Check for updates

Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes*

Jan Dreier
TU Wien
Wien, Austria
dreier@ac.tuwien.ac.at

Nikolas Mählmann
University of Bremen
Bremen, Germany
maehlmann@uni-bremen.de

Szymon Toruńczyk
University of Warsaw
Warsaw, Poland
szymtor@mimuw.edu.pl

ABSTRACT

A conjecture in algorithmic model theory predicts that the model-checking problem for first-order logic is fixed-parameter tractable on a hereditary graph class if and only if the class is *monadically dependent*. Originating in model theory, this notion is defined in terms of logic, and encompasses nowhere dense classes, monadically stable classes, and classes of bounded twin-width. Working towards this conjecture, we provide the first two combinatorial characterizations of monadically dependent graph classes. This yields the following dichotomy.

On the structure side, we characterize monadic dependence by a Ramsey-theoretic property called *flip-breakability*. This notion generalizes the notions of uniform quasi-wideness, flip-flatness, and bounded grid rank, which characterize nowhere denseness, monadic stability, and bounded twin-width, respectively, and played a key role in their respective model checking algorithms. Natural restrictions of flip-breakability additionally characterize bounded treewidth and cliquewidth and bounded treedepth and shrubdepth.

On the non-structure side, we characterize monadic dependence by explicitly listing few families of *forbidden induced subgraphs*. This result is analogous to the characterization of nowhere denseness via forbidden subdivided cliques, and allows us to resolve one half of the motivating conjecture: First-order model checking is AW[∗]-hard on every hereditary graph class that is monadically independent. The result moreover implies that hereditary graph classes which are small, have almost bounded twin-width, or have almost bounded flip-width, are monadically dependent.

Lastly, we lift our result to also obtain a combinatorial dichotomy in the more general setting of monadically dependent classes of binary structures.

*N.M. was supported by the German Research Foundation (DFG) with grant agreement No. 444419611. S.T. was supported by the project BOBR that is funded from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme with grant agreement No. 948057. The BOBR project also supported a stay of N.M. at the University of Warsaw, where part of this work was done.

STOC '24, June 24–28, 2024, Vancouver, BC, Canada

ACM ISBN 979-8-4007-0383-6/24/06
https://doi.org/10.1145/3618260.3649739

CCS CONCEPTS

• **Theory of computation → Finite Model Theory**; **Fixed parameter tractability**; • **Mathematics of computing → Graph theory**.

KEYWORDS

Monadically dependent, monadically NIP, first-order model checking, algorithmic model theory, structural graph theory

ACM Reference Format:
Jan Dreier, Nikolas Mählmann, and Szymon Toruńczyk. 2024. Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC '24), June 24–28, 2024, Vancouver, BC, Canada.* ACM, New York, NY, USA, 11 pages. https://doi.org/10.1145/3618260.3649739

1 INTRODUCTION

Algorithmic model theory studies the interplay between the computational complexity of computational problems defined using logic, and the structural properties of the considered instances. In this context, *algorithmic meta-theorems* [31, 33] are results that establish the tractability of entire families of computational problems, which are defined in terms of logic, while imposing structural restrictions on the input instances. The archetypical example is Courcelle's theorem, which states that every problem that can be expressed in monadic second order logic (MSO), can be solved in linear time, whenever the considered input graphs have bounded treewidth [3, 13]. This implies that the model checking problem for MSO is *fixed-parameter tractable* (fpt) on every class C of bounded treewidth. That is, there is an algorithm which determines whether a given input graph $G \in C$ satisfies a given MSO formula φ in time $f(|\varphi|) \cdot |V(G)|^c$ for some function $f\colon \mathbb{N} \to \mathbb{N}$ and some constant c (in this case $c = 1$). More generally, the model checking problem for MSO is fpt on all classes of bounded cliquewidth [14].

In this paper, we focus on the model-checking problem for *first-order logic* (FO), which allows to relax the structure of the input graphs greatly, at the cost of restricting the logic. Graph classes for which this problem is fpt include classes of bounded degree [42], the class of planar graphs [27], classes which exclude a minor [26], classes of bounded expansion [23], and more generally, nowhere dense classes [32], all of which are *sparse* graph classes (specifically, every n-vertex graph in such a class has $O(n^{1+\varepsilon})$ edges, for every fixed $\varepsilon > 0$). The problem is moreover fpt on proper hereditary classes of permutation graphs, some classes of bounded twin-width [9], structurally nowhere dense classes [18], monadically stable classes [17], and others [5]. The central question in the area, first phrased in [31, Sec. 8.2], is the following.

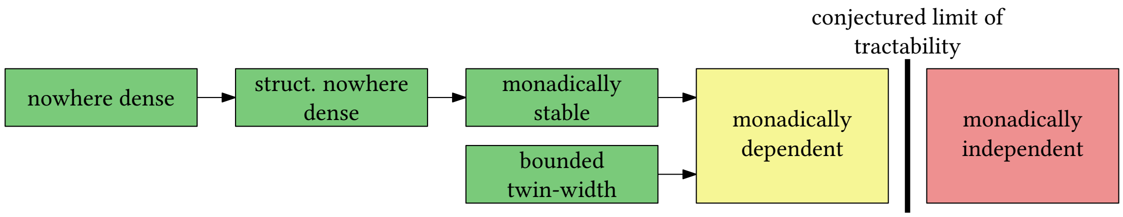

Figure 1: Hierarchy of selected properties of graph classes. Classes in green boxes are known do admit fpt model checking algorithms (with additional assumptions required in the case of bounded twin-width). *Monadically dependent* classes are known to generalize all these notions. It is conjectured that a hereditary class is monadically dependent if and only if its model checking problem is tractable.

> What are the structural properties that exactly characterize the hereditary[1] graph classes with fpt first-order model checking?

Sparsity theory [36], initiated by Nešetřil and Ossona de Mendez, has provided solid structural foundations, and a very general notion of structural tameness for sparse graph classes. More recently, *twin-width theory* [9] provides another notion of graph classes which are structurally tame, and not necessarily sparse. While *twin-width theory* and *sparsity theory* bear striking similarities, they are fundamentally incomparable in scope. The community eagerly anticipates a theory that unifies both frameworks, and answers the central question. *Stability theory* – an area in model theory developed initially by Shelah – provides very general notions of *logical* tameness of classes of graphs (or other structures), called (monadic) *stability* and *dependence*[2], which subsume the notions studied in sparsity theory and in twin-width theory, but are not easily amenable to combinatorial or algorithmic treatment. Let us briefly review *sparsity theory*, *twin-width theory*, and *stability theory*, which we build upon. Figure 1 shows the relationships between the properties of graph classes discussed in this paper.

Sparsity. The central notion of sparsity theory [36] is that of a *nowhere dense* graph class. This is a very general notion of structural tameness of sparse graph classes, and encompasses all classes with bounded degree, bounded treewidth, the class of planar graphs, and classes that exclude some graph as a minor. After a long sequence of prior work [16, 23, 26, 27, 42], the celebrated result of Grohe, Kreutzer and Siebertz [32] established that the model checking problem is fpt on every nowhere dense graph class. For classes that are *monotone* (closed under removal of vertices and edges) this is optimal [23, 33], yielding the following milestone result.

> A monotone graph class admits fpt model checking if and only if it is *nowhere dense* (assuming FPT $\neq$ AW$[*]$) [32].

Here, FPT $\neq$ AW$[*]$ is a standard complexity assumption in parameterized complexity, equivalent to the statement that model checking is not fpt on the class of all graphs. The major shortcoming of this result is that it only captures monotone classes, and thus *sparse* graph classes, while there are dense graph classes which are not monotone, and have fpt model checking. A trivial example is the class of cliques. More generally, all classes of bounded cliquewidth have fpt model checking. For many years, not many examples of graph classes for which the model checking problem is fpt – beyond nowhere dense classes and classes of bounded cliquewidth – were known.

A recent line of research extends sparsity theory beyond the sparse setting using *transductions*. We say a graph class $\mathcal{C}$ *transduces* a graph class $\mathcal{D}$ if there exists a first-order formula $\varphi(x, y)$ such that every graph in $\mathcal{D}$ can be obtained by the following four steps: (1) taking a graph $G \in \mathcal{C}$, (2) coloring the vertices of G, (3) replacing the edge set of G with $\{uv \mid u, v \in V(G), u \neq v, G \models \varphi(u, v) \vee \varphi(v, u)\}$, and (4) taking an induced subgraph. Step 3 can construct, for example, the *edge-complement* of G with $\varphi(x, y) = \neg E(x, y)$ or the *square* of G with $\varphi(x, y) = \exists z\ E(x, z) \wedge E(z, y)$. Note that the formula φ has access to the colors of G from step 2. Therefore, for example, the class of all graphs is transduced by the class of 1-subdivided cliques (by coloring the subdivision vertices that should be turned into edges), but is *not* transduced by the class of cliques. It was recently shown that model checking is fpt on *transductions of nowhere dense classes* [18], extending the work of Grohe, Kreutzer, and Siebertz [32] beyond the sparse setting. This result has been generalized very recently in [17], as described below.

