DE GRUYTER Adv. Calc. Var. 2024; 17(4): 1399–1444

Research Article

Marin Bužančić, Elisa Davoli* and Igor Velčić

Effective quasistatic evolution models for perfectly plastic plates with periodic microstructure: The limiting regimes

https://doi.org/10.1515/acv-2023-0020
Received March 1, 2023; accepted November 11, 2023

Abstract: We identify effective models for thin, linearly elastic and perfectly plastic plates exhibiting a microstructure resulting from the periodic alternation of two elastoplastic phases. We study here both the case in which the thickness of the plate converges to zero on a much faster scale than the periodicity parameter and the opposite scenario in which homogenization occurs on a much finer scale than dimension reduction. After performing a static analysis of the problem, we show convergence of the corresponding quasistatic evolutions. The methodology relies on two-scale convergence and periodic unfolding, combined with suitable measure-disintegration results and evolutionary Γ-convergence.

Keywords: Perfect plasticity, periodic homogenization, dimension reduction, quasistatic evolution, rate-independent processes, Γ-convergence

MSC 2020: 74C05, 74G65, 74K20, 49J45, 74Q09, 35B27

Communicated by: Irene Fonseca

1 Introduction

The main goal of this paper is to complete the study of limiting models stemming from the interplay of homogenization and dimension reduction in perfect plasticity which we have initiated in [7], as well as to show how the stress-strain approach introduced in [26] for the homogenization of elasto-perfect plasticity can be used to identify effective theories for composite plates. In our previous contribution, we considered a composite thin plate whose thickness h and microstructure-width ε_h were asymptotically comparable, namely, we assumed

$$\lim_{h\to 0} \frac{h}{\varepsilon_h} =: \gamma \in (0, +\infty).$$

In this work, instead, we analyze the two limiting regimes corresponding to the settings $\gamma = 0$ and $\gamma = +\infty$. These can be seen, roughly speaking, as situations in which homogenization and dimension reduction happen on different scales, so that the behavior of the composite plate should ideally approach either that obtained via homogenization of the lower-dimensional model or the opposite one in which dimension reduction is performed on the homogenized material.

To the authors' knowledge, apart from [7] there has been no other study of simultaneous homogenization and dimension reduction for inelastic materials. In the purview of elasticity, we single out the works [8, 14] (see also the book [44]) where first results were obtained in the case of linearized elasticity and under isotropy or

***Corresponding author: Elisa Davoli,** Institute of Analysis and Scientific Computing, TU Wien, Wiedner Hauptstr. 8–10, A-1040 Vienna, Austria, e-mail: elisa.davoli@tuwien.ac.at. https://orcid.org/0000-0002-1715-5004
Marin Bužančić, Faculty of Chemical Engineering and Technology, University of Zagreb, Trg Marka Marulića 19, 10000 Zagreb, Croatia, e-mail: buzancic@fkit.hr. https://orcid.org/0000-0002-9149-3706
Igor Velčić, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia, e-mail: igor.velcic@fer.hr. https://orcid.org/0000-0003-2494-2230

additional material symmetry assumptions, as well as [5] for the study of the general case without further constitutive restrictions and for an extension to some nonlinear models. A Γ-convergence analysis in the nonlinear case has been provided in [4, 10, 36, 43, 48], whereas the case of high-contrast elastic plates is the subject of [6].

We briefly review below the literature on dimension reduction in plasticity and that on the study of composite elastoplastic materials. Reduced models for homogeneous perfectly plastic plates have been characterized in [21, 22, 31, 40] in the quasistatic and dynamic settings, respectively, whereas the case of shallow shells is the focus of [41]. In the presence of hardening, an analogous study has been undertaken in [38, 39]. Further results in finite plasticity are the subject of [15, 16].

Homogenization of the elastoplastic equations in the small strain regime has been studied in [34, 35, 45]. We also refer to [28, 30] for a study of the Fleck and Willis model, and to [33] for the case of gradient plasticity. Static and partial evolutionary results for large-strain stratified composites in crystal plasticity have been obtained in [11, 12, 17, 20], whereas static results in finite plasticity are the subject of [18, 19]. Inhomogeneous perfectly plastic materials have been fully characterized in [27], an associated study of periodic homogenization is the focus of [26].

The main result of the paper, Theorem 6.2, is rooted in the theory of evolutionary Γ-convergence (see [42]) and consists in showing that rescaled three-dimensional quasistatic evolutions associated to the original composite plates converge, as the thickness and periodicity simultaneously go to zero, to the quasistatic evolution corresponding to suitable reduced effective elastic energies (identified by static Γ-convergence) and dissipation potentials, cf. Section 5.4. As one might expect, for $\gamma = 0$ the limiting driving energy and dissipation potential are homogenized versions of those identified in [21] where only dimension reduction was considered. In the $\gamma = \infty$ setting, instead, the key functionals are obtained by averaging the original ones in the periodicity cell.

Essential ingredients to identify the limiting models are to establish a characterization of two-scale limits of rescaled linearized strains, as well as to prove variants of the principle of maximal work in each of the two regimes. These are the content of Theorem 4.14, as well as Theorem 5.31 for the case $\gamma = 0$, and of Theorem 5.33 for $\gamma = +\infty$, respectively. A very delicate point consists in the identification of the limiting space of elastoplastic variables, for a fine characterization of the correctors arising in the two-scale limit passage needs to be established by delicate measure-theoretic disintegration arguments, cf. Section 4.

We finally mention that, for the regimes analyzed in this contribution, we obtain more restrictive results than in [7], for an additional assumption on the ordering of the phases on the interface, cf. Section 3.1 needs to be imposed in order to ensure lower semicontinuity of the dissipation potential, cf. Remark 3.3.

We briefly outline the structure of the paper. In Section 2 we introduce our notation and recall some preliminary results. Section 3 is devoted to the mathematical formulation of the problem, whereas Section 4 tackles compactness properties of sequences with equibounded energy and dissipation. In Section 5 we characterize the limiting model, we introduce the set of limiting deformations and stresses, and we discuss the duality between stress and strain. Eventually, in Section 6 we prove the main result of the paper, i.e., Theorem 6.2, where we show convergence of the quasistatic evolution of $3d$ composite thin plates to the quasistatic evolution associated to the limiting model. Similarly as in [7, 26], in the limiting model a decoupling of macroscopic and microscopic variables is not possible and both scales contribute to the description of the limiting evolution.

2 Notation and preliminary results

Points $x \in \mathbb{R}^3$ will be expressed as pairs (x', x_3), with $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$, whereas we will write $y \in \mathcal{Y}$ to identify points on a flat 2-dimensional torus. We will denote by I the open interval $I := (-\frac{1}{2}, \frac{1}{2})$. The notation $\nabla_{x'}$ will describe the gradient with respect to x'. Scaled gradients and symmetrized scaled gradients will be similarly denoted as follows:

$$\nabla_h v := \left[\nabla_{x'} v \,\middle|\, \tfrac{1}{h}\partial_{x_3} v \right], \quad E_h v := \operatorname{sym} \nabla_h v, \tag{2.1}$$

for $h > 0$, and for maps v defined on suitable subsets of $\mathbb{R}^3$. For $N = 2, 3$, we use the notation $\mathbb{M}^{N\times N}$ to identify the set of real $N \times N$ matrices. We will always implicitly assume this set to be endowed with the classical

Frobenius scalar product $A : B := \sum_{i,j} A_{ij}B_{ij}$ and the associated norm $|A| := \sqrt{A : A}$, for $A, B \in \mathbb{M}^{N\times N}$. The subspaces of symmetric and deviatoric matrices will be denoted by $\mathbb{M}^{N\times N}_{\mathrm{sym}}$ and $\mathbb{M}^{N\times N}_{\mathrm{dev}}$, respectively. For the trace and deviatoric part of a matrix $A \in \mathbb{M}^{N\times N}$ we will adopt the notation $\operatorname{tr} A$, and

$$A_{\mathrm{dev}} = A - \frac{1}{N}\operatorname{tr} A.$$

Given two vectors $a, b \in \mathbb{R}^N$, we will adopt standard notation for their scalar product and Euclidean norm, namely $a \cdot b$ and $|a|$. The dyadic (or tensor) product of a and b will be identified as by $a \otimes b$; correspondingly, the *symmetrized tensor product* $a \odot b$ will be the symmetric matrix with entries $(a \odot b)_{ij} := \frac{a_i b_j + a_j b_i}{2}$. We recall that $\operatorname{tr}(a \odot b) = a \cdot b$, and $|a \odot b|^2 = \frac{1}{2}|a|^2|b|^2 + \frac{1}{2}(a \cdot b)^2$, so that

$$\frac{1}{\sqrt{2}}|a||b| \le |a \odot b| \le |a||b|.$$

In analogy with the notation used for points $x \in \mathbb{R}^3$, given a vector $v \in \mathbb{R}^3$, we will use the notation v' to denote the 2-dimensional vector having its same first two components

$$v' := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

In the same way, for every $A \in \mathbb{M}^{3\times 3}$, we will use the notation A'' to identify the minor

$$A'' := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The natural embedding of $\mathbb{R}^2$ into $\mathbb{R}^3$ will be given by $\iota : \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$$\iota(v) := \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}.$$

We will adopt standard notation for the Lebesgue and Hausdorff measure, as well as for Lebesgue and Sobolev spaces, and for spaces of continuously differentiable functions. Given a set $U \subset \mathbb{R}^N$, we will denote its closure by $\overline{U}$ and its characteristic function by $\mathbb{1}_U$.

Let E be an Euclidean space. We will distinguish between the spaces $C^k_c(U; E)$ (C^k functions with compact support contained in U) and $C^k_0(U; E)$ (C^k functions "vanishing on ∂U"). The notation $C(\mathcal{Y}; E)$ will indicate the space of all continuous functions which are $[0, 1]^2$-periodic. Analogously, we will define

$$C^k(\mathcal{Y}; E) := C^k(\mathbb{R}^2; E) \cap C(\mathcal{Y}; E).$$

With a slight abuse of notation, $C^k(\mathcal{Y}; E)$ will be identified with the space of all C^k functions on the 2-dimensional torus.

We will frequently make use of the *standard mollifier* $\rho \in C^\infty(\mathbb{R}^N)$, defined by

$$\rho(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise}, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$, as well as of the associated family $\{\rho_\varepsilon\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}^N)$ defined with

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right).$$

Throughout the text, the letter C stands for generic positive constants whose value may vary from line to line.

A collection of all preliminary results which will be used throughout the paper can be found in [7, Section 2]. For an overview on basic notions in measure theory, functions of bounded variation (BV), as well as functions of bounded deformation (BD) and bounded Hessian (BH), we refer the reader to, e.g., [1, 2, 25], to the monograph [46], as well as to [23].

2.1 Convex functions of measures

Let U be an open set of $\mathbb{R}^N$ and X a finite-dimensional vector space, and denote by $\mathcal{M}_b(U;X)$ the set of finite X-valued Radon measures on the set U. For every $\mu \in \mathcal{M}_b(U;X)$ let $\frac{d\mu}{d|\mu|}$ be the Radon–Nikodym derivative of μ with respect to its variation $|\mu|$. Let $H : X \to [0,+\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H(\xi) \leq R|\xi| \quad \text{for every } \xi \in X, \tag{2.2}$$

where r and R are two constants, with $0 < r \leq R$.

Using the theory of convex functions of measures (see [32] and [24]) it is possible to define a nonnegative Radon measure $H(\mu) \in \mathcal{M}_b^+(U)$ as

$$H(\mu)(A) := \int_A H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset U$, as well as an associated functional $\mathcal{H} : \mathcal{M}_b(U;X) \to [0,+\infty)$ given by

$$\mathcal{H}(\mu) := H(\mu)(U) = \int_U H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

and being lower semicontinuous on $\mathcal{M}_b(U;X)$ with respect to weak* convergence, cf. [1, Theorem 2.38]).

Let $a, b \in [0,T]$ with $a \leq b$. The *total variation* of a function $\mu : [0,T] \to \mathcal{M}_b(U;X)$ on $[a,b]$ is defined as

$$\mathcal{V}(\mu; a, b) := \sup\Big\{\sum_{i=1}^{n-1} \|\mu(t_{i+1}) - \mu(t_i)\|_{\mathcal{M}_b(U;X)} : a = t_1 < t_2 < \cdots < t_n = b,\ n \in \mathbb{N}\Big\}.$$

Analogously, the $\mathcal{H}$-*variation* of a function $\mu : [0,T] \to \mathcal{M}_b(U;X)$ on $[a,b]$ is given by

$$\mathcal{D}_{\mathcal{H}}(\mu; a, b) := \sup\Big\{\sum_{i=1}^{n-1} \mathcal{H}(\mu(t_{i+1}) - \mu(t_i)) : a = t_1 < t_2 < \cdots < t_n = b,\ n \in \mathbb{N}\Big\}.$$

From (2.2) it follows that

$$r\mathcal{V}(\mu; a, b) \leq \mathcal{D}_{\mathcal{H}}(\mu; a, b) \leq R\mathcal{V}(\mu; a, b). \tag{2.3}$$

2.2 Generalized products

Let S and T be measurable spaces and let μ be a measure on S. Given a measurable function $f : S \to T$, we denote by $f_\#\mu$ the *push-forward* of μ under the map f, defined by

$$f_\#\mu(B) := \mu(f^{-1}(B)) \quad \text{for every measurable set } B \subseteq T.$$

In particular, for any measurable function $g : T \to \overline{\mathbb{R}}$ we have

$$\int_S g \circ f\, d\mu = \int_T g\, d(f_\#\mu).$$

Note that in the previous formula $S = f^{-1}(T)$.

Let $S_1 \subset \mathbb{R}^{N_1}$, $S_2 \subset \mathbb{R}^{N_2}$, for some $N_1, N_2 \in \mathbb{N}$, be open sets, and let $\eta \in \mathcal{M}_b^+(S_1)$. We say that a function $x_1 \in S_1 \mapsto \mu_{x_1} \in \mathcal{M}_b(S_2;\mathbb{R}^M)$ is η-measurable if $x_1 \in S_1 \mapsto \mu_{x_1}(B)$ is η-measurable for every Borel set $B \subseteq S_2$.

Given a η-measurable function $x_1 \mapsto \mu_{x_1}$ such that $\int_{S_1} |\mu_{x_1}|\, d\eta < +\infty$, then the *generalized product* $\eta \overset{\text{gen.}}{\otimes} \mu_{x_1}$ satisfies $\eta \overset{\text{gen.}}{\otimes} \mu_{x_1} \in \mathcal{M}_b(S_1 \times S_2;\mathbb{R}^M)$ and is such that

$$\langle \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}, \varphi\rangle := \int_{S_1}\Big(\int_{S_2} \varphi(x_1, x_2)\, d\mu_{x_1}(x_2)\Big)\, d\eta(x_1)$$

for every bounded Borel function $\varphi : S_1 \times S_2 \to \mathbb{R}$.

2.3 Traces of stress tensors

In this last subsection we collect some properties of classes of maps which will include our elastoplastic stress tensors.

We suppose here that U is an open bounded set of class C^2 in $\mathbb{R}^N$. If $\sigma \in L^2(U; \mathbb{M}^{N\times N}_{\mathrm{sym}})$ and $\operatorname{div}\sigma \in L^2(U; \mathbb{R}^N)$, then we can define a distribution $[\sigma\nu]$ on ∂U by

$$[\sigma\nu](\psi) := \int_U \psi \cdot \operatorname{div}\sigma\, dx + \int_U \sigma : E\psi\, dx \tag{2.4}$$

for every $\psi \in H^1(U; \mathbb{R}^N)$. It follows that $[\sigma\nu] \in H^{-1/2}(\partial U; \mathbb{R}^N)$ (see, e.g., [47, Chapter 1, Theorem 1.2]). If, in addition, $\sigma \in L^\infty(U; \mathbb{M}^{N\times N}_{\mathrm{sym}})$ and $\operatorname{div}\sigma \in L^N(U; \mathbb{R}^N)$, then (2.4) holds for $\psi \in W^{1,1}(U; \mathbb{R}^N)$. By Gagliardo's extension theorem [29, Theorem 1.II], in this case we have $[\sigma\nu] \in L^\infty(\partial U; \mathbb{R}^N)$, and

$$[\sigma_k\nu] \xrightarrow{*} [\sigma\nu] \quad \text{weakly* in } L^\infty(\partial U; \mathbb{R}^N),$$

whenever $\sigma_k \xrightarrow{*} \sigma$ weakly* in $L^\infty(U; \mathbb{M}^{N\times N}_{\mathrm{sym}})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^N(U; \mathbb{R}^N)$.

We will consider the normal and tangential parts of $[\sigma\nu]$, defined by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu, \quad [\sigma\nu]^\perp_\nu := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu.$$

Since $\nu \in C^1(\partial U; \mathbb{R}^N)$, we have that $[\sigma\nu]_\nu, [\sigma\nu]^\perp_\nu \in H^{-1/2}(\partial U; \mathbb{R}^N)$. If, in addition, $\sigma_{\mathrm{dev}} \in L^\infty(U; \mathbb{M}^{N\times N}_{\mathrm{dev}})$, then it was proved in [37, Lemma 2.4] that $[\sigma\nu]^\perp_\nu \in L^\infty(\partial U; \mathbb{R}^N)$ and

$$\|[\sigma\nu]^\perp_\nu\|_{L^\infty(\partial U;\mathbb{R}^N)} \le \frac{1}{\sqrt{2}}\|\sigma_{\mathrm{dev}}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\mathrm{dev}})}.$$

More generally, if U has Lipschitz boundary and is such that there exists a compact set $S \subset \partial U$ with $\mathcal{H}^{N-1}(S) = 0$ such that $\partial U \setminus S$ is a C^2-hypersurface, then arguing as in [27, Section 1.2] we can uniquely determine $[\sigma\nu]^\perp_\nu$ as an element of $L^\infty(\partial U; \mathbb{R}^N)$ through any approximating sequence $\{\sigma_n\} \subset C^\infty(\overline{U}; \mathbb{M}^{N\times N}_{\mathrm{sym}})$ such that

$$\begin{aligned} &\sigma_n \to \sigma \quad \text{strongly in } L^2(U; \mathbb{M}^{N\times N}_{\mathrm{sym}}),\\ &\operatorname{div}\sigma_n \to \operatorname{div}\sigma \quad \text{strongly in } L^2(U; \mathbb{R}^N),\\ &\|(\sigma_n)_{\mathrm{dev}}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\mathrm{dev}})} \le \|\sigma_{\mathrm{dev}}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\mathrm{dev}})}.\end{aligned}$$

3 Setting of the problem

We describe here our modeling assumptions and recall a few associated instrumental results. Unless otherwise stated, $\omega \subset \mathbb{R}^2$ is a bounded, connected, and open set with C^2 boundary. Given a small positive number $h > 0$, we assume

$$\Omega^h := \omega \times (hI)$$

to be the reference configuration of a linearly elastic and perfectly plastic plate.

We consider a nonzero Dirichlet boundary condition on the whole lateral surface, i.e., the Dirichlet boundary of Ω^h is given by $\Gamma^h_D := \partial\omega \times (hI)$.

We work under the assumption that the body is only submitted to a hard device on Γ^h_D and that there are no applied loads, i.e., the evolution is only driven by time-dependent boundary conditions. More general boundary conditions, together with volume and surface forces have been considered, e.g., in [13, 21, 27] but for simplicity of exposition will be neglected in this analysis.

3.1 Phase decomposition

We recall here some basic notation and assumptions from [26].

Recall that $\mathcal{Y} = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-dimensional torus, let $Y := [0,1)^2$ be its associated periodicity cell, and denote by $\mathcal{I} : \mathcal{Y} \to Y$ their canonical identification. For any $\mathcal{Z} \subset \mathcal{Y}$, we define

$$\mathcal{Z}_\varepsilon := \left\{ x \in \mathbb{R}^2 : \frac{x}{\varepsilon} \in \mathbb{Z}^2 + \mathcal{I}(\mathcal{Z}) \right\}, \tag{3.1}$$

and to every function $F : \mathcal{Y} \to X$ we associate the ε-periodic function $F_\varepsilon : \mathbb{R}^2 \to X$, given by

$$F_\varepsilon(x) := F(y_\varepsilon) \quad \text{for } \frac{x}{\varepsilon} - \left\lfloor \frac{x}{\varepsilon} \right\rfloor = \mathcal{I}(y_\varepsilon) \in Y.$$

With a slight abuse of notation we will also write $F_\varepsilon(x) = F(\frac{x}{\varepsilon})$.

The torus $\mathcal{Y}$ is assumed to be made up of finitely many phases $\mathcal{Y}_i$ together with their interfaces. We assume that those phases are pairwise disjoint open sets with Lipschitz boundary. Then we have $\mathcal{Y} = \bigcup_i \overline{\mathcal{Y}}_i$ and we denote the interfaces by

$$\Gamma := \bigcup_{i,j} \partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j.$$

We will write

$$\Gamma := \bigcup_{i \neq j} \Gamma_{ij},$$

where Γ_{ij} stands for the interface between $\mathcal{Y}_i$ and $\mathcal{Y}_j$.

Correspondingly, ω is composed of finitely many phases $(\mathcal{Y}_i)_\varepsilon$ and that ε is chosen small enough so that $\mathcal{H}^1(\bigcup_i (\partial\mathcal{Y}_i)_\varepsilon \cap \partial\omega) = 0$. Additionally, we assume that Ω^h is a specimen of a linearly elastic-perfectly plastic material having periodic elasticity tensor and dissipation potential.

We are interested in the situation when the period ε is a function of the thickness h, i.e., $\varepsilon = \varepsilon_h$, and we assume that the limit

$$\gamma := \lim_{h \to 0} \frac{h}{\varepsilon_h}$$

exists in $\{0, +\infty\}$. We additionally impose the following condition: there exists a compact set $\mathcal{S} \subset \Gamma$ with $\mathcal{H}^1(\mathcal{S}) = 0$ such that each connected component of $\Gamma \setminus \mathcal{S}$ is either a closed curve of class C^2 or an open curve with endpoints $\{a, b\}$ which is of class C^2 (excluding the endpoints).

We say that a multi-phase torus $\mathcal{Y}$ is *geometrically admissible* if it satisfies the above assumptions.

Remark 3.1. Notice that under the above assumptions, $\mathcal{H}^1$-almost every $y \in \Gamma$ is at the intersection of the boundaries of exactly two phases.

Remark 3.2. We point out that we assume greater regularity than that in [26], where the interface $\Gamma \setminus \mathcal{S}$ was allowed to be a C^1-hypersurface. Under such weaker assumptions, in fact, the tangential part of the trace of an admissible stress $[\sigma\nu]^\perp_\nu$ at a point x on $\Gamma \setminus \mathcal{S}$ would not be defined independently of the considered approximating sequence, cf. Section 2.3. By requiring a higher regularity of $\Gamma \setminus \mathcal{S}$, we will avoid dealing with this situation.

The set of admissible stresses. We assume that there exist convex compact sets $K_i \in \mathbb{M}^{3\times3}_{\mathrm{dev}}$ associated to each phase $\mathcal{Y}_i$ which will provide restrictions on the deviatoric part of the stress. We work under the assumption that there exist two constants r_K and R_K, with $0 < r_K \le R_K$, such that for every i,

$$\{\xi \in \mathbb{M}^{3\times3}_{\mathrm{dev}} : |\xi| \le r_K\} \subseteq K_i \subseteq \{\xi \in \mathbb{M}^{3\times3}_{\mathrm{dev}} : |\xi| \le R_K\}.$$

Finally, we define

$$K(y) := K_i \quad \text{for } y \in \mathcal{Y}_i.$$

We will require an ordering between the phases at the interface. Namely, we assume that at the point $y \in \Gamma$ where exactly two phases $\mathcal{Y}_i$ and $\mathcal{Y}_j$ meet we have that either $K_i \subset K_j$ or $K_j \subset K_i$.

We will call this requirement the assumption on the ordering of the phases.

Remark 3.3. The restrictive assumption on the ordering between the phases will allow us to use Reshetnyak's lower semicontinuity theorem to obtain lower semicontinuity of the dissipation functional, cf. the proof of Theorem 6.2. Notice that in the regime $\gamma \in (0, +\infty)$, see [7], we did not rely on such assumption (see also [26, 27]) and thus were able to prove the convergence to the limit model in the general case. In the regimes $\gamma \in \{0, \infty\}$ the general geometrical setting where no ordering between the phases is assumed remains an open problem.

The elasticity tensor. For every i, let $(\mathbb{C}_{\mathrm{dev}})_i$ and k_i be a symmetric positive definite tensor on $\mathbb{M}^{3\times3}_{\mathrm{dev}}$ and a positive constant, respectively, such that there exist two constants r_c and R_c, with $0 < r_c \le R_c$, satisfying

$$r_c|\xi|^2 \le (\mathbb{C}_{\mathrm{dev}})_i\xi : \xi \le R_c|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\mathrm{dev}}, \tag{3.2}$$

$$r_c \le k_i \le R_c. \tag{3.3}$$

Let $\mathbb{C}$ be the *elasticity tensor*, considered as a map from $\mathcal{Y}$ taking values in the set of symmetric positive definite linear operators, $\mathbb{C} : \mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to \mathbb{M}^{3\times3}_{\mathrm{sym}}$, defined as

$$\mathbb{C}(y)\xi := \mathbb{C}_{\mathrm{dev}}(y)\xi_{\mathrm{dev}} + (k(y)\operatorname{tr}\xi)I_{3\times3} \quad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3},$$

where $\mathbb{C}_{\mathrm{dev}}(y) = (\mathbb{C}_{\mathrm{dev}})_i$ and $k(y) = k_i$ for every $y \in \mathcal{Y}_i$.

Let $Q : \mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, and given by

$$Q(y, \xi) := \frac{1}{2}\mathbb{C}(y)\xi : \xi \quad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}}.$$

It follows that Q satisfies

$$r_c|\xi|^2 \le Q(y, \xi) \le R_c|\xi|^2 \quad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}}. \tag{3.4}$$

The dissipation potential. For each i, let $H_i : \mathbb{M}^{3\times3}_{\mathrm{dev}} \to [0, +\infty)$ be the support function of the set K_i, i.e.,

$$H_i(\xi) = \sup_{\tau \in K_i} \tau : \xi.$$

It follows that H_i is convex, positively 1-homogeneous, and satisfies

$$r_k|\xi| \le H_i(\xi) \le R_k|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\mathrm{dev}}. \tag{3.5}$$

The dissipation potential $H : \mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{dev}} \to [0, +\infty]$ is defined as follows:

(i) For every $y \in \mathcal{Y}_i$,

$$H(y, \xi) := H_i(\xi).$$

(ii) For a point $y \in \Gamma$ that is at interface of exactly two phases $\mathcal{Y}_i$ and $\mathcal{Y}_j$ we define

$$H(y, \xi) = \min_{i,j}\{H_i(y, \xi), H_j(y, \xi)\}.$$

(iii) For all other points we take

$$H(y, \xi) = \min_i H_i(y, \xi).$$

Remark 3.4. We point out that H is a Borel, lower semicontinuous function on $\mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{dev}}$. Furthermore, for each $y \in \mathcal{Y}$, the function $\xi \mapsto H(y, \xi)$ is positively 1-homogeneous and convex.

