots a_{k-1} y_{i_k} a_k$, $k \ge 1$, of the semiring-polynomials p_i, $1 \le i \le n$: replace each coefficient $a_j \ne 1$ by a new variable z and add an additional equation $z = a_j$; shorten now each product term $z_1 z_2 \dots z_k$, $k > 2$, to $z_1 u_1$ and add additional equations $u_1 = z_2 u_2, \dots, u_{k-2} = z_{k-1} z_k$, where $u_1, \dots, u_{k-2}$ are new variables. Then, by the Bekić-De Bakker-Scott rule, the components σ_i, $1 \le i \le n$, of σ are components of the least solution of the newly constructed A'-algebraic system in the canonical two form.

Theorem 2.4.1 *Each $a \in \mathfrak{Alg}(A')$ is a component of the least solution of an A'-algebraic system in the canonical two form.*

We will now consider a very useful transformation of an A'-algebraic system. In the next theorem, we write an A'-algebraic system in the form $y = My + P$, where $M \in {A'}^{n\times n}$ and $P \in A'(Y)^{n\times 1}$. Here the entries of My contain product terms of the form ay_i, $a \in A'$, $y_i \in Y$.

Theorem 2.4.2 *The least solutions of the A'-algebraic system $y = My + P$ and of the $\mathfrak{Rat}(A')$-algebraic system $y = M^*P$, where $M \in {A'}^{n\times n}$ and $P \in A'(Y)^{n\times 1}$, coincide.*

Proof. We use the proof method of Ésik, Leiß [46]. By the diagonal equation, Proposition 2.11, and by Theorem 3.4:

$$\mu y.(My + P(y)) = \mu y.\mu x.(Mx + P(y)) = \mu y.M^* P(y)\,.$$

$\square$

For the remainder of this section, our basic semiring will be $A\langle\langle\Sigma^*\rangle\rangle$, where A is *commutative*, and we will consider algebraic systems $y_i = p_i$, $1 \le i \le n$, where $p_i \in A\langle(\Sigma \cup Y)^*\rangle$.

Corollary 2.4.3 *The least solutions of the algebraic systems $y = My + P$ and $y = M^*P$, where $M \in (A\langle\varepsilon\rangle)^{n\times n}$ and $\mathrm{supp}(P_i) \subseteq (\Sigma \cup Y)^* - Y$, $1 \le i \le n$, coincide.*

Observe that the context-free grammar corresponding to the algebraic system $y = M^*P$ has no *chain rules*, i. e., has no productions of the type $y_i \to y_j$. (Compare with Salomaa [106], Theorem 6.3; Harrison [63], Theorem 4.3.2; Hopcroft, Ullman [65], Theorem 4.4.)

We now consider another useful transformation of an algebraic system. It corresponds to the transformation of a context-free grammar for deleting *ε-rules* (i. e., productions of the type $y_i \to \varepsilon$).

Theorem 2.4.4 *Let $y_i = p_i$, $1 \le i \le n$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, be an algebraic system with least solution σ. Let $\sigma = (\sigma, \varepsilon)\varepsilon + \tau$, where $(\tau, \varepsilon) = 0$. Then there exists an algebraic system $y_i = q_i$, $1 \le i \le n$, $q_i \in A\langle(\Sigma \cup Y)^*\rangle$, $(q_i, \varepsilon) = 0$, whose least solution is τ.*

Proof. Substitute $(\sigma_j, \varepsilon)\varepsilon + y_j$ for y_j, $1 \le j \le n$, into p_i and define

$$q_i(y) = \sum_{\alpha \in (\Sigma\cup Y)^+} (p_i((\sigma, \varepsilon)\varepsilon + y), \alpha)\alpha, \quad 1 \le i \le n.$$

The equalities

$$p_i((\sigma, \varepsilon)\varepsilon + \tau) = p_i(\sigma) = \sigma = (\sigma, \varepsilon)\varepsilon + \tau, \quad 1 \le i \le n,$$

imply, by comparing coefficients,

$$q_i(\tau) = \tau, \quad 1 \le i \le n.$$

Hence, τ is a solution of the algebraic system $y_i = q_i$, $1 \le i \le n$. Consider now an arbitrary solution τ' of $y_i = q_i$, $1 \le i \le n$. Then $\sigma' = (\sigma, \varepsilon)\varepsilon + \tau'$ is a solution of $y_i = p_i$, $1 \le i \le n$. Since σ is the least solution of $y_i = p_i$, $1 \le i \le n$, we infer that $\sigma \le \sigma'$. But this implies $\tau \le \tau'$. Hence, τ is the least solution of $y_i = q_i$, $1 \le i \le n$. $\square$

Observe that the context-free grammar corresponding to the algebraic system $y_i = q_i$, $1 \le i \le n$, has no ε-rules. (Compare with Salomaa [106], Theorem 6.2; Harrison [63], Theorem 4.3.1; Hopcroft, Ullman [65], Theorem 4.3.)

An algebraic system $y_i = p_i$, $1 \le i \le n$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, is termed *proper* iff $\mathrm{supp}(p_i) \subseteq (\Sigma \cup Y)^+ - Y$ for all $1 \le i \le n$. Proper algebraic systems correspond to context-free grammars without ε-rules and chain rules.

Corollary 2.4.5 *Let $r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$. Then there exists a proper algebraic system such that $\sum_{w \in \Sigma^+}(r, w)w$ is a component of its least solution.*

Proof. Apply the constructions of Theorem 4.4 and Corollary 4.3, in this order. □

Corollary 2.4.6 *For every context-free language L there exists a context-free grammar G without ε-rules and chain rules such that $L(G) = L - \{\varepsilon\}$.*

An algebraic system $y_i = p_i$, $1 \le i \le n$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, is termed *strict* iff $\mathrm{supp}(p_i) \subseteq \{\varepsilon\} \cup (\Sigma \cup Y)^*\Sigma(\Sigma \cup Y)^*$ for all $1 \le i \le n$. For a proof of the next result see Salomaa, Soittola [107], Theorem IV.1.1 and Kuich, Salomaa [88], Theorem 14.11.

Theorem 2.4.7 *Let $y_i = p_i$, $1 \le i \le n$, $p_i \in A\langle(\Sigma \cup Y)^*\rangle$, be an algebraic system with least solution σ.*

If $y_i = p_i$, $1 \le i \le n$, is a proper algebraic system then $(\sigma, \varepsilon) = 0$ and σ is the only solution with this property.

If $y_i = p_i$, $1 \le i \le n$, is a strict algebraic system then σ is its unique solution.

Example 2.4.1. Consider an $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$-algebraic system that can be written in matrix notation in the form $Z = M_1 Z M_2 + M$, where Z is an $n \times n$-matrix of variables and $M_1, M_2, M \in (A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{n \times n}$. Then, by computing the approximation sequence, it is easily seen that $S = \sum_{i \ge 0} {M_1}^i M {M_2}^i \in (A\langle\langle\Sigma^*\rangle\rangle)^{n \times n}$ is the least solution of $Z = M_1 Z M_2 + M$. (See also Berstel [4], Section V.6 and Kuich [75].)

Consider the linear context-free grammar

$$G = (\{y_1, y_2\}, \{a, b\}, \{y_1 \to a y_2, y_2 \to a y_2, y_2 \to a y_2 b, y_2 \to b\}, y_1)$$

and the corresponding strict algebraic system $y_1 = a y_2, y_2 = a y_2 + a y_2 b + b$. Denote its unique solution by $\sigma = (\sigma_1, \sigma_2)$. We infer by Theorem 4.2 that σ is the unique solution of the $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$-algebraic system $y_1 = a y_2, y_2 = a^+ y_2 b + a^* b$. Hence,

$$\sigma_2 = \sum_{n \ge 0} (a^+)^n a^* b^{n+1} = \sum_{n \ge 0} a^* (a^+)^n b^{n+1} \quad \text{and} \quad \sigma_1 = \sum_{n \ge 0} (a^+)^{n+1} b^{n+1}.$$

This implies

$$L(G) = \{a\}^* \bigcup_{n \ge 1} \{a\}^n \{b\}^n = \{a^m b^n \mid m \ge n \ge 1\}.$$

If $A = \mathbb{N}^\infty$, $\sigma_1 = \sum_{m \ge n \ge 1} \binom{m-1}{n-1} a^m b^n$. Hence, by Theorem 3.8, the word $a^m b^n \in L(G)$ has $\binom{m-1}{n-1}$ distinct leftmost derivations according to G. □

We are now ready to proceed to the various normal forms. Our next result deals with the transition to the Chomsky normal form. By definition, an algebraic system $y_i = p_i$, $1 \le i \le n$, is in the *Chomsky normal form* (Chomsky [20]) iff $\mathrm{supp}(p_i) \subseteq \Sigma \cup Y^2$, $1 \le i \le n$.

Theorem 2.4.8 *Let $r \in A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. Then there exists an algebraic system in the Chomsky normal form such that $\sum_{w \in \Sigma^+}(r, w)w$ is a component of its least solution.*

Proof. We assume, by Theorem 4.1, that r is a component of the least solution of an algebraic system in the canonical two form. Apply now the constructions of Theorem 5.4 and Corollary 5.3, in this order. The resulting algebraic system is in the Chomsky normal form. □

We now introduce operators w^{-1}, for $w \in \Sigma^*$, mapping $A\langle\langle \Sigma^* \rangle\rangle$ into $A\langle\langle \Sigma^* \rangle\rangle$. For $u \in \Sigma^*$, we define $uw^{-1} = v$ if $u = vw$, $uw^{-1} = 0$ otherwise. As usual we extend these mappings to power series $r \in A\langle\langle \Sigma^* \rangle\rangle$ by

$$rw^{-1} = \sum_{u \in \Sigma^*}(r, u)uw^{-1} = \sum_{v \in \Sigma^*}(r, vw)v.$$

Observe that, if $(r, \varepsilon) = 0$ then $r = \sum_{x \in \Sigma}(rx^{-1})x$. In language theory the mappings corresponding to w^{-1} are usually referred to as *right derivatives* with respect to the word w.

Our next result deals with the transition from the Chomsky normal form to the operator normal form. By definiton, an algebraic system $y_i = p_i$, $1 \le i \le n$, is in the *operator normal form* iff $\mathrm{supp}(p_i) \subseteq \{\varepsilon\} \cup Y\Sigma \cup Y\Sigma Y$. Operator normal forms are customarily defined in language theory to be more general: there are no two consecutive nonterminals on the right sides of the productions. (See Floyd [48], Harrison [63].)

Theorem 2.4.9 *Let $r \in A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. Then there exists an algebraic system in the operator normal form such that r is a component of its unique solution.*

Proof. By Theorem 4.8 we may assume that $\sum_{w \in \Sigma^+}(r, w)w$ is the first component of the least solution σ of an algebraic system $y_i = p_i$, $1 \le i \le n$, in the Chomsky normal form. We write this system as follows:

$$y_i = \sum_{x \in \Sigma}(p_i, x)x + \sum_{1 \le k,m \le n}(p_i, y_k y_m)y_k y_m, \quad 1 \le i \le n.$$

We now define a new algebraic system. The alphabet of new variables will be $Y' = \{z_0\} \cup \{z_i^x \mid x \in \Sigma,\ 1 \le i \le n\}$. The equations of the new algebraic system are

$$\begin{aligned} z_0 &= (r, \varepsilon)\varepsilon + \sum_{x \in \Sigma} z_1^x x, \\ z_i^x &= (p_i, x)\varepsilon + \sum_{x' \in \Sigma}\sum_{1 \le k,m \le n}(p_i, y_k y_m)z_k^{x'} x' z_m^x, \quad x \in \Sigma,\ 1 \le i \le n. \end{aligned}$$

We claim that the components of the unique solution of this new algebraic system are given by r (z_0-component) and $\sigma_i x^{-1}$ (z_i^x-component). The claim is proven by substituting the components of the unique solution into the equations:

$$(r,\varepsilon)\varepsilon + \sum_{x\in\Sigma}(\sigma_1 x^{-1})x = (r,\varepsilon)\varepsilon + \sigma_1 = (r,\varepsilon)\varepsilon + \sum_{w\in\Sigma^+}(r,w)w = r,$$
$$(p_i,x)\varepsilon + \sum_{1\le k,m\le n}(p_i,y_k y_m)(\sum_{x'\in\Sigma}(\sigma_k {x'}^{-1})x')(\sigma_m x^{-1}) =$$
$$(p_i,x)xx^{-1} + \sum_{1\le k,m\le n}(p_i,y_k y_m)(\sigma_k\sigma_m)x^{-1} = \sigma_i x^{-1}, \quad x\in\Sigma,\ 1\le i\le n.$$

Observe that the equalities are valid for σ because $(\sigma,\varepsilon)=0$. They are not valid for solutions τ of $y_i = p_i$, $1\le i\le n$, with $(\tau,\varepsilon)\ne 0$. □

By definition, an algebraic system $y_i = p_i$, $1\le i\le n$, is in the *Greibach normal form* iff $\mathrm{supp}(p_i)\subseteq\{\varepsilon\}\cup\Sigma\cup\Sigma Y\cup\Sigma YY$. (See Greibach [59], Rosenkrantz [100], Jacob [66], Urbanek [114], Ésik, Leiß [46].)

Theorem 2.4.10 *Let $r\in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$. Then there exists an algebraic system in the Greibach normal form such that r is a component of its unique solution.*

Proof. By Theorem 4.9 we may assume that r is the first component of the unique solution of an algebraic system $y_i = p_i$, $1\le i\le n$, in the operator normal form. We write this system as follows:

$$y^{\mathrm{T}} = y^{\mathrm{T}}M(y) + P,$$

where $y^{\mathrm{T}} = (y_1,\ldots,y_n)$ is the transpose of y, $M\in(A\langle(Y\cup\Sigma)^*\rangle)^{n\times n}$,

$$M_{j,i} = \sum_{x\in\Sigma}(p_i,y_j x)x + \sum_{1\le m\le n}\sum_{x\in\Sigma}(p_i,y_j x y_m)xy_m, \quad 1\le i,j\le n,$$

and $P=((p_1,\varepsilon)\varepsilon,\ldots,(p_n,\varepsilon)\varepsilon)$. Let Z be an $n\times n$-matrix whose (i,j)-entry is a new variable z_{ij}, $1\le i,j\le n$. We now consider the algebraic system in the Greibach normal form

$$\begin{aligned} y^{\mathrm{T}} &= PM(y)Z + PM(y) + P\\ Z &= M(y)Z + M(y). \end{aligned}$$

We show that $(\sigma, M(\sigma)^+)$ is its unique solution:

$$PM(\sigma)M(\sigma)^+ + PM(\sigma) + P = PM(\sigma)^* = \sigma^{\mathrm{T}},$$

by a row vector variant of Theorem 4.2, and

$$M(\sigma)M(\sigma)^+ + M(\sigma) = M(\sigma)^+.$$

Hence, $r=\sigma_1$ is a component of the unique solution of the new algebraic system in the Greibach normal form. □

In language theory, the two most important normal forms are the Chomsky normal form and the Greibach normal form. By definition, a context-free grammar $G=(Y,\Sigma,P,y_1)$ is in the Chomsky normal form iff all productions are of the two forms $y_i\to y_k y_m$ and $y_i\to x$, $x\in\Sigma$, $y_i,y_k,y_m\in Y$. It is in the Greibach normal form iff all productions are of the three forms $y_i\to xy_k y_m$, $y_i\to xy_k$ and $y_i\to x$, $x\in\Sigma$, $y_i,y_k,y_m\in Y$. (Usually, productions $y_i\to\varepsilon$ are not allowed in the Greibach normal form.)

Corollary 2.4.11 *For every context-free language L there exist a context-free grammar G_1 in the Chomsky normal form and a context-free grammar G_2 in the Greibach normal form such that $L(G_1) = L(G_2) = L - \{\varepsilon\}$.*

Proof. By Theorem 4.8, and by Theorem 4.10 together with Theorem 4.4. $\square$

If our basic semiring is $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle$, we can draw some even stronger conclusions by Theorem 3.8.

Corollary 2.4.12 *Let $d : \Sigma^* \to \mathbb{N}$. Then the following three statements are equivalent:*

(i) There exists a context-free grammar G with terminal alphabet Σ such that the number of distinct leftmost derivations of w, $w \in \Sigma^*$, from the start variable is given by $d(w)$.

(ii) There exists a context-free grammar G_1 in the Chomsky normal form with terminal alphabet Σ such that the number of distinct leftmost derivations of w, $w \in \Sigma^+$, from the start variable is given by $d(w)$.

(iii) There exists a context-free grammar G_2 in the Greibach normal form with terminal alphabet Σ such that the number of distinct leftmost derivations of w, $w \in \Sigma^+$, from the start variable is given by $d(w)$.

Corollary 2.4.13 *For every unambiguous context-free grammar G there exist an unambiguous context-free grammar G_1 in the Chomsky normal form and an unambiguous context-free grammar G_2 in the Greibach normal form such that $L(G_1) = L(G_2) = L(G) - \{\varepsilon\}$.*

Chapter 3

Pushdown automata and algebraic series

3.1 Introduction

In Section 2, we introduce (possibly infinite) matrices and show how the blocks of the star of an infinite matrix can be computed by applying the rational operations on the blocks of the infinite matrix. This can be considered to be a generalization of the matrix-star-equation of Theorem 1.2.18 to infinite matrices. In Section 3, we consider automata that may have an infinite state set and (possibly infinite) linear systems. We show the connection between these automata and linear systems. In Section 4, we introduce pushdown automata and prove the equivalence of pushdown automata and algebraic systems in the sense that both characterize the algebraic power series. In our last section, Section 5, we show a Kleene Theorem for algebraic power series that is a generalization of a result of Gruska [60].

3.2 Infinite matrices

We now introduce matrices in a more general manner as we did in Chapter 1. Consider two non-empty index sets I and I' and a set S. Mappings M of $I \times I'$ into S are called *matrices*. The values of M are denoted by $M_{i,i'}$, where $i \in I$ and $i' \in I'$. The values $M_{i,i'}$ are also referred to as the *entries* of the matrix M. In particular, $M_{i,i'}$ is called the (i,i')-*entry* of M. The collection of all matrices as defined above is denoted by $S^{I \times I'}$.

If both I and I' are finite, then M is called a *finite matrix*. If I or I' is a singleton, M is called a *row* or *column vector*, respectively. If $M \in S^{I \times 1}$ (resp. $M \in S^{1 \times I'}$) then we often denote the i-th entry of M, $i \in I$ (resp. $i \in I'$), by M_i instead of $M_{i,1}$ (resp. $M_{1,i}$).

We introduce some operations and special matrices inducing a monoid or semiring structure to matrices. For $M_1, M_2 \in A^{I\times I'}$ we define the *sum* $M_1 + M_2 \in A^{I\times I'}$ by $(M_1 + M_2)_{i,i'} = (M_1)_{i,i'} + (M_2)_{i,i'}$ for all $i \in I$, $i' \in I'$. Furthermore, we introduce the *zero matrix* $0 \in A^{I\times I'}$. All entries of the zero matrix 0 are 0. By these definitions, $\langle A^{I\times I'}, +, 0\rangle$ is a commutative monoid.

If I_2 is finite or if A is complete, then, for $M_1 \in A^{I_1\times I_2}$ and $M_2 \in A^{I_2\times I_3}$, we define the *product* $M_1M_2 \in A^{I_1\times I_3}$ by

$$(M_1M_2)_{i_1,i_3} = \sum_{i_2\in I_2} (M_1)_{i_1,i_2}(M_2)_{i_2,i_3} \quad \text{for all } i_1 \in I_1,\ i_3 \in I_3.$$

Furthermore, we introduce the *matrix of unity* $E \in A^{I\times I}$. The diagonal entries $E_{i,i}$ of E are equal to 1, the off-diagonal entries E_{i_1,i_2}, $i_1 \neq i_2$, of E are equal to 0, $i, i_1, i_2 \in I$.

It is easily shown that matrix multiplication is associative, the distribution laws are valid for matrix addition and multiplication, E is a multiplicative unit and 0 is a multiplicative zero. So we infer that $\langle A^{I\times I}, +, \cdot, 0, E\rangle$ is a semiring if I is finite or if A is complete.

If A is complete, infinite sums can be extended to matrices. Consider $A^{I\times I'}$ and define, for $M_j \in A^{I\times I'}$, $j \in J$, where J is an index set, $\sum_{j\in J} M_j$ by its entries:

$$\Big(\sum_{j\in J} M_j\Big)_{i,i'} = \sum_{j\in J} (M_j)_{i,i'}, \quad i \in I,\ i' \in I'.$$

By this definition, $A^{I\times I}$ is a complete semiring.

If A is ordered, the order on A is extended pointwise to matrices M_1 and M_2 in $A^{I\times I'}$:

$$M_1 \le M_2 \qquad \text{iff} \qquad (M_1)_{i,i'} \le (M_2)_{i,i'} \text{ for all } i \in I,\ i' \in I'.$$

If A is continuous then so is $A^{I\times I}$.

For the rest of this section we assume A to be a complete semiring. For the remainder of this book I (resp. Q), possibly provided with indices, denotes an arbitrary (resp. finite) index set.

We now introduce blocks of matrices. Consider a matrix M in $A^{I\times I}$. Assume the existence of a non-empty index set J and of non-empty index sets I_j for $j \in J$ such that $I = \bigcup_{j\in J} I_j$ and $I_{j_1} \cap I_{j_2} = \emptyset$ for $j_1 \neq j_2$. The mapping M, restricted to the domain $I_{j_1} \times I_{j_2}$, i. e., $M : I_{j_1} \times I_{j_2} \to A$ is, of course, a matrix in $A^{I_{j_1}\times I_{j_2}}$. We denote it by $M(I_{j_1}, I_{j_2})$ and call it the (I_{j_1}, I_{j_2})-*block* of M.

We can compute the blocks of the sum and the product of matrices M_1 and M_2 from the blocks of M_1 and M_2 in the usual way:

$$\begin{aligned}(M_1 + M_2)(I_{j_1}, I_{j_2}) &= M_1(I_{j_1}, I_{j_2}) + M_2(I_{j_1}, I_{j_2}),\\ (M_1M_2)(I_{j_1}, I_{j_2}) &= \sum_{j\in J} M_1(I_{j_1}, I_j)M_2(I_j, I_{j_2}).\end{aligned}$$

In a similar manner the matrices of $A^{I\times I'}$ can be partitioned into blocks. This yields the computational rule

$$(M_1 + M_2)(I_j, I'_{j'}) = M_1(I_j, I'_{j'}) + M_2(I_j, I'_{j'}).$$

If we consider matrices $M_1 \in A^{I \times I'}$ and $M_2 \in A^{I' \times I''}$ partitioned into compatible blocks, i. e., I' is partitioned into the same index sets for both matrices, then we obtain the computational rule

$$(M_1 M_2)(I_j, I''_{j''}) = \sum_{j' \in J'} M_1(I_j, I'_{j'}) M_2(I'_{j'}, I''_{j''}).$$

In the sequel we will need the following isomorphisms:

(i) The semirings

$$\left(A^{Q \times Q}\right)^{I \times I},\ A^{(I \times Q) \times (I \times Q)},\ A^{(Q \times I) \times (Q \times I)},\ \left(A^{I \times I}\right)^{Q \times Q}$$

are isomorphic by the correspondences between

$$\left(M_{i_1,i_2}\right)_{q_1,q_2},\ M_{(i_1,q_1),(i_2,q_2)},\ M_{(q_1,i_1),(q_2,i_2)},\ \left(M_{q_1,q_2}\right)_{i_1,i_2}$$

for all $i_1, i_2 \in I$, $q_1, q_2 \in Q$.

(ii) The semirings $A^{I \times I}\langle\langle \Sigma^* \rangle\rangle$ and $\left(A\langle\langle \Sigma^* \rangle\rangle\right)^{I \times I}$ are isomorphic by the correspondence between $(M, w)_{i_1,i_2}$ and (M_{i_1,i_2}, w) for all $i_1, i_2 \in I$, $w \in \Sigma^*$.

Observe that these correspondences are isomorphisms of complete semirings, i. e., they respect infinite sums. We will use these isomorphisms without further mention. Moreover, we will use the notation M_{i_1,i_2}, $i_1 \in I_1$, $i_2 \in I_2$, where $M \in A^{I_1 \times I_2}\langle\langle \Sigma^* \rangle\rangle$: M_{i_1,i_2} is the power series in $A\langle\langle \Sigma^* \rangle\rangle$ such that the coefficient (M_{i_1,i_2}, w) of $w \in \Sigma^*$ is equal to $(M, w)_{i_1,i_2}$. Similarly, we will use the notation (M, w), $w \in \Sigma^*$, where $M \in \left(A\langle\langle \Sigma^* \rangle\rangle\right)^{I_1 \times I_2}$: (M, w) is the matrix in $A^{I_1 \times I_2}$ whose (i_1, i_2)-entry $(M, w)_{i_1,i_2}$, $i_1 \in I_1$, $i_2 \in I_2$, is equal to (M_{i_1,i_2}, w).

The next theorem is central for automata theory and is a generalization of the matrix-star-equation to infinite matrices (see Conway [25], Lehmann [91], Kuich, Salomaa [88], Kuich [76], Kozen [71], Bloom, Ésik [10], the definition of the star of a matrix in Section 2 of Chapter 1). It allows to compute the blocks of the star of a matrix M by sum, product and star of the blocks of M. For notational convenience, we will denote $M(I_i, I_j)$ by $M_{i,j}$, $1 \le i, j \le 3$.

Theorem 3.2.1 *Let A be a complete starsemiring, let $M \in A^{I \times I}$ and $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$. Then*

$$\begin{aligned}
M^*(I_1, I_1) &= (M_{1,1} + M_{1,2} M_{2,2}^* M_{2,1})^*, \\
M^*(I_1, I_2) &= (M_{1,1} + M_{1,2} M_{2,2}^* M_{2,1})^* M_{1,2} M_{2,2}^*, \\
M^*(I_2, I_1) &= (M_{2,2} + M_{2,1} M_{1,1}^* M_{1,2})^* M_{2,1} M_{1,1}^*, \\
M^*(I_2, I_2) &= (M_{2,2} + M_{2,1} M_{1,1}^* M_{1,2})^*.
\end{aligned}$$

Proof. Consider the matrices

$$M_1 = \begin{pmatrix} M_{1,1} & 0 \\ 0 & M_{2,2} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & M_{1,2} \\ M_{2,1} & 0 \end{pmatrix}.$$

The computation of $(M_1 + M_2 M_1^* M_2)^* (E + M_2 M_1^*)$ and application of Lemma 1.2.25 prove our theorem. □

Corollary 3.2.2 *If $M_{2,1} = 0$ then*

$$M^* = \begin{pmatrix} M_{1,1}^* & M_{1,1}^* M_{1,2} M_{2,2}^* \\ 0 & M_{2,2}^* \end{pmatrix}.$$

Corollary 3.2.3 *If $M_{2,1} = 0$, $M_{3,1} = 0$ and $M_{3,2} = 0$ then*

$$M^* = \begin{pmatrix} M_{1,1}^* & M_{1,1}^* M_{1,2} M_{2,2}^* & M_{1,1}^* M_{1,2} M_{2,2}^* M_{2,3} M_{3,3}^* + M_{1,1}^* M_{1,3} M_{3,3}^* \\ 0 & M_{2,2}^* & M_{2,2}^* M_{2,3} M_{3,3}^* \\ 0 & 0 & M_{3,3}^* \end{pmatrix}$$

In the next theorem, I is partitioned into I_j, $j \in J$, and j_0 is a distinguished element in J.

Theorem 3.2.4 *Let A be a complete starsemiring and assume that the only non-null blocks of the matrix $M \in A^{I \times I}$ are $M(I_j, I_{j_0})$, $M(I_{j_0}, I_j)$ and $M(I_j, I_j)$, for all $j \in J$ and a fixed $j_0 \in J$. Then*

$$M^*(I_{j_0}, I_{j_0}) = \Big(M(I_{j_0}, I_{j_0}) + \sum_{j \in J,\, j \neq j_0} M(I_{j_0}, I_j) M(I_j, I_j)^* M(I_j, I_{j_0}) \Big)^*.$$

Proof. We partition I into I_{j_0} and $I' = I - I_{j_0}$. Then $M(I', I')$ is a block-diagonal matrix and $(M(I', I')^*)(I_j, I_j) = M(I_j, I_j)^*$ for all $j \in J - \{j_0\}$. By Theorem 2.1 we obtain

$$M^*(I_{j_0}, I_{j_0}) = \big(M(I_{j_0}, I_{j_0}) + M(I_{j_0}, I') M(I', I')^* M(I', I_{j_0}) \big)^*.$$

The computation of the right side of this equality proves our theorem. □

3.3 Automata and linear systems

In this section we generalize the finite automata and finite linear systems introduced in Chapter 1. These finite automata are generalized in the following direction: An infinite set of states will be allowed in the general definition. When dealing with pushdown automata in Section 4 this will enable us to store the contents of the pushdown tape in the states.

Our model of an automaton will be defined in terms of a (possibly infinite) transition matrix. As explained in Chapter 1, Section 3, the semiring element generated by the transition of the automaton from one state i to another state i' in exactly k computation steps equals the (i, i')-entry in the k-th power of the transition matrix. Consider now the star of the transition matrix. Then the semiring element generated by the automaton, also called the behavior of the automaton, can be expressed by the entries (multiplied by the initial and final weights of the states) of the star of the transition matrix.

In the sequel, A will denote a *continuous* semiring and A' will denote a subset of A containing 0 and 1.

An A'-*automaton*

$$\mathfrak{A} = (I, M, S, P)$$

is given by

(i) a non-empty set I of *states*,

(ii) a matrix $M \in {A'}^{I\times I}$, called the *transition matrix*,

(iii) $S \in {A'}^{1\times I}$, called the *initial state vector*,

(iv) $P \in {A'}^{I\times 1}$, called the *final state vector*.

The *behavior* $\|\mathfrak{A}\| \in A$ of the A'-automaton $\mathfrak{A}$ is defined by

$$\|\mathfrak{A}\| = \sum_{i_1,i_2\in I} S_{i_1}(M^*)_{i_1,i_2}P_{i_2} = SM^*P.$$

An A'-automaton is termed *finite* iff its state set is finite.

Usually, an automaton is depicted as a *directed graph*. The *nodes* of the graph correspond to the states of the automaton. A node corresponding to a state i with $S_i \neq 0$ (resp. $P_i \neq 0$) is called *initial* (resp. *final*). The *edges* (i, j) of the graph correspond to the transitions unequal to 0 and are labeled by $M_{i,j}$.

Consider the semiring $\mathbb{B}$. Then, for an arbitrary $\mathbb{B}$-automaton $\mathfrak{A}$, we obtain $\|\mathfrak{A}\| = 1$ iff there is a path in the graph from some initial node to some final node.

Let now $\mathbb{N}^\infty$ be the basic semiring and let $\mathfrak{A}$ be a $\{0,1\}$-automaton. Then $\|\mathfrak{A}\|$ is equal to the number (including ∞) of distinct paths in the graph from the initial nodes to the final nodes.

Assume that the basic semiring is one of the tropical semirings and consider an $\{\infty, 1, 0\}$-automaton $\mathfrak{A} = (I, M, S, P)$ such that the entries of M are in $\{\infty, 1\}$, and the entries of S and P are in $\{\infty, 0\}$. (Observe that a node i is initial or final if $S_i = 0$ or $P_i = 0$, respectively.) Then $\|\mathfrak{A}\|$ is equal to the length of the shortest path in the graph from some initial node to some final node. There is no such path iff $\|\mathfrak{A}\| = \infty$. (See Carré [19].)

Consider the semiring $\mathbb{R}_+^\infty$ and let $[0, 1] = \{a \in \mathbb{R}_+ \mid 0 \leq a \leq 1\}$. A $[0, 1]$-automaton, whose transition matrix is stochastic, can be considered as a Markov chain (see Paz [99], Seneta [110]).

We now generalize the finite A'-linear systems introduced in Chapter 2, Section 3. An *A'-linear system* is now of the form

$$y = My + P\,,$$

where y is a variable, M is a matrix in ${A'}^{I\times I}$ and P is a column vector in ${A'}^{I\times 1}$. A column vector $T \in A^{I\times 1}$ is called *solution* to $y = My + P$ iff $T = MT + P$. It is called *least solution* iff $T \leq T'$ for all solutions T'.

Theorem 3.3.1 *Let $y = My + P$ be an A'-linear system. Then M^*P is its least solution.*

Proof. A proof analogous to the proof of Theorem 2.3.4 shows that M^*P is the least solution. □

Corollary 3.3.2 *Let $\mathfrak{A} = (I, M, S, P)$ be an A'-automaton and let T be the least solution of the A'-linear system $y = My + P$. Then $\|\mathfrak{A}\| = ST$.*

A matrix $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{I\times I}$ is called *cycle-free* iff there exists an $n \geq 1$ such that $(M, \varepsilon)^n = 0$. An $A\langle\langle\Sigma^*\rangle\rangle$-linear system $y = My + P$ is called *cycle-free* iff M is cycle-free.

Theorem 3.3.3 (Kuich, Urbanek [89], Corollary 3.**)** *The cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system $y = My + P$ has the unique solution M^*P.*

An $A\langle\langle\Sigma^*\rangle\rangle$-automaton $\mathfrak{A} = (I, M, S, P)$ is called *cycle-free* iff M is cycle-free.

Corollary 3.3.4 *Let $\mathfrak{A} = (I, M, S, P)$ be a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-automaton and let T be the unique solution of the cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system $y = My+P$. Then $\|\mathfrak{A}\| = ST$.*

Example 3.3.1. (Kuich, Salomaa [88], Example 7.2.) Let $\Sigma = \{x_1, x_2, x_3\}$, $Q = \{q_1, q_2\}$,

$$C = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & x_3 \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} 0 & x_2^n \\ 0 & 0 \end{pmatrix}, \quad n \geq 0.$$

Define $M \in ((A\langle\Sigma^*\rangle)^{Q\times Q})^{\mathbb{N}\times\mathbb{N}}$, $S \in ((A\langle\varepsilon\rangle)^{1\times Q})^{1\times\mathbb{N}}$ and $P \in ((A\langle\varepsilon\rangle)^{Q\times 1})^{\mathbb{N}\times 1}$ by their non-null blocks:

$$M_{n,n+1} = C, \quad M_{n+1,n} = D, \quad M_{n,n} = B_n, \quad n \geq 0,$$
$$S_0 = \begin{pmatrix} \varepsilon & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}.$$

Consider the $A\langle\Sigma^*\rangle$-automaton $\mathfrak{A} = (I, M, S, P)$, where $I = \mathbb{N} \times Q$. (Strictly speaking, we should take the copies of M, S and P in $(A\langle\Sigma^*\rangle)^{(\mathbb{N}\times Q)\times(\mathbb{N}\times Q)}$, $(A\langle\varepsilon\rangle)^{1\times(\mathbb{N}\times Q)}$ and $(A\langle\varepsilon\rangle)^{(\mathbb{N}\times Q)\times 1}$, respectively.) Let

$$T_n = \begin{pmatrix} \sum_{j\geq n} x_1^{j-n} x_2^j x_3^j \\ x_3^n \end{pmatrix}, \quad n \geq 0,$$

be the n-th block of the column vector $T \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q\times 1})^{\mathbb{N}\times 1}$. We claim that T is a solution (and hence, the unique solution) of the cycle-free $A\langle\Sigma^*\rangle$-linear system $y = My+P$ ($(M, \varepsilon)^2 = 0$). This claim is proved by showing the equalities $(MT + P)_n = T_n$, $n \geq 0$:

$$\begin{aligned} (MT+P)_0 &= B_0T_0 + CT_1 + P_0 = T_0, \\ (MT+P)_n &= DT_{n-1} + B_nT_n + CT_{n+1} = T_n, \quad n \geq 1. \end{aligned}$$

This yields $\|\mathfrak{A}\| = ST = S_0T_0 = \sum_{j\geq 0} x_1^j x_2^j x_3^j$. □

3.4 Pushdown automata and algebraic systems

We now define A'-pushdown automata and consider their relation to A'-algebraic systems. It turns out that, for $a \in A$, $a \in \mathfrak{Alg}(A')$ iff it is the behavior of an A'-pushdown automaton. This generalizes the language theoretic result due to Chomsky [21] that a formal language is context-free iff it is accepted by a pushdown automaton.

A'-pushdown automata are finite automata (with state set Q) augmented by a pushdown tape. The contents of the pushdown tape is a word over the pushdown alphabet Γ. We consider an A'-pushdown automaton to be an A'-automaton in the sense of Section 3: the state set is given by $\Gamma^* \times Q$ and its transition matrix is in ${A'}^{(\Gamma^* \times Q)\times(\Gamma^* \times Q)}$. This allows us to store the contents of the pushdown tape and the states of the finite automaton in the states of the A'-pushdown automaton. Because of technical reasons, we do not work in the semiring $A^{(\Gamma^* \times Q)\times(\Gamma^* \times Q)}$ but in the isomorphic semiring $(A^{Q\times Q})^{\Gamma^* \times \Gamma^*}$. A matrix $M \in ({A'}^{Q\times Q})^{\Gamma^* \times \Gamma^*}$ is termed an *A'-pushdown transition matrix* iff

(i) for each $p \in \Gamma$ there exist only finitely many blocks $M_{p,\pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$M_{\pi_1,\pi_2} = \begin{cases} M_{p,\pi} & \text{if there exist } p \in \Gamma,\ \pi' \in \Gamma^* \text{ with} \\ & \pi_1 = p\pi' \text{ and } \pi_2 = \pi\pi', \\ 0 & \text{otherwise.} \end{cases}$$

The above definition implies that an A'-pushdown transition matrix has a finitary specification: it is completely specified by its non-null blocks of the form $M_{p,\pi}$, $p \in \Gamma$, $\pi \in \Gamma^*$. Item (ii) of the above definition shows that only the following transitions are possible: if the contents of the pushdown tape is given by $p\pi'$, the contents of the pushdown tape after a transition has to be of the form $\pi\pi'$; moreover, the transition does only depend on the leftmost (topmost) pushdown sympol p and not on π'. In this sense the A'-pushdown transition matrix represents a proper formalization of the principle “last in—first out”.

An *A'-pushdown automaton*

$$\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$$

is given by

(i) a finite set Q of *states*,

(ii) a finite alphabet Γ of *pushdown symbols*,

(iii) an *A'-pushdown transition matrix* $M \in ({A'}^{Q\times Q})^{\Gamma^* \times \Gamma^*}$,

(iv) $S \in {A'}^{1\times Q}$, called the *initial state vector*,

(v) $p_0 \in \Gamma$, called the *initial pushdown symbol*,

(vi) $P \in {A'}^{Q\times 1}$, called the *final state vector*.

The behavior $\|\mathfrak{P}\|$ of the A'-pushdown automaton $\mathfrak{P}$ is defined by

$$\|\mathfrak{P}\| = S(M^*)_{p_0,\varepsilon}P.$$

We now describe the computations of an A'-pushdown automaton. Initially, the pushdown tape contains the special symbol p_0. The A'-pushdown automaton now performs transitions governed by the A'-pushdown transition matrix until the pushdown tape is emptied. The result of these computations is given by $(M^*)_{p_0,\varepsilon}$. Multiplications by the initial state vector and by the final state vector yield the behavior of the A'-pushdown automaton.

Let now 2^{Σ^*} be our basic semiring. We connect our definition of an $2^{\Sigma\cup\{\varepsilon\}}$-pushdown automaton $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$ to the usual definition of a *pushdown automaton* $\mathfrak{P}' = (Q, \Sigma, \Gamma, \delta, q_0, p_0, F)$ (see e. g., Harrison [63]), where Σ is the *input alphabet*, δ, a function from $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$ to the set of all finite subsets of $Q \times \Gamma^*$, is the *transition function*, $q_0 \in Q$ is the *initial state* and $F \subseteq Q$ is the set of *final states*.

Assume that a pushdown automaton $\mathfrak{P}'$ is given as above. The transition function δ defines the pushdown transition matrix M of $\mathfrak{P}$ by

$$x \in (M_{p,\pi})_{q_1,q_2} \quad \text{iff} \quad (q_2, \pi) \in \delta(q_1, x, p)$$

for all $q_1, q_2 \in Q$, $p \in \Gamma$, $\pi \in \Gamma^*$, $x \in \Sigma\cup\{\varepsilon\}$. Let now $\vdash$ be the move relation over the *instantaneous descriptions* of $\mathfrak{P}'$ in $Q\times\Sigma^*\times\Gamma^*$. Then $(q_1, w, \pi_1) \vdash^k (q_2, \varepsilon, \pi_2)$ iff $w \in ((M^k)_{\pi_1,\pi_2})_{q_1,q_2}$ and $(q_1, w, \pi_1) \vdash^* (q_2, \varepsilon, \pi_2)$ iff $w \in ((M^*)_{\pi_1,\pi_2})_{q_1,q_2}$ for all $k \geq 0$, $q_1, q_2 \in Q$, $\pi_1, \pi_2 \in \Gamma^*$, $w \in \Sigma^*$. Hence, $(q_0, w, p_0) \vdash^* (q, \varepsilon, \varepsilon)$ iff $w \in ((M^*)_{p_0,\varepsilon})_{q_0,q}$. Define the initial state vector S and the final state vector P by $S_{q_0} = \{\varepsilon\}$, $S_q = \emptyset$ if $q \neq q_0$, $P_q = \{\varepsilon\}$ if $q \in F$, $P_q = \emptyset$ if $q \notin F$. Then a word w is accepted by the pushdown automaton $\mathfrak{P}'$ by both final state and empty store iff $w \in S(M^*)_{p_0,\varepsilon}P = \|\mathfrak{P}\|$.

In our first theorem we show that an A'-pushdown automaton can be regarded as an A'-automaton.

Theorem 3.4.1 *For each A'-pushdown automaton $\mathfrak{P}$ there exists an A'-automaton $\mathfrak{A}$ such that $\|\mathfrak{A}\| = \|\mathfrak{P}\|$.*

Proof. Let $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$. We define the A'-automaton $\mathfrak{A} = (\Gamma^* \times Q, M', S', P')$ by $M'_{(\pi_1,q_1),(\pi_2,q_2)} = (M_{\pi_1,\pi_2})_{q_1,q_2}$, $S'_{(p_0,q)} = S_q$, $S'_{(\pi,q)} = 0$, if $\pi \neq p_0$, $P'_{(\varepsilon,q)} = P_q$, $P'_{(\pi,q)} = 0$, if $\pi \neq \varepsilon$. Then

$$\begin{aligned}
\|\mathfrak{A}\| &= S'{M'}^*P' \\
&= \sum_{(\pi_1,q_1),(\pi_2,q_2)\in\Gamma^*\times Q} S'_{(\pi_1,q_1)}({M'}^*)_{(\pi_1,q_1),(\pi_2,q_2)}P'_{(\pi_2,q_2)}
\end{aligned}$$

$$= \sum_{q_1,q_2\in Q} S'_{(p_0,q_1)}(M'^*)_{(p_0,q_1),(\varepsilon,q_2)}P'_{(\varepsilon,q_2)} =$$
$$= \sum_{q_1,q_2\in Q} S_{q_1}((M^*)_{p_0,\varepsilon})_{q_1,q_2}P_{q_2} = S(M^*)_{p_0,\varepsilon}P = \|\mathfrak{P}\|.$$

□

Consider an A'-pushdown automaton with A'-pushdown transition matrix M and let $\pi = \pi_1\pi_2$ be a word over the pushdown alphabet Γ. Then our next proposition states that emptying the pushdown tape with contents π has the same effect (i. e., $(M^*)_{\pi,\varepsilon}$) as emptying first the pushdown tape with contents π_1 (i. e., $(M^*)_{\pi_1,\varepsilon}$) and afterwards (i. e., multiplying) the pushdown tape with contents π_2 (i. e., $(M^*)_{\pi_2,\varepsilon}$). (See also Kuich, Salomaa [88], Theorem 10.5.)

In the sequel, $F \in (A'^{Q\times Q})^{\Gamma^*\times 1}$ is defined by $F_\varepsilon = E$ and $F_\pi = 0$ if $\pi \in \Gamma^+$.

Proposition 3.4.2 *Let $M \in (A'^{Q\times Q})^{\Gamma^*\times\Gamma^*}$ be an A'-pushdown transition matrix. Then*

$$(M^*)_{\pi_1\pi_2,\varepsilon} = (M^*)_{\pi_1,\varepsilon}(M^*)_{\pi_2,\varepsilon}$$

holds for all $\pi_1, \pi_2 \in \Gamma^*$.

Proof. We consider the $A'^{Q\times Q}$-linear system $y = My + F$ and denote its approximation sequence by $(\tau^j \mid j \in \mathbb{N})$. Define $\tau = \sup(\tau^j \mid j \in \mathbb{N})$. Then, by Theorem 3.1, $\tau_\pi = (M^*)_{\pi,\varepsilon}$ for all $\pi \in \Gamma^*$. Hence, we have to prove that $\tau_{\pi_1\pi_2} = \tau_{\pi_1}\tau_{\pi_2}$ holds for all $\pi_1, \pi_2 \in \Gamma^*$. Clearly, the equation holds for $\pi_1 = \varepsilon$.

(i) Our first claim is that, for $\pi_1 \neq \varepsilon$, $\tau^j_{\pi_1\pi_2} \leq \tau_{\pi_1}\tau_{\pi_2}$, $j \geq 0$. We prove it by induction on j. Since the case $j = 0$ is trivial, we proceed by $j > 0$ and obtain $\tau^j_{\pi_1\pi_2} = (M\tau^{j-1})_{\pi_1\pi_2} = \sum_{\pi\in\Gamma^*} M_{\pi_1\pi_2,\pi}\tau^{j-1}_\pi = \sum_{\pi'\in\Gamma^*} M_{\pi_1\pi_2,\pi'\pi_2}\tau^{j-1}_{\pi'\pi_2} \leq \sum_{\pi'\in\Gamma^*} M_{\pi_1,\pi'}\tau_{\pi'}\tau_{\pi_2} = (M\tau)_{\pi_1}\tau_{\pi_2} = \tau_{\pi_1}\tau_{\pi_2}$. Here the third equality follows by $M_{\pi_1\pi_2,\pi} = 0$ if $\pi_1 \neq \varepsilon$ and $\pi \neq \pi'\pi_2$; the inequality follows by $M_{\pi_1\pi_2,\pi'\pi_2} = M_{\pi_1,\pi'}$ if $\pi_1 \neq \varepsilon$ and by induction hypothesis.

(ii) Our second claim is that, for $\pi_1 \neq \varepsilon$, $\tau^j_{\pi_1}\tau_{\pi_2} \leq \tau_{\pi_1\pi_2}$, $j \geq 0$. We prove it by induction on j. Since the case $j = 0$ is trivial, we proceed by $j > 0$ and obtain $\tau^j_{\pi_1}\tau_{\pi_2} = (M\tau^{j-1})_{\pi_1}\tau_{\pi_2} = \sum_{\pi\in\Gamma^*} M_{\pi_1,\pi}\tau^{j-1}_\pi\tau_{\pi_2} \leq \sum_{\pi\in\Gamma^*} M_{\pi_1\pi_2,\pi\pi_2}\tau_{\pi\pi_2} = \sum_{\pi'\in\Gamma^*} M_{\pi_1\pi_2,\pi'}\tau_{\pi'} = (M\tau)_{\pi_1\pi_2} = \tau_{\pi_1\pi_2}$.

Eventually, the first and the second claim imply $\tau_{\pi_1\pi_2} = \tau_{\pi_1}\tau_{\pi_2}$. □

Let $M \in (A'^{Q\times Q})^{\Gamma^*\times\Gamma^*}$ be an A'-pushdown transition matrix and let $\{y_p \mid p \in \Gamma\}$ be an alphabet of variables. We define $y_\varepsilon = \varepsilon$ and $y_{p\pi} = y_p y_\pi$ for $p \in \Gamma$, $\pi \in \Gamma^*$, and consider the $A'^{Q\times Q}$-algebraic system

$$y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi, \quad p \in \Gamma.$$

Given matrices $T_p \in A^{Q\times Q}$ for all $p \in \Gamma$, we define matrices $T_\pi \in A^{Q\times Q}$ for all $\pi \in \Gamma^*$ as follows: $T_\varepsilon = E$, $T_{p\pi} = T_pT_\pi$, $p \in \Gamma$, $\pi \in \Gamma^*$. By these matrices we define a matrix $\tilde{T} \in (A^{Q\times Q})^{\Gamma^*\times 1}$: the π-block of $\tilde{T}$ is given by T_π, $\pi \in \Gamma^*$, i. e., $\tilde{T}_\pi = T_\pi$.

Proposition 3.4.3 *If $(T_p)_{p\in\Gamma^*}$, $T_p \in A^{Q\times Q}$, is a solution of $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi$, $p \in \Gamma$, then $\tilde{T} \in (A^{Q\times Q})^{\Gamma^*\times 1}$ is a solution of $y = My + F$.*

Proof. Since M is an A'-pushdown transition matrix, we obtain, for all $p \in \Gamma$ and $\pi \in \Gamma^*$,

$$\begin{aligned}(M\tilde{T})_{p\pi} &= \sum_{\pi_1\in\Gamma^*} M_{p\pi,\pi_1}\tilde{T}_{\pi_1} = \sum_{\pi_2\in\Gamma^*} M_{p\pi,\pi_2\pi}\tilde{T}_{\pi_2\pi} = \\ &= \sum_{\pi_2\in\Gamma^*} M_{p,\pi_2}\tilde{T}_{\pi_2}\tilde{T}_\pi = (M\tilde{T})_p\tilde{T}_\pi .\end{aligned}$$

Since $(T_p)_{p\in\Gamma}$ is a solution of $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi$, $p \in \Gamma$, we infer that $\tilde{T}_p = T_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}T_\pi = \sum_{\pi\in\Gamma^*} M_{p,\pi}\tilde{T}_\pi = (M\tilde{T})_p$. Hence, $(M\tilde{T}+F)_{p\pi} = (M\tilde{T})_{p\pi} = \tilde{T}_p\tilde{T}_\pi = \tilde{T}_{p\pi}$, $p \in \Gamma$, $\pi \in \Gamma^*$. Additionally, we have $\tilde{T}_\varepsilon = E$ and $(M\tilde{T}+F)_\varepsilon = F_\varepsilon = E$. This implies that $\tilde{T}$ is a solution of $y = My + F$. □

Theorem 3.4.4 *The $A'^{Q\times Q}$-algebraic system $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi$ has the least solution $((M^*)_{p,\varepsilon})_{p\in\Gamma}$.*

Proof. We first show that $((M^*)_{p,\varepsilon})_{p\in\Gamma}$ is a solution of the $A'^{Q\times Q}$-algebraic system by substituting $(M^*)_{\pi,\varepsilon}$ for y_π:

$$\sum_{\pi\in\Gamma^*} M_{p,\pi}(M^*)_{\pi,\varepsilon} = (M^+)_{p,\varepsilon} = (M^*)_{p,\varepsilon}, \quad p \in \Gamma .$$

Assume now that $(T_p)_{p\in\Gamma}$ is a solution of $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi$. Then, by Proposition 4.3, $\tilde{T}$ is a solution of $y = My + F$. Since M^*F is the least solution of this $A'^{Q\times Q}$-linear system, we infer that $M^*F \le \tilde{T}$. This implies $(M^*F)_\pi = (M^*)_{\pi,\varepsilon} \le \tilde{T}_\pi = T_\pi$ for all $\pi \in \Gamma^*$. Hence, $(M^*)_{p,\varepsilon} \le T_p$ for all $p \in \Gamma$, and $((M^*)_{p,\varepsilon})_{p\in\Gamma}$ is the least solution of $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}y_\pi$, $p \in \Gamma$. □

Let $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$ be an A'-pushdown automaton and consider the A'-algebraic system

$$\begin{aligned} y_0 &= S y_{p_0} P, \\ y_p &= \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi, \quad p \in \Gamma, \end{aligned}$$

written in matrix notation: y_p is a $Q\times Q$-matrix whose (q_1,q_2)-entry is the variable $[q_1,p,q_2]$, $p\in\Gamma$, $q_1,q_2\in Q$; if $\pi = p_1\dots p_r$, $r\ge 1$, then the (q_1,q_2)-entry of y_π is given by the (q_1,q_2)-entry of $y_{p_1}\dots y_{p_r}$, $p_1,\dots,p_r\in\Gamma$; y_0 is a variable. Hence, the variables of the above A'-algebraic system are y_0, $[q_1,p,q_2]$, $p\in\Gamma$, $q_1,q_2\in Q$.

Corollary 3.4.5 *Let $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$ be an A'-pushdown automaton. Then $\|\mathfrak{P}\|$, $((M^*)_{p,\varepsilon})_{p\in\Gamma}$ is the least solution of the A'-algebraic system*

$$\begin{aligned} y_0 &= S y_{p_0} P, \\ y_p &= \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi, \quad p \in \Gamma. \end{aligned}$$

Corollary 3.4.6 *The behavior of an A'-pushdown automaton is an element of $\mathfrak{Alg}(A')$.*

Theorem 2.3.13 admits another corollary.

Corollary 3.4.7 *Let $\mathfrak{P}$ be an $\mathfrak{Alg}(A')$-pushdown automaton. Then $\|\mathfrak{P}\| \in \mathfrak{Alg}(A')$.*

We now want to show the converse of Corollary 4.6.

Theorem 3.4.8 *Let $a \in \mathfrak{Alg}(A')$. Then there exists an A'-pushdown automaton $\mathfrak{P}$ such that $\|\mathfrak{P}\| = a$.*

Proof. We assume, by Theorem 2.4.1, that a is the first component of the least solution σ of an A'-algebraic system in the canonical two form

$$y_i = \sum_{1 \le k,m \le n} a^i_{km} y_k y_m + \sum_{1 \le k \le n} a^i_k y_k + a_i, \quad 1 \le i \le n.$$

We define the A'-pushdown transition matrix (with $|Q| = 1$) $M \in {A'}^{Y^* \times Y^*}$ by

$$M_{y_i, y_k y_m} = a^i_{km}, \quad M_{y_i, y_k} = a^i_k, \quad M_{y_i, \varepsilon} = a_i, \quad 1 \le i, k, m \le n,$$

and write the above A'-algebraic system in the form

$$y_i = \sum_{1 \le k,m \le n} M_{y_i, y_k y_m} y_k y_m + \sum_{1 \le k \le n} M_{y_i, y_k} y_k + M_{y_i, \varepsilon}, \quad 1 \le i \le n.$$

By Theorem 4.4, the least solution of this A'-algebraic system is given by $((M^*)_{y_1,\varepsilon}, \ldots, (M^*)_{y_n,\varepsilon})$. Hence, $\sigma_i = (M^*)_{y_i,\varepsilon}$, $1 \le i \le n$. Consider now the A'-pushdown automata $\mathfrak{P}_i = (\{q\}, Y, M, 1, y_i, 1)$, $1 \le i \le n$. Then we obtain $\|\mathfrak{P}_i\| = (M^*)_{y_i,\varepsilon} = \sigma_i$, $1 \le i \le n$. Hence, $\|\mathfrak{P}_1\| = a$ and our theorem is proven. □

This completes the proof of the main result of Section 4:

Corollary 3.4.9 *Let $a \in A$. Then $a \in \mathfrak{Alg}(A')$ iff there exists an A'-pushdown automaton $\mathfrak{P}$ such that $\|\mathfrak{P}\| = a$.*

Corollary 3.4.10 *Let $r \in A\langle\langle \Sigma^* \rangle\rangle$. Then $r \in A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$ iff there exists an $A\langle \Sigma \cup \varepsilon \rangle$-pushdown automaton $\mathfrak{P}$ such that $||\mathfrak{P}|| = r$.*

If our basic semiring is 2^{Σ^*} then Corollary 4.9 is nothing else than the well-known characterization of the context-free languages by pushdown automata.

Corollary 3.4.11 *A formal language is context-free iff it is accepted by a pushdown automaton.*

Observe that the construction proving Corollary 4.5 is nothing else than the well-known triple construction. (See Hopcroft, Ullman [65], Theorem 5.4; Harrison [63], Theorem 5.4.3; Bucher, Maurer [17], Sätze 2.3.10, 2.3.30.) By the proof it is clear that $[q_1, p, q_2] \Rightarrow^* w$ in G iff $w \in ((M^*)_{p,\varepsilon})_{q_1,q_2}$ in $\mathfrak{P}$ iff $(q_1, w, p) \vdash^* (q_2, \varepsilon, \varepsilon)$ in $\mathfrak{P}'$, $q_1, q_2 \in Q$, $p \in \Gamma$, $w \in \Sigma^*$. This means that there exists a derivation of w from the variable $[q_1, p, q_2]$ iff w empties the pushdown tape with contents p by a computation from state q_1 to state q_2.

If our basic semiring is $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle$, we can draw some even stronger conclusions by Theorem 2.3.8. In Corollary 4.12 we consider, for a given pushdown automaton $\mathfrak{P}' = (Q, \Sigma, \Gamma, \delta, q_0, p_0, F)$, the number of distinct computations from the *initial instantaneous description* (q_0, w, p_0) *for* w to an *accepting instantaneous description* $(q, \varepsilon, \varepsilon)$, $q \in F$.

Corollary 3.4.12 *Let L be a formal language over Σ and let $d : \Sigma^* \to \mathbb{N}^\infty$. Then the following two statements are equivalent:*

(i) There exists a context-free grammar with terminal alphabet Σ such that the number (possibly ∞) of distinct leftmost derivations of w, $w \in \Sigma^*$, from the start variable is given by $d(w)$.

(ii) There exists a pushdown automaton with input alphabet Σ such that the number (possibly ∞) of distinct computations from the initial instantaneous description for w, $w \in \Sigma^*$, to an accepting instantaneous description is given by $d(w)$.

A pushdown automaton with input alphabet Σ is termed *unambiguous* iff, for each word $w \in \Sigma^*$ that is accepted, there exists a unique computation from the initial instantaneous description for w to some accepting instantaneous description.

Corollary 3.4.13 *A formal language is generated by an unambiguous context-free grammar iff it is accepted by an unambiguous pushdown automaton.*

Example 3.4.1. (Hopcroft, Ullmann [65], Example 5.3.) Let $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $\Gamma = \{p_0, p\}$ and $\mathfrak{P}' = (Q, \Sigma, \Gamma, \delta, q_0, p_0, \emptyset)$ be a pushdown automaton, where δ is given by

$$\begin{array}{lll} \delta(q_0, a, p_0) = \{(q_0, pp_0)\}, & \delta(q_0, a, p) = \{(q_0, pp)\}, & \delta(q_1, \varepsilon, p_0) = \{(q_1, \varepsilon)\}, \\ \delta(q_0, b, p) = \{(q_1, \varepsilon)\}, & \delta(q_1, b, p) = \{(q_1, \varepsilon)\}, & \delta(q_1, \varepsilon, p) = \{(q_1, \varepsilon)\}. \end{array}$$

We construct a $2^{\Sigma \cup \{\varepsilon\}}$-pushdown automaton $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$ such that $w \in \Sigma^*$ is accepted by $\mathfrak{P}'$ by empty store iff $w \in \|\mathfrak{P}\|$:

$$M_{p_0,pp_0} = M_{p,p^2} = \begin{pmatrix} \{a\} & \emptyset \\ \emptyset & \emptyset \end{pmatrix}, \quad M_{p,\varepsilon} = \begin{pmatrix} \emptyset & \{b\} \\ \emptyset & \{\varepsilon, b\} \end{pmatrix}, \quad M_{p_0,\varepsilon} = \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \{\varepsilon\} \end{pmatrix},$$

$$S = (\{\varepsilon\} \ \emptyset), \quad P = \begin{pmatrix} \{\varepsilon\} \\ \{\varepsilon\} \end{pmatrix}.$$

By the construction of Corollary 4.5, the following algebraic system in matrix notation corresponds to $\mathfrak{P}$:

$$y_0 = S y_{p_0} P, \quad y_{p_0} = M_{p_0,pp_0} y_p y_{p_0} + M_{p_0,\varepsilon}, \quad y_p = M_{p,p^2} y_p^2 + M_{p,\varepsilon}.$$

Hence, we obtain

$$
\begin{aligned}
&y_0 = [q_0, p_0, q_0] + [q_0, p_0, q_1] \\
&\begin{pmatrix} [q_0, p_0, q_0] & [q_0, p_0, q_1] \\ [q_1, p_0, q_0] & [q_1, p_0, q_1] \end{pmatrix} = \\
&\quad = \begin{pmatrix} \{a\}[q_0, p, q_0] & \{a\}[q_0, p, q_1] \\ \emptyset & \emptyset \end{pmatrix} \begin{pmatrix} [q_0, p_0, q_0] & [q_0, p_0, q_1] \\ [q_1, p_0, q_0] & [q_1, p_0, q_1] \end{pmatrix} + \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \{\varepsilon\} \end{pmatrix} \\
&\begin{pmatrix} [q_0, p, q_0] & [q_0, p, q_1] \\ [q_1, p, q_0] & [q_1, p, q_1] \end{pmatrix} = \\
&\quad = \begin{pmatrix} \{a\}[q_0, p, q_0] & \{a\}[q_0, p, q_1] \\ \emptyset & \emptyset \end{pmatrix} \begin{pmatrix} [q_0, p, q_0] & [q_0, p, q_1] \\ [q_1, p, q_0] & [q_1, p, q_1] \end{pmatrix} + \begin{pmatrix} \emptyset & \{b\} \\ \emptyset & \{\varepsilon, b\} \end{pmatrix}
\end{aligned}
$$

Inspection shows that the components of $[q_0, p_0, q_0]$, $[q_0, p, q_0]$, $[q_1, p_0, q_0]$, $[q_1, p, q_0]$ in the least solution are equal to $\emptyset$. This yields

$$
\begin{aligned}
&y_0 = [q_0, p_0, q_1], \\
&[q_0, p_0, q_1] = \{a\}[q_0, p, q_1][q_1, p_0, q_1], \qquad && [q_1, p_0, q_1] = \{\varepsilon\}, \\
&[q_0, p, q_1] = \{a\}[q_0, p, q_1][q_1, p, q_1] \cup \{b\}, && [q_1, p, q_1] = \{\varepsilon, b\}.
\end{aligned}
$$

Denoting $[q_0, p_0, q_1]$ and $[q_0, p, q_1]$ by y_1 and y_2, respectively, and simplifying yields the algebraic system

$$
y_1 = \{a\}y_2, \quad y_2 = \{a\}y_2\{\varepsilon, b\} \cup \{b\}.
$$

The context-free grammar of Example 2.4.1 corresponds to this algebraic system. Hence, $\mathfrak{P}'$ accepts exactly the words $a^m b^n$, $m \geq n \geq 1$, by empty store. Moreover, by Corollary 4.12, the number of distinct computations from the initial instantaneous description $(q_0, a^m b^n, p_0)$, $m \geq n \geq 1$, to the accepting instantaneous description $(q_1, \varepsilon, \varepsilon)$ is $\binom{m-1}{n-1}$. (There are no computations leading to $(q_0, \varepsilon, \varepsilon)$.) $\square$

3.5 A Kleene Theorem for algebraic power series

In this section we show a Kleene Theorem for algebraic power series. It is a generalization of a result of Gruska [60]. The presentation follows the lines of Kuich [79].

In the sequel, Σ_∞ denotes an infinite alphabet, Σ denotes a finite subalphabet of Σ_∞, and A denotes a continuous *commutative* semiring. All occuring symbols and variables are elements of Σ_∞. Our basic semiring will be $A\langle\langle \Sigma_\infty^* \rangle\rangle$.

We remind the reader to Proposition 2.2.7 and the remarks below this proposition, which justify the following notation. If $h : \Sigma_\infty \to A\langle\langle \Sigma_\infty^* \rangle\rangle$ is a mapping such that $h(x) = x$, $x \in \Sigma_\infty - \{y_1, \ldots, y_n\}$ where $y_1, \ldots, y_n$ are variables, $h^\# : A\langle\langle \Sigma_\infty^* \rangle\rangle \to A\langle\langle \Sigma_\infty^* \rangle\rangle$ is its extension to $A\langle\langle \Sigma_\infty^* \rangle\rangle$, and $r \in A\langle\langle \Sigma_\infty^* \rangle\rangle$, then we write $r = r(y_1, \ldots, y_n)$ and $h^{\#}(r) = r(h(y_1), \ldots, h(y_n))$. This is called the *substitution* of the power series $h(y_1), \ldots, h(y_n)$ into the variables $y_1, \ldots, y_n$, respectively.

For the convenience of the reader, we formulate Theorem 2.2.10 (Bekić-De Bakker-Scott rule) and Proposition 2.2.12 in the setting of $A\langle\langle\Sigma_\infty^*\rangle\rangle$.

Consider disjoint alphabets $\{y_1, \ldots, y_n\}$ and $\{z_1, \ldots, z_m\}$ of variables and let $\hat{\Sigma}_\infty = \Sigma_\infty - \{y_1, \ldots, y_n, z_1, \ldots, z_m\}$. Let $p_i(z_1, \ldots, z_m, y_1, \ldots, y_n)$, $1 \le i \le n$, and $q_j(z_1, \ldots, z_m, y_1, \ldots, y_n)$, $1 \le j \le m$, be power series in $A\langle\langle\Sigma_\infty^*\rangle\rangle$ and consider the system of equations

$$\begin{aligned} z_j &= p_j(z_1, \ldots, z_m, y_1, \ldots, y_n), \quad 1 \le j \le m\,, \\ y_i &= q_i(z_1, \ldots, z_m, y_1, \ldots, y_n)\,, \quad 1 \le i \le n\,. \end{aligned}$$

Let $(t_1(z_1, \ldots, z_m), \ldots, t_n(z_1, \ldots, z_m)) \in (A\langle\langle(\hat{\Sigma}_\infty \cup \{z_1, \ldots, z_m\})^*\rangle\rangle)^n$ and $(r_1, \ldots, r_m) \in (A\langle\langle\hat{\Sigma}_\infty^*\rangle\rangle)^n$ be the least solutions of the systems $y_i = q_i(z_1, \ldots, z_m, y_1, \ldots, y_n)$, $1 \le i \le n$, and $z_j = p_j(z_1, \ldots, z_m, t_1(z_1, \ldots, z_m), \ldots, t_n(z_1, \ldots, z_m))$, $1 \le j \le m$, respectively. Then $(r_1, \ldots, r_m, t_1(r_1, \ldots, r_m), \ldots, t_n(r_1, \ldots, r_m))$ is the least solution of the original system.

In the next proposition we use a vectorial notation: $z = (z_1, \ldots, z_m)$, $y = (y_1, \ldots, y_n)$, $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_n)$, etc.

Theorem 3.5.1 (Bekić-De Bakker-Scott rule) *Consider the system of equations*

$$z = p(z, y), \quad y = q(z, y).$$

Let $t(z)$ and r be the least solutions of the systems $y = q(z, y)$ and $z = p(z, t(z))$, respectively. Then $(r, t(r))$ is the least solution of the system $z = p(z, y)$, $y = q(z, y)$.

Moreover, r is the least solution of the system $z = p(z, t(r))$.

We introduce the following notation: Let $r(y_1, \ldots, y_i, \ldots, y_n) \in A\langle\langle\Sigma_\infty^*\rangle\rangle$, where $y_1, \ldots, y_i, \ldots, y_n$ are variables that may occur in r (besides, there may occur also other variables). We denote the least $\sigma \in A\langle\langle(\Sigma_\infty - \{y_i\})^*\rangle\rangle$ such that $r(y_1, \ldots, \sigma, \ldots, y_n) = \sigma$ by $\mu y_i.r(y_1, \ldots, y_i, \ldots, y_n)$, $1 \le i \le n$. This means that σ is the least solution of the equation $y_i = r(y_1, \ldots, y_i, \ldots, y_n)$ and μy_i is a fixed point operator. Observe that $\mu y_i.r(y_1, \ldots, y_i, \ldots, y_n) \in A\langle\langle(\Sigma_\infty - \{y_i\})^*\rangle\rangle$.

Proposition 3.5.2 *Let $r(y_1, \ldots, y_n, y) \in A\langle\langle\Sigma_\infty^*\rangle\rangle$ and $\sigma_i \in A\langle\langle(\Sigma_\infty - \{y\})^*\rangle\rangle$, $1 \le i \le n$. Let $s(y_1, \ldots, y_n) = \mu y.r(y_1, \ldots, y_n, y)$. Then*

$$s(\sigma_1, \ldots, \sigma_n) = \mu y.r(\sigma_1, \ldots, \sigma_n, y).$$

Proof. By Propositions 2.2.7 and 2.2.12. □

A subsemiring $\bar{A}$ of $A\langle\langle\Sigma_\infty^*\rangle\rangle$ is called *equationally closed* iff, for all $r \in \bar{A}$ and $y \in \Sigma_\infty$ the power series $\mu y.r$ is again in $\bar{A}$.

Let $A\{\Sigma_\infty^*\} = \{r \in A\langle\Sigma^*\rangle \mid \Sigma \subset \Sigma_\infty \text{ finite}\}$ and $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} = \{r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle \mid \Sigma \subset \Sigma_\infty \text{ finite}\}$. Denote by $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$ the least equationally closed semiring containing $A\{\Sigma_\infty^*\}$. We will prove in this section that $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\} = A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$.

Theorem 3.5.3 *Let* $t(y_1, \ldots, y_n)$, $\sigma_j \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$, $1 \le j \le n$. *Then* $t(\sigma_1, \ldots, \sigma_n) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$.

Proof. The proof is by induction on the number of applications of the operations $+$, $\cdot$ and μ to generate $t(y_1, \ldots, y_n)$.

(i) Let $t(y_1, \ldots, y_n) \in A\{\Sigma_\infty^*\}$, i. e., $t(y_1, \ldots, y_n) \in A\langle\Sigma^*\rangle$ for some $\Sigma \subset \Sigma_\infty$. Since $t(\sigma_1, \ldots, \sigma_n)$ is generated from $\sigma_1, \ldots, \sigma_n$ by applications of sum, product and scalar product, we infer that $t(\sigma_1, \ldots, \sigma_n) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$.

(ii) We only prove the case of the operator μ. Let $\sigma_1, \ldots, \sigma_n \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\} \cap A\langle\langle\Sigma^*\rangle\rangle$ for some Σ and choose a $y \in \Sigma_\infty$ that is not in $\Sigma \cup \{y_1, \ldots, y_n\}$. Without loss of generality assume that $t(y_1, \ldots, y_n) = \mu y.r(y_1, \ldots, y_n, y)$ (the variable y is "bound"), where $r(y_1, \ldots, y_n, y) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. By induction hypothesis, we obtain $r(\sigma_1, \ldots, \sigma_n, y) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. Hence, $t(\sigma_1, \ldots, \sigma_n) = \mu y.r(\sigma_1, \ldots, \sigma_n, y) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$ by Proposition 5.2. $\square$

Theorem 3.5.4 $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} \subseteq A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$.

Proof. The proof is by induction on the number of variables of algebraic systems. We use the following induction hypothesis: If $\tau \in (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^n$, $n \ge 1$, is the least solution of an algebraic system $y_i = q_i(y_1, \ldots, y_n)$, $1 \le i \le n$, with n variables $y_1, \ldots, y_n$ where $q_i \in A\{\Sigma_\infty^*\}$, then $\tau_i \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$.