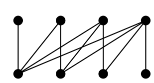

Figure 2: A half-graph of order 4.

Monadic Stability. Stability, and its variants, are logical tameness conditions arising in Shelah's classification program [43]. A graph class is *monadically*[3] *stable* if it does not transduce the class of all *half-graphs* (see Figure 2). Monadically stable classes include all structurally nowhere dense classes [2, 40]. In [17] it is shown that FO model checking is fpt on all monadically stable classes, and moreover obtains matching hardness bounds via a new characterization of such classes in terms of forbidden induced subgraphs.

[1] A graph class is *hereditary* if it is closed under vertex removal.
[2] In the literature, *dependence* is also referred to as *NIP*, which stands for *negation of the independence property*.

[3] *Monadically* refers to the coloring step of a transduction, which is an expansion of the graph with unary/monadic predicates. In classical model theory monadically stable (monadically dependent) classes are equivalently defined as those classes that remain stable (dependent) under unary expansions.

A class is *orderless* if it avoids some half-graph as a semi-induced[4] subgraph. This establishes monadically stable classes as the *limit of tractability* among orderless classes.

> An orderless, hereditary graph class admits fpt model checking if and only if it is *monadically stable* (assuming FPT ≠ AW[∗]) [17].

Twin-Width. Twin-width [9] is a recently introduced notion which has its roots in enumerative combinatorics and the Stanley-Wilf conjecture/Marcus-Tardos theorem. Graph classes with bounded twin-width include the class of planar graphs, all classes of bounded cliquewidth, the class of unit interval graphs, and every proper hereditary class of permutation graphs. It was shown that model checking is fpt on graph classes of bounded twin-width, assuming an appropriate decomposition of the graph (in form of a so-called *contraction sequence*) is given as part of the input [9]. On *ordered graphs* (that is, graphs equipped with a total order on the vertex set, which can be accessed by the formulas) a recent breakthrough result has lifted this additional proviso, and also characterized classes of bounded twin-width as the *limit of tractability* [7].

> A hereditary class of ordered graphs admits fpt model checking if and only if it has bounded *twin-width* (assuming FPT ≠ AW[∗]) [7].

Monadic Dependence. As discussed above, an exact characterization of classes with fpt model-checking has been established in three settings: for monotone graph classes, for the more general hereditary orderless graph classes, and for hereditary classes of ordered graphs, in terms of the notions of nowhere denseness, monadic stability, and bounded twin-width, respectively. Those notions turn out to be three facets of a single notion, again originating in stability theory. A graph class C is *monadically independent* if it transduces the class of all graphs [4]. Otherwise C is *monadically dependent.* In all three settings where we have a complete characterization of fpt model checking, monadic dependence precisely captures the limit of tractability.

- On monotone classes, nowhere denseness is equivalent to monadic dependence [2, 40].
- On orderless classes, monadic stability is equivalent to monadic dependence [37].
- On classes of ordered graphs, bounded twin-width is equivalent to monadic dependence [7].

This suggests that the known tractability limits [7, 17, 18, 32] are fragments of a larger picture, where monadically dependent classes unify sparsity theory, twin-width theory, and stability theory into a single theory of tractability.

Conjecture 1.1 (e.g., [1, 7, 18]). *Let C be a hereditary class of graphs. Then the model checking problem for first-order logic is fpt on C if and only if C is monadically dependent.*[5]

Originally stated in 2016 [1], the above conjecture is now the central open problem in the area. Both directions have been open.

[4]Semi-induced half-graphs look similar to Figure 2, but the connections within the top row and within the bottom row may be arbitrary.

[5]The conjecture stated in [1] mentions dependence instead of monadic dependence, but for hereditary classes, those notions are equivalent [12]. Furthermore, the conjecture in [1] states only one implication, but the other one was also posed at the same workshop as an open problem.

2 CONTRIBUTION

To approach Conjecture 1.1, a combinatorial characterization of monadic dependence is sought. Based on the development in the sparse, orderless, and ordered cases discussed above, it appears that what is needed are *combinatorial dichotomy results*, stating that all monadically dependent graph classes exhibit structure which can be used to design efficient algorithms, while all other graph classes exhibit a sufficient lack of structure, which can be used to prove hardness results. The three known restricted classifications of classes with fpt model checking were enabled in part (see discussion below) thanks to such combinatorial dichotomies for nowhere dense, monadically stable, and bounded twin-width graph classes. However, for monadically dependent classes not a single combinatorial characterization has been known. Previous characterizations of monadic dependence (for example, via indiscernible sequences or existentially defined canonical configurations [11, 12]) all have a logical, rather than combinatorial, aspect. This limits their algorithmic usefulness.

In this paper, we provide the first two *purely combinatorial* characterizations of monadically dependent classes, which together constitute a combinatorial dichotomy theorem for these classes.

- On the one hand, we show that monadically dependent graph classes have a Ramsey-like property called *flip-breakability*, which guarantees that any large set of vertices W contains two still-large subsets A, B that in a certain sense are strongly separated.
- On the other hand, we show that graph classes that are monadically independent contain certain highly regular patterns as induced subgraphs, which are essentially two highly interconnected sets.

As argued below, *flip-breakability* might be a crucial step towards establishing fixed-parameter tractability of the model checking problem for monadically dependent classes, and settling the tractability side of Conjecture 1.1. Moreover, we use the patterns of the second characterization to prove the hardness side of Conjecture 1.1: we show that first-order model checking is AW[∗]-hard on every hereditary graph class that is monadically independent (Theorem 2.7). We now present our two combinatorial characterizations of monadically dependent classes in more detail.

Flip-Breakability. Flips are a central emerging mechanism in the study of well-behaved graph classes (e.g., [5, 29, 30, 44]). A k*-flip* of a graph G is a graph H obtained from G by partitioning the vertex set into k parts and, for every pair X, Y of parts, either leaving the adjacency between X and Y intact, or complementing it.

We use flips to measure the interaction between vertex sets A, B at a fixed distance r in a graph G. Intuitively, if for some small value k there is a k-flip H of G in which the distance $\mathrm{dist}_H(A, B)$ is larger than r, then this witnesses that the sets A and B are "well-separated" at distance r.

Consider for example the graph G at the top of Figure 3, consisting of stacked half-graphs. Applying a flip between the blue squares and the blue circles (as depicted in the bottom of Figure 3), and between the orange squares and the orange circles, we obtain a 5-flip of G in which the sets A and B have distance at least 6.

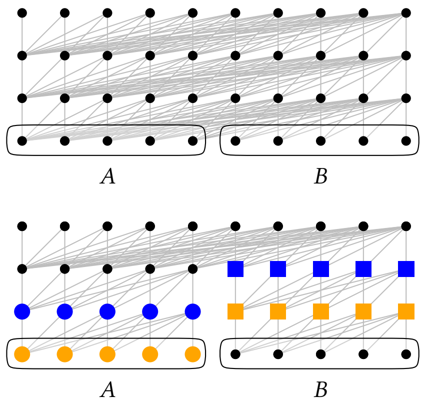

Figure 3: The sets A and B are far away after a 5-flip shown on the bottom. We flip between the blue squares and the blue circles, and between the orange squares and the orange circles.

Roughly, our first main result expresses that for graphs from a monadically dependent class, in every sufficiently large vertex set W, we can find two still-large subsets A and B whose distance in some $O(1)$-flip is larger than a given constant r. This is formally expressed as follows.

Definition 2.1 (Flip-Breakability)**.** A graph class C is *flip-breakable* if for every radius $r \in \mathbb{N}$ there exists a function $N_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in C$ and $W \subseteq V(G)$ with $|W| \geqslant N_r(m)$ there exist subsets $A, B \subseteq W$ with $|A|, |B| \geqslant m$ and a k_r-flip H of G such that:

$$\mathrm{dist}_H(A, B) > r.$$

Theorem 2.2. *A class of graphs is monadically dependent if and only if it is flip-breakable.*

Our proof is algorithmic: For a fixed monadically dependent class C and radius r, the subsets A and B and the witnessing flip H can be computed in time quadratic time.