Admissible triples and energy. On Γ_D^h we prescribe a boundary datum being the trace of a map $w^h \in H^1(\Omega^h; \mathbb{R}^3)$ with the following Kirchhoff–Love structure:

$$w^h(z) := \left(\bar{w}_1(z') - \frac{z_3}{h}\partial_1\bar{w}_3(z'), \bar{w}_2(z') - \frac{z_3}{h}\partial_2\bar{w}_3(z'), \frac{1}{h}\bar{w}_3(z')\right) \quad \text{for a.e. } z = (z', z_3) \in \Omega^h, \tag{3.6}$$

where $\bar{w}_\alpha \in H^1(\omega)$, $\alpha = 1, 2$, and $\bar{w}_3 \in H^2(\omega)$. The *set of admissible displacements and strains* for the boundary datum w^h is denoted by $\mathcal{A}(\Omega^h, w^h)$ and is defined as the class of all triples $(v, f, q) \in BD(\Omega^h) \times L^2(\Omega^h; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega^h; \mathbb{M}^{3\times3}_{\mathrm{dev}})$ satisfying

$$\begin{aligned} Ev &= f + q && \text{in } \Omega^h, \\ q &= (w^h - v) \odot \nu_{\partial\Omega^h}\mathcal{H}^2 && \text{on } \Gamma_D^h. \end{aligned}$$

The function v represents the *displacement* of the plate, while f and q are called the *elastic* and *plastic strain*, respectively.

For every admissible triple $(v, f, q) \in \mathcal{A}(\Omega^h, w^h)$ we define the associated *energy* as

$$\mathcal{E}_h(v, f, q) := \int_{\Omega^h} Q\Big(\frac{z'}{\varepsilon_h}, f(z)\Big)\, dz + \int_{\Omega^h \cup \Gamma_D^h} H\Big(\frac{z'}{\varepsilon_h}, \frac{dq}{d|q|}\Big)\, d|q|.$$

The first term represents the elastic energy, while the second term accounts for plastic dissipation.

3.2 The rescaled problem

As usual in dimension reduction problems, it is convenient to perform a change of variables in such a way to rewrite the system on a fixed domain independent of h. To this purpose, we consider the open interval $I = (-\frac{1}{2}, \frac{1}{2})$ and set

$$\Omega := \omega \times I, \quad \Gamma_D := \partial\omega \times I.$$

We consider the change of variables $\psi_h : \overline{\Omega} \to \overline{\Omega^h}$, defined as

$$\psi_h(x', x_3) := (x', hx_3) \quad \text{for every } (x', x_3) \in \overline{\Omega}, \tag{3.7}$$

and the linear operator $\Lambda_h : \mathbb{M}^{3\times3}_{\mathrm{sym}} \to \mathbb{M}^{3\times3}_{\mathrm{sym}}$ given by

$$\Lambda_h \xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{h}\xi_{13} \\ \xi_{21} & \xi_{22} & \frac{1}{h}\xi_{23} \\ \frac{1}{h}\xi_{31} & \frac{1}{h}\xi_{32} & \frac{1}{h^2}\xi_{33} \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}}. \tag{3.8}$$

To any triple $(v, f, q) \in \mathcal{A}(\Omega^h, w^h)$ we associate a triple $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ defined as follows:

$$u := (v_1, v_2, hv_3) \circ \psi_h, \quad e := \Lambda_h^{-1} f \circ \psi_h, \quad p := \frac{1}{h}\Lambda_h^{-1}\psi_h^{\#}(q).$$

Here the measure $\psi_h^{\#}(q) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times3})$ is the pull-back measure of q, satisfying

$$\int_{\Omega \cup \Gamma_D} \varphi : d\psi_h^{\#}(q) = \int_{\Omega^h \cup \Gamma_D^h} (\varphi \circ \psi_h^{-1}) : dq \quad \text{for every } \varphi \in C_0(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}).$$

According to this change of variable we have

$$\mathcal{E}_h(v, f, q) = h\mathcal{Q}_h(\Lambda_h e) + h\mathcal{H}_h(\Lambda_h p),$$

where

$$\mathcal{Q}_h(\Lambda_h e) = \int_{\Omega} Q\Big(\frac{x'}{\varepsilon_h}, \Lambda_h e\Big)\, dx \tag{3.9}$$

and

$$\mathcal{H}_h(\Lambda_h p) = \int_{\Omega \cup \Gamma_D} H\Big(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p}{d|\Lambda_h p|}\Big)\, d|\Lambda_h p|. \tag{3.10}$$

We also introduce the scaled Dirichlet boundary datum $w \in H^1(\Omega; \mathbb{R}^3)$, given by

$$w(x) := (\bar{w}_1(x') - x_3\partial_1 w_3(x'), \bar{w}_2(x') - x_3\partial_2 w_3(x'), w_3(x')) \quad \text{for a.e. } x \in \Omega.$$

By the definition of the class $\mathcal{A}(\Omega^h, w^h)$ it follows that the scaled triple (u, e, p) satisfies

$$Eu = e + p \quad \text{in } \Omega, \tag{3.11}$$

$$p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_D, \tag{3.12}$$

$$p_{11} + p_{22} + \frac{1}{h^2}p_{33} = 0 \quad \text{in } \Omega \cup \Gamma_D. \tag{3.13}$$

We are thus led to introduce the class $\mathcal{A}_h(w)$ of all triples $(u, e, p) \in \mathrm{BD}(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ satisfying (3.11)–(3.13), and to define the functional

$$\mathcal{J}_h(u, e, p) := \mathcal{Q}_h(\Lambda_h e) + \mathcal{H}_h(\Lambda_h p) \tag{3.14}$$

for every $(u, e, p) \in \mathcal{A}_h(w)$. In the following we will study the asymptotic behavior of the quasistatic evolution associated with $\mathcal{J}_h$, as $h \to 0$ and $\varepsilon_h \to 0$.

Notice that if $\bar{w}_\alpha \in H^1(\widetilde{\omega})$, $\alpha = 1, 2$, and $\bar{w}_3 \in H^2(\widetilde{\omega})$, where $\omega \subset \widetilde{\omega}$, then we can trivially extend the triple (u, e, p) to $\widetilde{\Omega} := \widetilde{\omega} \times I$ by

$$u = w, \quad e = Ew, \quad p = 0 \quad \text{on } \widetilde{\Omega} \setminus \overline{\Omega}.$$

In the following, with a slight abuse of notation, we will still denote this extension by (u, e, p), whenever such an extension procedure will be needed.

Kirchhoff–Love admissible triples and limit energy. We consider the set of *Kirchhoff–Love displacements*, defined as

$$\mathrm{KL}(\Omega) := \{u \in \mathrm{BD}(\Omega) : (Eu)_{i3} = 0 \text{ for } i = 1, 2, 3\}.$$

We note that $u \in \mathrm{KL}(\Omega)$ if and only if $u_3 \in \mathrm{BH}(\omega)$ and there exists $\bar{u} \in \mathrm{BD}(\omega)$ such that

$$u_\alpha = \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \alpha = 1, 2. \tag{3.15}$$

In particular, if $u \in \mathrm{KL}(\Omega)$, then

$$Eu = \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}. \tag{3.16}$$

If, in addition, $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ for some $1 \le p \le \infty$, then $\bar{u} \in W^{1,p}(\omega; \mathbb{R}^2)$ and $u_3 \in W^{2,p}(\omega)$. We call $\bar{u}, u_3$ the *Kirchhoff–Love components* of u.

For every $w \in H^1(\Omega; \mathbb{R}^3) \cap \mathrm{KL}(\Omega)$ we define the class $\mathcal{A}_{\mathrm{KL}}(w)$ of *Kirchhoff–Love admissible triples* for the boundary datum w as the set of all triples $(u, e, p) \in \mathrm{KL}(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ satisfying

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_D, \tag{3.17}$$

$$e_{i3} = 0 \quad \text{in } \Omega, \qquad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_D, \quad i = 1, 2, 3. \tag{3.18}$$

Note that the space

$$\{\xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : \xi_{i3} = 0 \text{ for } i = 1, 2, 3\}$$

is canonically isomorphic to $\mathbb{M}^{2\times2}_{\mathrm{sym}}$. Therefore, in the following, given a triple $(u, e, p) \in \mathcal{A}_{\mathrm{KL}}(w)$ we will usually identify e with a function in $L^2(\Omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ and p with a measure in $\mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{2\times2}_{\mathrm{sym}})$. Note also that the class $\mathcal{A}_{\mathrm{KL}}(w)$ is always nonempty as it contains the triple $(w, Ew, 0)$.

To provide a useful characterization of admissible triplets in $\mathcal{A}_{\mathrm{KL}}(w)$, let us first recall the definition of zero-th and first order moments of functions.

Definition 3.5. For $f \in L^2(\Omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ we denote by $\bar{f}, \widehat{f} \in L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ and $f^\perp \in L^2(\Omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ the following orthogonal components (with respect to the scalar product of $L^2(\Omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$) of f:

$$\bar{f}(x') := \int_I f(x', x_3)\, dx_3, \quad \widehat{f}(x') := 12 \int_I x_3 f(x', x_3)\, dx_3 \tag{3.19}$$

for a.e. $x' \in \omega$, and

$$f^\perp(x) := f(x) - \bar{f}(x') - x_3 \widehat{f}(x')$$

for a.e. $x \in \Omega$. We name $\bar{f}$ the *zero-th order moment* of f and $\widehat{f}$ the *first order moment* of f. More generally, we will also use the expressions (3.19) for any integrable function over I.

The coefficient in the definition of $\widehat{f}$ is chosen from the computation $\int_I x_3^2\, dx_3 = \frac{1}{12}$. It ensures that if f is of the form $f(x) = x_3 g(x')$ for some $g \in L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$, then $\widehat{f} = g$.

Analogously, we have the following definition of zero-th and first order moments of measures.

Definition 3.6. For $\mu \in M_b(\Omega \cup \Gamma_D; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ we define $\bar{\mu}, \hat{\mu} \in M_b(\omega \cup \gamma_D; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ and $\mu^\perp \in M_b(\Omega \cup \Gamma_D; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ as follows:

$$\int_{\omega\cup\gamma_D} \varphi : d\bar{\mu} := \int_{\Omega\cup\Gamma_D} \varphi : d\mu, \qquad \int_{\omega\cup\gamma_D} \varphi : d\hat{\mu} := 12 \int_{\Omega\cup\Gamma_D} x_3\varphi : d\mu$$

for every $\varphi \in C_0(\omega \cup \gamma_D; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, and

$$\mu^\perp := \mu - \bar{\mu} \otimes \mathcal{L}^1_{x_3} - \hat{\mu} \otimes x_3\mathcal{L}^1_{x_3},$$

where $\otimes$ is the usual product of measures, and $\mathcal{L}^1_{x_3}$ is the Lebesgue measure restricted to the third component of $\mathbb{R}^3$. We call $\bar{\mu}$ the *zero-th order moment* of μ and $\hat{\mu}$ the *first order moment* of μ.

We are now ready to recall the following characterization of $\mathcal{A}_{\mathrm{KL}}(w)$, given in [21, Proposition 4.3].

Proposition 3.7. *Let $w \in H^1(\Omega; \mathbb{R}^3) \cap \mathrm{KL}(\Omega)$ and let $(u, e, p) \in KL(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times 3}_{\mathrm{dev}})$. Then $(u, e, p) \in \mathcal{A}_{\mathrm{KL}}(w)$ if and only if the following three conditions are satisfied:*

(i) $E\bar{u} = \bar{e} + \bar{p}$ *in* ω *and* $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* γ_D,

(ii) $D^2u_3 = -(\hat{e} + \hat{p})$ *in* ω, $u_3 = w_3$ *on* γ_D, *and* $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* γ_D,

(iii) $p^\perp = -e^\perp$ *in* Ω *and* $p^\perp = 0$ *on* Γ_D.

3.3 The reduced problem

For a fixed $y \in \mathcal{Y}$, let $\mathbb{A}_y : \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ be the operator given by

$$\mathbb{A}_y\xi := \begin{pmatrix} \multicolumn{2}{c}{\xi} & \lambda_1^y(\xi) \\ & & \lambda_2^y(\xi) \\ \lambda_1^y(\xi) & \lambda_2^y(\xi) & \lambda_3^y(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where for every $\xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ the triple $(\lambda_1^y(\xi), \lambda_2^y(\xi), \lambda_3^y(\xi))$ is the unique solution to the minimum problem

$$\min_{\lambda_i\in\mathbb{R}} Q\left(y, \begin{pmatrix} \multicolumn{2}{c}{\xi} & \lambda_1 \\ & & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}\right). \tag{3.20}$$

We observe that for every $\xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, the matrix $\mathbb{A}_y\xi$ is given by the unique solution of the linear system

$$\mathbb{C}(y)\mathbb{A}_y\xi : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0 \quad \text{for every } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

This implies, in particular, for every $y \in \mathcal{Y}$ that $\mathbb{A}_y$ is a linear map.

Let $Q_r : \mathcal{Y} \times \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to [0, +\infty)$ be the map

$$Q_r(y, \xi) := Q(y, \mathbb{A}_y\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}.$$

By the properties of Q, we have that $Q_r(y, \cdot)$ is positive definite on symmetric matrices.

We also define the tensor $\mathbb{C}_r : \mathcal{Y} \times \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$, given by

$$\mathbb{C}_r(y)\xi := \mathbb{C}(y)\mathbb{A}_y\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}.$$

We remark that by (3.20) it holds

$$\mathbb{C}_r(y)\xi : \zeta = \mathbb{C}(y)\mathbb{A}_y\xi : \begin{pmatrix} \zeta'' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}, \zeta \in \mathbb{M}^{3\times 3}_{\mathrm{sym}},$$

and

$$Q_r(y, \xi) = \frac{1}{2}\mathbb{C}_r(y)\xi : \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}.$$

The reduced dissipation potential. The set $K_r(y) \subset \mathbb{M}^{2\times2}_{\mathrm{sym}}$ represents the set of admissible stresses in the reduced problem and can be characterized as follows (see [21, Section 3.2]):

$$\xi \in K_r(y) \iff \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi) I_{3\times3} \in K(y), \tag{3.21}$$

where $I_{3\times3}$ is the identity matrix in $\mathbb{M}^{3\times3}$.

The *plastic dissipation potential* $H_r : \mathcal{Y} \times \mathbb{M}^{2\times2}_{\mathrm{sym}} \to [0, +\infty)$ is given by the support function of $K_r(y)$, i.e.,

$$H_r(y, \xi) := \sup_{\sigma \in K_r(y)} \sigma : \xi \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{\mathrm{sym}}.$$

It follows that $H_r(y, \cdot)$ is convex and positively 1-homogeneous, and there are two constants $0 < r_H \le R_H$ such that

$$r_H|\xi| \le H_r(y, \xi) \le R_H|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{\mathrm{sym}}.$$

Therefore $H_r(y, \cdot)$ satisfies the triangle inequality

$$H_r(y, \xi_1 + \xi_2) \le H_r(y, \xi_1) + H_r(y, \xi_2) \quad \text{for every } \xi_1, \xi_2 \in \mathbb{M}^{2\times2}_{\mathrm{sym}}.$$

Finally, for a fixed $y \in \mathcal{Y}$, we can deduce the property

$$K_r(y) = \partial H_r(y, 0),$$

i.e., $K_r(y)$ is the convex subdifferential of the function $H_r(y, \cdot)$ at the point $0 \in \mathbb{M}^{2\times2}_{\mathrm{sym}}$.

3.4 Definition of quasistatic evolutions

For every $t \in [0, T]$ we prescribe a boundary datum $w(t) \in H^1(\Omega; \mathbb{R}^3) \cap \mathrm{KL}(\Omega)$ and we assume the map $t \mapsto w(t)$ to be absolutely continuous from $[0, T]$ into $H^1(\Omega; \mathbb{R}^3)$.

Definition 3.8. Let $h > 0$. An *h-quasistatic evolution* for the boundary datum $w(t)$ is a function

$$t \mapsto (u^h(t), e^h(t), p^h(t))$$

from $[0, T]$ into $\mathrm{BD}(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ that satisfies the following conditions:

$(\mathrm{qs1})_h$ for every $t \in [0, T]$ we have $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(w(t))$ and

$$\mathcal{Q}_h(\Lambda_h e^h(t)) \le \mathcal{Q}_h(\Lambda_h \eta) + \mathcal{H}_h(\Lambda_h \pi - \Lambda_h p^h(t))$$

for every $(v, \eta, \pi) \in \mathcal{A}_h(w(t))$,

$(\mathrm{qs2})_h$ the function $t \mapsto p^h(t)$ from $[0, T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ has bounded variation and for every $t \in [0, T]$,

$$\mathcal{Q}_h(\Lambda_h e^h(t)) + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) = \mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\Big(\frac{x'}{\varepsilon_h}\Big)\Lambda_h e^h(s) : E\dot{w}(s)\, dx\, ds.$$

The following existence result of a quasistatic evolution for a general multi-phase material can be found in [27, Theorem 2.7].

Theorem 3.9. *Assume* (3.2), (3.3), *and* (3.5). *Let* $h > 0$ *and let* $(u^h_0, e^h_0, p^h_0) \in \mathcal{A}_h(w(0))$ *satisfy the global stability condition* $(\mathrm{qs1})_h$. *Then there exists a two-scale quasistatic evolution* $t \mapsto (u^h(t), e^h(t), p^h(t))$ *for the boundary datum* $w(t)$ *such that* $u^h(0) = u_0$, $e^h(0) = e^h_0$, *and* $p^h(0) = p^h_0$.

Our goal is to study the asymptotics of the quasistatic evolution when h goes to zero. The main result is given by Theorem 6.2.

3.5 Two-scale convergence adapted to dimension reduction

We briefly recall some results and definitions from [26].

Definition 3.10. Let $\Omega \subset \mathbb{R}^3$ be an open set. Let $\{\mu^h\}_{h>0}$ be a family in $\mathcal{M}_b(\Omega)$ and consider $\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y})$. We say that

$$\mu^h \xrightarrow{2\text{-}*} \mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}),$$

if for every $\chi \in C_0(\Omega \times \mathcal{Y})$,

$$\lim_{h\to 0} \int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big)\, d\mu^h(x) = \int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\mu(x,y).$$

The convergence above is called *two-scale weak* convergence*.

Remark 3.11. Notice that the family $\{\mu^h\}_{h>0}$ determines the family of measures $\{\nu^h\}_{h>0} \subset \mathcal{M}_b(\Omega \times \mathcal{Y})$ obtained by setting

$$\int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\nu^h = \int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big)\, d\mu^h(x)$$

for every $\chi \in C_0^0(\Omega \times \mathcal{Y})$. Thus μ is simply the weak* limit in $\mathcal{M}_b(\Omega \times \mathcal{Y})$ of $\{\nu^h\}_{h>0}$.

We collect some basic properties of two-scale convergence in the proposition below (the first one is a direct consequence of Remark 3.11 and the second one follows from the definition). Before stating the proposition recall (3.1).

Proposition 3.12. *The following statements hold:*
(i) *Any sequence that is bounded in $\mathcal{M}_b(\Omega)$ admits a two-scale weakly* convergent subsequence.*
(ii) *Let $\mathcal{D} \subset \mathcal{Y}$ and assume that $\operatorname{supp}(\mu^h) \subset \Omega \cap (\mathcal{D}_{\varepsilon_h} \times I)$. If $\mu^h \xrightarrow{2\text{-}*} \mu$ two-scale weakly* in $\mathcal{M}_b(\Omega \times \mathcal{Y})$, then $\operatorname{supp}(\mu) \subset \Omega \times \overline{\mathcal{D}}$.*

4 Compactness results

In this section, we provide a characterization of two-scale limits of symmetrized scaled gradients. We will consider sequences of deformations $\{v^h\}$ such that $v^h \in BD(\Omega^h)$ for every $h > 0$, their L^1-norms are uniformly bounded (up to rescaling), and their symmetrized gradients Ev^h form a sequence of uniformly bounded Radon measures (again, up to rescaling). As already explained in Section 3.2, we associate to the sequence $\{v^h\}$ above a rescaled sequence of maps $\{u^h\} \subset BD(\Omega)$, defined as $u^h := (v_1^h, v_2^h, hv_3^h) \circ \psi_h$, where ψ_h is defined in (3.7). The symmetric gradients of the maps $\{v^h\}$ and $\{u^h\}$ are related as follows:

$$\frac{1}{h} Ev^h = (\psi_h)_\#(\Lambda_h Eu^h). \tag{4.1}$$

The boundedness of $\frac{1}{h}\|Ev^h\|_{\mathcal{M}_b(\Omega^h;\mathbb{M}^{3\times3}_{sym})}$ is equivalent to the boundedness of $\|\Lambda_h Eu^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times3}_{sym})}$. We will express our compactness result with respect to the sequence $\{u^h\}_{h>0}$.

We first recall a compactness result for sequences of non-oscillating fields (see [21]).

Proposition 4.1. *Let $\{u^h\}_{h>0} \subset BD(\Omega)$ be a sequence such that there exists a constant $C > 0$ for which*

$$\|u^h\|_{L^1(\Omega;\mathbb{R}^3)} + \|\Lambda_h Eu^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times3}_{sym})} \le C.$$

Then, there exist functions $\bar{u} = (\bar{u}_1, \bar{u}_2) \in BD(\omega)$ and $u_3 \in BH(\omega)$ such that, up to subsequences, there holds

$$\begin{aligned}
u_\alpha^h &\to \bar{u}_\alpha - x_3\partial_{x_\alpha} u_3 && \text{strongly in } L^1(\Omega), \quad \alpha \in \{1,2\},\\
u_3^h &\to u_3 && \text{strongly in } L^1(\Omega),\\
Eu^h &\xrightarrow{*} \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & 0 \\ 0 & 0 \end{pmatrix} && \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}).
\end{aligned}$$

Now we turn to identifying the two-scale limits of the sequence $\Lambda_h Eu^h$.

4.1 Corrector properties and duality results

In order to define and analyze the space of measures which arise as two-scale limits of scaled symmetrized gradients of BD functions, we will consider the following general framework (see also [3]).

Let V and W be finite-dimensional Euclidean spaces of dimensions N and M, respectively. We will consider kth order linear homogeneous partial differential operators with constant coefficients

$$\mathcal{A} : C_c^\infty(\mathbb{R}^n; V) \to C_c^\infty(\mathbb{R}^n; W).$$

More precisely, the operator $\mathcal{A}$ acts on functions $u : \mathbb{R}^n \to V$ as

$$\mathcal{A}u := \sum_{|\alpha|=k} A_\alpha \partial^\alpha u,$$

where the coefficients $A_\alpha \in W \otimes V^* \cong \mathrm{Lin}(V; W)$ are constant tensors, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ denotes the distributional partial derivative of order $|\alpha| = \alpha_1 + \dots + \alpha_n$.

We define the space

$$\mathrm{BV}^{\mathcal{A}}(U) = \{u \in L^1(U; V) : \mathcal{A}u \in \mathcal{M}_b(U; W)\}$$

of *functions with bounded $\mathcal{A}$-variations* on an open subset U of $\mathbb{R}^n$. This is a Banach space endowed with the norm

$$\|u\|_{\mathrm{BV}^{\mathcal{A}}(U)} := \|u\|_{L^1(U;V)} + |\mathcal{A}u|(U).$$

Here, the distributional $\mathcal{A}$-gradient is defined and extended to distributions via the duality

$$\int_U \varphi \cdot d\mathcal{A}u := \int_U \mathcal{A}^*\varphi \cdot u \, dx, \quad \varphi \in C_c^\infty(U; W^*),$$

where $\mathcal{A}^* : C_c^\infty(\mathbb{R}^n; W^*) \to C_c^\infty(\mathbb{R}^n; V^*)$ is the formal L^2-adjoint operator of $\mathcal{A}$

$$\mathcal{A}^* := (-1)^k \sum_{|\alpha|=k} A_\alpha^* \partial^\alpha.$$

The *total $\mathcal{A}$-variation* of $u \in L^1_{loc}(U; V)$ is defined as

$$|\mathcal{A}u|(U) := \sup\left\{ \int_U \mathcal{A}^*\varphi \cdot u \, dx : \varphi \in C_c^k(U; W^*), \ |\varphi| \le 1 \right\}.$$

Let $\{u_n\} \subset \mathrm{BV}^{\mathcal{A}}(U)$ and $u \in \mathrm{BV}^{\mathcal{A}}(U)$. We say that $\{u_n\}$ converges weakly* to u in $\mathrm{BV}^{\mathcal{A}}$ if $u_n \to u$ in $L^1(U; V)$ and $\mathcal{A}u_n \xrightarrow{*} \mathcal{A}u$ in $\mathcal{M}_b(U; W)$.

In order to characterize the two-scale weak* limit of scaled symmetrized gradients, we will generally consider two domains $\Omega_1 \subset \mathbb{R}^{N_1}$, $\Omega_2 \subset \mathbb{R}^{N_2}$, for some $N_1, N_2 \in \mathbb{N}$ and assume that the operator $\mathcal{A}_{x_2}$ is defined through partial derivatives only with respect to the entries of the n_2-tuple x_2. In the spirit of [26, Section 4.2], we will define the space

$$\mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1) := \{\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; V) : \mathcal{A}_{x_2}\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W), \ \mu(F \times \Omega_2) = 0 \text{ for every Borel set } F \subseteq \Omega_1\}.$$

We will assume that $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2)$ satisfies the following weak* compactness property:

Assumption 1. If $\{u_n\} \subset \mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2)$ is uniformly bounded in the $\mathrm{BV}^{\mathcal{A}_{x_2}}$-norm, then there exists a subsequence $\{u_m\} \subseteq \{u_n\}$ and a function $u \in \mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2)$ such that $\{u_m\}$ converges weakly* to u in $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2)$, i.e.,

$$u_m \to u \quad \text{in } L^1(\Omega_2; V) \qquad \text{and} \qquad \mathcal{A}_{x_2}u_m \xrightarrow{*} \mathcal{A}_{x_2}u \quad \text{in } \mathcal{M}_b(\Omega_2; W).$$

Furthermore, there exists a countable collection $\{U^k\}$ of open subsets of $\mathbb{R}^{n_2}$ that increases to Ω_2 (i.e., $\overline{U^k} \subset U^{k+1}$ for every $k \in \mathbb{N}$, and $\Omega_2 = \bigcup_k U^k$) such that $\mathrm{BV}^{\mathcal{A}_{x_2}}(U^k)$ satisfies the weak* compactness property above for every $k \in \mathbb{N}$.

The following theorem is our main disintegration result for measures in $\mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$, which will be instrumental to define a notion of duality for admissible two-scale configurations. The proof is an adaptation of the arguments in [26, Proposition 4.7] (see [7, Proposition 4.2]).

