Research Article

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Homogenization of high-contrast composites under differential constraints

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Abstract: We derive, by means of variational techniques, a limiting description for a class of integral functionals under linear differential constraints. The functionals are designed to encode the energy of a high-contrast composite, that is, a heterogeneous material which, at a microscopic level, consists of a periodically perforated matrix whose cavities are occupied by a filling with very different physical properties. Our main result provides a Γ-convergence analysis as the periodicity tends to zero, and shows that the variational limit of the functionals at stake is the sum of two contributions, one resulting from the energy stored in the matrix and the other from the energy stored in the inclusions. As a consequence of the underlying high-contrast structure, the study is faced with a lack of coercivity with respect to the standard topologies in L^p, which we tackle by means of two-scale convergence techniques. In order to handle the differential constraints, instead, we establish new results about the existence of potentials and of constraint-preserving extension operators for linear, k-th order, homogeneous differential operators with constant coefficients and constant rank.

Keywords: Homogenization, high-contrast, $\mathscr{A}$-quasiconvexity, Γ-convergence, two-scale convergence, periodic unfolding

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1 Introduction

The goal of this contribution is to derive a unified limiting description for a class of integral functionals that are inspired by the physics of high-contrast composites and that are evaluated on fields subject to certain differential constraints.

High-contrast materials are characterized by the property that their microscopic physical features may change abruptly from point to point. In the case of a binary periodic medium of this kind, we have two relevant microscales: the periodicity of the microstructure and the high-contrast parameter, which encodes the strong (high-contrast) difference in the properties of the two components. Typically, the scale at which periodicity is observed is much smaller than the size of the specimen. It is hence natural, from a mathematical viewpoint, to study the asymptotics of the quantities that describe the system in the limit of vanishing period. This procedure is called homogenization. It allows to make predictions on the effective character of heterogeneous media when no experiments are available, and eases numerical simulations by reducing the degrees

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of freedom of the problem. Therefore, mathematical homogenization has been a lively field of investigation for many decades. We refer, e.g., to the monographs [15, 33] for a thorough introduction to the subject.

In this paper we tackle the case in which both the periodicity and the high-contrast are quantified by a small number $\varepsilon > 0$ (see (1.3) and Remark 2.1). We assume that the vector fields that encode the significant quantities of the system satisfy a linear first-order differential constraint. The rationale behind this choice becomes transparent if one focuses on specific theories: in large strain elasticity the deformation gradient is a significant strain measure, while electromagnetism deals with the differential operators encoded by the Maxwell system. In other settings, the elastic behavior of some materials can be influenced by electric currents or magnetic fields and vice versa (see, e.g., [74] for a recent contribution). In all these cases, the key quantities lie in the kernel of a suitable differential operator. We therefore see that the need of combining homogenization and differential constraints naturally arises and leads to generalized notions of convexity (see [12] for an overview). It is well known that all the differential constraints above (and many more) can be treated in a unified way in the framework of $\mathscr{A}$-quasiconvexity (see [48] and references therein). Such a notion allows to extend various results which were originally available only for Sobolev maps and their gradients to the setting where admissible fields belong to the kernel of a linear first-order differential operator $\mathscr{A}$ of constant rank. We incidentally mention that highly heterogeneous media under differential constraints also appear in the homogenization approach to topology optimization, which plays a central, for instance, in the modeling of lattice materials for additive manufacturing [3, 4].

The novelty of our contribution consists in tackling homogenization of high-contrast materials in the setting of $\mathscr{A}$-quasiconvexity. In order to describe our main results, we need to introduce some basic notation.

For $d \in \mathbb{N}$, $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open set with Lipschitz boundary. We regard Ω as a reference configuration for a composite with a high-contrast microstructure obtained by periodically inserting inclusions of a specific kind in a surrounding matrix. For instance, in the context of elasticity, M. Cherdantsev and K. D. Cherednichenko [25] considered a stiff matrix with embedded soft fillings. In general, applications require that the properties of the inclusions and of the matrix are tuned so that the resulting effective behavior of the composite is the desired one.

To depict the fine texture of the material in mathematical terms, we consider the periodicity cell $Q := (0,1)^d \subset \mathbb{R}^d$ and an open connected subset with Lipschitz boundary $D_0 \Subset Q$ (see Figure 1), where the symbol $\Subset$ indicates that D_0 is compactly contained in Q. The high-contrast behavior of the composite sitting in Ω is identified by means of the sets D_0 and $D_1 := Q \setminus \bar{D}_0$ in the following way: we introduce a small parameter $\varepsilon > 0$ representing the periodicity scale and we define the sets

$$\Omega_{0,\varepsilon} := \bigcup_{z \in Z_\varepsilon} \varepsilon(D_0 + z) \quad \text{with } Z_\varepsilon := \{z \in \mathbb{Z}^d : \varepsilon(\bar{D}_0 + z) \subset \Omega\}, \tag{1.1}$$

and

$$\Omega_{1,\varepsilon} := \Omega \setminus \overline{\Omega}_{0,\varepsilon}, \tag{1.2}$$

which correspond respectively to the inclusions and to the surrounding matrix (see Figure 2).

Hereafter, we will systematically adopt the adjectives “soft” and “stiff” to refer respectively to $\Omega_{0,\varepsilon}$ and to $\Omega_{1,\varepsilon}$, as well as to their related quantities. This use is informal and just meant to convey ideas, though being justified by the possible interpretation of our model in the framework of elasticity. The reader should however bear in mind that our treatment is suited for a wider range of applications.

The main goal of this contribution is to characterize the asymptotic behavior as $\varepsilon \to 0$ in the sense of Γ-convergence [14, 37] of the energy functionals

$$\int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\varepsilon u)\,\mathrm{d}x + \int_{\Omega_{1,\varepsilon}} f_1\Big(\frac{x}{\varepsilon}, u\Big)\,\mathrm{d}x, \tag{1.3}$$

evaluated on fields $u \in L^p(\Omega; \mathbb{R}^N)$ satisfying $\mathscr{A}u = 0$, where $\mathscr{A}$ is a suitable linear first-order partial differential operator. The precise class of constraints is presented in Section 2.2 below and further investigated in Section 7. In the expression above, the family $\{f_{0,\varepsilon}\}$ represents the energy densities stored in the “soft” inclusions, while f_1 encodes the behavior of the “stiff” matrix. The assumptions on $\{f_{0,\varepsilon}\}_{\varepsilon>0}$ and f_1 are collected

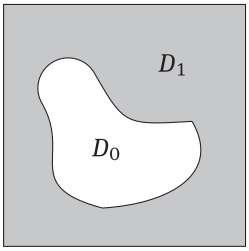

Figure 1: The subdivision of the unit cube Q into the "soft" region D_0 (white) and the "stiff" region D_1 (grey).

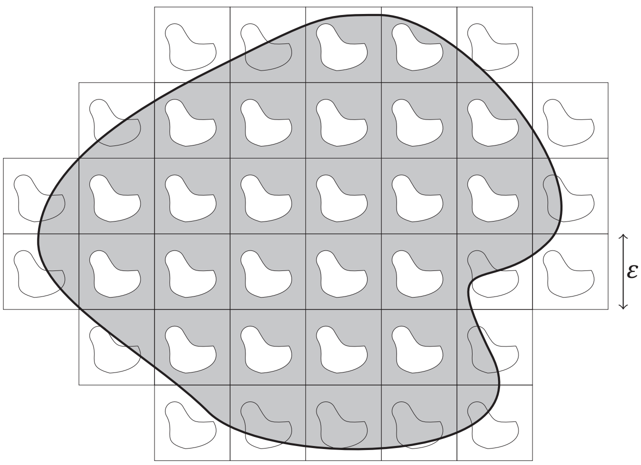

Figure 2: The reference set Ω and its microscopic structure. The collection of the "soft" inclusions $\Omega_{0,\varepsilon}$ is depicted in white, whereas the matrix $\Omega_{1,\varepsilon}$ is in grey.

in Section 2.1. We point out that considering a sequence of ε-dependent energy densities for the "soft" component is not merely a mathematical mannerism: on the contrary, it is a modeling necessity that arises when the admissible maps u are assumed to have the form $u = \nabla v$ for suitable fields v, i.e., when $\mathscr{A} = \operatorname{curl}$. We refer to [25, Remark 2] for a discussion on this point.

Even though $\{f_{0,\varepsilon}\}$ and f_1 in (1.3) are assumed to satisfy p-growth conditions from above and below, the fact that the "soft" energies are evaluated on εu rather than u results in a loss of coercivity. Consequently, classical weak and strong L^p-topologies are not capable to capture the asymptotic behavior of the problem and one needs to resort to an *ad hoc* notion of 'convergence in the high-contrast sense', which instead keeps track of the finest features of the microstructure, cf. Definition 2.5. Denoting by U_1 the set of maps in $L^p(\Omega;\mathbb{R}^N)$ such that $\mathscr{A}u = 0$ in Ω, our main result consists in showing that the limiting behavior of the energies in (1.3) is encoded by the functional $\mathscr{F}: U_1 \to \mathbb{R}$ defined as

$$\mathscr{F}(u) := \alpha_0 + \int_\Omega f_{\mathrm{hom}}(u(x))\,dx.$$

In the formula above, α_0 is a constant determined by f_0 (see (2.6)), f_{hom} is a suitable energy density related to f_1 (see (2.7)), and the condition $\mathscr{A}u = 0$ in Ω has to be interpreted in a distributional sense (see Section 2.2).

Our first main result is presented in Theorem 2.6, and, loosely speaking, states the following:

Theorem. *The functional $\mathscr{F}$ is the Γ-limit as $\varepsilon \to 0$ of the energies in (1.3) restricted to $\mathscr{A}$-free fields with respect to the high-contrast convergence.*

Key techniques to establish the theorem are p-equiintegrability arguments, as well as the notions of two-scale convergence [2] and of periodic unfolding [31, 32, 78, 79]. The proof relies essentially on the possibility of decoupling the behavior of the material in the "soft" and "stiff" contributions. The Γ-convergence of the "stiff" portion is a corollary of homogenization results in the setting of $\mathscr{A}$-quasiconvex [45]. The study of the "soft" part is instead more challenging, for the presence of the high-contrast microstructure and its interplay with the differential constraint result in the emergence of a second, hidden scale. This makes both the identification of a lower bound and the proof of its optimality need a delicate combination of two-scale results and weak-L^p compactness.

The decoupling of the system into regions having different material features hinges on careful compactness and splitting arguments. The latter are in turn based on two essential properties of the differential constraint $\mathscr{A}$: the fact that $\mathscr{A}$-free maps in the periodicity cell have null average, and the existence of a suitable extension operator on perforated domains (see Assumptions 1 and 2). In our general setting, it is not effortless to ensure such conditions, in contrast to the well-understood case in which the set Ω is contractible, $\mathscr{A}$ is the curl operator, and the $\mathscr{A}$-free fields are gradients. In particular, to the authors' knowledge, even for standard operators, it is still an open question whether extension operators preserving the $\mathscr{A}$-free constraint from a perforated domain to the 'filled' set exist. Similar results are available for gradients both in the Sobolev setting [1] and in the weaker one of antiplane fracture [20]. We also mention the recent result in [19] for the case of symmetric gradients in linearized fracture (in the space GSBD [38]).

The analysis of the differential constraints satisfying the aforementioned Assumptions 1 and 2 is the focus of Section 7. A special case in which the first requirement holds is that of operators $\mathscr{A}$ admitting "potentials", namely such that $\mathscr{A}u = 0$ if and only if $u = \mathscr{B}w$ for a suitable differential operator $\mathscr{B}$ and a field w. The second main contribution of this paper consists in showing that for a broad class of operators $\mathscr{A}$ existence of potentials and existence of extension operators are closely related. Roughly speaking, we prove the following.

Theorem. *Let Ω be a "nice" set and let $\mathscr{A}$ be a linear homogeneous differential operator with constant coefficients and constant rank. If an extension operator from $\mathscr{A}$-free maps on Ω to $\mathscr{A}$-free maps on the whole space exists, then $\mathscr{A}$ admits a potential on Ω. Conversely, if $\mathscr{A}$ admits a potential on Ω for which a suitable Korn-type inequality holds true, then there exists an extension operator from Ω to the whole space which preserves the property of being $\mathscr{A}$-free.*

We refer to Theorems 2.10 and 2.11 for the precise formulation of the result, as well as to Section 7 for a broader discussion on the topic. We point out that assuming the existence of potentials imposes some topological constraints on the set Ω. An example is easily provided by the case $\mathscr{A} = \operatorname{curl}$ for $d = 3$: one needs to require at least Ω to be simply connected, cf. Remark 7.2 and Examples 7.17–7.20. Essential tools for the proof of Theorems 2.10 and 2.11 are the theory of Fourier multipliers and the notion of generalized Moore–Penrose inverse.

Before proceeding with the mathematical details of our analysis, we conclude the section with some bibliographical notes.

The very first mathematical analysis of high-contrast materials was developed in the seminal work [8] by G. L. Auriault, who employed formal asymptotic expansions. In the early 2000, V. V. Zhikov [80] introduced a novel approach by extending classical two-scale techniques [2] to the setting of PDEs with rapidly oscillating coefficients. The study of high-contrast problems has been ever since at the center of an extensive scientific effort, with applications ranging from elastodynamics [75] to Maxwell's equations [13, 27]. A full characterization of the settings of linear elasticity and of conducting materials was provided in [21, 22], while the nonlinear elastic setting was considered in [25, 26] (see also [17]). The effects of microstructure on free-discontinuity functionals have been studied when high-contrast behavior appears in bulk terms only [10, 11], when it is just in surface terms [72, 73], and when both contributions are affected [18]. Mumford–Shah energies with high-contrast surface contributions have been recently characterized in [66]. The analysis of variational models for layered high-contrast materials was undertaken in [29, 30, 39].

As for the notion of $\mathscr{A}$-quasiconvexity, its introduction goes back to [35], while an extensive study was carried out in [48] for operators $\mathscr{A}$ with constant coefficients and constant rank (see (2.1) and (2.2) below). In [16] and [45], under p-growth assumptions on the energy density, relaxation and homogenization results were obtained for integral functionals evaluated on $\mathscr{A}$-free fields (i.e., fields in the kernel of $\mathscr{A}$); see also [44] for a related analysis on quasicrystals. Problems featuring simultaneously homogenization and dimension reduction were addressed in [59, 60], whereas oscillations and concentrations generated by $\mathscr{A}$-free fields were characterized in [46]. The case of nonpositive energy densities has been studied in [58]. A complete theory under linear growth assumptions on the energy density was established in [5, 7, 9, 62]. The characterization of $\mathscr{A}$-quasiconvexity was extended to operators with variable coefficients in [71]. We refer to [40, 41] for homogenization results in this purview, and to [42] for a corresponding relaxation formula. Applications to the theory of compressible Euler systems, as well as to adaptive image processing and to data-driven finite elasticity were the subject of [28], [43], and [34], respectively. To complete our review on $\mathscr{A}$-quasiconvexity, we mention the works [69, 70] on $\mathrm{BV}_{\mathscr{A}}$-spaces for elliptic and cancelling operators, [61] for a characterization of associated Young measures, as well as [49, 50, 67, 68] for the corresponding Sobolev analysis, and [53] for a compensated-compactness result.

1.1 Structure of the paper

The plan of the paper is as follows. In Section 2, we describe the setting of the problem and the related assumptions, and we formulate the precise statements of our main results, i.e., Theorems 2.6, 2.10, and 2.11. To lay the ground for the proofs, we collect auxiliary assertions and properties concerning measurable selection criteria, $\mathscr{A}$-free sequences, two-scale convergence, and Fourier analysis in Section 3. In Section 4 we study compactness properties of sequences of fields with equibounded energies, we detail the splitting argument, and present the proof of Theorem 2.6. Sections 5 and 6 contain the Γ-convergence analysis for the "soft" and "stiff" energy-contributions. Section 7 is focused on the description of the class of the differential operators that are admissible for our analysis, and deals with the proofs of Theorems 2.10 and 2.11.

2 Setting and main results

This section is devoted to the set-up of our analysis and to the presentation of the main achievements of this contribution. We will first fix the notation and the hypotheses used throughout the paper. The major results will be summarized in Section 2.3.

2.1 Energy functionals

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, $d \geq 2$, be a bounded, connected, open set with Lipschitz boundary. We denote by $\chi_{0,\varepsilon}$ and $\chi_{1,\varepsilon}$ the characteristic functions respectively of $\Omega_{0,\varepsilon}$ and $\Omega_{1,\varepsilon}$ in Ω (see (1.1) and (1.2)), i.e., for $i = 0, 1$,

$$\chi_{i,\varepsilon}(x) := \begin{cases} 1 & \text{if } x \in \Omega_{i,\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we let

$$\chi_i(y) := \begin{cases} 1 & \text{if } y \in D_i, \\ 0 & \text{otherwise,} \end{cases}$$

denote the characteristic function χ_i of D_i in Q.

We assume that $\{f_{0,\varepsilon}\}_{\varepsilon>0}$ and f_1 in (1.3) fulfill the following set of hypotheses:

(H1) Each $f_{0,\varepsilon} : \mathbb{R}^N \to \mathbb{R}$ is continuous and $f_1 : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.
(H2) The function $f_1(\,\cdot\,, \xi)$ is Q-periodic for every $\xi \in \mathbb{R}^N$.
(H3) There exist $a, \lambda, \Lambda > 0$ such that for a.e. $x \in \mathbb{R}^d$, for all $\xi \in \mathbb{R}^N$, and all $\varepsilon > 0$,

$$\lambda(-a + |\xi|^p) \leq f_{0,\varepsilon}(\xi) \leq \Lambda(1 + |\xi|^p),$$
$$\lambda(-a + |\xi|^p) \leq f_1(x, \xi) \leq \Lambda(1 + |\xi|^p),$$

for a fixed $p \in (1, +\infty)$.

(H4) There exists $\mu > 0$ such that for a.e. $x \in \Omega$, for all $\xi, \eta \in \mathbb{R}^N$, and $\varepsilon > 0$,

$$|f_{0,\varepsilon}(\xi) - f_{0,\varepsilon}(\eta)| \leq \mu(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|,$$
$$|f_1(x,\xi) - f_1(x,\eta)| \leq \mu(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|,$$

where p is the same as in (H3).

(H5) There exists $f_0 \colon \mathbb{R}^N \to \mathbb{R}$ such that for all $\xi \in \mathbb{R}^N$

$$\lim_{\varepsilon \to 0} f_{0,\varepsilon}(\xi) = f_0(\xi).$$

Remark 2.1 (Degeneracy of the "soft" component). We draw again attention to the fact that, in spite of the standard coercivity and growth of $f_{0,\varepsilon}$, the problem we address features some degeneracy, which is expressed by the factor ε multiplying the argument of the "soft" integrand (see (1.3)). From a modeling perspective, this ε accounts for the high-contrast between the two components, in that it makes the "soft" component less and less sensitive to external stimuli. This can be easily seen in the simple instance $f_{0,\varepsilon}(\xi) = f_1(\xi) = |\xi|^p$: formula (1.3) becomes

$$\varepsilon^p \int_{\Omega_{0,\varepsilon}} |u|^p \, dx + \int_{\Omega_{1,\varepsilon}} |u|^p \, dx,$$

and the ratio $\frac{1}{\varepsilon^p}$ between the "stiffness coefficients" of the two components blows up as ε vanishes.

Owing to the growth conditions prescribed by hypothesis (H3), the natural environment for our problem is the space L^p. We will often work with Q-periodic functions, i.e., those $u \colon \mathbb{R}^d \to \mathbb{R}^N$ such that $u(x+z) = u(x)$ for all $x \in \mathbb{R}^d$ and all $z \in \mathbb{Z}^d$. We will use the subscript 'per' to denote subspaces of Q-periodic maps; for instance,

$$L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N) := \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^N) : u \text{ is } Q\text{-periodic}\}.$$

We endow the previous space with the norm of $L^p(Q;\mathbb{R}^N)$. With a slight abuse of notation, we will tacitly identify $L^p(Q;\mathbb{R}^N)$ with $L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)$ by mapping the unit cube into the unit torus in $\mathbb{R}^d$ and extending the corresponding fields by periodicity. Analogously, we will denote the subspace of $L^p_{\mathrm{loc}}(\Omega \times \mathbb{R}^d;\mathbb{R}^N)$ containing all the functions that are Q-periodic with respect to their second argument by $L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ and implicitly identify this space with $L^p(\Omega \times Q;\mathbb{R}^N)$.

2.2 The differential constraint

Let $p \in (1,+\infty)$ be as in (H3) and (H4) above. We suppose that the constraint that we couple with the energy in (1.3) falls within the framework of $\mathscr{A}$-quasiconvexity as described by I. Fonseca and S. Müller [48]. Namely, for $N, M \in \mathbb{N} \setminus \{0\}$, we let $\mathscr{A}$ be a partial differential operator on $\mathbb{R}^d$ from $\mathbb{R}^N$ to $\mathbb{R}^M$ whose action on a function $u \colon \mathbb{R}^d \to \mathbb{R}^N$ is given by

$$\mathscr{A}u := \sum_{i=1}^{d} A^{(i)} \frac{\partial u}{\partial x_i}, \tag{2.1}$$

where $A^{(i)} \colon \mathbb{R}^N \to \mathbb{R}^M$ are linear maps for all $i = 1, \dots, d$. In other words, $\mathscr{A}$ is a linear first-order differential operator with constant coefficients.

Our fundamental requirement on $\mathscr{A}$ is the *constant rank* assumption introduced by F. Murat [64], that is, we suppose that there exists $r \in \mathbb{N}$ such that the *symbol* $\mathbb{A}$ of $\mathscr{A}$ satisfies the following: for all $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d \setminus \{0\}$, the operator

$$\mathbb{A}[\omega] := \sum_{i=1}^{d} \omega_i A^{(i)} \tag{2.2}$$

has rank equal to r.

We observe that for every open set $O \subset \mathbb{R}^d$, $\mathscr{A}$ can be regarded as an operator from $L^p(O;\mathbb{R}^N)$ to $W^{-1,p}(O;\mathbb{R}^M)$, where $W^{-1,p}(O;\mathbb{R}^M)$ denotes the dual of the Sobolev space $W^{1,p/(p-1)}_0(O;\mathbb{R}^M)$. To see this, we introduce the formal adjoint of $\mathscr{A}$, denoted by $\mathscr{A}^*$. For $v\colon O \to \mathbb{R}^M$, we set

$$\mathscr{A}^* v := -\sum_{i=1}^{d} A^{(i),*} \frac{\partial v}{\partial x_i},$$

where $A^{(i),*}$ is the adjoint (i.e., the transpose) of $A^{(i)}$. In this way,

$$\int_O \mathscr{A}\phi \cdot \psi \,\mathrm{d}x = \int_O \phi \cdot \mathscr{A}^*\psi \,\mathrm{d}x \quad \text{for all } \phi \in C^1_{\mathrm{c}}(O;\mathbb{R}^N) \text{ and } \psi \in C^1_{\mathrm{c}}(O;\mathbb{R}^M),$$

and for $u \in L^p(O;\mathbb{R}^N)$ we define the pairing

$$\langle \mathscr{A}u, v\rangle := \int_O u \cdot \mathscr{A}^* v \,\mathrm{d}x \quad \text{for all } v \in W^{1,p'}_0(O;\mathbb{R}^M),$$

where p' is the conjugate exponent to p. In what follows, if $u \in L^p(O;\mathbb{R}^N)$, equalities of the form $\mathscr{A}u = 0$ in O are always tacitly understood in the sense of $W^{-1,p}$, or, in other words, in the sense that

$$\int_O u \cdot \mathscr{A}^* v \,\mathrm{d}x = 0 \quad \text{for all } v \in W^{1,p'}_0(O;\mathbb{R}^M). \tag{2.3}$$

When such a relation holds, we say that u is *$\mathscr{A}$-free* on O. Similarly, if $u \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)$, we say that it is $\mathscr{A}$-free on the unit torus $\mathbb{T}^d$ when the equality in (2.3) is satisfied for $O = Q$ and for all $v \in W^{1,p'}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M)$; in particular, if $u \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)$ is $\mathscr{A}$-free on $\mathbb{T}^d$, then it is $\mathscr{A}$-free on Q as well.

