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Subsystems of Open Induction

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by

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Wien, 26. November 2024

Johannes Franz-Stefan Weiser

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Kurzfassung

In [Sho58] und [She63] wurde gezeigt, dass es einfache, manchmal endliche, alternative Axiomatisierungen von offener Induktion in verschiedenen Kontexten der Arithmetik gibt. Dies eröffnet zwei Fragen: Benötigt man die gesamte offene Induktion, um diese alternativen Axiomatisierungen zu beweisen oder genügt eine echte Teilmenge? Gibt es solche alternativen Axiomatisierungen von offener Induktion nur im arithmetischen Kontext oder auch für andere induktive Datentypen?

Wir zeigen in dieser Arbeit, dass unterschiedliche Teilsysteme der offenen Induktion oftmals gleich stark sind, in dem Sinne, dass sie die gleichen Formeln beweisen. Als Spezialfall sehen wir, dass die gesamte offene Induktion in verschiedene interessanten Fällen gleich viel beweist, wie Induktion über eine echte Teilmenge der Menge aller offenen Formeln (z.B. Literale). Darüber hinaus gibt es in Analogie zu den Resultaten von Shoenfield und Shepherdson in vielen der von uns betrachteten Fälle eine alternative *einfache* Axiomatisierung von offener Induktion.

Abstract

In [Sho58] and [She63] it was shown that there are simple, sometimes finite, alternative axiomatizations of open induction in the context of various arithmetical theories. This begs two questions: Does one need all open instances of the induction axiom to prove these alternative axiomatizations or do certain subsets suffice? Does this only work in the context of arithmetics or for other inductive data types as well?

In this thesis, we show that various subsystems of open induction are equally strong in the sense that they prove the same theorems. In particular, in multiple interesting cases, open induction collapses to induction over a proper subset of the set of all open formulas (e.g. literals). Moreover, in many of the cases, we considered, there are *simple* axiomatizations of open induction in analogy to the results of Shoenfield and Shepherdson.

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CHAPTER 1

Introduction

Induction is an important tool in mathematics and computer science. Additionally to using it in semantic proofs, we can also use it to axiomatize data types syntactically. For example, it is a rather important property of the natural numbers that every subset, which is closed under 0 and successor is the whole set of natural numbers. Switching from subsets to formulas, we can rephrase the axiom and say, that any formula $\varphi(x)$ with $\varphi(0)$ and $\varphi(x) \to \varphi(s(x))$ satisfies $\forall x \varphi(x)$. In PA, this leads to the axiom-scheme

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(s(x))) \to \forall x \varphi(x), \quad \text{for all formulas } \varphi.$$

As stated above this is not an axiom, but an axiom-scheme since we cannot quantify over formulas in FOL. This type of scheme can be adapted to fit different kinds of *inductive data types* - that is, data types, whose instances can be constructed inductively, such as k-ary trees or lists. Having axiom-schemes as described above, begs the question, what happens if we replace *for all formulas* with *for some formulas.* Of particular interest is the question, what happens, if we only allow open (i.e. quantifier-free) formulas or even smaller subsets of formulas (e.g. clauses). The interest in this question is due to the fact that modern automated theorem provers often can only deal with induction of quantifier-free formulas (if any) (cf. [Vie24]). For sufficiently strong arithmetical theories, it is well understood how different levels of induction relate to each other. If some non-arithmetical theory is strong enough to allow encoding of numbers, then there is not much of a difference to arithmetics.

However, there is a gap in the literature, when it comes to weak theories of inductive data types. Thus, in Chapter 3, we consider general inductive data types. We show that if the language is sufficiently uncomplicated, there are simple alternative axiomatizations of open induction.

In Chapter 4, we refine the results of [Sho58] and [She63]: Shoenfield and Shepherdson gave alternative axiomatizations for open induction in various arithmetical contexts. We consider the same theories and analyze, how the different subsystems of open induction relate to each other.

In Chapter 5, we combine the two previous chapters by adding a size function to some arbitrary inductive data type. Again, we give alternative axiomatizations of open induction. Moreover, we show that in this case, induction over all open formulas does not prove more than induction over literals regardless of the inductive data type at hand.

In Chapter 6 and Chapter 7, we consider lists and k-ary trees as special cases of inductive data types and apply our findings from the previous chapters.

CHAPTER 2

Preliminaries

In this whole thesis, we work in classical predicate logic with the usual connectives $\wedge, \vee, \neg$ and the quantifiers $\exists, \forall$. On the meta-level, we use $\equiv$ for syntactic equality of any kind of objects such as terms, formulas or variables. We use $\to$ and $\leftrightarrow$ defined as $\varphi \to \psi \equiv \neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$. Moreover, sometimes we may omit parentheses in order to increase the legibility. For this, we use the usual conventions that $\neg, \forall$ and $\exists$ all take precedence over $\wedge$, which again takes precedence over $\vee$. The semantics are defined in the usual way. We will write WFF for the set of well-formed formulas.

All of our languages will contain the relational symbol $=$, which is always interpreted as the equality with the usual axiomatization. If the logic in which we work is many sorted, there is an $=$ symbol for each sort, and the axiomatization is done accordingly. We will refrain from writing it down, each time we encounter a new language.

If the logic we work in has the sorts $S_1, \ldots, S_n$, then there are n different versions of $\forall$ and $\exists$. Formally, we should write something like $(\exists x : S_n)(\varphi(x))$ to denote that there is some x of the sort S_n, for which φ holds. In our cases, it will usually be clear from the context, over which sort we quantify and thus, we only write $\exists x : \varphi(x)$. In particular, we will write formulas of the form $\exists x_1, \ldots, x_n, y_1, \ldots, y_m : f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$ or even $\exists \overline{x}, \overline{y} : f(\overline{x}) = g(\overline{y})$ since n, m and the sort of every x_i and y_j can be inferred from the signatures of f and g.

We will analyze subsystems of open induction over various languages and theories. With this we mean the following: Let $\mathbf{I}(\varphi)$[1] be the scheme of induction over some language $\mathcal{L}$ with some base theory B. We explicitly allow that φ contains parameters. Since we work in FOL, we have to add an instance of $\mathbf{I}(\varphi)$ for any formula φ, we want to be able to use in the scheme. Now we can add all instances or just some. Depending on which instances we add, we have different names for the resulting sets:

[1] Note that we do not care how $\mathbf{I}(\varphi)$ is defined yet, we just accept that it is a formula scheme

- $IAtom = \{\mathbf{I}(\varphi) \mid \varphi \text{ is an atom}\}$
- $ILiteral = \{\mathbf{I}(\varphi) \mid \varphi \text{ is a literal}\}$
- $IClause = \{\mathbf{I}(\varphi) \mid \varphi \text{ is a clause}\}$
- $IDClause = \{\mathbf{I}(\varphi) \mid \varphi \text{ is a dual clause}\}$
- $IOpen = \{\mathbf{I}(\varphi) \mid \varphi \text{ is an open formula}\}$

These systems will be the main focus of our analysis. Note that these names are ambiguous and different schemes of induction I will yield different sets. However, the induction scheme is always clear from the context and we will therefore stick with these names.

A more subtle thing to note is that we use axiom-schemata and not rules as in in parts of [She63]. Consider the following axiom-schema of induction in the context of arithmetics

$$\varphi(0) \land \forall x : (\varphi(x) \to \varphi(sx)) \to \forall x : \varphi(x)$$

and compare it to the rule of induction in the same context

$$\frac{\varphi(0) \qquad \varphi(x) \to \varphi(sx)}{\forall x : \varphi(x)} \; .$$

Note that the axiom of induction is stronger in the sense that $\forall x : \varphi(x)$ holds in every model, where the preconditions are met, while the rule only gives us that $\forall x : \varphi(x)$ holds in all models if the antecedent holds in every model.

Since we want to talk about the expressivity of theories and axiomatizations, we need some notion to compare them. Given some set of formulas Γ, we write $Th(\Gamma)$ for the deductive closure of Γ given by $Th(\Gamma) = \{\varphi \in WFF \mid \Gamma \vdash \varphi\}$. Given two sets of formulas Γ, Δ, we define the relation $\preceq$ by $\Gamma \preceq \Delta$ if $Th(\Gamma) \subseteq Th(\Delta)$. We define $\Gamma \approx \Delta$ if $\Gamma \preceq \Delta$ and $\Delta \preceq \Gamma$. The respective strict relation is defined in the usual way: $\Gamma \precneq \Delta$ if $\Gamma \preceq \Delta$ and $\Gamma \not\approx \Delta$.

From the definitions above, it follows trivially that for any base theory B it holds that $B + IAtom \preceq B + ILiteral \preceq B + IClause \preceq B + IOpen$ and $B + ILiteral \preceq B + IDClause \preceq B + IOpen$. Graphically, this can be represented in the following way:

$$\begin{tikzcd}
 & B + IOpen \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
B + IDClause \arrow[dr, dash, dashed] & & B + IClause \arrow[dl, dash, dashed] \\
 & B + ILiteral \arrow[d, dash, dashed] & \\
 & B + IAtom \arrow[d, dash, dashed] & \\
 & B &
\end{tikzcd}$$

In the diagram above, dashed lines represent $\preceq$ and solid lines represent $\precneq$. Clearly, we cannot draw any solid lines yet, but we will need the distinction in later chapters. Note that depending on the base theories this diagram can collapse as induction over different sets of formulas might yield the same theorems and some base theories may even satisfy some form of induction.

CHAPTER 3

General Inductive Data Types

The following chapter contains, in some sense, the strongest results about open induction in this thesis. Although it is the first chapter, it was written as one of the last ones. This is due to the fact that it generalizes some results from later chapters. We will deal with inductive data types with constructors and selectors. Including arbitrary function symbols becomes so complicated that they lie outside of the scope of this thesis. Some function symbols that have common defining axioms are dealt with in the subsequent chapters.

One of the most interesting results is that some of the results fundamentally depend on the structure of the language, meaning that e.g. the arity or number of constructors has substantial influence on whether open induction can be axiomatized naturally and if the levels of induction actually differ.

3.1 General Frame

For defining an inductive data type D, we consider a (possibly) many-sorted logic with the sorts $\mathsf{D}, T_1, \ldots, T_n$. Our language is defined in the following way: We have some *constructors* $c_1, \ldots, c_k$, where each of the c_i has arity m_i and is a function symbol of type $\tau_i^1 \times \cdots \times \tau_i^{m_i} \to \mathsf{D}$ with $\tau_i^l \in \{\mathsf{D}, T_1, \ldots, T_n\}$. For each constructor c_i with $m_i \geqslant 1$, we add the *selectors* $d_i^1, \ldots, d_i^{m_i}$ to the language. Each selector d_i^j has the type $\mathsf{D} \to \tau_i^j$.

If a constructur does not take input of sort D, then we call it *static*. If it does take input of sort D, then we call it *dynamic*. In order to define induction, we need some well-founded order relation on the elements of the standard-model. This translates to the restriction that there is at least one static constructor c_i.

Without loss of generality, we assume that for every constructor c_i the first n (possibly 0) input-sorts are D. The other sorts T_l are ordered by their index i. This is an assumption, to simplify notation, but of course, all the results are independent on the ordering of the inputs of the c_i.

The following will be our base axioms:

$\mathbf{D_{i,j}}$	$c_i(\overline{x}) \neq c_j(\overline{y})$ for all $i \neq j, 1 \leqslant i, j \leqslant k$	Disjointness
$\mathbf{INJ_i}$	$c_i(\overline{x}) = c_i(\overline{y}) \rightarrow \overline{x} = \overline{y}$ for all $1 \leqslant i \leqslant k$	Injectivity
$\mathbf{INV^n_j}$	$d^n_j(c_j(x_1, \ldots, x_{m_j})) = x_n$ for all $1 \leqslant i \leqslant k, 1 \leqslant n \leqslant m_i$	Inverse

For the scheme of induction, we first define a shorthand:

$$\mathbf{LHS}(\varphi(x)) \quad \bigwedge_{i=1}^{k} \left(\forall x_1, \ldots, x_{m_i} \left(\bigwedge_{\substack{l \in \{1,\ldots,m_i\} \\ \tau_i^l = \mathsf{D}}} \varphi(x_l, \overline{z}) \rightarrow \varphi(c_i(x_1, \ldots, x_{m_i}), \overline{z}) \right)\right)$$

The scheme of induction now has the following form:

$$\mathbf{I}(\varphi) \quad \mathbf{LHS}(\varphi(x, \overline{z})) \rightarrow \forall x : \varphi(x, \overline{z})$$

The formula φ potentially contains parameters $\overline{z}$, which we will not explicitly mention in the following as from now on every formula contains parameters if not stated otherwise.

Now, we need to define two basic languages and theories:

Definition 3.1.1. *We define* $\mathcal{L}_0 = \{c_i \mid i \leqslant k\}$ *and* $\mathcal{L}_1 = \mathcal{L}_0 \cup \{d^j_i \mid m_i \geqslant 1, j \leqslant m_i\}$

Definition 3.1.2. *We define* $T_0 = \{D_{i,j} \mid i, j \leqslant k, i \neq j\} \cup \{INJ_i \mid i \leqslant k\}$ *and* $T_1 = T_0 \cup \{INV^n_j \mid j \leqslant k, m_j \geqslant n \geqslant 1\}$

Having these defined these basic things, it makes sense to talk about standard models of T_0 and T_1. Usually, there is not *one* standard model of T_0, but rather infinitely many, each being parameterized by the interpretation of the parameter sorts T_i.

Definition 3.1.3. *Let* $M_1, \ldots, M_n, Y$ *be sets. Then, the set* $\mathbb{T}(M_1, \ldots, M_n, Y)$ *of all ground terms is defined inductively:*

- $S_0 = \{c_i(m_1, \ldots, m_{m_i}) \mid c_i \textit{ is static}, m_l \in \tau^l_i\} \cup Y$
- $S_{n+1} = \{c_j(s_1, \ldots, s_{m_j}) \mid c_j \textit{ is dynamic}, \tau^l_j \neq \mathsf{D} \rightarrow s_l \in M_l, \tau^l_j = \mathsf{D} \rightarrow \exists k \leqslant n : s_l \in S_k\}$
- $\mathbb{T}(M_1, \ldots, M_n, Y) = \bigcup_{n \in \mathbb{N}} S_n$

Definition 3.1.4. *Let* $M_1, \ldots, M_n$ *be sets. The model* $\mathcal{M}^*(M_1, \ldots, M_n, Y)$ *of* T_0 *over the language* $\mathcal{L}_0$*, given that* $T_i^{\mathcal{M}^*} = M_i$*, is defined in the following way:*

- $\mathsf{D}^{\mathcal{M}^*} = \mathbb{T}(M_1, \ldots, M_n, Y)$

- *For any constructor c_i and any suitable tuple $(a_1, \dots, a_{m_i})$:* $c_i^{\mathcal{M}^*}(a_1, \dots, a_{m_i}) = c_i(a_1, \dots, a_{m_i})$

If $Y = \varnothing$, then we write $\mathcal{M}^*(M_1, \dots, M_n)$ instead of $\mathcal{M}^*(M_1, \dots, M_n, \varnothing)$ and call it the standard model w.r.t. $M_1, \dots, M_n$.

Definition 3.1.5. *For any set $I \subseteq \{1, \dots, n\}$, we write $\mathcal{M}_I^*(Y)$ for $\mathcal{M}^*(M_1, \dots, M_n, Y)$, where $M_i = \{i\}$ if $i \notin I$ and $M_i = \{a_i, b_i\}$ if $i \in I$. Moreover, we define $\mathcal{M}^*(Y) = \mathcal{M}_{\varnothing}^*(Y)$ and $\mathcal{M}_I^*(\varnothing) = \mathcal{M}_I^*$.*

Note that with the selectors being axiomatized in the way they are, there is a lot of ambiguity as to what a term of the form $d_j^n(c_i(\dots))$ is canonically interpreted as. Thus, if we work with $\mathcal{L}_1$, we will often refer to the standard models of T_0 over $\mathcal{L}_0$ and then extend the language.

Since, there are non-standard models, it makes sense, to define, what a standard element inside a non-standard model is:

Definition 3.1.6. *Let $\mathcal{M}$ be any model over the language $\{c_1, \dots, c_k\}$. The standard part $\mathcal{S}_{\mathcal{M}}$ of the model $\mathcal{M}$ is defined inductively:*

- $S_0 = \{c_i(m_1, \dots, m_{m_i}) \mid c_i$ *is static*, $m_l \in \tau_i^l\}$
- $S_{n+1} = \{c_j(s_1, \dots, s_{m_j}) \mid c_j$ *is dynamic*, $\tau_i j^l \neq \mathsf{D} \to s_l \in M_l, \tau_j^l = \mathsf{D} \to \exists k \leqslant n : s_l \in S_k\}$
- $\mathcal{S}_{\mathcal{M}} = \bigcup_{n \in \mathbb{N}} S_n$

An element $A \in \mathsf{D}^{\mathcal{M}}$ is a standard element if $A \in \mathcal{S}_{\mathcal{M}}$.

Often, it makes sense to consider the graph induced by the constructors c_i

Definition 3.1.7. *Let $\mathcal{M}$ be any model of $\varnothing$ over the language $\{c_1, \dots, c_k\}$. The directed graph $\mathcal{G} = (V, E)$ induced by $\mathcal{M}$ is given by $V = \mathsf{D}^{\mathcal{M}}$ and for any $A, B \in V$, $E(A, B)$ if there is some constructor c_i and elements $m_l \in (\tau_i^l)^{\mathcal{M}}$ s.t. $A = m_l$ for some of the l and $c_i(m_1, \dots, m_{m_i}) = B$. The connected components of the undirected educt of $\mathcal{G}$ are called comparison classes.*

Now let us define an additional axiom, we will need in the following sections:

$$\textbf{SUR} \quad \bigvee_{i=1}^{k} \exists \overline{y} : X = c_i(\overline{y}) \qquad \text{(Surjectivity)}$$

Note that *surjectivity* above does usually not mean that the interpretations of one of the constructors is surjective, but rather that $\mathsf{D}^{\mathcal{M}} = \bigcup_{i=1}^{k} c_i^{\mathcal{M}}((\tau_i^1)^{\mathcal{M}}, \dots, (\tau_i^{m_i})^{\mathcal{M}})$.

We will often refer to the following lemma:

Theorem 3.1.8. *Let $\mathcal{L}$ be any language extending $\mathcal{L}_0$. Then $\varnothing + ILiteral \vdash \mathrm{SUR}$.*

Proof. Take any model $\mathcal{M}$ of $\varnothing + ILiteral$. Assume that there is some element $B \in \mathsf{D}^{\mathcal{M}}$ s.t. B does not lie in the image of any $c_i^{\mathcal{M}}$. Consider the following literal $L(X) \equiv X \neq Z$ and the interpretation $\xi : Z \mapsto B$. Then $\mathcal{M}, \xi \vDash \mathbf{LHS}(A)$, but clearly, $\mathcal{M}, \xi \nvDash \forall X : A(X)$. Thus, $\mathcal{M}, \xi \nvDash ILiteral$, which is a contradiction. ■

First, we consider some interesting models:

3.2 Useful Models

We will now consider useful non-standard models for T_0 and T_1, which we will use in the following sections to separate theories.

3.2.1 The model $\mathcal{M}^*(\{\infty\})$

The following sums up our results about $\mathcal{M}^*(\{\infty\})$:

Theorem 3.2.1. *If there is exactly one constructor, then $\mathcal{M}^*(\{\infty\}) \nvDash IAtom$. If there is more than one constructor, then $\mathcal{M}^*(\{\infty\}) \vDash IAtom$. In any case, $\mathcal{M}^*(\{\infty\}) \nvDash \mathrm{SUR}$.*

This theorem follows from the following lemmas and observations:

Observation 3.2.2. $\mathcal{M}^*(\{\infty\}) \nvDash \mathrm{SUR}$

Lemma 3.2.3. *Let $\mathcal{M}$ be a model of T_0 or T_1 and assume that $\mathcal{M}$ contains exactly one standard element B and at least one non-standard element C. Then $\mathcal{M} \nvDash IAtom$.*

Proof. First, note that since there is only one standard element, there can only be one constructor, which has to be static. Consider the atom $A(X) \equiv X = Z$ and the interpretation $\xi : Z \mapsto B$. Then $\mathcal{M} \vDash \mathbf{LHS}(A)$, but $\mathcal{M} \nvDash \forall X : A(X)$. ■

Lemma 3.2.4. *If there is exactly one constructor c_i, then $\mathcal{M}^*(\{\infty\}) \nvDash IAtom$.*

Proof. This follows directly from Lemma 3.2.3. ■

The following Lemma applies to the case of dynamic constructors as well.

Lemma 3.2.5. *Take any model $\mathcal{M}$ of T_0, any comparison class $\mathcal{C} \subseteq \mathsf{D}^{\mathcal{M}}$ that does not contain cycles and two distinct standard elements $B, C \in \mathcal{C}$. For any term t, it holds that if there is some term s with $\mathcal{M} \vDash t(B) = s(B) \wedge t(C) = s(C)$, then $\mathcal{M} \vDash \forall X : t(X) = s(X)$.*

Proof. There are three cases:

1. If X appears in neither t nor s, then $\mathcal{M} \vDash (\exists X : t(X) = s(X)) \leftrightarrow (\forall X : t(X) = s(X))$. Therefore, the claim of the Lemma holds trivially.

2. If, w.l.o.g., X appears in t, but not in s, we claim that $\mathcal{M} \vDash t(B) \neq t(C)$ and thus, the conditions of the lemma are never met as $s^{\mathcal{M}}$ is constant. We prove the claim inductively: The base case is that $t \equiv X$. Then clearly, $\mathcal{M} \vDash t(B) \neq t(C)$. Assume that we have shown this property for some term t_1 that contains X and take some appropriate terms $t_2, \dots, t_{m_l}$. Consider the term $t = c_l(t_1, \dots, t_{m_i})$. By INJ_l and the induction hypothesis, $\mathcal{M} \vDash t(B) \neq t(C)$.

3. Assume that X appears in both s and t. We proceed with induction on the structure of t. The base case is that $t \equiv X$. By assumption, there are no cycles in $\mathcal{C}$. Thus, if s contains X and $\mathcal{M} \vDash t(B) = s(B)$, then $s \equiv X$ and $\mathcal{M} \vDash \forall X : t(X) = s(X)$. Now assume that t has the form $c_l(t_1, \dots, t_{m_l})$, where we have shown the claim for any term t_i. We can exclude the case that $s \equiv X$ by symmetry and the base case. We can also exclude the case that $s \equiv c_k(\overline{s})$ with $k \neq l$ by $D_{k,l}$. The only case left is that $s \equiv c_l(s_1, \dots, s_{m_l})$. By INJ_l, we obtain that $\mathcal{M} \vDash t_i(B) = s_i(B) \wedge t_i(C) = s_i(C)$. From the induction hypothesis, it follows that $\mathcal{M} \vDash \forall X : t_i(X) = s_i(X)$. Thus, $\mathcal{M} \vDash \forall X : t(X) = s(X)$. ∎

Corollary 3.2.6. *If some model $\mathcal{M}$ of T_0 contains no cycles at all, then $\mathcal{M} \vDash (Y \neq Z \wedge t(Y) = s(Y) \wedge t(Z) = s(Z)) \rightarrow (\forall X)(t(X) = s(X))$*

Corollary 3.2.7. *If there is more than one constructor, then $\mathcal{M}^*(\{\infty\}) \vDash IAtom$.*

Proof of Theorem 3.2.1. If there is exactly one constructor, then $\mathcal{M}^*(\{\infty\}) \nvDash IAtom$ by Lemma 3.2.4. If there is more than one constructor, then $\mathcal{M}^*(\{\infty\}) \vDash IAtom$ by Corollary 3.2.7. In any case, $\mathcal{M}^*(\{\infty\}) \nvDash$ SUR by Observation 3.2.2. ∎

3.2.2 The model $\mathcal{M}^*_\infty$

Definition 3.2.8. *We construct the model $\mathcal{M}^*_\infty$: Take the model $\mathcal{M}^*(\{\infty\})$ of T_0 over $\mathcal{L}_0$ and extend the language with the selectors. The selectors are interpreted in the following way*

- *If $C = c_i^{\mathcal{M}^*_\infty}(n_1, \dots, n_{m_i})$, then define $(d_i^l(C))^{\mathcal{M}^*_\infty} = n_l$*
- *If C is not in the image of $c_i^{\mathcal{M}}$ and $\tau_i^l \neq \mathsf{D}$, then there is only one possible way to interpret $(d_i^l(C))^{\mathcal{M}^*_\infty}$ since $|\tau_i^l| = 1$*
- *If C is not in the image of $c_i^{\mathcal{M}^*_\infty}$ and $\tau_i^l = \mathsf{D}$, then define $(d_i^l(C))^{\mathcal{M}^*_\infty} = C$*

The following sums up our findings about $\mathcal{M}^*_\infty$:

Theorem 3.2.9. *If there is exactly one constructor, then $\mathcal{M}^*_\infty \not\vDash IAtom$. If there is more than one constructor, then $\mathcal{M}^*_\infty \vDash IAtom$. In any case $\mathcal{M}^*_\infty \not\vDash \mathrm{SUR}$.*

This theorem is a consequence of the following observations and lemmas:

Observation 3.2.10. $\mathcal{M}^*_\infty \not\vDash \mathrm{SUR}$ *and* $\mathcal{M}^*_\infty \vDash T_1$.

Lemma 3.2.11. *If there is exactly one constructor, then $\mathcal{M}^*_\infty \not\vDash IAtom$.*

Proof. This follows directly from Lemma 3.2.3. ■

Lemma 3.2.12. *Assume that there is more than one constructor and that all constructors are static. Take any atom $A(X)$. If $\mathcal{M}^*_\infty \vDash A(E)$ for any standard element $E \in \mathsf{D}^{\mathcal{M}^*_\infty}$, then $\mathcal{M}^*_\infty \vDash \forall X : A(X)$.*

Proof. Since there are at least two constructors c_i and c_j, there are, by $D_{i,j}$, at least two distinct elements B, C in the standard part of $\mathcal{M}^*_\infty$.

We start by some preprocessing of the terms, we consider: Take any term t. If t contains some subterm of the form $d^l_i(t')$, then replace this subterm with the parameter z^l_i and set $(z^l_i)^{\mathcal{M}^*_\infty} = a$ with $\tau^l_i = \{a\}$. Thus, w.l.o.g., we can assume that the symbols d^l_i do not appear in any term, we consider.

Now take any atom $A(X) \equiv t_1 = t_2$. There are three cases:

1. If X appears on neither side of A, we are done.

2. If X appears on exactly on side of A, say t_1, then $t_1 \equiv X$ as no selectors may appear and all constructors are static and $t_2^{\mathcal{M}^*_\infty}$ is constant. Then, $\mathcal{M}^*_\infty \not\vDash A(B) \wedge A(C)$.

3. If X appears on both sides of A, then $t_1 \equiv t_2 \equiv X$ and A is an identity. Thus, $\mathcal{M}^*_\infty \vDash \forall X : A(X)$ ■

Lemma 3.2.13. *Let t be a term of the form $d^{l_1}_{i_1}(\dots d^{l_n}_{i_n}(X))$, where all the $\tau^{l_h}_{i_h}$ are D. Then, there is some term s_t with the following properties:*

- $\mathcal{M}^*_\infty \vDash \forall X : X = (t \circ s_t)(X)$
- *The interpretation $s_t^{\mathcal{M}^*_\infty}$ maps standard elements to standard elements*
- *All function symbols in s_t are dynamic constructors*
- *For any term t' of the form $d^{k_1}_{j_1}(\dots d^{k_m}_{j_m}(X))$, where all the $\tau^{k_h}_{j_h}$ are D and $\mathcal{M}^*_\infty \vDash \forall X : X = (t \circ s_{t'})$ and $\mathcal{M}^*_\infty \vDash \forall X : X = (t' \circ s_t)$, it holds that $\mathcal{M}^*_\infty \vDash t(X) = t'(X)$*

This term s_t is called right-inverse of t.

Proof. Take any such term t of the form $d^{l_1}_{i_1}(\dots d^{l_n}_{i_n}(X))$, where all the $\tau^{n_l}_{i_l}$ are D. We define the sequence $s_0, \dots, s_n$ inductively: $s_0 \equiv X$ and $s_{k+1} \equiv c_{i_{n-k+1}}(a_1, \dots, a_{m_{n-k+1}})$, where $a_{l_{n-k+1}} = s_k$ and the other a_l are arbitrary (fixed) parameters of the right sorts and if that sort is D, we choose it to be a standard element. Then, the term $s_t = s_n$ has the property that $T_1 \vdash \forall X : (t \circ s_t)(X) = X$. By construction $s_t^{\mathcal{M}^*_\infty}$ maps standard elements to standard elements.

Now take any $t' = d^{k_1}_{j_1}(\dots d^{k_m}_{j_m}(X))$ and its respective right-inverse $s_{t'}$. Assume that $\mathcal{M}^*_\infty \vDash \forall X : X = (t' \circ s_t)(X)$ and $\mathcal{M}^*_\infty \vDash \forall X : X = (t \circ s_{t'})(X)$. W.l.o.g., assume that $n \leqslant m$ and take the biggest $h \in \{1, \dots, n\}$ s.t. $d^{l_h}_{i_h} \neq d^{k_h}_{j_h}$ if such an h exists. There are two cases: If $i_h = j_h$, then $l_h \neq k_h$ and $(d_{i_h}(\dots d^{l_n}_{i_n}(s_{t'}(X))))^{M^*_\infty}$ is some fixed parameter by construction of the right-inverse. In particular, $((t \circ s_{t'})(X))^{\mathcal{M}^*_\infty}$ is constant, which contradicts our assumption. If $i_h \neq j_h$, then $\mathcal{M}^*_\infty \vDash \forall X : d^{l_h}_{i_h}(\dots d^{l_n}_{i_n}(s_{t'}(X))) = d^{l_{h+1}}_{i_{h+1}}(\dots d^{l_n}_{i_n}(s_{t'}(X)))$ by construction of $\mathcal{M}^*_\infty$. Thus, we can cut $d^{l_h}_{i_h}$ and reduce n by one for each time, we do this and call the result n'. In summary, we can assume that such an h does not exist and $d^{l_h}_{i_h} = d^{l_h}_{j_h}$ for any $h \leqslant n$. Now assume that $n' < m$. Then, $\mathcal{M}^*_\infty \vDash (t' \circ s_t)(X) = d^{k_1}_{j_1}(\dots d^{k_{m-n}}_{j_{m-n}}(X))$. However, there are standard elements E s.t. $\mathcal{M}^*_\infty \nvDash E = d^{k_1}_{j_1}(\dots d^{k_{m-n}}_{j_{m-n}}(E))$, which contradicts our assumptions. Thus, $n' = m$. This can only be the case if we never deleted any $d^{l_h}_{i_h}$. Thus, $n = m$ and for any $h \in \{1, \dots, n\}$ it holds that $d^{l_h}_{i_h} = d^{k_h}_{j_h}$. In particular $\mathcal{M}^*_\infty \vDash \forall X : t(X) = t'(X)$. ■

Lemma 3.2.14. *Assume that there is more than one constructor and that there is some dynamic constructor. Take any atom $A(X)$. If $\mathcal{M}^*_\infty \vDash A(E)$ for any standard element $E \in \mathsf{D}^{\mathcal{M}^*_\infty}$, then $\mathcal{M}^*_\infty \vDash \forall X : A(X)$.*

Proof. We start by some preprocessing of the terms, we consider: Take any term t. If t contains some subterm of the form $d^l_i(t')$, there are four cases:

- If X does not appear in t', then $d^l_i(t')$ has a fixed interpretation and we can replace it with some new parameter that we interpret as $(d^l_i(Z))^{\mathcal{M}}$.
- If $\tau^l_i \neq \mathsf{D}$, then there is only one possible interpretation of $d^l_i(t')$. Thus, replace $d^l_i(t')$ with some parameter z and interpret z as the only element in τ^l_i.
- If $\tau^l_i = \mathsf{D}$ and t' is of the form $c_j(s_1, \dots, s_{m_j})$, there are two cases: If $j = i$, then replace $d^l_i(t')$ with s_l. If $i \neq j$, then d^l_i is interpreted as the identity and we can replace $d^l_i(t')$ with t'.
- If the prior cases are not applicable, i.e. we have some term of the form $d^{l_1}_{i_1}(\dots d^{l_n}_{i_n}(X))$, we leave it as it is.

Thus, w.l.o.g., we can assume that the symbols d^l_i appear only stacked directly over X.

Now note that since there are at least two constructors, there are at least two distinct standard elements B, C by $D_{i,j}]$, where B lies in the image of $c_i^{\mathcal{M}_\infty^*}$ and C lies in the image of $c_j^{\mathcal{M}_\infty^*}$.

Now take any atom $A(X) \equiv t_1 = t_2$. There are three cases:

1. If X appears on neither side of A, we are done.

2. If X appears on exactly one side, say t_1, then $t_2^{\mathcal{M}}$ is constant. We show inductively that $t_1^{\mathcal{M}}$ is not constant in the standard part of the model and thus it does not hold that $\mathcal{M}_\infty^* \vDash A(E)$ for all standard elements E.

2.1. The first base case: If $t_1 \equiv X$, then $t_1^{\mathcal{M}_\infty^*}(B) \neq t_1^{\mathcal{M}_\infty^*}(C)$.

2.2. The second base case: If $t_1 \equiv d_{i_1}^{l_1}(\dots d_{i_n}^{l_n}(X))$, then consider the right-inverse s_{t_1} of t_1 from Lemma 3.2.13. It holds that $\mathcal{M} \nvDash A(s_{t_1}(B)) \wedge A(s_{t_1}(C))$

2.3. Assume that we have shown that the interpretations of the terms $s_1, \dots, s_{m_i}$ are not constant in the standard part. Then, by INJ_i, the interpretation of the term $c_i(s_1, \dots, s_{m_i})$ is not constant as well.