(1) Let $n = 1$ and assume that r is the least solution of the algebraic system $z = p(z)$. Then $r = \mu z.p(z) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$.

(2) Let $z, y_1, \ldots, y_n$ be variables and $p, q_1, \ldots, q_n$ be polynomials in $A\{\Sigma_\infty^*\}$, and consider the algebraic system $z = p(z, y)$, $y = q(z, y)$, where $y = (y_1, \ldots, y_n)$ and $q = (q_1, \ldots, q_n)$. Let $t(z) \in (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^n$ be the least solution of $y = q(z, y)$. By our induction hypothesis we obtain $t(z) \in (A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\})^n$. Since $p(z, y)$ is a polynomial, it is in $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. Hence, by Theorem 5.3, $p(z, t(z))$ is in $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. This implies $\mu z.p(z, t(z)) \in A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. Again, by Theorem 5.3, $t(\mu z.p(z, t(z)) \in (A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\})^n)$. By Theorem 5.1, $(\mu z.p(z, t(z)), t(\mu z.p(z, t(z))))$ is the least solution of the algebraic system $z = p(z, y)$, $y = q(z, y)$ and is, by Theorem 5.3, in $(A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\})^{n+1}$. Hence, the components of the least solution of this algebraic system are in $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\}$. $\square$

We now show the converse to Theorem 5.4.

Theorem 3.5.5 $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\} \subseteq A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$.

Proof. We show that $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ is an equationally closed semiring that contains $A\{\Sigma_\infty^*\}$. By Corollary 2.3.12, $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ is a semiring containing $A\{\Sigma_\infty^*\}$. Hence we have only to show that $\mu z.r$, $r \in A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ and $z \in \Sigma_\infty$, is in $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$.

Let $r \in A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ be the first component of the least solution of the algebraic system $y_i = p_i(y_1, \ldots, y_n, z)$, $1 \le i \le n$. Then, by Theorem 5.1, $\mu z.r$ is the z-component of the least solution of the algebraic system $z = y_1$, $y_i = p_i(y_1, \ldots, y_n, z)$, $1 \le i \le n$. $\square$

We have now achieved the main result of this section.

Theorem 3.5.6 *Let A be a continuous commutative semiring. Then $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\} = A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ and $A^{\mathrm{equ}}\{\{\Sigma_\infty^*\}\} \cap A\langle\langle\Sigma^*\rangle\rangle = A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$, $\Sigma \subset \Sigma_\infty$, Σ finite.*

Analogous to the rational expressions (see Chapter 2, Section 3) and similar to the context-free expressions of Gruska [60], we define algebraic expressions.

Assume that Σ_∞, A and $U = \{+, \cdot, \mu, [,]\}$ are mutually disjoint. A word E over $\Sigma_\infty \cup A \cup U$ is an *algebraic expression* over $A\{\Sigma_\infty^*\}$ iff

(i) E is a symbol of A, or

(ii) E is a symbol of Σ_∞, or else

(iii) E is of one of the forms $[E_1 + E_2]$, $[E_1 \cdot E_2]$, or $\mu y.E_1$, where E_1 and E_2 are algebraic expressions and y is a symbol of Σ_∞.

Each algebraic expression E over $A\{\Sigma_\infty^*\}$ *denotes* a formal power series $|E|$ in $A\{\{\Sigma_\infty^*\}\}$ according to the following conventions:

(i) The power series denoted by $a \in A$ is $a\varepsilon$ in $A\{\Sigma_\infty^*\}$.

(ii) The power series denoted by $x \in \Sigma_\infty$ is x in $A\{\Sigma_\infty^*\}$.

(iii) For algebraic expressions E_1, E_2 over $A\{\Sigma_\infty^*\}$ and $y \in \Sigma_\infty$, $|[E_1 + E_2]| = |E_1| + |E_2|$, $|[E_1 \cdot E_2]| = |E_1| \cdot |E_2|$, $|\mu y.E_1| = \mu y.|E_1|$.

Let ϕ be the mapping from the set of algebraic expressions over $A\{\Sigma_\infty^*\}$ into the finite sets of 2^{Σ_∞} defined by

(i) $\phi(a) = \emptyset$, $a \in A$.

(ii) $\phi(x) = \{x\}$, $x \in \Sigma_\infty$.

(iii) $\phi([E_1 + E_2]) = \phi([E_1 \cdot E_2]) = \phi(E_1) \cup \phi(E_2)$, $\phi(\mu y.E_1) = \phi(E_1) - \{y\}$, for algebraic expressions E_1, E_2 and $y \in \Sigma_\infty$.

Given an algebraic expression E, $\phi(E)$ contains the "free symbols" of E. This means that $|E|$ is a formal power series in $A\langle\langle\phi(E)^*\rangle\rangle$. Theorem 5.6 and the above definitions yield some corollaries.

Corollary 3.5.7 *A power series r is in $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ iff there exists an algebraic expression E over $A\{\Sigma_\infty^*\}$ such that $r = |E|$.*

Corollary 3.5.8 *Let A be a continuous commutative semiring and $r \in A\langle\langle\Sigma^*\rangle\rangle$. Then the following statements are equivalent:*

(i) $r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$,

(ii) r is the behavior of an $A\langle\Sigma \cup \varepsilon\rangle$-pushdown automaton,

(iii) there exists an algebraic expression E over $A\{\Sigma_\infty^*\}$, where $\phi(E) \subseteq \Sigma$, such that $r = |E|$.

Proof. By Corollary 4.10. □

Observe that $\mathbb{B}\{\{\Sigma_\infty^*\}\}$ is isomorphic to the semiring $\mathfrak{L}(\Sigma_\infty) = \{L \mid L \subseteq \Sigma^*, \Sigma \subset \Sigma_\infty\}$ of formal languages over Σ_∞. Hence, if $A = \mathbb{B}$ then each algebraic expresssion E over $\mathbb{B}\{\Sigma_\infty^*\}$ denotes by this isomorphism a formal language in $\mathfrak{L}(\Sigma_\infty)$ according to the following conventions:

(i) The language denoted by 0 or 1 is $\emptyset$ or $\{\varepsilon\}$, respectively.

(ii) The language denoted by $x \in \Sigma_\infty$ is $\{x\}$.

(iii) For algebraic expressions E_1, E_2 over $\mathbb{B}\{\Sigma_\infty^*\}$ and $y \in \Sigma_\infty$, $|[E_1 + E_2]| = |E_1| \cup |E_2|$, $|[E_1 \cdot E_2]| = |E_1| \cdot |E_2|$, $|\mu y.E_1| = \mu y.|E_1|$.

Corollary 3.5.9 (Gruska [60]) *A formal language L in $\mathfrak{L}(\Sigma_\infty)$ is context-free iff there exists an algebraic expression E over $\mathbb{B}\{\Sigma_\infty^*\}$ such that $L = |E|$.*

Corollary 3.5.10 *The following statements on a language $L \subseteq \Sigma^*$ are equivalent:*

(i) L is a context-free language,

(ii) L is accepted by a pushdown automaton,

(iii) there exists an algebraic expression E over $\mathbb{B}\{\Sigma_\infty^*\}$, where $\phi(E) = \Sigma$, such that $L = |E|$.

Proof. By Corollary 4.11. □

Chapter 4

Transducers and abstract families

4.1 Introduction

In Section 2 we generalize the concept of a rational transducer by replacing the rational representations in the usual definition by certain complete rational semiring morphisms. Moreover, we introduce algebraic transducers. It turns out that $\mathfrak{Rat}(A')$ and $\mathfrak{Alg}(A')$ are closed under rational and algebraic transductions, respectively.

In Section 3, these generalized rational transducers lead to the generalization of the concept of a full AFL (abbreviation for "abstract family of languages") to the concept of an AFE (abbreviation for "abstract family of elements"): these are fully rationally closed semirings that are also closed under the application of generalized rational transducers. These AFEs are then characterized by automata of a certain type. Additionally, the concept of an AFE generalizes the concepts of an AFL, an AFP (abbreviation for "abstract family of power series") and a full AFP.

The presentation of this chapter follows the lines of Kuich [78] and Karner, Kuich [69].

4.2 Transducers

In the sequel, A and $\hat{A}$ will denote continuous semirings. A semiring morphism $h : \hat{A} \to A$ is termed *complete semiring morphism* if, for all families $(a_i \mid i \in I)$ in $\hat{A}$, $h(\sum_{i \in I} a_i) = \sum_{i \in I} h(a_i)$. Note that a complete semiring morphism is a continuous mapping.

Given a mapping $h : \hat{A} \to A^{Q_1 \times Q_2}$, we define the mappings $h' : \hat{A}^{I_1 \times I_2} \to A^{(I_1 \times Q_1) \times (I_2 \times Q_2)}$ and $h'' : \hat{A}^{I_1 \times I_2} \to (A^{Q_1 \times Q_2})^{I_1 \times I_2}$ by

$$h'(M)_{(i_1,q_1),(i_2,q_2)} = \left(h''(M)_{i_1,i_2}\right)_{q_1,q_2} = h(M_{i_1,i_2})_{q_1,q_2}$$

for $M \in \hat{A}^{I_1 \times I_2}$, $i_1 \in I_1$, $i_2 \in I_2$, $q_1 \in Q_1$, $q_2 \in Q_2$. In the sequel, we use the same notation h for the mappings h, h' and h''.

Consider the two mappings $h : A \to A^{Q \times Q}$ and $h_1 : A \to A^{Q_1 \times Q_1}$. The *functional composition* $h \circ h_1 : A \to A^{(Q_1 \times Q) \times (Q_1 \times Q)}$ is defined by $(h \circ h_1)(a) = h(h_1(a))$ for $a \in A$.

Easy computations yield the following two technical results.

Theorem 4.2.1 *Let $h : \hat{A} \to A^{Q \times Q}$ be a complete semiring morphism. Then $h : \hat{A}^{I \times I} \to A^{(I \times Q) \times (I \times Q)}$ and $h : \hat{A}^{I \times I} \to (A^{Q \times Q})^{I \times I}$ are again complete semiring morphisms.*

Theorem 4.2.2 *Let $h : \hat{A} \to A$ be a complete semiring morphism. Then, for all $a \in \hat{A}$, $h(a^*) = h(a)^*$.*

In the sequel, A' and $\hat{A}'$ denote subsets of A and $\hat{A}$, respectively, both containing the respective neutral elements 0 and 1.

A semiring morphism $h : \hat{A} \to A^{Q \times Q}$ is called $(\hat{A}', A')$*-rational* or $(\hat{A}', A')$*-algebraic* if, for all $a \in \hat{A}'$, $h(a) \in \mathfrak{Rat}(A')^{Q \times Q}$ or $h(a) \in \mathfrak{Alg}(A')^{Q \times Q}$, respectively. If $\hat{A} = A$ and $\hat{A}' = A'$, these morphisms are called A'*-rational* or A'*-algebraic*, respectively.

Theorem 4.2.3 *Let $h : A \to A^{Q \times Q}$ and $h' : A \to A^{Q' \times Q'}$ be complete A'-rational semiring morphisms. Then the functional composition $h \circ h' : A \to A^{(Q' \times Q) \times (Q' \times Q)}$ is again an A'-rational semiring morphism.*

Proof. Clearly, $h'(a)_{q_1,q_2} \in \mathfrak{Rat}(A')$ for $a \in A'$, $q_1, q_2 \in Q'$. Since $\mathfrak{Rat}(A')$ is generated by the rational operations from A', Theorems 2.1 and 2.2 imply that the entries of $h(h'(a))$, $a \in A'$, are again in $\mathfrak{Rat}(A')$. □

Before we show a similar result for A'-algebraic semiring morphisms, some considerations on A'-algebraic systems are necessary.

Let $h : A \to A^{Q \times Q}$ be a semiring morphism and extend it to a mapping $h : A(Y) \to A^{Q \times Q}(Y)$ as follows:

(i) If a semiring-polynomial is represented by a product term $a_0 y_{i_1} a_1 \ldots a_{k-1} y_{i_k} a_k$, $k \geq 0$, where $a_j \in A$ and $y_{i_j} \in Y$, then its image is represented by the product term $h(a_0) y_{i_1} h(a_1) \ldots h(a_{k-1}) y_{i_k} h(a_k)$.

(ii) If a semiring-polynomial is represented by a sum of product terms $\sum_{1 \leq j \leq m} t_j$, then its image is represented by the sum of product terms $\sum_{1 \leq j \leq m} h(t_j)$.

Then, by the proof of Theorem 1.4.31 of Lausch, Nöbauer [90], this extension is well-defined and again a semiring morphism.

Theorem 4.2.4 *Let $h : A \to A^{Q \times Q}$ be a complete semiring morphism. Consider an A-algebraic system $y = p$ with least solution σ. Then $h(\sigma)$ is the least solution of the $A^{Q \times Q}$-algebraic system $y = h(p)$.*

Proof. Let $(\sigma^j \mid j \in \mathbb{N})$ be the approximation sequence of $y = p$. Then $(h(\sigma^j) \mid j \in \mathbb{N})$ is the approximation sequence of $y = h(p)$ and we obtain

$$\mathrm{fix}(h(p)) = \sup(h(\sigma^j) \mid j \in \mathbb{N}) = h(\sup(\sigma^j \mid j \in \mathbb{N})) = h(\sigma).$$

The second equality follows by the fact that a complete semiring morphism is a continuous mapping. Since $\mathrm{fix}(h(p))$ is the least solution of $y = h(p)$ by Theorem 2.3.3, we have proved our theorem. □

Corollary 4.2.5 *Let $h : A \to A^{Q\times Q}$ be a complete A'-algebraic semiring morphism. Then $h(a) \in \mathfrak{Alg}(A')^{Q\times Q}$ for $a \in \mathfrak{Alg}(A')$.*

Proof. By Theorem 2.4, $h(a)$, $a \in \mathfrak{Alg}(A')$, is a component of the least solution of an $\mathfrak{Alg}(A')^{Q\times Q}$-algebraic system. Hence, the entries of $h(a)$ are in $\mathfrak{Alg}(\mathfrak{Alg}(A'))$. Now apply Theorem 2.3.13. □

Theorem 4.2.6 *Let $h : A \to A^{Q\times Q}$ and $h' : A \to A^{Q'\times Q'}$ be complete A'-algebraic semiring morphisms. Then the functional composition $h \circ h' : A \to A^{(Q'\times Q)\times(Q'\times Q)}$ is again an A'-algebraic semiring morphism.*

Proof. Clearly, $h'(a)_{q_1,q_2} \in \mathfrak{Alg}(A')$ for $a \in A'$, $q_1, q_2 \in Q'$. Now, Corollary 2.5 implies that the entries of $h(h'(a))$, $a \in A'$, are again in $\mathfrak{Alg}(A')$. □

We are now ready to introduce the notions of a rational and an algebraic transducer.

An $(\hat{A}', A')$-*rational transducer*

$$\mathfrak{T} = (Q, h, S, P)$$

is given by

(i) a finite set Q of *states*,

(ii) a complete $(\hat{A}', A')$-rational semiring morphism $h : \hat{A} \to A^{Q\times Q}$,

(iii) $S \in \mathfrak{Rat}(A')^{1\times Q}$, called the *initial state vector*,

(iv) $P \in \mathfrak{Rat}(A')^{Q\times 1}$, called the *final state vector*.

The mapping $\|\mathfrak{T}\| : \hat{A} \to A$ *realized* by an $(\hat{A}', A')$-rational transducer $\mathfrak{T} = (Q, h, S, P)$ is defined by

$$\|\mathfrak{T}\|(a) = Sh(a)P, \qquad a \in \hat{A}.$$

A mapping $\tau : \hat{A} \to A$ is called an $(\hat{A}', A')$-*rational transduction* if there exists an $(\hat{A}', A')$-rational transducer $\mathfrak{T}$ such that $\tau(a) = \|\mathfrak{T}\|(a)$ for all $a \in \hat{A}$. In this case, we say that τ is *realized* by $\mathfrak{T}$. An (A', A')-rational transducer (in case $\hat{A} = A$ and $\hat{A}' = A'$) is called an A'-*rational transducer* and an (A', A')-rational transduction is called an A'-*rational transduction.*

An $(\hat{A}', A')$-*algebraic transducer* $\mathfrak{T} = (Q, h, S, P)$ is defined exactly as an $(\hat{A}', A')$-rational transducer except that h is now a complete $(\hat{A}', A')$-algebraic semiring morphism, and the entries of S and P are in $\mathfrak{Alg}(A')$. The definition of the notions of $(\hat{A}', A')$-*algebraic transduction*, A'-*algebraic transducer* and A'-*algebraic transduction* should be clear.

The next two theorems show that $(\hat{A}', A')$-rational (resp. $(\hat{A}', A')$-algebraic) transductions map $\mathfrak{Rat}(\hat{A}')$ (resp. $\mathfrak{Alg}(\hat{A}')$) into $\mathfrak{Rat}(A')$ (resp. $\mathfrak{Alg}(A')$).

Theorem 4.2.7 *Assume that $\mathfrak{T}$ is an $(\hat{A}', A')$-rational transducer and that $a \in \mathfrak{Rat}(\hat{A}')$. Then $\|\mathfrak{T}\|(a) \in \mathfrak{Rat}(A')$.*

Proof. Let a be the behavior of the finite $\hat{A}'$-automaton $\mathfrak{A} = (Q, M, S, P)$. Assume that $\mathfrak{T} = (Q', h, S', P')$. We consider now the finite $\mathfrak{Rat}(A')$-automaton $\mathfrak{A}' = (Q \times Q', h(M), S'h(S), h(P)P')$. Since $\mathfrak{Rat}(\mathfrak{Rat}(A')) = \mathfrak{Rat}(A')$ we obtain $\|\mathfrak{A}'\| \in \mathfrak{Rat}(A')$. Since $\|\mathfrak{A}'\| = S'h(S)h(M)^*h(P)P' = S'h(SM^*P)P' = \|\mathfrak{T}\|(\|\mathfrak{A}\|)$, our theorem is proved. □

Theorem 4.2.8 *Assume that $\mathfrak{T}$ is an $(\hat{A}', A')$-algebraic transducer and that $a \in \mathfrak{Alg}(\hat{A}')$. Then $\|\mathfrak{T}\|(a) \in \mathfrak{Alg}(A')$.*

Proof. Let a be the behavior of the $\hat{A}'$-pushdown automaton $\mathfrak{P} = (Q, \Gamma, M, S, p_0, P)$. Assume that $\mathfrak{T} = (Q', h, S', P')$. We consider now the $\mathfrak{Alg}(A')$-pushdown automaton $\mathfrak{P}' = (Q \times Q', \Gamma, h(M), S'h(S), p_0, h(P)P')$. By Corollary 3.4.9 and Theorem 2.3.13 we obtain $\|\mathfrak{P}'\| \in \mathfrak{Alg}(A')$.

Since $\|\mathfrak{P}'\| = S'h(S)(h(M)^*)_{p_0,\varepsilon}h(P)P' = S'h(S)h((M^*)_{p_0,\varepsilon})h(P)P' = S'h(S(M^*)_{p_0,\varepsilon}P)P' = \|\mathfrak{T}\|(\|\mathfrak{P}\|)$, our theorem is proved. □

We now consider the functional composition of A'-rational (resp. A'-algebraic) transductions.

Theorem 4.2.9 *The family of A'-rational (resp. A'-algebraic) transductions is closed under functional composition.*

Proof. Let $\mathfrak{T}_j = (Q_j, h_j, S_j, P_j)$, $j = 1, 2$, be two A'-rational (resp. A'-algebraic) transducers. We want to show that the mapping $\tau : A \to A$ defined by $\tau(a) = \|\mathfrak{T}_2\|(\|\mathfrak{T}_1\|(a))$, $a \in A$, is an A'-rational (resp. A'-algebraic) transduction.

Consider $\mathfrak{T} = (Q_1 \times Q_2, h_2 \circ h_1, S_2h_2(S_1), h_2(P_1)P_2)$. By Theorem 2.3 (resp. Theorem 2.6) the mapping $h_2 \circ h_1$ is a complete A'-rational (resp. A'-algebraic) semiring morphism. Furthermore, the entries of $S_2h_2(S_1)$ and $h_2(P_1)P_2$ are in $\mathfrak{Rat}(A')$ (resp. $\mathfrak{Alg}(A')$). Hence, $\mathfrak{T}$ is an A'-rational (resp. A'-algebraic) transducer. Since, for $a \in A$,

$$\begin{aligned} \|\mathfrak{T}\|(a) &= S_2h_2(S_1)h_2(h_1(a))h_2(P_1)P_2 = S_2h_2(S_1h_1(a)P_1)P_2 = \\ &= S_2h_2(\|\mathfrak{T}_1\|(a))P_2 = \|\mathfrak{T}_2\|(\|\mathfrak{T}_1\|(a)), \end{aligned}$$

our theorem is proved. □

Most of the definitions and results developed up to now in this section are due to Kuich [77]. They are generalizations of definitions and results which we will consider now. We specialize our definitions and results to semirings of formal power series (see Nivat [97], Jacob [66], Salomaa, Soittola [107], Kuich, Salomaa [88]). We make the following conventions for the remainder of this section: The set Σ_∞ is a fixed infinite alphabet, and Σ, possibly provided with indices, is a finite subalphabet of Σ_∞.

In connection with formal power series, our basic semiring will be $A\langle\langle\Sigma_\infty^*\rangle\rangle$, where A is *commutative*. The semiring containing all power series whose supports are contained in some Σ^* is denoted by $A\{\{\Sigma_\infty^*\}\}$, i. e.,

$$A\{\{\Sigma_\infty^*\}\} = \{r \in A\langle\langle\Sigma_\infty^*\rangle\rangle \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \text{ such that } \mathrm{supp}(r) \subseteq \Sigma^*\}.$$

It can be identified with a subsemiring of $A\langle\langle\Sigma_\infty^*\rangle\rangle$. For $\Sigma \subset \Sigma_\infty$, $A\langle\langle\Sigma^*\rangle\rangle$ is isomorphic to a subsemiring of $A\{\{\Sigma_\infty^*\}\}$. Hence, we may assume that $A\langle\langle\Sigma^*\rangle\rangle \subset A\{\{\Sigma_\infty^*\}\}$.

Furthermore, we define three subsemirings of $A\{\{\Sigma_\infty^*\}\}$, namely, the semiring of algebraic power series $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$, the semiring of rational power series $A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\}$ and the semiring of polynomials $A\{\Sigma_\infty^*\}$ by

$$\begin{aligned} A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} &= \{r \in A\{\{\Sigma_\infty^*\}\} \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \\ &\qquad \text{such that } r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle\}, \\ A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\} &= \{r \in A\{\{\Sigma_\infty^*\}\} \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \\ &\qquad \text{such that } r \in A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle\}, \text{ and} \\ A\{\Sigma_\infty^*\} &= \{r \in A\{\{\Sigma_\infty^*\}\} \mid \mathrm{supp}(r) \text{ is finite}\}. \end{aligned}$$

Moreover, we define $A\{\Sigma_\infty \cup \varepsilon\} = \{r \in A\{\Sigma_\infty^*\} \mid \mathrm{supp}(r) \subset \Sigma_\infty \cup \{\varepsilon\}\}$ and $A\{\Sigma_\infty\} = \{r \in A\{\Sigma_\infty^*\} \mid \mathrm{supp}(r) \subset \Sigma_\infty\}$. Observe that $A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\} = \mathfrak{Rat}(A\{\Sigma_\infty \cup \varepsilon\})$ and $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} = \mathfrak{Alg}(A\{\Sigma_\infty \cup \varepsilon\})$.

A multiplicative morphism $\mu : \Sigma_\infty^* \to (A\{\{\Sigma_\infty^*\}\})^{Q\times Q}$ is called a *representation* if there exists a Σ such that $\mu(x) = 0$ for $x \in \Sigma_\infty - \Sigma$. Observe that if μ is a representation, there exist only finitely many entries $\mu(x)_{q_1,q_2} \neq 0$, $x \in \Sigma_\infty$, $q_1, q_2 \in Q$. Hence, there is a Σ' such that $\mu(w)_{q_1,q_2} \in A\langle\langle\Sigma'^*\rangle\rangle$ for all $w \in \Sigma_\infty^*$. A representation can be extended to a mapping $\mu : A\langle\langle\Sigma_\infty^*\rangle\rangle \to (A\langle\langle\Sigma_\infty^*\rangle\rangle)^{Q\times Q}$ by the definition

$$\mu(r) = \mu\Big(\sum_{w\in\Sigma_\infty^*} (r,w)w\Big) = \sum_{w\in\Sigma_\infty^*} \mathrm{diag}((r,w))\mu(w), \quad r \in A\langle\langle\Sigma_\infty^*\rangle\rangle,$$

where $\mathrm{diag}(a)$ is the diagonal matrix whose diagonal entries all are equal to a. (Observe that $\mathrm{diag}((r,w))\mu(w) = (r,w) \otimes \mu(w)$, where $\otimes$ denotes the Kronecker product. For definitions and results in connection with the Kronecker product see Kuich, Salomaa [88].) Note that we are using the same notation "μ" for both mappings. However, this should not lead to any confusion. Observe that in fact μ is a mapping $\mu : A\langle\langle\Sigma_\infty^*\rangle\rangle \to (A\{\{\Sigma_\infty^*\}\})^{Q\times Q}$.

The following theorem states an important property of the extended mapping μ and is proved by an easy computation.

Theorem 4.2.10 *Let A be a commutative continuous semiring. If $\mu : \Sigma_\infty^* \to (A\{\{\Sigma_\infty^*\}\})^{Q\times Q}$ is a representation then the extended mapping $\mu : A\langle\langle\Sigma_\infty^*\rangle\rangle \to (A\{\{\Sigma_\infty^*\}\})^{Q\times Q}$ is a complete semiring morphism.*

A representation μ is called *rational* or *algebraic* if

$$\mu : \Sigma_\infty^* \to (A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\})^{Q\times Q} \quad \text{or} \quad \mu : \Sigma_\infty^* \to (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^{Q\times Q},$$

respectively.

Theorem 4.2.11 *Let A be a commutative continuous semiring and let $\mu_j : \Sigma_\infty^* \to (A\{\{\Sigma_\infty^*\}\})^{Q_j\times Q_j}$, $j = 1, 2$, be rational (resp. algebraic) representations. Then $\mu : \Sigma_\infty^* \to (A\{\{\Sigma_\infty^*\}\})^{(Q_1\times Q_2)\times(Q_1\times Q_2)}$, where $\mu(x) = \mu_2(\mu_1(x))$ for all $x \in \Sigma_\infty$, is again a rational (resp. an algebraic) representation.*

Furthermore, for all $r \in A\langle\langle\Sigma_\infty^*\rangle\rangle$, $\mu(r) = \mu_2(\mu_1(r))$.

Proof. By Theorem 2.3 (resp. Theorem 2.6), the entries of $\mu(x)$ are in $A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\}$ (resp. $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$) for all $x \in \Sigma_\infty$. Hence, μ is a rational (resp. an algebraic) representation.

We now prove the second part of our theorem. We deduce, for all $r \in A\langle\langle\Sigma_\infty^*\rangle\rangle$,

$$\begin{aligned}\mu_2(\mu_1(r)) &= \mu_2\Big(\sum_{w\in\Sigma_\infty^*} \mathrm{diag}((r,w))\mu_1(w)\Big) \\ &= \sum_{w\in\Sigma_\infty^*} \mathrm{diag}((r,w))\mu_2(\mu_1(w)) = \sum_{w\in\Sigma_\infty^*} \mathrm{diag}((r,w))\mu(w) = \mu(r).\end{aligned}$$

□

We now specialize the notions of A'-rational (resp. A'-algebraic) transducers and consider $A\{\Sigma_\infty \cup \varepsilon\}$-rational (resp. $A\{\Sigma_\infty \cup \varepsilon\}$-algebraic) transducers $\mathfrak{T} = (Q, \mu, S, P)$, where μ is a rational (resp. an algebraic) representation. Hence, there exist finite alphabets Σ and Σ' such that $\mu(x) = 0$ for $x \in \Sigma_\infty - \Sigma$ and the entries of $\mu(w)$, $w \in \Sigma^*$ are in $A^{\mathrm{rat}}\langle\langle\Sigma'^*\rangle\rangle$ (resp. $A^{\mathrm{alg}}\langle\langle\Sigma'^*\rangle\rangle$). Furthermore, we assume that the entries of S and P are in $A^{\mathrm{rat}}\langle\langle\Sigma'^*\rangle\rangle$ (resp. $A^{\mathrm{alg}}\langle\langle\Sigma'^*\rangle\rangle$). We call these $A\{\Sigma_\infty \cup \varepsilon\}$-rational (resp. $A\{\Sigma_\infty \cup \varepsilon\}$-algebraic) transducers simply *rational* (resp. *algebraic*) *transducers.*

A rational or an algebraic transducer $\mathfrak{T} = (Q, \mu, S, P)$ specified as above can be considered to be a finite automaton equipped with an output device. In a state transition from state q_1 to state q_2, $\mathfrak{T}$ reads a letter $x \in \Sigma$ and outputs the rational or algebraic power series $\mu(x)_{q_1,q_2}$. A sequence of state transitions outputs the product of the power series of the single state transitions. All sequences of length n of state transitions from state q_1 to state q_2 reading a word $w \in \Sigma^*$, $|w| = n$, output the power series $\mu(w)_{q_1,q_2}$. This output is multiplied with the correct components of the initial and the final state vector, and $S_{q_1}\mu(w)_{q_1,q_2}P_{q_2}$ is said to be the *translation* of w by transitions from q_1 to q_2. Summing up for all $q_1, q_2 \in Q$, $\sum_{q_1,q_2\in Q} S_{q_1}\mu(w)_{q_1,q_2}P_{q_2} = S\mu(w)P$ is said to be the *translation* of w by $\mathfrak{T}$. A power series $r \in A\langle\langle\Sigma_\infty^*\rangle\rangle$ is *translated* by $\mathfrak{T}$ to the power series

$$\begin{aligned}\|\mathfrak{T}\|(r) &= S\mu(r)P = S\Big(\sum_{w\in\Sigma_\infty^*} \mathrm{diag}((r,w))\mu(w)\Big)P \\ &= \sum_{w\in\Sigma_\infty^*} (r,w)S\mu(w)P = \sum_{w\in\Sigma_\infty^*} (r,w)\|\mathfrak{T}\|(w) \in A\langle\langle\Sigma'^*\rangle\rangle.\end{aligned}$$

Observe that $\|\mathfrak{T}\|(r) = \|\mathfrak{T}\|(r \odot \mathrm{char}(\Sigma^*))$. Hence, in fact, $\mathfrak{T}$ translates a power series in $A\langle\langle\Sigma^*\rangle\rangle$ to a power series in $A\langle\langle\Sigma'^*\rangle\rangle$.

Specializations of Theorems 2.7 and 2.8 yield the next result.

Corollary 4.2.12 *Assume that $\mathfrak{T}$ is a rational (resp. an algebraic) transducer and that $r \in A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ (resp. $r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$). Then $\|\mathfrak{T}\|(r) \in A^{\mathrm{rat}}\langle\langle\Sigma'^*\rangle\rangle$ (resp. $A^{\mathrm{alg}}\langle\langle\Sigma'^*\rangle\rangle$) for some Σ'.*

We now introduce the notion of a substitution. Assume that $\sigma : \Sigma_\infty^* \to A\{\{\Sigma_\infty^*\}\}$ is a representation, where $\sigma(x) = 0$ for $x \in \Sigma_\infty - \Sigma$ and the entries of $\sigma(x)$, $x \in \Sigma$, are in $A\langle\langle\Sigma'^*\rangle\rangle$. Then the mapping $\sigma : A\langle\langle\Sigma_\infty^*\rangle\rangle \to A\langle\langle\Sigma'^*\rangle\rangle$, where $\sigma(r) = \sum_{w\in\Sigma_\infty^*}(r,w)\sigma(w)$ for all $r \in A\langle\langle\Sigma_\infty^*\rangle\rangle$, is a complete semiring morphism. We call this complete semiring morphism a *substitution*. If $\sigma : \Sigma_\infty^* \to A^{\mathrm{rat}}\langle\langle\Sigma'^*\rangle\rangle$ or $\sigma : \Sigma_\infty^* \to A^{\mathrm{alg}}\langle\langle\Sigma'^*\rangle\rangle$ then we call the substitution *rational* or *algebraic*, respectively. Clearly, a rational or algebraic substitution is a particular rational or algebraic transduction, respectively.

Corollary 4.2.13 *If σ is a rational (resp. algebraic) substitution and r is a rational (resp. algebraic) power series then $\sigma(r)$ is again a rational (resp. algebraic) power series.*

We now turn to language theory (see Berstel [4]). The basic semiring is now $2^{\Sigma_\infty^*}$. Let $\mathfrak{L}(\Sigma_\infty)$ be the subset of $2^{\Sigma_\infty^*}$ containing all formal languages, i. e.,

$$\mathfrak{L}(\Sigma_\infty) = \{L \mid L \subseteq \Sigma^*,\ \Sigma \subset \Sigma_\infty\}.$$

A representation is now a multiplicative morphism $\mu : \Sigma_\infty^* \to \mathfrak{L}(\Sigma_\infty)^{Q\times Q}$. If the representation is rational or algebraic then the entries of $\mu(x)$, $x \in \Sigma_\infty$, are regular or context-free languages, respectively. A rational or an algebraic transducer $\mathfrak{T} = (Q, \mu, S, P)$ is specified by a rational or an algebraic representation μ as above; moreover, the entries of S and P are regular or context-free languages, respectively.

Corollary 4.2.14 *Assume that $\mathfrak{T}$ is a rational (resp. algebraic) transducer. Then $\|\mathfrak{T}\|(L)$ is a regular (resp. context-free) language if L is regular (resp. context-free).*

A substitution is now a complete semiring morphism $\sigma : \Sigma_\infty^* \to \mathfrak{L}(\Sigma_\infty)$ such that $\sigma(x) = \emptyset$ for $x \in \Sigma_\infty - \Sigma$. It is defined by the values of $\sigma(x) \subseteq \Sigma'^*$ for $x \in \Sigma$. Since σ is a complete semiring morphism, we obtain $\sigma(w) = \sigma(x_1)\ldots\sigma(x_n) \subseteq \Sigma'^*$ for $w = x_1 \ldots x_n$, $x_i \in \Sigma$, $1 \le i \le n$, and $\sigma(L) = \bigcup_{w\in L}\sigma(w) \subseteq \Sigma'^*$ for $L \subseteq \Sigma^*$.

A substitution is called *regular* or *context-free* if each symbol is mapped to a regular or context-free language, respectively.

Corollary 4.2.15 *A regular (resp. context-free) substitution maps a regular (resp. context-free) language to a regular (resp. context-free) language.*

4.3 Abstract families of elements

We start with some basic definitions. Given $A' \subseteq A$, we define $[A'] \subseteq A$ to be the least complete subsemiring of A that contains A'. The semiring $[A']$ is called the *complete semiring generated by* A'. Each element a of $[A']$ can be generated from elements of A' by multiplication and summation (including "infinite summation"):

$$a \in [A'] \quad \text{iff} \quad a = \sum_{i \in I} a_{i1} \dots a_{in_i},$$

where I is an index set, $a_{ij} \in A'$ and $n_i \geq 0$.

From now on, we assume that $[A'] = A$. Furthermore we make the notational convention that all sets Q, possibly provided with indices, are finite and non-empty, and are subsets of some fixed countably infinite set Q_∞ with the following property: if $q_1, q_2 \in Q_\infty$ then $(q_1, q_2) \in Q_\infty$.

Consider the family of all semiring morphisms $h : A \to A^{Q \times Q}$, $Q \subset Q_\infty$, Q finite, and let $\mathfrak{H}$ be a non-empty subfamily of this family. Then we define the subset $[\mathfrak{H}]$ of A as follows. For $h \in \mathfrak{H}$, $h : A \to A^{Q \times Q}$, let $B_h = [\{h(a)_{q_1,q_2} \mid a \in A, q_1, q_2 \in Q\}]$. Then $[\mathfrak{H}] = \bigcup_{h \in \mathfrak{H}} B_h$. A family of morphisms $\mathfrak{H}$ is called *closed under matricial composition* if the following conditions are satisfied for arbitrary morphisms $h : A \to A^{Q \times Q}$ and $h' : A \to A^{Q' \times Q'}$ in $\mathfrak{H}$:

(i) $A' \subseteq [\mathfrak{H}]$.

(ii) For each $a \in [\mathfrak{H}]$ there is an $h_a \in \mathfrak{H}$ with $h_a(a) = a$.

(iii) If $\bar{Q} \subset Q_\infty$ and there exists a bijection $\pi : \bar{Q} \to Q$, then $\bar{h} : A \to A^{\bar{Q} \times \bar{Q}}$, defined by $\bar{h}(a)_{q_1,q_2} = h(a)_{\pi(q_1),\pi(q_2)}$ for all $a \in A$, $q_1, q_2 \in \bar{Q}$, is in $\mathfrak{H}$.

(iv) The functional composition $h \circ h' : A \to A^{(Q' \times Q) \times (Q' \times Q)}$ is again in $\mathfrak{H}$.

(v) If $Q \cap Q' = \emptyset$ then the mapping $h + h' : A \to A^{(Q \cup Q') \times (Q \cup Q')}$ defined by

$$(h + h')(a) = \begin{pmatrix} h(a) & 0 \\ 0 & h'(a) \end{pmatrix}, \quad a \in A,$$

where the blocks are indexed by Q and Q', is again in $\mathfrak{H}$.

From now on, we assume that $\mathfrak{H}$ is a non-empty family of *complete A'-rational* semiring morphisms that is closed under matricial composition.

Next we deal with properties of $[\mathfrak{H}]$ and denote $\mathfrak{B} = \{B_h \mid h \in \mathfrak{H}\}$.

Lemma 4.3.1 *$\mathfrak{B}$ is directed by set inclusion and for every finite $F \subseteq [\mathfrak{H}]$ there exists an $h \in \mathfrak{H}$ such that $F \subseteq B_h$.*

Proof. Consider $B_{h_1}, B_{h_2} \in \mathfrak{B}$, where $h_1 : A \to A^{Q_1 \times Q_1}$ and $h_2 : A \to A^{Q_2 \times Q_2}$. Assume that $Q_1 \cap Q_2 = \emptyset$ by (iii). Then $B_{h_1}, B_{h_2} \subseteq B_{h_1+h_2}$. From this, $F \subseteq B_h$ follows directly. $\square$

Theorem 4.3.2 $[\mathfrak{H}]$ *is a starsemiring.*

Proof. Since $A' \subseteq [\mathfrak{H}]$, we infer that $0, 1 \in [\mathfrak{H}]$. The closure of $[\mathfrak{H}]$ under addition and multiplication follows from Lemma 3.1. Now consider $a \in [\mathfrak{H}]$. Then there is a $B \in \mathfrak{B}$ with $a \in B$. Since B is a complete subsemiring of A, $a^* \in B \subseteq [\mathfrak{H}]$. □

An $\mathfrak{H}$-A'-*rational transducer* is an A'-rational transducer $\mathfrak{T} = (Q, h, S, P)$ where $h : A \to A^{Q\times Q}$ is in $\mathfrak{H}$. A mapping $\tau : A \to A$ is called an $\mathfrak{H}$-A'-*rational transduction* if there exists an $\mathfrak{H}$-A'-rational transducer $\mathfrak{T}$ such that $\tau(a) = \|\mathfrak{T}\|(a)$ for all $a \in A$.

An $\mathfrak{H}$-A'-*family of elements* is just a subset of $[\mathfrak{H}]$. Let $\mathfrak{L}$ be an $\mathfrak{H}$-A'-family of elements. We define

$$\mathfrak{M}(\mathfrak{L}) = \{\tau(a) \mid a \in \mathfrak{L},\ \tau : A \to A \text{ is an } \mathfrak{H}\text{-}A'\text{-rational transduction}\}.$$

Note that we always have $\mathfrak{L} \subseteq \mathfrak{M}(\mathfrak{L})$ by (ii). The family $\mathfrak{L}$ is said to be *closed under* $\mathfrak{H}$-A'-*rational transductions* if $\mathfrak{M}(\mathfrak{L}) \subseteq \mathfrak{L}$.

The notation $\mathfrak{F}(\mathfrak{L})$ is used for the smallest substarsemiring of A that is closed under $\mathfrak{H}$-A'-rational transductions and contains $\mathfrak{L}$. Note that $\mathfrak{M}(\mathfrak{L}) \subseteq \mathfrak{F}(\mathfrak{L})$. (We have tried to use in our notation letters customary in AFL theory to aid the reader familiar with this theory. See Ginsburg [53].)

An $\mathfrak{H}$-A'-family of elements $\mathfrak{L}$ is called $\mathfrak{H}$-A'-*abstract family of elements* (briefly $\mathfrak{H}$-A'-AFE) if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$. These $\mathfrak{H}$-A'-abstract families of elements will now be characterized. We assume $\mathfrak{H}$ to be fixed. Recall that, by our convention, $\mathfrak{H}$ is a non-empty family of complete A'-rational semiring morphisms that is closed under matricial composition.

First, we consider an important special case.

Theorem 4.3.3 *If $\mathfrak{H}$ is a non-empty family of complete A'-rational semiring morphisms that is closed under matricial composition then $\mathfrak{Rat}(A')$ is a $\mathfrak{H}$-A'-AFE.*

Proof. By Theorems 2.7 and 3.2. □

Observe that every $\mathfrak{H}$-A'-AFE $\mathfrak{L}$ satisfies $\mathfrak{Rat}(A') \subseteq \mathfrak{L}$ by definition. Thus by Theorem 3.3, $\mathfrak{Rat}(A')$ is the smallest $\mathfrak{H}$-A'-AFE.

In the sequel, $\Delta = \{\mathbf{a} \mid a \in A\} \cup Z$ is an alphabet. Here $\{\mathbf{a} \mid a \in A\}$ is a copy of A and Z is an infinite alphabet of variables. A multiplicative monoid morphism $h : \Delta^* \to A^{Q\times Q}$ is *compatible with* $\mathfrak{H}$ if the following conditions are satisfied:

(i) The mapping $h' : A \to A^{Q\times Q}$ defined by $h'(a) = h(\mathbf{a})$, $a \in A$, is a complete A'-rational semiring morphism in $\mathfrak{H}$,

(ii) $h(\mathbf{a}), h(z) \in \mathfrak{Rat}(A')^{Q\times Q}$ for $a \in A'$, $z \in Z$, and $h(z) = 0$ for almost all variables $z \in Z$.

If $h : \Delta^* \to A^{Q\times Q}$ is compatible with $\mathfrak{H}$ and if $h_1 : A \to A^{Q_1\times Q_1}$ is a complete A'-rational semiring morphism in $\mathfrak{H}$ then $h_1 \circ h : \Delta^* \to A^{(Q\times Q_1)\times(Q\times Q_1)}$ is again compatible with $\mathfrak{H}$.

We introduce now the notions of a *type* T, a *T-matrix*, a *T-automaton* and the *automaton representing* T. Intuitively speaking this means the following. A T-automaton is a finite automaton with an additional working tape, whose contents are stored in the states of the T-automaton. The type T of the T-automaton indicates how information can be retrieved from the working tape. For instance, pushdown automata can be viewed as automata of a specific type.

A *type* is a quadruple
$$(\Gamma_T, \Delta_T, T, \pi_T),$$
where

(i) Γ_T is the set of *storage symbols*,

(ii) $\Delta_T \subseteq \{\mathbf{a} \mid a \in A'\} \cup Z$ is the alphabet of *instructions*,

(iii) $T \in (\mathbb{N}^\infty\{\Delta_T\})^{\Gamma_T^* \times \Gamma_T^*}$ is the *type matrix*,

(iv) $\pi_T \in \Gamma_T^*$ is the *initial contents of the working tape.*

In the sequel we often speak of the type T if Γ_T, Δ_T and π_T are understood.

A matrix $M \in (\mathfrak{Rat}(A')^{Q\times Q})^{\Gamma_T^* \times \Gamma_T^*}$ is called a *T-matrix* if there exists a monoid morphism $h : \Delta^* \to A^{Q\times Q}$ that is compatible with $\mathfrak{H}$ such that $M = h(T)$. If $M = h(T)$ is a T-matrix and $h' : A \to A^{Q'\times Q'}$ is a complete A'-rational semiring morphism in $\mathfrak{H}$ then, by Theorems 2.2.5 and 2.3, $h' \circ h$ is compatible with $\mathfrak{H}$ and $h'(M) = h'(h(T))$ is again a T-matrix.

A *T-automaton*
$$\mathfrak{A} = (Q, \Gamma_T, M, S, \pi_T, P)$$
is defined by

(i) a finite set Q of *states*,

(ii) a T-matrix M, called the *transition matrix*,

(iii) $S \in \mathfrak{Rat}(A')^{1\times Q}$, called the *initial state vector*,

(iv) $P \in \mathfrak{Rat}(A')^{Q\times 1}$, called the *final state vector*.

Observe that Γ_T and π_T are determined by T. The *behavior* of the T-automaton $\mathfrak{A}$ is given by
$$\|\mathfrak{A}\| = S(M^*)_{\pi_T,\varepsilon}P.$$

Clearly, for each such T-automaton $\mathfrak{A}$ there exists a $\mathfrak{Rat}(A')$-automaton $\mathfrak{A}' = (\Gamma_T^* \times Q, M', S', P')$ such that $\|\mathfrak{A}'\| = \|\mathfrak{A}\|$. This is achieved by choosing $M'_{(\pi_1,q_1),(\pi_2,q_2)} = (M_{\pi_1,\pi_2})_{q_1,q_2}$, $S'_{(\pi_T,q)} = S_q$, $S'_{(\pi,q)} = 0$, $\pi \neq \pi_T$, $P'_{(\varepsilon,q)} = P_q$, $P'_{(\pi,q)} = 0$, $\pi \neq \varepsilon$, $q_1, q_2, q \in Q$, $\pi_1, \pi_2, \pi \in \Gamma_T^*$.

The *automaton $\mathfrak{A}_T$ representing a type* $(\Gamma_T, \Delta_T, T, \pi_T)$ is an $\mathbb{N}^\infty\{\Delta_T\}$-automaton defined by
$$\mathfrak{A}_T = (\Gamma_T^*, T, S_T, P_T),$$

where $(S_T)_{\pi_T} = \varepsilon$, $(S_T)_\pi = 0$, $\pi \in \Gamma_T^*$, $\pi \neq \pi_T$, $(P_T)_\varepsilon = \varepsilon$, $(P_T)_\pi = 0$, $\pi \in \Gamma_T^*$, $\pi \neq \varepsilon$. The behavior of $\mathfrak{A}_T$ is $\|\mathfrak{A}_T\| = (T^*)_{\pi_T,\varepsilon}$.

In a certain sense, $\mathfrak{A}_T$ generates an A'-family of elements. Let $\hat{A} = \mathbb{N}^\infty\langle\langle\Delta_T^*\rangle\rangle$ and $\hat{A}' = \{d \mid d \in \Delta_T\} \cup \{\varepsilon, 0\}$ and consider $(\hat{A}', A')$-rational transducers $\mathfrak{T} = (Q, h, S, P)$, where $h : \Delta^* \to A^{Q\times Q}$ is a monoid morphism compatible with $\mathfrak{H}$. Given an $(\hat{A}', A')$-rational transducer $\mathfrak{T} = (Q, h, S, P)$, consider the T-automaton $\mathfrak{A} = (Q, \Gamma_T, M, S, \pi_T, P)$, where $M = h(T)$. We apply $\mathfrak{T}$ to $\|\mathfrak{A}_T\|$ and obtain

$$\|\mathfrak{T}\|(\|\mathfrak{A}_T\|) = Sh((T^*)_{\pi_T,\varepsilon})P = S(M^*)_{\pi_T,\varepsilon}P = \|\mathfrak{A}\|.$$

Conversely, for each T-automaton $\mathfrak{A}$ there exists an $(\hat{A}', A')$-rational transducer $\mathfrak{T}$ such that $\|\mathfrak{A}\| = \|\mathfrak{T}\|(\|\mathfrak{A}_T\|)$.

We define now the set

$$\mathfrak{Rat}_T(A') = \{\|\mathfrak{A}\| \mid \mathfrak{A} \text{ is a } T\text{-automaton}\} \subseteq A.$$

Hence, $\mathfrak{Rat}_T(A')$ contains exactly all elements $\|\mathfrak{T}\|(\|\mathfrak{A}_T\|)$, where $\mathfrak{T} = (Q, h, S, P)$ is an $(\hat{A}', A')$-rational transducer and $h : \Delta^* \to A^{Q\times Q}$ is compatible with $\mathfrak{H}$. Observe that in the definitions of a T-matrix, of a T-automaton and of $\mathfrak{Rat}_T(A')$, A' and $\mathfrak{H}$ are implicitly present.

It will turn out that $\mathfrak{Rat}_T(A')$ is an $\mathfrak{H}$-A'-AFE if T is a restart type. Here a type $(\Gamma_T, \Delta_T, T, \pi_T)$ is called a *restart type* if $\pi_T = \varepsilon$ and the non-null entries of T satisfy the conditions $T_{\varepsilon,\varepsilon} = z^0 \in Z$, $T_{\varepsilon,\pi} \in \mathbb{N}^\infty\{Z - \{z^0\}\}$, $T_{\pi,\pi'} \in \mathbb{N}^\infty\{\Delta_T - \{z^0\}\}$ for all $\pi \in \Gamma_T^+$, $\pi' \in \Gamma_T^*$, and for some distinguished instruction $z^0 \in \Delta_T$. Observe that the working tape is empty at the beginning of the computation.

Now we want to show that $\mathfrak{Rat}_T(A')$ is an $\mathfrak{H}$-A'-AFE.

Theorem 4.3.4 (Kuich [77], Theorems 4.2, 4.3, 4.4, 4.5) *If T is a restart type then $\mathfrak{Rat}_T(A')$ is a starsemiring containing $\mathfrak{Rat}(A')$ and closed under $\mathfrak{H}$-A'-rational transductions.*

Proof. We prove only closure under star and under $\mathfrak{H}$-A'-rational transductions.

Assume that $\mathfrak{A} = (Q, \Gamma_T, M, S, \varepsilon, P)$, where $M = h(T)$, is a T-automaton. We give the construction of a T-automaton $\mathfrak{A}' = (Q, \Gamma_T, M', S, \varepsilon, P)$ with $\|\mathfrak{A}'\| = \|\mathfrak{A}\|^+$.

Let $h' : \Delta^* \to A^{Q\times Q}$ be defined by $h'(z^0) = h(z^0) + PS$, $h'(d) = h(d)$, $d \in \Delta - \{z^0\}$. Then h' is compatible with $\mathfrak{H}$. Define $\tilde{M} \in (\mathfrak{Rat}_T(A')^{Q\times Q})^{\Gamma_T^*\times\Gamma_T^*}$ by $\tilde{M}_{\varepsilon,\varepsilon} = PS$, $\tilde{M}_{\pi_1,\pi_2} = 0$ for $(\pi_1, \pi_2) \neq (\varepsilon, \varepsilon)$. Let $M' = h'(T)$. Then we obtain $M' = M + \tilde{M}$ and $M'^* = (M^*\tilde{M})^* M^*$. We compute $(M^*\tilde{M})_{\varepsilon,\pi} = 0$ for $\pi \in \Gamma_T^+$, $(M^*\tilde{M})_{\varepsilon,\varepsilon} = (M^*)_{\varepsilon,\varepsilon}PS$, $((M^*\tilde{M})^*)_{\varepsilon,\varepsilon} = ((M^*)_{\varepsilon,\varepsilon}PS)^*$ and $((M^*\tilde{M})^*)_{\varepsilon,\pi} = 0$, $\pi \in \Gamma_T^+$. Hence,

$$(M'^*)_{\varepsilon,\varepsilon} = ((M^*\tilde{M})^*)_{\varepsilon,\varepsilon}(M^*)_{\varepsilon,\varepsilon} = ((M^*)_{\varepsilon,\varepsilon}PS)^*(M^*)_{\varepsilon,\varepsilon}$$

and

$$\|\mathfrak{A}'\| = S((M^*)_{\varepsilon,\varepsilon} PS)^*(M^*)_{\varepsilon,\varepsilon} P = \|\mathfrak{A}\|^+ .$$

Since $\mathfrak{Rat}(A')$ is a subset of $\mathfrak{Rat}_T(A')$ and $\mathfrak{Rat}_T(A')$ is closed under addition, $\|\mathfrak{A}\|^+ + 1 = \|\mathfrak{A}\|^*$ is in $\mathfrak{Rat}_T(A')$.

We now prove closure under $\mathfrak{H}$-A'-rational transductions. Assume that $\mathfrak{A} = (Q, \Gamma_T, M, S, \varepsilon, P)$, where $M = h(T)$, is a T-automaton and that $\mathfrak{T} = (Q', h', S', P')$ is an $\mathfrak{H}$-A'-rational transducer. Since $h : \Delta^* \to A^{Q\times Q}$ is compatible with $\mathfrak{H}$ and $h' : A \to A^{Q'\times Q'}$ is in $\mathfrak{H}$, the monoid morphism $h' \circ h : \Delta^* \to A^{(Q\times Q')\times(Q\times Q')}$ is again compatible with $\mathfrak{H}$. We prove now that the behavior of the T-automaton $\mathfrak{A}' = (Q \times Q', \Gamma_T, (h' \circ h)(T), S'h'(S), \varepsilon, h'(P)P')$ is equal to $\|\mathfrak{T}\|(\|\mathfrak{A}\|)$:

$$\begin{aligned} \|\mathfrak{A}'\| &= S'h'(S)(((h' \circ h)(T))^*)_{\varepsilon,\varepsilon} h'(P)P' = S'h'(S)h'((h(T)^*)_{\varepsilon,\varepsilon})h'(P)P' \\ &= S'h'(S(h(T)^*)_{\varepsilon,\varepsilon} P)P' = S'h'(\|\mathfrak{A}\|)P' = \|\mathfrak{T}\|(\|\mathfrak{A}\|). \end{aligned}$$

□

Theorem 4.3.5 *If T is a restart type then $\mathfrak{Rat}_T(A')$ is a $\mathfrak{H}$-A'-AFE.*

Proof. By Theorem 3.4, we have only to show that $\mathfrak{Rat}_T(A') \subseteq [\mathfrak{H}]$. Assume that $a = S(M^*)_{\pi_T,\varepsilon} P$. By Lemma 3.1, there are B_S and $B_P \in \mathfrak{B}$ containing the entries of S and P, respectively. Similarly, for every $z \in Z$, there is $B_z \in \mathfrak{B}$ containing all entries of $h(z)$, $z \in Z$. Moreover, $B_h \in \mathfrak{B}$ contains all entries of all $h(\mathbf{a})$, $a \in A$. Now by Lemma 3.1 there is a $B_0 \in \mathfrak{B}$ containing B_S, B_P, B_h, and all B_z, $z \in Z$, with $h(z) \neq 0$. (Note that we only have to consider finitely many B_z, $z \in Z$.) Then $a \in B_0 \subseteq [\mathfrak{H}]$. □

We now consider a second special case and show that $\mathfrak{Alg}(A')$ is a $\mathfrak{H}$-A'-AFE. In this section Γ_∞ denotes a countably infinite set of storage symbols containing the symbol p_0. We now introduce the two types T_Γ^{pd} and T_Γ^{rpd}, called pushdown type and reset pushdown type with pushdown alphabet Γ, respectively.

The *pushdown type with pushdown alphabet* Γ $(\Gamma, \Delta_\Gamma^{\mathrm{pd}}, T_\Gamma^{\mathrm{pd}}, p_0)$ is specified as follows: $\Gamma \subset \Gamma_\infty$ is a finite set of storage symbols containing p_0, $\Delta_\Gamma^{\mathrm{pd}} = \{d_{p,\pi} \mid p \in \Gamma,\ \pi \in \Gamma^*\} \subseteq Z$, and the non-null entries of T_Γ^{pd} are

$$(T_\Gamma^{\mathrm{pd}})_{p\pi',\pi\pi'} = d_{p,\pi} \text{ for all } p \in \Gamma,\ \pi \in \Gamma^* .$$

The *reset pushdown type with pushdown alphabet* Γ $(\Gamma, \Delta_\Gamma^{\mathrm{rpd}}, T_\Gamma^{\mathrm{rpd}}, \varepsilon)$ is specified as follows: $\Gamma \subset \Gamma_\infty$ is a finite non-empty set of storage symbols not necessarily containing p_0, $\Delta_\Gamma^{\mathrm{rpd}} = \{c_{p,\pi} \mid p \in \Gamma \cup \{\varepsilon\},\ \pi \in \Gamma^*\} \subseteq Z$, and the non-null entries of T_Γ^{rpd} are

$$(T_\Gamma^{\mathrm{rpd}})_{\varepsilon,\pi} = c_{\varepsilon,\pi}, \qquad (T_\Gamma^{\mathrm{rpd}})_{p\pi',\pi\pi'} = c_{p,\pi} \text{ for all } p \in \Gamma,\ \pi \in \Gamma^* .$$

Since $\Delta_\Gamma^{\mathrm{pd}} \subseteq Z$, the T_Γ^{pd}-automata are exactly the $\mathfrak{Rat}(A')$-pushdown automata with pushdown alphabet Γ. Hence,

$$\bigcup_{\Gamma \subset \Gamma_\infty \text{ finite},\ p_0 \in \Gamma} \mathfrak{Rat}_{T_\Gamma^{\mathrm{pd}}}(A') = \mathfrak{Alg}(A')$$

by Corollary 3.4.9 and Theorem 2.3.13. The T_Γ^{rpd}-automata are called $\mathfrak{Rat}(A')$-reset pushdown automata with pushdown alphabet Γ. Their computations are defined like the computations of the $\mathfrak{Rat}(A')$-pushdown automata (see Section 3.4) with the exception that they start with the empty tape and can, in a computational step, write a pushdown symbol on the empty tape. Observe that T_Γ^{rpd} is a restart type.

Lemma 4.3.6 *For all $\Gamma \subset \Gamma_\infty$, Γ finite and $p_0 \notin \Gamma$,*

$$\mathfrak{Rat}_{T_\Gamma^{\mathrm{rpd}}}(A') \subseteq \mathfrak{Rat}_{T_{\Gamma\cup\{p_0\}}^{\mathrm{pd}}}(A') .$$

Proof. Consider the type $(\Gamma \cup \{p_0\}, \Delta_T, T, p_0)$, where $\Delta_T = \{c_{p,\pi} \mid p \in \Gamma \cup \{\varepsilon\},\ \pi \in \Gamma^*\} \cup \{c\}$ and the non-null entries of T are

$$\begin{array}{l} T_{p\rho,\pi\rho} = c_{p,\pi}, \quad T_{p_0\rho,\pi p_0\rho} = c_{\varepsilon,\pi}, \quad T_{p_0\rho,\rho} = c \\ \text{for all } p \in \Gamma,\ \pi \in \Gamma^*, \rho \in (\Gamma \cup \{p_0\})^* . \end{array}$$

Observe that the following equalities for the $\Gamma^*\{p_0\} \times \Gamma^*\{p_0\}$-block and $\Gamma^*\{p_0\} \times ((\Gamma \cup \{p_0\})^* - (\Gamma^*\{p_0\} \cup \{\varepsilon\}))$-block of T hold:

$$T(\Gamma^*\{p_0\}, \Gamma^*\{p_0\})_{\pi_1 p_0, \pi_2 p_0} = (T_\Gamma^{\mathrm{rpd}})_{\pi_1,\pi_2} \text{ for all } \pi_1, \pi_2 \in \Gamma^* ,$$

and

$$T(\Gamma^*\{p_0\}, (\Gamma \cup \{p_0\})^* - (\Gamma^*\{p_0\} \cup \{\varepsilon\})) = 0 .$$

Hence, by a symmetric version of Corollary 3.2.2 we infer that

$$\begin{array}{l} (T^*)_{p_0,\varepsilon} = (T(\Gamma^*\{p_0\} \cup \{\varepsilon\}, \Gamma^*\{p_0\} \cup \{\varepsilon\})^*)_{p_0,\varepsilon} = \\ (T(\Gamma^*\{p_0\}, \Gamma^*\{p_0\})^* T(\Gamma^*\{p_0\}, \{\varepsilon\}))_{p_0,\varepsilon} = \\ (T(\Gamma^*\{p_0\}, \Gamma^*\{p_0\})^*)_{p_0,p_0} c = \\ ((T^{\mathrm{rpd}})^*)_{\varepsilon,\varepsilon} c . \end{array}$$

This implies the equality

$$||\mathfrak{A}_T|| = ||\mathfrak{A}_{T_\Gamma^{\mathrm{rpd}}}|| c .$$

Hence, we obtain

$$\mathfrak{Rat}_T(A') = \mathfrak{Rat}_{T_\Gamma^{\mathrm{rpd}}}(A') .$$

Consider now the type $(\Gamma \cup \{p_0\}, \Delta_{T'}, T', p_0)$ where $\Delta_{T'} = \{d_{p,\pi} \mid p \in \Gamma,\ \pi \in \Gamma^*\} \cup \{d_{p_0,\pi p_0} \mid \pi \in \Gamma^*\} \cup \{d_{p_0,\varepsilon}\}$ and the non-null entries of T' are

$$\begin{array}{l} T'_{p\rho,\pi\rho} = d_{p,\pi}, \quad T'_{p_0\rho,\pi p_0\rho} = d_{p_0,\pi p_0}, \quad T'_{p_0\rho,\rho} = d_{p_0,\varepsilon} \\ \text{for all } p \in \Gamma,\ \pi \in \Gamma^*,\ \rho \in (\Gamma \cup \{p_0\})^* . \end{array}$$

Observe that T' is obtained from T by relabeling. Hence, we infer

$$\mathfrak{Rat}_{T'}(A') = \mathfrak{Rat}_T(A') .$$

Moreover, for all $\rho_1, \rho_2 \in (\Gamma \cup \{p_0\})^*$,

$$T'_{\rho_1,\rho_2} = (T_{\Gamma\cup\{p_0\}}^{\mathrm{pd}})_{\rho_1,\rho_2} \odot \mathrm{char}(\Delta_{T'}^*) .$$

Hence, we infer

$$(T'^{*})_{p_0,\varepsilon} = ((T^{\mathrm{pd}}_{\Gamma\cup\{p_0\}})^{*})_{p_0,\varepsilon} \odot \mathrm{char}(\Delta^{*}_{T'})$$

and

$$||\mathfrak{A}_{T'}|| = ||\mathfrak{A}_{T^{\mathrm{pd}}_{\Gamma\cup\{p_0\}}}|| \odot \mathrm{char}(\Delta^{*}_{T'})\,.$$

The last equality implies

$$\mathfrak{Rat}_{T'}(A') \subseteq \mathfrak{Rat}_{T^{\mathrm{pd}}_{\Gamma\cup\{p_0\}}}(A')\,.$$

□

Lemma 4.3.7 *For all* $\Gamma \subset \Gamma_\infty$*,* Γ *finite and* $p_0 \in \Gamma$*,*

$$\mathfrak{Rat}_{T^{\mathrm{pd}}_{\Gamma}}(A') \subseteq \mathfrak{Rat}_{T^{\mathrm{rpd}}_{\Gamma}}(A')\,.$$

Proof. Consider the type $(\Gamma, \Delta_T, T, p_0)$, where $\Delta_T = \{c_{p,\pi} \mid p \in \Gamma,\ \pi \in \Gamma^*\}$ and the non-null entries of T are

$$T_{p\pi',\pi\pi'} = c_{p,\pi} \text{ for all } p \in \Gamma,\ \pi \in \Gamma^*\,.$$

Observe that T is obtained from T^{pd}_{Γ} by relabeling. Hence, we obtain

$$\mathfrak{Rat}_{T}(A') = \mathfrak{Rat}_{T^{\mathrm{pd}}_{\Gamma}}(A')\,.$$

Moreover, we have

$$T(\Gamma^+, \{\varepsilon\}) = T^{\mathrm{rpd}}_{\Gamma}(\Gamma^+, \{\varepsilon\}) \quad \text{and} \quad T(\Gamma^+, \Gamma^+) = T^{\mathrm{rpd}}_{\Gamma}(\Gamma^+, \Gamma^+)\,.$$

By Theorem 3.2.1, we obtain

$$\begin{aligned}&((T^{\mathrm{rpd}}_{\Gamma})^{*})_{\varepsilon,\varepsilon} = (c_{\varepsilon,\varepsilon} + T^{\mathrm{rpd}}_{\Gamma}(\{\varepsilon\}, \Gamma^+)T(\Gamma^+, \Gamma^+)^{*}T(\Gamma^+, \{\varepsilon\}))^{*} = \\ &(c_{\varepsilon,\varepsilon} + T^{\mathrm{rpd}}_{\Gamma}(\{\varepsilon\}, \Gamma^+)T^{*}(\Gamma^+, \{\varepsilon\}))^{*}\,.\end{aligned}$$

Hence, by Theorem 4.2.3, we infer that

$$\mathfrak{Rat}_{T}(A') \subseteq \mathfrak{Rat}_{T^{\mathrm{rpd}}_{\Gamma}}(A')\,,$$

i. e.,

$$\mathfrak{Rat}_{T}(A') = \mathfrak{Rat}_{T^{\mathrm{pd}}_{\Gamma}}(A') \subseteq \mathfrak{Rat}_{T^{\mathrm{rpd}}_{\Gamma}}(A')\,.$$

□

We now introduce two more types T^{pd} and T^{rpd}, called pushdown type and reset pushdown type, respectively. The *pushdown type* $(\Gamma_\infty, \Delta^{\mathrm{pd}}, T^{\mathrm{pd}}, p_0)$ is specified as follows: $\Delta^{\mathrm{pd}} = \{d_{p,\pi} \mid p \in \Gamma_\infty,\ \pi \in \Gamma^*_\infty\} \subseteq Z$, and the non-null entries of T^{pd} are

$$(T^{\mathrm{pd}})_{p\pi',\pi\pi'} = d_{p,\pi} \text{ for all } p \in \Gamma_\infty,\ \pi, \pi' \in \Gamma^*_\infty\,.$$

The *reset pushdown type* $(\Gamma_\infty, \Delta^{\mathrm{rpd}}, T^{\mathrm{rpd}}, \varepsilon)$ is specified as follows: $\Delta^{\mathrm{rpd}} = \{c_{p,\pi} \mid p \in \Gamma_\infty \cup \{\varepsilon\},\ \pi \in \Gamma_\infty^*\} \subseteq Z$, and the non-null entries of T^{rpd} are

$$(T^{\mathrm{rpd}})_{\varepsilon,\pi} = c_{\varepsilon,\pi}, \qquad (T^{\mathrm{rpd}})_{p\pi',\pi\pi'} = c_{p,\pi} \text{ for all } p \in \Gamma_\infty,\ \pi,\pi' \in \Gamma_\infty^*\,.$$

The Lemmas 3.6 and 3.7 imply the following theorem (see also Kuich, Salomaa [88], Theorem 13.15).

Theorem 4.3.8

$$\mathfrak{Rat}_{T^{\mathrm{rpd}}}(A') = \mathfrak{Rat}_{T^{\mathrm{rpd}}}(A') = \mathfrak{Alg}(A')\,.$$

Corollary 4.3.9 *If $\mathfrak{H}$ is a non-empty family of complete A'-rational semiring morphisms that is closed under matricial composition then $\mathfrak{Alg}(A')$ is a $\mathfrak{H}$-A'-AFE.*

Proof. The type T^{rpd} is a restart type. Our corollary follows then by Theorem 3.5. □

In order to get a complete characterization of $\mathfrak{H}$-A'-AFEs we need a result "converse" to Theorem 3.5. Let $\mathfrak{L} \subseteq [\mathfrak{H}]$ be an $\mathfrak{H}$-A'-AFE. Then we construct a restart type T such that $\mathfrak{L} = \mathfrak{Rat}_T(A')$. The construction will be relative to a fixed $\mathfrak{R} \subseteq \mathfrak{L}$ with $\mathfrak{F}(\mathfrak{R}) = \mathfrak{L}$. For each $b \in \mathfrak{R}$ there exists an index set I_b such that $b = \sum_{i \in I_b} a_{i1} \dots a_{in_i}$, $a_{ij} \in A'$, i. e.,

$$\mathfrak{R} = \{b \mid b = \sum_{i \in I_b} a_{i1} \dots a_{in_i}\}.$$

Such a representation of $\mathfrak{L}$ is possible since $\mathfrak{R} \subseteq \mathfrak{L} = \mathfrak{F}(\mathfrak{L}) \subseteq [\mathfrak{H}]$. The restart type $(\Gamma_T, \Delta_T, T, \varepsilon)$ is defined by

(i) $\Gamma_T = \bigcup_{b \in \mathfrak{L}} \Delta_b$, where $\Delta_b = \{\mathbf{a}_b \mid a \in A'\}$ is a copy of A' for $b \in \mathfrak{R}$,

(ii) $\Delta_T = \{\mathbf{a} \mid a \in A'\} \cup \{z^0\} \cup \{z_b \mid b \in \mathfrak{R}\}$,

(iii) $T \in (\mathbb{N}^\infty\{\Delta_T\})^{\Gamma_T^* \times \Gamma_T^*}$, where the non-null entries of T are
$T_{\varepsilon,\varepsilon} = z^0$,
$T_{\varepsilon,\mathbf{a}_b} = z_b$ for $\mathbf{a}_b \in \Delta_b$, $b \in \mathfrak{R}$,
$T_{\pi\mathbf{a}_b,\pi\mathbf{a}_b\mathbf{a}'_b} = \mathbf{a}$ for $\pi \in \Delta_b^*$, $\mathbf{a}_b, \mathbf{a}'_b \in \Delta_b$, $b \in \mathfrak{R}$,
$T_{\pi\mathbf{a}_b,\varepsilon} = (\sum 1)\mathbf{a}$ for $\pi \in \Delta_b^*$, $\mathbf{a}_b \in \Delta_b$, $b \in \mathfrak{R}$, where the summation ranges over all $i \in I_b$ such that $(\mathbf{a}_{i1})_b \dots (\mathbf{a}_{in_i})_b = \pi\mathbf{a}_b$.