Proposition 4.2. *Let Assumption 1 be satisfied. Let $\mu \in \mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$. Then there exist $\eta \in \mathcal{M}_b^+(\Omega_1)$ and a Borel map $(x_1, x_2) \in \Omega_1 \times \Omega_2 \mapsto \mu_{x_1}(x_2) \in V$ such that, for η-a.e. $x_1 \in \Omega_1$,*

$$\mu_{x_1} \in \mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2), \qquad \int_{\Omega_2} \mu_{x_1}(x_2)\,dx_2 = 0, \quad |\mathcal{A}_{x_2}\mu_{x_1}|(\Omega_2) \neq 0 \tag{4.2}$$

and

$$\mu = \mu_{x_1}(x_2)\eta \otimes \mathcal{L}^{n_2}_{x_2}. \tag{4.3}$$

Moreover, the map $x_1 \mapsto \mathcal{A}_{x_2}\mu_{x_1} \in \mathcal{M}_b(\Omega_2; W)$ is η-measurable and

$$\mathcal{A}_{x_2}\mu = \eta \overset{\text{gen.}}{\otimes} \mathcal{A}_{x_2}\mu_{x_1}.$$

Lastly, we give a necessary and sufficient condition with which we can characterize the $\mathcal{A}_{x_2}$-gradient of a measure, under the following two assumptions.

Assumption 2. For every $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathcal{A}^*_{x_2}\chi = 0$ (in the sense of distributions), there exists a sequence of smooth functions $\{\chi_n\} \subset C_c^\infty(\Omega_1 \times \Omega_2; W)$ such that $\mathcal{A}^*_{x_2}\chi_n = 0$ for every n, and $\chi_n \to \chi$ in $L^\infty(\Omega_1 \times \Omega_2; W)$.

Assumption 3. The following Poincaré–Korn-type inequality holds in $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2)$:

$$\Big\| u - \int_{\Omega_2} u\,dx_2 \Big\|_{L^1(\Omega_2;V)} \leq C|\mathcal{A}_{x_2}u|(\Omega_2) \quad \text{for all } u \in \mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2).$$

The proof of the following result is given in [7, Proposition 4.3].

Proposition 4.3. *Let Assumptions 1, 2 and 3 be satisfied. Let $\lambda \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W)$. Then the following items are equivalent:*

(i) *For every $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathcal{A}^*_{x_2}\chi = 0$ (in the sense of distributions) we have*

$$\int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\lambda(x_1, x_2) = 0.$$

(ii) *There exists $\mu \in \mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$ such that $\lambda = \mathcal{A}_{x_2}\mu$.*

Next we will apply these results to obtain auxiliary claims which we will use to characterize two-scale limits of scaled symmetrized gradients.

4.1.1 Case $\gamma = 0$

We consider $\mathcal{A}_{x_2} = E_y$, $\mathcal{A}^*_{x_2} = \mathrm{div}_y$, $\Omega_1 = \omega$, and $\Omega_2 = \mathcal{Y}$ (it can be easily seen that Proposition 4.2 and Proposition 4.3 are also valid if we take $\Omega_2 = \mathcal{Y}$). Then $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2) = \mathrm{BD}(\mathcal{Y})$ and we denote the associated corrector space by

$$\mathcal{X}_0(\omega) := \{\mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{R}^2) : E_y\mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}),\ \mu(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \omega\}.$$

Remark 4.4. We note that $\mathcal{X}_0(\omega)$ is the 2-dimensional variant of the set introduced in [26, Section 4.2], where its main properties have been characterized.

Analogously, let $\mathcal{A}_{x_2} = D^2_y$, $\mathcal{A}^*_{x_2} = \mathrm{div}_y\,\mathrm{div}_y$, $\Omega_1 = \omega$, and $\Omega_2 = \mathcal{Y}$, then $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2) = \mathrm{BH}(\mathcal{Y})$ and we denote the associated corrector space by

$$\Upsilon_0(\omega) := \{\kappa \in \mathcal{M}_b(\omega \times \mathcal{Y}) : D^2_y\kappa \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}),\ \kappa(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \omega\}.$$

Remark 4.5. It is known that Assumption 1 and Assumption 2 are satisfied in $\mathrm{BH}(\mathcal{Y})$, so we only need to justify Assumption 3. Owing to [23, Remarque 1.3], there exists a constant $C > 0$ such that

$$\|u - p(u)\|_{\mathrm{BH}(\mathcal{Y})} \leq C|D^2_y u|(\mathcal{Y}),$$

where $p(u)$ is given by

$$p(u) = \int_{\mathcal{Y}} \nabla_y u\,dy \cdot y + \int_{\mathcal{Y}} u\,dy - \int_{\mathcal{Y}} \nabla_y u\,dy \cdot \int_{\mathcal{Y}} y\,dy.$$

However, since integrating first derivatives of periodic functions over the periodicity cell provides a zero contribution, we precisely obtain the desired Poincaré–Korn-type inequality.

As a consequence of Proposition 4.2 and Proposition 4.3, we infer the following results.

Proposition 4.6. *Let $\mu \in \mathcal{X}_0(\omega)$ and $\kappa \in \Upsilon_0(\omega)$. Then there exist $\eta \in \mathcal{M}_b^+(\omega)$ and Borel maps $(x', y) \in \omega \times \mathcal{Y} \mapsto \mu_{x'}(y) \in \mathbb{R}^2$ and $(x', y) \in \omega \times \mathcal{Y} \mapsto \kappa_{x'}(y) \in \mathbb{R}$ such that, for η-a.e. $x' \in \omega$,*

$$\mu_{x'} \in \mathrm{BD}(\mathcal{Y}), \quad \int_{\mathcal{Y}} \mu_{x'}(y)\,dy = 0, \quad |E_y \mu_{x'}|(\mathcal{Y}) \neq 0,$$
$$\kappa_{x'} \in \mathrm{BH}(\mathcal{Y}), \quad \int_{\mathcal{Y}} \kappa_{x'}(y)\,dy = 0, \quad |D_y^2 \kappa_{x'}|(\mathcal{Y}) \neq 0,$$

and

$$\mu = \mu_{x'}(y)\eta \otimes \mathcal{L}_y^2, \quad \kappa = \kappa_{x'}(y)\eta \otimes \mathcal{L}_y^2.$$

Moreover, the maps $x' \mapsto E_y\mu_{x'} \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ and $x' \mapsto D_y^2\kappa_{x'} \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are η-measurable and

$$E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_{x'}, \quad D_y^2\kappa = \eta \overset{\text{gen.}}{\otimes} D_y^2\kappa_{x'}.$$

Proposition 4.7. *Let $\lambda \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$. The following items are equivalent:*
(i) *For every $\chi \in C_0(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ with $\mathrm{div}_y\, \chi(x', y) = 0$ (in the sense of distributions) we have*

$$\int_{\omega\times\mathcal{Y}} \chi(x', y) : d\lambda(x', y) = 0.$$

(ii) *There exists $\mu \in \mathcal{X}_0(\omega)$ such that $\lambda = E_y\mu$.*

Proposition 4.8. *Let $\lambda \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$. The following items are equivalent:*
(i) *For every $\chi \in C_0(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ with $\mathrm{div}_y\, \mathrm{div}_y\, \chi(x', y) = 0$ (in the sense of distributions) we have*

$$\int_{\omega\times\mathcal{Y}} \chi(x', y) : d\lambda(x', y) = 0.$$

(ii) *There exists $\kappa \in \Upsilon_0(\omega)$ such that $\lambda = D_y^2\kappa$.*

4.1.2 Case $\gamma = +\infty$

In this scaling regime, we consider $\mathcal{A}_{x_2} = E_y$, $\mathcal{A}_{x_2}^* = \mathrm{div}_y$, $\Omega_1 = \Omega$, and $\Omega_2 = \mathcal{Y}$. Then $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2) = \mathrm{BD}(\mathcal{Y})$ and we denote the associated corrector space by

$$\mathcal{X}_\infty(\Omega) := \{\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^2) : E_y\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}}),\ \mu(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \Omega\},$$

Further, we choose $\mathcal{A}_{x_2} = D_y$, $\mathcal{A}_{x_2}^* = \mathrm{div}_y$, $\Omega_1 = \Omega$, and $\Omega_2 = \mathcal{Y}$, so that $\mathrm{BV}^{\mathcal{A}_{x_2}}(\Omega_2) = \mathrm{BV}(\mathcal{Y})$ and the associated corrector space is given by

$$\Upsilon_\infty(\Omega) := \{\kappa \in \mathcal{M}_b(\Omega \times \mathcal{Y}) : D_y\kappa \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^2),\ \kappa(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \Omega\}.$$

Clearly Assumption 1, Assumption 2 and Assumption 3 are satisfied in $\mathrm{BD}(\mathcal{Y})$ and $\mathrm{BV}(\mathcal{Y})$. Thus, we can state the following propositions as consequences of Proposition 4.2 and Proposition 4.3.

Proposition 4.9. *Let $\mu \in \mathcal{X}_\infty(\Omega)$ and $\kappa \in \Upsilon_\infty(\Omega)$. Then there exist $\eta \in \mathcal{M}_b^+(\Omega)$ and Borel maps $(x, y) \in \Omega \times \mathcal{Y} \mapsto \mu_x(y) \in \mathbb{R}^2$ and $(x, y) \in \Omega \times \mathcal{Y} \mapsto \kappa_x(y) \in \mathbb{R}^2$ such that, for η-a.e. $x \in \Omega$,*

$$\mu_x \in \mathrm{BD}(\mathcal{Y}), \quad \int_{\mathcal{Y}} \mu_x(y)\,dy = 0, \quad |E_y\mu_x|(\mathcal{Y}) \neq 0,$$
$$\kappa_x \in \mathrm{BV}(\mathcal{Y}), \quad \int_{\mathcal{Y}} \kappa_x(y)\,dy = 0, \quad |D_y\kappa_x|(\mathcal{Y}) \neq 0,$$

and

$$\mu = \mu_x(y)\eta \otimes \mathcal{L}^2_y, \quad \kappa = \kappa_x(y)\eta \otimes \mathcal{L}^2_y.$$

Moreover, the maps $x \mapsto E_y\mu_x \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *and* $x \mapsto D_y\kappa_x \in \mathcal{M}_b(\mathcal{Y}; \mathbb{R}^2)$ *are η-measurable and*

$$E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_x, \quad D_y\kappa = \eta \overset{\text{gen.}}{\otimes} D_y\kappa_x.$$

Proposition 4.10. *Let* $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. *The following items are equivalent:*
(i) *For every* $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *with* $\operatorname{div}_y \chi(y) = 0$ *(in the sense of distributions) we have*

$$\int_{\mathcal{Y}} \chi(x,y) : d\lambda(x,y) = 0.$$

(ii) *There exists* $\mu \in \mathcal{X}_\infty(\Omega)$ *such that* $\lambda = E_y\mu$.

Proposition 4.11. *Let* $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^2)$. *The following items are equivalent:*
(i) *For every* $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{R}^2)$ *with* $\operatorname{div}_y \chi(y) = 0$ *(in the sense of distributions) we have*

$$\int_{\mathcal{Y}} \chi(x,y) : d\lambda(x,y) = 0.$$

(ii) *There exists* $\kappa \in \Upsilon_\infty(\Omega)$ *such that* $\lambda = D_y\kappa$.

4.2 Additional auxiliary results

4.2.1 Case $\gamma = 0$

In order to simplify the proof of the structure result for the two-scale limits of symmetrized scaled gradients, we will use the following lemma.

Lemma 4.12. *Let* $\{\mu^h\}_{h>0}$ *be a bounded family in* $\mathcal{M}_b(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *such that*

$$\mu^h \xrightarrow{2\text{-}*} \mu \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

for some $\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *as* $h \to 0$. *Assume that*
(i) $\bar{\mu}^h \xrightarrow{2\text{-}*} \lambda_1$ *two-scale weakly* in* $\mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, *for some* $\lambda_1 \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$,
(ii) *for every* $\chi \in C^\infty_c(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *such that* $\operatorname{div}_y \operatorname{div}_y \chi(x', y) = 0$ *we have*

$$\lim_{h\to 0} \int_{\omega} \chi\Big(x', \frac{x'}{\varepsilon_h}\Big) : d\hat{\mu}^h(x') = \int_{\omega\times\mathcal{Y}} \chi(x', y) : d\lambda_2(x', y),$$

for some $\lambda_2 \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$,
(iii) *there exists an open set* $\tilde{I} \supset I$ *which compactly contains* I *such that*

$$(\mu^h)^\perp \xrightarrow{2\text{-}*} 0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\omega \times \tilde{I} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Then, there exists $\kappa \in \Upsilon_0(\omega)$ *such that*

$$\mu = \lambda_1 \otimes \mathcal{L}^1_{x_3} + (\lambda_2 + D^2_y\kappa) \otimes x_3\mathcal{L}^1_{x_3}.$$

Proof. Every μ^h determines a measure ν^h on $\omega \times \tilde{I} \times \mathcal{Y}$ with the relation

$$\nu^h(B) := \mu^h(B \cap (\Omega \times \mathcal{Y}))$$

for every Borel set $B \subseteq \omega \times \tilde{I} \times \mathcal{Y}$. With a slight abuse of notation, we will still write μ^h instead of ν^h.

Let ν be the measure such that

$$\mu^h \xrightarrow{2\text{-}*} \nu \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega \times \tilde{I} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

We first observe that, from assumption (i) and (iii), it follows that $\bar{\nu} = \lambda_1$ and $\nu^\perp = 0$. Furthermore, $\mu^h \xrightarrow{2\text{-}*} \nu$ two-scale weakly* in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$.

Let $\chi \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. If we consider the orthogonal decomposition

$$\chi(x,y) = \bar{\chi}(x',y) + x_3\widehat{\chi}(x',y) + \chi^\perp(x,y),$$

then we have

$$\begin{aligned}\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\nu(x,y) &= \lim_{h\to 0}\int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big) : d\mu^h(x') \\ &= \lim_{h\to 0}\int_\omega \bar{\chi}\Big(x', \frac{x'}{\varepsilon_h}\Big) : d\bar{\mu}^h(x') + \frac{1}{12}\lim_{h\to 0}\int_\omega \widehat{\chi}\Big(x', \frac{x'}{\varepsilon_h}\Big) : d\widehat{\mu}^h(x') \\ &\quad + \lim_{h\to 0}\int_\Omega \chi^\perp\Big(x, \frac{x'}{\varepsilon_h}\Big) : d(\mu^h)^\perp(x) \\ &= \int_{\omega\times\mathcal{Y}} \bar{\chi}(x',y) : d\lambda_1(x',y) + \frac{1}{12}\lim_{h\to 0}\int_\omega \widehat{\chi}\Big(x', \frac{x'}{\varepsilon_h}\Big) : d\widehat{\mu}^h(x').\end{aligned}$$

Suppose now that $\chi(x,y) = x_3\widehat{\chi}(x',y)$ with $\operatorname{div}_y \operatorname{div}_y \widehat{\chi}(x',y) = 0$. Then the above equality yields

$$\int_{\omega\times\mathcal{Y}} \widehat{\chi}(x',y) : d\widehat{\nu}(x',y) = \lim_{h\to 0}\int_\omega \widehat{\chi}\Big(x', \frac{x'}{\varepsilon_h}\Big) : d\widehat{\mu}^h(x') = \int_{\omega\times\mathcal{Y}} \widehat{\chi}(x',y) : d\lambda_2(x',y).$$

By a density argument, we infer that

$$\int_{\omega\times\mathcal{Y}} \widehat{\chi}(x',y) : d(\widehat{\nu}(x',y) - \lambda_2(x',y)) = 0$$

for every $\widehat{\chi} \in C_0(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ with $\operatorname{div}_y \operatorname{div}_y \widehat{\chi}(x',y) = 0$ (in the sense of distributions). From this and Proposition 4.8 we conclude that there exists $\kappa \in \Upsilon_0(\omega)$ such that

$$\widehat{\nu} - \lambda_2 = D_y^2\kappa.$$

Since $\mu = \nu$ on $\Omega \times \mathcal{Y}$, we obtain the claim. □

4.2.2 Case $\gamma = +\infty$

The following result will be used in the proof of the structure result for the two-scale limits of symmetrized scaled gradients. We note, however, that this result is independent of the limit value γ.

Proposition 4.13. *Let $\{v^h\}_{h>0}$ be a bounded family in* $\mathrm{BD}(\Omega)$ *such that*

$$v^h \xrightharpoonup{*} v \quad \textit{weakly* in } \mathrm{BD}(\Omega)$$

for some $v \in \mathrm{BD}(\Omega)$. Then there exists $\mu \in \mathcal{X}_\infty(\Omega)$ such that

$$(Ev^h)'' \xrightharpoonup{2\text{-}*} E_{x'}v' \otimes \mathcal{L}_y^2 + E_y\mu \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Proof. The proof follows closely that of [26, Proposition 4.10].

By compactness, the exists $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that (up to a subsequence)

$$Ev^h \xrightharpoonup{2\text{-}*} \lambda \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Since $v^h \to v$ strongly in $L^1(\Omega; \mathbb{R}^3)$, we have componentwise

$$v_i^h \xrightharpoonup{2\text{-}*} v_i(x)\mathcal{L}_x^3 \otimes \mathcal{L}_y^2 \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}), \quad i = 1,2,3.$$

Consider $\chi \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that $\operatorname{div}_y \chi(x,y) = 0$. Then

$$\begin{aligned}\lim_{h\to 0}\int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big) : d(Ev^h)''(x) &= \lim_{h\to 0}\int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big) : dE_{x'}(v^h)'(x) \\ &= -\lim_{h\to 0}\int_\Omega (v^h)'(x) \cdot \operatorname{div}_{x'}\Big(\chi\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big)\,dx\end{aligned}$$

$$= -\lim_{h\to 0}\left(\int_\Omega (v^h)'(x)\cdot \operatorname{div}_{x'}\chi\Big(x,\frac{x'}{\varepsilon_h}\Big)\,dx + \frac{1}{\varepsilon_h}\int_\Omega (v^h)'(x)\cdot \operatorname{div}_y\chi\Big(x,\frac{x'}{\varepsilon_h}\Big)\,dx\right)$$
$$= -\lim_{h\to 0}\int_\Omega (v^h)'(x)\cdot \operatorname{div}_x\chi\Big(x,\frac{x'}{\varepsilon_h}\Big)\,dx$$
$$= -\int_{\Omega\times\mathcal{Y}} v'(x)\cdot \operatorname{div}_{x'}\chi(x,y)\,dx\,dy$$
$$= \int_{\Omega\times\mathcal{Y}} \chi(x,y) : d(E_{x'}v'\otimes\mathcal{L}^2_y).$$

By a density argument, we infer that

$$\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d(\lambda(x,y) - E_{x'}v'\otimes\mathcal{L}^2_y) = 0$$

for every $\chi\in C_0(\Omega\times\mathcal{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ with $\operatorname{div}_y\chi(x,y)=0$ (in the sense of distributions). In view of Proposition 4.10, we conclude that there exists $\mu\in\mathfrak{X}_\infty(\Omega)$ such that

$$\lambda - E_{x'}v'\otimes\mathcal{L}^2_y = E_y\mu.$$

This yields the claim. □

4.3 Two-scale limits of scaled symmetrized gradients

We are now ready to prove the main result of this section.

Theorem 4.14. *Let $\{u^h\}_{h>0}\subset \mathrm{BD}(\Omega)$ be a sequence such that there exists a constant $C>0$ for which*

$$\|u^h\|_{L^1(\Omega;\mathbb{R}^3)} + \|\Lambda_h E u^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times 3}_{sym})} \le C.$$

Then there exist

$$\bar u = (\bar u_1,\bar u_2)\in \mathrm{BD}(\omega),\quad u_3\in \mathrm{BH}(\omega),\quad \widetilde E\in\mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}),$$

and a (not relabeled) subsequence of $\{u^h\}_{h>0}$ which satisfy

$$\Lambda_h E u^h \xrightarrow{2\text{-}*} \begin{pmatrix} E\bar u - x_3 D^2 u_3 & 0\\ 0 & 0\end{pmatrix}\otimes\mathcal{L}^2_y + \widetilde E \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

(a) *If $\gamma = 0$, then there exist $\mu\in\mathfrak{X}_0(\omega)$, $\kappa\in\Upsilon_0(\omega)$ and $\zeta\in\mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{R}^3)$ such that*

$$\widetilde E = \begin{pmatrix} E_y\mu(x',y) - x_3 D^2_y\kappa(x',y) & \zeta'(x,y)\\ (\zeta'(x,y))^T & \zeta_3(x,y)\end{pmatrix}.$$

(b) *If $\gamma = +\infty$, then there exist $\mu\in\mathfrak{X}_\infty(\Omega)$, $\kappa\in\Upsilon_\infty(\Omega)$ and $\zeta\in\mathcal{M}_b(\Omega;\mathbb{R}^3)$ such that*

$$\widetilde E = \begin{pmatrix} E_y\mu(x,y) & \zeta'(x) + D_y\kappa(x,y)\\ (\zeta'(x)+D_y\kappa(x,y))^T & \zeta_3(x)\end{pmatrix}.$$

Proof. Owing to [46, Chapter II, Remark 3.3], we can assume without loss of generality that the maps u^h are smooth functions for every $h>0$. Further, the uniform boundedness of the sequence $\{Ev^h\}$ implies that

$$\int_\Omega |\partial_{x_\alpha}u^h_3 + \partial_{x_3}u^h_\alpha|\,dx \le Ch \quad \text{for } \alpha = 1,2, \tag{4.4}$$

$$\int_\Omega |\partial_{x_3}u^h_3|\,dx \le Ch^2. \tag{4.5}$$

In the following, we will consider $\lambda\in\mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\Lambda_h E u^h \xrightarrow{2\text{-}*} \lambda \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Step 1: We consider the case $\gamma = 0$, i.e., $\frac{h}{\varepsilon_h} \to 0$. By the Poincaré inequality in $L^1(I)$, there is a constant C independent of h such that

$$\int_I |u_3^h - \overline{u}_3^h|\,dx_3 \le C \int_I |\partial_{x_3} u_3^h|\,dx_3$$

for a.e. $x' \in \omega$. Integrating over ω, we obtain that

$$\int_\Omega |u_3^h - \overline{u}_3^h|\,dx \le C \int_\Omega |\partial_{x_3} u_3^h|\,dx \le Ch^2. \tag{4.6}$$

Set

$$\vartheta_3^h(x) := \frac{u_3^h(x) - \overline{u}_3^h(x')}{h^2}.$$

We have that $\{\vartheta_3^h\}_{h>0}$ is uniformly bounded in $L^1(\Omega)$. Correspondingly, we construct a sequence of antiderivatives $\{\theta_3^h\}_{h>0}$ by

$$\theta_3^h(x) := \int_{-\frac{1}{2}}^{x_3} \vartheta_3^h(x', z_3)\,dz_3 - C_{\vartheta_3^h},$$

where we choose $C_{\vartheta_3^h}$ such that $\overline{\theta}_3^h = 0$. Note that the constructed sequence is also uniformly bounded in $L^1(\Omega)$. Next, for $\alpha \in \{1, 2\}$, we construct sequences $\{\theta_\alpha^h\}_{h>0}$ by

$$\theta_\alpha^h(x) := \frac{u_\alpha^h(x) - \overline{u}_\alpha^h(x') + x_3 \partial_{x_\alpha} \overline{u}_3^h(x')}{h} + h\partial_{x_\alpha} \theta_3^h(x).$$

Then $\overline{\theta}_\alpha^h = 0$ and

$$\partial_{x_3} \theta_\alpha^h = \frac{\partial_{x_3} u_\alpha^h + \partial_{x_\alpha} \overline{u}_3^h}{h} + h\partial_{x_\alpha} \vartheta_3^h = \frac{\partial_{x_3} u_\alpha^h + \partial_{x_\alpha} u_3^h}{h},$$

since $\partial_{x_3} \theta_3^h = \vartheta_3^h$. Thus, by the Poincaré inequality in $L^1(I)$ and integrating over ω, we obtain that

$$\int_\Omega |\theta_\alpha^h|\,dx \le C \int_\Omega |\partial_{x_3} \theta_\alpha^h|\,dx \le C. \tag{4.7}$$

From the above constructions, we infer

$$u_\alpha^h(x) = \overline{u}_\alpha^h(x') - x_3 \partial_{x_\alpha} \overline{u}_3^h(x') + h^2 \partial_{x_\alpha} \theta_3^h(x) + h\theta_\alpha^h(x), \quad \alpha = 1, 2. \tag{4.8}$$

For the 2×2 minors of the scaled symmetrized gradients, a direct calculation shows

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y) : d\lambda''(x, y) = \lim_{h \to 0} \int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big) : (E(\overline{u}^h)'(x') - x_3 D^2 \overline{u}_3^h(x') + h^2 D_{x'}^2 \theta_3^h(x) + hE_{x'}(\theta^h)'(x))\,dx \tag{4.9}$$

for every $\chi \in C_c^\infty(\omega; C^\infty(I \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}))$. Notice that the last two terms in (4.9) are negligible in the limit. Indeed, we have

$$\begin{aligned}
\lim_{h \to 0} \int_\Omega \chi\Big(x, \frac{x'}{\varepsilon_h}\Big) : h^2 D_{x'}^2 \theta_3^h(x)\,dx &= \lim_{h \to 0} h^2 \int_\Omega \theta_3^h(x) \operatorname{div}_{x'} \operatorname{div}_{x'} \Big(\chi\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big)\,dx \\
&= \lim_{h \to 0} h^2 \sum_{\alpha,\beta=1,2} \int_\Omega \theta_3^h(x) \partial_{x_\alpha} \Big(\partial_{x_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big) + \frac{1}{\varepsilon_h} \partial_{y_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big)\,dx \\
&= \lim_{h \to 0} \sum_{\alpha,\beta=1,2} \int_\Omega \theta_3^h(x) \Big(h^2 \partial_{x_\alpha x_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big) + \frac{h^2}{\varepsilon_h} \partial_{y_\alpha x_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big) \\
&\qquad + \frac{h^2}{\varepsilon_h} \partial_{x_\alpha y_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big) + \frac{h^2}{\varepsilon_h^2} \partial_{y_\alpha y_\beta} \chi_{\alpha\beta}\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big)\,dx \\
&= 0.
\end{aligned} \tag{4.10}$$

Similarly, we compute

$$\begin{aligned}\lim_{h\to 0}\int_\Omega \chi\Big(x,\frac{x'}{\varepsilon_h}\Big):hE_{x'}(\theta^h)'(x)\,dx &= -\lim_{h\to 0} h\int_\Omega (\theta^h)'(x)\cdot \operatorname{div}_{x'}\Big(\chi\Big(x,\frac{x'}{\varepsilon_h}\Big)\Big)\,dx\\ &= -\lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\Omega \theta^h_\alpha(x)\Big(h\partial_{x_\beta}\chi_{\alpha\beta}\Big(x,\frac{x'}{\varepsilon_h}\Big)+\frac{h}{\varepsilon_h}\partial_{y_\beta}\chi_{\alpha\beta}\Big(x,\frac{x'}{\varepsilon_h}\Big)\Big)\,dx\\ &= 0. \end{aligned} \tag{4.11}$$

Thus, considering an open set $\widetilde{I}\supset I$ which compactly contains I, we infer

$$(E_{\alpha\beta}(u^h))^\perp \xrightarrow{2\text{-}*} 0 \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega\times\widetilde{I}\times\mathcal{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{4.12}$$