3. Now assume that X appears on both sides of A. We show by induction that for any term t it holds that if there is some term t' containing X with $\mathcal{M}_\infty^* \vDash t(E) = t'(E)$ for all standard elements E, then $\mathcal{M}_\infty^* \vDash \forall X : t(X) = t'(X)$.

3.1. The first base case: Assume that $t \equiv X$. If $t' \equiv c_l(\overline{s})$, then $\mathcal{M}_\infty^* \nvDash t(B) = t'(B) \wedge t(C) = t'(C)$ by $D_{l,i}$ or $D_{l,j}$. If t' is of the form $d_{i_1}^{l_1}(\dots d_{i_n}^{l_n}(X))$, then consider the right-inverse $s_{t'}$. Note that the outermost constructor c_l of $s_{t'}$ must be dynamic. Thus, if we take any static constructor c_m and define $E = s_{t'}(c_m^{\mathcal{M}_\infty^*}(\overline{a}))$ for some appropriate tuple $\overline{a}$, then $t'(E) = c_m^{\mathcal{M}_\infty^*}(\overline{m})$ and $\mathcal{M} \nvDash t(E) = t'(E)$. Thus, if the conditions are met, $t_2 \equiv X$.

3.2. The second base case: Assume that t is of the form $d_{i_1}^{l_1}(\dots d_{i_n}^{l_n}(X))$. By symmetry and case 3.1, we can exclude the case that $t_1 \equiv X$. Assume that $t_2 \equiv c_l(\overline{s})$. Note that c_l has to be dynamic since X must occur in it and no selector that maps into any sort other than D is allowed. Take the right-inverse s_t of t and any static constructor c_m. Define $E = c_m^{\mathcal{M}_\infty^*}(\overline{a})$. Then $t(E) = c_m^{\mathcal{M}_\infty^*}(\overline{a})$ and $\mathcal{M}_\infty^* \nvDash t(E) = t'(E)$. It follows that t' has the form $d_{j_1}^{k_1}(\dots d_{j_m}^{k_m}(X))$. Consider the right-inverse s_t of t. By assumption, $\mathcal{M}_\infty^* \vDash E = (t' \circ s_t)(E)$ and $\mathcal{M}_\infty^* \vDash E = (t \circ s_{t'})(E)$ for any standard element E. By the induction hypothesis, $\mathcal{M}_\infty^* \vDash \forall X : X = (t \circ s_{t'})(X)$ and $\mathcal{M}_\infty^* \vDash \forall X : X = (t' \circ s_t)(X)$. By Lemma 3.2.13, we obtain that $\mathcal{M}_\infty^* \vDash \forall X : t(X) = t'(X)$.

3.3. Assume that we have shown the claim for the terms $s_1, \dots, s_{m_i}$. Consider the term $t \equiv c_i(s_1, \dots, s_{m_i})$. It $\mathcal{M}_\infty^* \vDash t(E) = t'(E)$ for some term t' and all standard elements E, then from the base cases, we obtain that $t' \equiv c_l(s'_1, \dots, s'_{m_l})$. From $D_{i,l}$, it follows that $l = i$ and from INJ_i, we obtain that $\mathcal{M}_\infty^* \vDash s_l(E) = s'_l(E)$ for any standard element E and any l. From the induction hypothesis, we obtain that $\mathcal{M}_\infty^* \vDash \forall X : s_l(X) = s'_l(X)$ and thus $\mathcal{M}_\infty^* \vDash \forall X : t(X) = t'(X)$. ■

Corollary 3.2.15. *If there is more than one constructor, then $\mathcal{M}^*_\infty \vDash IAtom$.*

Proof. This follows from Lemma 3.2.12 and Lemma 3.2.14. ■

Proof of Theorem 3.2.9. If there is exactly one constructor, $\mathcal{M}^*_\infty \nvDash IAtom$ by Lemma 3.2.11. If there is more than one constructor, then $\mathcal{M}^*_\infty \vDash IAtom$ by Corollary 3.2.15. In any case $\mathcal{M}^*_\infty \nvDash$ SUR by Observation 3.2.10. ■

3.2.3 Models with cycles

We will later see that $IOpen$ proves acyclicty in the sense that for any term $t \not\equiv X$, $T_0 + IOpen \vdash X \neq t(X)$. However, it depends on the constructors, which level of induction we need for this. In the *worst case*, we need $IDClause$, while in some cases, $ILiteral$ suffices. For these cases, where induction over dual clauses is needed, we now give models with cycles, in which $ILiteral$ holds.

3.2.3.1 The model $\mathcal{M}_C$

For the rest of this subsubsection, we assume that there is at least one constructor $c_1 : \tau_i^1 \times \cdots \times \tau_i^{m_i} \to \mathsf{D}$, where at least one of the τ_i^l is D and $m_i \geqslant 2$. Note that τ_i^2 is either D or some T_k. In the second case, assume, w.l.o.g., that $k = 1$.

Definition 3.2.16. *Now, we make a case distinction based on τ_i^2:*

1. *If $\tau_i^2 = \mathsf{D}$, then start with the model $\mathcal{M} = \mathcal{M}^*(\{A, B\})$. Fix two standard elements $C_1, C_2 \in \mathsf{D}^{\mathcal{M}}$ and consider the set M defined as $\mathsf{D}^{\mathcal{M}}$ factorized by the equations $A = c_1(B, C_1, \overline{e})$ and $B = c_1(A, C_2, \overline{e})$, where $\overline{e}$ is some tuple of standard elements with appropriate sorts.*

2. *If $\tau_i^2 = T_1$, then start with the model $\mathcal{M} = \mathcal{M}^*_{\{1\}}(\{A, B\})$, where $T_1^{\mathcal{M}} = \{a_1, b_1\}$. Now, consider the set M defined as $\mathsf{D}^{\mathcal{M}}$ factorized by the equations $A = c_1(B, a_1, \overline{e})$ and $B = c_1(A, b_1, \overline{e})$, where $\overline{e}$ is some tuple of elements with appropriate sorts.*

Set $\mathsf{D}^{\mathcal{M}_C} = M$ and $T_i^{\mathcal{M}_C} = T_i^{\mathcal{M}}$. In any case, the constructors are interpreted canonically in the sense that $c_i^{\mathcal{M}_C}([f_1], \ldots, [f_{m_i}]) = [c_i^{\mathcal{M}}(f_1, \ldots, f_{m_i})]$. Note that this is well-defined and $\mathcal{M}_C \vDash T_0$.

Observation 3.2.17. *There is some term $t \not\equiv X$, which contains X with $\mathcal{M}_C \vDash \exists X : X = t(X)$.*

Lemma 3.2.18. $\mathcal{M}_C \vDash IClause$

Proof. Take any clause $C \equiv L_1 \vee \ldots L_n$ and assume that $\mathcal{M}_C \vDash \mathbf{LHS}(C)$. We start by preprocessing C:

Assume that there is some literal L_i that does not contain X. Then, $\mathcal{M}_C \vDash L_i \leftrightarrow \top$ or $\mathcal{M}_C \vDash L_i \leftrightarrow \bot$. In the first case, $\mathcal{M}_C \vDash C \leftrightarrow \top$. In the second case, $\mathcal{M}_C \vDash C \leftrightarrow C'$, where C' is obtained from C by deleting L_i. Thus, we can assume that every literal contains X.

Assume that there is some negated atom $L_i \equiv t_1 \neq t_2$ and both t_l have the outermost function symbol c_i. Then both t_l are of the form $c_i(t_l^1, \ldots, t_l^{m_i})$. By INJ_i, $\mathcal{M}_C \vDash L_i \leftrightarrow \bigvee_{k=1}^{m_i} t_1^k \neq t_2^k$. Thus, we can replace L_i in C with the clause $\bigvee_{k=1}^{m_i} t_1^k \neq t_2^k$ and obtain another equivalent clause.

Assume that all the L_i are atoms. Since the standard part of $\mathcal{M}_C$ is infinite, there is some L_i and distinct standard elements $B, C \in \mathsf{D}^{M_C}$ s.t. $\mathcal{M}_C \vDash L_i(B) \wedge L_i(C)$. By Lemma 3.2.5, $\mathcal{M}_C \vDash \forall X : L_i(X)$ and thus, $\mathcal{M}_C \vDash \forall X : C(X)$. In this case, we are done.

From now on, we assume that at least one of the $L_i \equiv t \neq s$ is a negated atom. W.l.o.g., assume that t_1 contains X.

We make a case distinction:

1. The case that both t and s share the same outermost function symbol is eliminated by our preprocessing.

2. If the outermost function symbols of t and s are c_i and c_j respectively with $i \neq j$, then $\mathcal{M}_C \vDash \forall X : L_i(X)$ by $D_{i,j}$ and thus, $\mathcal{M} \vDash \forall X : C(X)$. In this case, we are done.

3. If $t \equiv X$ and $t_2^{\mathcal{M}_C}$ is constant with some fixed interpretation B', then $\mathcal{M}_C \vDash L_i(B)$ for any $B \neq B'$. There are two cases: If B' is a standard element, then there has to be some L_j with $\mathcal{M}_C \vDash L_j(B')$ since $\mathcal{M}_C \vDash \mathbf{LHS}(C)$. If B' is a non-standard element, then by SUR, there is some dynamic constructor c_i and elements $B_1, \ldots, B_l, b_{l+1}, \ldots, b_{m_i}$ where all the B_j are of sort D and $B' = c_i^{\mathcal{M}_C}(B_1, \ldots, b_{m_i})$. As mentioned, $\mathcal{M}_C \vDash L_i(B_j)$ and thus, $\mathcal{M}_C \vDash C(B_j)$ for any j. Since $\mathcal{M}_C \vDash \mathbf{LHS}(C)$, we conclude that $\mathcal{M}_C \vDash C(B')$. Thus, in any case, $\mathcal{M}_C \vDash \forall X : C(X)$.

4. Assume that $t \equiv X$ and s contains X. If $s \equiv X$, then $\mathcal{M}_C \vDash \forall X : \neg L_i(X)$ and $\mathcal{M}_C \vDash C \leftrightarrow C'$, where C' is obtained from C by deleting L_i. In that case, we can start from the top with C'. Thus, we assume that $s \not\equiv X$. Assume that there is some $E \in \mathsf{D}^{\mathcal{M}_C}$ with $\mathcal{M} \nvDash L_i(E)$. Then $E = A$ or $E = B$. W.l.o.g., assume that $E = A$. It follows that $t \equiv c_i(s_1, s_2, \overline{e})$ with $s_2(E) = C_1$. Since C_1 is a standard element and A is not, $s_2^{\mathcal{M}_C}$ has to be constant. It follows that $\mathcal{M}_C \nvDash B = t(B)$ and thus $\mathcal{M}_C \vDash L_i(B)$. Since C_1 and all the e_i are standard elements, $\mathcal{M}_C \vDash L_i(E)$, and thus $\mathcal{M}_C \vDash C(E)$, for any E in $\{B, C_1\} \cup \{e_i \mid e_i \text{ is of sort } \mathsf{D}\}$. Thus, $\mathcal{M} \vDash C(A)$ since $\mathcal{M} \vDash \mathbf{LHS}(C)$. It follows that $\mathcal{M} \vDash \forall X : C(X)$. ■

3.2.3.2 The model $\mathcal{M}_C^u$

For the rest of this subsubsection, we assume that there are at least two dynamic constructors c_1, c_2 and all dynamic constructors are unary[1].

[1] The name $\mathcal{M}_C^u$ comes from *unary*

Definition 3.2.19. *We define the model $\mathcal{M}_C^u$ in the following way: Take the model $\mathcal{M} = \mathcal{M}^*(A, B)$ and define the set M to be $\mathsf{D}^{\mathcal{M}}$ factorized by the equations $A = c_1(B)$ and $B = c_2(A)$. Then set $\mathsf{D}^{\mathcal{M}_C^u} = M$ and $T_i^{\mathcal{M}_C^u} = T_i^{\mathcal{M}}$. The constructors are interpreted canonically in the sense that $c_i^{\mathcal{M}_C^u}([f_1], \ldots, [f_{m_i}]) = [c_i^{\mathcal{M}}(f_1, \ldots, f_{m_i})]$. Note that this is well-defined and $\mathcal{M}_C \vDash T_0$.*

Observation 3.2.20. *There is some term $t \not\equiv X$, which contains X with $\mathcal{M}_C^u \vDash \exists X : X = t(X)$.*

Lemma 3.2.21. *$\mathcal{M}_C^u \vDash ILiteral$.*

Proof. From Lemma 3.2.5, it follows that $\mathcal{M}_C \vDash IAtom$. Thus, we only have to deal with negated atoms. Fix the negated atom $L(X) \equiv t \neq s$. There are three cases:

1. If X appears on neither side of L, then $\mathcal{M}_C^u \vDash (\exists X : L(X)) \leftrightarrow (\forall X : L(X))$ and we are done.

2. If X appears on exactly one side of L, say in t_1, then $t_2^{\mathcal{M}_C^u}$ is constant. Assume that there is some element $E \in \mathsf{D}^{\mathcal{M}_C^u}$ s.t. $\mathcal{M}_C^u \nvDash L(E)$. If E lies in the image of the interpretation of some static constructor c_l, then $\mathcal{M} \nvDash \mathbf{LHS}(L)$. Assume that E does not lie in the image of the interpretation of any static constructor. Then, by construction of $\mathcal{M}_C^u$, E lies in the image of some dynamic constructor c_l and there is some $F \in \mathsf{D}^{\mathcal{M}}$ with $E = c_l(F)$. By repeated application of $D_{i,j}$ and INJ_k for any i, j, k, we obtain that $t_1(E)^{\mathcal{M}_C^u} \neq t_1(F)^{\mathcal{M}_C^u}$. Thus, $\vDash L(F)$ and hence, $\mathcal{M}_C^u \nvDash \mathbf{LHS}(L)$.

3. Assume that X appears on both sides of L and that $\mathcal{M}_C^u \vDash L(E)$ for any standard element E. Then $\neg L$ is not an identity. If $\mathcal{M}_C^u \nvDash \forall X : L(X)$, then, by construction of $\mathcal{M}_C^u$, the only possible elements E s.t. $\mathcal{M}_C^u \nvDash L(E)$ are A or B. W.l.o.g., assume that $\mathcal{M}_C^u \nvDash L(A)$. We proceed by induction on the structure of t and show that $\mathcal{M}_C^u \vDash L(B)$.

If $t \equiv X$, then, w.l.o.g., then $s \equiv c_1(s')$ since $\mathcal{M}_C^u \vDash A = s(A)$. By $D_{1,2}$, $\mathcal{M}_C^u \nvDash B = s(B)$ and thus, $\mathcal{M}_C^u \vDash L(B)$.

Assume that $t \equiv c_i(t')$ (by assumption, all dynamic constructors are unary). Since $\mathcal{M}_C^u \vDash t(A) = s(A)$, it follows that $s \equiv c_i(s')$. By INJ_i, we obtain that $t'(A) = s'(A)$. Both t' and s' must contain X. From the induction hypothesis, it follows that $\mathcal{M}_C^u \vDash t'(B) \neq s'(B)$ and consequently, by INJ_i, $\mathcal{M}_C^u \vDash t(B) \neq s(B)$. Thus, $\mathcal{M}_C^u \vDash L(B)$.

Thus, $\mathcal{M}_C^u \vDash L(B)$ if $\mathcal{M}_C^u \nvDash L(A)$. Since $A = c_1^{\mathcal{M}_C^u}(B)$, we conclude that $\mathcal{M}_C^u \nvDash \mathbf{LHS}(L)$. ■

3.3 Languages with Static Constructors only

In the following section we assume that our language contains only constructors that do not take any input of sort D. This is a special case, in which some things are easier.

The following theorem states exactly why this case is that easy:

Theorem 3.3.1. *Let $\mathcal{L}$ be any language extending $\{c_1, \dots, c_k\}$, where all the c_i are static. Then $\varnothing + \mathrm{SUR} \vdash I(\varphi)$ for any (not necessarily open) formula φ.*

Proof. We work in $\varnothing + \mathrm{SUR}$. Take any formula φ. Note that we only have constructors that do not take input of sort D. Thus, our induction axiom looks like this:

$$I(\varphi) \equiv \bigwedge_{i=1}^{k} (\forall x_1, \dots, x_{m_i} \, (\top \to \varphi(c_i(x_1, \dots, x_{m_i})))) \to \forall x : \varphi(x),$$

which is logically equivalent to

$$\left(\bigwedge_{i=1}^{k} \forall x_1, \dots, x_{m_i} : \varphi(c_i(x_1, \dots, x_{m_i})) \right) \to \forall x : \varphi(x).$$

Now we only need to note that by SUR every element is some instance of a constructor. Thus, if $\mathbf{LHS}(\varphi)$ holds, then $\forall X : \varphi(X)$ holds as well. ■

3.3.1 Constructors only

The following theorem will be the main result:

Theorem 3.3.2. *There are three cases:*

1. *There is more than one constructor*
2. *There is exactly one constructor c and c is not constant*
3. *There is exactly one constructor c and c is constant*

In the case 1, we obtain the following result:

$$\begin{aligned} T_0 &\approx T_0 + IAtom \\ &\lesssim T_0 + ILiteral \\ &\approx T_0 + IOpen \\ &\approx T_0 + \mathrm{SUR} \end{aligned}$$

This yields the following Hasse-Diagram:

$$\begin{tikzcd} T_0 + ILiteral \approx T_0 + IOpen \arrow[d, dash] \\ T_0 \approx T_0 + IAtom \end{tikzcd}$$

In case 2, we obtain the following result:

$$\begin{aligned} T_0 &\lesssim T_0 + IAtom \\ &\lesssim T_0 + ILiteral \\ &\approx T_0 + IOpen \\ &\approx T_0 + \mathrm{SUR} \end{aligned}$$

This yields the following Hasse-Diagram:

$$\begin{tikzcd} T_0 + ILiteral \approx T_0 + IOpen \arrow[d, dash] \\ T_0 + IAtom \arrow[d, dash] \\ T_0 \end{tikzcd}$$

In case 3, we obtain the following result:

$$\begin{aligned} T_0 &\lesssim T_0 + IAtom \\ &\approx T_0 + IOpen \\ &\approx T_0 + \mathrm{SUR} \end{aligned}$$

This yields the following Hasse-Diagram:

$$\begin{tikzcd} T_0 + IAtom \approx T_0 + IOpen \arrow[d, dash] \\ T_0 \end{tikzcd}$$

This theorem is a consequence of the following lemmas.

Lemma 3.3.3. *If there are static constructors only and $\mathcal{M}$ is a model of T_0, where there are at least two elements $B, C \in \mathsf{D}^{\mathcal{M}}$, which lie in the image of some (possibly the same) constructors $c_i^{\mathcal{M}}, c_j^{\mathcal{M}}$, then $\mathcal{M} \vDash IAtom$.*

Proof. Fix some model $\mathcal{M}$ with these properties and let $B, C \in \mathsf{D}^{\mathcal{M}}$ be two distinct elements in the image of some $c_i^{\mathcal{M}}, c_j^{\mathcal{M}}$. Take any atom $A(X) \equiv t_1 = t_2$. We make a case distinction:

1. If X appears on neither side of A, then $t_1^{\mathcal{M}}$ and $t_2^{\mathcal{M}}$ are both constant and $\mathcal{M} \vDash A(B) \leftrightarrow \forall X : A(X)$. Thus, $\mathcal{M} \vDash \mathbf{I}(A)$.

2. If X appears on both sides of A, then, since there are static constructors only, both t_1 and t_2 are syntactically identical to X. Thus, A is an identity and $\mathcal{M} \vDash \forall X : A(X)$ and consequently $\mathcal{M} \vDash \mathbf{I}(A)$.

3. If X appears on exactly one side of A, say in t_1, then $t_1 \equiv X$ and $t_2^{\mathcal{M}}$ is constant. Thus, $\mathcal{M} \vDash t_1(B) \neq t_1(C)$. Thus, there is some $E \in \{B, C\}$ with $\mathcal{M} \vDash t_1(E) \neq t_2$ and consequently $\mathcal{M} \vDash \neg A(E)$. Therefore, $\mathcal{M} \nvDash \mathbf{LHS}(A)$ and hence, $\mathcal{M} \vDash \mathbf{I}(A)$. ■

Corollary 3.3.4. *If there is more than one constructor, then $T_0 \vdash IAtom$*

Proof. Assume that there are two constructors c_i and c_j. By $D_{i,j}$, in any model $\mathcal{M}$ of T_0, the images of $c_i^{\mathcal{M}}$ and $c_j^{\mathcal{M}}$ are disjoint. Moreover, each of them contains at least one element and thus, Lemma 3.3.3 is applicable for $\mathcal{M}$. This holds for every model of T_0 and thus $T_0 \vdash IAtom$. ■

Lemma 3.3.5. *If there is more than one constructor, then $T_0 + IAtom \nvdash ILiteral$.*

Proof. Since $T_0 + ILiteral \vdash$ SUR, this follows from Theorem 3.2.1. ■

Lemma 3.3.6. *If there is exactly one constructor, then $T_0 \nvdash IAtom$.*

Proof. This follows from Theorem 3.2.1 ■

Lemma 3.3.7. *If there exactly one constructor c_i and c_i is not constant, then $T_0 + IAtom \nvdash ILiteral$*

Proof. It suffices to give a model of $T_0 + IAtom$, in which SUR does not hold. W.l.o.g. assume that c_i takes input of the sort T_1. Consider the model $\mathcal{M} = \mathcal{M}^*_{\{1\}}(\{B\})$. By definition, B does not lie in the image of $c_i^{\mathcal{M}}$ and thus, $\mathcal{M} \nvDash$ SUR.

It remains to be shown that $\mathcal{M} \vDash IAtom$. Note that, by definition, there are at least two elements in the image of $c_i^{\mathcal{M}}$. By Lemma 3.3.3, $\mathcal{M} \vDash IAtom$. ■

Lemma 3.3.8. *If there is exactly one constructor c and c is constant, then $\varnothing + IAtom \vdash$ SUR.*

Proof. Consider the atom $A(X) \equiv X = c$. Clearly, $\varnothing \vdash \mathbf{LHS}(A)$ and thus, $\varnothing + IAtom \vdash \forall X : A(X)$. If $\varnothing + IAtom \vdash \forall X : X = c$, then obviously, $\varnothing + IAtom \vdash$ SUR. ■

Proof of Theorem 3.3.2. The fact that $T_0 + \mathrm{SUR} \vdash IOpen$ is the content of Theorem 3.3.1. Moreover, from Theorem 3.1.8, it follows that $T_0 + ILiteral \vdash \mathrm{SUR}$ in any case.

Case 1 is a consequence of Corollary 3.3.4 and Lemma 3.3.5.

Case 2 is a consequence of Lemma 3.3.6 and Lemma 3.3.7.

Case 3 is a consequence of Lemma 3.3.6 and Lemma 3.3.8. ■

3.3.2 Constructors and Selectors

We now have a similar main result as in the last subsection:

Theorem 3.3.9. *There are three cases:*

1. *There is exactly one constructor*
2. *There is more than one constructor and at most one constructor is non-constant*
3. *There is more than one constructor and at least two constructors are non-constant*

In case 1, we have the following Hasse-Diagram:

$$\begin{tikzcd}
T_1 + IAtom \approx T_1 + IOpen \approx T_1 + \mathrm{SUR} \arrow[d, dash] \\
T_1
\end{tikzcd}$$

In case 2, we have the following Hasse-Diagram:

$$\begin{tikzcd}
T_1 + ILiteral \approx T_1 + IOpen \approx T_1 + \mathrm{SUR} \arrow[d, dash] \\
T_1 \approx T_1 + IAtom
\end{tikzcd}$$

In case 3, we have the following Hasse-Diagram:

$$\begin{tikzcd}
T_1 + \mathit{ILiteral} \approx T_1 + \mathit{IOpen} \approx T_1 + \mathrm{SUR} \arrow[d, dash] \\
T_1 + \mathit{IAtom} \arrow[d, dash] \\
T_1
\end{tikzcd}$$

This theorem will be a direct consequence of the following lemmas.

Now we need to work out, how IAtom behaves depending on the language. We have slightly different cases than in the last subsection:

Lemma 3.3.10. *If there is exactly one constructor c_1, then $T_1 \nvdash \mathit{IAtom}$*

Proof. This follows from Theorem 3.2.9. ■

Lemma 3.3.11. *If there is exactly one constructor c_1 and c_1 is constant, then $T_1 + \mathit{IAtom} \vdash \mathrm{SUR}$.*

Proof. This follows directly from Lemma 3.3.8 since $T_0 \subseteq T_1$ and $\mathcal{L}_0 \subseteq \mathcal{L}_1$. ■

Lemma 3.3.12. *If there is exactly one constructor c_1 and c_1 is not constant, then $T_1 + \mathit{IAtom} \vdash \mathrm{SUR}$. Consequently, $T_1 + \mathit{IAtom} \vdash \mathit{ILiteral}$.*

Proof. Take any model $\mathcal{M}$ of $T_1 + \mathit{IAtom}$ and assume that there is some element $B \in \mathsf{D}^{\mathcal{M}}$, which does not lie in the image of $c_1^{\mathcal{M}}$. Consider the atom $A(X) \equiv X = c_1(d_1^1(X), d_1^2(X), \dots, d_1^{m_1}(X))$. Since every element C in the standard part of $\mathcal{M}$ has the form $c_1^{\mathcal{M}}(a_1, \dots, a_{m_1})$, it holds by INV_1^l that $(d_1^l(C))^{\mathcal{M}} = a_l$ and thus, $\mathcal{M} \vDash A(C)$. We conclude that $\mathcal{M} \vDash \mathbf{LHS}(A)$. However, $B \in \mathsf{D}^{\mathcal{M}}$ does not lie in the image of $c_1^{\mathcal{M}}$ by assumption and thus $\mathcal{M} \nvDash \forall X : A(X)$, which contradicts our assumption that $\mathcal{M} \vDash T_1 + \mathit{IAtom}$. ■

Lemma 3.3.13. *If there is more than one constructor and all constructors are constant, then $T_1 \vdash \mathit{IAtom}$.*

Proof. Note that in this case that all constructors are constant, it holds that $\mathcal{L}_0 = \mathcal{L}_1$ and $T_0 = T_1$. Thus, this Lemma is equivalent to Corollary 3.3.4. ■

Lemma 3.3.14. *If there is more than one constructor and exactly one constructor is non-constant, then $T_1 \vdash \mathit{IAtom}$.*

Proof. Assume that c_1 is the only non-constant constructor and c_2 is some constant constructor. Take any model $\mathcal{M}$ of T_1. Before we deal with atomic induction, a small observation about the interpretation of terms in this model: Let $t(X)$ be some term that contains X. If there is some subterm in t of the form $d_1^l(c_1(\overline{t'}))$, we can identify this subterm with t_l' by INV_1^l. We assume that every term, we consider has already been simplified in this manner. Thus, if t contains X and since there is only one non-constant constructor, it can have the form X, $d_1^l(X)$ or $c_1(\overline{s})$, where each of the s_i has either constant interpretation or is of the form $d_1^l(X)$.

Now consider the atom $A(X) \equiv t_1 = t_2$ and make a case distinction:

1. If X appears on neither side of A, then $t_1^{\mathcal{M}}$ and $t_2^{\mathcal{M}}$ are both constant and there is nothing to prove.

2. If X appears on exactly one side of A, say in t_1, then $t_2^{\mathcal{M}}$ is constant. There are the three aforementioned possibilities.

2.1. If $t_1 \equiv X$, then, since $t_2^{\mathcal{M}}$ is constant, either $\mathcal{M} \nvDash A(c_1(\overline{a}))$ for some tuple $\overline{a}$ with $a_j \in \tau_1^j$ or $\mathcal{M} \nvDash A(c_2)$. Thus, $\mathcal{M} \nvDash \mathbf{LHS}(A)$.

2.2. If $t_1 \equiv d_1^l(X)$, then there are two cases: If $|\tau_1^l| = 1$, then $\mathcal{M} \vDash \forall X : A(X)$ trivially. If τ_1^l has at least two elements b_1, b_2, then consider the elements $B_i = c_1(\overline{m})$, where $m_j \in \tau_1^j$ and $m_l = b_i$. Then, because $t_2^{\mathcal{M}}$ is constant, $\mathcal{M} \nvDash A(B_1) \wedge A(B_2)$. Thus $\mathcal{M} \nvDash \mathbf{LHS}(A)$.

2.3. If $t_1 \equiv c_1(d_1^l(X), \overline{m})$, then we make the same case distinction: If there is τ_1^l has exactly one element, then replace $d_1^l(X)$ with some parameter and repeat the procedure from the top. If τ_1^l has at least two elements b_1, b_2, then construct the elements $B_i = c_1(\overline{m})$, where $m_j \in \tau_1^j$ and $m_l = b_i$. From INJ_1, it follows that $t_1(B_1) \neq t_1(B_2)$ and thus, $\mathcal{M} \nvDash A(B_1) \wedge A(B_2)$. Therefore, $\mathcal{M} \nvDash \mathbf{LHS}(A)$.

3. Assume that X appears on both sides of A. There are again three cases:

3.1. Assume $t_1 \equiv X$. If $t_2 \equiv d_1^l(X)$, then the atom A is not well-formed. If $t_2 \equiv c_1(\overline{s})$, then $\mathcal{M} \nvDash A(c_2)$ by $D_{1,2}$. Thus, if $\mathcal{M} \nvDash \mathbf{LHS}(A)$, we have that $t_2 \equiv X$.

3.2. Assume that $t_1 \equiv d_1^l(X)$. Then, there are two cases: If c_1 takes the input τ_1^l exactly once, then $t_2 \equiv d_1^l(X)$ for A to be well-formed. In that case, $\mathcal{M} \vDash \forall X : A(X)$. If c_1 takes the input τ_1^l more than once, then it is possible that $t_2 \equiv d_1^n(X)$, where $\tau_1^n = \tau_1^l$. There are two options: If $|\tau_1^l| = 1$, then $\mathcal{M} \vDash \forall X : A(X)$ trivially. If τ_1^l has at least two elements b_1, b_2, construct the element $B = c_1(\overline{m})$, where $m_j \in \tau_1^j$, $m_l = b_1$ and $m_n = b_2$. Then $\mathcal{M} \nvDash A(B)$ and thus, $\mathcal{M} \nvDash \mathbf{LHS}(A)$.

3.3. Assume that $t_1 \equiv c_1(\overline{s})$. From the cases 3.1 and 3.2, we conclude that $t_2 \equiv c_1(\overline{r})$. If $\mathcal{M} \vDash \mathbf{LHS}(A)$, then $\mathcal{M} \vDash \mathbf{LHS}(s_i = r_i)$ for any $i \in \{1, \ldots, m_1\}$ since we have static constructors only. From the cases 3.1 and 3.2, we obtain that $\mathcal{M} \vDash \forall X : s_i(X) = r_i(X)$ and thus, $\mathcal{M} \vDash \forall X : A(X)$. ■

Lemma 3.3.15. *If there is more than one constructor, at least two constructors are non-constant and no two constructors take input of the same sort, then $T_1 \nvDash IAtom$*

Proof. Fix the two constructors c_i, c_j and assume, w.l.o.g., that $\tau_i^1 = T_1$ and $\tau_j^1 = T_2$. Consider the following $\mathcal{M} = \mathcal{M}^*_{\{1,2\}}(\{B\})$ of T_0 over the language $\mathcal{L}_0$. We extend the language with the selectors and interpret them in the following way:

- For any constructor d_l^n, where $\tau_l^n \notin \{T_1, T_2\}$, there is only one element in τ_l^n, so there is only one way to interpret $d_l^n(A)$ for any element $A \in \mathsf{D}^{\mathcal{M}}$
- If $A = c_i^{\mathcal{M}}(m_1, \dots, m_{m_i})$, then $(d_i^1(A))^{\mathcal{M}} = m_1$.
- If $A = c_j^{\mathcal{M}}(m_1, \dots, m_{m_j})$, then $(d_j^1(A))^{\mathcal{M}} = m_1$
- For any standard element A, define $(d_i^1)^{\mathcal{M}}$ and $(d_j^1)^{\mathcal{M}}$ s.t. $(d_i^1(A))^{\mathcal{M}} = a_1$ iff $(d_j^1(A))^{\mathcal{M}} = a_2$, where $T_1 = \{a_1, b_1\}$ and $T_2 = \{a_2, b_2\}$.
- For the only non-standard element B, define $(d_i^1(B))^{\mathcal{M}} = a_1$ and $(d_j^1(B))^{\mathcal{M}} = b_2$

Note that no two selectors share any sort of their input and thus, this interpretation covers any case. Moreover, the axiom INV_l^n is satisfied for any $l \in \{1, \dots, k\}$ and $n \in \{1, \dots, m_l\}$.

Now consider the atom $A(X) \equiv d_i^1(X) = d_i^1(c_j(d_j^1(X), \overline{m}))$ for some appropriate tuple $\overline{m}$. By the definition above, we have that $A(C)$ holds iff $d_i^1(C) = a_1 \Leftrightarrow d_j^1(C) = a_2$. By construction, this holds for the whole standard part of the model and thus, $\mathcal{M} \vDash \mathbf{LHS}(A)$. However, $d_i^1(B) = a_1$ and $d_j^1(B) = b_1$. Thus, $\mathcal{M} \nvDash \forall X : A(X)$. ■

Lemma 3.3.16. *If there are at least two constructors and two of them share some input sort, then $T_1 \nvdash IAtom$*

Proof. Fix the two constructors c_1 and c_2 that share some input sort. W.l.o.g., we can assume that they share the sort T_1 and both of them take it as the first input.