Theorem 4.3.10 *Suppose that $\mathfrak{L}$ is an $\mathfrak{H}$-A'-AFE. Then, for the restart type T constructed above, it holds that*

$$\mathfrak{Rat}_T(A') = \mathfrak{L}\,.$$

Proof. We first compute $(T^*)_{\varepsilon,\varepsilon}$. This computation is easy if we consider the blocks of T according to the partition $\{\{\varepsilon\}\} \cup \{\Delta_b^+ \mid b \in \mathfrak{R}\} \cup \{\Gamma\}$, where $\Gamma = \Gamma_T^+ - \bigcup_{b \in \mathfrak{R}} \Delta_b^+$. The only non-null blocks according to this partition are

$\{\varepsilon\} \times \{\varepsilon\}$, $\{\varepsilon\} \times \Delta_b^+$, $\Delta_b^+ \times \{\varepsilon\}$, $\Delta_b^+ \times \Delta_b^+$, $b \in \mathfrak{R}$. Hence, by Theorem 3.2.4, we obtain

$$\begin{aligned}(T^*)_{\varepsilon,\varepsilon} &= \Big(T(\{\varepsilon\},\{\varepsilon\}) + \sum_{b\in\mathfrak{R}} T(\{\varepsilon\},\Delta_b^+)T(\Delta_b^+,\Delta_b^+)^*T(\Delta_b^+,\{\varepsilon\})\Big)^* \\ &= \Big(z^0 + \sum_{b\in\mathfrak{R}}\sum_{i\in I_b} z_b \mathbf{a}_{i1}\dots\mathbf{a}_{in_i}\Big)^*.\end{aligned}$$

We show now $\mathfrak{L} \subseteq \mathfrak{Rat}_T(A')$. Fix a $b \in \mathfrak{R}$. Since $\mathfrak{R} \subseteq \mathfrak{L} \subseteq [\mathfrak{H}]$ and $\mathfrak{H}$ is closed under matricial composition, there is an $h_b \in \mathfrak{H}$ with $h_b(b) = b$. Let now $h : \Delta^* \to A^{2\times 2}$ be the monoid morphism defined by

$$\begin{aligned}&h(\mathbf{a}) = \begin{pmatrix} h_b(a) & 0 \\ 0 & h_b(a)\end{pmatrix},\ a \in A, \qquad h(z_b) = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \\ &h(z_{b'}) = h(z^0) = 0 \text{ for } b' \in \mathfrak{R},\ b' \neq b.\end{aligned}$$

Since $\mathfrak{H}$ is closed under matricial composition, h is compatible with $\mathfrak{H}$.

We obtain

$$\begin{aligned}h((T^*)_{\varepsilon,\varepsilon}) &= \left(\sum_{i\in I_b}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix} h_b(a_{i1}\dots a_{in_i}) & 0 \\ 0 & h_b(a_{i1}\dots a_{in_i})\end{pmatrix}\right)^* \\ &= \begin{pmatrix} 0 & h_b(b) \\ 0 & 0\end{pmatrix}^* = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\end{aligned}$$

and infer that $b \in \mathfrak{Rat}_T(A')$. Hence, $\mathfrak{R} \subseteq \mathfrak{Rat}_T(A')$. Since $\mathfrak{Rat}_T(A')$ is an $\mathfrak{H}$-A'-AFE, we obtain $\mathfrak{L} = \mathfrak{F}(\mathfrak{R}) \subseteq \mathfrak{Rat}_T(A')$.

Conversely, we show now $\mathfrak{Rat}_T(A') \subseteq \mathfrak{L}$. Assume $a \in \mathfrak{Rat}_T(A')$. Then there exists a monoid morphism $h : \Delta^* \to A^{Q\times Q}$ compatible with $\mathfrak{H}$, and $S \in \mathfrak{Rat}(A')^{1\times Q}$, $P \in \mathfrak{Rat}(A')^{Q\times 1}$ such that $a = Sh((T^*)_{\varepsilon,\varepsilon})P$. Consider now the entries of this matrix product: The entries of $h(\mathbf{b})$, S, P, $h(z^0)$, and $h(z_b)$ are in $\mathfrak{L}$. Since only finitely many $h(z_b)$ are unequal to zero, the entries of $h(z^0) + \sum_{b\in\mathfrak{L}} h(z_b)h(\mathbf{b})$ are in $\mathfrak{L}$. Since $\mathfrak{L}$ is a starsemiring, the entries of $h((T^*)_{\varepsilon,\varepsilon})$ are in $\mathfrak{L}$. This implies $a \in \mathfrak{L}$. □

We have now achieved our main result of this section, a complete characterization of $\mathfrak{H}$-A'-closed semirings.

Corollary 4.3.11 *A semiring $\mathfrak{L}$ is an $\mathfrak{H}$-A'-AFE iff there exists a restart type T such that*

$$\mathfrak{L} = \mathfrak{Rat}_T(A').$$

We choose $\mathfrak{H}$ to be the family of all rational representations. By Theorem 2.11, the family of all mappings $\mu : A\langle\langle\Sigma_\infty^*\rangle\rangle \to (A\{\{\Sigma_\infty^*\}\})^{Q\times Q}$, $Q \subset Q_\infty$, Q finite, where μ is a rational representation, is closed under matricial composition. Moreover, it is easy to see that $[\mathfrak{H}] = A\{\{\Sigma_\infty^*\}\}$. Then the subsets of $A\{\{\Sigma_\infty^*\}\}$ are families of power series. A family of power series $\mathfrak{L}$ is now called a *full abstract family of power series* (abbreviated *full AFP*) if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$. Clearly,

$\mathfrak{L}$ is a full AFP iff $\mathfrak{F}(\mathfrak{L}) = \mathfrak{L}$. This means that now the $\mathfrak{H}$-A'-AFEs are just the full AFPs.

The next theorem is implied by the equalities $\mathfrak{Rat}(A\{\Sigma_\infty \cup \varepsilon\}) = A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\}$ and $\mathfrak{Alg}(A\{\Sigma_\infty \cup \varepsilon\}) = A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$, and by Theorem 3.3 and Corollary 3.9.

Theorem 4.3.12 *$A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\}$ and $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ are full AFPs.*

Theorem 4.3.13 *A family of power series $\mathfrak{L}$ is a full AFP iff there exists a restart type T such that $\mathfrak{L} = \mathfrak{Rat}_T(A\{\Sigma_\infty \cup \varepsilon\})$.*

Choose $\mathfrak{H}$ to be the family of regulated rational representations. By Theorems 6.12, 6.14 and 9.6 of Kuich, Salomaa [88], $\mathfrak{H}$ is closed under matricial composition. Again we obtain $[\mathfrak{H}] = A\{\{\Sigma_\infty^*\}\}$ and the $\mathfrak{H}$-A'-AFEs are just the AFPs in the sense of Kuich, Salomaa [88].

Theorem 4.3.14 *A family of power series $\mathfrak{L}$ is an AFP iff there exists a restart type $(\Gamma_T, \Delta_T, T, \varepsilon)$, where $\Delta_T \subseteq \{\mathbf{a} \mid a \in A\{\{\Sigma_\infty^*\}\},\ (a, \varepsilon) = 0\}$ such that $\mathfrak{L} = \mathfrak{Rat}_T(A\{\Sigma_\infty \cup \varepsilon\})$.*

Proof. The proofs of Theorem 3.4 and 3.5 do not depend on the form of Δ_T. Moreover, by Theorem 11.31 and Corollary 11.33 of Kuich, Salomaa [88], each AFP is generated by a family of quasiregular power series. Hence, in the construction of the type T of Theorem 3.10, symbols $\mathbf{a}$, where $(a, \varepsilon) \neq 0$, are not needed. □

Observe that the T-automata of Theorem 3.14 are cycle-free by Theorem 6.10 of Kuich, Salomaa [88].

Example 4.3.1. We consider the AFP $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$ (see Section 13 of Kuich, Salomaa [88]). Let $Z_2 = \{z_1, z_2\}$, $\bar{Z}_2 = \{\bar{z}_1, \bar{z}_2\}$, $z_1, z_2, \bar{z}_1, \bar{z}_2 \in \Sigma_\infty$, and consider the type *2-Dyck* defined by $(Z_2, Z_2 \cup \bar{Z}_2, D_2, \varepsilon)$, where the non-null entries of D_2 are

$$(D_2)_{v, z_i v} = z_i, \qquad (D_2)_{z_i v, v} = \bar{z}_i,\ i = 1, 2,\ v \in Z_2^*\,.$$

By Theorem 13.15 of Kuich, Salomaa [88], $(D_2^*)_{\varepsilon,\varepsilon}$ is a cone generator of $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$, i. e., $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} = \mathfrak{Rat}_{D_2}(Z_2 \cup \bar{Z}_2)$. Since $Z \cup \bar{Z} \subset \Sigma_\infty$, D_2 is a restart type of the form needed in Theorem 3.14.

For the purpose to give an example we now will construct the restart type $(\Gamma_T, \Delta_T, T, \varepsilon)$ defined before Theorem 3.10 with $\mathfrak{R} = \{(D_2^*)_{\varepsilon,\varepsilon}\}$ and $b = (D_2^*)_{\varepsilon,\varepsilon}$.

For $v \in (Z_2 \cup \bar{Z}_2)^*$, $v \neq \varepsilon$, we define, for $1 \leq j \leq |v|$, $a_{v,j}$ to be the j-th symbol of v. Then we obtain, for $I_b = b$,

$$b = \sum_{v \in I_b} a_{v,1} a_{v,2} \ldots a_{v,|v|}\,.$$

Moreover, we have $\Gamma_T = \Delta_b = \{\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2\}$, $\Delta_T = \{\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2, z^0, z_b\}$. We now define T by those states of $\mathfrak{A}_T$ that are strongly connected with the initial and final state ε in the graph of $\mathfrak{A}_T$:

$$\begin{aligned}
&T_{\varepsilon,\varepsilon} = z^0\,,\\
&T_{\varepsilon,\mathbf{z}_1} = T_{\varepsilon,\mathbf{z}_2} = z_b\,,\\
&T_{\mathbf{v},\mathbf{vz}} = \mathbf{r}(\mathbf{v}),\ z \in r(v),\ \text{if } v \text{ is a proper prefix of a word in } \mathrm{supp}((D_2^*)_{\varepsilon,\varepsilon})\,,\\
&T_{\mathbf{v},\mathbf{vz}_1} = T_{\mathbf{v},\mathbf{vz}_2} = T_{\mathbf{v},\varepsilon} = \mathbf{r}(\mathbf{v}) \text{ if } v \text{ is a word in } \mathrm{supp}((D_2^*)_{\varepsilon,\varepsilon}),\ v \neq \varepsilon\ .
\end{aligned}$$

Here, for a prefix of a word v in $\mathrm{supp}((D_2^*)_{\varepsilon,\varepsilon})$, $v \neq \varepsilon$, $r(v)$ is the rightmost symbol of v, $f(z_1) = f(\bar{z}_1) = \{z_1, \bar{z}_1, z_2\}$, $f(z_2) = f(\bar{z}_2) = \{z_1, z_2, \bar{z}_2\}$, and $\mathbf{v} = \mathbf{z}_{i_1} \ldots \mathbf{z}_{i_k}$ if $v = z_{i_1} \ldots z_{i_k}$, $z_{i_j} \in Z_2 \cup \bar{Z}_2$. Hence, if v is a proper prefix of a word in $\mathrm{supp}((D_2^*)_{\varepsilon,\varepsilon})$ then vz, $z \in r(v)$, is a prefix of a word in $\mathrm{supp}((D_2^*)_{\varepsilon,\varepsilon})$.

Clearly, T is a restart type and by Theorem 3.10, we obtain

$$\mathfrak{Rat}_T(Z_2 \cup \bar{Z}_2) = \mathfrak{Rat}_{D_2}(Z_2 \cup \bar{Z}_2) = A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}\,.$$

□

We now consider formal languages, i. e., our basic semiring is $2^{\Sigma_\infty^*}$. Each subset of $\mathfrak{L}(\Sigma_\infty)$ is called *family of languages*. A family of languages $\mathfrak{L}$ is called a *full abstract family of languages* (abbreviated *full AFL*) if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$. Clearly, $\mathfrak{L}$ is a full AFL iff $\mathfrak{F}(\mathfrak{L}) = \mathfrak{L}$. Theorems 3.12, 3.13 and 3.14 admit at once three corollaries.

Corollary 4.3.15 *The family of regular languages and the family of context-free languages are full AFLs.*

Corollary 4.3.16 *A family of languages $\mathfrak{L}$ is a full AFL iff there exists a restart type T such that $\mathfrak{L} = \mathfrak{Rat}_T(\{\{x\} \mid x \in \Sigma_\infty \cup \{\varepsilon\}\})$.*

Corollary 4.3.17 *A family of languages $\mathfrak{L}$ is an AFL iff there exists a restart type $(\Gamma_T, \Delta_T, T, \varepsilon)$, where $\Delta_T \subseteq \{\mathbf{a} \mid a \in \mathfrak{L}(\Sigma_\infty^*),\ a \cap \{\varepsilon\} = \emptyset\}$ such that $\mathfrak{L} = \mathfrak{Rat}_T(\{\{x\} \mid x \in \Sigma_\infty \cup \{\varepsilon\}\})$.*

Readers interested in AFL-theory should consult Ginsburg [53]. An excellent treatment of full AFLs is given in Berstel [4]. Advanced results can be found in Berstel, Boasson [5].

Chapter 5

Semiring-Semimodule pairs, and finite and infinite words

5.1 Introduction

In this chapter, we deal with semiring-semimodule pairs and finite automata over quemirings. Here semiring-semimodule pairs constitute a generalization of formal languages with finite and infinite words. The semiring models a formal language with finite words while the semimodule models a formal language with infinite words. The main result of this chapter is a generalization of the Kleene Theorem of Büchi [18] in the setting of semiring-semimodule pairs.

This chapter consists of this and seven more sections. In Section 2 we introduce the algebraic structures used in this chapter: semiring-semimodule pairs and quemirings.

In Section 3 we consider Conway semiring-semimodule pairs and prove that the matrix-omega equation is satisfied. Furthermore, given a Conway semiring-semimodule pair, we consider $n \times n$-matrices over the semiring and $n \times 1$-column vectors over the semimodule. Then we prove that the pairs consisting of the matrices and column vectors form again a Conway semiring-semimodule pair.

In Section 4 we define finite automata over quemirings. Given a starsemiring-omegasemimodule pair (A, V), where A is a Conway semiring and $0^\omega = 0$, we prove a Kleene Theorem for A'-finite automata, where A' is a subset of A: the collection of all behaviors of A'-finite automata coincides with the generalized starquemiring generated by A'. A special case of this Kleene Theorem is the result of Büchi [18].

In Secion 5 we consider linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations. We show a connection between certain solutions of these linear systems, the weights of finite and infinite derivations with respect to this grammar and the behavior of finite automata over quemirings.

In Section 6, ω-algebraic systems and ω-algebraic power series are considered. The solutions of order k of these ω-algebraic systems are characterized by behaviors of algebraic finite automata. The ω-algebraic systems and ω-algebraic power series are then connected in Section 7 to ω-context-free grammars and ω-context-free languages, respectively.

In Section 8 we consider morphisms of starsemiring-omegasemimodule pairs and their extension to matrices. We introduce rational and algebraic transducers and transductions over starsemiring-omegasemimodule pairs. We prove that the rational (resp. algebraic) transduction of a rational (resp. an algebraic) closure is again a rational (resp. an algebraic) closure. Then we specialize our results to rational and algebraic transducers in the classical sense and define abstract ω-families of languages.

The presentation of this chapter follows the lines of Ésik, Kuich [43, 42, 45].

We now give a typical example which will be helpful for readers with some background on Büchi automata over infinite words. Readers without this background should consult it when these automata are defined in the following sections.

Example 5.1.1. A *(finite) Büchi automaton*

$$\mathcal{A} = (Q, \Sigma, \delta, q, F)$$

is given by

(i) a finite *set of states* $Q = \{q_1, \ldots, q_n\}$, $n \geq 1$,

(ii) an *input alphabet* Σ,

(iii) a *transition function* $\delta : Q \times \Sigma \to 2^Q$,

(iv) an *initial state* $q \in Q$,

(v) a *set of repeated states* $F = \{q_1, \ldots, q_k\}$, $k \geq 0$.

A *run of* $\mathcal{A}$ on an infinite word $w \in \Sigma^\omega$, $w = a_1 a_2 a_3 \ldots$, is an infinite sequence of states $q(0), q(1), q(2), q(3), \ldots$ such that the following conditions are satisfied:

(i) $q(0) = q$,

(ii) $q(i) \in \delta(q(i-1), a_i)$ for $i \geq 1$.

A word $w \in \Sigma^\omega$ is *Büchi accepted by* $\mathcal{A}$ if there exists a run ρ of $\mathcal{A}$ on w and a repeated state in F occuring infinitely often in ρ.

The *behavior* $||\mathcal{A}|| \subseteq \Sigma^\omega$ of $\mathcal{A}$ is defined to be the set of infinite words that are Büchi accepted by $\mathcal{A}$ (see Büchi [18]).

Let now $\mathcal{A} = (Q, \Sigma, \delta, 2, \{1\})$ be a Büchi automaton, where $Q = \{1, 2\}$, $\Sigma = \{a, b, c, d\}$, and $\delta(1, a) = \{1\}$, $\delta(1, b) = \{2\}$, $\delta(2, c) = \{1\}$, $\delta(2, d) = \{2\}$ are the only non-empty images of δ. The graph of $\mathcal{A}$ is and the adjacency matrix of this graph is

$$M = \begin{pmatrix} \{a\} & \{b\} \\ \{c\} & \{d\} \end{pmatrix}.$$

(See Example 1.1.1.)

The language of inscriptions of paths from 1 to 1 not passing 1 is given by $\{a\} \cup \{b\}\{d\}^*\{c\}$. Hence, the ω-language of inscriptions of infinite paths starting in 1 and passing infinitely often through 1 is $(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega$ and the ω-language of inscriptions of infinite paths starting in 1, passing finitely often through 1 and infinitely often through 2 is $(\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega$. By symmetry, the ω-language of inscriptions of infinite paths starting in 2 and passing infinitely often through 2 (resp. finitely often through 2 and infinitely often through 1) is $(\{d\} \cup \{c\}\{a\}^*\{b\})^\omega$ (resp. $(\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega$).

We now define a column vector M^ω by

$$M^\omega = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \cup (\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega \\ (\{d\} \cup \{c\}\{a\}^*\{b\})^\omega \cup (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega \end{pmatrix},$$

where $(M^\omega)_1$ (resp. $(M^\omega)_2$) is the ω-language of inscriptions of all infinite paths starting in 1 (resp. 2). Observe that $(\{a\}\cup\{b\}\{d\}^*\{c\})^\omega\cap(\{a\}\cup\{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega = \emptyset$ and $(\{d\} \cup \{c\}\{a\}^*\{b\})^\omega \cap (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega = \emptyset$.

The ω-language of inscriptions of infinite paths starting in 2 and passing infinitely often through 1 is $\{d\}^*\{c\}(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega$. We define a column vector M^{ω_1} by

$$M^{\omega_1} = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \\ \{d\}^*\{c\}(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \end{pmatrix},$$

where $(M^{\omega_1})_1$ (resp. $(M^{\omega_1})_2$) is the ω-language of inscriptions of all infinite paths starting in 1 (resp. 2) and passing infinitely often through 1.

The ω-language $||\mathcal{A}||$ is the ω-language of all inscriptions of infinite paths starting in 2 and passing infinitely often through 1, i. e., $||\mathcal{A}|| = (M^{\omega_1})_2$. $\square$

5.2 Preliminaries

Suppose that A is a semiring and V is a commutative monoid written additively. We call V a (left) A-*semimodule* if V is equipped with a (left) action

$$\begin{array}{rcl} A \times V & \to & V \\ (s, v) & \mapsto & sv \end{array}$$

subject to the following rules:

$$\begin{array}{rcl} s(s'v) & = & (ss')v \\ (s+s')v & = & sv + s'v \\ s(v+v') & = & sv + sv' \end{array}$$

$$
\begin{aligned}
1v &= v \\
0v &= 0 \\
s0 &= 0,
\end{aligned}
$$

for all $s, s' \in A$ and $v, v' \in V$. When V is an A-semimodule, we call (A, V) a *semiring-semimodule pair*.

Suppose that (A, V) is a semiring-semimodule pair such that A is a starsemiring and A and V are equipped with an omega operation $^\omega : A \to V$. Then we call (A, V) a *starsemiring-omegasemimodule pair*. Following Bloom, Ésik [10], we call a starsemiring-omegasemimodule pair (A, V) a *Conway semiring-semimodule pair* if A is a Conway semiring and if the omega operation satisfies the *sum-omega equation* and the *product-omega equation*:

$$
\begin{aligned}
(a + b)^\omega &= (a^* b)^\omega + (a^* b)^* a^\omega \\
(ab)^\omega &= a(ba)^\omega,
\end{aligned}
$$

for all $a, b \in A$. It then follows that the *omega fixed-point equation* holds, i.e.,

$$
aa^\omega = a^\omega,
$$

for all $a \in A$.

Ésik, Kuich [44] define a *complete semiring-semimodule pair* to be a semiring-semimodule pair (A, V) such that A is a complete semiring, V is a complete monoid with

$$
\begin{aligned}
s(\sum_{i \in I} v_i) &= \sum_{i \in I} s v_i \\
(\sum_{i \in I} s_i) v &= \sum_{i \in I} s_i v,
\end{aligned}
$$

for all $s \in A$, $v \in V$, and for all families $s_i,\ i \in I$ over A and $v_i,\ i \in I$ over V. Moreover, it is required that an *infinite product operation*

$$
(s_1, s_2, \ldots) \mapsto \prod_{j \geq 1} s_j
$$

is given mapping infinite sequences over A to V subject to the following three conditions:

$$
\begin{aligned}
\prod_{i \geq 1} s_i &= \prod_{i \geq 1} (s_{n_{i-1}+1} \cdot \ldots \cdot s_{n_i}) \\
s_1 \cdot \prod_{i \geq 1} s_{i+1} &= \prod_{i \geq 1} s_i \\
\prod_{j \geq 1} \sum_{i_j \in I_j} s_{i_j} &= \sum_{(i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots} \prod_{j \geq 1} s_{i_j},
\end{aligned}
$$

where in the first equation $0 = n_0 \le n_1 \le n_2 \le \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets. Suppose that (A, V) is complete. Then we define

$$s^* = \sum_{i\ge 0} s^i$$
$$s^\omega = \prod_{i\ge 1} s,$$

for all $s \in A$. This turns (A, V) into a starsemiring-omegasemimodule pair. By Ésik, Kuich [44], each complete semiring-semimodule pair is a Conway semiring-semimodule pair. Observe that, if (A, V) is a complete semiring-semimodule pair, then $0^\omega = 0$.

A semiring-semimodule pair (A, V) is called *continuous* if (A, V) is a complete semiring-semimodule pair and A is a continuous semiring. A quemiring is called *continuous* if it is determined by a continuous semiring-semimodule pair.

A *star-omega semiring* is a semiring A equipped with unary operations * and $^\omega : A \to A$. A star-omega semiring A is called *complete* if (A, A) is a complete semiring-semimodule pair, i. e., if A is complete and is equipped with an infinite product operation that satisfies the three conditions stated above. A complete star-omega semiring A is called *commutative* if the semiring A is commutative and, for all bijections $\pi : \mathbb{N} \to \mathbb{N}$, and $s_j \in A$, $j \ge 0$, $\prod_{j\ge 0} s_{\pi(j)} = \prod_{j\ge 0} s_j$. Eventually, a complete star-omega semiring A is called *continuous* if the semiring A is continuous.

Example 5.2.1. Suppose that Σ is an alphabet. Let Σ^* denote the set of all finite words over Σ including the empty word ε, and let Σ^ω denote the set of all ω-words over Σ. The set 2^{Σ^*} of all subsets of Σ^*, equipped with the operations of set union as sum and concatenation as product is a semiring, where 0 is the empty set $\emptyset$ and 1 is the set $\{\varepsilon\}$. Moreover, equipped with the usual star operation, 2^{Σ^*} is a Conway semiring. Also, 2^{Σ^ω}, equipped with union as the sum operation and the empty set as 0 is a commutative idempotent monoid. Define an action of 2^{Σ^*} on 2^{Σ^ω} by $KL = \{uv \mid u \in K,\ v \in L\}$, for all $K \subseteq \Sigma^*$ and $L \subseteq \Sigma^\omega$. Moreover, for each sequence $(K_0, K_1, \ldots)$ over 2^{Σ^*}, let $\prod_{j\ge 0} K_j = \{u_0 u_1 \ldots \in \Sigma^\omega \mid u_i \in K_i,\ i \ge 0\}$. Then $(2^{\Sigma^*}, 2^{\Sigma^\omega})$ is a complete and continuous semiring-semimodule pair with idempotent module 2^{Σ^ω}. Note that in this example, $1^\omega = 0$, where $1 = \{\varepsilon\}$ and $0 = \emptyset$. □

Example 5.2.2. Consider the semiring $\mathbb{N}^\infty = N \cup \{\infty\}$, obtained by adjoining a top element ∞ to the semiring of the natural numbers. Note that $\mathbb{N}^\infty$ is a complete semiring where an infinite sum is ∞ iff either a summand is ∞ or the number of nonzero summands is infinite, moreover, ∞ multiplied with a nonzero element on either side gives ∞. Define an infinite product

$$(n_1, n_2, \ldots) \quad \mapsto \quad \prod_{j\ge 1} n_j$$

on $\mathbb{N}^\infty$ as follows. If some n_j is 0, then so is the product. Otherwise, if all but a finite number of the n_j are 1s, then the infinite product is the product of those

n_j with $n_j > 1$. In all remaining cases, the infinite product is ∞. Then $\mathbb{N}^\infty$ is a complete star-omega semiring, where * and $^\omega$ are defined as above.

Let Σ denote an alphabet. The semiring $A = \mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle$ of all power series over Σ^* with coefficients in $\mathbb{N}^\infty$ is a complete and continuous semiring. Now let $V = \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ be the collection of all formal power series over Σ^ω with coefficients in $\mathbb{N}^\infty$. Thus, the elements of V are formal sums of the sort

$$s = \sum_{w\in\Sigma^\omega}(s,w)w,$$

where each coefficient (s,w) belongs to $\mathbb{N}^\infty$. Now V can be turned into an A-semimodule by the pointwise sum operation and the action $(r,s)\mapsto rs$ defined by

$$(rs,w) = \sum_{u\in\Sigma^*, v\in\Sigma^\omega,\ uv=w}(r,u)(s,v),$$

where the infinite sum on the right-hand side exists since $\mathbb{N}^\infty$ is complete. We may also define an infinite product taking sequences over A to series in V. Given $s_1, s_2, \ldots$ in A, we define $\prod_{j\geq 1} s_j$ to be the series r in V with

$$(r,w) = \sum_{w=w_1w_2\ldots\in\Sigma^\omega}\prod_{j\geq 1}(s_j,w_j).$$

Then (A,V) is a complete and continuous semiring-semimodule pair and thus a Conway semiring-semimodule pair.

This can be generalized to a large extent. Suppose that A is a complete star-omega semiring. If Σ is a set, consider the complete semiring $A\langle\langle\Sigma^*\rangle\rangle$ and the complete monoid $A\langle\langle\Sigma^\omega\rangle\rangle$ of all series over Σ^ω with coefficients in A equipped with the pointwise sum operation. If we define the action sr of $s\in A\langle\langle\Sigma^*\rangle\rangle$ on $r\in A\langle\langle\Sigma^\omega\rangle\rangle$ by

$$(sr,w) = \sum_{w=uv}(s,u)(r,v),$$

then $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ becomes a semiring-semimodule pair. Now $A\langle\langle\Sigma^*\rangle\rangle$ is a starsemiring, and if we define the infinite product operation

$$(s_1,s_2,\ldots) \mapsto \prod_{j\geq 1}s_j \in A\langle\langle\Sigma^\omega\rangle\rangle$$

by

$$(\prod_{j\geq 1}s_j, w) = \sum_{w=w_1w_2\ldots\in\Sigma^\omega}\prod_{j\geq 1}(s_j,w_j),$$

then $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ becomes a complete semiring-semimodule pair, hence a Conway semiring-semimodule pair, satisfying $(a\varepsilon)^\omega = 0$ for all $a\in A$. If

A is a continuous semiring then $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ is a continuous semiring-semimodule pair. (See Theorem 5.5.) $\square$

Consider a starsemiring-omegasemimodule pair (A, V). Following Bloom, Ésik [10], we define a matrix operation ${}^\omega : A^{n\times n} \to V^{n\times 1}$ on a starsemiring-omegasemimodule pair (A, V) as follows. When $n = 0$, M^ω is the unique element of V^0, and when $n = 1$, so that $M = (a)$, for some $a \in A$, $M^\omega = (a^\omega)$. Assume now that $n > 1$ and write M as in (1). Then

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.1}$$

$$M^\omega = \begin{pmatrix} (a+bd^*c)^\omega + (a+bd^*c)^*bd^\omega \\ (d+ca^*b)^\omega + (d+ca^*b)^*ca^\omega \end{pmatrix}. \tag{5.2}$$

Following Ésik, Kuich [43], we define matrix operations ${}^{\omega_k} : A^{n\times n} \to V^{n\times 1}$, $0 \le k \le n$, as follows. Assume that $M \in A^{n\times n}$ is decomposed into blocks a, b, c, d as in (1), but with a of dimension $k \times k$ and d of dimension $(n-k) \times (n-k)$. Then

$$M^{\omega_k} = \begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \end{pmatrix} \tag{5.3}$$

Observe that $M^{\omega_0} = 0$ and $M^{\omega_n} = M^\omega$.

Suppose that (A, V) is a semiring-semimodule pair and consider $T = A \times V$. Define on T the operations

$$\begin{aligned}(s,u)\cdot(s',v) &= (ss', u+sv)\\ (s,u)+(s',v) &= (s+s', u+v)\end{aligned}$$

and constants $0 = (0,0)$ and $1 = (1,0)$. Equipped with these operations and constants, T satisfies the equations

$$(x+y)+z = x+(y+z) \tag{5.4}$$
$$x+y = y+x \tag{5.5}$$
$$x+0 = x \tag{5.6}$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{5.7}$$
$$x\cdot 1 = x \tag{5.8}$$
$$1\cdot x = x \tag{5.9}$$
$$(x+y)\cdot z = (x\cdot z)+(y\cdot z) \tag{5.10}$$
$$0\cdot x = 0. \tag{5.11}$$

Elgot[31] also defined the unary operation ¶ on T: $(s,u)¶ = (s,0)$. Thus, ¶ selects the "first component" of the pair (s,u), while multiplication with 0 on the right selects the "second component", for $(s,u)\cdot 0=(0,u)$, for all $u\in V$. The new operation satisfies:

$$x\P\cdot(y+z) \quad = \quad (x\P\cdot y)+(x\P\cdot z) \tag{5.12}$$
$$x \quad = \quad x\P+(x\cdot 0) \tag{5.13}$$
$$x\P\cdot 0 \quad = \quad 0 \tag{5.14}$$
$$(x+y)\P \quad = \quad x\P+y\P \tag{5.15}$$
$$(x\cdot y)\P \quad = \quad x\P\cdot y\P. \tag{5.16}$$

Note that when V is idempotent, also

$$x\cdot(y+z) \quad = \quad x\cdot y+x\cdot z$$

holds.

Elgot[31] defined a *quemiring* to be an algebraic structure T equipped with the above operations $\cdot,+,\P$ and constants $0,1$ satisfying the equations (5.4)–(5.11) and (5.12)–(5.16). A morphism of quemirings is a function preserving the operations and constants. It follows from the axioms that $x\P\P=x\P$, for all x in a quemiring T. Moreover, $x\P=x$ iff $x\cdot 0=0$.

When T is a quemiring, $A=T\P=\{x\P \mid x\in T\}$ is easily seen to be a semiring. Moreover, $V=T0=\{x\cdot 0\mid x\in T\}$ contains 0 and is closed under $+$, and, furthermore, $sx\in V$ for all $s\in A$ and $x\in V$. Each $x\in T$ may be written in a unique way as the sum of an element of $T\P$ and of an element of $T0$ as $x=x\P+x\cdot 0$. Sometimes, we will identify $A\times\{0\}$ with A and $\{0\}\times V$ with V. It is shown in Elgot [31] that T is isomorphic to the quemiring $A\times V$ determined by the semiring-semimodule pair (A,V).

Suppose now that (A,V) is a starsemiring-omegasemimodule pair. Then we define on $T=A\times V$ a *generalized star operation*:

$$(s,v)^{\otimes} \quad = \quad (s^*,s^\omega+s^*v) \tag{5.17}$$

for all $(s,v)\in T$. Note that the star and omega operations can be recovered from the generalized star operation, since s^* is the first component of $(s,0)^{\otimes}$ and s^ω is the second component. Thus:

$$(s^*,0) \quad = \quad (s,0)^{\otimes}\P$$
$$(0,s^\omega) \quad = \quad (s,0)^{\otimes}\cdot 0.$$

Observe that, for $(s,0)\in A\times\{0\}$, $(s,0)^{\otimes}=(s^*,0)+(0,s^\omega)$.

Suppose now that T is an (abstract) quemiring equipped with a generalized star operation $^{\otimes}$. As explained above, T as a quemiring is isomorphic to the quemiring $A\times V$ associated with the semiring-semimodule pair (A,V), where $A=T\P$ and $V=T0$, an isomorphism being the map $x\mapsto(x\P,x\cdot 0)$. It is clear that a generalized star operation $^{\otimes}:T\to T$ is determined by a star operation $^*:A\to A$ and an omega operation $^\omega:A\to V$ by (5.17) iff

$$x^{\otimes}\P \quad = \quad (x\P)^{\otimes}\P \tag{5.18}$$
$$x^{\otimes}\cdot 0 \quad = \quad (x\P)^{\otimes}\cdot 0+x^{\otimes}\P\cdot x\cdot 0 \tag{5.19}$$

hold. Indeed, these conditions are clearly necessary. Conversely, if (5.18) and (5.19) hold, then for any $x¶ \in T¶$ we may define

$$(x¶)^* = (x¶)^{\otimes}¶ \tag{5.20}$$
$$(x¶)^{\omega} = (x¶)^{\otimes} \cdot 0\,. \tag{5.21}$$

It follows that (5.17) holds. The definition of star and omega was forced.

Let us call a quemiring equipped with a generalized star operation $^{\otimes}$ a *generalized starquemiring.*

5.3 Conway semiring-semimodule pairs

Throughout this section we assume that (A, V) is a Conway semiring-semimodule pair and that $n \geq 1$. By induction on n, we will prove then that $(A^{n\times n}, V^n)$ is again a Conway semiring-semimodule pair. Furthermore, we will prove that the matrix-omega-equation is valid for Conway semiring-semimodule pairs. These results are due to Bloom, Ésik [10]. We will show the results directly by proofs that are similar to those of Chapter 1.

Firstly, we prove that some particular cases of the sum-omega-equation are satisfied.

Lemma 5.3.1 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $a, f \in A^{1\times 1}$, $g \in A^{1\times n}$, $h \in A^{n\times 1}$, $d, i \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} = \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} + \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^* \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^* \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{\omega}$$

Proof. The left side and the right side of the equality are equal to

$$\begin{pmatrix} \alpha^{\omega} + \alpha^* g(d+i)^{\omega} \\ \delta^{\omega} + \delta^* h(a+f)^{\omega} \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha'^{\omega} + \alpha'^* a^* g(d^*i)^{\omega} + \alpha'^* a^{\omega} + \alpha'^* a^* g(d^*i)^* d^{\omega} \\ \delta'^{\omega} + \delta'^* d^* h(a^*f)^{\omega} + \delta'^* d^* h(a^*f)^* a^{\omega} + \delta'^* d^{\omega} \end{pmatrix},$$

respectively, where $\alpha = a + f + g(d+i)^* h$, $\delta = d + i + h(a+f)^* g$, $\alpha' = a^*f + a^*g(d^*i)^*d^*h$, $\delta' = d^*i + d^*h(a^*f)^*a^*g$. We now obtain $\alpha^{\omega} = (a^*f + a^*g(d^*i)^*d^*h)^{\omega} + (a^*f + a^*g(d^*i)^*d^*h)^* a^{\omega} = \alpha'^{\omega} + \alpha'^* a^{\omega}$ and $\alpha^* g(d+i)^{\omega} = (a^*f + a^*g(d^*i)^*d^*h)^* a^* g((d^*i)^{\omega} + (d^*i)^* d^{\omega}) = \alpha'^* a^* g(d^*i)^{\omega} + \alpha'^* a^* g(d^*i)^* d^{\omega}$. The substitution $d \leftrightarrow a$, $i \leftrightarrow f$, $h \leftrightarrow g$ shows the symmetry of the proof for the second entries of the vectors. □

Observe that the equalities in Lemma 3.2 are particular cases of the sum-omega-equation since $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^{\omega} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^{\omega} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (In Conway semiring-semimodule pairs, $0^{\omega} = 0$.)

Lemma 5.3.2 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $f \in A^{1\times 1}$, $b, g \in A^{1\times n}$, $c, h \in A^{n\times 1}$, $i \in A^{n\times n}$, the following equalities are satisfied:*

$$\begin{aligned}
\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} &= \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^{*} \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega}, \\
\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} &= \left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}^{*} \begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega}
\end{aligned}$$

Proof. The left side and the right side of the first equality are equal to

$$\begin{pmatrix} \alpha^{\omega} + \alpha^{*}(g+b)i^{\omega} \\ \delta^{\omega} + \delta^{*}hf^{\omega} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha'^{\omega} + \alpha'^{*}(g+bi)i^{\omega} \\ \delta'^{\omega} + \delta'^{*}h(f+bh)^{\omega} \end{pmatrix},$$

respectively, where $\alpha = f + (g+b)i^{*}h$, $\delta = i + hf^{*}(g+b)$, $\alpha' = f + bh + (g+bi)i^{*}h$, $\delta' = i + h(f+bh)^{*}(g+bi)$. We now obtain $\alpha' = f + bh + gi^{*}h + bii^{*}h = f + gi^{*}h + bi^{*}h = \alpha$, $\alpha'^{*}(g+bi)i^{\omega} = \alpha^{*}(gi^{\omega} + bii^{\omega}) = \alpha^{*}(gi^{\omega} + bi^{\omega}) = \alpha^{*}(g+b)i^{\omega}$, $\delta'^{\omega} + \delta'^{*}h(f+bh)^{\omega} = (i + h(f^{*}bh)^{*}f^{*}g + h(f^{*}bh)^{*}f^{*}bi)^{\omega} + (i + h(f^{*}bh)^{*}f^{*}g + h(f^{*}bh)^{*}f^{*}bi)^{*}h((f^{*}bh)^{\omega} + (f^{*}bh)^{*}f^{\omega}) = ((hf^{*}b)^{*}hf^{*}g + (hf^{*}b)^{*}i)^{\omega} + ((hf^{*}b)^{*}hf^{*}g + (hf^{*}b)^{*}i)^{*}(hf^{*}b)^{\omega} + ((hf^{*}b)^{*}hf^{*}g + (hf^{*}b)^{*}i)^{*}(hf^{*}b)^{*}hf^{\omega} = (hf^{*}b + hf^{*}g + i)^{\omega} + (hf^{*}b + hf^{*}g + i)^{*}hf^{\omega} = \delta^{\omega} + \delta^{*}hf^{\omega}$.

The left side and the right side of the second equality are equal to

$$\begin{pmatrix} \alpha^{\omega} + \alpha^{*}gi^{\omega} \\ \delta^{\omega} + \delta^{*}(c+h)f^{\omega} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha'^{\omega} + \alpha'^{*}g(cg+i)^{\omega} \\ \delta'^{\omega} + \delta'^{*}(cf+h)f^{\omega} \end{pmatrix},$$

respectively, where $\alpha = f + gi^{*}(c+h)$, $\delta = i + (c+h)f^{*}g$, $\alpha' = f + g(cg + i)^{*}(cf+h)$, $\delta' = cg + i + (cf+h)f^{*}g$. The substitution $f \leftrightarrow i$, $h \leftrightarrow g$, $b \leftrightarrow c$, yielding $\alpha \leftrightarrow \delta$, $\alpha' \leftrightarrow \delta'$, shows the symmetry to the first equality of the lemma. □

Lemma 5.3.3 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $b \in A^{1\times n}$, $c \in A^{n\times 1}$ and $M \in A^{(n+1)\times(n+1)}$, the following equality is satisfied:*

$$\begin{aligned}
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^{\omega} = \\
&\quad \left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{*} M\right)^{\omega} + \left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{*} M\right)^{*} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{\omega}.
\end{aligned}$$

Proof.

$$
\begin{aligned}
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + M\right)^\omega = \left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + M\right)^\omega = \\
&\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^* M\right)^\omega = \\
&\left(\begin{pmatrix} bc & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^\omega = \\
&\left(\begin{pmatrix} bc & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^\omega = \\
&\left(\begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^\omega + \\
&\qquad \left(\begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & E \end{pmatrix} M\right)^* \cdot \\
&\qquad \begin{pmatrix} bc & 0 \\ 0 & 0 \end{pmatrix}^\omega = \\
&\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^\omega + \\
&\qquad \left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} (bc)^\omega \\ 0 \end{pmatrix} = \\
&\left(\begin{pmatrix} 1 & 0 \\ c & E \end{pmatrix} \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^\omega + \\
&\qquad \left(\begin{pmatrix} 1 & 0 \\ c & E \end{pmatrix} \begin{pmatrix} (bc)^* & (bc)^*b \\ 0 & E \end{pmatrix} M\right)^* \begin{pmatrix} 1 & 0 \\ c & E \end{pmatrix} \begin{pmatrix} (bc)^\omega \\ 0 \end{pmatrix} = \\
&\left(\begin{pmatrix} (bc)^* & (bc)^*b \\ c(bc)^* & (cb)^* \end{pmatrix} M\right)^\omega + \left(\begin{pmatrix} (bc)^* & (bc)^*b \\ c(bc)^* & (cb)^* \end{pmatrix} M\right)^* \begin{pmatrix} (bc)^\omega \\ (cb)^\omega \end{pmatrix} = \\
&\left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* M\right)^\omega + \left(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* M\right)^* \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^\omega .
\end{aligned}
$$

□

Theorem 5.3.4 *Let (A, V) be a Conway semiring-semimodule pair. Then the sum-omega-equation holds in the starsemiring-omegasemimodule pair $(A^{(n+1)\times(n+1)}, V^{n+1})$.*

Proof. Let $a \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d \in A^{n\times n}$, $M \in A^{(n+1)\times(n+1)}$. Then we obtain

$$
\begin{aligned}
&\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + M\right)^{\omega} = \\
&\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{*}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{*} M\right)^{\omega} + \\
&\quad \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{*}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{*} M\right)^{*}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{\omega} = \\
&\left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix} + \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^{\omega} + \\
&\quad \left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix} + \begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^{*}\begin{pmatrix} a^{\omega} \\ d^{\omega} \end{pmatrix} = \\
&\left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^{*}\begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^{\omega} + \\
&\quad \left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^{*}\begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^{*}\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^{\omega} + \\
&\quad \left(\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^{*}\begin{pmatrix} a^* & 0 \\ 0 & d^* \end{pmatrix} M\right)^{*}\begin{pmatrix} 0 & a^*b \\ d^*c & 0 \end{pmatrix}^{*}\begin{pmatrix} a^{\omega} \\ d^{\omega} \end{pmatrix} = \\
&\left(\begin{pmatrix} (a^*bd^*c)^*a^* & a^*b(d^*ca^*b)^*d^* \\ d^*c(a^*bd^*c)^*a^* & (d^*ca^*b)^*d^* \end{pmatrix} M\right)^{\omega} + \\
&\ \left(\begin{pmatrix} (a^*bd^*c)^*a^* & a^*b(d^*ca^*b)^*d^* \\ d^*c(a^*bd^*c)^*a^* & (d^*ca^*b)^*d^* \end{pmatrix} M\right)^{*}\begin{pmatrix} (a^*bd^*c)^{\omega} \\ (d^*ca^*b)^{\omega} \end{pmatrix} + \\
&\ \left(\begin{pmatrix} (a^*bd^*c)^*a^* & a^*b(d^*ca^*b)^*d^* \\ d^*c(a^*bd^*c)^*a^* & (d^*ca^*b)^*d^* \end{pmatrix} M\right)^{*} \cdot \\
&\ \begin{pmatrix} (a^*bd^*c)^* & a^*b(d^*ca^*b)^* \\ d^*c(a^*bd^*c)^* & (d^*ca^*b)^* \end{pmatrix}\begin{pmatrix} a^{\omega} \\ d^{\omega} \end{pmatrix} = \\
&\left(\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} M\right)^{\omega} + \\
&\ \left(\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix} M\right)^{*} \cdot \\
&\ \begin{pmatrix} (a^*bd^*c)^{\omega} + (a^*bd^*c)^*a^{\omega} + a^*b(d^*ca^*b)^*d^{\omega} \\ (d^*ca^*b)^{\omega} + d^*c(a^*bd^*c)^*a^{\omega} + (d^*ca^*b)^*d^{\omega} \end{pmatrix} = \\
&\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} M\right)^{\omega} + \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} M\right)^{*}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\omega} .
\end{aligned}
$$

□

Secondly, we prove that some particular cases of the product-omega-equation are satisfied.

Lemma 5.3.5 *Let (A,V) be a Conway semiring-semimodule pair. Then, for $a, f \in A^{1\times 1}$, $b \in A^{1\times n}$, $c \in A^{n\times 1}$, $d \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\right)^{\omega} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega} .$$

Proof. The left side and the right side of the equality are equal to

$$\begin{pmatrix} af & 0 \\ cf & 0 \end{pmatrix}^{\omega} = \begin{pmatrix} (af)^{\omega} \\ cf(af)^{\omega} \end{pmatrix} = \begin{pmatrix} (af)^{\omega} \\ c(fa)^{\omega} \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} fa & fb \\ 0 & 0 \end{pmatrix}^{\omega} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (fa)^{\omega} \\ 0 \end{pmatrix} =$$
$$\begin{pmatrix} a(fa)^{\omega} \\ c(fa)^{\omega} \end{pmatrix} = \begin{pmatrix} (af)^{\omega} \\ c(fa)^{\omega} \end{pmatrix},$$

respectively. □

Lemma 5.3.6 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $a, s \in A^{1\times 1}$, $b, t \in A^{1\times n}$, $c, u, h \in A^{n\times 1}$, $d, v \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{\omega} =$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega}.$$

Proof. The left side of the equality equals

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} th & 0 \\ vh & 0 \end{pmatrix}\right)^{\omega} = \begin{pmatrix} ath + bvh & 0 \\ cth + dvh & 0 \end{pmatrix}^{\omega} =$$
$$\begin{pmatrix} (ath + bvh)^{\omega} \\ (cth + dvh)(ath + bvh)^{\omega} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t(hat + hbv)^{\omega} \\ v(hat + hbv)^{\omega} \end{pmatrix}.$$

The right side of the equality without the first factor matrix equals

$$\begin{pmatrix} tha & thb \\ vha & vhb \end{pmatrix}^{\omega} =$$
$$\begin{pmatrix} (tha + thb(vhb)^* vha)^{\omega} + (tha + thb(vhb)^* vha)^* thb(vhb)^{\omega} \\ (vhb + vha(tha)^* thb)^{\omega} + (vhb + vha(tha)^* thb)^* vha(tha)^{\omega} \end{pmatrix} =$$
$$\begin{pmatrix} (t(hbv)^* ha)^{\omega} + (t(hbv)^* ha)^* thb(vhb)^{\omega} \\ (v(hat)^* hb)^{\omega} + (v(hat)^* hb)^* vha(tha)^{\omega} \end{pmatrix} =$$
$$\begin{pmatrix} t(hbv + hat)^{\omega} \\ v(hat + hbv)^{\omega} \end{pmatrix}.$$

□

Lemma 5.3.7 *Let (A, V) be a Conway semiring-semimodule pair. Let (A, V) be a Conway semiring-semimodule pair. Then, for $a, s \in A^{1\times 1}$, $b, t \in A^{1\times n}$, $c, u \in A^{n\times 1}$, $d, v, i \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} =$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega}.$$

Proof. The left side of the equality equals

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & ti \\ 0 & vi \end{pmatrix}\right)^{\omega} = \begin{pmatrix} 0 & ati+bvi \\ 0 & cti+dvi \end{pmatrix}^{\omega} =$$
$$\begin{pmatrix} (ati+bvi)(cti+dvi)^{\omega} \\ (cti+dvi)^{\omega} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t(ict+idv)^{\omega} \\ v(ict+idv)^{\omega} \end{pmatrix}.$$

The right side of the equality equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} 0 & ti \\ 0 & vi \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} tic & tid \\ vic & vid \end{pmatrix}^{\omega}.$$

The substitution $i \leftrightarrow h$, $a \leftrightarrow c$, $b \leftrightarrow d$ shows for the second factor matrix the symmetry to the equality of Lemma 3.6. □

Lemma 5.3.8 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $a, s \in A^{1\times 1}$, $b, g, t \in A^{1\times n}$, $c, u \in A^{n\times 1}$, $d, v \in A^{n\times n}$, the following equality is satisfied:*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\right)^{\omega} =$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega}.$$

Proof. The left and the right side of the equality are equal to

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & sg \\ 0 & ug \end{pmatrix}\right)^{\omega}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} 0 & sg \\ 0 & ug \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\omega},$$

respectively. The substitution $s \leftrightarrow t$, $g \leftrightarrow i$, $u \leftrightarrow v$ shows the symmetry to the equality of Lemma 3.7. □

Lemma 5.3.9 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $M, M' \in A^{(n+1)\times(n+1)}$, and $g \in A^{1\times n}$, $h \in A^{n\times 1}$, the following equality is satisfied:*

$$\left(MM'\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{\omega} = M\left(M'\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}M\right)^{\omega}.$$

Proof.

$$
\begin{aligned}
&\left(MM'\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{\omega} = \left(MM'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} + MM'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{\omega} = \\
&\left(\left(MM'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\right)^{*} MM'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{\omega} + \\
&\quad \left(\left(MM'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\right)^{*} MM'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{*} \left(MM'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\right)^{\omega} = \\
&\left(M\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{*} M'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{\omega} + \\
&\quad \left(M\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{*} M'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}\right)^{*} \cdot \\
&\quad M\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{\omega} = \\
&M\left(\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{*} M'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} M\right)^{\omega} + \\
&\quad M\left(\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{*} M'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} M\right)^{*} \cdot \\
&\quad \left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M\right)^{\omega} = \\
&M\left(M'\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} M + M'\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} M\right)^{\omega} = \\
&M\left(M'\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\right)^{\omega}.
\end{aligned}
$$

□

Lemma 5.3.10 *Let (A, V) be a Conway semiring-semimodule pair. Then, for $M \in A^{(n+1)\times(n+1)}$, and $f \in A^{1\times 1}$, $i \in A^{n\times n}$, the following equality is satisfied:*

$$\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} = M\left(\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega}\right.\,.$$

Proof.

$$
\begin{aligned}
&\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} = \left(M\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + M\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} = \\
&\left(\left(M\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\right)^{*} M\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} + \\
&\quad \left(\left(M\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\right)^{*} M\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{*} \left(M\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\right)^{\omega} = \\
&\left(M\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} + \\
&\quad \left(M\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\right)^{*} M\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{\omega} = \\
&M\left(\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega} + \\
&\quad M\left(\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} M\right)^{*} \left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M\right)^{\omega} = \\
&M\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} M + \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega} = M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega}.
\end{aligned}
$$

□

Theorem 5.3.11 *Let (A, V) be a Conway semiring-semimodule pair. Then the product-omega-equation is satisfied in the starsemiring-omegasemimodule pair $(A^{(n+1)\times(n+1)}, V^{n+1})$.*

Proof. Let $M \in A^{(n+1)\times(n+1)}$ and $f \in A^{1\times 1}$, $g \in A^{1\times n}$, $h \in A^{n\times 1}$, $i \in A^{n\times n}$. Then we prove that the equality

$$
\left(M\begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} = M\left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} M\right)^{\omega}
$$

is satisfied. We obtain

$$
\begin{aligned}
&\left(M\begin{pmatrix} f & g \\ h & i \end{pmatrix}\right)^{\omega} = \left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} + M\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{\omega} = \\
&\left(\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^{*} M\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{\omega} + \\
&\quad \left(\left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^{*} M\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{*} \left(M\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix}\right)^{\omega} = \\
&\left(M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{\omega} + \\
&\quad \left(M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix}\right)^{*} M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega} = \\
&M\left(\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\right)^{\omega} + \\
&\quad M\left(\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{*} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\right)^{*} \left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M\right)^{\omega} = \\
&M\left(\begin{pmatrix} f & 0 \\ 0 & i \end{pmatrix} M + \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} M\right)^{\omega} = M\left(\begin{pmatrix} f & g \\ h & i \end{pmatrix} M\right)^{\omega}.
\end{aligned}
$$

□

Corollary 5.3.12 (Bloom, Ésik [10]) *If (A, V) is a Conway semiring-semimodule pair then, for $n \geq 0$, $(A^{(n+1)\times(n+1)}, V^{n+1})$ again is a Conway semiring-semimodule pair.*

We prove now the matrix-omega-equation.

Theorem 5.3.13 (Bloom, Ésik [10]) *Let (A, V) be a Conway semiring-semimodule pair. Then the matrix-omega-equation holds in the starsemiring-omegasemimodule pair $(A^{(n+1)\times(n+1)}, V^{n+1})$.*

Proof. The proof is similar to the proof of the matrix-star-equation in Theorem 1.2.18. It is by induction on the dimension of the matrix. For 2×2-matrices there is no problem. Let $M \in A^{n\times n}$, $n \geq 3$, and partition M into nine blocks

$$M = \begin{pmatrix} f & g & h \\ i & a & b \\ j & c & d \end{pmatrix}$$

with dimensions $f \in A^{n_1\times n_1}$, $g \in A^{n_1\times n_2}$, $h \in A^{n_1\times n_3}$, $i \in A^{n_2\times n_1}$, $a \in A^{n_2\times n_2}$, $b \in A^{n_2\times n_3}$, $j \in A^{n_3\times n_1}$, $c \in A^{n_3\times n_2}$, $d \in A^{n_3\times n_3}$. The proof reduces then to showing that when we compute $^\omega$ of

$$M = \left(\begin{array}{c|cc} f & g & h \\ \hline i & a & b \\ j & c & d \end{array}\right) \quad \text{and} \quad M' = \left(\begin{array}{cc|c} f & g & h \\ i & a & b \\ \hline j & c & d \end{array}\right)$$

in the indicated ways we get the same result. Hence, we have to verify three equalities in nine variables.

(i) First we compute M^ω. We denote the blocks of M^ω by $(M^\omega)_i$, $1 \leq i \leq 3$. We obtain

$$\begin{aligned}
(M^\omega)_1 = {} & \left(f + (g\ h)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} i \\ j \end{pmatrix}\right)^\omega + \\
& \left(f + (g\ h)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} i \\ j \end{pmatrix}\right)^* (g\ h)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\omega = \\
& \left(f + (g\ h)\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\right)^\omega + \\
& \left(f + (g\ h)\begin{pmatrix} (a+bd^*c)^* & a^*b(d+ca^*b)^* \\ d^*c(a+bd^*c)^* & (d+ca^*b)^* \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\right)^* . \\
& (g\ h)\begin{pmatrix} (a+bd^*c)^\omega + (a+bd^*c)^*bd^\omega \\ (d+ca^*b)^\omega + (d+ca^*b)^*ca^\omega \end{pmatrix} = \\
& \alpha^\omega + \alpha^*(g(a+bd^*c)^\omega + g(a+bd^*c)^*bd^\omega + \\
& \quad h(d+ca^*b)^\omega + h(d+ca^*b)^*ca^\omega),
\end{aligned}$$

where $\alpha = f + g(a+bd^*c)^*i + ga^*b(d+ca^*b)^*j + hd^*c(a+bd^*c)^*i + h(d+ca^*b)^*j$,

$$\begin{aligned}
\begin{pmatrix} (M^\omega)_2 \\ (M^\omega)_3 \end{pmatrix} &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix} f^*(g\ h)\right)^\omega + \\
&\quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix} f^*(g\ h)\right)^* \begin{pmatrix} i \\ j \end{pmatrix} f^\omega = \\
&\begin{pmatrix} a+if^*g & b+if^*h \\ c+jf^*g & d+jf^*h \end{pmatrix}^\omega + \begin{pmatrix} a+if^*g & b+if^*h \\ c+jf^*g & d+jf^*h \end{pmatrix}^* \begin{pmatrix} if^\omega \\ jf^\omega \end{pmatrix} = \\
&\begin{pmatrix} \beta^\omega + \beta^*(b+if^*h)(d+jf^*h)^\omega \\ \gamma^\omega + \gamma^*(c+jf^*g)(a+if^*g)^\omega \end{pmatrix} + \\
&\quad \begin{pmatrix} \beta^* if^\omega + \beta^*(b+if^*h)(d+jf^*h)^* jf^\omega \\ \gamma^*(c+jf^*g)(a+if^*g)^* if^\omega + \gamma^* jf^\omega \end{pmatrix},
\end{aligned}$$

where $\beta = a+if^*g+(b+if^*h)(d+jf^*h)^*(c+jf^*g)$ and $\gamma = d+jf^*h+(c+jf^*g)(a+if^*g)^*(b+if^*h)$.

(ii) We now compute M'^ω. We denote the blocks of M'^ω by $(M'^\omega)_i$, $1 \le i \le 3$. We obtain

$$\begin{aligned}
\begin{pmatrix} (M'^\omega)_1 \\ (M'^\omega)_2 \end{pmatrix} &= \left(\begin{pmatrix} f & g \\ i & a \end{pmatrix} + \begin{pmatrix} h \\ b \end{pmatrix} d^*(j\ c)\right)^\omega + \\
&\quad \left(\begin{pmatrix} f & g \\ i & a \end{pmatrix} + \begin{pmatrix} h \\ b \end{pmatrix} d^*(j\ c)\right)^* \begin{pmatrix} h \\ b \end{pmatrix} d^\omega = \\
&\begin{pmatrix} f+hd^*j & g+hd^*c \\ i+bd^*j & a+bd^*c \end{pmatrix}^\omega + \begin{pmatrix} f+hd^*j & g+hd^*c \\ i+bd^*j & a+bd^*c \end{pmatrix}^* \begin{pmatrix} hd^\omega \\ bd^\omega \end{pmatrix} = \\
&\begin{pmatrix} \delta^\omega + \delta^*(g+hd^*c)(a+bd^*c)^\omega \\ \eta^\omega + \eta^*(i+bd^*j)(f+hd^*j)^\omega \end{pmatrix} + \\
&\quad \begin{pmatrix} \delta^* hd^\omega + \delta^*(g+hd^*c)(a+bd^*c)^* bd^\omega \\ \eta^*(i+bd^*j)(f+hd^*j)^* hd^\omega + \eta^* bd^\omega \end{pmatrix},
\end{aligned}$$

where $\delta = f+hd^*j+(g+hd^*c)(a+bd^*c)^*(i+bd^*j)$ and $\eta = a+bd^*c+(i+bd^*j)(f+hd^*j)^*(g+hd^*c)$.

$$\begin{aligned}
(M'^\omega)_3 &= \left(d + (j\ c)\begin{pmatrix} f & g \\ i & a \end{pmatrix}^* \begin{pmatrix} h \\ b \end{pmatrix}\right)^\omega + \\
&\quad \left(d + (j\ c)\begin{pmatrix} f & g \\ i & a \end{pmatrix}\begin{pmatrix} h \\ b \end{pmatrix}\right)^* (j\ c)\begin{pmatrix} f & g \\ i & a \end{pmatrix}^\omega = \\
&\left(d + (j\ c)\begin{pmatrix} (f+ga^*i)^* & f^*g(a+if^*g)^* \\ a^*i(f+ga^*i)^* & (a+if^*g)^* \end{pmatrix}\begin{pmatrix} h \\ b \end{pmatrix}\right)^\omega + \\
&\quad \left(d + (j\ c)\begin{pmatrix} (f+ga^*i)^* & f^*g(a+if^*g)^* \\ a^*i(f+ga^*i)^* & (a+if^*g)^* \end{pmatrix}\begin{pmatrix} h \\ b \end{pmatrix}\right)^*. \\
&\quad (j\ c)\begin{pmatrix} (f+ga^*i)^\omega + (f+ga^*i)^* ga^\omega \\ (a+if^*g)^\omega + (a+if^*g)^* if^\omega \end{pmatrix} = \\
&\chi^\omega + \chi^*(j(f+ga^*i)^\omega + j(f+ga^*i)^* ga^\omega + \\
&\quad c(a+if^*g)^\omega + c(a+if^*g)^* if^\omega),
\end{aligned}$$

where $\chi = d+j(f+ga^*i)^*h+jf^*g(a+if^*g)^*b+ca^*i(f+ga^*i)^*h+c(a+if^*g)^*b$.

(iii) We now show the equalities $(M^\omega)_i = (M'^\omega)_i$, $1 \le i \le 3$. We obtain $\alpha = \delta$ by Lemma 1.2.16. Hence, for the proof of $(M^\omega)_1 = (M'^\omega)_1$, we have to show the equality $g(a+bd^*c)^\omega + g(a+bd^*c)^*bd^\omega + h(d+ca^*b)^\omega + h(d+ca^*b)^*ca^\omega = (g+hd^*c)(a+bd^*c)^\omega + hd^\omega + (g+hd^*c)(a+bd^*c)^*bd^\omega$. On both sides, the terms $g(a+bd^*c)^\omega$ and $g(a+bd^*c)^*bd^\omega$ appear. The left side, without these terms, becomes $h(d^*ca^*b)^\omega + h(d^*ca^*b)^*d^\omega + h(d^*ca^*b)^*d^*ca^\omega$, while the right side, without these terms, becomes $hd^*c(a^*bd^*c)^\omega + hd^*c(a^*bd^*c)^*a^\omega + hd^*c(a^*bd^*c)^*a^*bd^\omega + hd^\omega$. It is easily checked that the first terms on both sides coincide, that the third term on the left side and the second term on the right side coincide, and that the second term on the left side and the sum of the third and the fourth term on the right side coincide. Hence, $(M^\omega)_1 = (M'^\omega)_1$.

We obtain $\beta = \eta$ by Lemma 1.2.17. Hence, for the proof of $(M^\omega)_2 = (M'^\omega)_2$, we have to show the equality $(b+if^*h)(d+jf^*h)^\omega + if^\omega + (b+if^*h)(d+jf^*h)^*jf^\omega = (i+bd^*j)(f+hd^*j)^\omega + (i+bd^*j)(f+hd^*j)^*hd^\omega + bd^\omega$. The left side becomes $b(d^*jf^*h)^\omega + if^*h(d^*jf^*h)^\omega + b(d^*jf^*h)^*d^\omega + if^*h(d^*jf^*h)^*d^\omega + if^\omega + b(d^*jf^*h)^*d^*jf^\omega + if^*h(d^*jf^*h)^*d^*jf^\omega$, while the right side becomes $i(f^*hd^*j)^\omega + bd^*j(f^*hd^*j)^\omega + i(f^*hd^*j)^*f^\omega + bd^*j(f^*hd^*j)^*f^\omega + i(f^*hd^*j)^*f^*hd^\omega + bd^*j(f^*hd^*j)^*f^*hd^\omega + bd^\omega$. We denote the seven terms on the left (resp. right) side read from left to right by $L_1, L_2, L_3, L_4, L_5, L_6, L_7$ (resp. $R_1, R_2, R_3, R_4, R_5, R_6, R_7$). The following equalities are easily checked: $L_1 = R_2$, $L_2 = R_1$, $L_3 = R_6 + R_7$, $L_4 = R_5$, $L_5 + L_7 = R_3$, $L_6 = R_4$. If we perform the substitution $f \leftrightarrow d$, $h \leftrightarrow j$, $g \leftrightarrow c$, $i \leftrightarrow b$ on the equation $(M^\omega)_1 = (M'^\omega)_1$, we get the equation $(M'^\omega)_3 = (M^\omega)_3$. Hence, we have proved our theorem. $\square$

5.4 Finite automata over quemirings and a Kleene Theorem

In this section we consider finite automata over quemirings and prove a Kleene Theorem. Throughout this section, (A, V) denotes a starsemiring-omegasemimodule pair and T denotes the generalized starquemiring $A \times V$. Moreover, A' denotes a subset of A.

A *finite A'-automaton* (*over the quemiring T*)

$$\mathfrak{A} = (n, I, M, P, k)$$

is given by

(i) a finite *set of states* $\{1, \ldots, n\}$, $n \ge 1$,

(ii) a *transition matrix* $M \in (A' \cup \{0,1\})^{n \times n}$,

(iii) an *initial state vector* $I \in (A' \cup \{0,1\})^{1 \times n}$,

(iv) a *final state vector* $P \in (A' \cup \{0,1\})^{n \times 1}$,

(v) a *set of repeated states* $\{1, \ldots, k\}$, $k \ge 0$.

The *behavior* of $\mathfrak{A}$ is an element of T and is defined by

$$||\mathfrak{A}|| = IM^*P + IM^{\omega_k}.$$

If $\mathfrak{A} = (n, (i_1\ \ i_2), \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, k)$, where $i_1 \in (A' \cup \{0,1\})^{1\times k}$, $i_2 \in (A' \cup \{0,1\})^{1\times(n-k)}$, $a \in (A' \cup \{0,1\})^{k\times k}$, $b \in (A' \cup \{0,1\})^{k\times(n-k)}$, $c \in (A' \cup \{0,1\})^{(n-k)\times k}$, $d \in (A' \cup \{0,1\})^{(n-k)\times(n-k)}$, $p_1 \in (A' \cup \{0,1\})^{k\times 1}$, $p_2 \in (A' \cup \{0,1\})^{(n-k)\times 1}$, we write also

$$\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k).$$

Let (A, V) be now a complete semiring-semimodule pair and consider a finite A'-automaton $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)$, the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$M^{\omega_k} = \begin{pmatrix} (a^*bd^*c)^\omega + (a^*bd^*c)^*a^\omega \\ d^*c(a^*bd^*c)^\omega + d^*c(a^*bd^*c)^*a^\omega \end{pmatrix},$$

and the directed graph of $\mathfrak{A}$ (see Chapter 1, Section 3). In the first summand of the entries of M^{ω_k}, the blocks b and c occur infinitely often, i. e., the i-the row of the first summand is the sum of the weights of all infinite paths starting in state i and passing infinitely often through the repeated states in $\{1, \ldots, k\}$ and the nonrepeated states in $\{k+1, \ldots, n\}$. In the second summand of the entries of M^{ω_k}, the block a occurs infinitely often and the blocks b and c occur only finitely often, i. e., the i-th row of the second summand is the sum of the weights of all infinite paths starting in state i and passing infinitely often through the repeated states in $\{1, \ldots, k\}$ and only finitely often through the nonrepeated states in $\{k+1, \ldots, n\}$. Hence, the i-th row of the first and the second summand of the entries of M^{ω_k} are sums of weights of disjoint sets of infinite paths. Moreover, each weight of an infinite path is counted at least once. Hence, we have the following result.

Theorem 5.4.1 *If (A, V) is a complete semiring-semimodule pair and $\mathfrak{A}$ is a finite A'-automaton then $||\mathfrak{A}|| = F + I$, where F is the sum of the weights of all finite paths from an initial state to a final state multiplied by the initial and final weights of these states and where I is the sum of the weights of all infinite paths starting at an initial state, passing infinitely often through repeated states, and multiplied by the initial weight of this initial state.*

By definition, $\omega\text{-}\mathfrak{Rat}(A')$ is the generalized starquemiring generated by A'.

For the remainder of this section we assume that (A, V) is a Conway semiring-semimodule pair. We now will prove a Kleene Theorem: Let $a \in A \times V$. Then $a \in \omega\text{-}\mathfrak{Rat}(A')$ iff a is the behavior of a finite A'-automaton. To achieve this result we need a few theorems and corollaries.

Let $\mathfrak{A} = (n, I, M, P, k)$ be a finite A'-automaton. It is called *normalized* if

(i) $n \geq 2$ and $k \leq n - 2$;

(ii) $I_{n-1} = 1$ and $I_j = 0$ for $j \neq n-1$;

(iii) $P_n = 1$ and $P_j = 0$ for $j \neq n$;

(iv) $M_{i,n-1} = 0$ and $M_{n,i} = 0$ for all $1 \leq i \leq n$.

Two finite A'-automata $\mathfrak{A}$ and $\mathfrak{A}'$ are *equivalent* if $||\mathfrak{A}|| = ||\mathfrak{A}'||$.

Theorem 5.4.2 *Each finite A'-automaton $\mathfrak{A} = (n, I, M, P, k)$ is equivalent to a normalized finite A'-automaton $\mathfrak{A}' = (n+2, I', M', P', k)$.*

Proof. We define $I' = (0\ 1\ 0)$, $M' = \begin{pmatrix} M & 0 & P \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let now $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)$. Then

$$M' = \begin{pmatrix} a & b & 0 & p_1 \\ c & d & 0 & p_2 \\ i_1 & i_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the first k entries of M'^{ω_k} are equal to

$$\left(a + (b\ 0\ p_1) \begin{pmatrix} d & 0 & p_2 \\ i_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix} \right)^\omega =$$
$$\left(a + (b\ 0\ p_1) \begin{pmatrix} d^* & 0 & d^* p_2 \\ i_2 d^* & 1 & i_2 d^* p_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix} \right)^\omega = (a + bd^*c)^\omega .$$

Hence, the last $n-k+2$ entries of M'^{ω_k} are equal to

$$\begin{pmatrix} d & 0 & p_2 \\ i_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix} (a + bd^*c)^\omega = \begin{pmatrix} d^*c \\ i_2 d^* c + i_1 \\ 0 \end{pmatrix} (a + bd^*c)^\omega$$

and we obtain $||\mathfrak{A}'|| = I'M'^*P' + I'M'^{\omega_k} = (M'^*)_{n+1,n+2} + (M'^{\omega_k})_{n+1} = IM^*P + (i_2 d^* c + i_1)(a + bd^*c)^\omega = IM^*P + IM^{\omega_k} = ||\mathfrak{A}||$. □

Lemma 5.4.3 *If $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)$ is a finite A'-automaton then*

$$\begin{aligned} ||\mathfrak{A}|| \quad &= \quad i_1(a + bd^*c)^*(p_1 + bd^*p_2) + i_2 d^* c(a + bd^*c)^*(p_1 + bd^*p_2) + \\ &\quad\ \ i_2 d^* p_2 + i_1(a + bd^*c)^\omega + i_2 d^* c(a + bd^*c)^\omega . \end{aligned}$$

Proof. We obtain

$$\begin{aligned} ||\mathfrak{A}|| \quad &= \quad (i_1\ i_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + (i_1\ i_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\omega_k} = \\ &\quad (i_1\ i_2) \begin{pmatrix} (a + bd^*c)^* & (a + bd^*c)^* bd^* \\ d^*c(a + bd^*c)^* & d^*c(a + bd^*c)^* bd^* + d^* \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \end{aligned}$$

$$
\begin{aligned}
&(i_1\ i_2)\begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \end{pmatrix} = \\
&i_1(a+bd^*c)^*p_1 + i_1(a+bd^*c)^*bd^*p_2 + i_2d^*c(a+bd^*c)^*p_1 + \\
&i_2d^*c(a+bd^*c)^*bd^*p_2 + i_2d^*p_2 + i_1(a+bd^*c)^\omega + i_2d^*c(a+bd^*c)^\omega\,.
\end{aligned}
$$

□

Let $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; f, g; m)$ and $\mathfrak{A}' = (n'; h, i; a', b', c', d'; p_1, p_2; k)$ be finite A'-automata. Then we define the finite A'-automata $\mathfrak{A} + \mathfrak{A}'$ and $\mathfrak{A} \cdot \mathfrak{A}'$ to be

$$
\begin{aligned}
\mathfrak{A} + \mathfrak{A}' \quad = \quad & (n+n'; (i_1\ h), (i_2\ i); \\
& \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}; \\
& \begin{pmatrix} f \\ p_1 \end{pmatrix}, \begin{pmatrix} g \\ p_2 \end{pmatrix}, m+k)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{A} \cdot \mathfrak{A}' \quad = \quad & (n+n'; (i_1\ 0), (i_2\ 0); \\
& \begin{pmatrix} a & fh \\ 0 & a' \end{pmatrix}, \begin{pmatrix} b & fi \\ 0 & b' \end{pmatrix}, \begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix}, \begin{pmatrix} d & gi \\ 0 & d' \end{pmatrix}; \\
& \begin{pmatrix} 0 \\ p_1 \end{pmatrix}, \begin{pmatrix} 0 \\ p_2 \end{pmatrix}, m+k)\,.
\end{aligned}
$$

For the definition of $\mathfrak{A} \cdot \mathfrak{A}'$ we assume that either $\begin{pmatrix} f \\ g \end{pmatrix}(h\ i) \in (A' \cup \{0,1\})^{n \times n'}$ or $\mathfrak{A}'$ is normalized. Observe that the definitions of $\mathfrak{A} + \mathfrak{A}'$ and $\mathfrak{A} \cdot \mathfrak{A}'$ (and of $\mathfrak{A}^\otimes$ which is defined below) are the usual ones except that certain rows and columns are permuted. These permutations are needed since the set of repeated states of a finite A'-automaton is always a set $\{1, \ldots, k\}$.