Since $\{(\overline{u}^h)'\}$ is bounded in $BD(\omega)$ with $(\overline{u}^h)' \xrightarrow{*} \bar{u}$ weakly* in $BD(\omega)$, by [26, Proposition 4.10] (the result follows by duality argument, using Proposition 4.7) there exists $\mu\in\mathcal{X}_0(\omega)$ such that

$$E(\overline{u}^h)' \xrightarrow{2\text{-}*} E\bar{u}\otimes\mathcal{L}^2_y + E_y\mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega\times\mathcal{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{4.13}$$

From Proposition 4.1 there holds

$$\begin{aligned} u^h_\alpha &\to \bar{u}_\alpha - x_3\partial_{x_\alpha}u_3 && \text{strongly in } L^1(\Omega),\quad \alpha=1,2,\\ u^h_3 &\to u_3 && \text{strongly in } L^1(\Omega). \end{aligned}$$

thus we infer that

$$\overline{u}^h_3 \xrightarrow{2\text{-}*} u_3(x')\mathcal{L}^2_{x'}\otimes\mathcal{L}^2_y \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega\times\mathcal{Y}) \tag{4.14}$$

Further, multiplying (4.8) with x_3 and integrating over ω, we obtain

$$\partial_{x_\alpha}\overline{u}^h_3(x') = -\widehat{u}^h_\alpha(x') + h^2\partial_{x_\alpha}\widehat{\theta}^h_3(x') + h\widehat{\theta}^h_\alpha(x'),\quad \alpha=1,2.$$

Using similar calculations as in (4.10) and (4.11), we obtain that only the first term is not negligible in the limit, from which we conclude that, for any $\varphi\in C^\infty_c(\omega\times\mathcal{Y})$,

$$\lim_{h\to 0}\int_\omega \partial_{x_\alpha}\overline{u}^h_3(x')\varphi\Big(x',\frac{x'}{\varepsilon_h}\Big)\,dx' = \int_{\omega\times\mathcal{Y}}\partial_{x_\alpha}u_3(x')\varphi(x',y)\,dx'\,dy,\quad \alpha=1,2. \tag{4.15}$$

Consider now $\chi\in C^\infty_c(\omega\times\mathcal{Y};\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that $\operatorname{div}_y\operatorname{div}_y\chi(x',y)=0$. Then

$$\begin{aligned} &\lim_{h\to 0}\int_\omega \chi\Big(x',\frac{x'}{\varepsilon_h}\Big):D^2\overline{u}^h_3(x')\,dx'\\ &= \lim_{h\to 0}\int_\omega \overline{u}^h_3(x')\operatorname{div}_{x'}\operatorname{div}_{x'}\Big(\chi\Big(x',\frac{x'}{\varepsilon_h}\Big)\Big)\,dx'\\ &= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}^h_3(x')\Big(\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)+\frac{1}{\varepsilon_h}\partial_{y_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\\ &\qquad + \frac{1}{\varepsilon_h}\partial_{x_\alpha y_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)+\frac{1}{\varepsilon_h^2}\partial_{y_\alpha y_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\Big)\,dx'\\ &= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}^h_3(x')\Big(\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)+\frac{2}{\varepsilon_h}\partial_{y_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\Big)\,dx'\\ &= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\int_\omega \overline{u}^h_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\,dx' + 2\int_\omega\Big(\partial_{x_\alpha}\Big(\overline{u}^h_3(x')\partial_{x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\Big)\\ &\qquad - \partial_{x_\alpha}\overline{u}^h_3(x')\partial_{x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big) - \overline{u}^h_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\Big)\,dx'\\ &= \lim_{h\to 0}\sum_{\alpha,\beta=1,2}\Big(-\int_\omega \overline{u}^h_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\,dx' - 2\int_\omega \partial_{x_\alpha}\overline{u}^h_3(x')\partial_{x_\beta}\chi_{\alpha\beta}\Big(x',\frac{x'}{\varepsilon_h}\Big)\,dx'\Big), \end{aligned}$$

where in the last equality we used Green's theorem. Passing to the limit, by (4.14) and (4.15), we have

$$\begin{aligned}\lim_{h\to 0}\int_\omega \chi\Big(x',\frac{x'}{\varepsilon_h}\Big) : D^2\overline{u}_3^h(x')\,dx' &= \sum_{\alpha,\beta=1,2}\Big(-\int_{\omega\times\mathcal{Y}} u_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}(x',y)\,dx'\,dy - 2\int_{\omega\times\mathcal{Y}} \partial_{x_\alpha}u_3(x')\partial_{x_\beta}\chi_{\alpha\beta}(x',y)\,dx'\,dy\Big)\\ &= \sum_{\alpha,\beta=1,2}\Big(-\int_{\omega\times\mathcal{Y}} u_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}(x',y)\,dx'\,dy\\ &\qquad - 2\int_{\omega\times\mathcal{Y}} (\partial_{x_\alpha}(u_3(x')\partial_{x_\beta}\chi_{\alpha\beta}(x',y)) - u_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}(x',y))\,dx'\,dy\Big)\\ &= \sum_{\alpha,\beta=1,2}\int_{\omega\times\mathcal{Y}} u_3(x')\partial_{x_\alpha x_\beta}\chi_{\alpha\beta}(x',y)\,dx'\,dy\\ &= \int_{\omega\times\mathcal{Y}} \chi(x',y) : d(D^2u_3\otimes\mathcal{L}^2_y). \end{aligned}\tag{4.16}$$

From (4.12), (4.13), (4.16) and Lemma 4.12, we conclude that

$$\lambda'' = E\bar{u}\otimes\mathcal{L}^2_y + E_y\mu - x_3D^2u_3\otimes\mathcal{L}^2_y - x_3D^2_y\kappa,$$

where $\mu\in\mathfrak{X}_0(\omega)$, $\kappa\in\Upsilon_0(\omega)$. Finally, consider the vector $\zeta^h(x)$ given by the third column of Λ_hEu^h, for every $h>0$. The boundedness of the sequence of functions $v^h\in \mathrm{BD}(\Omega^h)$ implies that $\{\zeta^h\}_{h>0}$ is a uniformly bounded sequence in $L^1(\Omega;\mathbb{R}^3)$. Consequently, we can extract a subsequence which two-scale weakly* converges in $\mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{R}^3)$ such that

$$\frac{1}{h}E_{\alpha 3}(u^h)\xrightarrow{2\text{-}*}\zeta_\alpha\quad\text{two-scale weakly* in }\mathcal{M}_b(\Omega\times\mathcal{Y}),\quad \alpha=1,2,$$
$$\frac{1}{h^2}E_{33}(u^h)\xrightarrow{2\text{-}*}\zeta_3\quad\text{two-scale weakly* in }\mathcal{M}_b(\Omega\times\mathcal{Y}),$$

for a suitable $\zeta\in\mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{R}^3)$. This concludes the proof in the case $\gamma=0$.

Step 2: Consider the case $\gamma=+\infty$, i.e., $\frac{\varepsilon_h}{h}\to 0$. For the 2×2 minors of two-scale limit, by Proposition 4.13 and the proof of Proposition 4.1, we have that there exists $\mu\in\mathfrak{X}_\infty(\Omega)$ such that

$$\lambda'' = (E\bar{u} - x_3D^2u_3)\otimes\mathcal{L}^2_y + E_y\mu.$$

Let $\chi^{(1)}\in C_c^\infty(\Omega)$ and $\chi^{(2)}\in C^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\int_{\mathcal{Y}}\chi^{(2)}\,dy = 0.$$

We consider a test function

$$\chi(x,y) = \chi^{(1)}(x)\chi^{(2)}\Big(\frac{x'}{\varepsilon_h}\Big)$$

such that

$$\int_{\Omega\times\mathcal{Y}}\chi(x,y) : d\lambda(x,y) = \lim_{h\to 0}\int_\Omega \chi^{(1)}(x)\chi^{(2)}\Big(\frac{x'}{\varepsilon_h}\Big) : d(\Lambda_hEu^h(x)).$$

For each $i=1,2,3$, let G_i denote the unique solution in $C^\infty(\mathcal{Y})$ to the Poisson's equation

$$-\Delta_yG_i = \chi^{(2)}_{3i},\qquad \int_{\mathcal{Y}}G_i\,dy = 0.$$

Then, observing that

$$\int_{\Omega\times\mathcal{Y}}\chi_{33}(x,y) : d\lambda_{33}(x,y) = \lim_{h\to 0}\frac{1}{h^2}\int_\Omega \partial_{x_3}u_3^h(x)\chi^{(1)}(x)\chi^{(2)}_{33}\Big(\frac{x'}{\varepsilon_h}\Big)\,dx,$$

we find

$$\begin{aligned}
&\int_{\Omega\times\mathcal{Y}} \chi_{33}(x,y) : d\lambda_{33}(x,y)\\
&\quad = -\lim_{h\to0}\frac{1}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_3}u_3^h(x)\chi^{(1)}(x)\partial_{y_\alpha y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad = \lim_{h\to0}\frac{1}{h^2}\sum_{\alpha=1,2}\int_\Omega u_3^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad = \lim_{h\to0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\bigg(\int_\Omega u_3^h(x)\partial_{x_\alpha}\Big(\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\Big)\,dx - \int_\Omega u_3^h(x)\partial_{x_\alpha x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\bigg)\\
&\quad = \lim_{h\to0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\bigg(-\int_\Omega \partial_{x_\alpha}u_3^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx + \int_\Omega \partial_{x_3}u_3^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\bigg).
\end{aligned}$$

Recalling (4.4) and (4.5), we deduce

$$\begin{aligned}
&\int_{\Omega\times\mathcal{Y}} \chi_{33}(x,y) : d\lambda_{33}(x,y)\\
&\quad = \lim_{h\to0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_3}u_\alpha^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad = -\lim_{h\to0}\frac{\varepsilon_h}{h^2}\sum_{\alpha=1,2}\int_\Omega u_\alpha^h(x)\partial_{x_3x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad = -\lim_{h\to0}\frac{{\varepsilon_h}^2}{h^2}\sum_{\alpha=1,2}\bigg(\int_\Omega u_\alpha^h(x)\partial_{x_\alpha}\Big(\partial_{x_3x_3}\chi^{(1)}(x)G_3\Big(\frac{x'}{\varepsilon_h}\Big)\Big)\,dx - \int_\Omega u_3^h(x)\partial_{x_\alpha x_3x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\bigg)\\
&\quad = \lim_{h\to0}\frac{{\varepsilon_h}^2}{h^2}\sum_{\alpha=1,2}\int_\Omega \partial_{x_\alpha}u_\alpha^h(x)\partial_{x_3x_3}\chi^{(1)}(x)G_3\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad = 0.
\end{aligned}\tag{4.17}$$

Thus, recalling that

$$\int_{\mathcal{Y}} \chi_{33}^{(2)}\,dy = 0,$$

and since for arbitrary test function we can subtract their mean value over $\mathcal{Y}$ to obtain a function with mean value zero, we infer that there exists $\zeta_3 \in \mathcal{M}_b(\Omega)$ such that

$$\lambda_{33} = \zeta_3 \otimes \mathcal{L}_y^2.$$

Similarly, from the observation that

$$\int_{\Omega\times\mathcal{Y}} \chi_{13}(x,y) : d\lambda_{13}(x,y) + \int_{\Omega\times\mathcal{Y}} \chi_{23}(x,y) : d\lambda_{23}(x,y) = \lim_{h\to0}\frac{1}{2h}\sum_{\alpha=1,2}\int_\Omega (\partial_{x_\alpha}u_3^h(x) + \partial_{x_3}u_\alpha^h(x))\chi^{(1)}(x)\chi_{3\alpha}^{(2)}\Big(\frac{x'}{\varepsilon_h}\Big)\,dx,$$

we deduce

$$\begin{aligned}
&\int_{\Omega\times\mathcal{Y}} \chi_{13}(x,y) : d\lambda_{13}(x,y) + \int_{\Omega\times\mathcal{Y}} \chi_{23}(x,y) : d\lambda_{23}(x,y)\\
&\quad = \lim_{h\to0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\bigg(\int_\Omega \partial_{x_\alpha}u_3^h(x)\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx + \int_\Omega \partial_{x_3}u_\alpha^h(x)\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\bigg).
\end{aligned}\tag{4.18}$$

Suppose now that $\operatorname{div}_y \chi_{3\alpha}^{(2)} = 0$, i.e.,

$$\sum_{\alpha,\beta=1,2}\partial_{y_\alpha y_\beta y_\beta}G_\alpha = 0.$$

Then we have

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_\alpha}u_3^h(x)\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\Big(-\int_{\Omega}u_3^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx-\frac{1}{\varepsilon_h}\int_{\Omega}u_3^h(x)\chi^{(1)}(x)\partial_{y_\alpha y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&\quad=-\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}u_3^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_{\Omega}\partial_{x_\beta}u_3^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx+\int_{\Omega}u_3^h(x)\partial_{x_\alpha x_\beta}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&\quad=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_\beta}u_3^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&=-\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_3}u_\beta^h(x)\partial_{x_\alpha}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}u_\beta^h(x)\partial_{x_\alpha x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&=0.
\end{aligned}\tag{4.19}$$

Furthermore,

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_3}u_\alpha^h(x)\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=-\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}u_\alpha^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_{\Omega}\partial_{x_\beta}u_\alpha^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx+\int_{\Omega}u_\alpha^h(x)\partial_{x_\beta x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&\quad=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_\beta}u_\alpha^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=-\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}\partial_{x_\alpha}u_\beta^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_{\Omega}u_\beta^h(x)\partial_{x_\alpha x_3}\chi^{(1)}(x)\partial_{y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)+\frac{1}{\varepsilon_h}\int_{\Omega}u_\beta^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&\quad=\lim_{h\to 0}\frac{1}{2h}\sum_{\alpha,\beta=1,2}\int_{\Omega}u_\beta^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha y_\beta}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\\
&\quad=-\lim_{h\to 0}\frac{\varepsilon_h}{2h}\sum_{\alpha,\beta=1,2}\Big(\int_{\Omega}\partial_{x_\beta}u_\beta^h(x)\partial_{x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx+\int_{\Omega}u_\beta^h(x)\partial_{x_\beta x_3}\chi^{(1)}(x)\partial_{y_\alpha}G_\alpha\Big(\frac{x'}{\varepsilon_h}\Big)\,dx\Big)\\
&\quad=0.
\end{aligned}\tag{4.20}$$

From (4.18), (4.19) and (4.20), and Proposition 4.11, and recalling that $\int_{\mathcal{Y}}\chi_{13}^{(2)}\,dy=0$ and $\int_{\mathcal{Y}}\chi_{23}^{(2)}\,dy=0$, we conclude that there exist $\kappa\in\Upsilon_\infty(\Omega)$ and $\zeta'\in\mathcal{M}_b(\Omega;\mathbb{R}^2)$ such that

$$\begin{pmatrix}\lambda_{13}\\ \lambda_{23}\end{pmatrix}=\zeta'\otimes\mathcal{L}_y^2+D_y\kappa.$$

This concludes the proof of the theorem. □

5 Two-scale statics and duality

In this section we define a notion of stress-strain duality and analyze the two-scale behavior of our functionals. The main goal is to prove the principle of maximum plastic work in Section 5.4, which we will use in Section 6 to prove the global stability of the limiting model. In Section 5.1 we characterize the duality between stress and strain on the torus $\mathcal{Y}$, the admissible two-scale configurations are discussed in Section 5.2, while the admissible two-scale stresses are the subject of Section 5.3.

5.1 Stress-plastic strain duality on the cell

5.1.1 Case $y = 0$

Definition 5.1. The set $\mathcal{K}_0$ of admissible stresses is defined as the set of all elements $\Sigma \in L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$ satisfying:
(i) $\Sigma_{i3}(x_3, y) = 0$ for $i = 1, 2, 3$,
(ii) $\Sigma_{\text{dev}}(x_3, y) \in K(y)$ for $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3, y) \in I \times \mathcal{Y}$,
(iii) $\operatorname{div}_y \bar{\Sigma} = 0$ in $\mathcal{Y}$,
(iv) $\operatorname{div}_y \operatorname{div}_y \hat{\Sigma} = 0$ in $\mathcal{Y}$,
where $\bar{\Sigma}, \hat{\Sigma} \in L^2(\mathcal{Y}; \mathbb{M}^{2\times2}_{\text{sym}})$ are the zero-th and the first order moments of the 2×2 minor of Σ.

Recalling (3.21), by conditions (i) and (ii) we may identify $\Sigma \in \mathcal{K}_0$ with an element of $L^\infty(I \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\text{sym}})$ such that $\Sigma(x_3, y) \in K_r(y)$ for $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3, y) \in I \times \mathcal{Y}$. Thus, in this regime it will be natural to define the family of admissible configurations by means of conditions formulated on $\mathbb{M}^{2\times2}_{\text{sym}}$.

Definition 5.2. The family $\mathcal{A}_0$ of admissible configurations is given by the set of quadruplets

$$\bar{u} \in \mathrm{BD}(\mathcal{Y}), \quad u_3 \in \mathrm{BH}(\mathcal{Y}), \quad E \in L^2(I \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\text{sym}}), \quad P \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\text{sym}})$$

such that

$$E_y \bar{u} - x_3 D^2_y u_3 = E \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P \quad \text{in } I \times \mathcal{Y}. \tag{5.1}$$

Recalling the definitions of zero-th and first order moments of functions and measures (see Definition 3.5 and Definition 3.6), we introduce the following analogue of the duality between moments of stresses and plastic strains.

Definition 5.3. Let $\Sigma \in \mathcal{K}_0$ and let $(\bar{u}, u_3, E, P) \in \mathcal{A}_0$. We define the distributions $[\bar{\Sigma} : \bar{P}]$ and $[\hat{\Sigma} : \hat{P}]$ on $\mathcal{Y}$ by

$$[\bar{\Sigma} : \bar{P}](\varphi) := -\int_{\mathcal{Y}} \varphi \bar{\Sigma} : \bar{E}\, dy - \int_{\mathcal{Y}} \bar{\Sigma} : (\bar{u} \odot \nabla_y \varphi)\, dy, \tag{5.2}$$

$$[\hat{\Sigma} : \hat{P}](\varphi) := -\int_{\mathcal{Y}} \varphi \hat{\Sigma} : \hat{E}\, dy + 2\int_{\mathcal{Y}} \hat{\Sigma} : (\nabla_y u_3 \odot \nabla_y \varphi)\, dy + \int_{\mathcal{Y}} u_3 \hat{\Sigma} : \nabla^2_y \varphi\, dy \tag{5.3}$$

for every $\varphi \in C^\infty(\mathcal{Y})$.

Remark 5.4. Note that the second integral in (5.2) is well defined since $\mathrm{BD}(\mathcal{Y})$ is embedded into $L^2(\mathcal{Y}; \mathbb{R}^2)$. Similarly, the second and third integrals in (5.3) are well defined since $\mathrm{BH}(\mathcal{Y})$ is embedded into $H^1(\mathcal{Y})$. Moreover, the definitions are independent of the choice of (u, E), so (5.2) and (5.3) define a meaningful distributions on $\mathcal{Y}$ (this is valid for arbitrary $\bar{\Sigma}, \hat{\Sigma} \in L^\infty(\mathcal{Y}; \mathbb{M}^{2\times2}_{\text{sym}})$ that satisfy the properties (iii) and (iv) of Definition 5.1). Arguing as in [21, Section 7], one can prove that $[\bar{\Sigma} : \bar{P}]$ and $[\hat{\Sigma} : \hat{P}]$ are bounded Radon measures on $\mathcal{Y}$. For $\bar{\Sigma}$ of class C^1 and $\hat{\Sigma}$ of class C^2 it can be shown by integration by parts (see, e.g., [27] and [22, Remark 7.1, Remark 7.4] that

$$\int_{\mathcal{Y}} \varphi d[\bar{\Sigma} : \bar{P}] = \int_{\mathcal{Y}} \varphi \bar{\Sigma} d\bar{P}, \quad \int_{\mathcal{Y}} \varphi d[\hat{\Sigma} : \hat{P}] = \int_{\mathcal{Y}} \varphi \hat{\Sigma} d\hat{P}. \tag{5.4}$$

From this it follows that for $\bar{\Sigma}$ of class C^1 and $\widehat{\Sigma}$ of class C^2 we have

$$|[\bar{\Sigma} : \bar{P}]| \le \|\bar{\Sigma}\|_{L^\infty}|\bar{P}|, \quad |[\widehat{\Sigma} : \widehat{P}]| \le \|\widehat{\Sigma}\|_{L^\infty}|\widehat{P}|, \quad \varphi \in C(\mathcal{Y}). \tag{5.5}$$

Through the approximation by convolution (5.4) then extends to arbitrary continuous $\bar{\Sigma}$, $\widehat{\Sigma}$ and (5.5) applies to arbitrary $\bar{\Sigma}, \widehat{\Sigma} \in L^\infty(\mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ satisfying the properties (iii) and (iv) of Definition 5.1).

Remark 5.5. If α is a simple C^2 curve in $\mathcal{Y}$, then

$$[\bar{\Sigma} : \bar{P}] = \Sigma \nu^1_\alpha \cdot (\bar{u}_1 - \bar{u}_2)\mathcal{H}^1, \tag{5.6}$$

where ν^1_α is a unit normal on the curve α while $\bar{u}_1$ and $\bar{u}_2$ are the traces on α of $\bar{u}$ ($\bar{u}_1$ is from the side toward which normal is pointing, $\bar{u}_2$ is from the opposite side). This can be obtained from (5.4) and approximation by convolution, see, e.g., [27, Lemma 3.8].

From (2.4) it follows that if U is an open set in $\mathcal{Y}$ whose boundary is of class C^2 and $\bar{\Sigma}_n \in L^\infty(U; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ a bounded sequence such that $\bar{\Sigma}_n \to \bar{\Sigma}$ almost everywhere (and thus in $L^p(U)$, for every $p < \infty$) and $\operatorname{div}_y \bar{\Sigma}_n \to 0$ strongly in $L^2(U)$, then $\bar{\Sigma}_n \nu^1_\alpha \xrightharpoonup{*} \bar{\Sigma}\nu^1_\alpha$, weakly* in $L^\infty(K \cap \alpha)$ for any compact set $K \subset U$.

Remark 5.6. It can be shown that if $\alpha \subset \mathcal{Y}$ is simple C^2 closed or non-closed C^2 curve with endpoints $\{a, b\}$, then there exists $b_1(\widehat{\Sigma}) \in L^\infty_{\mathrm{loc}}(\alpha)$ such that

$$[\widehat{\Sigma} : \widehat{P}] = b_1(\widehat{\Sigma})\partial_{\nu_\alpha} u_3^{1,2}\mathcal{H}^1 \quad \text{on } \alpha, \tag{5.7}$$

where ν_α is a unit normal of α and $\partial_{\nu_\alpha} u_3^{1,2}$ is a jump in the normal derivative of u_3 (from the side in the opposite direction of the normal), which is an $L^1_{\mathrm{loc}}(\alpha)$ function. This is a direct consequence of (5.3) and [23, Théoreme 2], see also [22, Remark 7.4] and the fact that $|[\widehat{\Sigma} : \widehat{P}]|\{a, b\} = 0$ (see (5.5)).

From [23, Théoreme 2 and Appendice, Théoreme 1] it follows that if U is an open set in $\mathcal{Y}$ whose boundary is of class C^2 and $\widehat{\Sigma}_n \in L^\infty(U; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ a bounded sequence such that $\widehat{\Sigma}_n \to \widehat{\Sigma}$ almost everywhere (and thus in $L^p(U)$, for every $p < \infty$) and $\operatorname{div}_y \operatorname{div}_y \widehat{\Sigma}_n \to 0$ strongly in $L^2(U)$, then $b_1(\widehat{\Sigma}_n) \xrightharpoonup{*} b_1(\widehat{\Sigma})$, weakly* in $L^\infty(K \cap \alpha)$ for any compact set $K \subset U$.

We are now in a position to introduce a duality pairing between admissible stresses and plastic strains.

Definition 5.7. Let $\Sigma \in \mathcal{K}_0$ and let $(\bar{u}, u_3, E, P) \in \mathcal{A}_0$. Then we can define a bounded Radon measure $[\Sigma : P]$ on $I \times \mathcal{Y}$ by setting

$$[\Sigma : P] := [\bar{\Sigma} : \bar{P}] \otimes \mathcal{L}^1_{x_3} + \frac{1}{12}[\widehat{\Sigma} : \widehat{P}] \otimes \mathcal{L}^1_{x_3} - \Sigma^\perp : E^\perp,$$

so that

$$\int_{I\times\mathcal{Y}} \varphi \, d[\Sigma : P] = -\int_{I\times\mathcal{Y}} \varphi\Sigma : E \, dx_3 \, dy - \int_{\mathcal{Y}} \bar{\Sigma} : (\bar{u} \odot \nabla_y\varphi) \, dy + \frac{1}{6}\int_{\mathcal{Y}} \widehat{\Sigma} : (\nabla_y u_3 \odot \nabla_y\varphi) \, dy + \frac{1}{12}\int_{\mathcal{Y}} u_3\widehat{\Sigma} : \nabla^2_y\varphi \, dy \tag{5.8}$$

for every $\varphi \in C^2(\mathcal{Y})$.

Remark 5.8. Notice that

$$\overline{[\Sigma : P]} := [\bar{\Sigma} : \bar{P}] + \frac{1}{12}[\widehat{\Sigma} : \widehat{P}] - \overline{\Sigma^\perp : E^\perp}.$$

The following proposition will be used in Section 5.4 to prove the main result of this section.

Proposition 5.9. *Let $\Sigma \in \mathcal{K}_0$ and $(\bar{u}, u_3, E, P) \in \mathcal{A}_0$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus, under the assumption on the ordering of the phases we have*

$$\overline{H_r\Big(y, \frac{dP}{d|P|}\Big)|P|} \ge \overline{[\Sigma : P]}. \tag{5.9}$$

Proof. The proof is divided into two steps.