Consider the model $\mathcal{M} = \mathcal{M}^*_{\{1\}}(\{B\})$ of T_0 over the language $\mathcal{L}_0$. We extend the language with the selectors and interpret them in the following way:

- For any selector d_i^l that maps into some sort T_j other than T_1 there is only one possible interpretation for terms of the form $d_i^l(t')$
- If $C \in \mathsf{D}^{\mathcal{M}}$ has the form $c_i^{\mathcal{M}}(m_1, \dots, m_{m_i})$ with $\tau_i^l = T_1$, then $(d_i^l(C))^{\mathcal{M}} = m_l$
- For any standard element C and we interpret $d_1^1(C)$ and $d_2^1(C)$ to coincide
- For B, we interpret $d_1^1(B) = a_1$ and $d_2^1(B) = b_1$, where $T_1 = \{a_1, b_1\}$
- If some case was not covered above, define $d_i^l(C)$ arbitrarily

Note that $\mathcal{M} \models T_1$.

Consider the atom $A(X) \equiv d_1^1(X) = d_2^1(X)$. Then, $\mathcal{M} \models \mathbf{LHS}(A)$, but $\mathcal{M} \not\models \forall X : A(X)$. ■

Lemma 3.3.17. *If there is more than one constructor, then $T_1 + IAtom \nvdash ILiteral$*

Proof. Since $T_1 + ILiteral \vdash \mathrm{SUR}$, this follows from Theorem 3.2.9. ■

Proof of Theorem 3.3.9. The fact that $T_0 + \mathrm{SUR} \vdash IOpen$ is the content of Theorem 3.3.1. Moreover, from Theorem 3.1.8, it follows that $T_1 + ILiteral \vdash \mathrm{SUR}$ in any case.

Case 1 follows from Lemma 3.3.10, Lemma 3.3.11 and Lemma 3.3.12.

Case 2 follows from Lemma 3.3.13, Lemma 3.3.14 and Lemma 3.3.17.

Case 3 follows from Lemma 3.3.15, Lemma 3.3.16 and Lemma 3.3.17. ■

3.4 Languages with Dynamic Constructors

In the following section, we will restrict ourselves to the case, where there is at least one constructor that does take some input of sort D. This is arguably the more interesting case as the induction is now more than just the base case.

3.4.1 Constructors only

First we need some definitions:

Definition 3.4.1. *The D-depth $d(t)$ of a term t is defined inductively: For any static constructor c_j and variables X we define $d(c_j(\overline{y})) = d(X) = 0$. For any dynamic constructor c_l, we define $d(c_l(t_1, \dots, t_n)) = 1 + \max\{d(t_1), \dots, d(t_n)\}$.*

Definition 3.4.2. *Consider the set of all terms of D-depth $n \geqslant 1$. We define the set M_n by taking all the terms t with D-depth n that satisfy the following properties:*

- *One of the variables of sort D is X*
- *Every variable is used exactly once*
- *Any subterm that does not contain X is a variable*

Now consider the binary relation $\sim$ on the set M_n defined by $t \sim s$ if $t \equiv s$ up to renaming of variables and define the set $M_n' = (M_n)_{/\sim}$. Note that M_n' is finite. We define the set S_n by choosing a representative for each class in M_n' and $\mathcal{S} = \bigcup_{n \geqslant 1} S_n$.

Now we can define a new set of axioms

$\mathbf{G_t}$ $X \neq t(X)$ for all $t \in \mathcal{S}$ (Acyclicity)

The following will be the main result of this section.

Theorem 3.4.3. *We have the following cases:*

1. *There is only one dynamic constructor and this dynamic constructor is unary*
2. *There is more than one dynamic constructor or there is exactly one dynamic constructor and this constructor is not unary*
    a) *For every dynamic constructor c_i, there is exactly one l with $\tau_i^l = \mathsf{D}$*
    b) *There is some dynamic constructor c_i with $l_1 \neq l_2$ and $\tau_i^{l_1} = \tau_i^{l_2} = \mathsf{D}$*

In case 1, we obtain the following Hasse-Diagram:

$$\begin{tikzcd}
T_0 + \mathit{ILiteral} \approx T_0 + \mathit{IOpen} \approx T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \arrow[d, dash] \\
T_0 \approx T_0 + \mathit{IAtom}
\end{tikzcd}$$

In case 2a, we obtain the following Hasse Diagram:

$$\begin{tikzcd}
T_0 + \mathit{IDClause} \approx T_0 + \mathit{IOpen} \approx T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \arrow[d, dash] \\
T_0 + \mathit{ILiteral} \approx T_0 + \mathit{IClause} \arrow[d, dash] \\
T_0 \approx T_0 + \mathit{IAtom}
\end{tikzcd}$$

In case 2b, we obtain the following Hasse Diagram:

$$\begin{tikzcd}
T_0 + \mathit{IDClause} \approx T_0 + \mathit{IOpen} \approx T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \arrow[d, dash] \\
T_0 + \mathit{IClause} \arrow[d, dash, dashed] \\
T_0 + \mathit{ILiteral} \arrow[d, dash] \\
T_0 \approx T_0 + \mathit{IAtom}
\end{tikzcd}$$

This theorem will a consequence of the following lemmas:

Lemma 3.4.4. $T_0 \vdash \mathit{IAtom}$

Proof. This follows directly from Lemma 3.2.5 since there are at least two constructors in our language. ■

Lemma 3.4.5. $T_0 + \mathit{IAtom} \nvdash \mathit{ILiteral}$

Proof. Since $T_0 + \mathit{ILiteral} \vdash \mathrm{SUR}$, this follows directly from Theorem 3.2.1. ■

Definition 3.4.6. *Let $\mathcal{M}$ be any model of T_0. A finite sequence $(B_0, \dots, B_n) \in (\mathsf{D}^{\mathcal{M}})^{n+1}$ is called a cycle if $B_0 = B_n$ and for all $l \in \{1, \dots, n\}$, there is some term $t_l(X) \equiv c_{j_l}(s_1^l, \dots, s_{m_{j_l}}^l)$ s.t. $\mathcal{M} \vDash B_l = t_l(B_{l-1})$ and at least one of the s_k^l is identical to X. A cycle C can have two types:*

1. *There is some term $t_C(X) \equiv c_i(s_1, \dots, s_{m_i})$ where at least one of the $s_l \equiv X$ - X may appear in other s_m as well - s.t. $\mathcal{M} \vDash B_{i+1} = t_C(B_i)$ for any $B_i \in C$, $i \leqslant n-1$.*
2. *There is no such term*

Lemma 3.4.7. *Let $\mathcal{M}$ be any model of $T_0 + \mathit{ILiteral}$ and $C = (B_0, \dots, B_n)$ a cycle in $\mathcal{M}$. Then C is not of type 1.*

Proof. Assume that C is of type 1. Then there is a term $t_C \equiv c_i(s_1, \dots, s_{m_i})$ s.t. $\mathcal{M} \vDash B_{i+1} = t_C(B_i)$. Consider the sequence of terms $t_1, \dots, t_n$, where $t_1 \equiv t_C$ and $t_{i+1} \equiv t_C(t_i)$. Then $B_i = t_i(B_0)$ and $B_0 = t_n(B_0)$. Consider the literal $L(X) \equiv X \neq t_n(X)$. Clearly, $\mathcal{M} \vDash L(c_j(\overline{a}))$ for any static constructor c_j and tuple $\overline{a}$. Assume that $\mathcal{M} \nvDash L(A)$ for some element A. Then, $A = c_i^{\mathcal{M}}(A_1, \dots, A_k, a_{k+1}, \dots, a_{m_i}) = c_i^{\mathcal{M}}(s_1(t_{n-1}, \dots, s_{m_i}(t_{n-1}))$ for some elements A_l, a_j. By INJ_i, $A_l = s_l^{\mathcal{M}}(t_{n-1}(A))$. By assumption, there is one $s_l \equiv X$ and thus, there is one $A_l = t_{n-1}^{\mathcal{M}}(A)$. By construction of t_C, $A = t_C(t_{n-1}(A)) = t_C(A_l)$. Thus, $A_l = t_{n-1}(t_C(A_l)) = t_n(A_l)$. Thus, $\mathcal{M} \vDash \mathbf{LHS}(L)$. By $ILiteral$, $\mathcal{M} \vDash \forall X : A(X)$. This, however, contradicts the assumption that such a cycle C exists. ■

Lemma 3.4.8. *Assume that every dynamic constructor takes the sort* D *as input exactly once. Let* $\mathcal{M}$ *be a model of* $T_0 + ILiteral$ *and* $C = (B_1, \dots, B_n)$ *a cycle in* $\mathcal{M}$*. If* $\mathcal{M} \vDash B_i = t(B_i)$ *for some term* $t \not\equiv X$ *that contains* X*, then* $\mathcal{M} \nvDash B_l = t(B_l)$ *for some* B_l*.*

Proof. By Lemma 3.4.7, C is not of type 1. In particular, there is no term $s = c_l(X, \overline{a})$ s.t. $\mathcal{M} \vDash B_i = s(B_{i-1})$ for all B_i. Now take any term $t \not\equiv X$ containing X with $\mathcal{M} \vDash B_i = t(B_i)$. t has the form $c_i(X, \overline{b})$. There is some B_l s.t. $B_l = c_j(B_{l-1}, \overline{e})$, where either $j \neq i$ or $\overline{b} \neq \overline{e}$. From $D_{i,j}$ or INJ_i, it follows that $\mathcal{M} \nvDash B_l = t(B_l)$. ■

Lemma 3.4.9. *If every dynamic constructor takes the sort* D *as input exactly once, then* $T_0 + ILiteral \vdash IClause$*.*

Proof. Note that by Theorem 3.1.8, we have that $T_0 + ILiteral \vdash \mathrm{SUR}$.

Now take any model $\mathcal{M}$ of $T_0 + ILiteral$ and some clause $C(X) \equiv L_1 \vee \cdots \vee L_n$ and assume that $\mathcal{M} \vDash \mathbf{LHS}(C)$. We start by preprocessing C:

Assume that there is some literal L_i that does not contain X. Then, $\mathcal{M} \vDash L_i \leftrightarrow \top$ or $\mathcal{M} \vDash L_i \leftrightarrow \bot$. In the first case, $\mathcal{M} \vDash C \leftrightarrow \top$. In the second case, $\mathcal{M} \vDash C \leftrightarrow C'$, where C' is obtained from C by deleting L_i. Thus, we can assume that every literal contains X.

Assume that there is some negated atom $L_i \equiv t_1 \neq t_2$ and both t_l have the outermost function symbol c_i. Then both t_l are of the form $c_i(t_l^1, \dots, t_l^{m_i})$. By INJ_i, $\mathcal{M} \vDash L_i \leftrightarrow \bigvee_{k=1}^{m_i} t_1^k \neq t_2^k$. Thus, we can replace L_i in C with the clause $\bigvee_{k=1}^{m_i} t_1^k \neq t_2^k$ and obtain another equivalent clause.

Assume that all the L_i are atoms. Since the standard part of $\mathcal{M}$ is infinite, there is some L_i and distinct standard elements $B, C \in \mathsf{D}^{\mathcal{M}}$ s.t. $\mathcal{M} \vDash L_i(B) \wedge L_i(C)$. By Lemma 3.2.5, $\mathcal{M} \vDash \forall X : L_i(X)$ and thus, $\mathcal{M} \vDash \forall X : C(X)$. In this case, we are done.

From now on, we assume that at least one of the $L_i \equiv t \neq s$ is a negated atom. W.l.o.g., assume that t_1 contains X.

We make a case distinction:

1. The case that both t and s share the same outermost function symbol is eliminated by our preprocessing.

2. If the outermost function symbols of t and s are c_i and c_j respectively with $i \neq j$, then $\mathcal{M} \vDash \forall X : L_i(X)$ and thus, $\mathcal{M} \vDash \forall X : C(X)$. In this case, we are done.

3. If $t \equiv X$ and $t_2^{\mathcal{M}}$ is constant with some fixed interpretation B_Y, then $\mathcal{M} \vDash L_i(B)$ for any $B \neq B_Y$. There are two cases: If B_Y is a standard element, then there has to be

some L_j with $\mathcal{M} \vDash L_j(B_Y)$ by assumption. If B_Y is a non-standard element, then by SUR, there is some dynamic constructor c_i and elements $B_1, \ldots, B_l, b_{l+1}, \ldots, b_{m_i}$ where all the B_j are of sort D and $B_Y = c_i^{\mathcal{M}}(B_1, \ldots, b_{m_i})$. As mentioned, $\mathcal{M} \vDash L_i(B_j)$ and thus, $\mathcal{M} \vDash C(B_j)$ for any j. Since $\mathcal{M} \vDash \mathbf{LHS}(C)$, we conclude that $\mathcal{M} \vDash C(B_Y)$. Thus, in any case, $\mathcal{M} \vDash \forall X : C(X)$.

4. Assume that $t \equiv X$ and s contains X. If $s \equiv X$, then $\mathcal{M} \vDash \forall X : \neg L_i(X)$ and $\mathcal{M} \vDash C \leftrightarrow C'$, where C' is obtained from C by deleting L_i. In that case, we can start from the top with C'. Thus, we assume that $s \not\equiv X$. Assume that there is some $B_0 \in \mathsf{D}^{\mathcal{M}}$ with $\mathcal{M} \nvDash L(B)$. Then B_0 lies in a cycle $E = (B_0, \ldots, B_n)$. By Lemma 3.4.8, there is some B_l with $\mathcal{M} \vDash L(B_l)$ and thus, $\mathcal{M} \vDash C(B_l)$. Since every constructor takes the sort D exactly once as input and $\mathcal{M} \vDash \mathbf{LHS}(C)$, we obtain that $\mathcal{M} \vDash L(B_{l+1})$ and inductively for all B_i in E. Thus, $\mathcal{M} \vDash \forall X : C(X)$. ■

We have not succeeded in proving that $T_0 + \mathit{ILiteral} \vdash \mathit{IClause}$ if there is an constructor that takes the sort D as input twice. This leads to the following open problem:

Open Problem 3.4.10. *Assume that there is some dynamic constructor c_i s.t. there are $l_1 \neq l_2$ with $\tau_i^{l_1} = \tau_i^{l_2} = \mathsf{D}$. Does it hold that $T_0 + \mathit{ILiteral} \vdash \mathit{IClause}$?*

Building up on the definitions of S_n and $\mathcal{S}$, we need a new definition:

Definition 3.4.11. *For this definition, fix $\mathsf{D} = T_0$. Fix some $k \geqslant 1$. Define $Q_i^k = \{y \mid$ the variable y is of sort T_i and occurs in some term in $S_k\}$ for $i \geqslant 0$. Let $\mathcal{F}_i^k$ be the set of functions $f : Q_i^k \to Q_i^k$. Note that every $\mathcal{F}_i^k$ is finite since S_k is finite. Moreover, for any variable x, let $i(x)$ be the i s.t. $x \in T_i$. For any term $t(x_1, \ldots, x_n)$ assume that the variables of t are fully indicated. Define the set $R_k = \{t[x_1/f_1(x_1), \ldots, x_n/f_n(x_n)] \mid t \in S_n, f_i \in F_{i(x_i)}^k\}$. Again, R_k is finite.*

Lemma 3.4.12. *$\{G_t \mid t \in \mathcal{S}\} \vdash X \neq t(X)$ for any term t, which contains X, but is not identical to X.*

Proof. Take any model $\mathcal{M}$ of $\{G_t \mid t \in \mathcal{S}\}$. Assume that there is some term t, which contains X, but is not identical to X and some element $A \in \mathsf{D}^{\mathcal{M}}$ s.t. $\mathcal{M} \vDash A = t(A)$. Now define the term s by picking one occurrence of X in t and changing every other occurrence to a different, unique parameter. Define s' by changing every subterm of s, which does not contain X to some new parameter. Modulo renaming $s' \in \mathcal{S}$ and by interpreting the fresh parameters accordingly, we obtain that $\mathcal{M} \vDash A = s'(A)$. ■

Lemma 3.4.13. *$T_0 + \mathit{IDClause} \vdash G_t$ for any $t \in \mathcal{S}$.*

Proof. Take any model $\mathcal{M}$ of $T_0 + \mathit{IDClause}$ and any $k \geqslant 1$. Define the dual clause $D(X) \equiv \bigwedge_{t \in R_k} X \neq t(X)$. We show that $\mathcal{M} \vDash \forall X : D(X)$. By the definition of $\mathcal{S}$, the claim then follows directly.

It is clear that $\mathcal{M} \vDash D(c_j(\overline{a}))$ for any static constructor c_j and tuple of elements $\overline{a}$. Now, assume that there is some element A s.t. $\mathcal{M} \nvDash D(A)$. Then there is a term $t \in R_k$ with $\mathcal{M} \vDash A = t(A)$. Since $d(t) \geqslant 1$, t has the form $c_i(s_1, \ldots, s_{m_i})$ and all s_l except for one are parameters. W.l.o.g., assume that s_1 is not a parameter and all $s_l \equiv a_k$ for $l \geqslant 2$ are parameters occurring in the respective Q_i^k. It follows that $A = c_i(A_1, a_2, \ldots, a_{m_i})$ for some element A_1. From INJ_i, it follows that $A_1 = s_1(A) = s_1(c_i(A_1, a_2, \ldots, a_{m_i}))$. Note that $d(t) = 1 + \max\{s_1, \ldots, s_{m_i}\} = 1 + \max\{s_1, 0, \ldots, 0\} = 1 + s_1$. Thus, and because any subterm of s_1, which does not contain X is a parameter and has D-depth 0, the term $s(X) \equiv s_1(c_1(X, a_2, \ldots, a_{m_i}))$ satisfies that $d(s) = 1 + d(s_1) = d(t)$. Since all the a_l are in the respective Q_i^k, it follows that $s \in R_k$. Thus, $\mathcal{M} \nvDash D(A_1)$ and hence, $\mathcal{M} \vDash \mathbf{LHS}(D)$. Since $\mathcal{M} \vDash IDClause$, it follows that $\mathcal{M} \vDash \forall X : D(X)$. ■

Lemma 3.4.14. *If there is only one dynamic constructor c_1 and c_1 is unary, then $T_0 + ILiteral \vdash G_t$ for any $t \in \mathcal{S}$.*

Proof. Since there is only one dynamic constructor c_1 and c_1 is unary, any term t of D-depth k, which contains X has the form $c_1^k(X)$. Define the literal $L(X) \equiv X \neq c^n(X)$ and take any model $\mathcal{M}$ of $T_0 + ILiteral$. Clearly, $\mathcal{M} \vDash L(c_j(\overline{a}))$ for any static constructor c_j and tuple of elements $\overline{a}$. Assume that there is some $A \in \mathsf{D}^{\mathcal{M}}$ with $\mathcal{M} \nvDash L(A)$. Then, $A = (c_1^n(A))^{\mathcal{M}} = (c_1(c_1^{n-1}(A)))^{\mathcal{M}} = (c_1(B))^{\mathcal{M}}$ for some B. By INJ_1, it follows that $B = (c_1^{n-1}(A))^{\mathcal{M}} = (c_1^{n-1}(c_1(B)))^{\mathcal{M}} = (c_1^n(B))^{\mathcal{M}}$. Thus, $\mathcal{M} \nvDash L(B)$ and hence, $\mathcal{M} \vDash \mathbf{LHS}(L)$. Since $\mathcal{M} \vDash ILiteral$, $\mathcal{M} \vDash \forall X : L(X)$. Since $S_k = R_k = \{c_1^n(X)\}$, this proves the claim. ■

Lemma 3.4.15. *If there is more than one dynamic constructor or there is one dynamic constructor, which is not unary, then $T_0 + IClause \nvdash IDClause$.*

Proof. We need to give a model $\mathcal{M}$ of $T_0 + IClause$ in which there is some cycle. There are two mutually exclusive cases:

1. If there is some non-unary dynamic constructor, then by Lemma 3.2.18, the model $\mathcal{M}_C$ does exactly what we want.

2. If there is more than one dynamic constructor and every dynamic constructor is unary, then $T_0 + ILitera \vdash IClause$. It suffices to give a model $\mathcal{M}$ of $T_0 + ILiteral$. By Lemma 3.2.21, the model $\mathcal{M}_C^u$ does exactly what we want. ■

Lemma 3.4.16. *Let $A(X) \equiv t_1 = t_2$ be an atom. Take any model $\mathcal{M}$ of $T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\}$, where the sort D is interpreted as the set M. Then the set $S = \{X \in M \mid \mathcal{M} \vDash A(X)\}$ is either empty, has cardinality 1 or is equal to M.*

Proof. Take any model $\mathcal{M}$ of $T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\}$ and any atom $A(X) \equiv t_1 = t_2$.

There are three cases:

1. If neither side contains X, then S is either empty or equal to M.

2. If only one term, say t_1, contains X, then $t_2^{\mathcal{M}}$ is constant. We show inductively that there is at most one $B \in \mathsf{D}^{\mathcal{M}}$ with $\mathcal{M} \vDash A(B)$. If $t_1 \equiv X$, the claim holds trivially. Assume that $t_1 \equiv c_1(s_1, \ldots, s_{m_i})$. If there is some element B with $\mathcal{M} \vDash t_1(B) = t_2$, then there are elements $A_1, \ldots, A_j, a_{j+1}, \ldots a_{m_i}$ with $t_2^M = c_i^{\mathcal{M}}(A_1, \ldots, a_{m_i})$. At least one of the s_l has to contain X and from the induction hypothesis it follows that there is at most one B s.t. $\mathcal{M} \vDash s_l(B) = a_l$. From INJ_i, it follows that there is at most one B s.t. $\mathcal{M} \vDash t_1(B) = t_2$.

3. Assume that both t_1 and t_2 contain X. We proceed by induction on the structure of t_1:

3.1. Assume that $t_1 \equiv X$. If $t_2 \not\equiv X$, then $S = \varnothing$ by G_{t_2}. If $t_2 \equiv X$, then $S = M$.

3.2. Assume that $t_1 = c_1(s_1, \ldots, s_{m_i})$. If there is some element B with $\mathcal{M} \vDash t_1(B) = t_2(B)$, then t_2 has the form $c_i(r_1, \ldots, r_{m_i})$ by $D_{i,j}$, $i \neq j$. From INJ_i, it follows that $S = \bigcap_{k=1}^{m_i} S_k$ with $S_k = \{B \in \mathsf{D}^{\mathcal{M}} \mid \mathcal{M} \vDash s_k(B) = r_k(B)\}$. From the induction hypothesis and the cases 1 and 2, it follows that every S_k is either empty, has cardinality 1 or is equal to M. Thus, as the intersection of the S_k, S has this property as well. ■

Lemma 3.4.17. *Let M, N be two finite or cofinite sets. Then $M \cap N$ and $M \cup N$ are also finite or co-finite.*

Proof. Since a set is finite or cofinite iff its complement is and $(M \cup N)^c = M^c \cap N^c$, it suffices to deal with one of the two cases.

Consider $M \cup N$ and assume that it is infinite. Then at least one of the two sets has to infinite and, therefore, co-finite, w.l.o.g. assume that M is co-finite. Since $(M \cup N)^c \subseteq M^c$, it follows that also $M \cup N$ has to be co-finite. ■

Lemma 3.4.18. *Let $F(X)$ be an open formula with $X \in \mathsf{D}$. In any model $\mathcal{M}$ of $T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\}$, it holds that $S_F = \{X \in D^{\mathcal{M}} \mid \mathcal{M} \vDash F(X)\}$ is finite or co-finite.*

Proof. We proceed inductively: The base is that F is an atom. From Lemma 3.4.16 it follows that S is finite or cofinite in this case. If $F \equiv \neg G$, then $S_F = S_G^c$ and thus, S_F is finite or cofinite if S_G is. If $F \equiv G_1 \vee G_2$ or $F \equiv G_1 \wedge G_2$, then it follows from Lemma 3.4.17 that S_F is finite or cofinite if both S_{G_1} and S_{G_2} are. ■

Theorem 3.4.19. $T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \vdash$ *IOpen*

Proof. Take any model $\mathcal{M}$ of $T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\}$ and any open formula $F(X)$. Assume that $\mathcal{M} \vDash \mathbf{LHS}(F)$. By Lemma 3.4.18, the set S_F is finite or cofinite. Since the standard part of $\mathcal{M}$ is infinite and $\mathcal{M} \vDash F(B)$ for any element $B \in \mathsf{D}^{\mathcal{M}}$, S_F is cofinite and $|S_F^c| = n \in \mathbb{N}$. Now define a partial function $p : \mathsf{D}^{\mathcal{M}} \hookrightarrow \mathsf{D}^{\mathcal{M}}$ on the set S_F^c. Let B be any element in S_F^c. Since $\mathcal{M} \vDash \mathbf{LHS}(F)$, B cannot be a standard element and, by SUR, there is some dynamic constructor c_i with elements $B_1, \ldots, B_j, b_{j+1}, \ldots, b_{m_i}$ s.t. $B = c_i^M(B_1, \ldots, b_{m_i})$. For at least one of the B_l it holds that $\mathcal{M} \nvDash F(B_l)$. Define $p(B)$ by picking any such B_l with $\mathcal{M} \nvDash F(B_l)$. Assume that S_F^c is not empty and that there is some B with $\mathcal{M} \nvDash F(B)$. Then, $\mathcal{M} \nvDash F(p^l(B))$ for any $l \in \mathbb{N}$. In particular, $\mathcal{M} \nvDash F(p^n(B))$. Since $|S_F^c| = n$, there has to be some $E \in S_F^c$ and $k \leqslant n$ with $p^l(E) = E$. By definition of p, this implies that there is a term t with $\mathcal{M} \vDash E = t(E)$, which contradicts the assumption that $\mathcal{M} \vDash \{G_t \mid t \in \mathcal{S}\}$. Thus, $S_F^c = \varnothing$ and $\mathcal{M} \vDash \forall X : F(X)$. ■

Proof of Theorem 3.4.3. In all three cases, it follows from Lemma 3.4.4 that $T_0 \vdash IAtom$, from Lemma 3.4.5 that $T_0 + IAtom \nvdash ILiteral$ and from Theorem 3.4.19 that $T_0 + \mathrm{SUR} + \{G_t \mid t \in S_n\} \vdash IOpen$.

In case 1, it follows from Lemma 3.4.14 that $T_0 + ILiteral \vdash IOpen$.

In both cases 2a and 2b, it follows from Lemma 3.4.15 that $T_0 + IClause \nvdash IDClause$.

In case 2a, it follows from Lemma 3.4.9 that $ILiteral \vdash IClause$. ■

3.4.2 Constructors and Selectors

The following will be our main result:

Theorem 3.4.20. *The are two cases:*

1. *There is exactly one dynamic constructor and all other constructors are constants.*
2. *There is more than one dynamic constructor or some static constructor, which is not constant.*

In the first case, we obtain he following (partial) Hasse-Diagram:

$$\begin{tikzcd}
& T_1 + IOpen \arrow[dl, dashed, no head] \arrow[dr, dashed, no head] & \\
T_1 + IDClause \arrow[dr, dashed, no head] & & T_1 + IClause \arrow[dl, dashed, no head] \\
& T_1 + ILiteral \arrow[d, no head] & \\
& T_1 \approx T_1 + IAtom &
\end{tikzcd}$$

In the second case, we obtained the following (partial) Hasse-Diagram:

$$\begin{tikzcd}
 & T_1 + \mathit{IOpen} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
T_1 + \mathit{IDClause} \arrow[dr, dash, dashed] & & T_1 + \mathit{IClause} \arrow[dl, dash, dashed] \\
 & T_1 + \mathit{ILiteral} \arrow[d, dash] & \\
 & T_1 + \mathit{IAtom} \arrow[d, dash] & \\
 & T_1 &
\end{tikzcd}$$

This will be a direct consequence of the following lemmas:

Lemma 3.4.21. $T_1 + \mathit{ILiteral} \vdash \mathrm{SUR}$

Proof. This follows directly from Theorem 3.1.8. ■

Lemma 3.4.22. $T_1 + \mathit{IDClause} \vdash G_t$ *for any term* $t \in \mathcal{S}$*.*

Proof. This follows directly from Lemma 3.4.13 since $T_1 \supseteq T_0$. ■

Lemma 3.4.23. *If there is only one dynamic constructor* c *and* c *is unary, then* $T_1 + \mathit{ILiteral} \vdash G_t$ *for any term* t *in* S_n *with* $n \geqslant 1$

Proof. This follows directly from Lemma 3.4.14 because $T_1 \supseteq T_0$. ■

Lemma 3.4.24. *If there are at least two dynamic constructors, then* $T_1 \nvdash \mathit{IAtom}$

Proof. Fix two dynamic constructors c_1, c_2 and one selector $d_2^1 : D \to \mathsf{D}$ of d_2^1. Now start with the model $\mathcal{M} = \mathcal{M}^*(\{\infty\})$ of T_0 over language $\mathcal{L}_0$. We need to interpret the selectors. For this, we fix two distinct elements $B_1, B_2 \in \mathsf{D}^{\mathcal{M}}$ and proceed with a case distinction:

- If $B \in \mathsf{D}^{\mathcal{M}}$ can be written as $c_i^{\mathcal{M}}(\bar{b})$, then define $(d_i^k(B))^{\mathcal{M}} = b_k$
- If $B = c_1^{\mathcal{M}}(\bar{b})$ and B is a standard element, then define $(d_2^1(B))^{\mathcal{M}} = B_1$

- If $B = c_1^{\mathcal{M}}(\bar{b})$ and B is a non-standard element, then define $(d_2^1(B))^{\mathcal{M}} = B_2$
- If none of the previous cases is applicable and $\tau_i^k \neq \mathsf{D}$, then there is only one possible interpretation for $d_i^k(B)$
- If none of the previous cases is applicable and $\tau_i^k = \mathsf{D}$, then define $(d_i^k(B))^{\mathcal{M}} = B$

Note that by construction $\mathcal{M} \models T_1$.

Consider the atom $A(X) \equiv d_2^1(c_1(X)) = Z$ and the interpretation $\xi : Z \mapsto B_1$. By construction $\mathcal{M} \models A(B)$ iff B is a standard element. Thus, $\mathcal{M} \models \mathbf{LHS}(A)$, but $\mathcal{M} \not\models \forall X : A(X)$. ■

Lemma 3.4.25. *If there is exactly one dynamic constructor, but there is some non-constant static constructor, then $T_1 \nvdash IAtom$.*

Proof. Fix the dynamic constructor c_1 and the non-constant static constructor c_2. W.l.o.g., assume that $\tau_2^1 = T_1$ and thus, $d_2^1 : D \to T_1$. Now take the model $\mathcal{M} = \mathcal{M}_{\{1\}}(\{\infty\})$ of T_0 over the language $\mathcal{L}_0$. We need to interpret the selectors. For this, fix the elements a_1, b_1 with $T_1^{\mathcal{M}} = \{a_1, b_1\}$ and proceed with a case distinction:

- If $B \in \mathsf{D}^{\mathcal{M}}$ can be written as $c_i^{\mathcal{M}}(\bar{b})$, then define $(d_i^k(B))^{\mathcal{M}} = b_k$
- If $B = c_1^{\mathcal{M}}(\bar{b})$ and B is a standard element, then define $(d_2^1(B))^{\mathcal{M}} = a_1$
- If $B = c_1^{\mathcal{M}}(\bar{b})$ and B is a standard element, then define $(d_2^1(B))^{\mathcal{M}} = b_1$
- If none of the previous cases is applicable and $\tau_i^k = T_1$, then define $(d_i^k(B))^{\mathcal{M}} = a_1$
- If none of the previous cases is applicable and $\tau_i^k = \mathsf{D}$, then define $(d_i^k(B))^{\mathcal{M}} = B$
- If none of the previous cases is applicable and $\tau_i^k \notin \{\mathsf{D}, T_1\}$, then there is only one possible interpretation for $d_i^k(B)$

Note that by construction $\mathcal{M} \models T_1$.

Consider the atom $A(X) \equiv d_2^1(c_1(X)) = z$ and the interpretation $\xi : z \mapsto a_1$. By construction $\mathcal{M} \models A(B)$ iff B is a standard element. Thus, $\mathcal{M} \models \mathbf{LHS}(A)$, but $\mathcal{M} \not\models \forall X : A(X)$. ■

Remark 3.4.26. *Prior to this subsection, our standard approach to prove that e.g. $T + IAtom \nvdash ILiteral$, was to find some formula φ with $T + ILiteral \vdash \varphi$ and show that $T + IAtom \nvdash \varphi$. In the current setting it is much more difficult to use this approach due to the added complexity of the selectors in combination with dynamic constructors. However, there is some other approach, we used in two previous Lemmas that we think could be fruitful if someone decides to try to find solutions to the questions that we did not manage to answer: If there are at least two constructors c_1 and c_2 that take some* *input, then the axioms of T_1 make no statement about how the selectors of c_1 should be interpreted on elements of the form $c_2^{\mathcal{M}}(\dots)$ in some model $\mathcal{M}$ of T_1. Using this, one might be able to define a model with different patterns that prove that some level of induction is stronger than some other level. Moreover, in this case, it seems very difficult to give a simple alternative axiomatization of open induction and after spending some thought on this, we think that it might not even be possible.*

This leads to the following conjecture:

Conjecture 3.4.27. *If there is at least one dynamic constructor and some other constructor that takes input of some sort, then $T_1 + ILiteral \nvdash IClause$, $T_1 + ILiteral \nvdash IDClause$, $T_1 + IClause \nvdash IDClause$ and $T_1 + IDClause \nvdash IClause$.*

Lemma 3.4.28. *Assume that there is only one dynamic constructor c_1 and that every other constructor is constant. Let $\mathcal{M}$ be any model of T_1 and t be a term of the form $d_1^{l_1}(\dots d_1^{l_n}(X))$, where all the $\tau_1^{l_h}$ are D. Then, there is some term s_t with the following properties:*

- $\mathcal{M} \vDash \forall X : X = (t \circ s_t)(X)$
- $d(t) = d(s_t)$
- *All function symbols in s_t are dynamic constructors*
- *The interpretation $s_t^{\mathcal{M}}$ maps standard elements to standard elements*
- *For any term t' of the form $d_{j_1}^{k_1}(\dots d_{j_m}^{k_m}(X))$, where all the $\tau_{j_h}^{k_h}$ are D and $\mathcal{M} \vDash \forall X : X = (t \circ s_{t'})$ and $\mathcal{M} \vDash \forall X : X = (t' \circ s_t)$, it holds that $\mathcal{M} \vDash t(X) = t'(X)$*

This term s_t is called right-inverse of t.