Theorem 5.4.4 *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be finite A'-automata. Then $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$ and $||\mathfrak{A} \cdot \mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be defined as above. We first show $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$ and compute $||\mathfrak{A} + \mathfrak{A}'|| \cdot 0$. The transition matrix of $\mathfrak{A} + \mathfrak{A}'$ is given by

$$
M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.
$$

We now compute the first $m+k$ entries of $M^{\omega_{m+k}}$. This column vector of dimension $m+k$ is given by

$$
\left(\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}^* \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} \right)^\omega =
$$

$$
\begin{pmatrix} a+bd^*c & 0 \\ 0 & a'+b'd'^*c' \end{pmatrix}^\omega = \begin{pmatrix} (a+bd^*c)^\omega \\ (a'+b'd'^*c')^\omega \end{pmatrix}.
$$

The last $n+n'-(m+k)$ entries of $M^{\omega_{m+k}}$ are given by the product of

$$\begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}^* \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} = \begin{pmatrix} d^*c & 0 \\ 0 & d'^*c' \end{pmatrix}$$

with the column vector computed above. Hence, we obtain by Lemma 4.3

$$\begin{aligned} &||\mathfrak{A}+\mathfrak{A}'||\cdot 0 = (i_1\ h\ i_2\ i)M^{\omega_{m+k}} = i_1(a+bd^*c)^\omega + h(a'+b'd'^*c')^\omega + \\ &i_2d^*c(a+bd^*c)^\omega + id'^*c'^*(a'+b'd'^*c')^\omega = (||\mathfrak{A}||+||\mathfrak{A}'||)\cdot 0\,. \end{aligned}$$

We now compute $||\mathfrak{A}+\mathfrak{A}'||¶$. If, in the transition matrix M of $\mathfrak{A}+\mathfrak{A}'$ we commute the $m+1,\ldots,m+k$ row and column with the $m+k+1,\ldots,n+k$ row and column, and do the same with the initial and final vector we obtain by the star permutation equation (see Conway [25], Ésik, Kuich [39])

$$\begin{aligned} &||\mathfrak{A}+\mathfrak{A}'||¶ = (i_1\ i_2\ h\ i)\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}^* \begin{pmatrix} f \\ g \\ p_1 \\ p_2 \end{pmatrix} = \\ &(i_1\ i_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} f \\ g \end{pmatrix} + (h\ i)\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^* \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \\ &(||\mathfrak{A}||+||\mathfrak{A}'||)¶\,. \end{aligned}$$

Hence, $||\mathfrak{A}+\mathfrak{A}'|| = ||\mathfrak{A}||+||\mathfrak{A}'||$.

We now show $||\mathfrak{A}\cdot\mathfrak{A}'|| = ||\mathfrak{A}||\cdot||\mathfrak{A}'||$ and compute $||\mathfrak{A}\cdot\mathfrak{A}'||\cdot 0$. The transition matrix of $\mathfrak{A}\cdot\mathfrak{A}'$ is given by

$$M = \begin{pmatrix} a & fh & b & fi \\ 0 & a' & 0 & b' \\ c & gh & d & gi \\ 0 & c' & 0 & d' \end{pmatrix}.$$

We now compute the first $m+k$ entries of $M^{\omega_{m+k}}$. This column vector of dimension $m+k$ is given by

$$\begin{aligned} &\left(\begin{pmatrix} a & fh \\ 0 & a' \end{pmatrix} + \begin{pmatrix} b & fi \\ 0 & b' \end{pmatrix}\begin{pmatrix} d^* & d^*gid'^* \\ 0 & d'^* \end{pmatrix}\begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix}\right)^\omega = \\ &\begin{pmatrix} a+bd^*c & (f+bd^*g)(h+id'^*c') \\ 0 & a'+b'd'^*c' \end{pmatrix}^\omega = \\ &\begin{pmatrix} (a+bd^*c)^\omega + (a+bd^*c)^*(f+bd^*g)(h+id'^*c')(a'+b'd'^*c')^\omega \\ (a'+b'd'^*c')^\omega \end{pmatrix}. \end{aligned}$$

The last $n+n'-(m+k)$ entries of $M^{\omega_{m+k}}$ are given by the product of

$$\begin{pmatrix} d & gi \\ 0 & d' \end{pmatrix}^* \begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix} = \begin{pmatrix} d^*c & d^*g(h+id'^*c') \\ 0 & d'^*c' \end{pmatrix}$$

with the column vector computed above. Hence, we obtain

$$\begin{aligned}
&||\mathfrak{A}\cdot\mathfrak{A}'||\cdot 0 = (i_1\ 0\ i_2\ 0)M^{\omega_{m+k}} = \\
&i_1(a+bd^*c)^\omega + i_1(a+bd^*c)^*(f+bd^*g)(h+id'^*c)(a'+b'd'^*c')^\omega + \\
&i_2d^*c(a+bd^*c)^\omega + i_2d^*c(a+bd^*c)^*(f+bd^*g)(h+id'^*c')(a'+b'd'^*c')^\omega + \\
&i_2d^*g(h+id'^*c')(a'+b'd'^*c')^\omega\,.
\end{aligned}$$

On the other side, we obtain by Lemma 4.3

$$\begin{aligned}
&||\mathfrak{A}||\cdot||\mathfrak{A}'||\cdot 0 = ||\mathfrak{A}||\cdot 0 + ||\mathfrak{A}||\P\cdot||\mathfrak{A}'||\cdot 0 = \\
&i_1(a+bd^*c)^\omega + i_2d^*c(a+bd^*c)^\omega + (i_1(a+bd^*c)^*(f+bd^*g) + \\
&i_2d^*c(a+bd^*c)^*(f+bd^*g) + i_2d^*g)(h+id'^*c')(a'+b'd'^*c')^\omega\,.
\end{aligned}$$

Hence, $||\mathfrak{A}\cdot\mathfrak{A}'||\cdot 0 = ||\mathfrak{A}||\cdot||\mathfrak{A}'||\cdot 0$.

We now compute $||\mathfrak{A}\cdot\mathfrak{A}'||\P$. If, in the transition matrix M of $\mathfrak{A}\cdot\mathfrak{A}'$ we commute the $m+1,\ldots,m+k$ row and column with the $m+k+1,\ldots,n+k$ row and column, and do the same with the initial and final vector we obtain by the star permutation equation (see Conway [25], Ésik, Kuich [39])

$$\begin{aligned}
&||\mathfrak{A}\cdot\mathfrak{A}'||\P = (i_1\ i_2\ 0\ 0)\begin{pmatrix} a & b & fh & fi \\ c & d & gh & gi \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}^* \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_2 \end{pmatrix} = \\
&(i_1\ i_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} f \\ g \end{pmatrix} (h\ i) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^* \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \\
&||\mathfrak{A}||\P\cdot||\mathfrak{A}'||\P = ||\mathfrak{A}||\cdot||\mathfrak{A}'||\P\,.
\end{aligned}$$

Hence, $||\mathfrak{A}\cdot\mathfrak{A}'|| = ||\mathfrak{A}||\cdot||\mathfrak{A}'||$. □

Let $\mathfrak{A} = (n; h, i; a, b, c, d; f, g; k)$ be a finite A'-automaton and write $I = (h\ i)$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P = \begin{pmatrix} f \\ g \end{pmatrix}$. Then we define the finite A'-automaton $\mathfrak{A}^\otimes$ to be

$$\begin{aligned}
\mathfrak{A}^\otimes \quad = \quad & (1+n+n; (1\ 0), (0\ 0); \begin{pmatrix} 0 & h \\ 0 & a \end{pmatrix}, \begin{pmatrix} i & I \\ b & 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 & c \\ P & 0 \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & M \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}; 1+k)\,.
\end{aligned}$$

Theorem 5.4.5 *Let $\mathfrak{A}$ be a finite A'-automaton. Then $||\mathfrak{A}^\otimes|| = ||\mathfrak{A}||^\otimes$.*

Proof. Let $\mathfrak{A}$ be defined as above. Let

$$M' = \begin{pmatrix} 0 & h & i & I \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ P & 0 & 0 & M \end{pmatrix}.$$

We first compute $||\mathfrak{A}^{\otimes}||\P$. Observe that M' can be written as

$$M' = \begin{pmatrix} 0 & I & I \\ 0 & M & 0 \\ P & 0 & M \end{pmatrix}$$

and that $||\mathfrak{A}^{\otimes}||\P = (M'^*)_{11}$. We obtain

$$\begin{aligned}(M'^*)_{11} = &\left((I\ I) \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}^* \begin{pmatrix} 0 \\ P \end{pmatrix} \right)^* = \\ &(IM^*P)^* = (||\mathfrak{A}||\P)^* = ||\mathfrak{A}||^{\otimes}\P\,.\end{aligned}$$

We now compute the first $1+k$ entries of $M'^{\omega_{1+k}}$. This column vector of dimension $1+k$ is given by

$$\begin{aligned}&\left(\begin{pmatrix} 0 & h \\ 0 & a \end{pmatrix} + \begin{pmatrix} i & I \\ b & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & M \end{pmatrix}^* \begin{pmatrix} 0 & c \\ P & 0 \end{pmatrix} \right)^{\omega} = \\ &\begin{pmatrix} IM^*P & h+id^*c \\ 0 & a+bd^*c \end{pmatrix}^{\omega}.\end{aligned}$$

Hence, $||\mathfrak{A}^{\otimes}|| \cdot 0 = (M'^{\omega_{1+k}})_1 = (IM^*P)^{\omega} + (IM^*P)^*(h+id^*c)(a+bd^*c)^{\omega}$. By definition, $||\mathfrak{A}||^{\otimes} \cdot 0 = (||\mathfrak{A}||\P)^{\omega} + (||\mathfrak{A}||\P)^*||\mathfrak{A}|| \cdot 0$. Thus $||\mathfrak{A}||^{\otimes} \cdot 0 = (IM^*P)^{\omega} + (IM^*P)^*(h+id^*c)(a+bd^*c)^{\omega} = ||\mathfrak{A}^{\otimes}|| \cdot 0$ and we obtain $||\mathfrak{A}^{\otimes}|| = ||\mathfrak{A}||^{\otimes}$. □

Theorem 5.4.6 *Let $\mathfrak{A} = (n, I, M, P, k)$ be a finite A'-automaton. Then there exists a finite A'-automaton $\mathfrak{A}\P$ such that $||\mathfrak{A}\P|| = ||\mathfrak{A}||\P$.*

Proof. $||\mathfrak{A}||\P = (n, I, M, P, 0)$. □

Theorem 5.4.7 *Let $a \in A' \cup \{0,1\}$. Then there exists a finite A'-automaton $\mathfrak{A}_a$ such that $||\mathfrak{A}_a|| = a$.*

Proof. Let $\mathfrak{A}_a = (2, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, (1\ 0), \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0)$. Then

$$||\mathfrak{A}_a|| = (1\ 0) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\,.$$

□

Corollary 5.4.8 *The behaviors of finite A'-automata form a generalized starquemiring that contains A'.*

Theorem 5.4.9 (Kleene Theorem) *Let (A, V) be a Conway semiring-semimodule pair. Then the following statements are equivalent for $(s, v) \in A \times V$:*

(i) $(s, v) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite A'-automaton,*

(ii) $(s, v) \in \omega\text{-}\mathfrak{Rat}(A')$,

(iii) $s \in \mathfrak{Rat}(A')$ *and* $v \in \sum_{1\le k\le m} s_k t_k^m$ *with* $s_k, t_k \in \mathfrak{Rat}(A')$.

Proof. (ii) $\Rightarrow$ (iii): Each entry in M^{ω_k} is of the form $(s, \sum_{1\le k\le m} s_k t_k^m)$ with $s, s_k, t_k \in \mathfrak{Rat}(A')$.

(iii) $\Rightarrow$ (ii): $(s,v) = (s,0) + (0,v)$. Since $(s,0)$ is in $\mathfrak{Rat}(A') \subseteq \omega\text{-}\mathfrak{Rat}(A')$ and $(0,v) = (0, \sum_{1\le k\le m} s_k t_k^m)$ is in $\omega\text{-}\mathfrak{Rat}(A')$, (s,v) is in $\omega\text{-}\mathfrak{Rat}(A')$.

(ii) $\Rightarrow$ (i): By Corollary 4.8. □

5.5 Linear systems over quemirings

In this section we consider linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations. Before dealing with these linear systems we prove two matrix theorems, Theorems 5.1 and 5.4 for Conway semiring-semimodule pairs, and two theorems on complete semiring-semimodule pairs, Theorems 5.5 and 5.6.

Theorem 5.5.1 *Let (A,V) be a Conway semiring-semimodule pair. Then, for $0 \le k \le n$,*

$$MM^{\omega_k} = M^{\omega_k}.$$

Proof. Let M be partitioned as in (1), but with a of dimension $k \times k$ and d of dimension $(n-k)\times(n-k)$. Then

$$MM^{\omega_k} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \end{pmatrix} = M^{\omega_k}.$$

□

Two lemmas are needed before the proof of Theorem 5.4.

Lemma 5.5.2 *Let (A,V) be a Conway semiring-semimodule pair and $0 \le k \le n$. Let $a \in A^{k\times k}$, $b_0, b_1 \in A^{k\times(n-k)}$, $c \in A^{(n-k)\times k}$, $d_0, d_1 \in A^{(n-k)\times(n-k)}$. Furthermore, let $M_0 = \begin{pmatrix} 0 & b_0 \\ 0 & d_0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}$. Then*

$$(M_0 + M_1)^{\omega_k} = (M_0^* M_1)^{\omega_k}.$$

Proof.

$$(M_0^*M_1)^{\omega_k} = \left(\begin{pmatrix} E & b_0d_0^* \\ 0 & d_0^* \end{pmatrix}\begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}\right)^{\omega_k} =$$

$$\begin{pmatrix} a + b_0d_0^*c & b_1 + b_0d_0^*d_1 \\ d_0^*c & d_0^*d_1 \end{pmatrix}^{\omega_k} =$$

$$\begin{pmatrix} (a + b_0d_0^*c + (b_1+b_0d_0^*d_1)(d_0^*d_1)^*d_0^*c)^\omega \\ (d_0^*d_1)^*d_0^*c(a + b_0d_0^*c + (b_1+b_0d_0^*d_1)(d_0^*d_1)^*d_0^*c)^\omega \end{pmatrix}.$$

The upper block equals $(a+b_0d_0^*c+b_1(d_0+d_1)^*c+b_0(d_0^*d_1)(d_0^*d_1)^*d_0^*c)^\omega = (a+(b_0+b_1)(d_0+d_1)^*c)^\omega$. The lower block equals $(d_0+d_1)^*c(a+(b_0+b_1)(d_0+d_1)^*c)^\omega$. Hence, our lemma is proven. □

Lemma 5.5.3 *Let (A, V) be a Conway semiring-semimodule pair, and $0 \le k \le n$. Let $a_0, a_1 \in A^{k\times k}$, $b \in A^{k\times(n-k)}$, $c_0, c_1 \in A^{(n-k)\times k}$, $d \in A^{(n-k)\times(n-k)}$, and assume $a_0^\omega = 0$. Furthermore, let $M_0 = \begin{pmatrix} a_0 & 0 \\ c_0 & 0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} a_1 & b \\ c_1 & d \end{pmatrix}$. Then*

$$(M_0 + M_1)^{\omega_k} = M_0^*(M_1 M_0^*)^{\omega_k} .$$

Proof.

$$(M_1 M_0^*)^{\omega_k} = \left(\begin{pmatrix} a_1 & b \\ c_1 & d \end{pmatrix} \begin{pmatrix} a_0^* & 0 \\ c_0 a_0^* & E \end{pmatrix} \right)^{\omega_k} =$$

$$\begin{pmatrix} a_1 a_0^* + b c_0 a_0^* & b \\ c_1 a_0^* + d c_0 a_0^* & d \end{pmatrix}^{\omega_k} =$$

$$\begin{pmatrix} (a_1 a_0^* + b c_0 a_0^* + b d^*(c_1 a_0^* + d c_0 a_0^*))^\omega \\ d^*(c_1 a_0^* + d c_0 a_0^*)(a_1 a_0^* + b c_0 a_0^* + b d^*(c_1 a_0^* + d c_0 a_0^*))^\omega \end{pmatrix} =$$

$$\begin{pmatrix} ((a_1 + b d^*(c_0 + c_1)) a_0^*)^\omega \\ (d^* c_1 + d^* d c_0)(a_0^*(a_1 + b d^*(c_0 + c_1)))^\omega \end{pmatrix} .$$

Hence,

$$M_0^*(M_1 M_0^*)^{\omega_k} = \begin{pmatrix} (a_0^*(a_1 + b d^*(c_0 + c_1)))^\omega \\ d^*(c_0 + c_1)(a_0^*(a_1 + b d^*(c_0 + c_1)))^\omega \end{pmatrix} =$$

$$\begin{pmatrix} (a_0 + a_1 + b d^*(c_0 + c_1))^\omega \\ d^*(c_0 + c_1)(a_0 + a_1 + b d^*(c_0 + c_1))^\omega \end{pmatrix} .$$

In the last step we applied the sum-omega equation and used the assumption $a_0^\omega = 0$. Hence, our lemma is proven. $\square$

Theorem 5.5.4 *Let (A, V) be a Conway semiring-semimodule pair, and $0 \le k \le n$. Let $a_0, a_1 \in A^{k\times k}$, $b_0, b_1 \in A^{k\times(n-k)}$, $c_0, c_1 \in A^{(n-k)\times k}$, $d_0, d_1 \in A^{(n-k)\times(n-k)}$, and assume $(a_0 + b_0 d_0^* c_0)^\omega = 0$. Furthermore, let*

$$M_{01} = \begin{pmatrix} 0 & b_0 \\ 0 & d_0 \end{pmatrix}, \quad M_{02} = \begin{pmatrix} a_0 & 0 \\ c_0 & 0 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} .$$

Then

$$(M_{01}^* M_{02})^*(M_{01}^* M_1 (M_{01}^* M_{02})^*)^* M_{01}^* = (M_{01} + M_{02} + M_1)^*$$

and

$$(M_{01}^* M_{02})^*(M_{01}^* M_1 (M_{01}^* M_{02})^*)^{\omega_k} = (M_{01} + M_{02} + M_1)^{\omega_k} .$$

Proof. The left side of the first equality equals $(M_{01}^*(M_{02}+M_1))^* M_{01}^* = (M_{01} + M_{02} + M_1)^*$.

The left upper block of $M_{01}^* M_{02}$ equals $a_0 + b_0 d_0^* c_0$. Hence, by Lemma 5.3, the left side of the second equality equals $(M_{01}^*(M_{02} + M_1))^{\omega_k}$, which is, by Lemma 5.2, equal to $(M_{01} + M_{02} + M_1)^{\omega_k}$. $\square$

For the remainder of this section we consider a complete star-omega semiring A. In the next two theorems Σ_∞ is a finite or infinite alphabet.

Theorem 5.5.5 *Let A be a complete star-omega semiring. Then $(A\langle\langle\Sigma_\infty^*\rangle\rangle, A\langle\langle\Sigma_\infty^\omega\rangle\rangle)$ is a complete semiring-semimodule pair.*

Proof. It is clear that $A\langle\langle\Sigma_\infty^\omega\rangle\rangle$ is a left $A\langle\langle\Sigma_\infty^*\rangle\rangle$-semimodule, and moreover, the action distributes over all sums in both arguments. Given series $s_1, s_2, \dots$ in $A\langle\langle\Sigma_\infty^*\rangle\rangle$, define $s = s_1 s_2 \dots$ by

$$(s, w) \quad = \quad \sum_{w=w_1 w_2 \dots} (s_1, w_1)(s_2, w_2) \dots$$

for all $w \in \Sigma_\infty^*$. We verify that the infinite product satisfies the three conditions on the infinite product operation in the definition of a complete semiring-semimodule pair in Section 2.

Suppose that $s_0, s_1, s_2, \dots$ are in $A\langle\langle\Sigma_\infty^*\rangle\rangle$. If $1 \le k_1 \le k_2 \dots$, then for all $w \in \Sigma_\infty^\omega$,

$$\begin{aligned}
&((s_1 \dots s_{k_1})(s_{k_1+1} \dots s_{k_2}) \dots, w) = \\
&\textstyle\sum_{w=w_1' w_2' \dots} (s_1 \dots s_{k_1}, w_1')(s_{k_1+1} \dots s_{k_2}, w_2') \dots = \\
&\textstyle\sum_{w=w_1' w_2' \dots} \sum_{w_1' = w_1 \dots w_{k_1}} (s_1, w_1) \dots (s_{k_1}, w_{k_1}) \sum_{w_2' = w_{k_1+1} \dots w_{k_2}} (s_{k_1+1}, w_{k_1+1}) \dots (s_{k_2}, w_{k_2}) \dots = \\
&\textstyle\sum_{w=w_1' w_2' \dots} \sum_{w_1' = w_1 \dots w_{k_1}, w_2' = w_{k_1+1} \dots w_{k_2}, \dots} (s_1, w_1) \dots (s_{k_1}, w_{k_1})(s_{k_1+1}, w_{k_1+1}) \dots = \\
&\textstyle\sum_{w=w_1 w_2 \dots} (s_1, w_1)(s_2, w_2) \dots = (s_1 s_2 \dots, w),
\end{aligned}$$

proving the first condition. Also,

$$\begin{aligned}
&(s_0 \cdot (s_1 s_2 \dots), w) = \\
&\textstyle\sum_{w=w_0 w'} (s_0, w_0)(s_1 s_2 \dots, w') = \\
&\textstyle\sum_{w=w_0 w'} (s_0, w_0) \sum_{w' = w_1 w_2 \dots} (s_1, w_1)(s_2, w_2) \dots = \\
&\textstyle\sum_{w=w_0 w'} \sum_{w' = w_1 w_2 \dots} (s_0, w_0)(s_1, w_1)(s_2, w_2) \dots = \\
&\textstyle\sum_{w=w_0 w_1 w_2 \dots} (s_0, w_0)(s_1, w_1)(s_2, w_2) \dots = (s_0 s_1 s_2 \dots, w),
\end{aligned}$$

proving the second condition. Finally, suppose that $s_{i_j}^j \in A\langle\langle\Sigma_\infty^*\rangle\rangle$ for all $i_j \in I_j$, $j \ge 1$, where each I_j is an arbitrary index set. Then, for each $w \in \Sigma_\infty^*$,

$$\begin{aligned}
&(\textstyle\sum_{i_1 \in I_1} s_{i_1}^1 \sum_{i_2 \in I_2} s_{i_2}^2 \dots, w) = \\
&\textstyle\sum_{w=w_1 w_2 \dots} (\sum_{i_1 \in I_1} s_{i_1}^1, w_1)(\sum_{i_2 \in I_2} s_{i_2}^2, w_2) \dots = \\
&\textstyle\sum_{w=w_1 w_2 \dots} \sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} (s_{i_1}^1, w_1)(s_{i_2}^2, w_2) \dots = \\
&\textstyle\sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} \sum_{w=w_1 w_2 \dots} (s_{i_1}^1, w_1)(s_{i_2}^2, w_2) = \\
&\textstyle\sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} (s_{i_1}^1 s_{i_2}^2 \dots, w),
\end{aligned}$$

proving the third condition. □

Theorem 5.5.6 *Let A be a complete star-omega semiring. For $M^{(j)} \in (A\langle\langle\Sigma_\infty^*\rangle\rangle)^{n \times n}$, $j \ge 1$, define $\prod_{j \ge 1} M^{(j)}$ by*

$$(\prod_{j \ge 1} M^{(j)})_i = \sum_{1 \le i_1, i_2, \dots \le n} M_{i i_1}^{(1)} M_{i_1 i_2}^{(2)} M_{i_2 i_3}^{(3)} \dots, \qquad 1 \le i \le n\,.$$

Then $((A\langle\langle\Sigma_\infty^*\rangle\rangle)^{n \times n}, (A\langle\langle\Sigma_\infty^\omega\rangle\rangle)^n)$ is a complete semiring-semimodule pair.

Proof. We only prove the third condition on the infinite product operation in the definition of a complete semiring-semimodule pair in Section 2.

Let $M^{(i_j)} \in (A\langle\langle\Sigma_\infty^*\rangle\rangle)^{n\times n}$ for $j \geq 1$. Then we obtain, for $1 \leq k \leq n$,

$$
\begin{aligned}
&(\textstyle\prod_{j\geq 1}(\sum_{i_j\in I_j} M^{(i_j)}))_k = \\
&\textstyle\sum_{1\leq k_1,k_2,\ldots\leq n}(\sum_{i_1\in I_1} M^{(i_1)})_{kk_1}(\sum_{i_2\in I_2} M^{(i_2)})_{k_1k_2}\ldots = \\
&\textstyle\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\sum_{1\leq k_1,k_2,\ldots\leq n} M^{(i_1)}_{kk_1}M^{(i_2)}_{k_1k_2}\ldots = \\
&\textstyle\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}(\prod_{j\geq 1} M^{(i_j)})_k\,,
\end{aligned}
$$

verifying the third condition. □

An A'*-linear system* (*with variables* $z_1,\ldots,z_n$*, over the quemiring* $A\times V$) is a system of equations

$$My + P = y \tag{5.22}$$

where $M \in (A' \cup \{0,1\})^{n\times n}$, $P \in (A' \cup \{0,1\})^{n\times 1}$, $y = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$. A column vector $\sigma \in (A\times V)^{n\times 1}$ is called a *solution* to the system (22) if

$$M\sigma + P = \sigma\,.$$

Theorem 5.5.7 *Let* (A,V) *be a Conway semiring-semimodule pair. Consider an* A'*-linear system*

$$My + P = y\,,$$

where $M \in (A'\cup\{0,1\})^{n\times n}$*,* $P \in (A'\cup\{0,1\})^{n\times 1}$*, and* $y = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ *is a column vector of variables. Then, for each* $0 \leq k \leq n$*,* $M^{\omega_k} + M^*P$ *is a solution of* $My + P = y$*.*

Proof. We obtain, by Theorem 5.1, for each $0 \leq k \leq n$,

$$M(M^{\omega_k} + M^*P) + P = M^{\omega_k} + M^*P\,.$$

□

Let $\mathfrak{A}_i = (n, e_i, M, P, k)$, $1 \leq i \leq n$, be finite A'-automata, where e_i is the i-th vector of unity. Then $||\mathfrak{A}_i||$ is the i-th component of a solution given in Theorem 5.1 of the A'-linear system $My + P = y$. Therefore, we call the solution $\begin{pmatrix} ||\mathfrak{A}_1|| \\ \vdots \\ ||\mathfrak{A}_n|| \end{pmatrix} = M^{\omega_k} + M^*P$ of $My+P = y$ the *k-th automata-theoretic solution* of $My + P = y$.

Theorem 5.5.8 *Let (A, V) be a Conway semiring-semimodule pair and $A' \subseteq A$. Let $\mathfrak{A} = (n, I, M, P, k)$ be a finite A'-automaton. Then $||\mathfrak{A}|| = I\sigma$, where σ is the k-th automata-theoretic solution of the A'-linear system $My + P = y$.*

Let A be a complete star-omega semiring and consider an A'-linear system $My+P = y$ over the quemiring $A\langle\langle\Sigma^*\rangle\rangle \times A\langle\langle\Sigma^\omega\rangle\rangle$ as defined before Theorem 5.7 for $A' = A\langle\Sigma \cup \varepsilon\rangle$. Write this system in the form

$$z_i = \sum_{1\le j\le n} \sum_{x\in\Sigma\cup\varepsilon} (M_{ij}, x)xz_j + \sum_{x\in\Sigma\cup\{\varepsilon\}} (P_i, x)x\,, \quad 1 \le i \le n\,.$$

We associate to this system the rightlinear grammars $G_i = (\{z_1, \ldots, z_n\}, \Sigma, R, z_i)$, $1 \le i \le n$, with weights in the semiring A, where $R = \{z_i \to (M_{ij}, x)xz_j \mid 1 \le j \le n,\ x \in \Sigma \cup \{\varepsilon\}\} \cup \{z_i \to (P_i, x)x \mid x \in \Sigma \cup \{\varepsilon\}\}$. Here (M_{ij}, x) and (P_i, x) are the weights of the productions $z_i \to xz_j$ and $z_i \to x$, respectively. (Compare with Chapter 2 before Theorem 2.3.8.) Furthermore, let $\mathfrak{A}_i^k = (n, e_i, M, P, k)$ be finite A'-automata, $1 \le i \le n$, for some fixed $k \in \{0, \ldots, n\}$, where e_i is the i-th row vector of unity.

Consider now a finite derivation with respect to G_i:

$$z_i \Rightarrow (M_{i,i_1}, x_1)x_1z_{i_1} \Rightarrow \ldots \Rightarrow (M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)x_1\ldots x_m z_{i_m} \Rightarrow (M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)(P_{i_m}, x_{m+1})x_1\ldots x_m x_{m+1}$$

generating the word $x_1 \ldots x_m x_{m+1}$ with weight

$$(M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)(P_{i_m}, x_{m+1})\,.$$

This finite derivation corresponds to the following finite path in the directed graph of $\mathfrak{A}_i^k$:

$$(z_i, x_1, z_{i_1}), \ldots, (z_{i_{m-1}}, x_m, z_{i_m})$$

with weight

$$(M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)\,,$$

initial weight 1 and final weight $(P_{i_m}, x_{m+1})x_{m+1}$.

Consider now an infinite derivation with respect to G_i:

$$z_i \Rightarrow (M_{i,i_1}, x_1)x_1z_{i_1} \Rightarrow \ldots \Rightarrow (M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)x_1\ldots x_m z_{i_m} \Rightarrow \ldots$$

generating the infinite word $x_1x_2 \ldots x_m \ldots$ with weight $(M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)\ldots$. This infinite derivation corresponds to the following infinite path in the directed graph of $\mathfrak{A}_i^k$:

$$(z_i, x_1, z_{i_1}), \ldots, (z_{i_{m-1}}, x_m, z_{i_m}), \ldots$$

with weight $(M_{i,i_1}, x_1)\ldots(M_{i_{m-1},i_m}, x_m)\ldots$ and initial weight 1.

Hence, we obtain, by Theorems 4.1 and 5.5, the following result for G_i and $\mathfrak{A}_i^k$ as defined above.

Theorem 5.5.9 *If A is a complete star-omega semiring and $1 \leq i \leq n$, $0 \leq k \leq n$, then, for $w \in \Sigma^*$, $(||\mathfrak{A}_i^k||, w) = ((M^*P)_i, w)$ is the sum of the weights of all finite derivations of w with respect to G_i; and for $w \in \Sigma^\omega$, $(||\mathfrak{A}_i^k||, w) = ((M^{\omega_k})_i, w)$ is the sum of the weights of all infinite derivations of w with respect to G_i such that at least one of the variables of $\{z_1, \ldots, z_k\}$ appears infinitely often in these infinite derivations.*

In particular, if $A = \mathbb{N}^\infty$ and $(M_{ij}, x), (P_i, x) \in \{0, 1\}$, $x \in \Sigma \cup \{\varepsilon\}$, $1 \leq i, j \leq n$, then we get the following result.

Theorem 5.5.10 *For $w \in \Sigma^*$, $(||\mathfrak{A}_i^k||, w) = ((M^*P)_i, w)$ is the number of finite derivations of w with respect to G_i; and for $w \in \Sigma^\omega$, $(||\mathfrak{A}_i^k||, w) = ((M^{\omega_k})_i, w)$ is the number of all infinite derivations of w with respect to G_i such that at least one of the variables of $\{z_1, \ldots, z_k\}$ appears infinitely often in these infinite derivations.*

We now want to delete ε-moves in finite $A\langle\Sigma \cup \varepsilon\rangle$-automata without changing their behavior.

Theorem 5.5.11 *Let $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair, where $(a\varepsilon)^\omega = 0$ for all $a \in A$, and consider a finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton $\mathfrak{A} = (n, I, M, P, k)$. Then there exists an finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton $\mathfrak{A}' = (n, I', M', P', k)$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$ satisfying the following conditions:*

(i) $M' \in (A\langle\Sigma\rangle)^{n \times n}$,

(ii) $I' \in (A\langle\varepsilon\rangle)^{1 \times n}$,

(iii) $P' \in (A\langle\varepsilon\rangle)^{n \times 1}$.

Proof. Without loss of generality we assume by Theorem 4.2 that $I \in (A\langle\varepsilon\rangle)^{1 \times n}$ and $P \in (A\langle\varepsilon\rangle)^{n \times 1}$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a is $k \times k$ and d is $(n-k) \times (n-k)$. Let $a = a_0 + a_1$, $b = b_0 + b_1$, $c = c_0 + c_1$, $d = d_0 + d_1$, such that the supports of the entries of a_0, b_0, c_0, d_0 (resp. a_1, b_1, c_1, d_1) are subsets of $\{\varepsilon\}$ (resp. Σ). Since $\varepsilon^\omega = 0$, we obtain $(a_0 + b_0 d_0^* c_0)^\omega = 0$.

Define the matrices M_{01}, M_{02} and M_1 to be $M_{01} = \begin{pmatrix} 0 & b_0 \\ 0 & d_0 \end{pmatrix}$, $M_{02} = \begin{pmatrix} a_0 & 0 \\ c_0 & 0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. We now specify the finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton $\mathfrak{A}'$: $I' = I(M_{01}^* M_{02})^*$, $M' = M_{01}^* M_1 (M_{01}^* M_{02})^*$ and $P' = M_{01}^* P$. The behavior of $\mathfrak{A}'$ is then given by

$$
\begin{aligned}
&||\mathfrak{A}'|| = I'M'^*P' + I'M'^{\omega_k} = \\
&I(M_{01}^* M_{02})^* (M_{01}^* M_1 (M_{01}^* M_{02})^*)^* M_{01}^* P + \\
&I(M_{01}^* M_{02})^* (M_{01}^* M_1 (M_{01}^* M_{02})^*)^{\omega_k} = \\
&I(M_{01} + M_{02} + M_1)^* P + I(M_{01} + M_{02} + M_1)^{\omega_k} = \\
&IM^*P + IM^{\omega_k} = ||\mathfrak{A}||\,.
\end{aligned}
$$

Here we have applied Theorem 5.4 in the third equality. □

Theorem 5.5.12 *Let $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair, where $(a\varepsilon)^\omega = 0$ for all $a \in A$, and consider a finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton $\mathfrak{A} = (n, I, M, P, k)$. Then there exists an finite $A\langle\Sigma \cup \varepsilon\rangle$-automaton $\mathfrak{A}' = (n+1, I', M', P', k)$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$ satisfying the following conditions:*

(i) $M' \in (A\langle\Sigma\rangle)^{(n+1)\times(n+1)}$,

(ii) $I'_j = 0$, $1 \le j \le n$, *and* $I'_{n+1} = \varepsilon$,

(iii) $P' \in (A\langle\varepsilon\rangle)^{(n+1)\times 1}$.

Proof. We assume that $\mathfrak{A}$ satisfies the conditions of Theorem 5.11. We specify $\mathfrak{A}'$ by $I' = (0\ \varepsilon)$, $M' = \begin{pmatrix} M & 0 \\ IM & 0 \end{pmatrix}$ and $P' = \begin{pmatrix} P \\ IP \end{pmatrix}$. We compute $M'^* = \begin{pmatrix} M^* & 0 \\ IMM^* & 1 \end{pmatrix}$ and, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $I = (i_1\ i_2)$,

$$IM'^{\omega_k} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ i_1 a + i_2 c & i_1 b + i_2 d & 0 \end{pmatrix}^{\omega_k} =$$

$$\begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \\ (i_1(a+bd^*c) + i_2 d^*c)(a+bd^*c)^\omega \end{pmatrix} = \begin{pmatrix} M^{\omega_k} \\ IM^{\omega_k} \end{pmatrix}.$$

Hence, $||\mathfrak{A}'|| = IMM^*P + IP + IM^{\omega_k} = ||\mathfrak{A}||$. □

In the case of the Boolean semiring, the finite $\mathbb{B}\langle\Sigma \cup \varepsilon\rangle$-automata of Theorem 5.12 are nothing else than the finite automata introduced by Büchi [18].

In the case of the semiring $\mathbb{N}^\infty$ we get the following result.

Theorem 5.5.13 *The constructions of Theorems 5.11 and 5.12 do not change, for $w \in \Sigma^*$ (resp. for $w \in \Sigma^\omega$), in the digraphs of the finite automata, the number of finite paths with label w from an initial state to a final state (resp. the number of infinite paths with label w starting in an initial state and passing infinitely often through repeated states).*

Given a rightlinear grammar $G_i = (\{z_1, \ldots, z_n\}, \Sigma, R, z_i)$, $1 \le i \le n$, as above, and $k \in \{0, \ldots, n\}$, $L(G_i)_k$ is defined to be the weighted language

$$L(G_i)_k = \{((M^*P)_i, w)w \mid w \in \Sigma^*\} \cup \{((M^{\omega_k})_i, w)w \mid w \in \Sigma^\omega\}.$$

The next theorem, Theorem 5.14, shows that such weighted languages can be generated by rightlinear grammars with weights in the semiring A which have only two types of productions:

$$z_i \to axz_j \quad \text{and} \quad z_i \to a\varepsilon,$$

where $a \in A$ and $x \in \Sigma$. Hence, in such rightlinear grammars there are no productions $z_i \to az_j$. Corollary 5.15 shows then, that the two types of productions can be chosen as

$$z_i \to axz_j \quad \text{and} \quad z_i \to ax,$$

where $a \in A$ and $x \in \Sigma$. (Of course, ε is no longer derived.)

Theorem 5.5.14 *Let $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair, where $(a\varepsilon)^\omega = 0$ for all $a \in A$, consider an $A\langle\Sigma\cup\varepsilon\rangle$-linear system $My + P = y$, where $M \in (A\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$, $P \in (A\langle\Sigma\cup\varepsilon\rangle)^{n\times 1}$, and $y = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ and let $i \in \{1,\ldots,n\}$. Then there exists an $A\langle\Sigma\cup\varepsilon\rangle$-linear system $M'y' + P' = y'$, where $M' \in (A\langle\Sigma\rangle)^{(n+1)\times(n+1)}$, $P' \in (A\langle\varepsilon\rangle)^{(n+1)\times 1}$, and $y' = \begin{pmatrix} y \\ z_{n+1} \end{pmatrix}$ such that*

$$(M'^{\omega_k} + M'^* P')_{n+1} = (M^{\omega_k} + M^* P)_i\,.$$

Proof. Consider the finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton $\mathfrak{A}_i^k = (n, e_i, M, P, k)$, whose behavior is $||\mathfrak{A}_i^k|| = (M^*P)_i + (M^{\omega_k})_i$. Starting with $\mathfrak{A}_i^k$, perform the constructions of Theorems 5.11 and 5.12. This yields a finite $A\langle\Sigma\cup\varepsilon\rangle$-automaton $\mathfrak{A}' = (n+1, e_{n+1}, M', P', k)$ with behavior $||\mathfrak{A}'|| = (M'^*P')_{n+1} + (M'^{\omega_k})_{n+1} = ||\mathfrak{A}_i^k||$. $\square$

Corollary 5.5.15 *Let $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair, where $(a\varepsilon)^\omega = 0$ for all $a \in A$, consider an $A\langle\Sigma\cup\varepsilon\rangle$-linear system $My + P = y$, where $M \in (A\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$, $P \in (A\langle\Sigma\cup\varepsilon\rangle)^{n\times 1}$, and $y = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ and let $i \in \{1,\ldots,n\}$. Then there exists an $A\langle\Sigma\cup\varepsilon\rangle$-linear system $M'y' + P' = y'$, where $M' \in (A\langle\Sigma\rangle)^{(n+1)\times(n+1)}$, $P' \in (A\langle\Sigma\rangle)^{(n+1)\times 1}$, and $y' = \begin{pmatrix} y \\ z_{n+1} \end{pmatrix}$ such that*

$$(M'^{\omega_k} + M'^* P')_{n+1} = (M^{\omega_k} + MM^* P)_i\,.$$

Proof. Let $M'y' + P'' = y'$ be the $A\langle\Sigma\cup\varepsilon\rangle$-linear system constructed according to Theorem 5.14 from $My + P = y$. Consider the $A\langle\Sigma\cup\varepsilon\rangle$-linear system $M'y' + P' = y'$, where $P' = M'P''$. Then $(M'^{\omega_k} + M'^*P')_{n+1} = (M'^{\omega_k} + M'^*M'P'')_{n+1} = (M^{\omega_k} + MM^*P)_i$. $\square$

If we consider $\mathbb{N}^\infty\langle\Sigma\cup\varepsilon\rangle$-linear systems we obtain the following result about the derivations with respect to the rightlinear grammars G_i defined above.

Theorem 5.5.16 *The constructions of Theorem 5.14 and Corollary 5.15 do not change, for $w \in \Sigma^+$ (resp. for $w \in \Sigma^\omega$), the number of finite derivations of w with respect to G_i (resp. the number of infinite derivations of w with respect to G_i such that at least one of the variables of $\{z_1,\ldots,z_n\}$ appears infinitely often in these infinite derivations).*

Hence, the constructions transform unambiguous grammars into unambiguous grammars.

5.6 ω-Algebraic systems and ω-context-free grammars

In the sequel, T is a quemiring, $Y = \{y_1, \ldots, y_n\}$ is a set of (quemiring) variables, $T¶ = A$ and $T0 = V$. A *product term* t has the form $t(y_1, \ldots, y_n) = s_0 y_{i_1} s_1 \ldots s_{k-1} y_{i_k} s_k$, $k \geq 0$, where $s_j \in A - \{0\}$, $0 \leq j < k$, $s_k \in A$, and $y_{i_j} \in Y$. The elements s_j are referred to as *coefficients* of the product term. If $k \geq 1$, we do not write down coefficients that are equal to 1.

A *sum-product term* p is a finite sum of product terms t_j, i. e.,

$$p(y_1, \ldots, y_n) = \sum_{1 \leq j \leq m} t_j(y_1, \ldots, y_n)\,.$$

The coefficients of all the product terms t_j, $1 \leq j \leq m$, are referred to as the *coefficients* of the sum-product term p. Observe that each sum-product term represents a polynomial of the *polynomial quemiring over the quemiring T in the set of variables Y* in the sense of Lausch, Nöbauer [90], Chapter 1.4. For a subset $A' \subseteq A$, we denote the collection of all sum-product terms with coefficients in A' by $A'(Y)$. Observe that the sum-product terms in $A(Y)$ represent exactly the polynomials of the subquemiring of the polynomial quemiring that is generated by $A \cup Y$.

We are only interested in the mappings induced by sum-product terms. These mappings are *polynomial functions on T* in the sense of Lausch, Nöbauer [90], Chapter 1.6.

Each product term t (resp. sum-product term p) with variables $y_1, \ldots, y_n$ induces a mapping $\bar{t}$ (resp. $\bar{p}$) from T^n into T. For a product term t represented as above, the mapping $\bar{t}$ is defined by

$$\bar{t}(\tau_1, \ldots, \tau_n) = s_0 \tau_{i_1} s_1 \ldots s_{k-1} \tau_{i_k} s_k\,,$$

and for a sum-product term p, represented by a finite sum of product terms t_j as above, the mapping $\bar{p}$ is defined by

$$\bar{p}(\tau_1, \ldots, \tau_n) = \sum_{1 \leq j \leq m} \bar{t_j}(\tau_1, \ldots, \tau_n)$$

for all $(\tau_1, \ldots, \tau_n) \in T^n$.

Let (A, V) be a semiring-semimodule pair and let $A \times V$ be the quemiring determined by it. Let $A' \subseteq A$. An *A'-algebraic system (with variables $y_1, \ldots, y_n$) over the quemiring $A \times V$* is a system of equations

$$y_i = p_i,\ 1 \leq i \leq n\,,$$

where each p_i is a sum-product term in $A'(Y)$. A *solution* to this A'-algebraic system is given by $(\tau_1, \ldots, \tau_n) \in T^n$ such that $\tau_i = \bar{p}_i(\tau_1, \ldots, \tau_n)$, $1 \leq i \leq n$.

Often it is convenient to write the A'-algebraic system $y_i = p_i$, $1 \le i \le n$, in matrix notation. Defining the two column vectors

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

we can write $y_i = p_i$, $1 \le i \le n$, in the matrix notation

$$y = p(y) \quad \text{or} \quad y = p\,.$$

A *solution* to $y = p(y)$ is now given by $\tau \in T^n$ such that $\tau = \bar{p}(\tau)$ with $\bar{p} = (\bar{p}_i)_{1 \le i \le n}$.

Consider now a product term $t(y_1, \ldots, y_n) = s_0 y_{i_1} s_1 \ldots s_{k-1} y_{i_k} s_k$ and let $\tau_i = (\sigma_i, \omega_i) \in A \times V$, $1 \le i \le n$. Then

$$\begin{aligned}&\bar{t}(\tau_1, \ldots, \tau_n) = s_0(\sigma_{i_1}, \omega_{i_1}) s_1 \ldots s_{k-1}(\sigma_{i_k}, \omega_{i_k}) s_k = \\ &(s_0 \sigma_{i_1} s_1 \ldots s_{k-1} \sigma_{i_k} s_k, s_0 \omega_{i_1} + s_0 \sigma_{i_1} s_1 \omega_{i_2} + \ldots + s_0 \sigma_{i_1} s_1 \ldots s_{k-2} \sigma_{i_{k-1}} s_{k-1} \omega_{i_k})\,.\end{aligned}$$

By definition, for $\sigma = (\sigma_1, \ldots, \sigma_n) \in A^n$,

$$t_\sigma(z_1, \ldots, z_n) = s_0 z_{i_1} + s_0 \sigma_{i_1} s_1 z_{i_2} + \ldots + s_0 \sigma_{i_1} s_1 \ldots s_{k-2} \sigma_{i_{k-1}} s_{k-1} z_{i_k}$$

and, if $p(y_1, \ldots, y_n) = \sum_{1 \le j \le m} t_j(y_1, \ldots, y_n)$,

$$p_\sigma(z_1, \ldots, z_n) = \sum_{1 \le j \le m} (t_j)_\sigma(z_1, \ldots, z_n)\,.$$

Here $z_1, \ldots, z_n$ are variables over the semimodule V. We now obtain

$$\bar{t}(\tau_1, \ldots, \tau_n) = \bar{t}(\sigma_1, \ldots, \sigma_n) + \bar{t}_\sigma(\omega_1, \ldots, \omega_n)$$

and

$$\bar{p}(\tau_1, \ldots, \tau_n) = \bar{p}(\sigma_1, \ldots, \sigma_n) + \bar{p}_\sigma(\omega_1, \ldots, \omega_n)\,.$$

Moreover,

$$\bar{p}(\tau_1, \ldots, \tau_n)\P = \bar{p}(\sigma_1, \ldots, \sigma_n) \quad \text{and} \quad \bar{p}(\tau_1, \ldots, \tau_n).0 = \bar{p}_\sigma(\omega_1, \ldots, \omega_n)\,.$$

In the next theorem, y (resp. x and z) denotes a column vector $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ (resp. $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$), where the y_i (resp. x_i and z_i) are variables over $A \times V$ (resp. A and V).

In the sequel, A' will always denote a subset of A containing 0 and 1.

Theorem 5.6.1 *Let $A \times V$ be a quemiring and let $y = p(y)$ be an A'-algebraic system over $A \times V$. Then $(\sigma, \omega) \in (A \times V)^n$ is a solution of $y = p(y)$ iff σ is a solution of the A'-algebraic system $x = p(x)$ over A and ω is a solution of the $\mathfrak{Alg}(A')$-linear system $z = p_\sigma(z)$ over V.*

Proof. $\tau = (\sigma, \omega)$ is a solution $\Leftrightarrow \tau = \bar{p}(\tau) = \bar{p}(\sigma) + \bar{p}_\sigma(\omega) \Leftrightarrow \sigma = \bar{p}(\sigma)$ and $\omega = \bar{p}_\sigma(\omega)$. □

Consider an A'-algebraic system $y = p(y)$ over a continuous quemiring $A \times V$. Then the least solution of the A'-algebraic system $x = p(x)$ over A, say σ, exists. Moreover, write the $\mathfrak{Alg}(A')$-linear system $z = p_\sigma(z)$ over V in the form $z = Mz$, where M is an $n \times n$-matrix. Then, by Theorem 5.5, M^{ω_k} for $0 \le k \le n$ is a solution of $z = p_\sigma(z)$. Hence, by Theorem 6.1, (σ, M^{ω_k}), $0 \le k \le n$, is a solution of $y = p(y)$. Given a $k \in \{0, 1, \ldots, n\}$, we call this solution the *solution of order k of $y = p(y)$*. By ω-$\mathfrak{Alg}(A')$ we denote the collection of all components of solutions of all orders k of A'-algebraic systems over $A \times V$.

We now consider a star-omega semiring A where the semiring A is additionally commutative and continuous, and an alphabet Σ. By Theorem 5.5, $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ is a continuous semiring-semimodule pair.

Let $A\Sigma^* = \{sw \mid s \in A,\ w \in \Sigma^*\}$. Then ω-$\mathfrak{Alg}(A\Sigma^*)$ is equal to the collection of the components of the solutions of order k of $A\Sigma^*$-algebraic systems over $A\langle\langle\Sigma^*\rangle\rangle \times A\langle\langle\Sigma^\omega\rangle\rangle$ $y_i = p_i$, $1 \le i \le n$, where p_i is a polynomial in $A\langle(\Sigma \cup Y)^*\rangle$. This is due to the commutativity of A: any polynomial function that is induced by a sum-product term of $A\Sigma^*(Y)$ is also induced by a polynomial of $A\langle(\Sigma \cup Y)^*\rangle$ and vice versa. We denote ω-$\mathfrak{Alg}(A\Sigma^*)$ by $A^{\omega\text{-alg}}\langle\langle\Sigma^*, \Sigma^\omega\rangle\rangle$. The $A\Sigma^*$-algebraic systems are called *ω-algebraic systems* (*over A and* Σ) and the power series in $A^{\omega\text{-alg}}\langle\langle\Sigma^*, \Sigma^\omega\rangle\rangle$ are called *ω-algebraic power series* (*over A and* Σ).

Consider now a product term in $A\langle(\Sigma \cup Y)^*\rangle$

$$t(y_1, \ldots, y_n) = sw_0 y_{i_1} w_1 \ldots w_{k-1} y_{i_k} w_k\,,$$

where $s \in A$ and $w_i \in \Sigma^*$, $1 \le i \le k$. By definition, for $x = (x_i)_{1 \le i \le n}$, $t_x(x_1, \ldots, x_n, z_1, \ldots, z_n) = sw_0 z_{i_1} + sw_0 x_{i_1} w_1 z_{i_2} + \ldots + sw_0 x_{i_1} w_1 \ldots w_{k-2} x_{i_{k-1}} w_{k-1} z_{i_k}$, and, if $p(y_1, \ldots, y_n) = \sum_{1 \le j \le m} t_j(y_1, \ldots, y_n)$, then

$$p_x(x_1, \ldots, x_n, z_1, \ldots, z_n) = \sum_{1 \le j \le m} (t_j)_x(x_1, \ldots, x_n, z_1, \ldots, z_n)\,.$$

Here $x_1, \ldots, x_n$ (resp. $z_1, \ldots, z_n$) are variables over A (resp. V). Observe that, for $\sigma \in (A\langle\langle\Sigma^*\rangle\rangle)^n$, we obtain $p_x(\sigma_1, \ldots, \sigma_n, z_1, \ldots, z_n) = p_\sigma(z_1, \ldots, z_n)$.

Given an ω-algebraic system $y = p(y)$ over $A\langle\langle\Sigma^*\rangle\rangle \times A\langle\langle\Sigma^\omega\rangle\rangle$, we call $x = p(x)$, $z = p_x(x, z)$ the *mixed ω-algebraic system over $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ induced by $y = p(y)$*.

Write $z = p_x(x, z)$ in the form $z = M(x)z$, where $M(x)$ is an $n \times n$-matrix. Then $(\sigma, M(\sigma)^{\omega_k})$ for $0 \le k \le n$ is a solution of $x = p(x)$, $z = p_x(x, z)$. Moreover, it is the solution of order k of $y = p(y)$.

A *mixed ω-context-free grammar*

$$G = (n, \Sigma, P, j, k)$$

is given by

(i) an alphabet $X = \{x_1, \ldots, x_n\}$ of *variables for finite derivations* and an alphabet $Z = \{z_1, \ldots, z_n\}$ of *variables for infinite derivations*, $n \geq 1$, $X \cap Z = \emptyset$;

(ii) an alphabet Σ of *terminal symbols*, $\Sigma \cap (X \cup Z) = \emptyset$;

(iii) a finite set of *productions* of the form $x \to \alpha$, $x \in X$, $\alpha \in (X \cup \Sigma)^*$, or $z \to \alpha z'$, $z, z' \in Z$, $\alpha \in (X \cup \Sigma)^*$;

(iv) the *startvariable* x_j (resp. z_j) for finite (resp. infinite) derivations, $1 \leq i \leq n$;

(v) the set of *repeated variables for infinite derivations* $\{z_1, \ldots, z_k\}$, $0 \leq k \leq n$.

A *finite leftmost derivation* (with respect to G) $\alpha \Rightarrow_L^* w$, $\alpha \in (X \cup \Sigma)^*$, $w \in \Sigma^*$, is defined as usual. An *infinite leftmost derivation* (with respect to G) $\pi : z \Rightarrow_L^\omega w$, $z \in Z$, $w \in \Sigma^\omega$, is defined as follows:

$$\begin{aligned} &\pi : z \Rightarrow_L \alpha_1 z_{i_1} \Rightarrow_L^* w_1 z_{i_1} \Rightarrow_L w_1 \alpha_2 z_{i_2} \Rightarrow_L^* w_1 w_2 z_{i_2} \Rightarrow_L \ldots \Rightarrow_L^* \\ &w_1 w_2 \ldots w_m z_{i_m} \Rightarrow_L w_1 w_2 \ldots w_m \alpha_{m+1} z_{i_{m+1}} \Rightarrow_L^* \ldots, \end{aligned}$$

where $z \to \alpha_1 z_{i_1}, z_{i_1} \to \alpha_2 z_{i_2}, \ldots, z_{i_m} \to \alpha_{m+1} z_{i_{m+1}}, \ldots \in P$, $w_1, w_2, \ldots, w_m, \ldots \in \Sigma^*$ and $w = w_1 w_2 \ldots w_m \ldots$. Let $\mathrm{INV}(\pi) = \{z \in Z \mid z$ is infinitely often rewritten in $\pi\}$. Then $L(G) = \{w \in \Sigma^* \mid x_j \Rightarrow_L^* w\} \cup \{w \in \Sigma^\omega \mid \pi : z_j \Rightarrow_L^\omega w,\ \mathrm{INV}(\pi) \cap \{z_1, \ldots, z_k\} \neq \emptyset\}$.

We now discuss the connection between mixed ω-algebraic systems over $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$, where A is $\mathbb{B}$ or $\mathbb{N}^\infty$, and mixed ω-context-free grammars (see also Chapter 2, before Theorem 2.3.8). We associate to a given mixed ω-context-free grammar $G_{j,k} = (n, \Sigma, P, j, k)$, $1 \leq j \leq n$, $0 \leq k \leq n$, the mixed ω-algebraic system $x_i = p_i(x_1, \ldots, x_n)$, $z_i = q_i(x_1, \ldots, x_n, z_1, \ldots, z_n)$, $1 \leq i \leq n$, over $(A\langle\langle\Sigma^*\rangle\rangle, A\langle\langle\Sigma^\omega\rangle\rangle)$ by

$$\begin{aligned} &(p_i, \alpha) = 1 \text{ if } x_i \to \alpha \in P, \quad (p_i, \alpha) = 0 \text{ otherwise}, \\ &(q_i, \alpha) = 1 \text{ if } z_i \to \alpha \in P, \quad (q_i, \alpha) = 0 \text{ otherwise}. \end{aligned}$$

Conversely, we associate to a mixed ω-algebraic system $x_i = p_i(x_1, \ldots, x_n)$, $z_i = q_i(x_1, \ldots, x_n, z_1, \ldots, z_n)$, $1 \leq i \leq n$, the mixed ω-context-free grammars $G_{j,k} = (n, \Sigma, P, j, k)$, $1 \leq j \leq n$, $0 \leq k \leq n$, by $x_i \to \alpha \in P$ iff $(p_i, \alpha) \neq 0$ and $z_i \to \alpha \in P$ iff $(z_i, \alpha) \neq 0$. Whenever we speak of a mixed ω-context-free grammar associated to a mixed ω-algebraic system or vice versa, then we mean the correspondence in the sense of the above definition.

In the next theorem we use the isomorphism between $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle$ and $2^{\Sigma^*} \times 2^{\Sigma^\omega}$.

Theorem 5.6.2 *Let $G_{j,k} = (n, \Sigma, P, j, k)$, $1 \le j \le n$, $0 \le k \le n$, be a mixed ω-context-free grammar and $x_i = p_i(x_1, \ldots, x_n)$, $z_i = q_i(x_1, \ldots, x_n, z_1, \ldots, z_n)$, $1 \le i \le n$, be the mixed ω-algebraic system over $(\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle)$ corresponding to it. Let (σ, τ) be the solution of order k, $0 \le k \le n$, of $x_i = p_i$, $z_i = q_i$, $1 \le i \le n$. Then $L(G_{j,k}) = \sigma_j + \tau_j$, $1 \le j \le n$, $0 \le i \le k$.*

Proof. By Theorem 2.3.6, we obtain $\sigma_j = \{w \in \Sigma^* \mid x_j \Rightarrow_L^* w\}$, $1 \le j \le n$, and by Theorem 5.9 applied to $A = \mathbb{B}$ we obtain $\tau_j = \{w \in \Sigma^\omega \mid \pi : z_j \Rightarrow_L^* w,\ \mathrm{INV}(\pi) \cap \{z_1, \ldots, z_k\} \ne \emptyset\}$, $1 \le j \le n$, $0 \le k \le n$. □

If our basic quemiring is $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle \times \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ we can draw some stronger conclusions.

Theorem 5.6.3 *Let $G_{j,k} = (n, \Sigma, P, j, k)$, $1 \le j \le n$, $0 \le k \le n$, be a mixed ω-context-free grammar and $x_i = p_i(x_1, \ldots, x_n)$, $z_i = q_i(x_1, \ldots, x_n, z_1, \ldots, z_n)$, $1 \le i \le n$ be the mixed ω-algebraic system over $(\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle, \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle)$ corresponding to it. Let (σ, τ) be the solution of order k, $0 \le k \le n$, of $x_i = p_i$, $z_i = q_i$, $1 \le i \le n$. Denote by $d_j(w)$, $w \in \Sigma^*$ (resp. $w \in \Sigma^\omega$) the number (possibly ∞) of distinct finite leftmost derivations (resp. infinite leftmost derivations π with INV$(\pi) \cap \{z_1, \ldots, z_k\} \ne \emptyset$) from the variable x_j (resp. z_j), $1 \le j \le n$. Then*

$$\sigma_j = \sum_{w \in \Sigma^*} d_j(w)w \quad \textit{and} \quad \tau_j = \sum_{w \in \Sigma^\omega} d_j(w)w\,, \qquad 1 \le j \le n\,.$$

Proof. By Theorem 2.3.8 and by Theorem 5.9 applied to $A = \mathbb{N}^\infty$. □

An *ω-context-free grammar* (*with repeated variables*) $G = (\Phi, \Sigma, P, A, F)$ is a usual context-free grammar (Φ, Σ, P, A) augmented by a *set $F \subseteq \Phi$ of repeated variables*. (See also Cohen, Gold [23].)

An *infinite leftmost derivation π with respect to G*, starting from some string α is given by

$$\pi : \alpha \Rightarrow_L \alpha_1 \Rightarrow_L \alpha_2 \Rightarrow_L \ldots\,,$$

where $\alpha, \alpha_i \in (\Phi \cup \Sigma)^*$ and $\Rightarrow_L$ is defined as usual. This infinite leftmost derivation π can be uniquely written as

$$\begin{aligned}\alpha = \beta_0 B_0 \gamma_0 &\Rightarrow_L^* v_0 B_0 \gamma_0 \Rightarrow_L v_0 \beta_1 B_1 \gamma_1 \gamma_0 \Rightarrow_L^* \\ v_0 v_1 B_1 \gamma_1 \gamma_0 &\Rightarrow_L v_0 v_1 \beta_2 B_2 \gamma_2 \gamma_1 \gamma_0 \Rightarrow_L^* \ldots\,,\end{aligned}$$

where $v_i \in \Sigma^*$, $\beta_i, \gamma_i \in (\Phi \cup \Sigma)^*$, $B_i \to \beta_{i+1} B_{i+1} \gamma_{i+1} \in P$, $\beta_i \Rightarrow_L^* v_i$, the specific occurence of the variable B_i is not rewritten in the subderivation $\beta_i B_i \gamma_i \Rightarrow_L^* v_i B_i \gamma_i$ and the variables of γ_i are never rewritten in the infinite leftmost derivation π. This occurence of the variable B_i is called the *i-th significant variable* of π. (Observe that the infinite derivation tree of π has a unique infinite path determining the B_i's.) We write also, for this infinite leftmost derivation, $\pi : \alpha \Rightarrow_L^\omega w$ for $w = w_0 w_1 \ldots w_n \ldots$. By definition, INV$(\pi) = \{\Sigma \in \Phi \mid \Sigma$ is rewritten infinitely often in $\pi\}$. The *ω-language $L(G)$ generated by the ω-context-free grammar G* is defined by

$$L(G) = \{w \in \Sigma^* \mid A \Rightarrow_L^* w\} \cup \{w \in \Sigma^\omega \mid \pi : A \Rightarrow_L^\omega w,\ \mathrm{INV}(\pi) \cap F \ne \emptyset\}\,.$$

An ω-*language* L is called ω-*context-free* if it is generated by an ω-context-free grammar. (Usually, an ω-language is a subset of Σ^ω. Here it is a subset of $\Sigma^* \cup \Sigma^\omega$.)

The connection between an ω-algebraic system over $A\langle\langle\Sigma^*\rangle\rangle \times A\langle\langle\Sigma^\omega\rangle\rangle$ and an ω-context-free grammar is as usual (see also Chapter 2, before Theorem 2.3.8). We associate to a given ω-context-free grammar $G_j = (\{y_1, \dots, y_n\}, \Sigma, P, y_j, \{y_1, \dots, y_k\})$ the ω-algebraic system $y_i = p_i(y_1, \dots, y_n)$, $1 \le i \le n$, over $A\langle\langle\Sigma^*\rangle\rangle \times A\langle\langle\Sigma^\omega\rangle\rangle$ by $(p_i, \alpha) = 1$ if $y_i \to \alpha \in P$, $(p_i, \alpha) = 0$ otherwise. Conversely, we associate to an ω-algebraic system $y_i = p_i(y_1, \dots, y_n)$, $1 \le i \le n$, the ω-context-free grammars $G_{j,k} = (\{y_1, \dots, y_n\}, \Sigma, P, y_j, \{y_1, \dots, y_k\})$, $1 \le j \le n$, $0 \le k \le n$, by $y_i \to \alpha$ iff $(p_i, \alpha) \ne 0$.

Each ω-context-free grammar G *induces* a mixed ω-context-free grammar G' as follows. Let $G = (\Phi, \Sigma, P, A, F)$, where without loss of generality, $\Phi = \{y_1, \dots, y_n\}$, $A = y_j$, and $F = \{y_1, \dots, y_k\}$. Then $G' = (n, \Sigma, P', j, k)$, where P' is defined as follows. Let $y_i \to \alpha = w_0 y_{i_1} w_1 \dots w_{t-1} y_{i_t} w_t \in P$, where $y_i, y_{i_1}, \dots, y_{i_t} \in \Phi$ and $w_0, w_1, \dots, w_t \in \Sigma^*$. Then we define the following set of productions

$$
\begin{aligned}
U_{y_i \to \alpha} \quad = \quad & \{x_i \to w_0 x_{i_1} w_1 \dots w_{t-1} x_{i_t} w_t\} \cup \\
& \{z_i \to w_0 z_{i_1}, z_i \to w_0 x_{i_1} w_1 z_{i_2}, \dots, z_i \to w_0 x_{i_1} w_1 x_{i_2} \dots w_{t-1} z_{i_t}\},
\end{aligned}
$$

and, moreover,

$$
P' = \bigcup_{y_i \to \alpha \in P} U_{y_i \to \alpha} .
$$

It is clear that, for a finite leftmost derivation $y_i \Rightarrow_L^* w$, $w \in \Sigma^*$ in G, there exists a finite leftmost derivation $x_i \Rightarrow_L^* w$ in G' using only the x-productions. Moreover, for each infinite leftmost derivation in G

$$
\begin{aligned}
& y_i \Rightarrow_L \beta_1 y_{i_1} \gamma_1 \Rightarrow_L^* w_1 y_{i_1} \gamma_1 \Rightarrow_L w_1 \beta_2 y_{i_2} \gamma_2 \gamma_1 \Rightarrow_L^* \\
& w_1 w_2 y_{i_2} \gamma_2 \gamma_1 \Rightarrow_L w_1 w_2 \beta_3 y_{i_3} \gamma_3 \gamma_2 \gamma_1 \Rightarrow_L^* \dots
\end{aligned}
$$

where y_i is the 0-th, and y_{i_j} is the j-th significant variable, there exists the following infinite leftmost derivation in G':

$$
z_i \Rightarrow_L \bar\beta_1 z_{i_1} \Rightarrow_L^* w_1 z_{i_1} \Rightarrow_L w_1 \bar\beta_2 z_{i_2} \Rightarrow_L^* w_1 w_2 z_{i_2} \Rightarrow_L w_1 w_2 \bar\beta_3 z_{i_3} \Rightarrow_L^* \dots ,
$$

where, if in β_i the y's are replaced by x's, we get $\bar\beta_i$. Here $z_i \to \bar\beta_1 z_{i_1} \in U_{y_i \to \beta_1 y_{i_1} \gamma_1}$ and $z_{i_j} \to \bar\beta_{j+1} z_{i_{j+1}} \in U_{y_{i_j} \to \beta_{j+1} y_{i_{j+1}} \gamma_{j+1}}$. Both infinite leftmost derivations generate $w_1 w_2 w_3 \dots \in \Sigma^\omega$.

Vice versa, to each infinite leftmost derivation $z_i \Rightarrow_L^\omega w$ in G' there exists, in the same manner, an infinite leftmost derivation in G $y_i \Rightarrow_L^\omega w$, $w \in \Sigma^\omega$. Moreover, if P' is the disjoint union of the $U_{y_i \to \alpha}$ for all $y_i \to \alpha \in P$, then the correspondence between infinite leftmost derivations in G and in G' is one-to-one.

For an infinite leftmost derivation π in an ω-context-free grammar G, define $\mathrm{INSV}(\pi) = \{y_i \in \Phi \mid y_i \text{ appears infinitely often as a significant variable in } \pi\}$.

Clearly, if for all infinite leftmost derivations π of the ω-context-free grammar $G = (\Phi, \Sigma, P, A, F)$, $\mathrm{INV}(\pi) \cap F \neq \emptyset$ iff $\mathrm{INSV}(\pi) \cap F \neq \emptyset$, then $L(G') = L(G)$, where G' is the mixed ω-context-free grammar induced by G.

Theorem 5.6.4 *Let $G_{j,k} = (\{y_1, \ldots, y_n\}, \Sigma, P, y_j, \{y_1, \ldots, y_k\})$, $1 \leq j \leq n$, $0 \leq k \leq n$, be an ω-context-free grammar and $y_i = p_i(y_1, \ldots, y_n)$, $1 \leq i \leq n$, be the ω-algebraic system over $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle$ corresponding to it. Assume that, for each infinite leftmost derivation π, $INV(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$ iff $INSV(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$. Let (σ, τ) be the solution of order k, $0 \leq k \leq n$, of the ω-algebraic system over $(\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle)$ induced by $y_i = p_i$, $1 \leq i \leq n$. Then $L(G_{j,k}) = \sigma_j + \tau_j$, $1 \leq j \leq n$, $0 \leq i \leq k$.*

Theorem 5.6.5 *Let $G_{j,k} = (\{y_1, \ldots, y_n\}, \Sigma, P, y_j, \{y_1, \ldots, y_k\})$, $1 \leq j \leq n$, $0 \leq k \leq n$, be an ω-context-free grammar and $y_i = p_i(y_1, \ldots, y_n)$, $1 \leq i \leq n$, be the ω-algebraic system over $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle \times \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$ corresponding to it. Assume that, for each infinite leftmost derivation π, $INV(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$ iff $INSV(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$. Denote by $d_j(w)$, $w \in \Sigma^*$ (resp. $w \in \Sigma^\omega$) the number (possibly ∞) of distinct finite leftmost derivations (resp. infinite leftmost derivations π with $INSV(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$) from the variable y_j, $1 \leq j \leq n$. Then*

$$\sigma_j = \sum_{w \in \Sigma^*} d_j(w)w \quad \textit{and} \quad \tau_j = \sum_{w \in \Sigma^\omega} d_j(w)w\,, \qquad 1 \leq j \leq n\,.$$

Observe, that if $k = n$ or $n = 1$, then the assumption $\mathrm{INV}(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$ iff $\mathrm{INSV}(\pi) \cap \{y_1, \ldots, y_k\} \neq \emptyset$ for all π is satisfied.