Theorem 2.3. *For every monadically dependent class C and radius $r \in \mathbb{N}$, there exists an unbounded function $f_r : \mathbb{N} \to \mathbb{N}$, a constant $k_r \in \mathbb{N}$, and an algorithm that, given a graph $G \in C$ and $W \subseteq V(G)$, computes in time $O_{C,r}(|V(G)|^2)$ two subsets $A, B \subseteq W$ with $|A|, |B| \geqslant f_r(|W|)$ and a k_r-flip H of G such that:*

$$\mathrm{dist}_H(A, B) > r.$$

Relation of Flip-Breakability to Other Work. The model checking algorithms for nowhere dense classes [32] and monadically stable classes [17, 18] are respectively based on winning strategies for *pursuit-evasion games* called the *splitter game* [32] and the *flipper game* [29]. These winning strategies were obtained from a characterization of nowhere dense classes in terms of *uniform quasi-wideness* [15, 35] and a characterization of monadically stable classes in terms of *flip-flatness* [19]. Both are Ramsey-like properties which are very similar to flip-breakability. The definitions of these three notions match the same template:

> A class of graphs C is … if for every radius $r \in \mathbb{N}$ there exists a function $N_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in C$ and $W \subseteq V(G)$ with $|W| \geqslant N_r(m)$ there exist …

In all three cases we continue the definition by stating the existence of a large number of vertices that are in a certain sense "well-separated". However, the sentence is completed in an increasingly more general way for the three notions. The crucial differences are highlighted in bold.

- uniform quasi-wide [15, 35]: …a set $S \subseteq V(G)$ of at most **k_r vertices** and $A \subseteq W \setminus S$ with $|A| \geqslant m$, such that in $G \setminus S$, all vertices from A have **pairwise distance greater than r**.
- flip-flat [19]: …a **k_r-flip H** of G and $A \subseteq W$ with $|A| \geqslant m$, such that in H, all vertices from A have **pairwise distance greater than r**.
- flip-breakable: …a **k_r-flip H** of G and $A, B \subseteq W$ with $|A|, |B| \geqslant m$, such that in H, **all vertices in A have distance greater than r from all vertices in B**.

In the context of ordered graphs, the model checking algorithm for classes of bounded twin-width crucially relied on a characterization of these classes in terms of *bounded grid rank* [7, Sec. 3.4]. Rephrasing this characterization in the language of this paper, it reads as follows.

> A class of ordered graphs C has *bounded grid rank* if there exists $k \in \mathbb{N}$ such that for all $G \in C$ and ordered sequences of vertices $a_1 < \cdots < a_k \in V(G)$, $b_1 < \cdots < b_k \in V(G)$, there exists a k-flip H of G and indices $i, j \in [k-1]$ defining ranges $A = \{a \in V(G) \mid a_i \leqslant a \leqslant a_{i+1}\}$, $B = \{b \in V(G) \mid b_j \leqslant b \leqslant b_{j+1}\}$, such that there are no edges incident to both A and B in H.

Flip-breakability combines the distance-r-based aspects of uniform quasi-wideness and flip-flatness with the "no edges incident to both A and B" criterion of grid rank. Using our *insulation property*, one can easily reprove the characterization of nowhere dense classes in terms of uniform quasi-wideness [35], of monadically stable classes in terms of flip-flatness [19], and of classes of ordered graphs with bounded twin-width in terms of grid rank [7].

Given that flip-breakability characterizes monadically dependent graph classes and naturally generalizes the previous notions, we believe that it will also play a crucial role in a future model checking algorithm for monadically dependent classes: Mirroring the situation for nowhere dense and monadically stable classes, flip-breakability might lead to a characterization of monadically dependent classes in terms of a *pursuit-evasion game*. Just like the previous two algorithms [17, 18, 32], winning strategies for this game might then be useful for model checking, and settling the backward direction of Conjecture 1.1. (Note however that in each of these cases, the games where only one of the main ingredients of the model checking algorithm).

Binary Structures. Our results apply in the more general setting of binary structures, rather than graphs, that is, of structures equipped with one or more binary relation. In particular, we prove that monadically dependent classes of binary structures can be equivalently characterized in terms of flip-breakability, defined suitably for binary structures. As an application, we derive a key result

Table 1: Variants of flip-breakability. Corresponding definitions and proofs can be found in the full version [20] of this paper.

		flatness	breakability
dist-r	flip-	*monadic stability* [19]	*monadic dependence*
	deletion-	*nowhere denseness* [15, 35]	*nowhere denseness*
dist-∞	flip-	*bounded shrubdepth*	*bounded cliquewidth*
	deletion-	*bounded treedepth*	*bounded treewidth*

of [7]: that monadically dependent classes of ordered graphs have bounded grid rank (see definition above).

Theorem 2.4 ([7]). *Let C be a monadically dependent class of ordered graphs. Then C has bounded grid rank.*

Variants of Flip-Breakability. The previously highlighted similarities between flip-breakability, flip-flatness, and uniform quasi-wideness suggest the following natural variations of flip-breakability. We allow to modify the graph using either *(1) flips or vertex deletions* and demand that the resulting subset is either *(2) flat or broken*, that is, either pairwise separated or separated into two large sets. We further additionally parameterize the type of separation to be either *(3) distance-r or distance-*∞. While *distance-r* separation corresponds to the usual kind given in Definition 2.1, *distance-*∞ separation demands sets to be in different connected components of the graph. This is formalized by the following definition.

Definition 2.5. A class of graphs C is *distance-*∞ *flip-breakable*, if there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a constant $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in C$ and $W \subseteq V(G)$ with $|W| \geqslant N(m)$ there exist subsets $A, B \subseteq W$ with $|A|, |B| \geqslant m$ and a k-flip H of G such that in H, no two vertices $a \in A$ and $b \in B$ are in the same connected component.

Using this terminology, uniform quasi-wideness, for example, becomes *distance-r deletion-flatness*. For formal definitions of these variants, we refer to the full version [20] of this paper. As summarized in Table 1, each of the eight possible combinations of (1), (2) and (3) characterizes a well-studied property of graph classes.

Notably, the right column of Table 1 corresponds to conjectured tractability limits of the model checking problem, where the distance-r and distance-∞ variants correspond to, respectively, first-order and monadic second-order logic, and the flip and deletion variants correspond to hereditary and monotone classes. Furthermore, it is interesting to see that the seemingly more general deletion-breakability collapses to deletion-flatness, as both properties characterize nowhere denseness.

Forbidden Induced Subgraphs. Imposing bounds on the size of certain patterns is a common and powerful mechanism for defining well-behaved graph classes. For example, by definition, a class C is *nowhere dense* if and only if for every $r \in \mathbb{N}$, C avoids some r-subdivided clique as a subgraph. In similar spirit, it is shown [17] that an orderless class C is *monadically stable* if and only if for every $r \in \mathbb{N}$, the class C avoids all r-flips of the r-subdivided clique, and its line graph, as an induced subgraph. A similar characterization for monadically dependent classes has so far been elusive. In this paper, we show that a class C is *monadically dependent* if and only if for every $r \in \mathbb{N}$, the class C avoids certain variations of the r-subdivision of some complete bipartite graph, as an induced subgraph. Let us start by defining these patterns.

For $r \geqslant 1$, the *star r-crossing* of order n is the r-subdivision of $K_{n,n}$. More precisely, it consists of *roots* $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ together with r-vertex paths $\{\pi_{i,j} \mid i, j \in [n]\}$ that are pairwise vertex-disjoint (see Figure 4). We denote the two endpoints of a path $\pi_{i,j}$ by $\mathrm{start}(\pi_{i,j})$ and $\mathrm{end}(\pi_{i,j})$. We require that roots appear on no path, that each root a_i is adjacent to $\{\mathrm{start}(\pi_{i,j}) \mid j \in [n]\}$, and that each root b_j is adjacent to $\{\mathrm{end}(\pi_{i,j}) \mid i \in [n]\}$. The *clique r-crossing* of order n is the graph obtained from the star r-crossing of order n by turning the neighborhood of each root into a clique. Moreover, we define the *half-graph r-crossing* of order n similarly to the star r-crossing of order n, where each root a_i is instead adjacent to $\{\mathrm{start}(\pi_{i',j}) \mid i', j \in [n], i \leqslant i'\}$, and each root b_j is instead adjacent to $\{\mathrm{end}(\pi_{i,j'}) \mid i, j' \in [n], j \leqslant j'\}$. Each of the three r-crossings contains no edges other than the ones described. At last, the *comparability grid* of order n consists of vertices $\{a_{i,j} \mid i, j \in [n]\}$ and edges between vertices $a_{i,j}$ and $a_{i',j'}$ if and only if either $i = i'$, or $j = j'$, or $i < i' \Leftrightarrow j < j'$.