Proof. Take any such term t of the form $d_{i_1}^{l_1}(\dots d_{i_n}^{l_n}(X))$, where all the $\tau_{i_l}^{n_l}$ are D. We define the sequence $s_0, \dots, s_n$ inductively: $s_0 \equiv X$ and $s_{k+1} \equiv c_{i_{n-k+1}}(a_1, \dots, a_{m_{n-k+1}})$, where $a_{l_{n-k+1}} = s_k$ and the other a_l are arbitrary (fixed) parameters of the right sorts and if that sort is D, we choose it to be a standard element. Then, the term $s_t = s_n$ has the property that $T_1 \vdash \forall X : (t \circ s_t)(X) = X$. By construction $s_t^{\mathcal{M}}$ maps standard elements to standard elements.

Now take any $t' = d_1^{k_1}(\dots d_1^{k_m}(X))$ and its respective right-inverse $s_{t'}$. Assume that $\mathcal{M} \vDash \forall X : X = (t' \circ s_t)(X)$ and $\mathcal{M} \vDash \forall X : X = (t \circ s_{t'})(X)$. W.l.o.g., assume that $n \leqslant m$ and take the biggest $h \in \{1, \dots, n\}$ s.t. $d_1^{l_h} \neq d_1^{k_h}$ if such an h exists. It holds that $l_h \neq k_h$ and $(d_1^{l_h}(\dots d_{i_n}^{l_n}(s_{t'}(X))))^{M_\infty^*}$ is some fixed parameter by construction of the right-inverse. In particular, $((t \circ s_{t'})(X))^{\mathcal{M}_\infty^*}$ is constant, which contradicts our assumption. Thus, such an h cannot exist and for all $h \leqslant n$, it holds that $d_1^{l_h} = d_1^{k_h}$. Assume that $n < m$. Then define $r \equiv d_1^{k_1}(\dots, d_1^{k_{m-n}}(X))$. It holds that $\mathcal{M} \vDash \forall X : X = (t' \circ s_t)(X)$ and

$\mathcal{M} \vDash \forall X : (t' \circ s_t)(X) = r(X)$. Thus, $\mathcal{M} \vDash \forall X : X = r(X)$. However, there are standard elements E s.t. $\mathcal{M} \nvDash E = r(E)$. Thus, $m = n$, t and t' are identical and $\mathcal{M} \vDash \forall X : t(X) = t'(X)$. ■

Lemma 3.4.29. *Assume that there is only one dynamic constructor and all other constructors take no input at all. Let $\mathcal{M}$ be any model of T_0 s.t. $T_i^{\mathcal{M}}$ contains at least two elements a_i, b_i. If there are two terms t and s of sort T_i, which consist of selectors only and contain X, then they are either syntactically identical or there is some standard element $B \in \mathsf{D}^{\mathcal{M}}$ with $t(B)^{\mathcal{M}} \neq s(B)^{\mathcal{M}}$.*

Proof. It holds that t has the form $d_1^{l_1}(\dots, d_1^{l_m}(X))$ and s has the form $d_1^{k_1}(\dots(d_1^{k_m}(X))$, where the selectors $d_1^{l_1}$ and $d_1^{k_1}$ map into the sort T_i and all other selectors occurring in t or s map into the sort D. W.l.o.g., assume that $m \geqslant n$. First, we fix some constant constructor c and make a case distinction:

1. If $d_1^{l_1} \neq d_1^{k_1}$, then define the following sequence of standard elements $B_0 = c_1^{\mathcal{M}}(\bar{b})$, where $b_j = c$ if $\tau_1^j = \mathsf{D}, b_{l_1} = a_1, b_{k_1} = a_2$ and b_j is some arbitrary parameter otherwise. $B_{k+1} = c_1^{\mathcal{M}}(\bar{b})$, where $b_j = B_k$ if $\tau_1^j = \mathsf{D}, b_{l_1} = a_1, b_{k_1} = a_2$ and b_j is some arbitrary parameter otherwise. By construction, it follows that $t(B_m)^{\mathcal{M}} = a_1$ and $s(B_m)^{\mathcal{M}} = a_2$ as $m \geqslant n$.

2. Now assume that $d_1^{l_1} = d_1^{k_1}$. We define some auxiliary sequence first: $B_0 = c_1^{\mathcal{M}}(\bar{b})$, where $b_j = c$ if $\tau_1^j = \mathsf{D}$, $b_j = a_1$ if $\tau_1^j = \tau_1^{l_1}$ and b_j is some arbitrary parameter otherwise. $B_{k+1} = c_1^{\mathcal{M}}(\bar{b})$, where $b_j = B_k$ if $\tau_j^l = \mathsf{D}$, $b_j = a_1$ if $\tau_1^j = \tau_1^{l_1}$ and b_j is some arbitrary parameter otherwise. Fix the element B_m. Note that any term r, which consists of selectors only, has D-depth $\leqslant m$ has the property and is of sort T_i, that $r(B_m) = a_1$. Now define the sequence E_k in the following way: $E_0 = c_1^{\mathcal{M}}(\bar{e})$, where $e_{l_n} = a_2$, $e_j = B_m$ if $\tau_1^j = \mathsf{D}$ and e_j is some arbitrary parameter otherwise. $E_{k+1} = c_1^{\mathcal{M}}(\bar{e})$, where $e_{l_{n-k}} = E_k$, $e_j = B_m$ if $\tau_1^j = \mathsf{D}$ and e_j is some parameter otherwise. It holds that $t(E_n)^{\mathcal{M}} = a_2$. If there is some there is some $d_1^{l_j} \neq d_1^{k_j}$ or $m > n$, then we can write $s \equiv r \circ q$ with $q(E_n)^{\mathcal{M}} = B_m$ and thus $r(E_n)^{\mathcal{M}} = a_1$.

In any case, we obtain that there is some standard element B wiht $\mathcal{M} \nvDash t(B) = s(B)$ if they are not syntactically identical. ■

Lemma 3.4.30. *Assume that there is only one dynamic constructor and all other constructors take no input at all. Take any model $\mathcal{M}$ of T_1 and atom $A(X) \equiv t_1 = t_2$. If $\mathcal{M} \vDash A(E)$ for any standard element E in $\mathsf{D}^{\mathcal{M}}$, then $\mathcal{M} \vDash \forall X : A(X)$.*

Proof. Let c_1 be the only dynamic constructor. Assume that $\mathcal{M} \vDash \mathbf{LHS}(A)$. First note that since there is only one dynamic constructor and every other constructor is constant, we can assume that there are no subterms of the form $d_1^k(c_1(\dots))$.

There are three cases:

1. If X appears on neither side of A, then $\mathcal{M} \vDash (\exists X : A(X)) \leftrightarrow (\forall X : A(X))$ and we are done.

2. If X appears on exactly one side of A, say in t_1, then $s^{\mathcal{M}}$ is constant. There are two cases:

2a. Assume $t_1(X)$ is of sort T_i. If $T_i1^{\mathcal{M}}$ has exactly one element, then $t_1^{\mathcal{M}}$ is constant and $\mathcal{M} \vDash \forall X : t_1(X) = t_2(X)$. Assume that $T_i^{\mathcal{M}}$ contains the distinct elements a_1, a_2. Note that t_1 has the form $d_1^{l_1}(\dots d_1^{l_n}(X))$, where $\tau_1^{l_1} = T_i$ and for $i \geqslant 2$, $\tau_1^{l_i} = \mathsf{D}$. Define $t_1' = d_1^{l_2}(\dots d_1^{l_n}(X))$ Then by 3.4.28, there is some term $s_{t_1'}$ with $\mathcal{M} \vDash \forall X : X = (t_1' \circ s_{t_1'})(X)$ and $t_1 \equiv d_1^{l_1} \circ t'$. It follows that $\mathcal{M} \vDash \forall X : t_1(s_{t_1'}(X)) = d_1^{l_1}(X)$. Now define the elements $B_j = c_1(\overline{e^j})$, where $e_{l_1}^j = a_j$ for $j \in \{1, 2\}$. Thus, $\mathcal{M} \vDash t_1(s_{t_1'}(B_1)) = a_1$ and $\mathcal{M} \vDash t_1(s_{t_1'}(B_2)) = a_2$. Thus, it does not hold that $\mathcal{M} \vDash t_1(E) = t_2(E)$ for any standard element E.

2b. Assume that $t_1(X)$ is of the sort D. We claim that t_1 is not constant in the standard part and thus, it does not hold that $\mathcal{M} \vDash A(E)$ for any standard element E. We prove the claim inductively:

2b.i. The first base case: If $t_1 \equiv X$, the claim is trivially true.

2b.ii. The second base case: If $t_1 \equiv d_1^{l_1}(\dots(d_1^{l_n}(X)))$, then, by Lemma 3.4.28, there is a right-inverse term s_{t_1} with $\mathcal{M} \vDash \forall X : X = (t_1 \circ s_{t_1})(X)$. In particular, t_1 is not constant in the standard part of $\mathcal{M}$.

2b.iii. For the induction step, assume that t_1 has the form $c_1(s_1, \dots, s_{m_1})$. At least one of the s_l has to contain X and by the induction hypothesis and case 2a, all the terms s_l that contain X have either constant interpretation $s_l^{\mathcal{M}}$ in the whole model or there are elements B, E s.t. $\mathcal{M} \vDash s_l(B) \neq s_l(E)$. If all the $s_l^{\mathcal{M}}$ are constant, then so is $t_1^{\mathcal{M}}$. If there is some s_l and standard elements B, E with $\mathcal{M} \vDash s_l(B) \neq s_l(E)$, then, by INJ_1, $\mathcal{M} \vDash t_1(B) \neq t_1(E)$.

3. Assume that X appears on both sides of A. There are two cases:

3a. Assume that t_1 and t_2 are of the sort $T_i \neq \mathsf{D}$. There are two cases:

3a.i. If $T_i^{\mathcal{M}}$ has exactly one element, then $\mathcal{M} \vDash \forall X : t_1(X) = t_2(X)$ trivially.

3a.ii. If $T_i^{\mathcal{M}}$ contains at least two elements, then it follows from Lemma 3.4.29 that t_1 and t_2 are either syntactically identical or there is some standard element B with $\mathcal{M} \vDash t_1(B) \neq t_2(B)$.

3b. Assume that t_1 and t_2 are of the sort D. We proceed by induction on the structure of t.

3b.i. The first base case: Assume that $t \equiv X$. If the outermost connective of s is some constructor c_i, then it does not hold that $\mathcal{M} \vDash A(E)$ for any standard element E since we have at least one other constructor c_j and $\mathcal{M} \vDash D_{i,j}$. If the outermost connective of s is some selector $d_1^{k_1}$, then s has the form $d_1^{k_1}(\dots(d_1^{k_n}(X)))$. By Lemma 3.4.28, there is some term $r \equiv s_s$ with $\mathcal{M} \vDash \forall X : s(X) = r(X)$. If $\mathcal{M} \vDash t(E) = s(E)$ for any standard element E, then $\mathcal{M} \vDash (t \circ r)(E) = (s \circ r)(E)$ for any standard element E since $r^{\mathcal{M}}$ maps standard elements to standard elements. Since $\mathcal{M} \vDash \forall X : (s \circ r)(X) = X$ and $t \equiv X$, this implies that $\mathcal{M} \vDash r(E) = E$ for any standard element E. We are now in the first case that $t \equiv X$ and the outermost function symbol of s is a constructor (with switched sides). Thus, it does not hold that $\mathcal{M} \vDash r(E) = E$ for any standard element E, which is a contradiction. We conclude that if $t \equiv X$ and $\mathcal{M} \vDash t(E) = s(E)$ for any standard element E, then $s \equiv X$ and $\mathcal{M} \vDash \forall X : t(X) = s(X)$.

3b.ii. The second base case: Assume that $t \equiv d_1^{l_1}(\ldots d_1 l_n(X))$ and that $\mathcal{M} \vDash t(E) = s(E)$ for any standard element E. By 3.4.28, there is some term s_t with $\mathcal{M} \vDash \forall X : X = (t \circ s_t)(X)$. Consequently, $\mathcal{M} \vDash E = (s \circ s_t)(E)$ for any standard element E. From the case 3b, it follows that $\mathcal{M} \vDash \forall X : X = (s \circ s_t)(X)$. By Lemma 3.4.28, $\mathcal{M} \vDash \forall X : t(X) = s(X)$.

3c.iii. The induction step: Assume that $t \equiv c_1(s_1, \ldots, s_{m_1})$. From the cases 3a and 3b, it follows that $s \equiv c_1(r_1, \ldots, r_{m_i})$. If $\mathcal{M} \vDash t(E) = s(E)$ for any standard element E, then, by INJ_1, $\mathcal{M} \vDash r_i(E) = s_i(E)$ for any standard element E and from the induction hypothesis and the cases 1, 2, and 3a, it follows that $\mathcal{M} \vDash \forall X : r_i(X) = s_i(X)$. Consequently, $\mathcal{M} \vDash \forall X : t(X) = s(X)$. ■

Corollary 3.4.31. *If there is only one dynamic constructor and all other constructors take no input at all, then* $T_1 \vdash IAtom$.

Lemma 3.4.32. $T_0 + IAtom \nvdash ILiteral$

Proof. Since there are at least two constructors, a static one and a dynamic one, this follows directly from Theorem 3.2.9 and the fact that $T_1 + ILiteral \vdash \mathrm{SUR}$. ■

Proof of Theorem 3.4.20. In either case, it follows from Lemma 3.4.32 that $T_1 + IAtom \nvdash ILiteral$.

In case 1, it follows from Corollary 3.4.31 that $T_1 \vdash IAtom$.

In case 2, it follows from Lemma 3.4.24 and Lemma 3.4.24 that $T_1 \nvdash IAtom$. ■

CHAPTER 4

Arithmetics

Open Induction in the context of arithmetics has already been studied in [Sho58] and [She63]. However, they only considered induction over all open formulas and not over particular subsets of them. The goal of this chapter is to analyze *how much* induction is needed for the known results and prove them in the respective subsystems. The structure of this chapter closely follows the one of [She63]. We consider systems of arithmetics up to and including multiplication.

4.1 General Frame

In the following we will consider a one-sorted logic. We work with the language $\Sigma = \{0, s, p, +, \cdot\}$ or subsets of it. 0 is 0-ary, s and p are unary and $+$ and $\cdot$ are binary. Furthermore, we use $=$ as a binary predicate with the usual axiomatization. The base axioms are

A1 $s(x) \neq 0$

A2 $p(0) = 0$

A3 $p(s(x)) = x$

A3a $sx = sy \rightarrow x = y$

A4 $x + 0 = x$

A5 $x + s(y) = s(x + y) \neq 0$

A6 $x \cdot 0 = 0 \neq 0$

A7 $x \cdot s(y) = x \cdot y + x$

For the sake of readability, we will often write sx and px instead of $s(x)$ and $p(x)$ respectively. If it is clear from the context, we might also drop the $\cdot$ and write xy instead of $x \cdot y$.

Lastly, we have the following scheme for induction:

$$\begin{array}{ll} \mathbf{LHS}(\varphi(x,\overline{z})) & \varphi(0,\overline{z}) \wedge \forall x : \varphi(x,\overline{z}) \to \varphi(sx,\overline{z}) \\ \mathbf{I}(\varphi(x,\overline{z})) & \mathbf{LHS}(\varphi(x,\overline{z}) \to \forall x : \varphi(x,\overline{z}) \end{array}$$

In the scheme above $\overline{z}$ is a parameter in the formula φ. Again, for the purpose of legibility, we might sometimes not mention parameters explicitly as all formulas may contain them if not stated otherwise.

Note that in list of the axioms above, we have the axioms $A3$ and $A3a$. The first one, does not state directly that s is injective, but it follows trivially from it. Thus, whenever, we have the axiom $A3$, we can also use $A3a$ freely.

Having established the general context, we can consider various theories over Σ and its subsets in the following sections. We start by taking the empty theory and expand it gradually. Also the language will be enriched step-by-step.

4.2 Useful models

In this section, we define two useful non-standard models of arithmetics. We will then prove some properties of them and refer to them in the later parts of this chapter.

4.2.1 The model $\mathbb{N}_\infty$

In this subsection, we will define a model that is in some sense the *least non-standard model* of the natural numbers. We will prove some properties of this model and make heavy use of this model in course of this chapter.

In this section we consider the language $\mathcal{L} = \{0, s, p, +, \cdot\}$ with the axiomatization $T = \{A1, \ldots, A7\}$.

Definition 4.2.1. *The model $\mathbb{N}_\infty$ is constructed in the following way: The domain is given by $\mathbb{N} \cup \{\infty\}$. The symbols are interpreted in the following way:*

- *0 is interpreted as 0*
- *All the symbols $s, p, +, \cdot$ are interpreted canonically in the standard part of the model*
- *$s(\infty) = p(\infty) = \infty$*
- *For any x in the domain, $\infty + x = x + \infty = \infty$*
- *$\infty \cdot 0 = 0 \cdot \infty = 0$ and for any $x \neq 0$, $x \cdot \infty = \infty \cdot x = \infty$*

Lemma 4.2.2. $\mathbb{N}_\infty \vDash T$

Proof. This holds trivially by construction. ■

Lemma 4.2.3. *Let $\underline{n}$ be the numeral $s^n(0)$ and $A(x) \equiv t_1 = t_2$ any atom. If $\mathbb{N}_\infty \vDash A(\underline{n})$ for all $n \in \mathbb{N}$, then $\mathbb{N}_\infty \vDash \forall x : A(x)$.*

Proof. 1. If x does not appear in either t_1 nor t_2, then $A(0) \leftrightarrow A(\infty)$.

2. If x appears on exactly one side of A, say t_1, then t_2 is constant in x and there are two cases:

2a. If some parameter in t_1 is interpreted as ∞, then $t_1(x) = \infty$ for any x. Thus, $A(\infty) \leftrightarrow A(0)$.

2b. If no parameter in t_1 evaluates to ∞, then t_1 is evaluated as a polynomial in x almost everywhere in the standard part of the model. In particular, it cannot be constant there and the conditions are not met.

3. If x appears on both sides, then, by construction of the model, both terms $t_i(\infty)$ evaluate to ∞. Thus, $A(\infty)$ holds. ■

Lemma 4.2.4. *For any reduct $\mathbb{N}_\infty^*$ of $\mathbb{N}_\infty$ (that still contains 0 and s) it holds that $\mathbb{N}_\infty^* \vDash IAtom$*

Proof. W.l.o.g. assume that $\mathbb{N}_\infty^* = \mathbb{N}_\infty$. The claim follows directly from Lemma 4.2.3 as $\mathbb{N}_\infty \vDash A(\underline{n})$ for any $n \in \mathbb{N}$ if $\mathbb{N}_\infty \vDash A(0) \wedge A(x) \rightarrow A(sx)$. ■

Lemma 4.2.5. *For any reduct $\mathbb{N}_\infty^*$ of $\mathbb{N}_\infty$ (that still contains 0 and s) it holds that $\mathbb{N}_\infty^* \nvDash ILiteral$*

Proof. Consider the literal $L(x) \equiv x \neq z$. where the parameter z is interpreted as ∞. Clearly, $\mathbb{N}_\infty^* \vDash L(0) \wedge L(x) \rightarrow L(sx)$, but obviously $\mathbb{N}_\infty^* \nvDash \forall x : L(x)$. Note that this works regardless of the concrete reduct as L uses no symbols of the language. ■

Theorem 4.2.6. *Let $T' \subseteq T$ be some theory over the language $\mathcal{L}' \subseteq \mathcal{L}$. Then $T' + IAtom \nvDash ILiteral$*

Proof. By Lemma 4.2.2 the appropriate reduct $\mathbb{N}_\infty^*$ of $\mathbb{N}_\infty$ is a model of T'. By Lemma 4.2.4 $\mathbb{N}_\infty^*$ is a model of $IAtom$, but by Lemma 4.2.5 $\mathbb{N}_\infty^*$ is not a model of $ILiteral$. ■

4.2.2 The model $\mathbb{N}_{a,b}$

In this subsection, we will define another very common non-standard model of natural numbers. We will use this model, to separate atomic induction from the base theory in some cases.

Although the model below is slightly different, it is very similar to the model in [Het24, page 38].

Definition 4.2.7. *The model $\mathbb{N}_{a,b}$ is constructed in the following way: The domain is given by $\mathbb{N} \cup \{a, b\}$. The symbols are interpreted as follows:*

- *0 is interpreted as* 0
- *For every standard element n, $s^{\mathbb{N}_{a,b}}n = n + 1$, $s^{\mathbb{N}_{a,b}}a = a$ and $s^{\mathbb{N}_{a,b}}b = b$*
- *For every standard element $n \neq 0$, $p^{\mathbb{N}_{a,b}}n = n - 1$, $p^{\mathbb{N}_{a,b}}0 = 0$, $p^{\mathbb{N}_{a,b}}a = a$, $p^{\mathbb{N}_{a,b}}b = b$*
- *$+$ and $\cdot$ are interpreted according to the tables below*

$+$	0	1	2	…	a	b
0	0	1	2	…	b	a
1	1	2	3	…	b	a
2	2	3	4	…	b	a
⋮	⋮	⋮	⋮	⋱	⋮	⋮
a	a	a	a	…	a	a
b	b	b	b	…	b	b

$\cdot$	0	1	2	…	a	b
0	0	0	0	…	b	a
1	0	1	2	…	b	a
2	0	2	4	…	b	a
⋮	⋮	⋮	⋮	⋱	⋮	⋮
0	b	b	b	…	a	a
0	a	a	a	…	b	b

Lemma 4.2.8. $\mathbb{N}_{a,b} \models \{A1, A2, A3, A3a, A4, A5, A6, A7\}$

Proof. It was shown in [Het24, page 38] that $\mathbb{N}_{a,b} \models \{A1, A3a, A4, A5, A6, A7\}$. Note that $A2$ and $A3$ hold by construction. ■

Lemma 4.2.9. $\mathbb{N}_{a,b} \not\models x + y = y + x$

Proof. Consider $a + 0 = a \neq b = 0 + a$. ■

4.3 0 and Successor only

In this section, we fix the language $\mathcal{L} = \{0, s\}$

Definition 4.3.1. *We define two very basic theories:*

- $T_0 = \varnothing$
- $T_1 = \{A1\}$

Additionally, we define the following auxiliary axioms:

$$\mathbf{B_{n,m}} \quad s^n 0 = s^{m+1} 0 \to \forall x \bigvee_{k=0}^{m} x = s^k 0 \text{ for any } n, m \in \mathbb{N} \text{ and } n \leqslant m$$

Theorem 4.3.2. *In [She63], it was shown that the following holds:*

- $T_0 + IOpen \approx T_0 + \{B_{n,m} \mid n, m \in \mathbb{N}, n \leqslant m\}$

- $T_1 + IOpen \approx T_1 + \{B_{n,m} \mid n, m \in \mathbb{N}, n \leqslant m\}$

We can strengthen this by the following theorem:

Theorem 4.3.3. *For any $i \in \{1, 2\}$ the following holds:*

$$\begin{aligned} T_i &\lnsim T_i + IAtom \\ &\lnsim T_i + ILiteral \\ &\lnsim T_i + IDClause \\ &\approx T_i + IClause \\ &\approx T_i + IOpen \approx T_i + \{B_{n,m} \mid n, m \in \mathbb{N}, n \leqslant m\} \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd} T_i + IDClause \approx T_i + IClause \approx T_i + IOpen \arrow[d, no head] \\ T_i + ILiteral \arrow[d, no head] \\ T_i + IAtom \arrow[d, no head] \\ T_i \end{tikzcd}$$

Proof. This Theorem will follow directly from the following Lemmas. ■

Remark 4.3.4. *Note that T_1 and T_2 are defined over the same language with $T_0 \subseteq T_1$. Thus, by showing $T_1 \nvdash \varphi$ for some formula φ, we obtain that $T_0 \nvdash \varphi$ for free and vice-versa for $T_0 \vdash \varphi$.*

Lemma 4.3.5. *$T_0 + IClause \vdash B_{n,m}$ for any $n, m \in \mathbb{N}$ with $n \leqslant m$*

Proof. We work in $T_0 + IClause$. Fix suitable n, m and assume that $s^n 0 = s^{m+1} 0$. Consider the clause $C(x) \equiv \bigvee_{k=0}^{m} x = s^k 0$. Clearly, $C(0)$ is logically valid. Now, assume that $C(x)$ holds. There is some $k \leqslant m$ s.t. $x = s^k 0$. We make a case distinction:

- If $k < m$, then $k + 1 \leqslant n$ and $sx = s^{k+1} 0$ makes the clause true

- If $k = m$, then $s^n 0 = s^{k+1} 0$ by assumption. Since $n \leqslant m$, we set $k' = n \leqslant m$ and obtain $sx = s^{k+1} 0 = s^n 0 = s^{k'} 0$.

Applying the scheme of induction on C, yields that $B_{n,m}$ holds. ■

Lemma 4.3.6. $T_1 \nvdash IAtom$

Proof. Consider the following model $\mathcal{M}$: The domain is given by $\{0, 1, 2, a, b, c\}$ and the symbols are interpreted in the following way:

- 0 is interpreted as 0
- $s0 = 1, s1 = 2, s2 = 1$
- $sa = b, sb = c, sc = a$

Since 0 has no predecessor in this model, all axioms of T_1 hold. Consider the atom $A(x) \equiv sx = s^3 x$. Then A holds for the elements $0, 1, 2$, but not for a, b, c. In particular, $\mathcal{M} \vDash \mathbf{LHS}(A)$, but $\mathcal{M} \nvDash \forall x : A(x)$. Thus, $\mathcal{M} \nvDash \mathbf{I}(A)$. ■

Lemma 4.3.7. $T_1 + IAtom \nvdash ILiteral$

Proof. This follows directly from Theorem 4.2.6. ■

Lemma 4.3.8. *Let $\mathcal{M}$ be any model of $T_0 + ILiteral$. Then, $\mathcal{M}$ contains no element $z \neq 0$ s.t. $sy = z \rightarrow y = z$.*

Proof. Assume that there is such an element z and consider the literal $L(x) \equiv x \neq z$. Then, clearly, $\mathcal{M} \vDash L(0) \wedge L(x) \rightarrow L(sx)$. Thus, by induction on L, $\forall x : x \neq z$, which is a contradiction. Such an element cannot exist. ■

Lemma 4.3.9. *Let $\mathcal{M}$ be any model of $T_0 + IDClause$ and $z \in \mathcal{M}$ any element that is not a successor of 0. Then for every $n \in \mathbb{N}$, there is a sequence of distinct elements $y_0, \ldots, y_n$ s.t. $sy_i = y_{i+1}$ and $y_n = z$.*

Proof. Assume that there is a non-standard element z and some n s.t. such a sequence does not exist for z. Let m be the biggest number s.t. such a sequence $y_0, \ldots, y_m$ does exist for z and consider the dual clause $D(x) \equiv \bigwedge_{i=0}^{m} x \neq y_i$. Since z is non-standard, $\mathcal{M} \vDash D(0)$. If $D(x)$ holds and $D(sx)$ does not, then $sx = y_0$. Then, however, $x, y_0, \ldots, y_m$ would be the sequence for m, which contradicts our assumption of m being maximal. Thus, $\mathcal{M} \vDash D(x) \rightarrow D(sx)$. Induction on D yields $\mathcal{M} \vDash \forall x : D(x)$, which contradicts $\mathcal{M} \vDash z = z$. ■

Lemma 4.3.10. $T_0 + IDClause \vdash IClause$.

Proof. From Theorem 3.1.8, it follows that $T_0 + \mathit{ILiteral} \vdash x = 0 \vee \exists y : x = sy$. Now let us analyse, how models of $T_0 + \mathit{IDClause}$ look like: Let $\mathcal{M}$ be any model with the domain M, $G = (M, E)$ be the graph induced by s and $\mathcal{C} \subseteq 2^M$ the set of all connected components of G. We fix C_0 to be the component containing 0 and distinguish a few cases: 1. If $s0 = 0$, consider the atom $A(x) \equiv sx = x$. Clearly, $\mathcal{M} \vDash A(0) \wedge A(x) \rightarrow A(sx)$. Thus, by applying the induction scheme on A, we obtain that every element is its own successor. By Lemma 4.3.8, 0 can be the only such element. In particular, the domain of $\mathcal{M}$ is $\{0\}$ and induction over all formular clearly holds.

2. Assume that $s(0) \neq 0$ and there are some n, m with $m \neq 0$ s.t. $s^n0 = s^{n+m}0$. Consider the atom $s^n x = s^{n+m}x$. Clearly, $\mathcal{M} \vDash A(0)$. Assume that $\mathcal{M} \vDash A(x)$. Then $s^n(sx) = s(s^n x) = s(s^{n+m}x) = s^{n+m}(sx)$ and thus $A(sx)$. By induction, we obtain that $\mathcal{M} \vDash \forall x : A(x)$. W.l.o.g., we assume that n and m are minimal with those properties. Now assume that there is some element x, which is not a successor of 0. Then there are three cases:

2a. If x has some predecessor y that does not lie in a cycle. Then, since every element other than 0 is a successor, there have to be $n + m$ distinct predecessors of x. Let y be the predecessor that $s^{n+m}y = x$. Then, $s^n y \neq s^{n+m}y$, which is a contradiction. Thus, such an x cannot exist.

2b. Assume that x is its own only predecessor. By Lemma 4.3.8 this cannot be.

2c. We are left with the case that every non-standard element lies in a cycle of length $k \geqslant 2$. For any $k \neq m$, consider the literal $L(x) \equiv x \neq s^k x$. Clearly, $\mathcal{M} \vDash L(0) \wedge L(x) \rightarrow L(sx)$. Thus, every non-standard element lies in a cycle of length m.

By Lemma 4.3.9 there cannot be any finite non-standard cycle. Thus, every element in $\mathcal{M}$ is a successor of 0 and induction over all formulas holds.

3. Assume that there are no n, m with $m \neq 0$ s.t. $s^n0 = s^{n+m}0$. Then, all of the $B_{n,m}$ trivially hold. By Theorem 4.3.2, open induction has to hold in $\mathcal{M}$ and in particular, induction over clauses. ■

Lemma 4.3.11. *$T_1 + \mathit{ILiteral} \nvdash \mathit{IClause}$ and $T_1 + \mathit{ILiteral} \nvdash \mathit{IDClause}$.*

Proof. For the first part, it suffices, by Lemma 4.3.5, to give a model of $T_1 + \mathit{ILiteral}$, which does not satisfy $B_{n,m}$ for some suitable n, m. For the second part, we give a dual clause D, over which induction does not work in the same model.

Consider the following model $\mathcal{M}$: The domain is given by $\{0, 1, 2, a, b\}$. The symbols are interpreted in the following way:

- 0 is interpreted as 0
- $s0 = 1, s1 = 2, s2 = 1$
- $sa = b, sb = a$

As 0 is no successor, T_1 holds. Moreover, $B_{1,2}$ does not hold.

Consider the dual clause $D(x) \equiv x \neq y_1 \wedge x \neq y_2$, where y_1 and y_2 are interpreted as a and b respectively. Clearly, $\mathcal{M} \vDash D(0)$ and by construction of the model, $\mathcal{M} \vDash D(x) \rightarrow D(sx)$ as well. However, $\mathcal{M} \nvDash D(a)$. Thus, induction over dual clauses does not hold.

It remains to be shown that $ILiteral$ holds.

First take any atom $A(x) \equiv t_1 = t_2$. Note that any term t has the form $s^n(y)$, where y is either x, some parameter, or 0. We make a case distinction:

1. If x appears in neither t_i, then $A \leftrightarrow \bot$ or $A \leftrightarrow \top$. Induction over A clearly holds.

2. If x appears in only one term, say t_1, then t_2 is constant in x. However, t_1 is not constant in the whole standard part of the model. Induction over A holds, as the left-hand side of the scheme is not satisfied.

3. If x appears on both sides, then A has the form $s^n x = s^m x$. There are three cases:

3a. If $n = m = 0$, then A holds trivially in the whole model.

3b. If $n = 0 \neq m$, then $A(0)$ does not hold.

3c. If $n \neq 0 \neq m$, then $\mathcal{M} \vDash A(0) \leftrightarrow A(x)$ for any x. In particular, induction over A holds.

Now take any negated atom $L(x) \equiv t_1 \neq t_2$. We make a case distinction:

4. If x appears on neither side, induction over L holds for the same reasons as above.

5. If x appears in exactly on term, say t_1, then t_2 has a constant interpretation and t_1 has the form $s^n x$. There are three cases:

5a. If t_2 is interpreted as 0 and $n = 0$, then $L(0)$ does not hold.

5b. If t_2 is interpreted as 0 and $n \neq 0$, then $L(x)$ holds for any x.

5c. If t_2 is interpreted as $y \neq 0$, then there is some y', z' s.t. $sy' = y$ and $s^n z' = y'$. However, $s^n sz' = sy' = y$. Thus, the left-hand side of the scheme of induction is not met. ■

4.4 Injective Successor

We define the new theory T_2:

Definition 4.4.1. *$T_2 = \{A1, A3a\}$ over the language $\{0, s\}$.*

Now, we need some supplementary axioms:

Definition 4.4.2. *We define the following axiom for every natural $n \geqslant 1$:*

$\mathbf{B_n}$ *$x \neq s^n x$ for any $n \geqslant 1$*

In [She63] the following was shown:

Theorem 4.4.3. *$T_2 + IOpen$ is equivalent to $T_2 + \{B_n \mid n \in \mathbb{N}, n \geqslant 1\}$.*

We can also use Theorem 3.4.3:

Theorem 4.4.4.

$$\begin{aligned} T_2 &\approx T_2 + IAtom \\ &\lnsim T_2 + IAtom \\ &\approx T_2 + IOpen \\ &\approx T_2 + \mathrm{SUR} + \{B_n \mid n \geqslant 1\} \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd} T_2 + ILiteral \approx T_2 + IOpen \arrow[d, dash] \\ T_2 \approx T_2 + IAtom \end{tikzcd}$$

We conclude that SUR is superfluous in this context.