Example 5.6.1 (see also Cohen, Gold [23], Example 3.1.6). Consider the ω-algebraic system over $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle$ where $\Sigma = \{a, b\}$: $y_1 = ay_1b + ab$, $y_2 = y_1y_2$. It induces the mixed ω-algebraic system over $(\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathbb{B}\langle\langle\Sigma^\omega\rangle\rangle)$ $x_1 = ax_1b + ab$, $x_2 = x_1x_2$, $z_1 = az_1$, $z_2 = z_1 + x_1z_2$. The least solution of $x_1 = ax_1b + ab$, $x_2 = x_1x_2$ is given by $\sigma = \left(\sum_{n \geq 1} a^n b^n, 0\right)^T$. The z-equations can be written in the form $z = Mz$, where $M = \begin{pmatrix} a & 0 \\ \varepsilon & x_1 \end{pmatrix}$. We obtain $M^{\omega_1} = \begin{pmatrix} a^\omega \\ x_1^* a^\omega \end{pmatrix}$ and $M^{\omega_2} = \begin{pmatrix} a^\omega \\ x_1^\omega + x_1^* a^\omega \end{pmatrix}$.

The ω-context-free grammar G corresponding to the ω-algebraic system has productions $y_1 \to ay_1b$, $y_1 \to ab$, $y_2 \to y_1y_2$. The infinite leftmost derivations are

(i) $y_1 \Rightarrow_L ay_1b \Rightarrow_L aay_1bb \Rightarrow_L \ldots \Rightarrow_L a^n y_1 b^n \Rightarrow_L \ldots$, i.e., $y_1 \Rightarrow_L^\omega a^\omega$, with repeated variable y_1;

(ii) $y_2 \Rightarrow_L y_1y_2 \Rightarrow_L^* a^{n_1}b^{n_1}y_2 \Rightarrow_L a^{n_1}b^{n_1}y_1y_2 \Rightarrow_L^* a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t}y_2 \Rightarrow_L \ldots$, i.e., $y_1 \Rightarrow_L^\omega a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t} \ldots$, with repeated variables y_1, y_2;

(iii) $y_2 \Rightarrow_L^* a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t}y_2 \Rightarrow_L a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t}y_1y_2 \Rightarrow_L^\omega a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t}a^\omega$, i.e., $y_2 \Rightarrow_L^\omega a^{n_1}b^{n_1} \ldots a^{n_t}b^{n_t}a^\omega$, $t \geq 0$, with repeated variable y_1.

If y_1 is the only repeated variable, and y_1 or y_2 is the start variable, then $L(G_{1,1}) = \sum_{n\geq 1} a^n b^n + a^\omega$ or $L(G_{2,1}) = \left(\sum_{n\geq 1} a^n b^n\right)^\omega \cup \left(\sum_{n\geq 1} a^n b^n\right)^* a^\omega$, respectively. If the repeated variables are y_1 and y_2, and y_1 or y_2 is the start variable then we obtain again $L(G_{1,2}) = \sum_{n\geq 1} a^n b^n + a^\omega$ or $L(G_{2,2}) = \left(\sum_{n\geq 1} a^n b^n\right)^\omega \cup \left(\sum_{n\geq 1} a^n b^n\right)^* a^\omega$, respectively. Compare this with the solutions of order 1 or 2 of the ω-algebraic system $y_1 = ay_1b + ab$, $y_2 = y_1y_2$: $\left(\sum_{n\geq 1} a^n b^n, 0\right)^T + \left(a^\omega, \left(\sum_{n\geq 1} a^n b^n\right)^\omega a^\omega\right)^T$ or $\left(\sum_{n\geq 1} a^n b^n, 0\right)^T + \left(a^\omega, \left(\sum_{n\geq 1} a^n b^n\right)^\omega + \left(\sum_{n\geq 1} a^n b^n\right)^* a^\omega\right)^T$, respectively. If y_1 is the only repeated variable and y_2 is the start variable then $\left(\sum_{n\geq 1} a^n b^n\right)^\omega$ is missing. That is due to the fact that in the derivations (ii) each y_1 derives a finite word $a^{n_j} b^{n_j}$ by a finite leftmost subderivation $y_1 \Rightarrow_L^* a^{n_j} b^{n_j}$ and never is a significant variable.

If all variables are repeated variables that does not matter: each infinite leftmost derivation contributes to the generated language. Hence, if the repeated variables are y_1, y_2 and the start variable is y_1 or y_2, the infinite parts of the solutions of order 1 or 2 correspond to the generated languages by Theorem 6.4.

□

In the next example there is only one variable. Hence, we can apply Theorems 6.4 and 6.5.

Example 5.6.2. Consider the ω-algebraic system $y_1 = ay_1y_1 + b$ over $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle \times \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle$, where $\Sigma = \{a, b\}$. The least solution of the algebraic system $x_1 = ax_1x_1 + b$ over $\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle$ is given by $\sigma = D^*b$, where D is the characteristic series of the restricted Dyck language (see Berstel [4]). The mixed ω-algebraic system over $(\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle, \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle)$ $x_1 = ax_1x_1 + b$, $z_1 = az_1 + ax_1z_1$ has the solution of order 1 $(D^*b, (a + ax_1)^\omega(D^*b)) = (D^*b, (a + aD^*b)^\omega) = (D^*b, (a + D)^\omega)$, since $aD^*b = D$.

The ω-context-free grammar corresponding to $y_1 = ay_1y_1 + b$ has productions $y_1 \to ay_1y_1$, $y_1 \to b$ and generates the language $D^*b + (a+D)^\omega = D^*b + (a^*D)^\omega + (a^*D)^*a^\omega$.

Since each word in $(a^*D)^*$ and in $(a^*D)^\omega$ has a unique factorization into words of a^*D, all coefficients of $D^*b + (a+D)^\omega$ are 0 or 1, i. e., the ω-context-free grammar with productions $y_1 \to ay_1y_1$, $y_1 \to b$ is an "unambiguous" ω-context-free grammar. □

Let (A, V) be a continuous starsemiring-omegasemimodule pair and inspect the solutions of order k: If (σ, ω) is a solution of order k of an A'-algebraic system over $A \times V$ then $\sigma \in \mathfrak{Alg}(A')$ and ω is the k-th automata theoretic solution of a finite $\mathfrak{Alg}(A')$-linear system. Hence, by Theorem 4.9, ω is of the form $\omega = \sum_{1\leq j\leq m} s_j t_j^\omega$ with $s_j, t_j \in \mathfrak{Rat}(\mathfrak{Alg}(A')) = \mathfrak{Alg}(A')$. Hence, again by Theorem 4.9 we obtain the following result.

Theorem 5.6.6 *Let (A, V) be a continuous starsemiring-omegasemimodule pair. Then the following statements are equivalent for $(s, v) \in A \times V$:*

(i) $(s, v) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite $\mathfrak{Alg}(A')$-automaton,*

(ii) $(s, v) \in \omega\text{-}\mathfrak{Alg}(A')$,

(iii) $s \in \mathfrak{Alg}(A')$ *and* $v = \sum_{1 \le k \le m} s_k t_k^\omega$, *where* $s_k, t_k \in \mathfrak{Alg}(A')$.

Theorem 5.6.7 *Let* (A, V) *be a continuous starsemiring-omegasemimodule pair. Then* $\omega\text{-}\mathfrak{Alg}(A')$ *is a generalized starquemiring.*

Proof. Since, by assumption, $0, 1 \in A'$ we infer that $0, 1 \in \omega\text{-}\mathfrak{Alg}(A')$. Assume now that (σ_1, ω_1) and (σ_2, ω_2) are in $\omega\text{-}\mathfrak{Alg}(A')$. Then, by Theorem 6.6, $\sigma_1, \sigma_2 \in \mathfrak{Alg}(A')$ and $\omega_1 = \sum_{1 \le k \le m_1} s_k^1 {t_k^1}^\omega$, $\omega_2 = \sum_{1 \le k \le m_2} s_k^2 {t_k^2}^\omega$ for some $s_k^1, s_k^2, t_k^1, t_k^2 \in \mathfrak{Alg}(A')$. We obtain

$$(\sigma_1, \omega_1) + (\sigma_2, \omega_2) = (\sigma_1 + \sigma_2, \sum_{1 \le k \le m_1} s_k^1 {t_k^1}^\omega + \sum_{1 \le k \le m_2} s_k^2 {t_k^2}^\omega)$$

and

$$(\sigma_1, \omega_1) \cdot (\sigma_2, \omega_2) = (\sigma_1 \sigma_2, \sum_{1 \le k \le m_1} s_k^1 {t_k^1}^\omega + \sigma_1 \cdot \sum_{1 \le k \le m_2} s_k^2 {t_k^2}^\omega)\,.$$

Hence, $(\sigma_1, \omega_1) + (\sigma_2, \omega_2)$ and $(\sigma_1, \omega_1) \cdot (\sigma_2, \omega_2)$ are again in $\omega\text{-}\mathfrak{Alg}(A')$.

Moreover, we obtain

$$(\sigma_1, \omega_1)\P = (\sigma_1, 0)$$

and

$$(\sigma_1, \omega_1)^\otimes = (\sigma_1^*, \sigma_1^\omega + \sigma_1^* \cdot \sum_{1 \le k \le m_1} s_k^1 {t_k^1}^\omega)\,.$$

Hence, $(\sigma_1, \omega_1)\P$ and $(\sigma_1, \omega_1)^\otimes$ are again in $\omega\text{-}\mathfrak{Alg}(A')$ and $\omega\text{-}\mathfrak{Alg}(A')$ is rationally closed. $\square$

Notation 3.1.5, Definition 2.2.1 and Theorem 4.1.8(a) of Cohen, Gold [23] and Theorem 6.6(iii) yield the next result.

Theorem 5.6.8 $CFL_\omega = \{L0 \subseteq \Sigma^\omega \mid L0 \in \mathbb{B}^{\omega\text{-}alg}\langle\langle \Sigma^*, \Sigma^\omega \rangle\rangle$, Σ *an alphabet*$\}$.

Let $t \in \mathbb{B}^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. Then t is the x_2-component of the least solution of an algebraic system $x_i = p_i(x_2, \ldots, x_n)$, $2 \le i \le n$, over $\mathbb{B}^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. Consider the ω-algebraic system over $\mathbb{B}\langle\langle \Sigma^* \rangle\rangle \times \mathbb{B}\langle\langle \Sigma^\omega \rangle\rangle$:

$$y_1 = y_2 y_1\,, \qquad y_i = p_i(y_2, \ldots, y_n),\ 2 \le i \le n\,,$$

and consider the induced mixed ω-algebraic system over $(\mathbb{B}\langle\langle \Sigma^* \rangle\rangle, \mathbb{B}\langle\langle \Sigma^\omega \rangle\rangle)$:

$$\begin{aligned} &z_1 = z_2 + x_2 z_1\,, \quad && z_i = (p_i)_x(x_1, \ldots, x_n, z_1, \ldots, z_n),\ 2 \le i \le n\,,\\ &x_1 = x_2 x_1\,, && x_i = p_i(x_2, \ldots, x_n),\ 2 \le i \le n\,. \end{aligned}$$

The first component of the least solution of $x_1 = x_2 x_1$, $x_i = p_i(x_2, \ldots, x_n)$, $2 \le i \le n$, is 0. We now compute the solution of order 1 of $z_1 = z_2 + x_2 z_1$, $z_i = (p_i)_x(x_1, \ldots, x_n, z_1, \ldots, z_n)$, $2 \le i \le n$. We write the system in the form

$z = Mz$ and obtain $M = \left(\begin{array}{c|c} x_2 & \varepsilon\ 0\ \ldots\ 0 \\ \hline 0 & \\ \vdots & M' \\ 0 & \end{array}\right)$. Hence, the first component of M^{ω_1} is x_2^ω and the first component of the solution of order 1 is given by $(0, t^\omega)$.

Consider now the ω-context-free grammar G corresponding to $y_1 = y_2 y_1$, $y_i = p_i$, $2 \le i \le n$, with the set of repeated variables $\{y_1\}$ and start variable y_1. The only infinite leftmost derivations π, where y_1 appears infinitely often, are of the form

$$\pi : y_1 \Rightarrow_L y_2 y_1 \Rightarrow_L^* w_1 y_1 \Rightarrow_L w_1 y_2 y_1 \Rightarrow_L^* w_1 w_2 y_1 \Rightarrow_L \ldots .$$

The only significant variable of such a derivation π is y_1, i.e., $\mathrm{INSV}(\pi) = \{y_1\}$, and $\mathrm{INSV}(\pi) \cap \{y_1\} \neq \emptyset$ iff $\mathrm{INV}(\pi) \cap \{y_1\} \neq \emptyset$. Hence, $L(G_{1,1}) = t^\omega$ by Theorem 3.3.

The usual constructions yield then, for $s + v$, where $v = \sum_{1 \le k \le n} s_k t_k^\omega$, $s, s_k, t_k \in \mathbb{B}^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$, an ω-context-free grammar G' such that $L(G') = s + v$.

Hence, we have given a construction proving again Theorem 6.8. But additionally, G' has the nice property that for each infinite leftmost derivation π, we obtain $\mathrm{INSV}(\pi) \cap F \neq \emptyset$ iff $\mathrm{INV}(\pi) \cap F \neq \emptyset$, where F is the set of repeated variables of G'.

5.7 Transductions and abstract ω-families of power series

In the sequel, (A, V) and $(\hat{A}, \hat{V})$ denote semiring-semimodule pairs, and Q and J (resp. I), possibly provided with indices, denote finite (resp. arbitrary) index sets. A *mapping* (h_A, h_V) *of semiring-semimodule pairs (from* $(\hat{A}, \hat{V})$ *into* (A, V)*)* is given by mappings $h_A : \hat{A} \to A$ and $h_V : \hat{V} \to V$. Let $(\hat{A}, \hat{V})$ and (A, V) be starsemiring-omegasemimodule pairs. Then such a mapping (h_A, h_V) is called a *morphism of starsemiring-omegasemimodule pairs (from* $(\hat{A}, \hat{V})$ *into* (A, V)*)* if $h_A : \hat{A} \to A$ is a semiring morphism, $h_V : \hat{V} \to V$ is a monoid morphism, and the following conditions are satisfied for all $s \in \hat{A}$, $v \in \hat{V}$:

(i) $h_A(s) h_V(v) = h_V(sv)$,

(ii) $h_A(s)^* = h_A(s^*)$,

(iii) $h_A(s)^\omega = h_V(s^\omega)$.

Let now $(\hat{A}, \hat{V})$ and (A, V) be complete semiring-semimodule pairs. Then a mapping (h_A, h_V) of semiring-semimodule pairs (from $(\hat{A}, \hat{V})$ into (A, V)) is called a *morphism of complete semiring-semimodule pairs* (*from* $(\hat{A}, \hat{V})$ *into* (A, V)) if $h_A : \hat{A} \to A$ is a semiring morphism, $h_V : \hat{V} \to V$ is a monoid morphism and the following conditions are satisfied for all $s, s_i \in \hat{A}$, $v \in \hat{V}$:

(i) $h_A(s) h_V(v) = h_V(sv)$,

(ii) $\sum_{i\in I} h_A(s_i) = h_A(\sum_{i\in I} s_i)$,

(iii) $\prod_{i\geq 1} h_A(s_i) = h_V(\prod_{i\geq 1} s_i)$.

A mapping (h_A, h_V), where $h_A : \hat{A} \to A^{Q_1'\times Q_2'}$ and $h_V : \hat{V} \to V^{Q'}$, can be extended to a mapping (h_A', h_V'), where $h_A' : \hat{A}^{Q_1\times Q_2} \to A^{(Q_1\times Q_1')\times(Q_2\times Q_2')}$ and $h_V' : \hat{V}^Q \to V^{Q\times Q'}$ by $h_A'(M)_{(q_1,q_1'),(q_2,q_2')} = h_A(M_{q_1,q_2})_{q_1',q_2'}$ and $h_V'(P)_{(q,q')} = h_V(P_q)_{q'}$ for $M \in \hat{\Sigma}^{Q_1\times Q_2}$, $P \in \hat{V}^Q$, $q_1 \in Q_1$, $q_2 \in Q_2$, $q_1' \in Q_1'$, $q_2' \in Q_2'$, $q \in Q$, $q' \in Q'$.

Let $Q_1 = \bigcup_{j_1\in J_1} Q^1_{j_1}$, $Q^1_{j_1} \cap Q^1_{j_1'} = \emptyset$ for $j_1 \neq j_1'$, $Q_2 = \bigcup_{j_2\in J_2} Q^2_{j_2}$, $Q^2_{j_2} \cap Q^2_{j_2'} = \emptyset$ for $j_2 \neq j_2'$, and partition the matrix $M \in \hat{\Sigma}^{Q_1\times Q_2}$ according to the partitions of Q_1 and Q_2 into blocks $M(Q^1_{j_1}, Q^2_{j_2})$, i. e., write $M = (M(Q^1_{j_1}, Q^2_{j_2}))_{j_1\in J_1, j_2\in J_2}$. It is then easily shown that $h_A'(M) = (h_A'(M(Q^1_{j_1}, Q^2_{j_2})))_{j_1\in J_1, j_2\in J_2}$. If $P \in \hat{\Sigma}^Q$ is partitioned into blocks $P(Q^1_{j_1})$, i. e., $P = (P(Q^1_{j_1}))_{j_1\in J_1}$, then $h_V'(P) = (h_V'(P(Q^1_{j_1})))_{j_1\in J_1}$.

In the sequel, we use the same notation for the mappings (h_A, h_V) and (h_A', h_V').

Theorem 5.7.1 *Let (A, V) and $(\hat{A}, \hat{V})$ be starsemiring-omegasemimodule pairs and let (h_A, h_V), where $h_A : \hat{A} \to A^{n'\times n'}$ and $h_V : \hat{V} \to V^{n'}$, be a morphism of starsemiring-omegasemimodule pairs. Then the extended mapping (h_A, h_V), where $h_A : \hat{A}^{n\times n} \to A^{(n\times n')\times(n\times n')}$ and $h_V : \hat{V}^n \to V^{(n\times n')}$ is again a morphism of starsemiring-omegasemimodule pairs.*

Proof. It is easily proven that $h_A : \hat{A}^{n\times n} \to A^{(n\times n')\times(n\times n')}$ is a semiring morphism and $h_V : \hat{V}^n \to V^{n\times n'}$ is a monoid morphism. Item (i) of the conditions of the definition of a morphism of starsemiring-omegasemimodule pairs is also easily proven. We prove only items (ii) and (iii) by induction on n.

For $n = 0, 1$, the theorem is clear. If $n > 1$, partition $M \in \hat{A}^{n\times n}$ according to (1). We obtain

$$h_A(M)^* = \begin{pmatrix} h_A(a) & h_A(b) \\ h_A(c) & h_A(d) \end{pmatrix}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = h_A(\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}) = h_A(M^*)\,,$$

where

$$\begin{aligned}
&\alpha = (h_A(a) + h_A(b)h_A(d)^*h_A(c))^* = h_A(\alpha'), && \alpha' = a + bd^*c,\\
&\delta = (h_A(d) + h_A(c)h_A(a)^*h_A(b))^* = h_A(\delta'), && \delta' = d + ca^*b,\\
&\beta = h_A(a)^*h_A(b)\delta = h_A(\beta'), && \beta' = a^*b\delta' \text{ and}\\
&\gamma = h_A(d)^*h_A(c)\alpha = h_A(\gamma'), && \gamma' = d^*c\alpha',
\end{aligned}$$

and

$$h_A(M)^\omega = \begin{pmatrix} h_A(a) & h_A(b) \\ h_A(c) & h_A(d) \end{pmatrix}^\omega = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = h_V(\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}) = h_V(M^\omega)\,,$$

where

$$\begin{aligned}
\alpha = {} & (h_A(a) + h_A(b)h_A(d)^* h_A(c))^\omega + \\
& (h_A(a) + h_A(b)h_A(d)^* h_A(c))^* h_A(b)h_A(d)^\omega = \\
& h_A(\alpha')^\omega + h_A(\alpha'')h_A(d)^\omega = h_V(\alpha'^\omega) + \\
& h_A(\alpha'')h_V(d^\omega) = h_V(\alpha'^\omega) + h_V(\alpha'' d^\omega) = h_V(\alpha'^\omega + \alpha'' d^\omega), \\
\alpha' = {} & a + bd^*c, \qquad \alpha'' = (a + bd^*c)^* b, \\
\beta = {} & (h_A(d) + h_A(c)h_A(a)^* h_A(b))^\omega + \\
& (h_A(d) + h_A(c)h_A(a)^* h_A(b))^* h_A(c)h_A(a)^\omega = \\
& h_A(\beta')^\omega + h_A(\beta'')h_A(a)^\omega = h_V(\beta'^\omega) + h_A(\beta'')h_V(a^\omega) = \\
& h_V(\beta'^\omega) + h_V(\beta'' a^\omega) = h_V(\beta'^\omega + \beta'' a^\omega), \\
\beta' = {} & (d + ca^*b)^\omega, \qquad \beta'' = (d + ca^*b)^* c.
\end{aligned}$$

□

Corollary 5.7.2 *Let (A, V) and $(\hat{A}, \hat{V})$ be starsemiring-omegasemimodule pairs and let (h_A, h_V), where $h_A : \hat{A} \to A$ and $h_V : \hat{V} \to V$, be a morphism of starsemiring-omegasemimodule pairs. Then the extended mapping (h_A, h_V), where $h_A : \hat{A}^{n\times n} \to A^{n\times n}$ and $h_V : \hat{V}^n \to V^n$, is again a morphism of starsemiring-omegasemimodule pairs.*

Proof. Take $n' = 1$ in Theorem 7.1. □

In the next theorem we consider a matrix $M'^{\omega_{(k,n')}}$ for $M' \in A^{(n\times n')\times(n\times n')}$, $0 \le k \le n$. This matrix is defined as follows (in accordance with (1)): We partition M' into blocks $M' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a is $(k \times n') \times (k \times n')$ and contains the entries of M' with indices $(1,1), \ldots, (1,n'), \ldots, (k,1), \ldots, (k,n')$; and where d is $(n-k, n') \times (n-k, n')$, and contains the entries of M' with indices $(k+1,1), \ldots, (k+1,n'), \ldots, (n,1), \ldots, (n,n')$. Then

$$M'^{\omega_{(k,n')}} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}.$$

Theorem 5.7.3 *Let (A, V) and $(\hat{A}, \hat{V})$ be starsemiring-omegasemimodule pairs and let (h_A, h_V), where $h_A : \hat{A} \to A^{n'\times n'}$ and $h_V : \hat{V} \to V^{n'}$, be a morphism of starsemiring-omegasemimodule pairs. Then $h_A(M)^{\omega_{(k,n')}} = h_V(M^{\omega_k})$ for $M \in \hat{A}^{n\times n}$.*

Proof. Partition $M \in \hat{A}^{n\times n}$ into $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a is $k \times k$ and d is $(n-k) \times (n-k)$. Then we obtain

$$h_A(M) = \begin{pmatrix} h_A(a) & h_A(b) \\ h_A(c) & h_A(d) \end{pmatrix} \in A^{(n\times n')\times(n\times n')}$$

and

$$h_A(M)^{\omega_{(k,n')}} = \begin{pmatrix} (h_A(a) + h_A(b)h_A(d)^* h_A(c))^\omega \\ h_A(d)^* h_A(c)(h_A(a) + h_A(b)h_A(d)^* h_A(c))^\omega \end{pmatrix} =$$

$$\begin{pmatrix} h_A(a + bd^*c)^\omega \\ h_A(d^*c)h_A(a + bd^*c)^\omega \end{pmatrix} = \begin{pmatrix} h_V((a + bd^*c)^\omega) \\ h_A(d^*c)h_V((a + bd^*c)^\omega) \end{pmatrix} =$$

$$\begin{pmatrix} h_V((a + bd^*c)^\omega) \\ h_V(d^*c(a + bd^*c)^\omega) \end{pmatrix} = h_V(\begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}) = h_V(M^{\omega_k}) .$$

$\square$

In the sequel, (A, V) and $(\hat{A}, \hat{V})$ denote starsemiring-omegasemimodule pairs, and A', $\hat{A}'$ denote subsets of A and $\hat{A}$.

A morphism (h_A, h_V) of starsemiring-omegasemimodule pairs is called $(\hat{A}', A')$-*rational* or $(\hat{A}', A')$-*algebraic* if, for all $s \in \hat{A}'$, $h_A(s) \in \mathfrak{Rat}(A')^{Q\times Q}$ or $h_A(s) \in \mathfrak{Alg}(A')^{Q\times Q}$, respectively. If $\hat{A} = A$, $\hat{V} = V$ and $\hat{A}' = A'$, these morphisms are called A'-*rational* or A'-*algebraic*, respectively.

We are now ready to introduce the notions of a rational and an algebraic transducer over starsemiring-omegasemimodule pairs.

An $(\hat{A}', A')$-*rational transducer* (*over starsemiring-omegasemimodule pairs* $(\hat{A}, \hat{V})$ *and* (A, V))

$$\mathfrak{T} = (n', I', (h_A, h_V), P')$$

is given by

(i) a finite *set of states* $\{1, \ldots, n'\}$, $n' \geq 1$,

(ii) an $(\hat{A}', A')$-rational morphism (h_A, h_V) of starsemiring-omegasemimodule pairs, where $h_A : \hat{A} \to A^{n'\times n'}$ and $h_V : \hat{V} \to V^{n'}$,

(iii) $I' \in \mathfrak{Rat}(A')^{1\times n'}$, called the *initial state vector*,

(iv) $P' \in \mathfrak{Rat}(A')^{n'\times 1}$, called the *final state vector*.

The mapping $||\mathfrak{T}|| : \hat{A}\times\hat{V} \to A\times V$ from the quemiring $\hat{A}\times\hat{V}$ into the quemiring $A \times V$ *realized* by an $(\hat{A}', A')$-rational transducer $\mathfrak{T} = (n', I', (h_A, h_V), P')$ is defined by

$$||\mathfrak{T}||((s, v)) = (I'h_A(s)P', I'h_V(v)) ,$$

$s \in \hat{A}$, $v \in \hat{V}$. We use also the notation

$$||\mathfrak{T}||(s + v) = I'h_A(s)P' + I'h_V(v) .$$

Observe that $||\mathfrak{T}||(s, 0) = (I'h_A(s)P', 0)$ and $||\mathfrak{T}||(0, v) = (0, I'h_V(v))$. Hence, $||\mathfrak{T}||(s + v) = ||\mathfrak{T}||(s) + ||\mathfrak{T}||(v)$.

A mapping $\tau : \hat{A} \times \hat{V} \to A \times V$ is called an $(\hat{A}', A')$-*rational transduction* if there exists an $(\hat{A}', A')$-rational transducer $\mathfrak{T}$ such that $\tau((s, v)) = ||\mathfrak{T}||((s, v))$ for all $s \in \hat{A}$, $v \in \hat{V}$. In this case, we say that τ is *realized* by $\mathfrak{T}$. An (A', A')-rational transducer (in case $\hat{A} = A$ and $\hat{A}' = A$) is called an A'-*rational transducer* and an (A', A')-rational transduction is called an A'-*rational transduction*.

An $(\hat{A}', A')$-*algebraic transducer* $\mathfrak{T} = (n', I', (h_A, h_V), P')$ is defined exactly as an $(\hat{A}', A')$-rational transducer except that (h_A, h_V) is now an $(\hat{A}', A')$-algebraic morphism of starsemiring-omegasemimodule pairs, and the entries of I' and P' are in $\mathfrak{Alg}(A')$. The definitions of the notions of $(\hat{A}', A')$-*algebraic transduction*, A'-*algebraic transducer* and A'-*algebraic transduction* should be clear.

We now show that $(\hat{A}', A')$-rational transductions map ω-$\mathfrak{Rat}(\hat{A}')$ into ω-$\mathfrak{Rat}(A')$.

Theorem 5.7.4 *Let $(\hat{A}, \hat{V})$ and (A, V) be Conway semiring-semimodule pairs. Assume that $\mathfrak{T}$ is an $(\hat{A}', A')$-rational transducer and that $(s, v) \in \omega$-$\mathfrak{Rat}(\hat{A}')$. Then $||\mathfrak{T}||((s, v)) \in \omega$-$\mathfrak{Rat}(A')$.*

Proof. Let (s, v) be the behavior of the finite $\hat{A}'$-automaton $\mathfrak{A} = (n, I, M, P, k)$. Assume that $\mathfrak{T} = (n', I', (h_A, h_V), P')$. We consider now the finite A-automaton $\mathfrak{A}' = (n \times n', I'h_A(I), h_A(M), h_A(P)P', k \times n')$. Since the entries of $h_A(s)$ are in $\mathfrak{Rat}(A')$ for $s \in \hat{A}'$, $\mathfrak{A}'$ is in fact a finite $\mathfrak{Rat}(A')$-automaton. Hence, by (iii) of Theorem 4.9, there exist $s, t_k, s_k \in \mathfrak{Rat}(\mathfrak{Rat}(A')) = \mathfrak{Rat}(A')$ such that $||\mathfrak{A}|| = s + \sum_{1 \le k \le m} s_k t_k^\omega$. This implies that $||\mathfrak{A}'|| \in \omega$-$\mathfrak{Rat}(A')$. We obtain

$$\begin{aligned}
&||\mathfrak{A}'|| = I'h_A(I)h_A(M)^* h_A(P)P' + I'h_A(I)h_A(M)^{\omega_{(k\times n')}} = \\
&I'h_A(I)h_A(M^*)h_A(P)P' + I'h_A(I)h_V(M^{\omega_k}) = \\
&I'h_A(IM^*P)P' + I'h_V(IM^{\omega_k}) = ||\mathfrak{T}||(||\mathfrak{A}||)\,.
\end{aligned}$$

Here the second equality follows by Theorem 7.1 and the third equality follows by Theorems 7.1 and 7.3. Hence, $||\mathfrak{T}||(||\mathfrak{A}||) \in \mathfrak{Rat}(A')$. $\square$

We now consider the functional composition of A'-rational transductions.

Theorem 5.7.5 *Let (A, V) be a starsemiring-omegasemimodule pair. Then the family of A'-rational transductions is closed under functional composition.*

Proof. Let $\mathfrak{T}' = (n', I', (h'_A, h'_V), P')$ and $\mathfrak{T}'' = (n'', I'', (h''_A, h''_V), P'')$ be two A'-rational transducers. We want to show that the mapping $\tau : A \times V \to A \times V$ defined by $\tau((s, v)) = ||\mathfrak{T}''||(||\mathfrak{T}'||((s, v)))$, $s \in A$, $v \in V$, is again an A'-rational transduction.

Consider $\mathfrak{T} = (n' \times n'', I''h''_A(I'), (h''_A \circ h'_A, h''_V \circ h'_V), h''_A(P')P'')$. By Theorem 4.2.3 and by Theorem 7.3 the mapping $(h''_A \circ h'_A, h''_V \circ h'_V)$ is an A'-rational morphism of starsemiring-omegasemimodule pairs. Furthermore, the entries of $I''h''_A(I')$ and $h''_A(P')P''$ are in $\mathfrak{Rat}(A')$. Hence, $\mathfrak{T}$ is a rational A'-transducer over starsemiring-omegasemimodule pairs. We obtain, for $s \in A$, $v \in V$,

$$\begin{aligned}
&||\mathfrak{T}||(s + v) = I''h''_A(I')h''_A(h'_A(s))h''_A(P')P'' + I''h''_A(I')h''_V(h'_V(v)) = \\
&I''h''_A(I'h'_A(s)P')P'' + I''h''_V(I'h'_V(v)) = \\
&I''h''_A(||\mathfrak{T}'||(s))P'' + I''h''_V(||\mathfrak{T}'||(v)) = ||\mathfrak{T}''||(||\mathfrak{T}'||(s + v))\,.
\end{aligned}$$

Hence, our theorem is proved. $\square$

Assume that (A_i, V_i) are starsemiring-omegasemimodule pairs and $A'_i \subseteq A_i$, $i = 1, 2, 3$. Then, by a proof similar to that of Theorem 7.5, we obtain the following result: *If* τ_1 *is an* (A'_1, A'_2)*-rational transduction and* τ_2 *is an* (A'_2, A'_3)*-rational transduction then the composite of* τ_1 *and* τ_2 *is an* (A'_1, A'_3)*-rational transduction.*

We now show that $(\hat{A}', A')$-algebraic transductions map ω-$\mathfrak{Alg}(\hat{A}')$ into ω-$\mathfrak{Alg}(\hat{A}')$. The proof is similar to that of Theorem 7.4.

Theorem 5.7.6 *Let* $(\hat{A}, \hat{V})$ *and* (A, V) *be continuous semiring-semimodule pairs. Assume that* $\mathfrak{T}$ *is an* $(\hat{A}', A')$*-algebraic transducer and that* $(s, v) \in \omega$-$\mathfrak{Alg}(\hat{A}')$*. Then* $||\mathfrak{T}||((s, v)) \in \omega$-$\mathfrak{Alg}(A')$*.*

Proof. Let (s, v) be the behavior of the finite $\hat{A}'$-automaton $\mathfrak{A} = (n, I, M, P, k)$. Assume that $\mathfrak{T} = (n', I', (h_A, h_V), P')$. We consider now the finite A-automaton $\mathfrak{A}' = (n \times n', I' h_A(I), h_A(M), h_A(P)P', k \times n')$. Since the entries of $h_A(s)$ are in $\mathfrak{Alg}(A')$ for $s \in \hat{A}'$, $\mathfrak{A}'$ is in fact a finite $\mathfrak{Alg}(A')$-automaton. Hence, by (iii) of Theorem 6.6, there exist $s, t_k, s_k \in \mathfrak{Alg}(\mathfrak{Alg}(A')) = \mathfrak{Alg}(A')$ such that $||\mathfrak{A}|| = s + \sum_{1 \leq k \leq m} s_k t_k^\omega$. This implies that $||\mathfrak{A}'|| \in \omega$-$\mathfrak{Alg}(A')$. We obtain $||\mathfrak{A}'|| = ||\mathfrak{T}||(||\mathfrak{A}||)$ as in the proof of Theorem 7.4. Hence, $||\mathfrak{T}||(||\mathfrak{A}||) \in \mathfrak{Alg}(A')$. □

We now consider the functional composition of A'-algebraic transductions. The proof of the next theorem is similar to that of Theorem 7.5.

Theorem 5.7.7 *Let* (A, V) *be a continuous semiring-semimodule pair. Then the family of* A'*-algebraic transductions is closed under functional composition.*

Proof. Let $\mathfrak{T}' = (n', I', (h'_A, h'_V), P')$ and $\mathfrak{T}'' = (n'', I'', (h''_A, h''_V), P'')$ be two A'-algebraic transducers. We want to show that the mapping $\tau : A \times V \to A \times V$ defined by $\tau((s, v)) = ||\mathfrak{T}''||(||\mathfrak{T}'||((s, v)))$, $s \in A$, $v \in V$, is again an A'-algebraic transduction.

Consider $\mathfrak{T} = (n' \times n'', I'' h''_A(I'), (h''_A \circ h'_A, h''_V \circ h'_V), h''_A(P')P'')$. By Theorem 4.2.6 and by Theorem 7.3 the mapping $(h''_A \circ h'_A, h''_V \circ h'_V)$ is an A'-algebraic morphism of starsemiring-omegasemimodule pairs. Furthermore, the entries of $I'' h''_A(I')$ and $h''_A(P')P''$ are in $\mathfrak{Alg}(A')$. Hence, $\mathfrak{T}$ is a algebraic A'-transducer over starsemiring-omegasemimodule pairs. We obtain, for $s \in A$, $v \in V$, $||\mathfrak{T}||(s + v) = ||\mathfrak{T}''||(||\mathfrak{T}'||(s + v))$ as in the proof of Theorem 7.5. □

Assume that (A_i, V_i) are continuous semiring-semimodule pairs and $A'_i \subseteq A_i$, $i = 1, 2, 3$. Then, by a proof similar to that of Theorem 7.7, we obtain the following result: *If* τ_1 *is an* (A'_1, A'_2)*-algebraic transduction and* τ_2 *is an* (A'_2, A'_3)*-algebraic transduction then the composite of* τ_1 *and* τ_2 *is an* (A'_1, A'_3)*-algebraic transduction.*

For the rest of this section, A is a *complete* star-omega semiring, Σ_∞ is an *infinite* alphabet and $\Sigma \subseteq \Sigma_\infty$ is a *finite subalphabet* of Σ. All items may be indexed. Hence, by Theorem 5.5, $(A\langle\langle \Sigma_\infty^* \rangle\rangle, A\langle\langle \Sigma_\infty^\omega \rangle\rangle)$ will be a *complete* semiring-semimodule pair for the rest of this section.

We define

$$A\{\{\Sigma_\infty^*\}\} = \{s \in A\langle\langle \Sigma_\infty^* \rangle\rangle \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \text{ such that } \mathrm{supp}(s) \subseteq \Sigma^*\},$$

and

$$A\{\{\Sigma_\infty^\omega\}\} = \{v \in A\langle\langle\Sigma_\infty^\omega\rangle\rangle \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \text{ such that } \mathrm{supp}(v) \subseteq \Sigma^\omega\},$$

Then $(A\{\{\Sigma_\infty^*\}\}, A\{\{\Sigma_\infty^\omega\}\})$ is a starsemiring-omegasemimodule pair. We assume that $A\langle\langle\Sigma^*\rangle\rangle \subset A\{\{\Sigma_\infty^*\}\}$ and $A\langle\langle\Sigma^\omega\rangle\rangle \subset A\{\{\Sigma_\infty^\omega\}\}$.

For the rest of this section, A denotes a *commutative* complete star-omega semiring. Now, we define, additionally to similar definitions in Chapter 4,

$$\begin{aligned}
A^{\mathrm{alg}}\{\{\Sigma_\infty^\omega\}\} &= \{v \in A\{\{\Sigma_\infty^\omega\}\} \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \\
&\qquad \text{and } s_k, t_k \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle \text{ such that } v = \textstyle\sum_{1\le k\le m} s_k t_k^\omega\},\\
A^{\mathrm{rat}}\{\{\Sigma_\infty^\omega\}\} &= \{v \in A\{\{\Sigma_\infty^\omega\}\} \mid \text{there exists a finite alphabet } \Sigma \subset \Sigma_\infty \\
&\qquad \text{and } s_k, t_k \in A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle \text{ such that } v = \textstyle\sum_{1\le k\le m} s_k t_k^\omega\},\\
A\{\Sigma_\infty^\omega\} &= \{s \in A\langle\Sigma^\omega\rangle \mid \Sigma \subset \Sigma_\infty \text{ finite}\},
\end{aligned}$$

A pair (h_A, h_V) of mappings $h_A : \Sigma_\infty^* \to (A\langle\langle\Sigma_\infty^*\rangle\rangle)^{n'\times n'}$ and $h_V : \Sigma_\infty^\omega \to (A\langle\langle\Sigma_\infty^\omega\rangle\rangle)^{n'}$ is called a *representation* if the following conditions are satisfied:

(i) the mapping h_A is a multiplicative monoid morphism such that there exists a finite $\Sigma \subset \Sigma_\infty$ with $h_A(x) = 0$ for $x \in \Sigma_\infty - \Sigma$,

(ii) $h_A(w)h_V(u) = h_V(wu)$ for all $w \in \Sigma_\infty^*$, $u \in \Sigma_\infty^\omega$,

(iii) $\prod_{i\ge 0} h_A(w_i) = h_V(\prod_{i\ge 0} w_i)$ for all $w_i \in \Sigma_\infty^*$, $i \ge 0$.

Observe that if (h_A, h_V) is a representation, there exist only finitely many entries $h_A(x)_{i,j} \ne 0$, $x \in \Sigma_\infty$. Hence, there is a finite $\Sigma' \subset \Sigma_\infty$ such that $h_A(w)_{i,j} \in A\langle\langle\Sigma'^*\rangle\rangle$ for all $w \in \Sigma_\infty^*$. A representation (h_A, h_V) is called *rational* (resp. *algebraic*) if $h_A : \Sigma_\infty^* \to (A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\})^{n'\times n'}$ (resp. $h_A : \Sigma_\infty^* \to (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^{n'\times n'}$).

A representation (h_A, h_V) can be *extended* to a mapping (μ_A, μ_V), where $\mu_A : A\langle\langle\Sigma_\infty^*\rangle\rangle \to (A\{\{\Sigma_\infty^*\}\})^{n'\times n'}$ and $\mu_V : A\langle\langle\Sigma_\infty^\omega\rangle\rangle \to (A\{\{\Sigma_\infty^\omega\}\})^{n'}$ by the definitions

$$\mu_A(s) = \mu_A(\sum_{w\in\Sigma_\infty^*} (s,w)w) = \sum_{w\in\Sigma_\infty^*} (s,w) \otimes h_A(w), \qquad s \in A\langle\langle\Sigma_\infty^*\rangle\rangle$$

and

$$\mu_V(v) = \mu_V(\sum_{u\in\Sigma_\infty^\omega} (v,u)u) = \sum_{u\in\Sigma_\infty^\omega} (v,u) \otimes h_V(u), \qquad v \in A\langle\langle\Sigma_\infty^\omega\rangle\rangle\,.$$

Here $\otimes$ denotes the Kronecker product (See Kuich, Salomaa [88]). If (h_A, h_V) is a representation then its extension is always denoted by (μ_A, μ_V). This applies also to indexed h's and μ's.

Observe that in the next theorem, $((A\{\{\Sigma_\infty^*\}\})^{n'\times n'}, (A\{\{\Sigma_\infty^\omega\}\})^{n'})$ is a complete semiring-semimodule pair by Theorem 5.6.

Theorem 5.7.8 *Let A be a commutative complete star-omega semiring. If (h_A, h_V) is a representation then (μ_A, μ_V) is a morphism of complete semiring-semimodule pairs from $(A\{\{\Sigma_\infty^*\}\}, A\{\{\Sigma_\infty^\omega\}\})$ into $((A\{\{\Sigma_\infty^*\}\})^{n'\times n'}, (A\{\{\Sigma_\infty^\omega\}\})^{n'})$.*

Proof. It is easily shown that μ_A is a semiring morphism and μ_V is a monoid morphism. Moreover, we obtain for all $s, s_i \in A\{\{\Sigma_\infty^*\}\}$ and $v \in A\{\{\Sigma_\infty^\omega\}\}$:

(i) $$\begin{aligned}&\mu_A(s)\mu_V(v) = \\ &(\textstyle\sum_{w\in\Sigma_\infty^*}(s,w)\otimes h_A(w))(\sum_{u\in\Sigma_\infty^\omega}(v,u)\otimes h_V(u)) = \\ &\textstyle\sum_{w\in\Sigma_\infty^*}\sum_{u\in\Sigma_\infty^\omega}(s,w)(v,u)\otimes h_A(w)h_V(u) = \\ &\textstyle\sum_{w\in\Sigma_\infty^*}\sum_{u\in\Sigma_\infty^\omega}(s,w)(v,u)\otimes h_V(wu)\,;\end{aligned}$$

here we have applied Theorem 4.33 of Kuich, Salomaa [88] in the second equality; and
$$\begin{aligned}&\mu_V(sv) = \mu_V((\textstyle\sum_{w\in\Sigma_\infty^*}(s,w)w)(\sum_{u\in\Sigma_\infty^\omega}(v,u)u)) = \\ &\mu_V(\textstyle\sum_{w\in\Sigma_\infty^*}\sum_{u\in\Sigma_\infty^\omega}(s,w)(v,u)wu) = \\ &\textstyle\sum_{w\in\Sigma_\infty^*}\sum_{u\in\Sigma_\infty^\omega}(s,w)(v,u)\otimes h_V(wu)\,,\end{aligned}$$

i. e., $\mu_A(s)\mu_V(v) = \mu_V(sv)$.

(ii) $$\begin{aligned}&\textstyle\sum_{i\in I}\mu_A(s_i) = \\ &\textstyle\sum_{i\in I}\mu_A(\sum_{w\in\Sigma_\infty^*}(s_i,w)w) = \\ &\textstyle\sum_{i\in I}\sum_{w\in\Sigma_\infty^*}(s_i,w)\otimes h_A(w) = \\ &\textstyle\sum_{w\in\Sigma_\infty^*}(\sum_{i\in I}(s_i,w))\otimes h_A(w) = \\ &\mu_A(\textstyle\sum_{i\in I}s_i)\,.\end{aligned}$$

(iii) $$\begin{aligned}&\textstyle\prod_{i\geq 1}\mu_A(s_i) = \\ &\textstyle\prod_{i\geq 1}\sum_{w_i\in\Sigma_\infty^*}((s_i,w_i)\otimes h_A(w_i)) = \\ &\textstyle\sum_{(w_1,w_2,\ldots)\in\Sigma_\infty^*\times\Sigma_\infty^*\times\ldots}\prod_{i\geq 1}((s_i,w_i)\otimes h_A(w_i)) = \\ &\textstyle\sum_{(w_1,w_2,\ldots)\in\Sigma_\infty^*\times\Sigma_\infty^*\times\ldots}\prod_{i\geq 1}(s_i,w_i)\otimes\prod_{i\geq 1}h_A(w_i)\end{aligned}$$

and
$$\begin{aligned}&\mu_V(\textstyle\prod_{i\geq 1}s_i) = \\ &\mu_V(\textstyle\prod_{i\geq 1}\sum_{w_i\in\Sigma_\infty^*}(s_i,w_i)w_i) = \\ &\mu_V(\textstyle\sum_{(w_1,w_2,\ldots)\in\Sigma_\infty^*\times\Sigma_\infty^*\times\ldots}\prod_{i\geq 1}((s_i,w_i)w_i)) = \\ &\mu_V(\textstyle\sum_{(w_1,w_2,\ldots)\in\Sigma_\infty^*\times\Sigma_\infty^*\times\ldots}\prod_{i\geq 1}(s_i,w_i)\prod_{i\geq 1}w_i) = \\ &\textstyle\sum_{(w_1,w_2,\ldots)\in\Sigma_\infty^*\times\Sigma_\infty^*\times\ldots}\prod_{i\geq 1}(s_i,w_i)\otimes h_V(\prod_{i\geq 1}w_i)\,,\end{aligned}$$

i. e.,
$$\prod_{i\geq 1}\mu_A(s_i) = \mu_V(\prod_{i\geq 1}s_i)\,.$$

□

We now specialize the notions of A'-rational and A'-algebraic transducers for a fixed semiring A and a fixed alphabet Σ_∞. A *rational transducer*

$$\mathfrak{T} = (n', I', (h_A, h_V), P')$$

is given by

(i) a finite *set of states* $\{1, \ldots, n'\}$, $n' \geq 1$,

(ii) a *rational representation* (h_A, h_V),

(iii) $I' \in (A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\})^{1\times n'}$, called the *initial state vector*,

(iv) $P' \in (A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\})^{n'\times 1}$, called the *final state vector*.

The mapping $||\mathfrak{T}|| : A\langle\langle\Sigma_\infty^*\rangle\rangle \times A\langle\langle\Sigma_\infty^\omega\rangle\rangle \to A\langle\langle\Sigma_\infty^*\rangle\rangle \times A\langle\langle\Sigma_\infty^\omega\rangle\rangle$ *realized* by $\mathfrak{T}$ is then given by

$$\begin{aligned}
&||\mathfrak{T}||(s+v) = I'\mu_A(s)P' + I'\mu_V(v) = \\
&I'\textstyle\sum_{w\in\Sigma_\infty^*}((s,w)\otimes h_A(w))P' + I'\sum_{u\in\Sigma_\infty^\omega}(v,u)\otimes h_V(u) = \\
&\textstyle\sum_{w\in\Sigma_\infty^*}(s,w)I'h_A(w)P' + \sum_{u\in\Sigma_\infty^\omega}(v,u)I'h_V(u) = \\
&\textstyle\sum_{w\in\Sigma_\infty^*}(s,w)||\mathfrak{T}||(w) + \sum_{u\in\Sigma_\infty^\omega}(v,u)||\mathfrak{T}||(u)\,.
\end{aligned}$$

Observe that there exists a finite $\Sigma \subset \Sigma_\infty$ such that $h_A(x) = 0$ for $x \in \Sigma_\infty - \Sigma$, $h_A(x) \in (A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{n'\times n'}$ for $x \in \Sigma$, $I' \in (A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{1\times n'}$ and $P' \in (A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{n'\times 1}$. Hence, in fact, $||\mathfrak{T}||$ is a mapping $A\langle\langle\Sigma^*\rangle\rangle\times A\langle\langle\Sigma^\omega\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle\times A\langle\langle\Sigma^\omega\rangle\rangle$. *Algebraic transducers* with an algebraic representation (h_A, h_V) and $I' \in (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^{1\times n'}$, $P' \in (A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\})^{n'\times 1}$, are defined in the same way.

A rational or an algebraic transducer $\mathfrak{T}$ as specified above can be considered to be a finite automaton equipped with an output device. In a state transition from state i to state j, $\mathfrak{T}$ reads a letter $x \in \Sigma$ and outputs the rational or algebraic power series $h_A(x)_{i,j}$. A finite or infinite sequence of state transitions outputs the product of the power series of the single state transitions.

All finite sequences of n state transitions from state i to state j reading a word $w \in \Sigma^*$, $|w| = n$, output the power series $h_A(w)_{i,j}$. This output is multiplied with the correct components of the initial and final state vector, and $I'_i h_A(w)_{i,j} P'_j$ is said to be the *translation of w by finite sequences of transitions from i to j*. Summing up, for all $i, j \in \{1, \ldots, n'\}$, $\sum_{1\leq i,j\leq n'} I'_i h_A(w)_{i,j} P'_j = I'h_A(w)P' = ||\mathfrak{T}||(w)$ is said to be the *translation of w by $\mathfrak{T}$*. A power series $s \in A\langle\langle\Sigma_\infty^*\rangle\rangle$ is then *translated by* $\mathfrak{T}$ to $\sum_{w\in\Sigma^*}(s,w)I'h_A(w)P' = I'\mu_A(s)P' = ||\mathfrak{T}||(s)$.

All infinite sequences of state transitions starting in state i and reading a word $u \in \Sigma^\omega$ output the power series $h_V(u)_i$. This output is multiplied with the correct component of the initial state vector, and $I'h_V(u)_i$ is said to be the *translation of u by infinite sequences of transitions starting in i*. Summing up, for all $i \in \{1, \ldots, n'\}$, $\sum_{1\leq i\leq n'} I'_i h_V(u)_i = I'h_V(u) = ||\mathfrak{T}||(u)$ is said to be the *translation of u by $\mathfrak{T}$*. Observe here that, if $u = \prod_{i\geq 0} x_i$, $x_i \in \Sigma_\infty$, then $I'h_V(u) = I'\prod_{i\geq 0} h_A(x_i)$. A power series $v \in A\langle\langle\Sigma_\infty^\omega\rangle\rangle$ is then *translated by* $\mathfrak{T}$ to $\sum_{u\in\Sigma}(v,u)I'h_V(u) = I'\mu_V(v) = ||\mathfrak{T}||(v)$.

A power series pair $s+v$ of the quemiring $A\langle\langle\Sigma_\infty^*\rangle\rangle\times A\langle\langle\Sigma_\infty^\omega\rangle\rangle$ is then *translated by* $\mathfrak{T}$ to the sum of the translations of s and v by $\mathfrak{T}$, i. e., to $||\mathfrak{T}||(s) + ||\mathfrak{T}||(v) = ||\mathfrak{T}||(s+v)$.

Specializations of Theorems 7.4 and 7.6 yield the next result.

Theorem 5.7.9 *Let A be a commutative complete (resp. continuous) star-omega semiring. Assume that $\mathfrak{T}$ is a rational (resp. an algebraic) transducer and that $s \in A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\}$, $v \in A^{\mathrm{rat}}\{\{\Sigma_\infty^\omega\}\}$ (resp. $s \in A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\}$, $v \in A^{\mathrm{alg}}\{\{\Sigma_\infty^\omega\}\}$). Then $||\mathfrak{T}||((s,v)) \in A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{rat}}\{\{\Sigma_\infty^\omega\}\}$ (resp. $||\mathfrak{T}||((s,v)) \in A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{alg}}\{\{\Sigma_\infty^\omega\}\}$).*

Observe that if $A = \mathbb{B}$ then Theorem 7.9 is a statement on formal languages.

Theorem 5.7.10 *Let A be a commutative complete star-omega semiring. Let (h'_A, h'_V) and (h''_A, h''_V) be rational representations with extensions (μ'_A, μ'_V) and (μ''_A, μ''_V), respectively. Then $(h_A, h_V) = (\mu''_A \circ h'_A, \mu''_V \circ h'_V)$ is again a rational representation and its extension (μ_A, μ_V) satisfies $\mu_A(s) = \mu''_A(\mu'_A(s))$, $s \in A\langle\langle \Sigma_\infty^* \rangle\rangle$, and $\mu_V(v) = \mu''_V(\mu'_V(v))$, $v \in A\langle\langle \Sigma_\infty^\omega \rangle\rangle$.*

Proof. We show that the three conditions in the definition of a rational representation are satisfied for (h_A, h_V).

(i) We obtain $h_A(\varepsilon) = \mu''_A(h'_A(\varepsilon)) = \mu''_A(E) = E'$, where E and E' are the matrices of unity of suitable dimensions, and, for $w_1, w_2 \in \Sigma_\infty^*$, $h_A(w_1 w_2) = \mu''_A(h'_A(w_1 w_2)) = \mu''_A(h'_A(w_1) h'_A(w_2)) = \mu''_A(h'_A(w_1)) \mu''_A(h'_A(w_2)) = h_A(w_1) h_A(w_2)$. Moreover, there exists an $\Sigma \subset \Sigma_\infty$ with $h_A(x) = 0$ for $x \in \Sigma_\infty - \Sigma$. Eventually, since the entries of $h'_A(x)$ are rational power series, we infer by Theorem 7.4 that the entries of $h_A(x) = \mu''_A(h'_A(x))$, $x \in \Sigma_\infty$, are again rational power series.

(ii) For $w \in \Sigma_\infty^*$ and $u \in \Sigma_\infty^\omega$ we obtain $h_A(w) h_V(u) = \mu''_A(h'_A(w)) \mu''_V(h'_V(u)) = \mu''_V(h'_A(w) h'_V(u)) = \mu''_V(h'_V(wu)) = h_V(wu)$.

(iii) For $w_i \in \Sigma_\infty^*$, $i \geq 0$, we obtain $\prod_{i \geq 0} h_A(w_i) = \prod_{i \geq 0} \mu''_A(h'_A(w_i)) = \mu''_V(\prod_{i \geq 0} h'_A(w_i)) = \mu''_V(h'_V(\prod_{i \geq 0} w_i)) = h_V(\prod_{i \geq 0} w_i)$.

We now prove the last part of our theorem: for $s \in A\langle\langle \Sigma_\infty^* \rangle\rangle$ and $v \in A\langle\langle \Sigma_\infty^\omega \rangle\rangle$ we obtain $\mu''_A(\mu'_A(s)) = \mu''_A(\sum_{w \in \Sigma_\infty^*} (s,w) \otimes h'_A(w)) = \sum_{w \in \Sigma_\infty^*} (s,w) \otimes \mu''_A(h'_A(w)) = \mu_A(s)$ and $\mu''_V(\mu'_V(v)) = \mu''_V(\sum_{u \in \Sigma_\infty^\omega} (v,u) \otimes h'_V(u)) = \sum_{u \in \Sigma_\infty^\omega} (v,u) \otimes \mu''_V(h'_V(u)) = \mu_V(u)$. $\square$

Corollary 5.7.11 *Let A be a commutative complete star-omega semiring and let $\mathfrak{T}'$ and $\mathfrak{T}''$ be rational transducers. Then there exists a rational transducer $\mathfrak{T}$ such that, for each $s+v \in A\langle\langle \Sigma_\infty^* \rangle\rangle \times A\langle\langle \Sigma_\infty^\omega \rangle\rangle$, $||\mathfrak{T}||(s+v) = ||\mathfrak{T}''||(||\mathfrak{T}'||(s+v))$.*

We now introduce the notion of an abstract ω-family of power series. Before, some additional definitions are necessary. Any subset $\mathfrak{L}$ of the quemiring $A\{\{\Sigma_\infty^*\}\} \times A\{\{\Sigma_\infty^\omega\}\}$ is called *ω-family of power series.* Let now $\mathfrak{T}$ be a rational transducer. Then, for each $s + v \in A\{\{\Sigma_\infty^*\}\} \times A\{\{\Sigma_\infty^\omega\}\}$, we obtain $||\mathfrak{T}||(s+v) \in A\{\{\Sigma_\infty^*\}\} \times A\{\{\Sigma_\infty^\omega\}\}$. Hence, for an ω-family $\mathfrak{L}$ of power series,

$$\hat{\mathfrak{M}}(\mathfrak{L}) = \{||\mathfrak{T}||(s+v) \mid s+v \in \mathfrak{L},\ \mathfrak{T} \text{ a rational transducer}\}$$

is again an ω-family of power series. By Theorem 7.10 we obtain $\hat{\mathfrak{M}}(\hat{\mathfrak{M}}(\mathfrak{L})) = \hat{\mathfrak{M}}(\mathfrak{L})$. Hence, if $\mathfrak{L} = \hat{\mathfrak{M}}(\mathfrak{L})$, the ω-family $\mathfrak{L}$ of power series is said to be *closed under rational transductions* and is then called *full cone.* Theorem 7.9 admits at once the following result.

Theorem 5.7.12 *Let A be a commutative complete (resp. continuous) star-omega semiring. Then the quemiring $A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{rat}}\{\{\Sigma_\infty^\omega\}\}$ (resp. $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{alg}}\{\{\Sigma_\infty^\omega\}\}$) is a full cone.*

Let $(A\langle\langle\Sigma_\infty^*\rangle\rangle, A\langle\langle\Sigma_\infty^\omega\rangle\rangle)$ be a complete semiring-semimodule pair. Given an ω-family $\mathfrak{L}$ of power series, the notation $\hat{\mathfrak{F}}(\mathfrak{L})$ is used for the *smallest ω-rationally closed quemiring that is closed also under rational transductions and contains* $\mathfrak{L}$. Clearly, $\hat{\mathfrak{F}}(\mathfrak{L})$ is again an ω-family of power series. An ω-family $\mathfrak{L}$ of power series is called *full abstract ω-family of power series* if $\mathfrak{L} = \hat{\mathfrak{F}}(\mathfrak{L})$. This definition yields the last result of this chapter. It is implied by Theorems 7.12, 4.9 and 6.7.

Theorem 5.7.13 *Let A be a commutative complete (resp. continuous) star-omega semiring. Then the quemiring $A^{\mathrm{rat}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{rat}}\{\{\Sigma_\infty^\omega\}\}$ (resp. $A^{\mathrm{alg}}\{\{\Sigma_\infty^*\}\} \times A^{\mathrm{alg}}\{\{\Sigma_\infty^\omega\}\}$) is a full abstract ω-family of power series.*

Chapter 6

Formal tree series

6.1 Introduction

We assume that the reader has some basic knowledge of tree languages and tree automata (see Gécseg, Steinby [51, 52], Comon, Dauchet, Gilleron, Jaquemard, Lugiez, Tison, Tommasi [24]). Formal tree series were introduced by Berstel, Reutenauer [6], and then extensively studied by Bozapalidis [11, 12, 13, 14, 15], Bozapalidis, Rahonis [16], Kuich [80, 81, 82, 83, 85], Block, Griffing [8], Engelfriet, Fülöp, Vogler [33] and Fülöp, Vogler [50].

This chapter consists of this and seven more sections. In Section 2, we define distributive Σ-algebras, where Σ is any signature. We introduce tree series and characterize the distributive Σ-algebras of tree series (with coefficients in a continuous semiring) by a universal property. We use this characterization to derive properties of tree series substitutions.

In Section 3 we define tree automata and systems of equations whose right sides consist of tree series. These notions form a framework for considering finite tree automata and pushdown tree automata, and polynomial systems. The main result of this section is that (finite, polynomial) tree automata and (finite, polynomial) systems are equivalent mechanisms.

In Section 4 we prove a Kleene Theorem for recognizable tree series that allows also the definition of recognizable tree series by expressions which are analogous to regular expressions.

In Section 5, pushdown tree automata and algebraic tree systems are introduced and it is shown that these mechanisms are equivalent. Moreover, a Kleene Theorem for algebraic tree series is proved.

Top-down tree series transducers are introduced in Section 6. We concentrate on (top-down) nondeterministic simple recognizable tree series transducers and prove that they preserve recognizability of tree series. Full abstract families of tree series (briefly, full AFTs) are families of tree series closed under nondeterministic simple recognizable tree series transductions and certain specific "rational" operations.

These full AFTs are introduced in Section 7. It is shown that the families of recognizable tree series and of algebraic tree series are full AFTs.

The last section exhibits connections of formal tree series to formal power series. We first show that the macro power series (a generalization of the indexed languages) are exactly the yields of algebraic tree series. Moreover, we prove a Kleene Theorem for macro power series (and indexed languages). Then we show that algebraic power series are exactly the yields of recognizable tree series. Finally, we prove the important result that the yield of a full AFT forms a full abstract family of power series.

The presentation of this chapter follows the lines of Ésik, Kuich [38].

6.2 Preliminaries

In this section we first consider distributive algebras. The definitions and results on distributive algebras are heavily influenced by Bozapalidis [14], especially by his notion of a K-Γ-algebra. He noticed that the multilinear mappings of his well ω-additive K-Γ-algebras assure that certain important mappings induced by formal power series are continuous. (See Theorem 2 of Bozapalidis [14].) We have tried in the forthcoming definition of a distributive algebra to simplify the type of algebra used while saving the important results. The results on distributive algebras are generalizations of the results on semirings given in the Preliminaries of Chapter 2.

In the second part of this section we introduce formal tree series. These formal tree series form a distributive algebra.

In the final part of this section we consider some important mappings connected with formal tree series and show that they are continuous.

A *signature* is a non-empty set Σ, whose elements are called *operation symbols*, together with a mapping $\Sigma \to \mathbb{N}$, called the *arity function*, assigning to each operation symbol its *arity* ($\mathbb{N}$ denotes the nonnegative integers). We write $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_k \cup \ldots$, where Σ_k, $k \geq 0$, contains the operation symbols of arity k.

Let Σ be a signature. Recall that a Σ-*algebra* $\langle A, \Sigma \rangle$ consists of a non-empty set A and a family of operations $\{\sigma_A : A^k \to A \mid \sigma \in \Sigma_k,\ k \geq 0\}$. As usual, we denote this family of operations again by Σ, and the family of k-ary operations by Σ_k, $k \geq 0$. (See Gécseg, Steinby [51], Grätzer [58], Lausch, Nöbauer [90], Wechler [115].) The algebra $\langle A, +, 0, \Sigma \rangle$, where $\langle A, +, 0 \rangle$ is a commutative monoid and $\langle A, \Sigma \rangle$ is a Σ-algebra, is called a *distributive* Σ-*algebra* iff the following two conditions are satisfied for all $\sigma_A \in \Sigma_k$ and all $a, a_1, \ldots, a_k \in A$, $k \geq 1$:

(i) $\sigma_A(a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_k) = 0$ for all $1 \leq j \leq k$,

(ii) $\sigma_A(a_1, \ldots, a_{j-1}, a_j + a, a_{j+1}, \ldots, a_k) =$
$\sigma_A(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) + \sigma_A(a_1, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_k)$
for all $1 \leq j \leq k$.

A morphism of distributive Σ-algebras preserves both the monoid structure and the operations in Σ. In the sequel, $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_k \cup \ldots$ will always denote a signature. In connection with trees, a signature will be called a *ranked alphabet*, where the *rank* of an operation symbol is its arity.

A distributive Σ-algebra $\langle A, +, 0, \Sigma\rangle$ is briefly denoted by A if $+$, 0 and Σ are understood. The notion of a distributive Σ-algebra was introduced in Kuich [80] by the name "distributive multioperator monoid". Idempotent distributive Σ-algebras, i. e., Σ-algebras $\langle A, +, 0, \Sigma\rangle$ where $a + a = a$ for all $a \in A$, were introduced in Courcelle [26] by the name "distributive F-magma". Moreover, distributive Σ-algebras are a "reduced" version of the K-Γ-algebras of Bozapalidis [14].

A distributive Σ-algebra $\langle A, +, 0, \Sigma\rangle$ is termed *ordered* iff $\langle A, +, 0\rangle$ is ordered and each operation $\sigma_A \in \Sigma$ preserves the order in each argument. When the order is the natural order, this latter condition holds by distributivity. A morphism of ordered distributive Σ-algebras is an order preserving distributive Σ-algebra morphism.

A distributive Σ-algebra $\langle A, +, 0, \Sigma\rangle$ is *complete* iff $\langle A, +, 0\rangle$ is complete and the following additional condition is satisfied for all $\sigma_A \in \Sigma_k$, index sets $I_1, \ldots, I_k$, and $a_{i_1}, \ldots, a_{i_k} \in A$, $i_1 \in I_1$, $\ldots$, $i_k \in I_k$, $k \geq 1$:

$$\sigma_A(\sum_{i_1 \in I_1} a_{i_1}, \ldots, \sum_{i_k \in I_k} a_{i_k}) = \sum_{i_1 \in I_1} \ldots \sum_{i_k \in I_k} \sigma_A(a_{i_1}, \ldots, a_{i_k}).$$

Finally, an ordered distributive Σ-algebra $\langle A, +, 0, \Sigma\rangle$ is called *continuous* iff $\langle A, +, 0\rangle$ is continuous and if the operations $\sigma_A \in \Sigma_k$ are continuous: For all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in A$, $1 \leq i \leq k$, and for each directed set $D \subseteq A$,

$$\sigma_A(a_1, \ldots, \sup D, \ldots, a_k) = \sup \sigma_A(a_1, \ldots, D, \ldots, a_k)\,.$$

A morphism of complete (resp. continuous) distributive Σ-algebras is both a complete (resp. continuous) monoid morphism and a distributive Σ-algebra morphism. From Proposition 2.2.1 we easily derive:

Proposition 6.2.1 *Any continuous distributive Σ-algebra is complete. Any morphism of continuous distributive Σ-algebras is a morphism of complete distributive Σ-algebras.*

Example 6.2.1. Let $\Sigma = \bigcup_{k\geq 0} \Sigma_k$, $\Sigma_k = \{\omega_k\}$, $k \geq 0$. Consider a semiring $\langle A, +, \cdot, 0, 1\rangle$ and define ω_k, $k \geq 0$, to be the following k-ary operations: the constant ω_0 is 1, the unary operation ω_1 is the identity mapping and the k-ary operation ω_k is the k-fold product, i. e., $\omega_k(a_1, \ldots, a_k) = a_1 \cdots a_k$, $k \geq 2$. Then $\langle A, +, 0, \Sigma\rangle$ is a distributive Σ-algebra. If $\langle A, +, \cdot, 0, 1\rangle$ is a continuous semiring then $\langle A, +, 0, \Sigma\rangle$ is a continuous distributive Σ-algebra. □

Example 6.2.2. Consider a semiring $\langle A, +, \cdot, 0, 1\rangle$. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{\omega\}$, $\Sigma_1 = \{\omega_a \mid a \in A\}$. Then the semiring A can be "simulated" by the distributive Σ-algebra $\langle A, +, 0, \Sigma\rangle$. Here ω is the constant 1 and, for all $a, b \in A$, $\omega_a(b) = a\cdot b$.

In addition to the laws of a distributive Σ-algebra, the following laws are satisfied for all $a, a_1, a_2, b \in A$:

$$\omega_{a_1}(\omega_{a_2}(b)) = \omega_{a_1 \cdot a_2}(b), \quad \omega_{a_1+a_2}(b) = \omega_{a_1}(b) + \omega_{a_2}(b),$$
$$\omega_0(b) = 0, \quad \omega_1(b) = b, \quad \omega_a(1) = a.$$

□

In the sequel, X will denote an *alphabet of leaf symbols*, disjoint from Σ. (Observe that an alphabet may be infinite.) By $T_\Sigma(X)$ we denote the set of *trees* formed over $\Sigma \cup X$. This set $T_\Sigma(X)$ is the smallest set formed according to the following conventions:

(i) if $\sigma \in \Sigma_0 \cup X$ then $\sigma \in T_\Sigma(X)$,

(ii) if $\sigma \in \Sigma_k$, $k \geq 1$, and $t_1, \ldots, t_k \in T_\Sigma(X)$ then $\sigma(t_1, \ldots, t_k) \in T_\Sigma(X)$.

If $\Sigma_0 \neq \emptyset$ then X may be the empty set ($\emptyset$ denotes the empty set).

If Σ is a finite ranked alphabet and X is a finite alphabet of leaf symbols, then $T_\Sigma(X)$ is generated by the context-free grammar $G = (\{S\}, \Sigma \cup X, P, S)$, where $P = \{S \to \omega(\underbrace{S, \ldots, S}_{k}) \mid \omega \in \Sigma_k,\ k \geq 1\} \cup \{S \to \omega \mid \omega \in \Sigma_0 \cup X\}$.

Sometimes it is more suggestive to employ a pictorial representation: The tree $\omega \in \Sigma_0 \cup X$ represents the rooted tree with just a single node labeled by ω; the tree $\omega(t_1, \ldots, t_k)$, $\omega \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(X)$, $k \geq 1$, represents the rooted ordered tree where the root is labeled by ω and has sons $t_1, \ldots, t_k$ (in this order).

The set $T_\Sigma(X)$ may be turned into a Σ-algebra by defining, for each $\sigma \in \Sigma_k$ and all $t_1, \ldots, t_k \in T_\Sigma(X)$, $\sigma_{T_\Sigma(X)}(t_1, \ldots, t_k)$ to be the tree $\sigma(t_1, \ldots, t_k)$. It is well-known that equipped with these operations, $T_\Sigma(X)$ is *freely generated by* X: Each function $h : X \to D$, where D is a Σ-algebra, extends to a unique Σ-algebra morphism $T_\Sigma(X) \to D$.

We now turn to formal tree series. They will form a distributive Σ-algebra. Let A be a semiring. Then we denote by $A\langle\langle T_\Sigma(X)\rangle\rangle$ the set of *formal tree series* over $T_\Sigma(X)$, i. e., the set of mappings $s : T_\Sigma(X) \to A$ written in the form $\sum_{t \in T_\Sigma(X)} (s, t)t$, where the *coefficient* (s, t) is the value of s for the tree $t \in T_\Sigma(X)$. For a formal tree series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$, we define the *support* of s, $\mathrm{supp}(s) = \{t \in T_\Sigma(X) \mid (s, t) \neq 0\}$. By $A\langle T_\Sigma(X)\rangle$ we denote the set of tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ that have finite support. A power series with finite support is called *polynomial*.

We first define, for $s_1, s_2 \in A\langle\langle T_\Sigma(X)\rangle\rangle$, the sum $s_1 + s_2 \in A\langle\langle T_\Sigma(X)\rangle\rangle$ by

$$s_1 + s_2 = \sum_{t \in T_\Sigma(X)} ((s_1, t) + (s_2, t))t\,.$$

The zero tree series 0 is defined to be the tree series having all coefficients equal to 0. Clearly, $\langle A\langle\langle T_\Sigma(X)\rangle\rangle, +, 0\rangle$ is a commutative monoid.