Step 1. In this step we consider a phase $\mathcal{Y}_i$ for arbitrary i. Regularizing Σ just by convolution with respect to y, we obtain a sequence $\{\Sigma_n\}$ satisfying

$$\Sigma_n \to \Sigma \quad \text{strongly in } L^2(I \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}), \qquad \operatorname{div}_y \bar{\Sigma}_n = 0, \qquad \operatorname{div}_y \operatorname{div}_y \widehat{\Sigma}_n = 0.$$

We also have that for every $\varepsilon > 0$ there exists $n(\varepsilon)$ large enough such that $(\Sigma_n(x_3, y))_{\mathrm{dev}} \in K_i$ for a.e. $x_3 \in I$ and every $y \in \mathcal{Y}_i$ that are distanced from $\partial\mathcal{Y}_i$ more than ε, for every $n \geq n(\varepsilon)$. Consider the orthogonal decomposition

$$P = \bar{P} \otimes \mathcal{L}^1_{x_3} + \widehat{P} \otimes x_3 \mathcal{L}^1_{x_3} + P^{\perp},$$

where $\bar{P}, \widehat{P} \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ and $P^{\perp} \in L^2(I \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. We infer that $|P|$ is absolutely continuous with respect to the measure

$$\Pi := |\bar{P}| \otimes \mathcal{L}^1_{x_3} + |\widehat{P}| \otimes \mathcal{L}^1_{x_3} + \mathcal{L}^3_{x_3,y}.$$

As a consequence, for $|\Pi|$-a.e. $(x_3, y) \in I \times \mathcal{Y}_i$ such that $\mathrm{dist}(y, \partial\mathcal{Y}_i) > \varepsilon$ we have

$$H_r\Big(y, \frac{dP}{d|\Pi|}\Big) \geq \Sigma_n : \frac{dP}{d|\Pi|}$$

for every $n \geq n(\varepsilon)$. Thus for every $\varphi \in C_c(\mathcal{Y}_i)$, such that $\varphi \geq 0$, we obtain

$$\begin{aligned}\int_{I\times\mathcal{Y}_i} \varphi(y) H_r\Big(y, \frac{dP}{d|P|}\Big)\, d|P| &= \int_{I\times\mathcal{Y}_i} \varphi H_r\Big(y, \frac{dP}{d|\Pi|}\Big)\, d|\Pi| \\ &\geq \int_{I\times\mathcal{Y}_i} \varphi \Sigma_n : \frac{dP}{d|\Pi|}\, d|\Pi| = \int_{I\times\mathcal{Y}_i} \varphi \Sigma_n : \frac{dP}{d|P|}\, d|P| = \int_{I\times\mathcal{Y}_i} \varphi\, d[\Sigma_n : P]\end{aligned}$$

for n large enough. Since $\bar{\Sigma}_n$, $\widehat{\Sigma}_n$ and $(\Sigma_n)^{\perp}$ are smooth with respect to y, from (5.2), (5.3) and (5.5) we conclude that

$$\begin{aligned}[\bar{\Sigma}_n : \bar{P}] &\overset{*}{\rightharpoonup} [\bar{\Sigma} : \bar{P}] \quad \text{weakly* in } \mathcal{M}_b(\mathcal{Y}), \\ [\widehat{\Sigma}_n : \widehat{P}] &\overset{*}{\rightharpoonup} [\widehat{\Sigma} : \widehat{P}] \quad \text{weakly* in } \mathcal{M}_b(\mathcal{Y}), \\ \int_{I\times\mathcal{Y}_i} \varphi (\Sigma_n)^{\perp} : P^{\perp}\, dx_3\, dy &\to \int_{I\times\mathcal{Y}_i} \varphi (\Sigma)^{\perp} : P^{\perp}\, dx_3\, dy.\end{aligned}$$

Passing to the limit, we have

$$\int_{I\times\mathcal{Y}_i} \varphi(y) H_r\Big(y, \frac{dP}{d|P|}\Big)\, d|P| \geq \int_{I\times\mathcal{Y}_i} \varphi\, d[\Sigma : P].$$

This proves (5.9) on every phase.

Step 2. In this step we consider a curve α that is of class C^2 (together with its possible endpoints) and that is the connected component of $\Gamma \setminus S$. The points on α (with the exception of the possible endpoints) belong to the intersection of the boundary of exactly two phases $\partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j$. From the assumption on the ordering of the phases, without loss of generality we can assume that $K_i \subset K_j$. By (5.1) (cf. Proposition 3.7) as well as by the continuity of u_3, we find

$$\bar{P} = (\bar{u}_j - \bar{u}_i) \odot \nu^i_\alpha \mathcal{H}^1, \quad \widehat{P} = (\nabla u^i_3 - \nabla u^j_3) \odot \nu^i_\alpha \mathcal{H}^1 = \partial_{\nu^i_\alpha} u^{i,j}_3 \nu^i_\alpha \odot \nu^i_\alpha \mathcal{H}^1 \quad \text{on } \alpha \tag{5.10}$$

and

$$P = \bar{P} + x_3 \widehat{P} \quad \text{on } \alpha, \tag{5.11}$$

where $\bar{u}_i, \bar{u}_j$ are traces of $\bar{u}$ on α from $\mathcal{Y}_i$ and $\mathcal{Y}_j$ respectively and $\partial_{\nu^i_\alpha} u^{i,j}_3$ is a jump in the normal derivative of u_3. From (5.6) and (5.7) (cf. Remark 5.8) we deduce

$$\overline{[\Sigma : P]} = (\Sigma \nu^i_\alpha \cdot (\bar{u}_j - \bar{u}_i) + b_1(\widehat{\Sigma}) \partial_{\nu^i_\alpha} u^{i,j}_3) \mathcal{H}^1 \quad \text{on } \alpha. \tag{5.12}$$

Since, for each i, $\mathcal{Y}_i$ is a bounded open set with piecewise C^2 boundary (in particular, with Lipschitz boundary), by [9, Proposition 2.5.4] there exists a finite open covering $\{\mathcal{U}^{(i)}_k\}$ of $\overline{\mathcal{Y}}_i$ such that $\mathcal{Y}_i \cap \mathcal{U}^{(i)}_k$ is (strongly) star-shaped with Lipschitz boundary (the construction is simple and those $\mathcal{U}^{(i)}_k$ that intersect the boundary have cylindrical form up to rotation). We take only those members of the covering that have nonempty intersection with α. We can easily modify these cylindrical sets $\mathcal{Y}_i \cap \mathcal{U}^{(i)}_k$ to be of class C^2. Let $\{\psi^{(i)}_k\}$ be a partition of unity of α subordinate to the covering $\{\mathcal{U}^{(i)}_k\}$, i.e., $\psi^{(i)}_k \in C(\alpha)$, with $0 \leq \psi^{(i)}_k \leq 1$, such that $\mathrm{supp}(\psi^{(i)}_k) \subset \mathcal{U}^{(i)}_k$ and $\sum_k \psi^{(i)}_k = 1$

on α and let $\varphi \in C_c(\alpha)$ be an arbitrary nonnegative function. For each k we define an approximation of the stress Σ on $\mathcal{Y}_i \cap \mathcal{U}_k^{(i)}$ by

$$\Sigma_{n,k}^{(i)}(x_3, y) := ((\Sigma \circ d_{n,k}^{(i)})(x_3, \cdot) * \rho_{\frac{1}{n+1}})(y), \tag{5.13}$$

where $d_{n,k}^{(i)}(x_3, y) = (x_3, \frac{n}{n+1}(y - y_k^{(i)}) + y_k^{(i)})$ and $y_k^{(i)}$ is the point with respect to which $\mathcal{Y}_i \cap \mathcal{U}_k^{(i)}$ is star shaped. Obviously one has for every k,

$$\begin{aligned}
&\Sigma_{n,k}^{(i)} \in (K_i)_r && \text{for } |\Pi|\text{-a.e. } (x_3, y) \in I \times (\overline{\mathcal{Y}}_i \cap \mathcal{U}_k^{(i)}),\\
&\|\Sigma_{n,k}^{(i)}\|_{L^\infty} \le \|\Sigma\|_{L^\infty(\mathcal{Y}_i \cap \mathcal{U}_k^{(i)})},\\
&\Sigma_{n,k}^{(i)} \to \Sigma, \quad \bar{\Sigma}_{n,k}^{(i)} \to \bar{\Sigma} \quad \hat{\Sigma}_{n,k}^{(i)} \to \hat{\Sigma} && \text{strongly in } L^2(\overline{\mathcal{Y}}_i \cap \mathcal{U}_k^{(i)}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}),\\
&\operatorname{div}_y \bar{\Sigma}_{n,k}^{(i)} = 0, \quad \operatorname{div}_y \operatorname{div}_y \hat{\Sigma}_{n,k}^{(i)} = 0.
\end{aligned}$$

From these and by using Remark 5.4, Remark 5.5 and (5.12) we conclude for every k,

$$\begin{aligned}
&\int_{I\times\alpha} \psi_k^i(y)\varphi(y) H_r\Big(y, \frac{dP}{d|P|}\Big)\, d|P|\\
&= \int_{I\times\alpha} \psi_k^i(y)\varphi(y) H_r\Big(y, \frac{dP}{d|\Pi|}\Big)\, d|\Pi| \ge \int_{I\times\alpha} \psi_k^i \varphi \Sigma_{n,k}^{(i)} : \frac{dP}{d|\Pi|}\, d|\Pi|\\
&= \int_\alpha \psi_k^i \varphi (\Sigma_{n,k}^{(i)} \nu_\alpha^i \cdot (\bar{u}_j - \bar{u}_i) + b_1(\hat{\Sigma}_{n,k}^{(i)}) \partial_{\nu_\alpha^i} u_3^{i,j})\, d\mathcal{H}^1 \to \int_\alpha \psi_k^i \varphi (\Sigma \nu_\alpha^i \cdot (\bar{u}_j - \bar{u}_i) + b_1(\hat{\Sigma}) \partial_{\nu_\alpha^i} u_3^{i,j})\, d\mathcal{H}^1.
\end{aligned}$$

By summing over k we infer (5.9) on α.

The final claim goes by combining Step 1 and Step 2 and using the fact that both measures in (5.9) are zero on $\mathcal{S}$ as a consequence of (5.1) and (5.5). □

5.1.2 Case $\gamma = +\infty$

We first define the set of admissible stresses and configurations on the torus.

Definition 5.10. The set $\mathcal{K}_\infty$ of admissible stresses is defined as the set of all elements $\Sigma \in L^2(\mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ satisfying:
(i) $\operatorname{div}_y \Sigma = 0$ in $\mathcal{Y}$,
(ii) $\Sigma_{\mathrm{dev}}(y) \in K(y)$ for $\mathcal{L}_y^2$-a.e. $y \in \mathcal{Y}$.

Notice that in (i) we neglect the third column of Σ.

Definition 5.11. The family $\mathcal{A}_\infty$ of admissible configurations is given by the set of quintuplets

$$\bar{u} \in BD(\mathcal{Y}), \quad u_3 \in BV(\mathcal{Y}), \quad v \in \mathbb{R}^3, \quad E \in L^2(\mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}), \quad P \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{dev}})$$

such that

$$\begin{pmatrix} E_y \bar{u} & v' + D_y u_3 \\ (v' + D_y u_3)^T & v_3 \end{pmatrix} = E\mathcal{L}_y^2 + P \quad \text{in } \mathcal{Y}. \tag{5.14}$$

We also define a notion of stress-strain duality on the torus.

Definition 5.12. Let $\Sigma \in \mathcal{K}_\infty$ and let $(\bar{u}, u_3, v, E, P) \in \mathcal{A}_\infty$. We define the distribution $[\Sigma_{\mathrm{dev}} : P]$ on $\mathcal{Y}$ by

$$\begin{aligned}
[\Sigma_{\mathrm{dev}} : P](\varphi) := -\int_{\mathcal{Y}} \varphi \Sigma : E\, dy - \int_{\mathcal{Y}} \Sigma'' : (\bar{u} \odot \nabla_y \varphi)\, dy - 2\int_{\mathcal{Y}} u_3 \begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy\\
+ 2v' \cdot \int_{\mathcal{Y}} \varphi \begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} dy + v_3 \int_{\mathcal{Y}} \varphi \Sigma_{33}\, dy
\end{aligned} \tag{5.15}$$

for every $\varphi \in C^\infty(\mathcal{Y})$.

Remark 5.13. Note that the integrals in (5.15) are well defined since $\mathrm{BD}(\mathcal{Y})$ and $\mathrm{BV}(\mathcal{Y})$ are both embedded into $L^2(\mathcal{Y};\mathbb{R}^2)$. Moreover, the definition is independent of the choice of $(\bar{u}, u_3, v, E)$, so (5.15) defines a meaningful distribution on $\mathcal{Y}$.

The following proposition provides an estimate on the total variation of $[\Sigma_{\mathrm{dev}} : P]$. As a consequence, we find that $[\Sigma_{\mathrm{dev}} : P]$ depends indeed only on the deviatoric part of Σ.

Proposition 5.14. *Let $\Sigma \in \mathcal{K}_\infty$ and $(\bar{u}, u_3, v, E, P) \in \mathcal{A}_\infty$. Then $[\Sigma_{\mathrm{dev}} : P]$ can be extended to a bounded Radon measure on $\mathcal{Y}$, whose variation satisfies*

$$|[\Sigma_{\mathrm{dev}} : P]| \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})}|P| \quad \text{in } \mathcal{M}_b(\mathcal{Y}).$$

Proof. Using a convolution argument we construct a sequence $\{\Sigma_n\} \subset C^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\begin{aligned}
&\Sigma_n \to \Sigma \quad \text{strongly in } L^2(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}}),\\
&\operatorname{div}_y \Sigma_n = 0 \quad \text{in } \mathcal{Y},\\
&\|(\Sigma_n)_{\mathrm{dev}}\|_{L^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{dev}})} \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{dev}})}.
\end{aligned}$$

According to the integration by parts formulas for $\mathrm{BD}(\mathcal{Y})$ and $\mathrm{BV}(\mathcal{Y})$, we have for every $\varphi \in C^1(\mathcal{Y})$

$$\int_{\mathcal{Y}} \varphi \operatorname{div}_y(\Sigma_n)'' \cdot \bar{u}\, dy + \int_{\mathcal{Y}} \varphi(\Sigma_n)'' : dE_y\bar{u} + \int_{\mathcal{Y}} (\Sigma_n)'' : (\bar{u} \odot \nabla_y \varphi)\, dy = 0,$$

$$\int_{\mathcal{Y}} \varphi u_3 \operatorname{div}_y \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + \int_{\mathcal{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y u_3 + \int_{\mathcal{Y}} u_3 \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy = 0.$$

From these two equalities, together with the above convergence and the expression in (5.15), we compute

$$\begin{aligned}
[\Sigma_{\mathrm{dev}} : P](\varphi) &= \lim_n \Bigg[-\int_{\mathcal{Y}} \varphi \Sigma_n : E\, dy - \int_{\mathcal{Y}} (\Sigma_n)'' : (\bar{u} \odot \nabla_y \varphi)\, dy - 2\int_{\mathcal{Y}} u_3 \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy \\
&\qquad + 2v' \cdot \int_{\mathcal{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathcal{Y}} \varphi(\Sigma_n)_{33}\, dy \Bigg] \\
&= \lim_n \Bigg[-\int_{\mathcal{Y}} \varphi \Sigma_n : E\, dy + \int_{\mathcal{Y}} \varphi \operatorname{div}_y(\Sigma_n)'' \cdot \bar{u}\, dy + \int_{\mathcal{Y}} \varphi(\Sigma_n)'' : dE_y\bar{u} + 2\int_{\mathcal{Y}} \varphi u_3 \operatorname{div}_y \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy \\
&\qquad + 2\int_{\mathcal{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y u_3 + 2v' \cdot \int_{\mathcal{Y}} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathcal{Y}} \varphi(\Sigma_n)_{33}\, dy \Bigg] \\
&= \lim_n \Bigg[\int_{\mathcal{Y}} \varphi \operatorname{div}_y(\Sigma_n) \cdot \begin{pmatrix} \bar{u} \\ u_3 \end{pmatrix} dy + \int_{\mathcal{Y}} \varphi \Sigma_n : dP \Bigg] \\
&= \lim_n \int_{\mathcal{Y}} \varphi(\Sigma_n)_{\mathrm{dev}} : dP.
\end{aligned}$$

In view of the L^∞-bound on $\{(\Sigma_n)_{\mathrm{dev}}\}$, passing to the limit yields

$$|[\Sigma_{\mathrm{dev}} : P]|(\varphi) \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{sym}})} \int_{\mathcal{Y}} |\varphi|\, d|P|,$$

from which the claims follow. □

The following proposition characterizes $[\Sigma_{\mathrm{dev}} : P]$ on the interface. Before the statement we recall Remark 3.1

Proposition 5.15. *Let $\Sigma \in \mathcal{K}_\infty$. Assume that $\mathcal{Y}$ is a geometrically admissible multi-phase torus. Then, for $\mathcal{H}^1$-a.e. $y \in \partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j$,*

$$[\Sigma\iota(\nu^i)]^\perp_{\iota(\nu^i)}(y) \in ((K_i \cap K_j)\iota(\nu^i))^\perp_{\iota(\nu^i)}. \tag{5.16}$$

Furthermore, if $(\bar{u}, u_3, v, E, P) \in \mathcal{A}_\infty$*, then for every* $i \neq j$*,*

$$[\Sigma_{\text{dev}} : P]\lfloor\Gamma_{ij} = \left([\Sigma'' \nu^i]^{\perp}_{\nu^i} \cdot (\bar{u}^i - \bar{u}^j) + 2\left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu^i \right)(u_3^i - u_3^j) \right) \mathcal{H}^1 \lfloor \Gamma_{ij}, \tag{5.17}$$

where $\bar{u}^i, u_3^i$ *and* $\bar{u}^j, u_3^j$ *are the traces on* Γ_{ij} *of the restrictions of* $\bar{u}, u_3$ *to* $\mathcal{Y}_i$ *and* $\mathcal{Y}_j$ *respectively, assuming that* ν^i *points from* $\mathcal{Y}_j$ *to* $\mathcal{Y}_i$*.*

Proof. To prove (5.17), let $\varphi \in C^1(\mathcal{Y})$ be such that its support is contained in $\mathcal{Y}_i \cup \mathcal{Y}_j \cup \Gamma_{ij}$. Let $\mathcal{U} \subset\subset \mathcal{Y}$ be a compact set containing $\operatorname{supp}(\varphi)$, and consider any smooth approximating sequence $\{\Sigma_n\} \subset C^\infty(\mathcal{U}; \mathbb{M}^{3\times3}_{\text{sym}})$ such that

$$\Sigma_n \to \Sigma \quad \text{strongly in } L^2(\mathcal{U}; \mathbb{M}^{3\times3}_{\text{sym}}), \tag{5.18}$$

$$\operatorname{div}_y \Sigma_n = 0 \quad \text{in } \mathcal{U}, \tag{5.19}$$

$$\|(\Sigma_n)_{\text{dev}}\|_{L^\infty(\mathcal{U};\mathbb{M}^{3\times3}_{\text{dev}})} \leq \|\Sigma_{\text{dev}}\|_{L^\infty(\mathcal{U};\mathbb{M}^{3\times3}_{\text{dev}})}. \tag{5.20}$$

Note that $((\Sigma_n)'' \nu^i)^{\perp}_{\nu^i} = ((\Sigma_n)''_{\text{dev}} \nu^i)^{\perp}_{\nu^i}$ and

$$((\Sigma_n)''_{\text{dev}} \nu^i)^{\perp}_{\nu^i} \xrightharpoonup{*} [\Sigma''_{\text{dev}} \nu^i]^{\perp}_{\nu^i} \quad \text{weakly* in } L^\infty(\Gamma_{ij}; \mathbb{R}^2).$$

Since $\varphi\bar{u} \in BD(\mathcal{Y})$ and $\varphi u_3 \in BD(\mathcal{Y})$, with

$$E_y(\varphi\bar{u}) = \varphi E_y \bar{u} + \bar{u} \odot \nabla_y \varphi,$$
$$D_y(\varphi u_3) = \varphi D_y u_3 + u_3 \nabla_y \varphi,$$

we compute using (5.14)

$$\begin{aligned}
[\Sigma_{\text{dev}} : P](\varphi) = \lim_n \Bigg[&- \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \Sigma_n : E\, dy - \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} (\Sigma_n)'' : (\bar{u} \odot \nabla_y \varphi)\, dy - 2 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} u_3 \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nabla_y \varphi\, dy \\
&+ 2v' \cdot \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi (\Sigma_n)_{33}\, dy \Bigg] \\
= \lim_n \Bigg[&- \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \Sigma_n : E\, dy - \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} (\Sigma_n)'' : dE_y(\varphi\bar{u}) + \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi (\Sigma_n)'' : E_y \bar{u} \\
&- 2 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y(\varphi u_3) + 2 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y u_3 \\
&+ 2v' \cdot \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} dy + v_3 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi (\Sigma_n)_{33}\, dy \Bigg] \\
= \lim_n \Bigg[&- \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} (\Sigma_n)'' : dE_y(\varphi\bar{u}) - 2 \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot dD_y(\varphi u_3) + \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi \Sigma_n : dP \Bigg].
\end{aligned}$$

Owing to the assumption on $\operatorname{supp}(\varphi)$, we have that the only relevant part of the boundary of $\mathcal{Y}_i \cup \mathcal{Y}_j$ is Γ_{ij}. Thus, an integration by parts yields

$$[\Sigma_{\text{dev}} : P](\varphi) = \lim_n \Bigg[\int_{\Gamma_{ij}} \varphi ((\Sigma_n)'' \nu^i) \cdot (\bar{u}^i - \bar{u}^j)\, d\mathcal{H}^1 + 2 \int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} (\Sigma_n)_{13} \\ (\Sigma_n)_{23} \end{pmatrix} \cdot \nu^i \right)(u_3^i - u_3^j)\, d\mathcal{H}^1 + \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi (\Sigma_n)_{\text{dev}} : dP \Bigg].$$

Now

$$P \lfloor \Gamma_{ij} = \begin{pmatrix} E_y \bar{u} & D_y u_3 \\ (D_y u_3)^T & 0 \end{pmatrix} \lfloor \Gamma_{ij} = \begin{pmatrix} (\bar{u}^i - \bar{u}^j) \odot \nu^i & (u_3^i - u_3^j)\nu^i \\ (u_3^i - u_3^j)(\nu^i)^T & 0 \end{pmatrix} \mathcal{H}^1$$

and $\operatorname{tr} P = 0$ imply that $\bar{u}^i(y) - \bar{u}^j(y) \perp \nu^i(y)$ for $\mathcal{H}^1$-a.e. $y \in \Gamma_{ij}$. The above computation then yields

$$[\Sigma_{\text{dev}} : P](\varphi) = \int_{\Gamma_{ij}} \varphi [\Sigma'' \nu^i]^{\perp}_{\nu^i} \cdot (\bar{u}^i - \bar{u}^j)\, d\mathcal{H}^1 + 2 \int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu^i \right)(u_3^i - u_3^j)\, d\mathcal{H}^1 + \lim_n \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi (\Sigma_n)_{\text{dev}} : dP. \tag{5.21}$$

By defining $\lambda_n \in \mathcal{M}_b(\mathcal{Y}_i \cup \mathcal{Y}_j \cup \Gamma_{ij})$ as

$$\lambda_n(\varphi) := \int_{\mathcal{Y}_i \cup \mathcal{Y}_j} \varphi(\Sigma_n)_{\mathrm{dev}} : dP,$$

the L^∞-bound on $\{(\Sigma_n)_{\mathrm{dev}}\}$ ensures that it satisfies

$$|\lambda_n| \leq C|P|\lfloor(\mathcal{Y}_i \cup \mathcal{Y}_j),$$

and we infer from (5.21) that

$$\lambda_n \xrightharpoonup{*} \lambda \quad \text{weakly* in } \mathcal{M}_b(\mathcal{Y}_i \cup \mathcal{Y}_j \cup \Gamma_{ij})$$

for a suitable $\lambda \in \mathcal{M}_b(\mathcal{Y}_i \cup \mathcal{Y}_j \cup \Gamma_{ij})$ with

$$|\lambda| \leq C|P|\lfloor(\mathcal{Y}_i \cup \mathcal{Y}_j), \tag{5.22}$$

and

$$[\Sigma_{\mathrm{dev}} : P](\varphi) = \int_{\Gamma_{ij}} \varphi[\Sigma'' \nu^i]^{\perp}_{\iota(\nu^i)} \cdot (\bar{u}^i - \bar{u}^j)\, d\mathcal{H}^1 + 2 \int_{\Gamma_{ij}} \varphi \left(\begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \cdot \nu^i \right) (u_3^i - u_3^j)\, d\mathcal{H}^1 + \lambda(\varphi).$$

Since (5.22) implies $\lambda\lfloor\Gamma_{ij} = 0$, the result directly follows. To prove (5.16), we first notice that as a consequence of [27, Section 1.2] there holds $[\Sigma\iota(\nu^i)]^{\perp}_{\iota(\nu^i)} \in L^\infty(\Gamma)$. We locally approximate Σ at every point $y \in \partial\mathcal{Y}_i$ by dilation and convolution as in the proof of Proposition 5.9, see (5.13), so that the approximating sequence $\{\Sigma_n\}$ consequently satisfies (5.18)–(5.20) and also $\Sigma_n \in K_i$. Since we have that $[\Sigma_n\iota(\nu^i)]^{\perp}_{\iota(\nu^i)} \xrightharpoonup{*} [\Sigma\iota(\nu^i)]^{\perp}_{\iota(\nu^i)}$ the claim follows from the convexity of K_i. □

The following proposition is analogous to Proposition 5.9 and will also be used in Section 5.4 to prove the main result of this section.

Proposition 5.16. *Let $\Sigma \in \mathcal{K}_\infty$ and $(\bar{u}, u_3, v, E, P) \in \mathcal{A}_\infty$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus and the assumption on the ordering of the phases is satisfied, we have*

$$H\Big(y, \frac{dP}{d|P|}\Big)|P| \geq [\Sigma_{\mathrm{dev}} : P] \quad \textit{in } \mathcal{M}_b(\mathcal{Y}).$$

Proof. To establish the stated inequality, we consider the behavior of the measures on each phase $\mathcal{Y}_i$ and interface Γ_{ij} respectively. First, consider an open set $\mathcal{U}$ such that $\overline{\mathcal{U}} \subset \mathcal{Y}_i$ for some i. Regularizing by convolution, we obtain a sequence $\Sigma_n \in C^\infty(\mathcal{U}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that

$$\begin{aligned} \Sigma_n &\to \Sigma && \text{strongly in } L^2(\mathcal{U}; \mathbb{M}^{3\times3}_{\mathrm{sym}}), \\ \operatorname{div}_y \Sigma_n &= 0 && \text{in } \mathcal{U}. \end{aligned}$$

Furthermore, $(\Sigma_n(y))_{\mathrm{dev}} \in K_i$ for every $y \in \mathcal{U}$. As a consequence, for $|P|$-a.e. $y \in \mathcal{U}$ we have

$$H\Big(y, \frac{dP}{d|P|}\Big) = H_i\Big(\frac{dP}{d|P|}\Big) \geq \Sigma_n : \frac{dP}{d|P|}.$$

Thus for every $\varphi \in C(\mathcal{U})$ such that $\varphi \geq 0$, we obtain

$$\int_{\mathcal{U}} \varphi H\Big(y, \frac{dP}{d|P|}\Big)\, d|P| \geq \int_{\mathcal{U}} \varphi \Sigma_n : \frac{dP}{d|P|}\, d|P| = \int_{\mathcal{U}} \varphi\, d[\Sigma_n : P].$$

Since Σ_n is smooth, we conclude that

$$[\Sigma_n : \bar{P}] \xrightharpoonup{*} [\Sigma : \bar{P}] \quad \text{weakly* in } \mathcal{M}_b(\mathcal{U}).$$

Passing to the limit we have

$$\int_{\mathcal{U}} \varphi H\Big(y, \frac{dP}{d|P|}\Big)\, d|P| \geq \int_{\mathcal{U}} \varphi\, d[\Sigma : P].$$

The inequality on the phase $\mathcal{Y}_i$ now follows by considering a collection of open subsets that increases to $\mathcal{Y}_i$. Next, for every $i \neq j$,

$$H\Big(y, \frac{dP}{d|P|}\Big)|P|\lfloor\Gamma_{ij} = \min\{H_i, H_j\}\left(\begin{pmatrix} (\bar{u}^j - \bar{u}^i) \odot \nu & (u_3^j - u_3^i)\nu \\ (u_3^j - u_3^i)\nu^T & 0 \end{pmatrix}\right)\mathcal{H}^1\lfloor\Gamma_{ij},$$

where $\bar{u}^i$, u_3^i and $\bar{u}^j$, u_3^j are the traces on Γ_{ij} of the restrictions of $\bar{u}$, u_3 to $\mathcal{Y}_i$ and $\mathcal{Y}_j$ respectively, assuming that ν points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$. The claim then directly follows in view of Proposition 5.15. □

5.2 Disintegration of admissible configurations

Let $\widetilde{\omega} \subseteq \mathbb{R}^2$ be an open and bounded set such that $\omega \subset \widetilde{\omega}$ and $\widetilde{\omega} \cap \partial\omega = \gamma_D$. We also denote by $\widetilde{\Omega} = \widetilde{\omega} \times I$ the associated reference domain. In order to make sense of the duality between the two-scale limits of stresses and plastic strains, we will need to disintegrate the two-scale limits of the kinematically admissible fields in such a way to obtain elements of $\mathcal{A}_0$ and $\mathcal{A}_\infty$, respectively.