It is easy to see that these patterns are logically complicated: For every fixed r, the four graph classes containing these four types of patterns are monadically independent.

We also need to consider certain flips of the above patterns. To this end, we partition the vertices of star, clique, and half-graph r-crossings into *layers*: The zeroth layer consists of the vertices $\{a_1, \ldots, a_n\}$. The lth layer, for $l \in [r]$, consists of the lth vertices of the paths $\{\pi_{i,j} \mid i, j \in [n]\}$ (that is, the 1st and rth layer, respectively, are $\{\mathrm{start}(\pi_{i,j}) \mid i, j \in [n]\}$ and $\{\mathrm{end}(\pi_{i,j}) \mid i, j \in [n]\}$). Finally, the $(r+1)$th layer consists of the vertices $\{b_1, \ldots, b_n\}$. A *flipped* star/clique/half-graph r-crossing is a graph obtained from a star/clique/half-graph r-crossing by performing a flip where the parts of the specifying partition are the layers of the r-crossing. Note that while there is only one star/clique/half-graph r-crossing of order n, there are multiple flipped star/clique/half-graph r-crossings of order n. Their number is however bounded by $2^{(r+2)^2}$: an upper bound for the number of possible flips specified by a single partition of size $(r+2)$.

Theorem 2.6. *Let C be a graph class. Then C is monadically dependent if and only if for every $r \geqslant 1$ there exists $k \in \mathbb{N}$ such that C excludes as induced subgraphs*

- *all flipped star r-crossings of order k, and*
- *all flipped clique r-crossings of order k, and*
- *all flipped half-graph r-crossings of order k, and*
- *the comparability grid of order k.*

This characterization by forbidden induced subgraphs opens the door for various algorithmic, logical, and combinatorial hardness

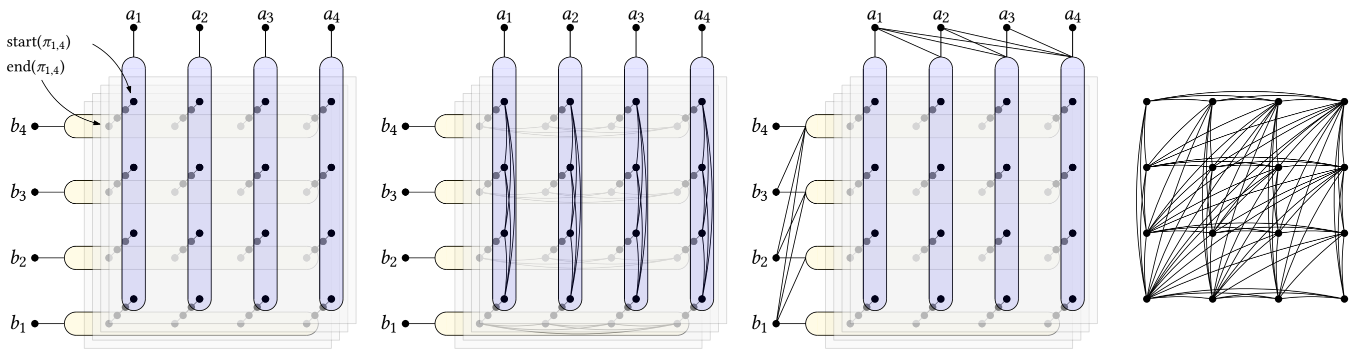

Figure 4: (*i*) star 4-crossing of order 4. (*ii*) clique 4-crossing of order 4. (*iii*) half-graph 4-crossing of order 4. (*iv*) comparability grid of order 4. In (*i*), (*ii*), (*iii*), the roots are adjacent to all vertices in their respective colorful strip.

results. On the algorithmic side, we prove the hardness part of Conjecture 1.1.

Theorem 2.7. *The first-order model checking problem is* AW[∗]-*hard on every hereditary, monadically independent graph class.*

The result builds on the following logical hardness result, which is of independent interest.

Theorem 2.8. *Every hereditary, monadically independent graph class efficiently interprets the class of all graphs.*

Recall that monadically independent classes were defined as those which transduce the class of all graphs. For hereditary classes, the above result strengthens this definition: *interpretations* are a more restrictive version of transductions, that do not include a nondeterministic coloring step. Moreover, *efficient* interpretations come with a polynomial time algorithm which, given an arbitrary input graph G, outputs a graph in C in which G is encoded. This allows to reduce the model checking problem for all graphs to the model checking problem on the class C, thus proving Theorem 2.7.

On the combinatorial side, we show that no hereditary, monadically independent graph class is *small*, has *almost bounded twin-width*, or has *almost bounded flip-width*. Let us quickly explain the three notions.

A graph class C is *small* if it contains at most $n!c^n$ distinct labeled n-vertex graphs, for some constant c. This notion has been studied in enumerative combinatorics [34, 38]. It is known that all classes of bounded twin-width are small [10]. The converse implication was conjectured, and was subsequently refuted [6]. In the context of ordered graphs, it was shown that all small classes of ordered graphs have bounded twin-width [7]. We prove the following.

Theorem 2.9. *Every hereditary, small graph class is monadically dependent.*

Say that a class C of graphs has *almost bounded twin-width* if for every $\varepsilon > 0$ the twin-width of every n-vertex graph $G \in C$ is bounded by $O_{\varepsilon,C}(n^{\varepsilon})$. We prove the following.

Theorem 2.10. *Every hereditary, almost bounded twin-width graph class is monadically dependent.*

In [44], a family of graph-width parameters called *flip-width of radius* r, for $r \geqslant 1$, is introduced, together with the ensuing notion of classes of *almost bounded flip-width*. Classes of almost bounded flip-width include all nowhere dense classes, all classes of bounded twin-width, and more generally, and all classes of almost bounded twin-width. It is conjectured that for hereditary graph classes, almost bounded flip-width coincides with monadic dependence:

Conjecture 2.11. *A hereditary graph class* C *has almost bounded flip-width if and only if it is monadically dependent.*

We prove one implication of this conjecture.

Theorem 2.12. *Every hereditary, almost bounded flip-width graph class is monadically dependent.*

Relation of the Patterns to Other Work. Our forbidden patterns characterization Theorem 2.6 generalizes similar previous characterizations. Recall that monadic dependence is captured by monadic stability, for hereditary orderless graph classes, and by bounded twin-width, for hereditary classes of ordered graphs. The results [7, 17] characterizing monadic dependence in those two settings can be restated as follows.

- For orderless graph classes, a class C is monadically dependent if and only if for every $r \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that no graph in C contains a flipped star r-crossing or clique r-crossing of order k as an induced subgraph [17].
- For classes of ordered graphs, a class C is monadically dependent if and only if there exists $k \in \mathbb{N}$ and a specific, finite family of ordered graphs with $2k$ vertices (similar to a matching on $2k$ vertices ordered suitably) which are avoided by C as semi-induced ordered subgraphs [7].

Our approach towards proving the forbidden patterns characterization and model checking hardness originates in [17] and [7]. As a result, we reprove some of their results. A subset of the patterns identified in this paper are sufficient to characterize monadic dependence in the setting of orderless graph classes. As orderless classes cannot contain large half-graph crossings or comparability grids, Theorem 2.6 implies the result of [17] characterizing monadically stable classes in terms of forbidden induced subgraphs. It is worth noting that in [17], for hereditary, orderless, monadically independent graph classes, hardness is shown even for *existential* model checking. Adapting our results to the setting of binary structures, we show how to derive the characterization from [7] of monadically dependent classes of ordered graphs in terms of forbidden patterns.

A recent paper by Braunfeld and Laskowski shows that a hereditary class of relational structures is monadically dependent if and only if it is dependent [12]. Dependence is a generalization of monadic dependence studied in model theory. For the special case of graph classes, our Theorem 2.8 reproves that result. Similarly to our paper, the proof of Braunfeld and Laskowski exhibits certain large configurations (called pre-coding configurations) in classes that are monadically independent. As pre-coding configurations are defined in terms of formulas with tuples of free variables, these results are not a purely combinatorial characterization of monadically dependent graph classes. In particular, they seem insufficient for obtaining algorithmic hardness results. However, we believe that one could also prove Theorem 2.9, stating that small, hereditary classes are dependent, using the results of [12].