For $\omega \in \Sigma_k$, $k \geq 0$, we define the mapping $\bar{\omega} : (A\langle\langle T_\Sigma(X)\rangle\rangle)^k \to A\langle\langle T_\Sigma(X)\rangle\rangle$ by

$$\bar{\omega}(s_1, \ldots, s_k) = \sum_{t_1,\ldots,t_k \in T_\Sigma(X)} (s_1, t_1) \ldots (s_k, t_k)\omega(t_1, \ldots, t_k)\,,$$

$s_1, \ldots, s_k \in A\langle\langle T_\Sigma(X)\rangle\rangle$.

Clearly, $\langle A\langle\langle T_\Sigma(X)\rangle\rangle, +, 0, \bar{\Sigma}\rangle$, where $\bar{\Sigma} = (\bar{\omega} \mid \omega \in \Sigma)$, is a distributive Σ-algebra, as is $\langle A\langle T_\Sigma(X)\rangle, +, 0, \bar{\Sigma}\rangle$ with the same operations. If A is (naturally) ordered (resp. complete or continuous) then $A\langle\langle T_\Sigma(X)\rangle\rangle$ is again a (naturally) ordered (resp. complete or continuous) distributive Σ-algebra. The order on $A\langle\langle T_\Sigma(X)\rangle\rangle$ is the pointwise order. Also, when A is ordered, $A\langle T_\Sigma(X)\rangle$ is an ordered distributive Σ-algebra.

Example 6.2.3. Formal tree series have the advantage that the coefficient of a tree in a series can be used to give information about some quantity connected with that tree.

(i) (See Example 2.1 of Berstel, Reutenauer [6].) Define the height $h(t)$ of a tree t in $T_\Sigma(X)$ as follows:

$$h(t) = \begin{cases} 0 & \text{if } t \in \Sigma_0 \cup X, \\ 1 + \max\{h(t_i) \mid 1 \leq i \leq k\} & \text{if } t = \omega(t_1, \ldots, t_k),\ k \geq 1. \end{cases}$$

Now height is a formal tree series in $\mathbb{N}\langle\langle T_\Sigma(X)\rangle\rangle$ defined as

$$\text{height} = \sum_{t \in T_\Sigma(X)} h(t)t\,.$$

(ii) Consider formal tree series s in $\mathbb{R}_+\langle\langle T_\Sigma(X)\rangle\rangle$ such that $0 \leq (s,t) \leq 1$ for all $t \in T_\Sigma(X)$. Then (s,t) can be interpreted as a probability associated with the tree t. Here $\mathbb{R}_+$ is the semiring of nonnegative reals.

(iii) Consider formal tree series s in $\mathbb{N}^\infty\langle\langle T_\Sigma(X)\rangle\rangle$, where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$. Then the coefficient (s,t) of $t \in T_\Sigma(X)$ can be interpreted as the number (possibly ∞) of distinct computations of t by some mechanism. (See Theorem 3.1.)

More examples can be found in Berstel, Reutenauer [6]. □

We now exhibit a universal property of the above constructions. Note that $A\langle\langle T_\Sigma(X)\rangle\rangle$ may be equipped with a *scalar multiplication* $A \times A\langle\langle T_\Sigma(X)\rangle\rangle \to A\langle\langle T_\Sigma(X)\rangle\rangle$, $(a,s) \mapsto as$, defined by $(as,t) = a(s,t)$, for all $t \in T_\Sigma(X)$. When $s \in A\langle T_\Sigma(X)\rangle$, then also $as \in A\langle T_\Sigma(X)\rangle$. This operation satisfies the following equations:

$$a(bs) = (ab)s \tag{6.1}$$
$$1s = s \tag{6.2}$$
$$(a+b)s = as + bs \tag{6.3}$$
$$a(s+s') = as + as' \tag{6.4}$$
$$a0 = 0\,, \tag{6.5}$$

for all $a, b \in A$ and $s, s' \in A\langle\langle T_\Sigma(X)\rangle\rangle$. It follows that

$$0s \;=\; 0\,,$$

for all s. Moreover, when A is commutative, we also have that

$$\bar{\omega}(a_1 s_1, \ldots, a_k s_k) \;=\; a_1 \ldots a_k \bar{\omega}(s_1, \ldots, s_k)\,, \tag{6.6}$$

for all $\omega \in \Sigma_k$, $k \geq 0$, and for all $a_i \in A$, $s_i \in A\langle\langle T_\Sigma(X)\rangle\rangle$, $1 \leq i \leq k$.

Theorem 6.2.2 *Suppose that A is a commutative semiring and D is a distributive Σ-algebra equipped with a scalar multiplication $A \times D \to D$, $(a, d) \mapsto ad$, which satisfies the equations (6.1)–(6.6). Then any function $\varphi : X \to D$ extends to a unique distributive Σ-algebra morphism $\varphi^\sharp : A\langle T_\Sigma(X)\rangle \to D$ preserving scalar multiplication.*

Proof. It is well-known that φ extends to a unique Σ-algebra morphism $\bar{\varphi}$: $T_\Sigma(X) \to D$. We further extend $\bar{\varphi}$ to $\varphi^\sharp$ by defining

$$\varphi^\sharp(s) = \sum_{t \in T_\Sigma(X)} (s, t)\bar{\varphi}(t)\,,$$

for all $s \in A\langle T_\Sigma(X)\rangle$. It is a routine matter to show that $\varphi^\sharp$ extends φ and is a distributive Σ-algebra morphism that preserves scalar multiplication. Since the definition of $\varphi^\sharp$ was forced, the extension is unique. □

A similar result holds when A is a complete semiring, so that $A\langle\langle T_\Sigma(X)\rangle\rangle$ is a complete distributive Σ-algebra.

Theorem 6.2.3 *Suppose that A is a complete commutative semiring and D is a complete distributive Σ-algebra equipped with a scalar multiplication $A \times D \to D$, $(a, d) \mapsto ad$, which satisfies the equations (6.1)–(6.6). Moreover, assume that*

$$(\sum_{i \in I} a_i)d \;=\; \sum_{i \in I} a_i d \tag{6.7}$$

$$a \sum_{i \in I} d_i \;=\; \sum_{i \in I} a d_i\,, \tag{6.8}$$

for all $a, a_i \in A$ and $d, d_i \in D$, $i \in I$, where I is any index set. Then any function $\varphi : X \to D$ extends to a unique complete distributive Σ-algebra morphism $\varphi^\sharp : A\langle\langle T_\Sigma(X)\rangle\rangle \to D$ preserving scalar multiplication.

Proof. The proof of this result parallels that of Theorem 2.2. First we extend φ to $\overline{\varphi} : T_\Sigma(X) \to D$, and then define

$$\varphi^\sharp(s) \;=\; \sum_{t \in T_\Sigma(X)} (s, t)\overline{\varphi}(t),$$

for all $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$. This sum makes sense since D is complete. The details of the proof that $\varphi^\sharp$ is a complete distributive Σ-algebra homomorphism preserving scalar multiplication are routine. The definition of $\varphi^\sharp$ was again forced. □

When A is ordered by $\leq$, we may order $A\langle\langle T_\Sigma(X)\rangle\rangle$, and thus $A\langle T_\Sigma(X)\rangle$, by the pointwise order: We define $s \leq s'$ for $s, s' \in A\langle\langle T_\Sigma(X)\rangle\rangle$ iff $(s,t) \leq (s',t)$ for all $t \in T_\Sigma(X)$. Equipped with this order, both $A\langle\langle T_\Sigma(X)\rangle\rangle$ and $A\langle T_\Sigma(X)\rangle$ are ordered distributive Σ-algebras. Moreover, scalar multiplication preserves the order in both arguments. Finally, if A is a continuous semiring, then $A\langle\langle T_\Sigma(X)\rangle\rangle$ is also continuous, and scalar multiplication preserves least upper bounds of directed sets in both arguments.

Corollary 6.2.4 *Suppose that A is an ordered commutative semiring and D is an ordered distributive Σ-algebra equipped with an order preserving scalar multiplication $A \times D \to D$, $(a,d) \mapsto ad$, which satisfies the equations (6.1)–(6.6). Then any function $\varphi : X \to D$ extends to a unique distributive Σ-algebra morphism $\varphi^\sharp : A\langle T_\Sigma(X)\rangle \to D$ preserving scalar multiplication. Moreover, when A is a continuous commutative semiring and D is a continuous distributive Σ-algebra equipped with a continuous scalar multiplication $A \times D \to D$, $(a,d) \mapsto ad$, which satisfies the above equations, then any function $\varphi : X \to D$ extends to a unique continuous distributive Σ-algebra morphism $\varphi^\sharp : A\langle\langle T_\Sigma(X)\rangle\rangle \to D$ preserving scalar multiplication.*

In the sequel, A will denote a *continuous (complete) commutative semiring* where sums are defined by Proposition 2.2.1. Let s be a formal tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$, and let D denote a continuous distributive Σ-algebra equipped with a scalar multiplication $A \times D \to D$ satisfying (6.1)–(6.6) which is also continuous. The set D^X of all functions $X \to D$ is also a continuous distributive Σ-algebra by the pointwise operations and ordering as is the set of all continuous functions $D^X \to D$. Moreover, it is equipped with the pointwise scalar multiplication which again satisfies (6.1)–(6.6) and is continuous. Now s induces a mapping $s^D : D^X \to D$, $h \mapsto h^\sharp(s)$ for $h \in D^X$.

Proposition 6.2.5 *The function s^D is continuous. Moreover, the assignment $s \to s^D$ defines a continuous function of s.*

Proof. It is known that when $t \in T_\Sigma(X)$, then the function $t^D : D^X \to D$ induced by t is continuous, since it can be constructed from continuous functions (namely, the projections and the continuous operations of D corresponding to the symbols in Σ) by function composition, see, e.g., Guessarian [61]. Since scalar multiplication and $+$ are continuous, so is any function induced by a series in $A\langle T_\Sigma(X)\rangle$. But s^D is the pointwise supremum of the functions induced by the polynomials $\sum_{t \in F}(s,t)t$, where F is a finite subset of $T_\Sigma(X)$. Since the pointwise supremum of continuous functions is continuous, see Guessarian [61], the result follows.

To show that the assignment $s \mapsto s^D$ defines a continuous function, let S denote a directed set in $A\langle\langle T_\Sigma(X)\rangle\rangle$. We need to prove that

$$(\sup_{s \in S} s)^D = \sup_{s \in S} s^D.$$

But for any $h : X \to D$,

$$\begin{aligned}(\sup_{s\in S} s)^D(h) &= h^\sharp(\sup_{s\in S} s)\\ &= \sup_{s\in S} h^\sharp(s)\\ &= \sup_{s\in S} s^D(h)\\ &= (\sup_{s\in S} s^D)(h).\end{aligned}$$

□

From now on we will write just h for $h^\sharp$ and denote s^D by just s.

In particular, formal tree series induce continuous mappings called *substitutions* as follows. Let Y denote a non-empty set of variables, where $Y \cap (\Sigma \cup X) = \emptyset$, and consider a mapping $h : Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. This mapping can be extended to a mapping $h : T_\Sigma(X \cup Y) \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ by setting first $h(x) = x$, $x \in X$. Now, by the above result, for any series $s \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, the mapping $h \mapsto h(s)$ is a continuous function of h. By the arguments outlined above, $h(s)$ can be constructed as follows. First, extend h to trees by defining

$$h(\omega(t_1,\ldots,t_k)) = \bar{\omega}(h(t_1),\ldots,h(t_k)) = \\ \textstyle\sum_{t'_1,\ldots,t'_k\in T_\Sigma(X\cup Y)}(h(t_1),t'_1)\ldots(h(t_k),t'_k)\omega(t'_1,\ldots,t'_k)\,,$$

for $\omega \in \Sigma_k$ and $t_1,\ldots,t_k \in T_\Sigma(X \cup Y)$, $k \geq 0$. One more extension of h yields a mapping $h : A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ by defining $h(s) = \sum_{t\in T_\Sigma(X\cup Y)}(s,t)h(t)$. Now $s(h)$ is just the value of this extended function on s. If $Y = \{y_1,\ldots,y_n\}$ is finite, we use the following notation: $h : Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, where $h(y_i) = s_i$, $1 \leq i \leq n$, is denoted by $(s_i,\ 1 \leq i \leq n)$ or $(s_1,\ldots,s_n)$ and the value of s with argument h is denoted by $s(s_i,\ 1 \leq i \leq n)$ or $s(s_1,\ldots,s_n)$. Intuitively, this is simply the *substitution* of the formal tree series $s_i \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ into the variables y_i, $1 \leq i \leq n$, of $s \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. By Proposition 2.5, the mapping $s : (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, i. e., the substitution of formal tree series into the variables of Y, is a continuous mapping. Moreover, $s(s_1,\ldots,s_n)$ is also continuous in s. (So it is continuous in s and in each s_i.) Observe that $s(s_1,\ldots,s_n) = \sum_{t\in T_\Sigma(X\cup Y)}(s,t)t(s_1,\ldots,s_n)$.

In certain situations, formulae are easier to read if we use the notation $s[s_i/y_i,\ 1 \leq i \leq n]$ for the substitution of the formal tree series s_i into the variables y_i, $1 \leq i \leq n$, of s instead of the notation $s(s_i,\ 1 \leq i \leq n)$. So we will use sometimes this notation $s[s_i/y_i,\ 1 \leq i \leq n]$.

In the same way, $s \in A\langle\langle T_\Sigma(X\cup Y)\rangle\rangle$ also induces a mapping $s : (A\langle\langle T_\Sigma(X)\rangle\rangle)^Y \to A\langle\langle T_\Sigma(X)\rangle\rangle$.

Our substitution on formal tree series is a generalization of the OI-substitutions on formal tree languages. We do not consider generalizations of the IO-substitution. Bozapalidis [15], Engelfriet, Fülöp, Vogler [33] and Fülöp, Vogler [50] consider these generalizations to formal tree series. For precise definitions of the OI- and IO-substitutions see Engelfriet, Schmidt [34], Definition 2.1.1.

The construction of tree series and the above freeness results can be generalized to a great extent. Suppose that D is any Σ-algebra and A is any complete semiring. Then the set of functions $D \to A$, denoted $A\langle\langle D\rangle\rangle$, is a complete distributive Σ-algebra. We call the elements of $A\langle\langle D\rangle\rangle$ series and denote them as $\sum_{d\in D}(s,d)d$, or $\sum_{d\in \mathrm{supp}(s)}(s,d)d$. The sum of any family of series is their pointwise sum. The zero series serves as zero. Moreover, for each $\omega \in \Sigma_k$, $k \geq 1$, and for each $s_1,\ldots,s_k \in A\langle\langle D\rangle\rangle$,

$$\omega(s_1,\ldots,s_k) = \sum_{d\in D}(\sum_{d=\omega(d_1,\ldots,d_k)}(s_1,d_1)\ldots(s_k,d_k))d\,.$$

Note also that $A\langle\langle D\rangle\rangle$ is equipped with a scalar multiplication $A \times A\langle\langle D\rangle\rangle \to A\langle\langle D\rangle\rangle$. Note that equations (6.1)–(6.6) and (6.7), (6.8) hold. When A is a continuous semiring then, equipped with the pointwise order, $A\langle\langle D\rangle\rangle$ is a continuous distributive Σ-algebra and scalar multiplication is continuous. We are now ready to state the promised generalization of Theorem 2.3.

Theorem 6.2.6 *Suppose that A is a complete commutative semiring and D' is a distributive Σ-algebra equipped with a scalar multiplication $A\times D' \to D'$ which satisfies the equations (6.1)–(6.6) as well as (6.7) and (6.8). Moreover, assume that D is a Σ-algebra. Then any Σ-algebra morphism $\varphi : D \to D'$ extends to a unique complete distributive Σ-algebra morphism $\varphi^\sharp : A\langle\langle D\rangle\rangle \to D'$ preserving scalar multiplication. When A is a continuous commutative semiring and D' is a continuous distributive Σ-algebra and the scalar multiplication $A \times D' \to D'$ is continuous, then so is the function $\varphi^\sharp$.*

Proof. Given φ, we are forced to define

$$\varphi^\sharp(s) \quad = \quad \sum_{d\in D}(s,d)\varphi(d), \tag{6.9}$$

for all $s \in A\langle\langle D\rangle\rangle$. On the other hand, it is a routine matter to verify that (6.9) defines a complete distributive Σ-algebra morphism $\varphi^\sharp : A\langle\langle D\rangle\rangle \to D'$ that extends φ.

Suppose now that A and D are continuous and that the scalar multiplication $A\times D' \to D'$ is also continuous. In order to prove that $\varphi^\sharp$ is continuous, suppose that S is a directed set in $A\langle\langle D\rangle\rangle$. Then for each $d \in D$, the set $\{(s,d) : s \in S\}$ is also directed, moreover, $(\sup S, d) = \sup_{s\in S}(s,d)$. Using this, and the continuity of scalar multilication and summation, we have that

$$\begin{aligned}
\varphi^\sharp(\sup S) \quad &= \quad \sum_{d\in D}(\sup S,d)\varphi(d)\\
&= \quad \sum_{d\in D}(\sup_{s\in S}(s,d))\varphi(d)\\
&= \quad \sum_{d\in D}\sup_{s\in S}(s,d)\varphi(d)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{s\in S} \sum_{d\in D} (s,d)\varphi(d) \\
&= \sup\{\varphi^\sharp(s) : s \in S\} \\
&= \sup \varphi^\sharp(S).
\end{aligned}
$$

□

Theorem 2.2 can be generalized in the same way. For more on series over Σ-algebras we refer the reader to Kuich [84, 86].

In the sequel, Y, Y', Z will denote sets of variables that are disjoint from Σ and X, and Y_k, $k \geq 1$, will denote the set of variables $\{y_1, \ldots, y_k\}$. Moreover, $Y_0 = \emptyset$. Furthermore, I and I' will denote arbitrary index sets.

Given a set S, $S^{I_1\times I_2}$ will denote the set of matrices indexed by $I_1 \times I_2$ with entries in S. (E. g., $(A\langle\langle T_\Sigma(X)\rangle\rangle)^{I'\times I^k}$ denotes the set of matrices M, such that the $(i,(i_1,\ldots,i_k))$-entry of M is in $A\langle\langle T_\Sigma(X)\rangle\rangle$, $i \in I'$, $i_1,\ldots,i_k \in I$.) A matrix $M \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I_1\times I_2}$ is *row finite* iff, for all $i_1 \in I$, $M_{i_1,i_2} \neq 0$ for only finitely many $i_2 \in I$.

Our tree automata will be defined by transition matrices. A *matrix* $M \in (A\langle\langle T_\Sigma(X\cup Y_k)\rangle\rangle)^{I'\times I^k}$, $k \geq 1$, I' and I arbitrary index sets, *induces a mapping*

$$M : (A\langle\langle T_\Sigma(X\cup Y')\rangle\rangle)^{I\times 1} \times \ldots \times (A\langle\langle T_\Sigma(X\cup Y')\rangle\rangle)^{I\times 1} \to (A\langle\langle T_\Sigma(X\cup Y')\rangle\rangle)^{I'\times 1}$$

(there are k argument vectors), defined by the entries of the resulting vector as follows: For $P_1,\ldots,P_k \in (A\langle\langle T_\Sigma(X\cup Y')\rangle\rangle)^{I\times 1}$ we define, for all $i \in I'$,

$$
\begin{aligned}
&M(P_1,\ldots,P_k)_i = \textstyle\sum_{i_1,\ldots,i_k\in I} M_{i,(i_1,\ldots,i_k)}((P_1)_{i_1},\ldots,(P_k)_{i_k}) = \\
&\textstyle\sum_{i_1,\ldots,i_k\in I} \sum_{t\in T_\Sigma(X\cup Y_k)} (M_{i,(i_1,\ldots,i_k)},t)t((P_1)_{i_1},\ldots,(P_k)_{i_k}) .
\end{aligned}
$$

Hence, $M(P_1,\ldots,P_k)_i$ is defined to be the result of substituting the components $(P_1)_{i_1},\ldots,(P_k)_{i_k}$ of $P_1,\ldots,P_k$ for $y_1,\ldots,y_k$, respectively, in $M_{i,(i_1,\ldots,i_k)}$, and then summing over all possible $(i_1,\ldots,i_k) \in I^k$.

In the next theorem the *Kronecker symbol* $\delta_{i,j} \in A$ over I is used: for $i, j \in I$, $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Theorem 6.2.7 *Let* $M \in (A\langle\langle T_\Sigma(X\cup Y_k)\rangle\rangle)^{I'\times I^k}$, $k \geq 1$. *Define* $\bar{M} \in (A\langle\langle T_\Sigma(X\cup Y_k)\rangle\rangle)^{I'\times I^m}$, $m > k$, *by*

$$\bar{M}_{i,(i_1,\ldots,i_m)} = \delta_{i,i_m}\delta_{i_m,i_{m-1}}\cdots\delta_{i_{k+2},i_{k+1}}M_{i,(i_1,\ldots,i_k)} ,$$

for $i \in I'$, $i_1,\ldots,i_m \in I$.

Then, for $P_1,\ldots,P_m \in (A\langle\langle T_\Sigma(X\cup Y')\rangle\rangle)^{I\times 1}$,

$$\bar{M}(P_1,\ldots,P_m) = M(P_1,\ldots,P_k) .$$

Proof.

$$
\begin{aligned}
&\bar{M}(P_1,\ldots,P_m)_i = \\
&\textstyle\sum_{i_1,\ldots,i_m\in I} \bar{M}_{i,(i_1,\ldots,i_m)}((P_1)_{i_1},\ldots,(P_m)_{i_m}) = \\
&\textstyle\sum_{i_1,\ldots,i_m\in I} \delta_{i,i_m}\delta_{i_m,i_{m-1}}\cdots\delta_{i_{k+2},i_{k+1}}M_{i,(i_1,\ldots,i_k)}((P_1)_{i_1},\ldots,(P_k)_{i_k}) = \\
&\textstyle\sum_{i_1,\ldots,i_k\in I} M_{i,(i_1,\ldots,i_k)}((P_1)_{i_1},\ldots,(P_k)_{i_k}) = \\
&M(P_1,\ldots,P_k)_i, \quad i \in I' .
\end{aligned}
$$

□

For the definition of the tree series transducers we will need a generalization of the substitution defined by a matrix in $(A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I' \times I^k}$, $k \geq 1$. A matrix $M \in (A\langle\langle T_\Sigma(X \cup Y_m)\rangle\rangle)^{I' \times (I \times Z_k)^m}$, $Z_k = \{z_1, \ldots, z_k\}$, $k \geq 1$, *induces a mapping*

$$M : (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{I \times 1} \times \cdots \times (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{I \times 1} \to (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{I' \times 1}$$

(there are k argument vectors) defined by the entries of the resulting vector as follows: For $P_1, \ldots, P_k \in (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{I \times 1}$ we define, for all $i \in I'$,

$$M[P_1, \ldots, P_k]_i = \sum_{i_1, \ldots, i_m \in I,\ 1 \leq j_1, \ldots, j_m \leq k} M_{i,((i_1, z_{j_1}), \ldots, (i_m, z_{j_m}))}((P_{j_1})_{i_1}, \ldots, (P_{j_m})_{i_m})\,.$$

Theorem 6.2.8 *Let* $C = \{((i_1, z_1), \ldots, (i_k, z_k)) \mid i_1, \ldots, i_k \in I\}$ *and* $M \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I' \times (I \times Z_k)^k}$, $k \geq 1$, *such that* $M_{i,\alpha} = 0$ *for* $i \in I'$ *and* $\alpha \in (I \times Z_k)^k - C$. *Define* $\bar{M} \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I' \times I^k}$ *by* $\bar{M}_{i,(i_1, \ldots, i_k)} = M_{i,((i_1, z_1), \ldots, (i_k, z_k))}$ *for* $i \in I'$, $i_1, \ldots, i_k \in I$. *Then, for* $P_1, \ldots, P_k \in (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{I \times 1}$,

$$M(P_1, \ldots, P_k) = \bar{M}[P_1, \ldots, P_k]\,.$$

Proof.

$$\begin{aligned}
&M[P_1, \ldots, P_k]_i = \\
&\textstyle\sum_{i_1, \ldots, i_k \in I} M_{i,((i_1, z_1), \ldots, (i_k, z_k))}((P_1)_{i_1}, \ldots, (P_k)_{i_k}) = \\
&\textstyle\sum_{i_1, \ldots, i_k \in I} \bar{M}_{i,(i_1, \ldots, i_k)}((P_1)_{i_1}, \ldots, (P_k)_{i_k}) = \\
&\bar{M}(P_1, \ldots, P_k)_i, \quad i \in I'\,.
\end{aligned}$$

□

6.3 Tree automata and systems of equations

In this section we define tree automata and systems of equations (over semirings). These notions are a framework for considering finite tree automata and pushdown tree automata, and polynomial systems. The definitions are slightly modified from Kuich [80, 85]. The main result of this section is that (finite, polynomial) tree automata and (finite, polynomial) systems are equivalent mechanisms.

Our tree automata are a generalization of the nondeterministic root-to-frontier tree recognizers. (See Gécseg, Steinby [51, 52] and Kuich [80].) A *tree automaton* (with input alphabet Σ and leaf alphabet X over the semiring A)

$$\mathfrak{A} = (I, M, S, P)$$

is given by

(i) a non-empty set I of *states*,

(ii) a *transition matrix* $M \in (A\langle\langle T_\Sigma(X \cup Y_m)\rangle\rangle)^{I\times I^m}$, for some $m \geq 1$,

(iii) a row finite row vector $S \in (A\langle\langle T_\Sigma(X \cup Y_1)\rangle\rangle)^{1\times I}$, called the *initial state vector*,

(iv) a column vector $P \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, called the *final state vector*.

The *approximation sequence* $(\sigma^j \mid j \in \mathbb{N})$, $\sigma^j \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, $j \geq 0$, *associated with* $\mathfrak{A}$ is defined as follows:

$$\sigma^0 = 0, \qquad \sigma^{j+1} = M(\sigma^j, \ldots, \sigma^j) + P, \quad j \geq 0\,.$$

(There are m argument vectors σ^j.) The *behavior* $||\mathfrak{A}|| \in A\langle\langle T_\Sigma(X)\rangle\rangle$ of the tree automaton $\mathfrak{A}$ is defined by

$$||\mathfrak{A}|| = \sum_{i\in I} S_i(\sigma_i) = S(\tau)\,,$$

where $\tau \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$ is the least upper bound of the approximation sequence associated with $\mathfrak{A}$. Since $\sigma^j \leq \sigma^{j+1}$ for all j (by continuity of substitution), and since $(A\langle\langle T_\Sigma(X \cup Y_m)\rangle\rangle)^{I\times I^m}$ has all directed least upper bounds with respect to the pointwise order, it follows that this least upper bound and, hence, the behavior of $\mathfrak{A}$ exist.

Observe that Σ may be infinite and there may be no bound on the rank of symbols in Σ. But in any case, only finitely many variables $y_1, \ldots, y_m$ are allowed in the entries of M.

Our tree automata are slightly modified from Kuich [85]. In Kuich [85] a tree automaton (with a finite sequence of transition matrices) $\mathfrak{A}' = (I, M', S, P)$ is defined as here, except that M' is a *sequence* of transition matrices $M' = (M'_k \mid 1 \leq k \leq m)$, where $M_k \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I\times I^k}$, $1 \leq k \leq m$, and the approximation sequence $(\sigma'^j \mid j \in \mathbb{N})$, $\sigma'^j \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, $j \geq 0$, is defined by

$$\sigma'^0 = 0, \qquad \sigma'^{j+1} = \sum_{1\leq k\leq m} M'_k(\sigma'^j, \ldots, \sigma'^j) + P\,.$$

(There are k argument vectors σ'^j in M'_k.) Given a tree automaton $\mathfrak{A}' = (I, M', S, P)$ according to Kuich [85], we construct an equivalent tree automaton $\mathfrak{A} = (I, M, S, P)$ according to our definition: We define $M_k \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I\times I^m}$, $1 \leq k \leq m$, by its entries

$$(M_k)_{i,(i_1,\ldots,i_m)} = \delta_{i,i_m}\delta_{i_m,i_{m-1}}\delta_{i_{k+2},i_{k+1}}(M'_k)_{i,(i_1,\ldots,i_k)}\,, \qquad i, i_1, \ldots, i_m \in I\,,$$

and $M \in (A\langle\langle T_\Sigma(X \cup Y_m)\rangle\rangle)^{I\times I^m}$ by

$$M = \sum_{1\leq k\leq m} M_k\,.$$

We claim that the approximation sequences of $\mathfrak{A}$ and $\mathfrak{A}'$ coincide, i. e., $\sigma^j = \sigma'^j$ for all $j \geq 0$, and prove it by induction on j. The case $j = 0$ being clear, we proceed by $j > 0$. For all $i \in I$, we obtain by Theorem 2.7

$$\sigma_i^j = (M(\sigma^{j-1}, \ldots, \sigma^{j-1}) + P)_i = \sum_{1 \leq k \leq m} M_k(\sigma^{j-1}, \ldots, \sigma^{j-1})_i + P_i =$$
$$\sum_{1 \leq k \leq m} M'_k(\sigma'^{j-1}, \ldots, \sigma'^{j-1})_i + P_i = \sigma_i'^j .$$

Hence, we have proved $||\mathfrak{A}|| = ||\mathfrak{A}'||$. We state this as a *Remark*.

Remark. The definitions of tree automata given here and in Kuich [85] (with a finite sequence of transition matrices) are equivalent with respect to the behaviors of these tree automata. □

A tree automaton $\mathfrak{A} = (I, M, S, P)$ is called *finite* iff I is finite. A tree automaton $\mathfrak{A} = (I, M, S, P)$ is called *simple* iff the entries of the transition matrix M, of the initial state vector S and of the final state vector P have the following specific forms:

(i) the entries of M are of the form $\sum_{1 \leq k \leq m} \sum_{\omega \in \Sigma_k} a_\omega \omega(y_1, \ldots, y_k) + \sum_{\omega \in \Sigma_0 \cup X} a_\omega \omega + a y_1$, $a_\omega, a \in A$;

(ii) the entries of P are of the form $\sum_{\omega \in \Sigma_0 \cup X} a_\omega \omega$, $a_\omega \in A$;

(iii) the entries of S are of the form $d y_1$, $d \in A$.

It is called *proper* iff there are no terms $a y_1$ in (i). Observe that the term $a y_1$ in (i) corresponds to ε-moves in ordinary automata.

Intuitively, a simple tree automaton $\mathfrak{A}$ recognizes a tree $t \in T_\Sigma(X)$ with coefficient $(||\mathfrak{A}||, t)$ as follows in a nondeterministic way.

At the root of t, $\mathfrak{A}$ may be in any *initial state* $i \in I$, i.e., in a state with $(S, i) \neq 0$. We now describe a *computation* starting in the initial state $i_0 \in I$ and its *weight*. If in the recognition procedure $\mathfrak{A}$ analyzes the root of a subtree of t of the form $\omega(t_1, \ldots, t_k)$, $\omega \in \Sigma_k$, $k \geq 1$, in state i and $(M_{i,(i_1,\ldots,i_m)}, \omega(y_1, \ldots, y_k)) = a_i \neq 0$ then $\mathfrak{A}$ proceeds in parallel in states $i_1, \ldots, i_k$ at the roots of $t_1, \ldots, t_k$, respectively. If, in the recognition procedure, $\mathfrak{A}$ analyzes a leaf of t labelled by $\omega \in \Sigma_0 \cup X$ in state i and $(P_i, \omega) = a_i \neq 0$ or $(M_{i,(i_1,\ldots,i_m)}, \omega) = a_i \neq 0$ for some $i_1, \ldots, i_m \in I$, then $\mathfrak{A}$ terminates this branch of its computation. If, in the recognition procedure, $\mathfrak{A}$ analyzes the root of a subtree of t in state i and $(M_{i,(i_1,\ldots,i_m)}, y_1) = a_i \neq 0$ then $\mathfrak{A}$ moves to state i_1 and analyzes again the root of this subtree.

The weight of such a computation starting in the initial state i_0 is (S, i_0) multiplied with all the semiring elements a_i occuring in the procedure described above. The coefficient $(||\mathfrak{A}||, t)$ is then the sum of all weights of all possible computations.

Consider the case that A is the semiring $\mathbb{N}^\infty$, i. e., we consider tree series in $\mathbb{N}^\infty\langle\langle T_\Sigma(X)\rangle\rangle$. A simple tree automaton is called 1-*simple* iff all the coefficients a_ω, a in (i), a_ω in (ii) and d in (iii) belong to $\{0, 1\}$. By Seidl [109], Proposition 3.1, the coefficient $(||\mathfrak{A}||, t)$ of the behavior of $\mathfrak{A}$ is the number (possibly ∞) of distinct computations for t.

Theorem 6.3.1 *Consider a 1-simple tree automaton $\mathfrak{A}$ and let $d(t)$, $t \in T_\Sigma(X)$, be the number (possibly ∞) of distinct computations of $\mathfrak{A}$ for t. Then*

$$||\mathfrak{A}|| = \sum_{t \in T_\Sigma(X)} d(t)t \in \mathbb{N}^\infty\langle\langle T_\Sigma(X)\rangle\rangle .$$

We now turn to systems.

A *system* (with variables in $Z = \{z_i \mid i \in I\}$) is a system of formal equations $z_i = p_i$, $i \in I$, I an arbitrary index set, where each p_i is in $A\langle\langle T_\Sigma(X \cup Z_i)\rangle\rangle$. Here Z_i is, for each $i \in I$, a finite subset of Z with $|Z_i| \leq m$ for some $m \geq 0$. The system can be written in matrix notation as $z = p(z)$. Here z and p denote vectors, whose i-th component is z_i and p_i, $i \in I$, respectively. A *solution* to the system $z = p(z)$ is given by $\sigma \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$ such that $\sigma_i = p_i[\sigma/z]$, $i \in I$. A solution σ of $z = p(z)$ is called *least* solution iff $\sigma \leq \tau$ for all solutions τ of $z = p(z)$.

The *approximation sequence* $(\sigma^j \mid j \in \mathbb{N})$, $\sigma^j \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, $j \geq 0$, *associated with the system* $z = p(z)$ is defined as follows:

$$\sigma_i^0 = 0, \qquad \sigma_i^{j+1} = p_i[\sigma/z], \quad j \geq 0 .$$

Since $\sigma^j \leq \sigma^{j+1}$ for all j, and since $(A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$ has least upper bounds of all directed sets, the least upper bound $\sigma = \sup(\sigma^j \mid j \in \mathbb{N})$ of this approximation sequence exists. Moreover, it is the least solution of the system $z = p(z)$.

Our systems are a generalization of the systems of linear equations of Berstel, Reutenauer [6]: we allow infinitely many equations and the right sides of the equations are tree series instead of simple tree polynomials. A system $z_i = p_i$, $i \in I$, is called *proper* iff $(p_i, z_j) = 0$ for all $z_j \in Z_i$, $i \in I$. It is called *finite* iff I is finite.

Theorem 6.3.2 *For each system there exists a proper one with the same least solution. A proper system has a unique solution.*

Proof. The construction is as follows. Consider a system $z = p$ as defined above. Write it in the form $z = Mz + r$, where $M \in A^{I\times I}$ and $(r_i, z_j) = 0$ for $z_j \in Z_i$, $i \in I$. Then, by the diagonal equation (Proposition 2.2.10), the systems $z = Mz + r$ and $z = M^*r$ have the same least solution and, by construction, $z = M^*r$ is a proper system. A modification of the proof of Proposition 6.1 of Berstel, Reutenauer [6] proves the second sentence of our theorem. Clearly, this unique solution is at the same time the least solution. □

We now show that tree automata and systems are mechanisms of equal power. For a given tree automaton $\mathfrak{A} = (I, M, S, P)$ as defined above we construct the system with variables in $Z = \{z_i \mid i \in I\}$

$$z_i = \sum_{i_1,\ldots,i_m \in I} M_{i,(i_1,\ldots,i_m)}(z_{i_1}, \ldots, z_{i_m}) + P_i , \quad i \in I .$$

Here we have substituted the variables $z_{i_1}, \ldots, z_{i_m}$ for the variables $y_1, \ldots, y_m$ in $M_{i,(i_1,\ldots,i_m)}(y_1, \ldots, y_m)$. In matrix notation, this system can be written as

$$z = M(z, \ldots, z) + P\,.$$

Here z is an $I \times 1$-vector whose i-th component is the variable z_i, $i \in I$.

As before, the approximation sequences associated with this system and to the tree automaton $\mathfrak{A}$ coincide. Consider now the system with variables in $\{z_0\} \cup Z$

$$z_0 = \sum_{i \in I} S_i(z_i), \qquad z = M(z, \ldots, z) + P\,.$$

Then the z_0-component of its least solution is equal to $||\mathfrak{A}||$.

Conversely, consider a system $z = p(z)$ as defined above. Let $Z_i = \{z_{i_1}, \ldots, z_{i_k}\}$, $i \in I$, and $p_i = p_i(z_{i_1}, \ldots, z_{i_k})$, $k \le m$. Construct now the tree automaton $\mathfrak{A} = (Z, M, S, 0)$, where, for all $i, j_1, \ldots, j_m \in I$,

$$M_{z_i,(z_{j_1},\ldots,z_{j_m})}(y_1, \ldots, y_m) = \delta_{i,j_m}\delta_{j_m,j_{m-1}} \cdots \delta_{j_{k+2},j_{k+1}}\delta_{j_k,i_k} \cdots \delta_{j_1,i_1} p_i(y_1, \ldots, y_k)\,.$$

Moreover, choose a $z_{i_0} \in Z$ and let $S_{z_{i_0}}(y_1) = y_1$, $S_{z_i}(y_1) = 0$ for $z_i \ne z_{i_0}$.

Let $(\sigma^j \mid j \in \mathbb{N})$ and $(\tau^j \mid j \in \mathbb{N})$ be the approximation sequences associated to $z = p(z)$ and $\mathfrak{A}$, respectively. We claim that $\sigma^j = \tau^j$ for $j \ge 0$ and show it by induction on j. The case $j = 0$ being clear, we proceed with $j > 0$. Then we obtain, for all $i \in I$,

$$\begin{aligned}
&\tau^j_{z_i} = M(\tau^{j-1}, \ldots, \tau^{j-1})_{z_i} = \\
&\textstyle\sum_{j_1,\ldots,j_m \in I} M_{z_i,(z_{j_1},\ldots,z_{j_m})}(\tau^{j-1}_{z_{j_1}}, \ldots, \tau^{j-1}_{z_{j_m}}) = \\
&\textstyle\sum_{j_1,\ldots,j_m \in I} \delta_{i,j_m}\delta_{j_m,j_{m-1}} \cdots \delta_{j_{k+2},j_{k+1}}\delta_{j_k,i_k} \cdots \delta_{j_1,i_1} p_i(\tau^{j-1}_{z_{j_1}}, \ldots, \tau^{j-1}_{z_{j_k}}) = \\
&p_i(\tau^{j-1}_{z_{i_1}}, \ldots, \tau^{j-1}_{z_{i_k}}) = p_i(\sigma^{j-1}_{z_{i_1}}, \ldots, \sigma^{j-1}_{z_{i_k}}) = \sigma^j_i\,.
\end{aligned}$$

Hence, $||\mathfrak{A}||$ is equal to the z_{i_0}-component of the least solution of $z = p(z)$.

Observe that we could place, for $k = 0$, p_i into P_{z_i} instead into $M_{z_i,(z_i,\ldots,z_i)}$.

Theorem 3.3 summarizes the above considerations. (See also Kuich [85], Theorem 2.3 and Kuich [80], Corollary 3.9.)

Theorem 6.3.3 *A power series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ is a component of the least solution of a system iff s is the behavior of a tree automaton.*

We now consider polynomial tree automata and polynomial systems and show that they are mechanisms of equal power.

A tree automaton $\mathfrak{A} = (I, M, S, P)$ is called *polynomial* iff the following conditions are satisfied:

(i) The entries of M are *polynomials* in $A\langle T_\Sigma(X \cup Y_m)\rangle$.

(ii) The entries of the initial state vector S are of the form $S_i = d_i y_1$, $d_i \in A$, $i \in I$.

(iii) The entries of the final state vector P are *polynomials* in $A\langle T_\Sigma(X)\rangle$.

A system (with variables in Z) $z_i = p_i$, $i \in I$, is called *polynomial* iff each p_i is a *polynomial* in $A\langle T_\Sigma(X \cup Z_i)\rangle$, $i \in I$.

The same constructions that proved Theorem 3.3 prove also the next theorem. (See also Kuich [85], Theorem 2.4 and Kuich [80], Corollary 4.4.)

Theorem 6.3.4 *A power series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ is a component of the least solution of a polynomial system iff s is the behavior of a polynomial tree automaton.*

A system $z_i = p_i$, $i \in I$, is called *simple* iff p_i is a sum of terms of the following form:

(i) $a\omega(z_{i_1}, \ldots, z_{i_k})$, $a \in A$, $\omega \in \Sigma_k$, $i_1, \ldots, i_k \in I$, $1 \leq k \leq m$, for some $m \geq 1$,

(ii) $a\omega$, $a \in A$, $\omega \in \Sigma_0 \cup X$,

(iii) az_i, $a \in A$, $i \in I$.

Compare the next theorem with Theorem 4.2 of Kuich [80].

Theorem 6.3.5 *Let $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ be a component of the least solution of a finite polynomial system. Then there exists a finite polynomial system that is simple and proper such that s is a component of its least solution.*

Proof. By Theorem 3.2 and Lemma 6.3 of Berstel, Reutenauer [6]. □

Corollary 6.3.6 *The following statements on a formal tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent:*

(i) s is a component of the least solution of a finite polynomial system;

(ii) s is a component of the unique solution of a finite polynomial system that is also simple and proper;

(iii) s is the behavior of a finite polynomial tree automaton;

(iv) s is the behavior of a finite polynomial tree automaton with one initial state of weight 1 that is also simple and proper.

Proof. By Theorems 3.5 and 3.4, the statements (i), (ii) and (iii) are equivalent. By the construction in the proof of Theorem 3.3, statement (iv) is implied by statement (iii). □

If a formal tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ satisfies one and, hence, all statements of Corollary 3.6 we call it *recognizable*. In case of fields, it is the same notion of recognizability as introduced by Berstel, Reutenauer [6]. The collection of all recognizable tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ is denoted by $A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$. In the theory of tree languages, recognizable tree languages are defined only for *finite* alphabets Σ and X. We allow also *infinite* alphabets Σ and X. This is justified by the observation that, for $s \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$, there exist finite alphabets Σ' and X', $\Sigma' \subseteq \Sigma$, $X' \subseteq X$, such that $\mathrm{supp}(s) \subseteq T_{\Sigma'}(X')$. Moreover

$$A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle = \bigcup_{\Sigma' \subseteq \Sigma \text{ finite},\ X' \subseteq X \text{ finite}} A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X')\rangle\rangle \,.$$

Example 6.3.1.[1] (See Berstel, Reutenauer [6], Examples 6.2 and 4.2.) Our basic semiring is $\mathbb{Z}$, the semiring of integers. Let $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 = \{\ominus\}$, $\Sigma_2 = \{\oplus, \otimes\}$. We will evaluate arithmetic expressions with operators $\ominus, \oplus, \otimes$, and operands in the leaf alphabet X.

Define an interpretation *eval* of the elements of X, i. e., $\mathrm{eval} : X \to \mathbb{Z}$. Extend it to a mapping $\mathrm{eval} : T_\Sigma(X) \to \mathbb{Z}$ by $\mathrm{eval}(\ominus(t)) = -\mathrm{eval}(t)$, $\mathrm{eval}(\oplus(t_1, t_2)) = \mathrm{eval}(t_1) + \mathrm{eval}(t_2)$, $\mathrm{eval}(\otimes(t_1, t_2)) = \mathrm{eval}(t_1) \cdot \mathrm{eval}(t_2)$ for $t, t_1, t_2 \in T_\Sigma(X)$. Then $\mathrm{eval} = \sum_{t \in T_\Sigma(X)} \mathrm{eval}(t)t$ is a formal tree series in $\mathbb{Z}\langle\langle T_\Sigma(X)\rangle\rangle$.

Consider the proper system

$$\begin{aligned} z_1 &= \oplus(z_1, z_2) + \oplus(z_2, z_1) + \otimes(z_1, z_1) + (-1) \ominus (z_1) + \textstyle\sum_{x \in X} \mathrm{eval}(x)x \,, \\ z_2 &= \oplus(z_2, z_2) + \otimes(z_2, z_2) + \ominus(z_2) + \textstyle\sum_{x \in X} x \,. \end{aligned}$$

Let (σ_1, σ_2) be its unique solution. Then we claim that $\sigma_1 = \mathrm{eval}$, $\sigma_2 = \mathrm{char}$, where $\mathrm{char} = \sum_{t \in T_\Sigma(X)} t$. The claim is proven by substituting $(\mathrm{eval}, \mathrm{char})$ into the equations of the system:

$$\begin{aligned} &\oplus(\mathrm{eval}, \mathrm{char}) + \oplus(\mathrm{char}, \mathrm{eval}) + \otimes(\mathrm{eval}, \mathrm{eval}) - \ominus(\mathrm{eval}) + \textstyle\sum_{x \in X} \mathrm{eval}(x)x = \\ &\textstyle\sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_1) \oplus (t_1, t_2) + \sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_2) \oplus (t_1, t_2) + \\ &\textstyle\sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_1)\mathrm{eval}(t_2) \otimes (t_1, t_2) + \sum_{t \in T_\Sigma(X)} -\mathrm{eval}(t) \ominus t + \\ &\textstyle\sum_{x \in X} \mathrm{eval}(x)x = \\ &\textstyle\sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(\oplus(t_1, t_2)) \oplus (t_1, t_2) + \sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(\otimes(t_1, t_2)) \otimes (t_1, t_2) + \\ &\textstyle\sum_{t \in T_\Sigma(X)} \mathrm{eval}(\ominus(t)) \ominus t + \sum_{x \in X} \mathrm{eval}(x)x = \\ &\textstyle\sum_{t \in T_\Sigma(X)} \mathrm{eval}(t)t = \mathrm{eval} \,, \end{aligned}$$

$$\begin{aligned} &\oplus(\mathrm{char}, \mathrm{char}) + \otimes(\mathrm{char}, \mathrm{char}) + \ominus(\mathrm{char}) + \textstyle\sum_{x \in X} x = \\ &\textstyle\sum_{t_1, t_2 \in T_\Sigma(X)} \oplus(t_1, t_2) + \\ &\textstyle\sum_{t_1, t_2 \in T_\Sigma(X)} \otimes(t_1, t_2) + \sum_{t \in T_\Sigma(X)} \ominus(t) + \sum_{x \in X} x = \mathrm{char} \,. \end{aligned}$$

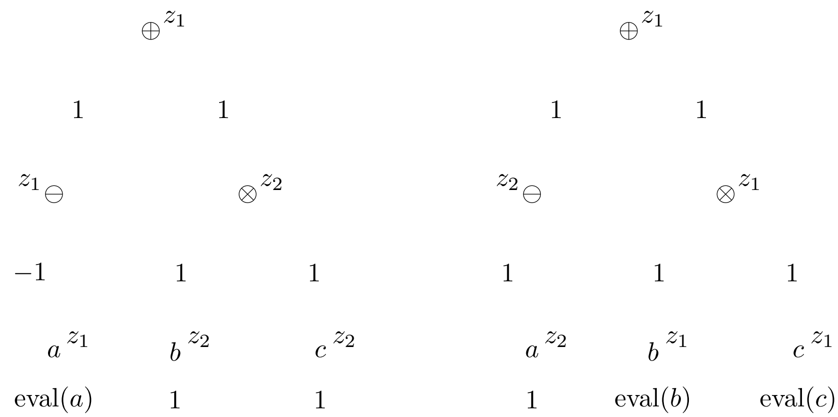
Consider now the finite tree automaton $\mathfrak{A} = (Q, M, S, P)$, where $Q = \{z_1, z_2\}$, $S_{z_1} = y_1$, $S_{z_2} = 0$, $P_{z_1} = \sum_{x \in \Sigma} \mathrm{eval}(x)x$, $P_{z_2} = \sum_{x \in \Sigma} x$, and the nonnull entries of M are given by

$$\begin{aligned} M_{z_1,(z_1,z_1)} &= (-1) \ominus (y_1), & M_{z_2,(z_2,z_2)} &= \ominus(y_1), \\ M_{z_1,(z_1,z_1)} &= \otimes(y_1, y_2), & M_{z_1,(z_1,z_2)} &= \oplus(y_1, y_2), \\ M_{z_1,(z_2,z_1)} &= \oplus(y_1, y_2), & M_{z_2,(z_2,z_2)} &= \oplus(y_1, y_2) + \otimes(y_1, y_2) \,. \end{aligned}$$

Then we obtain $||\mathfrak{A}|| = \sigma_1 = \mathrm{eval}$.

Let $X = \{a, b, c\}$ and $t = \oplus(\ominus a, \otimes(b, c))$. Then there are two computations for t starting at z_1 and none starting from z_2. These two computations are given in the following pictorial form:

[1] In the examples we often drop our convention that the basic semiring is continuous.



Hence, $(||\mathfrak{A}||, t) = -\mathrm{eval}(a) + \mathrm{eval}(b)\mathrm{eval}(c)$. $\square$

Example 3.1 gives also an intuitive feeling how a finite nondeterministic root-to-frontier tree recognizer is simulated by a finite tree automaton over the semiring $\mathbb{B}$.

Theorem 6.3.7 (Kuich [80], Theorem 3.6) *For each finite nondeterministic root-to-frontier tree recognizer* $\mathbf{A}$ *in the sense of Gécseg, Steinby [51] there exists a simple proper finite polynomial tree automaton* $\mathfrak{A}$ *over the Boolean semiring* $\mathbb{B}$ *such that* $||\mathfrak{A}|| = T(\mathbf{A})$*, and vice versa.*

6.4 Closure properties and a Kleene Theorem for recognizable tree series

In this section we prove a Kleene Theorem for recognizable tree series. (See Thatcher, Wright [113], Bozapalidis [14], Gruska [60], Gécseg, Steinby [51, 52], Kuich [79].) By this Kleene Theorem expressions which are analogous to regular expressions can be defined that characterize the recognizable tree series.

We first show that $A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ is a distributive Σ-algebra. This result is a specialization of Theorem 6.5 of Kuich [80].

Theorem 6.4.1 $\langle A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle, +, 0, \bar{\Sigma}\rangle$ *is a distributive* Σ*-algebra that contains* $A\langle T_\Sigma(X)\rangle$ *and is closed under scalar product.*

Proof. Let $s_j \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ be the first components of the unique solution of the simple proper finite polynomial systems (written in matrix notation) $z^j = p^j(z^j)$, $1 \le j \le m$, with pairwise disjoint variable alphabets. Let σ^j be the unique solution of $z^j = p^j(z^j)$, $1 \le j \le m$, with $\sigma_1^j = s_j$.

(i) Consider the system

$$z_0 = p_1^1(z^1) + p_1^2(z^2), \quad z^1 = p^1(z^1), \quad z^2 = p^2(z^2)\,.$$

It is again simple and proper. We claim that its unique solution is given by $(s_1 + s_2, \sigma^1, \sigma^2)$ and show this by substituting $s_1 + s_2$ for z_0, σ^1 for z^1 and σ^2

for z^2:

$$p_1^1[\sigma^1/z^1] + p_1^2[\sigma^2/z^2] = \sigma_1^1 + \sigma_1^2 = s_1 + s_2, \qquad p^j[\sigma^j/z^j] = \sigma^j, \quad j = 1, 2\,.$$

(ii) Let $\omega \in \Sigma_k$, $k \geq 0$, and consider the system

$$z_0 = \omega(z_1^1, \ldots, z_1^k), \qquad z^j = p^j(z^j), \quad 1 \leq j \leq k\,.$$

It is again simple and proper. We claim that its unique solution is given by $(\bar{\omega}(s_1, \ldots, s_k), \sigma^1, \ldots, \sigma^k)$ and show this again by substituting the components of the claimed solution into the equations:

$$\bar{\omega}(\sigma_1^1, \ldots, \sigma_1^k) = \bar{\omega}(s_1, \ldots, s_k), \qquad p^j[\sigma^j/z^j] = \sigma^j, \quad 1 \leq j \leq k\,.$$

(iii) Let $a \in A$ and consider the system

$$z_0 = ap_1^1(z^1), \quad z^1 = p^1(z^1)\,.$$

It is again simple and proper. We claim that its unique solution is given by (as_1, σ^1) and show this by substitution:

$$ap_1^1[\sigma^1/z^1] = as_1, \quad p^1[\sigma^1/z^1] = \sigma^1\,.$$

(iv) For $s \in A\langle T_\Sigma(X)\rangle$, s is the unique solution of the system $z_0 = s$.

Since $A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ contains $A\langle T_\Sigma(X)\rangle$ and is closed under addition and top catenation, it is a Σ-subalgebra of $A\langle\langle T_\Sigma(X)\rangle\rangle$. Hence, it is a distributive Σ-algebra. $\square$

In the sequel, $Z = \{z_j \mid j \geq 1\}$, $Z_n = \{z_j \mid 1 \leq j \leq n\}$, $Z_0 = \emptyset$. We introduce the following notation: Let $s \in A\langle\langle T_\Sigma(X \cup Z_n)\rangle\rangle$. Then we denote the least $\sigma \in A\langle\langle T_\Sigma(X \cup \{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\})\rangle\rangle$ such that $s(z_1, \ldots, z_{i-1}, \sigma, z_{i+1}, \ldots, z_n) = \sigma$ by $\mu z_i.s(z_1, \ldots, z_n)$, $1 \leq i \leq n$. This means that $\mu z_i.s$ is the least fixed point solution of the system $z_i = s(z_1, \ldots, z_i, \ldots, z_n)$; this system consists of one equation only and its single variable is z_i.

The operation $\mu z.s$, where $z \in Z$ and $s \in A\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$, is a slight modification and generalization (to semirings) of the Kleene star operation for tree languages as defined by Gécseg, Steinby [52]. The connection of these two operations in the case of the Boolean semiring is explained below Theorem 4.7.

A distributive Σ-algebra $\langle V, +, 0, \bar{\Sigma}\rangle$, $V \subseteq A\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is called *rationally closed* iff V is closed under scalar product and for all $s \in V$ and $z \in Z$ the formal tree series $\mu z.s$ is again in V. By definition, $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is the least rationally closed distributive Σ-algebra containing $A\langle T_\Sigma(X \cup Z)\rangle$. Observe that for each $s \in A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ there exists an $m \geq 0$ such that $\mathrm{supp}(s) \subseteq T_\Sigma(X \cup Z_m)$.

We will prove that $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle = A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Before proving the main result of this section we apply a few results of the fixed point theory of continuous functions to systems (see the Preliminaries of Chapter 2).

(1) *The parameter identity.* Let $r \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ and denote $r' = \mu y.r$, $y \in Y$. Let $y_i \neq y$ and $\tau_i \in A\langle\langle T_\Sigma(X \cup (Y - \{y\}))\rangle\rangle$, $1 \leq i \leq n$. Then $r'[\tau_1/y_1, \ldots, \tau_n/y_n] = \mu y.(r[\tau_1/y_1, \ldots, \tau_n/y_n])$.

(2) *The Bekić-De Bakker-Scott rule.* Consider the system $y_i = r_i$, $1 \leq i \leq n$, $r_i \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ with variables $y_1, \ldots, y_n$ and $m \in \{1, \ldots, n-1\}$. Let $(\sigma_{m+1}, \ldots, \sigma_n)$ be the least solution of the system $y_i = r_i$, $m+1 \leq i \leq n$. Furthermore, let $(\tau_1, \ldots, \tau_m)$ be the least solution of the system $y_i = r_i[\sigma_{m+1}/y_{m+1}, \ldots, \sigma_n/y_n]$, $1 \leq i \leq m$. Then

$$(\tau_1, \ldots, \tau_m, \sigma_{m+1}[\tau_1/y_1, \ldots, \tau_m/y_m], \ldots, \sigma_n[\tau_1/y_1, \ldots, \tau_m/y_m])$$

is the least solution of the original system $y_i = r_i$, $1 \leq i \leq n$.

We first show that $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is closed under substitution.

Theorem 6.4.2 *Assume that $s(z_1, \ldots, z_n)$ and σ_j, $1 \leq j \leq n$, are in $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Then $s(\sigma_1, \ldots, \sigma_n)$ is again in $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.*

Proof. The proof is by induction on the number of applications of the operations $\bar{\omega} \in \bar{\Sigma}$, $+$, scalar product and μ to generate $s(z_1, \ldots, z_n)$ from polynomials.

(i) Let $s(z_1, \ldots, z_n) \in A\langle T_\Sigma(X \cup Z)\rangle$. Since $s(\sigma_1, \ldots, \sigma_n)$ is generated from $\sigma_1, \ldots, \sigma_n$ by application of sum, $\bar{\omega} \in \bar{\Sigma}$ and scalar product, we infer that $s(\sigma_1, \ldots, \sigma_n) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.

(ii) The cases of addition, top catenation and scalar product are clear. Thus, we only prove the case of the operator μ. Assume that, for $1 \leq j \leq n$, $\text{supp}(\sigma_j) \subseteq T_\Sigma(X \cup Z_m)$ for some $m \geq 0$. Choose a $z = z_k \in Z$ with $k > m$. Without loss of generality assume that $s(z_1, \ldots, z_n) = \mu z.r(z_1, \ldots, z_n, z)$ (the variable z is "bound"), where $r(z_1, \ldots, z_n, z) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. By induction hypothesis, we have $r(\sigma_1, \ldots, \sigma_n, z) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Hence, $s(\sigma_1, \ldots, \sigma_n) = \mu z.r(\sigma_1, \ldots, \sigma_n, z) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ by the parameter identity. $\square$

Theorem 6.4.3 (Bozapalidis [14], Section 5.) $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle = A^{\text{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.

Proof. (i) We show $A^{\text{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle \subseteq A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. The proof is by induction on the number of variables of finite polynomial systems. We use the following induction hypothesis: If $\tau = (\tau_1, \ldots, \tau_n)$, $\tau_i \in A^{\text{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$, $1 \leq i \leq n$, is the least solution of a finite polynomial system $z_i = q_i(z_1, \ldots, z_n)$, $1 \leq i \leq n$, with n variables $z_1, \ldots, z_n$, where $q_i(z_1, \ldots, z_n) \in A\langle T_\Sigma(X \cup Z)\rangle$ then $\tau_i \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.

(1) Let $n = 1$ and assume that $s \in A^{\text{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is the least solution of the finite polynomial system $z_1 = p(z_1)$. Since $p(z_1)$ is a polynomial, $s = \mu z_1.p(z_1) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.

(2) Consider the finite polynomial system $z_i = q_i(z_1, \ldots, z_{n+1})$, $1 \leq i \leq n+1$, $n \geq 1$. Let $\tau(z_1) = (\tau_2(z_1), \ldots, \tau_{n+1}(z_1))$, $\tau_i(z_1) \in A^{\text{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$, $2 \leq i \leq n+1$, be the least solution of the finite polynomial system $z_i = q_i(z_1, \ldots, z_{n+1})$, $2 \leq i \leq n+1$. By our induction hypothesis we infer that $\tau_i(z_1) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Since $q_1(z_1, \ldots, z_{n+1})$ is a polynomial, it is in $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Hence, by Theorem 4.2, $p(z_1) = q_1(z_1, \tau_2(z_1), \ldots, \tau_{n+1}(z_1))$ is in $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. This implies that $\mu z_1.p(z_1)$ is in $A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Again by Theorem 4.2, $\tau_i(\mu z_1.p(z_1)) \in A^{\text{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$, $2 \leq i \leq n+1$. By the Bekić-De Bakker-Scott rule, $(\mu z_1.p(z_1), \tau_2(\mu z_1.p(z_1)), \ldots, \tau_{n+1}(\mu z_1.p(z_1)))$ is the least solution of

the finite polynomial system $z_i = q_i(z_1, \ldots, z_{n+1})$, $1 \le i \le n+1$. Hence, the components of the least solution of $z_i = q_i(z_1, \ldots, z_{n+1})$, $1 \le i \le n+1$, are in $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$.

(ii) We show that $A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is a rationally closed distributive Σ-algebra that contains $A\langle T_\Sigma(X \cup Z)\rangle$. This will imply $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle \subseteq A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. By Theorem 4.1 (with $X \cup Z$ instead of X), $A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ is a distributive Σ-algebra closed under scalar product that contains $A\langle T_\Sigma(X \cup Z)\rangle$. Hence, we have only to show that $\mu z.s$, $s \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$, is in $A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Let $(\tau_2(z_1), \ldots, \tau_{n+1}(z_1))$ be the least solution of the finite polynomial system $z_i = p_i(z_1, \ldots, z_{n+1})$, $2 \le i \le n+1$, and take $s = \tau_2(z_1)$. Consider now the finite polynomial system $z_1 = p_2(z_1, \ldots, z_{n+1})$, $z_i = p_i(z_1, \ldots, z_{n+1})$, $2 \le i \le n+1$. Then, by the Bekić-De Bakker-Scott rule, $\mu z_1.\tau_2(z_1)$ is the first component of its least solution. $\square$

Analogously to the rational expressions (see Chapter 2, Section 3) and to the ΣZ-expressions (see Gécseg, Steinby [52]) we define now rational tree series expressions. Assume that A, Σ, X, Z and $U = \{+, \cdot, \mu, [,]\}$ are pairwise disjoint. A word E over $A \cup \Sigma \cup X \cup Z \cup U$ is a *rational tree series expression* over (A, Σ, X, Z) iff

(i) E is a symbol of $X \cup Z$, or

(ii) E is of one of the forms $[E_1 + E_2]$, $\omega(E_1, \ldots, E_k)$, aE_1, or $\mu z.E_1$, where $E_1, E_2, \ldots, E_k$ are rational tree series expressions over (A, Σ, X, Z), $\omega \in \Sigma_k$, $k \ge 0$, $a \in A$, and $z \in Z$.

Each rational tree series expression E over (A, Σ, X, Z) denotes a formal tree series $|E|$ in $A\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$ according to the following conventions:

(i) If E is in $X \cup Z$ then E denotes the tree series E, i. e., $|E| = E$.

(ii) For rational tree series expressions $E_1, \ldots, E_k$ over (A, Σ, X, Z), $\omega \in \Sigma_k$, $k \ge 0$, $a \in A$, $z \in Z$, we define

$$\begin{aligned}
&|[E_1 + E_2]| = |E_1| + |E_2|\,,\\
&|\omega(E_1, \ldots, E_k)| = \bar{\omega}(|E_1|, \ldots, |E_k|)\,,\\
&|aE_1| = a|E_1|\,,\\
&|\mu z.E_1| = \mu z.|E_1|\,.
\end{aligned}$$

Let φ_1 and φ_2 be mappings from the set of rational tree series expressions over (A, Σ, X, Z) into the set of finite subsets of $X \cup Z$ defined by

(i) $\varphi_1(x) = \emptyset$, $\varphi_2(x) = \{x\}$, $x \in X$,
$\varphi_1(z) = \{z\}$, $\varphi_2(z) = \emptyset$, $z \in Z$.

(ii) $\varphi_j([E_1 + E_2]) = \varphi_j(E_1) \cup \varphi_j(E_2)$,
$\varphi_j(\omega(E_1, \ldots, E_k)) = \varphi_j(E_1) \cup \ldots \cup \varphi_j(E_k)$,
$\varphi_j(aE_1) = \varphi_j(E_1)$, $a \ne 0$, $\varphi_j(0E_1) = \emptyset$, $a = 0$,
$\varphi_j(\mu z.E_1) = \varphi_j(E_1) - \{z\}$, $j = 1, 2$
for rational tree series expressions $E_1, E_2, \ldots, E_k$ over (A, Σ, X, Z), $\omega \in \Sigma_k$, $k \ge 0$, $a \in A$, and $z \in Z$.

Given a rational tree series expression E over (A, Σ, X, Z), $\varphi_1(E) \subseteq Z$ contains the "free variables" of E, while $\varphi_2(E) \subseteq X$ contains the used symbols of the leaf alphabet X. This means that $|E|$ is a formal tree series in $A\langle\langle T_\Sigma(\varphi_2(E) \cup \varphi_1(E))\rangle\rangle$. Theorem 4.3 and the above definitions yield some corollaries.

Corollary 6.4.4 *A tree series s is in $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle \cap A\langle\langle T_\Sigma(X' \cup Z')\rangle\rangle$, where $X' \subseteq X$, $Z' \subseteq Z$, iff there exists a rational tree series expression E over (A, Σ, X, Z) such that $s = |E|$, where $\varphi_2(E) = X'$ and $\varphi_1(E) = Z'$.*

Corollary 6.4.5 *A tree series s is in $A^{\mathrm{rat}}\langle\langle T_\Sigma(X \cup Z)\rangle\rangle \cap A\langle\langle T_\Sigma(X')\rangle\rangle$, $X' \subseteq X$, iff there exists a rational tree series expression E over (A, Σ, X, Z) such that $s = |E|$, where $\varphi_1(E) = \emptyset$ and $\varphi_2(E) = X'$.*

Corollary 6.4.6 *A tree series s is in $A^{\mathrm{rec}}\langle\langle T_\Sigma(X')\rangle\rangle$, $X' \subseteq X$, iff there exists a rational tree series expression E over (A, Σ, X, Z) such that $s = |E|$, where $\varphi_1(E) = \emptyset$ and $\varphi_2(E) = X'$.*

Observe that our Corollary 4.4 is stronger than "Kleene's Theorem" of Bozapalidis [14], Section 5, since we can use our Theorem 4.2 and do not need "closure under substitution" in the definition of a rationally closed distributive Σ-algebra.

We summarize our results in a Kleene-like theorem (see Bozapalidis [14]).

Theorem 6.4.7 *The following statements on a power series $r \in A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent:*

(i) r is a component of the least solution of a finite polynomial system;

(ii) r is the behavior of a finite polynomial tree series automaton;

(iii) there exists a rational tree series expression E over (A, Σ, X, Z), where $\varphi_1(E) = \emptyset$, such that $r = |E|$.

Proof. By Corollary 3.6 and Theorem 4.3. □

In the characterization of the recognizable tree languages, Gécseg, Steinby [52] use the following closure operation for a tree language $r(y_1, \ldots, y_n, y) \in \mathbb{B}\langle\langle T_\Sigma(X \cup \{y_1, \ldots, y_n, y\})\rangle\rangle$, $y = y_{n+1}$:

$$\begin{aligned} r^{0,y}(y_1, \ldots, y_n, y) &= \{y\}, \\ r^{j+1,y}(y_1, \ldots, y_n, y) &= r(y_1, \ldots, y_n, r^{j,y}(y_1, \ldots, y_n, y)) \cup r^{j,y}(y_1, \ldots, y_n, y),\ j \geq 0, \\ r^{*y}(y_1, \ldots, y_n, y) &= \textstyle\bigcup_{j \geq 0} r^{j,y}(y_1, \ldots, y_n, y). \end{aligned}$$

(Here we use the isomorphism between $2^{T_\Sigma(Y)}$ and $\mathbb{B}\langle\langle T_\Sigma(Y)\rangle\rangle$.) Consider now the finite polynomial system over $\mathbb{B}$ with just one equation and variable y_0

$$y_0 = r(y_1, \ldots, y_n, y_0) + y\,,$$

and denote by $(\tau^j(y_1,\dots,y_n,y) \mid j \geq 0)$ its approximation sequence

$$\begin{aligned} \tau^0(y_1,\dots,y_n,y) &= 0\,, \\ \tau^{j+1}(y_1,\dots,y_n,y) &= r(y_1,\dots,y_n,\tau^j(y_1,\dots,y_n,y)) + y, \qquad j \geq 0\,. \end{aligned}$$

Using the equality

$$r^{j+1,y}(y_1,\dots,y_n,y) = r(y_1,\dots,y_n,r^{j,y}(y_1,\dots,y_n,y)) + y, \quad j \geq 0,$$

easy proofs by induction on the elements of the approximation sequence show that, for $j \geq 0$,

(i) $y \leq r^{j,y}(y_1,\dots,y_n,y)$, $y \leq \tau^{j+1}(y_1,\dots,y_n,y)$,

(ii) $\tau^j(y_1,\dots,y_n,y) \leq r^{j,y}(y_1,\dots,y_n,y)$,

(iii) $r^{j,y}(y_1,\dots,y_n,y) \leq \tau^{j+1}(y_1,\dots,y_n,y)$.

Since $r^{j,y}(y_1,\dots,y_n,y) = \sum_{0\leq i\leq j} r^{i,y}(y_1,\dots,y_n,y)$, we obtain $\sup(\tau^j(y_1,\dots,y_n,y) \mid j \geq 0) = r^{*y}(y_1,\dots,y_n,y)$. Hence, $r^{*y}(y_1,\dots,y_n,y)$ is the least solution of the equation $y_0 = r(y_1,\dots,y_n,y_0) + y$, i. e., $\mu y_0.(r(y_1,\dots,y_n,y_0) + y) = r^{*y}(y_1,\dots,y_n,y)$. Observe that these considerations are valid not only for $\mathbb{B}$ but for all idempotent semirings.

Bozapalidis [14] had the idea to replace $\mu y_0.(r(y_1,\dots,y_n,y_0) + y)$ by $\mu y.r(y_1,\dots,y_n,y)$. (For context-free languages, Gruska [60] used implicitely this closure operator; see Kuich [79].) We have used this closure operator of Bozapalidis [14] in this chapter. The essential difference of the two closure operators is that $r^{*y}(y_1,\dots,y_n,y) \in \mathbb{B}\langle\langle T_\Sigma(X \cup Y_n \cup \{y\})\rangle\rangle$, while $\mu y.r(y_1,\dots,y_n,y) \in \mathbb{B}\langle\langle T_\Sigma(X \cup Y_n)\rangle\rangle$.

By the parameter identity we can even say more:

$$\mu y.r(y_1,\dots,y_n,y) = r^{*y}(y_1,\dots,y_n,0).$$

Hence, our interpretation of a rational tree series expression over $(\mathbb{B},\Sigma,X,Z)$ that is given below is different from that by Gécseg, Steinby [52].

Each rational tree series expression E over $(\mathbb{B},\Sigma,X,Z)$ denotes a tree language $|E| \subseteq T_\Sigma(X \cup Z)$ according to the following conventions:

(o) The tree language denoted by 0 is $\emptyset$.

(i) The tree language denoted by $x \in X$ is $\{x\}$.

(ii) The tree language denoted by $z \in Z$ is $\{z\}$.

(iii) For rational tree series expressions $E_1, E_2, \dots, E_k$ over $(\mathbb{B},\Sigma,X,Z)$, $\omega \in \Sigma_k$, $k \geq 0$, and $z \in Z$,

$$\begin{gathered} |[E_1 + E_2]| = |E_1| \cup |E_2|, \\ |\omega(E_1,\dots,E_k)| = \bar{\omega}(|E_1|,\dots,|E_k|), \\ |\mu z.E_1| = \mu z.|E_1|. \end{gathered}$$

In the next theorem we use the notation of Gécseg, Steinby [52].

Theorem 6.4.8 *The following statements on a tree language $L \subseteq T_\Sigma(X)$ are equivalent:*

(i) L is generated by a regular ΣX-grammar;

(ii) L is recognized by a nondeterministic finite root-to-frontier ΣX-recognizer;

(iii) $L = |E|$, where E is a rational tree series expression over $(\mathbb{B}, \Sigma, X, Z)$ and $\varphi_1(E) = \emptyset$.