In addition to the constant rank hypothesis, we need to consider further restrictions on the class of operators $\mathscr{A}$ for which our analysis will be performed. We formulate two *ad hoc* assumptions:

Assumption 1 (Null-average). For all $u \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)$ such that $u = 0$ on D_1 and $\mathscr{A}u = 0$ on $\mathbb{T}^d$, it holds $\int_Q u(y)\,\mathrm{d}y = 0$.

Assumption 2 ($\mathscr{A}$-free extension). There exist an ε-independent constant $c > 0$ and a sequence of operators $\{\mathsf{E}^\varepsilon_{\mathscr{A}}\}$ with $\mathsf{E}^\varepsilon_{\mathscr{A}}\colon L^p(\Omega;\mathbb{R}^N) \to L^p(\Omega;\mathbb{R}^N)$ such that the following holds: for all $\mathscr{A}$-free $u \in L^p(\Omega;\mathbb{R}^N)$,
(1) $\mathsf{E}^\varepsilon_{\mathscr{A}} u = u$ a.e. in $\Omega_{1,\varepsilon}$,
(2) $\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)}$,
(3) $\mathscr{A}(\mathsf{E}^\varepsilon_{\mathscr{A}} u) = 0$ on Ω, and
(4) if $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ is p-equiintegrable, then $\{\mathsf{E}^\varepsilon_{\mathscr{A}} u_\varepsilon\}$ is also p-equiintegrable in $L^p(\Omega;\mathbb{R}^N)$.

We recall that $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ is p-equiintegrable if for every $\eta > 0$ there exists $m > 0$ such that

$$\int_E |u_\varepsilon|^p \,\mathrm{d}x < \eta \quad \text{for all } \varepsilon > 0$$

whenever $E \subset \Omega$ satisfies $\mathscr{L}^d(E) < m$. Thanks to the Dunford–Pettis Theorem, this is equivalent to the fact that $\{|u_\varepsilon|^p\}$ admits a subsequence that is weakly convergent in $L^1(\Omega)$. We point out that in Assumption 2 it is essential to start with maps u which are $\mathscr{A}$-free in the whole set Ω, for this is related to the existence of “potentials” for the operator $\mathscr{A}$. We refer to the considerations after Theorem 2.11 in Section 2.3 for a discussion on this point, while we collect some comments on Assumption 2 in the next remark.

Remark 2.2 (On Assumption 2). In problems involving perforated media the use of extension techniques is fairly common (see, e.g., [1] or [15, Chapter 19]), and Assumption 2 is akin to others in the literature. For comparison, we mention [46, Definition 1.4]. The main differences consist in: (1) the fact that the extension operator here is assumed to exist on a periodically perforated domain, with constants independent of the periodicity parameter ε, (2) the fact that the existence of an extended p-equiintegrable sequence here is only required when starting from a sequence that also had p-equiintegrability properties. We refer to [46] for a general discussion on the connections between the existence of extension operators from an open set to a bigger surrounding domain, and the theory of DiPerna–Majda measures.

A notion of $\mathscr{A}$-free extension domain was also considered in [58, Definition 4.4]. There, however, no p-equiintegrability conditions are prescribed. As pointed out in [58, Section 4.2], such extension results are, in general, nontrivial. Additionally, they depend both on the topology of the set and on the operator $\mathscr{A}$ under consideration. In particular, no such $\mathscr{A}$-free extension is possible when $\mathscr{A}$ is the operator associated to the Cauchy-Riemann system, not even in the case of very regular sets Ω.

As far as our analysis is concerned, items (1)–(3) in Assumption 2 are among the main ingredients in the proof of Proposition 4.1, and without them we would not be able to establish high-contrast compactness for sequences with equibounded energy (see Definition 2.5). On the contrary, the preservation of p-equiintegrability is not required at this stage, whereas it is used in the proof of our main Γ-convergence result (see the proofs of Propositions 2.9 and 4.4). Nonetheless, point (4) in Assumption 2 could be dispensed with in some simplified versions of our problem, e.g., when $f_{0,\varepsilon} = f_0$ for every $\varepsilon > 0$ and Ω is a rectangle.

For the sake of brevity, it is convenient to give a name to the class of operators addressed in our study:

Definition 2.3. Let $\mathscr{A}$ be a linear first-order differential operator as in (2.1). We say that $\mathscr{A}$ is *admissible* if and only if it is of constant rank and it satisfies Assumptions 1 and 2.

It is well known that the family of constant rank operators is quite large [48]; as for the subclass of the admissible ones, we prove that it is nonempty in Section 7 where, in particular, we analyze the cases of the curl, of the curl curl operator, of the divergence, as well as the setting associated with higher-order gradients.

2.3 Main results

The first main result of this paper, Theorem 2.6 below, is the characterization of the asymptotic behavior as $\varepsilon \to 0$ of the functionals in (1.3) when the family is restricted to $\mathscr{A}$-free fields. In order to deduce also some information on the related energy minimizers, we resort to a variational kind of convergence, namely Γ-convergence (see, e.g., [14, 37]). Here, we consider a quite abstract version of its definition: indeed, we consider functionals defined on a certain set for which it is only known that some of its sequences are converging, the limit point being declared too; no topological structure is provided, instead. We summarize this situation by saying that such a set is endowed with a notion of convergence.

Definition 2.4. Let X be a set endowed with a notion of convergence. We say that the family $\{F_\varepsilon\}_{\varepsilon>0}$ of functions on X with values in $[-\infty, +\infty]$ Γ*-converges* as $\varepsilon \to 0$ to the function $F\colon X \to [-\infty, +\infty]$ if for any $x \in X$ and for any sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ the following holds:

(1) for any sequence $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset X$ such that $x_{\varepsilon_k} \to x$, we have

$$F(x) \le \liminf_{k\to+\infty} F_{\varepsilon_k}(x_{\varepsilon_k}),$$

(2) there exists a sequence $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset X$ such that $x_{\varepsilon_k} \to x$ and

$$\limsup_{k\to+\infty} F_{\varepsilon_k}(x_{\varepsilon_k}) \le F(x).$$

We have already remarked that the natural domain of the functionals (2.8) is contained in $L^p(\Omega;\mathbb{R}^N)$. However, in our setting, the two kinds of convergence in this space that are most frequently considered, the strong and the weak one, are not well suited to perform a Γ-convergence analysis. Indeed, they do not match the high-contrast nature of the problem and would therefore not yield useful compactness results for sequences with equibounded energy. This leads us to introduce a peculiar notion of convergence.

Definition 2.5. We say that a family $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ *converges in the high-contrast sense* to $u \in L^p(\Omega;\mathbb{R}^N)$ (relatively to the sequence of sets $\{\Omega_{1,\varepsilon}\}$) if $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$ and if there exists a second family $\{\tilde{u}_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that $\mathscr{A}\tilde{u}_\varepsilon = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^M)$, $u_\varepsilon = \tilde{u}_\varepsilon$ in $\Omega_{1,\varepsilon}$ and $\tilde{u}_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$.

Some comments are in order. The reason why the definition above is made up of two requirements is that there is no standard convergence which is able to capture alone the behaviors of both components. On one hand, the convergence of extensions is related to the coercivity of the "stiff" part, and it is a somehow customary requirement in homogenization (see Section 2.4); on the other, the convergence $\varepsilon u \rightharpoonup 0$ is needed to keep track of the "soft" component. Here, in the light of Assumption 1, the 0 vector turns out to be the average of some two-scale limit.

If we select $\mathscr{A}$ = curl and we assume that Ω is simply connected, the problem may be recast in $W^{1,p}(\Omega)$. In this setting, as far as the asymptotics of the "stiff" part is concerned, Rellich–Kondrachov's theorem allows to switch from weak convergence in $W^{1,p}(\Omega)$ to strong convergence in $L^p(\Omega)$. Therefore, the use of high-contrast convergence on the matrix leads to the standard homogenization results for perforated media. At the same time, it allows to quantify the change in the effective energy of the perforated material that occurs when "soft" inclusions are added (cf. α_0 in (2.5)). Indeed, it can be readily shown that $\varepsilon\nabla w_\varepsilon \rightharpoonup 0$ if $\mathscr{F}_\varepsilon(\nabla w_\varepsilon) \le c$ by well-known two-scale properties of gradients.

As a last remark on Definition 2.5, note that it is obvious that the weak convergence in $L^p(\Omega;\mathbb{R}^N)$ of a sequence $\{u_\varepsilon\}$ of $\mathscr{A}$-free maps entails high-contrast convergence (it suffices to set $\tilde{u}_\varepsilon := u_\varepsilon$). In particular, bounded sequences in L^p are pre-compact in the high-contrast sense. As Proposition 4.1 proves, however, a much weaker control is enough (see (4.9)), at least for admissible differential operators.

It will be convenient to have a special notation for spaces of functions satisfying the differential constraint encoded by $\mathscr{A}$. For any open $\Omega' \subset \Omega$, we define

$$U_0(\Omega') := \{u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)) : u = 0 \text{ if } y \in D_1 \text{ and } \mathscr{A}_y u = 0 \text{ in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \text{ for a.e. } x \in \Omega'\} \tag{2.4}$$

and

$$U_1 := \{u \in L^p(\Omega;\mathbb{R}^N) : \mathscr{A}u = 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^M)\}.$$

Note that limit points of high-contrast convergent sequences are automatically in U_1. Hereafter, we use the subscripts x and y to denote that the operator $\mathscr{A}$ acts on the variables x and y, respectively. We will prove that the limiting behavior of the ε-dependent energies constrained to $\mathscr{A}$-free fields is described by the functional $\mathscr{F} : U_1 \to \mathbb{R}$ defined as

$$\mathscr{F}(u) := \alpha_0 + \int_\Omega f_{\mathrm{hom}}(u(x))\,\mathrm{d}x, \tag{2.5}$$

where

$$\alpha_0 := \sup_{\Omega' \Subset \Omega} \inf_{u \in U_0(\Omega')} \int_{\Omega'}\int_{D_0} f_0(u(x,y))\,\mathrm{d}y\,\mathrm{d}x \tag{2.6}$$

and

$$\begin{aligned} f_{\mathrm{hom}}(\xi) := \liminf_{k\to+\infty} \inf\Big\{ \sum_{z\in\mathbb{Z}^d} \int_{Q\cap k^{-1}(D_1+z)} f_1(ky, \xi + v(y))\,\mathrm{d}y : v \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N), \int_Q v(z)\,\mathrm{d}z = 0, \\ \mathscr{A}v = 0 \text{ in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M)\Big\}. \end{aligned} \tag{2.7}$$

We are now ready to state our main convergence result.

Theorem 2.6. *Let $\mathscr{F} : L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)) \to \mathbb{R}\cup\{+\infty\}$ be the functional in formula* (2.5) *and, for $\varepsilon > 0$, let $\mathscr{F}_\varepsilon : L^p(\Omega;\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ be given by*

$$\mathscr{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\varepsilon u)\,\mathrm{d}x + \int_{\Omega_{1,\varepsilon}} f_1\Big(\frac{x}{\varepsilon}, u\Big)\,\mathrm{d}x & \text{if } u \in U_1, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^N). \end{cases} \tag{2.8}$$

Let us assume that $\{f_{0,\varepsilon}\}$ and f_1 fulfill (H1)–(H5)*, and that $\mathscr{A}$ is an admissible partial differential operator in the sense of Definition 2.3. Then the following properties hold:*

(1) *If a family $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ satisfies $\mathscr{F}_\varepsilon(u_\varepsilon) \le c$ for all $\varepsilon > 0$ and for some $c \ge 0$, then it admits a subsequence that is convergent in the high-contrast sense.*

(2) *For any $u \in U_1$ and for any sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ that converges to u in the high-contrast sense, it holds that*

$$\mathscr{F}(u) \le \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon).$$

(3) *For any $u \in U_1$ there exists a sequence $\{u_\varepsilon\} \subset U_1$ that converges to u in the high-contrast sense and satisfies*

$$\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \le \mathscr{F}(u).$$

By standard Γ-convergence techniques, we find the following corollary of Theorem 2.6:

Corollary 2.7. *Under the same assumptions and notation of Theorem 2.6, if $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ is a sequence of almost-minimizers for $\{\mathscr{F}_\varepsilon\}$, i.e., if*

$$\lim_{\varepsilon\to 0}\Big(\mathscr{F}_\varepsilon(u_\varepsilon) - \inf_{u\in U_1}\mathscr{F}_\varepsilon(u)\Big) = 0,$$

then there exists a subsequence of $\{u_\varepsilon\}$ that is convergent in the high-contrast sense to a minimum point of $\mathscr{F}$. Moreover,

$$\inf_{u\in L^p(\Omega;\mathbb{R}^N)} \mathscr{F}_\varepsilon(u) \to \min_{u\in U_1}\mathscr{F}(u).$$

Our proof of Theorem 2.6 relies on four main intermediate results:
(1) a compactness analysis combined with the characterization of high-contrast limits of $\mathscr{A}$-free fields, cf. Proposition 4.1,
(2) a splitting argument that allows to reduce the study of the Γ-limit of $\{\mathscr{F}_\varepsilon\}$ to the study of two independent problems, concerning the asymptotic behavior of the "soft" and of the "stiff" energy contributions, respectively, cf. Lemma 4.2 and Proposition 4.4,
(3) the identification of an optimal lower bound for the "soft" part of the energy, cf. Proposition 2.8,
(4) the identification of the limiting description for the "stiff" part, cf. Proposition 2.9.

The proof of Theorem 2.6 is exposed in Section 4.

Let us now present shortly the results that we obtain for the "soft" and "stiff" parts of the energy. The splitting procedure in Section 4 yields the functionals

$$\mathscr{F}_{0,\varepsilon}(u) := \begin{cases} \displaystyle\int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\varepsilon u)\,\mathrm{d}x & \text{if } u \in U_1, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^N), \end{cases} \tag{2.9}$$

$$\mathscr{F}_{1,\varepsilon}(u) := \begin{cases} \displaystyle\int_{\Omega_{1,\varepsilon}} f_1\Big(\frac{x}{\varepsilon}, u\Big)\,\mathrm{d}x & \text{if } u \in U_1, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^N), \end{cases} \tag{2.10}$$

where the densities $\{f_{0,\varepsilon}\}$ and f_1 satisfy (H1)–(H5). We will show, respectively in Sections 5 and 6, that $\{\mathscr{F}_{0,\varepsilon}\}$ and $\{\mathscr{F}_{1,\varepsilon}\}$ Γ-converge to

$$\mathscr{F}_0(u) := \begin{cases} \alpha_0 & \text{if } u = 0, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^N), \end{cases} \tag{2.11}$$

$$\mathscr{F}_1(u) := \begin{cases} \displaystyle\int_{\Omega} f_{\mathrm{hom}}(u(x))\,\mathrm{d}x & \text{if } u \in U_1, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^N), \end{cases} \tag{2.12}$$

Precisely, we establish the following.

Proposition 2.8 (Γ-limit of the "soft" component). *Let $\mathscr{F}_{0,\varepsilon}, \mathscr{F}_0 : L^p(\Omega;\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ be as above. If hypotheses* (H1) *and* (H3)–(H5) *are satisfied, and if $\mathscr{A}$ is an admissible differential operator as in Definition 2.3, then for all $u \in L^p(\Omega;\mathbb{R}^N)$ the following hold:*
(1) *for every p-equiintegrable sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that $u_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$ and that $\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$, we have*

$$\mathscr{F}_0(u) \le \liminf_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon),$$

(2) *there exists a sequence $\{u_\varepsilon\} \subset U_1$ with the properties that $u_\varepsilon = 0$ in $\Omega_{1,\varepsilon}$ for every $\varepsilon > 0$, $\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$, and*

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) \le \mathscr{F}_0(u).$$

We point out that the p-equiintegrability condition in the first part of the previous statement does not affect the generality of the result, as Proposition 4.4 will prove. For more details on this point, the reader is referred to the splitting argument in Section 4 below.

For what concerns the functionals $\mathscr{F}_{1,\varepsilon}$, the analysis carried out by I. Fonseca and S. Krömer in [45] yields:

Proposition 2.9 (Γ-limit of the "stiff" component). *Let $\mathscr{F}_{1,\varepsilon}, \mathscr{F}_1 : L^p(\Omega;\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ be as above. If hypotheses* (H1)–(H4) *are satisfied and if $\mathscr{A}$ is an admissible differential operator as in Definition 2.3, then the Γ-limit of $\{\mathscr{F}_{1,\varepsilon}\}$ with respect to the weak $L^p(\Omega;\mathbb{R}^N)$-convergence is $\mathscr{F}_1$.*

The last section is focused on the study of admissible differential operators. Differently from the other parts of the paper, the analysis of Section 7 encompasses all linear, k-th order, homogeneous differential operators with constant coefficients and constant rank. It will be useful to consider those operators $\mathscr{A}$ for which there exists a second operator $\mathscr{B}$ with the property that $u = \mathscr{B}w$ for some Sobolev function w whenever $\mathscr{A}u = 0$; in this case we say that $\mathscr{B}$ is a potential for $\mathscr{A}$. Our second main result proves that, for $\mathscr{A}$ to admit a potential on a certain open set Ω, it is sufficient that $\mathscr{A}$-free maps on Ω can be extended to $\mathscr{A}$-free maps on the whole space in such a way that a control on the L^p norm is ensured. We dub a set Ω with this property an $\mathscr{A}$-extension domain, see Definition 7.1. As Remark 7.2 shows, an $\mathscr{A}$-extension domain has to comply with some topological requirements; in particular, it cannot have holes, in general.

Theorem 2.10 (Existence of potentials for $\mathscr{A}$-free maps). *Let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank. If $\Omega \subset \mathbb{R}^d$ is a bounded, connected, open set with Lipschitz boundary which is also an $\mathscr{A}$-extension domain, then there exists $\ell \in \mathbb{N}$, and a differential operator $\mathscr{B}$ of order ℓ satisfying the following: for all $\mathscr{A}$-free maps $u \in L^p(\Omega;\mathbb{R}^N)$ there is a function $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ such that $u = \mathscr{B}w$ almost everywhere in Ω.*

An immediate consequence of this result is that Assumption 1 holds whenever the unit cube Q is an $\mathscr{A}$-extension domain (cf. Corollary 7.3).

Our third and last main result shows that for operators admitting a potential satisfying a suitable Korn-type inequality it is possible to define an extension operator preserving $\mathscr{A}$-free maps.

Theorem 2.11. *Let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank. Let also $\mathscr{B}$ be a linear, ℓ-th order, homogeneous differential operator with constant coefficients such that*

$$\ker \mathbb{A}[\omega] = \operatorname{im} \mathbb{B}[\omega] \quad \text{for all } \omega \in \mathbb{R}^d \setminus \{0\},$$

where $\mathbb{A}$ and $\mathbb{B}$ are the symbols of $\mathscr{A}$ and $\mathscr{B}$, respectively. We further assume that:
- *for all $\mathscr{A}$-free $u \in L^p(\Omega;\mathbb{R}^N)$ there exists $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ satisfying $u = \mathscr{B}w$,*
- *for any bounded, connected and open set $D \subset \mathbb{R}^d$ with Lipschitz boundary there exist a projection operator on the subspace of $\mathscr{B}$-free maps $\Pi_{\mathscr{B}} : W^{\ell,p}(D;\mathbb{R}^M) \to W^{\ell,p}(D;\mathbb{R}^M)$, as well as a constant $c > 0$ such that*

$$\|\nabla^\ell(w - \Pi_{\mathscr{B}}w)\|_{L^p(D;\mathbb{R}^{N\times d^\ell})} \le c\|\mathscr{B}w\|_{L^p(D;\mathbb{R}^N)} \quad \text{for all } w \in W^{\ell,p}(D;\mathbb{R}^M). \tag{2.13}$$

Then there exist a constant $c := c(d, p, D_1) > 0$ independent of ε and Ω, as well as a sequence of maps $\{\mathsf{E}^\varepsilon_{\mathscr{A}}\}$, with $\mathsf{E}^\varepsilon_{\mathscr{A}} : L^p(\Omega;\mathbb{R}^N) \to L^p(\Omega;\mathbb{R}^N)$, satisfying Assumption 2.

It is fundamental to observe that the maps under consideration in the theorem above are already *a priori* in the kernel of the operator $\mathscr{A}$ on the whole set Ω. For this reason, it is meaningful to assume the existence of potentials $\mathscr{B}$. In the case in which, instead, our fields were in the kernel of the operator $\mathscr{A}$ only in the perforated domain, its lack of contractibility would prevent the existence of a potential to hold even in the simple setting $d = 3$ and $\mathscr{A} = \operatorname{curl}$. We refer once again to Remark 7.2 for a counterexample.

2.4 Comparison with other works

Before getting to the heart of the matter, we take the chance to compare our main convergence result, Theorem 2.6, with similar ones in the literature.

When $\mathscr{A}$ is the curl operator, our analysis is akin to the one performed by M. Cherdantsev and K. D. Cherednichenko in [25]. However, our conclusions differ from theirs, also in the simple situation of a contractible Ω. Indeed, even though under this topological assumption curl-free maps coincide with gradients, the notion of convergence that we use is much weaker than the strong two-scale convergence considered in [25]. Another key difference with [25] is that in our setting convergence of minimizers follows directly from Theorem 2.6 (see Corollary 2.7), whereas in [25] it had to be shown *a posteriori* (see [25, Section 10]). Actually, the introduction of high-contrast convergence is motivated exactly by the fact that sequences with uniformly bounded energy $\mathscr{F}_\varepsilon$ are sequentially precompact in that sense.

This notion of convergence is inspired by the one considered by X. Pellet, L. Scardia and C. I. Zeppieri in [66] to deal with high-contrast Mumford–Shah energies. Unlike that contribution, aside from the role of the differential constraint $\mathscr{A}$, we require additionally that $\{\varepsilon u_\varepsilon\}$ converges to zero, so as to comply with the delicate two-scale characterization in Proposition 4.1.

We also stress that, in the absence of growth conditions from below on the "soft" part of the energy, the weak L^p-topology alone would not ensure convergence of minimizers. We refer to [25, Section 4] and [17, Section 4] for a discussion on this topic.