5.2.1 Case $\gamma = 0$

Definition 5.17. Let $w \in H^1(\widetilde{\Omega}; \mathbb{R}^3) \cap \mathrm{KL}(\widetilde{\Omega})$. We define the class $\mathcal{A}_0^{\mathrm{hom}}(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u, E, P) with

$$u \in \mathrm{KL}(\widetilde{\Omega}), \quad E \in L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}), \quad P \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$$

such that

$$u = w, \quad E = Ew, \quad P = 0 \quad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y},$$

and also such that there exist $\mu \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$ with

$$Eu \otimes \mathcal{L}^2_y + E_y\mu - x_3 D^2_y\kappa = E\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P \quad \text{in } \widetilde{\Omega} \times \mathcal{Y}. \tag{5.23}$$

The following lemma gives the disintegration result that will be used in the proof of Proposition 5.30.

Lemma 5.18. *Let $(u, E, P) \in \mathcal{A}_0^{\mathrm{hom}}(w)$ with the associated $\mu \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$, let $\bar{u} \in \mathrm{BD}(\widetilde{\omega})$ and $u_3 \in \mathrm{BH}(\widetilde{\omega})$ be the Kirchhoff–Love components of u. Then there exists $\eta \in \mathcal{M}_b^+(\widetilde{\omega})$ such that the following disintegrations hold true:*

$$Eu \otimes \mathcal{L}^2_y = (A_1(x') + x_3 A_2(x'))\eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y, \tag{5.24}$$

$$E\mathcal{L}^3_x \otimes \mathcal{L}^2_y = C(x')E(x,y)\eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y \tag{5.25}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_{x'}. \tag{5.26}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and $C : \widetilde{\omega} \to [0, +\infty]$ are respective Radon–Nikodym derivatives of $E\bar{u}$, $-D^2 u_3$ and $\mathcal{L}^2_{x'}$ with respect to η, $E(x, y)$ is a Borel representative of E, and $P_{x'} \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for η-a.e. $x' \in \widetilde{\omega}$. Furthermore, we can choose Borel maps $(x', y) \in \widetilde{\omega} \times \mathcal{Y} \mapsto \mu_{x'}(y) \in \mathbb{R}^2$ and $(x', y) \in \widetilde{\omega} \times \mathcal{Y} \mapsto \kappa_{x'}(y) \in \mathbb{R}$ such that, for η-a.e. $x' \in \widetilde{\omega}$,

$$\mu = \mu_{x'}(y)\eta \otimes \mathcal{L}^2_y, \quad E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_{x'}, \tag{5.27}$$

$$\kappa = \kappa_{x'}(y)\eta \otimes \mathcal{L}^2_y, \quad D^2_y\kappa = \eta \overset{\text{gen.}}{\otimes} D^2_y\kappa_{x'}, \tag{5.28}$$

where $\mu_{x'} \in \mathrm{BD}(\mathcal{Y})$, $\int_{\mathcal{Y}} \mu_{x'}(y)\,dy = 0$ and $\kappa_{x'} \in \mathrm{BH}(\mathcal{Y})$, $\int_{\mathcal{Y}} \kappa_{x'}(y)\,dy = 0$.

Proof. The proof is a consequence of Proposition 4.6 and follows along the lines of [7, Lemma 5.8]. □

Remark 5.19. From the above disintegration, we have that, for η-a.e. $x' \in \widetilde{\omega}$,

$$E_y\mu_{x'} - x_3 D^2_y\kappa_{x'} = [C(x')E(x,y) - (A_1(x') + x_3 A_2(x'))]\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P_{x'} \quad \text{in } I \times \mathcal{Y}.$$

Thus, the quadruplet

$$(\mu_{x'}, \kappa_{x'}, [C(x')E(x,y) - (A_1(x') + x_3 A_2(x'))], P_{x'})$$

is an element of $\mathcal{A}_0$.

5.2.2 Case $\gamma = +\infty$

Definition 5.20. Let $w \in H^1(\widetilde{\Omega};\mathbb{R}^3) \cap \mathrm{KL}(\widetilde{\Omega})$. We define the class $\mathcal{A}^{\mathrm{hom}}_{\infty}(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u, E, P) with

$$u \in \mathrm{KL}(\widetilde{\Omega}), \quad E \in L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}), \quad P \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{dev}})$$

such that

$$u = w, \quad E = Ew, \quad P = 0 \quad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y},$$

and also such that there exist $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$ with

$$Eu \otimes \mathcal{L}^2_y + \begin{pmatrix} E_y\mu & \zeta' + D_y\kappa \\ (\zeta' + D_y\kappa)^T & \zeta_3 \end{pmatrix} = E\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P \quad \text{in } \widetilde{\Omega} \times \mathcal{Y}. \tag{5.29}$$

The following lemma provides a disintegration result in this regime and will be instrumental for Proposition 5.32.

Lemma 5.21. *Let $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}_{\infty}(w)$ with the associated $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathcal{M}_b(\Omega;\mathbb{R}^3)$, let $\bar{u} \in \mathrm{BD}(\widetilde{\omega})$ and $u_3 \in \mathrm{BH}(\widetilde{\omega})$ be the Kirchhoff–Love components of u. Then there exists $\eta \in \mathcal{M}^+_b(\widetilde{\Omega})$ such that the following disintegrations hold true:*

$$Eu \otimes \mathcal{L}^2_y = (A_1(x') + x_3A_2(x'))\eta \otimes \mathcal{L}^2_y, \tag{5.30}$$

$$\zeta \otimes \mathcal{L}^2_y = z(x)\eta \otimes \mathcal{L}^2_y, \tag{5.31}$$

$$E\mathcal{L}^3_x \otimes \mathcal{L}^2_y = C(x)E(x,y)\eta \otimes \mathcal{L}^2_y \tag{5.32}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_x. \tag{5.33}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, $z : \widetilde{\omega} \to \mathbb{R}^3$ and $C : \widetilde{\Omega} \to [0,+\infty]$ are the respective Radon–Nikodym derivatives of $E\bar{u}$, $-D^2u_3$, ζ and $\mathcal{L}^3_x$ with respect to η, $E(x,y)$ is a Borel representative of E, and $P_x \in \mathcal{M}_b(\mathcal{Y};\mathbb{M}^{3\times 3}_{\mathrm{dev}})$ for η-a.e. $x \in \widetilde{\Omega}$. Furthermore, we can choose Borel maps $(x,y) \in \widetilde{\Omega} \times \mathcal{Y} \mapsto \mu_x(y) \in \mathbb{R}^2$ and $(x,y) \in \widetilde{\Omega} \times \mathcal{Y} \mapsto \kappa_x(y) \in \mathbb{R}$ such that, for η-a.e. $x \in \widetilde{\Omega}$,

$$\mu = \mu_x(y)\eta \otimes \mathcal{L}^2_y, \quad E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_x, \tag{5.34}$$

$$\kappa = \kappa_x(y)\eta \otimes \mathcal{L}^2_y, \quad D^2_y\kappa = \eta \overset{\text{gen.}}{\otimes} D^2_y\kappa_x, \tag{5.35}$$

where $\mu_x \in \mathrm{BD}(\mathcal{Y})$, $\int_{\mathcal{Y}} \mu_x(y)\,dy = 0$ and $\kappa_x \in \mathrm{BV}(\mathcal{Y})$, $\int_{\mathcal{Y}} \kappa_x(y)\,dy = 0$.

Proof. The proof builds upon Proposition 4.9 and follows along [7, Lemma 5.8]. □

Remark 5.22. From the above disintegration, we have that, for η-a.e. $x \in \widetilde{\Omega}$,

$$\begin{pmatrix} E_y\mu_x & z' + D_y\kappa_x \\ (z' + D_y\kappa_x)^T & z_3 \end{pmatrix} = \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right] \mathcal{L}^2_y + P_x \quad \text{in } \mathcal{Y}.$$

Thus, the quintuplet

$$\left(\mu_x, \kappa_x, z, \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \right], P_x \right)$$

is an element of $\mathcal{A}_\infty$.

5.3 Admissible stress configurations and approximations

For every $e^h \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})$ we define $\sigma^h(x) := \mathbb{C}(\frac{x'}{\varepsilon_h})\Lambda_h e^h(x)$. We introduce the set of stresses for the rescaled h problems:

$$\mathcal{K}_h = \left\{ \sigma^h \in L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}) : \operatorname{div}_h \sigma^h = 0 \text{ in } \Omega,\ \sigma^h\nu = 0 \text{ in } \partial\Omega \setminus \overline{\Gamma}_D,\ \sigma^h_{\mathrm{dev}}(x', x_3) \in K\Big(\frac{x'}{\varepsilon_h}\Big) \text{ for a.e. } x' \in \omega,\ x_3 \in I \right\}.$$

We recall some properties of the limiting stress that can be found in [21].

If we consider the weak limit $\sigma \in L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ of the sequence $\sigma^h \in \mathcal{K}_h$ as $h \to 0$, then $\sigma_{i3} = 0$ for $i = 1, 2, 3$. Furthermore, since the uniform boundedness of the sets $K(y)$ implies that the deviatoric part of the weak limit, i.e., $\sigma_{\mathrm{dev}} = \sigma - \frac{1}{3}\operatorname{tr}\sigma I_{3\times3}$, is bounded in $L^\infty(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}})$, we have that the components $\sigma_{\alpha\beta}$ are all bounded in $L^\infty(\Omega)$. Lastly,

$$\operatorname{div}_{x'} \bar{\sigma} = 0 \text{ in } \omega, \quad \text{and} \quad \operatorname{div}_{x'} \operatorname{div}_{x'} \widehat{\sigma} = 0 \text{ in } \omega.$$

In the following, we further characterize the sets of two-scale limits of sequences of elastic stresses $\{\sigma^h\}$, depending on the regime.

5.3.1 Case $\gamma = 0$

We first introduce the set of limiting two-scale stress.

Definition 5.23. The set $\mathcal{K}_0^{\mathrm{hom}}$ is the set of all elements $\Sigma \in L^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ satisfying:
(i) $\Sigma_{i3}(x, y) = 0$ for $i = 1, 2, 3$,
(ii) $\Sigma_{\mathrm{dev}}(x, y) \in K(y)$ for $\mathcal{L}^3_x \otimes \mathcal{L}^2_y$-a.e. $(x, y) \in \Omega \times \mathcal{Y}$,
(iii) $\operatorname{div}_y \bar{\Sigma}(x', \cdot) = 0$ in $\mathcal{Y}$ for a.e. $x' \in \omega$,
(iv) $\operatorname{div}_y \operatorname{div}_y \widehat{\Sigma}(x', \cdot) = 0$ in $\mathcal{Y}$ for a.e. $x' \in \omega$,
(v) $\operatorname{div}_{x'} \bar{\sigma} = 0$ in ω,
(vi) $\operatorname{div}_{x'} \operatorname{div}_{x'} \widehat{\sigma} = 0$ in ω,
where $\bar{\Sigma}, \widehat{\Sigma} \in L^\infty(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of Σ, $\sigma := \int_{\mathcal{Y}} \Sigma(\cdot, y)\, dy$, and $\bar{\sigma}, \widehat{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ.

The following proposition motivates the above definition.

Proposition 5.24. *Let $\{\sigma^h\}$ be a bounded family in $L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that $\sigma^h \in \mathcal{K}_h$ and*

$$\sigma^h \xrightharpoonup{2} \Sigma \quad \textit{two-scale weakly in } L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Then $\Sigma \in \mathcal{K}_0^{\mathrm{hom}}$.

Proof. Properties (v) and (vi) follow from Section 5.3.

To prove (i), let $\psi \in C_c^\infty(\omega; C^\infty(\bar{I} \times \mathcal{Y}; \mathbb{R}^3))$ and consider the test function $h\psi(x, \frac{x'}{\varepsilon_h})$. We find that

$$\nabla_h\Big(h\psi\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big) = \Big[\, h\nabla_{x'}\psi\Big(x, \frac{x'}{\varepsilon_h}\Big) + \frac{h}{\varepsilon_h}\nabla_y\psi\Big(x, \frac{x'}{\varepsilon_h}\Big) \,\Big|\, \partial_{x_3}\psi\Big(x, \frac{x'}{\varepsilon_h}\Big) \,\Big]$$

converges strongly in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3})$. Hence, taking such a test function in $\operatorname{div}_h \sigma^h = 0$ and passing to the limit, we get

$$\int_{\Omega\times\mathcal{Y}} \Sigma(x, y) : \begin{pmatrix} 0 & 0 & \partial_{x_3}\psi_1(x, y) \\ 0 & 0 & \partial_{x_3}\psi_2(x, y) \\ \partial_{x_3}\psi_1(x, y) & \partial_{x_3}\psi_2(x, y) & \partial_{x_3}\psi_3(x, y) \end{pmatrix} dx\, dy = 0,$$

which is sufficient to conclude that $\Sigma_{i3}(x, y) = 0$ for $i = 1, 2, 3$.

To prove (ii), we define

$$\Sigma^h(x, y) = \sum_{i \in I_{\varepsilon_h}(\widetilde{\omega})} \mathbb{1}_{Q^i_{\varepsilon_h}}(x')\sigma^h(\varepsilon_h i + \varepsilon_h \mathcal{I}(y), x_3), \tag{5.36}$$

and consider the set

$$S = \{\Xi \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}) : \Xi_{\mathrm{dev}}(x, y) \in K(y) \text{ for } \mathcal{L}^3_x \otimes \mathcal{L}^2_y\text{-a.e. } (x, y) \in \Omega \times \mathcal{Y}\}.$$

The construction of Σ^h from $\sigma^h \in \mathcal{K}_h$ ensures that $\Sigma^h \in S$ and that $\Sigma^h \rightharpoonup \Sigma$ weakly in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$. Properties (i) and (ii) imply that $\Sigma \in L^\infty$.

Since compactness of $K(y)$ implies that S is convex and weakly closed in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$, we have that $\Sigma \in S$, which concludes the proof.

Finally, to prove (iii) and (iv), let $\phi \in C_c^\infty(\omega; C^\infty(\mathcal{Y}; \mathbb{R}^3))$ and consider the test function

$$\varphi(x) = \varepsilon_h \begin{pmatrix} \phi_1(x', \frac{x'}{\varepsilon_h}) \\ \phi_2(x', \frac{x'}{\varepsilon_h}) \\ 0 \end{pmatrix} + \varepsilon_h{}^2 \begin{pmatrix} -x_3 \partial_{x_1} \phi_3(x', \frac{x'}{\varepsilon_h}) - \frac{x_3}{\varepsilon_h} \partial_{y_1} \phi_3(x', \frac{x'}{\varepsilon_h}) \\ -x_3 \partial_{x_2} \phi_3(x', \frac{x'}{\varepsilon_h}) - \frac{x_3}{\varepsilon_h} \partial_{y_2} \phi_3(x', \frac{x'}{\varepsilon_h}) \\ \frac{1}{h} \phi_3(x', \frac{x'}{\varepsilon_h}) \end{pmatrix}.$$

By a direct computation we infer

$$E_h \varphi(x) \to \begin{pmatrix} E_y \phi'(x', y) - x_3 D_y^2 \phi_3(x', y) & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 0 \end{pmatrix} \quad \text{strongly in } L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Hence, taking such a test function in $\operatorname{div}_h \sigma^h = 0$ and passing to the limit, we get

$$\int_{\Omega\times\mathcal{Y}} \Sigma(x, y) : \begin{pmatrix} E_y \phi' - x_3 D_y^2 \phi_3 & 0 \\ 0 & 0 \end{pmatrix} dx\, dy = 0.$$

Suppose now that $\phi(x', y) = \psi^{(1)}(x')\psi^{(2)}(y)$ for $\psi^{(1)} \in C_c^\infty(\omega)$ and $\psi^{(2)} \in C^\infty(\mathcal{Y}; \mathbb{R}^3)$. Then

$$\int_\omega \psi^{(1)}(x') \bigg(\int_{I\times\mathcal{Y}} \Sigma(x, y) : \begin{pmatrix} E_y(\psi^{(2)})'(y) - x_3 D_y^2 \psi_3^{(2)}(y) & 0 \\ 0 & 0 \end{pmatrix} dx_3\, dy \bigg) dx' = 0,$$

from which we deduce that, for a.e. $x' \in \omega$,

$$\begin{aligned} 0 &= \int_{I\times\mathcal{Y}} \Sigma(x, y) : \begin{pmatrix} E_y(\psi^{(2)})'(y) - x_3 D_y^2 \psi_3^{(2)}(y) & 0 \\ 0 & 0 \end{pmatrix} dx_3\, dy \\ &= \int_{\mathcal{Y}} \bar{\Sigma}(x', y) : E_y(\psi^{(2)})'(y)\, dy - \frac{1}{12} \int_{\mathcal{Y}} \hat{\Sigma}(x', y) : D_y^2 \psi_3^{(2)}(y)\, dy \\ &= -\int_{\mathcal{Y}} \operatorname{div}_y \bar{\Sigma}(x', y) \cdot (\psi^{(2)})'(y)\, dy - \frac{1}{12} \int_{\mathcal{Y}} \operatorname{div}_y \operatorname{div}_y \hat{\Sigma}(x', y) \cdot \psi_3^{(2)}(y)\, dy. \end{aligned}$$

Thus, $\operatorname{div}_y \bar{\Sigma}(x', \cdot) = 0$ in $\mathcal{Y}$ and $\operatorname{div}_y \operatorname{div}_y \hat{\Sigma}(x', \cdot) = 0$ in $\mathcal{Y}$. □

The following lemma approximates the limiting stresses with respect to the macroscopic variable and will be used in Proposition 5.30. It is proved under the assumption that the domain is star-shaped.

Lemma 5.25. *Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathcal{K}_0^{\mathrm{hom}}$. Then there exists a sequence $\Sigma_n \in L^\infty(\mathbb{R}^2 \times I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that the following holds:*
(a) *$\Sigma_n \in C^\infty(\mathbb{R}^2; L^\infty(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^p(\omega \times I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ for $1 \le p < +\infty$,*
(b) *$(\Sigma_n)_{i3}(x, y) = 0$ for $i = 1, 2, 3$,*
(c) *$(\Sigma_n(x, y))_{\mathrm{dev}} \in K(y)$ for every $x' \in \mathbb{R}^2$ and $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3, y) \in I \times \mathcal{Y}$,*
(d) *$\operatorname{div}_y \bar{\Sigma}_n(x', \cdot) = 0$ in $\mathcal{Y}$ for every $x' \in \omega$,*
(e) *$\operatorname{div}_y \operatorname{div}_y \hat{\Sigma}_n(x', \cdot) = 0$ in $\mathcal{Y}$ for every $x' \in \omega$,*
where $\bar{\Sigma}_n, \hat{\Sigma}_n \in L^\infty(\omega \times \mathcal{Y}; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of Σ_n. Further, if we set

$$\sigma_n(x) := \int_{\mathcal{Y}} \Sigma_n(x, y)\, dy,$$

and $\bar{\sigma}_n, \hat{\sigma}_n \in L^\infty(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:
(f) *$\sigma_n \in C^\infty(\mathbb{R}^2 \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\sigma_n \to \sigma$ strongly in $L^p(\omega \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ for $1 \le p < +\infty$,*
(g) *$\operatorname{div}_{x'} \bar{\sigma}_n = 0$ in ω,*
(h) *$\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}_n = 0$ in ω.*

Proof. The approximation is done by dilation and convolution and is analogous to [7, Lemma 5.13]. □

5.3.2 Case $\gamma = +\infty$

In this regime, the set of limiting two-scale stresses is defined as follows.

Definition 5.26. The set $\mathcal{K}_\infty^{\mathrm{hom}}$ is the set of all elements $\Sigma \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ satisfying:
(i) $\operatorname{div}_y \Sigma(x,\cdot) = 0$ in $\mathcal{Y}$ for a.e. $x \in \Omega$,
(ii) $\Sigma_{\mathrm{dev}}(x,y) \in K(y)$ for $\mathcal{L}^3_x \otimes \mathcal{L}^2_y$-a.e. $(x,y) \in \Omega \times \mathcal{Y}$,
(iii) $\sigma_{i3}(x) = 0$ for $i = 1,2,3$,
(iv) $\operatorname{div}_{x'} \bar{\sigma} = 0$ in ω,
(v) $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$ in ω,
where $\sigma := \int_{\mathcal{Y}} \Sigma(\cdot, y)\, dy$, and $\bar{\sigma}, \hat{\sigma} \in L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ.

The previous definition is motivated by the following.

Proposition 5.27. *Let $\{\sigma^h\}$ be a bounded family in $L^2(\Omega; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that $\sigma^h \in \mathcal{K}_h$ and*

$$\sigma^h \xrightharpoonup{2} \Sigma \quad \textit{two-scale weakly in } L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Then $\Sigma \in \mathcal{K}_\infty^{\mathrm{hom}}$.

Proof. Properties (iii), (iv) and (v) follow in view of Section 5.3. To prove (i), we consider the test function $\varepsilon_h \phi(x, \frac{x'}{\varepsilon_h})$ for $\phi \in C_c^\infty(\omega; C^\infty(\bar{I} \times \mathcal{Y}; \mathbb{R}^3))$. We see that

$$\nabla_h\Big(\varepsilon_h \phi\Big(x, \frac{x'}{\varepsilon_h}\Big)\Big) = \Big[\ \varepsilon_h \nabla_{x'} \phi\Big(x, \frac{x'}{\varepsilon_h}\Big) + \nabla_y \phi\Big(x, \frac{x'}{\varepsilon_h}\Big) \ \Big|\ \frac{\varepsilon_h}{h} \partial_{x_3} \phi\Big(x, \frac{x'}{\varepsilon_h}\Big)\ \Big]$$

converges strongly in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3})$. Hence, taking such a test function in $\operatorname{div}_h \sigma^h = 0$ and passing to the limit, we get

$$\int_{\Omega\times\mathcal{Y}} \Sigma(x,y) : E_y \phi(x,y)\, dx\, dy = 0.$$

Suppose now that $\phi(x,y) = \psi^{(1)}(x)\psi^{(2)}(y)$ for $\psi^{(1)} \in C_c^\infty(\omega; C^\infty(\bar{I}))$ and $\psi^{(2)} \in C^\infty(\mathcal{Y}; \mathbb{R}^3)$. Then

$$\int_\Omega \psi^{(1)}(x)\Big(\int_{\mathcal{Y}} \Sigma(x,y) : E_y \psi^{(2)}(y)\, dy\Big)\, dx = 0,$$

from which we can deduce that $\operatorname{div}_y \Sigma(x,\cdot) = 0$ in $\mathcal{Y}$ for a.e. $x \in \Omega$.

To conclude the proof, it remains to show the stress constraint $\Sigma_{\mathrm{dev}}(x,y) \in K(y)$ for $\mathcal{L}^3_x \otimes \mathcal{L}^2_y$-a.e. $(x,y) \in \Omega \times \mathcal{Y}$. To do this, we can define the approximating sequence (5.36) and argue as in the proof of Proposition 5.24. □

The following lemma is analogous to Lemma 5.25.

Lemma 5.28. *Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathcal{K}_\infty^{\mathrm{hom}}$. Then, there exists a sequence $\Sigma_n \in L^2(\mathbb{R}^2 \times I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ such that the following holds:*
(a) *$\Sigma_n \in C^\infty(\mathbb{R}^3; L^2(\mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^2(\omega \times I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$,*
(b) *$\operatorname{div}_y \Sigma_n(x,\cdot) = 0$ on $\mathcal{Y}$ for every $x \in \mathbb{R}^3$,*
(c) *$(\Sigma_n(x,y))_{\mathrm{dev}} \in K(y)$ for every $x \in \mathbb{R}^3$ and $\mathcal{L}^2_y$-a.e. $y \in \mathcal{Y}$.*
Further, if we set

$$\sigma_n(x) := \int_{\mathcal{Y}} \Sigma_n(x,y)\, dy,$$

and $\bar{\sigma}_n, \hat{\sigma}_n \in L^2(\omega; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:
(d) *$\sigma_n \in C^\infty(\mathbb{R}^2 \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\sigma_n \to \sigma$ strongly in $L^2(\omega \times I; \mathbb{M}^{3\times3}_{\mathrm{sym}})$,*
(e) *$\operatorname{div}_{x'} \bar{\sigma}_n = 0$ in ω,*
(f) *$\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}_n = 0$ in ω.*

Proof. The proof is again analogous to [7, Lemma 5.13]. The only difference is that the convolution and dilation used to define Σ_n are taken in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. □

5.4 The principle of maximum plastic work

We introduce the following functionals: Let $\gamma \in \{0, +\infty\}$. For $(u, E, P) \in \mathcal{A}_\gamma^{\mathrm{hom}}(w)$ we define

$$\mathcal{Q}_0^{\mathrm{hom}}(E) := \int_{\Omega\times\mathcal{Y}} Q_r(y, E)\,dx\,dy, \quad \mathcal{Q}_{+\infty}^{\mathrm{hom}}(E) := \int_{\Omega\times\mathcal{Y}} Q(y, E)\,dx\,dy \tag{5.37}$$

and

$$\mathcal{H}_0^{\mathrm{hom}}(P) := \int_{\overline{\Omega}\times\mathcal{Y}} H_r\Big(y, \frac{dP}{d|P|}\Big)\,d|P|, \quad \mathcal{H}_{+\infty}^{\mathrm{hom}}(P) := \int_{\overline{\Omega}\times\mathcal{Y}} H\Big(y, \frac{dP}{d|P|}\Big)\,d|P|. \tag{5.38}$$

The aim of this subsection is to prove the following inequality between two-scale dissipation and plastic work, which in turn will be essential to prove the global stability condition of two-scale quasistatic evolutions. It is used in Step 2 of the proof of Theorem 6.2 and its proof is a direct consequence of Theorem 5.31 for the case $\gamma = 0$, and of Theorem 5.33 for the case $\gamma = +\infty$.

Corollary 5.29. *Let $\gamma \in \{0, +\infty\}$. Then*

$$\mathcal{H}_\gamma^{\mathrm{hom}}(P) \geq -\int_{\Omega\times\mathcal{Y}} \Sigma : E\,dx\,dy + \int_\omega \bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_\omega \hat{\sigma} : D^2 w_3\,dx'$$

for every $\Sigma \in \mathcal{K}_\gamma^{\mathrm{hom}}$ and $(u, E, P) \in \mathcal{A}_\gamma^{\mathrm{hom}}(w)$.