4.5 Adding the Predecessor

We define the theory T_3:

Definition 4.5.1. $T_3 = \{A1, A2, A3\}$ *over the language* $\{0, s, p\}$.

We need another auxiliary axiom:

Definition 4.5.2. *We define the axiom:*

B1 $x \neq 0 \rightarrow x = spx$

In [She63] the following was proven:

Theorem 4.5.3. *$T_3 + IOpen$ is equivalent to $T_3 + \{B_n \mid n \in \mathbb{N}, n \geqslant 1\} + \{B1\}$.*

Again, we can strengthen this result:

Theorem 4.5.4.

$$\begin{aligned} T_3 &\approx T_3 + IAtom \\ &\lnsim T_3 + ILiteral \\ &\approx T_3 + IOpen \\ &\approx T_3 + \{B_n \mid n \in \mathbb{N}, n \geqslant 1\} + \{B1\} \end{aligned}$$

This yields the following Hasse diagram:

$$\begin{tikzcd} T_3 + ILiteral \approx T_3 + IOpen \arrow[d, dash] \\ T_3 \approx T_3 + IAtom \end{tikzcd}$$

Lemma 4.5.5. *$T_3 + ILiteral \vdash B_n$ for all $n \geqslant 1$.*

Proof. We work in $T_3 + ILiteral$. Fix any $n \geqslant 1$ and consider the literal $L_1(x) \equiv x \neq s^n x$. By A1 we have that $L_1(0)$ holds. By A3 we have $sx = s^{n+1}x \to x = psx = ps^{n+1}x = ps^n x$ and by counterposition $L_1(x) \to L_1(sx)$. By induction on L_1, we obtain $\forall x : x \neq s^n x$. ■

Lemma 4.5.6. *$T_3 + ILiteral \vdash B1$*

Proof. From Theorem 3.1.8 it follows that $T_3 + ILiteral \vdash x = 0 \vee \exists y : x = sy$. Assume that $x \neq 0$. There is some y s.t. $x = sy$. It follows that $x = sy \overset{A3}{=} s(psy) = (sp)sy = spx$ ■

Lemma 4.5.7. *$T_3 \vdash IAtom$*

Proof. We work in T_3. Take any atom $A(x) \equiv t_1 = t_2$. Assume that both terms have the form st'_i. Then by A3, we obtain that $t'_1 = pst'_1 = pst'_2 = t'_2$. We can therefore cancel the leading s and assume that at most one of the terms starts with s. If neither term contains x, then $A \leftrightarrow \top$ or $A \leftrightarrow \bot$ - in any case, the induction over A holds. If one of the terms does not contain x, then the other is interpreted as a fixed value, regardless of x. In particular, either $A(0)$ or $A(s0)$ does not hold since $s0 \neq 0$ by A1. Thus, induction over A holds. We are left with the case where both terms contain x. If one of the terms, in fact, starts with an s, then $A(0)$ does not hold by A1. It follows that $A(x) \leftrightarrow p^n x = p^m x$. Assume that $A(0)$ and $A(x) \to A(sx)$ holds. If $m - n = k > 0$, then $A(x) \to A(s^m x)$ and thus $A(x)$ implies $s^k x = p^n s^m x = p^m s^m x = x$. In particular, $A(0)$ implies that $0 = s^k 0$, which contradicts A1. By symmetry, we conclude that $m = n$ and $A(x) \leftrightarrow p^n x = p^n x \leftrightarrow \top$, which proves the claim. ■

Lemma 4.5.8. *$T_3 \nvdash ILiteral$*

Proof. This follows directly from Theorem 4.2.6. ■

4.6 Linear Arithmetic

We define the theory T_4:

Definition 4.6.1. $T_4 = \{A1, A2, A3, A4, A5\}$ *over the language* $\{0, s, p, +\}$

Now again, some auxiliary axioms:

Definition 4.6.2. *We define the following axioms:*

B2 $x + y = y + x$
B3 $(x + y) + z = x + (y + z)x$
B4 $x + y = x + z \rightarrow y = z$

In [Sho58] the following was shown:

Theorem 4.6.3. $T_4 + IOpen$ *is equivalent to* $T_4 + \{B1, B2, B3, B4\}$

We can strengthen this result in the usual way:

Theorem 4.6.4.

$$\begin{aligned} T_4 &\lnsim T_4 + IAtom \\ &\lnsim ILiteral \\ &\approx T_4 + IOpen \\ &\approx T_4 + \{B1, B2, B3, B4\} \end{aligned}$$

This yields the Hasse diagram:

$$\begin{tikzcd} T_4 + ILiteral \approx T_4 + IOpen \arrow[d, dash] \\ T_4 + IAtom \arrow[d, dash] \\ T_4 \end{tikzcd}$$

Lemma 4.6.5. $T_4 + ILiteral \vdash B1$

Proof. This follows directly from Lemma 4.5.6 and the fact that T_4 is a superset of T_3. ■

Lemma 4.6.6. *$T_4 + IAtom \vdash B2$.*

Proof. Consider the atom $A_1(x) \equiv 0 + x = x$. By A4, we have that $A_1(0)$ holds. Assume that $A_1(x)$ holds and $0 + x = x$. Then $0 + sx \overset{A5}{=} s(0 + x) \overset{IH}{=} sx$. By induction on A_1, we obtain $\forall x : 0 + x = x$.

Now consider the literal $A_2(x) \equiv sy + x = s(y + x)$. $A_2(0)$ holds because $sy + 0 \overset{A4}{=} sy \overset{A4}{=} s(y + 0)$. Assume that $A_2(x)$ holds and $sy + x = s(y + x)$, then $sy + sx \overset{A5}{=} s(sy + x) \overset{IH}{=} s(s(y + x)) \overset{A5}{=} s(y + sx)$. By induction on A_2 and universal quantification, we obtain $\forall y \forall x : sy + x = s(y + x)$.

Now consider the atom $L_3(x) \equiv x + y = y + x$. $L_3(0)$ holds because $0 + y \overset{A_1}{=} y \overset{A4}{=} y + 0$. Assume that $L_3(x)$ holds and $x + y = y + x$. Then $sx + y \overset{A_2}{=} s(x + y) \overset{IH}{=} s(y + x) \overset{A5}{=} y + sx$. By induction on L_3 and universal quantification, we obtain $\forall y \forall x : x + y = y + x$. ■

Lemma 4.6.7. *$T_4 + IAtom \vdash B3$.*

Proof. Note that by Lemma 4.6.6, we already know that $x + y = y + x$. Now consider the atom $A(x) \equiv (x + y) + z = x + (y + z)$. $A(0)$ holds because $(0 + y) + z \overset{B2}{=} (y + 0) + z \overset{A4}{=} y + z \overset{A4}{=} (y + z) + 0 \overset{B2}{=} 0 + (y + z)$. Assume that $A(x)$ holds and $(x + y) + z = x + (y + z)$. Then,

$$\begin{aligned}(sx + y) + z &\overset{B2}{=} z + (y + sx) \overset{A4}{=} s(z + (y + x)) \overset{B2}{=} s((x + y) + z) \overset{IH}{=} s(x + (y + z)) \\ &\overset{B2}{=} s((y + z) + x) \overset{A5}{=} (y + z) + sx \overset{B2}{=} sx + (y + z)\end{aligned}$$

and $A(sx)$ holds as well. By induction on A and universal quantification, we obtain $\forall z \forall y \forall x : (x + y) + z = x + (y + z)$. ■

Lemma 4.6.8. *$T_4 + ILiteral \vdash B4$*

Proof. We work in $T_4 + ILiteral$. Note that by Lemma 4.6.6, we can use commutativity.

Assume that $y \neq z$ and consider the literal $L(x) \equiv x + y \neq x + z$. $L(0)$ holds because $0 + y \overset{B2}{=} y + 0 \overset{A4}{=} y \neq z \overset{A4}{=} z + 0 \overset{B2}{=} 0 + z$. Assume that $L(x)$ holds and $x + y \neq x + z$. Then $sx + y \overset{B2}{=} y + sx \overset{A5}{=} s(y + x) \overset{B2}{=} s(x + y) \overset{A3a}{\neq} s(x + z) \overset{B2}{=} s(z + x) \overset{A5}{=} z + sx \overset{B2}{=} sx + z$. By induction over L, we obtain $\forall x : x + y \neq x + z$. ■

Lemma 4.6.9. *$T_4 \nvdash IAtom$*

Proof. By Lemma 4.6.6, it suffices to give a model of T_4, in which $+$ is not commutative.

The claim now follows directly from Lemma 4.2.9 if we take the appropriate reduct of $\mathbb{N}_{a,b}$ from Definition 4.2.7. ■

Lemma 4.6.10. *$T_4 + IAtom \nvdash ILiteral$*

Proof. This follows directly from Theorem 4.2.6. ■

4.7 Polynomials

We define the theory T_5:

Definition 4.7.1. $T_5 = \{A1 - A7\}$ *over the language* $\{0, s, p, +, \cdot\}$.

We define new auxiliary axioms:

Definition 4.7.2. *We define the following axioms:*

B5 $xy = yx$

B6 $x(yz) = (xy)z$

B7 $x(y + z) = xy + xz \rightarrow y = z$

$\mathbf{C'_d}$ $dy = dz \rightarrow \bigvee_{i=0}^{d-1} (x + i)y = (x + i)z$ *for any* $d = 2, 3, \ldots$

The following result was postulated in [She63] and proven in [She67].

Theorem 4.7.3. $T_5 + IOpen$ *is equivalent to* $T_5 + \{B1 - B7\} + \{C'_d \mid d \in \mathbb{N}, d \geqslant 2\}$.

We can strengthen this result:

Theorem 4.7.4.

$$\begin{aligned} T_5 &\lnsim T_5 + IAtom \\ &\lnsim T_5 + ILiteral \\ &\approx T_5 + IDClause \\ &\lnsim T_5 + IClause \\ &\approx T_5 + IOpen \\ &\approx T_5 + \{B1 - B7\} + \{C'_d \mid d \in \mathbb{N}, d \geqslant 2\} \end{aligned}$$

This yields the following Hasse diagram:

$$\begin{tikzcd}
T_5 + IClause \approx T_5 + IOpen \arrow[d, dash] \\
T_5 + ILiteral \approx T_5 + IDClause \arrow[d, dash] \\
T_5 + IAtom \arrow[d, dash] \\
T_5
\end{tikzcd}$$

This theorem will be a consequence of the following lemmas in combination with Theorem 4.7.3.

Lemma 4.7.5. $T_5 + IAtom \vdash \{B2, B3, B5, B6, B7\}$

Proof. Since $T_5 \supseteq T_4$, it follows from the Lemma 4.6.6 and Lemma 4.6.7 that $T_5 + IAtom \vdash B2$ and $T_5 + IAtom \vdash B3$. For the remaining proof, we work in $T_5 + IAtom$.

B5 : We need three atoms:

1. Consider the atom $A_1(x) \equiv 0x = 0$. $A_1(0)$ holds because $0 \cdot 0 \overset{A6}{=} 0$. Assume that $A_1(x)$ holds and $0x = 0$. Then $0sx \overset{A7}{=} 0x + 0 \overset{IH}{=} 0 + 0 \overset{A4}{=} 0$ and $A_1(sx)$ holds as well. By induction over A_1, we obtain that $\forall x : 0x = 0$.

2. Consider the atom $A_2(x) \equiv (sy)x = yx + x$. $A_2(0)$ holds because $(sy)0 \overset{A6}{=} 0 \overset{A4}{=} 0 + 0 \overset{A6}{=} y0 + 0$. Assume that $A_2(x)$ holds and $(sy)x = yx + x$. Then,

$$\begin{aligned}
(sy)(sx) &\overset{A7}{=} (sy)x + sy \overset{A5}{=} s((sy)x + y) \overset{IH}{=} s((yx + x) + y) \overset{B2}{=} s(y + (yx + x)) \\
&\overset{B3}{=} s((y + yx) + x) \overset{A5}{=} (y + yx) + sx \overset{B2}{=} (yx + y) + sx \overset{A7}{=} y(sx) + sx
\end{aligned}$$

and $A_2(sx)$ holds as well. By induction over A_2 and universal quantification, we obtain $\forall y \forall x : (sy)x = yx + y$.

3. Now consider the atom $A_3(x) \equiv xy = yx$. $A_3(0)$ holds because $0y \overset{A_1}{=} 0 \overset{A6}{=} y0$. Assume that $A_3(x)$ holds and $xy = yx$. Then $(sx)y \overset{A_2}{=} xy + y \overset{IH}{=} yx + y \overset{A7}{=} y(sx)$ and $A_3(sx)$ holds as well. By induction over A_3 and universal quantification, we obtain that $\forall y \forall x : xy = yx$.

B7 : Consider the atom $A_4(x) \equiv x(y+z) = xy + xz$. $A_4(0)$ holds because $0(x+y) \overset{A_1}{=} 0 \overset{A4}{=} 0 + 0 \overset{A_1}{=} 0x + 0y$. Assume that $A_4(x)$ holds and $x(y+z) = xy + xz$. Then,

$$\begin{aligned}(sx)(y+z) &\overset{B5}{=} (y+z)(sx) \overset{A7}{=} (y+z)x + (y+z) \overset{B5}{=} x(y+z) + (y+z)\\ &\overset{IH}{=} (xy + xz) + (y+z) \overset{B2,B3}{=} (xy+y) + (xz+z) \overset{B5}{=} (yx+y) + (zx+z)\\ &\overset{A7}{=} y(sx) + z(sx) \overset{B5}{=} (sx)y + (sx)z\end{aligned}$$

and $A_4(sx)$ holds as well. By induction over A_4 and universal quantification, we obtain that $\forall z \forall y \forall x : x(y+z) = xy + xz$

B6 : Consider the atom $A_5(x) \equiv (xy)z = x(yz)$. $A_6(0)$ holds because $(0y)z \overset{A_1}{=} 0 \overset{A6}{=} (yz)0 \overset{B5}{=} 0(yz)$. Assume that $A_5(x)$ holds. By A7, commutativity of $\cdot$, distributivity of $+$ and $\cdot$ and the induction hypothesis, it follows that

$$\begin{aligned}((sx)y)z &\overset{B5}{=} (y(sx))z \overset{A7}{=} (yx+y)z \overset{B5}{=} z(yx+y) \overset{B7}{=} z(yx) + zy\\ &\overset{B5}{=} (xy)z + yz \overset{IH}{=} x(yz) + yz \overset{A7}{=} (yz)(sx) \overset{B5}{=} (sx)(yz)\end{aligned}$$

Induction on A_5 yields the desired result. ■

Lemma 4.7.6. *$T_5 + ILiteral \vdash \{B1, B4\}$*

Proof. Since $T_5 \supseteq T_4$, this claim follows directly from Lemma 4.6.5 and Lemma 4.6.8. ■

Lemma 4.7.7. *$T_5 + IClause \vdash C'_d$ for any $d \geqslant 2$.*

Proof. We work in $T_5 + IClause$. Fix any $d \geqslant 2$ and consider the clause $C(x) \equiv dy = dz \to \bigvee_{k=0}^{d-1}(s^k x)y = (s^k x)z$. By A6 and commutativity of $\cdot$, we have that $0y = 0 = 0z$ and in particular that $C(0)$ holds. Now assume that $C(x)$ holds. Take any y, z s.t. $dy = dz$. If there is some $k > 0$ s.t. $(s^k x)y = (s^k x)z$, then $(s^{k-1}sx)y = (s^{k-1}sx)z$ and in particular $C(sx)$ holds. If $k = 0$, then $xy = xz$. Consider the term $(s^d x)y$. It holds that $(s^d x)y = (s^{d-1}x)y + y = \cdots = (s^0 x)y + dy = xy + dy = xz + dz = (s^d x)z$ and thus $C(sx)$. Induction on C yields the desired result. ■

Lemma 4.7.8. *$T_5 \nvdash IAtom$*

Proof. By Lemma 4.7.5 it suffices to give a model of T_5, where $+$ is not commutative. The claim now follows directly from Lemma 4.2.9. ■

Lemma 4.7.9. *$T_5 + IAtom \nvdash ILiteral$*

Proof. This follows directly from Theorem 4.2.6. ■

We now proceed similarly as in [She67], but fill the gaps:

Lemma 4.7.10. *The models of $T_5 + \{B1 - B7\}$ are exactly the ones obtained by taking a commutative ring with* 1*, $\mathcal{R} = (R, +, -, 0, \cdot, 1)$, and then taking some subset M of R that does not contain -1, is closed under $0, +, \cdot, x \mapsto x+1$ and is closed under $x \mapsto x-1$ for all $x \neq 0$. We then define the operations $+, \cdot$ as in $\mathcal{R}$, $sx = x+1$ and $px = x-1$ if $x \neq 0$ and* 0 *otherwise.*

Proof. The first direction is trivial: Any subset of a commutative ring with 1 with the given properties is a model of $T_5 + \{B1 - B7\}$.

For the other direction, take any model $\mathcal{M}$ of $T_5 + \{B1 - B7\}$ with domain M. Consider the set $R = M^2_{/\sim}$, where $\sim$ is equivalence relation defined by $(x, y) \sim (a, b) :\Leftrightarrow x + b = a + y$[1]. The operations $+, \cdot, -$ are defined canonically on R:

- $[(x, y)]_\sim + [(a, b)]_\sim = [(x + a, y + b)]_\sim$
- $[(x, y)]_\sim \cdot [(a, b)]_\sim = [(xa + yb, ab + ya)]_\sim$
- $[(x, y)]_\sim - [(a, b)]_\sim = [(x, y)]_\sim + [(b, a)]_\sim$

Note that these operations are well-defined. Then, $[(0, 0)]_\sim$ is neutral w.r.t. $+$ and $[s0, 0)]_\sim$ is neutral w.r.t to $\cdot$. These elements are thus our 0 and 1 elements. The ring axioms hold.

Define the function $\varphi : M \to R : x \mapsto [(x, 0)]_\sim$. φ is clearly a homomorphism w.r.t. $0, 1, +, \cdot$. Assume that $x \neq 0$ and let y be s.t. $x = sy$. Then $px = y$ and since $x + 0 = x = sy = y + 1$, $\varphi(px) = [(y, 0)]_\sim = [(x, 1)]_\sim = [(x, 0)]_\sim - [(1, 0)]_\sim$. Thus, a copy of $\mathcal{M}$ lies in R. ■

Note that any such ring has characteristic 0 because $0 \neq s^n 0$ for any $n \geqslant 1$.

Definition 4.7.11. *Let $\mathcal{R}$ be a commutative ring with unit and characteristic* 0*. $\mathcal{R}$ induces a graph with the map $x \mapsto x + 1$. The connected components of this graph are called comparison classes.*

Definition 4.7.12. *Let R be a commutative ring with* 1*. For any given natural number d, we define $I_d = \{x \in R \mid dy = 0 \to xy = 0\}$.*

Lemma 4.7.13. *For each natural number d, I_d is an ideal of $\mathcal{R}$.*

Proof. We have to show that I_d is an additive subgroup and that for any $z \in R$ it holds that $zI_d \subseteq I_d$.

I_d is trivially closed under 0 and $-$. To show the closure under $+$, consider two elements x, x' in I_d and any element y s.t. $dy = 0$. By distributivity, we have $(x + x')y = xy + x'y = 0 + 0 = 0$.

[1] Note that this is basically the construction of $\mathbb{Z}$

For the second part, take any $z \in R$, any $x \in I_d$, and any $y \in R$ s.t. $dy = 0$. By associativity, it holds that $(zx)y = z(xy) = z0 = 0$. ■

Lemma 4.7.14. *For each natural number d, for each element $x \in R$ we have $x \equiv k\ (I_d)$ for some $k \in \{0, \dots, d-1\}$.*

Proof. [She67, Lemma 2] ■

Lemma 4.7.15. *Let $f \in \mathcal{R}[x]$ be a polynomial. If f has degree n and more than n roots in one comparison class, then there is some natural number d s.t. the set of roots of f are the union of certain equivalence classes modulo I_d.*

Proof. [She67, Lemma 3] ■

Lemma 4.7.16. *Let $\mathcal{M}$ be any model of $T_5 + \{B1 - B7\}$ with domain M and t some term. Identify M with its embedding in the ring R. There is some polynomial $f \in M[x]$ s.t. $t(x)$ is interpreted as $f(x)$ for almost all elements in M. The (finitely many) elements, where the evaluation differs are all standard elements.*

Proof. f is obtained by replacing every occurrence of px with $x - 1$. Note that $x - 1$ and px have the same evaluation if $x \neq 0$. By the commutativity, associativity and distributivity of $+$ and $\cdot$, f is, w.l.o.g., a polynomial. If n is the largest number s.t. $p^n(t')$ appears in t, then f and t have a different evaluation, at most, at the first n successors of 0. ■

Lemma 4.7.17. *Let $\mathcal{M}$ be any model of $T_0 + \{B1, \dots, B7\}$ and A any atom. If $\mathcal{M} \vDash A(s^n 0)$ for any $n \geqslant 0$, then $\mathcal{M} \vDash \forall x : A(x)$.*

Proof. Take any such model $\mathcal{M}$ and atom A. By Lemma 4.7.16, for every term t, there is a polynomial g_t s.t. $t(x) = g_t(x)$ for almost all elements in $\mathcal{M}$ except for maybe finitely many standard elements. Now take any atom $A(x) \equiv t_1 = t_2$. Define the polynomial $f = g_{t_1} - g_{t_2}$. Then, for almost all $x \in M$, except for maybe finitely many standard elements, $\mathcal{M} \vDash A(x)$ iff $f(x) = 0$. By assumption, f has infinitely many roots in the comparison class of 0. By Lemma 4.7.15, there is some $d \in \mathbb{N}$ s.t. the set of roots of f is the union of certain equivalence classes modulo I_d. Fix this d. We claim that f has a root in every equivalence class of I_d and thus, $f = 0$. Note that $x \equiv x + d \mod I_d$ is equivalent to $d \equiv 0 \mod d$, which holds by definition. Thus, and since $\mathcal{M} \vDash A(0) \wedge A(x) \rightarrow A(sx)$, we conclude that f has a zero in the equivalence classes of $0, 1, \dots, d-1$. By Lemma 4.7.14, these cover all the equivalence classes. Since $\mathcal{M} \vDash A(x)$ iff $f(x) = 0$ for any non-standard element x, we have $\mathcal{M} \vDash \forall x : A(x)$. ■

Theorem 4.7.18. *$T_5 + \{B1 - B7\} \vdash ILiteral$.*

Proof. Take any model $\mathcal{M}$ of $T_5 + \{B1 - B7\}$. By Lemma 4.7.16, for every term t, there is a polynomial g_t s.t. $t(x) = g_t(x)$ for almost all elements in $\mathcal{M}$ except for maybe finitely many standard elements.

It follows directly from Lemma 4.7.17 that $\mathcal{M}$ satisfies induction over atoms.

Now take any any negated atom $L(x) \equiv t_1 \neq t_2$ and assume that $\mathcal{M} \vDash L(0) \wedge L(x) \rightarrow L(sx)$. Again, if $f = g_{t_1} - g_{t_2}$, then $\mathcal{M} \vDash L(x)$ iff $f(x) \neq 0$ for almost all elements, except for maybe finitely many standard elements. If there is some y s.t. $L(y)$ does not hold, then this y has to be a non-standard element and $f(y - m) = 0$ for any $m \geqslant 0$. Thus, f has infinitely many zeros in the comparison class of y. By Lemma 4.7.15, there is some $d \in \mathbb{N}$ s.t. the set of roots of f is the union of certain equivalence classes modulo I_d. In particular, for every element y' in the equivalence class of y it holds that $f(y') = 0$. By Lemma 4.7.14, there is some $k \in \{0, \ldots, d-1\}$ s.t. $k \equiv y \mod I_d$. Note that $x \equiv x + d \mod I_d$ is equivalent to $0 \equiv d \mod I_d$, which holds by definition. Thus, $f(k + md) = 0$ for any $d \in \mathbb{Z}$ and f has infinitely many roots in the standard part of the model, which contradicts our assumption of $\mathcal{M} \vDash L(0) \wedge L(x) \rightarrow L(sx)$. ■

Lemma 4.7.19. $T_5 + ILiteral \nvdash IClause$

Proof. By Lemma 4.7.7 it suffices to give a model of $T_5 + ILiteral$, where some C'_d does not hold.

We construct a suitable model. Take the ring $\mathbb{Z}[u, v]$ and some prime number $p \in \mathbb{P}$. Consider the factor ring $\mathcal{R} = \mathbb{Z}[u, v]_{/p \cdot (u-v)}$. $\mathcal{R}$ is still a commutative ring with 1. For the domain of the model consider the following subset $M = \{[f] \in \mathcal{R} \mid$ all coefficients of highest degree of f are non-negative$\}$. Note that in some equivalence class $[f]$, there can be some polynomials that satisfy the conditions and some that do not - we pick any class that contains at least one polynomial that satisfies the conditions. We observe that M is closed under $0, 1, +$ and $\cdot$. Take any polynomial $g \in \mathbb{Z}[u, v]$. Any polynomial of the fort $g \cdot p \cdot (u - v) - 1$ cannot have only non-negative coefficients of highest degree. It follows that $[-1] \notin M$. However, if $[f] \neq [0]$, then $[f] - [1] \in M$. The model $\mathcal{M}$ is obtained by taking M as the domain and defining the operations canonically. By Lemma 4.7.10, $\mathcal{M}$ is a model of $T_5 + \{B1, \ldots, B7\}$. By Theorem 4.7.18, $\mathcal{M}$ is a model of $ILiteral$.

Now consider $x = y = [u]$ and $z = [v]$. Then $p[u] = [pu] = [pv] = p[v]$, but $([u] + [k])[u] = [(u + k)u] \neq [(u + k)v] = ([u] + [k])[v]$ for any $k \in \{0, \ldots, p-1\}$ as this would imply that $(u + k)(u - v)$ is divisible by p, which cannot be the case. Thus, C'_p does not hold in $\mathcal{M}$. ■

Lemma 4.7.20. $T_5 + ILiteral \vdash IDClause$

Proof. Take any model $\mathcal{M}$ of $T_5 + ILiteral$ with domain M. By Lemma 4.7.10, we know that $\mathcal{M}$ extends to a commutative ring with 1. and by Lemma 4.7.16 we know that for every term t, there is a polynomial g_t s.t. $g(x)$ and $t(x)$ have the same interpretation for almost all elements in $\mathcal{M}$, except for maybe finitely many standard elements.

Take any dual clause $D(x) \equiv L_1 \wedge \cdots \wedge L_n$. Assume that $\mathcal{M} \vDash D(0) \wedge D(x) \rightarrow D(sx)$ If any of the L_i is an atom, then $\mathcal{M} \vDash L_i(s^n 0)$ for any $n \geqslant 0$ and by Lemma 4.7.17, $\mathcal{M} \vDash \forall x : L_i(x)$. Thus, $D \leftrightarrow D'$, with D' being obtained by deleting L_i from D. We can therefore assume that every literal in D is a negated atom $L_i(x) \equiv t_i^1 \neq t_i^2$. For any $i \leqslant n$, let f_i be the associated polynomial s.t. $\mathcal{M} \vDash L_i(x)$ iff $f(x) = 0$ for almost all x except for maybe finitely many standard elements.

Assume that there is some non-standard element $y \in M$ s.t. $\mathcal{M} \nvDash D(y)$. Then, $\mathcal{M} \nvDash D(y - m)$ for any $m \geqslant 0$. For any of these infinitely many $y - m$, one of the L_i cannot hold. Thus, there is some L_i with $\mathcal{M} \nvDash L_i(y - m)$ for infinitely many $m \geqslant 0$. This is equivalent to f having infinitely many roots in the comparison class of y. By Lemma 4.7.15, there is some $d \in \mathbb{N}$ s.t. the set of roots of f is the union of certain equivalence classes modulo I_d. Pick any root and call it y'. By Lemma 4.7.14, $y' \equiv k \mod I_d$ for some $k \in \{0, \ldots, d-1\}$. Since $k \equiv k + md$ for any $m \in \mathbb{Z}$, we obtain that $f_i(k + md) = 0$ for any $m \in \mathbb{Z}$. In particular, f_i has infinitely many roots in the standard part of $\mathcal{M}$, which contradicts our assumption that $\mathcal{M} \vDash D(0) \wedge D(x) \rightarrow D(sx)$. Thus, such a y cannot exist and $\mathcal{M} \vDash \forall x : D(x)$. ■

CHAPTER 5

General Inductive Data Types with a Size Function

We now take an arbitrary general inductive data type as defined in Chapter 3. In contrast to Chapter 3, we will consider, for this sort, constructors only. We will add another sort Nat to this data type, to mimic the natural numbers and add a size function to connect the sorts. This will allow us, to reduce the level of induction we need. In analogy to [Sho58] and [She63], we give a simple alternative axiomatizations of open induction in this context.

This chapter is now closer to reality as one rarely considers only one inductive data type, but usually combines them.

About the outline of the chapter: We start by defining, what we mean with *induction* in the context of two inductive data types. Then, we will prove some general lemmas about these notions of induction.

5.1 General Frame

We start similar as in Section 3.1: We consider a (possibly) many-sorted logic with the sorts $\mathsf{D}, T_1, \ldots, T_n$. The first part of our language consists of the constructors $c_1, \ldots, c_k$, where each of the c_i has arity m_i and is a function symbol of type $\tau_i^1 \times \cdots \times \tau_i^{m_i} \to \mathsf{D}$ with $\tau_i^l \in \{\mathsf{D}, T_1, \ldots, T_n\}$.

The definition of *static* and *dynamic* constructors is taken from Section 3.1. In order to define induction, we need some well-founded order relation on the elements of the standard-model. This translates to the restriction that there is at least one static constructor c_i.

Without loss of generality, we assume that for every constructor c_i the first n_i (possibly 0) input-sorts are D. The other sorts T_l are ordered by their index i.

The following will be the first part of our base axioms (cf. Section 3.1):

$\mathbf{D_{i,j}}$ $c_i(\overline{x}) \neq c_j(\overline{y})$ for all $i \neq j, 1 \leqslant i, j \leqslant k$ Disjointness

$\mathbf{INJ_i}$ $c_i(\overline{x}) = c_i(\overline{y}) \rightarrow \overline{x} = \overline{y}$ for all $1 \leqslant i \leqslant k$ Injectivity

Now we add another sort Nat and enrich our language with the symbols $0 \in \mathsf{Nat}, s : \mathsf{Nat} \rightarrow \mathsf{Nat}, + : \mathsf{Nat} \times \mathsf{Nat} \rightarrow \mathsf{Nat}, l : \mathsf{D} \rightarrow \mathsf{Nat}$. We add the following to our base axioms for the sort Nat (cf. Section 4.1):

A1 $s(x) \neq 0$

A3a $sx = sy \rightarrow x = y$

A4 $x + 0 = x$

A5 $x + s(y) = s(x + y) \neq 0$

Furthermore, we need to axiomatize the size-function l:

E1 $l(c_i(\overline{x})) = 0$ for any static constructor c_i

E2 $l(c_i(X_1, \ldots, X_j, \overline{x})) = s(\sum_{i=1}^{j} l(X_i))$ for any dynamic constructor, where exactly the first j input-sorts of c_i are D

Those are all the base axioms. We also need a definition for our case distinction:

Definition 5.1.1. *A constructor $c_i : \tau_i^1 \times \cdots \times \tau_i^{m_i}$ is essentially unary iff exactly one of the τ_i^l is D. Otherwise, it is not essentially unary.*

Lastly, we define some auxiliary axioms, which we will need later. Note that these axioms are not chosen arbitrarily. The ones relating to arithmetics were used in Chapter 4 to give an alternative axiomatization of open induction in the context of arithmetics. The ones concerning the data type D were used in Chapter 3 to give an alternative axiomatization of open induction in the general context. Using them as a starting point, in order to find an alternative axiomatization of open induction seems like a logical step. In fact, we will see that they suffice.

$\mathbf{G_t}$ $X \neq t(X)$ for any $t \in \mathcal{S}$ (Acyclicity for D)

SUR $\bigvee_{i=1}^{k} \exists \overline{y} : X = c_i(\overline{y})$ (Surjectivity for D)

$\mathbf{B_n}$ $x \neq s^n x$ for any $n \geqslant 1$ (Acyclicity for Nat)

B1a $x = 0 \vee \exists y : x = sy$ (Surjectivity for Nat)

B2 $x + y = y + x$

B3 $x + (y + z) = (x + y) + z$

B4 $x + y = x + z \rightarrow y = z$

Regarding the notation, we define a shorthand: Instead of writing $x+x+\cdots+x$ (n-times), we write nx. Since we do not have the symbol $\cdot$ in our language, this should not lead to any misunderstanding.

We overload the name of our language and base theory:

Definition 5.1.2. *If there are essentially unary constructors only, then* $\mathcal{L}_0 = \{c_1, \dots, c_k\} \cup \{0, s, l\}$ *and* $T_0 = \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{D2_i \mid 1 \leqslant i \leqslant k\} \cup \{A1, A3a, E1, E2\}$*.*

If there is at least one constructor, which is not essentially unary, then $\mathcal{L}_0 = \{c_1, \dots, c_k\} \cup \{0, s, l, +\}$ *and* $T_0 = \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{D2_i \mid 1 \leqslant i \leqslant k\} \cup \{A1, A3a, A5, A5, E1, E2\}$

Note that the definition of $\mathcal{L}_0$ is perfectly fine in both cases as we do not need $+$ if there are only essentially unary constructors.

5.1.1 The Scheme of Induction

In this case, with two sorts that satisfy some form of induction (at least in their respective standard model), it is a priori unclear how the scheme of induction looks like. We present two possibilities, that we will deal with in the following sections.