For a general differential constraint $\mathscr{A}$, our approach does not correspond to the ones in [17, 25]. If $\mathscr{A}$ does not admit a of suitable "potential" $\mathscr{B}$ satisfying a Korn-type inequality (cf. Theorem 2.10 below), then $\mathscr{A}$-free maps and fields u of the form $u = \mathscr{B}w$ are not interchangeable. Another *caveat* involves the choice of the "soft" effective energy. Looking at the results in [25], one could expect that in our case the $\mathscr{A}$-quasiconvex envelope of f_0 is to be retrieved in the Γ-limit. We recall that for a continuous function $g\colon \mathbb{R}^N \to \mathbb{R}$ with p-growth the $\mathscr{A}$-quasiconvex envelope of g is defined as

$$Q_{\mathscr{A}} g(\xi) := \inf\bigg\{ \int_Q g(\xi + v(z))\,\mathrm{d}z : v \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N),\ \int_Q v(z)\,\mathrm{d}z = 0,\ \mathscr{A}v = 0 \text{ in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \bigg\}.$$

The example below shows that $Q_{\mathscr{A}} f_0$ in not the correct limiting energy density.

Example 2.12. Let $\{f_{0,\varepsilon}\}$ satisfy hypotheses (H1)–(H5), let $\Omega' \Subset \Omega$, and let $D \subset D_0$ be open. In general, if a family $\{w_\varepsilon\} \subset L^p(\Omega'; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ satisfies

$$\begin{aligned} &\{w_\varepsilon\} \text{ is } p\text{-equiintegrable},\\ &w_\varepsilon \rightharpoonup 0 \text{ weakly in } L^p(\Omega' \times Q;\mathbb{R}^N),\\ &\mathscr{A}_y w_\varepsilon = 0 \text{ in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \text{ for a.e. } x \in \Omega' \end{aligned}$$

we cannot conclude that for all $\xi_0 \in \mathbb{R}^N$ it holds

$$\int_{\Omega'}\int_D Q_{\mathscr{A}} f_0(\xi_0)\,\mathrm{d}y\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \int_{\Omega'}\int_D f_{0,\varepsilon}(\xi_0 + w_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x, \tag{2.14}$$

as the following counterexample proves.

For $d = 2$, $N = 2 \times 2$ and $p = 2$, up to translations and dilations, we may assume $\Omega' := (0,1) \times (0,1) = Q$. We focus on the case $\mathscr{A} = \mathrm{curl}$, in which the $\mathscr{A}$-quasiconvex envelope $Q_{\mathscr{A}} f$ of the function f coincides with the well-know quasiconvex envelope Qf (see [36, Section 6.3]). Given $\lambda > 0$, as energy densities we set

$$f_{0,\varepsilon}(\xi) = f_0(\xi) = \psi_\lambda(\xi) - \det(\xi)$$

for every $\varepsilon > 0$, where $\psi_\lambda\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is a convex function such that

$$\psi_\lambda(\xi) = \begin{cases} 0 & \text{if } |\xi| < 2,\\ \left(\frac{1}{2} + \lambda\right)|\xi|^2 & \text{if } |\xi| \ge 3. \end{cases}$$

In this way, requirements (H1) and (H3)–(H5) are met, and

$$Qf_{0,\varepsilon} = Qf_0 = f_0,$$

because $f_{0,\varepsilon} = f_0$ is quasiconvex.

In order to disprove (2.14), we pick $\xi_0 = 0$ and, for $x = (x_1, x_2) \in (0,1)\times(0,1)$ and $y \in D$, we define

$$a(x) := \begin{cases} -\mathbb{I} & \text{if } x_1 \in \left(0, \frac{1}{2}\right], \\ \mathbb{I} & \text{if } x_1 \in \left(\frac{1}{2}, 1\right), \end{cases} \quad \text{and} \quad w_\varepsilon(x,y) := a\left(\frac{x}{\varepsilon}\right) + \varepsilon\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$ and a is extended by Q-periodicity. Then $\{w_\varepsilon\}$ is p-equiintegrable and $w_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega'; \mathbb{R}^{2\times 2})$. Besides, $\mathscr{A}_y w_\varepsilon = \operatorname{curl}_y w_\varepsilon = 0$, because the sequence does not depend on y.

For such particular choices, we see that the left-hand side of (2.14) equals 0, while for the right-hand side we find

$$\lim_{\varepsilon\to 0} \int_{\Omega'} \int_D [\psi_\lambda(w_\varepsilon(x,y)) - \det(w_\varepsilon(x,y))]\,dy\,dx = -\lim_{\varepsilon\to 0} \int_{\Omega'} \int_D \det(w_\varepsilon(x,y))\,dy\,dx = -\mathscr{L}^2(D) < 0.$$

3 Preliminaries

We gather in this section some preliminary definitions and results to be used later in the paper.

3.1 Measurable selection arguments

We recall here a measurable selection criterion that we will invoke in the proof of Corollary 7.3. For a thorough discussion on the topic we refer to the book by C. Castaing and M. Valadier [24].

Proposition 3.1 ([24, Theorem III.6]). *Let S be a multifunction defined on the measurable space $\mathscr{O}$ and taking values in the collection of nonempty complete subsets of the separable metric space X. If for all open $O \subset X$ the set $\{\omega \in \mathscr{O} : S(\omega) \cap O \neq \emptyset\}$ is measurable, then S admits a measurable selection, that is, there exists a measurable function $s\colon \mathscr{O} \to X$ such that $s(x) \in S(x)$ for all $x \in \mathscr{O}$.*

3.2 $\mathscr{A}$-free fields and two-scale limits

In this subsection we include some useful tools for the variational analysis of problems under differential constraints.

We recall two results about (asymptotically) $\mathscr{A}$-free fields. The operator $\mathscr{A}$ is always assumed to be of the form (2.1) and of constant rank.

Lemma 3.2 ($\mathscr{A}$-free decomposition, [48, Lemma 2.15]). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain, and let $p \in (1, +\infty)$. Let $\{u_k\} \subset L^p(\Omega; \mathbb{R}^N)$ be a bounded sequence such that $\mathscr{A}u_k \to 0$ strongly in $W^{-1,p}(\Omega; \mathbb{R}^M)$. Then there exist a subsequence $\{u_{j_k}\}$ and a sequence $\{v_k\} \subset L^p(\Omega; \mathbb{R}^N)$ such that the following holds:*
(1) *$\{v_k\}$ is bounded, $\mathscr{A}$-free, and p-equiintegrable,*
(2) *$u_{j_k} - v_k \to 0$ strongly in $L^q(\Omega; \mathbb{R}^N)$ for every $q \in [1, p)$.*

Lemma 3.3 ($\mathscr{A}$-free periodic extension, [45, Lemma 2.8]). *Let $D \subset Q$ be open, and let $p \in (1, +\infty)$. Let $\{u_k\} \subset L^p(D; \mathbb{R}^N)$ be a p-equiintegrable sequence such that $u_k \rightharpoonup 0$ weakly in $L^p(D; \mathbb{R}^N)$ and $\mathscr{A}u_k \to 0$ in $W^{-1,p}(D; \mathbb{R}^M)$. Then there exists an $\mathscr{A}$-free sequence $\{v_k\} \subset L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N)$ which is p-equiintegrable in Q and satisfies*

$$v_k - u_k \to 0 \quad \text{strongly in } L^p(D; \mathbb{R}^N),$$
$$v_k \to 0 \quad \text{strongly in } L^p(Q \setminus D; \mathbb{R}^N),$$
$$\int_Q v_k(x,y)\,dy = 0, \quad \|v_k\|_{L^p(Q;\mathbb{R}^N)} \le c(\mathscr{A}, D)\|u_k\|_{L^p(D;\mathbb{R}^N)} \quad \text{for all } k \in \mathbb{N}.$$

To perform our analysis, we need to track how the differential constraint behaves along sequences that converge in the sense of Definition 2.5 above. In this respect, the notion of two-scale convergence [2, 65] will be crucial:

Definition 3.4. We say that a sequence $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ *weakly two-scale converges in* L^p to a function $u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ if for all $v \in L^{p'}(\Omega; C_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ it holds

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) \cdot v\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Q u(x, y) \cdot v(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

In this case we write $u_\varepsilon \xrightarrow{2} u$.

We say that $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ *strongly two-scale converges in* L^p to $u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ if $u_\varepsilon \xrightharpoonup{2} u$ in L^p and $\|u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \to \|u\|_{L^p(\Omega;L^p(Q;\mathbb{R}^N))}$. In this case we write $u_\varepsilon \xrightarrow{2} u$ strongly in L^p.

For the sake of completeness, we record in the following lines the properties of two-scale convergence to be exploited in this work; we refer to [2, 78] for further reading.

Lemma 3.5. *Let $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ be a sequence.*
1. *If $\{u_\varepsilon\}$ is weakly two-scale convergent, then it is bounded in $L^p(\Omega; \mathbb{R}^N)$; conversely, if $\{u_\varepsilon\}$ is bounded in $L^p(\Omega; \mathbb{R}^N)$, then, it possesses a subsequence which is weakly two-scale convergent.*
2. *If $u_\varepsilon \xrightharpoonup{2} u$ weakly two-scale in L^p, then $u_\varepsilon \rightharpoonup \int_Q u(\,\cdot\,, y)\,\mathrm{d}y$ weakly in $L^p(\Omega; \mathbb{R}^N)$.*
3. *If $u_\varepsilon \xrightharpoonup{2} u$ weakly two-scale in L^p and $\{v_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ is another sequence with the property that $v_\varepsilon \xrightarrow{2} v$ strongly two-scale in L^p, then $u_\varepsilon v_\varepsilon \xrightharpoonup{2} uv$ weakly two scale in L^p.*
4. *If $u \in L^p(\Omega; C_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ or $u \in C(\bar{\Omega}; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$, then the sequence $\{u_\varepsilon\}$ defined as*

$$u_\varepsilon(x) := u\Big(x, \frac{x}{\varepsilon}\Big) \quad \textit{for a.e. } x \in \Omega$$

is p-equiintegrable and $u_\varepsilon \xrightarrow{2} u$ strongly in L^p.

It is well known that two-scale convergence in L^p can be related to L^p convergence by means of the unfolding operator.

Lemma 3.6. *For $\varepsilon > 0$, let us define the* unfolding operator $\mathsf{S}_\varepsilon : L^p(\Omega) \to L^p(\mathbb{R}^d; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ *as*

$$\mathsf{S}_\varepsilon v(x, y) := \tilde{v}\Big(\varepsilon\Big\lfloor \frac{x}{\varepsilon} \Big\rfloor + \varepsilon y\Big),$$

where $\tilde{v}$ denotes the extension of v by 0 outside Ω. Then if $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ is a bounded sequence, the following holds:
1. *$u_\varepsilon \xrightharpoonup{2} u$ weakly two-scale in L^p if and only if $\mathsf{S}_\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^p(\mathbb{R}^d \times Q; \mathbb{R}^N)$,*
2. *$u_\varepsilon \xrightarrow{2} u$ strongly two-scale in L^p if and only if $\mathsf{S}_\varepsilon u_\varepsilon \to u$ strongly in $L^p(\mathbb{R}^d \times Q; \mathbb{R}^N)$.*

Moreover, if $\{u_\varepsilon\}$ is p-equiintegrable, the sequence of unfoldings $\{\mathsf{S}_\varepsilon u_\varepsilon\}$ is p-equiintegrable too in $L^p(\mathbb{R}^d \times Q; \mathbb{R}^N)$.

We refer to [31, 32, 78, 79] for further properties of the unfolding operator.

A characterization of weak two-scale limits of $\mathscr{A}$-free sequences was established by I. Fonseca and S. Krömer.

Proposition 3.7 ([45, Theorem 1.2]). *A function $u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ is the weak two-scale limit of an $\mathscr{A}$-free sequence $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ if and only if*

$$\mathscr{A}_x\Big(\int_Q u(x, y)\,\mathrm{d}y\Big) = 0 \quad \textit{in } W^{-1,p}(\Omega; \mathbb{R}^M),$$
$$\mathscr{A}_y u(x, y) = 0 \quad \textit{in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M) \textit{ for a.e. } x \in \Omega.$$

As a last technical tool, we show that the unfolding of an $\mathscr{A}$-free map is in turn $\mathscr{A}$-free with respect to the periodicity variable. We preliminarily introduce the set

$$\hat{\Omega}_\varepsilon := \bigcup_{z\in\hat{Z}_\varepsilon} \varepsilon(Q + z) \quad \text{with } \hat{Z}_\varepsilon := \{z \in \mathbb{Z}^d : \varepsilon(Q + z) \subset \Omega\}. \tag{3.1}$$

Note that $\hat{Z}_\varepsilon \subset Z_\varepsilon$, where Z_ε is the collection of indices in (1.1).

Lemma 3.8. *Let $v \in L^p(\Omega;\mathbb{R}^N)$ be such that $\mathscr{A}v = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^M)$. Then, for every $\varepsilon > 0$, there holds*

$$\mathscr{A}_y(\mathsf{S}_\varepsilon v) = 0 \quad \text{in } W^{-1,p}(Q;\mathbb{R}^M) \text{ for a.e. } x \in \hat{\Omega}_\varepsilon. \tag{3.2}$$

Moreover, if also $v = 0$ in $\Omega_{1,\varepsilon}$, then

$$\mathscr{A}_y(\mathsf{S}_\varepsilon v) = 0 \quad \text{in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \text{ for a.e. } x \in \hat{\Omega}_\varepsilon.$$

Proof. Let $\eta \in C_c^\infty(\hat{\Omega}_\varepsilon)$ and $\psi \in W_0^{1,p'}(Q;\mathbb{R}^M)$. A change of variables yields

$$\begin{aligned}\int_{\hat{\Omega}^\varepsilon}\int_Q (\mathsf{S}_\varepsilon v)(x,y)\cdot\eta(x)\mathscr{A}^*\psi(y)\,dy\,dx &= \int_{\hat{\Omega}_\varepsilon}\int_Q v\Big(\varepsilon\Big\lfloor\frac{x}{\varepsilon}\Big\rfloor + \varepsilon y\Big)\cdot\eta(x)\mathscr{A}^*\psi(y)\,dy\,dx \\ &= \frac{1}{\varepsilon^{d-1}}\int_{\hat{\Omega}_\varepsilon}\eta(x)\int_{\varepsilon(Q+\lfloor\varepsilon^{-1}x\rfloor)} v(z)\cdot\mathscr{A}_z^*\psi_\varepsilon^x(z)\,dz\,dx,\end{aligned} \tag{3.3}$$

where

$$\psi_\varepsilon^x(z) := \psi(\varepsilon^{-1}z - \lfloor\varepsilon^{-1}x\rfloor) \quad \text{for a.e. } z \in \varepsilon(Q+\lfloor\varepsilon^{-1}x\rfloor).$$

Since $\psi \in W_0^{1,p'}(Q;\mathbb{R}^M)$, it follows that ψ_ε^x belongs to $W_0^{1,p'}(\varepsilon(Q+\lfloor\varepsilon^{-1}x\rfloor);\mathbb{R}^M)$ and it can be regarded as an element of $W_0^{1,p'}(\Omega;\mathbb{R}^M)$ by extending it to 0 outside $\varepsilon(Q+\lfloor\varepsilon^{-1}x\rfloor)$. In this step (3.1) needs to be used. The conclusion follows now from (3.3), because v is $\mathscr{A}$-free in Ω.

Assume further that $v = 0$ in $\Omega_{1,\varepsilon}$. By the definition of the unfolding operator, we have $\mathsf{S}_\varepsilon v(x,y) = 0$ for all $(x,y) \in \hat{\Omega}_\varepsilon \times D_1$, so that for any $\psi \in W_{\mathrm{per}}^{1,p'}(\mathbb{R}^d;\mathbb{R}^M)$

$$\int_Q \mathsf{S}_\varepsilon v(x,y)\cdot\mathscr{A}^*\psi(y)\,dy = \int_{D_0}\mathsf{S}_\varepsilon v(x,y)\cdot\mathscr{A}^*\psi(y)\,dy \quad \text{for all } x \in \hat{\Omega}_\varepsilon.$$

Let now $\eta \in C_c^\infty(Q;[0,1])$ be a cut-off function which is constantly 1 on D_0. From (3.2) we conclude that

$$\begin{aligned}\int_Q \mathsf{S}_\varepsilon v(x,y)\cdot\mathscr{A}^*\psi(y)\,dy &= \int_{D_0}\mathsf{S}_\varepsilon v(x,y)\cdot\mathscr{A}^*(\eta(y)\psi(y))\,dy \\ &= \int_Q \mathsf{S}_\varepsilon v(x,y)\cdot\mathscr{A}^*(\eta(y)\psi(y))\,dy = 0\end{aligned}$$

for almost every $x \in \hat{\Omega}_\varepsilon$. □

3.3 Fourier analysis

The study of the class of admissible operators in Section 7 is grounded on the theory of Fourier multipliers. For a comprehensive treatment of the matter, we refer to the monographs [51, 76]; here we limit ourselves to a short recollection of useful properties.

We let $\mathscr{S}$ denote the Schwartz space of rapidly decreasing functions and for $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$, we let

$$\mathcal{F}u(\omega) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\omega\cdot x}u(x)\,dx \tag{3.4}$$

be its Fourier transform. We also denote by $\mathcal{F}^{-1}$ the inverse transform and by $\mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^N)$ the space of linear maps from $\mathbb{R}^N$ to $\mathbb{R}^N$. We recall that a measurable function $m\colon \mathbb{R}^d \to \mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^N)$ is said to be an *L^p-multiplier* if the linear operator T_m defined as

$$\mathsf{T}_m u := \mathcal{F}^{-1}(m(\mathcal{F}u)) \quad \text{for } u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$$

can be extended to a bounded operator from L^p to L^p. Given a sufficiently smooth m, S. G. Mikhlin's multipliers theorem provides a condition for it to be a multiplier in terms of the decay of its derivatives. We firstly introduce some notation: when i is a d-dimensional multi-index, i.e., $i := (i_1, \dots, i_d) \in \mathbb{N}^d$, we set $|i| := \sum_j i_j$ and

$$\partial_i u(x) := \frac{\partial^{|i|} u}{\partial^{i_1} x_1 \cdots \partial^{i_d} x_d}(x).$$

In the scalar case, the criterion reads:

Proposition 3.9 ([63, Theorem 2 in Appendix]). *Let $m\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a function of class C^k with $k > \frac{d}{2}$. If there exists $c \geq 0$ such that*

$$|\partial_i m(x)| \leq \frac{c}{|x|^{|i|}} \quad \text{whenever } |i| \leq k,$$

then m is an L^p-multiplier.

A helpful consequence of the previous result is:

Corollary 3.10. *Let $m\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be a function of class C^k with $k > d/2$. If m is 0-homogeneous, then it is an L^p-multiplier for all $p \in (1, +\infty)$.*

Multipliers stand as basics examples of pseudo-differential operators. Their theory, in turn, can be used to characterize Sobolev spaces as follows. Given $k \in \mathbb{N}$, we introduce for $\omega \in \mathbb{R}^d$ the symbol $\mathbb{S}[\omega] := (1 + |\omega|^2)^{\frac{k}{2}}$ and the associated pseudo-differential operator $(I - \Delta)^{\frac{k}{2}}$:

$$(I - \Delta)^{\frac{k}{2}} u := \mathcal{F}^{-1}(\mathbb{S}(\mathcal{F}u)).$$

We then say that a distribution u belongs to $W^{k,p}(\mathbb{R}^d)$ if

$$(I - \Delta)^{\frac{k}{2}} u \in L^p(\mathbb{R}^d)$$

and we endow the space with the norm $\|u\|_{W^{k,p}(\mathbb{R}^d)} := \|(I - \Delta)^{\frac{k}{2}} u\|_{L^p(\mathbb{R}^d)}$. It turns out that this definition of $W^{k,p}$ is equivalent to the one given in terms of weak derivatives and that the norms are comparable too. The same approach shows that the dual space $W^{-k,p'}(\mathbb{R}^d)$ coincides with the space of distributions for which it holds

$$(I - \Delta)^{-\frac{k}{2}} u \in L^{p'}(\mathbb{R}^d)$$

and that the norms $\|u\|_{W^{-k,p'}(\mathbb{R}^d)}$ and $\|(I - \Delta)^{-\frac{k}{2}} u\|_{L^{p'}(\mathbb{R}^d)}$ coincide. Here, we recall that $p' := \frac{p}{p-1}$ and $(I - \Delta)^{-\frac{k}{2}}$ is the pseudo-differential operator defined by the symbol $(1 + |\omega|^2)^{-\frac{k}{2}}$.

4 Proof of Theorem 2.6

This section is devoted to the proof of Theorem 2.6, which stands as our principal result about the asymptotics of the energy functionals $\mathscr{F}_\varepsilon$ in (2.8). For the moment being, we assume that the limiting behaviors of the "soft" and "stiff" contributions are known, i.e., we suppose that Propositions 2.8 and 2.9 hold true. Their proofs are dealt with in Sections 5 and 6 below. Our main task in the current section is then to show that sequences with equibounded energies are precompact in a suitable sense and that the asymptotics of the global energy $\mathscr{F}_\varepsilon$ is determined by that of the functionals accounting for the "soft" and "stiff" parts. In this respect, it is useful to regard the total energy $\mathscr{F}_\varepsilon(u)$ of $u \in L^p(\Omega; \mathbb{R}^N)$ as the sum of $\mathscr{F}_{0,\varepsilon}(u)$ and $\mathscr{F}_{1,\varepsilon}(u)$, which were defined in (2.9) and (2.10).

The section is organized as follows. We first address the compactness result in Proposition 4.1, second we show how to reduce our problem to the sub-problems regarding the "soft" and the "stiff" components, then we prove the Γ-convergence statement in Theorem 2.6.

4.1 Compactness

The next proposition shows that sequences which are equibounded in energy have converging subsequences with respect to the high-contrast of convergence introduced in Definition 2.5. Note that, for the result to hold, neither boundary condition nor analogues of loading terms need to be taken into account.

Since according to position (2.8) only $\mathscr{A}$-free fields give rise to finite energy configurations, we can prove here that high-contrast limits of $\mathscr{A}$-free sequences inherit differential constraints.