Observe that Theorem 4.8 is stronger than Proposition 9.3 (Kleene's Theorem) of Gécseg, Steinby [52], since we do not need "closure under substitution" for our tree expressions over $(\mathbb{B}, \Sigma, X, Z)$.

Example 6.4.1. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{c, d\}$, $\Sigma_1 = \{g\}$, $\Sigma_2 = \{f\}$, $z, z_1 \in Z$ and consider the rational tree series expression $[g(c) + \mu z_1.[f(c, z_1) + z]]$ over $(\mathbb{B}, \Sigma, X, Z)$. It denotes

$$|[g(c)+\mu z_1.[f(c,z_1)+z]]| = g(c)+z+f(c,z)+f(c,f(c,z))+f(c,f(c,f(c,z)))+\dots$$

Moreover,

$$|[g(c) + \mu z_1.[f(c, z_1) + d]]| = |[g(c) + \mu z_1.[f(c, z_1) + z]]|[d/z]\,.$$

Compare this with the second paragraph on page 21 of Gécseg, Steinby [52]. □

6.5 Pushdown tree automata, algebraic tree systems, and a Kleene Theorem

In this section we consider pushdown tree automata and algebraic tree systems. Moreover, we prove a Kleene Theorem due to Bozapalidis [15].

Guessarian [62] introduced the notion of a (top-down) pushdown tree automaton and showed that these pushdown tree automata recognize exactly the class of context-free tree languages. Here a tree language is called context-free iff it is generated by a context-free tree grammar (using OI derivation mode). Moreover, she showed that pushdown tree automata are equivalent to restricted pushdown tree automata, i. e., to pushdown automata, whose pushdown store is linear.

Kuich [85] generalized these results of Guessarian [62] to formal tree series. He defined pushdown tree automata whose behaviors are formal tree series and showed that the class of behaviors of these pushdown tree automata coincides with the class of algebraic tree series. Here a tree series is called algebraic iff it is the initial component of the least solution of an algebraic tree system with initial function variable. These algebraic tree systems are a generalization of the context-free tree grammars (see Rounds [102] and Gécseg, Steinby [52]). They are a particular instance of the second-order systems of Bozapalidis [15]. The presentation follows Kuich [85].

A *pushdown tree automaton* (with input alphabet Σ and leaf alphabet X) over the semiring A

$$\mathfrak{P} = (Q, \Gamma, Z, Y, M, S, p_0, P)$$

is given by

(i) a finite non-empty set Q of *states*;

(ii) a finite ranked alphabet $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_{\bar{m}}$ of *pushdown symbols*;

(iii) a finite alphabet $Z = \{z_1, \ldots, z_{\bar{m}}\}$ of *pushdown variables*;

(iv) a finite alphabet $Y = \{y_1, \ldots, y_k\}$ of *variables*;

(v) a *pushdown tree transition matrix M of order k*;

(vi) $S \in (A\langle T_\Sigma(X \cup Y_1)\rangle)^{1\times Q}$, where $S_q = d_q y_1$, $d_q \in A$, $q \in Q$, called the *initial state vector*;

(vii) $p_0 \in \Gamma_0$, called the *initial pushdown symbol*;

(viii) a finite family $P = (P_{g(z_1,\ldots,z_m)} \mid g \in \Gamma_m,\ 0 \le m \le \bar{m})$ of *final state vectors* $P_{g(z_1,\ldots,z_m)} \in (A\langle T_\Sigma(X)\rangle)^{Q\times 1}$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$.

Here a *pushdown tree transition matrix of order k* is a matrix

$$M \in ((A\langle T_\Sigma(X \cup Y_k)\rangle)^{Q\times Q^k})^{T_\Gamma(Z)\times T_\Gamma(Z)^k}$$

which satisfies the following two conditions:

(i) for all $t, t_1, \ldots, t_k \in T_\Gamma(Z)$

$$M_{t,(t_1,\ldots,t_k)} = \begin{cases} \sum M_{g(z_1,\ldots,z_m),(v_1(z_1,\ldots,z_m),\ldots,v_k(z_1,\ldots,z_m))} \\ \quad \text{where the sum extends over all } v_1,\ldots,v_k \in T_\Gamma(Z_m), \\ \quad \text{such that } t_j = v_j(u_1,\ldots,u_m),\ 1 \le j \le k, \\ \text{if } t = g(u_1,\ldots,u_m),\ g \in \Gamma_m,\ u_1,\ldots,u_m \in T_\Gamma(Z_m)\,; \\ 0\,, \quad \text{otherwise}\,. \end{cases}$$

(ii) M is row finite, i. e., for each $g \in \Gamma_m$, $0 \le m \le \bar{m}$, there exist only finitely many blocks $M_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}$, where $v_1, \ldots, v_k \in T_\Gamma(Z_m)$, that are unequal to zero;

Observe that if the root of t is labeled by $g \in \Gamma_m$, then $M_{t,(t_1,\ldots,t_k)} \neq 0$ implies $t, t_1, \ldots, t_k \in T_\Gamma(Z_m)$.

Intuitively, the definition of the pushdown tree transition matrix means that the action of the pushdown tree automaton with tree $t = g(u_1, \ldots, u_m)$ on its pushdown store depends only on the label g of the root of t. Observe that a pushdown tree transition matrix of order k is defined by its finitely many nonnull blocks of the form $M_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}$, $g \in \Gamma_m$.

Observe that our definition of pushdown tree automata differs slightly from the definition given in Kuich [85]: there M is a sequence of pushdown tree transition matrices. But by the Remark given below the definition of tree automata in Section 3, both types of pushdown tree automata are equivalent with respect to their behaviors.

Let now $Z_Q = \{(z_i)_q \mid 1 \leq i \leq \bar{m},\ q \in Q\}$ be an alphabet of variables and denote $Z_Q^m = \{(z_i)_q \mid 1 \leq i \leq m,\ q \in Q\}$, $1 \leq m \leq \bar{m}$, $Z_Q^0 = \emptyset$. Define $F \in ((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ by its entries as follows:

(i) $(F_t)_q = (P_{g(z_1,\dots,z_m)})_q$ if $t = g(u_1,\dots,u_m)$, $g \in \Gamma_m$, $0 \leq m \leq \bar{m}$, $u_1,\dots,u_m \in T_\Gamma(Z_m)$, $q \in Q$;

(ii) $(F_{z_i})_q = (z_i)_q$, $1 \leq i \leq \bar{m}$, $q \in Q$;

(iii) $(F_t)_q = 0$, otherwise.

Hence, F_{z_i}, $1 \leq i \leq \bar{m}$, is a column vector of dimension Q whose q-entry, $q \in Q$, is the variable $(z_i)_q$.

The *approximation sequence* $(\tau^j \mid j \in \mathbb{N})$, $\tau^j \in ((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$, $j \geq 0$, *associated with* $\mathfrak{P}$ is defined as follows:

$$\tau^0 = 0, \qquad \tau^{j+1} = M(\tau^j,\dots,\tau^j) + F, \quad j \geq 0\,.$$

This means that, for all $t \in T_\Gamma(Z)$, the block vectors τ_t^j of τ^j are defined by

$$\tau_t^0 = 0, \quad \tau_t^{j+1} = \sum_{t_1,\dots,t_k \in T_\Gamma(Z)} M_{t,(t_1,\dots,t_k)}(\tau_{t_1}^j,\dots,\tau_{t_k}^j) + F_t,\ j \geq 0\,.$$

Moreover, for all $t \in T_\Gamma(Z)$, $q \in Q$,

$$\begin{aligned}
&(\tau_t^0)_q = 0\,,\\
&(\tau_t^{j+1})_q = \textstyle\sum_{t_1,\dots,t_k \in T_\Gamma(Z)} \sum_{q_1,\dots,q_k \in Q}\\
&\qquad\qquad (M_{t,(t_1,\dots,t_k)})_{q,(q_1,\dots,q_k)}((\tau_{t_1}^j)_{q_1},\dots,(\tau_{t_k}^j)_{q_k}) + (F_t)_q\ j \geq 0\,.
\end{aligned}$$

Hence, for all $g \in \Gamma_m$, $0 \leq m \leq \bar{m}$, and all $u_1,\dots,u_m \in T_\Gamma(Z_m)$, we obtain, for all $j \geq 0$,

$$\begin{aligned}
\tau_{g(u_1,\dots,u_m)}^{j+1} = &\textstyle\sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)}\\
&M_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}(\tau_{v_1(u_1,\dots,u_m)}^j,\dots,\tau_{v_k(u_1,\dots,u_m)}^j)\\
&+P_{g(z_1,\dots,z_m)}
\end{aligned}$$

and

$$\tau_{z_i}^{j+1} = F_{z_i}, \quad z_i \in Z\,.$$

Let $\tau \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ be the least upper bound of the approximation sequence associated with $\mathfrak{P}$. Then the *behavior* $||\mathfrak{P}||$ of the pushdown tree automaton $\mathfrak{P}$ is defined by

$$||\mathfrak{P}|| = S(\tau_{p_0}) = \sum_{q \in Q} S_q((\tau_{p_0})_q) = \sum_{q \in Q} d_q(\tau_{p_0})_q\,.$$

Observe that $||\mathfrak{P}||$ is a tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$. Furthermore, observe that $(\tau_t)_q \in A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle$, $t \in T_\Gamma(Z)$, $q \in Q$, induces a mapping from $(A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{\bar{m}|Q|}$ into $A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle$.

We now construct a polynomial tree automaton $\mathfrak{A}$ that has the same behavior as the pushdown tree automaton $\mathfrak{P}$. Let $\hat{M} \in (A\langle T_\Sigma(X \cup Y_k)\rangle)^{(T_\Gamma(Z)\times Q)\times(T_\Gamma(Z)\times Q)^k}$ and $\hat{F} \in (A\langle T_\Sigma(X \cup Z_Q)\rangle)^{(T_\Gamma(Z)\times Q)\times 1}$ be the isomorphic copies of M and F, respectively. Observe that $\hat{M}$ is row finite. Furthermore define $\hat{S} \in (A\langle T_\Sigma(X \cup Y_1)\rangle)^{1\times(T_\Gamma(Z)\times Q)}$ by $\hat{S}_{(p_0,q)} = S_q$, $\hat{S}_{(t,q)} = 0$, $t \neq p_0$, $q \in Q$. Specify the polynomial tree automaton $\mathfrak{A}$ with input alphabet Σ and leaf alphabet $X \cup Z_Q$ by

$$\mathfrak{A} = (T_\Gamma(Z) \times Q, \hat{M}, \hat{S}, \hat{F})\,.$$

Then it is clear that $||\mathfrak{A}|| = ||\mathfrak{P}||$, i. e., our pushdown tree automaton fits into the general definition of a polynomial tree automaton, but for technical reasons, we prefer to work with the transition matrix M in $((A\langle T_\Sigma(X \cup Y_k)\rangle)^{Q\times Q^k})^{T_\Gamma(Z)\times T_\Gamma(Z)^k}$ and with the final state vector F in $((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$.

Clearly, this means that all notions concerning tree automata (e. g., simple tree automata) are also notions for pushdown tree automata.

Observe that we have adapted the definition of a pushdown tree automaton as given in Kuich [85] to fit into our general definition of a polynomial tree automaton.

Consider now the polynomial system constructed from $\mathfrak{A}$ as in the proof of Theorem 3.3 and transfer it isomorphically to a system that “belongs” to $\mathfrak{P}$, i. e.,

$$y = M(y, \ldots, y) + F\,. \qquad (*)$$

Here $y \in (\{(y_t)_q \mid t \in T_\Gamma(Z),\ q \in Q\}^{Q\times 1})^{T_\Gamma(Z)\times 1}$ is a vector of variables $(y_t)_q$, $t \in T_\Gamma(Z)$, $q \in Q$, such that $(y_t)_q$ is the t-q-entry of y.

The equations of the linear system $(*)$ are, in block notation, for $t \in T_\Gamma(Z)$,

$$y_t = \sum_{t_1,\ldots,t_k \in T_\Gamma(Z)} M_{t,(t_1,\ldots,t_k)}(y_{t_1}, \ldots, y_{t_k}) + F_t\,,$$

where y_t is a $Q \times 1$-vector, whose q-entry is the variable $(y_t)_q$, $q \in Q$; and for $t \in T_\Gamma(Z)$, $q \in Q$,

$$\begin{aligned}(y_t)_q = {} & \textstyle\sum_{t_1,\ldots,t_k \in T_\Gamma(Z)} \sum_{q_1,\ldots,q_k \in Q} \\ & (M_{t,(t_1,\ldots,t_k)})_{q,(q_1,\ldots,q_k)}((y_{t_1})_{q_1}, \ldots, (y_{t_k})_{q_k}) + (F_t)_q\,.\end{aligned}$$

Hence, for all $g \in \Gamma_m$, $0 \leq m \leq \bar{m}$, and all $u_1, \ldots, u_m \in T_\Gamma(Z_m)$, the equations in matrix notation are

$$\begin{aligned}y_{g(u_1,\ldots,u_m)} = {} & \textstyle\sum_{v_1,\ldots,v_k \in T_\Gamma(Z_m)} \\ & M_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}(y_{v_1(u_1,\ldots,u_m)}, \ldots, y_{v_k(u_1,\ldots,u_m)}) \\ & + P_{g(z_1,\ldots,z_m)}\,.\end{aligned}$$

and, for $z_i \in Z$,

$$y_{z_i} = F_{z_i}\,.$$

Here $v_i(u_1, \ldots, u_m)$, $1 \le i \le k$, denotes $v_i[u_j/z_j,\ 1 \le j \le m]$. The least solution of this polynomial system is the least upper bound of the approximation sequence associated with $\mathfrak{P}$.

An example will illustrate the notions connected with pushdown tree automata. This example is due to Guessarian [62], Example 3, and was already given in Kuich [85].

Example 6.5.1. The pushdown tree automaton M of Example 3 of Guessarian [62] is specified by our concepts as follows: The input alphabet is $F = \{b, c_1, c_2\}$, $\mathrm{ar}(b) = 2$, $\mathrm{ar}(c_i) = 0$, $i = 1, 2$; X is the empty set. $\mathfrak{P} = (Q, \Gamma, \{z\}, \{y_1, y_2\}, M, S, Z_0, P)$, where $Q = \{q_0, q_1, q_2\}$, $\Gamma = \{G, C, Z_0\}$, $\mathrm{ar}(G) = 1$, $\mathrm{ar}(C) = \mathrm{ar}(Z_0) = 0$, $P = (P_C, P_{Z_0}, P_{G(z)})$, and the pushdown transition matrix M of order 2 is defined by

(0) $(M_{Z_0,(G(C),Z_0)})_{q_0,(q_0,q_0)} = y_1$,

(1) $(M_{G(z),(G(G(z)),G(z))})_{q_0,(q_0,q_0)} = y_1$,

(2) $(M_{G(z),(z,z)})_{q_0,(q_1,q_2)} = b(y_1, y_2)$,

(3) $(M_{G(z),(z,z)})_{q_i,(q_i,q_i)} = b(y_1, y_2)$, $i = 1, 2$,

(4) $(P_C)_{q_i} = c_i$, $i = 1, 2$.

All other entries of the Z_0, C and $G(z)$ block row of M_1 and M_2 are zero. Moreover, $(P_C)_{q_0} = 0$ and $P_{Z_0} = 0$, $P_{G(z)} = 0$. Furthermore, $S_{q_0} = y_1$, $S_{q_1} = S_{q_2} = 0$.

The important entries of the vectors of the approximation sequence associated with $\mathfrak{P}$ are defined as follows for all $u \in T_\Gamma(\{z\})$ and $j \ge 0$:

$$\begin{aligned}
&(\tau^{j+1}_{Z_0})_{q_0} = (\tau^j_{G(C)})_{q_0}, \qquad (\tau^{j+1}_{Z_0})_{q_i} = 0,\ i = 1,2\,;\\
&(\tau^{j+1}_{C})_{q_0} = 0,\ (\tau^{j+1}_{C})_{q_i} = c_i,\ i = 1,2\,;\ (\tau^{j+1}_{z})_{q_i} = z_{q_i},\ i = 0,1,2\,;\\
&(\tau^{j+1}_{G(u)})_{q_0} = (\tau^j_{G(G(u))})_{q_0} + b((\tau^j_u)_{q_1}, (\tau^j_u)_{q_2})\,,\\
&(\tau^{j+1}_{G(u)})_{q_i} = b((\tau^j_u)_{q_i}, (\tau^j_u)_{q_i}),\ i = 1,2\,.
\end{aligned}$$

Let $G^k(C) \in T_\Gamma(\emptyset)$ be defined by $G^0(C) = C$, $G^{k+1}(C) = G(G^k(C))$, $k \ge 0$, and consider the equations for $G^k(C)$, $k \ge 0$, $j \ge 0$, $i = 1, 2$:

$$\begin{aligned}
&(\tau^{j+1}_{G^0(C)})_{q_0} = 0,\ (\tau^{j+1}_{G^0(C)})_{q_i} = c_i\,;\\
&(\tau^{j+1}_{G^k(C)})_{q_0} = (\tau^j_{G^{k+1}(C)})_{q_0} + b((\tau^j_{G^{k-1}(C)})_{q_1}, (\tau^j_{G^{k-1}(C)})_{q_2})\,, \quad k \ge 1\,;\\
&(\tau^{j+1}_{G^k(C)})_{q_i} = b((\tau^j_{G^{k-1}(C)})_{q_i}, (\tau^j_{G^{k-1}(C)})_{q_i})\,, \quad k \ge 1\,.
\end{aligned}$$

Let $\tau = \sup(\tau^j \mid j \in \mathbb{N})$. Then, for $k \ge 1$, $i = 1, 2$, $k \ge 1$,

$$\begin{aligned}
&(\tau_{G^0(C)})_{q_0} = 0,\ (\tau_{G^0(C)})_{q_i} = c_i\,;\\
&(\tau_{G^k(C)})_{q_0} = (\tau_{G^{k+1}(C)})_{q_0} + b((\tau_{G^{k-1}(C)})_{q_1}, (\tau_{G^{k-1}(C)})_{q_2})\,;\\
&(\tau_{G^k(C)})_{q_i} = b((\tau_{G^{k-1}(C)})_{q_i}, (\tau_{G^{k-1}(C)})_{q_i})\,.
\end{aligned}$$

Also, $(\tau_{Z_0})_{q_0} = (\tau_{G(C)})_{q_0}$.

Hence, $(\tau_{G^k(C)} \mid k \geq 0)$ is the least solution of the *polynomial* system

$$\begin{aligned}
&(z_0)_{q_0} = 0, \quad (z_0)_{q_i} = c_i,\ i = 1, 2\,;\\
&(z_k)_{q_0} = (z_{k+1})_{q_0} + b((z_{k-1})_{q_1}, (z_{k-1})_{q_2})\,, \quad k \geq 1\,;\\
&(z_k)_{q_i} = b((z_{k-1})_{q_i}, (z_{k-1})_{q_i})\,, \quad i = 1, 2,\ k \geq 1\,.
\end{aligned}$$

By Theorem 3.2, $(\tau_{G^k(C)} \mid k \geq 0)$ is also the least solution of the system

$$\begin{aligned}
&(z_0)_{q_0} = 0, \quad (z_0)_{q_i} = c_i,\ i = 1, 2\,;\\
&(z_k)_{q_0} = \textstyle\sum_{j \geq k-1} b((z_j)_{q_1}, (z_j)_{q_2})\,, \quad k \geq 1\,;\\
&(z_k)_{q_i} = b((z_{k-1})_{q_i}, (z_{k-1})_{q_i})\,, \quad i = 1, 2,\ k \geq 1\,.
\end{aligned}$$

This system is *proper* and has the unique solution $(\tau_{G^k(C)} \mid k \geq 0)$. Observe that this system is *not* polynomial.

Define now the trees $t_i^j \in T_F(\emptyset)$, $i = 1, 2$, $j \geq 0$, by

$$t_i^0 = c_i, \quad t_i^{j+1} = b(t_i^j, t_i^j),\ i = 1, 2,\ j \geq 0\,.$$

Let

$$(s_0)_{q_0} = 0,\ (s_0)_{q_i} = c_i,\ i = 1, 2\,, \quad (s_k)_{q_0} = \sum_{j \geq k-1} b(t_1^j, t_2^j),\ (s_k)_{q_i} = t_i^k,\ k \geq 1.$$

Then $((s_k)_{q_i} \mid k \geq 0,\ i = 0, 1, 2)$ is a solution of this proper system and, hence, $(s_k)_{q_i} = (\tau_{G^k(C)})_{q_i}$, $k \geq 0$, $i = 0, 1, 2$. Since $||\mathfrak{P}|| = (\tau_{Z_0})_{q_0} = (\tau_{G(C)})_{q_0}$, we infer that $||\mathfrak{P}|| = (s_1)_{q_0} = \sum_{j \geq 0} b(t_1^j, t_2^j)$.

This example indicates also a method to prove in a mathematically rigorous manner that the behavior of a pushdown tree automaton equals a certain formal tree series. $\square$

We now will refer to a result for pushdown tree automata that is analogous to Proposition 3.4.2 for pushdown automata. Intuitively, it states that the computations of the pushdown tree automaton governed by a pushdown store with contents $t(u_1, \ldots, u_m)$ (i. e., $\tau_{t(u_1, \ldots, u_m)}$), where $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$ and $u_i \in T_\Gamma(Z_m)$, $1 \leq i \leq m$, are the same as the computations governed by a pushdown store with contents $t(z_1, \ldots, z_m)$ (i. e., $\tau_{t(z_1, \ldots, z_m)}$) applied to the computations governed by pushdown stores with contents $u_1, \ldots, u_m$ (i. e., $\tau_{t(z_1, \ldots, z_m)}[\tau_{u_i}/F_{z_i}, 1 \leq i \leq m]$).

Theorem 6.5.1 (Kuich [85], Theorem 3.5) *Let τ be the least solution of the polynomial linear system $(*)$. Then, for all $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \leq m \leq \bar{m}$, and $u_i \in T_\Gamma(Z_m)$, $1 \leq i \leq m$,*

$$\tau_{t(u_1, \ldots, u_m)} = \tau_{t(z_1, \ldots, z_m)}[\tau_{u_i}/F_{z_i}, 1 \leq i \leq m]\,.$$

We now introduce algebraic tree systems. The definitions follow Kuich [85]. Let $\Phi = \{G_1, \dots, G_n\}$, $\Phi \cap \Sigma = \emptyset$, be a finite ranked alphabet of *function variables*, where G_i has rank r_i, $1 \le i \le n$, and $\bar{m} = \max\{r_i \mid 1 \le i \le n\}$.

Let $D = A\langle\langle T_\Sigma(X \cup Z_{r_1})\rangle\rangle \times \dots \times A\langle\langle T_\Sigma(X \cup Z_{r_n})\rangle\rangle$ and consider tree series $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$. Then each s_i induces a function $\bar{s}_i : D \to A\langle\langle T_\Sigma(X \cup Z_{r_i})\rangle\rangle$. For $(\tau_1, \dots, \tau_n) \in D$, we define inductively $\bar{s}_i(\tau_1, \dots, \tau_n)$ to be

(i) z_m if $s_i = z_m$, $1 \le m \le r_i$; x if $s_i = x$, $x \in X$;

(ii) $\bar{\omega}(\bar{t}_1(\tau_1, \dots, \tau_n), \dots, \bar{t}_r(\tau_1, \dots, \tau_n))$ if $s_i = \omega(t_1, \dots, t_r)$, $\omega \in \Sigma_r$, $t_1, \dots, t_r \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(iii) $\tau_j(\bar{t}_1(\tau_1, \dots, \tau_n), \dots, \bar{t}_{r_j}(\tau_1, \dots, \tau_n))$ if $s_i = G_j(t_1, \dots, t_{r_j})$, $G_j \in \Phi$, $t_1, \dots, t_{r_j} \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(iv) $a\bar{t}(\tau_1, \dots, \tau_n)$ if $s_i = at$, $a \in A$, $t \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(v) $\sum_{j\in J} \bar{r}_j(\tau_1, \dots, \tau_n)$ if $s_i = \sum_{j\in J} r_j$, $r_j \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $j \in J$, for an arbitrary index set J.

The mappings $\bar{s}_i$, $1 \le i \le n$, and the mapping $\bar{s} : D \to D$, where $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$ are continuous. This can be shown for $s_i \in T_{\Sigma\cup\Phi}(X \cup Z_r)$ by induction on the structure of s_i. If s_i is of the form $G_j(t_1, \dots, t_{r_j})$, the continuity of $\bar{s}_i$ follows from the continuity of substitution as shown by Proposition 2.5. Since scalar multiplication is continuous, it follows now that each $s_i = at$, where $a \in A$ and $t \in T_{\Sigma\cup\Phi}(X \cup Z_r)$ also induces a continuous function. The general case $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_r)\rangle\rangle$ can now be handled using the fact that summations preserve least upper bounds of directed sets. Hence, $\bar{s}$ has a least fixed point in D. (See also Lemmas 4.24 and 4.3 of Guessarian [61]; Lemmas 3.1 and 3.2 of Engelfriet, Schmidt [34]; Bloom, Ésik [10]; Ésik [35], Kuich [85], Lemma 3.6.) In certain situations, formulae are easier to read if we use the notation $s_i[\tau_1/G_1, \dots, \tau_n/G_n]$ instead of the notation $\bar{s}_i(\tau_1, \dots, \tau_n)$.

An *algebraic tree system* $\mathfrak{S} = (\Phi, Z, \Sigma, E)$ (with *function variables* in Φ, *variables* in Z and *terminal symbols* in Σ) has a set E of *formal equations*

$$G_i(z_1, \dots, z_{r_i}) = s_i(z_1, \dots, z_{r_i}), \quad 1 \le i \le n\,,$$

where each s_i is in $A\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle$. A *solution* to the algebraic tree system $\mathfrak{S}$ is given by $(\tau_1, \dots, \tau_n) \in D$ such that $\tau_i = \bar{s}_i(\tau_1, \dots, \tau_n)$, $1 \le i \le n$, i. e., by any fixed point $(\tau_1, \dots, \tau_n)$ of $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$. A solution $(\sigma_1, \dots, \sigma_n)$ of the algebraic tree system $\mathfrak{S}$ is called *least solution* iff $\sigma_i \sqsubseteq \tau_i$, $1 \le i \le n$, for all solutions $(\tau_1, \dots, \tau_n)$ of $\mathfrak{S}$. Since the least solution of $\mathfrak{S}$ is nothing else than the least fixed point of $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$, the least solution of the algebraic system $\mathfrak{S}$ exists in D. (See Wechler [115], Section 1.5.)

Theorem 6.5.2 *Let $\mathfrak{S} = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ be an algebraic tree system, where $s_i \in A\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle$. Then the least solution of this algebraic tree system $\mathfrak{S}$ exists in D and equals*

$$\mathrm{fix}(\bar{s}) = \sup(\bar{s}^j(0) \mid j \in \mathbb{N})\,,$$

where $\bar{s}^j$ is the j-th iterate of the mapping $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n \rangle : D \to D$ and $\bar{s}^0$ is the identity.

Theorem 5.2 indicates how we can compute an approximation to the least solution of an algebraic tree system. The *approximation sequence* $(\tau^j \mid j \in \mathbb{N})$, where each $\tau^j \in D$, *associated with* the algebraic tree system $\mathfrak{S} = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ is defined as follows:

$$\tau^0 = 0, \quad \tau^{j+1} = \bar{s}(\tau^j),\ j \in \mathbb{N}\,.$$

Since $\tau^j = \bar{s}^j(0)$ for all $j \in \mathbb{N}$, the least solution $\mathrm{fix}(\bar{s})$ of $\mathfrak{S}$ is equal to $\sup(\tau^j \mid j \in \mathbb{N})$. An algebraic tree system $\mathfrak{S} = (\Phi_0, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\}, G_0)$ (with function variables in $\Phi_0 = \{G_0, G_1, \dots, G_n\}$, variables in Z, terminal symbols in Σ) with *initial function variable* G_0 is an algebraic tree system $(\Phi_0, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$ such that G_0 has rank 0. Let $(\tau_0, \tau_1, \dots, \tau_n)$ be the least solution of $(\Phi_0, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$. Then τ_0 is called the *initial component* of the least solution. Observe that $\tau_0 \in A\langle\langle T_\Sigma(X)\rangle\rangle$ contains no variables of Z.

Our algebraic tree systems are second-order systems in the sense of Bozapalidis [15] and are a generalization of the context-free tree grammars. (See Rounds [102], and Engelfriet, Schmidt [34], especially Theorem 3.4, Wechler [115], Section 1.5, Courcelle [26], Section 9.5.)

A tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ is called *algebraic* iff it is the initial component of the least solution of an algebraic tree system with initial function variable. The collection of all these initial components is denoted by $A^{\mathrm{alg}}\langle\langle T_\Sigma(X)\rangle\rangle$. There is no restriction of the alphabets Σ and X in the definiton of an algebraic tree series, i. e., they may be infinite. This is due to the fact that, for any $s \in A^{\mathrm{alg}}\langle\langle T_\Sigma(X)\rangle\rangle$, there exist finite alphabets Σ' and X', $\Sigma' \subseteq \Sigma$, $X' \subseteq X$, such that $\mathrm{supp}(s) \subseteq T_{\Sigma'}(X')$. Hence,

$$A^{\mathrm{alg}}\langle\langle T_\Sigma(X)\rangle\rangle = \bigcup_{\Sigma' \subseteq \Sigma \text{ finite},\ X' \subseteq X \text{ finite}} A^{\mathrm{alg}}\langle\langle T_{\Sigma'}(X')\rangle\rangle\,.$$

We shall now show how an equivalent algebraic tree system $\mathfrak{S} = (\Phi_0, Z_Q, \Sigma, E, y_0)$ with initial function variable y_0 can be constructed from any given a pushdown tree automaton $\mathfrak{P} = (Q, \Gamma, Z, Y, M, S, p_0, P)$. (The construction follows Kuich [85].) Here $\Phi_0 = \{y_0\} \cup \{(y_{g(z_1,\dots,z_m)})_q \mid g \in \Gamma_m,\ 0 \le m \le \bar{m},\ q \in Q\}$. The function variable $(y_{g(z_1,\dots,z_m)})_q$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, $q \in Q$, has the rank $m|Q|$. By definition, the $Q \times 1$-vector $y_{g(z_1,\dots,z_m)}$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, is the column vector with q-component $(y_{g(z_1,\dots,z_m)})_q$, $q \in Q$.

For the specification of the formal equations in E we have to introduce, for $t \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, vectors $\hat{y}_t$ in $(T_\Phi(Z_Q^m))^{Q \times 1}$ as follows:

$$\begin{aligned} &\hat{y}_{g(u_1,\dots,u_m)} = y_{g(z_1,\dots,z_m)}(\hat{y}_{u_1}, \dots, \hat{y}_{u_m})\,, \\ &\qquad\qquad g \in \Gamma_m,\ u_1, \dots, u_m \in T_\Gamma(Z_m),\ 1 \le m \le \bar{m}\,; \\ &\hat{y}_g = y_g,\ g \in \Gamma_0\,; \quad (\hat{y}_{z_i})_q = (z_i)_q,\ 1 \le i \le \bar{m},\ q \in Q\,. \end{aligned}$$

Written componentwise, the first equation reads

$$(\hat{y}_{g(u_1,\dots,u_m)})_q = (y_{g(z_1,\dots,z_m)})_q((\hat{y}_{u_i})_{q'},\ 1 \le i \le m,\ q' \in Q)$$

for $g \in \Gamma_m$, $u_1, \dots, u_m \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, $q \in Q$. Observe that

$$(\hat{y}_{g(z_1,\dots,z_m)})_q = (y_{g(z_1,\dots,z_m)})_q((z_i)_{q'},\ 1 \le i \le m,\ q' \in Q)$$

for $g \in \Gamma_m$, $1 \le m \le \bar{m}$, $q \in Q$. Observe further that $y_{g(z_1,\dots,z_m)}(\hat{y}_{u_1}, \dots, \hat{y}_{u_m})$ means $y_{g(z_1,\dots,z_m)}[\hat{y}_{u_i}/F_{z_i},\ 1 \le i \le m]$ and $(y_{g(z_1,\dots,z_m)})_q((\hat{y}_i)_{q'},\ 1 \le i \le m,\ q' \in Q)$ means $(y_{g(z_1,\dots,z_m)})_q[(\hat{y}_i)_{q'}/(z_i)_{q'},\ 1 \le i \le m,\ q' \in Q]$. The formal equations in E are now given in matrix notation:

$$\begin{aligned}
& y_0 = S(y_{p_0})\,, \\
& y_{g(z_1,\dots,z_m)}((z_i)_{q'},\ 1 \le i \le m,\ q' \in Q) = \qquad\qquad (**)\\
& \quad (M(\hat{y}, \dots, \hat{y}) + F)_{g(z_1,\dots,z_m)} = \\
& \quad \textstyle\sum_{t_1,\dots,t_k \in T_\Gamma(Z_m)} M_{g(z_1,\dots,z_m),(t_1,\dots,t_k)}(\hat{y}_{t_1}, \dots, \hat{y}_{t_k}) + P_{g(z_1,\dots,z_m)}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad g \in \Gamma_m,\ 0 \le m \le \bar{m}\,.
\end{aligned}$$

We now give explicitely the formal equations, except the first one, with index $q \in Q$.

$$\begin{aligned}
& (y_{g(z_1,\dots,z_m)})_q((z_i)_{q'},\ 1 \le i \le m,\ q' \in Q) = \\
& \textstyle\sum_{t_1,\dots,t_k \in T_\Gamma(Z_m)} \sum_{q_1,\dots,q_k \in Q} \\
& \quad (M_{g(z_1,\dots,z_m),(t_1,\dots,t_k)})_{q,(q_1,\dots,q_k)}((\hat{y}_{t_1})_{q_1}, \dots, (\hat{y}_{t_k})_{q_k}) + (P_{g(z_1,\dots,z_m)})_q\,, \\
& \qquad\qquad\qquad\qquad\qquad\qquad g \in \Gamma_m,\ 0 \le m \le \bar{m},\ q \in Q\,.
\end{aligned}$$

Observe that indexing by $q \in Q$ is needed only in examples. In theoretical considerations, we save the indexing by states $q, q_1, \dots, q_n$, i. e., we use the form given further above.

The next theorem is a key result for proving the equivalence of pushdown tree automata and algebraic tree systems with initial function variable.

Theorem 6.5.3 (Kuich [85], Theorem 3.13) *If τ is the least solution of the polynomial linear system $(*)$ then $(\tau_{g(z_1,\dots,z_m)} \mid g \in \Gamma_m,\ 0 \le m \le \bar{m})$ is the least solution of the algebraic tree system $(**)$.*

Corollary 6.5.4 *The initial component of the least solution of the algebraic tree system $\mathfrak{S}$ coincides with $||\mathfrak{P}||$.*

Corollary 6.5.5 *The behavior of a pushdown tree automaton is an algebraic tree series.*

Example 6.5.1 (continued). We now construct step-by-step for the pushdown tree automaton $\mathfrak{P}$ the algebraic tree system $\mathfrak{S}$ with initial function variable such that $||\mathfrak{P}||$ is the initial component of its least solution. We first consider the linear system $(*)$ written in the form

$$\hat{y} = M(\hat{y}, \hat{y}) + F$$

and write down explicitly the equations for $\hat{y}_{G(z)}$, $\hat{y}_{Z_0}$ and $\hat{y}_C$:

$$\begin{aligned}
&(\hat{y}_{G(z)})_{q_0} = (\hat{y}_{G(G(z))})_{q_0} + b((\hat{y}_z)_{q_1}, (\hat{y}_z)_{q_2}),\\
&(\hat{y}_{G(z)})_{q_i} = b((\hat{y}_z)_{q_i}, (\hat{y}_z)_{q_i}),\ i = 1,2,\\
&(\hat{y}_{Z_0})_{q_0} = (\hat{y}_{G(C)})_{q_0}, \quad (\hat{y}_{Z_0})_{q_i} = 0,\ i = 1,2,\\
&(\hat{y}_C)_{q_0} = 0, \quad (\hat{y}_C)_{q_i} = c_i,\ i = 1,2\,.
\end{aligned}$$

Now we express the components of $\hat{y}$ by $y_{G(z)}$, y_{Z_0} and y_C; and obtain the algebraic system $(**)$:

$$\begin{aligned}
&(y_{G(z)})_{q_0}(z_{q_0}, z_{q_1}, z_{q_2}) = (y_{G(z)})_{q_0}((y_{G(z)})_{q_0}(z_{q_0}, z_{q_1}, z_{q_2}),\\
&\qquad (y_{G(z)})_{q_1}(z_{q_0}, z_{q_1}, z_{q_2}), (y_{G(z)})_{q_2}(z_{q_0}, z_{q_1}, z_{q_2})) + b(z_{q_1}, z_{q_2}),\\
&(y_{G(z)})_{q_i}(z_{q_0}, z_{q_1}, z_{q_2}) = b(z_{q_i}, z_{q_i}),\ i = 1,2,\\
&(y_{Z_0})_{q_0} = (y_{G(z)})_{q_0}((y_C)_{q_0}, (y_C)_{q_1}, (y_C)_{q_2}),\\
&(y_{Z_0})_{q_i} = 0,\ i = 1,2,\\
&(y_C)_{q_0} = 0,\\
&(y_C)_{q_i} = c_i,\ i = 1,2.
\end{aligned}$$

The algebraic tree system $\mathfrak{S} = (\Phi_0, Z, F, E, y_0)$ is now specified by
$\Phi_0 = \{(y_{G(z)})_{q_i}, (y_{Z_0})_{q_i}, (y_C)_{q_i} \mid i = 0,1,2\} \cup \{y_0\}$, where the ranks of $(y_{G(z)})_{q_i}$, $(y_{Z_0})_{q_i}$, $(y_C)_{q_i}$ are $3,0,0$, respectively, for $i = 0,1,2$;
$Z = \{z_{q_0}, z_{q_1}, z_{q_2}\}$;
E is the set of equations specified above augmented by the additional equation $y_0 = (y_{Z_0})_{q_0}$.

Observe that the construction of $\mathfrak{P}$ from $\mathfrak{S}$ is essentially the same construction as given by Guessarian [62] in her proof of Theorem 1. □

The converse of Corollary 5.5 can be proved and yields the main result of Kuich [85]. It is also the main result of this section.

Theorem 6.5.6 (Kuich [85], Corollary 3.17) *The following statements on a formal tree series s in $A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent:*

(i) s is an algebraic tree series;

(ii) s is the behavior of a pushdown tree automaton;

(iii) s is the behavior of a simple pushdown tree automaton with one initial state of weight 1.

Observe that the proof of Corollary 3.17 in Kuich [85] is given for pushdown tree automata defined by a finite sequence of transition matrices. But by the Remark given below the definition of tree automata in Section 3, this proof can easily be rewritten for pushdown tree automata according to our definition.

If our basic semiring is $\mathbb{N}^\infty$, i. e., if we consider tree series in $\mathbb{N}^\infty\langle\langle T_\Sigma(X)\rangle\rangle$, we can draw some stronger conclusions.

Let $G = (\Phi, Z, \Sigma, R)$ be a context-free tree grammar, where $\Phi = \{G_1, \ldots, G_n\}$ and R is the set of rules

$$G_i(z_1, \ldots, z_{r_i}) \to t_i^j, \quad 1 \le j \le n_i,\ 1 \le i \le n\,.$$

Denote by $d_i(t)$, $1 \le i \le n$, the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X \cup Z_{r_i})$ with respect to G starting from G_i. Let $\mathfrak{S} = (\Phi, Z, \Sigma, E)$ be an algebraic tree system, where E is the set of formal equations

$$G_i(z_1, \ldots, z_{r_i}) = \sum_{1 \le j \le n_i} t_i^j, \quad 1 \le i \le n\,.$$

Theorem 6.5.7 (Bozapalidis [15], Theorem 11 ii)) *Let $G = (\Phi, Z, \Sigma, R)$ and $\mathfrak{S} = (\Phi, Z, \Sigma, E)$ be the context-free tree grammar and the algebraic tree system, respectively, considered above. Let $d_i(t)$, $1 \le i \le n$, be the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X \cup Z_{r_i})$ with respect to G starting from G_i. Then the least solution of $\mathfrak{S}$ is given by*

$$\Big(\sum_{t \in T_\Sigma(X \cup Z_{r_i})} d_i(t)t \mid 1 \le i \le n \Big)\,.$$

Theorems 5.7, 3.1 and Corollary 5.6 yield the following theorem.

Theorem 6.5.8 *Let $d : T_\Sigma(X) \to \mathbb{N}^\infty$. Then the following statements are equivalent:*

(i) There exists a context-free tree grammar with an initial function variable, and with terminal alphabet Σ and leaf alphabet X such that the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X)$ from the initial function variable is given by $d(t)$.

(ii) There exists a 1-simple pushdown tree automaton with input alphabet Σ and leaf alphabet X such that the number (possibly ∞) of distinct accepting computations for $t \in T_\Sigma(X)$ is given by $d(t)$.

A context-free tree grammar with initial function variable, and with terminal alphabet Σ and leaf alphabet X is called *unambiguous* iff for all $t \in T_\Sigma(X)$ the number of distinct leftmost derivations of t with respect to G is either 1 or 0. A 1-simple pushdown tree automaton with terminal alphabet Σ and leaf alphabet X is called *unambiguous* iff for all $t \in T_\Sigma(X)$ the number of distinct accepting computations for t is either 1 or 0.

Corollary 6.5.9 *Let $L \subseteq T_\Sigma(X)$ be a tree language. Then L is generated by an unambiguous context-free tree grammar iff $\sum_{t \in L} t$ is the behavior of an unambiguous 1-simple pushdown tree automaton.*

A pushdown tree automaton $\mathfrak{P} = (Q, \Gamma, Z, Y, M, S, p_0, P)$ is called *restricted* iff $\Gamma = \{p_0\} \cup \Gamma_1$, i. e., except for the initial pushdown symbol p_0 of rank 0, all other pushdown symbols have rank 1.

The next theorem augments the list of equivalent statements of Corollary 5.6.

Theorem 6.5.10 (Kuich [85], Corollary 4.8) *The following statements on a formal tree series s in $A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent*

(i) s is an algebraic tree series;

(ii) s is the behavior of a restricted pushdown tree automaton;

(iii) s is the behavior of a simple restricted pushdown tree automaton.

We now turn to formal tree series in $\mathbb{N}^\infty\langle\langle T_\Sigma(X)\rangle\rangle$.

Theorem 6.5.11 *Let $d : T_\Sigma(X) \to \mathbb{N}^\infty$. Then the following statement is equivalent to the statements of Theorem 5.8:*

(iii) There exists a 1-simple restricted pushdown tree automaton with input alphabet Σ and leaf alphabet X such that the number (possibly ∞) of distinct accepting computations for $t \in T_\Sigma(X)$ is given by $d(t)$.

Corollary 6.5.12 *Let $L \subseteq T_\Sigma(X)$ be a tree language. Then L is generated by an unambiguous context-free tree grammar iff $\sum_{t\in L} t$ is the behavior of an unambiguous 1-simple restricted pushdown tree automaton.*

We now prove a Kleene Theorem due to Bozapalidis [15]. For the remainder of this chapter, $\Phi_\infty = \{G_i \mid i \geq 0\}$ denotes an infinite ranked alphabet of function variables, where G_i has rank r_i, $i \geq 0$, and for each $r \geq 0$ there are infinitely many function variables with rank r. Let $\hat{\Sigma} = \Sigma \cup \{G_{k_1}, \ldots, G_{k_m}\}$ and $\hat{D} = A\langle\langle T_{\hat{\Sigma}}(X \cup Z_{r_{i_1}})\rangle\rangle \times \cdots \times A\langle\langle T_{\hat{\Sigma}}(X \cup Z_{r_{i_n}})\rangle\rangle$ for some pairwise different $i_1, \ldots, i_n, k_1, \ldots, k_m \geq 0$. Consider a tree series $s \in A\langle\langle T_{\hat{\Sigma}\cup\{G_{i_1},\ldots,G_{i_n}\}}(X \cup Z_r)\rangle\rangle$. (The function variables $G_{k_1}, \ldots, G_{k_m}$ are considered here to be ranked symbols like in Σ.) Then s induces a function $\bar{s} : \hat{D} \to A\langle\langle T_{\hat{\Sigma}}(X \cup Z_r)\rangle\rangle$ as defined above.

We now consider an algebraic tree system $\mathfrak{S} = (\{G_{i_1}, \ldots, G_{i_n}\}, Z, \hat{\Sigma}, E)$, where E is $G_{i_j}(z_1, \ldots, z_{r_{i_j}}) = s_j(z_1, \ldots, z_{r_{i_j}}, G_{i_1}, \ldots, G_{i_n})$, $1 \leq j \leq n$, and $s_j \in A\langle T_{\hat{\Sigma}\cup\{G_{i_1},\ldots,G_{i_n}\}}(X \cup Z_{r_{i_j}})\rangle$.

The least solution of $\mathfrak{S}$ is in $\hat{D}$. The collection of components of least solutions of all such algebraic systems (with free choice of pairwise different $i_1, \ldots, i_n, k_1, \ldots, k_m \geq 0$) is denoted by $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. Observe that each power series in $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ is in fact a power series in $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi}(X\cup Z_r)\rangle\rangle$ for some finite $\Phi \subset \Phi_\infty$ and some $r \geq 0$.

Before proving our results we apply a few results of the fixed point theory of continuous functions to algebraic tree systems (see the Preliminaries of Chapter 2). An *extended* algebraic tree system $\mathfrak{S} = (\Phi, Z, \Sigma, E)$ and its least solution are defined as an algebraic tree system and its least solution with the exception that the right sides of the equations $G_i(z_1, \ldots, z_{r_i}) = s_i(z_1, \ldots, z_{r_i})$, $1 \le i \le n$, are now in $A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$.

(1) *The parameter identity.* Let $r \in A\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$, and denote $r' = \mu G.r$, $G \in \Phi_\infty$. Let $G_i \ne G$, $\tau_i \in A\langle\langle T_{\Sigma\cup(\Phi_\infty-\{G\})}(X \cup Z)\rangle\rangle$, $1 \le i \le n$, and $\sigma_j \in A\langle\langle T_\Sigma(X\cup Z)\rangle\rangle$, $1 \le j \le k$. Then $r'[\sigma_1/z_1, \ldots, \sigma_k/z_k, \tau_1/G_1, \ldots, \tau_n/G_n] = \mu G.(r[\sigma_1/z_1, \ldots, \sigma_k/z_k, \tau_1/G_1, \ldots, \tau_n/G_n])$.

(2) *The Bekić-De Bakker-Scott rule.* Consider the equations $G_i(z_1, \ldots, z_{r_i}) = s_i(z_1, \ldots, z_{r_i})$, $1 \le i \le n$, $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$ of an extended algebraic tree system $\mathfrak{S} = (\Phi, Z, \Sigma, E)$, where $\Phi = \{G_1, \ldots, G_n\}$, and let $m \in \{1, \ldots, n-1\}$. Let $(\tau_{m+1}, \ldots, \tau_n)$ be the least solution of the extended algebraic tree system $\mathfrak{S}' = (\Phi', Z, \Sigma, E')$, where $\Phi' = \{G_{m+1}, \ldots, G_n\}$ and $E' = \{G_i(z_1, \ldots, z_{r_i}) = s_i(z_1, \ldots, z_{r_i}) \mid m+1 \le i \le n\}$. Hence, $\tau_j \in A\langle\langle T_{\Sigma\cup\{G_1,\ldots,G_m\}}(X \cup Z_{r_j})\rangle\rangle$, $m+1 \le i \le n$. Furthermore, let $(\tau_1, \ldots, \tau_m)$ be the least solution of the extended algebraic system $\mathfrak{S}'' = (\Phi'', Z, \Sigma, E'')$, where $\Phi'' = \{G_1, \ldots, G_m\}$ and $E'' = \{G_i(z_1, \ldots, z_{r_i}) = s_i(z_1, \ldots, z_{r_i})[\tau_{m+1}/G_{m+1}, \ldots, \tau_n/G_n] \mid 1 \le i \le m\}$. Then

$$(\tau_1, \ldots, \tau_m, \tau_{m+1}[\tau_1/G_1, \ldots, \tau_m/G_m], \ldots, \tau_n[\tau_1/G_1, \ldots, \tau_m/G_m])$$

is the least solution of the original extended algebraic tree system.

We now proceed analogously to Section 4.

Theorem 6.5.13 *$\langle A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle, +, 0, \bar\Sigma\cup\bar\Phi_\infty\rangle$ is a distributive $\Sigma\cup\Phi_\infty$-algebra that contains $A\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle$ and is closed under scalar product.*

Hence, for any $G \in \Phi_\infty$ of rank r and any $\sigma_1, \ldots, \sigma_r \in A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$, $\bar G(\sigma_1, \ldots, \sigma_r)$ is again in $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$.

Proof. We only prove the second sentence. The proof of the first sentence is analogous to the proof of Theorem 4.1.

Let $\sigma_1, \ldots, \sigma_r \in A\langle\langle T_{\Sigma\cup\{G_{k_1},\ldots,G_{k_m}\}}(X \cup \{z_1, \ldots, z_k\})\rangle\rangle$. Then there exist r algebraic tree systems $G_{tj}(z_1, \ldots, z_{i_{tj}}) = s_{tj}$, $1 \le t \le r$, $1 \le j \le n_r$, where the rank of G_{t1} is k, such that the first components of their least solutions are σ_t.

Consider now the algebraic tree system

$$\begin{aligned} &H(z_1, \ldots, z_k) = G(G_{11}(z_1, \ldots, z_k), \ldots, G_{r1}(z_1, \ldots, z_k))\,, \\ &G_{tj}(z_1, \ldots, z_{i_{tj}}) = s_{tj}, \quad 1 \le t \le r,\ 1 \le j \le n_r\,. \end{aligned}$$

By the Bekić-De Bakker-Scott rule, the H-component of its least solution is then given by $\bar G(\sigma_1, \ldots, \sigma_r)$. □

A distributive $\Sigma \cup \Phi_\infty$-algebra $\langle V, +, 0, \bar\Sigma \cup \bar\Phi_\infty\rangle$, $V \subseteq A\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$ is called *equationally closed* iff V is closed under scalar product, and for all $s \in V$ and $G \in \Phi_\infty$ the formal tree series $\mu G.s$ is again in V. Here $\mu G.s$ denotes the least solution of $G(z_1, \ldots, z_r) = s$, where r is the rank of G. By definition, $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$ is the least equationally closed distributive $\Sigma \cup \Phi_\infty$-algebra containing $A\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle$. Observe that each power series in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ is in fact a power series in $A\langle\langle T_{\Sigma\cup\Phi}(X\cup Z_r)\rangle\rangle$ for some finite $\Phi \subset \Phi_\infty$ and $r \ge 0$.

We will prove that $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle = A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. We first show that $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$ is closed under substitution for function variables.

Theorem 6.5.14 *Consider tree series s and σ_j, $1 \le j \le n$, in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ and assume that $s(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n}) \in A\langle\langle T_{\hat\Sigma\cup\{G_{i_1},\dots,G_{i_n}\}}(X \cup Z_r)\rangle\rangle$ and $\sigma_j \in A\langle\langle T_{\hat\Sigma}(X\cup Z_{r_{i_j}})\rangle\rangle$, where $\hat\Sigma = \Sigma\cup\{G_{k_1},\dots,G_{k_m}\}$ and $i_1,\dots,i_n,k_1,\dots,k_m \ge 0$ are pairwise disjoint.*

Then $\bar s(z_1,\dots,z_r,\sigma_1,\dots,\sigma_n)$ is again in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$.

Proof. The proof is by induction on the number of applications of the operations $\bar\omega \in \bar\Sigma$, $\bar G \in \bar\Phi_\infty$, sum, scalar product and μ to generate $s(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n})$ from polynomials.

(i) Let $s(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n}) \in A\langle T_{\hat\Sigma\cup\{G_{i_1},\dots,G_{i_n}\}}(X\cup Z_r)\rangle$. Since $\bar s(z_1,\dots,z_r,\sigma_1,\dots,\sigma_n)$ is generated from $\sigma_1,\dots,\sigma_n$ and $z_1,\dots,z_r$ by application of sum, $\bar\omega \in \bar\Sigma$, $\bar G_{k_1},\dots,\bar G_{k_m}$, substitution into $\sigma_1,\dots,\sigma_n$ (which is handled by a theorem similar to Theorem 4.2), and scalar product, we infer that $\bar s(z_1,\dots,z_r,\sigma_1,\dots,\sigma_n) \in A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$.

(ii) We only prove the case of the operator μ. Choose a $G \in \Phi_\infty$ with rank r that is distinct from $G_{i_1},\dots,G_{i_n},G_{k_1},\dots,G_{k_m}$. Without loss of generality we assume that $s(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n}) = \mu G.s'(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n},G)$. By induction hypothesis, we have that $s'(z_1,\dots,z_r,G_{i_1},\dots,G_{i_n},G)$ is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$. Hence $\bar s(z_1,\dots,z_r,\sigma_1,\dots,\sigma_n) = \mu G.\bar s'(z_1,\dots,z_r,\sigma_1,\dots,\sigma_n,G)$ is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ by the parameter identity. □

Theorem 6.5.15 (Bozapalidis [15], Section 6.) $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle = A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$

Proof. (i) We show that $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle \subseteq A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. The proof is by induction on the number of variables of algebraic systems. We use the following induction hypothesis:

If $(\tau_1,\dots,\tau_n)$, where $\tau_j \in A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$, $1 \le j \le n$, is the least solution of an algebraic system $G_j(z_1,\dots,z_{r_j}) = s_j(z_1,\dots,z_{r_j},G_1,\dots,G_n)$, $1 \le j \le n$, with n function variables $G_1,\dots,G_n$, where $s_j(z_1,\dots,z_{r_j},G_1,\dots,G_n) \in A\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle$, then $\tau_j \in A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$.

(1) Let $n = 1$ and assume that $s \in A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$ is the least solution of the algebraic system $G_1(z_1,\dots,z_{r_1}) = p(z_1,\dots,z_{r_1},G_1)$. Since $p(z_1,\dots,z_{r_1},G_1)$ is a polynomial, $s = \mu G_1.p(z_1,\dots,z_{r_1},G_1) \in A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$.

(2) Consider the algebraic system $G_j(z_1,\dots,z_{r_j}) = s_j(z_1,\dots,z_{r_j},G_1,\dots,G_{n+1})$, $1 \le j \le n+1$, $n \ge 1$. Let $(\tau_2(G_1),\dots,\tau_{n+1}(G_1))$, $\tau_j(G_1) \in A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$, $2 \le j \le n+1$, be the least solution of the algebraic system $G_j(z_1,\dots,z_{r_j}) = s_j(z_1,\dots,z_{r_j},G_1,\dots,G_{n+1})$, $2 \le j \le n+1$. By our induction hypothesis we infer that $\tau_j(G_1) \in A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. Hence, by Theorem 5.14, $p(G_1) = \bar s_1(z_1,\dots,z_{r_1},G_1,\tau_2(G_1),\dots,\tau_{n+1}(G_1))$ is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. This implies that $\mu G_1.p(G_1)$ is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$. Again by Theorem 5.14

$\bar{\tau}_j(\mu G_1.p(G_1)) \in A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$, $2 \le j \le n+1$. By the Bekić-De Bakker-Scott rule,

$$(\mu G_1.p(G_1), \bar{\tau}_2(\mu G_1.p(G_1)), \ldots, \bar{\tau}_{n+1}(\mu G_1.p(G_1)))$$

is the least solution of the algebraic system $G_j(z_1, \ldots, z_{r_j}) = s_j(z_1, \ldots, z_{r_j}, G_1, \ldots, G_{n+1})$, $1 \le j \le n+1$. Hence, the components of the least solution of this algebraic system are in $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$.

(ii) We show that $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ is an equationally closed distributive $\Sigma \cup \Phi_\infty$-algebra that contains $A\langle T_\Sigma(X \cup Z\rangle$. This will imply $A^{\mathrm{equ}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle \subseteq A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$. By Theorem 5.13, we have only to show that $\mu G.s$, $s \in A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$ is in $A^{\mathrm{alg}}\langle\langle T_{\Sigma\cup\Phi_\infty}(X\cup Z)\rangle\rangle$. Let $(\tau_2(G_1), \ldots, \tau_{n+1}(G_1))$ be the least solution of the algebraic system $G_j(z_1, \ldots, z_{r_j}) = s_j(z_1, \ldots, z_{r_j}, G_1, \ldots, G_{n+1})$, $2 \le j \le n+1$, let G_1 be of rank r_2 and take $s = \tau_2$. Consider now the algebraic system $G_1(z_1, \ldots, z_{r_2}) = s_2(z_1, \ldots, z_{r_2}, G_1, \ldots, G_{n+1})$, $G_j(z_1, \ldots, z_{r_j}) = s_j(z_1, \ldots, z_{r_j}, G_1, \ldots, G_{n+1})$, $2 \le j \le n+1$. Then, by the Bekić-De Bakker-Scott rule, $\mu G_1.\bar{s}_2(z_1, \ldots, z_{r_2}, G_1, \tau_2(G_1), \ldots, \tau_{n+1}(G_1)) = \mu G_1.\tau_2(G_1)$ is the first component of its least solution. □

We now introduce algebraic tree series expressions. Assume that $A, \Sigma, X, Z, \Phi_\infty$ and $U = \{+, \cdot, \mu, [,]\}$ are pairwise disjoint. A word E over $A\cup\Sigma\cup X\cup Z\cup\Phi_\infty\cup U$ is an *algebraic tree series expression over* $(A, \Sigma, X, Z, \Phi_\infty)$ iff

(i) E is in $X \cup Z$, or

(ii) E is of one of the forms $[E_1 + E_2]$, $\omega(E_1, \ldots, E_k)$, $G(E_1, \ldots, E_k)$, aE_1, or $\mu G.E_1$, where $E_1, \ldots, E_k$ are algebraic tree series expressions over $(A, \Sigma, X, Z, \Phi_\infty)$, for $\omega \in \Sigma$ of rank k, $G \in \Phi_\infty$ of rank k, $k \ge 0$, and $a \in A$.

Each algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$ denotes a formal tree series $|E|$ in $A\langle\langle T_{\Sigma\cup\Phi_\infty}(X \cup Z)\rangle\rangle$ according to the following conventions:

(i) If E is in $X \cup Z$ then E denotes the tree series E, i. e., $|E| = E$.

(ii) For algebraic tree series expressions $E_1, \ldots, E_k$ over $(A, \Sigma, X, Z, \Phi_\infty)$, $\omega \in \Sigma$ of rank k, $G \in \Phi_\infty$ of rank k, $k \ge 0$, $a \in A$, we define

$$\begin{aligned}
&|[E_1 + E_2]| = |E_1| + |E_2|\,,\\
&|\omega(E_1, \ldots, E_k)| = \bar{\omega}(|E_1|, \ldots, |E_k|)\,,\\
&|G(E_1, \ldots, E_k)| = \bar{G}(|E_1|, \ldots, |E_k|)\,,\\
&|aE_1| = a|E_1|\,,\\
&|\mu G.E_1| = \mu G.|E_1|\,.
\end{aligned}$$

Let $\varphi_1, \varphi_2, \varphi_3$ be the mappings from the set of algebraic tree series expressions over $(A, \Sigma, X, Z, \Phi_\infty)$ into the set of finite subsets of $X \cup Z \cup \Phi_\infty$ defined by

(i) $\varphi_1(x) = \emptyset$, $\varphi_2(x) = \{x\}$, $\varphi_3(x) = \emptyset$, $x \in X$,
$\varphi_1(z) = \{z\}$, $\varphi_2(z) = \emptyset$, $\varphi_3(z) = \emptyset$, $z \in Z$.

(ii) $\varphi_j([E_1 + E_2]) = \varphi_j(E_1) + \varphi_j(E_2)$, $j = 1, 2, 3$,
$\varphi_j(\omega(E_1, \dots, E_k)) = \varphi_j(E_1) \cup \dots \cup \varphi_j(E_k)$, $j = 1, 2, 3$,
$\varphi_j(G(E_1, \dots, E_k)) = \varphi_j(E_1) \cup \dots \cup \varphi_j(E_k)$, $j = 1, 2$,
$\varphi_3(G(E_1, \dots, E_k)) = \varphi_3(E_1) \cup \dots \cup \varphi_3(E_k) \cup \{G\}$,
$\varphi_j(aE_1) = \varphi_j(E_1)$, $a \neq 0$, $\varphi_j(0E_1) = \emptyset$, $a = 0$, $j = 1, 2, 3$,
$\varphi_j(\mu G.E_1) = \varphi_j(E_1) - \{G\}$ $j = 1, 2, 3$,
for algebraic tree series expressions $E_1, \dots, E_k$ over $(A, \Sigma, X, Z, \Phi_\infty)$, $\omega \in \Sigma$ of rank k, $G \in \Phi_\infty$ of rank k, $k \geq 0$, $a \in A$.

Given an algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$, $\varphi_1(E) \subseteq Z$ contains the variables of E, $\varphi_2(E) \subseteq X$ contains the used symbols of the leaf alphabet X and $\varphi_3(E) \subseteq G$ contains the "free function variables" of E. This means that $|E|$ is a formal tree series in $A\langle\langle T_{\Sigma \cup \varphi_3(E)}(\varphi_2(E) \cup \varphi_1(E))\rangle\rangle$.

Theorem 5.15 and the above definitions yield some corollaries.

Corollary 6.5.16 *A tree series s is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma \cup \Phi_\infty}(X \cup Z)\rangle\rangle \cap A\langle\langle T_{\Sigma \cup \Phi'}(X' \cup Z')\rangle\rangle$, with $X' \subseteq X$, $Z' \subseteq Z$ and $\Phi' \subseteq \Phi_\infty$, iff there is an algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$ such that $s = |E|$, where $\varphi_2(E) = X'$, $\varphi_1(E) = Z'$ and $\varphi_3(E) = \Phi'$.*

Corollary 6.5.17 *A tree series s is in $A^{\mathrm{equ}}\langle\langle T_{\Sigma \cup \Phi_\infty}(X \cup Z)\rangle\rangle \cap A\langle\langle T_\Sigma(X')\rangle\rangle$, with $X' \subseteq X$, iff there is an algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$ such that $s = |E|$, where $\varphi_1(E) = \varphi_3(E) = \emptyset$ and $\varphi_2(E) = X'$.*

Corollary 6.5.18 *A tree series is in $A^{\mathrm{alg}}\langle\langle T_\Sigma(X')\rangle\rangle$, with $X' \subseteq X$, iff there exists an algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$ such that $s = |E|$, where $\varphi_1(E) = \varphi_3(E) = \emptyset$ and $\varphi_2(E) = X'$.*

We summarize our results in a Kleene-like theorem.

Theorem 6.5.19 *The following statements on a power series $r \in A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent.*

(i) r is an algebraic tree series,

(ii) r is the behavior of a simple pushdown tree automaton,

(iii) r is the behavior of a simple restricted pushdown tree automaton,

(iv) there exists an algebraic tree series expression E over $(A, \Sigma, X, Z, \Phi_\infty)$, where $\varphi_1(E) = \varphi_3(E) = \emptyset$, such that $r = |E|$.

If we interpret algebraic tree series expressions in $\mathbb{B}\langle\langle T_{\Sigma \cup \Phi_\infty}(X \cup Z)\rangle\rangle$ then we get analogous results on formal tree languages.

Example 6.5.2. Consider the algebraic tree system $\mathfrak{S} = (\Phi, Z, \Sigma, E, G_0)$ with initial function variable G_0, specified by $\Phi = \Phi_0 \cup \Phi_2$, $\Phi_0 = \{G_0\}$, $\Phi_2 = \{G\}$, $\Sigma = \Sigma_2 = \{b\}$, $X = \{c_1, c_2\}$, and $E = \{G_0 = G(c_1, c_2), G(z_1, z_2) =$

$G(b(z_1, z_1), b(z_2, z_2)) + b(z_1, z_2)\}$. (This algebraic system is a simplified version of that in Example 5.1.) The initial component of its least solution is given by

$$|\mu G.[G(b(c_1, c_1), b(c_2, c_2)) + b(c_1, c_2)]| = \sum_{j \geq 0} b(t_1^j, t_2^j),$$

where $t_i^0 = c_i$, $t_i^{j+1} = b(t_i^j, t_i^j)$, $i = 1, 2$, $j \geq 0$. □

6.6 Tree series transducers

Tree transducers have been introduced in Rounds [101, 103] and Thatcher [111, 112]. (See also Fülöp, Vogler [49].) Kuich [81] generalized a restricted form of top-down tree transducers to tree series transducers which map formal tree series into formal tree series. Engelfriet, Fülöp, Vogler [33] and Fülöp, Vogler [50] generalized this approach and defined bottom-up and top-down tree series transducers as generalization of frontier-to-root and root-to-frontier tree transducers in the sense of Gécseg, Steinby [51, 52].

In this section we only consider the case of top-down tree series transducers. (The bottom-up tree series transducers use a generalization of IO-substitutions and are difficult to handle.) Our definition of top-down tree series transducers is different but equivalent to the definition of Engelfriet, Fülöp, Vogler [33].

We then define nondeterministic simple recognizable tree series transducers and show that they preserve recognizability of tree series.

A tree $t \in T_\Sigma(X \cup Y_m)$, $m \geq 1$, is called *linear* iff the variable y_j appears at most once in t, $1 \leq j \leq m$. A tree $t \in T_\Sigma(X \cup Y_m)$, $m \geq 1$, is called *nondeleting* iff the variable y_j appears at least once in t, $1 \leq j \leq m$. A tree series $s \in A\langle\langle T_\Sigma(X \cup Y_m)\rangle\rangle$, $m \geq 1$, is called *linear* or *nondeleting* iff all $t \in \mathrm{supp}(s)$ are linear or nondeleting, respectively.

We define

$$(A\langle\langle T_\Sigma(X)\rangle\rangle)^{I_1 \times I_2^*} = \bigcup_{m \geq 0} (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I_1 \times I_2^m}.$$

A *tree representation* μ (with *state set* Q, *ranked input alphabet* Σ, *input leaf alphabet* X, *ranked output alphabet* Σ', *output leaf alphabet* X', *over the semiring* A) is a family $\mu = (\mu_k \mid k \geq 0)$ of mappings

$$\begin{aligned} &\mu_k : \Sigma_k \to (A\langle\langle T_{\Sigma'}(X' \cup Y)\rangle\rangle)^{Q \times (Q \times Z_k)^*}, \ k \geq 1, \\ &\mu_0 : \Sigma_0 \cup X \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}, \end{aligned}$$

such that, if $\mu_k(\omega) \in (A\langle\langle T_{\Sigma'}(X' \cup Y)\rangle\rangle)^{Q \times (Q \times Z_k)^m}$ for some $m \geq 0$ and $\omega \in \Sigma_k$, $k \geq 1$, then $\mu_k(\omega) \in (A\langle\langle T_{\Sigma'}(X' \cup Y_m)\rangle\rangle)^{Q \times (Q \times Z_k)^m}$ and every entry of $\mu_k(\omega)$ is linear and nondeleting. Observe that $\mu_k(\omega)$ with $\omega \in \Sigma_k$, $k \geq 1$, induces a mapping

$$\mu_k(\omega) : (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1} \times \cdots \times (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1} \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}$$

(there are k argument vectors; see the definition before Theorem 2.8).

Since $\langle (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q\times 1}, (\mu_k(\omega) \mid \omega \in \Sigma_k,\ k \geq 0)\rangle$ is a Σ-algebra, the mapping

$$\mu_0 : X \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q\times 1}$$

can be uniquely extended to a morphism

$$\mu : T_\Sigma(X) \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q\times 1}$$

by

$$\mu(\omega(t_1, \ldots, t_k)) = \mu_k(\omega)[\mu(t_1), \ldots, \mu(t_k)]\,,$$

for $\omega \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(X)$, $k \geq 0$.

One more extension yields

$$\mu : A\langle\langle T_\Sigma(X)\rangle\rangle \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q\times 1}$$

by

$$\mu(s) = \sum_{t\in T_\Sigma(X)} (s,t) \otimes \mu(t), \quad s \in A\langle\langle T_\Sigma(X)\rangle\rangle\,,$$

where $\otimes$ denotes the Kronecker product. (See Kuich, Salomaa [88], Section 4.) In our case this means that each entry of $\mu(t)$ is multiplied by (s,t). Hence, for $q \in Q$,

$$\mu(s)_q = \sum_{t\in T_\Sigma(X)} (s,t)\mu(t)_q, \quad s \in A\langle\langle T_\Sigma(X)\rangle\rangle\,.$$

We have denoted a tree representation μ and the mapping $\mu : A\langle\langle T_\Sigma(X)\rangle\rangle \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q\times 1}$ induced by it by the same letter μ. This should not lead to any confusion.

A (*top-down*) *tree series transducer* (with *state set* Q, *ranked input alphabet* Σ, *input leaf alphabet* X, *ranked output alphabet* Σ', *output leaf alphabet* X', *over the semiring* A)

$$\mathfrak{T} = (Q, \mu, S)$$

is given by

(i) a non-empty finite set Q of *states*,

(ii) a *tree representation* μ with $Q, \Sigma, X, \Sigma', X'$ over A,

(iii) an *initial state vector* $S \in (A\langle T_{\Sigma'}(Y_1)\rangle)^{1\times Q}$, where $S_q = a_q y_1$, $a_q \in A$, $q \in Q$.

The mapping

$$||\mathfrak{T}|| : A\langle\langle T_\Sigma(X)\rangle\rangle \to A\langle\langle T_{\Sigma'}(X')\rangle\rangle$$

realized by a tree series transducer $\mathfrak{T} = (Q, \mu, S)$ is defined by

$$\begin{aligned}&||\mathfrak{T}||(s) = S(\mu(s)) = \textstyle\sum_{q\in Q}(S_q, y_1)\mu(s)_q = \\ &\textstyle\sum_{q\in Q}\sum_{t\in T_\Sigma(X)} a_q(s,t)\mu(t)_q, \quad s \in A\langle\langle T_\Sigma(X)\rangle\rangle\,.\end{aligned}$$

A tree representation μ is called *polynomial* iff the entries of the images of μ_k, $k \geq 0$, are polynomials. A tree series transducer $\mathfrak{T} = (Q, \mu, S)$ is called *polynomial* iff μ is a polynomial tree representation.

Example 6.6.1. (See Example IV.1.6 of Gécseg, Steinby [51].) Let $Q = \{a_0, a_1, a_2\}$, $\Sigma = \Sigma_1 = \{\sigma\}$, $X = \{x\}$, $\Sigma' = \Sigma'_1 \cup \Sigma'_2$, $\Sigma'_1 = \{\omega_1\}$, $\Sigma'_2 = \{\omega_2\}$, $X' = \{x'_1, x'_2\}$.