In [25], Eppstein and McCarty prove that many different types of geometric graphs have unbounded flip-width, and in fact [25, App. A], form monadically independent graph classes. These include interval graphs, permutation graphs, circle graphs, and others. It is shown that these classes contain large *interchanges*, which are similar to 1-subdivided cliques, and to our patterns. Containing interchanges of arbitrarily large order is a sufficient, but not necessary condition to monadic independence.

3 TECHNICAL OVERVIEW

Due to space constraints, (almost) all proofs and definitions are deferred to the full version [20] of this paper. In the following, we give a high level overview of the methods and statements used to derive our results. All our proofs are fully constructive and only use elementary tools such as Ramsey's theorem.

Insulators. In order to prove flip-breakability for a graph G and a large set $W \subseteq V(G)$, we try to enclose a sizeable subset $W_\star \subseteq W$ in a structure $\mathcal{A}$ that "insulates" the elements of $W_\star$ from each other, and from external vertices that are not enclosed in $\mathcal{A}$. We call this structure $\mathcal{A}$ an *insulator*. It is a grid-like partition of a subset of $V(G)$. An example is shown in the top of Figure 5 (in general, each cell may contain more than one vertex).

Roughly speaking, there is a coloring of the vertex set, using a bounded number of colors, such that the adjacency between a pair of vertices in non-adjacent rows of the grid is determined only by their colors. For a pair of vertices in adjacent rows, the adjacency is determined by their colors and the order of their columns. Finally, the adjacencies between vertices in the exterior of the grid and the vertices inside the grid (without the first and last column, and last row) are determined by their colors. Exceptionally, the connections within the top row and between adjacent cells of the grid may be arbitrary.

The large subset $W_\star \subseteq W$ is distributed in the bottom row of the insulator, in a way such that every vertex of $W_\star$ is contained in a different column. Assume the insulator has height at least r. We can choose two large subsets A and B of $W_\star$, such that for any two vertices $v \in A$ and $w \in B$, it takes at least r steps along edges which are not controlled by the insulator to get from v to w.

Given $r \in \mathbb{N}$, we strive for insulators of height r, where the number of colors is bounded by some constant, depending on r.

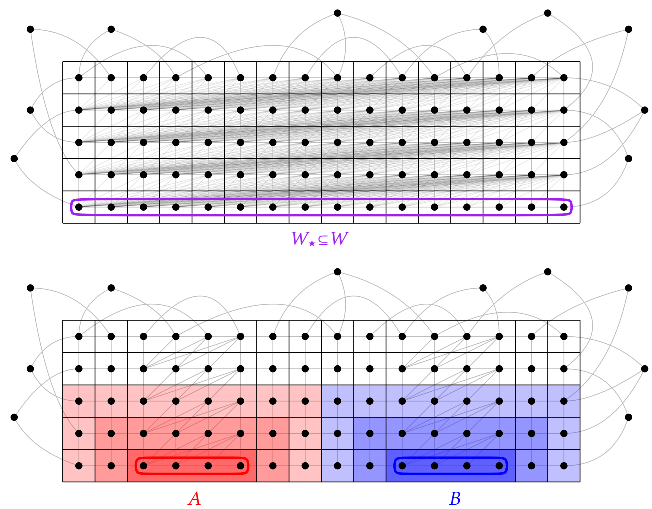

Figure 5: Top: An insulator surrounding the set $W_\star$. The shown stacked half-graph layout is easily described by a bounded number of colors and the column order. Bottom: A bounded number of flips are sufficient to ensure the highlighted subsets $A \subseteq W_\star$ and $B \subseteq W_\star$ have distance larger than 4. The shades of red and blue highlight upper bounds on the distance of other vertices to A and B. In particular, all unshaded vertices have distance larger than two to these sets.

Thus, if we embed large sets A, B in such an insulator and coarsen the columns as shown on the bottom of Figure 5, we can ensure with a bounded number of flips (depending on the number of colors) that edges only go between adjacent cells, and thus A and B have distance at least r. Hence, we can use the insulators to obtain flip-breakability.

Constructing Insulators or Prepatterns. One can trivially construct an insulator of height one that embeds a set W: Build a single row by placing every vertex of W into a distinct cell. The central step of our construction, presented in Part I of the full version [20], takes a large insulator of height r, and adds another row on top to obtain a still-large insulator of height $r + 1$. To this end, we build upon and significantly extend techniques developed in the context of flip-flatness [19], based on *indiscernible sequences*, a fundamental tool in model theory.

Given a large insulator of height r in any graph G, the key step in our construction shows that there is either *enough disorder* to connect the topmost row of the grid into a preliminary pattern, called *prepattern*, resembling a crossing (Figure 6, left), or *enough order* to add an additional row $r + 1$ on top (Figure 6, right). While both cases reduce the number of columns in the insulator, there still remains a large number of them.

We define *prepattern-free* classes by excluding certain prepatterns. In such classes, we therefore always find *enough order* to construct *large insulators* row-by-row. We say that such classes have the *insulation property*. By the arguments outlined in Figure 5, such classes are easily shown to be *flip-breakable*. Lastly, using

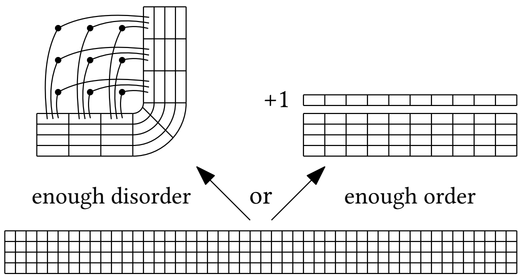

Figure 6: In every graph, we can either increase the height of an insulator of find a large prepattern.

the locality property of first-order logic, it is easy to show that *flip-breakable* classes are monadically dependent (Proposition 4.1). Thus, any graph class satisfies the following chain of implications.

prepattern-free
⇒ insulation property
⇒ flip-breakable
⇒ monadically dependent

A proof of the last implication is given in Section 4.

Cleaning Up Prepatterns. Afterwards, in Part II of the full paper [20], we clean up the prepatterns. This involves heavy use of Ramsey's theorem to regularize the patterns until we either obtain a flipped star/clique/half-graph crossing, or a comparability grid. From such patterns one can trivially transduce all graphs. This is summarized by the following chain of implications.

not prepattern-free
⇔ large prepatterns
⇒ large patterns
⇒ monadically independent

Hence, by contrapositive, monadic dependence implies prepattern-freeness. Together with the previous chain of implications, we obtain the desired equivalences.

insulation property
⇔ flip-breakable
⇔ no large patterns
⇔ monadically dependent

Hardness. Consider a hereditary class C that is monadically independent. As discussed, C contains large patterns witnessing this. To obtain hardness of model checking, we reduce from the model checking problem in general graphs. This requires an encoding of an arbitrary graph into a large pattern, using a first-order formula. We know that when adding colors, such an encoding is possible: classes that are monadically independent transduce the class of all graphs. However, for our reduction we require encodings that do not use colors: we want to show that every hereditary class that is monadically independent *interprets* the class of all graphs. Here, an *interpretation* is a transduction that does not use colors and where instead of taking arbitrary induced subgraphs, the vertex set of the interpreted graph must be definable in the original graph by a formula $\delta(x)$. Heavily relying on the fact that in hereditary classes we can take induced subgraphs *before* applying the interpretation, it is not too hard to show that for each $r \geqslant 1$, the hereditary closures of the class of all comparability grids and the classes of all *non-flipped* star/clique/half-graph r-crossings interpret the class of all graphs. The main challenge is to "reverse" the flips using first-order formulas in the case of *flipped* r-crossings. This is achieved by using sets of twins to mark layers of the r-crossing.

4 FLIP-BREAKABILITY IMPLIES MONADIC DEPENDENCE

In this section, we prove the simpler of the two directions that characterize monadic dependence in terms of flip-breakability. We assume familiarity with basic notation in graph theory and logic. For a graph H and vertex set A, we denote in the following by $N_r^H(A)$ all vertices with distance at most r from A in H.

Proposition 4.1. *Every flip-breakable class of graphs is monadically dependent.*

Let $\varphi(x,y)$ be a formula over the signature of colored graphs, G^+ be a colored graph, and W be a set of vertices in G^+. We say that φ *shatters* W in G^+, if there exists vertices $(a_R)_{R\subseteq W}$ such that for all $b \in P$ and $R \subseteq W$,

$$G^+ \models \varphi(a_R, b) \Leftrightarrow b \in R.$$

Let G be an (uncolored) graph, and W be a set of vertices in G. We say that φ *monadically shatters* W in G, if there exists a coloring G^+ of G in which φ shatters W.