Proposition 4.1 (High-contrast limits of $\mathscr{A}$-free sequences). *Let $\mathscr{A}$ be a constant rank differential operator of the form (2.1). Let also $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ be such that there exists $c \geq -a\lambda\mathscr{L}^d(\Omega)$ for which $\sup_{\varepsilon>0}\mathscr{F}_\varepsilon(u_\varepsilon) \leq c$. Then the following statements hold:*

(1) *There exist a (non-relabeled) subsequence and two maps $u_0 \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ and $u_1 \in L^p(\Omega;\mathbb{R}^N)$ such that $\{\varepsilon\chi_{0,\varepsilon}u_\varepsilon\}$ converges to u_0 weakly two-scale and that $\{\chi_{1,\varepsilon}u_\varepsilon\}$ converges to u_1 weakly in $L^p(\Omega;\mathbb{R}^N)$. Moreover, up to subsequences, $\{\varepsilon u_\varepsilon\}$ and $\{\chi_{1,\varepsilon}u_\varepsilon\}$ weakly two-scale converge in L^p, respectively, to u_0 and to a map $v_1 \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ that satisfy*

$$u_0(x,y) = 0 \text{ if } y \in D_1 \quad \text{and} \quad \int_Q v_1(\,\cdot\,,y)\,\mathrm{d}y = u_1. \tag{4.1}$$

(2) *The following differential constraints are fulfilled:*

$$\mathscr{A}_x\Big(\int_{D_0} u_0(x,y)\,\mathrm{d}y\Big) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^M), \tag{4.2}$$

$$\mathscr{A}_y u_0(x,y) = 0 \quad \text{in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \text{ for a.e. } x \in \Omega, \tag{4.3}$$

$$\mathscr{A}_y v_1(x,y) = 0 \quad \text{in } W^{-1,p}(D_1;\mathbb{R}^M) \text{ for a.e. } x \in \Omega. \tag{4.4}$$

(3) *If $\mathscr{A}$ satisfies Assumption 2, setting $\tilde{u}_\varepsilon := \mathsf{E}^\varepsilon_{\mathscr{A}}u_\varepsilon$, there exists $\tilde{u} \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ such that, up to subsequences, $\{\tilde{u}_\varepsilon\}$ weakly two-scale converges to $\tilde{u}$ in L^p, and there holds*

$$v_1(x,y) = \chi_{D_1}(y)\tilde{u}(x,y) \qquad \text{almost everywhere in } \Omega\times Q, \tag{4.5}$$

$$\mathscr{A}u_1 = -\mathscr{A}_x\Big(\int_{D_0}\tilde{u}(x,y)\,\mathrm{d}y\Big) \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^M), \tag{4.6}$$

$$\mathscr{A}_y(\chi_0\tilde{u}) = 0 \qquad \text{in } W^{-1,p}(Q;\mathbb{R}^M) \text{ for a.e. } x \in \Omega. \tag{4.7}$$

(4) *If additionally $\mathscr{A}$ satisfies Assumption 1, then*

$$\mathscr{A}u_1 = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^M)$$

and $\tilde{u}_\varepsilon \rightharpoonup u_1$ weakly in $L^p(\Omega;\mathbb{R}^N)$. In particular, up to subsequences, $\{u_\varepsilon\}$ converges to u_1 in the high-contrast sense.

Proof. If $\mathscr{F}_\varepsilon(u_\varepsilon) \leq c$, then $\mathscr{F}_{0,\varepsilon}(u_\varepsilon) \leq c$ and $\mathscr{F}_{1,\varepsilon}(u_\varepsilon) \leq c$ (cf. (2.9)–(2.10)). The growth assumptions in (H3) yield

$$\|\varepsilon\chi_{0,\varepsilon}u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \leq c, \quad \|\chi_{1,\varepsilon}u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \leq c. \tag{4.8}$$

(1) By Lemma 3.5, since bounded sequences in L^p are weakly two-scale precompact, there exist three maps $u, u_0, v_1 \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ such that, up to a (non-relabeled) subsequence,

$$\varepsilon u_\varepsilon \xrightharpoonup{2} u, \quad \varepsilon\chi_{0,\varepsilon}u_\varepsilon \xrightharpoonup{2} u_0, \quad \chi_{1,\varepsilon}u_\varepsilon \xrightharpoonup{2} v_1 \quad \text{weakly two-scale in } L^p. \tag{4.9}$$

For what concerns (4.1), we observe that, thanks to the relation between weak two-scale convergence and weak L^p-convergence, we have

$$\chi_{1,\varepsilon}u_\varepsilon \rightharpoonup u_1 := \int_Q v_1(\,\cdot\,,y)\,\mathrm{d}y \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N). \tag{4.10}$$

Further, it holds that

$$\chi_{i,\varepsilon} \xrightarrow{2} \chi_i \quad \text{strongly two-scale in } L^p, \tag{4.11}$$

whence, by the first two convergences in (4.9), we find

$$u_0(x,y) = \chi_0(y)u(x,y) \quad \text{almost everywhere in } \Omega\times Q.$$

In particular, also the first equality in (4.1) is satisfied.

By linearity, the previous considerations also entail that $\varepsilon\chi_{1,\varepsilon}u_\varepsilon \xrightarrow{2} (1-\chi_0)u$ weakly two-scale in L^p. On the other hand, from (4.8) we infer that $\{\varepsilon\chi_{1,\varepsilon}u_\varepsilon\}$ must actually converge to 0 strongly in $L^p(\Omega;\mathbb{R}^N)$ and, *a fortiori*, in the strong two-scale sense. Thus, $(1-\chi_0)u = 0$, which implies that $u(x,y) = 0$ if $y \in D_1$. We therefore conclude that $u = u_0$.

(2) Since $\mathscr{A}u_\varepsilon = 0$ in Ω for all $\varepsilon > 0$, Proposition 3.7 yields immediately (4.2) and (4.3). As for (4.4), given any $\psi \in W_0^{1,p'}(D_1;\mathbb{R}^M)$, we extend it to the whole $\mathbb{R}^d$ by setting $\psi = 0$ in D_0 and $\psi(x+z) = \psi(x)$ for all $x \in Q$ and $z \in \mathbb{Z}^d$. Fix a function $\eta \in C_c^1(\Omega)$. Since $\mathscr{A}u_\varepsilon = 0$ in Ω for every $\varepsilon > 0$, denoting by $\mathbb{A}^*$ the symbol of $\mathscr{A}^*$ (see Section 2.2), we have

$$\begin{aligned} 0 &= \Big\langle \mathscr{A}u_\varepsilon, \varepsilon\eta(\cdot)\psi\Big(\frac{\cdot}{\varepsilon}\Big)\Big\rangle_{W^{-1,p},W_0^{1,p'}} \\ &= \int_\Omega u_\varepsilon(x)\cdot\Big(\varepsilon\mathbb{A}^*[\nabla\eta(x)]\psi\Big(\frac{x}{\varepsilon}\Big) + \eta(x)\mathscr{A}^*\psi\Big(\frac{x}{\varepsilon}\Big)\Big)\,dx \\ &= \int_\Omega \chi_{1,\varepsilon}(x)u_\varepsilon(x)\cdot\Big(\varepsilon\mathbb{A}^*[\nabla\eta(x)]\psi\Big(\frac{x}{\varepsilon}\Big) + \eta(x)\mathscr{A}^*\psi\Big(\frac{x}{\varepsilon}\Big)\Big)\,dx, \end{aligned}$$

where the latter equality is due to the choice of the support of ψ. Therefore, the third convergence in (4.9) yields

$$\int_\Omega \eta(x)\Big(\int_Q v_1(x,y)\cdot\mathscr{A}^*\psi(y)\,dy\Big)\,dx = 0,$$

which in turn implies (4.4).

(3) For every $\varepsilon > 0$ let $\tilde{u}_\varepsilon$ denote the $\mathscr{A}$-free extension of u_ε provided by Assumption 2. Owing to (4.8), the sequence $\{\tilde{u}_\varepsilon\}$ is bounded in $L^p(\Omega;\mathbb{R}^N)$ and there exists $\tilde{u} \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ such that (up to subsequences)

$$\begin{aligned} &\tilde{u}_\varepsilon \xrightarrow{2} \tilde{u} \quad \text{weakly two-scale in } L^p, \\ &\mathscr{A}_x\Big(\int_Q \tilde{u}(x,y)\,dy\Big) = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^M). \end{aligned} \tag{4.12}$$

Since Assumption 2 grants that $\chi_{1,\varepsilon}\tilde{u}_\varepsilon = \chi_{1,\varepsilon}u_\varepsilon$ almost everywhere in Ω, we infer (4.5) from (4.9) and (4.11). Relationship (4.5) in turn rewrites as

$$\mathscr{A}u_1 = \mathscr{A}_x\Big(\int_Q v_1(x,y)\,dy\Big) = \mathscr{A}_x\Big(\int_{D_1}\tilde{u}(x,y)\,dy\Big) = -\mathscr{A}_x\Big(\int_{D_0}\tilde{u}(x,y)\,dy\Big),$$

i.e., (4.6) is proved. Finally, to obtain (4.7), we combine the fact that $\tilde{u}_\varepsilon = \chi_{1,\varepsilon}u_\varepsilon + \chi_{0,\varepsilon}\tilde{u}_\varepsilon$ almost everywhere in Ω with the assumption that $\mathscr{A}\tilde{u}_\varepsilon = 0$ for every $\varepsilon > 0$. Using again Proposition 3.7, as well as (4.4) and (4.9)–(4.12), we complete the proof of the third statement.

(4) By (4.7) and Assumption 1, we deduce that

$$\int_{D_0}\tilde{u}(x,y)\,dy = 0 \quad \text{for almost every } x \in \Omega.$$

Then, by (4.5), $\mathscr{A}u_1 = 0$ in Ω. Besides, in view of its weak two-scale convergence, $\{\tilde{u}_\varepsilon\}$ converges weakly in $L^p(\Omega;\mathbb{R}^N)$ to

$$\int_Q \tilde{u}(\cdot,y)\,dy = \int_Q \chi_{D_1}(y)\tilde{u}(\cdot,y)\,dy = \int_Q v_1(\cdot,y)\,dy = u_1(\cdot),$$

where we used (4.5) and (4.1). This also shows that $\{u_\varepsilon\}$ admits an extension on the “soft” part that converges to u_1 weakly in L^p. Therefore, to grant that there is a subsequence of $\{u_\varepsilon\}$ that converges to u_1 in the high-contrast sense, we are only left to observe that $\varepsilon u_\varepsilon \rightharpoonup \int_Q u_0(\cdot,y)\,dy$ and that such average vanishes because of (4.1), (4.3), and Assumption 1. The proof is now concluded. □

As already anticipated in Remark 2.2, we note that the preservation of p-equiintegrability granted by Assumption 2 is not needed to prove item (3) of the previous proposition. We will instead resort to it to establish Proposition 4.4 below.

4.2 Splitting

In the light of the of previous subsection, we know that those sequences which are equibounded in energy are precompact with respect to the high-contrast convergence, and that their cluster points fulfill suitable differential constraints. Following the approach of M. Cherdantsev and K. D. Cherednichenko [25], the next step is to show that the asymptotic behavior of the energy along such sequences coincides with the sum of those of the energies of two decoupled systems, one sitting on the “soft” inclusions, the other on the “stiff” matrix.

To favor intuition, consider a family $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that $\sup_{\varepsilon>0} \mathscr{F}_\varepsilon(u_\varepsilon) \le c$ for some $c \ge -a\lambda\mathscr{L}^d(\Omega)$, and assume that the operator $\mathscr{A}$ is admissible in the sense of Definition 2.3. As Proposition 4.1 proves, the growth condition (H3) on the energy densities entails $\|\chi_{1,\varepsilon}u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le c$. Hence, by exploiting Assumption 2, we retrieve the functions $\tilde{u}_\varepsilon := \mathsf{E}^\varepsilon_{\mathscr{A}} u_\varepsilon \in L^p(\Omega;\mathbb{R}^N)$ such that for all ε,
(1) $\chi_{1,\varepsilon}\tilde{u}_\varepsilon = \chi_{1,\varepsilon}u_\varepsilon$ almost everywhere in Ω,
(2) $\|\tilde{u}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le c$,
(3) $\mathscr{A}\tilde{u}_\varepsilon = 0$ in Ω,
(4) $\{\tilde{u}_\varepsilon\}$ is p-equiintegrable if so is $\{u_\varepsilon\}$.
Setting $v_\varepsilon := u_\varepsilon - \tilde{u}_\varepsilon$, we rewrite

$$\mathscr{F}_\varepsilon(u_\varepsilon) = \mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon) + \mathscr{E}_\varepsilon(u_\varepsilon),$$

where $\mathscr{F}_{0,\varepsilon}$ and $\mathscr{F}_{1,\varepsilon}$ are as in (2.9)–(2.10) and

$$\mathscr{E}_\varepsilon(u_\varepsilon) := \int_{\Omega_{0,\varepsilon}} [f_{0,\varepsilon}(\varepsilon u_\varepsilon) - f_{0,\varepsilon}(\varepsilon v_\varepsilon)]\,dx \tag{4.13}$$

(notice that $v_\varepsilon(x) = 0$ when $x \in \Omega_{1,\varepsilon}$). In the lemma below we show that the error made by substituting $\mathscr{F}_\varepsilon(u_\varepsilon)$ by the sum $\mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon)$, i.e., $\mathscr{E}_\varepsilon(u_\varepsilon)$, is asymptotically negligible whenever high-contrast convergence holds.

Lemma 4.2 (Splitting). *Let $\mathscr{A}$ be a constant rank differential operator of the form* (2.1). *Let also $u \in L^p(\Omega;\mathbb{R}^N)$ be the high-contrast limit of a family $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$. Explicitly, assume that $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$ and that there is a family $\{\tilde{u}_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ with the properties that $\mathscr{A}\tilde{u}_\varepsilon = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^M)$, $\tilde{u}_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$ and $u_\varepsilon = \tilde{u}_\varepsilon$ in $\Omega_{1,\varepsilon}$. If $\sup_{\varepsilon>0} \mathscr{F}_\varepsilon(u_\varepsilon) \le c$ for some $c \ge -a\lambda_0\mathscr{L}^d(\Omega)$, and if $v_\varepsilon := u_\varepsilon - \tilde{u}_\varepsilon$, then the following hold:*

$$\begin{aligned} \mathscr{A}u &= 0 \quad && \text{in } W^{-1,p}(\Omega;\mathbb{R}^M),\\ v_\varepsilon &= 0 && \text{in } \Omega_{1,\varepsilon},\\ \mathscr{A}v_\varepsilon &= 0 && \text{in } W^{-1,p}(\Omega;\mathbb{R}^M),\\ \varepsilon v_\varepsilon &\rightharpoonup 0 && \text{weakly in } L^p(\Omega;\mathbb{R}^N), \end{aligned}$$

and

$$\liminf_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \liminf_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon) \le \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon), \tag{4.14}$$
$$\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \le \limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon).$$

Proof. As a consequence of the fact that $\mathscr{A}\tilde{u}_\varepsilon = 0$ in Ω for all ε, also the weak limit of $\{\tilde{u}_\varepsilon\}$, i.e., u, must be $\mathscr{A}$-free. As immediate consequences of the definition, we also find that $v_\varepsilon = 0$ in $\Omega_{1,\varepsilon}$ and that $\mathscr{A}v_\varepsilon = 0$ in Ω. Being $\{\tilde{u}_\varepsilon\}$ bounded in $L^p(\Omega;\mathbb{R}^N)$, we also deduce

$$\varepsilon v_\varepsilon = \varepsilon(u_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N).$$

Only the estimates involving the semilimits are now left to prove. It suffices to show that $\lim_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_\varepsilon) = 0$, with $\mathscr{E}_\varepsilon$ as in (4.13). To this end, we observe that in view of (H4), for almost every x we have

$$|f_{0,\varepsilon}(\varepsilon u_\varepsilon) - f_{0,\varepsilon}(\varepsilon v_\varepsilon)| \le \mu(1 + |\varepsilon v_\varepsilon|^{p-1} + |\varepsilon u_\varepsilon|^{p-1})|\varepsilon\tilde{u}_\varepsilon|.$$

Hölder’s inequality yields $|\mathscr{E}_\varepsilon(u_\varepsilon)| \le c\varepsilon\|\tilde{u}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)}$, and the conclusion is achieved by exploiting again the boundedness of $\{\tilde{u}_\varepsilon\}$ in $L^p(\Omega;\mathbb{R}^N)$. □

We conclude this section by showing that, thanks to the $\mathscr{A}$-free decomposition procedure in Lemma 3.2, we can always reduce to the case in which the sequence $\{v_\varepsilon\}$ on the left-hand side of (4.14) is p-equiintegrable. This motivates the Γ-liminf inequality contained in Proposition 5.3. We premise a lemma.

Lemma 4.3. *Let $\{f_{0,\varepsilon}\}$ and f_1 satisfy hypotheses* (H1), (H3), *and* (H4). *Let also $\{u_\varepsilon\}, \{v_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ be bounded sequences such that $u_\varepsilon - v_\varepsilon \to 0$ in measure, and that $\{v_\varepsilon\}$ is p-equiintegrable. Then*

$$\liminf_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}} [f_{0,\varepsilon}(u_\varepsilon) - f_{0,\varepsilon}(v_\varepsilon)]\, dx \geq 0, \tag{4.15}$$

$$\liminf_{\varepsilon \to 0} \int_{\Omega_{1,\varepsilon}} \Big[f_1\Big(\frac{x}{\varepsilon}, u_\varepsilon\Big) - f_1\Big(\frac{x}{\varepsilon}, v_\varepsilon\Big)\Big]\, dx \geq 0.$$

Proof. Despite the dependence of the energy density f_1 on the oscillating variable, up to a different notational realization, the “stiff” case is completely analogous to the “soft” one. For this reason, we detail the proof of (4.15) only

We first note that (4.15) is left unchanged if we replace $f_{0,\varepsilon}$ with $f_{0,\varepsilon} - f_{0,\varepsilon}(0)$, and hence we may assume that $f_{0,\varepsilon}(0) = 0$. Upon extraction of a (non-relabeled) subsequence, we may also suppose that the left-hand side in (4.15) is a limit. In view of [47, Lemma 8.13], for every $\varepsilon > 0$, we can decompose the elements of such subsequence as $u_\varepsilon = u_\varepsilon^o + u_\varepsilon^c$, where $\{u_\varepsilon^o\}$ (the “oscillating” part) is p-equiintegrable and $\{u_\varepsilon^c\}$ (the “concentrating” part) converges to zero in measure. If we let $R_\varepsilon := \{x \in \Omega : u_\varepsilon^o \neq u_\varepsilon\}$, we have that $\mathscr{L}^d(R_\varepsilon) \to 0$, whence, by the current assumptions, $\{u_\varepsilon^o - v_\varepsilon\}$ is p-equiintegrable and converges to zero in measure. Thanks to Vitali's convergence theorem (see, e.g., [47, Theorem 2.24]), it follows that $u_\varepsilon^o - v_\varepsilon \to 0$ strongly $L^p(\Omega; \mathbb{R}^N)$, and, in view of (H4), we deduce

$$\lim_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}} |f_{0,\varepsilon}(u_\varepsilon^o) - f_{0,\varepsilon}(v_\varepsilon)|\, dx = 0. \tag{4.16}$$

By employing the definition of R_ε, (H3), and (H4), we find the estimate

$$\begin{aligned}
\Big|\int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(u_\varepsilon)\, dx - \int_{\Omega_{0,\varepsilon}} [f_{0,\varepsilon}(u_\varepsilon^o) + f_{0,\varepsilon}(u_\varepsilon^c)]\, dx\Big| &= \Big|\int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} f_{0,\varepsilon}(u_\varepsilon^o + u_\varepsilon^c)\, dx - \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} [f_{0,\varepsilon}(u_\varepsilon^o) + f_{0,\varepsilon}(u_\varepsilon^c)]\, dx\Big| \\
&\leq \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} |f_{0,\varepsilon}(u_\varepsilon^o + u_\varepsilon^c) - f_{0,\varepsilon}(u_\varepsilon^c)|\, dx + \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} |f_{0,\varepsilon}(u_\varepsilon^o)|\, dx \\
&\leq \mu \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} (1 + |u_\varepsilon|^{p-1} + |u_\varepsilon^c|^{p-1})|u_\varepsilon^o|\, dx + \Lambda \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} (1 + |u_\varepsilon^o|^p)\, dx.
\end{aligned}$$

Since $\{u_\varepsilon^o\}$ and $\{u_\varepsilon^c\}$ are bounded in $L^p(\Omega; \mathbb{R}^N)$ and $\{u_\varepsilon^o\}$ is p-equiintegrable, we infer that the last term tends to zero as $\varepsilon \to 0$. Therefore, we see that

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}} [f_{0,\varepsilon}(u_\varepsilon) - f_{0,\varepsilon}(v_\varepsilon)]\, dx &= \lim_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}} [f_{0,\varepsilon}(u_\varepsilon^o) + f_{0,\varepsilon}(u_\varepsilon^c) - f_{0,\varepsilon}(v_\varepsilon)]\, dx \\
&= \lim_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} f_{0,\varepsilon}(u_\varepsilon^c)\, dx. \\
&\geq \liminf_{\varepsilon \to 0} \int_{\Omega_{0,\varepsilon}\cap R_\varepsilon} \lambda(-a + |u_\varepsilon^c|^p)\, dx \\
&\geq 0,
\end{aligned}$$

where the second equality is a consequence of (4.16), the second to last inequality is due to (H3), and the last one follows from $\mathscr{L}^d(R_\varepsilon) \to 0$. □

The improved variant of Lemma 4.2 for the liminf inequality in (4.14) reads as follows.

Proposition 4.4 (Refined splitting for the liminf). *Suppose $\mathscr{A}$ is an admissible differential operator and let $u \in L^p(\Omega;\mathbb{R}^N)$. Consider two sequences $\{u_\varepsilon\}, \{\tilde u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ with the property that $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$, $\mathscr{A}\tilde u_\varepsilon = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^M)$, $\tilde u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$, and $u_\varepsilon = \tilde u_\varepsilon$ in $\Omega_{1,\varepsilon}$. If $\sup_{\varepsilon>0} \mathscr{F}_\varepsilon(u_\varepsilon) \le c$ for some $c \ge -a\lambda_0\mathscr{L}^d(\Omega)$, then there exists $\{\tilde v_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that the following hold:*

$$
\begin{aligned}
&\tilde v_\varepsilon = 0 \quad \text{in } \Omega_{1,\varepsilon},\\
&\mathscr{A}\tilde v_\varepsilon = 0 \quad \text{in } W^{-1,p}(\Omega;\mathbb{R}^M),\\
&\{\varepsilon\tilde v_\varepsilon\} \quad \text{is } p\text{-equiintegrable},\\
&\varepsilon\tilde v_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N), && (4.17)\\
&\liminf_{\varepsilon\to0} \mathscr{F}_{0,\varepsilon}(\tilde v_\varepsilon) + \liminf_{\varepsilon\to0} \mathscr{F}_{1,\varepsilon}(\tilde u_\varepsilon) \le \liminf_{\varepsilon\to0} \mathscr{F}_\varepsilon(u_\varepsilon). && (4.18)
\end{aligned}
$$

Proof. The equiboundedness in energy entails that

$$\|\varepsilon u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le c \quad \text{and} \quad \|\chi_{1,\varepsilon} u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le c. \tag{4.19}$$

By Lemma 3.2, there exist a (not relabeled) subsequence of $\{u_\varepsilon\}$ and a sequence $\{w_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that
(1) $\{\varepsilon w_\varepsilon\}$ is bounded, $\mathscr{A}$-free, and p-equiintegrable,
(2) $\varepsilon(w_\varepsilon - u_\varepsilon) \to 0$ strongly in $L^q(\Omega;\mathbb{R}^N)$ for every $q \in [1,p)$,
(3) $\varepsilon\chi_{1,\varepsilon}(w_\varepsilon - u_\varepsilon) \to 0$ strongly in $L^p(\Omega;\mathbb{R}^N)$.
Note that the last property is not mentioned explicitly in the statement of Lemma 3.2, but it is a byproduct of the construction in its proof (see [48, Lemma 2.15]).