The proof relies on the approximation argument given in Lemmas 5.25 and 5.28 and on two-scale duality, which can be established only for smooth stresses by disintegration and duality pairings between admissible stresses and plastic strains (given by (5.8) and (5.15)). The problem is that the measure η defined in Lemmas 5.18 and 5.21 can concentrate on the points where the stress (which is only in L^2) is not well-defined. The difference with respect to [26, Proposition 5.11] is that one can rely only on the approximation given by Lemmas 5.25 and 5.28, which are given for star-shaped domains. To prove the corresponding result for general domains we rely on the localization argument (see Step 2 of the proof of Proposition 5.30 and the proof of Theorem 5.31, as well as Proposition 5.32 and Theorem 5.33).

5.4.1 Case $\gamma = 0$

The following proposition defines the measure λ through two-scale stress-strain duality based on the approximation argument.

Proposition 5.30. *Let $\Sigma \in \mathcal{K}_0^{\mathrm{hom}}$ and $(u, E, P) \in \mathcal{A}_0^{\mathrm{hom}}(w)$ with the associated $\mu \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$. There exists an element $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$*

$$\begin{aligned}\langle \lambda, \varphi \rangle = &-\int_{\Omega\times\mathcal{Y}} \varphi(x')\Sigma : E\,dx\,dy + \int_\omega \varphi\bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_\omega \varphi\hat{\sigma} : D^2 w_3\,dx' - \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx' \\ &- \frac{1}{6}\int_\omega \hat{\sigma} : (\nabla(u_3 - w_3) \odot \nabla\varphi)\,dx' - \frac{1}{12}\int_\omega (u_3 - w_3)\hat{\sigma} : \nabla^2\varphi\,dx'.\end{aligned}$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega} \times \mathcal{Y}) = -\int_{\Omega\times\mathcal{Y}} \Sigma : E\,dx\,dy + \int_\omega \bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_\omega \hat{\sigma} : D^2 w_3\,dx'. \tag{5.39}$$

Proof. The proof is divided into two steps.

Step 1. Suppose that ω is star-shaped with respect to one of its points. Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathcal{Y}; \mathbb{M}_{\mathrm{sym}}^{3\times 3}))$ be the sequence given by Lemma 5.25. We define the sequence

$$\lambda_n := \eta \overset{\mathrm{gen.}}{\otimes} [\Sigma_n(x', \cdot) : P_{x'}] \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}),$$

where η is given by Lemma 5.18 and the duality $[\Sigma_n(x',\cdot):P_{x'}]$ is a well defined bounded measure on $I\times\mathcal{Y}$ for η-a.e. $x'\in\widetilde{\omega}$. Further, in view of Remark 5.19, (5.8) gives

$$\int_{I\times\mathcal{Y}}\psi\,d[\Sigma_n(x',\cdot):P_{x'}] = -\int_{I\times\mathcal{Y}}\psi(y)\Sigma_n(x,y):[C(x')E(x,y)-(A_1(x')+x_3A_2(x'))]\,dx_3\,dy$$
$$-\int_{\mathcal{Y}}\bar{\Sigma}_n(x',y):(\mu_{x'}(y)\odot\nabla_y\psi(y))\,dy+\frac{1}{6}\int_{\mathcal{Y}}\widehat{\Sigma}_n(x',y):(\nabla_y\kappa_{x'}(y)\odot\nabla_y\psi(y))\,dy$$
$$+\frac{1}{12}\int_{\mathcal{Y}}\kappa_{x'}(y)\widehat{\Sigma}_n(x',y):\nabla_y^2\psi(y)\,dy$$

for every $\psi\in C^2(\mathcal{Y})$, and

$$|[\Sigma_n(x',\cdot):P_{x'}]|\le\|\Sigma_n(x',\cdot)\|_{L^\infty(I\times\mathcal{Y};\mathbb{M}^{2\times2}_{\mathrm{sym}})}|P_{x'}|\le C|P_{x'}|,$$

where the last inequality stems from item (c) in Lemma 5.25. This in turn implies that

$$|\lambda_n| = \eta\overset{\text{gen.}}{\otimes}|[\Sigma_n(x',\cdot):P_{x'}]|\le C\eta\overset{\text{gen.}}{\otimes}|P_{x'}| = C|P|,$$

from which we conclude that is $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I}\supset I$ be an open set which compactly contains I. We extend these measures by zero on $\widetilde{\omega}\times\widetilde{I}\times\mathcal{Y}$. Let ξ be a smooth cut-off function with $\xi\equiv1$ on I, with support contained in $\widetilde{I}$. Finally, we consider a test function $\phi(x,y):=\varphi(x')\xi(x_3)$, for $\varphi\in C_c^\infty(\widetilde{\omega})$. Then, since $\nabla_y\phi(x,y)=0$ and $\nabla_y^2\phi(x,y)=0$, we have

$$\begin{aligned}\langle\lambda_n,\phi\rangle &= \int_{\widetilde{\omega}}\Bigg(\int_{I\times\mathcal{Y}}\phi(x,y)\,d[\Sigma_n(x',\cdot):P_{x'}]\Bigg)\,d\eta(x')\\ &= -\int_{\widetilde{\Omega}\times\mathcal{Y}}\varphi(x')\Sigma_n(x,y):[C(x')E(x,y)-(A_1(x')+x_3A_2(x'))]\,d(\eta\otimes\mathcal{L}^1_{x_3}\otimes\mathcal{L}^2_y)\\ &= -\int_{\widetilde{\Omega}\times\mathcal{Y}}\varphi(x')\Sigma_n(x,y):E(x,y)\,dx\,dy+\int_{\widetilde{\Omega}}\varphi(x')\sigma_n(x):(A_1(x')+x_3A_2(x'))\,d(\eta\otimes\mathcal{L}^1_{x_3})\\ &= -\int_{\widetilde{\Omega}\times\mathcal{Y}}\varphi(x')\Sigma_n(x,y):E(x,y)\,dx\,dy+\int_{\widetilde{\Omega}}\varphi(x')\sigma_n(x):dEu(x).\end{aligned}$$

Since $u\in\mathrm{KL}(\widetilde{\Omega})$, we have

$$\int_{\widetilde{\Omega}}\varphi(x')\sigma_n(x):dEu(x)=\int_{\widetilde{\omega}}\varphi(x')\bar{\sigma}_n(x'):dE\bar{u}(x')-\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\widehat{\sigma}_n(x'):dD^2u_3(x'),$$

where $\bar{u}\in\mathrm{BD}(\widetilde{\omega})$ and $u_3\in\mathrm{BH}(\widetilde{\omega})$ are the Kirchhoff–Love components of u. From the characterization given in Proposition 3.7, we can thus conclude that

$$\begin{aligned}\int_{\widetilde{\Omega}}\varphi(x')\sigma_n(x):dEu(x) &= \int_{\widetilde{\omega}}\varphi(x')\bar{\sigma}_n(x'):\bar{e}(x')\,dx'+\int_{\widetilde{\omega}}\varphi(x')\bar{\sigma}_n(x'):d\bar{p}(x')\\ &\quad+\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\widehat{\sigma}_n(x'):\widehat{e}(x')\,dx'+\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\widehat{\sigma}_n(x'):d\widehat{p}(x')\\ &= \int_{\widetilde{\omega}}\varphi(x')\bar{\sigma}_n(x'):\bar{e}(x')\,dx'+\int_{\widetilde{\omega}}\varphi(x')\,d[\bar{\sigma}_n:\bar{p}](x')\\ &\quad+\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\widehat{\sigma}_n(x'):\widehat{e}(x')\,dx'+\frac{1}{12}\int_{\widetilde{\omega}}\varphi(x')\,d[\widehat{\sigma}_n:\widehat{p}](x'),\end{aligned}$$

where in the last equality we used that $\bar{\sigma}_n$ and $\widehat{\sigma}_n$ are smooth functions. Notice that, since $\bar{p}\equiv0$ and $\widehat{p}\equiv0$ outside of $\omega\cup\gamma_D$, we have

$$\int_{\widetilde{\omega}}\varphi\,d[\bar{\sigma}_n:\bar{p}]=\int_{\omega\cup\gamma_D}\varphi\,d[\bar{\sigma}_n:\bar{p}],\qquad\int_{\widetilde{\omega}}\varphi\,d[\widehat{\sigma}_n:\widehat{p}]=\int_{\omega\cup\gamma_D}\varphi\,d[\widehat{\sigma}_n:\widehat{p}].$$

Furthermore, since $e = E = E\bar{w} - x_3 D^2 w_3$ on $\widetilde{\Omega} \setminus \Omega$, we can conclude that

$$\begin{aligned}\langle \lambda_n, \phi \rangle &= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\Sigma_n : E\,dx\,dy + \int_{\widetilde{\omega}} \varphi\bar{\sigma}_n : \bar{e}\,dx' + \frac{1}{12}\int_{\widetilde{\omega}} \varphi\widehat{\sigma}_n : \widehat{e}\,dx' + \int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12}\int_{\omega\cup\gamma_D} \varphi\,d[\widehat{\sigma}_n : \widehat{p}]\\ &= -\int_{\Omega\times\mathcal{Y}} \varphi(x')\Sigma_n : E\,dx\,dy + \int_{\omega} \varphi\bar{\sigma}_n : \bar{e}\,dx' + \frac{1}{12}\int_{\omega} \varphi\widehat{\sigma}_n : \widehat{e}\,dx' + \int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12}\int_{\omega\cup\gamma_D} \varphi\,d[\widehat{\sigma}_n : \widehat{p}].\end{aligned}$$

Taking into account that $\operatorname{div}_{x'} \bar{\sigma}_n = 0$ in ω, by integration by parts (see also [21, Proposition 7.2]) we have for every $\varphi \in C^1(\overline{\omega})$,

$$\int_{\omega\cup\gamma_D} \varphi\,d[\bar{\sigma}_n : \bar{p}] + \int_{\omega} \varphi\bar{\sigma}_n : (\bar{e} - E\bar{w})\,dx' + \int_{\omega} \bar{\sigma}_n : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx' = 0.$$

Likewise taking into account that $\operatorname{div}_{x'}\operatorname{div}_{x'} \widehat{\sigma}_n = 0$ in ω and $u_3 = w_3$ on γ_D, by integration by parts (see also [21, Proposition 7.6]), we have for every $\varphi \in C^2(\overline{\omega})$,

$$\int_{\omega\cup\gamma_D} \varphi\,d[\widehat{\sigma}_n : \widehat{p}] + \int_{\omega} \varphi\widehat{\sigma}_n : (\widehat{e} + D^2 w_3)\,dx' + 2\int_{\omega} \widehat{\sigma}_n : (\nabla(u_3 - w_3) \odot \nabla\varphi)\,dx' + \int_{\omega} (u_3 - w_3)\widehat{\sigma}_n : \nabla^2\varphi\,dx' = 0.$$

Let now $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ be such that (up to a subsequence)

$$\lambda_n \xrightharpoonup{*} \lambda \quad \text{weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}).$$

By items (a) and (f) in Lemma 5.25, we have in the limit

$$\begin{aligned}\langle \lambda, \phi \rangle &= \lim_n \langle \lambda_n, \phi \rangle\\ &= \lim_n \Big[-\int_{\Omega\times\mathcal{Y}} \varphi(x')\Sigma_n : E\,dx\,dy + \int_{\omega} \varphi\bar{\sigma}_n : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega} \varphi\widehat{\sigma}_n : D^2 w_3\,dx' - \int_{\omega} \bar{\sigma}_n : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx'\\ &\qquad - \frac{1}{6}\int_{\omega} \widehat{\sigma}_n : (\nabla(u_3 - w_3) \odot \nabla\varphi)\,dx' - \frac{1}{12}\int_{\omega} (u_3 - w_3)\widehat{\sigma}_n : \nabla^2\varphi\,dx' \Big]\\ &= -\int_{\Omega\times\mathcal{Y}} \varphi(x')\Sigma : E\,dx\,dy + \int_{\omega} \varphi\bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega} \varphi\widehat{\sigma} : D^2 w_3\,dx' - \int_{\omega} \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx'\\ &\qquad - \frac{1}{6}\int_{\omega} \widehat{\sigma} : (\nabla(u_3 - w_3) \odot \nabla\varphi)\,dx' - \frac{1}{12}\int_{\omega} (u_3 - w_3)\widehat{\sigma} : \nabla^2\varphi\,dx'.\end{aligned}$$

Taking $\varphi \nearrow \mathbb{1}_{\overline{\omega}}$, we deduce (5.39).

Step 2. If ω is not star-shaped, then since ω is a bounded C^2 domain (in particular, with Lipschitz boundary) by [9, Proposition 2.5.4] there exists a finite open covering $\{U_i\}$ of $\overline{\omega}$ such that $\omega \cap U_i$ is (strongly) star-shaped with Lipschitz boundary. Again, since the sets which are intersecting $\partial\omega$ are cylindrical up to a rotation, we can slightly change them such that they become C^2.

Let $\{\psi_i\}$ be a smooth partition of unity subordinate to the covering $\{U_i\}$, i.e., $\psi_i \in C^\infty(\overline{\omega})$, with $0 \le \psi_i \le 1$, such that $\operatorname{supp}(\psi_i) \subset U_i$ and $\sum_i \psi_i = 1$ on $\overline{\omega}$.

For each i, let

$$\Sigma^i(x, y) := \begin{cases} \Sigma(x, y) & \text{if } x' \in \omega \cap U_i,\\ 0 & \text{otherwise.} \end{cases}$$

Since $\Sigma^i \in \mathcal{K}_0^{\mathrm{hom}}$, the construction in Step 1 yields that there exist sequences $\{\Sigma_n^i\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ and

$$\lambda_n^i := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\mathrm{dev}}(x', \cdot) : P_{x'}] \in \mathcal{M}_b((\omega \cap U_i) \times I \times \mathcal{Y})$$

such that

$$\lambda_n^i \xrightharpoonup{*} \lambda^i \quad \text{weakly* in } \mathcal{M}_b((\omega \cap U_i) \times I \times \mathcal{Y})$$

with

$$\langle \lambda^i, \varphi \rangle = -\int_{(\omega\cap U_i)\times I\times\mathcal{Y}} \varphi(x')\Sigma : E\,dx\,dy + \int_{\omega\cap U_i} \varphi\bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_{\omega\cap U_i} \varphi\hat{\sigma} : D^2 w_3\,dx' - \int_{\omega\cap U_i} \bar{\sigma} : ((\bar{u}-\bar{w})\odot\nabla\varphi)\,dx'$$
$$- \frac{1}{6}\int_{\omega\cap U_i} \hat{\sigma} : (\nabla(u_3 - w_3)\odot\nabla\varphi)\,dx' - \frac{1}{12}\int_{\omega\cap U_i} (u_3 - w_3)\hat{\sigma} : \nabla^2\varphi\,dx'$$

for every $\varphi \in C_c^2(\overline{\omega} \cap U_i)$. This allows us to define measures on $\widetilde{\Omega} \times \mathcal{Y}$ by letting, for every $\phi \in C_0(\widetilde{\Omega} \times \mathcal{Y})$,

$$\langle \lambda_n, \phi \rangle := \sum_i \langle \lambda_n^i, \psi_i(x')\phi \rangle$$

and

$$\langle \lambda, \phi \rangle := \sum_i \langle \lambda^i, \psi_i(x')\phi \rangle.$$

Then we can see that $\lambda_n \xrightharpoonup{*} \lambda$ weakly* in $\mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$, and λ satisfies all the required properties. □

The following theorem provides a two-scale Hill's principle (cf. [26, Theorem 5.12]).

Theorem 5.31. *Let $\Sigma \in \mathcal{K}_0^{\text{hom}}$ and $(u, E, P) \in \mathcal{A}_0^{\text{hom}}(w)$ with the associated $\mu \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa \in \Upsilon_0(\widetilde{\omega})$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus, under the assumption on the ordering of phases we have*

$$\overline{H_r\Big(y, \frac{dP}{d|P|}\Big)|P|} \geq \bar{\lambda},$$

where $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ is given by Proposition 5.30.

Proof. Take $\varphi \in C_c(\widetilde{\omega} \times \mathcal{Y})$ nonnegative. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in Step 2 of the proof of Proposition 5.30. Item (c) in Lemma 5.25 implies that

$$(\Sigma_n^i)_{\text{dev}}(x, y) \in K(y) \quad \text{for every } x' \in \omega \text{ and } \mathcal{L}_{x_3}^1 \otimes \mathcal{L}_y^2\text{-a.e. } (x_3, y) \in I \times \mathcal{Y}.$$

By Proposition 5.9, we have for η-a.e. $x' \in \widetilde{\omega}$,

$$\int_{I\times\mathcal{Y}} \varphi(x', y) H_r\Big(y, \frac{dP_{x'}}{d|P_{x'}|}\Big)\,d|P_{x'}| \geq \int_{I\times\mathcal{Y}} \varphi(x', y)\,d[\Sigma_n^i : P_{x'}] \quad \text{for every } \varphi \in C(\mathcal{Y}), \varphi \geq 0.$$

Since $\frac{dP}{d|P|}(x, y) = \frac{dP_{x'}}{d|P_{x'}|}(x_3, y)$ for $|P_{x'}|$-a.e. $(x_3, y) \in I \times \mathcal{Y}$ by [7, Proposition 2.2], we can conclude that

$$\begin{aligned} H_r\Big(y, \frac{dP}{d|P|}\Big)|P| &= \eta \overset{\text{gen.}}{\otimes} H_r\Big(y, \frac{dP}{d|P|}\Big)|P_{x'}| \\ &= \eta \overset{\text{gen.}}{\otimes} H_r\Big(y, \frac{dP_{x'}}{d|P_{x'}|}\Big)|P_{x'}| \\ &= \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} H_r\Big(y, \frac{dP_{x'}}{d|P_{x'}|}\Big)|P_{x'}|. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(\cdot, y) H_r\Big(y, \frac{dP}{d|P|}\Big)\,d|P| &= \sum_i \int_{\widetilde{\omega}} \psi_i(x')\bigg(\int_{I\times\mathcal{Y}} \varphi(x', y) H_r\Big(y, \frac{dP_{x'}}{d|P_{x'}|}\Big)|P_{x'}|\bigg)\,d\eta(x') \\ &\geq \sum_i \int_{\widetilde{\omega}} \psi_i(x')\bigg(\int_{I\times\mathcal{Y}} \varphi(x', y)\,d[\Sigma_n^i : P_{x'}]\bigg)\,d\eta(x') \\ &= \sum_i \int_{\widetilde{\Omega}\times\mathcal{Y}} \psi_i(x')\varphi(x', y)\,d\lambda_n^i(x, y) = \int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi\,d\lambda_n. \end{aligned}$$

By passing to the limit, we infer the desired inequality. □

5.4.2 Case $\gamma = +\infty$

The following proposition is the analogue of Proposition 5.30.

Proposition 5.32. *Let $\Sigma \in \mathcal{K}_\infty^{\mathrm{hom}}$ and let $(u, E, P) \in \mathcal{A}_\infty^{\mathrm{hom}}(w)$ with the associated $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, and $\zeta \in \mathcal{M}_b(\Omega; \mathbb{R}^3)$. There exists an element $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$*

$$\langle \lambda, \varphi \rangle = -\int_{\Omega\times\mathcal{Y}} \varphi(x')\Sigma : E\,dx\,dy + \int_\omega \varphi\bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_\omega \varphi\hat{\sigma} : D^2 w_3\,dx' - \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\,dx'$$
$$- \frac{1}{6}\int_\omega \hat{\sigma} : (\nabla(u_3 - w_3) \odot \nabla\varphi)\,dx' - \frac{1}{12}\int_\omega (u_3 - w_3)\hat{\sigma} : \nabla^2\varphi\,dx'.$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega} \times \mathcal{Y}) = -\int_{\Omega\times\mathcal{Y}} \Sigma : E\,dx\,dy + \int_\omega \bar{\sigma} : E\bar{w}\,dx' - \frac{1}{12}\int_\omega \hat{\sigma} : D^2 w_3\,dx'. \tag{5.40}$$

Proof. Suppose that ω is star-shaped with respect to one of its points.

Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^3; L^2(\mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ be the sequence given by Lemma 5.28. We define the sequence

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x] \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}),$$

where η is given by Lemma 5.21 and the duality $[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x]$ is a well defined bounded measure on $\mathcal{Y}$ for η-a.e. $x \in \widetilde{\Omega}$. Further, in view of Remark 5.22, (5.15) gives

$$\int_{\mathcal{Y}} \psi\, d[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x] = -\int_{\mathcal{Y}} \psi\Sigma_n : \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right] dy$$
$$- \int_{\mathcal{Y}} (\Sigma_n)''(x,y) : (\mu_x(y) \odot \nabla_y\psi(y))\,dy - \sum_{\alpha=1,2}\int_{\mathcal{Y}} \kappa_x(y)(\Sigma_n)_{\alpha3}(x,y)\partial_{y_\alpha}\psi(y)\,dy$$
$$+ \sum_{i=1,2,3} z_i \int_{\mathcal{Y}} \psi(y)(\Sigma_n)_{i3}(x,y)\,dy$$

for every $\psi \in C^1(\mathcal{Y})$, and

$$|[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x]| \le \|(\Sigma_n)_{\mathrm{dev}}(x,\cdot)\|_{L^\infty(\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})}|P_x| \le C|P_x|,$$

where the last inequality stems from item (c) in Lemma 5.28. This in turn implies that

$$|\lambda_n| = \eta \overset{\text{gen.}}{\otimes} |[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x]| \le C\eta \overset{\text{gen.}}{\otimes} |P_x| = C|P|,$$

from which we conclude that is $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I} \supset I$ be an open set which compactly contains I and extend the above measures by zero on $\widetilde{\omega} \times \widetilde{I} \times \mathcal{Y}$. Let ξ be a smooth cut-off function with $\xi \equiv 1$ on I, with support contained in $\widetilde{I}$. Finally, we consider a test function $\phi(x) := \varphi(x')\xi(x_3)$ for $\varphi \in C_c^\infty(\widetilde{\omega})$. Then, since $\nabla_y\phi(x) = 0$, $\partial_{y_\alpha}\phi(x) = 0$ and $\int_{\mathcal{Y}} (\Sigma_n)_{i3}(x,y)\,dy = 0$, we have

$$\langle \lambda_n, \phi \rangle = \int_{\widetilde{\Omega}} \left(\int_{\mathcal{Y}} \phi(x,y)\, d[(\Sigma_n)_{\mathrm{dev}}(x,\cdot) : P_x] \right) d\eta(x)$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\Sigma_n(x,y) : \left[C(x)E(x,y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right] d(\eta \otimes \mathcal{L}_y^2)$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\Sigma_n(x,y) : E(x,y)\,dx\,dy + \int_{\widetilde{\Omega}} \varphi(x')\sigma_n(x) : (A_1(x') + x_3 A_2(x'))\,d\eta$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\Sigma_n(x,y) : E(x,y)\,dx\,dy + \int_{\widetilde{\Omega}} \varphi(x')\sigma_n(x) : dEu(x)$$

From this point on, the proof is exactly the same as the proof of Proposition 5.30 by defining in the analogous way Σ_n^i, λ_n^i, i.e., Σ_i, λ_i. □

The following theorem is analogous to Theorem 5.31.

Theorem 5.33. *Let $\Sigma \in \mathcal{K}_\infty^{\mathrm{hom}}$ and $(u, E, P) \in \mathcal{A}_\infty^{\mathrm{hom}}(w)$ with the associated $\mu \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\kappa \in \mathcal{X}_\infty(\widetilde{\Omega})$, $\zeta \in \mathcal{M}_b(\Omega; \mathbb{R}^3)$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus, under the assumption on the ordering of phases we have*

$$H\Big(y, \frac{dP}{d|P|}\Big)|P| \geq \lambda,$$

where $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ is given by Proposition 5.32.

Proof. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in the proof of Proposition 5.32. Item (c) in Lemma 5.28 implies that

$$(\Sigma_n^i)_{\mathrm{dev}}(x, y) \in K(y) \quad \text{for every } x \in \Omega \text{ and } \mathcal{L}_y^2\text{-a.e. } y \in \mathcal{Y}.$$

By Proposition 5.16, we have for η-a.e. $x \in \widetilde{\Omega}$,

$$H\Big(y, \frac{dP_x}{d|P_x|}\Big)|P_x| \geq [(\Sigma_n^i)_{\mathrm{dev}}(x, \cdot) : P_x] \quad \text{as measures on } \mathcal{Y}.$$

Since $\frac{dP}{d|P|}(x, y) = \frac{dP_x}{d|P_x|}(y)$ for $|P_x|$-a.e. $y \in \mathcal{Y}$ by [7, Proposition 2.2], we can conclude that

$$\begin{aligned} H\Big(y, \frac{dP}{d|P|}\Big)|P| &= \eta \overset{\text{gen.}}{\otimes} H\Big(y, \frac{dP}{d|P|}\Big)|P_x| = \eta \overset{\text{gen.}}{\otimes} H\Big(y, \frac{dP_x}{d|P_x|}\Big)|P_x| \\ &= \sum_i \psi_i(x')\eta \overset{\text{gen.}}{\otimes} H\Big(y, \frac{dP_x}{d|P_x|}\Big)|P_x| \\ &\geq \sum_i \psi_i(x')\eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\mathrm{dev}}(x, \cdot) : P_x] \\ &= \sum_i \psi_i(x')\lambda_n^i = \lambda_n. \end{aligned}$$

By passing to the limit, we have the desired inequality. □

6 Two-scale quasistatic evolutions

The associated $\mathcal{H}^{\mathrm{hom}}$-variation of a function $P : [0, T] \to \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}_{\mathrm{dev}}^{3\times3})$ on $[a, b]$ is then defined as

$$\mathcal{D}_{\mathcal{H}_\gamma^{\mathrm{hom}}}(P; a, b) := \sup\Big\{\sum_{i=1}^{n-1} \mathcal{H}_\gamma^{\mathrm{hom}}(P(t_{i+1}) - P(t_i)) : a = t_1 < t_2 < \cdots < t_n = b,\ n \in \mathbb{N}\Big\}.$$

In this section we prescribe for every $t \in [0, T]$ a boundary datum $w(t) \in H^1(\widetilde{\Omega}; \mathbb{R}^3) \cap \mathrm{KL}(\widetilde{\Omega})$ and we assume the map $t \mapsto w(t)$ to be absolutely continuous from $[0, T]$ into $H^1(\widetilde{\Omega}; \mathbb{R}^3)$.

We now give the notion of the limiting quasistatic elastoplastic evolution.