Fact 4.2 ([4]). *A class of graphs $\mathcal{C}$ is monadically dependent if and only if for every formula $\varphi(x,y)$ over the signature of colored graphs, there exists a bound m such that φ monadically shatters no set of size m in any graph of $\mathcal{C}$.*

Proposition 4.1 will be implied by the following.

Lemma 4.3. *Let $\varphi(x,y)$ be a formula and $k \in \mathbb{N}$. There exist $r_\varphi, m_{\varphi,k} \in \mathbb{N}$, where r_φ depends only on φ, such that no graph G contains a set of at least $m_{\varphi,k}$ vertices W such that*

- *φ monadically shatters W in G, and*
- *W is (r_φ, k)-flip-breakable in G.*

In order to prove Lemma 4.3, we use the following statement, which is an immediate consequence of Gaifman's locality theorem [28]. For an introduction of the locality theorem see for example [31, Sec. 4.1].

Corollary 4.4 (of [28, Main Theorem]). *Let $\varphi(x,y)$ be a formula. Then there are numbers $r, t \in \mathbb{N}$, where r depends only on the quantifier-rank of φ and t depends only on the signature and quantifier-rank of φ, such that every (colored) graph G can be vertex-colored using t colors in such a way that for any two vertices $u, v \in V(G)$ with distance greater than r in G, $G \models \varphi(u,v)$ depends only on the colors of u and v. We call r the* Gaifman radius *of φ.*

Proof of Lemma 4.3. We set r_φ to be the Gaifman radius of φ. Let s be the number of colors used by φ. As stated in Corollary 4.4, let t be the number of colors needed to determine the truth value of formulas in the signature of $(s \cdot k)$-colored graphs that have the same quantifier-rank as $\varphi(x, y)$. Let $m_{\varphi,k} := 3(t+1)$.

Assume now towards a contradiction the existence of an (r_φ, k)-flip-breakable in set W in G of size $m_{\varphi,k}$ such that φ monadically shatters W in G. Then there exists an s-coloring G^+ of G in which φ shatters W. We apply flip-breakability which yields a k-flip H of G together with two disjoint sets $A, B \subseteq W$ each of size at least $t+1$, such that $N_r^H(A) \cap N_r^H(B) = \emptyset$. By using k colors to encode the flip, we can rewrite φ to a formula ψ with the same quantifier-rank as φ, such that there exists a $(s \cdot k)$-coloring H^+ of H where for all $u, v \in V(G)$,

$$G^+ \models \varphi(u, v) \Leftrightarrow H^+ \models \psi(u, v).$$

In particular, ψ shatters W in H^+. Since ψ has the same quantifier-rank as φ and is a formula over the signature of $s \cdot k$-colored graphs, by Corollary 4.4 there exists a coloring of H^+ with t colors such that the truth of $\psi(u, v)$ in H^+ only depends on the colors of u and v for all vertices u and v with distance greater than r in H^+. Recall that A and B each have size $t+1$. By the pigeonhole principle, there exist two distinct vertices $a_1, a_2 \in A$ that are assigned the same color and two distinct vertices $b_1, b_2 \in B$ that are assigned the same color. Since ψ shatters W in H^+, there exists a vertex $v \in V(G)$ such that

$$H^+ \models \psi(v, a_1) \wedge \neg\psi(v, a_2) \wedge \psi(v, b_1) \wedge \neg\psi(v, b_2).$$

By Corollary 4.4, v must be contained in $N_r^H(A) \cap N_r^H(B)$, as the truth of ψ is inhomogeneous among both v and $\{a_1, a_2\}$ and among v and $\{b_1, b_2\}$. This is a contradiction to $N_r^H(A) \cap N_r^H(B) = \emptyset$. □

5 CONCLUSIONS AND FUTURE WORK

In this paper, we obtain the first combinatorial characterizations of monadically dependent graph classes, which open the way to generalizing the results of sparsity theory to the setting of hereditary graph classes. Our main results, Theorem 2.2 and Theorem 2.6, may be seen as analogues of the result [35] characterizing nowhere dense graph classes as exactly the *uniformly quasi-wide* classes [15], and the result [2, 40] characterizing nowhere dense classes as exactly those, whose monotone closure is monadically dependent.

Central results in sparsity theory, characterizing nowhere dense classes, can be usually grouped into two types: *qualitative characterizations* and *quantitative characterizations*. Qualitative characterizations typically say that for every radius r, some quantity is bounded by a constant depending on r for all graphs in the class. Our two main results fall within this category.

On the other hand, quantitative characterizations of nowhere dense classes are phrased in terms of a fine analysis of densities of some parameters, such as degeneracy, minimum degree, weak coloring numbers, neighborhood complexity, or VC-density; those results almost always involve bounds of the form n^ε or $n^{1+\varepsilon}$, where n is the number of vertices of the considered graph, and $\varepsilon > 0$ can be chosen arbitrarily small. For instance, given a nowhere dense graph class C, for every fixed $r \in \mathbb{N}$ and $\varepsilon > 0$, all graphs G whose r-subdivision is a subgraph of a graph in C satisfy

$$|E(G)| \leqslant O_{C,r,\varepsilon}(|V(G)|^{1+\varepsilon}).$$

Similarly, a parameter called the *weak r-coloring number* (for any fixed $r \in \mathbb{N}$, $\varepsilon > 0$), is bounded by $O_{C,r,\varepsilon}(n^\varepsilon)$, for every n-vertex graph G in a nowhere dense class.

Both the qualitative and quantitative results are of fundamental importance in sparsity theory, and lifting them to the setting of monadically dependent graph classes is therefore desirable. Some of the most elaborate results of sparsity theory combine both aspects of the theory. In particular for nowhere dense model checking [32], the *splitter game* [32] – a qualitative characterization – is combined with the quantitative characterization in terms of weak coloring numbers. The characterization in terms of the splitter game relies on uniform quasi-wideness, and has been extended to the setting of monadically stable graph classes, in terms of the flipper game [29], which in turn relies on the characterization in terms of flip-flatness. Basing on this, we believe that flip-breakability may be a first step towards obtaining a game characterization of monadic dependence.

As we observe now, our results also provide a first quantitative characterization of monadically dependent classes. As an analogue of the notion of *containing the r-subdivision of a graph as a subgraph*, we introduce the following concept of *radius-r encodings*. Fix an integer $r \geqslant 1$. Let $G = (A, B, E)$ be a bipartite graph with $|A| = |B| = n$ for some n, and let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. Consider a graph H which is a star/clique/half-graph r-crossing with roots $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Recall that $V(H)$ can be partitioned into $r+2$ *layers*, and there are n^2 distinguished r-vertex *paths* $\pi_{i,j}$ connecting a_i and b_j, for $i, j \in [n]$. Let H' be a graph obtained from H by:

(1) adding arbitrary edges within each layer of H,
(2) removing all vertices of the paths $\pi_{i,j}$ for $i, j \in [n]$ such that $\{a_i, b_j\} \notin E(G)$,
(3) flipping pairs of layers arbitrarily.

We call H' a *radius-r encoding of* G. In particular, every flipped star/clique/half-graph r-crossing of order n is a radius-r encoding of the complete bipartite graph $K_{n,n}$. Moreover, the comparability grid of order $n+1$ contains as an induced subgraph a radius-1 encoding of the complete bipartite graph $K_{n,n}$, which can be obtained from the half-graph 1-crossing of order $n+1$ by adding edges within the three layers.

The following result is proven in the full version of our paper, where it easily follows from our non-structure characterization of monadic dependence combined with results of Dvořák [21, 22].

Theorem 5.1. *Let C be a hereditary graph class. The following conditions are equivalent:*

(1) *C is monadically dependent,*
(2) *for every real $\varepsilon > 0$ and integer $r \geqslant 1$, for every bipartite graph G such that C contains some radius-r encoding of G, we have that $|E(G)| \leqslant O_{C,r,\varepsilon}(|V(G)|^{1+\varepsilon})$,*
(3) *for every integer $r \geqslant 1$ there is an integer $N \geqslant 1$ such that for every bipartite graph G with $|V(G)| > N$ such that C contains some radius-r encoding of G, we have that $|E(G)| < \frac{1}{4}|V(G)|^2$.*

Theorem 5.1 may be a first step towards developing a quantitative theory of monadically dependent classes. A challenging goal here is to generalize the characterization of nowhere denseness in terms of weak coloring numbers, to monadically dependent classes.