For every $\varepsilon > 0$, we define $\tilde w_\varepsilon := \mathsf{E}^\varepsilon_{\mathscr{A}} w_\varepsilon$, with $\mathsf{E}^\varepsilon_{\mathscr{A}}$ as in Assumption 2. We then set $\tilde v_\varepsilon := w_\varepsilon - \tilde w_\varepsilon$. In view of Assumption 2, we immediately obtain that $\tilde v_\varepsilon$ vanishes on $\Omega_{1,\varepsilon}$, that it is $\mathscr{A}$-free in Ω, and that $\{\varepsilon\tilde v_\varepsilon\}$ is p-equiintegrable. To prove (4.17), we consider the identity

$$\varepsilon\tilde v_\varepsilon = \varepsilon(w_\varepsilon - u_\varepsilon) - \varepsilon(\tilde w_\varepsilon - u_\varepsilon).$$

Thanks to (2) above, $\varepsilon(w_\varepsilon - u_\varepsilon) \to 0$ strongly in $L^q(\Omega;\mathbb{R}^N)$. Additionally, by Assumption 2,

$$\varepsilon\|\tilde w_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le c\varepsilon\|w_\varepsilon\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)} \le c\varepsilon\|w_\varepsilon - u_\varepsilon\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)} + c\varepsilon\|u_\varepsilon\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)} \to 0,$$

the convergence to 0 following from (3) above and (4.19). In view of the assumptions on $\{u_\varepsilon\}$ and of (2), (4.17) is inferred.

To complete the proof of the corollary, it suffices now show that for $q \in [1,p)$,

$$\varepsilon(\tilde v_\varepsilon - v_\varepsilon) \to 0 \quad \text{strongly in } L^q(\Omega;\mathbb{R}^N), \tag{4.20}$$

where $v_\varepsilon := u_\varepsilon - \tilde u_\varepsilon$. Indeed, once this is proven, (4.18) is deduced from Lemma 4.2 and Lemma 4.3. To prove (4.20), we notice that, by the definitions of v_ε and $\tilde v_\varepsilon$, as well as by Hölder's inequality, for every $q \in [1,p)$ it holds

$$\|\varepsilon(\tilde v_\varepsilon - v_\varepsilon)\|_{L^q(\Omega;\mathbb{R}^N)} \le \varepsilon\|w_\varepsilon - u_\varepsilon\|_{L^q(\Omega;\mathbb{R}^N)} + c\varepsilon\|\mathsf{E}^\varepsilon_{\mathscr{A}}(w_\varepsilon - u_\varepsilon)\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Then, in view of (2), of Assumption 2, and of (3), we deduce (4.20). □

4.3 Γ-convergence

In this short subsection we tackle the proofs of Theorem 2.6 and Corollary 2.7.

Proof of Theorem 2.6. Statement (1), concerned with the high-contrast compactness of families with equibounded energy, follows from Proposition 4.1.

Let now $u \in U_1$ be fixed and assume that there are two sequences $\{u_\varepsilon\}, \{\tilde u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ with the property that $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$, $\mathscr{A}\tilde u_\varepsilon = 0$ in Ω, $\tilde u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$, and $u_\varepsilon = \tilde u_\varepsilon$ in $\Omega_{1,\varepsilon}$. If the lower limit of $\{\mathscr{F}_\varepsilon(u_\varepsilon)\}$ is not finite, the estimate of point (2) holds trivially. Otherwise, we define $v_\varepsilon := u_\varepsilon - \tilde u_\varepsilon$ and, owing to Proposition 4.4, by applying Proposition 2.8 to $\{v_\varepsilon\}$, as well as Proposition 2.9 to $\{\tilde{u}_\varepsilon\}$, we deduce

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) + \liminf_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(u_\varepsilon)\\ &= \liminf_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \liminf_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon)\\ &\geq \alpha_0 + \mathscr{F}_1(u) =: \mathscr{F}(u).\end{aligned}$$

Finally, we turn to (3). For $u \in U_1$, again in the light of Propositions 2.8 and 2.9, we can find two sequences $\{v_\varepsilon\}, \{\tilde{u}_\varepsilon\} \subset U_1$ such that

$$\begin{aligned} v_\varepsilon &= 0 && \text{in } \Omega_{1,\varepsilon},\\ \varepsilon v_\varepsilon &\rightharpoonup 0 && \text{weakly in } L^p(\Omega;\mathbb{R}^N),\\ \tilde{u}_\varepsilon &\rightharpoonup u && \text{weakly in } L^p(\Omega;\mathbb{R}^N),\\ \limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_{0,\varepsilon}) &\leq \alpha_0, \\ \limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon) &\leq \mathscr{F}_1(u).\end{aligned}$$

Then, setting $u_\varepsilon := v_\varepsilon + \tilde{u}_\varepsilon$, by Lemma 4.2 we deduce the desired limsup inequality from the ones satisfied by $\{\mathscr{F}_{0,\varepsilon}(v_\varepsilon)\}$ and $\{\mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon)\}$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) &\leq \limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) + \limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(u_\varepsilon)\\ &= \limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(v_\varepsilon) + \limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(\tilde{u}_\varepsilon)\\ &\leq \alpha_0 + \mathscr{F}_1(u) =: \mathscr{F}(u).\end{aligned}$$ □

At this stage, the convergence of infima and of minimizers easily follows.

Proof of Corollary 2.7. The equicoercivity of the energies with respect to the high-contrast convergence and the convergence result in Theorem 2.6 yield the conclusion by standard Γ-convergence arguments. □

5 Asymptotics for the soft component

This section is devoted to the proof of Proposition 2.8. We address the asymptotic analysis of the energy stored in the "soft" component of the system, that is, $\{\mathscr{F}_{0,\varepsilon}\}$ in (2.9), and we prove that the limiting behavior is encoded by the functional $\mathscr{F}_0$ in (2.11). We recall that $\mathscr{F}_0$ is finite only when $u = 0$ and in that case the value of the functional equals the constant α_0 given by

$$\alpha_0 := \sup_{\Omega' \Subset \Omega} \inf_{u \in U_0(\Omega')} \int_{\Omega'} \int_{D_0} f_0(u(x,y))\, dy\, dx,$$

where the supremum is meant to run over all open sets that are compactly contained in Ω, and, for any open $\Omega' \subset \Omega$, $U_0(\Omega')$ is as in (2.4).

Remark 5.1 (On the constant α_0). By the definition of $U_0(\Omega')$ in (2.4), we see that

$$U_0(\Omega) \subset U_0(\Omega'_2) \subset U_0(\Omega'_1) \quad \text{whenever } \Omega'_1 \subset \Omega'_2 \subset \Omega. \tag{5.1}$$

Since $\lambda(-a + |\xi|^p) \leq f_0(\xi)$ for every $\xi \in \mathbb{R}^N$, from (5.1) we deduce that

$$\alpha_0 \leq \inf_{u \in U_0} \int_\Omega \int_{D_0} f_0(u(x,y))\, dy\, dx,$$

and, more generally, that

$$\inf_{u \in U_0 \in (\Omega'_1)} \int_{\Omega'_1} \int_{D_0} f_0(u(x,y))\, dy\, dx \leq \inf_{u \in U_0(\Omega'_2)} \int_{\Omega'_2} \int_{D_0} f_0(u(x,y))\, dy\, dx \tag{5.2}$$

when $\Omega_1' \subset \Omega_2' \subset \Omega$. In particular, the supremum in the definition of α_0 is not a maximum, in general. Indeed, suppose that $\Omega_1' \Subset \Omega$ achieves such supremum. Then we would get a contradiction whenever there exists $\Omega_2' \Subset \Omega$ containing Ω_1' such that the inequality in (5.2) is strict.

We separate the discussion of the Γ-liminf and of the Γ-limsup inequalities.

5.1 Γ-liminf inequality

The proof of the Γ-liminf inequality relies on the compactness and splitting results of Section 4, on unfolding techniques, as well as on the following intermediate statement.

Lemma 5.2. *Let $\{f_{0,\varepsilon}\}$ satisfy* (H1) *and* (H3)–(H5). *Suppose that $\{w_\varepsilon\} \subset L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ is a p-equiintegrable family satisfying the following properties:*

$$w_\varepsilon = 0 \quad \text{if } y \in D_1, \tag{5.3}$$

and for all open $\Omega' \Subset \Omega$ there exists $\varepsilon' := \varepsilon'(\Omega')$ such that

$$\mathscr{A}_y w_\varepsilon = 0 \quad \text{in } W^{-1,p}(\mathbb{T}^d;\mathbb{R}^M) \quad \text{for a.e. } x \in \Omega' \text{ if } \varepsilon < \varepsilon'. \tag{5.4}$$

Then it holds

$$\alpha_0 \le \liminf_{\varepsilon\to 0} \int_\Omega \int_{D_0} f_{0,\varepsilon}(w_\varepsilon(x,y))\,dy\,dx.$$

Proof. The proof is divided into two steps.

Step 1: A simplified setting. We start by establishing an inequality for the case in which the densities on the right-hand side do not depend on ε (cf. [25, Lemma 20]), that is, we first prove that

$$\alpha_0 \le \liminf_{\varepsilon\to 0} \int_\Omega \int_{D_0} f_0(w_\varepsilon(x,y))\,dy\,dx. \tag{5.5}$$

According to the definition of α_0 and to Remark 5.1, for an arbitrary $\delta > 0$, there is an open set $\Omega'_\delta \Subset \Omega$ such that $\operatorname{dist}(\partial\Omega, \partial\Omega'_\delta) < \delta$ and

$$\alpha_0 \le \inf_{u\in U_0(\Omega'_\delta)} \int_{\Omega'_\delta} \int_{D_0} f_0(u(x,y))\,dy\,dx + \delta.$$

In view of (5.3) and (5.4), $w_\varepsilon \in U_0(\Omega'_\delta)$ for $\varepsilon > 0$ sufficiently small, whence

$$\begin{aligned}\alpha_0 &\le \liminf_{\varepsilon\to 0} \int_{\Omega'_\delta} \int_{D_0} f_0(w_\varepsilon(x,y))\,dy\,dx + \delta\\ &= \liminf_{\varepsilon\to 0} \int_\Omega \int_{D_0} f_0(w_\varepsilon(x,y))\,dy\,dx - \int_{\Omega\setminus\Omega'_\delta} \int_{D_0} f_0(w_\varepsilon(x,y))\,dy\,dx + \delta.\end{aligned}$$

Thanks to the p-equiintegrability of $\{w_\varepsilon\}$, the conclusion (5.5) now follows by letting δ vanish.

Step 2: The general case. We adapt the argument in the proof of [37, Theorem 5.14], relying on the almost everywhere convergence of $\{f_{0,\varepsilon}\}$ to f_0, as well as on the p-Lipschitz continuity of these energy densities (see (H4) and (H5)).

Fix $\delta > 0$. By the p-equiintegrability of $\{w_\varepsilon\}$ and by the p-growth assumptions on f_0, there exists $r > 0$ such that

$$\sup_{\varepsilon>0} \int_{\{(x,y)\in\Omega\times D_0 \,:\, |w_\varepsilon(x,y)|>r\}} f_0(w_\varepsilon(x,y))\,dx\,dy \le \delta. \tag{5.6}$$

Besides, since $f_{0,\varepsilon}$ and f_0 are p-Lipschitz, we find $\rho > 0$ such that

$$|f_0(\xi) - f_0(\eta)| + \sup_{\varepsilon>0} |f_{0,\varepsilon}(\xi) - f_{0,\varepsilon}(\eta)| \leq \delta \quad \text{for every } \xi, \eta \in B(0,\rho). \tag{5.7}$$

Let now $\xi_1, \dots, \xi_K \in B(0,r)$ be such that

$$\overline{B(0,r)} \subset \bigcup_{k=1}^{K} B(\xi_k, \rho). \tag{5.8}$$

In view of (H5), for any such ξ_k there exist $\bar{\varepsilon}_k > 0$ such that $|f_{0,\varepsilon}(\xi_k) - f_0(\xi_k)| \leq \delta$ if $\varepsilon < \bar{\varepsilon}_k$. By letting now $\bar{\varepsilon} := \min\{\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_K\}$, it follows that for any $k = 1, \dots, K$,

$$|f_{0,\varepsilon}(\xi_k) - f_0(\xi_k)| \leq \delta \quad \text{if } \varepsilon < \bar{\varepsilon}. \tag{5.9}$$

By (5.8), for every $\eta \in \overline{B(0,r)}$ there exists $k \in \{1, \dots, K\}$ such that $\eta \in B(\xi_k, \rho)$. For this particular k, the combination of the triangle inequality, (5.7), and (5.9) yields

$$|f_{0,\varepsilon}(\eta) - f_0(\eta)| \leq |f_{0,\varepsilon}(\eta) - f_{0,\varepsilon}(\xi_k)| + |f_{0,\varepsilon}(\xi_k) - f_0(\xi_k)| + |f_0(\eta) - f_0(\xi_k)| \leq 3\delta \tag{5.10}$$

for every $\eta \in \overline{B(0,r)}$ and every $\varepsilon < \bar{\varepsilon}$.

In view of (5.5) we deduce

$$\begin{aligned}
\alpha_0 &\leq \liminf_{\varepsilon\to 0} \int_\Omega \int_{D_0} f_0(w_\varepsilon(x,y))\,dy\,dx \\
&\leq \liminf_{\varepsilon\to 0} \int_{\{(x,y)\in\Omega\times D_0 : |w_\varepsilon(x,y)|\leq r\}} f_0(w_\varepsilon(x,y))\,dx\,dy + \delta \\
&\leq \liminf_{\varepsilon\to 0} \int_\Omega \int_{D_0} f_{0,\varepsilon}(w_\varepsilon(x,y))\,dy\,dx + 3\delta \mathscr{L}^{2d}(\Omega\times D_0) + \delta,
\end{aligned}$$

where the first inequality is due to (5.6), and the second one to (5.10). The arbitrariness of $\delta > 0$ yields the conclusion. □

We are now ready to tackle the Γ-liminf inequality for the "soft" contribution; the argument is comparable to the one in [25, Lemma 21].

Proposition 5.3. *Let $\{f_{0,\varepsilon}\}_\varepsilon$ satisfy assumptions* (H1) *and* (H3)–(H5)*, and let $u \in L^p(\Omega;\mathbb{R}^N)$. Then, for every p-equiintegrable sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that $u_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$, $\mathscr{A}u_\varepsilon = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^M)$ for every $\varepsilon > 0$, and $\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$, we have*

$$\mathscr{F}_0(u) \leq \liminf_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon).$$

Proof. If the lower limit of $\mathscr{F}_{0,\varepsilon}(u_\varepsilon)$ is not finite, then there is nothing to prove. Otherwise, without loss of generality, we focus on the case in which the lower limit of $\mathscr{F}_{0,\varepsilon}(u_\varepsilon)$ is finite, and we assume that it is a limit. Then it follows by Proposition 4.1 and Assumption 1 that necessarily $u = 0$. Therefore, we are left to prove that

$$\alpha_0 \leq \lim_{\varepsilon\to 0} \int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\varepsilon u_\varepsilon(x))\,dx,$$

where α_0 is given by (2.6) and $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$.

We exhibit a formula for $\mathscr{F}_{0,\varepsilon}(u_\varepsilon)$ that involves the unfolding operator introduced in Lemma 3.6. We let $\hat{u}_\varepsilon$ be the unfolded map of $\varepsilon u_\varepsilon$:

$$\hat{u}_\varepsilon(x,y) := \varepsilon \mathsf{S}_\varepsilon u_\varepsilon(x,y),$$

whence

$$\hat{u}_\varepsilon(x,y) = 0 \quad \text{for almost every } x \in \Omega \text{ and } y \in D_1. \tag{5.11}$$

Also, since the unfolding procedure preserves p-equiintegrability, $\{\hat{u}_\varepsilon\} \subset L^p(\Omega \times Q; \mathbb{R}^N)$ is p-equiintegrable. Finally, Lemma 3.8 grants that

$$\mathscr{A}_y \hat{u}_\varepsilon = 0 \quad \text{in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M) \quad \text{for a.e. } x \in \hat{\Omega}_\varepsilon,$$

where $\hat{\Omega}_\varepsilon$ is as in (3.1). In particular, if $\Omega' \subset \Omega$ is an open set such that the distance $\delta := \operatorname{dist}(\partial\Omega', \partial\Omega)$ is strictly positive, it is clear that $\Omega' \subset \hat{\Omega}_\varepsilon$ if $\sqrt{d}\varepsilon < \delta$. Thus, for all open $\Omega' \Subset \Omega$

$$\mathscr{A}_y \hat{u}_\varepsilon = 0 \quad \text{in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M) \quad \text{for a.e. } x \in \Omega' \text{ if } \varepsilon < \varepsilon', \tag{5.12}$$

where $\varepsilon' > 0$ is a suitable threshold depending on Ω'.

By the definition of $\Omega_{0,\varepsilon}$ (see (1.1)), we have

$$\mathscr{F}_{0,\varepsilon}(u_\varepsilon) = \varepsilon^d \sum_{z \in Z_\varepsilon} \int_{D_0} f_{0,\varepsilon}(\varepsilon u_\varepsilon(\varepsilon(y+z)))\,\mathrm{d}y.$$

Being z an integer-valued vector, it holds $\hat{u}_\varepsilon(\varepsilon z, y) = \varepsilon u_\varepsilon(\varepsilon(y+z))$ for every $z \in Z_\varepsilon$ and $y \in D_0$. Hence, we obtain

$$\begin{aligned}
\mathscr{F}_{0,\varepsilon}(u_\varepsilon) &= \varepsilon^d \sum_{z \in Z_\varepsilon} \int_{D_0} f_{0,\varepsilon}(\hat{u}_\varepsilon(\varepsilon z, y))\,\mathrm{d}y \\
&= \sum_{z \in Z_\varepsilon} \int_{\varepsilon(Q+z)} \int_{D_0} f_{0,\varepsilon}\Big(\hat{u}_\varepsilon\Big(\varepsilon\Big\lfloor \frac{x}{\varepsilon} \Big\rfloor, y\Big)\Big)\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \int_{\hat{\Omega}_\varepsilon} \int_{D_0} f_{0,\varepsilon}(\hat{u}_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

because for all $x \in \varepsilon(Q+z)$, $\lfloor \frac{x}{\varepsilon} \rfloor = z$. We can rewrite the last estimate as follows:

$$\mathscr{F}_{0,\varepsilon}(u_\varepsilon) \geq \int_{\Omega} \int_{D_0} f_{0,\varepsilon}(\hat{u}_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x - \int_{\Omega \setminus \hat{\Omega}_\varepsilon} \int_{D_0} f_{0,\varepsilon}(\hat{u}_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

From this, owing to (5.11) and (5.12), we can invoke Lemma 5.2 to infer

$$\mathscr{F}_{0,\varepsilon}(u_\varepsilon) \geq \alpha_0 - \int_{\Omega \setminus \hat{\Omega}_\varepsilon} \int_{D_0} f_{0,\varepsilon}(\hat{u}_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

In view of the p-equiintegrability of $\{\hat{u}_\varepsilon\}$, the desired liminf inequality is now obtained by taking the limit as $\varepsilon \to 0$. □

5.2 Γ-limsup inequality

We now turn to the Γ-limsup inequality for the energy functional associated to the "soft" portion of the material. The optimality of the lower bound identified in Proposition 5.3 hinges upon the next proposition.

Proposition 5.4. *Let $\{f_{0,\varepsilon}\}$ satisfy (H1) and (H3)–(H5), and let $\Omega' \Subset \Omega$ be open. Then, for all $w \in U_0(\Omega')$, there exists a family $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^N)$ that satisfies the following:*
(1) *$u_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$ and u_ε is $\mathscr{A}$-free in Ω for all $\varepsilon > 0$,*
(2) *$\varepsilon u_\varepsilon \xrightarrow{2} w'$ strongly two-scale in L^p, where $w'(x,y) := \chi_{\Omega'}(x) w(x,y)$,*
(3) *it holds that*

$$\limsup_{\varepsilon \to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) \leq \int_{\Omega'} \int_{D_0} f_0(w(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

We postpone the proof of Proposition 5.4 to the end of the section, and we discuss next how the Γ-limsup inequality is deduced from it.

Corollary 5.5. *Let $\{f_{0,\varepsilon}\}$ satisfy hypotheses* (H1) *and* (H3)–(H5), *and let $u \in L^p(\Omega;\mathbb{R}^N)$. Then there exists a family $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that the following holds:*
(1) *$u_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$ and u_ε is $\mathscr{A}$-free in Ω for all $\varepsilon > 0$,*
(2) *$\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^N)$,*
(3) *the upper limit inequality*

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) \le \mathscr{F}_0(u)$$

is satisfied.

Proof. We first remark that if $u \neq 0$, there is nothing to prove. Thus, we assume in what follows that $u = 0$ and $\mathscr{F}_0(u) = \alpha_0$.

Fix now $\delta > 0$ and $\Omega' \Subset \Omega$. By the definition of α_0, there exists a map $w_\delta \in U_0(\Omega')$ such that

$$\int_{\Omega'}\int_{D_0} f_0(w_\delta(x,y))\,dy\,dx \le \alpha_0 + \delta.$$

Proposition 5.4 yields a sequence $\{u_\varepsilon\}$ such that
(1) $u_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$ and u_ε is $\mathscr{A}$-free in Ω for all $\varepsilon > 0$,
(2) $\{\varepsilon u_\varepsilon\}$ converges strongly two-scale in L^p to $\chi_{\Omega'}(x)w_\delta(x,y)$,
(3) the inequality

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) \le \int_{\Omega'}\int_{D_0} f_0(w_\delta(x,y))\,dy\,dx$$

is satisfied.

Lemma 3.5 and Assumption 1 entail that $\varepsilon u_\varepsilon \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^N)$. Besides, the last estimate and the definition of w_δ get

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{0,\varepsilon}(u_\varepsilon) \le \alpha_0 + \delta.$$

The conclusion hence follows then by the arbitrariness of δ and by standard properties of Γ-convergence (see, e.g., [14, Section 1.2]). □

We now address the proof of Proposition 5.4.

Proof of Proposition 5.4. We construct a family $\{\tilde{u}_\varepsilon\}$ that strongly two-scale converges to

$$w'(x,y) := \chi_{\Omega'}(x)w(x,y) \quad \text{in } L^p,$$

so that $\{u_\varepsilon\}$ will be given by $u_\varepsilon := \frac{\tilde{u}_\varepsilon}{\varepsilon}$. However, because of measurability issues, we cannot deal directly with a general $w \in U_0(\Omega')$. The argument is thus subdivided in three parts: firstly, we define $\{\tilde{u}_\varepsilon\}$ under some extra regularity assumption on w; then, we prove the limsup inequality when w is regular; lastly, we recover the general statement by means of an approximation argument.

Step 1: Construction of $\{\tilde{u}_\varepsilon\}$ when w is regular. In this step we assume that w possesses some extra regularity in the x variable, namely we take $w \in U_0(\Omega') \cap L^p_{\mathrm{per}}(\mathbb{R}^d; C(\bar{\Omega};\mathbb{R}^N))$. We construct a sequence $\{\tilde{u}_\varepsilon\}$ of $\mathscr{A}$-free maps such that $\tilde{u}_\varepsilon = 0$ on $\Omega_{1,\varepsilon}$ for all $\varepsilon > 0$ and that $\tilde{u}_\varepsilon \xrightarrow{2} w'$ strongly two scale in L^p. Below, when we write $w(x,y)$, we consider the second entry y to be the periodic variable.