The first possibility, is to take the two single schemes of induction and take the union of the induced sets of axioms. In that case, we have two left-hand-sides for the respective axiom of induction:

$$\mathbf{LHS}_{\mathsf{D}}(\varphi(x)) \quad \bigwedge_{i=1}^{k} \left(\forall x_1, \dots, x_{m_i} \left(\bigwedge_{\substack{l \in \{1, \dots, m_i\} \\ \tau_i^l = \mathsf{D}}} \varphi(x_l, \overline{z}) \to \varphi(c_i(x_1, \dots, x_{m_i}), \overline{z}) \right)\right)$$

$$\mathbf{LHS}_{\mathsf{Nat}}(\psi(x)) \ \ \psi(0) \wedge (\forall x : A)(\psi(x) \to \psi(sx))$$

The schemes of induction now have the following form:

$$\mathbf{I}_{\mathsf{D}}(\varphi) \quad \mathbf{LHS}_{\mathsf{D}}(\varphi(x, \overline{z})) \to (\forall X : D)(\varphi(X, \overline{z}))$$
$$\mathbf{I}_{\mathsf{Nat}}(\psi) \ \ \mathbf{LHS}_{\mathsf{Nat}}(\psi(x, \overline{z})) \to (\forall x : A)(\psi(x, \overline{z}))$$

The formulas φ and ψ potentially contain parameters $\overline{z}$, which we will not explicitly mention in the following as from now on every formula contains parameters if not stated otherwise.

In the subscript of each scheme, it says, which sort it applies to.

Definition 5.1.3. *We define* $\mathbf{I_2} = \mathbf{I}_{\mathsf{Nat}} + \mathbf{I}_{\mathsf{D}}$ *to be the union of the two single schemes of induction.*

It should be stated that $\mathbf{I_2}$ is the standard approach in the literature. However, there is a problem with this form of induction if we restrict it to open formulas: Assume that we want to show that $T \vdash (\forall X : D)(\forall x : A)(F(x, X))$ for some base theory T and open formula F. We could apply the single schemes sequentially. The problem, however, is that after the first application, we obtain $\forall x : \varphi(X, x)$, which is not open anymore.

This leads to the second option. Let C be the set of all static constructors and C' the set of all dynamic constructors. We define a new left-hand-side:

$\mathbf{LHS_1}(\varphi(x, X, \overline{z}))$

$$\bigwedge_{c_i \in C} \forall \overline{x} : \varphi(c_i(\overline{x}), 0, \overline{z}) \wedge \bigwedge_{c_i \in C'} \left(\forall \overline{x} \left(\bigwedge_{\tau_i^k = \mathsf{D}} \varphi(x_k, u, \overline{z}) \rightarrow \varphi(x_1, s(u), \overline{z}) \wedge \varphi(c_i(\overline{x}), u, \overline{z}) \right)\right)$$

Then define the scheme of induction:

$\mathbf{I_1}(\varphi)$ $\mathbf{LHS_1}(\varphi(x, X, \overline{z})) \rightarrow (\forall X : D)(\forall x : A)(\varphi(x, X, \overline{z}))$

This scheme introduces two universal quantifiers at once and thus, avoids the aforementioned problem.

The following observation seems obvious, but needs to be formulated nonetheless:

Observation 5.1.4. *For any language $\mathcal{L}$ and base theory T, it holds that if some formula φ can be shown in $T + \mathbf{I_2}(\Gamma)$, where $\Gamma \in \{IAtom, ILiteral, IClause, IDClause, IOpen\}$, then φ can be shown in $T + \mathbf{I_1}(\Gamma)$.*

5.1.2 General Lemmas

We will now present some general lemmas that will come in handy later. Consider the following example as motivation:

Example 5.1.5. *Consider the language $\{0, s, c_1, c_2, l, +\}$ and the sorts* D *and* Nat. $0 \in \mathsf{Nat}, s : \mathsf{Nat} \rightarrow \mathsf{Nat}, + : \mathsf{Nat} \times \mathsf{Nat} \rightarrow \mathsf{Nat}, c_1 \in D, c_2 : D \rightarrow \mathsf{D}$, $l : D \rightarrow \mathsf{Nat}$. *Let* $T = \{D1_{1,2}, D2_1, A1, A3a, A4, A5, E1, E2\}$.

The standard model $\mathcal{M}$ of this theory is two copies of $\mathbb{N}$ - one for Nat *and one for* D. *Consider any atom $B(x, X)$ in this standard model. There is some affine function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ with integer coefficients s.t. $\mathcal{M} \vDash B(\underline{n}, \underline{m})$ iff $f(n, m) = 0$ for all natural numbers n, m. Let $S_f = \{(n, m) \in \mathbb{N}^2 \mid f(n, m) = 0\}$ be the set of solutions of f. We identify f with its extension in $\mathbb{R}^{\mathbb{R} \times \mathbb{R}}$. The set $P_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$ now describes a plane in $\mathbb{R}^3$. If there are solutions of B in $\mathcal{M}$ of the form $(\underline{l}, \underline{c}), (\underline{l}, \underline{c}), (\underline{a}, \underline{m}), (\underline{b}, \underline{m})$ with $a \neq b$ and $l \neq n$, then $P_f = \{(x, y, 0) \mid (x, y) \in \mathcal{R}^2\}$, f is constantly 0 and $\mathcal{M} \vDash (\forall x : \mathsf{Nat})(\forall X : \mathsf{D})(B(x, X))$.*

The idea of the following lemmas is to abstract the observation of the example above and use it, to prove that only trivial atoms have *many* solutions in the standard part of the models, we consider.

Lemma 5.1.6. *Let $(A_i)_{i=1}^n$ be a family of subsets of $\mathbb{N} \times \mathbb{N}$ s.t. $\bigcup_{i=1}^n A_i = \{0, \dots, n\}^2$. Then there is some A_i and $a, b, c, x, y, z \leqslant n$ s.t. $a \neq b$, $x \neq y$ and $\{(a, z), (b, z), (c, x), (c, y)\} \subseteq A_i$.*

Proof. Fix such sets $(A_i)_{i=1}^n$ and assume that there is no A_i and a, b, c, x, y, z with the desired properties. We fix the terminology, that two points a, b *lie on a line* if either the first or the second coordinate are the same (i.e. we ignore diagonals). We show, that under these assumptions - for each A_i there is at most one line with two points on it - every A_i contains at most $n+1$ elements. Fix any A_i and assume that A_i contains $n+2$ elements of the form (u_l, v_l). Since there are $n+1$ rows and $n+1$ columns, by the pigeonhole principle, there have to be u and v s.t. there are elements $(u, v_0), (u, v_1), (u_0, v), (u_1, v)$ in A_i with $v_0 \neq v_1$ and $u_0 \neq u_1$. Thus, for any A_i it holds that $|A_i| \leqslant n+1$.

It follows that $|\bigcup_{i=1}^n A_i| \leqslant n(n+1) < (n+1)^2 = |\{0, \dots, n\}^2|$, which contradicts our assumptions. Thus, there exists some A_i and some a, b, c, x, y, z with the desired properties. ∎

Lemma 5.1.7. *Assume that there is at least one dynamic constructor c_i. Let $\mathcal{L} \supseteq \{c_1, \dots, c_k, s\}$ be a language, $T \supseteq \{A1, A3a\} \cup \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{D2_i \mid 1 \leqslant i \leqslant k\}$ a theory and $\mathcal{M}$ a model of T. Let $F(x, X) \equiv F_1 \vee \cdots \vee F_n$ be a disjunction of other formulas. If $\mathcal{M} \vDash F(a, A)$ for all standard elements $a \in \mathsf{Nat}^{\mathcal{M}}$ and $A \in \mathsf{D}^{\mathcal{M}}$, then there is some F_i and some $a, b, c \in \mathsf{Nat}^{\mathcal{M}}$ and $A, B, C \in \mathsf{D}^{\mathcal{M}}$ with the following properties:*

1. *$a = s^n b$ for some $n \neq 1$*
2. *$A = t(B)$ for some non-constant term t that is not the identity*
3. *$\mathcal{M} \vDash F_i(a, C) \wedge F_i(b, C) \wedge F_i(c, A) \wedge F_i(c, B)$*

Proof. First, fix some instance $c' = c_j(\overline{a})$ for some static constructor c_j. Now, fix the dynamic constructor c_i. We define as a shorthand $c(E) = c_i(E, \overline{b})$, where E is of sort D, every other occurrence of some variable of sort D is substituted with c', and the rest of $\overline{b}$ is some tuple of elements of appropriate sorts. Then, by $A3a$ and $D2_i$, it holds that $s^n 0 = s^m 0 \Rightarrow n = m$ and $c^n(c') = c^m(c') \Rightarrow m = n$. Thus, there is a natural bijection between $\mathbb{N} \times \mathbb{N}$ and $\{(s^n 0, c^m(c')) \mid n, m \in \mathbb{N}\}$. Since all these elements $s^n 0$ and $c^n(c')$ are standard elements, $\mathcal{M} \vDash F(s^n 0, c^m(c'))$ for any $n, m \in \mathbb{N}$. Define the function $f : \{1, \dots, n\} \to \mathcal{P}(\mathbb{N} \times \mathbb{N}) : i \mapsto \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \mathcal{M} \vDash F_i(s^n 0, c^m(c'))\}$. By assumption, $\mathbb{N} \times \mathbb{N} = \bigcup_{i=1}^n f(i)$ and thus, $\{0, \dots, n\}^2 = \{0, \dots, n\}^2 \cap \bigcup_{i=1}^n f(i) = \bigcup_{i=1}^n \left(f(i) \cap \{0, \dots, n\}^2\right)$. Now, we apply Lemma 5.1.6 and obtain that there is some $i \in \{1, \dots, n\}$ s.t. $f(i) \cap \{0, \dots, n\}^2$ contains elements $(j, r), (l, r), (k, p), (k, q)$ with $j \neq l$ and $p \neq q$. W.l.o.g., $j > l$ and $p > q$. By definition, this means that $\mathcal{M} \vDash F_i(s^j 0, c^r(c')) \wedge F_i(s^l 0, c^r(c')) \wedge F_i(s^k 0, c^p(c')) \wedge F_i(s^l 0, c^q(c'))$. Setting $a = s^j 0, b = s^l 0, c = s^k 0, A = c^p(c'), B = c^q(c'), C = c^r(c')$ proves the claim. ∎

Lemma 5.1.8. *Assume that there is at least one dynamic constructor c_i. Let $\mathcal{L} \supseteq \{c_1, \ldots, c_k, s\}$ be a language, $T \supseteq \{\mathrm{SUR}, B1a\} \cup \{G_t \mid t \in \mathcal{S}\} \cup \{B_n \mid n \geqslant 1\}$ a theory and $\mathcal{M}$ a model of $T + \mathbf{I_2}(Open)$. Let $F(x, X) \equiv D_1 \vee \cdots \vee D_n$ be a formula in DNF s.t. one of the dual clauses $D_i \equiv L_1 \wedge \cdots \wedge L_k$ consists of negated atoms only. Assume that $\mathcal{M} \vDash \mathbf{LHS_1}(F)$ and that there are non-standard elements $a \in \mathsf{Nat}^{\mathcal{M}}$ and $A \in \mathsf{D}^{\mathcal{M}}$ s.t. $\mathcal{M} \nvDash F(a, A)$. Then, there is some negated atom L_j in D_i and some $a, b, c \in \mathsf{Nat}^{\mathcal{M}}$ and $A, B, C \in \mathsf{D}^{\mathcal{M}}$ with the following properties:*

1. *$a = s^n b$ for some $n \neq 1$*
2. *$A = t(B)$ for some non-constant term t that is not the identity*
3. *$\mathcal{M} \nvDash L_j(a, C) \vee L_j(b, C) \vee L_j(c, A) \vee L_j(c, B)$*

Proof. Our goal will be to apply Lemma 5.1.6. First, we define partial function $q : \mathsf{D}^{\mathcal{M}} \hookrightarrow \mathsf{D}^{\mathcal{M}}$ on the set of all $B \in \mathsf{D}^{\mathcal{M}}$ s.t. $\mathcal{M} \nvDash F(a, B)$. Fix such a B. Note that the set $Q_B = \{C \in \mathsf{D}^{\mathcal{M}} \mid \mathcal{M} \nvDash F(a, C)$, there is some c_i and $\bar{c}$ s.t. $B = c_i^{\mathcal{M}}(C, \bar{c})\}$ is not empty since $\mathcal{M} \vDash \mathbf{LHS_1}(F)$ and $\mathcal{M} \vDash \mathrm{SUR}$. For any standard element $E \in \mathsf{D}^{\mathcal{M}}$ consider the formula $F'(x) \equiv F(x, E)$. It holds that $\mathcal{M} \vDash \mathbf{LHS_1}(F')$ and thus, the ones of $\mathbf{I_2}$. Since $\mathcal{M} \vDash \mathbf{I_2}(Open)$, we conclude that $\mathcal{M} \vDash \forall x : F(x, E)$. In particular, $\mathcal{M} \vDash F(a, E)$ for any standard element E and the set Q_B contains only non-standard elements. Now define $q(B)$ by choosing any element from Q_B.

Analogously, we define the partial function $p : \mathsf{Nat}^{\mathcal{M}} \hookrightarrow \mathsf{Nat}^{\mathcal{M}}$ on the set of all $B \in \mathsf{D}^{\mathcal{M}}$ with $\mathcal{M} \nvDash F(b, A)$. Let $P_b = \{c \in \mathsf{Nat}^{\mathcal{M}} \mid \mathcal{M} \nvDash F(c, A) \wedge b = sc\}$. For the same reasons as above P_b is not empty and contains only non-standard elements. Define $p(b)$ by choosing some element in P_b.

From G_t and B_n, it follows that $p^n a \neq a$ and $q^n a \neq a$ for any $n \geqslant 1$. Assume that there is some $n, m \in \mathbb{N}$ s.t. $\mathcal{M} \vDash F(p^n a, q^m A)$. Then, since $\mathcal{M} \vDash \mathbf{LHS_1}(F)$, $\mathcal{M} \vDash F(a, q^m A)$, which contradicts the definition of q. Thus, for every $n, m \in \mathbb{N}$, it holds that $\mathcal{M} \nvDash F(p^n a, q^m A)$. Since, we work in a model and every parameter is fixed, we obtain $\mathcal{M} \vDash \neg F(p^n a, q^m A)$ and, by classical logic, $\mathcal{M} \vDash \neg D_i(p^n a, q^m A)$. Define the function $f : \{1, \ldots, k\} \to \mathcal{P}(\mathbb{N} \times \mathbb{N}) : i \mapsto \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \mathcal{M} \vDash \neg L_i(p^n a, q^m A)\}$. By assumption, $\bigcup_{i=1}^{k} f(i) = \mathbb{N} \times \mathbb{N}$ and $\{0, \ldots, k\}^2 \cap \bigcup_{i=1}^{k} f(i) = \{0, \ldots, k\}^2$. We apply Lemma 5.1.6 and obtain that there is some j s.t. $\{0, \ldots, k\}^2 \cap f(j)$ contains elements k, l, m, r, s, t with $(k, t), (l, t), (r, m), (s, m) \in f(j)$. This means that $\mathcal{M} \nvDash L_j(p^k a, q^t A) \vee L_j(p^l a, q^t A) \vee L_j(p^r a, q^m A) \vee L_j(p^s a, q^m A)$, which proves the claim. ∎

Definition 5.1.9. *Let $\mathcal{L} \supseteq \{c_1, \ldots, c_n, 0, s\}$ be a language and T a theory over $\mathcal{L}$. We say that T has the anchor property if the following implication holds: Take any atom $A(x, X)$ with the variables $x \in \mathsf{Nat}$ and $X \in \mathsf{D}$ and any model $\mathcal{M}$ of T. Assume that there are elements $e, f, g \in \mathsf{Nat}^{\mathcal{M}}$ and $E, F, G \in \mathsf{D}^{\mathcal{M}}$ s.t. $a = s^n b$, $A = t(B)$, where t contains constructors only and $A \neq B$, and $\mathcal{M} \vDash A(e, G) \wedge A(f, G) \wedge A(g, F) \wedge A(g, G)$. Then, it holds that $\mathcal{M} \vDash \forall x \forall X : A(x, X)$.*

Lemma 5.1.10. *Let $\mathcal{L} \supseteq \{c_1, \ldots, c_n, 0, s\}$ be a language with a least one dynamic constructor c_i and and $T \supseteq \{A1, A3a, \mathrm{SUR}, B1a\} + \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} + \{D2_i \mid 1 \leqslant i \leqslant k\} + \{G_t \mid t \in \mathcal{S}\} + \{B_n \mid n \geqslant 1\}$ a theory with the anchor property. Then $T \vdash \mathbf{I_2}(Open)$.*

Proof. We need to consider two induction schemes: For Nat and for D. Take any model $\mathcal{M}$ of T. Now, we make a case distinction:

1. Take any formula $F(x) \equiv D_1 \vee \cdots \vee D_n$ in DNF and assume that $\mathcal{M} \vDash F(0) \wedge F(x) \to F(sx)$. Assume that every dual clause D_i contains some positive A_i and consider the formula $F' \equiv A_1 \vee \cdots \vee A_n$. By Lemma 5.1.7, there are elements l, m, n, L, M, N and some atom A_i s.t. $\mathcal{M} \vDash A_i(l, N) \wedge A_i(m, N) \wedge A_i(n, L) \wedge A_i(n, M)$ and $l = s^k n, L \neq M, L = t(M)$. By the anchor property, $\mathcal{M} \vDash \forall x \forall X : A_i(x, X)$. Note that the X in $A_i(x, X)$ does not actually appear anywhere. There are two cases:

- If $D_i \equiv A_i$, then $\mathcal{M} \vDash \forall x : F(x)$ and we are done
- If $D_i \equiv A_i \wedge D'_i$, then $\mathcal{M} \vDash F \leftrightarrow G$, where G is obtained from F by replacing D_i with D'_i. We can restart the procedure with G.

Thus, w.l.o.g., we can assume that there is some dual clause $D_i \equiv L_1 \wedge \cdots \wedge L_k$ that contains negated atoms only. Assume that there is some literal L_j with $\mathcal{M} \vDash \forall x : \neg L_j(x)$. There are two cases:

- If $F \equiv D_i$, then $\mathcal{M} \vDash F \leftrightarrow \bot$, which contradicts our assumption of $\mathcal{M} \vDash F(0)$
- If $F \equiv D_i \vee G$, then $\mathcal{M} \vDash F \leftrightarrow G$ and we can restart the procedure with G

Thus, w.l.o.g., we can assume that for every L_j in D_i, it holds that $\mathcal{M} \vDash \exists x : L_j(x)$. We define the partial function $p : \mathsf{Nat}^{\mathcal{M}} \hookrightarrow \mathsf{Nat}^{\mathcal{M}}$ on the set $S = \{n \in \mathsf{Nat}^{\mathcal{M}} \mid \mathcal{M} \nvDash F(n)\}$. Pick any element $m \in S$. m has to be a non-standard element and in particular $m \neq 0$. By $B1a$, the set $P_m = \{n \in \mathsf{Nat}^{\mathcal{M}} \mid sn = m\}$ is not empty. By $A3a$, P_m contains exactly one element n. Define $p(m) = n$. Note that $\mathcal{M} \nvDash F(n)$ since $\mathcal{M} \vDash F(x) \to F(sx)$. Thus, it makes sense to write $p^k m$ for the k-th application of p on m. By B_n, $l \neq k$ implies that $p^l m \neq p^k m$. By the pigeonhole principle, there is some L_j and $l \neq n, 1 \leqslant l, n \leqslant k+1$ with $\mathcal{M} \vDash \neg L_j(p^l m) \wedge \neg L_j(p^n m)$. Since X does not appear in L_j, $\mathcal{M} \vDash L_j(m, A) \leftrightarrow L_j(m, B)$ for any $A, B \in \mathsf{D}^{\mathcal{M}}$. In particular, the conditions of the anchor property are triggered and $\mathcal{M} \vDash \forall x : \neg L_j(x)$. This, however, contradicts our assumption above. Thus, such an $m \in \mathsf{Nat}^{\mathcal{M}}$ does not exist.

2. Take any formula $F(X) \equiv D_1 \vee \cdots \vee D_n$ in DNF and assume that $\mathcal{M} \vDash \mathbf{LHS}_{\mathsf{D}}(F)$. For the same reasons as above, we can assume w.l.o.g. that there is some dual clause $D_i \equiv L_1 \vee \cdots \vee L_k$ that consists of negated atoms only and $\mathcal{M} \vDash \exists X : L_j(X)$ for any j. We define a partial function $q : \mathsf{D}^{\mathcal{M}} \hookrightarrow \mathsf{D}^{\mathcal{M}}$ on the set $S' = \{N \in \mathsf{D}^{\mathcal{M}} \mid \mathcal{M} \nvDash F(N)\}$. Take any element $A \in S'$. By SUR, the set $Q_A = \{B \in \mathsf{D}^{\mathcal{M}} \mid$ There is some c_i and tuple $\overline{y}$ s.t. $A = c_i(B, \overline{y}\}$ is not empty. Since $\mathcal{M} \models \mathbf{LHS_D}(F)$, there is some element $B \in Q_A$ with $\mathcal{M} \not\models F(B)$. Define $q(A) = B$. Again, it makes sense to write $q^k(A)$ for the k-th application of q. By G_t, it holds that $k \neq l$ implies that $q^k(A) \neq q^l(A)$. Now assume that there is in fact some $A \in S'$. By the pigeonhole principle, there is some L_j in D_i and some $k \neq l \in \mathbb{N}$ with $\mathcal{M} \not\models L_j(q^k(A)) \vee L_j(q^l(A))$. Again, the anchor property is triggered and $\mathcal{M} \models \forall X : \neg L_j(X)$, which contradicts our assumption. Thus $S' = \varnothing$. ∎

Theorem 5.1.11. *Let $\mathcal{L} \supseteq \{c_1, \dots, c_n, 0, s\}$ a language with a least one dynamic constructor c_i and and $T \supseteq \{A1, A3a, \mathrm{SUR}, B1a\} + \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} + \{D2_i \mid 1 \leqslant i \leqslant k\} + \{G_t \mid t \in \mathcal{S}\} + \{B_n \mid n \geqslant 1\}$ a theory with the anchor property. Then $T \vdash \mathbf{I_1}(Open)$.*

Proof. First, note that by Lemma 5.1.10, $\Gamma \vdash \mathbf{I_2}(Open)$.

Now, take any model $\mathcal{M}$ of T and any formula $F(x, X) \equiv D_1 \wedge \dots \wedge D_n$ in DNF. Assume that $\mathcal{M} \models \mathbf{LHS_1}(F)$. If every dual clause D_i contains an atom A_i, then consider the formula $G = A_1 \vee \dots \vee A_n$. Clearly, $\vdash F \rightarrow G$. Thus, $\mathcal{M} \models G(a, A)$ for any standard two elements $a \in \mathsf{Nat}^{\mathcal{M}}$ and $A \in \mathsf{D}^{\mathcal{M}}$. Lemma 5.1.7 is applicable and there is some A_i and elements $b, c, e \in \mathsf{Nat}^{\mathcal{M}}$ and $B, C, E \in \mathsf{D}^{\mathcal{M}}$ with $b = s^n c$, $B = t(C), B \neq C$ and $\mathcal{M} \models A_i(b, E) \wedge A_i(c, E) \wedge A_i(e, B) \wedge A_i(e, C)$. By the anchor property, $\mathcal{M} \models \forall x \forall X : A_i(x, X)$. We make a case distinction:

- If $D_i = A_i$, then $\mathcal{M} \models \forall x \forall X : F(x, X)$ and induction over F holds
- If $D_i = A_i \wedge D_i'$, then $\mathcal{M} \models F \leftrightarrow F'$, where F' is obtained from F by replacing D_i with D_i'. We can restart with F'

Thus, w.l.o.g., we can assume that there is one dual clause D_i, which only contains negated atoms. Now fix this $D_i \equiv L_1 \wedge \dots L_k$. Assume that there is some L_j with $\mathcal{M} \models \forall x \forall X : \neg L_j(x, X)$. We make another case distinction:

- If $F \equiv D_i$, then $\mathcal{M} \models \forall x \forall X : \neg F(x, X)$, which is contradictory to our assumption
- If $F \equiv D_i \vee F'$, then $\mathcal{M} \models F \leftrightarrow F'$ and we can restart with F'

Thus, w.l.o.g., we can assume that for every literal L_j, $\mathcal{M} \models \exists x \exists X : L_j(x, X)$. Assume that there are some elements $a \in \mathsf{Nat}^{\mathcal{M}}$ and $A \in \mathsf{D}^{\mathcal{M}}$ s.t. $\mathcal{M} \models \neg F(a, A)$. Then, Lemma 5.1.8 is applicable and we obtain some L_j and elements $b, c, e \in \mathsf{Nat}^{\mathcal{M}}$ and $B, C, E \in \mathsf{D}^{\mathcal{M}}$ with $b = s^n c$, $B = t(C), B \neq C$ and $\mathcal{M} \models \neg L_j(b, E) \wedge \neg L_j(c, E) \wedge \neg L_j(e, B) \wedge \neg L_j(e, C)$. Note that $\neg L_j$ is an atom and by the anchor property, we obtain that $\mathcal{M} \models \forall x \forall X : \neg L_j(x, X)$, which contradicts our assumption. Thus, such elements $a \in \mathsf{Nat}^{\mathcal{M}}$ and $A \in \mathsf{D}^{\mathcal{M}}$ cannot exist and $\mathcal{M} \models \forall x \forall X : F(x, X)$. ∎

5.2 Useful models

We will now define some models, which will prove useful in the later parts of this chapter. One model will be for the case, where there are only essentially unary constructors and two will be for the case, where there are not essentially-unary constructors as well.

5.2.1 The model $\mathcal{M}_\infty^u$

In this subsection, assume that every dynamic constructor is essentially unary[1]. We fix the language $\mathcal{L}_0$ and the base theory T_0:

Definition 5.2.1. $\mathcal{L}_0 = \{c_1, \ldots, c_k\} \cup \{0, s, l\}$ *and* $T_0 = \{D1_{i,j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{D2_i \mid 1 \leqslant i \leqslant k\} \cup \{A1, A3a, E1, E2\}$

Note that the definition of $\mathcal{L}$ is perfectly fine as we do not need $+$ because every dynamic constructor is essentially unary.

Now, we define the model $\mathcal{M}_\infty^u$:

Definition 5.2.2. *We start with the interpretation of the sorts:*

- $T_i^{\mathcal{M}_\infty^u} = \{i\}$
- $\mathsf{D}^{\mathcal{M}_\infty^u} = \mathbb{T}(T_1^{\mathcal{M}_\infty^u}, \ldots, T_n^{\mathcal{M}_\infty^u})$
- $\mathsf{Nat}^{\mathcal{M}_\infty^u} = \mathbb{N} \cup \{\infty\}$

The constructors are interpreted canonically:

- $c_i(\overline{a})^{\mathcal{M}_\infty^u} = c_i(\overline{a})$ *for any static constructor* c_i
- $c_i^{\mathcal{M}_\infty^u}(t, a_2, \ldots, a_{m_i})) = c_i(t_1, a_2 \ldots, a_{m_i})$ *for any dynamic constructor* c_i
- $0^{\mathcal{M}_\infty^u} = 0$
- $s^{\mathcal{M}_\infty^u} n = n + 1,\ s^{\mathcal{M}_\infty^u} \infty = \infty$

The size function l *is interpreted according to* $E1$ *and* $E2$*:*

- $l^{\mathcal{M}_\infty^u}(c_i^{\mathcal{M}_\infty^u}(\overline{a})) = 0$ *for any static constructor* c_i
- $l^{\mathcal{M}_\infty^u}(c_i(t, a_2, \ldots, a_{m_i})) = s^{\mathcal{M}_\infty^u} l^{\mathcal{M}_\infty^u}(t)$ *for any dynamic constructor* c_i

Observation 5.2.3. $\mathcal{M}_\infty^u \vDash T_0$

Observation 5.2.4. $\mathcal{M}_\infty^u \nvDash x \neq s^n x$ *for any* $n \geqslant 1$

[1] The u in $\mathcal{M}_\infty^u$ comes from *unary*

5.2.2 The model $\mathcal{M}_\infty$

In this subsection assume that there is at least one constructor, which are not essentially unary. Now, we define the model $\mathcal{M}_\infty$:

Definition 5.2.5. *We start with the interpretation of the sorts:*

- $T_i^{\mathcal{M}_\infty} = \{i\}$
- $\mathsf{D}^{\mathcal{M}_\infty} = \mathbb{T}(T_1^{\mathcal{M}_\infty}, \dots, T_n^{\mathcal{M}_\infty})$
- $\mathsf{Nat}^{\mathcal{M}_\infty} = \mathbb{N} \cup \{\infty\}$

The constructors are interpreted canonically:

- $c_i^{\mathcal{M}_\infty}(\overline{a}) = c_i(\overline{a})$ *for any static constructor* c_i
- $c_i^{\mathcal{M}_\infty}(t_1, \dots, t_{k_i}, a_{k_i+1}, \dots, a_{m_i}) = c_i(t_1, \dots, t_{m_i})$ *for any dynamic constructor* c_i*, where exactly the first* k_i *inputs are of sort* D
- $0^{\mathcal{M}_\infty} = 0$
- $s^{\mathcal{M}_\infty} n = n + 1$, $s^{\mathcal{M}_\infty} \infty = \infty$

The symbol $+$ *is interpreted as in* $\mathbb{N}_\infty$*:*

- $(n + m)^{\mathcal{M}_\infty} = n + m$
- $(n + \infty)^{\mathcal{M}_\infty} = (\infty + n)^{\mathcal{M}_\infty} = \infty$

The size function l *is interpreted according to* $E1$ *and* $E2$*:*

- $l^{\mathcal{M}_\infty}(c_i(\overline{a})) = 0$ *for any static constructor* c_i
- $l^{\mathcal{M}_\infty}(c_i(t_1, \dots, t_{k_i}, a_{k_i+1}, \dots a_{m_i})) = s^{\mathcal{M}_\infty}((l(t_1)) + \dots + l(t_{k_i}))^{\mathcal{M}_\infty}$ *for any dynamic constructor* c_i*, where exactly the first* k_i *inputs are of sort* D

Observation 5.2.6. $\mathcal{M}_\infty \vDash T_0$

Observation 5.2.7. $\mathcal{M}_\infty \nvDash x \neq s^n x$ *for any* $s \geqslant 1$

Observation 5.2.8. *In* $\mathcal{M}_\infty$*,* $+$ *is associative and commutative. Thus, for any term* $t(x, X)$ *of sort* Nat*, there is another term* $s(x, X) \equiv nl(X) + mx + s'$ *s.t. neither* x *nor* X *appear in* s' *and* $\mathcal{M}_\infty \vDash (\forall x : \mathsf{Nat})(\forall X : \mathsf{D})(t(x, X) = s(x, X))$*. Therefore, we can assume that any term of sort* Nat *has this form.*

Lemma 5.2.9. $\mathcal{M}_\infty \vDash \mathbf{I_1}(\mathit{Atom})$

Proof. Take any atom $A(x, X) \equiv t_1 = t_2$ and assume that $\mathcal{M} \vDash \mathbf{LHS_1}(A)$. There are two cases:

1. Both terms t_i are of the sort D. Then, x cannot appear in A. By construction of $\mathcal{M}_\infty$, induction on the sort D and thus, on A, holds.

2. Both terms are of the sort Nat. W.l.o.g., assume that each t_i has the form $n_i l(X) + m_i x + t'_i$ for some natural numbers n_i, m_i. Fix some static constructor c_i and some dynamic constructor c_j. We define $B = c_i^{\mathcal{M}_\infty}(\overline{b})$ for some tuple $\overline{b}$ and $C = c_j^{\mathcal{M}_\infty}(\overline{c})$ for some tuple $\overline{c}$, where each element in $\overline{c}$ of sort D is B. Note that $l^{\mathcal{M}_\infty}(B) = 0$ and $l^{\mathcal{M}_\infty}(C) = s0$. By assumption, $\mathcal{M}_\infty \vDash A(0, B) \wedge A(0, C) \wedge A(s0, B)$. Thus, $\mathcal{M}_\infty \vDash t'_1 = n_1 0 + m_1 0 + t'_1 = t_1(0, B) = t_2(0, B) = n_2 0 + m_2 0 + t'_2 = t'_2$. Define $t' \equiv t'_1$. There is now another case distinction: If $t'^{\mathcal{M}_\infty} = \infty$, then $\mathcal{M} \vDash \forall x \forall X : A(x, X)$ since ∞ *absorbs* everything. Assume that $t'^{\mathcal{M}_\infty} \neq \infty$. Note that in the standard part of $\mathbb{N}_\infty$ cancellation w.r.t. $+$ holds. Then, $m_1 s0 + t' = t_1(s0, B) = t_2(s0, B) = m_2 s0 + t'$. By cancellation, we obtain $m_1 s0 = m_2 s0$. Repeated application of cancellation and $A1$ yields that $m_1 = m_2 = m$. Lastly, $n_1 s0 + m0 + t' = t_1(0, C) = t_2(0, C) = n_2 s0 + m0 + t_1$. Cancellation yields, $n_1 s0 = n_2 s0$ and thus $n_1 = n_2 = n$. Thus, A is an identity and $\mathcal{M}_\infty \vDash \forall x \forall X : A(x, X)$. ■

5.2.3 The model $\mathcal{M}_{a,b}$

In this subsection assume that there is at least one constructor, which are not essentially unary. We adapt the model $\mathbb{N}_{\{a,b\}}$, which is again based on [Het24, page 38]:

Definition 5.2.10. *We start with the interpretation of the sorts:*

- $T_i^{\mathcal{M}_{\{a,b\}}} = \{i\}$
- $\mathsf{D}^{\mathcal{M}_{\{a,b\}}} = \mathbb{T}(T_1^{\mathcal{M}_{\{a,b\}}}, \ldots, T_n^{\mathcal{M}_{\{a,b\}}})$
- $\mathsf{Nat}^{\mathcal{M}_{\{a,b\}}} = \mathbb{N} \cup \{a, b\}$

The constructors are interpreted canonically:

- $c_i^{\mathcal{M}_{\{a,b\}}}(\overline{a}) = c_i(\overline{a})$ *for any static constructor* c_i
- $c_i^{\mathcal{M}_{\{a,b\}}}(t_1, \ldots, t_{k_i}, a_{k_i+1}, \ldots, a_{m_i}) = c_i(t_1, \ldots, t_{m_i})$ *for any dynamic constructor* c_i*, where exactly the first* k_i *inputs are of sort* D
- $0^{\mathcal{M}_{\{a,b\}}} = 0$
- $s^{\mathcal{M}_{\{a,b\}}} n = n + 1$, $s^{\mathcal{M}_{\{a,b\}}} a = a$, $s^{\mathcal{M}_{\{a,b\}}} b = b$

$+$ *is interpreted according to the following table:*

The size function l *is interpreted according to* $E1$ *and* $E2$*:*

$+$	0	1	2	$\dots$	a	b
0	0	1	2	$\dots$	b	a
1	1	2	3	$\dots$	b	a
2	2	3	4	$\dots$	b	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
a	a	a	a	$\dots$	a	a
b	b	b	b	$\dots$	b	b

- $l^{\mathcal{M}_\infty}(c_i(\overline{a})) = 0$ *for any static constructor* c_i
- $l^{\mathcal{M}_\infty}(c_i(t_1, \dots, t_{k_i}, a_{k_i+1}, \dots a_{m_i})) = s^{\mathcal{M}_\infty}((l(t_1)) + \dots + l(t_{k_i}))^{\mathcal{M}_\infty}$ *for any dynamic constructor* c_i*, where exactly the first* k_i *inputs are of sort* D

Observation 5.2.11. $\mathcal{M}_{a,b} \vDash T_0$

Observation 5.2.12. $\mathcal{M}_{a,b} \nvDash x + y = y + x$

5.3 Essentially Unary Constructors Only

In this section, we consider the case that every dynamic constructor of D is unary.