Definition 6.1. A *two-scale quasistatic evolution* for the boundary datum $w(t)$ is a function $t \mapsto (u(t), E(t), P(t))$ from $[0, T]$ into $\mathrm{KL}(\widetilde{\Omega}) \times L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}_{\mathrm{sym}}^{3\times3}) \times \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}_{\mathrm{dev}}^{3\times3})$ which satisfies the following conditions:

$(\mathrm{qs1})_\gamma^{\mathrm{hom}}$ for every $t \in [0, T]$ we have $(u(t), E(t), P(t)) \in \mathcal{A}_\gamma^{\mathrm{hom}}(w(t))$ and

$$\mathcal{Q}_\gamma^{\mathrm{hom}}(E(t)) \leq \mathcal{Q}_\gamma^{\mathrm{hom}}(H) + \mathcal{H}_\gamma^{\mathrm{hom}}(\Pi - P(t))$$

for every $(v, H, \Pi) \in \mathcal{A}_\gamma^{\mathrm{hom}}(w(t))$,

$(\mathrm{qs2})_\gamma^{\mathrm{hom}}$ the function $t \mapsto P(t)$ from $[0, T]$ into $\mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}_{\mathrm{dev}}^{3\times3})$ has bounded variation and for every $t \in [0, T]$,

$$\mathcal{Q}_0^{\mathrm{hom}}(E(t)) + \mathcal{D}_{\mathcal{H}_0^{\mathrm{hom}}}(P; 0, t) = \mathcal{Q}_0^{\mathrm{hom}}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}_r(y)E(s) : E\dot{w}(s)\, dx\, dy\, ds$$

for $\gamma = 0$ and

$$\mathcal{Q}_{+\infty}^{\mathrm{hom}}(E(t)) + \mathcal{D}_{\mathcal{H}_{+\infty}^{\mathrm{hom}}}(P; 0, t) = \mathcal{Q}_{+\infty}^{\mathrm{hom}}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E(s) : E\dot{w}(s)\, dx\, dy\, ds$$

for $\gamma = +\infty$.

Recalling the definition of a h-quasistatic evolution for the boundary datum $w(t)$ given in Definition 3.8, we are in a position to formulate the main result of the paper.

Theorem 6.2. *Let $t \mapsto w(t)$ be absolutely continuous from $[0,T]$ into $H^1(\widetilde{\Omega};\mathbb{R}^3) \cap \mathrm{KL}(\widetilde{\Omega})$. Let $\mathcal{Y}$ be a geometrically admissible multi-phase torus and let the assumption on the ordering of phases be satisfied. Assume also* (3.2), (3.3) *and* (3.5) *and that there exists a sequence of triples $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(w(0))$ such that*

$$u_0^h \xrightarrow{*} u_0 \quad \textit{weakly* in } \mathrm{BD}(\widetilde{\Omega}), \tag{6.1}$$

$$\Lambda_h e_0^h \xrightarrow{2} E_0 \quad \textit{two-scale strongly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}), \tag{6.2}$$

$$\Lambda_h p_0^h \xrightarrow{2\text{-}*} P_0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{dev}}) \tag{6.3}$$

for $(u_0, E_0, P_0) \in \mathcal{A}^{\mathrm{hom}}_\infty(w(0))$ if $\gamma = +\infty$, and $(u_0, E_0'', P_0'') \in \mathcal{A}^{\mathrm{hom}}_0(w(0))$ with $E_0 = \mathbb{A}_y E_0''$ if $\gamma = 0$. For every $h > 0$, let

$$t \mapsto (u^h(t), e^h(t), p^h(t))$$

be a h-quasistatic evolution in the sense of Definition 3.8 for the boundary datum w such that $u^h(0) = u_0^h$, $e^h(0) = e_0^h$, and $p^h(0) = p_0^h$. Then there exists a two-scale quasistatic evolution

$$t \mapsto (u(t), E(t), P(t))$$

for the boundary datum $w(t)$ such that $u(0) = u_0$, $E(0) = E_0$, and $P(0) = P_0$, and such that (up to subsequence) for every $t \in [0,T]$,

$$u^h(t) \xrightarrow{*} u(t) \quad \textit{weakly* in } \mathrm{BD}(\widetilde{\Omega}), \tag{6.4}$$

$$\Lambda_h e^h(t) \xrightarrow{2} E(t) \quad \textit{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}), \tag{6.5}$$

$$\Lambda_h p^h(t) \xrightarrow{2\text{-}*} P(t) \quad \textit{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{dev}}) \tag{6.6}$$

in case $\gamma = +\infty$, and

$$u^h(t) \xrightarrow{*} u(t) \qquad \textit{weakly* in } \mathrm{BD}(\widetilde{\Omega}), \tag{6.7}$$

$$\Lambda_h e^h(t) \xrightarrow{2} \mathbb{A}_y E(t) \qquad \textit{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}), \tag{6.8}$$

$$p^h(t) \xrightarrow{2\text{-}*} \begin{pmatrix} P(t) & 0 \\ 0 & 0 \end{pmatrix} \qquad \textit{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}) \tag{6.9}$$

in case $\gamma = 0$.

Proof. The proof is divided into several steps, in the spirit of evolutionary Γ-convergence and it follows the lines of [7, Theorem 6.2]. We present the proof in the case $\gamma = 0$, while the argument for the case $\gamma = +\infty$ is identical upon replacing the appropriate structures in the statement of Theorem 4.14 and definition of $\mathcal{A}^{\mathrm{hom}}_\gamma(w)$.

Step 1: Compactness. First, we prove that there exists a constant C, depending only on the initial and boundary data, such that

$$\sup_{t\in[0,T]} \|\Lambda_h e^h(t)\|_{L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})} \le C \quad \text{and} \quad \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, T) \le C \tag{6.10}$$

for every $h > 0$. Indeed, the energy balance of the h-quasistatic evolution $(\mathrm{qs2})_h$ and (3.4) imply

$$\begin{aligned} & r_c \|\Lambda_h e^h(t)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) \\ & \le R_c \|\Lambda_h e^h(0)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})} + 2R_c \sup_{t\in[0,T]} \|\Lambda_h e^h(t)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})} \int_0^T \|E\dot{w}(s)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}})}\, ds, \end{aligned}$$

where the last integral is well defined as $t \mapsto E\dot{w}(t)$ belongs to $L^1([0,T]; L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{sym}}))$. In view of the boundedness of $\Lambda_h e_0^h$ that is implied by (6.2), property (6.10) now follows by the Cauchy–Schwarz inequality.

Second, from the latter inequality in (6.10) and (3.5), we infer that

$$r_k \|\Lambda_h p^h(t) - \Lambda_h p_0^h\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\mathrm{dev}})} \le \mathcal{H}_h(\Lambda_h p^h(t) - \Lambda_h p_0^h) \le \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) \le C$$

for every $t \in [0,T]$, which together with (6.3) implies

$$\sup_{t\in[0,T]} \|\Lambda_h p^h(t)\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})} \le C. \tag{6.11}$$

Next, we note that $\|\cdot\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})}$ is a continuous seminorm on $\mathrm{BD}(\widetilde{\Omega})$ which is also a norm on the set of rigid motions. Then, using a variant of Poincaré–Korn's inequality (see [46, Chapter II, Proposition 2.4]) and the fact $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(w(t))$, we conclude that, for every $h > 0$ and $t \in [0,T]$,

$$\begin{aligned}
\|u^h(t)\|_{\mathrm{BD}(\widetilde{\Omega})} &\le C(\|u^h(t)\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \|Eu^h(t)\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})})\\
&\le C(\|w(t)\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \|e^h(t)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + \|p^h(t)\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})})\\
&\le C(\|w(t)\|_{L^2(\widetilde{\Omega};\mathbb{R}^3)} + \|\Lambda_h e^h(t)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + \|\Lambda_h p^h(t)\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})}).
\end{aligned}$$

In view of the assumption $w \in H^1(\widetilde{\Omega};\mathbb{R}^3)$, from (6.11) and the former inequality in (6.10) it follows that the sequences $\{u^h(t)\}$ are bounded in $\mathrm{BD}(\widetilde{\Omega})$ uniformly with respect to t.

Owing to (2.3), we infer that $\mathcal{D}_{\mathcal{H}_h}$ and $\mathcal{V}$ are equivalent norms, which immediately implies

$$\mathcal{V}(\Lambda_h p^h; 0, T) \le C \tag{6.12}$$

for every $h > 0$. Hence, by a generalized version of Helly's selection theorem (see [13, Lemma 7.2]), there exists a (not relabeled) subsequence, independent of t, and $P \in \mathrm{BV}(0,T;\mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{dev}}))$ such that

$$\Lambda_h p^h(t) \xrightarrow{2\text{-}*} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{dev}})$$

for every $t \in [0,T]$, and $\mathcal{V}(P;0,T) \le C$. We extract a further subsequence (possibly depending on t),

$$\begin{aligned}
u^{h_t}(t) &\xrightarrow{*} u(t) \quad \text{weakly* in } \mathrm{BD}(\widetilde{\Omega}),\\
\Lambda_{h_t} e^{h_t}(t) &\xrightarrow{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})
\end{aligned}$$

for every $t \in [0,T]$. From Proposition 4.1, we can conclude for every $t \in [0,T]$ that $u(t) \in \mathrm{KL}(\widetilde{\Omega})$. Furthermore, according to Theorem 4.14, one can choose the above subsequence in a way such that there exist $\mu(t) \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa(t) \in \Upsilon_0(\widetilde{\omega})$ and $\zeta(t) \in \mathcal{M}_b(\widetilde{\Omega}\times\mathcal{Y};\mathbb{R}^3)$ such that

$$\Lambda_h E u^{h_t}(t) \xrightarrow{2\text{-}*} Eu(t)\otimes\mathcal{L}^2_y + \begin{pmatrix} E_y\mu(t) - x_3 D^2_y\kappa(t) & \zeta'(t)\\ (\zeta'(t))^T & \zeta_3(t)\end{pmatrix}.$$

Since $\Lambda_{h_t} E u^{h_t}(t) = \Lambda_{h_t} e^{h_t}(t) + \Lambda_{h_t} p^{h_t}(t)$ in $\widetilde{\Omega}$ for every $h > 0$ and $t \in [0,T]$, we deduce that

$$(u(t), E''(t), P''(t)) \in \mathcal{A}^{\text{hom}}_0(w(t)).$$

Lastly, we consider for every $t \in [0,T]$,

$$\sigma^{h_t}(t) := \mathbb{C}\Big(\frac{x'}{\varepsilon_{h_t}}\Big)\Lambda_{h_t} e^{h_t}(t).$$

Then we can choose a (not relabeled) subsequence such that

$$\sigma^{h_t}(t) \xrightarrow{2} \Sigma(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}}), \tag{6.13}$$

where $\Sigma(t) := \mathbb{C}(y)E(t)$. Since $\sigma^{h_t}(t) \in \mathcal{K}_{h_t}$ for every $t \in [0,T]$, by Proposition 5.24 we can conclude $\Sigma(t) \in \mathcal{K}^{\text{hom}}_0$. From this it follows that $E(t) = \mathbb{A}_y E''(t)$.

Step 2: Global stability. From Step 1 we have $(u(t), E''(t), P''(t)) \in \mathcal{A}^{\text{hom}}_0(w(t))$ with the associated $\mu(t) \in \mathcal{X}_0(\widetilde{\omega})$, $\kappa(t) \in \Upsilon_0(\widetilde{\omega})$. Then for every $(v, H, \Pi) \in \mathcal{A}^{\text{hom}}_0(w(t))$ with the associated $\nu(t) \in \mathcal{X}_0(\widetilde{\omega})$, $\lambda(t) \in \Upsilon_0(\widetilde{\omega})$ we have

$$(v - u(t), H - E''(t), \Pi - P''(t)) \in \mathcal{A}^{\text{hom}}_0(0).$$

Furthermore, since from the first step of the proof $\mathbb{C}_r(y)E''(t) \in \mathcal{K}^{\text{hom}}_0$, by Corollary 5.29 we have

$$\mathcal{H}^{\text{hom}}_0(\Pi - P''(t)) \ge -\int_{\omega\times I\times\mathcal{Y}} \mathbb{C}_r(y)E''(t) : (H - E''(t))\,dx\,dy = \mathcal{Q}^{\text{hom}}_0(E''(t)) + \mathcal{Q}^{\text{hom}}_0(H - E''(t)) - \mathcal{Q}^{\text{hom}}_0(H),$$

where the last equality is a straightforward computation. From the above, we immediately deduce

$$\mathcal{H}_0^{\text{hom}}(\Pi - P''(t)) + \mathcal{Q}_0^{\text{hom}}(H) \geq \mathcal{Q}_0^{\text{hom}}(E''(t)) + \mathcal{Q}_0^{\text{hom}}(H - E''(t)) \geq \mathcal{Q}_0^{\text{hom}}(E''(t)),$$

hence the global stability of the two-scale quasistatic evolution $(\text{qs1})_y^{\text{hom}}$.

We proceed by proving that the limit functions $u(t)$ and $E(t)$ do not depend on the subsequence. Since $E(t) = \mathbb{A}_y E''(t)$, it is enough to conclude that $E''(t)$ is unique. Assume $(v(t), H(t), P(t)) \in \mathcal{A}_0^{\text{hom}}(w(t))$ with the associated $\nu(t) \in \mathcal{X}_0(\widetilde{\omega})$, $\lambda(t) \in \Upsilon_0(\widetilde{\omega})$ also satisfy the global stability of the two-scale quasistatic evolution. By the strict convexity of $\mathcal{Q}_0^{\text{hom}}$, we immediately obtain that

$$H(t) = E''(t).$$

Identifying $Eu(t), Ev(t)$ with elements of $\mathcal{M}_b(\widetilde{\Omega}; \mathbb{M}^{2\times 2}_{\text{sym}})$ and using (5.23), we have that

$$\begin{aligned} Ev(t) \otimes \mathcal{L}_y^2 + E_y\nu(t) - x_3 D_y^2\lambda(t) &= H(t)\mathcal{L}_x^3 \otimes \mathcal{L}_y^2 + P(t) \\ &= E(t)\mathcal{L}_x^3 \otimes \mathcal{L}_y^2 + P(t) \\ &= Eu(t) \otimes \mathcal{L}_y^2 + E_y\mu(t) - x_3 D_y^2\kappa(t). \end{aligned}$$

Integrating over $\mathcal{Y}$, we obtain

$$Ev(t) = Eu(t).$$

Using the variant of Poincaré–Korn's inequality as in Step 1, we can infer that $v(t) = u(t)$ on $\widetilde{\Omega}$.

This implies that the whole sequences converge without depending on t, i.e.,

$$\begin{aligned} &u^h(t) \xrightharpoonup{*} u(t) && \text{weakly* in } BD(\widetilde{\Omega}), \\ &\Lambda_h e^h(t) \xrightharpoonup{2} E(t) = \mathbb{A}_y E''(t) && \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}). \end{aligned}$$

Step 3: Energy balance. In order to prove the energy balance of the two-scale quasistatic evolution $(\text{qs2})_y^{\text{hom}}$, it is enough (by arguing as in, e.g., [13, Theorem 4.7] and [27, Theorem 2.7]) to prove the energy inequality

$$\mathcal{Q}_0^{\text{hom}}(E''(t)) + \mathcal{D}_{\mathcal{H}_0^{\text{hom}}}(P''; 0, t) \leq \mathcal{Q}_0^{\text{hom}}(E''(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}_r(y)E''(s) : E\dot{w}(s)\,dx\,dy\,ds. \tag{6.14}$$

For a fixed $t \in [0, T]$, let us consider a subdivision $0 = t_1 < t_2 < \cdots < t_n = t$ of $[0, t]$. In view of the lower semicontinuity of $\mathcal{Q}_0^{\text{hom}}$ and $\mathcal{H}_0^{\text{hom}}$ as a consequence of the convexity of Q and Reshetnyak lower-semicontinuity (see [1, Theorem 2.38] and Remark 3.11, see also [26, Lemma 4.6]) from $(\text{qs2})_h$ we have

$$\begin{aligned} \mathcal{Q}_0^{\text{hom}}(E(t)) + \sum_{i=1}^n \mathcal{H}_0^{\text{hom}}(P(t_{i+1}) - P(t_i)) &\leq \liminf_h \Big(\mathcal{Q}_h(\Lambda_h e^h(t)) + \sum_{i=1}^n \mathcal{H}_h(\Lambda_h p^h(t_{i+1}) - \Lambda_h p^h(t_i)) \Big) \\ &\leq \liminf_h (\mathcal{Q}_h(\Lambda_h e^h(t)) + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t)) \\ &= \liminf_h \Big(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\Big(\frac{x'}{\varepsilon_h}\Big)\Lambda_h e^h(s) : E\dot{w}(s)\,dx\,ds \Big). \end{aligned}$$

In view of the strong convergence assumed in (6.2) and (6.13), by the Lebesgue's dominated convergence theorem we infer

$$\lim_h \Big(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\Big(\frac{x'}{\varepsilon_h}\Big)\Lambda_h e^h(s) : E\dot{w}(s)\,dx\,ds \Big) = \mathcal{Q}_0^{\text{hom}}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}_r(y)E''(s) : E\dot{w}(s)\,dx\,dy\,ds.$$

Hence, we have

$$\mathcal{Q}_0^{\text{hom}}(E(t)) + \sum_{i=1}^n \mathcal{H}_0^{\text{hom}}(P''(t_{i+1}) - P''(t_i)) \leq \mathcal{Q}_0^{\text{hom}}(E''(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}_r(y)E''(s) : E\dot{w}(s)\,dx\,dy\,ds.$$

Taking the supremum over all partitions of $[0, t]$ yields (6.14), which concludes the proof, after replacement of E with E'' and P with P''. □

Remark 6.3. The prevailing effect of dimension reduction for the case $\gamma = 0$ can be argued in the following way. The potentials $\mathbb{C}_r$ and H_r are the ones that are obtained by performing a dimension reduction for perfectly plastic homogeneous plates (see [21]). Note that in our limiting model for $\gamma = 0$ such quantities, though, depend on the microscopic variable y, as if to hint that, roughly speaking, a dimension reduction occurred and was immediately followed by a homogenization procedure. This is also suggested by the result on correctors given in Theorem 4.14, which is analogous to the one obtained in, e.g., [43]. In the case $\gamma \in (0, +\infty)$ the limit energy and dissipation potential are not of this type and one cannot obtain them by minimizing third row and column like in the case $\gamma = 0$ (see Section 3.3).

The prevailing effect of homogenization in the regime $\gamma = +\infty$ is more difficult to explain. However, the corrector result for the case $\gamma = +\infty$ is again analogous to the one obtained in, e.g., [43], where it is known that this regime corresponds to the case when we firstly do the homogenization and then dimension reduction. Also, part (i) of Definition 5.26, where the two-scale limit stress is defined, suggests this interplay, since x_3 is kept fixed and the equation is divergence free in y (see, for comparison, again the case $\gamma \in (0, +\infty)$, analyzed in [7]).

We emphasize the fact that, to the best of our knowledge, neither a homogenization of the plate model in [21] (the model derived there is for homogeneous material) nor a dimension reduction of the homogenized model obtained in [26] have been studied in the literature.

Funding: M. Bužančić and I. Velčić were supported by the Croatian Science Foundation under Grant Agreement no. IP-2018-01-8904 (Homdirestroptcm). The research of E. Davoli was supported by the Austrian Science Fund (FWF) projects F65, V 662, Y1292, and I 4052. All authors are thankful for the support from the OeAD-WTZ project HR 08/2020.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Oxford University, New York, 2000.

[2] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Math. 1694, Springer, Berlin, 1998.

[3] D. Breit, L. Diening and F. Gmeineder, On the trace operator for functions of bounded $\mathbb{A}$-variation, *Anal. PDE* **13** (2020), no. 2, 559–594.

[4] L. Bufford, E. Davoli and I. Fonseca, Multiscale homogenization in Kirchhoff's nonlinear plate theory, *Math. Models Methods Appl. Sci.* **25** (2015), no. 9, 1765–1812.

[5] M. Bukal and I. Velčić, On the simultaneous homogenization and dimension reduction in elasticity and locality of Γ-closure, *Calc. Var. Partial Differential Equations* **56** (2017), no. 3, Paper No. 59.

[6] M. Bužančić, K. Cherednichenko, I. Velčić and J. Žubrinić, Spectral and evolution analysis of composite elastic plates with high contrast, *J. Elasticity* **152** (2022), no. 1–2, 79–177.

[7] M. Bužančić, E. Davoli and I. Velčić, Effective quasistatic evolution models for perfectly plastic plates with periodic microstructure, preprint (2022), https://arxiv.org/abs/2212.02116.

[8] D. Caillerie, Thin elastic and periodic plates, *Math. Methods Appl. Sci.* **6** (1984), no. 2, 159–191.

[9] L. Carbone and R. De Arcangelis, *Unbounded Functionals in the Calculus of Variations: Representation, Relaxation, and Homogenization*, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. 125, Chapman & Hall/CRC, Boca Raton, 2019.

[10] M. Cherdantsev and K. Cherednichenko, Bending of thin periodic plates, *Calc. Var. Partial Differential Equations* **54** (2015), no. 4, 4079–4117.

[11] F. Christowiak and C. Kreisbeck, Homogenization of layered materials with rigid components in single-slip finite crystal plasticity, *Calc. Var. Partial Differential Equations* **56** (2017), no. 3, Paper No. 75.

[12] F. Christowiak and C. Kreisbeck, Asymptotic rigidity of layered structures and its application in homogenization theory, *Arch. Ration. Mech. Anal.* **235** (2020), no. 1, 51–98.

[13] G. Dal Maso, A. DeSimone and M. G. Mora, Quasistatic evolution problems for linearly elastic-perfectly plastic materials, *Arch. Ration. Mech. Anal.* **180** (2006), no. 2, 237–291.

[14] A. Damlamian and M. Vogelius, Homogenization limits of the equations of elasticity in thin domains, *SIAM J. Math. Anal.* **18** (1987), no. 2, 435–451.

[15] E. Davoli, Linearized plastic plate models as Γ-limits of 3D finite elastoplasticity, *ESAIM Control Optim. Calc. Var.* **20** (2014), no. 3, 725–747.

[16] E. Davoli, Quasistatic evolution models for thin plates arising as low energy Γ-limits of finite plasticity, *Math. Models Methods Appl. Sci.* **24** (2014), no. 10, 2085–2153.

[17] E. Davoli, R. Ferreira and C. Kreisbeck, Homogenization in BV of a model for layered composites in finite crystal plasticity, *Adv. Calc. Var.* **14** (2021), no. 3, 441–473.
[18] E. Davoli, C. Gavioli and V. Pagliari, A homogenization result in finite plasticity, preprint (2022), https://arxiv.org/abs/2204.09084.
[19] E. Davoli, C. Gavioli and V. Pagliari, Homogenization of high-contrast media in finite-strain elastoplasticity, preprint (2022), https://arxiv.org/abs/2301.02170.
[20] E. Davoli and C. Kreisbeck, On static and evolutionary homogenization in crystal plasticity for stratified composites, in: *Research in Mathematics of Materials Science*, Assoc. Women Math. Ser. 31, Springer, Cham (2022), 159–183.
[21] E. Davoli and M. G. Mora, A quasistatic evolution model for perfectly plastic plates derived by Γ-convergence, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **30** (2013), no. 4, 615–660.
[22] E. Davoli and M. G. Mora, Stress regularity for a new quasistatic evolution model of perfectly plastic plates, *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 2581–2614.
[23] F. Demengel, Fonctions à hessien borné, *Ann. Inst. Fourier (Grenoble)* **34** (1984), no. 2, 155–190.
[24] F. Demengel and R. Temam, Convex functions of a measure and applications, *Indiana Univ. Math. J.* **33** (1984), no. 5, 673–709.
[25] I. Fonseca and G. Leoni, *Modern Methods in the Calculus of Variations: L^p Spaces*, Springer Monogr. Math., Springer, New York, 2007.
[26] G. Francfort and A. Giacomini, On periodic homogenization in perfect elasto-plasticity, *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 3, 409–461.
[27] G. A. Francfort and A. Giacomini, Small-strain heterogeneous elastoplasticity revisited, *Comm. Pure Appl. Math.* **65** (2012), no. 9, 1185–1241.
[28] G. A. Francfort, A. Giacomini and A. Musesti, On the Fleck and Willis homogenization procedure in strain gradient plasticity, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), no. 1, 43–62.
[29] , E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in *n* variabili, *Rend. Semin. Mat. Univ. Padova* **27** (1957), 284–305.
[30] A. Giacomini and A. Musesti, Two-scale homogenization for a model in strain gradient plasticity, *ESAIM Control Optim. Calc. Var.* **17** (2011), no. 4, 1035–1065.
[31] P. Gidoni, G. B. Maggiani and R. Scala, Existence and regularity of solutions for an evolution model of perfectly plastic plates, *Commun. Pure Appl. Anal.* **18** (2019), no. 4, 1783–1826.
[32] C. Goffman and J. Serrin, Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964), 159–178.
[33] H. Hanke, Homgenization in gradient plasticity, *Math. Models Methods Appl. Sci.* **21** (2011), no. 8, 1651–1684.
[34] M. Heida and B. Schweizer, Non-periodic homogenization of infinitesimal strain plasticity equations, *ZAMM Z. Angew. Math. Mech.* **96** (2016), no. 1, 5–23.
[35] M. Heida and B. Schweizer, Stochastic homogenization of plasticity equations, *ESAIM Control Optim. Calc. Var.* **24** (2018), no. 1, 153–176.
[36] P. Hornung, S. Neukamm and I. Velčić, Derivation of a homogenized nonlinear plate theory from 3d elasticity, *Calc. Var. Partial Differential Equations* **51** (2014), no. 3–4, 677–699.
[37] R. Kohn and R. Temam, Dual spaces of stresses and strains, with applications to Hencky plasticity, *Appl. Math. Optim.* **10** (1983), no. 1, 1–35.
[38] M. Liero and A. Mielke, An evolutionary elastoplastic plate model derived via Γ-convergence, *Math. Models Methods Appl. Sci.* **21** (2011), no. 9, 1961–1986.
[39] M. Liero and T. Roche, Rigorous derivation of a plate theory in linear elastoplasticity via Γ-convergence, *NoDEA Nonlinear Differential Equations Appl.* **19** (2012), no. 4, 437–457.
[40] G. B. Maggiani and M. G. Mora, A dynamic evolution model for perfectly plastic plates, *Math. Models Methods Appl. Sci.* **26** (2016), no. 10, 1825–1864.
[41] G. B. Maggiani and M. G. Mora, Quasistatic evolution of perfectly plastic shallow shells: A rigorous variational derivation, *Ann. Mat. Pura Appl. (4)* **197** (2018), no. 3, 775–815.
[42] A. Mielke, T. Roubíček and U. Stefanelli, Γ-limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differential Equations* **31** (2008), no. 3, 387–416.
[43] S. Neukamm and I. Velčić, Derivation of a homogenized von-Kármán plate theory from 3D nonlinear elasticity, *Math. Models Methods Appl. Sci.* **23** (2013), no. 14, 2701–2748.
[44] G. Panasenko, *Multi-Scale Modelling for Structures and Composites*, Springer, Dordrecht, 2005.
[45] B. Schweizer and M. Veneroni, Homogenization of plasticity equations with two-scale convergence methods, *Appl. Anal.* **94** (2015), no. 2, 376–399.
[46] R. Temam, *Problèmes mathématiques en plasticité*, Math. Methods Inform. Sci. 12, Gauthier-Villars, Paris, 1985.
[47] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, American Mathematical Society, Providence, 2001.
[48] I. Velčić, On the derivation of homogenized bending plate model, *Calc. Var. Partial Differential Equations* **53** (2015), no. 3–4, 561–586.