Recalling (1.1), we define

$$\hat{\Omega}'_\varepsilon := \bigcup_{z\in\hat{Z}'_\varepsilon} \varepsilon(Q+z), \qquad \hat{Z}'_\varepsilon := \{z \in \mathbb{Z}^d : \varepsilon(Q+z) \subset \Omega'\} \subset Z_\varepsilon,$$

and, for $\varepsilon > 0$ and $(\bar{x},\bar{y}) \in \Omega \times \mathbb{R}^d$, we consider the averages of $w(\cdot,\bar{y})$ on the cubes that compound $\hat{\Omega}'_\varepsilon$:

$$w_\varepsilon(\bar{x},\bar{y}) := \begin{cases} \displaystyle\fint_{\varepsilon(Q+z)} w(x,\bar{y})\,dx & \text{if } \bar{x} \in \varepsilon(Q+z) \text{ for some } z \in \hat{Z}'_\varepsilon, \\ 0 & \text{for any other } \bar{x} \in \Omega. \end{cases}$$

In this way, $w_\varepsilon(\cdot, y)$ is piecewise constant for all $y \in Q$ and $w_\varepsilon(x, \cdot)$ is Q-periodic for almost every $x \in \Omega$. The position

$$\tilde{u}_\varepsilon(x) := w_\varepsilon\Big(x, \frac{x}{\varepsilon}\Big) \quad \text{for every } x \in \Omega,$$

defines a measurable function which vanishes on $\Omega_{1,\varepsilon}$, because $w = 0$ on $\Omega \times D_1$.

We firstly check that $\{\tilde{u}_\varepsilon\}$ converges strongly two-scale in L^p to $w'(x, y) := \chi_{\Omega'}(x)w(x, y)$. To prove the claim, we start by showing that $\{\tilde{u}_\varepsilon\}$ is bounded in $L^p(\Omega; \mathbb{R}^N)$. The properties of $\{w_\varepsilon\}$ and a change of variables grant that the following identities hold:

$$\begin{aligned}
\int_\Omega |\tilde{u}_\varepsilon(x)|^p \,dx &= \int_{\Omega_{0,\varepsilon}} \Big|w_\varepsilon\Big(x, \frac{x}{\varepsilon}\Big)\Big|^p \,dx \\
&= \sum_{z \in Z_\varepsilon} \int_{\varepsilon(D_0+z)} \Big|w_\varepsilon\Big(x, \frac{x}{\varepsilon}\Big)\Big|^p \,dx \\
&= \sum_{z \in Z_\varepsilon} \varepsilon^d \int_{D_0} |w_\varepsilon(\varepsilon(z + y), y)|^p \,dy \\
&= \sum_{z \in \hat{Z}'_\varepsilon} \varepsilon^d \int_{D_0} \Big| \fint_{\varepsilon(Q+z)} w(x, y) \,dx\Big|^p \,dy.
\end{aligned}$$

From Jensen's inequality we then deduce

$$\int_\Omega |\tilde{u}_\varepsilon(x)|^p \,dx \le \sum_{z \in \hat{Z}'_\varepsilon} \varepsilon^d \int_{D_0} \fint_{\varepsilon(Q+z)} |w(x, y)|^p \,dx\,dy = \int_{D_0} \int_{\hat{\Omega}'_\varepsilon} |w(x, y)|^p \,dx\,dy. \tag{5.13}$$

As for the convergence of the sequence $\{\tilde{u}_\varepsilon\}$, in view of Lemma 3.5, formula (5.13) yields the existence of a map $\tilde{w} \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ such that $\tilde{u}_\varepsilon \xrightharpoonup{2} \tilde{w}$ weakly two-scale in L^p. To identify the limit $\tilde{w}$, we consider $\phi \in C(\bar{\Omega}; C_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$ and, by similar computations to the ones above, we find

$$\begin{aligned}
\int_\Omega \tilde{u}_\varepsilon(x) \cdot \phi\Big(x, \frac{x}{\varepsilon}\Big) \,dx &= \int_{\Omega_{0,\varepsilon}} w_\varepsilon\Big(x, \frac{x}{\varepsilon}\Big) \cdot \phi\Big(x, \frac{x}{\varepsilon}\Big) \,dx \\
&= \sum_{z \in Z_\varepsilon} \int_{\varepsilon(D_0+z)} w_\varepsilon\Big(x, \frac{x}{\varepsilon}\Big) \cdot \phi\Big(x, \frac{x}{\varepsilon}\Big) \,dx \\
&= \varepsilon^d \sum_{z \in Z_\varepsilon} \int_{D_0} w_\varepsilon(\varepsilon(z + y), y) \cdot \phi(\varepsilon(z + y), y) \,dy \\
&= \sum_{z \in \hat{Z}'_\varepsilon} \int_{D_0} \int_{\varepsilon(Q+z)} w(x, y) \cdot \phi(\varepsilon(z + y), y) \,dx\,dy \\
&= \int_{\hat{\Omega}'_\varepsilon} \int_{D_0} w(x, y) \cdot \tilde{\phi}_\varepsilon(x, y) \,dy\,dx,
\end{aligned}$$

where $\tilde{\phi}_\varepsilon(x, y) := \phi(\varepsilon(y + z), y)$ if $x \in \varepsilon(Q + z)$ with $z \in \hat{Z}'_\varepsilon$. By the dominated convergence theorem, we infer

$$\lim_{\varepsilon\to0} \int_\Omega \tilde{u}_\varepsilon(x) \cdot \phi\Big(x, \frac{x}{\varepsilon}\Big) \,dx = \int_{\Omega'} \int_{D_0} w(x, y) \cdot \phi(x, y) \,dy\,dx,$$

that is, $\tilde{u}_\varepsilon \xrightharpoonup{2} w'$ weakly two-scale in L^p (recall that $w(x, y) = 0$ if $y \in D_1$). In turn, the weak two-scale convergence implies

$$\|w\|_{L^p(\Omega'; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))} \le \liminf_{\varepsilon\to0} \|\tilde{u}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)},$$

which, combined with (5.13), ensures that

$$\lim_{\varepsilon\to0} \|\tilde{u}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} = \|w\|_{L^p(\Omega'; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))}.$$

In view of the definition of strong two-scale convergence, cf. Definition 3.4, we conclude that $\tilde{u}_\varepsilon \xrightarrow{2} w'$ strongly two-scale in L^p.

Finally, we show that $\{\tilde u_\varepsilon\}$ is $\mathscr{A}$-free on Ω. To this end, we fix $\psi \in W_0^{1,p'}(\Omega;\mathbb{R}^M)$ and, by repeating the steps already seen above, we obtain

$$\int_\Omega \tilde u_\varepsilon(x)\cdot \mathscr{A}^*\psi(x)\,\mathrm{d}x = \sum_{z\in\hat Z'_\varepsilon}\int_{\varepsilon(Q+z)}\int_{D_0} w(x,y)\cdot\mathscr{A}^*\psi(\varepsilon(z+y))\,\mathrm{d}y\,\mathrm{d}x.$$

If $\eta \in C_c^\infty(Q;[0,1])$ is a cut-off function which equals 1 on D_0, thanks to the fact that $w(x,\cdot)$ vanishes on D_1 and is $\mathscr{A}$-free on $\mathbb{T}^d$ for almost every $x\in\hat\Omega'_\varepsilon$, we conclude

$$\begin{aligned}\int_\Omega \tilde u_\varepsilon(x)\cdot \mathscr{A}^*\psi(x)\,\mathrm{d}x &= \sum_{z\in\hat Z'_\varepsilon}\int_{\varepsilon(Q+z)}\int_{D_0} w(x,y)\cdot\mathscr{A}^*\psi(\varepsilon(z+y))\,\mathrm{d}y\,\mathrm{d}x\\ &= \sum_{z\in\hat Z'_\varepsilon}\int_{\varepsilon(Q+z)}\int_{D_0} w(x,y)\cdot\mathscr{A}^*[\eta(y)\psi(\varepsilon(z+y))]\,\mathrm{d}y\,\mathrm{d}x\\ &= \sum_{z\in\hat Z'_\varepsilon}\int_{\varepsilon(Q+z)}\int_{Q} w(x,y)\cdot\mathscr{A}^*[\eta(y)\psi(\varepsilon(z+y))]\,\mathrm{d}y\,\mathrm{d}x\\ &= 0.\end{aligned}$$

Step 2: Limsup inequality when w is regular. We have so far shown that, if $w\in U_0(\Omega')\cap L^p_{\mathrm{per}}(\mathbb{R}^d;C(\bar\Omega;\mathbb{R}^N))$ and $\{\tilde u_\varepsilon\}$ is the sequence of Step 1, then

$$\varepsilon u_\varepsilon = \tilde u_\varepsilon \xrightarrow{2} w' \quad \text{strongly two-scale in } L^p,$$

where $w'(x,y) := \chi_{\Omega'}(x)w(x,y)$. Moreover, $u_\varepsilon =$ on $\Omega_{0,\varepsilon}$ and $\mathscr{A}u_\varepsilon = 0$ on Ω for all $\varepsilon>0$. Here, we prove the inequality

$$\limsup_{\varepsilon\to0}\mathscr{F}_{0,\varepsilon}(u_\varepsilon) = \limsup_{\varepsilon\to0}\int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\tilde u_\varepsilon(x))\,\mathrm{d}x \le \int_{\Omega'}\int_{D_0} f_0(w(x,y))\,\mathrm{d}y\,\mathrm{d}x. \tag{5.14}$$

We preliminarily notice that, by construction, $\tilde u_\varepsilon$ vanishes outside the set $\hat\Omega'_\varepsilon$. Therefore, we have the identities

$$\mathscr{F}_{0,\varepsilon}(u_\varepsilon) = \int_{\hat\Omega'_\varepsilon} f_{0,\varepsilon}(\tilde u_\varepsilon(x))\,\mathrm{d}x = \varepsilon^d\sum_{z\in\hat Z'_\varepsilon}\int_Q f_{0,\varepsilon}(\tilde u_\varepsilon(\varepsilon(y+z)))\,\mathrm{d}y.$$

Since $z\in\mathbb{Z}^d$, we have that $\tilde u_\varepsilon(\varepsilon(y+z)) = \mathsf{S}_\varepsilon\tilde u_\varepsilon(\varepsilon z,y)$, S_ε being the unfolding operator. Therefore, by exploiting the properties of S_ε, the energy of $\tilde u_\varepsilon$ is rewritten as follows:

$$\begin{aligned}\mathscr{F}_{0,\varepsilon}(u_\varepsilon) &= \varepsilon^d\sum_{z\in\hat Z'_\varepsilon}\int_Q f_{0,\varepsilon}(\mathsf{S}_\varepsilon\tilde u_\varepsilon(\varepsilon z,y))\,\mathrm{d}y\\ &= \sum_{z\in\hat Z'_\varepsilon}\int_{\varepsilon(Q+z)}\int_{D_0} f_{0,\varepsilon}\Big(\mathsf{S}_\varepsilon\tilde u_\varepsilon\Big(\varepsilon\Big\lfloor\frac{x}{\varepsilon}\Big\rfloor,y\Big)\Big)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{\hat\Omega'_\varepsilon}\int_{D_0} f_{0,\varepsilon}(\mathsf{S}_\varepsilon\tilde u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{\Omega'}\int_{D_0} f_{0,\varepsilon}(\mathsf{S}_\varepsilon\tilde u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

where the last identity is due to the fact that $\mathsf{S}_\varepsilon\tilde u_\varepsilon$ vanishes for $x\notin\hat\Omega'_\varepsilon$.

In the light of the previous equalities, we have

$$\begin{aligned}\mathscr{F}_{0,\varepsilon}(u_\varepsilon) = {}&\int_{\Omega'}\int_{D_0} f_{0,\varepsilon}(\mathsf{S}_\varepsilon\tilde u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x - \int_{\Omega'}\int_{D_0} f_{0,\varepsilon}(w(x,y))\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega'}\int_{D_0} f_{0,\varepsilon}(w(x,y))\,\mathrm{d}y\,\mathrm{d}x\\ &- \int_{\Omega'}\int_{D_0} f_0(w(x,y))\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega'}\int_{D_0} f_0(w(x,y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Keeping in mind Lemma 3.5, we observe that $\mathsf{S}_\varepsilon\tilde u_\varepsilon \to w$ strongly in $L^p(\Omega'\times D_0;\mathbb{R}^N)$, and consequently, owing to (H4) and (H5), the limits as $\varepsilon\to0$ of the first and of the second term on the right-hand side equal 0. We thereby conclude that (5.14) holds.

Step 3: Recovering the general statement. Let $w \in U_0(\Omega')$. We first extend it to the whole space setting $w = 0$ for $x \in \mathbb{R}^d \setminus \Omega$, and then we mollify it with respect to the variable x. This procedure ensures that $\tilde{w}$, the regularization of w, belongs to $U_0(\Omega') \cap L^p_{\mathrm{per}}(\mathbb{R}^d; C(\bar{\Omega}; \mathbb{R}^N))$ and it is close to w in $L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N))$. Now, Steps 1–2 yield a recovery sequence for $\tilde{w}$, and, by means of a diagonal argument and of the continuity of the right-hand side of (5.14) in L^p, a recovery sequence for w satisfying all the desired requirements can be constructed. □

On the whole, the results above form a proof of the Γ-convergence of the energies $\{\mathscr{F}_{0,\varepsilon}\}$.

Proof of Proposition 2.8. The fact that $\mathscr{F}_0$ is a lower bound for the asymptotic behavior of the sequence $\{\mathscr{F}_{0,\varepsilon}\}$ is established in Proposition 5.3. The optimality of such lower bound is proven in Corollary 5.5. □

6 Asymptotics for the stiff component

In this section, we characterize the limiting behavior of the family $\{\mathscr{F}_{1,\varepsilon}\}$ in (2.10), which accounts for the energy of the "stiff" component of the system, and we prove that $\{\mathscr{F}_{1,\varepsilon}\}$ Γ-converges to the functional $\mathscr{F}_1$ in (2.12). We rely on the contribution by I. Fonseca and S. Krömer about homogenization of multiple integrals under differential constraints:

Theorem 6.1 ([45, Theorem 1.1]). *Let $g\colon \mathbb{R}^d \times \mathbb{R}^N \to [0, +\infty)$ be a Carathéodory function which is Q-periodic with respect to its first argument. Assume that there exist $c \ge 0$, $\Lambda > 0$, and $p > 1$ satisfying*

$$-c \le g(y, \xi) \le \Lambda(1 + |\xi|^p) \quad \text{for a.e. } y \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{R}^N.$$

Then, for all $u \in L^p(\Omega; \mathbb{R}^N)$ such that $\mathscr{A}u = 0$, the functional

$$\mathscr{G}_\varepsilon(u) := \int_\Omega g\Big(\frac{x}{\varepsilon}, u(x)\Big)\,\mathrm{d}x$$

Γ-converges with respect to the weak L^p-convergence to

$$\mathscr{G}_{\mathrm{hom}}(u) := \int_\Omega g_{\mathrm{hom}}(u(x))\,\mathrm{d}x,$$

where, for all $\xi \in \mathbb{R}^N$,

$$g_{\mathrm{hom}}(\xi) := \liminf_{k\to+\infty} \inf\Big\{\int_Q g(ky, \xi + v(y))\,\mathrm{d}y : v \in L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N),\ \int_Q v(z)\,\mathrm{d}z = 0,\ \mathscr{A}v = 0 \text{ in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M)\Big\}.$$

As a corollary of the theorem above, we derive the asymptotics of $\{\mathscr{F}_{1,\varepsilon}\}$.

Proof of Proposition 2.9. For $\varepsilon > 0$, we introduce the set

$$\tilde{\Omega}_{1,\varepsilon} := \Omega \cap \bigcup_{z\in\mathbb{Z}^d} \varepsilon(D_1 + z),$$

and we observe that $\tilde{\Omega}_{1,\varepsilon} \subset \Omega_{1,\varepsilon}$ (recall (1.2)). We let

$$\tilde{\mathscr{F}}_{1,\varepsilon}(u) := \begin{cases} \displaystyle\int_{\tilde{\Omega}_{1,\varepsilon}} \tilde{f}_1\Big(\frac{x}{\varepsilon}, u\Big)\,\mathrm{d}x & \text{if } u \in U_1, \\ +\infty & \text{otherwise in } L^p(\Omega; \mathbb{R}^N), \end{cases}$$

with $\tilde{f}_1\colon \mathbb{R}^d \times \mathbb{R}^N \to [-a\lambda, +\infty)$ defined as

$$\tilde{f}_1(y, \xi) := f_1(y, \xi) \sum_{z\in\mathbb{Z}^d} \chi_{D_1+z}(y),$$

where a and λ are the constants in (H3), and χ_{D_1+z} is the characteristic function of the set $D_1 + z$. When u is $\mathscr{A}$-free we can hence write

$$\mathscr{F}_{1,\varepsilon}(u) = \tilde{\mathscr{F}}_{1,\varepsilon}(u) + \int_{\Omega_{1,\varepsilon}\setminus\tilde{\Omega}_{1,\varepsilon}} f_1\Big(\frac{x}{\varepsilon}, u(x)\Big)\,\mathrm{d}x. \tag{6.1}$$

We observe that Theorem 6.1 applies to $\tilde{\mathscr{F}}_{1,\varepsilon}$, yielding

$$\mathscr{F}_1(u) = \Gamma - \lim_{\varepsilon\to 0} \tilde{\mathscr{F}}_{1,\varepsilon}(u)$$

in the weak L^p-topology. As for the difference between $\mathscr{F}_{1,\varepsilon}$ and $\tilde{\mathscr{F}}_{1,\varepsilon}$, by (H3) one has

$$-a\lambda\mathscr{L}^d(\Omega_{1,\varepsilon} \setminus \tilde{\Omega}_{1,\varepsilon}) \le \int_{\Omega_{1,\varepsilon}\setminus\tilde{\Omega}_{1,\varepsilon}} f_1\Big(\frac{x}{\varepsilon}, u(x)\Big)\,dx \le \Lambda\Big(\mathscr{L}^d(\Omega_{1,\varepsilon} \setminus \tilde{\Omega}_{1,\varepsilon}) + \int_{\Omega_{1,\varepsilon}\setminus\tilde{\Omega}_{1,\varepsilon}} |u(x)|^p\,dx\Big).$$

Notice that the term $\mathscr{L}^d(\Omega_{1,\varepsilon} \setminus \tilde{\Omega}_{1,\varepsilon})$ is at most of order ε. Thus, the lower bound for $\{\mathscr{F}_{1,\varepsilon}\}$ follows from (6.1) and Theorem 6.1:

$$\mathscr{F}_1(u) \le \liminf_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(u_\varepsilon)$$

for all $u \in L^p(\Omega;\mathbb{R}^N)$ such that $\mathscr{A}u = 0$ and all $\mathscr{A}$-free sequences $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ that converge to u weakly. For what concerns the upper bound, let us fix $u \in L^p(\Omega;\mathbb{R}^N)$ in the kernel of $\mathscr{A}$ and consider a recovery sequence $\{u_\varepsilon\}$ for $\{\tilde{\mathscr{F}}_{1,\varepsilon}\}$ given by Theorem 6.1. Thanks to Lemma 3.2, we get a p-equiintegrable family $\{v_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ such that $\{u_\varepsilon - v_\varepsilon\}$ converges to 0 strongly in $L^q(\Omega;\mathbb{R}^N)$ for all $q \in [1,p)$. Therefore,

$$\lim_{\varepsilon\to 0} \int_{\Omega_{1,\varepsilon}\setminus\tilde{\Omega}_{1,\varepsilon}} |v_\varepsilon(x)|^p\,dx = 0.$$

Let now $\{\varepsilon_n\}$ be such that

$$\limsup_{\varepsilon\to 0} \tilde{\mathscr{F}}_{1,\varepsilon}(v_\varepsilon) = \lim_{n\to+\infty} \tilde{\mathscr{F}}_{1,\varepsilon_n}(v_{\varepsilon_n}).$$

By Lemma 4.3, we obtain the desired upper limit inequality:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(v_\varepsilon) = \limsup_{\varepsilon\to 0} \tilde{\mathscr{F}}_{1,\varepsilon}(v_\varepsilon) &= \lim_{n\to+\infty} \tilde{\mathscr{F}}_{1,\varepsilon_n}(v_{\varepsilon_n}) \\ &\le \lim_{n\to+\infty} \tilde{\mathscr{F}}_{1,\varepsilon_n}(u_{\varepsilon_n}) \le \limsup_{\varepsilon\to 0} \tilde{\mathscr{F}}_{1,\varepsilon}(u_\varepsilon) \le \mathscr{F}_1(u).\end{aligned}$$

□

7 Admissible differential constraints

In what follows we deepen our analysis of constant rank operators. We investigate the existence of Sobolev potentials for $\mathscr{A}$-free maps and of extension operators on perforated domains. In particular, we prove Theorems 2.10 and 2.11.

In contrast to the previous sections, here we take into account also differential operators of order greater than 1: we treat the broader class of linear, k-th order, homogeneous differential operators with constant coefficients. To fix the notation, given $k, N, M \in \mathbb{N} \setminus \{0\}$, we consider linear maps $A^{(i)}: \mathbb{R}^N \to \mathbb{R}^M$ for all d-dimensional multi-indices i with $|i| = k$ and we focus on the operators whose action on a function $u: \mathbb{R}^N \to \mathbb{R}^M$ is given by

$$\mathscr{A}u := \sum_{|i|=k} A^{(i)}\partial_i u.$$

Recall that if $i := (i_1, \ldots, i_d) \in \mathbb{N}^d$ is a multi-index, then

$$\partial_i u(x) := \frac{\partial^{|i|}u}{\partial_{i_1}x_1 \cdots \partial_{i_d}x_d}(x).$$

We keep in place the constant rank condition, which currently reads: there exists $r \in \mathbb{N}$ such that the rank of $\mathbb{A}[\omega]$ equals r for all $\omega \in \mathbb{R}^d \setminus \{0\}$, where

$$\mathbb{A}[\omega] := \sum_{|i|=k} \omega^i A^{(i)}$$

is the symbol of $\mathscr{A}$ and $\omega^i = \Pi_{j=1}^d \omega_j^{i_j}$ if $i = (i_1, \ldots, i_d)$ is a multi-index. Note that when $k = 1$, we recover the setting of Section 2.2. As before, we say that a map u is $\mathscr{A}$-free in Ω if $\mathscr{A}u = 0$ in $W^{-k,p}(\Omega;\mathbb{R}^M)$.

7.1 Existence of potentials

This subsection is devoted to the proof of Theorem 2.10. Loosely speaking, our goal is to prove that, given an operator $\mathscr{A}$ of constant rank, there exists a second differential operator $\mathscr{B}$ such that any $\mathscr{A}$-free map $u \in L^p$ satisfies $u = \mathscr{B}w$ for a suitable Sobolev function w. When such a relation holds, we will say for brevity that w is a *potential* for u and that $\mathscr{B}$ is a *potential* for $\mathscr{A}$. We are able to prove existence of potentials for those functions that are $\mathscr{A}$-free on a certain class of subsets of $\mathbb{R}^d$, which we introduce in the next definition.