The nonnull entries of μ_0 and μ_1 are given by

$$\begin{aligned}
&\mu_0(x)_{a_1} = x'_1, \quad \mu_0(x)_{a_2} = x'_2\,,\\
&\mu_1(\sigma)_{a_0,((a_1,z_1),(a_2,z_1))} = \omega_2(y_1, y_2)\,,\\
&\mu_1(\sigma)_{a_1,(a_1,z_1)} = \omega_1(y_1), \quad \mu_1(\sigma)_{a_2,(a_2,z_1)} = \omega_1(y_1)\,.
\end{aligned}$$

Let $S = (y_1, 0, 0)$ and consider the polynomial tree series transducer $\mathfrak{T} = (Q, (\mu_0, \mu_1), S)$. We claim that, for $n \geq 0$,

$$\mu(\sigma^n(x))_{a_i} = \omega_1^n(x'_i), \quad i = 1, 2\,,$$

and prove it by induction on n.

We have

$$\mu(x)_{a_i} = \mu_0(x)_{a_i} = x'_i\,,$$

and, for $n > 0$,

$$\begin{aligned}
&\mu(\sigma^n(x))_{a_i} = \mu_1(\sigma)[\mu(\sigma^{n-1}(x))]_{a_i} =\\
&\mu_1(\sigma)_{a_i,(a_i,z_1)}[\omega_1^{n-1}(x'_i)] =\\
&\omega_1(y_1)(\omega_1^{n-1}(x'_i)) = \omega_1^n(x'_i), \quad i = 1, 2\,.
\end{aligned}$$

Hence, we obtain for $n \geq 1$,

$$\begin{aligned}
&||\mathfrak{T}||(\sigma^n(x)) = \mu(\sigma^n(x))_{a_0} =\\
&\mu_1(\sigma)[\mu(\sigma^{n-1}(x))]_{a_0} =\\
&\mu_1(\sigma)_{a_0,((a_1,z_1),(a_2,z_1))}(\mu(\sigma^{n-1}(x))_{a_1}, \mu(\sigma^{n-1}(x))_{a_2}) =\\
&\omega_2(\omega_1^{n-1}(x'_1), \omega_1^{n-1}(x'_2))\,.
\end{aligned}$$

Given a formal tree series

$$s = (s, x)x + \sum_{n \geq 1} (s, \sigma^n(x))\sigma^n(x)\,,$$

we obtain

$$||\mathfrak{T}||(s) = \sum_{n \geq 1} (s, \sigma^n(x))\omega_2(\omega_1^{n-1}(x'_1), \omega_1^{n-1}(x'_2))\,.$$

$\square$

In connection with Example IV.1.6 of Gécseg, Steinby [51], Example 6.1 gives also an intuitive feeling, how a root-to-frontier tree transducer in the sense of Gécseg, Steinby [51] is simulated by a top-down tree series transducer over the semiring $\mathbb{B}$.

Consider the trees in the supports of the entries of $\mu_k(\omega)$, with $\omega \in \Sigma_k$, $k \geq 1$, as given in the definition of the tree representation. Then observe that the restriction of Engelfriet, Fülöp, Vogler [33], page 27, that in a top-down tree representation the variables in these trees occur in order $y_1, \ldots, y_m$ from left to right is irrelevant to the computational power. Hence, we have the following theorem.

Theorem 6.6.1 (Engelfriet, Fülöp, Vogler [33], Lemmas 4.10 and 4.12). *A mapping is realized by a root-to-frontier tree transducer iff it is realized by a polynomial top-down tree series transducer over the semiring* $\mathbb{B}$.

Let $Q'_i = \{(q, z_i) \mid q \in Q\}$, $1 \le i \le k$. Then there is a one-to-one correspondence between $Q'_1 \times \cdots \times Q'_k$ and Q^k given by

$$((q_1, z_1), \ldots, (q_k, z_k)) \Leftrightarrow (q_1, \ldots, q_k)\,,$$

$q_1, \ldots, q_k \in Q$. A tree representation $(\mu_k \mid k \ge 0)$ is called *nondeterministic simple* iff

$$\mu_k : \Sigma_k \to (A\langle\langle T_{\Sigma'}(X' \cup Y_k)\rangle\rangle)^{Q \times (Q'_1 \times \cdots \times Q'_k)}, \quad k \ge 1\,.$$

If $(\mu_k \mid k \ge 0)$ is a nondeterministic simple tree representation, we work with the isomorphic copies $\mu_k(\omega)'$ of $\mu_k(\omega)$ in $(A\langle\langle T_{\Sigma'}(X' \cup Y_k)\rangle\rangle)^{Q \times Q^k}$, $k \ge 0$. By Theorem 2.8,

$$\mu_k(\omega)'(P_1, \ldots, P_k) = \mu_k(\omega)[P_1, \ldots, P_k]$$

for $P_j \in (A\langle\langle T_\Sigma(X \cup Y')\rangle\rangle)^{Q \times 1}$, $1 \le j \le k$, and $\omega \in \Sigma_k$, $k \ge 0$. Hence, we can define a nondeterministic simple tree representation to be a family of mappings $(\mu_k \mid k \ge 0)$, where

$$\begin{aligned}
&\mu_k : \Sigma_k \to (A\langle\langle T_{\Sigma'}(X' \cup Y_k)\rangle\rangle)^{Q \times Q^k}, \quad k \ge 1\,,\\
&\mu_0 : \Sigma_0 \cup X \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}\,.
\end{aligned}$$

The morphic extension of μ_0 is again defined by $\mu(\omega(t_1, \ldots, t_k)) = \mu_k(\omega)(\mu(t_1), \ldots, \mu(t_k))$, $\omega \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(X)$, $k \ge 1$, and, for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$, we define again $\mu(s) = \sum_{t \in T_\Sigma(X)} (s, t) \otimes \mu(t)$.

A *nondeterministic simple tree series transducer* is now a tree series transducer $\mathfrak{T} = (Q, \mu, S)$, where μ is a nondeterministic simple tree representation and $||\mathfrak{T}||(s) = S(\mu(s)) = \sum_{q \in Q} (S_q, y_1)\mu(s)_q$ for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$.

In Kuich [81], page 139, it is explained how a nondeterministic simple tree series transducer over the semiring $\mathbb{B}$ is connected with a nondeterministic simple root-to-frontier tree transducer in the sense of Gécseg, Steinby [51], Exercise IV.4.

Theorem 6.6.2 (Kuich [81], Theorem 6) *Let, for some* $k \ge 1$, $s \in A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle$ *be linear and nondeleting, and* $s_{i_j} \in A\langle\langle T_\Sigma(X)\rangle\rangle$, $a_{i_j} \in A$ *for* $i_j \in I_j$, $1 \le j \le k$. *Then*

$$s(\sum_{i_1 \in I_1} a_{i_1} s_{i_1}, \ldots, \sum_{i_k \in I_k} a_{i_k} s_{i_k}) = \sum_{i_1 \in I_1} \ldots \sum_{i_k \in I_k} a_{i_1} \ldots a_{i_k} s(s_{i_1}, \ldots, s_{i_k})\,.$$

Theorem 6.6.3 *Let $\omega \in \Sigma_k$, $k \geq 1$, $s_1, \ldots, s_k \in A\langle\langle T_\Sigma(X)\rangle\rangle$, and μ be a nondeterministic simple tree representation with state set Q. Then*

$$\mu_k(\omega)(\mu(s_1), \ldots, \mu(s_k)) = \mu(\bar{\omega}(s_1, \ldots, s_k))\,.$$

Proof. We first compute the left side of the equality for index $q \in Q$:

$$\begin{aligned}
&\mu_k(\omega)(\mu(s_1), \ldots, \mu(s_k))_q = \\
&\sum_{q_1,\ldots,q_k \in Q} \mu_k(\omega)_{q,(q_1,\ldots,q_k)}(\mu(s_1)_{q_1}, \ldots, \mu(s_k)_{q_k}) = \\
&\sum_{q_1,\ldots,q_k \in Q} \mu_k(\omega)_{q,(q_1,\ldots,q_k)}(\sum_{t_1 \in T_\Sigma(X)} (s_1, t_1)\mu(t_1)_{q_1}, \ldots, \sum_{t_k \in T_\Sigma(X)} (s_k, t_k)\mu(t_k)_{q_k}) = \\
&\sum_{q_1,\ldots,q_k \in Q} \sum_{t_1,\ldots,t_k \in T_\Sigma(X)} (s_1, t_1) \cdots (s_k, t_k)\mu_k(\omega)_{q,(q_1,\ldots,q_k)}(\mu(t_1)_{q_1}, \ldots, \mu(t_k)_{q_k}) = \\
&\sum_{t_1,\ldots,t_k \in T_\Sigma(X)} (s_1, t_1) \cdots (s_k, t_k)\mu(\omega(t_1, \ldots, t_k))_q\,.
\end{aligned}$$

Here the third equality follows by the assumption that μ is a nondeterministic simple tree representation and by Theorem 6.2. We now compute the right side of the equality for index $q \in Q$:

$$\begin{aligned}
&\mu(\bar{\omega}(s_1, \ldots, s_k))_q = \\
&\mu(\sum_{t \in T_\Sigma(X)} (\sum_{\omega(t_1,\ldots,t_k)=t} (s_1, t_1) \cdots (s_k, t_k))t)_q = \\
&\sum_{t \in T_\Sigma(X)} (\sum_{\omega(t_1,\ldots,t_k)=t} (s_1, t_1) \cdots (s_k, t_k))\mu(t)_q \\
&\sum_{t_1,\ldots,t_k \in T_\Sigma(X)} (s_1, t_1) \cdots (s_k, t_k)\mu(\omega(t_1, \ldots, t_k))_q\,.
\end{aligned}$$

Since the two sides of the equation coincide, the theorem is proven. □

If the Boolean semiring $\mathbb{B}$ is the basic semiring, it is easy to see by Example 6.1 that our polynomial tree series transducers do not preserve the recognizability of tree series. (See also the example in the last paragraph of page 18 of Gécseg, Steinby [52].) On the other hand, linear root-to-frontier tree transducers do preserve recognizability of tree languages. (See Thatcher [111]; and Gécseg, Steinby [51], Theorem IV.2.7, Lemma IV.6.5 and Corollary IV.6.6.) In the rest of this section we show that nondeterministic simple recognizable tree transducers do preserve recognizability of tree series. We show this by a construction based on finite recognizable systems.

A system $z_i = p_i$, $1 \leq i \leq n$ is called *recognizable* iff each p_i is in $A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z_n)\rangle\rangle$.

We show that the least solution of a finite recognizable system has recognizable components.

Theorem 6.6.4 *Let $z_i = p_i$, $1 \leq i \leq n$, be a finite recognizable system with least solution σ. Then $\sigma_i \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ for all $1 \leq i \leq n$.*

Proof. Without loss of generality let $z_i = p_i$, $1 \le i \le n$, be a proper finite recognizable system. Since $p_i \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X \cup Z_n)\rangle\rangle$, $1 \le i \le n$, there exist proper finite polynomial systems $y_{ij} = q_{ij}$, $1 \le j \le m_i$, $m_i \ge 1$, where the y_{ij} are new variables and $q_{ij} \in A\langle T_\Sigma(X \cup Z_n \cup \{y_{i1}, \ldots, y_{im_i}\})\rangle$, such that the y_{i1}-components of their least solutions τ_i are equal to p_i. Consider now the proper finite polynomial system $z_i = q_{i1}(z_1, \ldots, z_n, y_{i1}, \ldots, y_{im_i})$, $y_{ij} = q_{ij}(z_1, \ldots, z_n, y_{i1}, \ldots, y_{im_i})$, $1 \le j \le m_i$, $1 \le i \le n$, and observe that it has a unique solution. We claim that this unique solution is given by $\sigma \cup ((\tau_i)_j(\sigma_1, \ldots, \sigma_n) \mid 1 \le j \le m_i,\ 1 \le i \le n)$. Substitution of this vector yields, for $1 \le j \le m_i$, $1 \le i \le n$,

$$\begin{aligned} &q_{i1}(\sigma_1, \ldots, \sigma_n, (\tau_i)_1(\sigma_1, \ldots, \sigma_n), \ldots, (\tau_i)_{m_i}(\sigma_1, \ldots, \sigma_n)) = \\ &\qquad\qquad (\tau_i)_1(\sigma_1, \ldots, \sigma_n) = p_i(\sigma_1, \ldots, \sigma_n) = \sigma_i\,, \\ &q_{ij}(\sigma_1, \ldots, \sigma_n, (\tau_i)_1(\sigma_1, \ldots, \sigma_n), \ldots, (\tau_i)_{m_i}(\sigma_1, \ldots, \sigma_n)) = (\tau_i)_j(\sigma_1, \ldots, \sigma_n)\,. \end{aligned}$$

Hence $\sigma \cup ((\tau_i)_j(\sigma_1, \ldots, \sigma_n) \mid 1 \le j \le m_i,\ 1 \le i \le n)$ is the unique solution of the proper finite polynomial system and $\sigma \in (A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle)^{n \times 1}$. $\square$

Consider a finite system $y_i = p_i(y_1, \ldots, y_n)$, $1 \le i \le n$, where $p_i \in A\langle\langle T_\Sigma(X \cup Y_n)\rangle\rangle$, and a nondeterministic simple tree representation $\mu = (\mu_k \mid k \ge 0)$ with state set Q, where $\mu_k : \Sigma_k \to (A\langle\langle T_{\Sigma'}(X' \cup Z_k)\rangle\rangle)^{Q \times Q^k}$, $k \ge 1$, and $\mu_0 : \Sigma_0 \cup X \to (A\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}$. Let $(y_i)_q$, $1 \le i \le n$, $q \in Q$, be new variables and denote $Y_Q^k = \{(y_i)_q \mid 1 \le i \le k,\ q \in Q\}$. Extend the definition of μ to the domain $\Sigma \cup X \cup Y_n$, by

$$\mu_0 : Y_n \to (A\langle\langle T_{\Sigma'}(Y_Q^n)\rangle\rangle)^{Q \times 1}\,,$$

where $\mu(y_j)_q = (y_j)_q$, $1 \le j \le n$, $q \in Q$. By this extension, we obtain the mapping

$$\mu : T_\Sigma(X \cup Y_n) \to (A\langle\langle T_{\Sigma'}(X' \cup Y_Q^n)\rangle\rangle)^{Q \times 1}\,.$$

Lemma 6.6.5 *Consider $s(y_1, \ldots, y_n) \in A\langle\langle T_\Sigma(X \cup Y_n)\rangle\rangle$ and a nondeterministic simple tree representation μ with domain $\Sigma \cup X \cup Y_n$. Let $s_1, \ldots, s_n \in A\langle\langle T_\Sigma(X)\rangle\rangle$. Then*

$$\mu(s)[\mu(s_j)_q/(y_j)_q,\ 1 \le j \le n,\ q \in Q] = \mu(s(s_1, \ldots, s_n))\,.$$

Proof. We first consider a tree $t \in T_\Sigma(X \cup Y_n)$ and show by induction on the form of t that $\mu(t)[\mu(s_j)_q/(y_j)_q,\ 1 \le j \le n,\ q \in Q] = \mu(t(s_1, \ldots, s_n))$.
(i) For $t = y_i$, $1 \le i \le n$, we get $\mu(y_i)[\mu(s_j)/\mu(y_j),\ 1 \le j \le n] = \mu(s_i) = \mu(y_i(s_1, \ldots, s_n))$.
(ii) For $t = x$, $x \in \Sigma_0 \cup X$, we obtain $\mu(x)[\mu(s_j)/\mu(y_j),\ 1 \le j \le n] = \mu(x) = \mu(x(s_1, \ldots, s_n))$.
(iii) For $t = \omega(t_1, \ldots, t_k)$, $\omega \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(X \cup Y_n)$, $k \ge 1$, we obtain

$$\begin{aligned} &\mu(\omega(t_1, \ldots, t_k))[\mu(s_j)/\mu(y_j),\ 1 \le j \le n] = \\ &\mu_k(\omega)(\mu(t_1)[\mu(s_j)/\mu(y_j),\ 1 \le j \le n], \ldots, \mu(t_k)[\mu(s_j)/\mu(y_j),\ 1 \le j \le n]) = \\ &\mu_k(\omega)(\mu(t_1(s_1, \ldots, s_n)), \ldots, \mu(t_k(s_1, \ldots, s_n))) = \\ &\mu(\bar\omega(t_1(s_1, \ldots, s_n), \ldots, t_k(s_1, \ldots, s_n))) = \\ &\mu((\bar\omega(t_1, \ldots, t_k))(s_1, \ldots, s_n))\,. \end{aligned}$$

Here we have applied the induction hypothesis in the second equality and Theorem 6.3 in the third equality.

Finally, we obtain

$$\begin{aligned}
&\mu(s)[\mu(s_j)/\mu(y_j),\ 1 \leq j \leq n] = \\
&\textstyle\sum_{t \in T_\Sigma(X \cup Y_n)} (s,t) \otimes \mu(t)[\mu(s_j)/\mu(y_j),\ 1 \leq j \leq n] = \\
&\textstyle\sum_{t \in T_\Sigma(X \cup Y_n)} (s,t) \otimes \mu(t(s_1, \ldots, s_n)) = \\
&\mu(\textstyle\sum_{t \in T_\Sigma(X \cup Y_n)} (s,t) t(s_1, \ldots, s_n)) = \mu(s(s_1, \ldots, s_n))\,.
\end{aligned}$$

□

Theorem 6.6.6 *Consider a nondeterministic simple tree representation μ with domain $\Sigma \cup X \cup Y_n$. Let $y_i = p_i(y_1, \ldots, y_n)$, $1 \leq i \leq n$, where $p_i \in A\langle\langle T_\Sigma(X \cup Y_n)\rangle\rangle$, be a finite system with least solution σ. Then $\mu(\sigma)$ is the least solution of the finite system $\mu(y_i) = \mu(p_i(y_1, \ldots, y_n))$, $1 \leq i \leq n$.*

Proof. Let $(\sigma^j \mid j \in \mathbb{N})$ and $(\tau^j \mid j \in \mathbb{N})$ be the approximation sequences of $y_i = p_i(y_1, \ldots, y_n)$, $1 \leq i \leq n$, and $\mu(y_i) = \mu(p_i(y_1, \ldots, y_n))$, $1 \leq i \leq n$, respectively. We claim that $\tau_i^j = \mu(\sigma_i^j)$, $1 \leq i \leq n$, $j \geq 0$, and show this by induction on j. The case $j = 0$ is clear. Let $j \geq 0$. Then, for $1 \leq i \leq n$,

$$\begin{aligned}
&\tau_i^{j+1} = \mu(p_i(y_1, \ldots, y_n))[\tau_k^j/\mu(y_k),\ 1 \leq k \leq n] = \\
&\mu(p_i(y_1, \ldots, y_n))[\mu(\sigma_k^j)/\mu(y_k),\ 1 \leq k \leq n] = \\
&\mu(p_i(\sigma_1^j, \ldots, \sigma_n^j)) = \mu(\sigma_i^{j+1})\,.
\end{aligned}$$

Here we have applied the induction hypothesis in the second equality and Lemma 6.5 in the third equality. The claim now implies our theorem. □

A nondeterministic simple tree representation $\mu = (\mu_k \mid k \geq 0)$ is called *recognizable* iff $\mu_k(\omega) \in (A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X' \cup Z_k)\rangle\rangle)^{Q \times Q^k}$ for every $\omega \in \Sigma_k$, $k \geq 1$, and $\mu_0(\omega) \in (A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}$ for every $\omega \in \Sigma_0 \cup X$. A nondeterministic simple tree series transducer $\mathfrak{T} = (Q, \mu, S)$ is *recognizable* iff μ is a nondeterministic simple recognizable tree representation.

Theorem 6.6.7 *Consider a nondeterministic simple recognizable tree representation μ. Let s be in $A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$. Then $\mu(s)$ is in $(A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}$.*

Proof. By Corollary 3.6, s is a component of a finite simple polynomial system $y_i = p_i$, $1 \leq i \leq n$. By Theorem 6.6, $\mu(s)$ is a component of the finite recognizable system $\mu(y_i) = \mu(p_i)$, $1 \leq i \leq n$. Hence, Theorem 6.4 proves our theorem.

□

Corollary 6.6.8 (Kuich [81], Corollary 14) *Consider a nondeterministic simple recognizable tree series transducer $\mathfrak{T}$ and a recognizable tree series s. Then $||\mathfrak{T}||(s)$ is again recognizable.*

Corollary 6.6.9 (Thatcher [111], and Gécseg, Steinby [51], Chapter IV, Corollary 6.6.) *Linear root-to-frontier tree transducers preserve recognizability.*

6.7 Full abstract families of tree series

Full abstract families of tree series (briefly, full AFTs) are families of tree series closed under nondeterministic simple recognizable tree series transductions and certain other specific operations. We will show that the families of recognizable tree series and of algebraic tree series are full AFTs. Our first construction will show that the mappings realized by nondeterministic simple recognizable tree series transducers are closed under functional composition. The construction is analogous to the construction of Engelfriet [32] in Lemma 4.2 (see also Gécseg, Steinby [51], Theorem IV.3.15).

Recall that $Z_Q = \{(z_i)_q \mid i \geq 1,\ q \in Q\}$ and $Z_Q^k = \{(z_i)_q \mid 1 \leq i \leq k,\ q \in Q\}$ for $k \geq 1$. We now define, for $r_1, \ldots, r_k \in Q$, the operator

$$\varphi_{r_1,\ldots,r_k} : A\langle\langle T_\Sigma(X \cup Z_Q^k)\rangle\rangle \to A\langle\langle T_\Sigma(X \cup \{(z_1)_{r_1}, \ldots, (z_k)_{r_k}\})\rangle\rangle$$

as follows: For $s \in A\langle\langle T_\Sigma(X \cup Z_Q^k)\rangle\rangle$ and $t \in T_\Sigma(X \cup \{(z_1)_{r_1}, \ldots, (z_k)_{r_k}\})$,

$$(\varphi_{r_1,\ldots,r_k}(s), t) = \begin{cases} (s,t) & \text{iff each of the variables } (z_1)_{r_1}, \ldots, (z_k)_{r_k} \\ & \text{appears exactly once in } t, \\ 0 & \text{otherwise.} \end{cases}$$

Let μ' be a nondeterministic simple recognizable tree representation with state set Q_1 mapping $\Sigma \cup X$ into matrices with entries in $A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X' \cup Z)\rangle\rangle$. Furthermore, let μ'' be an extended nondeterministic simple recognizable tree representation with state set Q_2 mapping $\Sigma' \cup X' \cup Z$ into matrices with entries in $A^{\mathrm{rec}}\langle\langle T_{\Sigma''}(X'' \cup Z \cup Z_{Q_2})\rangle\rangle$. Define the recognizable tree representation μ with state set $Q_1 \times Q_2$ mapping $\Sigma \cup X$ into matrices with entries in $A^{\mathrm{rec}}\langle\langle T_{\Sigma''}(X'' \cup Z)\rangle\rangle$ by

$$\begin{aligned} &\mu_0(x)_{(q_1,q_2)} = \mu''(\mu'_0(x)_{q_1})_{q_2}, \quad \text{for } x \in \Sigma_0 \cup X,\ q_1 \in Q_1,\ q_2 \in Q_2\,, \\ &\mu_k(\omega)_{(q_1,q_2),((r_1,s_1),\ldots,(r_k,s_k))} = \\ &\qquad \varphi_{s_1,\ldots,s_k}(\mu''(\mu'_k(\omega)_{q_1,(r_1,\ldots,r_k)})_{q_2})[z_1/(z_1)_{s_1}, \ldots, z_k/(z_k)_{s_k}]\,, \\ &\qquad \text{for } \omega \in \Sigma_k,\ k \geq 1,\ q_1, r_1, \ldots, r_k \in Q_1,\ q_2, s_1, \ldots, s_k \in Q_2\,. \end{aligned}$$

Then, by Kuich [82], Lemma 2.3, $(\mu_k \mid k \geq 0)$ is a nondeterministic simple recognizable tree representation and, for $t \in T_\Sigma(X)$ and $q_1 \in Q_1$, $q_2 \in Q_2$,

$$\mu(t)_{(q_1,q_2)} = \mu''(\mu'(t)_{q_1})_{q_2}\,.$$

This construction yields the first theorem of this section.

Theorem 6.7.1 (Kuich [78], Theorem 2.4) *Let μ' (resp. μ'') be a nondeterministic simple recognizable tree representation with state set Q_1 (resp. Q_2) mapping $\Sigma \cup X$ (resp. $\Sigma' \cup X'$) into matrices with entries in $A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X' \cup Z)\rangle\rangle$ (resp. $A^{\mathrm{rec}}\langle\langle T_{\Sigma''}(X'' \cup Z)\rangle\rangle$). Let $\mathfrak{T}_1 = (Q_1, \mu', S_1)$ and $\mathfrak{T}_2 = (Q_2, \mu'', S_2)$ be nondeterministic simple recognizable tree series transducers. Then there exists a nondeterministic simple recognizable tree series transducer $\mathfrak{T}$ such that $||\mathfrak{T}||(s) = ||\mathfrak{T}_2||(||\mathfrak{T}_1||(s))$ for all $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$.*

Proof. The nondeterministic simple recognizable tree series transducer $\mathfrak{T} = (Q_1 \times Q_2, \mu, S_1 \odot S_2)$ is defined by the nondeterministic simple recognizable tree representation μ constructed above.

Let $(S_1)_{q_1} = a_{q_1} z_1$, $a_{q_1} \in A$, $q_1 \in Q_1$, and $(S_2)_{q_2} = b_{q_2} z_1$, $b_{q_2} \in A$, $q_2 \in Q_2$. Then $(S_1 \odot S_2)_{(q_1,q_2)} = a_{q_1} b_{q_2} z_1$ for $q_1 \in Q_1$, $q_2 \in Q_2$. We now obtain, for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$,

$$\begin{aligned}
&||\mathfrak{T}_2||(||\mathfrak{T}_1||(s)) = \textstyle\sum_{q_2\in Q_2} b_{q_2} \sum_{t_2\in T_{\Sigma'}(X')}(||\mathfrak{T}_1||(s), t_2)\mu''(t_2)_{q_2} = \\
&\textstyle\sum_{q_2\in Q_2} b_{q_2} \sum_{t_2\in T_{\Sigma'}(X')}(\sum_{q_1\in Q_1} a_{q_1} \sum_{t_1\in T_\Sigma(X)}(s,t_1)\mu'(t_1)_{q_1}, t_2)\mu''(t_2)_{q_2} = \\
&\textstyle\sum_{q_1\in Q_1}\sum_{q_2\in Q_2} a_{q_1} b_{q_2} \sum_{t_1\in T_\Sigma(X)}(s,t_1)\sum_{t_2\in T_{\Sigma'}(X')}(\mu'(t_1)_{q_1}, t_2)\mu''(t_2)_{q_2} = \\
&\textstyle\sum_{(q_1,q_2)\in Q_1\times Q_2} a_{q_1} b_{q_2} \sum_{t_1\in T_\Sigma(X)}(s,t_1)\mu(t_1)_{(q_1,q_2)} = \\
&||\mathfrak{T}||(s)\,.
\end{aligned}$$

□

We make the following convention for the rest of Section 7: The set Σ_∞ (resp. X_∞) is a fixed *infinite* ranked alphabet (resp. *infinite* alphabet) and Σ, Σ' (resp. X, X'), possibly provided with indices, are *finite* subalphabets of Σ_∞ (resp. X_∞). Our basic semiring will be $A\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$.

Any non-empty subset of $\bigcup_{\Sigma\subset\Sigma_\infty}\bigcup_{X\subset X_\infty} A\langle\langle T_\Sigma(X)\rangle\rangle$ is called *family of tree series.* A mapping

$$\tau : \bigcup_{\Sigma\subset\Sigma_\infty}\bigcup_{X\subset X_\infty} A\langle\langle T_\Sigma(X)\rangle\rangle \to \bigcup_{\Sigma\subset\Sigma_\infty}\bigcup_{X\subset X_\infty} A\langle\langle T_\Sigma(X)\rangle\rangle$$

is called a *nondeterministic simple recognizable tree series transduction* iff there exist Σ, X, Σ', X' such that $\tau(s) \in A\langle\langle T_{\Sigma'}(X')\rangle\rangle$ for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ and $\tau(s) = 0$ for $s \notin A\langle\langle T_\Sigma(X)\rangle\rangle$, and there exists a nondeterministic simple recognizable tree series transducer $\mathfrak{T}$ such that $\tau(s) = ||\mathfrak{T}||(s)$ for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$.

For a family $\mathfrak{L}$ of tree series, we define

$$\mathfrak{M}(\mathfrak{L}) = \{\tau(s) \mid s \in \mathfrak{L} \text{ and } \tau \text{ is a nondeterministic simple recognizable tree series transduction}\}\,.$$

Observe that, by Theorem 7.1, $\mathfrak{M}(\mathfrak{M}(\mathfrak{L})) = \mathfrak{M}(\mathfrak{L})$. A family $\mathfrak{L}$ of tree series is said to be *closed under nondeterministic simple recognizable tree series transductions*, and is called a *recognizable tree series cone* iff $\mathfrak{L} = \mathfrak{M}(\mathfrak{L})$.

We first consider recognizable tree series. Theorem 6.7 yields at once the next theorem.

Theorem 6.7.2 *$A^{\mathrm{rec}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is a recognizable tree series cone.*

Theorem 6.7.3 *Let $\mathfrak{L}$ be a recognizable tree series cone and assume that $\mathfrak{L}$ contains some tree series s such that $(s, x) = 1$ for some $x \in X_\infty$. Then $A^{\mathrm{rec}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle \subseteq \mathfrak{L}$.*

Proof. Consider a recognizable tree series r and the nondeterministic simple recognizable tree series transducer $\mathfrak{T} = (\{q\}, (\mu_k \mid k \geq 0), z_1)$, where $\mu_0(x) = r$,

$\mu_0(x') = 0$ for $x' \neq x$, $x' \in X_\infty$, and $\mu_k(\omega) = 0$, $\omega \in \Sigma_\infty$, of rank $k \geq 0$. Then $||\mathfrak{T}||(s) = r$. $\square$

We introduce analogous to the REC-closed families of tree series of Bozapalidis, Rahonis [16] equationally closed families of tree series. A family $\mathfrak{L}$ of tree series is called *equationally closed* whenever the following conditions are satisfied:

(i) $0 \in \mathfrak{L}$.

(ii) If $s_1, s_2 \in \mathfrak{L}$ then $s_1 + s_2 \in \mathfrak{L}$.

(iii) If $\omega \in \Sigma_\infty$ is of rank $k \geq 0$ and $s_1, \ldots, s_k \in \mathfrak{L}$ then $\bar{\omega}(s_1, \ldots, s_k) \in \mathfrak{L}$; if $x \in X_\infty$ then $x \in \mathfrak{L}$.

(iv) If $s \in \mathfrak{L}$ and $x \in X_\infty$ then the least solution $\mu x.s$ of the equation $x = s$ is in $\mathfrak{L}$.

Hence, a family $\mathfrak{L}$ of tree series is equationally closed iff $\langle \mathfrak{L}, +, 0, (\bar{\omega} \mid \omega \in \Sigma_\infty) \cup X_\infty \rangle$ is a distributive $\Sigma_\infty \cup X_\infty$-algebra that satisfies condition (iv), i. e., our "rational" operations are 0, addition, top-catenation and least solutions of equations. (Observe that we do not ask for the closure under substitution as Bozapalidis, Rahonis [16] do for their REC-closed families of tree languages.)

Theorem 6.7.4 *$A^{\mathrm{rec}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is an equationally closed family of tree series.*

Proof. By Theorem 4.3. $\square$

We are now ready to introduce full AFTs. We use the notation $\hat{\mathfrak{F}}(\mathfrak{L})$, where $\mathfrak{L}$ is a family of tree series, for the smallest equationally closed family of tree series that is closed under nondeterministic simple recognizable tree series transductions and contains $\mathfrak{L}$. A family $\mathfrak{L}$ of tree series is called a *full AFT* iff $\mathfrak{L} = \hat{\mathfrak{F}}(\mathfrak{L})$.

Theorem 6.7.5 (Kuich [82], Theorem 3.5) *$A^{\mathrm{rec}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is a full AFT.*

Proof. By Theorems 7.2 and 7.4. $\square$

We now consider algebraic tree series.

Theorem 6.7.6 *$A^{\mathrm{alg}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is an equationally closed family of tree series.*

Proof. By Theorem 5.15. $\square$

We will show that $A^{\mathrm{alg}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is a full AFT closed under nondeterministic simple algebraic tree series transductions. Some definitions and results are needed before that result.

A nondeterministic simple tree representation $\mu = (\mu_k \mid k \geq 0)$ is called *algebraic* iff $\mu_k(\omega) \in (A^{\mathrm{alg}}\langle\langle T_{\Sigma'}(X' \cup Z_k)\rangle\rangle)^{Q \times Q^k}$ for $\omega \in \Sigma_k$, $k \geq 1$, and $\mu_0(\omega) \in (A^{\mathrm{alg}}\langle\langle T_{\Sigma'}(X')\rangle\rangle)^{Q \times 1}$ for $\omega \in \Sigma_0 \cup X$. A nondeterministic simple tree series transducer $\mathfrak{T} = (Q, \mu, S)$ is called *algebraic* iff μ is an algebraic tree representation. *Nondeterministic simple algebraic tree series transductions* are defined analogously to nondeterministic simple recognizable tree series transductions.

Theorem 6.7.7 (Kuich [83], Corollary 3.6.) *Let $\mathfrak{T}$ be a nondeterministic simple algebraic tree series transducer and s be an algebraic tree series. Then $||\mathfrak{T}||(s)$ is again algebraic.*

Theorem 6.7.8 *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be nondeterministic simple algebraic tree series transducers. Then there exists a nondeterministic simple algebraic tree series transducer $\mathfrak{T}$ such that $||\mathfrak{T}||(s) = ||\mathfrak{T}_2||(||\mathfrak{T}_1||(s))$ for all $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$.*

Proof. The construction of $\mathfrak{T}$ from $\mathfrak{T}_1$ and $\mathfrak{T}_2$ is analogous to the construction in the proof of Theorem 7.1. Theorem 7.7 proves that μ is algebraic. □

For a family $\mathfrak{L}$ of tree series, we define

$$\mathfrak{M}^{\mathrm{alg}}(\mathfrak{L}) = \{\tau(s) \mid \begin{array}{l} s \in \mathfrak{L} \text{ and } \tau \text{ is a nondeterministic simple} \\ \text{algebraic tree series transduction } \}. \end{array}$$

Observe that by Theorem 7.8, $\mathfrak{M}^{\mathrm{alg}}(\mathfrak{M}^{\mathrm{alg}}(\mathfrak{L})) = \mathfrak{M}^{\mathrm{alg}}(\mathfrak{L})$. A family $\mathfrak{L}$ of tree series is said to be *closed under nondeterministic simple algebraic tree series transductions*, and is called an *algebraic tree series cone* iff $\mathfrak{L} = \mathfrak{M}^{\mathrm{alg}}(\mathfrak{L})$.

Theorem 6.7.9 *If $\mathfrak{L}$ is an algebraic tree series cone and $\mathfrak{L}$ contains some tree series s such that $(s, x) = 1$ for some $x \in X_\infty$ then $A^{\mathrm{alg}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle \subseteq \mathfrak{L}$.*

Proof. Similar to the proof of Theorem 7.3. □

Theorem 6.7.10 *$A^{\mathrm{alg}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is an algebraic tree series cone.*

Proof. By Theorems 7.6 and 7.7. □

Corollary 6.7.11 (Kuich [83], Theorem 4.4) *$A^{\mathrm{alg}}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$ is a full AFT that is closed under nondeterministic simple algebraic tree series transductions.*

6.8 Connections to formal power series

The application of the yield-mapping to formal tree series yields formal power series. We will first show that the macro power series are exactly the yield of algebraic tree series. Here, macro power series are introduced as a generalization of the OI languages of Fischer [47] and the indexed languages of Aho [2]. Moreover, we show a Kleene Theorem for macro power series and indexed languages. Then we show that the algebraic power series are exactly the yield of recognizable tree series. Finally, we prove that the yield of a full abstract family of tree series is a full abstract family of power series.

We now introduce macro power series. Let $\Phi = \{G_1, \ldots, G_n\}$, $\Phi \cap X = \emptyset$, where G_i has rank r_i, $1 \le i \le n$, be a finite ranked alphabet of *function variables*. We define $T(\Phi, X)$ to be the set of words over $\Phi \cup X \cup \{(\} \cup \{)\} \cup \{,\}$ satisfying the following conditions:

(i) $X \cup \{\varepsilon\} \subset T(\Phi, X)$;

(ii) if $t_1, t_2 \in T(\Phi, X)$ then $t_1 t_2 \in T(\Phi, X)$;

(iii) if $G \in \Phi$, where G is of rank $r \geq 0$, and $t_1, \dots, t_r \in T(\Phi, X)$ then $G(t_1, \dots, t_r) \in T(\Phi, X)$.

The words of $T(\Phi, X)$ are called *terms over* Φ and X. By $A\langle\langle T(\Phi, X)\rangle\rangle$ (resp. $A\langle T(\Phi, X)\rangle$) we denote the set of power series whose supports are subsets (resp. finite subsets) of $T(\Phi, X)$.

Let $D' = A\langle\langle (X \cup Z_{r_1})^*\rangle\rangle \times \ldots \times A\langle\langle (X \cup Z_{r_n})^*\rangle\rangle$ and consider power series $s_i \in A\langle\langle T(\Phi, X \cup Z_{r_i})\rangle\rangle$, $1 \leq i \leq n$. Then each s_i induces a function $\bar{s}_i : D' \to A\langle\langle (X \cup Z_{r_i})^*\rangle\rangle$. For $(\tau_1, \dots, \tau_n) \in D'$, we define inductively $\bar{s}_i(\tau_1, \dots, \tau_n)$ to be

(i) z_m if $s_i = z_m$, $1 \leq m \leq r_i$; x if $s_i = x$, $x \in X$;

(ii) $\bar{t}_1(\tau_1, \dots, \tau_n)\bar{t}_2(\tau_1, \dots, \tau_n)$ if $s_i = t_1 t_2$, $t_1, t_2 \in T(\Phi, X \cup Z_{r_i})$;

(iii) $\tau_j(\bar{t}_1(\tau_1, \dots, \tau_n), \dots, \bar{t}_{r_j}(\tau_1, \dots, \tau_n))$ if $s_i = G_j(t_1, \dots, t_{r_j})$, $G_j \in \Phi$, $t_1, \dots, t_{r_j} \in T(\Phi, X \cup Z_{r_i})$;

(iv) $a \cdot \bar{t}(\tau_1, \dots, \tau_n)$ if $s_i = at$, $a \in A$, $t \in T(\Phi, X \cup Z_{r_i})$;

(v) $\sum_{j \in J} \bar{r}_j(\tau_1, \dots, \tau_n)$ if $s_i = \sum_{j \in J} r_j$, $r_j \in A\langle\langle T(\Phi, X \cup Z_{r_i})\rangle\rangle$, $j \in J$, where J is an arbitrary index set.

The mappings $\bar{s}_i$, $1 \leq i \leq n$, are continuous and the mapping $\bar{s} : D' \to D'$, where $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$, is again continuous. This is proved similarly as the continuity of the mappings defined in connection with algebraic tree systems (below Theorem 5.1).

A *macro system* $\mathfrak{S} = (\Phi, Z, X, E)$ (with *function variables* in Φ, *variables* in Z and *terminal symbols* in X) has a set E of *formal equations*

$$G_i(z_1, \dots, z_{r_i}) = s_i(z_1, \dots, z_{r_i}),\ 1 \leq i \leq n\,,$$

where each s_i is in $A\langle T(\Phi, X \cup Z_{r_i})\rangle$.

A *solution* to the macro system $\mathfrak{S}$ is given by $(\tau_1, \dots, \tau_n) \in D'$ such that $\tau_i = \bar{s}_i(\tau_1, \dots, \tau_n)$, $1 \leq i \leq n$, i. e., by any fixed point $(\tau_1, \dots, \tau_n)$ of $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$. A solution $(\sigma_1, \dots, \sigma_n)$ of the macro system $\mathfrak{S}$ is called *least solution* iff $\sigma_i \leq \tau_i$, $1 \leq i \leq n$, for all solutions $(\tau_1, \dots, \tau_n)$ of $\mathfrak{S}$. Since the least solution of $\mathfrak{S}$ is nothing else than the least fixpoint of $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle$, the least solution of the macro system $\mathfrak{S}$ exists in D'.

Theorem 6.8.1 (Kuich [85], Theorem 5.1.) *Let $\mathfrak{S} = (\Phi, Z, X, \{G_i = s_i \mid 1 \leq i \leq n\})$ be a macro system, where $s_i \in A\langle T(\Phi, X \cup Z_{r_i})\rangle$. Then the least solution of this macro system $\mathfrak{S}$ exists in D' and equals*

$$\mathrm{fix}(\bar{s}) = \sup(\bar{s}^i(0) \mid i \in \mathbb{N})\,,$$

where $\bar{s}^i$, is the i-th iterate of the mapping $\bar{s} = \langle \bar{s}_1, \dots, \bar{s}_n\rangle : D' \to D'$.

Theorem 8.1 indicates how we can compute an approximation to the least solution of a macro system. The *approximation sequence* $(\tau^j \mid j \in \mathbb{N})$, where each $\tau^j \in D'$, *associated with* the macro system $\mathfrak{S} = (\Phi, Z, X, \{G_i = s_i \mid 1 \leq i \leq n\})$ is defined as follows:

$$\tau^0 = 0, \quad \tau^{j+1} = \bar{s}(\tau^j),\ j \in \mathbb{N}\,.$$

Clearly, the least solution $\mathrm{fix}(\bar{s})$ of $\mathfrak{S}$ is equal to $\sup(\tau^j \mid j \in \mathbb{N})$. A *macro system with an initial function variable* $\mathfrak{S} = (\Phi \cup \{G_0\}, Z, X, \{G_i = s_i \mid 0 \leq i \leq n\}, G_0)$ (with function variables in $\Phi \cup \{G_0\}$, variables in Z, terminal symbols in X) is a macro system $(\Phi \cup \{G_0\}, Z, X, \{G_i = s_i \mid 0 \leq i \leq n\})$ and G_0 is the initial function variable of rank 0. Let $(\tau_0, \tau_1, \ldots, \tau_n)$ be the least solution of $(\Phi \cup \{G_0\}, Z, X, \{G_i = s_i \mid 0 \leq i \leq n\})$. Then τ_0 is called the *initial component* of the least solution. Observe that $\tau_0 \in A\langle\langle X^* \rangle\rangle$ contains no variables of Z.

A power series r in $A\langle\langle X^* \rangle\rangle$ is called *macro power series* iff r is the initial component of the least solution of a macro system with an initial function variable.

Analogously to the proof of Theorem 3.4 of Engelfriet, Schmidt [34] it can be shown that, in the case of the Boolean semiring, $r \in \mathbb{B}\langle\langle X^* \rangle\rangle$ is a macro power series iff $\mathrm{supp}(r) \in X^*$ is an OI language in the sense of Definition 3.10 of Fischer [47]. Moreover, by Theorem 5.3 of Fischer [47], $r \in \mathbb{B}\langle\langle X^* \rangle\rangle$ is a macro power series iff $\mathrm{supp}(r) \in X^*$ is an indexed language (see Aho [2]).

We now define a mapping $\mathrm{yd} : A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z) \rangle\rangle \to A\langle\langle T(\Phi, X \cup Z) \rangle\rangle$. For $s \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z) \rangle\rangle$, $\mathrm{yd}(s)$ is called the *yield* of s; $\mathrm{yd}(s)$ is defined inductively to be

(i) z_m if $s = z_m \in Z$; x if $s = x$, $x \in X$;

(ii) $\mathrm{yd}(t_1) \ldots \mathrm{yd}(t_r)$ if $s = \omega(t_1, \ldots, t_r)$, $\omega \in \Sigma_r$, $t_1, \ldots, t_r \in T_{\Sigma\cup\Phi}(X \cup Z)$, $r \geq 0$; (observe that $\mathrm{yd}(\omega) = \varepsilon$ if $\omega \in \Sigma_0$);

(iii) $G_i(\mathrm{yd}(t_1), \ldots, \mathrm{yd}(t_{r_i}))$ if $s = G_i(t_1, \ldots, t_{r_i})$, $t_1, \ldots, t_{r_i} \in T_{\Sigma\cup\Phi}(X \cup Z)$, $1 \leq i \leq n$;

(iv) $\sum_{t \in T_{\Sigma\cup\Phi}(X\cup Z)} (s, t)\mathrm{yd}(t)$ if $s = \sum_{t \in T_{\Sigma\cup\Phi}(X\cup Z)} (s, t)t$.

Observe that $\mathrm{yd}(s) \in A\langle\langle (X \cup Z)^* \rangle\rangle$ if $s \in A\langle\langle T_\Sigma(X \cup Z) \rangle\rangle$. Hence, our mapping yd is an extension of the usual yield-mapping (see Gécseg, Steinby [52], Section 14).

We will connect algebraic tree series and macro power series by the yield-mapping in our next theorem.

Given an algebraic tree system $\mathfrak{S} = (\Phi, Z, \Sigma, \{G_i(z_1, \ldots, z_{r_i}) = s_i \mid 1 \leq i \leq n\})$, we define the macro system $\mathrm{yd}(\mathfrak{S})$ to be $\mathrm{yd}(\mathfrak{S}) = (\Phi, Z, X, \{G_i(z_1, \ldots, z_{r_i}) = \mathrm{yd}(s_i) \mid 1 \leq i \leq n\})$.

Theorem 6.8.2 (Kuich [85], Theorem 5.5.) *If* $(\tau_1, \ldots, \tau_n)$ *is the least solution of the algebraic tree system* $\mathfrak{S}$ *then* $(\mathrm{yd}(\tau_1), \ldots, \mathrm{yd}(\tau_n))$ *is the least solution of the macro system* $\mathrm{yd}(\mathfrak{S})$.

Corollary 6.8.3 *If s is an algebraic tree series then* $\mathrm{yd}(s)$ *is a macro power series.*

Theorem 6.8.4 *Let $\{\bullet, e\} \subseteq \Sigma$, where $\bullet$ and e have rank 2 and 0, respectively. Then a power series $r \in A\langle\langle X^* \rangle\rangle$ is a macro power series iff there exists an algebraic tree series $s \in A\langle\langle T_\Sigma(X) \rangle\rangle$ such that* $\mathrm{yd}(s) = r$.

Proof. Assume that r is the initial component of the least solution of the macro system with initial function variable $\mathfrak{S} = (\Phi \cup \{G_0\}, Z, X, \{G_i = s_i \mid 0 \le i \le n\}, G_0)$. We construct an algebraic tree system $\mathfrak{S}' = (\Phi \cup \{G_0\}, Z, \{\bullet, e\}, \{G_i = s'_i \mid 0 \le i \le n\}, G_0)$ such that $\mathrm{yd}(\mathfrak{S}') = \mathfrak{S}$ by constructing for each word w in $T(\Phi \cup \{G_0\}, X \cup Z)$ a tree $t(w)$ in $T_{\{\bullet, e\} \cup \Phi \cup \{G_0\}}(X \cup Z)$:

(i) $t(\varepsilon) = e$, $t(x) = x$, $x \in X$ and $t(z) = z$, $z \in Z$;

(ii) if $w_1, w_2 \in T(\Phi \cup \{G_0\}, X \cup Z)$ then $t(w_1 w_2) = \bullet(t(w_1), t(w_2))$;

(iii) if $G \in \Phi \cup \{G_0\}$, where G is of rank $r \ge 0$ and $w_1, \dots, w_r \in T(\Phi \cup \{G_0\}, X \cup Z)$ then $t(G(w_1, \dots, w_n)) = G(t(w_1), \dots, t(w_n))$.

Clearly, we obtain $\mathrm{yd}(t(w)) = w$ for all $w \in T(\Phi \cup \{G_0\}, X \cup Z)$.

Define now $s'_i = \sum_{w \in A\langle\langle T(\Phi \cup \{G_0\}, X \cup Z) \rangle\rangle} (s_i, w) t(w)$. Then $\mathrm{yd}(\mathfrak{S}') = \mathfrak{S}$. Assume now that s is the initial component of the least solution of $\mathfrak{S}'$. Then $r = \mathrm{yd}(s)$. □

Example 6.8.1. Let $\mathfrak{S} = (\Phi, Z, \Sigma, E, Z_0)$ be the algebraic tree system with initial function variable Z_0 specified by

(i) $\Phi = \{G_0, G_1, G_2, Z_0\}$, where the ranks of G_0, G_1, G_2 are 3 and the rank of Z_0 is 0;

(ii) $Z = \{z_0, z_1, z_2\}$;

(iii) $\Sigma = \Sigma_2 = \{b\}$, $X = \{c_1, c_2\}$;

(iv) the formal equations of E are

$$\begin{aligned}
&G_0(z_0, z_1, z_2) = G_0(G_0(z_0, z_1, z_2), G_1(z_0, z_1, z_2), G_2(z_0, z_1, z_2)) + b(z_1, z_2),\\
&G_i(z_0, z_1, z_2) = b(z_i, z_i),\ i = 1, 2,\\
&Z_0 = G_0(0, c_1, c_2).
\end{aligned}$$

Then the initial component of the least solution of $\mathfrak{S}$ is $\sum_{j \ge 0} b(t_1^j, t_2^j)$, where t_1^j and t_2^j, $j \ge 0$, are defined in Example 5.1. The macro system $\mathrm{yd}(\mathfrak{S}) = (\Phi, Z, X, E', Z_0)$ with initial function variable Z_0 is specified by the following formal equations of E':

$$\begin{aligned}
&G_0(z_0, z_1, z_2) = G_0(G_0(z_0, z_1, z_2), G_1(z_0, z_1, z_2), G_2(z_0, z_1, z_2)) + z_1 z_2,\\
&G_i(z_0, z_1, z_2) = z_i z_i,\ i = 1, 2,\\
&Z_0 = G_0(0, c_1, c_2).
\end{aligned}$$

The initial component of the least solution of $\mathrm{yd}(\mathfrak{S})$ is $\sum_{j\geq 0} c_1^{2^j} c_2^{2^j}$. $\square$

We now introduce macro power series expressions. Assume that A, X, Z, Φ_∞ and $U = \{+, \cdot, \mu, [,]\}$ are pairwise disjoint. A word E over $A \cup X \cup Z \cup \Phi_\infty \cup U$ is a *macro power series expression over* (A, X, Z, Φ_∞) iff

(i) E is in $X \cup Z \cup \{\varepsilon\}$, or

(ii) E is of one of the forms $[E_1 + E_2]$, $[E_1 E_2]$, $G(E_1, \dots, E_k)$, aE_1 or $\mu G.E_1$, where $E_1, \dots, E_k$ are macro power series expressions over (A, X, Z, Φ_∞), $G \in \Phi_\infty$ is of rank k, $k \geq 0$, and $a \in A$.

Each macro power series expression E over (A, X, Z, Φ_∞) denotes a formal power series $|E|$ in $A\langle\langle T(\Phi, X \cup Z)\rangle\rangle$, where Φ is some suitable finite subset of Φ_∞, according to the following conventions:

(i) If E is in $X \cup Z \cup \{\varepsilon\}$ then E denotes the formal power series E, i. e., $|E| = E$.

(ii) For macro power series expressions $E_1, \dots, E_k$ over (A, X, Z, Φ_∞), $G \in \Phi_\infty$ of rank k, $k \geq 0$, $a \in A$, we define
$|[E_1 + E_2]| = |E_1| + |E_2|$,
$|[E_1 E_2]| = |E_1||E_2|$,
$|G(E_1, \dots, E_k)| = \sum_{t_1,\dots,t_k \in T(\Phi, X\cup Z)} (|E_1|, t_1) \dots (|E_k|, t_k) G(t_1, \dots, t_k)$,
$|aE_1| = a|E_1|$,
$|\mu G.E_1| = \mu G.|E_1|$.

We now define a "yield-mapping" Y, that maps algebraic tree series expressions over $(A, \Sigma, X, Z, \Phi_\infty)$ to macro power series expressions over (A, X, Z, Φ_∞), in the following manner:

(i) if E is in $X \cup Z$ then $\mathrm{Y}(E) = E$,

(ii) for algebraic tree series expressions $E_1, \dots, E_k$ over $(A, \Sigma, X, Z, \Phi_\infty)$, $\omega \in \Sigma$ of rank k, $G \in \Phi_\infty$ of rank k, $k \geq 0$, $a \in A$ we define
$\mathrm{Y}([E_1 + E_2]) = [\mathrm{Y}(E_1) + \mathrm{Y}(E_2)]$,
$\mathrm{Y}(\omega(E_1, \dots, E_k)) = [\dots[\mathrm{Y}(E_1)\mathrm{Y}(\mathrm{E}_2)] \cdots \mathrm{Y}(E_k)]$
(including $\mathrm{Y}(\omega) = \varepsilon$ for $k = 0$, $\mathrm{Y}(\omega(E_1)) = \mathrm{Y}(E_1)$ for $k = 1$),
$\mathrm{Y}(G(E_1, \dots, E_k)) = G(\mathrm{Y}(E_1), \dots, \mathrm{Y}(E_k))$,
$\mathrm{Y}(aE_1) = a\mathrm{Y}(E_1)$,
$\mathrm{Y}(\mu G.E_1) = \mu G.\mathrm{Y}(E_1)$.

We claim that $\mathrm{yd}(|E|) = |\mathrm{Y}(E)|$ for an algebraic tree series expression over $(A, \Sigma, X, Z, \Phi_\infty)$. The proof is by induction of the form of E. We only show the case $E = \mu G.E_1$. We obtain

$$\mathrm{yd}(|E|) = \mathrm{yd}(\mu G.|E_1|) = \mu G.\mathrm{yd}(|E_1|) = \mu G.|\mathrm{Y}(E_1)| = |\mathrm{Y}(\mu G.E_1)| = |\mathrm{Y}(E)|\,.$$

Here the second equality follows by the continuity of the mapping yd and the third equality follows by the induction hypothesis.

We now define the mappings $\varphi_1, \varphi_2, \varphi_3$ analogous to these mappings in Section 5.

These considerations, together with Corollaries 5.18 and 8.4 imply the following result. It can be considered as a Kleene Theorem for macro power series.

Theorem 6.8.5 *A power series $r \in A\langle\langle X^*\rangle\rangle$ is a macro power series iff there exists a macro power series expression E over (A, X, Z, Φ_∞) such that $r = |E|$, where $\varphi_1(E) = \varphi_3(E) = \emptyset$.*

If the basic semiring is $\mathbb{B}$, then Theorem 8.5 can be considered as a Kleene Theorem for indexed languages.

Example 6.8.2. Consider the macro system $\mathfrak{M} = (\Phi, Z, X, E, G_0)$ with initial function variable G_0, specified by $\Phi = \Phi_0 \cup \Phi_2$, $\Phi_0 = \{G_0\}$, $\Phi_2 = \{G\}$, $X = \{c_1, c_2\}$ and $E = \{G_0 = G(c_1, c_2), G(z_1, z_2) = G(z_1^2, z_2^2) + z_1 z_2\}$. Since $\mathfrak{M} = \mathrm{yd}(\mathfrak{S})$, where $\mathfrak{S}$ is defined in Example 5.2, we obtain that the initial component of the least solution of $\mathfrak{M}$ is given by $\sum_{j\geq 0} c_1^{2^j} c_2^{2^j} = |\mu G.[G([c_1c_1], [c_2c_2]) + [c_1c_2]]|$. Observe that this macro power series expression is $\mathrm{Y}(E)$ where E is the algebraic tree series expression given in Example 5.2. □

We now show that algebraic power series are the yield of recognizable tree series.

Let $z_i = p_i$, $p_i \in A\langle\langle T_\Sigma(X \cup Z_n)\rangle\rangle$, $1 \leq i \leq n$, be a simple proper finite polynomial system with least solution $(\sigma_1, \ldots, \sigma_n)$. Consider the proper algebraic system $z_i = \mathrm{yd}(p_i)$, $\mathrm{yd}(p_i) \in A\langle\langle (X \cup Z_n)^*\rangle\rangle$, $1 \leq i \leq n$. Then it is easily proved that its least solution is given by $(\mathrm{yd}(\sigma_1), \ldots, \mathrm{yd}(\sigma_n))$. This proves the next theorem.

Theorem 6.8.6 *If s is a recognizable tree series then $\mathrm{yd}(s)$ is an algebraic power series.*

Corollary 6.8.7 *Let $\{\bullet, e\} \subseteq \Sigma$, where $\bullet$ and e have rank 2 and 0, respectively. Then a power series $r \in A\langle\langle X^*\rangle\rangle$ is algebraic iff there exists a tree series in $A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ such that $\mathrm{yd}(s) = r$.*

For $A = \mathbb{N}^\infty$, Theorem 8.6 and Theorem 3.9 of Kuich [78] imply the following wellknown result of formal language theory. (See also Bucher, Maurer [17], Section 3.3, Gécseg, Steinby [52], Section 14, and Seidl [109].)

Theorem 6.8.8 *Let G be a context-free grammar. Then for $w \in L(G)$ there are $d(w)$ different leftmost derivations for w in G iff there are $d(w)$ nonisomorphic derivation trees of G with result w.*

The Kleene Theorems of Section 4 imply by Corollary 8.7 the Kleene Theorem 3.5.6 for algebraic power series and context-free languages. (See Kuich [79], Gruska [60].)

We now turn to the theory of full abstract families of tree series and make the following convention for the rest of Section 8: The set Σ_∞ (resp. X_∞) is a fixed *infinite* ranked alphabet (resp. *infinite* alphabet) and Σ (resp. X), possibly provided with indices, is a *finite* subalphabet of Σ_∞ (resp. X_∞). Moreover, Σ_∞ contains a symbol $\bullet$ of rank 2 and a symbol e of rank 0.

We will show that, for a full AFT $\mathfrak{L}$, yield($\mathfrak{L}$) is a full abstract family of power series (briefly, AFP). Here $\mathrm{yield}(\mathfrak{L}) = \{\mathrm{yd}(s) \mid s \in \mathfrak{L}\}$.

Theorem 6.8.9 *Let $\mathfrak{L}$ be an equationally closed family of tree series. Then* yield($\mathfrak{L}$) *is closed under addition, multiplication and star and contains 0 and 1.*

Proof. (i) Let $r_1, r_2 \in \mathrm{yield}(\mathfrak{L})$. Then there exist $s_1, s_2 \in \mathfrak{L}$ such that $\mathrm{yd}(s_i) = r_i$, $i = 1, 2$. Since $\mathfrak{L}$ is closed under addition, $s = s_1 + s_2 \in \mathfrak{L}$ and $\mathrm{yd}(s) = r_1 + r_2 \in \mathrm{yield}(\mathfrak{L})$. Since $\mathfrak{L}$ is closed under top-catenation, $s' = \bullet(s_1, s_2) \in \mathfrak{L}$ and $\mathrm{yd}(s') = r_1 r_2 \in \mathrm{yield}(\mathfrak{L})$.

(ii) Let $s \in \mathfrak{L}$ and assume that $x \in X_\infty$ does not appear in s. Consider the equation $x = \bullet(s, x) + e$. Its least solution $\mu x.(\bullet(s, x) + e)$ is in $\mathfrak{L}$. Hence, the least solution $\mu x.\mathrm{yd}(\bullet(s, x) + e) = \mu x.(\mathrm{yd}(s)x + \varepsilon) = \mathrm{yd}(s)^*$ of $x = \mathrm{yd}(\bullet(s, x) + e) = \mathrm{yd}(s)x + \varepsilon$ is in yield($\mathfrak{L}$). Moreover, $\mathrm{yd}(0) = 0$ and $0^* = 1$ are in yield($\mathfrak{L}$). □

A multiplicative morphism

$$\nu : X^* \to (A\langle\langle X'^* \rangle\rangle)^{Q \times Q}$$

is called a *power series representation.* A power series representation ν is called *rational* (resp. *algebraic*, *macro*) iff the entries of $\nu(x)$, $x \in X$, are rational (resp. algebraic, macro) power series. A power series transducer $\mathfrak{Z} = (Q, \nu, S, P)$ is called *rational* (resp. *algebraic*, *macro*) iff ν is a rational (resp. an algebraic, a macro) power series representation and the entries of S and P are rational (resp. algebraic, macro) power series. A power series transduction is called *rational* (resp. *algebraic*, *macro*) iff it is realized by a rational (resp. an algebraic, a macro) power series transducer.

Lemma 6.8.10 *Let ν be an algebraic power series representation defined by $\nu : X \to (A^{\mathrm{alg}}\langle\langle X'^* \rangle\rangle)^{Q \times Q}$. Then there exists a nondeterministic simple recognizable tree representation μ with state set $Q \times Q$ mapping $\Sigma \cup X$ into matrices with entries in $A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X' \cup Z) \rangle\rangle$, $\Sigma' = \{\bullet, e\}$, such that, for all $s \in A\langle\langle T_\Sigma(X) \rangle\rangle$ and $q_1, q_2 \in Q$,*

$$\mathrm{yd}(\mu(s)_{(q_1,q_2)}) = \nu(\mathrm{yd}(s))_{q_1,q_2} \, .$$

Proof. We construct $\mu = (\mu_k \mid k \geq 0)$:

(i) For $x \in X$ and $q_1, q_2 \in Q$ we construct $\mu_0(x)_{(q_1,q_2)}$ according to Corollary 8.7 with the property that $\mathrm{yd}(\mu_0(x)_{(q_1,q_2)}) = \nu(x)_{q_1,q_2}$.

(ii) For $\omega \in \Sigma_0$ and $q_1, q_2 \in Q$ we define $\mu_0(\omega)_{(q_1,q_2)} = \delta_{q_1,q_2} e$, where δ is the Kronecker symbol; hence, $\mathrm{yd}(\mu(\omega)_{(q_1,q_2)}) = \delta_{q_1,q_2}\varepsilon = \nu(\varepsilon)_{q_1,q_2}$.

(iii) For $\omega \in \Sigma_k$, $k \geq 1$, and $q_1, q_2, r_1, \ldots, r_k, s_1, \ldots, s_k \in Q$, we define

$$\begin{aligned} \mu_k(\omega)_{(q_1,q_2),((r_1,s_1),\ldots,(r_k,s_k))} = {} & \delta_{q_1,r_1}\delta_{s_1,r_2} \ldots \delta_{s_{k-1},r_k}\delta_{s_k,q_2} \\ & \bullet(z_1, \bullet(z_2, \bullet(\ldots \bullet(z_{k-1}, z_k) \ldots))) \, . \end{aligned}$$

We first consider a tree $t \in T_\Sigma(X)$ and show that $\mathrm{yd}(\mu(t)_{(q_1,q_2)}) = \nu(\mathrm{yd}(t))_{q_1,q_2}$, $q_1, q_2 \in Q$. The proof is by induction on the structure of trees in $T_\Sigma(X)$. The induction basis is true by (i) and (ii). Let now $t = \omega(t_1, \ldots, t_k)$, $\omega \in \Sigma_k$, $k \geq 1$, $t_1, \ldots, t_k \in T_\Sigma(X)$. Then we obtain, for $q_1, q_2 \in Q$,

$$\begin{aligned}
&\mathrm{yd}(\mu(t)_{(q_1,q_2)}) = \mathrm{yd}(\mu(\omega(t_1, \ldots, t_k))_{(q_1,q_2)}) = \\
&\mathrm{yd}(\textstyle\sum_{r_1,\ldots,r_k \in Q} \sum_{s_1,\ldots,s_k \in Q} \mu(\omega)_{(q_1,q_2),((r_1,s_1),\ldots,(r_k,s_k))} \\
&\qquad\qquad [\mu(t_1)_{(r_1,s_1)}/z_1, \ldots, \mu(t_k)_{(r_k,s_k)}/z_k]) = \\
&\mathrm{yd}(\textstyle\sum_{r_1,\ldots,r_k \in Q} \sum_{s_1,\ldots,s_k \in Q} \delta_{q_1,r_1}\delta_{s_1,r_2} \ldots \delta_{s_{k-1},r_k}\delta_{s_k,q_2} \\
&\ \bullet(\mu(t_1)_{(r_1,s_1)}, \bullet(\mu(t_2)_{(r_2,s_2)}, \bullet(\ldots, \bullet(\mu(t_{k-1})_{(r_{k-1},s_{k-1})}, \mu(t_k)_{(r_k,s_k)}) \ldots)))) = \\
&\textstyle\sum_{s_1,\ldots,s_{k-1} \in Q} \mathrm{yd}(\mu(t_1)_{(q_1,s_1)})\mathrm{yd}(\mu(t_2)_{(s_1,s_2)}) \ldots \\
&\qquad\qquad \ldots \mathrm{yd}(\mu(t_{k-1})_{(s_{k-2},s_{k-1})})\mathrm{yd}(\mu(t_k)_{(s_{k-1},q_2)}) = \\
&\textstyle\sum_{s_1,\ldots,s_{k-1} \in Q} \nu(\mathrm{yd}(t_1))_{q_1,s_1}\nu(\mathrm{yd}(t_2))_{s_1,s_2} \ldots \\
&\qquad\qquad \ldots \nu(\mathrm{yd}(t_{k-1}))_{s_{k-2},s_{k-1}}\nu(\mathrm{yd}(t_k))_{s_{k-1},q_2} = \\
&\nu(\mathrm{yd}(t_1) \ldots \mathrm{yd}(t_k))_{q_1,q_2} = \nu(\mathrm{yd}(t))_{q_1,q_2}\,.
\end{aligned}$$

Hence, for $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ and $q_1, q_2 \in Q$,

$$\begin{aligned}
&\mathrm{yd}(\mu(s)_{(q_1,q_2)}) = \textstyle\sum_{t \in T_\Sigma(X)} (s,t)\mathrm{yd}(\mu(t)_{(q_1,q_2)}) = \\
&\textstyle\sum_{t \in T_\Sigma(X)} (s,t)\nu(\mathrm{yd}(t))_{q_1,q_2} = \nu(\mathrm{yd}(s))_{q_1,q_2}\,.
\end{aligned}$$

□

A non-empty family of power series is called an *algebraic cone* iff it is closed under algebraic power series transductions. Observe that each algebraic cone is a (rational) cone, i. e., a family of power series closed under rational power series transductions.

Theorem 6.8.11 *Let $\mathfrak{L}$ be a full AFT. Then* yield($\mathfrak{L}$) *is an algebraic cone.*

Proof. Let $s \in \mathfrak{L}$, $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$, $r = \mathrm{yd}(s)$, and $\mathfrak{Z} = (Q, \nu, S, P)$ be an algebraic transducer. We will show that $||\mathfrak{Z}||(r) \in A\langle\langle X'^*\rangle\rangle$ is again in yield($\mathfrak{L}$). Observe that $||\mathfrak{Z}||(r) = S\nu(r)P = \sum_{q_1,q_2 \in Q} S_{q_1}\nu(r)_{q_1,q_2}P_{q_2}$, where $S_q, P_q \in A^{\mathrm{alg}}\langle\langle X'^*\rangle\rangle$, $q \in Q$. By Corollary 8.7 there exist $s_q, p_q \in A^{\mathrm{rec}}\langle\langle T_{\Sigma'}(X')\rangle\rangle$, $\bullet, e \in \Sigma'$, such that $\mathrm{yd}(s_q) = S_q$, $\mathrm{yd}(p_q) = P_q$, $q \in Q$. By Lemma 8.10 there exists a nondeterministic simple recognizable tree representation μ with state set $Q \times Q$ such that $\mathrm{yd}(\mu(s)_{(q_1,q_2)}) = \nu(r)_{q_1,q_2}$ for all q_1, q_2. Since $\mathfrak{L}$ is equationally closed, $\sum_{q_1,q_2 \in Q} \bullet(s_{q_1}, \bullet(\mu(s)_{(q_1,q_2)}, p_{q_2}))$ is in $\mathfrak{L}$. Hence,

$$\begin{aligned}
&\mathrm{yd}(\textstyle\sum_{q_1,q_2 \in Q} \bullet(s_{q_1}, \bullet(\mu(s)_{(q_1,q_2)}, p_{q_2}))) = \\
&\textstyle\sum_{q_1,q_2 \in Q} \mathrm{yd}(s_{q_1})\mathrm{yd}(\mu(s)_{(q_1,q_2)})\mathrm{yd}(p_{q_2}) = \\
&\textstyle\sum_{q_1,q_2 \in Q} S_{q_1}\nu(r)_{q_1,q_2}P_{q_2} = ||\mathfrak{Z}||(r)
\end{aligned}$$

is in yield($\mathfrak{L}$). □

Corollary 6.8.12 *Let $\mathfrak{L}$ be a full AFT. Then* yield($\mathfrak{L}$) *is a full AFP that is closed under algebraic transductions.*

Corollary 6.8.13 *The family of algebraic power series is a full AFP closed under algebraic transductions.*

Corollary 6.8.14 *The family of algebraic power series is a full AFP closed under substitutions.*

Theorem 6.8.15 *Let $\mathfrak{L}$ be a full AFT closed under algebraic tree series transductions. Then* yield($\mathfrak{L}$) *is a full AFP closed under macro power series transductions.*

Proof. Similar to the proof of Theorem 8.11. □

Corollary 6.8.16 *The family of macro power series is a full AFP closed under macro power series transductions.*

Corollary 6.8.17 *The family of macro power series is a full AFP closed under substitution.*

We now turn to the language case, i. e., our basic semiring is now $2^{T_{\Sigma_\infty}(X_\infty)}$. We use without mentioning the isomorphism between $2^{T_{\Sigma_\infty}(X_\infty)}$ and $\mathbb{B}\langle\langle T_{\Sigma_\infty}(X_\infty)\rangle\rangle$.

A family $\mathfrak{L}$ of tree languages is called *equationally closed* iff $\langle \mathfrak{L}, \cup, \emptyset, (\bar{\omega} \mid \omega \in \Sigma_\infty) \cup X_\infty \rangle$ is a distributive $\Sigma_\infty \cup X_\infty$-algebra that satisfies the following condition:

> If $L \in \mathfrak{L}$ and $x \in X_\infty$ then the least solution $\mu x.L$ of the tree language equation $x = L$ is in $\mathfrak{L}$.

Define $\hat{\mathfrak{F}}(\mathfrak{L})$ to be the smallest equationally closed family of tree languages that is closed under nondeterministic simple recognizable tree series transductions and contains $\mathfrak{L}$. A family $\mathfrak{L}$ of tree languages is called a *full abstract family of tree languages* iff $\mathfrak{L} = \hat{\mathfrak{F}}(\mathfrak{L})$.

We now connect our full abstract families of tree languages with full AFLs (see Salomaa [106], Ginsburg [53] and Berstel [4]).

Theorem 6.8.18 *Let $\mathfrak{L}$ be a full abstract family of tree languages. Then* yield($\mathfrak{L}$) *is a full AFL that is closed under algebraic transductions.*

A substitution σ is called *context-free* iff $\sigma(x)$ is a context-free language for each $x \in X$.

Corollary 6.8.19 *Let $\mathfrak{L}$ be a full abstract family of tree languages. Then* yield($\mathfrak{L}$) *is a full AFL that is closed under context-free substitutions.*

Corollary 6.8.20 *The family of context-free languages is an AFL closed under substitution.*

Corollary 6.8.21 (Aho [2], Theorem 3.4.) *The family of indexed languages is an AFL closed under substitution.*

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