Flip-width. The *flip-width* parameters were introduced [44] with the aim of obtaining quantitative characterizations of monadic dependence. The flip-width at radius $r \geqslant 1$, denoted $\mathrm{fw}_r(\cdot)$, is an analogue of the weak r-coloring number, and the notion of classes of bounded flip-width, and almost bounded flip-width, generalize classes of bounded expansion and nowhere dense classes to the dense setting. A graph class $\mathcal{C}$ has *bounded flip-width* if for every integer $r \geqslant 1$ there is a constant k_r such that $\mathrm{fw}_r(G) \leqslant k_r$ for all graphs $G \in \mathcal{C}$. A hereditary graph class $\mathcal{C}$ has *almost bounded flip-width* if for every integer $r \geqslant 1$ and real $\varepsilon > 0$ we have $\mathrm{fw}_r(G) \leqslant O_{\mathcal{C},r,\varepsilon}(n^\varepsilon)$ for all n-vertex graphs $G \in \mathcal{C}$. Conjecture 2.11 – equating monadic dependence with almost bounded flip-width – therefore postulates a quantitative characterization of monadic dependence, parallel to the characterization of nowhere denseness in terms of weak coloring numbers.

As mentioned, our results imply the forward implication of Conjecture 2.11: that every hereditary class of almost bounded flip-width is monadically dependent. Theorem 5.1 might be an initial step towards resolving the backwards direction in Conjecture 2.11, as by Theorem 5.1, Conjecture 2.11 is equivalent to the following:

Conjecture 5.2. *Let $\mathcal{C}$ be a hereditary graph class. Then the following conditions are equivalent:*

(1) *for every real $\varepsilon > 0$ and integer $r \geqslant 1$, for every bipartite graph G such that $\mathcal{C}$ contains some radius-r encoding of G, we have:*

$$\frac{|E(G)|}{|V(G)|} \leqslant O_{\mathcal{C},r,\varepsilon}(|V(G)|^\varepsilon),$$

(equivalently, by Theorem 5.1, $\mathcal{C}$ is monadically dependent)

(2) *$\mathcal{C}$ has almost bounded flip-width: For every real $\varepsilon > 0$ and integer $r \geqslant 1$, and graph $G \in \mathcal{C}$, we have:*

$$\mathrm{fw}_r(G) \leqslant O_{\mathcal{C},r,\varepsilon}(|V(G)|^\varepsilon).$$

Note that the implication (2)→(1) in Conjecture 5.2 holds, by Theorem 5.1 and Theorem 2.12. The following conjecture would imply the converse implication.

Conjecture 5.3. *For every $r \geqslant 1$ there are integers $s, k \geqslant 1$ such that for every graph G we have:*

$$\mathrm{fw}_r(G) \leqslant \max_H \left(\frac{|E(H)|}{|V(H)|}\right)^k,$$

where the maximum ranges over all bipartite graphs H such that G contains some radius-s encoding of H as an induced subgraph.

An analogue of Conjecture 5.3 holds in the sparse setting. There, the flip-width parameters are replaced by weak coloring numbers, and the maximum ranges over all graphs H such that G contains some s-subdivision of H as a subgraph. Conjecture 5.3 would furthermore imply the following characterization of classes of bounded flip-width, analogous to a known characterization of classes with bounded expansion in terms of weak coloring numbers [45].

Conjecture 5.4. *Let $\mathcal{C}$ be a hereditary graph class. Then the following conditions are equivalent:*

(1) *for every integer $r \geqslant 1$ there is a constant k_r such that for every bipartite graph G such that $\mathcal{C}$ contains some radius-r encoding of G, we have that $|E(G)| \leqslant k_r \cdot |V(G)|$,*
(2) *$\mathcal{C}$ has bounded flip-width: For every integer $r \geqslant 1$ there is a constant k_r such that for every graph $G \in \mathcal{C}$, we have that $\mathrm{fw}_r(G) \leqslant k_r$.*

Note that the implication (2)→(1) follows from the results of [44] (that every weakly sparse transduction of a class of bounded flip-width has bounded expansion). The converse implication would resolve Conjecture 11.4 from [44], which predicts that if a class $\mathcal{C}$ has unbounded flip-width, then $\mathcal{C}$ transduces a weakly sparse class of unbounded expansion.

Near-twins. A specific goal, not involving flip-width, which would be implied by the above conjectures, can be phrased in terms of *near-twins*. Say that two distinct vertices u, v in a graph G are k-near-twins, where $k \in \mathbb{N}$, if the symmetric difference of the neighborhoods of u and of v consists of at most k vertices. It is known [44] that every graph G with more than one vertex contains a pair of $(2\,\mathrm{fw}_1(G))$-near-twins. Consequently, for every class $\mathcal{C}$ of bounded flip-width there is a constant k such that every graph $G \in \mathcal{C}$ with more than one vertex contains a pair of k-near-twins. Similarly, if $\mathcal{C}$ has almost bounded flip-width then every n-vertex graph $G \in \mathcal{C}$ with $n > 1$ contains a pair of $O_{\mathcal{C},\varepsilon}(n^\varepsilon)$-near-twins.

Therefore, a first step towards Conjecture 2.11 is to prove that for all monadically dependent classes, every n-vertex graph $G \in \mathcal{C}$ with $n > 1$ contains a pair of $O_{\mathcal{C},\varepsilon}(n^\varepsilon)$-near-twins. Similarly, a step towards Conjecture 5.4 is to prove the following consequence of Conjecture 5.3 (in the case $r = 1$), stated below.

Conjecture 5.5. *There is an integer $s \geqslant 1$ and an unbounded function $f : \mathbb{N} \to \mathbb{N}$ such that every graph G with more than one vertex and no pair of d-near twins, for some $d \in \mathbb{N}$, contains as an induced subgraph a radius-s encoding of some bipartite graph H with $|E(H)|/|V(H)| \geqslant f(d)$.*

VC-density and Neighborhood Complexity. Another, related conjecture [17, Conj. 2] bounds the *neighborhood complexity*, or *VC-density* of set systems defined by neighborhoods in graphs from a monadically dependent graph class, and is phrased as follows.

Conjecture 5.6 ([17]). *Let $\mathcal{C}$ be a monadically dependent graph class and let $\varepsilon > 0$ be a real. Then for every graph $G \in \mathcal{C}$ and set $A \subseteq V(G)$, we have that*

$$|\{N(v) \cap A : v \in V(G)\}| \leqslant O_{\mathcal{C},\varepsilon}(|A|^{1+\varepsilon}).$$

This conjecture is confirmed for all nowhere dense classes [24, 39], for all monadically stable classes [17], and for all classes of bounded twin-width [8, 41].

ACKNOWLEDGEMENTS

N.M. thanks Michał Pilipczuk for hosting a stay at the University of Warsaw, where part of this work was done.

REFERENCES

[1] 2016. Algorithms, Logic and Structure Workshop in Warwick – Open Problem Session. https://warwick.ac.uk/fac/sci/maths/people/staff/daniel_kral/alglogstr/openproblems.pdf. [Online; accessed 23-Jan-2023].
[2] Hans Adler and Isolde Adler. 2014. Interpreting nowhere dense graph classes as a classical notion of model theory. *European Journal of Combinatorics* 36 (2014), 322–330.
[3] Stefan Arnborg, Jens Lagergren, and Detlef Seese. 1991. Easy Problems for Tree-Decomposable Graphs. *J. Algorithms* 12, 2 (1991), 308–340. https://doi.org/10.1016/0196-6774(91)90006-K
[4] John T Baldwin and Saharon Shelah. 1985. Second-order quantifiers and the complexity of theories. *Notre Dame Journal of Formal Logic* 26, 3 (1985), 229–303.
[5] Édouard Bonnet, Jan Dreier, Jakub Gajarský, Stephan Kreutzer, Nikolas Mählmann, Pierre Simon, and Szymon Toruńczyk. 2022. Model Checking on

Interpretations of Classes of Bounded Local Cliquewidth. In *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022*, Christel Baier and Dana Fisman (Eds.). ACM, 54:1–54:13. https://doi.org/10.1145/3531130.3533367

[6] Édouard Bonnet, Colin Geniet, Romain Tessera, and Stéphan Thomassé. 2022. Twin-width VII: groups. *arXiv preprint arXiv:2204.12330* (2022).