Definition 7.1. Let $\mathscr{A}$ be a homogeneous operator of order k. An open set $\Omega \subset \mathbb{R}^d$ is an *$\mathscr{A}$-extension domain* if there exist $\mathsf{E}_{\mathscr{A}} : L^p(\Omega; \mathbb{R}^N) \to L^p(\mathbb{R}^d; \mathbb{R}^N)$ and $c > 0$ such that the following holds: for all $u \in L^p(\Omega; \mathbb{R}^N)$ such that $\mathscr{A}u = 0$ in $W^{-k,p}(\Omega; \mathbb{R}^M)$, we have
(1) $\mathsf{E}_{\mathscr{A}} u = u$ a.e. in Ω,
(2) $\|\mathsf{E}_{\mathscr{A}} u\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega;\mathbb{R}^N)}$, and
(3) $\mathscr{A}(\mathsf{E}_{\mathscr{A}} u) = 0$ on $\mathbb{R}^d$.

Analogous definitions concerning extensions of sequences that are (asymptotically) $\mathscr{A}$-free are found in [46, Definition 1.4] and [45, Lemma 2.8] (i.e., Lemma 3.3 above), while the reader is referred to Section 7.2 below for a more detailed discussion on extension problems. We also refer to [55] for a related study in the stochastic setting. It is important to notice that the property of being an $\mathscr{A}$-extension domain carries some topological consequences.

Remark 7.2 (On the topology of $\mathscr{A}$-extension domains). The requirements of Definition 7.1 may act as implicit restrictions on the topology of the set Ω, according to the specific choice of $\mathscr{A}$. Let us consider, for instance, the case in which $d = 3$, $\mathscr{A}$=curl, and $\Omega = \mathbb{R}^3 \setminus \mathbb{R}e_3$, $\{e_1, e_2, e_3\}$ being the canonical basis in $\mathbb{R}^3$. We assume by contradiction that Ω is an $\mathscr{A}$-extension domain. Then, if for every $x = (x_1, x_2, x_3) \in \Omega$ we let

$$u(x_1, x_2, x_3) := \Big(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0\Big),$$

we find that $\operatorname{curl} u = 0$ almost everywhere in Ω, whence $\operatorname{curl}(\mathsf{E}_{\mathscr{A}} u) = 0$ in $\mathbb{R}^3$, if $\mathsf{E}_{\mathscr{A}}$ is the extension operator in Definition 7.1 for $\mathscr{A} = \operatorname{curl}$. This fact, in turn, yields a potential w such that $\mathsf{E}_{\mathscr{A}} u = \nabla w$ in $\mathbb{R}^3$ and, from the requirement that $\mathsf{E}_{\mathscr{A}} u = u$ on Ω, we conclude that $u = \nabla w$ on Ω. This leads to a contradiction, for the field u is not conservative.

For the proof of Theorem 2.10 we rely heavily on some recent contributions by B. Raiţă and coauthors, who established existence of potentials for $\mathscr{A}$-free smooth maps and Korn-type inequalities for constant rank operators. Further related bibliographical comments are provided when we discuss Propositions 7.4 and 7.5 below. Before dealing with the proof, we pinpoint here that Theorem 2.10 constitutes a sufficient condition for Assumption 1 to hold:

Corollary 7.3. *Assume that $\mathscr{A}$ is a first order operator satisfying the assumptions of Theorem 2.10 and that the unit cube Q is an $\mathscr{A}$-extension domain. Then, for any open set $\Omega' \subset \Omega$, the space $U_0(\Omega')$ in (2.4) is characterized as follows:*

$$U_0(\Omega') = \Big\{u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N)) : u = 0 \textit{ if } y \in D_1,\ \mathscr{A}_y u = 0 \textit{ in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M) \textit{ for a.e. } x \in \Omega',$$
$$\int_Q u(x, y)\,\mathrm{d}y = 0 \textit{ for a.e. } x \in \Omega'\Big\}.$$

Proof. For the desired equality to hold, we just need to prove that

$$U_0(\Omega') \subset \Big\{u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d; \mathbb{R}^N)) : u = 0 \text{ if } y \in D_1,\ \mathscr{A}_y u = 0 \text{ in } W^{-1,p}(\mathbb{T}^d; \mathbb{R}^M) \text{ for a.e. } x \in \Omega',$$
$$\int_Q u(x, y)\,\mathrm{d}y = 0 \text{ for a.e. } x \in \Omega'\Big\}.$$

Let us fix $u \in U_0(\Omega')$. Theorem 2.10 grants that for a.e. $x \in \Omega'$ there exists $w_x \in W^{\ell,p}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M)$ such that $u(x,\cdot) = \mathscr{B}_y w_x$ a.e. in Q. To the aim of defining a measurable selection of the maps $\{w_x\}_{x\in\Omega'}$, we exploit Proposition 3.1.

We first observe that for every $u \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N))$ and $x \in \Omega'$ the set

$$W(x) := \{w \in W^{\ell,p}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M) : \mathscr{B}_y w(y) = u(x,y) \text{ for a.e. } y \in \mathbb{R}^d\}$$

is either empty or a closed (and hence complete) subspace of $W^{\ell,p}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M)$. Besides,

$$x \mapsto u(x,\,\cdot\,) \in L^p_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^N)$$

being measurable, then for every open set $O \subset W^{\ell,p}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M)$, the set

$$\{x \in \Omega' : W(x) \cap O \neq \emptyset\}$$

is measurable as well. It follows that the multifunction $x \mapsto W(x)$ admits a measurable selection, which we denote by $w(x,\,\cdot\,)$.

Summing up, we recovered a measurable function w from Ω' to $W^{\ell,p}_{\mathrm{per}}(\mathbb{R}^d;\mathbb{R}^M)$ with the property that $u = \mathscr{B}_y w$ for almost every $x \in \Omega'$ and $y \in \mathbb{R}^d$. We conclude that

$$\int_Q u(x,y)\,\mathrm{d}y = \sum_{|i|=\ell} B^{(i)} \int_Q \partial_{y^i} w(x,y)\,\mathrm{d}y = 0,$$

where the latter inequality follows from the periodicity of w in its second variable. □

We devote the remainder of the section to the proof of Theorem 2.10 and to the pertaining tools. We first highlight that B. Raiţă [69] has recently pointed out that the kernel of the symbol of *any* constant rank operator coincides with the image of the symbol of a suitable operator $\mathscr{B}$, and that the latter is actually a potential for $\mathscr{A}$ when tools from Fourier analysis are applicable. Precisely, we have:

Proposition 7.4 ([69, Theorem 1 and Lemma 2]). *Let $\mathscr{A}$ be a linear homogeneous differential operator on $\mathbb{R}^d$ with constant coefficients. Then $\mathscr{A}$ is of constant rank if and only if there exists a linear homogeneous differential operator $\mathscr{B}$ on $\mathbb{R}^d$ with constant coefficients such that*

$$\ker \mathbb{A}[\omega] = \operatorname{im} \mathbb{B}[\omega] \quad \text{for all } \omega \in \mathbb{R}^d \setminus \{0\}, \tag{7.1}$$

where $\mathbb{A}$ and $\mathbb{B}$ are the symbols of $\mathscr{A}$ and $\mathscr{B}$, respectively.

Moreover, suppose that $\mathscr{A}$ and $\mathscr{B}$ are, respectively, operators from $\mathbb{R}^N$ to $\mathbb{R}^M$ and from $\mathbb{R}^M$ to $\mathbb{R}^N$. If (7.1) *holds, then, for all $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ such that $\mathscr{A}u = 0$, there exists $w \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^M)$ such that $u = \mathscr{B}w$.*

Observe that if the operator $\mathscr{B}$ satisfies (7.1), then it is in turn of constant rank; precisely, $\operatorname{rank}\mathbb{B}[\omega] = N - r$ if $\operatorname{rank}\mathbb{A}[\omega] = r$ for all $\omega \in \mathbb{R}^d \setminus \{0\}$. Proposition 7.4 builds a unified framework that encompasses some well-studied special cases, such as the one of the curl and of the divergence. Results in the same spirit had been previously obtained by J. Van Schaftingen [77, Proposition 4.2] for elliptic operators, which correspond to the subclass of operators whose symbol is injective.

A second relevant finding related to constant rank operators was obtained by A. Guerra and B. Raiţă [52], who showed that this class is exactly the one in which a sort of Korn inequality holds. To state their result, we need to introduce the *projection* $\Pi_{\mathscr{A}}$ associated with the operator $\mathscr{A}$. For $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$, it is defined as

$$\Pi_{\mathscr{A}} u(x) := (\mathcal{F}^{-1}(\mathbb{P}_{\mathscr{A}}\mathcal{F}u))(x), \tag{7.2}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse (see (3.4)), and the map

$$\mathbb{P}_{\mathscr{A}} : \mathbb{R}^d \setminus \{0\} \to \mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^N)$$

associates to each $\omega \in \mathbb{R}^d \setminus \{0\}$ the orthogonal projection operator onto $\ker \mathbb{A}[\omega]$. Here and in the rest of the paper, $\mathrm{Lin}(V;W)$ is the set of linear operators from the vector space V to the vector space W. We have the following proposition.

Proposition 7.5 (Korn-type inequality for constant rank operators [52]). *Let $p \in (1, \infty)$ and let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients. Then $\mathscr{A}$ is of constant rank if and only if there exists $c := c(d, p)$ such that*

$$\|\nabla^k(\phi - \Pi_{\mathscr{A}}\phi)\|_{L^p(\mathbb{R}^d;\mathbb{R}^{N\times d^k})} \leq c\|\mathscr{A}\phi\|_{L^p(\mathbb{R}^d;\mathbb{R}^M)} \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^N).$$

For future use, we remark that the conclusion remains valid when $\phi \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$, as the Fourier analysis on which the proof is based is still viable in this case, or even when $\phi \in W^{k,p}(\mathbb{R}^d;\mathbb{R}^N)$, by approximation. The sufficiency of the constant rank condition had been already observed in the literature, see for instance the bibliography in [52] or the paper by D. Gustafson [54], where a proof for first order operators is given.

To the purpose of constructing Sobolev potentials for $\mathscr{A}$-free fields, we first need to extend the projection operator $\Pi_{\mathscr{A}}$ in (7.2) to nonsmooth maps. We invoke the following result by T. Kato [56], which we present in a form due to F. Murat:

Lemma 7.6 ([64, Lemma 3.7]). *Let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank. Then the operator $\Pi_{\mathscr{A}}$ in (7.2) can be extended to a bounded linear operator from $L^p(\mathbb{R}^d;\mathbb{R}^N)$ to $L^p(\mathbb{R}^d;\mathbb{R}^N)$.*

In the proof of the lemma the constant rank assumption is fundamental. Indeed, it ensures that the map $\omega \mapsto \mathbb{P}_{\mathscr{A}}[\omega]$ introduced above is analytic on $\mathbb{R}^d \setminus \{0\}$ (see [64]); this, in combination with the 0-homogeneity of $\mathbb{P}_{\mathscr{A}}$, allows the use of S. G. Mikhlin's multipliers theorem (cf. Corollary 3.10).

Remark 7.7 (Self-adjointness of the projection operator). As a consequence of Parseval's formula and of the self-adjointness of $\mathbb{P}_{\mathscr{A}}$, by density there holds

$$\int_{\mathbb{R}^d} \Pi_{\mathscr{A}}u \cdot v \,dx = \int_{\mathbb{R}^d} u \cdot \Pi_{\mathscr{A}}v \,dx \quad \text{for all } u, v \in L^p(\mathbb{R}^d;\mathbb{R}^N). \tag{7.3}$$

We next clarify why $\Pi_{\mathscr{A}}$ is entitled to be named projection: Lemma 7.9 below shows that for an L^p-function u one has that $\mathscr{A}(\Pi_{\mathscr{A}}u) = 0$, and $\Pi_{\mathscr{A}}u = u$ if $\mathscr{A}u = 0$. We premise the instrumental notion of *Moore–Penrose generalized inverse* $\Lambda^\dagger$ of a linear map Λ.

Lemma 7.8 (Properties of the generalized inverse [23, 52]). *Given $\Lambda \in \mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^M)$, we let $\Lambda^\dagger \in \mathrm{Lin}(\mathbb{R}^M;\mathbb{R}^N)$ be defined as*

$$\Lambda^\dagger := (\Lambda\lfloor_{(\ker\Lambda)^\perp})^{-1} \circ \mathbb{P}_{\mathrm{im}\,\Lambda},$$

where $(\ker\Lambda)^\perp = \mathrm{im}\,\Lambda^*$ is the orthogonal complement of the kernel of Λ and $\mathbb{P}_{\mathrm{im}\,\Lambda}$ is the orthogonal projection on the image of Λ. Then the following hold:
1. *$\Lambda^\dagger$ is the unique element in $\mathrm{Lin}(\mathbb{R}^M;\mathbb{R}^N)$ such that $\Lambda^\dagger \circ \Lambda = \mathbb{P}_{\mathrm{im}\,\Lambda^*}$ and $\Lambda \circ \Lambda^\dagger = \mathbb{P}_{\mathrm{im}\,\Lambda}$.*
2. *Let $\mathbb{A} : O \to \mathrm{Lin}(\mathbb{R}^N;\mathbb{R}^M)$ be a smooth map on the open set $O \subset \mathbb{R}^d$. If $\mathrm{rank}\,\mathbb{A}[\omega]$ is constant for all $\omega \in O$, then the map $O \ni \omega \mapsto (\mathbb{A}[\omega])^\dagger$ is locally bounded and smooth.*
3. *If $\mathscr{A}$ is a linear, k-th order, homogeneous differential operator with constant coefficients, then the map $\omega \mapsto (\mathbb{A}[\omega])^\dagger$ is $(-k)$-homogeneous.*

We can now characterize the properties of the projection $\Pi_{\mathscr{A}}$. Note that they are comparable to the ones presented by I. Fonseca and S. Müller [48, Lemma 2.14] for the periodic setting. We also refer to [60, Section 2.8] for an alternative projection operator on the unit torus for which no null-average conditions are imposed.

Lemma 7.9. *Let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank. For every $u \in L^p(\mathbb{R}^d;\mathbb{R}^N)$, there holds:*
1. *For all $\psi \in C_c^1(\mathbb{R}^d;\mathbb{R}^M)$, we have $\Pi_{\mathscr{A}}(\mathscr{A}^*\psi) = 0$ and, as a consequence, $\mathscr{A}(\Pi_{\mathscr{A}}u) = 0$.*
2. *If $h = 0$ and $u \in L^p(\mathbb{R}^d;\mathbb{R}^N)$, or if $h = 1, \dots, k-1$ and $u \in W^{h,p}(\mathbb{R}^d;\mathbb{R}^N)$, there exists a positive constant $c := c(p, h)$ such that*

$$\|\nabla^h(u - \Pi_{\mathscr{A}}u)\|_{L^p(\mathbb{R}^d;\mathbb{R}^{N\times d^h})} \leq c\|\mathscr{A}u\|_{W^{-(k-h),p}(\mathbb{R}^d;\mathbb{R}^N)}. \tag{7.4}$$

In particular, when $\mathscr{A}u = 0$ in $\mathbb{R}^d$, then $\Pi_{\mathscr{A}}u = u$ a.e.

Proof. When $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$, the facts that $\mathscr{A}(\Pi_{\mathscr{A}} u) = 0$ and $\Pi_{\mathscr{A}} u = u$ if $\mathscr{A} u = 0$ are simple consequences of (7.2). To extend these properties to the case of a nonsmooth u, we regard $\psi \in C^1_c(\Omega;\mathbb{R}^M)$ as a Schwartz function on the whole space and we find

$$\Pi_{\mathscr{A}}(\mathscr{A}^*\psi) = \mathcal{F}^{-1}(\mathbb{P}_{\mathscr{A}}(\mathbb{A}^*\mathcal{F}\psi)) = 0,$$

because the image of $\mathbb{A}^*[\omega]$ is orthogonal to $\ker \mathbb{A}[\omega]$ for all $\omega \in \mathbb{R}^d$. It follows by (7.3) that

$$\int_{\mathbb{R}^d} \Pi_{\mathscr{A}} u \cdot \mathscr{A}^*\psi \, dx = \int_{\mathbb{R}^d} u \cdot \Pi_{\mathscr{A}} \mathscr{A}^*\psi \, dx = 0$$

and statement (1) holds.

We now turn to point (2). We argue for $h = 0$ as the case $h > 0$ is analogous. We consider at first $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ and we compute the Fourier transform of $u - \Pi_{\mathscr{A}} u$. Setting $\hat{u} = \mathcal{F}u$, we find

$$\hat{u}[\omega] - \mathbb{P}_{\mathscr{A}}[\omega]\hat{u}[\omega] = \mathbb{P}_{\operatorname{im}\mathbb{A}[\omega]^*}\hat{u}[\omega] = \mathbb{A}[\omega]^\dagger(\mathbb{A}[\omega]\hat{u}[\omega]) \quad \text{for all } \omega \in \mathbb{R}^d,$$

where in the latter inequality we exploited statement (1) in Lemma 7.8.

Since $\mathbb{A}$ is k-homogeneous and $\mathbb{A}^\dagger$ is $(-k)$-homogeneous, we infer by Corollary 3.10 that $\mathbb{A}^\dagger\mathbb{A}$ is an L^p-Fourier multiplier. Therefore, the operator T defined for $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ as $\mathsf{T}u := \mathcal{F}^{-1}(\hat{u} - \mathbb{P}_{\mathscr{A}}\hat{u})$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d;\mathbb{R}^N)$ to $L^p(\mathbb{R}^d;\mathbb{R}^N)$, which we still denote by T.

For every $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ we write

$$\begin{aligned}\mathcal{F}(\mathsf{T}u)[\omega] &= \mathbb{A}^\dagger[\omega](\mathcal{F}(\mathscr{A}u)[\omega]) \\ &= (1+|\omega|^2)^{\frac{k}{2}}\mathbb{A}^\dagger[\omega]((1+|\omega|^2)^{-\frac{k}{2}}\mathcal{F}(\mathscr{A}u)[\omega]),\end{aligned}$$

whence, by the $(-k)$-homogeneity of $\mathscr{A}^\dagger$, Mikhlin's multipliers theorem yields

$$\|\mathsf{T}u\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|(I-\Delta)^{-\frac{k}{2}}(\mathscr{A}u)\|_{L^p(\Omega;\mathbb{R}^N)}.$$

From the definition of T and the characterization of $W^{-k,p}$ recalled in Section 3.3, inequality (7.4) follows for $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$. The general assertion is then obtained by density. ☐

Eventually, we are able to establish the existence of potentials in a nonsmooth setting and prove the first main result of this section. We will use the following variant of Poincaré–Wirtinger's inequality, which can be derived as a corollary of Rellich–Kondrachov's theorem.

Lemma 7.10. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open set with Lipschitz boundary. There exists a constant $c > 0$ such that for every $u \in W^{\ell,p}(\Omega;\mathbb{R}^N)$,*

$$\|u - \Pi_{\nabla^\ell} u\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|\nabla^\ell u\|_{L^p(\Omega;\mathbb{R}^{N\times d^\ell})}, \tag{7.5}$$

where Π_{∇^ℓ} is the projection on the kernel of the operator ∇^ℓ.

Proof of Theorem 2.10. Let us fix $u \in L^p(\Omega;\mathbb{R}^N)$ satisfying $\mathscr{A}u = 0$ in $W^{-k,p}(\Omega;\mathbb{R}^M)$. Since we postulate that Ω is an $\mathscr{A}$-extension domain, there exists an operator $\mathsf{E}_{\mathscr{A}}: L^p(\Omega;\mathbb{R}^N) \to L^p(\mathbb{R}^d;\mathbb{R}^N)$ as in Definition 7.1. In particular, if we let $\tilde{u} := \mathsf{E}_{\mathscr{A}}u$, then $\tilde{u}$ is $\mathscr{A}$-free on $\mathbb{R}^d$.

As a first step, we approximate $\tilde{u}$ by maps in the image of $\mathscr{B}$. By the definition of the projection $\Pi_{\mathscr{A}}$, there exist a sequence $\{u_k\} \subset \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ such that

$$u_k \to \tilde{u} \quad \text{and} \quad \tilde{u}_k := \Pi_{\mathscr{A}} u_k \to \Pi_{\mathscr{A}}\tilde{u} \quad \text{strongly in } L^p(\mathbb{R}^d;\mathbb{R}^N).$$

By construction, the functions $\tilde{u}_k$ belong to $\mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ and, in view of Proposition 7.4, we recover a sequence $\{w_k\} \subset \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ such that $\tilde{u}_k = \mathscr{B}w_k$. Therefore, recalling that $\Pi_{\mathscr{A}}\tilde{u} = \tilde{u}$ because $\tilde{u}$ is $\mathscr{A}$-free, we deduce

$$\mathscr{B}w_k \to \tilde{u} \quad \text{strongly in } L^p(\mathbb{R}^d;\mathbb{R}^N). \tag{7.6}$$

Next, we proceed by applying Proposition 7.5: since $\mathscr{B}$ has constant rank,

$$\|\nabla^\ell(w_k - \Pi_{\mathscr{B}}w_k)\|_{L^p(\mathbb{R}^d;\mathbb{R}^M)} \le c\|\tilde{u}_k\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)},$$

where ℓ is the order of $\mathscr{B}$. Note that the right-hand side is uniformly bounded in k because $\{\tilde{u}_k\}$ is convergent in $L^p(\Omega;\mathbb{R}^N)$ by (7.6). As a consequence, inequality (7.5) yields a function $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ such that (up to subsequences)

$$\tilde{w}_k := w_k - \Pi_{\mathscr{B}} w_k - \Pi_{\nabla^\ell}(w_k - \Pi_{\mathscr{B}} w_k) \to w \quad \text{strongly in } L^p(\Omega;\mathbb{R}^M).$$

Since the functions w_k are smooth, the equality $\mathscr{B}w_k = \mathscr{B}\tilde{w}_k$ holds pointwise and we deduce from (7.6) that, for all $\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$,

$$\int_\Omega u(y)\cdot\phi(y)\,\mathrm{d}y = \lim_{k\to+\infty}\int_\Omega \tilde{w}_k \cdot \mathscr{B}^*\phi(y)\,\mathrm{d}y = \int_\Omega w(y)\cdot\mathscr{B}^*\phi(y)\,\mathrm{d}y,$$

where $\mathscr{B}^*$ is the adjoint of $\mathscr{B}$. The conclusion is then achieved. □

Remark 7.11. Let $u \in L^p(\Omega;\mathbb{R}^N)$ be an $\mathscr{A}$-free map and let $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ be a potential for u in the sense of Theorem 2.10. Then there exist constants $c_0, c_1 > 0$ such that

$$\frac{1}{c_0}\|u\|_{L^p(\Omega;\mathbb{R}^N)} \le \|w\|_{W^{\ell,p}(\Omega;\mathbb{R}^N)} \le c_1\|u\|_{L^p(\Omega;\mathbb{R}^N)}. \tag{7.7}$$

The first inequality easily follows from the identity $u = \mathscr{B}w$. Conversely, keeping in force the notations in the proof of Theorem 2.10, we have

$$\|\tilde{w}_k\|_{W^{\ell,p}(\Omega;\mathbb{R}^M)} \le c_1\|\tilde{u}_k\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)},$$

for a suitable $c_1 > 0$, whence, by taking the limit $k \to +\infty$ and recalling requirement (2) in Definition 7.1, the second estimate in (7.7) is achieved.