Now, before we dive into the scheme of induction, we will have a closer look at our theory combined with our auxiliary axioms.

Definition 5.3.1. *We define* $\Gamma = T_0 + \mathrm{SUR} + B1a + \{G_t \mid t \in \mathcal{S}\} + \{B_n \mid n \geqslant 1\}$

It will be part of the next theorem that, analogously to [Sho58] and [She63], Γ is, in fact, an alternative axiomatization of open induction in this context:

Theorem 5.3.2.

$$\begin{aligned} T_0' &\approx T_0' + \mathbf{I_2}(Atom) \\ &\lnapprox T_0' + \mathbf{I_2}(Literal) \\ &\approx T_0' + \mathbf{I_2}(Open) \\ &\approx \Gamma \end{aligned} \qquad\qquad \begin{aligned} T_0' &\approx T_0' + \mathbf{I_1}(Atom) \\ &\lnapprox T_0' + \mathbf{I_1}(Literal) \\ &\approx T_0' + \mathbf{I_1}(Open) \\ &\approx \Gamma \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd} T_0 + \mathbf{I_2}(Literal) \approx T_0 + \mathbf{I_1}(Literal) \approx T_0 + \mathbf{I_2}(Open) \approx T_0 + \mathbf{I_1}(Open) \arrow[d, dash] \\ T_0 \approx T_0 + \mathbf{I_2}(Atom) \approx T_0 + \mathbf{I_1}(Atom) \end{tikzcd}$$

Observation 5.3.3. *Let $t(X)$ be a term of sort Nat that contains X. Then, $T_0 \vdash t(X) = s^n(l(X))$ for some $n \in \mathbb{N}$.*

Lemma 5.3.4. *Take any model $\mathcal{M}$ of Γ and any atom $A(x, X) \equiv t_1 = t_2$. If there are elements $B, C, D \in \mathsf{D}^{\mathcal{M}}$ and $b, c, d \in \mathsf{Nat}^{\mathcal{M}}$ s.t. $\mathcal{M} \vDash A(b, D) \wedge A(c, D) \wedge A(d, B) \wedge A(d, C)$, $b = s^n c$ for some $n \geqslant 1$ and $B = t(C)$ for some term $t \not\equiv X$, then $\mathcal{M} \vDash \forall x \forall X : A(x, X)$. In other words, Γ has the anchor property.*

Proof. There are five cases:

1. If A contains neither x nor X, we are done.

2. Assume that A contains X on exactly one side, say t_1, and that there there are two elements $B \neq C$ with $B = t(C)$ and $\mathcal{M} \vDash A(B) \wedge A(C)$. There are two cases: If both terms are of sort Nat, then, w.l.o.g., t_1 has the form $s^k(l(X))$ and $t_2^{\mathcal{M}} \in \mathsf{Nat}^{\mathcal{M}}$ is constant in X. Note that $\mathcal{M} \vDash l(B) = s^k(l(C))$ for some $k \geqslant 1$. From $A3a$ and $\mathcal{M} \vDash s^n(l(C)) = s^{n+l}(l(C))$, it follows that $\mathcal{M} \vDash l(C) = s^k(l(C))$, which contradicts B_k. In the other case - both terms are of the sort D - it follows similarly that there is a cycle in $\mathsf{D}^{\mathcal{M}}$, which contradicts G_t.

3. Assume that A contains x on exactly one side, say t_1, and that there are elements $b \neq c$ with $b = s^n c$ and $\mathcal{M} \vDash A(b) \wedge A(c)$. W.l.o.g., t_1 has the form $s^k x$ and $t_2^{\mathcal{M}}$ is constant in x. It follows that $\mathcal{M} \vDash s^k c = s^{n+k} c$. From $A3a$ it follows that $\mathcal{M} \vDash c = s^k c$, which contradicts B_n.

4. Assume that A contains X on both sides and that there are $B \neq C \in \mathsf{D}^{\mathcal{M}}$ with $B = t(C)$ for some non-constant term t and $\mathcal{M} \vDash A(B) \wedge A(C)$. There are two cases: If A is of the sort D, then both t_1 and t_2 contain only the constructors c_i. Since $\mathcal{M}$ does not contain any cycles, it follows from 3.2.6 that $\mathcal{M} \vDash \forall X : t_1(X) = t_2(X)$. Assume that A is of the sort Nat. Then, w.l.o.g., t_1 and t_2 have the form $s^{n_i}(l(X))$. It follows from $A3a$ that $n_1 = n_2$. Thus, $\mathcal{M} \vDash (\forall X : D) t_1(X) = t_2(X)$.

5. Assume that A contains x on both sides and there are $b, c \in \mathsf{Nat}^{\mathcal{M}}$ with $b = s^n c$ and $\mathcal{M} \vDash A(b) \wedge A(c)$. Then both t_i have the form $s^{n_i} x$. Since $\mathcal{M}$ does not contain any cycles, it follows from 3.2.6 that $\mathcal{M} \vDash (\forall x : \mathsf{Nat})(t_1(x) = t_2(x))$. ∎

5.3.1 Two Schemes of Induction

Now, we consider a concrete scheme of induction, namely, the two single schemes for arithmetics and general data types.

Lemma 5.3.5. *It holds that*

- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash x \neq s^n x$
- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash x = 0 \vee \exists y : x = sy$
- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash \mathrm{SUR}$

Proof. This follows directly from Theorem 3.1.8 and Section 4.6. ■

The crucial Lemma is the following:

Lemma 5.3.6. *$T_0 + \mathbf{I_2}(Literal) \vdash G_t$ for any $t \in \mathcal{S}$.*

Proof. Assume that there is some $t \in \mathcal{S}$ and X s.t. $X = t(X)$. Then, since every t is not identical to X and there are only unary constructors, there is some $m \geqslant 1$ s.t. $l(X) = s^m(l(X))$, which contradicts Lemma 5.3.5 ■

Corollary 5.3.7. *$T_0 + \mathbf{I_2}(Literal) \vdash \Gamma$*

Lemma 5.3.8. *$\Gamma \vdash \mathbf{I_2}(Open)$*

Proof. This follows directly from Lemma 5.3.4 and Lemma 5.1.10. ■

Lemma 5.3.9. *$T_0 \vdash \mathbf{I_2}(Atom)$*

Proof. Take any model $\mathcal{M}$ of T_0. Fix the atom $A(x, X)$ with the variables x of sort Nat and X of sort D. We want to show that for any $b \in \mathsf{Nat}^{\mathcal{M}}$ and $B \in \mathsf{D}^{\mathcal{M}}$, the respective scheme of induction works for $A(b, X)$ and $A(x, B)$. Note that we can restrict ourselves to the case, where there is some occurrence of the function symbol l in A. Assume that there is not such occurrence. Then, A is a well-formed atom in either $\{0, s\}$ or $\{c_1, \ldots, c_k\}$. By Lemma 3.4.4, there are proofs of the respective induction axiom, instantiated with $A(b, X)$ and $A(x, B)$, in either $\{A1, A3a\}$ or $\{D1_{i,j} \mid i \neq j, 1 \leqslant i, j \leqslant k\} \cup \{D2_j \mid 1 \leqslant j \leqslant k\}$, which clearly remains valid in T_0.

Now, assume that l occurs somewhere in A. Both terms have to be of the sort Nat. We make a case distinction:

1. If we consider the unary atom $A(x, B)$, then x cannot appear inside l. Thus, we can replace every subterm of the form $l(s_i)$ with some parameter z_i. We obtain an equivalent atom A' that does not contain l. Thus, induction over A holds.

2. Now, let us consider the atom $A(b, X)$. There are three cases:

2a. If X does not appear in A, we are done.

2b. If X appears on exactly one side, say t_1, then modulo T_0, t_1 can be written as $s^n l(X)$ for some $n \geqslant 0$. $t_2^{\mathcal{M}}$ is constant. Since, t_1 is clearly not constant in the standard part of the model, the left hand side of the induction scheme cannot hold.

2c. If X appears on both sides of A, then both terms can be written, modulo T_0, as $s^{n_i} l(X)$. Since there is a static constructor c_i with $l(c_i(\overline{y})) = 0$ for any tuple $\overline{y}$ and all successors of 0 are distinct, we conclude that if $A(b, c_i(\overline{y}))$ holds, then $n_1 = n_2$ and $A(b, X)$ holds for all X. The left hand side of the induction scheme, clearly implies $A(b, c_i(\overline{y}))$ and thus, induction over $A(b, X)$ holds. ■

Lemma 5.3.10. $T_0 + \mathbf{I_2}(\mathit{Atom}) \nvdash \mathbf{I_2}(\mathit{Literal})$

Proof. By Lemma 5.3.9 it suffices to give a model $\mathcal{M}$ of T_0 s.t. $\mathit{ILiteral}$ does not hold in $\mathcal{M}$. By Lemma 5.3.5, it suffices if $\mathcal{M} \nvDash B1a$. The model $\mathcal{M}^u_\infty$ does exactly that. ■

5.3.2 One Combined Scheme of Induction

Now, we consider the case that we have one combined induction scheme of the following form:

$$\mathbf{I}(\varphi)\ \bigwedge_{c_i \in C} \forall \overline{x} : \varphi(c_i(\overline{x}), 0, \overline{z}) \wedge \bigwedge_{c_i \in C'} \left(\forall \overline{x} \left(\varphi(\bigwedge_{\tau_i^k = \mathsf{D}} \varphi(x_k, u, \overline{z}) \rightarrow \varphi(x_1, s(u), \overline{z}) \wedge \varphi(c_i(\overline{x}), u, \overline{z})\right)\right) \rightarrow (\forall X : \mathsf{D})(\forall u : \mathsf{Nat})\varphi(X, u, \overline{z})$$

We collect known results:

Lemma 5.3.11. *It holds that*

- $T_0 + \mathbf{I_1}(\mathit{Literal}) \vdash x = 0 \vee \exists y : x = sx$
- $T_0 + \mathbf{I_1}(\mathit{Literal}) \vdash x \neq s^n x$
- $T_0 + \mathbf{I_1}(\mathit{Literal}) \vdash \mathrm{SUR}$
- $T_0 + \mathbf{I_1}(\mathit{Literal}) \vdash G_t$ *for any* $t \in \mathcal{S}$

Proof. This follows directly from Lemma 5.3.5, Lemma 5.3.6 and the Observation 5.1.4 that the two individual schemes of induction are subsumed by the combined one. ■

Corollary 5.3.12. $T_0 + \mathbf{I_1}(\mathit{Literal}) \vdash \Gamma$

Lemma 5.3.13. $\Gamma \vdash \mathbf{I_1}(\mathit{Open})$

Proof. This follows directly from Lemma 5.3.4 and Theorem 5.1.11. ■

Lemma 5.3.14. $T_0 \vdash \mathbf{I_1}(\mathit{Atom})$

Proof. Let $\mathcal{M}$ be any model of T_0. Take any atom $A(x, X) \equiv t_1 = t_2$. Note that if A does not contain both x and X, then this case is subsumed by Lemma 5.3.9. We are left with the case that A contains both x and X. W.l.o.g., assume that $t_1 = s^n l(t(X))$ and $t_2 = s^m x$. Modulo T_0, we can write t_1 as $s^k l(X)$. Assume that $\mathcal{M} \vDash \mathbf{LHS_1}(A)$. Thus, $T_0 \vdash A(0, c_i(\overline{y}))$ and $T_0 \vdash A(s0, c_i(\overline{y}))$ for any static constructor c_i and tuple $\overline{y}$. By $E1$, $t_1(c_i(\overline{y})) = s^k 0$. Thus, $s^m 0 = s^k 0 = s^m(s0) = s^{m+1}0$ and $m = k = m + 1$, which cannot be. Therefore, A cannot satisfy the left hand side of the scheme of induction. ■

Lemma 5.3.15. $T_0 + \mathbf{I_1}(\mathit{Atom}) \nvdash \mathbf{I_1}(\mathit{Literal})$

Proof. By Lemma 5.3.9 it suffices to give a model $\mathcal{M}$ of T_0 s.t. $\mathit{ILiteral}$ does not hold in $\mathcal{M}$. By Lemma 5.3.5, it suffices if $\mathcal{M} \nvDash B1a$. The model $\mathcal{M}^u_\infty$ does exactly that. ■

5.4 Not Only Essentially Unary Constructors

In this subsection, we consider the case that there is some constructor c_i, which is not essentially unary. Note that our arguments would work otherwise just as well, but we would not add + to our language and could receive different results (cf. Section 5.3).

Definition 5.4.1. *We define the theory* $\Gamma = T_0 + \{B1a, B2, B3, B4, \mathrm{SUR}\} + \{G_t \mid t \in \mathcal{S}\}$.

Observation 5.4.2. *Let* $t(x, X)$ *be any term in* $\mathcal{L}_0$. *If* t *is of sort* Nat, *then* $\Gamma \vdash t(x, X) = nl(X) + mx + t'$ *for some* $n, m \in \mathbb{N}$ *and* t', *which contains neither* x *nor* X.

The following will be our main result:

Theorem 5.4.3.

$$\begin{array}{ll} \begin{aligned} T_0 &\lnsim T_0 + \mathbf{I_2}(Atom) \\ &\lnsim T_0 + \mathbf{I_2}(Literal) \\ &\approx T_0 + \mathbf{I_2}(Open) \\ &\approx \Gamma \end{aligned} & \qquad \begin{aligned} T_0 &\lnsim T_0 + \mathbf{I_1}(Atom) \\ &\lnsim T_0 + \mathbf{I_1}(Literal) \\ &\approx T_0 + \mathbf{I_1}(Open) \\ &\approx \Gamma \end{aligned} \end{array}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd}
T_0 + \mathbf{I_2}(ILiteral) \approx T_0 + \mathbf{I_1}(Literal) \approx T_0 + \mathbf{I_2}(Open) \approx T_0 + \mathbf{I_1}(Open) \arrow[d, dash] \\
T_0 + \mathbf{I_1}(Atom) \arrow[d, dash, dashed] \\
T_0 + \mathbf{I_2}(Atom) \arrow[d, dash] \\
T_0
\end{tikzcd}$$

Lemma 5.4.4. $T_0 + B4 \vdash x \neq s^n x$ *for any* $n \geqslant 1$.

Proof. Assume that there is some x s.t. $s = s^n x$. Then $0 + x = x = s^n x = s^n 0 + x$, which contradicts $A1$ or $B4$. ■

Lemma 5.4.5. $T_0 + B1a \vdash x + y = 0 \rightarrow x = 0 \wedge y = 0$

Proof. Assume that $x + y = 0$. If $y \neq 0$, then there is some z s.t. $y = sz$. Thus, $0 = x + sz = s(x + z)$, which cannot be. Thus, $y = 0$ and $0 = x + y = x + 0 = x$. ■

Lemma 5.4.6. *Let $\mathcal{M}$ be a model of Γ and $A(x, X)$ an atom with the variables $x \in \mathsf{Nat}$ and $X \in \mathsf{D}$. Assume that there are values $b, c, d \in \mathsf{Nat}^{\mathcal{M}}$ and $B, C, D \in \mathsf{D}^{\mathcal{M}}$ s.t. $b = s^l c$ ($l \geqslant 1$) and $B = t(C)$ with $t \neq \mathrm{id}$ and $\mathcal{M} \models A(b, D) \wedge A(c, D) \wedge A(d, B) \wedge A(d, C)$. Then, $\mathcal{M} \models \forall x \forall X : A(x, X)$. In other words, Γ has the anchor property.*

Proof. There are two cases:

1. Both term t_i are of the sort D. Then both terms contain only constructors c_j. Since $\mathcal{M}$ does not contain any cycles, it follows from Corollary 3.2.6 that $\mathcal{M} \models (\forall X : \mathsf{D})(A(d, X))$, which is equivalent to $\mathcal{M} \models (\forall x : \mathsf{Nat})(\forall X : \mathsf{D})(A(x, X))$ as x does not appear in A.

2. Both terms t_i are of the sort Nat. Then, w.l.o.g., they both have the form $n_i l(X) + m_i x + t_i'$. Since $\mathcal{M} \models A(d, B) \wedge A(d, C)$ and $\mathcal{M} \models l(B) = sl(C) + t_C$, it follows that $\mathcal{M} \models t_1(d, B) = n_1 l(B) + m_1 d + t_1' = n_1(sl(C) + t_C) + m_1 d + t_1' = n_1(s0 + t_C) + (n_1 l(C) + m_1 d + t_1') = n_1(s0 + t_C) + t_1(d, C)$. Analogously, $\mathcal{M} \models t_2(d, B) = n_2(s0 + t_C) + t_2(d, C)$. It follows that $\mathcal{M} \models n_1(s0 + t_C) + t_1(d, C) = n_2(s0 + t_C) + t_2(d, C) = n_2(s0 + t_C) + t_1(d, C)$. By cancellation w.r.t. $+$, we obtain $\mathcal{M} \models n_1 s0 = n_2 s0$ and thus, $n_1 = n_2$. We define $n = n_1$. Since $\mathcal{M} \models A(b, D) \wedge A(c, D)$ and $\mathcal{M} \models b = s^k c$, it follows that $\mathcal{M} \models t_1(b, D) = nl(D) + m_1 s^k c + t_1' = nl(D) + m_1 s^k 0 + m_1 c + t_1' = m_1 s^k 0 + t_1(c, D)$. Analogously, it follows that $\mathcal{M} \models t_2(b, D) = m_2 s^k 0 + t_2(c, D)$. It follows that $\mathcal{M} \models m_1 s^k 0 + t_1(c, D) = m_2 s^k 0 + t_1(c, D)$. By cancellation w.r.t. $+$, we obtain $\mathcal{M} \models m_1 s^k 0 = m_2 s^k 0$. Thus, $m_1 = m_2$. We define $m = m_1$. By assumption, t_1' and t_2' contain neither x nor X. Since $\mathcal{M} \models A(b, D)$, we conclude that $\mathcal{M} \models nl(D) + mb + t_1' = nl(D) + mb + t_2'$. By cancellation w.r.t. $+$, we obtain $\mathcal{M} \models t_1' = t_2'$. Thus, $\mathcal{M} \models (\forall x : \mathsf{Nat})(\forall X : \mathsf{D})(A(x, X))$. ■

5.4.1 Two Schemes of Induction

Again, we start by considering the combination of the two schemes of induction - one for the sort Nat and one for the sort D.

Lemma 5.4.7. *The following holds:*

- $T_0 + \mathbf{I_2}(\mathit{Atom}) \vdash x + y = y + x$
- $T_0 + \mathbf{I_2}(\mathit{Atom}) \vdash x + (y + z) = (x + y) + z$
- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash x + y = x + z \rightarrow y = z$
- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash \mathrm{SUR}$
- $T_0 + \mathbf{I_2}(\mathit{Literal}) \vdash x = 0 \vee \exists y : x = sy$

Proof. This follows directly from Theorem 3.1.8 and Section 4.6. ■

Lemma 5.4.8. *$T_0 + \mathbf{I_2}(Literal) \vdash G_t$ for any $t \in \mathcal{S}$.*

Proof. Take any $n \geqslant 1$ and $t \in S_n$. Then, since t is not identical to X and by $E2$, we can write $l(t(X))$ as $l(X) + s(t')$, where t' is some term of sort Nat (possibly containing X). If there is some X s.t. $X = t(X)$, then, by $B4$, we obtain $0 = s(t')$, which contradicts $A1$. ■

Lemma 5.4.9. $\Gamma \vdash \mathbf{I_2}(Open)$

Proof. This follows directly from Lemma 5.4.6 and Lemma 5.1.10. ■

Lemma 5.4.10. $\Gamma \nvdash \mathbf{I_2}(Atom)$

Proof. It suffices to give a model $\mathcal{M}$ of T_0 with $\mathcal{M} \nvDash x + y = y + x$. The model $\mathcal{M}_{a,b}$ from the subsection 5.2.3 does exactly that. ■

Lemma 5.4.11. $\Gamma + \mathbf{I_2}(Atom) \nvdash \mathbf{I_2}(Literal)$

Proof. It suffices to give a model $\mathcal{M}$ of $T_0 + \mathbf{I_2}(Atom)$ with $\mathcal{M} \nvDash x \neq s^n x$ for some $n \geqslant 1$. Consider the model $\mathcal{M}_\infty$ from the Subsection 5.2.2. $\mathcal{M}_\infty \vDash \mathbf{I_1}(Atom)$. Thus, any by Observation 5.1.4, $\mathcal{M} \vDash \mathbf{I_2}(Atom)$. Also, $\mathcal{M} \nvDash x \neq s^n x$ for every $n \in \mathbb{N}$. ■

5.4.2 One Combined Scheme of Induction

Now, we use one combined scheme of induction $\mathbf{I_1}$.

Lemma 5.4.12. $T_0 + \mathbf{I_1}(Literal) \vdash \Gamma$

Proof. This follows directly from Lemma 5.4.7 and Observation 5.1.4. ■

Lemma 5.4.13. $\Gamma \vdash \mathbf{I_1}(Open)$

Proof. This follows directly from Lemma 5.4.6 and Theorem 5.1.11. ■

Lemma 5.4.14. $T_0 \nvdash \mathbf{I_1}(Atom)$

Proof. It suffices to give a model $\mathcal{M}$ of T_0 with $\mathcal{M} \nvDash x + y = y + x$. The model $\mathcal{M}_{a,b}$ from the subsection 5.2.3 does exactly that. ■

Lemma 5.4.15. $T_0 + \mathbf{I_1}(Atom) \nvdash \mathbf{I_1}(Literal)$

Proof. It suffices to give a model $\mathcal{M}$ of $T_0 + \mathbf{I_1}(Atom)$ with $\mathcal{M} \nvDash x \neq s^n x$ for some $n \geqslant 1$. The mode $\mathcal{M}_\infty$ from does exactly that. ■

Open Problem 5.4.16. *It is yet unclear, whether $T_0 + \mathbf{I_2}(Atom) \vdash \mathbf{I_2}(Atom)$ in the case with not only essentially unary constructors.*

CHAPTER 6

Lists

In this chapter, we deal with the structure of lists in more depth. This seems to be the logical next step as numbers can be considered a special case of lists, where the lists contain only one element (possibly multiple times). We will see that theories of lists are, in fact, much more complicated than arithmetic theories.

After defining the general frame, we work in, we will apply theorems from Chapters 3 and 5 to the special case of lists. After that, we consider list concatenation and see that it makes things considerably more difficult.

6.1 General Frame

In the following, we will consider a two-sorted logic to represent lists: The sort ι is the sort of the list elements, and L is the sort of the actual lists. To avoid confusion, we will capitalize lists and list-variables and write elements and element-variables in lowercase. The only exception will be the empty list nil.

The language $\mathcal{L}$ consists of the following symbols $nil, [\cdot|\cdot], +$, where nil is a constant symbol of sort L, $[\cdot|\cdot]$ is a function symbol of type $\iota \times \mathsf{L} \to \mathsf{L}$, and $+$ is a function symbol of type $\mathsf{L} \times \mathsf{L} \to \mathsf{L}$ to denote list concatenation. Additionally, we have the equality relation for each sort with the usual axiomatization. The other axioms are the following:

L1 $nil \neq [x|X]$

L2 $[x|X] = [y|Y] \to x = y \land X = Y$

L3 $nil + X = X$

L4 $[x|X] + Y = [x|X + Y]$

For the sake of readability, we use the shorthand $[x_1, x_2, \dots, x_n|X]$ for $[x_1|[x_2|[\dots|[x_n|X]]]]$, where $[\varnothing|X]$ is defined to be X.

The scheme of induction axiom is given by

LHS$(\varphi(x, \overline{z}))$ $\varphi(nil, \overline{z}) \wedge \forall X \forall x : \varphi(X, \overline{z}) \to \varphi([x|X], \overline{z})$

I$(\varphi(x, \overline{z}))$ **LHS**$(\varphi(x, \overline{z}) \to \forall X : \varphi(X, \overline{z}))$

Again, for the sake of readability, we will often refrain from writing the parameters of a formula explicitly. If not explicitly stated otherwise, all our formulas in this chapter may possibly contain parameters.

Remark 6.1.1. *While dealing with lists, we thought about interesting properties of them. One that should be mentioned is the property that lists are not periodic, in the sense that no period $x_1, \dots, x_n$ is repeated infinitely many times. This follows clearly from the fact that lists are finite. However, this property cannot be formulated in FOL: Assume there is a formula, $\psi(X)$ s.t. ψ expresses that X is not periodic. Now extend the language with the new constant symbol c. We define new formulas $\varphi_n(X) \equiv (\exists Y : \mathsf{D})(\exists x_1, x_2 : \iota)(X = [x_1, x_2, x_1, x_2, \dots, x_1, x_2|Y]$, where the period x_1, x_2 is repeated n times. Consider the set $\Gamma = \{\varphi_n(c) \mid n \in \mathbb{N}\}$. $T_0 + \Gamma + \psi(c)$ is finitely satisfiable (in any standard model of lists). However, $\psi(c) + \Gamma$ is inconsistent. Thus, ψ cannot exist.*

The even more interesting part is the following: The standard model of lists over the alphabet ι can be considered to be the set ι^ω. If we think about non-standard models of lists, it is thus natural to consider the subsets of ι^α for some limit-ordinal α with the canonical interpretation of $[\cdot|\cdot]$ and $+$ (cf. [HV24, Section 2.3]). In these models, all lists are aperiodic in the sense from above. However, if we consider the set Γ from above, then $T_0 + \Gamma + \{\mathbf{I}(\varphi) \mid \varphi \in \mathrm{WFF}\}$ is finitely satisfiable in any standard model. Thus, it is satisfiable by compactness and there is a non-standard model of lists, which satisfies induction and contains a periodic list. This shows that there are non-standard models of lists with induction that are fundamentally different and harder to grasp than the ones we might consider.

6.2 Constructors only

As usual, we start with the theory $T_0 = \{L1, L2\}$ and the language consisting of only nil and $[\cdot|\cdot]$.

We overload the axioms SUR and G_t to fit into the context of lists. Note that we only have rather simple terms in this context. Thus, there is exactly one term t in $S_n \subseteq \mathcal{S}$ for any n. Therefore, we can write G_n instead of G_t:

SUR $\forall Y : Y = nil \vee \exists X \exists x : Y = [x|X]$

$\mathbf{G_n}$ $X \neq [x_1, \dots, x_n|X]$ for all $n \geqslant 1$

We have already dealt with this case in Chapter 3:

Theorem 6.2.1.

$$\begin{aligned} T_0 &\approx T_0 + IAtom \\ &\lnsim T_0 + ILiteral \\ &\approx T_0 + IClause \\ &\lnsim T_0 + IDClause \\ &\approx T_0 + IOpen \\ &\approx T_0 + \mathrm{SUR} + \{G_n \mid n \geqslant 1\} \end{aligned}$$

We receive the following Hasse Diagram:

$$\begin{tikzcd} T_0 + IDClause \approx T_0 + IOpen \arrow[d, dash] \\ T_0 + ILiteral \approx T_0 + IClause \arrow[d, dash] \\ T_0 \approx T_0 + IAtom \end{tikzcd}$$

Proof. This is a direct consequence of Theorem 3.4.3. ∎

6.3 Constructors only and a Size Function

We now add the sort Nat to the lists to represent natural numbers. Additionally, we add the symbols $0 \in \mathsf{Nat}, s : \mathsf{Nat} \rightarrow \mathsf{Nat}, l : L \rightarrow \mathsf{Nat}$ with the following axioms:

A1. $0 = sx$

A3a. $sx = sy \rightarrow x = y$

E1. $l(nil) = 0$

E2. $l([x|X]) = sl(X)$

We will also need these auxiliary axioms:

B1a. $x = 0 \vee \exists y : x = sy$

$\mathbf{B_n}$. $x \neq s^n x$

Definition 6.3.1. $T_0' = T_0 + \{A1, A3a, E1, E2\}$ *over the language* $\mathcal{L}' = \{nil, [\cdot|\cdot], 0, s, l\}$

Definition 6.3.2. $\Gamma = T_0' + \mathrm{SUR} + B1a + \{B_n, G_n \mid n \geqslant 1\}$

Theorem 6.3.3.

$$\begin{aligned} T_0' &\approx T_0' + \mathbf{I_2}(Atom) \\ &\lnsim T_0' + \mathbf{I_2}(Literal) \\ &\approx T_0' + \mathbf{I_2}(Open) \\ &\approx \Gamma \end{aligned} \qquad \begin{aligned} T_0' &\approx T_0' + \mathbf{I_1}(Atom) \\ &\lnsim T_0' + \mathbf{I_1}(Literal) \\ &\approx T_0' + \mathbf{I_1}(Open) \\ &\approx \Gamma \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd} T_0' + \mathbf{I_2}(Literal) \approx T_0' + \mathbf{I_1}(Literal) \approx T_0' + \mathbf{I_2}(Open) \approx T_0' + \mathbf{I_1}(Open) \arrow[d, dash] \\ T_0' \approx T_0' + \mathbf{I_2}(Atom) \approx T_0' + \mathbf{I_1}(Atom) \end{tikzcd}$$

Proof. This follows directly from Theorem 5.3.2. ■

6.4 Concatenation

We now consider the language $nil, [\cdot|\cdot]$ and $+$ with the theory $T_1 = \{L1, L2, L3, l4\}$. We use the induction scheme from Section 6.2.

Again, we define some additional axioms, which we will derive in this section:

M1. $X + nil = X$

M2. $X + (Y + Z) = (X + Y) + Z$

M3. $X + Y = X + Z \to Y = Z$

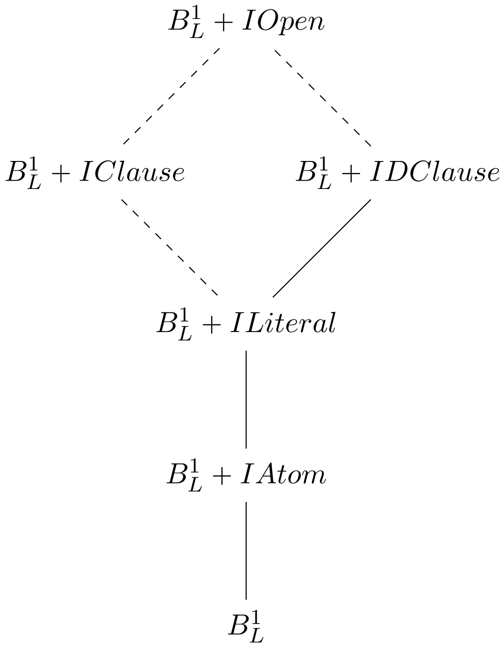
M4. $X + Y = nil \to X = nil \land Y = nil$

The following will be our main result:

Theorem 6.4.1.

$$\begin{aligned} T_1 &\lnsim T_1 + IAtom \\ &\lnsim T_1 + ILiteral \\ &\lnsim T_1 + IDClause \end{aligned}$$

We obtain the following (partial) Hasse Diagram:



We start by collecting the results from the previous sections:

Lemma 6.4.2. *The following holds:*

- $T_1 + ILiteral \vdash \mathrm{SUR}$
- $T_1 + IDClause \vdash G_n$ *for any* $n \geqslant 1$

Proof. These results are special cases of Theorem 3.1.8 and Lemma 3.4.13. ■

We also need to prove some new results:

Lemma 6.4.3. $T_1 + IAtom \vdash X + nil = X$.