[7] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. 2022. Twin-width IV: ordered graphs and matrices. In *Proceedings of the 54th Annual ACM Symposium on Theory of Computing (STOC 2022)*. 924–937.

[8] Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, Stéphan Thomassé, and Rémi Watrigant. 2022. Twin-width and Polynomial Kernels. *Algorithmica* 84, 11 (2022), 3300–3337. https://doi.org/10.1007/S00453-022-00965-5

[9] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. 2021. Twin-width I: tractable FO model checking. *ACM Journal of the ACM (JACM)* 69, 1 (2021), 1–46.

[10] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. 2022. Twin-width II: Small classes. *Combinatorial Theory* 2, 2 (2022). https://doi.org/10.5070/c62257876

[11] Samuel Braunfeld and Michael Laskowski. 2021. Characterizations of monadic NIP. *Transactions of the American Mathematical Society, Series B* 8, 30 (2021), 948–970.

[12] Samuel Braunfeld and Michael C. Laskowski. 2022. Existential characterizations of monadic NIP. arXiv:2209.05120 [math.LO]

[13] Bruno Courcelle. 1990. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and computation* 85, 1 (1990), 12–75.

[14] Bruno Courcelle, Johann A Makowsky, and Udi Rotics. 2000. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems* 33, 2 (2000), 125–150.

[15] Anuj Dawar. 2010. Homomorphism preservation on quasi-wide classes. *J. Comput. System Sci.* 76, 5 (2010), 324–332.

[16] Anuj Dawar, Martin Grohe, and Stephan Kreutzer. 2007. Locally excluding a minor. In *22nd Annual IEEE Symposium on Logic in Computer Science (LICS 2007)*. IEEE, 270–279.

[17] Jan Dreier, Ioannis Eleftheriadis, Nikolas Mählmann, Rose M. McCarty, Michał Pilipczuk, and Szymon Toruńczyk. 2023. First-order model checking on monadically stable graph classes. *arXiv preprint arXiv:2311.18740* (2023).

[18] Jan Dreier, Nikolas Mählmann, and Sebastian Siebertz. 2023. First-Order Model Checking on Structurally Sparse Graph Classes. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing* (Orlando, FL, USA) *(STOC 2023)*. Association for Computing Machinery, New York, NY, USA, 567–580. https://doi.org/10.1145/3564246.3585186

[19] Jan Dreier, Nikolas Mählmann, Sebastian Siebertz, and Szymon Toruńczyk. 2023. Indiscernibles and Flatness in Monadically Stable and Monadically NIP Classes. In *50th International Colloquium on Automata, Languages, and Programming, ICALP 2023, July 10-14, 2023, Paderborn, Germany (LIPIcs, Vol. 261)*, Kousha Etessami, Uriel Feige, and Gabriele Puppis (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 125:1–125:18. https://doi.org/10.4230/LIPIcs.ICALP.2023.125

[20] Jan Dreier, Nikolas Mählmann, and Szymon Toruńczyk. 2024. Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes. arXiv:2403.15201 [math.CO]

[21] Zdenek Dvořák. 2007. *Asymptotical structure of combinatorial objects.* Ph. D. Dissertation. Charles University, Faculty of Mathematics and Physics.

[22] Zdeněk Dvořák. 2018. Induced subdivisions and bounded expansion. *European Journal of Combinatorics* 69 (2018), 143–148.

[23] Zdeněk Dvořák, Daniel Král, and Robin Thomas. 2013. Testing first-order properties for subclasses of sparse graphs. *J. ACM* 60, 5 (2013), 36:1–36:24. https://doi.org/10.1145/2499483

[24] Kord Eickmeyer, Archontia C. Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. 2017. Neighborhood Complexity and Kernelization for Nowhere Dense Classes of Graphs. In *ICALP (LIPIcs, Vol. 80)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 63:1–63:14.

[25] David Eppstein and Rose McCarty. 2023. Geometric Graphs with Unbounded Flip-Width. In *Proceedings of the 35th Canadian Conference on Computational Geometry, CCCG 2023, Montreal, Canada, July 31-August 4, 2023*, Denis Pankratov (Ed.). 197–206. Available at https://arxiv.org/abs/2306.12611.

[26] Jörg Flum and Martin Grohe. 2001. Fixed-parameter tractability, definability, and model-checking. *SIAM J. Comput.* 31, 1 (2001), 113–145.

[27] Markus Frick and Martin Grohe. 2001. Deciding first-order properties of locally tree-decomposable structures. *Journal of the ACM (JACM)* 48, 6 (2001), 1184–1206.

[28] Haim Gaifman. 1982. On Local and Non-Local Properties. In *Proceedings of the Herbrand Symposium*. Stud. Logic Found. Math., Vol. 107. Elsevier, 105 – 135.

[29] Jakub Gajarský, Nikolas Mählmann, Rose McCarty, Pierre Ohlmann, Michał Pilipczuk, Wojciech Przybyszewski, Sebastian Siebertz, Marek Sokołowski, and Szymon Toruńczyk. 2023. Flipper Games for Monadically Stable Graph Classes. In *50th International Colloquium on Automata, Languages, and Programming (ICALP 2023) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 261)*, Kousha Etessami, Uriel Feige, and Gabriele Puppis (Eds.). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 128:1–128:16. https://doi.org/10.4230/LIPIcs.ICALP.2023.128

[30] Robert Ganian, Petr Hliněný, Jaroslav Nesetril, Jan Obdrzálek, and Patrice Ossona de Mendez. 2019. Shrub-depth: Capturing Height of Dense Graphs. *Log. Methods Comput. Sci.* 15, 1 (2019). https://doi.org/10.23638/LMCS-15(1:7)2019

[31] Martin Grohe. 2008. Logic, graphs, and algorithms. *Logic and Automata* 2 (2008), 357–422.

[32] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. 2017. Deciding first-order properties of nowhere dense graphs. *Journal of the ACM (JACM)* 64, 3 (2017), 1–32.

[33] Stephan Kreutzer. 2011. *Algorithmic meta-theorems*. Cambridge University Press, 177–270.

[34] Colin McDiarmid, Angelika Steger, and Dominic J.A. Welsh. 2005. Random planar graphs. *Journal of Combinatorial Theory, Series B* 93, 2 (2005), 187–205. https://doi.org/10.1016/j.jctb.2004.09.007

[35] Jaroslav Nešetřil and Patrice Ossona De Mendez. 2011. On nowhere dense graphs. *European Journal of Combinatorics* 32, 4 (2011), 600–617.

[36] Jaroslav Nešetřil and Patrice Ossona De Mendez. 2012. *Sparsity: graphs, structures, and algorithms*. Vol. 28. Springer Science & Business Media.

[37] Jaroslav Nešetřil, Patrice Ossona de Mendez, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. 2021. Rankwidth meets stability. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2014–2033.

[38] Serguei Norine, Paul Seymour, Robin Thomas, and Paul Wollan. 2006. Proper minor-closed families are small. *Journal of Combinatorial Theory, Series B* 96, 5 (2006), 754–757. https://doi.org/10.1016/j.jctb.2006.01.006

[39] Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. 2018. On the number of types in sparse graphs. In *33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018*. ACM, 799–808. https://doi.org/10.1145/3209108.3209178

[40] Klaus-Peter Podewski and Martin Ziegler. 1978. Stable graphs. *Fundamenta Mathematicae* 100, 2 (1978), 101–107. http://eudml.org/doc/210953

[41] Wojciech Przybyszewski. 2023. Distal Combinatorial Tools for Graphs of Bounded Twin-Width. In *LICS*. 1–13. https://doi.org/10.1109/LICS56636.2023.10175719

[42] Detlef Seese. 1996. Linear time computable problems and first-order descriptions. *Mathematical Structures in Computer Science* 6, 6 (1996), 505–526.

[43] Saharon Shelah. 1986. Monadic logic: Hanf numbers. In *Around classification theory of models*. Springer, 203–223.

[44] Szymon Toruńczyk. 2023. Flip-width: Cops and Robber on dense graphs. In *2023 IEEE 64th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE Computer Society, Los Alamitos, CA, USA, 663–700. https://doi.org/10.1109/FOCS57990.2023.00045 Full version available at https://arxiv.org/abs/2302.00352.

[45] Xuding Zhu. 2009. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics* 309, 18 (2009), 5562–5568. https://doi.org/10.1016/j.disc.2008.03.024 Combinatorics 2006, A Meeting in Celebration of Pavol Hell's 60th Birthday (May 1–5, 2006).

Received 13-NOV-2023; accepted 2024-02-11