7.2 $\mathscr{A}$-free extensions

The Γ-convergence analysis that we developed in Sections 4–6 is grounded on the splitting argument contained in Lemma 4.2, which in turn requires Assumption 2. The latter is designed to tackle the periodically perforated structure featured by our problem. As a preliminary step towards Theorem 2.11, we tackle the simpler scenario in which the parameter ε is neglected. We establish the following:

Theorem 7.12 (Existence of $\mathscr{A}$-free extensions). *Let $D, O \subset \mathbb{R}^d$ be open sets such that D is connected, O is bounded, and $\partial D \cap \bar{O}$ is a Lipschitz boundary. Let also $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank, and let $\mathscr{B}$ be a linear, ℓ-th order, homogeneous differential operator with constant coefficients such that* (7.1) *holds. We further assume that*

- *for all $\mathscr{A}$-free $u \in L^p(D;\mathbb{R}^N)$ there exists $w \in W^{\ell,p}(D;\mathbb{R}^M)$ satisfying $u = \mathscr{B}w$,*
- *there exist a projection operator on the subspace of $\mathscr{B}$-free maps $\Pi_{\mathscr{B}}\colon W^{\ell,p}(D;\mathbb{R}^M) \to W^{\ell,p}(D;\mathbb{R}^M)$ and a constant $c > 0$ such that*

$$\|\nabla^\ell(w - \Pi_{\mathscr{B}}w)\|_{L^p(D;\mathbb{R}^{N\times d^\ell})} \le c\|\mathscr{B}w\|_{L^p(D;\mathbb{R}^M)} \quad \text{for all } w \in W^{\ell,p}(D;\mathbb{R}^N). \tag{7.8}$$

Then there exist a map

$$\mathsf{E}_{\mathscr{A}}\colon L^p(D;\mathbb{R}^N) \to L^p(O;\mathbb{R}^N)$$

and a constant $c := c(d,p,\mathscr{A},D,O)$ such that, for all $u \in L^p(D;\mathbb{R}^N)$ with $\mathscr{A}u = 0$ in $W^{-k,p}(D\cap O;\mathbb{R}^N)$,

(1) *$\mathsf{E}_{\mathscr{A}}u = u$ a.e. in $D \cap O$,*
(2) *$\|\mathsf{E}_{\mathscr{A}}u\|_{L^p(O;\mathbb{R}^N)} \le c\|u\|_{L^p(D;\mathbb{R}^N)}$, and*
(3) *$\mathscr{A}(\mathsf{E}_{\mathscr{A}}u) = 0$ on O.*

Here, the projection $\Pi_{\mathscr{B}}$ on the kernel of $\mathscr{B}$ has to be understood as an analogue of the one in (7.2) and Lemma 7.6. The main difference is that $\Pi_{\mathscr{B}}$ acts on functions defined on a domain, and not on the whole space.

Remark 7.13 (On Korn-type inequalities). After the first version of this manuscript was completed, A. Arroyo-Rabasa proved in the preprint [6] that inequalities of the form (2.13) (that is, (7.8) for every open bounded D)

hold whenever $\mathscr{B}$ meets a suitable maximal rank requirement, which entails, in particular, that $\mathscr{B}$ has constant rank. Therefore, if $\mathscr{A}$ admits a potential $\mathscr{B}$ in such class, the assumptions in Theorem 7.12 are satisfied. An example of maximal rank operator is the divergence, see Example 7.20 below.

We will comment on the relationships between the previous theorem and the theory developed above at the end of this section, see Remark 7.16. For the moment being, let us just highlight that the hypothesis concerning existence of potentials for $\mathscr{A}$-free fields on D enables us to recast the problem in terms of extension of Sobolev maps. For the latter, an adaptation of well-established arguments yields the following:

Lemma 7.14 (cf. [1, Lemma 2.6]). *Let $D, O \subset \mathbb{R}^d$ be open sets. If D is connected, O is bounded, and $\partial D \cap \bar{O}$ is a Lipschitz boundary, there exist a bounded linear map*

$$\mathsf{E}\colon W^{\ell,p}(D;\mathbb{R}^N) \to W^{\ell,p}(O;\mathbb{R}^N)$$

and a constant $c := c(d,p,D,O)$ such that
(1) $\mathsf{E}u = u$ *a.e. in* $D \cap O$,
(2) $\|\mathsf{E}u\|_{L^p(O;\mathbb{R}^N)} \le c\|u\|_{L^p(D;\mathbb{R}^N)}$, *and*
(3) $\|\nabla^\ell(\mathsf{E}u)\|_{L^p(O;\mathbb{R}^{N\times d^\ell})} \le c\|\nabla^\ell u\|_{L^p(D;\mathbb{R}^{N\times d^\ell})}$.

Proof. The proof follows the same lines of [1, Lemma 2.6]. Note that, in order to recover item (3) when $\ell > 1$, Poincaré's inequality has to be replaced with (7.5). □

We are now in a position to prove the first main result of this section.

Proof of Theorem 7.12. Let us fix $u \in L^p(D;\mathbb{R}^N)$ such that $\mathscr{A}u = 0$. The current assumptions grant that there is a Sobolev potential $w \in W^{\ell,p}(D;\mathbb{R}^M)$ for u, i.e., $u = \mathscr{B}w$, and that it satisfies

$$\|\nabla^\ell(w - \Pi_{\mathscr{B}}w)\|_{L^p(D;\mathbb{R}^{M\times d^\ell})} \le c\|u\|_{L^p(D;\mathbb{R}^N)}. \tag{7.9}$$

If E is the extension operator in Lemma 7.14, we set

$$\mathsf{E}_{\mathscr{A}}u := \mathscr{B}(\mathsf{E}(w - \Pi_{\mathscr{B}}w)).$$

By construction, then, $\mathsf{E}_{\mathscr{A}}u = u$ almost everywhere in $D \cap O$. Additionally, $\mathscr{A}(\mathsf{E}_{\mathscr{A}}u) = 0$ on O by the definition of $\mathscr{A}$-free maps and (7.1).

To conclude, we need to show that $\|\mathsf{E}_{\mathscr{A}}u\|_{L^p(O;\mathbb{R}^N)} \le c\|u\|_{L^p(D;\mathbb{R}^N)}$. By the definition of $\mathsf{E}_{\mathscr{A}}u$, we have

$$\|\mathsf{E}_{\mathscr{A}}u\|_{L^p(O;\mathbb{R}^N)} = \|\mathscr{B}(\mathsf{E}(w - \Pi_{\mathscr{B}}w))\|_{L^p(O;\mathbb{R}^N)} \le c\|\nabla^\ell(\mathsf{E}(w - \Pi_{\mathscr{B}}w))\|_{L^p(O;\mathbb{R}^{M\times d^\ell})},$$

c being a constant depending on $\mathscr{B}$ (and hence on $\mathscr{A}$). Thanks to Lemma 7.14, we obtain a bound in terms of the potential of u:

$$\|\mathsf{E}_{\mathscr{A}}u\|_{L^p(O;\mathbb{R}^N)} \le c\|\nabla^\ell(w - \Pi_{\mathscr{B}}w)\|_{L^p(D;\mathbb{R}^{M\times d^\ell})}$$

for some $c := c(d,p,\mathscr{A},D,O)$. The conclusion is achieved by combining the above inequality with (7.9). □

Arguing as in the proof of Theorem 7.12, we ground the study about extensions from perforated domains on the corresponding result for Sobolev functions. When the perforations are detached from the boundary, the following holds:

Proposition 7.15 (cf. [25, Lemma 8] and references therein). *Let $\Omega_{1,\varepsilon}$ be as in position* (1.2). *Then there exist a positive constant $c := c(d,p,D)$ independent of ε and Ω, as well as a sequence of operators $\{\mathsf{E}^\varepsilon\}$, with $\mathsf{E}^\varepsilon\colon W^{\ell,p}(\Omega_{1,\varepsilon};\mathbb{R}^N) \to W^{\ell,p}(\Omega;\mathbb{R}^N)$, such that*
(1) $\mathsf{E}^\varepsilon u = u$ *a.e. in* $\Omega_{1,\varepsilon}$,
(2) $\|\mathsf{E}^\varepsilon u\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)}$,
(3) $\|\nabla^\ell(\mathsf{E}^\varepsilon u)\|_{L^p(\Omega;\mathbb{R}^{N\times d^\ell})} \le c\|\nabla^\ell u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^{N\times d^\ell})}$, *and*
(4) *if $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^N)$ is bounded and $\{\nabla^\ell u_\varepsilon\}$ is p-equiintegrable, then $\{\nabla^\ell(\mathsf{E}^\varepsilon u_\varepsilon)\}$ is p-equiintegrable as well.*

The proposition may be proved by adapting the strategy in the seminal work by E. Acerbi, V. Chiadò Piat, G. Dal Maso and D. Percivale [1]. Their result addresses only the case $\ell = 1$, and the analogue of (4) is not mentioned; we omit nonetheless the proof in the case $\ell > 1$, which is a natural adaptation of [1, proof of

Theorem 2.1]. As for point (4), it is a mere consequence of the construction of E^ε: it suffices to note that reflections, dilations, and patching by partitions of unity preserve p-equiintegrability.

Thanks to the previous result, we obtain that Assumption 2 is fulfilled by an operator $\mathscr{A}$ as soon as it admits a potential for which a Korn-type inequality holds. Note that it is fundamental that we start from fields u which are $\mathscr{A}$-free in the whole set Ω: in principle, the existence of a potential would be false if we worked with maps that are $\mathscr{A}$-free just on the perforated set $\Omega_{1,\varepsilon}$, cf. Remark 7.2.

Proof of Theorem 2.11. Let $u \in L^p(\Omega;\mathbb{R}^N)$ be an $\mathscr{A}$-free map and let $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ be its potential. In the same spirit of Theorem 7.12, we set

$$\mathsf{E}^\varepsilon_{\mathscr{A}} u := \mathscr{B}(\mathsf{E}^\varepsilon(w - \Pi_{\mathscr{B}} w)),$$

where E^ε is the extension operator in Proposition 7.15. As in the proof of Theorem 7.12, it easy to check that (1) and (3) in Assumption 2 hold. We now turn to item (2), that is,

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)} \quad \text{for all } \mathscr{A}\text{-free } u \in L^p(\Omega;\mathbb{R}^N).$$

By the definition of $\Omega_{0,\varepsilon}$ in (1.1), there exists an open set $\Omega' \subset \Omega$ with Lipschitz boundary such that $\Omega_{0,\varepsilon} \subset \Omega'$ and that $\delta := \operatorname{dist}(\partial\Omega', \partial\Omega) > 0$. In particular, recalling (3.1), $\Omega' \subset \hat{\Omega}_\varepsilon$ if $\sqrt{d}\varepsilon < \delta$, and $\{\hat{\Omega}_\varepsilon, \Omega \setminus \bar{\Omega}'\}$ is an open cover of Ω. We observe that

$$\mathsf{E}^\varepsilon w = w \quad \text{a.e. in } \Omega \setminus \bar{\Omega}' \subset \Omega_{1,\varepsilon},$$

whence, by the definition on $\mathsf{E}^\varepsilon_{\mathscr{A}}$,

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\Omega\setminus\Omega';\mathbb{R}^N)} = \|u\|_{L^p(\Omega\setminus\Omega';\mathbb{R}^N)} \le \|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)} \tag{7.10}$$

For what concerns the contribution on $\hat{\Omega}_\varepsilon$, if we let

$$\hat{\Omega}_{1,\varepsilon} := \bigcup_{z\in\hat{Z}_\varepsilon} \varepsilon(D_1 + z) \quad \text{with } \hat{Z}_\varepsilon := \{z \in \mathbb{Z}^d : \varepsilon(Q + z) \subset \Omega\},$$

we have

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\hat{\Omega}_\varepsilon;\mathbb{R}^N)} \le c\|\nabla^\ell(\mathsf{E}^\varepsilon(w - \Pi_{\mathscr{B}} w))\|_{L^p(\hat{\Omega}_\varepsilon;\mathbb{R}^{M\times d^\ell})} \le c\|\nabla^\ell(w - \Pi_{\mathscr{B}} w)\|_{L^p(\hat{\Omega}_{1,\varepsilon};\mathbb{R}^{M\times d^\ell})}.$$

The second inequality is a consequence of the construction of E^ε, which is obtained by patching together the extension operators from each "stiff" unit $\varepsilon(D_1 + z)$ to the cell $\varepsilon(Q + z)$. Thanks to the definition of $\hat{\Omega}_{1,\varepsilon}$ we can further bound the last quantity from above by invoking Korn's inequality

$$\|\nabla^\ell(w - \Pi_{\mathscr{B}} w)\|_{L^p(D_1;\mathbb{R}^{M\times d^\ell})} \le c\|\mathscr{B} w\|_{L^p(D_1;\mathbb{R}^N)},$$

which holds by assumption. We obtain

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\hat{\Omega}_\varepsilon;\mathbb{R}^N)} \le c\|u\|_{L^p(\hat{\Omega}_{1,\varepsilon};\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)}.$$

On the whole, the last estimate and (7.10) yield

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\Omega;\mathbb{R}^N)} \le \|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\hat{\Omega}_\varepsilon;\mathbb{R}^N)} + \|\mathsf{E}^\varepsilon_{\mathscr{A}} u\|_{L^p(\Omega\setminus\Omega';\mathbb{R}^N)} \le c\|u\|_{L^p(\Omega_{1,\varepsilon};\mathbb{R}^N)}.$$

Eventually, we turn to (4). For any $\varepsilon > 0$, let w_ε be the potential of u_ε. We fix $\eta > 0$ arbitrarily and we let $E \subset \Omega$ be a Lebesgue measurable set. As a first step, we prove that there exists $m > 0$ such that $\mathscr{L}^d(E) < m$ implies

$$\|\nabla^\ell(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon)\|^p_{L^p(E;\mathbb{R}^{M\times d^\ell})} < \eta.$$

Since $\{u_\varepsilon\}$ is p-equiintegrable, there exists $\tilde{m} > 0$ with the property that $\|u_\varepsilon\|_{L^p(F;\mathbb{R}^N)} < \eta$ whenever $\mathscr{L}^d(F) < 2\tilde{m}$. Thanks to the outer regularity of the Lebesgue measure, we can select a finite union of open hyperrectangles $U \supset E$ such that $\mathscr{L}^d(U) < \mathscr{L}^d(E) + \tilde{m}$. Thus, if we set $m := \tilde{m}$ and we assume that $\mathscr{L}^d(E) < m$, thanks to (2.13) we deduce

$$\|\nabla^\ell(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon)\|^p_{L^p(E;\mathbb{R}^{M\times d^\ell})} \le \|\nabla^\ell(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon)\|^p_{L^p(U;\mathbb{R}^{M\times d^\ell})} \le c\|u_\varepsilon\|^p_{L^p(U;\mathbb{R}^N)} < c\eta,$$

where the last inequality is a consequence of the p-equiintegrability of $\{u_\varepsilon\}$.

In conclusion, we infer the p-equiintegrability of $\{\mathsf{E}^\varepsilon_{\mathscr{A}} u_\varepsilon\}$ from the one of $\{\nabla^\ell(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon)\}$. According to the definition of $\mathsf{E}^\varepsilon_{\mathscr{A}} u$, we have

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u_\varepsilon\|_{L^p(E;\mathbb{R}^N)} = \|\mathscr{B}(\mathsf{E}^\varepsilon(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon))\|_{L^p(E;\mathbb{R}^N)} \leq c\|\nabla^\ell(\mathsf{E}^\varepsilon(w_\varepsilon - \Pi_{\mathscr{B}} w_\varepsilon))\|_{L^p(E;\mathbb{R}^{M\times d^\ell})},$$

whence, owing to point (4) in Proposition 7.15,

$$\|\mathsf{E}^\varepsilon_{\mathscr{A}} u_\varepsilon\|_{L^p(E;\mathbb{R}^N)} < \eta$$

if $\mathscr{L}^d(E)$ is sufficiently small, as desired. □

In light of our analysis, existence of potentials and of extension operators turn out to be almost equivalent; a key role in this respect is played by (7.8). We elaborate on this point in the next remark.

Remark 7.16 (Relations between existence of potentials and of extension operators). Here, we compare the main results of the current section.

We let the operator $\mathscr{A}$ be as above, notably we assume that it has constant rank. Thanks to Theorem 2.10, we know that if $D \subset \mathbb{R}^d$ is a bounded, connected, open set with Lipschitz boundary which is also an $\mathscr{A}$-extension domain in the sense of Definition 7.1, then $\mathscr{A}$-free maps on D admit potentials. In short, for the class of operators under consideration and for sufficiently "nice" open sets D, it holds

$$D \text{ is an } \mathscr{A}\text{-extension domain} \implies \text{existence of potentials for } \mathscr{A}\text{-free fields on } D. \tag{7.11}$$

Conversely, if $D \subset \mathbb{R}^d$ is an open bounded set with Lipschitz boundary such that all $\mathscr{A}$-free fields on D admit Sobolev potentials through the operator $\mathscr{B}$ and if for the latter (7.8) holds, then D is an $\mathscr{A}$-extension domain. Indeed, by a slight adaptation of the proof, a variant of Theorem 7.12 for the case $O = \mathbb{R}^d$ can be established. Schematically, again for sufficiently "nice" open sets D, we have

$$\left.\begin{array}{r}\text{existence of potentials for } \mathscr{A}\text{-free fields on } D\\ \text{Korn-type inequality for } \mathscr{B} \text{ on } D\end{array}\right\} \implies D \text{ is an } \mathscr{A}\text{-extension domain.}$$

All in all, we see that, given a constant rank operator $\mathscr{A}$ and a bounded, connected, open set with Lipschitz boundary D, the existence of Sobolev potentials for $\mathscr{A}$-free maps on D and the existence of an $\mathscr{A}$-free extension operator from D to $\mathbb{R}^d$ are nearly equivalent. More specifically, they would actually be equivalent as soon as we knew that the generalized Korn inequality (7.8) holds when D is a "nice" open set and $\mathscr{B}$ is a constant rank operator. In conclusion, we believe that investigations about the validity of (7.8) constitute a very interesting line of research.

We conclude with a parade of examples.

Example 7.17 (Curl). For the choice $\mathscr{A}$ = curl, we have (classically) $d = N = M = 3$ and

$$\operatorname{curl} u = \sum_{i=1}^{3} A^{(i)} \frac{\partial u}{\partial x_i},$$

with

$$A^{(1)} := \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0 \end{pmatrix}, \quad A^{(2)} := \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0 \end{pmatrix}, \quad A^{(3)} := \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

The symbol of curl is then

$$\mathbb{A}[\omega] = \begin{pmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

and rank $\mathbb{A}[\omega] = 2$ for all $\omega \in \mathbb{R}^d \setminus \{0\}$.

It is easy to check that Assumption 1 holds. Besides, when Ω is simply connected, in view of Proposition 7.15 all the conditions in Assumption 2 are fulfilled too.

Example 7.18 (Operators associated with higher-order gradients). Let Ω be simply connected. Then, for any $k \in \mathbb{N} \setminus \{0\}$, a constant rank differential operator $\mathscr{A}$ can be constructed such that $\mathscr{A}u = 0$ if and only if $u = \nabla^k w$ for a suitable w [48]. Then, similarly to the previous example, for such an operator $\mathscr{A}$ Assumptions 1 and 2 are consequences, respectively, of a simple check and of Proposition 7.15.

Example 7.19 (The curl curl operator). Let $d = M = N = 3$ and, for all $i, j = 1, 2, 3$, let $E^{(i,j)}$ be the matrix whose entries all 0, but for the one in position (i, j), which equals 1. We consider the operator $\mathscr{A} = \operatorname{curl}\operatorname{curl}$, defined as

$$\operatorname{curl}\operatorname{curl} u := \sum_{i,j=1}^{3} A^{(i,j)} \frac{\partial^2 u}{\partial x_j \partial x_i},$$

where

$$A^{(i,i)} = -E^{(i+1,i+1)} - E^{(i+2,i+2)}$$

(the indices are computed modulo 3) and

$$A^{(i,j)} = E^{(i,j)}.$$

The symbol of this operator is

$$\mathbb{A}[\omega] = \begin{pmatrix} -\omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_1^2 - \omega_3^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_3\omega_3 & -\omega_1^2 - \omega_2^2 \end{pmatrix}$$

and $\operatorname{rank} \mathbb{A}[\omega] = 2$ for all $\omega \in \mathbb{R}^d \setminus \{0\}$.

When $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with Lipschitz boundary, we have that $\mathscr{A}u = 0$ if and only if $u = \mathscr{B}w$ for a suitable potential w, where $\mathscr{B}$ is the symmetric gradient $\mathscr{B}w := \frac{1}{2}(\nabla w + (\nabla w)^{\mathrm{t}})$. As a corollary of a recent result by F. Cagnetti, A. Chambolle, M. Perugini and L. Scardia [19, Theorem 1.1], every such Ω is a curl curl-extension domain, whence Assumption 1 is satisfied. The classical Korn's inequality grants also that Assumption 2 holds.

Example 7.20 (Divergence). Let us choose $\mathscr{A} = \operatorname{div}$, div being the standard divergence operator on $\mathbb{R}^d$. Then $N = d$, $M = 1$, and

$$\operatorname{div} u = \sum_{i=1}^{d} e_i^{\mathrm{t}} \cdot \frac{\partial u}{\partial x_i},$$

where, for $i = 1, \ldots, d$, e_i is the i-th element of the canonical basis of $\mathbb{R}^d$ and e_i^{t} is its transpose. The symbol of div is

$$\mathbb{A}[\omega] = \sum_{i=1}^{d} \omega_i e_i^{\mathrm{t}},$$

thus $\operatorname{rank} \mathbb{A}[\omega] = 1$ for all $\omega \in \mathbb{R}^d \setminus \{0\}$. For what concerns Assumption 1 and the existence of extension operators, we resort to a result by T. Kato, M. Mitrea, G. Ponce and M. Taylor, which we present in a simplified setting:

Proposition 7.21 (Extensions and potentials for divergence-free vector fields [57]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected set with Lipschitz boundary. Then the following holds:*
(1) *Ω is a* div*-extension domain in the sense of Definition 7.1.*
(2) *For every $u \in L^p(\Omega; \mathbb{R}^d)$ such that* $\operatorname{div} u = 0$ *in Ω, there exists $w \in W^{1,p}(\Omega; \operatorname{Antisym}(d \times d))$ satisfying*

$$u = \operatorname{Div} w := \sum_{i,j} \frac{\partial w_{i,j}}{\partial x_j} e_i \quad \text{in } \Omega,$$

where $\operatorname{Antisym}(d \times d)$ *is the space of $d \times d$ antisymmetric matrices. Besides, w can be selected in such a way that the map $u \mapsto w$ from $L^p(\Omega; \mathbb{R}^d)$ to $W^{1,p}(\Omega; \mathbb{R}^d)$ is linear and bounded.*

It is interesting to notice that the authors of [57] derive item (2) from (1), that is, they prove an implication of the form (7.11).

It is proved in [6, Section 4.1] that Div satisfies (2.13). Hence, Theorem 7.12 holds for $\mathscr{A} = \operatorname{div}$ and Assumption 2 is fulfilled too.

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