Proof. We work in $T_1 + IAtom$. Consider the atom $A(X) \equiv X + nil = X$. By $L3$ it holds that $nil + nil = nil$ and thus $A(nil)$. Now assume that A holds for some X. Let $x \in \iota$ be arbitrary. By $L4$ and the induction hypothesis, it follows that $[x|X] + nil = [x|X + nil] = [x|X]$. By applying the induction scheme on A, we obtain $\forall X : X + nil = X$. ■

Lemma 6.4.4. $T_1 + IAtom \vdash X + (Y + Z) = (X + Y) + Z$

Proof. Take any model $\mathcal{M}$ of $T_1 + IAtom$ and consider the atom $A(X) \equiv X + (Y + Z) = (X + Y) + Z$. Fix any interpretation $\xi : Y \mapsto B, Z \mapsto C$. From $L3$ it follows that $\mathcal{M}, \xi \vDash nil + (Y + Z) = Y + Z = (nil + Y) + Z$ and thus $\mathcal{M}, \xi \vDash A(nil)$. Now assume that $\mathcal{M}, \xi \vDash A(E)$ for some element E and pick an arbitrary element $e \in \iota^{\mathcal{M}}$. From $L3$ and the induction hypothesis, it follows that $\mathcal{M}, \xi \vDash [e|E] + (Y + Z) = [e|(E + (Y + Z)] = [e|(E + Y) + Z] = [e|E + Y] + Z = ([e|E] + Y) + Z$. Thus, $\mathcal{M}, \xi \vDash A([e|E])$. Induction yields $\mathcal{M}, \xi \vDash \forall X : X + (Y + Z) = (X + Y) + Z$. Since ξ was arbitrary, it follows from the semantics of classical FOL that $\mathcal{M} \vDash \forall Z \forall Y \forall X : X + (Y + Z) = (X + Y) + Z$. ∎

Lemma 6.4.5. $T_0 + \mathrm{SUR} \vdash X + Y = nil \rightarrow X = nil \wedge Y = nil$. *Thus,* $T_0 + ILiteral \vdash X + Y = nil \rightarrow X = nil \wedge Y = nil$.

Proof. We work in $T_0 + \mathrm{SUR}$. Take any X, Y s.t. $X + Y = nil$. Assume that $X \neq nil$. By SUR there is some element Z s.t. $X = [x|Z]$. By $L4$, it holds that $X + Y = [x|Z] + Y = [x|Z + Y]$. By $L1$, this cannot be equal to nil. Thus, $X = nil$. Assume that $Y \neq nil$. By SUR there is some element Z s.t. $Y = [x|Z]$. By $L3$, it holds that $X + Y = nil + Y = Y = [x|Z]$. Again, by $L1$ this cannot be equal to nil. Thus, $Y = nil$. ∎

Remark 6.4.6. *The Lemma above will not be particularly useful, but it is a nice property that can be shown with a low level of induction. Moreover, it emphasizes the following: In previous Lemmas, we have seen that a formula can be an implication, but if it contains* X *on only one side of the implication, it can be dealt with like a literal by fixing the parameters (e.g.* $M3$*). The situation now is different: Although* $X + Y = nil \rightarrow X = nil \wedge Y = nil$ *it is not even a clause or a dual clause and contains* X *in various places, it follows from a level of induction much lower than general open induction.*

The more general question is: What is the relation between two formulas F *and* G *if* $\mathbf{I}(G) \vdash F$*? This question goes much deeper than the Lemma above and this thesis in general, but this seemed like a good place to mention it.*

Lemma 6.4.7. $T_0 + ILiteral \vdash X + Y = X + Z \rightarrow Y = Z$

Proof. We work in $T_0 + ILiteral$. Fix any two $Y \neq Z$ and consider the literal $L(X) \equiv X + Y \neq X + Z$. From $L3$ it follows that $nil + Y = Y \neq Z = nil + Z$ and thus $A(nil)$. Now assume that A holds for some X and let $x \in \iota$ be arbitrary. By counterposition of $L2$, we obtain $[x|X + Y] \neq [x|X + Z]$. By induction on the literal L, we obtain that it holds for all X. ∎

Remark 6.4.8. *The previous Lemma shows that left-cancellation holds in all models of* $T_0 + ILiteral$*. Interestingly, in [HV24, Chapter 4] it was shown that right-cancellation (i.e.* $Y + X = Z + X \rightarrow Y = Z$*) cannot be shown with open induction at all. This is characteristic for the problems we encountered with lists: Since list concatenation is in general not commutative it is very difficult to deal with formulas, where the induction variable appears on the right side of some (in)equality.*

Lemma 6.4.9. $T_1 \nvdash IAtom$

Proof. By Lemma 6.4.3 it suffices to give a model of T_1 that does not satisfy $M1$. For this consider the following model $\mathcal{M}$:

- $\iota^{\mathcal{M}} = \{1\}$
- $nil^{\mathcal{M}} = 0$
- $\mathsf{L}^{\mathcal{M}} = \mathbb{N} \cup \{a, b\}$
- $[1|n]^{\mathcal{M}} = n + 1$, $[1|a]^{\mathcal{M}} = a$, $[1|b]^{\mathcal{M}} = b$
- $(n + m)^{\mathcal{M}} = n + m$, $(n + a)^{\mathcal{M}} = a$, $(n + b)^{\mathcal{M}} = b$, $(a + n)^{\mathcal{M}} = b$, $(b + n)^{\mathcal{M}} = a$, $(a + b)^{\mathcal{M}} = (b + a)^{\mathcal{M}} = (a + a)^{\mathcal{M}} = (b + b)^{\mathcal{M}} = a$

Then, $\mathcal{M} \vDash T_1$, but $\mathcal{M} \nvDash x + nil = x$ since $\mathcal{M} \vDash a + nil = b$. ■

Lemma 6.4.10. *$T_1 + IAtom \nvdash ILiteral$*

Proof. By Lemma 6.4.2 it suffices to give a model of $T_0 + IAtom$, where SUR does not hold.

Consider the following model $\mathcal{M}$:

- $\iota^{\mathcal{M}} = \{1\}$
- $\mathsf{L}^{\mathcal{M}} = \mathbb{N} \times \{0, 1\}$
- $nil^{\mathcal{M}} = (0, 0)$
- $[1|(n, m)]^{\mathcal{M}} = (n + 1, m)$
- $((a, b) + (n, m))^{\mathcal{M}} = (a + n, b + m \mod 2)$

$\mathcal{M} \vDash T_1$.

We still need to show that induction over atoms holds. For this take any atom $A(X) \equiv t_1 = t_2$. If neither t_1 nor t_2 contains X, then A either holds in the whole model or it holds nowhere. Assume that exactly one term, say t_1, contains X. Note that the reduct $(\mathsf{L}, +)$ of our model is the product of two commutative monoids. Since commutative monoids form a variety, they are closed under products. In particular, $(L, +)$ is a commutative monoid. Thus, w.l.o.g., we can write t_1 in the form of $nX + t$, where t is some fixed parameter and nX is the usual abbreviation for $\sum_{i=1}^{n} X$. Obviously, $t_1(nil) = t \neq (n, 0) + t = t_1((1, 0))$. Thus, $\mathcal{M} \nvDash \mathbf{LHS}(A)$. We are left with the case that X appears in both sides of A. Thus, $t_1 = nX + t$ and $t_2 = mX + t'$. Assume that $A(nil)$ holds. Then $t = t'$. Thus, $t_2 = mX + t$. Let $t = (a, b)$. If $m \neq n$, then $t_1((1, 0)) = (n + a, b) \neq (m + y, b) = t_2((1, 0))$. Thus, under the assumption that $A((0, 0))$ and $A((1, 0))$ hold, A has to hold for all elements in $\mathsf{L}^{\mathcal{M}}$. Since this condition is clearly implied if $\mathcal{M} \vDash \mathbf{LHS}(A)$, we are done.

Lastly, we need to state the obvious: Since $(0, 1)$ is not in the image of $[1|\cdot]$, the axiom SUR and thus induction over literals cannot hold. ■

Lemma 6.4.11. $T_1 + \mathit{ILiteral} \nvdash \mathit{IDClause}$

Proof. By Lemma 6.4.2 it suffices to give a model of $T_1 + \mathit{ILiteral}$, where some G_n does not hold. Consider the alphabet $\Sigma = \{a, b, c, d\}$. We define the model $\mathcal{M}$:

- $\iota^{\mathcal{M}} = \{a, b\}$
- $\mathsf{L}^{\mathcal{M}} = \{a, b\}^* \cup \{c, d\} \cup \{a, b\}^* bc \cup \{a, b\}^* ad$
- $[a|c] = d$ and $[b|d] = c$
- $[x|w] = xw$ for any other combination of x, w
- $\varepsilon + w = w + \varepsilon = w$
- $a + w = [a|w]$ and $b + w = [b|w]$
- $c + w = d + w = w$[1]
- $v + w$ for some composite word v is defined according to $L4$

Note that $L1$ and $L3$ hold by construction. For $L2$, note that $[a|v] \neq [b|w]$ for any $v, w \in L$. Moreover, $[a|v] = [a|w]$ iff $v = w$ for any $v, w \in L$. $L4$ again holds by construction. Note that G_n does not hold and as a consequence, induction over dual clauses cannot hold.

It remains to be shown that induction over literals holds.

Take any atom $A(X) \equiv t_1 = t_2$. If neither t_1 nor t_2 contain X, then A is true in the whole model or false in the whole model. If only t_1 contains X, then $t_2^{\mathcal{M}}$ is constant. Note the following: If $t_1(\mathit{nil})^{\mathcal{M}} \neq c$ and $t_1(\mathit{nil})^{\mathcal{M}} \neq d$, then $t_1([a|\mathit{nil}])^{\mathcal{M}} \neq t_1(\mathit{nil})$. If $t_1^{\mathcal{M}} = c$, then $t_1([a|\mathit{nil}])^{\mathcal{M}} \neq t_1(\mathit{nil})^{\mathcal{M}}$. Analogously for $t_1(\mathit{nil})^{\mathcal{M}} = d$. In any case, $\mathcal{M} \nvDash A(\mathit{nil}) \wedge \forall x \forall X (A(X) \rightarrow A([x|X]))$

We are left with the case that both sides contain X. Assume that $A(\mathit{nil})$ and $A(X) \rightarrow A([x|X])$ hold. Again, since $+$ is associative in this model, we can write the terms as sums $t_1 = \sum_{i=1}^{n} Y_i$ and $t_2 = \sum_{j=1}^{m} Z_j$, where at least one of the Y_i and Z_j is identical X. W.l.o.g., we assume that every second Y_i and Z_j is identical to X. Assume $Y_1 \equiv X$ and $Z_1 \not\equiv X$. W.l.o.g., we can assume that $\mathcal{M} \vDash Z_1 \neq c$ and $\mathcal{M} \vDash Z_1 \neq d$, as we could pull it to the next term otherwise. Thus, $(Z_1)^{\mathcal{M}}$ starts with either a or b and either $\mathcal{M} \nvDash A(b)$ or $\mathcal{M} \nvDash A(a)$. Assume that neither Y_1 nor Z_1 is identical to X. Then $Y_2 \equiv Z_2 \equiv X$. Assume that $\mathcal{M} \vDash Y_1 \neq Y_2$. By choosing X as a or b appropriately, we can again conclude that $\mathcal{M} \nvDash A(a)$ or $\mathcal{M} \nvDash A(b)$. In summary, we have shown that $\mathcal{M} \vDash Y_1 = Z_1$. Since left-cancellation holds in our model, it follows that $\mathcal{M} \vDash A(X) \rightarrow \sum_{i=2}^{n} Y_i(X) = \sum_{j=2}^{m} Z_j(X)$. By proceeding inductively, we can assume w.l.o.g. that $t_1 \equiv X$. However, if $t_1 \equiv X$ and $A(\mathit{nil})$ holds, $t_2 \equiv X$ as well. Thus, $\mathcal{M} \vDash \forall X : A(X)$.

[1] Note that c and d are not elements of ι.

Now, take any negated atom $L(X) \equiv t_1 \neq t_2$. Again, we can exclude the case that neither t_1 nor t_2 contains X. Assume that exactly one term, say t_1, contains X and that $L(nil)$ holds. If there is some element B s.t. $\mathcal{M} \nvDash L(B)$, then there is some C and c s.t. $B = [c|C]$ and by similar reasoning as above, $t_1(C)^{\mathcal{M}} \neq t_1(B)^{\mathcal{M}} = t_2$. Thus, $\mathcal{M} \nvDash \mathbf{LHS}(L)$.

We are left with the case that both terms contain X. Assume that $L(nil)$ and $L(X) \to L([x|X])$ both hold. From this assumption, it follows that $\mathcal{M} \vDash L(B)$ for any standard element B in $\mathsf{L}^{\mathcal{M}}$. If there is some element non-standard element B in $\mathsf{L}^{\mathcal{M}}$ s.t. $\mathcal{M} \nvDash L(B)$, then $\mathcal{M} \nvDash L(c)$ or $\mathcal{M} \nvDash L(d)$ since one of them is a predecessor of B. Assume that this is the case.

Again, since $+$ is associative in this model, we can write the terms as sums $t_1 = \sum_{i=1}^{n} Y_i$ and $t_2 = \sum_{j=1}^{m} Z_j$, where at least one of the Y_i and Z_j is identical to X. W.l.o.g., we assume that every second Y_i and Z_j is equal to X. Assume $Y_1 \equiv X$ and $Z_1 \not\equiv X$. W.l.o.g., we can assume that $\mathcal{M} \vDash Z_1 \neq c$ and $\mathcal{M} \vDash Z_1 \neq d$, as we could pull it to the next term otherwise. Consider Y_n and Z_m. Assume that $Y_n \equiv X \not\equiv Z_m$. If $\mathcal{M} \nvDash L(c)$, then $\mathcal{M} \nvDash L(d)$ as well. However, in that case $(Z_m)^{\mathcal{M}}$ would have to end with c and d, which is not possible. Now assume that $Y_n \not\equiv X \not\equiv Z_m$, then $\mathcal{M} \vDash t_1(c) = t_1(nil) \neq t_2(nil) = t_2(c)$ and $\mathcal{M} \vDash t_1(d) = t_1(nil) \neq t_2(nil) = t_2(d)$, which contradicts our assumption. Lastly, assume that $Y_n \equiv X \equiv Z_m$. Assume that $\mathcal{M} \nvDash L(c)$. Then $\mathcal{M} \vDash t_1(c) = t_1'(c) + c = t_2'(c) + c = t_2(c)$, where t_i' is just the term t_i cut off s.t. the last summand not identical to X. $\mathcal{M} \nvDash L(d)$ as well and $\mathcal{M} \vDash t_1'(d) + d = t_2'(d) + d$. Now note that since the last summand of t_i' is not X, we have that $\mathcal{M} \vDash t_i'(nil) = t_i'(c) = t_i'(d) = t_i^*$ for $i \in \{1, 2\}$. Thus, $\mathcal{M} \vDash t_1^* + c = t_2^* + c$ and $\mathcal{M} \vDash t_1^* + d = t_2^* + d$, which can only be the case if $\mathcal{M} \vDash t_1^* = t_2^* = nil$. In that case, however, $t_1 \equiv t_2 \equiv X$ and $\mathcal{M} \vDash t_1(nil) = t_2(nil)$, which contradicts the assumption that $\mathcal{M} \vDash L(nil)$. ■

6.5 Concatenation and a Size Function

We proceed analogously to Section 6.3: We consider the language consisting of the symbols $0, s, +_{\mathsf{Nat}}, nil, [\cdot|\cdot], +_L$ and l. Since it is always clear from the context, whether $+_{\mathsf{Nat}}$ or $+_L$ is used, we usually just write $+$.

We define the following additional axioms:

A4. $x + 0 = x$

A5. $x + sy = s(x + y)$

E3 $l(X + Y) = l(X) + l(Y)$

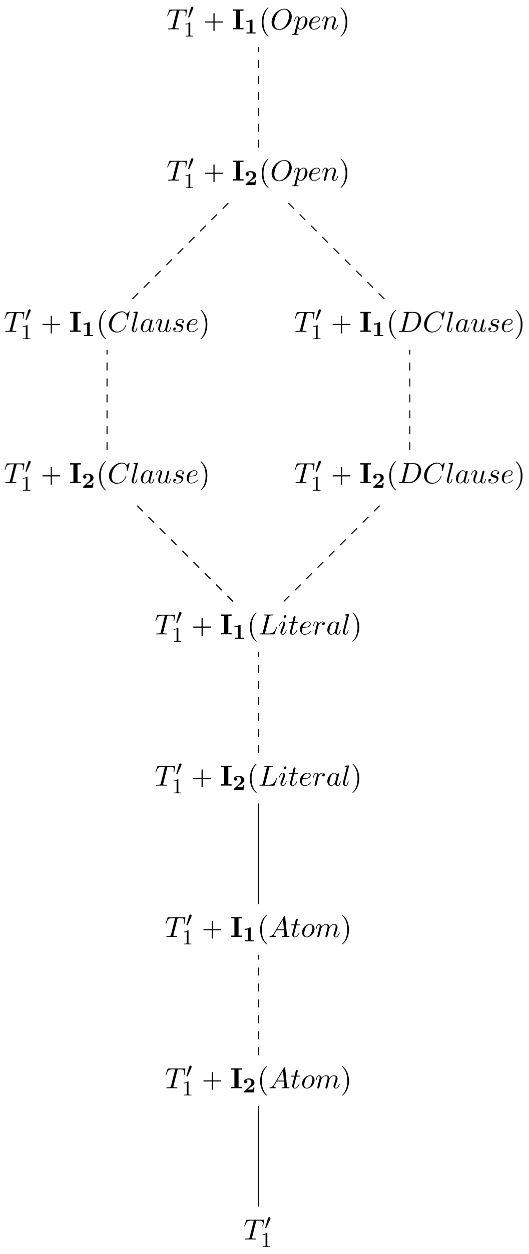
Definition 6.5.1. $T_1' = T_1 + \{A1, A3a, A4, A5, E1, E2, E3\}$ *over the language* $\mathcal{L}_1' = \{nil, [\cdot|\cdot], +_L, 0, s, +_{\mathsf{Nat}}, l\}$

The main result will be the following:

Theorem 6.5.2.

$$\begin{aligned} T_1' &\lnsim T_1' + \mathbf{I_2}(Atom) \\ &\lnsim T_1' + \mathbf{I_2}(Literal) \end{aligned} \qquad\qquad \begin{aligned} T_1' &\lnsim T_1' + \mathbf{I_1}(Atom) \\ &\lnsim T_1' + \mathbf{I_1}(ILiteral) \end{aligned}$$

This yields the following (partial) Hasse Diagram:


$T_1' + \mathbf{I_1}(Open)$
$T_1' + \mathbf{I_2}(Open)$
$T_1' + \mathbf{I_1}(Clause)$
$T_1' + \mathbf{I_1}(DClause)$
$T_1' + \mathbf{I_2}(Clause)$
$T_1' + \mathbf{I_2}(DClause)$
$T_1' + \mathbf{I_1}(Literal)$
$T_1' + \mathbf{I_2}(Literal)$
$T_1' + \mathbf{I_1}(Atom)$
$T_1' + \mathbf{I_2}(Atom)$
T_1'

Lemma 6.5.3. *The following holds:*

- $T_1' + \mathbf{I_2}(Atom) \vdash X + nil = X$
- $T_1' + \mathbf{I_2}(Atom) \vdash X + (Y + Z) = (X + Y) + Z$
- $T_1' + \mathbf{I_2}(Literal) \vdash X + Y = X + Z \rightarrow Y = Z$
- $T_1' + \mathbf{I_2}(Literal) \vdash X = nil \vee \exists Y \exists y : X = [y|Y]$
- $T_1' + \mathrm{SUR} \vdash X + Y = nil \rightarrow X = nil \wedge Y = nil$
- $T_1' + \mathbf{I_2}(ILiteral) \vdash G_n$

Proof. This follows directly from the Sections 6.4 and 6.3 and 4.4 since $T_1' \supseteq T_1$ and the formulas above only contain symbols that are part of the original language. ■

Lemma 6.5.4. $T_1' + \mathbf{I_2}(Literal) \vdash x = 0 \vee \exists y : sy = x$

Proof. This follows directly from Lemma 5.4.7. ■

The following lemma is not very important here, but it is a nice example of how the second sort Nat can come in handy: Using only lists, we failed to show the following property. Using the additional sort Nat, it becomes almost trivial.

Lemma 6.5.5. *Let $t(X)$ be any term that contains X. Then $T_1' + ILiteral \vdash t(X) \neq t([\overline{x}|X])$ for any non-empty tuple $\overline{x}$.*

Proof. Let n be the length of $\overline{x}$ and m the number of occurrences of X in t. From associativity and commutativity of $+_{\mathsf{Nat}}$ (cf. Section 4.6) and $A5$ it follows that $l(t([\overline{x}|X])) = s^{nm}(l(t(X)))$. By assumption $n \neq 0 \neq m$ and thus, by B_n, $l(t([\overline{x}|X])) \neq l(t(X))$. Since l is a function symbol, $t([\overline{x}|X])$ and $t(X)$ cannot coincide either. ■

Lemma 6.5.6. $T_1' \nvdash \mathbf{I_2}(Atom)$

Proof. It suffices to give a model $\mathcal{M}$ of T_1', where $+_{\mathsf{Nat}}$ is not commutative. Consider the following model $\mathcal{M}$:

- $\iota^{\mathcal{M}} = \{1\}$
- $\mathsf{L}^{\mathcal{M}} = \mathbb{N}$
- $\mathsf{Nat}^{\mathcal{M}} = \mathbb{N} \cup \{a, b\}$
- $nil^{\mathcal{M}} = 0 \in \mathsf{L}^{\mathcal{M}}$
- $[1|n] = n$

- $l(n) = n$
- The symbols $0, s, +$ are interpreted as in $\mathbb{N}_{a,b}$

$\mathcal{M} \vDash T_1'$, but $\mathcal{M} \nvDash (\forall x : \mathsf{Nat})(\forall y : \mathsf{Nat})(x + y = y + x)$ ■

Lemma 6.5.7. $T_1' + \mathbf{I_1}(\mathit{Atom}) \nvdash \mathbf{I_2}(\mathit{Literal})$

Proof. By Lemma 6.5.3 it suffices to give a model of $T_0 + \mathbf{I_1}(\mathit{Atom})$, where SUR does not hold.

Consider the following model $\mathcal{M}$: The sort ι is interpreted as $\{1\}$, L is interpreted as $\{0, 1\} \times \mathbb{N}$ and Nat is interpreted as $\mathbb{N}$. We interpret the function symbols in the following way:

- $\mathit{nil} = (0, 0)$, $0 = 0$
- $[1|(n, m)] = (n + 1, m)$
- $(x, y) + (a, b) = (x + a, y + b \mod 2)$
- $sn = n + 1$
- $n + m = n + m$
- $l(n, m) = n$

The axioms of T_1' clearly hold. Also, the element $(0, 1)$ is not in the image of $[\cdot|\cdot]$ and not nil either. It follows that SUR and thus induction over literals cannot hold.

It remains to be shown that induction over atoms does hold. For this take an arbitrary atom $A(x, X) \equiv t_1 = t_2$ with the variables $x \in \mathsf{Nat}$ and $X \in L$. We make the following observation: Both $+_{\mathsf{Nat}}$ and $+_L$ are commutative and associative. Now, make a case distinction:

1. Assume that both terms are of the sort Nat. Then, we can write each term $t_i = n_i l(X) + m_i x + t_i'$, where n_i and m_i can potentially be 0. Assume that $\mathcal{M} \vDash \mathbf{LHS(A)}$. If $A(0, \mathit{nil})$ holds, it follows that $t_1' = t_2'$. In our model, right cancellation holds and thus $n_1 l(X) + m_1 x = n_2 l(X) + m_2 x$. Since A is closed under successors, we conclude that $A(0, [1|\mathit{nil}])$ and $A(s0, \mathit{nil})$ both have to hold. This, in turn, entails that $n_1 = n_2$ and $m_1 = m_2$. Thus, A holds in the whole model and induction over it works.

2. Assume that both terms are of the sort L. We can write each term t_i as $n_i X + t_i'$. Assume that $\mathcal{M} \vDash \mathbf{LHS(A)}$. From $A(0, \mathit{nil})$, we obtain $t_1' = t_2'$. And from right cancellation and $A(0, [1|\mathit{nil}])$, we obtain $n_1 = n_2$. Thus, A holds in the whole model.

Since A was arbitrary, induction over all atoms holds in this model. ■

Open Problem 6.5.8. *In previous sections, we used the approach to add a simple inductive data type, which is well understood, to a more complicated one, in order to understand it better. This worked to some extent, but it could be formalized and extended. Theorems of the form* $T_0 + IDClause \vdash F \Leftrightarrow T_0' + ILiteral \vdash F$ *have not been shown, but could be true.*

CHAPTER 7

K-ary Trees

In this chapter, we will deal with the structure of k-ary trees. Again, we will see that trees are more complicated than arithmetics. Moreover, we will include the structures of natural numbers.

7.1 General Frame

In the following, we will use a two sorted logic with the sorts ι and T, where ι is the sort of the labels of the nodes of the trees and T is the sort of the trees. Again, to avoid confusion, we will capitalize variables of sort T. Variables of sort ι will be written in lowercase letters.

The language $\mathcal{L}$ consist of the function symbols $nil \in \mathsf{T}$ and $c : \mathsf{T}^k \times \iota \to \mathsf{T}$. The intention is that nil is the empty tree and the function c takes k trees $X_1, \dots, X_k$ and an element z in ι and maps them to the tree that has a root node labelled by z with the children $X_1, \dots, X_k$. Then the axioms are as follows:

T1. $nil \neq c(X_1, \dots, X_k, z)$

T2. $c(X_1, \dots, X_k, z_1) = c(Y_1, \dots, Y_k, , z_2) \to z_1 = z_2 \wedge \bigwedge_{i=1}^{k} X_i = Y_i$

We define the scheme of induction similarly as in the previous chapters:

$$\mathbf{LHS}(\varphi(X, \overline{z})) \quad \varphi(nil, \overline{z}) \wedge \forall X_1, \dots, X_k, y(\bigwedge_{i=1}^{k} \varphi(X_i, \overline{z}) \to \varphi(c(X_1, \dots, X_k, y), \overline{z}))$$

$$\mathbf{I}(\varphi(X, \overline{z})) \qquad \mathbf{LHS}(\varphi(X, \overline{z}) \to \forall X : \varphi(X, \overline{z}))$$

For the sake of readability, we will often refrain from writing the parameters of a formula explicitly. If not explicitly stated otherwise, all our formulas in this chapter may possibly contain parameters.

We define some important axioms (cf. Chapter 3):

$\mathbf{G_t}$. $X \neq t(X)$ for any $t \in \mathcal{S}$

SUR. $X = nil \vee \exists X_1, \ldots X_k \exists z : X = c(X_1, \ldots, X_k, z)$

7.2 Constructors only

As usual, we start with the theory $T_0 = \{T1, T2\}$ and the full language nil, c.

Implicitly, we have already dealt with this case in Chapter 3. By applying Theorem 3.4.3 to this case, we obtain the following result:

Theorem 7.2.1.

$$\begin{aligned} T_0 &\approx T_0 + IAtom \\ &\lesssim T_0 + ILiteral \\ &\lesssim T_0 + IDClause \\ &\approx T_0 + IOpen \\ &\approx T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd}
T_0 + IDClause \approx T_0 + IOpen \approx T_0 + \mathrm{SUR} + \{G_t \mid t \in \mathcal{S}\} \arrow[d, dash] \\
T_0 + IClause \arrow[d, dash, dashed] \\
T_0 + ILiteral \arrow[d, dash] \\
T_0 \approx T_0 + IAtom
\end{tikzcd}$$

7.3 Constructors only and a Size Function

We now add the sort Nat to the lists to represent natural numbers. Additionally, we add the symbols $0 \in \mathsf{Nat}, s : \mathsf{Nat} \to \mathsf{Nat}, + : \mathsf{Nat} \times \mathsf{Nat} \to \mathsf{Nat}, l : L \to \mathsf{Nat}$ with the following axioms:

A1. $0 = sx$

A3a. $sx = sy \rightarrow x = y$

A4. $x + 0 = x$

A5. $x + sy = s(x + y)$

E1. $l(nil) = 0$

E2. $l(c(X_1, \ldots, X_n, z)) = s(\sum_{i=1}^{k} l(X_i))$

We will also need these auxiliary axioms:

B1a. $x = 0 \vee \exists y : x = sy$

B2. $x + (y + z) = (x + y) + z$

B3. $x + y = y + x$

B4. $x + y = x + z \rightarrow y = z$

Definition 7.3.1. $T'_0 = T_0 + \{A1, A3a, A4, A5, E1, E2\}$ *over the language* $\mathcal{L}' = \{nil, c, 0, s, +, l\}$

Definition 7.3.2. $\Gamma = T'_0 + \text{SUR} + \{G_t \mid t \in \mathcal{S}\} + B1a + B2 + B3 + B4$

Theorem 7.3.3.

$$\begin{aligned} T'_0 &\precnapprox T'_0 + \mathbf{I_2}(Atom) \\ &\precnapprox T'_0 + \mathbf{I_2}(Literal) \\ &\approx T'_0 + \mathbf{I_2}(Open) \\ &\approx \Gamma \end{aligned} \qquad\qquad \begin{aligned} T'_0 &\precnapprox T'_0 + \mathbf{I_1}(Atom) \\ &\precnapprox T'_0 + \mathbf{I_1}(Literal) \\ &\approx T'_0 + \mathbf{I_1}(Open) \\ &\approx \Gamma \end{aligned}$$

This yields the following Hasse Diagram:

$$\begin{tikzcd}
T'_0 + \mathbf{I_2}(ILiteral) \approx T'_0 + \mathbf{I_1}(Literal) \approx T'_0 + \mathbf{I_2}(Open) \approx T'_0 + \mathbf{I_1}(Open) \arrow[d, dash] \\
T'_0 + \mathbf{I_1}(Atom) \arrow[d, dash, dashed] \\
T'_0 + \mathbf{I_2}(Atom) \arrow[d, dash] \\
T'_0
\end{tikzcd}$$

Proof. This follows directly from Theorem 5.4.3. ■

CHAPTER 8

Conclusion

We analyzed how open induction behaves in different contexts. The two main questions considered are:

1. How do the subsystems of open induction relate to each other?
2. Can open induction be axiomatized by other *non-induction* axioms?

Regarding question 1, we saw that sometimes it is not that straightforward, which theory proves which theorems. Consider $T_1 + ILiteral$ from Section 6.4 (lists with concatenation). Obviously, there are some literals L (e.g. $L(X) \equiv X \neq [x|X]$) s.t. $T_1 + \mathbf{I}(L) \vdash \forall X : L(X)$ and thus, $T_1 + ILitearl \vdash \forall X : L(X)$. Less straightforward, yet unsurprising: There are literals L (e.g. $L(X) \equiv X \neq [x_1, x_2|X]$) s.t. every model $\mathcal{M}$ of lists satisfies $\mathcal{M} \vDash L(E)$ for any standard element $E \in \mathsf{L}^{\mathcal{M}}$, but $T_1 + ILiteral \nvdash \forall X : L(X)$. This is not surprising since it was shown in [She65] that the irrationality of $\sqrt{2}$ (which can be expressed by the open formula $p \neq 0 \rightarrow p \cdot p + p \cdot p \neq q \cdot q$) cannot be shown with open induction.

However, on a more positive note, there are formulas $F(X)$, which are not literals, but can be proven in $ILiteral$. The most interesting example for this is probably the formula $F(X) \equiv X + Y = nil \rightarrow X = nil \wedge Y = nil$ since it is neither a clause nor a dual clause and if we write it in CNF as $C_1 \wedge C_2$, then the induction variable X appears in both C_1 and C_2. Still $T_1 + ILiteral \vdash \forall Y \forall X : F(X)$.

Moreover, we saw two important things: First, depending on the language and base theory, there often is some level of induction, which is seemingly weaker than open induction, but entails open induction nonetheless. As an example consider the base theory T_2 from Subsection 4.4 (arithmetics with $0 \neq sx$ and injective successor). It holds that, $T_0 + ILiteral \vdash IOpen$. Secondly, this level of induction is not monotone in the complexity of the language or theory in the following sense: If we consider the, less complicated, empty theory over the same language as T_2, then $\varnothing + \mathit{ILiteral} \nvdash \mathit{IClause}$. However, if we make the language and theory more complicated by switching to T_5 from Section 4.7 (arithmetics with addition, multiplication and the usual axioms), we have that $T_5 + \mathit{ILiteral} \nvdash \mathit{IClause}$ as well.

In particular, we saw in Chapter 5 that adding some inductive data type to another one drastically increases the complexity of the language and theory, but can have the effect that the open induction collapses to a lower level than it would if there was only one data type.

Regarding question 2, the most important discovery is, the connection between open induction and the sets definable by open formulas in the language at hand. If one can show that, modulo some theory T, open formulas only define very simple sets, this might allow one to prove that $T \vdash \mathit{IOpen}$. This connection was used in [Sho58] and [She63], but we saw that this goes deeper than just arithmetics and also works for general inductive data types (cf. Chapter 3), where all the sets definable with open formulas are even finite or cofinite. The sets we can define become progressively more complicated if we add complexity to the language as we could see in Section 4.7. In this case, we could apply ring theory to show that open induction has an alternative characterization. This might be more difficult in a non-arithmetical context if the underlying structures are not as well understood.

We did not succeed in giving an alternative axiomatization of open induction for every base theory, we considered. While there there are theories, for which we believe it to be unlikely that there is an alternative *simple* axiomatization of open induction (cf. Conjecture 3.4.27), there are other theories, for which this seems more likely to us (e.g. T_1' from Section 6.5).

There are three major directions, in which future work on this topic could be headed. The first option is to work with general inductive data types and try to prove obtain results in the most general setting. A good starting point for this would be to prove Conjecture 3.4.27. The second option is to consider some data structure in more depth. Here, one could start by considering lists with concatenation. Lastly, one could consider the interplay of several data types. We considered a size function for general inductive data types. The next step could be to connect trees and lists by tree-traversal functions.

Overview of Generative AI Tools Used

1. The free version of the Overleaf Add-on Writeful (`https://www.writefull.com/writefull-for-overleaf`) was used throughout the thesis, to try to improve the readability of the thesis. Since the free version has a cap on the suggestions per day, one can see, the extent, to which it was used, is not substantial.

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